4-1986

Microscopic Optical Potentials

Michael R. Sloothaak
Western Michigan University

Follow this and additional works at: http://scholarworks.wmich.edu/masters_theses

Part of the Nuclear Commons

Recommended Citation
http://scholarworks.wmich.edu/masters_theses/1445

This Masters Thesis-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Master's Theses by an authorized administrator of ScholarWorks at WMU. For more information, please contact maira.bundza@wmich.edu.
MICROSCOPIC OPTICAL POTENTIALS

by

Michael R. Sloothaak

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Master of Arts
Department of Physics

Western Michigan University
Kalamazoo, Michigan
April 1986
MICROSCOPIC OPTICAL POTENTIALS

Michael R. Sloothaak, M.A.
Western Michigan University, 1986

The Schroedinger equation for nucleon scattering may be reduced to

\[ E \chi(r) = \left\{ \left( -\frac{\hbar^2}{2m} \right) + U_{\text{direct}} + U_{\text{coulomb}} \right\} \chi(r) + U_{\text{exc}}(\chi(r)) \]

The direct and exchange potentials are functions of the two-body nucleon-nucleon potential. The two-body potential employed in this work is a density dependent potential derived from the Hamada-Johnston potential. The interaction is expressed as a sum of Yukawa potentials. By folding this interaction, complex potentials are produced. The results are then compared to experimental data. In general, the results are slightly below, and more diffractive than the experimental values.
INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.

2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.

3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of "sectioning" the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.

5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.
Sloothaak, Michael Robert

MICROSCOPIC OPTICAL POTENTIALS

Western Michigan University  M.A.  1985

University Microfilms International  300 N. Zeeb Road, Ann Arbor, MI 48106
PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark √.

1. Glossy photographs or pages
2. Colored illustrations, paper or print
3. Photographs with dark background
4. Illustrations are poor copy
5. Pages with black marks, not original copy
6. Print shows through as there is text on both sides of page
7. Indistinct, broken or small print on several pages
8. Print exceeds margin requirements
9. Tightly bound copy with print lost in spine
10. Computer printout pages with indistinct print
11. Page(s) missing when material received, and not available from school or author.
12. Page(s) seem to be missing in numbering only as text follows.
14. Curling and wrinkled pages
15. Dissertation contains pages with print at a slant, filmed as received
16. Other

______________________________________________________________
______________________________________________________________

University Microfilms International

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
# TABLE OF CONTENTS

**INTRODUCTION** ........................................   1

**THEORY** .................................................. 3

- The Exchange Potential .................................... 5
- The Slater Approximation .................................. 7
- The Local Energy Approximation ............................ 8
- The Effective Interaction .................................. 9

**COMPUTATIONS** ............................................. 11

**RESULTS** .................................................. 14

- Comparison of Potentials .................................. 14
- Comparison of Exchange Approximations .................. 14
- Comparison of Cross Sections ............................... 15

**CONCLUSIONS** ............................................... 25

**APPENDIX** ................................................ 28

- Spin-Orbit Derivation ...................................... 28

**BIBLIOGRAPHY** .............................................. 36
Attempts to explain elastic nucleon scattering data with a nucleon-nucleus interaction were made as early as 1935 (Hodgson, 1965). Different problems rendered many preliminary forms inadequate until the development of the optical model. The optical model, so called because it replaces the nucleus with a 'cloudy crystal ball' of complex refractive index, has been successful for all but the lightest nuclei in the energy region of 10 to 100 MeV. (At energies below 10 MeV resonances become more apparent, and at energies above 100 MeV relativistic effects become important.) The model assumes the main bulk of the nucleus is at constant density which is a good approximation for all but the lightest nuclei. Typical optical model calculations and results can be found in the article by Percy et al. (1976) listed in the bibliography.

The success of the optical model in reproducing data brought about only a limited understanding of the nuclear forces and structure. Fitting the data required adjustment of a set of potential well parameters. Thus only general conclusions about the nucleus can be made, and scattering cross sections could not be predicted before the experiment. These shortcomings show the need for a less phenomenological understanding of nuclear properties. Many efforts have been made to devise optical model potentials from a more fundamental nucleon-nucleon force. Such an approach assumes a nucleon-nu-
cleon interaction, uses it to develop a nuclear model, and reproduces
data with the qualitative properties of the experimental cross-sec-
tions. The cross sections can, therefore, be directly related to the
properties of the nucleon-nucleon force. The purpose of this work is
to investigate one such approach.

In the following chapters a theory is developed which produces
optical model potentials by folding a two-body interaction developed
from nuclear matter calculations (Von Geramb, 1983). Approximations
are made to reduce the more involved potential terms to a calculable
form. Computer codes are then used to evaluate the potential well
quantitatively, and the resulting scattering cross-sections are com-
pared to experiment.

The results, although not as good as the parametric fits, still
are encouraging. The calculated potentials exhibit a shape and an
energy dependence much like the phenomenological potentials. The
cross sections from the calculated potentials do not agree quite as
well with experimental data as do those from the parametric poten-
tials; the calculated cross sections consistantly falling below the
data, indicating a weakness in magnitude of the two-body interaction
used. However, the qualitative success of the procedures used in
this work certainly provides motivation for constructing additional
density dependent two-body interactions.
CHAPTER II
THEORY

Most physical systems can be represented by the eigenvalue equation

\[ H \psi = E \psi \]

where \( H \) is the hamiltonian operator for the system, \( \psi \) is the eigenfunction, and \( E \) is the eigenvalue. The nuclear hamiltonian can be expressed as

\[ H = \sum_i T_i + \sum_{i<j} V_{ij} \]

where \( T_i \) is the kinetic energy of particle \( i \), and \( V_{ij} \) is the potential between the \( i^{th} \) and \( j^{th} \) particles.

