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Integrity of Digraphs

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INTEGRITY OF DIGRAPHS

by

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The *vertex-integrity* of a digraph D, denoted I(D), is defined to be the minimum over alll subsets X of the vertex set of D for the quantity $|X|$ + m(D - X), where $|X|$ is the number of vertices in X and m(D - X) is the maximum order of a strong component in the digraph D - X. In a like manner, the *arc-integrity* of the digraph D, denoted I'(D), is defined to be the minimum over all subsets Y of the arc set of D for the quantity $|Y|$ + m(D - Y), where $|Y|$ is the number of arcs inY. These two measures of the vulnerability of a digraph are analogous to the undirected concepts, which were introduced by Barefoot, Entringer and Swart in 1987.

This investigation of these two parameters centers on the vertex-integrity and arc-integrity for orientations of graphs in several interesting families, including complete graphs, complete bipartite graphs, cartesian products of paths, and hypercubes. Because different orientations of the same graph may lead to different values of the parameters, we can only hope to bound these values for a given graph or class of graphs. Every graph has an acyclic orientation, where the largest strong component is of order 1. Both the vertex-integrity and arc-integrity for such an orientation are 1. This being the case, we focus on the maximum vertex-(arc-)integrity which can be attained by some orientation of the graph.

If one can find an induced subgraph H of a graph G for which any orientation of H has vertex-integrity 1, then the maximum vertex-integrity which an orientation could attain is at most $|G - H| + 1$. To that end, we define the *decycling number* of a graph G, denoted $\nabla(G)$, to be the minimum order of a subset S of the vertices of G, such that G - S is a forest. Since G - S is acyclic, then for any orientation of G - S, the order of the largest strong component is 1. Therefore the maximum vertex-integrity over all orientations of G is at most (G) + 1. This parameter is investigated for the families of graphs we study in the chapter on vertex-integrity.

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Robert Charles Vandell

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CHAPTER I

INTRODUCTION

1.1 Connectivity

Networks are becoming more and more of a necessity for the daily existence of business because of a national need for overnight delivery services or telecommunications. In addition there is the more localized networking of office machines. The vulnerability to disruption of such systems must be a concern to both the user and provider of such services. Since these networks can be modeled by graphs or digraphs, another look at some of the established parameters that measure vulnerability, as well as perhaps some new ones, seems in order.

The first vulnerability parameter that one usually encounters is connectivity. Following [1.2], we define the *connectivity* $\kappa(G)$ of a graph G as the minimum number of vertices one must remove from G so that the remaining graph is either disconnected or a single vertex. In the same vein, the *edge-connectivity* $\kappa_1(G)$ is the minimum number of edges which must be removed so that the remaining graph is either disconnected or trivial. Although these parameters are easy to work with and many results about them are known, they are often not very discriminating in describing the vulnerability of a graph. For example, the graphs $K_{1,8}$ and P_9 shown in Figure 1.1 have the same order (number of vertices), the same size (number of edges), and the same connectivity, but they display very

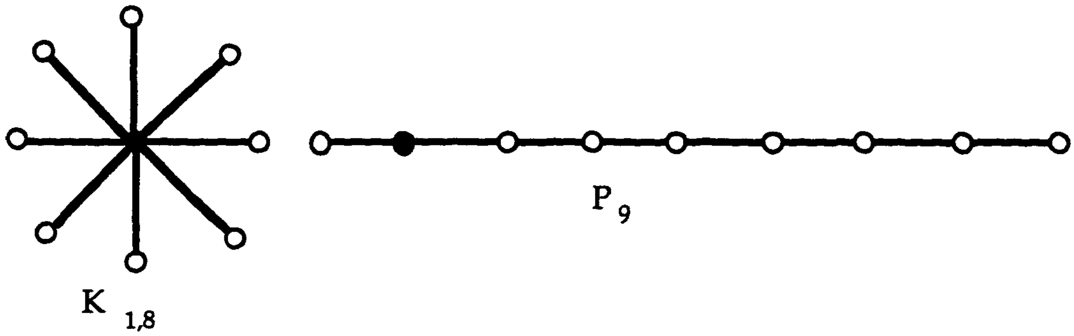


Figure 1.1

different connectedness behaviors when vertices are deleted from them. If the central vertex of $K_{1,8}$ is removed, then no pair of remaining vertices may communicate with one another, whereas if a penultimate vertex of P_9 is removed, seven vertices can still transfer information and only one cannot (see Figure 1.1). These two graphs also have the same edge-connectivity, but removal of any edge from $K_{1,8}$ still leaves eight vertices in contact while removal of a central edge of P_9 means that one group of four vertices cannot communicate with the other five vertices (see Figure 1.2).

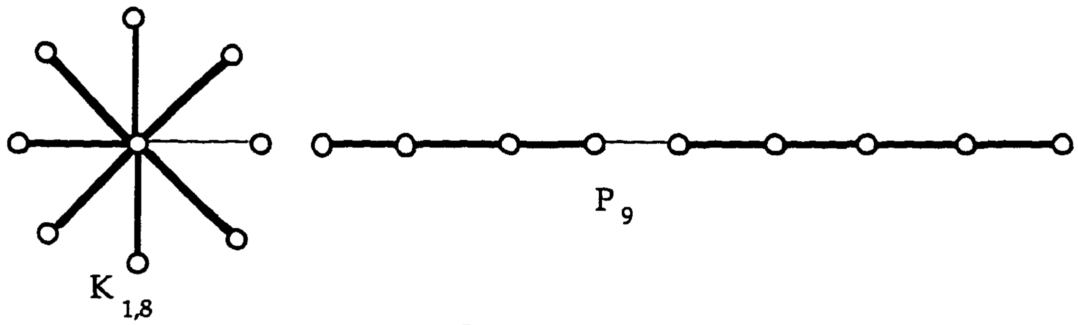


Figure 1.2

1.2 Integrity

In an effort to overcome the limitations of the connectivity parameters, Barefoot, Entringer, and Swart [1.1] introduced the corresponding integrity parameters. The *vertex-integrity* (or simply the *integrity*) *I(G)* of a graph G is defined as

$$I(G) = \min\{|X| + m(G - X) : X \subset V(G)\}$$

where m(G) is the maximum order of a component of G. The *edge-integrity* is

$$I'(G) = \min\{|Y| + m(G - Y) : Y \subseteq E(G)\}.$$

For example, $I(K_{1,8}) = 2$ and $I(P_9) = 5$ (see Figure 1.3, where the dark vertices are a set for which the integrity is achieved), while $I'(K_{1,8}) = 9$ and

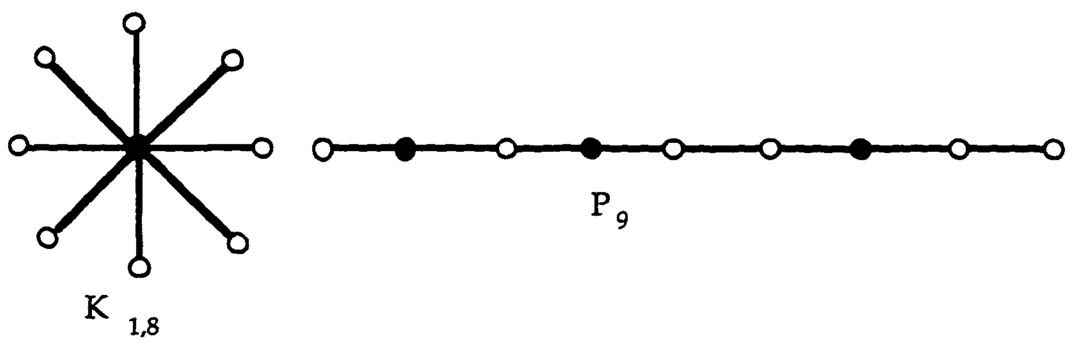


Figure 1.3

$I'(P_9) = 5$ (see Figure 1.4, where the light edges are a set for which the edge-integrity is achieved). In each case the graph that is the least susceptible to disruption has the greater (edge-) integrity.

There are many known results on both integrity and edge-integrity, some of which compare them to other graphical parameters. For the most part we will be more interested in their values for specific classes of

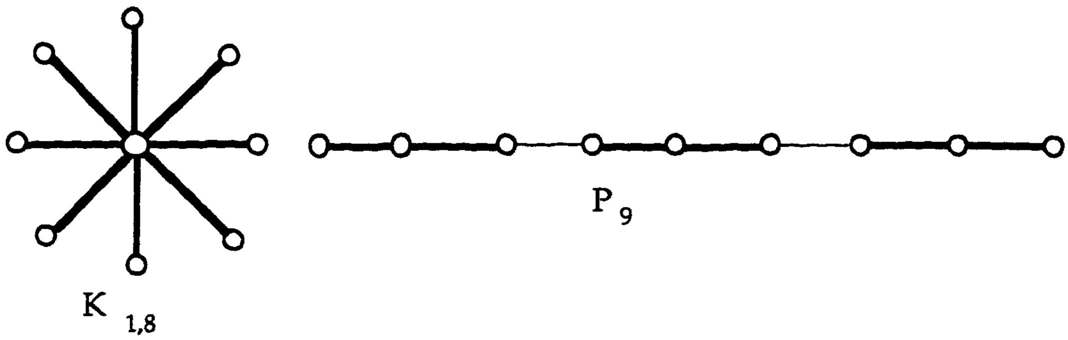


Figure 1.4

graphs. In their original work, Barefoot, Entringer, and Swart determined the integrity and edge-integrity for several families, some of which are given in Table 1. We will use some of these values later to help obtain new results, while other results will be stated as needed.

Table 1

The Vertex-integrity and Edge-integrity of Some Graphs

Graph G	I(G)	I'(G)
Complete graph K_n	n	n
Path P_n	$\lceil 2\sqrt{n+1}\rceil - 2$	$\lceil 2\sqrt{n}\rceil - 1$
Cycle C_n	$\lceil 2\sqrt{n}\rceil - 1$	$\lceil 2\sqrt{n}\rceil$
Complete Bipartite Graph $K_{m,n}$	$1 + \min\{m,n\}$	$m + n$
n-Cube Q_n	$O(2^{n-1})$	2^n

1.3 Directed Integrity

A *directed graph* or *digraph* D is a finite, but nonempty, set of vertices together with a set of ordered pairs of vertices, called *arcs*. The digraph with vertex set {w, x, y, z} and arc set {(w, x), (w, y), (y, w), (z, x)} is pictured in Figure 1.5. Many networks have a natural flow direction

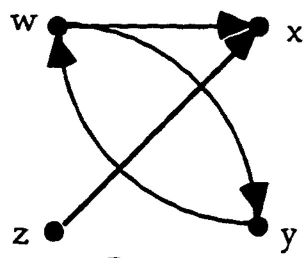


Figure 1.5

associated with each of the connections the edges represent, so they can be modeled by directed graphs. We are also interested in how vulnerable these networks are to disruption. We replace the idea of a component that is used in graphs with the notion of a *strongly connected component* (or more simply *strong component*), which is defined as a maximal subdigraph of the digraph in which there is a directed path from each of its vertices to each of the others. The maximum order of a strong component of D is denoted by m(D). One important difference between strong components of digraphs and components of graphs which we will utilize is that each vertex in a nontrivial strong component lies on a directed cycle completely contained in the strong component.

One of the problems encountered in the study of digraphs is the relative lack of families that are easy to work with, contrary to the situation with graphs. We can, however, create a digraph D from a graph G by replacing each edge uv in G by either (u,v) or (v,u) but not both. Then D is called an *orientation of* G, and G is known as the *underlying graph of* D. We note here that the number of vertices in a nontrivial strong component is at least 2 for a digraph, 3 for an orientation of a graph, and 4 for an orientation of a bipartite graph.

One goal in this dissertation is to investigate the directed counterparts to integrity. We define the *vertex- integrity I(D)* of a digraph D, as

$$I(D) = \min\{|X| + m(D - X) : X \subset V(D)\},$$

and the *arc-integrity I'(D)* as

$$I'(D) = \min\{|Y| + m(D - Y) : Y \subseteq E(D)\}.$$

Much of this dissertation deals with orientations of graphs and finding upper bounds for the values of some vulnerability parameters for them. For some important families, we focus on the integrity and arc-integrity of these orientations.

1.4 Decycling Number

An upper bound for the integrity of a graph G can be described in terms of the *covering number* $\beta(G)$, which equals the minimum number of vertices in a set X such that each edge of G is incident with some vertex of X. For such a set X, the graph G - X is edgeless; hence

$$I(G) \leq m(G - X) + \beta(G) = 1 + |X|.$$

We can apply this idea to the integrity of orientations of graphs. Every orientation of a forest is acyclic and hence contains no strong component of order greater than 1. Thus, if we find a maximal forest F that is an induced subgraph of G, then for each orientation D of G,

$$I(D) \leq |G| - |F| + 1.$$

where $|G|$ denotes the order of a graph G.

In order to utilize this idea we define a new graphical parameter, which we call the decycling number. The *decycling number* $\nabla(G)$ of a graph G is the minimum cardinality of a set S of vertices of G such that G - S is acyclic. Consequently, for each orientation D of G, $I(D) \leq 1 + \nabla(G)$.

The upper bound $1 + \nabla(G)$ for the integrity of a graph is only an estimate, and in certain cases we will be able to improve upon it by taking advantage of nontrivial strong components.

1.5 Definitions, Notation, and Labeling

Each of the subsequent three chapters will be dealing with the same families of graphs, so rather than repeat the definitions, notation, and labeling schemes in each instance, we will state them here.

We use $\alpha(G)$ to denote the *independence number* (order of the largest independent set of vertices) of a graph G and $\beta(G)$ will denote the covering number, as defined in Section 1.4. Using this notation a theorem of Gallai states that for every graph G

$$\alpha(G) + \beta(G) = |G|$$

We also let c(G) denote the number of components of a graph G.

We will consider three binary operations on graphs. First, the *union* $G \cup H$ of two graphs G and H has vertex set $V(G) \cup V(H)$ and edge set E(G)

∪ E(H) (see Figure 1.6).

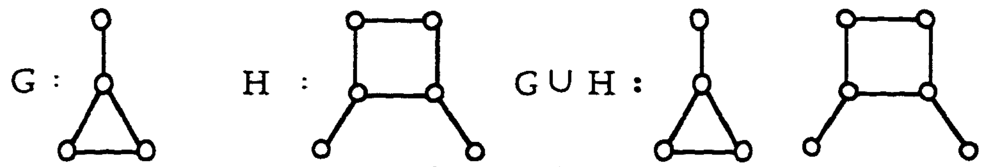


Figure 1.6

The *join* $G + H$ of G and H has vertex set V(G) ∪ V(H) and edge set E(G) ∪ E(H) ∪ {uv | u ∈ V(G) and v ∈ V(H)} (see Figure 1.7). We are

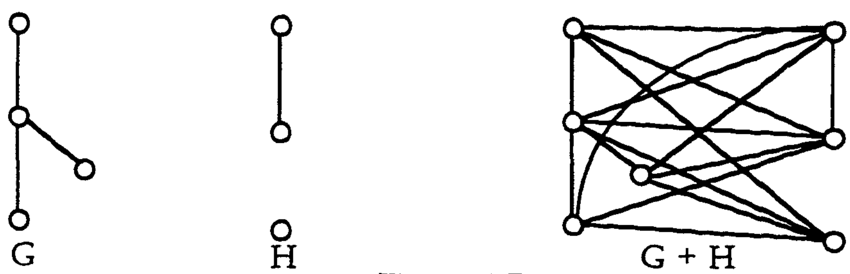


Figure 1.7

particularly interested in this operation when the two graphs are K_1 and C_n or when they are the empty graphs $\overline{K_m}$ and $\overline{K_n}$. The graph $K_1 + C_n$ is known as the *n-wheel* W_n (see Figure 1.8a), while the graph $\overline{K_m} + \overline{K_n}$ is called the *complete bipartite graph* $K_{m,n}$ (see Figure 1.8b). The sets $V(\overline{K_m})$

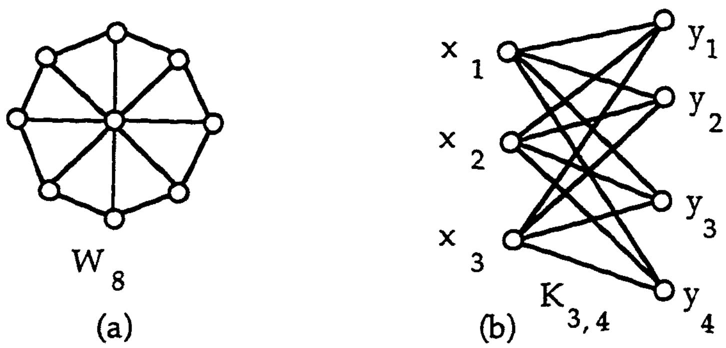


Figure 1.8

and $V(\overline{K_n})$ are called the partite sets, which we denote by X and Y, respectively. The vertices of X will be labeled $x_1,\ldots, x_m$ and the vertices of Y will be labeled $y_1,\ldots,y_n$. Note that the smallest cycle in $K_{m,n}$ has four vertices.

The *Cartesian product* $G \times H$ of graphs G and H has vertex set $V(G) \times V(H)$, with two vertices (u_1,u_2) and (v_1,v_2) being adjacent if and only if either (a) $u_1 = v_1$ and u_2v_2 is an edge in H or (b) $u_2 = v_2$ and u_1v_1 is an edge in G (see Figure 1.9). Another way to think of the Cartesian product is that

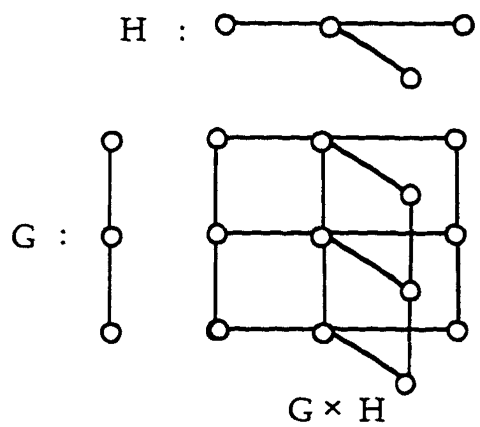


Figure 1.9

each vertex of G is replaced by a copy of H, and each edge uv of G is replaced by $|H|$ edges of the form $(u,x)(v,x)$ for every vertex x of H.

In addition, we will need a standard labeling for the vertex set of the Cartesian product of two graphs. Label the vertices of graphs G and H as $g_1, g_2,\ldots, g_s$ and $h_1, h_2,\ldots, h_t$, respectively. Then the label $v_{i,j}$ will represent the vertex (g_i,h_j) in $G \times H$. This labeling is reminiscent of the notation for the entries of a matrix, where the copy of H at g_i corresponds to the i^{th} row, while the copy of G at h_j corresponds to the j^{th} column. This idea is particularly helpful when we study the products of paths and cycles, and hence we introduce the following notation. Let V(*,j) represent the copy of G at h_j and V(i,*) the copy of H at g_i. When we need all the copies of G in the product from the one at h_i to the one at h_j, we denote this by V(*,i:j) (see Figure 1.10 for examples of this notation in the graph $P_3 \times P_4$).

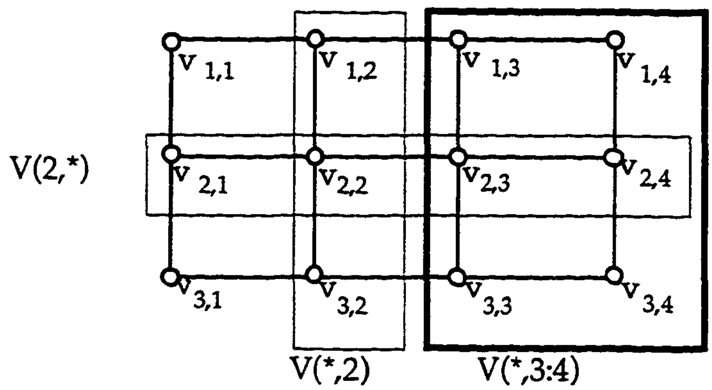


Figure 1.10

The final Cartesian product for which a labeling is defined is the hypercube Q_n. We define the labeling inductively (see Figure 1.11). Recall

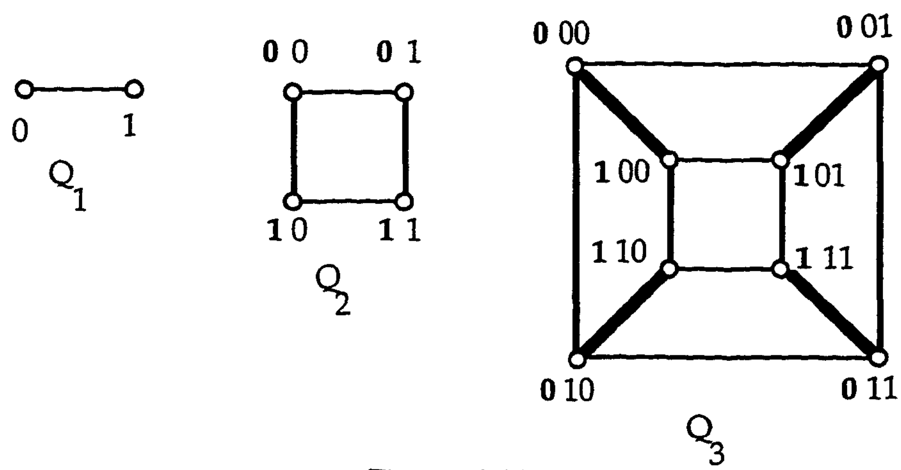
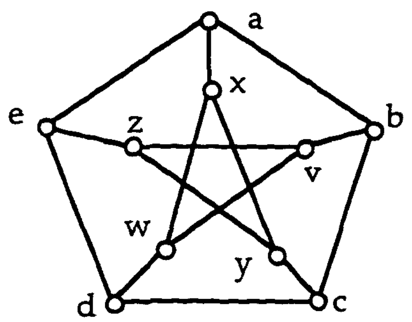


Figure 1.11

that $Q_n = K_2 \times Q_{(n-1)}$ for $n \geq 2$, where $Q_1 = K_2$. Label the vertices of Q_1 by 0 and 1; then we produce a labeling for the vertices of Q_n by adding a prefix of 0 to the labels of one copy of Q_{n-1} and a prefix of 1 to the other labels. It will be useful to focus on specific smaller cubes within a given n-cube. When possible, we use the common prefix to label the smaller cube. For example, for the cube Q_3 of Figure 1.11, we will denote the 2-cube with vertex set {000, 001, 010, 011} by Q0, while Q11 will represent the 1-cube Q_1 induced by the vertices 110 and 111.

The graph we use as one of the first examples in each of the subsequent chapters is the Petersen graph P (see Figure 1.12). The 5-cycle



Petersen graph
Figure 1.12

induced by the vertices a, b, c, d, e we will call the pentagon, while the 5-cycle induced by the vertices v, w, x, y, z will be called the pentagram.

There are certain notations that are unique to digraphs. One class of digraphs that we use is the family of *circulant digraphs* D(n, S). For a subset S of {1, 2, ..., n - 1}, the digraph D(n, S) has vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and contains the arc (v_i, v_j) if and only if $j - i \equiv s \pmod n$ for some $s \in S$ (see Figure 1.13 for the circulant D(6, {1, 4}).

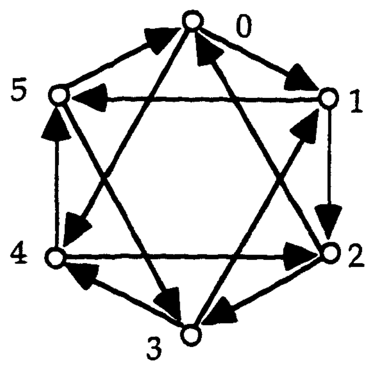


Figure 1.13

As in [1.2] we denote the degree of a vertex v by *deg v*, and the notation for the minimum and maximum degree for any vertex of a graph G will be $\delta(G)$ and $\Delta(G)$ respectively (or simply δ and Δ when the graph is obvious). To extend this to digraphs, we denote the out(in)degree of a vertex v of a digraph D by deg^+v (deg^-v) and the minimum and maximum out(in)degree of a digraph D by $\delta^+(D)$ ($\delta^-(D)$) and $\Delta^+(D)$ ($\Delta^-(D)$), respectively (or simply δ^+ (δ^-) and Δ^+ (Δ^-) when the digraph D is clear).

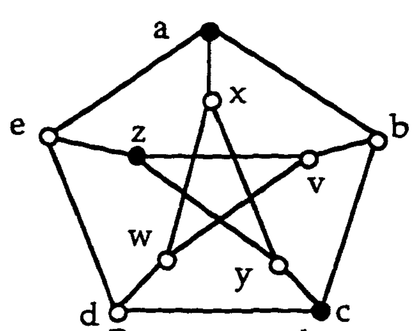
CHAPTER II

THE DECYCLING NUMBER OF GRAPHS

2.1 Definitions

A *decycling set* of a graph G is a set S of vertices of G such that G - S is acyclic. The *decycling number,* $\nabla(G)$ is the minimum cardinality of a decycling set of G. Thus, $\nabla(G) = 0$ if and only if G is a forest. Moreover, if G is a unicyclic graph, then $\nabla(G) = 1$. The converse of this last statement is not true, however, since, for example, the fan graph $K_1 + nK_2$ $(n \geq 2)$ has n cycles yet has decycling number 1. A decycling set S of cardinality $\nabla(G)$ is called a *∇-set* of G. We *decycle* a graph when a subset S of the vertices of G is removed and G - S is acyclic.

For an example of a graph with decycling number greater than 1, we consider the Petersen graph P (see Figure 2.1). Certainly, to decycle this



Petersen graph
Figure 2.1

graph one must remove at least one vertex from each of the 5-cycles

<a,b,c,d,e> and <v,w,x,y,z>. Hence $\nabla(P) \geq 2$. On the other hand, if we remove any two nonadjacent vertices from P, we obtain a graph of order 8 and size 9; while if we remove any two adjacent vertices from P, we arrive at a graph of order 8 and size 10. In either case,the resulting graph contains cycles since the maximum size of a forest of order p is p - 1. However, P - {a, c, z} is an acyclic graph; so $\nabla(P) = 3$.

Certainly, the decycling number is defined for every graph G and is at most $|G| - 2$. There are a number of other interpretatons of this parameter. If G is a graph of order p, the decycling number $\nabla(G)$ is also (a) the minimum number of elements in a set $S \subseteq V(G)$ so that every cycle of G meets S and (b) p less the maximum order of an induced forest in G.

For a graph G, define t(G) as the maximum cardinality of a set of vertices of a graph G which induces a tree. This concept was investigated by Erdös, Saks, and Sós in [2.6]. Their work, however, centered on the relationship between t(G) and other parameters, rather than evaluating the parameter for specific classes of graphs. The digraphical counterpart to the decycling set of a graph, called a *vertex feedback set* of a digraph, has been studied by several researchers.

The notion of the maximum order of an induced forest in a graph leads to an important tool for this study. Recall that a forest on p vertices has at most p - 1 edges, from which we derive the following simple but useful result.

Lemma 2.1.1: Let G be a graph with p vertices, q edges, and nonincreasing degree sequence $d_1, d_2, \ldots, d_p$. If S is a decycling set for G, then

$$|S| \le p - q + \sum_{i=1}^{|S|} d_i - 1.$$

Proof: Removing a vertex of degree d from G destroys d edges in the graph. Hence, removing a set of k vertices from G destroys at most $\sum_{i=1}^{k} d_i$ edges, leaving at least $q - \sum_{i=1}^{k} d_i$ edges in G - S. But, if G - S is a forest, then it must have at most p - k - 1 edges. □

Corollary 2.1.2: Let G be a graph with p vertices, q edges, and maximum degree Δ. If S is a decycling set of G, then

$$|S|(\Delta - 1) \ge q - p + 1.$$

Corollary 2.1.3: Let G be a graph with p vertices and q edges. If S is a decycling set of cardinality k whose vertices have degrees $s_1, s_2, \ldots, s_k$, then

$$p - k - 1 \ge q - \sum_{i=1}^{k} s_i .$$

Earlier we noted that $\nabla(G) = 0$ if and only if G is a forest. The graphs with decycling number 1 can also be quite easily characterized.

Theorem 2.1.4: Let G be a graph. Then $\nabla(G) = 1$ if and only if some vertex v of G is contained in each cycle of G.

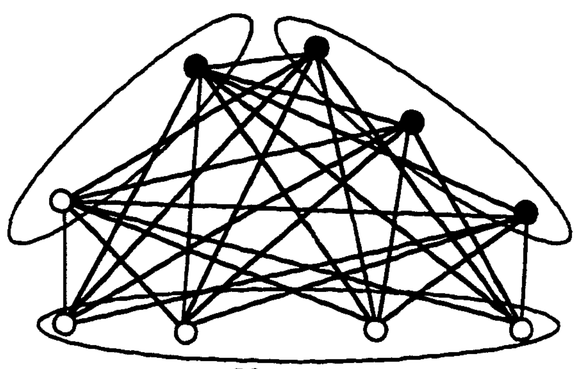
Proof: If $\nabla(G) = 1$ and no vertex lies on all cycles of G, then there is no vertex v for which G - v is acyclic, which implies that $\nabla(G) > 1$. On the other hand, if some vertex v lies on all cycles of G, then the graph G - v is acyclic and hence $\nabla(G) = 1$. □

We now investigate the parameter $\nabla(G)$ for certain families of graphs and with respect to some binary operations on graphs.

2.2 Complete Multipartite Graphs and Complete Graphs

Theorem 2.2.1: If G is the complete multipartite graph $K(m_1, m_2, \ldots, m_n)$ of order p with $m = \max_i \{m_i\}$, $\sum_{i=1}^{n} m_i = p$, and $n > 1$, then $\nabla(G) = p - m - 1$.

Proof: Any two pairs of vertices from different partite sets form a cycle, as do any three vertices from different partite sets. From these facts we see that any decycling set must include all of the vertices from n-1 of the partite sets with at most one exception. A minimum decycling set occurs when the remaining vertices induce a star of maximum order (see Figure 2.2 for an example where the dark vertices form a minimum decycling set). □



$K_{2,3,4}$

Figure 2.2

Corollary 2.2.2: For $p \geq 2$, $\nabla(K_p) = p - 2$.