If one lets the projectile be particle number 0 and the bound nucleons be particles 1 through \( A \), then one can express \( H \) as

\[ T_0 + \sum_{i=1}^{A} V_{0i} + \sum_{i=1}^{A} T_i + \sum_{i<j}^{*} V_{ij} \]

where the summation over \( i<j \) is now understood to exclude the \( V_{0i} \) term. The wave function for \( A+1 \) particles may be written as \( \oplus \phi_n(x_0) \) where \( \phi_n(x_1,...,x_A) \) is the wave function for the nucleus, \( \phi_n(x_0) \) is the scattered wave, and \( \oplus \) is the antisymmetry operator. If the ground state of the nucleus is defined to be and to remain at zero energy, the last two terms of the target hamiltonian will produce zero when acting on the total wave function. Since the integration is over only \( r_1,...,r_A \), evaluation of this matrix element will become
a function of $r_0$. The matrix elements can be evaluated by writing $\phi_n$ and $\phi_n^{\sigma o}$ as Slater determinants. Using the properties of $@$, one can write the remaining potential term as

$$A < \phi_n | V_{0i} | \phi_n^{\sigma o} >.$$  

The potential is now only a function of $r_0$ and $r_1$, and orthonormality allows only two terms to result when the determinants are expanded:

$$\sum_j < \phi_j^{(1)} | V_{0i} | \phi_j^{(1)} \chi(0) > = U_{s\mu} (\Phi(r_0)),$$

the direct term, and

$$\sum_j < \phi_j^{(1)} | V_{0i} | \phi_j^{(0)} \chi(1) > = U_{s\mu} (\Phi(r_0)),$$

the exchange term.

Neglecting the spin-orbit interaction, one can express $V_{01}$ as

$$V_{01} = V_{\text{center}} + V_{\text{iso}} + V_{\text{spin}} \vec{r}_1 \cdot \vec{r}_2 + V_{\text{spin}} \vec{r}_1 \cdot \vec{r}_2.$$  

Each of these terms should be considered separately. Fortunately a sufficient number of nuclei exist which have zero spin to limit the examination to them. This allows the elimination of the $\sigma s \sigma o$ term. But nuclei with nonzero isospin will be studied, so we must consider the isospin term. The operator $\vec{r}_1 \cdot \vec{r}_2$ acts in the following way:

$$\vec{r}_1 \cdot \vec{r}_2 | np > = - | np > , \vec{r}_1 \cdot \vec{r}_2 | pp > = + | pp >$$  

so if one is scattering protons, the direct term of the isospin potential can be expressed as

$$\sum_p < \phi_p^{(1)} | V_{\text{iso}} | \phi_p^{(1)} \chi(0) > - \sum_n < \phi_n^{(1)} | V_{\text{iso}} | \phi_n^{(1)} \chi(0) >$$
where \( p \) and \( n \) are the indexes over protons and neutrons, respectively. This can be expressed as

\[
\mathcal{X}(r_0) \int \left( \rho_p(r_i) - \rho_n(r_i) \right) V_{i5o} \left( \vec{r}_i - \vec{r}_0 \right) \, dr_i
\]

where

\[
\rho_p(r) = \sum_{i=p} \phi_i^* (r) \phi_i (r).
\]

Throughout this work it is assumed \( Z_p = N_p \), where \( Z_p \) is found from nuclear scattering. In a similar way one can write the central potential term as

\[
\mathcal{X}(r_0) \int \rho(r_i) V_{i5n+\omega} \left( \vec{r}_i - \vec{r}_0 \right) \, dr_i
\]

where \( \rho(r_i) = \rho_n + \rho_p \). The forms of the central and isospin parts are so similar that only the central potential derivation will be continued in evaluating the exchange potential.

### The Exchange Potential

The exchange term

\[
\mathcal{V}(\mathcal{X}(r_0)) = \sum_j < \phi_j (i) | V_{01} | \phi_j (0) \mathcal{X}(i) >
\]

may be approximated by a more desirable local form using the exchange two body effective interaction \( U(r_0-r_i) \), which is \( V(r_0-r_i) \) with negative odd components, and the one body mixed density \( \rho(r_i, r_0) \). The mixed density term

\[
\rho(r_i, r_0) = \sum_j \phi_j^* (r_i) \phi_j (r_0)
\]
can be approximated by assuming the $\phi_j$'s are plane waves, as is the case in infinite nuclear matter. The sum over occupied states should also be replaced by an integral over the Fermi sphere. Then

$$p_{\text{mixed}} = \frac{1}{2\pi^3} \int_0^{k_f} \frac{d^3k}{(k^2)^{3/2}} \mathbf{2} \cdot (\mathbf{r}_0 - \mathbf{r}_i)$$

where $k_f$ is the Fermi momentum. If one defines $r = r_0 - r_i$ and expands the exponential in terms of spherical harmonics (Messiah, 1962), the mixed density can be expressed as

$$p_{\text{mixed}} = \frac{1}{2\pi^3} \int_0^{k_f} \sum_{\ell,m} Y_\ell^m(kr) Y_\ell^m(\hat{r}) k^2 \, d\Omega$$

To reduce this expression introduce

$$Y_0^0(k) = \frac{1}{\sqrt{4\pi}}.$$

The orthomormality of the $Y_{l,m}$'s reduces the sum to just the one term where $l=m=0$. Thus

$$p_{\text{mixed}} = \frac{1}{2\pi^3} \int_0^{k_f} \int_0^{k_f} j_0(kr) k^2 \, d\Omega$$

integrating over $\Omega$, one obtains

$$p_{\text{mixed}} = \left( \frac{2}{\pi^2} \right) \int_0^{k_f} j_0(kr) k^2 \, d\Omega$$

and integrating over $k$ results in

$$p_{\text{mixed}} = \left( \frac{2}{3\pi^2} \right) k_f^3$$

The constant density of nuclear matter,

$$\rho_{nm} = \left( \frac{2}{3\pi^2} \right) k_f^3$$

can be introduced giving

$$p_{\text{mixed}} = \rho_{nm} \left( \frac{3}{k_f r} \right) j_1(k_f r)$$
The exchange integral, given the results of the previous paragraph, can now be written as

\[ U_e \chi_L(\chi(r_0)) = \int \frac{3 \rho_{nm}}{k_f r} j_1(k_f r) U(r) \chi(\vec{r} + \vec{r}_0) \, dr, \]

Two different methods are used to approximate this integral, the Slater approximation and the local energy approximation.

The Slater Approximation

In the Slater approximation (Love, 1978) the term \( \chi(\vec{r} + \vec{r}_0) \) is assumed to be a plane wave,

\[ \chi(\vec{r} + \vec{r}_0) = e^{i \vec{k} \cdot \vec{r}} e^{i \vec{k} \cdot \vec{r}_0}. \]

The term involving \( r_0 \) can be factored out of the exchange integral giving

\[ U_e \chi_L(\chi(r_0)) = \int \frac{3}{k_f r} j_1(k_f r) U(r, \rho) \rho_{nm} e^{i \vec{k} \cdot \vec{r}} \, dr \chi(\vec{r}_0), \]

where \( \rho \) is an expression for density. Expanding \( \exp(i \vec{k} \cdot \vec{r}) \) into spherical harmonics as before and substituting in the nuclear density derived from electron scattering for \( \rho_{nm} \) one has

\[ U_e \chi_L(\chi(r_0)) = \int \frac{3}{k_f r} j_1(k_f r) U(r, \rho) \rho(r) j_0(k_f r) \, dr \chi(r_0). \]

A numerical evaluation of this integral gives the Slater approximation.
The Local Energy Approximation

In addition to the Slater approximation, one other method is used here, the local energy approximation. In this case we start with the exchange integral as first used in the previous method, and define (Love, 1978)

\[ f(r) = \frac{3f_{NM}}{k_r} j_1(k_r r) U(r, \rho). \]

The scattered wave is Fourier transformed to give

\[ \chi(r_0 + \bar{r}) = 2\pi^{-3/2} \int \chi(k) e^{i \bar{r} k} d\bar{k}. \]

Inserting this into \( U_{\text{exc}}(\chi(r_0)) \), one has

\[ U_{\text{exc}}(\chi(r_0)) = 2\pi^{-3/2} \int \chi(k) e^{i \bar{r} k} f(k) e^{i \bar{r} k} d\bar{k} d\bar{r}. \]

The order of integration can be reversed giving

\[ 2\pi^{-3/2} \int \chi(k) e^{i \bar{r} k} f(k) d\bar{k}. \]

If one assumes \( f(k) = f(k^2) \) and expands \( f(k^2) \) about some \( \lambda^2 \) in a truncated Taylor series,

\[ f(k^2) = f(\lambda^2) + f'(\lambda^2) (k^2 - \lambda^2). \]

The integral can now be evaluated and

\[ U_{\text{exc}}(\chi(r_0)) = \left\{ f(\lambda^2) + f'(\lambda^2) (\nabla^2 + \lambda^2) \right\} \chi(r_0). \]

The local energy approximation comes from choosing

\[ \lambda^2(r) = \frac{2z}{k_r^2} \left\{ E_{cM} - U(r) \right\} \]

thus making the second term approximately zero. If one makes the assumption that \( k_f \) is small, then

\[ \left( \frac{3}{k_f r} \right) \left\{ j_1(k_f r) \right\} \rightarrow 1, \]

and

\[ f(\lambda^2 r) = \rho_{NM} U(\lambda^2 r). \]
If one substitutes the nuclear density for the nuclear matter density, one has

\[ U_{exc}(\chi(r_0)) = \rho(r) U(\chi^2(r)) \chi(r_0). \]

This is equivalent to solving the exchange integral using \(f(\lambda^2(r))\delta(r)\) in place of \(U(r)\).

The Effective Interaction

The effective interaction \(V(r_1 - r_0)\) is expressed as a finite sum of Yukawa potentials. These terms were found in tabular form and developed using nuclear matter calculations (Von Geramb, 1979). The \(v_i\) will be complex, because evaluation of the nucleon self-energy in nuclear matter involves integration over a singularity. This will result in \(U_{total}(r)\) being complex.

In the theory of nuclear matter the effective interaction operator \(t(w)\) can be expressed using the Bethe-Goldstone equation as

\[ t(w) = v + v \Sigma^+ G^+(w) t(w), \]

where \(w\) is the total energy, \(v\) is the free nucleon-nucleon interaction, and \(G^+\) is the propagator. One would prefer a coordinate space expression for the effective interaction \(t(r)\) where

\[ t(r) \simeq \sum_{\sigma T} A^T_0 (r) \begin{pmatrix} P^S P^T + \sum_T A^T_1 (r) \bar{L} \cdot \bar{S} \end{pmatrix} P^T + \sum_T A^T_2 r^2 S_{12} P^T. \]

These three terms represent the central, spin-orbit, and tensor terms.
respectively. The coefficients $A_i$ can be found numerically, but to tabulate the effective interaction operator, an analytic form is desired. This is accomplished by Von Geramb (1979) through the use of Yukawa potentials. Here
\[ t(r,p) = \sum_{i=1}^{N} V_i(r) \exp \left( \frac{\mu_i r}{r} \right) \]
where the $\mu_i$ are chosen to cover the necessary range and the $V_i$ terms are found (as a function of density) by fitting them to the numerical solution involving the coefficients $A_i$. The values for $V_i$ and $\mu_i$ are then published for widespread use. (Von Geramb, 1979)

Using the above methods one can express the Schrödinger equation for the scattered wave in a local form:
\[ \left( \frac{-\hbar^2}{2m^*} \nabla^2 + U_{\text{TOTAL}}(r) \right) \chi(r) = E \chi(r) \]
where
\[ U_{\text{TOTAL}}(r) = U_{\text{dir}}(r) + U_{\text{xc}}(r) + U_{\text{coulomb}}(r) \]
The next chapter deals with computer programs to evaluate the potentials, $\chi(r)$, and the scattering cross-sections.
CHAPTER III
COMPUTATIONS

RSPACE (Halderson) is the computer program that evaluates the central (and isospin) part of the interaction. As in the previous chapter, the central interaction can be broken into direct and exchange terms. The direct term, where

\[ \int \rho(r) V_{\text{central}}(r, \rho(r_m)) \, dr \]

and

\[ r_n = \frac{r_1 - r_0}{2}, \quad r_1 = \sqrt{r^2 + r_0^2} + r_0 \cos \theta \]

is evaluated using the Yukawa potentials explained earlier and one of two forms used for the nuclear density term. The two nuclear density parameterizations used here are the harmonic oscillator model and the two parameter Fermi model. In the harmonic oscillator model

\[ \rho(r) = \frac{3 \Lambda}{\pi \alpha^2} \alpha (2 + \frac{3}{\alpha}) \left( 1 + \alpha \frac{r}{\alpha} e^{-(r/\alpha)^2} \right) \]

and in the Fermi model

\[ \rho(r) = \frac{3 \Lambda}{4 \pi c^2} \left( 1 + \frac{\alpha^2}{c^2} \right) \left( 1 - e^{-(r/c)^2} \right) \]

where the parameters \( \alpha, a, c, z \) are taken from electron scattering fits (de Jager, 1974).