2.3 Binary Operations

Once a parameter has been evaluated for several classes of graphs, we are often interested in the value of that parameter when these graphs are combined. Here we look at the operations of union, join, and Cartesian product (with varying degrees of success).

Observe that for any graph G the decycling number is the sum of the decycling numbers of its components. Consequently, we have the following result.

Theorem 2.3.1: For disjoint graphs G and H,

$$\nabla(G \cup H) = \nabla(G) + \nabla(H).$$

Another commonly used binary operation on graphs is the join G + H of disjoint graphs G and H, which we defined in Section 1.5 to have vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G)$ and $v \in V(H)\}$.

Theorem 2.3.2: For two graphs G and H:

$$\nabla(G + H) = \min\{|G| + \nabla(H), |H| + \nabla(G)\},$$

unless one of the graphs (say G) is edgeless and $\nabla(H)$ is at least $|H| - |G|$, in which case it is $|H| - 1$.

Proof: In the graph G + H, any set of four vertices, two from each of the constituent graphs, induces a cycle; so in order to decycle the graph, we can have at most a single vertex remain from one of the graphs. However, if any one vertex from one of the graphs does remain, then we must delete all but an independent set of vertices from the other graph in order for the remaining graph to be acyclic. For a graph G without isolated vertices, $\alpha(G) + \beta(G) = |G|$. If we keep the previous fact in mind, then $\beta(G) > \nabla(G)$ in a graph G with no isolated vertices since the graph induced by an independent set of vertices and any one other vertex is acyclic. This means that the most efficient way to decycle the join of graphs is to remove all of the vertices of one graph and a decycling set of minimum cardinality from the other, unless one of the graphs (say G) is edgeless and $\nabla(H)$ is at least $|H| - |G|$, in which case it is more effective to leave a single vertex from H and all those in G. This gives us

$$\nabla(G + H) = \min\{|G| + \nabla(H), |H| + \nabla(G)\}$$

in the first case and $|H| - 1$ in the second. □

Corollary 2.3.3: For $n \geq 3$, the decycling number of the wheel W_n is $\nabla(W_n) = 2$.

Note that Theorem 2.2.1 also follows from Theorem 2.3.2.

The focus of the remainder of this chapter is the *Cartesian product* $G \times H$ of graphs G and H which was defined in Section 1.5.

Unfortunately, there is no simple formula for the decycling number of the Cartesian product of two graphs in terms of the decycling numbers of the two graphs. For example, $\nabla(P_4) = 0$ and $\nabla(P_n) = 0$; yet $\nabla(P_4 \times P_n) = n$.

One result which will be helpful later is the decycling number for the graph $P_2 \times G$, for which sharp bounds can be found. Recall that $\alpha(G)$ is the independence number of G and $\beta(G)$ is the covering number.

Theorem 2.3.4: For every graph G,

$$2\nabla(G) \le \nabla(P_2 \times G) \le \nabla(G) + \beta(G).$$

Proof: The lower bound of $2\nabla(G)$ is easily seen since both copies of G in $P_2 \times G$ must be decycled. To derive the upper bound, let S be a ∇-set for one copy of G with cardinality $\nabla(G)$, and let T be an independent set of G with cardinality $\alpha(G)$. Then G - S is acyclic and the vertices of T have degree 1 in the subgraph induced by the vertices in $(G - S) \cup T$. Hence, that graph is acyclic. Using Gallai's theorem this gives us $\nabla(P_2 \times G) \le \nabla(G) + \beta(G)$. □

The lower bound can be shown to be sharp since $\nabla(P_2 \times C_3) = 2 = 2 \cdot 1 = 2\nabla(C_3)$, while the upper bound is shown to be sharp by the graph $P_2 \times P_n$ since $\nabla(P_2 \times P_n)$ will be shown to be $\lfloor n/2 \rfloor$, which equals $\nabla(P_2) + \beta(P_n)$.

2.4 Grid Graphs

We next consider the family of Cartesian products of paths. We will use the labeling of $P_m \times P_n$ described in Section 1.5 (see Figure 2.3 for an example). Under this labeling, the degree of vertex $v_{i,j}$ is (a) 4 if $1 < i < m$ and $1 < j < n$; (b) 2 if $i = 1, m$ and $j = 1, n$; and (c) 3 otherwise. Also, for $P_m \times P_n$ we will need to consider specific 4-cycles in the graph. Let $C_{i,j}$ denote the 4-cycle induced by the vertices $v_{i,j}$, $v_{i,j+1}$, $v_{i+1,j+1}$, and $v_{i+1,j}$ (see Figure 2.3).

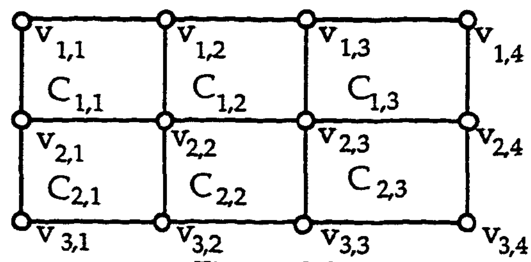


Figure 2.3

Theorem 2.4.1: For $n \geq 2$, $\nabla(P_2 \times P_n) = \lfloor n/2 \rfloor$.

Proof: Since $\nabla(P_n) = 0$ and $\alpha(P_n) = \lfloor n/2 \rfloor$, we have $\lfloor n/2 \rfloor$ as an upper bound from Theorem 2.3.3. Let S be a set of fewer than $\lfloor n/2 \rfloor$ vertices; then by the pigeon-hole principle, for some $j = 1,\dots,n-1$, $V(*,j{:}j+1) \cap S = \varnothing$, so the cycle $C_{1,j}$ is not destroyed. □

Theorem 2.4.2: For $n \geq 3$, $\nabla(P_3 \times P_n) = \lfloor 3n/4 \rfloor$.

Proof: The set

$$S = \{v_{1,j} : j = 3 + 4k \text{ and } k = 0,1,\dots,\lceil (n-3)/4 \rceil\} \cup$$
$$\{v_{2,j} : j = 2k \ \&\ k = 1,2,\dots,\lfloor n/2 \rfloor\}$$

(see Figure 2.4 for an example) is a decycling set of cardinality $\lfloor 3n/4 \rfloor$; hence $\nabla(P_3 \times P_n) \leq \lfloor 3n/4 \rfloor$.

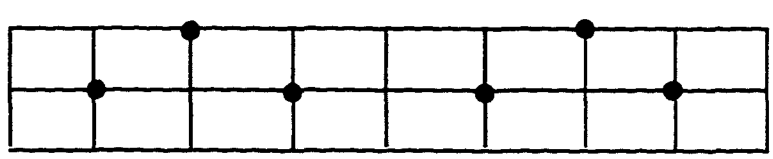


Figure 2.4

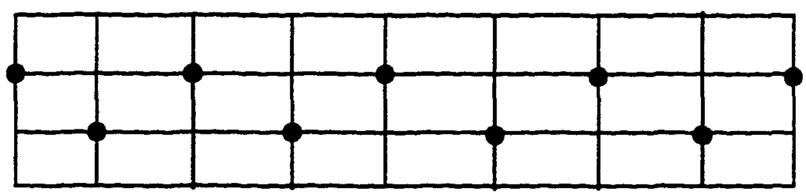
For the lower bound, we use a greedy approach with respect to destroying edges to select the vertices of a decycling set. We want to choose as many vertices of large degree as possible in order to destroy as many edges as we can. To decycle the graph $P_3 \times P_n$, we can choose at most $\lfloor (n-1)/2 \rfloor$ independent vertices of degree 4, since otherwise some edge will be covered twice and our edge count as the sum of the degrees won't be exact. The other t vertices in the decycling set S must be of degree at most 3 in G - S. Using Corollary 2.1.3, we obtain the inequality

$$3n - \lfloor (n-1)/2 \rfloor - t - 1 \geq 5n - 3 - 4\lfloor (n-1)/2 \rfloor - 3t.$$

Applying some algebra on the four cases $n \equiv 0, 1, 2, 3 \pmod 4$, we find that $k = \lfloor (n-1)/2 \rfloor + t \geq \lfloor 3n/4 \rfloor$, which completes the proof. □

Theorem 2.4.3: For $n \geq 2$, $\nabla(P_4 \times P_n) = n$.

Proof: The set $S = \{v_{2,1}, v_{3,2}, v_{2,3}, v_{3,4}, \ldots, v_{t,n} : t = 2$ if n is odd and $t = 3$ if n is even$\}$ is a decycling set of cardinality n (see Figure 2.5).



$P_4 \times P_9$

Figure 2.5

We establish the lower bound using Corollary 2.1.3. Let S be a ∇-set of cardinality k. Any decycling set must have at least one vertex of degree at most 3, since otherwise, the vertices of degree at most 3 form a cycle.

Optimally the remaining vertices have degree 4. Then Corollary 2.1.3 implies $4n - k - 1 \geq 7n - 4 - 4(k - 1) - 3$. Solving this inequality for k, we find that $k \geq n - 2/3$. Hence, $\nabla(P_4 \times P_n) \geq n$ and the theorem follows. □

By Lemma 2.1.1, if we are given a decycling set S of cardinality k in a graph G which has order p, size q, and non-increasing degree sequence $d_1, d_2, \ldots, d_p$, then

$$p - k - 1 \geq q - \sum_{i=1}^{k} d_i .$$

There may be many decycling sets which satisfy this inequality, but the ones in which we will be most interested are those that come as close as possible to making it an equality. With this idea in mind, we use the device of adding in a slack function $\mu(G,k)$, which we will call the *margin of G at k*. This gives us the equation

$$p - k - 1 = q - \sum_{i=1}^{k} d_i + \mu(G,k).$$

We then solve to get

$$\mu(G,k) = p - k - 1 - q + \sum_{i=1}^{k} d_i .$$

The use of this function will become apparent shortly.

A subset T of the vertex set of a graph G with $|T| = \nabla(G)$ need not be a decycling set even though it satisfies the inequality $p - k - 1 \geq q - \sum_{i=1}^{k} s_i$ of Corollary 2.1.3. There are three types of occurrences that might happen within the set T that would violate assumptions made about the parameters in the inequality and perhaps render the inequality untrue. First, G - T may have more than one component, which would decrease the larger side of the inequality. Then, there may be some edges in G having both of their incident vertices in T, thereby increasing the lesser side, and finally the sum of the degrees of the vertices in T may be less than the sum of the $|T|$ largest degrees, which would also add to the value of the smaller side. This means that a minimum decycling set S of cardinality k can't have too many of the aforementioned problems. To keep track of these we define the *outlay* $\theta(S)$ of a set S of vertices. For a set S of cardinality k whose vertices have degrees $s_1, s_2, \ldots, s_k$, we define $\theta(S)$ as the sum of (a) $c(G - S) - 1$, recalling that $c(G)$ is the number of components of G; (b) $\varepsilon(S)$, which we define as the number of edges of G both of whose incident vertices are in S; and (c) $\sum_{i=1}^{k} d_i - \sum_{i=1}^{k} s_i$, where the d_i are the k largest degrees of the vertices of G.

Lemma 2.4.4: If S is any decycling set of $P_m \times P_{2r}$ with $m \geq 4$, then $\theta(S) \geq 2$.

Proof: There are r vertex-disjoint 4-cycles in both V(1:2,*) and V(m - 1:m,*) (see Figure 2.6), so we need to remove at least r vertices from the

V(1:2,*)	X		X		X		X		X		X
V(4:5,*)	X		X		X		X		X		X

Figure 2.6

first and the last pairs of paths P_{2r}. This means that from each of these pairs of paths, we must either choose a vertex of degree at most 3 or a pair of adjacent vertices. Thus we must select at least two such vertices, and the inequality follows. □

Lemma 2.4.5: Let G be a graph and S a set of k vertices of G. If $\theta(S) > \mu(G,k)$, then S is not a decycling set of G.

Proof: Let G be a graph of order p, size q, and non-increasing degree sequence $d_1, d_2, \ldots, d_p$, and let S be a set of k vertices with degrees $s_1, s_2, \ldots, s_k$. Assume S is a decycling set for which $\theta(S) > \mu(G,k)$. By substitution we derive the inequality

$$c(G - S) - 1 + \varepsilon(S) + \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} s_i > p - k - 1 - q + \sum_{i=1}^{k} d_i .$$

This inequality can be algebraically reduced to

$$q - \sum_{i=1}^{k} s_i - \varepsilon(S) > p - k - c(G - S).$$

Both sides of the inequality are expressions for the number of edges in G - S, and hence are equal. Thus we have a contradiction to the original assumption, and the lemma is proved. □

We will use this result along with the value

$$\mu(P_m \times P_n, k) = 3k - (m - 1)(n - 1)$$

(which is $3k - 4n + 4$ for the case when $m = 5$) to complete some of the proofs that follow.

Lemma 2.4.6: For n > 1,

$$\nabla(P_5 \times P_n) \geq \left\lceil \frac{4n}{3} \right\rceil \text{ when } n \equiv 0(\text{mod } 6)$$
$$\nabla(P_5 \times P_n) \geq \left\lceil \frac{4n}{3} \right\rceil - 1 \text{ otherwise.}$$

Proof: The graph $P_5 \times P_n$ has 5n vertices and 9n - 5 edges. If we try to choose a decycling set S in a greedy manner with respect to edge removal, then one vertex in S must have degree at most 3, since the vertices of degree 2 and 3 form a cycle. The remaining vertices may be of degree 4. If $|S| = k$, then Lemma 2.1.1 implies that $5n - k - 1 \geq 9n - 5 - 4(k-1) - 3$, so $k \geq \lceil 4n/3 \rceil - 1$.

Assume that $n \equiv 0(\text{mod } 6)$. If $k = \lceil 4n/3 \rceil - 1$, then $\mu(P_5 \times P_n, k) = 1$, but, by Lemma 2.4.4, we know $\theta(S) \geq 2$; hence $k \geq \lceil 4n/3 \rceil$. □

Lemma 2.4.7: For n = 2, 3, 4, 5, 7, 8, 9,

$$\nabla(P_5 \times P_n) = \lceil 4n/3 \rceil - 1 \text{ and } \nabla(P_5 \times P_6) = 8.$$

Proof: We have already established this result for n = 2, 3, 4 and have obtained the given values as lower bounds for the other values of n; so the examples of decycling sets shown in Figures 2.7.a and 2.7.b complete the

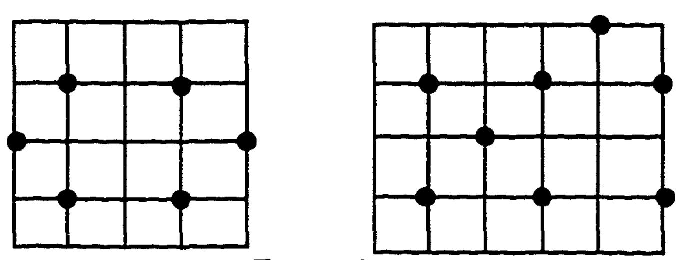
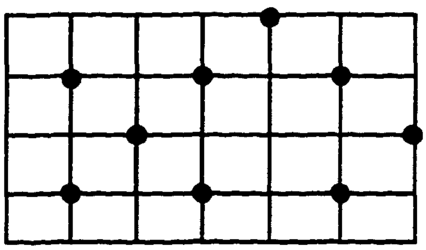
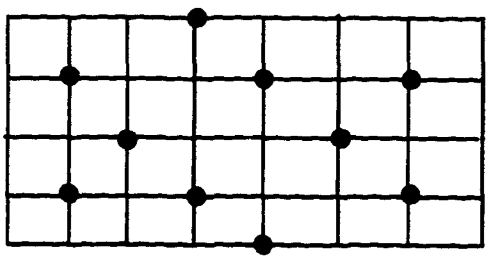


Figure 2.7.a





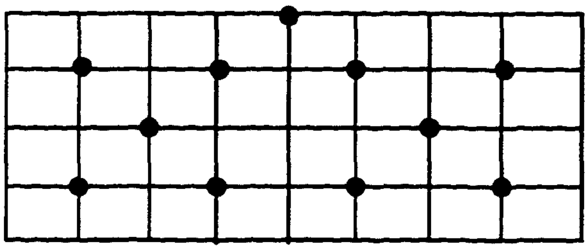


Figure 2.7.b

proof. □

Lemma 2.4.8: For $P_5 \times P_9$, a decycling set of cardinality 11 is unique up to symmetry.