Integration is accomplished by first reducing the integral over \( r \) to integrations over \( \phi, \theta, r \). Because of the \( \phi \) independence, integration over \( \phi \) results in \( 2\pi \). Integration over \( \theta \) and \( r \) is accomplished simultaneously using a ten point Gaussian quadrature for \( \theta \), and two eight point Gaussian quadratures for \( r \). The \( r \) integration is broken into two regions so that the region between 0 and 1.2 fm, where more accuracy is required, can be done separately from the
Figure 1. Definition of Positional Terms.
region 1.2 to 7 fm.

ALWORLD (Carr, 1983) is the program that evaluates the spin-orbit part of the potential. A method of folding is used in momentum space whose description is similar to that found in Carr (1983) (Appendix).

The program DWUCK (Kundz) uses the optical model potential well produced by RSPACE and ALWORLD to find the phase shifts for the various partial waves starting with $\ell=0$ and going to $\ell=17$. These phase shifts are then used to find the differential scattering cross-sections to compare with those found by experiment. Such comparisons are the subject of the next chapter.
CHAPTER IV

RESULTS

The following plots were computer generated using the Calcomp Electromechanical Plotter at Western Michigan University.

Comparison of Potentials

In Figures 2-4, the microscopic potentials used for \(^{40}\text{Ca}\) are compared to the parametric optical potential of Becchetti and Greenlees (Perey, 1976). Both potentials have a similar energy dependence. As the energy of the projectile increases, the real part of both potentials decreases, and the imaginary or inelastic parts become deeper. This similar energy dependence provides a theoretical basis for the parametric fits.

Comparison of Exchange Approximations

In figure 5 the Slater approximation and the local energy approximation are compared. The Slater approximation then was chosen for use in all subsequent comparisons. Although both these approximations follow equally well the experimental data, the Slater approximation is more palatable theoretically. \(^{40}\text{Ca}\) was chosen for the comparisons because it is the heaviest doubly magic nucleus, and also because it has a very complete set of experimental data.
Comparison of Cross Sections

Figure 6 shows the fits of $^{40}\text{Ca}(p,p)$ at 40 MeV, 61.4 MeV, and 80.2 MeV. Figure 7 shows $^{40}\text{Ca}(n,n)$ at 40 MeV. Theoretical results often are below the experimental data and are more diffractive than the experimental data. Similar results came about in the $^{90}\text{Zr}$ comparisons (Figure 8).

$^{208}\text{Pb}$ results (Figure 9) are of similar quality except somewhat more out of phase. An attempt to correct this by adjusting the Coulomb radius proved unsuccessful.

$^{12}\text{C}(p,p)$ results (Figure 10) are not surprising if one considers the approximations are partly the result of the use of the theory of infinite nuclear matter and $^{12}\text{C}$'s small size would not lend itself to such assumptions.
Figure 2. Comparison of Potential Wells at 40 MeV.
Figure 3. Comparison of Potential Wells at 61.4 MeV.
Figure 4. Comparison of Potential Wells at 80.2 MeV.
\( \Theta : 40 \text{ MeV} \) (Blumberg, 1966)

- - - - : Local energy approximation

- - - : Slater approximation

Figure 5. \( ^{40}\text{Ca}(p,p) \)
Figure 6. $^{40}\text{Ca}(p,p)$

- □: 40 MeV (Blumberg, 1966)
- ○: 61.4 MeV (Flumer, 1969)
- △: 80.2 MeV (Nadasen, 1981)
Figure 7. $^{40}\text{Ca}(n,n)$

\[\frac{d\sigma}{d\Omega} \text{ (mb/sr)}\]

$\Theta$: 40 MeV (DeVito, 1981)
Figure 8. $^{90}$Zr$(p,p)$

- $\bigcirc$: 40 MeV (Blumberg, 1966)
- $\triangle$: 61.4 MeV (Flumer, 1969)
- $\oplus$: 79.9 MeV (Nadasen, 1981)
- $\times$: 100.4 MeV (Kwiatkowski, 1978)
Figure 9. $^{208}\text{Pb}(p,p)$

- ⊙: 40 MeV (Blumberg, 1966)
- △: 61.4 MeV (Flumer, 1969)
- †: 79.9 MeV (Nadasen, 1981)
- ×: 100.4 MeV (Kwiatkowski, 1978)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 10. $^{12}\text{C}(p,p)$

- □: 40 MeV (Blumberg, 1966)
- ○: 61.4 MeV (Flumer, 1969)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER V

CONCLUSIONS

By assuming a zero energy constant nuclear ground state, using Slater determinants, and limiting the study to zero spin nuclei, the Schrödinger equation is reduced to an evaluation of the direct and exchange potentials. Evaluating the exchange potential is begun by using the two-body effective interaction and concepts from the theory of infinite nuclear matter. At this point two different methods of approximation are used to assist in the simplification of the resultant integral and the Slater approximation was found more theoretically agreeable and retained for subsequent use. A method is then explained to evaluate the effective interaction in a numerical form thus making the complete numerical evaluation of the Schrödinger equation possible.

The results, given the many approximations employed, show that the methods used have many valid assumptions. Their validity does not rest in fits superior to those of the parametric optical models, but in the fact that the physical assumptions and approximations used produce potentials very much like those obtained from parametric fits.

The results also show that approximations such as the local two-body effective interaction, the Slater approximation, concepts from the theory of nuclear matter, and many other methods used appear reasonable.
The general trend of the derived cross-sections being lower than that of the experimental data could very well be traced to the weak magnitude of the two-body interaction derived from the Hamada-Johnston potential (Hamada, 1968). In binding energy calculations too, other investigators have encountered this situation. Its replacement with a new more accurate potential in the Hamada-Johnston calculations would be a likely first step in an attempt to improve on or extend this work.
This appendix derives an expression for the exchange contribution to the spin-orbit potential as calculated in ALWORLD.

\[ \mathbf{V}_{S-O} = -\mathbf{V}^{L_S}_{(r)} \ \mathbf{\hat{L}} \cdot \mathbf{\hat{S}} \]

\[ \mathbf{\hat{L}} = \mathbf{\hat{r}} \times \mathbf{\hat{p}} \]

\[ \mathbf{\hat{L}} \cdot \mathbf{\hat{S}} = \frac{1}{2} \mathbf{\hat{r}} \times \mathbf{\hat{p}} \cdot \mathbf{\hat{S}} = \frac{1}{2} \mathbf{\hat{r}} \cdot (\mathbf{\hat{p}}_{10} \times \mathbf{\hat{S}}) = -\frac{1}{4} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}})^2 \gamma_1(r) \cdot \left[ \mathbf{\hat{p}}_{10} \otimes \mathbf{\hat{S}} \right] \]

since

\[ \mathbf{z} = \sqrt{\frac{4\pi}{3}} \gamma_1(r) \gamma_{10} \]

\[ \mathbf{\hat{r}} = \frac{1}{\mathbf{\hat{r}}^2} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} \]

\[ \mathbf{\hat{p}}_{10} = \mathbf{\hat{r}} \times \mathbf{\hat{p}}_{1} \]

\[ \mathbf{\hat{p}}_{1} = \mathbf{\hat{r}} \times \mathbf{\hat{p}} \]

\[ \mathbf{\hat{p}}_{10} \otimes \mathbf{\hat{S}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \gamma_{10} \gamma_1(r) \]

\[ \mathbf{\hat{p}} \mathbf{\hat{S}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \gamma_{10} \gamma_1(r) \]

\[ \mathbf{\hat{p}}_{10} \times \mathbf{\hat{S}} = \mathbf{p}_{01} \mathbf{\hat{S}}_{y} - \mathbf{p}_{01} \mathbf{\hat{S}}_{x} \]

\[ \mathbf{\hat{p}}_{01} \otimes \mathbf{\hat{S}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{p}}_{01} \mathbf{\hat{S}}^0 \]

\[ \mathbf{\hat{p}}_{01} \mathbf{\hat{S}}^0 = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{p}}_{01} \mathbf{\hat{S}}^0 \]

\[ \mathbf{\hat{p}}_{10} \times \mathbf{\hat{S}} = \mathbf{p}_{01} \mathbf{\hat{S}}_{y} - \mathbf{p}_{01} \mathbf{\hat{S}}_{x} \]

\[ \mathbf{\hat{p}}_{01} \otimes \mathbf{\hat{S}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{p}}_{01} \mathbf{\hat{S}}^0 \]

\[ \mathbf{\hat{p}}_{10} \times \mathbf{\hat{S}} = \mathbf{p}_{01} \mathbf{\hat{S}}_{y} - \mathbf{p}_{01} \mathbf{\hat{S}}_{x} \]

\[ \mathbf{\hat{p}}_{01} \otimes \mathbf{\hat{S}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{p}}_{01} \mathbf{\hat{S}}^0 \]

\[ \mathbf{\hat{p}}_{10} \times \mathbf{\hat{S}} = \mathbf{p}_{01} \mathbf{\hat{S}}_{y} - \mathbf{p}_{01} \mathbf{\hat{S}}_{x} \]

\[ \mathbf{\hat{p}}_{01} \otimes \mathbf{\hat{S}} = \frac{1}{\sqrt{2}} (\mathbf{\hat{S}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{p}}_{01} \mathbf{\hat{S}}^0 \]
For exchange one needs the integral

\[
\int \phi^{*}(\vec{r}_{1}) \nabla_{L_{5}} (\mathbf{r}) \cdot \vec{E} \cdot \chi(\vec{r}_{0}) \varphi(\vec{r}_{0}) \, d\mathbf{r}_{1}
\]

\[
= \frac{1}{(2\pi)^{12}} \int \phi^{*}(\vec{k}_{1}) e^{-i\vec{k}_{1} \cdot \vec{r}_{1}} \nabla_{L_{5}} (\mathbf{q}) e^{i\vec{q} \cdot (\vec{r}_{0} - \vec{r}_{1})} \varphi(\vec{k}_{3}) \chi(\vec{k}_{3}) \varphi(\vec{k}_{3}) \, d\vec{k}_{1} \, d\vec{k}_{2} \, d\vec{k}_{3} \, d\vec{q} \, d\mathbf{r}_{1}
\]

\[
= \frac{1}{(2\pi)^{12}} \int \phi^{*}(\vec{k}) e^{-i\vec{k} \cdot \vec{r}} \nabla_{L_{5}} (\mathbf{q}) e^{i\vec{q} \cdot (\vec{r}_{0} - \vec{r})} (-i\hbar \vec{\nabla}) \cdot \vec{E} \chi(\vec{k}_{2}) \varphi(\vec{k}_{3})
\]

\[
e^{-i\vec{k} \cdot \vec{r}} e^{i\vec{q} \cdot (\vec{r}_{0} - \vec{r})} 2\, d\vec{k} \, d\vec{q} \, d\vec{k}_{1} \, d\vec{q}_{1} \, d\mathbf{r}
\]

\[
k_{2} = \frac{1}{2}(\vec{k} + \vec{k})
\]

\[
k_{3} = \frac{1}{2}(\vec{k} - \vec{k})
\]

\[
\nabla_{L_{5}} (\vec{r}) = \nabla_{L_{5}} (\mathbf{r}) \vec{r}
\]

\[
= \frac{1}{(2\pi)^{12}} \int \phi^{*}(\vec{k}) e^{-i\vec{k} \cdot \vec{r}} \nabla_{L_{5}} (\mathbf{q}) e^{i\vec{q} \cdot (\vec{r}_{0} - \vec{r})} [(-i\hbar)(-\vec{k})] \cdot \vec{E}
\]

\[
\chi(\vec{k}) \varphi(\vec{k}_{3}) e^{-i\vec{k}_{2} \cdot \vec{r}_{1}} + i\vec{k} \cdot \vec{r}_{0} \, d\vec{k}_{1} \, d\vec{k}_{2} \, d\vec{q} \, d\mathbf{r}_{1}
\]