Proof: For the graph $G = P_5 \times P_9$, $\mu(G,11) = 1$. Therefore our decycling set must contain eleven independent vertices, ten of which have degree 4 and the other degree 3. Since we may only use one vertex of degree 3, any decycling set S must contain at least three of the vertices $v_{2,2}$, $v_{4,2}$, $v_{2,8}$ and $v_{4,8}$ (for otherwise one of the 4-cycles $C_{1,1}$, $C_{4,1}$, $C_{1,n-1}$, $C_{4,n-1}$ will remain). If one of these four vertices is missing, say $v_{2,2}$, then the 4-cycle $C_{1,1}$ must intersect S in either $v_{1,2}$ or $v_{2,1}$.

It follows that if we choose $v_{1,2}$, then the 4-cycle $C_{2,1}$ remains; while choosing $v_{2,1}$ leaves the 8-cycle that includes $v_{5,1}$ and $v_{3,3}$. We are then forced to include the vertices $v_{3,3}$ and $v_{3,7}$ in S, since otherwise one of the four 8-cycles which enclose either $v_{2,2}$, $v_{4,2}$, $v_{2,8}$ or $v_{4,8}$ will remain.

The four 4-cycles $C_{1,3}$, $C_{4,3}$, $C_{1,6}$, $C_{4,6}$ must intersect S in at least three of the vertices $v_{2,4}$, $v_{4,4}$, $v_{2,6}$, and $v_{4,6}$. If one vertex, say $v_{2,4}$, is not in S, then we must find one vertex of degree 3 and one of degree 4 which lie in all of the remaining cycles, and these two vertices must not be adjacent to any of the others. The only way to complete the decycling set is to choose $v_{2,4}$ and either $v_{1,5}$ or $v_{5,5}$ (see Figure 2.7). The two different sets we get are clearly symmetric. □

We use the decycling set of Lemma 2.4.8 as a building block in constucting decycling sets for grids of the form $P_5 \times P_{8n+1}$.

Lemma 2.4.9: For each positive integer n, the minimum decycling sets for $P_5 \times P_{8n+1}$ have cardinality 11n and are of the form

$$S = \{v_{i,j} \mid i = 2 \text{ or } 4 \text{ and } j = 2t \text{ for } t = 1,2,\ldots,4n\} \cup$$
$$\{v_{i,j} \mid i = 3 \text{ and } j = 4t - 1 \text{ for } t = 1,2,\ldots 2n\} \cup$$
$$\{v_{i,j} \mid j = 8t - 3 \text{ for } t = 1,2,\ldots,n \text{ and } i = 1 \text{ or } 5\}.$$

(see Figure 2.8 for example)

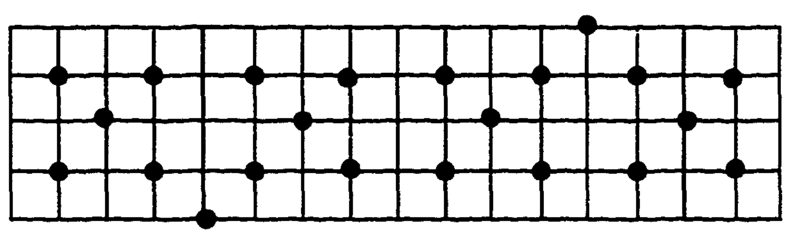


Figure 2.8

Proof: The anchor for an induction proof is given in Lemma 2.4.8. Assume that the lemma holds for all $n < k$. Let S be a minimum decycling set for $P_5 \times P_{8k+1}$. We complete the proof by considering the three forms that a decycling set S can take.

Case 1: $S \cap V(*,8n + 1) = \emptyset$ for some n = 1, 2, ..., k-1. Then $S \cap V(*,1{:}8n + 1)$ and $S \cap V(*,8n + 1{:}8k + 1)$ must be of the prescribed form; hence the lemma is true.

Case 2: $|S \cap V(*,1{:}8n + 1)| = 1$ for some n = 1, 2, ..., k-1. By the induction hypothesis, the decycling set is not minimal for $V(*,1{:}8n + 1)$ and $V(*,8n + 1{:}8k + 1)$; so $|S \cap V(*,1{:}8n + 1)| > 11n$ and $|S \cap V(*,8n + 1{:}8k + 1)| > 11(k\text{-}n)$. This implies that $|S| > 11k$, which is a contradiction.

Case 3: $|S \cap V(*,8n + 1)| \geq 2$ for n = 1, 2, ...,k-1. Here we know that $\mu(P_5 \times P_{8k+1},11k) = k$, so all we need show is that $\theta(S) \geq k + 1$.

Since $\nabla(P_5 \times P_5) = 6$, we know that $V(*,1{:}5) \cap S$ contains at least six vertices. Regardless of how we place these six vertices, we must increase the outlay by at least 1. The largest independent set of vertices of degree 4 in $V(*,1{:}5)$ in the original graph has order 6. However, choosing such an independent set for inclusion in S adds another component to G - S. Whether we use an independent set or not, we thus add at least 2 to the outlay. Using the same argument, we must add 2 to the outlay if $|V(*,1{:}5) \cap S| = 7$.

If the outlay is only increased by 1 for $V(*,1{:}5) \cap S$, then it must be increased by another in $V(*,6{:}9) \cap S$. In this case, $|V(*,6{:}8) \cap S| \geq 4$ since $\nabla(P_5 \times P_8) = 10$ and only 6 vertices were used in $V(*,1{:}5)$. Since $|V(*,9) \cap S| \geq 2$ by assumption, either the outlay is increased by one, or $v_{9,2}$ and $v_{9,4}$ are in S (see Figure 2.9). This leaves only seven vertices in $V(*,6{:}8)$ (see Figure 2.9 This does not automatically increase the outlay, but regardless of how

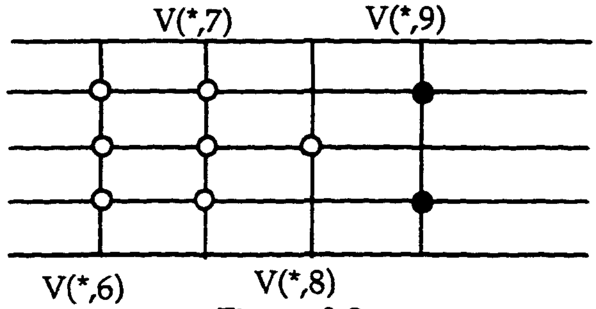


Figure 2.9

we choose four of the seven, we end up increasing the outlay by either isolating a vertex or choosing a vertex adjacent to another in S.

For each $V(*,8n + 2,8n + 8) \cap S$, where $n = 1,\dots,k - 1$, we must also increase the outlay by 1. Recall that $\nabla(P_5 \times P_7) = 9$ and for each of these subgraphs we can use at most nine vertices in a decycling set or we get a contradiction. If $|V(*,8n + 5) \cap S| > 2$, then the outlay increases by 1. When $|V(*,8n + 5) \cap S| = 2$, then either (a) the outlay is increased; (b) $V(*,8n + 5) \cap S = \{v_{8n+5,2}, v_{8n+5,4}\}$ and $|V(*,8n + 2{:}8n + 4) \cap S|$ is at least 4; or (c) $V(*,8n + 5) \cap S = \{v_{8n+5,2}, v_{8n+5,4}\}$ and $|V(*,8n + 6{:}8n + 8) \cap S|$ is at least 4. In any of the cases, there are only seven vertices whose inclusion in S does not automatically increase the outlay, but any choice of four of them will increase the outlay. If $|V(*,8n + 5) \cap S| = 1$, then either $v_{8n+5,2}$ (equivalently $v_{8n+5,4}$) or $v_{8n+5,3}$ is in S. If $v_{8n+5,2}$ is in S, then regardless of how vertices are chosen for S from $V(*,8n + 4)$ and $V(*,8n + 6)$, either a cycle remains or the outlay is increased by one. On the other hand, if $v_{8n+5,3}$ is chosen for S, then we must also choose $v_{8n+4,2}$, $v_{8n+4,4}$, $v_{8n+6,2}$, and $v_{8n+6,4}$ or leave a cycle. Thus, however a vertex is chosen from $V(*,8n + 7)$ or $V(*,8n + 3)$ for S, the outlay increases, but if none are chosen, then a cycle remains.

Thus, the outlay of the set is at least $k + 2$, which is more than the k allowed. □

Theorem 2.4.10: Let m, n, and q be integers such that

$$1 \le m \le 8,\ q \ge 0 \text{ and } n = 8q + m. \text{ Then}$$
$$\nabla(P_5 \times P_n) = 11q + \nabla(P_5 \times P_m).$$

Proof: We prove the lower bound by induction on n. The result is trivially true for $1 \le n \le 8$. Assume it to be true for all $n \le k$, where $k \ge 8$. If $n = k + 1 = 8q + m$ $(m < 8)$ and the set S is a decycling set of minimum cardinality, then we consider the sets $V(*,8t+1)$ for $t = 1,2,\ldots,q$, and their intersections with S.

Case 1: At least one of the intersections (say $S \cap V(*,8r + 1)$) is empty. There must be at least $11r$ vertices of $V(*,1:8r + 1)$ in S by Lemma 2.4.9 and at least $11(q - r) + \nabla(P_5 \times P_m)$ vertices of $V(*,8r + 1:8q + m)$ by the induction hypothesis. Hence $\nabla(P_5 \times P_n) \ge 11q + \nabla(P_5 \times P_m)$.

Case 2: $|V(*,8r + 1) \cap S| = 1$ for some r. Then S must contain at least $11r$ vertices of $V(*,1:8r)$ since this along with $V(*,8r + 1)$ forms a $P_5 \times P_{8r+1}$ whose decycling set is not of the type in Corollary 2.4.9, and hence is not minimal. Therefore it contains at least $11r + 1$ vertices. In S, $V(*,8r + 1:8q + m)$ must contain at least $11(q-r) + \nabla(P_5 \times P_m)$ vertices, so $\nabla(P_5 \times P_n) \ge 11q + \nabla(P_5 \times P_m)$.

Case 3: For each r, $|S \cap V(*,8r + 1)| \ge 2$. In this case, S must contain at least $11r - 1$ vertices in $V(*,1:8r)$ since the decycling set for $V(*,1:8r + 1)$ is not minimal.

If $m = 0, 1, 3, 5, 7$, then the last $P_5 \times P_{8(q-1)+m-1}$ must contain at least

$11(q - r) + \nabla(P_5 \times P_{m-1}) = 11(q - r) + \nabla(P_5 \times P_m) - 1$ vertices of S, which gives the desired result.

If $m = 2, 4, 6$, then $\mu(G, 11q + \nabla(P_5 \times P_m) - 1) = q - 3, q$, or $q+1$, respectively, and using an argument similar to that of Case 3 in Lemma 2.4.9, we get contradictions for all three cases.

In each of the above cases, the upper bound can be shown to hold by constructing a decycling set as the union of appropriate decycling sets from Figures 2.7 and 2.8. □

For $P_6 \times P_n$ we already know the decycling number for $n < 6$, so we need only consider $n \geq 6$.

Theorem 2.4.11: For $n \geq 6$, $\nabla(P_6 \times P_n) = \left\lceil \frac{5n}{3} \right\rceil$.

Proof: The decycling set

$$S = \{v_{2,i} \mid i \equiv 1 (\text{mod } 2)\} \cup \{v_{3,i} \mid i \equiv 0 \text{ or } 2 (\text{mod } 6)\} \cup$$
$$\{v_{4,i} \mid i \equiv 3 \text{ or } 5 (\text{mod } 6)\} \cup \{v_{5,i} \mid i \equiv 0 (\text{mod } 2)\}$$

establishes the upper bound (see Figure 2.10).

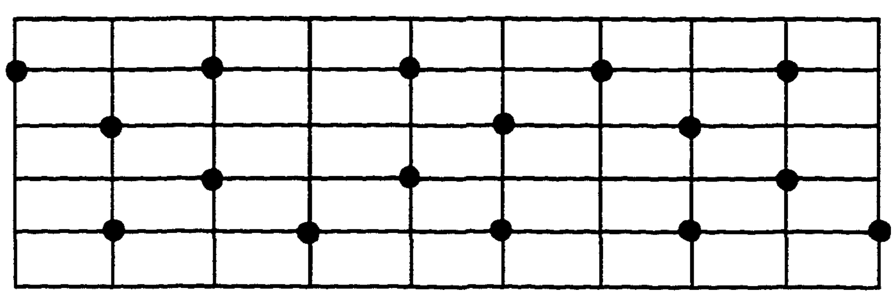


Figure 2.10

The proof of the lower bound depends on the n. Let S be a ∇-set of G. Since 6 is even, Lemma 2.4.4 implies that S must use at least two vertices that contribute at most 3 to the edge count in Lemma 2.1.1. Consequently, when $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$, we have the lower bound of $\left\lceil \frac{5n}{3} \right\rceil$. This leaves only the case $n \equiv 0 \pmod 3$; say $n = 6t$, whence $\left\lceil \frac{5n}{3} \right\rceil = 10t$. Suppose that S has cardinality $k = 10t - 1$ (it cannot be less by Lemma 2.1.1). Then $m(P_6 \times P_n, k) = 2$. Lemma 2.4.4 implies that $\theta(S) = 2$. In fact, V(*,1:2) must either contain a vertex x of degree 3 or a pair x, x' of adjacent vertices of degree 4 (in this case x is either $v_{3,2}$ or $v_{4,2}$). We take a vertex of V(*,n - 1:n) analogous to x and call it y. (In passing, we note that at least one of x and y has degree 3.) Let S' = S - {x,y}. It follows that S' must consist of 10t - 3 independent vertices of degree 4. Note that the removal of any independent set of six degree-4 vertices from three consecutive columns leaves an isolated vertex. This would increase the number of components of G - S, and thereby the outlay; so S' cannot contain six such vertices. The largest independent set of degree-4 vertices in $P_6 \times P_n$ that does not isolate a vertex is of cardinality 10t - 3. This set can be shown to be unique up to isomorphism as follows. Let *N(i)* (respectively *N(i:j)*) denote the number of vertices in S - {x,y} ∩ V(*,i) (resp. S - {x,y} ∩ V(*,i:j)). Then in order to decycle V(*,1:2), then, we must have N(2) = 2. We consider two cases depending on the distance d between the two vertices of S' in V(*,2).