\[
= \frac{1}{(2\pi)^{12}} \int (-\vec{k}) \phi^{*}(\vec{k}) e^{-i\vec{k} \cdot \vec{r}} \nabla_{L_{5}} (\mathbf{q}) e^{i\vec{q} \cdot (\vec{r}_{0} - \vec{r})} (\vec{k}_{2} - \vec{k}_{3})
\]

\[
\chi(\vec{k}_{2}) \varphi(\vec{k}_{3}) e^{i\vec{k}_{2} \cdot \vec{r}_{0} + i\vec{k}_{3} \cdot \vec{r}_{1}}
\]

\[
\cdot \mathbf{S} \chi(\vec{k}_{2}) \phi(\vec{k}_{3}) e^{-i\vec{k}_{3} \cdot (\vec{r}_{0} - \vec{r})} + i\vec{k}_{3} \cdot (\vec{r}_{0} - \vec{r}) \, d\vec{k}_{3} \, d\mathbf{r}_{1}
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
These are the formulae coded in ALWORLD for inelastic scattering.

To see the contribution to elastic scattering where the target is $0^+ \rightarrow 0^+$, look at the term

$$O_{el} = \frac{1}{4} \sum_n \omega_n \cdot f'(k_n) \left[ \mathcal{D}_0 \cdot \mathcal{Z}_0 \right]$$

where $\lambda$ must be 0

$$= \frac{1}{4} \sum_n \omega_n \cdot f'(k_n) \mathcal{Z}_0 \mathcal{P}_0 \left[ \mathcal{D}_0 \cdot \mathcal{Z}_0 \mathcal{P}_0 \right]$$

$$\nu \mathcal{D}_0 (k_r) = \frac{1}{2} \frac{\sin kr}{kr} \hat{r} = \frac{k^2 \cos kr - k \sin kr}{k^2 r^2} \hat{r}$$

$$= \left( \frac{\cos kr - k \sin kr}{kr} \right) \hat{r} = k \left( \frac{\cos kr - \sin kr}{kr^2} \right) \hat{r} = -k \mathcal{J}_0 (k_r) \hat{r}$$

$$O_{el} = \frac{1}{4} \sum_n \omega_n \cdot f'(k_n) \mathcal{Z}_0 \mathcal{P}_0 \left[ \mathcal{D}_0 \cdot \mathcal{Z}_0 \mathcal{P}_0 \right]$$

One needs the integral

$$\int \phi^* (\vec{r}) \mathcal{V}_{el} (\vec{r}) \cdot \hat{k} \cdot \hat{s} \phi (\vec{r}) \, d\vec{r}_i$$

$$= \int \phi^* \frac{1}{4} \sum_n \omega_n^2 \left( - g_i (k_n) \right) \mathcal{Z}_0 (\vec{r}) \left( - \frac{k \mathcal{J}_0 (k_r)}{r} \right) \mathcal{P}_0 \cdot \mathcal{P}_0 \phi (\vec{r}_i)$$

$$= \frac{1}{4} \sum_n \omega_n^2 \left( - g_i (k_n) \right) \int \phi^* \mathcal{Z}_0 (\vec{r}) \frac{\mathcal{J}_0 (k_r)}{r} \phi \, d\vec{r}_i \mathcal{P}_0 \cdot \mathcal{P}_0$$

$$= \frac{1}{4} \sum_n \omega_n^2 \left( - g_i (k_n) \right) \left( \int \phi (\vec{r}) \mathcal{Z}_0 (\vec{r}) \left( \frac{\mathcal{J}_0 (k_r)}{r} \right) \phi \, d\vec{r}_i \right) \mathcal{P}_0 \cdot \mathcal{P}_0$$
\[ A^1 \left\{ \frac{1}{\sqrt{4\pi}} \sum_{\ell^\prime \ell J} i^{\ell^\prime-1} \omega_n^2 \, g'(k_n) \hat{L} \hat{L}' C^{\ell \ell^\prime J}_{000} \right\} [J] \]

\[ (-1)^{\ell+\ell^\prime+J} \frac{1}{\sqrt{3}} \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) + \]

\[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \right\} \{ \ell^\prime J \} + \frac{3}{\sqrt{4\pi}} \sum_{n \ell} \omega_n f'(k_n) \]

\[ \left[ \left( \nabla \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \right) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \right] \frac{2}{V_2} \]

where \( \mathbf{p}_{\mathbf{J}} = \left[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \right] J \) and \( \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) = \left[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \right] J \)

\[ = A^1 \left\{ \frac{1}{\sqrt{4\pi}} \sum_{\ell^\prime \ell J} i^{\ell^\prime-1} \omega_n^2 \, g'(k_n) \hat{L} \hat{L}' C^{\ell \ell^\prime J}_{000} \right\} [J] \]

\[ \mathbf{w}(\ell \ell'JJ) \cdot \left[ \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \right] \]

\[ + \frac{c}{\sqrt{4\pi}} \sum_{n \ell} \omega_n f'(k_n) \left[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \right] \]

where \( \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) = \left( \nabla \times \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \right) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \)

\[ \mathbf{A}^1 \frac{1}{\sqrt{4\pi}} i^{\ell^\prime J} \hat{L} \hat{L}' C^{\ell \ell^\prime J}_{000} \sqrt{3} (\ell J) \omega_n f'(k_n) \mathbf{w}(\ell \ell'JJ) \]

\[ = \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \]

\[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \]

\[ + \frac{1}{\sqrt{4\pi}} \sum_{n \ell} \omega_n f'(k_n) \left[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \right] \]

\[ \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{J}_{\ell J} \mathbf{J}(\mathbf{\bar{r}}_o) \cdot \mathbf{p}_{\mathbf{J}}(\mathbf{\bar{r}}_o) \]
Likewise
\[
\sqrt{\frac{3}{4\pi}} \nabla r_0 \sum_{nL} \omega_n \mathcal{F}'(k_n) \left( \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right)
\]
\[= g_0(r) \gamma_{10}(r) \]

Take part of (2) and use (3) and (4) in part of (2)
\[
\sqrt{\frac{3}{4\pi}} \nabla r_0 \sum_{nL} \omega_n \mathcal{F}'(k_n) \nabla r_0 \left[ \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right] \cdot \left[ \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right] \cdot \left[ \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right]
\]
\[= A \left\{ \frac{1}{\sqrt{4\pi}} \sum_{nL} \omega_n \mathcal{F}'(k_n) \nabla r_0 \left[ \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right] \cdot \left[ \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right] \cdot \left[ \mathbf{J}_L(r_0) \cdot \mathbf{J}_L(r) \right] \right\}
\]
Consider the $z$-component of
\[ \nabla_{\vec{r}_0} \sum_{nL} \omega_n^2 f'(k_n) \vec{z}_n(\vec{r}_0) \cdot \vec{z}_n(\vec{r}) \sqrt{\frac{3}{4\pi}} = R \]
where \( f'(k_n) = -g_1(k_n) / k_n \)
\[ R = \sqrt{\frac{3}{4\pi}} \nabla_{\vec{r}_0} \sum_{nL} \omega_n^2 f'(k_n) (-1)^l \hat{\ell} \left[ \vec{z}_n \otimes \vec{z}_n \right]_o \]
\[ = \sqrt{\frac{3}{4\pi}} \nabla_{\vec{r}_0} \sum_{nL} \omega_n^2 f'(k_n) (-1)^l \hat{\ell} C_{o,o,o} \left[ \vec{z}_n \otimes \vec{z}_n \right]_o \]
\[ = \nabla_{\vec{r}_0}^2 \left\{ \sqrt{3} \left[ \frac{1}{4\pi} \sum_{nL} \omega_n^2 f'(k_n) \frac{\hat{\ell}}{i} C_{o,o,o} \left[ \vec{z}_n \otimes \vec{z}_n \right]_o \right] \right\} \]
\[ = \nabla_{\vec{r}_0} \left\{ \frac{1}{2\pi^3} \int f'(k) Y_{00}(\hat{k}) e^{i \vec{k} \cdot \vec{r}_0} e^{-i \vec{k} \cdot \vec{r}} \, d\hat{k} \right\} \]
\[ = \sqrt{3} \frac{1}{(2\pi)^3} \int f'(k) \frac{1}{V_{4\pi}} (i k_z) e^{i \vec{k} \cdot \vec{r}_0} e^{-i \vec{k} \cdot \vec{r}} \, d\hat{k} \]
\[ = \frac{1}{(2\pi)^3} \int f'(k) i \left( \sqrt{\frac{3}{4\pi}} k_z \right) e^{i \vec{k} \cdot \vec{r}_0} e^{-i \vec{k} \cdot \vec{r}} \, d\hat{k} \]
\[ = \frac{1}{(2\pi)^3} \int \left[ -g_1(k) \frac{1}{k} \right] i \left( \sqrt{\frac{3}{4\pi}} k_z \right) e^{i \vec{k} \cdot \vec{r}_0} e^{-i \vec{k} \cdot \vec{r}} \, d\hat{k} \]
\[ = \frac{1}{(2\pi)^3} \int \left[ -g_1(k) \right] i \left( \sqrt{\frac{3}{4\pi}} \frac{k_z}{k} \right) i^{-1} (i) e^{i \vec{k} \cdot \vec{r}_0} e^{-i \vec{k} \cdot \vec{r}} \, d\hat{k} \]
\[ = \frac{1}{(2\pi)^3} \int g_1(k) Y_{10}(k) i^{-1} e^{i \vec{k} \cdot \vec{r}_0} e^{-i \vec{k} \cdot \vec{r}} \, d\hat{k} \]
\[ = g_1(r) Y_{10}(\hat{r}) \quad (v) \]
\[\frac{1}{(2\pi)^2} \int g(k) \, Y_{LM}(k) \, e^{i\mathbf{k} \cdot \mathbf{r}} \, e^{-i\mathbf{k} \cdot \mathbf{r}_0} \, dk = \frac{1}{(2\pi)^2} \int g(k) \, Y_{LM}(k) \sum_{\ell_m} \sum_{\ell_m'} (4\pi)^2 \, e^{i \ell_m' \cdot r_0} \, j_{\ell_m'}(kr_0) \, j_{\ell_m}(kr_0) \, d\mathbf{k},\]

\[Y_{\ell_m}^{'*}(k) \, Y_{\ell_m}(r_0) \, Y_{\ell_m'}^{'*}(l) \, Y_{\ell_m'}(r) \, d\mathbf{k},\]

\[Y_{\ell_m}^{'*}(k) \, Y_{\ell_m}(r_0) \, Y_{\ell_m'}^{'*}(l) \, Y_{\ell_m'}(r) \, d\mathbf{k} = \sum_{\ell_m'} \sum_{\ell_m''} (4\pi)^2 \, e^{i \ell_m' \cdot r_0} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}}, \]

\[g_c(r) \, Y_{LM}(r) = \frac{i}{(2\pi)^2} \int g(k) \sum_{\ell_m} \sum_{\ell_m'} (4\pi)^2 \, e^{i \ell_m' \cdot r_0} \, j_{\ell_m'}(kr_0) \, j_{\ell_m}(kr_0) \, Y_{\ell_m}^{'*}(l) \, Y_{\ell_m}(r_0) \, Y_{\ell_m'}^{'*}(l) \, Y_{\ell_m'}(r) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, dk,\]

\[\frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, k^2 \, dk = \frac{1}{(2\pi)^2} \sum_{\ell_m'} \sum_{\ell_m} \, e^{i \ell_m' \cdot r_0} \, j_{\ell_m'}(kr_0) \, j_{\ell_m}(kr_0) \, Y_{\ell_m}^{'*}(l) \, Y_{\ell_m}(r_0) \, Y_{\ell_m'}^{'*}(l) \, Y_{\ell_m'}(r) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, k^2 \, dk,\]

\[= \frac{1}{\sqrt{4\pi}} \sum_{\ell_m'} \sum_{\ell_m} \, e^{i \ell_m' \cdot r_0} \, \left( \frac{k_m \Delta k}{(2\pi)^2} \right) g(k_m) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, \Delta k \, dk,\]