Case 1: d = 3. In this case the vertices are $v_{2,2}$ and $v_{5,2}$, which means that either $v_{3,3}$ or $v_{4,3}$ is the only vertex in S ∩ V(*,3). However, this then forces $\{v_{4,2}, v_{4,5}\}$ = S ∩ V(*,4). The 2-1-2-1-... pattern is forced for the rest of the independent set; hence $N(2:n\text{-}1) < \frac{5n}{3} - 3$, which is a contradiction.

Case 2: $d = 2$. Without loss of generality, the vertices are $v_{3,2}$, and $v_{5,2}$. Now S must contain $v_{4,3}$ (or a cycle will remain) and $v_{2,3}$. Otherwise, $S \cap V(*,4) = \{v_{2,4}, v_{5,4}\}$, which we have seen in Case 1 leads to a contradiction. This implies that $S \cap V(*,4) = \{v_{5,4}\}$. If not, then either a vertex is isolated, which excessively increases the outlay of S, or a cycle remains. We are forced to choose $v_{2,5}$ and $v_{4,5}$ in $V(*,5)$ or leave a cycle intact, which essentially brings us back to where we started. This 2-2-1-2-2-1... pattern must continue, and for each $V(*,i)$ the choice of vertices is forced. When $n = 3t$, this forces $V(*,n\text{-}1:n)$ to contain at least four vertices of S, which would make the outlay of S too large.

In either case, for $n = 3t$, $5n/3 - 1$ vertices is not enough to decycle the graph $P_6 \times P_n$, and hence the equality holds. □

Theorem 2.4.12: $\nabla(P_{3s+1} \times P_{2t}) = s(2t - 1) + 1$.

Proof: We get the expression for the lower bound using an argument similar to those in previous proofs, while the upper bound is shown by the set

$$\{v_{2i,2j} \mid i=1,\ldots,\left\lfloor \frac{3s+1}{2} \right\rfloor, 2i \equiv 1 \pmod 3, \text{ and } j=1,\ldots,t-1\} \cup$$

$$\{v_{2i+1,2j+1} \mid i=1,\ldots,\left\lfloor \frac{3s+1}{2} \right\rfloor - 1, 2i+1 \equiv 1 \pmod 3, \text{ and } j = 1,\ldots,t-1\} \cup$$

$$\{v_{3i+1,j} \mid i = 0,\ldots,s, j = 2t \text{ if } i \text{ odd, and } j = 2t-1 \text{ if } i \text{ even}\}$$ □

Theorem 2.4.13: For $n \geq 7$, $\nabla(P_7 \times P_n) = 2n\text{-}1$.

Proof: For n even, this follows from Theorem 2.4.12. If n is odd, each decycling set must contain at least one vertex of degree at most 3, for otherwise those vertices form a cycle. Adjusting the inequality accordingly, we get $\nabla(P_7 \times P_n) \geq \lceil (6n - 5)/3 \rceil = 2n - 1$. The upper bound is shown by the decycling set

$$S = \{v_{2,i} \mid i \equiv 0 \pmod 2\} \cup \{v_{3,i} \mid i > 1 \ \&\ i \equiv 1 \pmod 2\} \cup$$
$$\{v_{4,2}\} \cup \{v_{5,i} \mid i > 1 \ \&\ i \equiv 1 \pmod 2\} \cup$$
$$\{v_{6,i} \mid i \equiv 0 \pmod 2\}$$

(see Figure 2.11). □

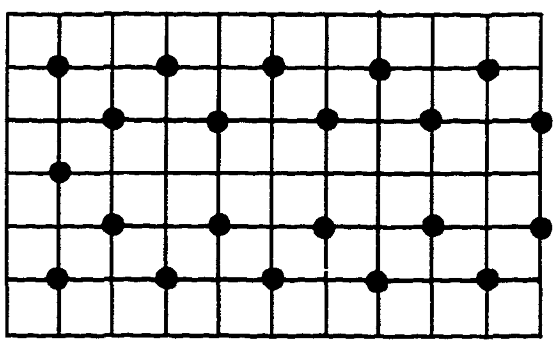


Figure 2.11

Theorem 2.4.14: For $n \geq 8$, $\nabla(P_{10} \times P_n) = 3n - 2$.

Proof: Once again the even case follows from Theorem 2.4.12. When n is odd, we get the lower bound as before and the upper bound from the set

$$\{v_{i,2j} \mid i = 2, 6, \text{ or } 9 \text{ and } j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\} \cup$$
$$\{v_{i,2j+1} \mid i = 3, 5, \text{ or } 8 \text{ and } j = 1, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor\} \cup \{v_{4,2}, v_{4,n}, v_{7,1}, v_{8,n}\}$$

(see Figure 2.12). □

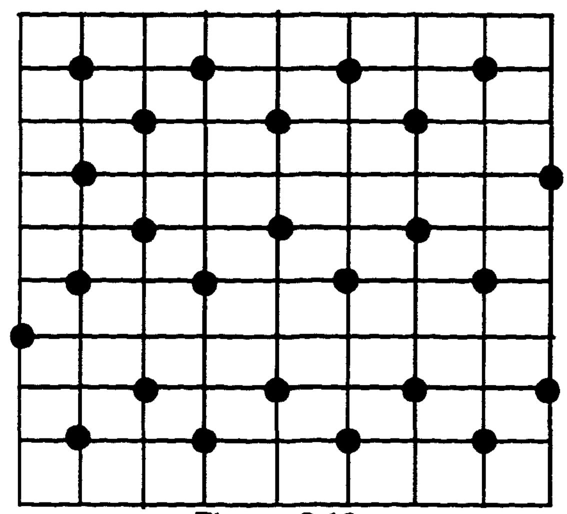


Figure 2.12

Theorem 2.4.15: For $n \geq 11$, $\nabla(P_{13} \times P_n) = 4n - 3$.

Proof: Theorem 2.4.12 establishes the even case. The lower bound for the odd case is shown as before. The upper bound is demonstrated by the set

$$S = \{v_{2i,2j} \mid i = 1,3,4, \text{ or } 6 \text{ and } j = 1,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\} \cup$$

$$\{v_{i,2j+1} \mid i = 3,5,9 \text{ or } 11 \text{ and } j = 1,\ldots,\left\lfloor \frac{n-2}{2} \right\rfloor\} \cup$$

$$\{v_{4,2}, v_{4,n}, v_{7,n}, v_{10,2}, v_{10,n}\}$$

(see Figure 2.13). □

In looking at the decycling sets of Theorems 2.4.12 and 2.4.13, one can see how the decycling set for $P_7 \times P_n$ is used to construct the sets for $P_{13} \times P_n$. This idea can be extended to the general case of $P_m \times P_n$ to produce a very good upper bound for the decycling number.

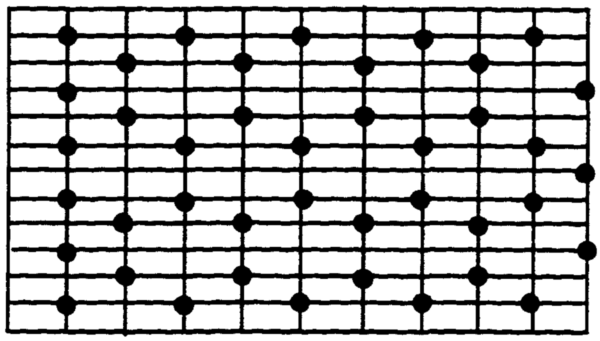


Figure 2.13

Theorem 2.4.16: Let $m = 6q + r$ and $n = 6s + t$ with $1 \le r, t \le 6$. Then $\nabla(P_m \times P_n) \le \min\{q(2n - 1) + \nabla(P_r \times P_n), s(2m - 1) + \nabla(P_m \times P_t)\}$.

2.5 Other Cartesian Products

A natural next step is to consider the product of paths and cycles. We start with the prisms $P_2 \times C_n$.

Theorem 2.5.1: For $n \ge 3$, $\nabla(P_2 \times C_n) = \lfloor n/2 \rfloor + 1$.

Proof: If the vertices of $G = P_2 \times C_n$ are labeled as described in Section 1.5, then the set $S = \{v_{1,1}, v_{2,2}, v_{2,4}, v_{2,6}, \ldots, v_{2,j-2}, u_j : j = 2\lfloor n/2 \rfloor\}$ is a decycling set

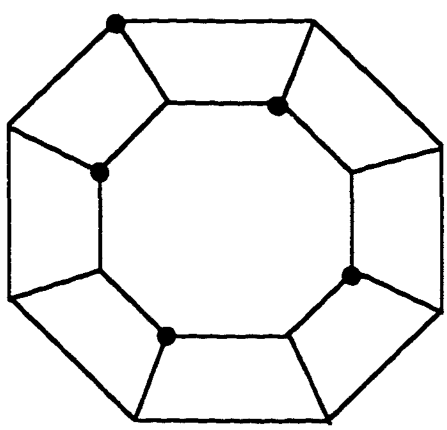


Figure 2.14

of cardinality $\lfloor n/2 \rfloor + 1$ (see Figure 2.14). Since G is 3-regular of order 2n, every decycling set S of cardinality k must contain at least $\lfloor n/2 \rfloor + 1$ vertices. This is because the size of a tree on $|G - S|$ vertices is $2n - k - 1 \geq 3n - 3k$ which is the minimum size of G - S. □

Theorem 2.5.2: For $n \geq 3$, $\nabla(P_3 \times C_n) = \left\lfloor \frac{3n + 5}{4} \right\rfloor$.

Proof: We label the vertices of $P_3 \times C_n$ as described in Section 1.5. Then if $n \equiv 0, 2,$ or 3 (mod 4), the set

$$S = \{v_{1,i} : i = 3 + 4k \;\&\; k = 0,1,\ldots,\left\lfloor \frac{n}{4} \right\rfloor\} \cup$$
$$\{v_{2,j} : j = 2k \;\&\; k = 1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\} \cup \{v_{3,1}\}$$

is a decycling set of G of cardinality $\left\lfloor \frac{3n+5}{4} \right\rfloor$. If $n \equiv 1$ (mod 4), the set

$$S = \{v_{1,i} : i = 3 + 4k \;\&\; k = 0,1,\ldots,\left\lfloor \frac{n}{4} \right\rfloor\} \cup$$
$$\{v_{2,j} : j = 2k \;\&\; k = 1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\} \cup \{v_{3,1}, v_{2,n}\}$$

is a decycling set of cardinality $\left\lfloor \frac{3n+5}{4} \right\rfloor$. The lower bound is shown through an argument by cases which involves the inequality on the size used in several of the previous proofs. □

Theorem 2.5.3: For $n \geq 3$,

$$\nabla(P_4 \times C_n) = \begin{cases} n+1 & \text{if } n \equiv 1 (\text{mod } 2) \\ n+2 & \text{if } n \equiv 0 (\text{mod } 2) \end{cases}.$$

Proof: Any decycling set of $P_4 \times C_n$ must contain at least two vertices of degree 3, for otherwise a cycle of degree-3 vertices remains. A lower bound comes from another edge counting argument similar to that employed in the proof of Corollary 2.1.3. If n is odd, this bound is shown to be sharp by the set (seeFigure 2.15) $S = \{v_{1,1}, v_{1,4}, v_{2,3}, v_{3,2}, v_{4,2}, v_{5,3}, \ldots v_{n-1,3}, v_{n,2}\}$.

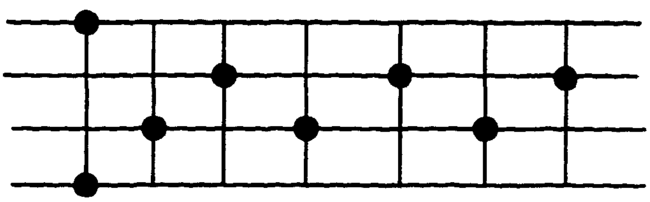


Figure 2.15

If n is even, then $\mu(P_4 \times C_n, n+1) = 2$, so any decycling set must consist of exactly two vertices of degree 3 and an independent set of n-1 degree-4 vertices. The independent set must be similar to $\{v_{2,3}, v_{3,2}, v_{4,2}, v_{5,3}, \ldots v_{n-1,2}, v_{n,3}\}$, which cannot be completed as a decycling set using only two vertices of degree 3; therefore if n is even, then $\nabla(P_4 \times C_n) > n + 1$. The upper bound is demonstrated by the set

$$S = \{v_{1,4}, v_{2,1}, v_{1,2}, v_{2,3}, v_{3,2}, v_{4,2}, v_{5,3}, \ldots v_{n-1,2}, v_{n,3}\}$$

(see Figure 2.16). □

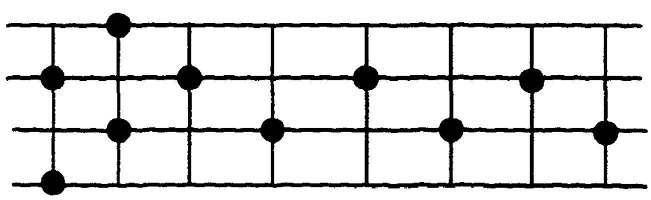


Figure 2.16

The next product that we consider is that of two cycles. From Corollary 2.1.2, we obtain the folowing preliminary result for such products.

Lemma 2.5.4: For $m,n \geq 3$, $\nabla(C_m \times C_n) \geq \left\lceil \frac{mn+1}{3} \right\rceil$.

Theorem 2.5.5: For $n \geq 3$, $\nabla(C_3 \times C_n) = n + 1$.

Proof: The lower bound follows from Lemma 2.5.4. If the vertices are labeled as described in Section 1.5, then $\{v_{i,j} : j \equiv i(\text{mod } 3)\} \cup \{v_{1,3}\}$ is a decycling set of cardinality n + 1 (see Figure 2.17). This establishes the upper bound, and hence the result. □

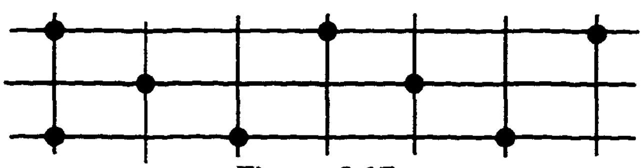


Figure 2.17

Theorem 2.5.6: For $n \geq 3$, $\nabla(C_4 \times C_n) = \left\lceil \frac{3n}{2} \right\rceil$.

Proof: From Theorem 2.5.1, we have $\nabla(P_2 \times C_4) = 3$. Hence any decycling set must meet one of the pairs of 4-cycles $C_{i,j}$ and $C_{i,j+1}$ or $C_{i,j}$ and $C_{i+1,j}$ in at least three vertices, which gives the lower bound of $\left\lceil \frac{3n}{2} \right\rceil$. Label the vertices as described in Section 1.5. Then the set

$$\{v_{i,j} \mid j \equiv i(\text{mod } 4)\} \cup \{v_{1,j} \mid j \equiv 3(\text{mod } 4)\} \cup \{v_{3,j} \mid j \equiv 1(\text{mod } 4)\}$$

of vertices (see Figure 2.18) is a decycling set of cardinality $\left\lceil \frac{3n}{2} \right\rceil$, which completes the proof. □

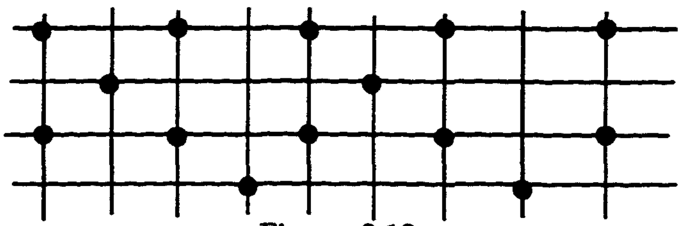


Figure 2.18

Another interesting product is $K_m \times K_n$. Here we think in terms of the maximum order of an induced forest. Clearly, an induced forest cannot contain three vertices in the same complete graph, and hence no vertex of degree 3 (otherwise two of its neighbors lie in the same complete graph). Also, if the forest is not connected, it is not maximal as we can add a common neighbor of any pair of vertices of degree 1 from different components of the forest. From the two preceding statements, it can be seen that a maximal induced forest in $K_m \times K_n$ will have similar characteristics to an n-snake (an induced path of maximum length) in the n-dimensional cube Q_n (see below).

Theorem 2.5.7: For $n \geq m \geq 1$,

$$\nabla(K_n \times K_m) = \begin{cases} m(n-2) & \text{if } n > m \\ m(n-2)+1 & \text{if } n = m \end{cases}.$$

Proof: If $n > m$, than each induced path contains at most two vertices from each copy of K_n. If the vertices have the standard labeling, then the set $S = \{v_{1,1}, v_{2,1}, v_{2,2}, v_{3,2}, v_{3,3},\ldots,v_{m-1,m-2}, v_{m-1,m-1}, v_{m,m-1}, v_{m,m}\}$ induces a path of order 2m. If, on the other hand, n = m, then any set of 2m vertices induces a cycle in the product while $S - \{v_m u_m\}$ induces a path. □

2.6 The n-Dimensional Cube

The final Cartesian product of graphs that we consider here is the n-cube Q_n. As mentioned before, one type of maximal tree in Q_n is a path which we call the *n-snake*. Unfortunately, as n gets large, an n-snake is not an induced forest of maximum order. There are two fairly standard ways of describing the n-cube, both of which will be used here. First, one can take the vertices to be binary sequences of length n. Two vertices are adjacent if their sequences differ in exactly one position. This perspective lends itself nicely to coding theory, which we will use to find large induced forests and thus some upper bounds for the decycling number. The more traditional graph theory approach is to define Q_n inductively as $K_2 \times Q_{n-1}$ with $Q_1 = K_2$. Label the vertices of Q_1 as 0 and 1; then we obtain an inductive labeling for the vertices of $Q_n = K_2 \times Q_{n-1}$ by adding a prefix of 0 to the labels of one copy of Q_{n-1} and a prefix of 1 to the other labels. Using either concept we arrive at the same graph with the same labeling.

For $n \leq 8$, the value of $\nabla(Q_n)$ is demonstrated by the following collection of lemmas and previously known results.

Lemma 2.6.1: $\nabla(Q_3) = 3$.

Proof: From Lemma 2.1.1, if the cardinality of a minimum decycling set is k, then $2^3 - k - 1 \geq 3(2^{3-1}) - 3k$; so $k \geq 3$. Let F be a maximum induced forest in Q_3. If $\Delta(F) = 2$, then F is isomorphic to P_5; and if $\Delta(F) = 3$, then F is the graph $K_{1,3} \cup K_1$ (otherwise we get a cycle). Each of these forests has five vertices; so $|S| \leq 3$. Hence $\nabla(Q_3) = 3$. □

Lemma 2.6.2: $\nabla(Q_4) = 6$.

Proof: An implication of Theorem 2.3.4 (on the decycling number of $P_2 \times G$) is that a decycling set for Q_4 must have at least six vertices. We get our maximum induced forest using coding theory. Let A(n,d) be the maximum number M of codewords in any (n,M,d) code (with words of length n such that any pair of codewords differ in at least d positions). The value of A(4,4) is 2. Any set of codewords in a (4,2,4) code along with the vertices adjacent to them (e.g., {<u>0000</u>,0001,0010,0100,1000} ∪ {<u>1111</u>,1110,1101,1011,0111}) induces a forest of order 10 (since the distance between codewords is 4, we obtain a union of stars); hence the upper bound for $\nabla(Q_4)$ is 6. □

These maximal forests and the associated minimum decycling sets of Q_4 are unique up to isomorphism. Since we require this fact in the proof of the next case, we justify it in the following lemma.

Lemma 2.6.3: In Q_4, every maximum induced forest is of the form $2K_{1,4}$, where the distance between the central vertices is 4.

Proof: Consider two copies of Q_3 making up Q_4. Each copy must be decycled by removing three vertices. This leaves a copy of either P_5 or $K_{1,3} \cup K_1$ in each Q_3. Not both can be P_5 since, by the pigeon-hole principle, one path must have at least two vertices adjacent to vertices in the other path, and these force an induced cycle. If only one is a P_5, then either it has two vertices adjacent to the $K_{1,3}$, or three vertices adjacent to the associated decycling set. In either case, a cycle is induced. In the final case when the forests in each copy of Q_3 are $K_{1,3} \cup K_1$, then when the distances between the vertices of degree 3 in the two $K_{1,3}$ is less than 4, one star has a pair of pendant vertices adjacent to the same vertex of the other $K_{1,3}$ (see Figure 2.19). □

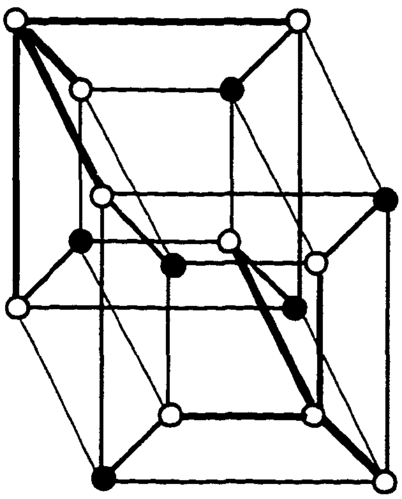


Figure 2.19

Lemma 2.6.4: $\nabla(Q_5) \geq 14$.

Proof: Let S be a decycling set for Q_5. Lemma 2.1.1 states that if the cardinality of S is k, then

$$2^5 - k - 1 \geq 5(2^{5-1}) - 5k,$$

so $k \geq 13$. Assume that S has cardinality 13, and recall that $Q_5 = K_2 \times Q_4 = C_4 \times Q_3$. The vertices with the first two coordinates 00 induce a Q_3 as do those which begin 01, 10, and 11. For convenience we will call these 3-cubes Q00, Q01, Q10, and Q11, respectively. Since k = 13, S must intersect one of the 3-cubes (say Q11) in exactly four vertices while it meets the other three in precisely three vertices. Once the decycling set for Q00 is chosen, then from Lemma 2.6.3, the decycling sets for Q01 and Q10 are forced upon us and have the same labels except for the first two coordinates. By symmetry, each of the five vertices in the Q10 forest are adjacent to the corresponding vertex (same final three coordinates) in Q01 in the graph (Q_4 - Q11) - S, so no matter how we remove the four vertices from Q11, a cycle will remain. This implies that $\nabla(Q_5) \geq 14$. (The upper bound is established by the decycling set {00011, 00110, 00101, 01000, 01001, 01010, 01100, 10100, 10010, 10001, 10111, 11011, 11110, 11101}) □

Lemma 2.6.5: $\nabla(Q_8) \leq 112$.

Proof: The lower bound of 112 comes from Theorem 2.3.4 while an upper bound of 112 comes from coding theory. It is known that A(8,4) = 16. The subgraph F of Q_8 induced by the vertices associated with an (8,16,4) code and the neighbors of those vertices is a forest of order 144; hence $V(Q_8)$ - F is a decycling set of cardinality 112. □

For n = 6, 7, the decycling numbers are derived from Theorem 2.3.4 and the cases when n = 5 and n = 8.

Theorem 2.6.6: For the n-cube, $\nabla(Q_n)$ is

n	1	2	3	4	5	6	7	8
$\nabla(Q_n)$	0	1	3	6	14	28	56	112

For cubes of dimension $n > 8$ Theorem 2.3.3, along with the fact that the independence number of Q_n is 2^{n-1}, give the bounds $112 \cdot 2^{n-8} \leq \nabla(Q_n) \leq 112 + 2^7 + \ldots + 2^{n-2}$. These bounds are not very close. When n is smaller, coding theory improves the upper bound a little. Recall that a distance-4 code along with its adjacent vertices induce a forest, so $\nabla(Q_n) \leq 2^n - (n+1)A(n,4)$. It is known that $A(9,4) = 20$, $38 \leq A(10,4) \leq 40$, $72 \leq A(11,4) < 80$, $144 \leq A(12,4) < 160$, and $A(13,4) = 256$. We can use the lower bounds mentioned here to derive upper bounds for the decycling numbers, and extend our results for $\nabla(Q_n)$ as shown in Table 2.

Table 2

Bounds on $\nabla(Q_n)$

n	Lower Bound	Upper Bound
8	112	112
9	224	312
10	448	606
11	896	1184
12	1792	2224
13	3584	4680

2.7 Edge Decycling Index

It is natural to consider the analogous concept for edge deletion. The *edge decycling index* $\nabla'(G)$ of G is defined as the minimum cardinality of a set S of edges of G for which G - S is acyclic. However this concept is already known as the cycle rank r(G) of the graph, and if G is a (p,q) graph, then $r(G) = \nabla'(G)$ is known to be $q - p + 2 - c(G)$, where c(G) is the number of components of G.

CHAPTER III

THE INTEGRITY OF ORIENTED GRAPHS

3.1 Definitions

Analogous to the graphical definition, the *vertex-integrity*, or simply *integrity, I(D)* of a digraph D is the minimum value over all proper subsets S of the vertex set of the sum m(D - S) + |S|, where m(D - S) denotes the order of a largest strong component of D - S. We will call a set S an *I-set* of D if I(D) = |S| + m(D - S).

3.2 Integrity of Digraphs

We begin with a preliminary result on the integrity of digraphs.

Proposition 3.2.1: If F is a subdigraph of D, then I(F) ≤ I(D).

Proof: Let S be any I-set for D and let T = S ∩ V(F). Then m(D - S) ≥ m(F - T) since the vertices of any strong component of F - T are contained in a strong component of D - S, and |S| ≤ |T|. Therefore I(D) = m(D - S) + |S| ≥ m(F - T) + |T| ≥ I(F). □

Other links between the integrity of a digraph and its subdigraphs involve the maximum order of a strong component of the digraph and the integrity of certain subdigraphs.

Proposition 3.2.2: Let D be a nontrivial digraph. Then (a) I(D) ≤ m(D), and

(b) $I(D) \le 1 + I(D - v)$ for any vertex v in D.

Proof: By definition $I(D) = \min_{S \subseteq V(D)} \{m(D - S) + |S|\} \le m(D - \emptyset) + |\emptyset|$, hence part (a) is true.

Let S be an I-set for D - v. Then

$$I(D) \le |S \cup \{v\}| + m(D - (S \cup \{v\})) \le$$
$$1 + |S| + m((D - v) - S) = 1 + I(D - v)$$

which proves part (b). □

Induced subdigraphs can also help define the integrity of a digraph recursively.

Theorem 3.2.3: For a nontrivial digraph D,

$$I(D) = \min\{m(D), 1 + \min_{v \in V(D)} I(D - v)\}.$$

Proof: By Lemma 3.2.2, $I(D) \le m(D)$ and

$$I(D) \le 1 + \min_{v \in V(D)} I(D - v).$$

For the reverse inequality, assume that S is an I-set for D of smallest order. If $S = \emptyset$, then $I(D) = m(D)$, so the desired result holds. So assume that $S \neq \emptyset$. Then $I(D) < m(D)$. Let v be a vertex in S and let $S' = S - v$. Then it follows that

$$I(D) = |S| + m(D\text{-}S) = 1 + |S'| + m((D - v) - S')$$
$$\ge 1 + I(D - v) \ge 1 + \min_{v \in V(D)} I(D - v).$$

Now suppose that for some $x \notin S$, $I(D - x) > I(D) - 1$. Let R be an I-set of $D - x$ and let $R' = R \cup \{x\}$. Then

$$I(D) < I(D - x) + 1 = 1 + |R| + m((D - x) - R) =$$
$$|R'| + m(D - R') \leq I(D),$$

which is a contradiction. Hence, if $S \neq \emptyset$, then $I(D) = 1 + \min_{v \in V(D)} I(D\text{-}v)$. □

Corollary 3.2.4: For a nontrivial strongly connected digraph D, $I(D) = 1 + \min_{v \in V(D)} I(D\text{-}v)$.

For a digraph D, let D' denote the converse digraph, that is, the digraph in which all the arcs of D are reversed. The two digraphs D and D' have the same partition of their vertex sets into strong components, which leads to the following proposition.

Proposition 3.2.5: $I(D) = I(D')$.

Although a characterization of digraphs with integrity 1 is clear, the case where the integrity is 2 is more interesting.

Proposition 3.2.6: A digraph has integrity 1 if and only if it is acyclic.

Theorem 3.2.7: A directed graph has integrity 2 if and only if it is not acyclic and either some vertex is on every dicycle or $m(D) = 2$.

Proof: To prove the first condition, we assume $I(D) = 2$ and $m(D) > 2$. If S is an I-set, then $|S| = m(D - S) = 1$. Consequently, some vertex v lies on every dicycle in D.

In order to show the converse, assume $m(D) > 2$ and some vertex v lies on every dicycle. Then $D - v$ is acyclic, and so $m(D - v) = 1$. If, on the other hand, $m(D) = 2$, then $I(D) < 3$, but the removal of any single vertex will leave at least one vertex. In either case $I(D) = 2$. □

Recall from Section 1.5 that $\delta^+(D)$ (respectively $\delta^-(D)$) is the minimum out(in)degree among the vertices in D.

Theorm 3.2.8: $I(D) \geq \max\{\delta^+(D), \delta^-(D)\} + 1$.

Proof: Let $r = \max\{\delta^+(D), \delta^-(D)\}$. Then D contains a strong component of order at least $r + 1$, otherwise in the strong component which dominates all other strong components there must be a vertex v with $\deg^-(v) < r$. For any I-set S, $\max\{\delta^+(D - S), \delta^-(D - S)\} \geq \max\{\delta^+(D), \delta^-(D)\} - |S| = r - |S|$. Hence $D - S$ contains a strong component of order at least $r - |S| + 1$. Therefore, $I(D) = |S| + m(D - S) \geq |S| + r - |S| + 1 = r + 1$. This demonstrates the inequality. □

3.3 Orientations of Graphs

As was mentioned in Chapter I, one of the problems we need to deal with is the relative lack of nice classes of digraphs to work with. We can, however, create a digraph from a graph by letting the vertex set of the digraph be the vertex set of the graph, and (a) replacing each edge uv with both arcs $\vec{uv}$ and $\vec{vu}$, or (b) replacing each edge uv with exactly one of the arcs $\vec{uv}$ or $\vec{vu}$. In case (a) the integrity of the digraph is the integrity of the graph, so nothing more need be said.