\[= \frac{1}{\sqrt{4\pi}} \sum_{\ell_m'} \sum_{\ell_m} \, e^{i \ell_m' \cdot r_0} \, \left( \frac{k_m \Delta k}{(2\pi)^2} \right) g(k_m) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, \Delta k \, dk,\]

\[= \frac{1}{\sqrt{4\pi}} \sum_{\ell_m} \, e^{i \ell_m \cdot r_0} \, \left( \frac{k_m \Delta k}{(2\pi)^2} \right) g(k_m) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, \Delta k \, dk,\]

\[= \frac{1}{\sqrt{4\pi}} \sum_{\ell_m} \, e^{i \ell_m \cdot r_0} \, \left( \frac{k_m \Delta k}{(2\pi)^2} \right) g(k_m) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, \Delta k \, dk,\]

\[= \frac{1}{\sqrt{4\pi}} \sum_{\ell_m} \, e^{i \ell_m \cdot r_0} \, \left( \frac{k_m \Delta k}{(2\pi)^2} \right) g(k_m) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, \Delta k \, dk,\]

\[= \frac{1}{\sqrt{4\pi}} \sum_{\ell_m} \, e^{i \ell_m \cdot r_0} \, \left( \frac{k_m \Delta k}{(2\pi)^2} \right) g(k_m) \, \frac{\hat{\ell}_m' \cdot \hat{\ell}_m}{|\mathbf{k}|} \, C_{\ell_m \ell_m''}^{\ell_m'} \, C_{\ell_m' \ell_m''}^{\ell_m^{*}} \, \Delta k \, dk,\]

where

\[Z_{\ell_m}(r) = \frac{1}{\pi} \int j_{\ell_m}(kr) \, Y_{\ell_m}(r) \]
\[
\frac{1}{r^2} \left[ 2 \mathbf{p}_0 \cdot \mathbf{i}_y - 2 \mathbf{i} \cdot \mathbf{p}_0 \mathbf{\Omega} \mathbf{\Omega} \right] = \frac{1}{r^2} \mathbf{i} \left( \mathbf{p}_0 \cdot \mathbf{\Omega} - \mathbf{p}_0 \mathbf{\Omega} \right)
\]

... (\mathbf{p}_0 \mathbf{\Omega}) = \frac{1}{r} \left[ \mathbf{p}_0 \mathbf{\Omega} \right]_0

\[
\begin{align*}
[ \mathbf{p}_0 \mathbf{\Omega}]^T &= \mathbf{p}_0 \mathbf{\Omega}^T - \mathbf{p}_0 \mathbf{\Omega} - \mathbf{\Omega} \mathbf{p}_0 \\
&= \mathbf{p}_0 \mathbf{\Omega}^T - \mathbf{p}_0 \mathbf{\Omega} - \mathbf{\Omega} \mathbf{p}_0 \\
A^T &= -\frac{i}{\sqrt{1/3}} \
g_1(r) &= r^{-\frac{1}{2}} \mathbf{\nu} r
\end{align*}
\]

\[
\mathbf{\nu} r = A^T g_1(r) \mathbf{Y}_1(r)
\]

\[
\begin{align*}
[ \mathbf{p}_0 \mathbf{\Omega}]^T + [ \mathbf{p}_0 \mathbf{\Omega}^T] + [ \mathbf{\Omega} \mathbf{p}_0 ]^T - [ \mathbf{p}_0 \mathbf{\Omega}, ]^T
\end{align*}
\]

Consider

\[
\int g_L(r) Y_{LM}(\mathbf{k}) e^{-i \mathbf{k} \cdot \mathbf{r}} \, d\mathbf{r}
\]

\[
= \frac{1}{r} \int g_L(r) Y_{LM}(\mathbf{k}) e^{-i \mathbf{k} \cdot \mathbf{r}} \, d\mathbf{r}
\]

\[
= \int g_L(r) \frac{4\pi}{i} j_L(kr) Y_{LM}(\mathbf{k}) r^2 \, d\mathbf{r}
\]

\[
= \int g_L(r) 4\pi \frac{1}{i} j_L(kr) r^2 \, d\mathbf{r} Y_{LM}(\mathbf{k})
\]

\[
= i^{-L} g(k) Y_{LM}(k)
\]

\[
g_L(r) Y_{LM}(\mathbf{k}) = \int i^{-L} g(k) Y_{LM}(k) e^{i \mathbf{k} \cdot \mathbf{r}} \, d\mathbf{r} = \frac{1}{(2\pi)^3}
\]
The direct term would look like
\[
\frac{1}{(2\pi)^4} \left( -\frac{i}{2} \right) \int \phi^*(k) e^{-i (\vec{E}_k \cdot \vec{r}_1)} \nabla \phi \cdot \vec{r} e^{i (\vec{k}_2 \cdot \vec{r}_1 + \vec{k}_3 \cdot \vec{r}_2)} e^{i (\vec{k}_2 \cdot \vec{r}_0 + \vec{k}_3 \cdot \vec{r}_0)} \frac{1}{(2\pi)^4} \left( -\frac{i}{2} \right) \int \phi^*(k) e^{-i (\vec{E}_k \cdot \vec{r}_1)} \nabla \phi \cdot \vec{r} e^{i (\vec{k}_2 \cdot \vec{r}_1 + \vec{k}_3 \cdot \vec{r}_1)} \frac{1}{i} \int \frac{d\vec{k}_2}{d\vec{r}_1}
\]

Therefore the exchange term is like having an additional direct term with a potential whose Fourier transform is
\[
\int e^{-i \vec{k} \cdot \vec{r}} (\nabla \phi \cdot \vec{r}) d\vec{r} = \nabla \phi \cdot \vec{r} (k_{cm}) = \text{constant}
\]
much like central exchange.

L. N. Blumberg et al., Phys. Rev. 147, 812 (1966)


B. D. Day, Rev. Mod. Phys. 39, 719 (1967)


D. Halderson, Western Michigan University, unpublished.

T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1968)


P. D. Kundz, University of Colorado, unpublished.


A. Messiah, Quantum Mechanics, (North Holland Press, Amsterdam, 1962) p. 357


F. G. Perey et al., Atomic Data and Nucl. Data Tables 16, 1 (1976)


H. Von Geramb, Table of Effective Density and Energy Dependent Interactions", 1979, unpublished