Any digraph D derived as in (b) is called an orientation of the graph

G, and G is called the underlying graph of D. We focus our attention on orientations of certain families of graphs.

Since the integrity values for several classes of graphs have been found, it will be useful to know how the integrity of a graph G is related to the integrity of an orientation D of G.

Proposition 3.3.1: $I(D) \leq I(G)$.

Proof: For any I-set S of G,

$$|S| + m(D - S) \leq |S| + m(G - S)$$

since the vertices of any strong component of D must lie wihtin a connected component of G. □

There are classes of graphs for which each graph has an orientation which demonstrates the sharpness of this bound. The set of null (edgeless) graphs on n vertices is a trivial example. A less trivial example is the following: take $n \geq 2$ copies of the circulant digraph D(7,{1,2,4}), which has connectivity 3 [3.9], and identify the n vertices labelled 0 (see Figure 3.1 for the case n = 2). The digraphs and the underlying graphs all have integrity 7.

Another fact peculiar to oriented graphs is that there are no strong components of order 2. For any graph with q edges there are 2^q orientations of the edges and, if the graph contains a cycle, distinct orientations can have distinct integrities. This means that, with the exception of forests, we can only hope to obtain a range for the integrity of the orientations of a particular graph. Every graph has an

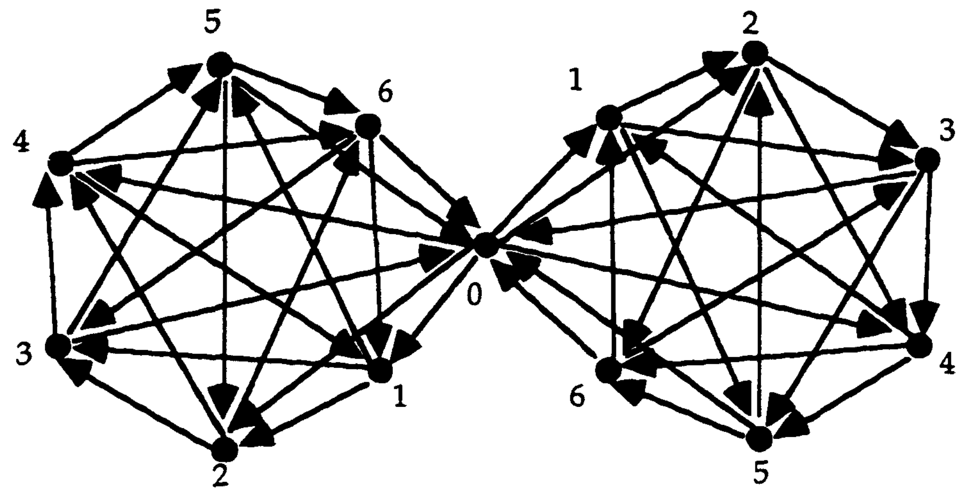


Figure 3.1

acyclic orientation, so the (sharp) lower bound is always 1. We will therefore focus our attention on the maximum integrity of any orientation of a graph G, which we denote $\bar{I}(G)$. The integrity for several classes of graphs was given in Section 1.2, so we have upper bounds for the integrity of orientations of the graphs in these classes. In addition, given a graph G with decycling set S, then G - S is acyclic, as is any orientation of G - S. This gives the following result:

Proposition 3.3.2: $\bar{I}(G) \leq \nabla(G) + 1$.

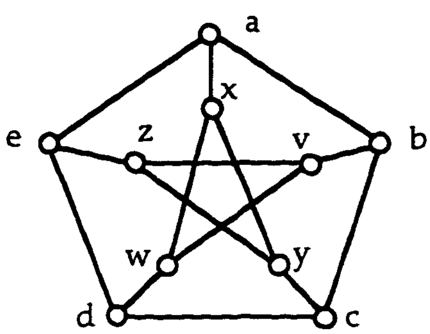
Thus, the results of Chapter II provide upper bounds for the integrity of several classes of graphs. We will use orientations of these classes of graphs as a starting point for our study.

The following observations on the integrity of orientations of forests and cycles are corollaries of Proposition 3.2.6 and Theorem 3.2.7.

Proposition 3.3.3: The integrity of any orientation of a forest is 1.

Proposition 3.3.4: If D is an orientation of a cycle, then I(D) = 2 if D is a directed cycle and I(D) = 1 otherwise.

As in Chapter II, we use the Petersen graph as an example. Using the labelling of Figure 3.2, we note that at



Petersen Graph
Figure 3.2

least three of the edges incident with both the pentagon and the pentagram must be oriented the same way (say from the pentagon to the pentagram). Hence, at most two of the arcs (say a_i and a_j) are oriented the other way. If we remove the vertex in the pentagon away from a_i and the vertex on the pentagram away from a_j, then there can be no directed cycles remaining in the digraph, and so $\bar{I}(P) \leq 3$. This bound can be attained when both the pentagon and the pentagram are oriented cycles and exactly two edges are oriented from the pentagon to the pentagram, and hence $\bar{I}(P)$ is 3.

3.4 Tournaments

We next discuss bounds for the integrity of tournaments of small order. It follows from Proposition 3.2.6 that a tournament T has integrity 1 if and only if it is transitive. The maximum integrity of tournaments of small order will be found using known results about the structure of tournaments and circulant digraphs. A most helpful result is the following theorem of Parker and Reid [3.6] on the minimum number of vertices that a tournament must have in order to guarantee that it contain a transitive n-tournament.

Theorem 3.4.1: Let f(n) denote the minimum number r such that every tournament of order r contains a transitive n-tournament. Then the values of f(n) are given in Table 3 below.

Table 3

Values of f(n) for Small n

n order of transitive subtournament	f(n) order of tournament
1	1
2	2
3	4
4	8
5	14
6	28

and $2^{(n-1)/2} \leq f(n) \leq 7 \cdot 2^{n-4}$, for $n > 6$. □

This theorem leads to the following upper bound for the integrity of an orientation of a graph.

Theorem 3.4.2: If D is an orientation of a graph G on n vertices for which $f(a) \leq n < f(a+1)$, then

$$I(D) \leq n - a + 1.$$

Proof: Let D be an orientation of a graph G of order n. Let T be a tournament containing D. Since there is some set of n - a vertices S for which T - S is a transitive a-tournament, then $n - a + 1 \geq I(T)$. Since $I(T) \geq I(D)$, the result follows. □

The value of $\bar{I}(K_n)$ for $n \leq 28$ will be determined through a series of lemmas.

Lemma 3.4.3: For all n, $\bar{I}(K_{n-1}) \leq \bar{I}(K_n) \leq \bar{I}(K_{n-1})+1$.

Proof: The first inequality follows from Proposition 3.2.1. Suppose that the second inequality is false. Then for some tournament T on n vertices $I(T) = \bar{I}(K_n) \geq \bar{I}(K_{n-1}) + 2$. Let v be any vertex in T, let $T^* = T - v$, and let S^* be any I-set for T^*. Now, $I(T) \leq m(T - (S^* \cup \{v\})) + |S^* \cup \{v\}| = m(T^* - S^*) + |S^*| + 1 = I(T^*) + 1 < \bar{I}(K_n)$, which is a contradiction. □

Corollary 3.4.4: If r and s are positive integers for which $\bar{I}(K_{r+s}) = \bar{I}(K_r) + s$, then for $0 < i < s$, $\bar{I}(K_{r+i}) = \bar{I}(K_r) + i$. □

Lemma 3.4.5: $\bar{I}(K_1) = \bar{I}(K_2) = 1$.

Proof: Every tournament on either one or two vertices is acyclic. □

Lemma 3.4.6: $\bar{I}(K_3) = \bar{I}(K_4) = 2$.

Proof: The oriented 3-cycle and Theorem 3.2.7 show that $\bar{I}(K_3) = 2$, while Theorem 3.4.1 implies that $\bar{I}(K_4) \le 2$. Lemma 3.4.2 completes the proof. □

Lemma 3.4.7: $\bar{I}(K_7) = \bar{I}(K_8) = 5$.

Proof: Let T be the rotational tournament D(7,{1,2,4}). Since it has connectivity 3 [3.9] and contains no transitive 4-tournament, it has a set S of three vertices such that T - S is not strong. It follows that T - S must have two strong components, a 3-cycle and a single vertex, one of which must dominate the other. Removal of fewer than three vertices from T leaves a strong component of order at least 5, taking away three vertices leaves a strong component of order at least 3, but removal of three vertices which disconnect the digraph along with one of the vertices in the remaining 3-cycle leaves a transitive 3-tournament, so $\bar{I}(K_7) \ge 5$.

It follows from Theorem 3.4.1 that $\bar{I}(K_8) \le 5$, and so by Theorem 3.4.2, $\bar{I}(K_7) = \bar{I}(K_8) = 5$. □

Lemma 3.4.8: $\bar{I}(K_{13}) = \bar{I}(K_{14}) = 10$.

Proof: Let T be the tournament D(13,{1,2,3,5,6,9}). Then from [3.9] we know k(T) = 6 but contains no transitive 5-tournament. Therefore, if I(T) < 10, it must contain a set S of six vertices whose removal leaves no strong component of order greater than 3. Since T contains no transitive 5-tournament, T - S must have two strong components of order 3 and one of order 1 on the seven remaining vertices. The condensation of these components must form a transitive 3-tournament [3.4]. If two vertices are chosen from each of the components of order 3, then those four vertices, along with the component of order 1, form a transitive 5-tournament, and this is a contradiction. Therefore, when six vertices are removed, some strong component must have order 4 or greater. When seven or eight vertices are removed, some strong component must have order at least 3 (given that there is no transitive 5-tournament), so $I(T) \geq 10$. Hence $\bar{I}(K_{13})$ is at least 10.

Now let T be any tournament of order 14. By Theorem 3.4.1, T has a transitive 5-tournament. Therefore it must contain a set S of nine vertices whose removal leaves a transitive 5-tournament. Hence $I(T) \leq 10$. Since T was arbitrary, this implies that $\bar{I}(K_{14}) \leq 10$, and the result follows. □

Lemma 3.4.9: $\bar{I}(K_{27}) = \bar{I}(K_{28}) = 23$.

Proof: From Theorem 3.4.1, $\bar{I}(K_{28}) \leq 23$. To prove the lower bound, note that the quadratic residue tournament QT_{27} contains no transitive 6-tournament. We now demonstrate that $I(QT_{27}) \geq 23$.

Assume that $I(QT_{27}) \leq 22$, and let S be an I-set. We will show that regardless of the order of S the tournament QT_{27} - S must contain a transitive 6-tournament, and this yields a contradiction. Let T^* denote QT_{27} - S. Then $m(T^*) = I(QT_{27}) - |S|$.

Case 1: $|S| = 21$ or 20. Then $m(T^*)$ must be 1, and so T^* contains a transitive 6-tournament.

Case 2: $|S| = 19$. Then $m(T^*) \leq 3$. The subtournament induced by any two vertices from each strong component of order 3 together with the vertices from the trivial strong components is a transitive tournament and has at least order 6.

Case 3: $|S| \leq 18$. Let H be a strong component of maximum order in QT_{27} - S and let F = QT_{27} - S - H. Because of the constraints on $|S|$ and $I(QT_{27})$, the order of F must be at least 4. Each vertex of F either has arcs to all of the vertices in H or from all of them. If $|H| > 3$, then both H and F contain transitive 3-tournaments, and those six vertices either induce a transitive 6-tournament or contradict the fact that H was of maximum order. If, on the other hand, $|H| < 4$, then any set of two vertices from each strong component of order 3 along with the vertices in the strong components of order 1 in D - S induce a transitive tournament of order at least 6. In either case, a transitive 6-tournament is forced, which contradicts the structure of the digraph.

All of the above cases combined imply that $\bar{I}(K_{27}) = \bar{I}(K_{28}) = 23$. □

Using Lemmas 3.5.3 - 3.5.9, along with Corollary 3.5.4 to fill in the gaps, we obtain the following theorem on the maximum integrity for tournaments of order 28 or less:

Theorem 3.4.10: For $2 \leq p \leq 28$, and

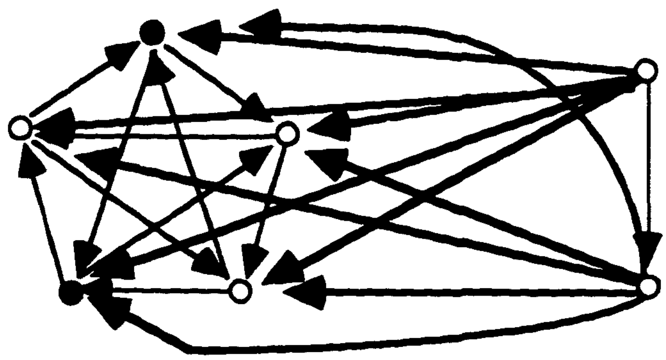
$$f(n) \leq p < f(n+1), \text{ then } \bar{I}(K_p) = p - n + 1.$$

If we compare the previous result to Theorem 3.4.2 we note that for small p, $\bar{I}(K_p)$ has the maximum integrity attainable for any graph on p vertices. If we recall that a graph G has integrity $|G|$ (which is the maximum possible integrity) if and only if G is complete, then it seems natural to expect this to extend to $\bar{I}(K_p)$ for all values of p, that is:

Conjecture: For $p \geq 2$, if $f(n) \leq p \leq f(n + 1)$, then $\bar{I}(K_p) = p - n + 1$.

Theorem 3.4.11: For any positive integers n and a with $1 \leq a \leq \bar{I}(K_n)$, there exists a tournament T on n vertices with $I(T) = a$.

Proof: From Lemma 3.5.5 we know $\bar{I}(K_2) = 1$, while Lemma 3.5.3 states $\bar{I}(K_{n-1}) \leq \bar{I}(K_n) \leq \bar{I}(K_{n-1}) + 1$. Combining these two lemmas gives us the fact that there is some tournament T on at most n vertices with $I(T) = a$. Complete T by adding an (n - a)-transitive tournament R for which all the arcs between R and T are oriented into T (see Figure 3.3 for an example). □



A tournament on 7 vertices with integrity 3
Figure 3.3

3.5 Bipartite Tournaments

We now consider orientations of complete bipartite graphs $K_{m,n}$ with partite sets X and Y of order m and n, respectively. Without loss of generality, we henceforth assume $m \leq n$. These digraphs, which we denote by $T_{m,n}$, are known as *bipartite tournaments*. In such digraphs, the order of any non-trivial strong component must be at least 4. One way in which the lower bound for the integrity of these tournaments can be achieved is for all arcs to be oriented from one partite set to the other. The upper bound $\bar{I}(K_{m,n})$ is determined in the following theorem.

Theorem 3.5.1: If $m \leq n$, then $\bar{I}(K_{m,n}) = m$.

Proof: From Theorem 2.2.1, we know that $\nabla(K_{m,n}) = m - 1$, and hence by Proposition 3.3.2, $I(T_{m,n}) \leq m$. To show the reverse inequality, we use a construction. We construct the digraph D as follows: Let $X = \{ x_1, x_2, \ldots, x_m \}$ and $Y = \{ y_1, y_2, \ldots, y_n \}$. Orient the edge x_i,y_j as (y_j,x_i) unless $i = j$ in which case orient the edge as (x_i,y_j). Since the removal of any one vertex can destroy at most one arc from X into Y, and since any pair of arcs from X into Y lie in a directed 4-cycle, for any subset S of the vertex set with $|S| = k < m - 1$, $m(D - S) \geq 2(m - k)$, and so $|S| + (D - S) > m - 1$. This implies that $I(T_{m,n}) \geq m$, and the theorem follows. □

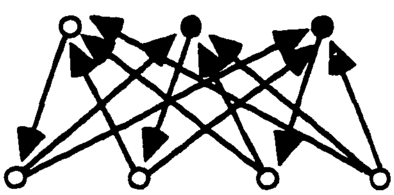


Figure 3.4

Results from the previous two sections can be combined to give rough bounds for the maximum integrity for the orientations of complete multipartite graphs.

Theorem 3.5.2: When $r_1 \le r_2 \le ... \le r_k$ and $n = \sum_{i=1}^{k} r_i$, then

$$r_1 = \bar{I}(K(r_1, n - r_1)) \le \bar{I}(K(r_1, r_2, ..., r_k)) \le \min\{\bar{I}(K_n), n - r_k\}.$$

3.6 Binary Operations

For orientations of the union of graphs, the bounds derived for integrity of unions of graphs turn out to be the best possible bounds.

Theorem 3.6.1: For $G = \bigcup_{i=1}^{n} G_i$,

$$\max_i \bar{I}(G_i) \le \bar{I}(G) \le \sum_{i=1}^{n} \bar{I}(G_i) - n + 1.$$

Proof: The lower bound comes from Proposition 3.2.1, while the upper bound is is derived from Proposition 3.3.2 and the corresponding bound for graphs. □

The graph $G \cup (n-1)K_2$ demonstrates the sharpness of the lower bound while nC_{n+1} is an example in which the upper bound is attained.

An upper bound for the integrity of an orientation of the join of two graphs is implied by the graphical bounds, and can be shown to be sharp by wheel graphs.

Theorem 3.6.2: $\bar{I}(G + H) \le \min\{\bar{I}(G) + |H|, \bar{I}(H) + |G|\}$.

Proof: Let D be an orientation of G + H for which $I(D) = \bar{I}(G + H)$, let S be an I-set of D, and let F be an I-set for G. Now $I(D) = m(D - S) + |S| \le m(G -$

F) + $|V(F) \cup V(H)|$ since S is an I-set of G + H. However, m(G - F) + $|V(F) \cup V(H)|$ = m(G - F) + $|F|$ + $|H|$ = I(G) + $|H| \le \bar{I}(G) + |H|$. The same argument shows that $\bar{I}(G + H) \le \bar{I}(H) + |G|$, which completes the proof. □

Corollary 3.6.3: For the wheel W_n with n > 3, $\bar{I}(W_n) = 3$.

Proof: By Theorem 3.6.2, $\bar{I}(W_n) \le 3$. The oriented wheel in which the n-cycle is a directed cycle and two edge disjoint triangles are oriented as cycles shows that the

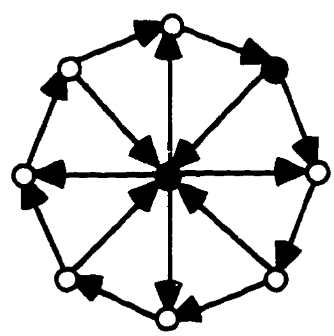


Figure 3.5

upper bound is sharp. □

3.7 Products of Paths and Cycles

From Chapter II and Proposition 3.3.2 we have $\nabla(P_m \times P_n) + 1$ as an upper bound for $\bar{I}(P_m \times P_n)$. This bound turns out to be exact for small n, but as n increases it is more efficient to allow nontrivial strong components, of relatively small order. We will see in this section that the value for $\bar{I}(P_m \times P_n)$ will be bounded above by the minimum of several functions. The decycling number of $(P_m \times P_n)$ plus 1 will usually be sharp for small n, while allowing larger strong components will give a pair of bounds. One reason for this is the fact that since $P_m \times P_n$ is bipartite, there can be no strong components of order 2 or 3.

Theorem 3.7.1:

$$\bar{I}(P_2 \times P_n) = \min\left\{\left\lfloor \frac{n}{2}+1 \right\rfloor, \left\lceil \sqrt{2n} \right\rceil + 2\left\lceil \frac{n+1}{\left\lceil \sqrt{2n} \right\rceil} \right\rceil - 3, \left\lfloor \sqrt{2n} \right\rfloor + 2\left\lceil \frac{n+1}{\left\lfloor \sqrt{2n} \right\rfloor} \right\rceil - 3\right\}.$$

Proof: Label the vertices and 4-cycles of the graph as in Chapter II. Then D, the orientation in which the 4-cycles are oriented alternately clockwise and counter-clockwise (see Figure 3.6), will be shown to attain the maximum

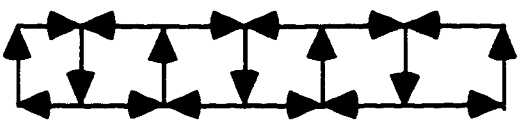


Figure 3.6

integrity. Because the underlying graph is bipartite, strong components are either trivial or of order at least 4. In fact, if we use the notation of Chapter II, each strong component is either trivial or a $P_2 \times P_r$ of the form V(*,i:j). Given an I-set S, a strong component of D - S is either of the form V(*,i:j), in which case m(D - S) = 2(j - i + 1) or the maximum order of a strong component is 1. If m(D - S) = 1, then since D has $\lfloor n/2 \rfloor$ vertex disjoint 4-cycles, $|S| \geq \lfloor n/2 \rfloor$. Hence $I(D) = \lfloor n/2 \rfloor + 1$, which is the general upper bound of $\nabla(P_2 \times P_n) + 1$ from Chapter II. If m(D - S) > 1, then assume D - S has k + 1 strong components of near uniform order. If we remove the vertex $v_{2,i}$ from the graph, then V(*,1:i) is disconnected from the rest of the digraph (see Figure 3.6). This means that we need to remove only k vertices to get k + 1 strong components. To get nearly uniform strong components, the order of largest strong component will be $2\lceil (n-k)/(k+1) \rceil$. This gives the integrity

$$I(D) = k + 2\lceil (n-k)/(k+1) \rceil =$$
$$k + 2\lfloor n/(k+1) \rfloor \le k + 2n/(k+1)$$

(recall that $\lceil a/b \rceil = \lfloor (a+b-1)/b \rfloor$). For fixed n, the minimum value of the expression can be found by differentiating with respect to k, setting the derivative equal to 0, and solving for k in terms of n. When we do that, we get $k = \sqrt{2n} - 1$, which may be a non-integer. Since we want k to be an integer, we can choose either $k = \lceil \sqrt{2n} - 1 \rceil$ or $k = \lfloor \sqrt{2n} - 1 \rfloor$. As can be seen by the graph in Figure 3.7, the value we should choose for k to minimize

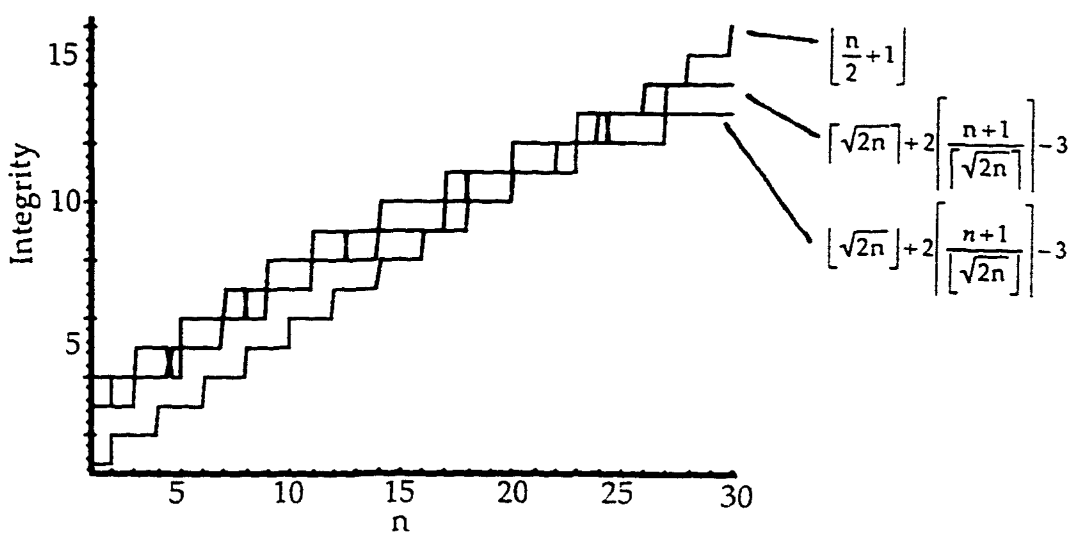


Figure 3.7

the expression $|S| + m(D - S)$ depends upon n. Thus I(D) is the minimum of the three exprssions stated in the theorem, which demonstrates the lower bound.

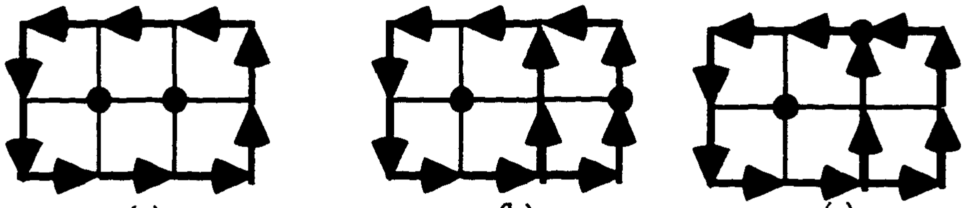
To show the upper bound, let $t = \lfloor n/k \rfloor$, $k = \lceil \sqrt{2n} - 1 \rceil$, $r = \left\lceil \frac{n-t}{t+1} \right\rceil$, $j = \lfloor \sqrt{2n} - 1 \rfloor$, and $s = \left\lceil \frac{n-j}{j+1} \right\rceil$. Then one of the sets $S_1 = \{v_{1,2}, v_{1,4}, \ldots v_{1,2t}\}$, $S_2 = \{v_{1,r+1}, v_{1,2r+2}, \ldots, v_{1,kr+k}\}$, or $S_3 = \{v_{1,s+1}, v_{1,2s+2}, \ldots, v_{1,js+j}\}$ attains the upper bound. □

Later we use a technique of the next theorem to get a general upper bound for $\bar{I}(P_m \times P_n)$, so here we go through all the details as an illustration of the method.

Theorem 3.7.2:

$$\bar{I}(P_3 \times P_n) \le \min\left\{\left\lceil \frac{3n}{4} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor + 4,\right.$$
$$\left. \lfloor \sqrt{3n} \rfloor + 3\left\lceil \frac{n+1}{\lfloor \sqrt{3n} \rfloor} \right\rceil - 2,\ \lceil \sqrt{3n} \rceil + 3\left\lceil \frac{n+1}{\lceil \sqrt{3n} \rceil} \right\rceil - 2 \right\}.$$

Proof: From Chapter II and Theorem 3.3.2 we know that $\bar{I}(P_3 \times P_n) \le \left\lfloor \frac{3n}{4} \right\rfloor + 1$. This can be improved upon when $n \equiv 0 \pmod 4$ to $\frac{3n}{4}$ as follows. When $n = 4$, remove $v_{2,2}$ and $v_{2,3}$. If a strong component remains, it must be a directed 10-cycle (see Figure 3.8(a)). Now to remove $v_{2,2}$ and $v_{2,4}$. If a strong component remains, it must be a directed 8-cycle with the orientation of the arcs determined by the orientation of the 10-cycle (see Figure 3.8(b)). It follows that removing $v_{2,2}$ and $v_{1,3}$ leaves no



(a) (b) (c)

Figure 3.8

nontrivial strong component (see Figure 3.8(c)). Thus $\bar{I}(P_3 \times P_4) \leq 3$. When $n = 4t$ for some positive integer t, then the decycling set $S = \{v_{2,i} \mid i \equiv 0 \pmod 2\} \cup \{[v_{3,j} \mid j \equiv 3 \pmod 4\}$ for V(*,1:4t - 4) along with two vertices chosen as in the $n = 4$ case for V(*,4t - 3:4t) demonstrate the upper bound of 3t. Recall that when $n \equiv 0 \pmod 4$, then $\lfloor 3n/4 \rfloor + 1 = \lceil 3n/4 \rceil$, and hence $\bar{I}(P_3 \times P_n) \leq \lceil 3n/4 \rceil$. For n = 3, 4, 5, 6, 8, and 9, these bounds will be shown to be sharp. However, for larger n, we get a better bound if we allow nontrivial strong components. Since two arcs between V(*,i) and V(*,i + 1) must be oriented in the same direction, there can be at most one oriented the other way. If we remove the vertex in V(*,i) of such an arc, then V(*,1:i) is disconnected from the rest of the digraph (see Figure 3.9). This means that

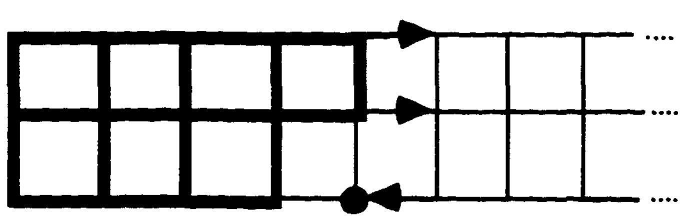


Figure 3.9

a set S need only contain k vertices in order that D - S have at least k + 1 strong components. Note that the other two vertices in V(*,i) could still be in a strong component containing V(*,1:i - 1), and if we try to keep the components of approximately uniform order, then the order of the largest strong component of D - S would be at most either 4 (when the vertices are chosen from alternating P_3's) or $3\lceil (n-k)/(k+1) \rceil + 2$. This gives an upper bound for the integrity of either $\lfloor n/2 \rfloor + 4$ when $m(D - S) = 4$, or $k + 3\lceil (n - k)/(k + 1) \rceil + 2 = k + 3\lfloor n/(k + 1) \rfloor + 2 \leq k + 3n/(k + 1) + 2$ otherwise. Using calculus as before, we find that for fixed n this second function has a minimum value when $k = \sqrt{3n} - 1$. Again, for our purposes we need k to be an integer, so k can be either $\lfloor \sqrt{3n} - 1 \rfloor$ or $\lceil \sqrt{3n} - 1 \rceil$, and hence the theorem follows. □

Corollary 3.7.3: For $3 \leq n \leq 9$, $\bar{I}(P_3 \times P_n)$ is given in Table 4.

Table 4

Maximum Integrity for Orientations of $P_3 \times P_n$

n	$\bar{I}(P_3 \times P_n)$
3	3
4	3
5	4
6	5
7	5
8	6
9	7

Proof: The upper bounds for all but n = 7 were demonstrated in Theorem

3.7.2. The lower bounds will be shown by giving an example of an orientation which attains the integrity (see Figure 3.10).

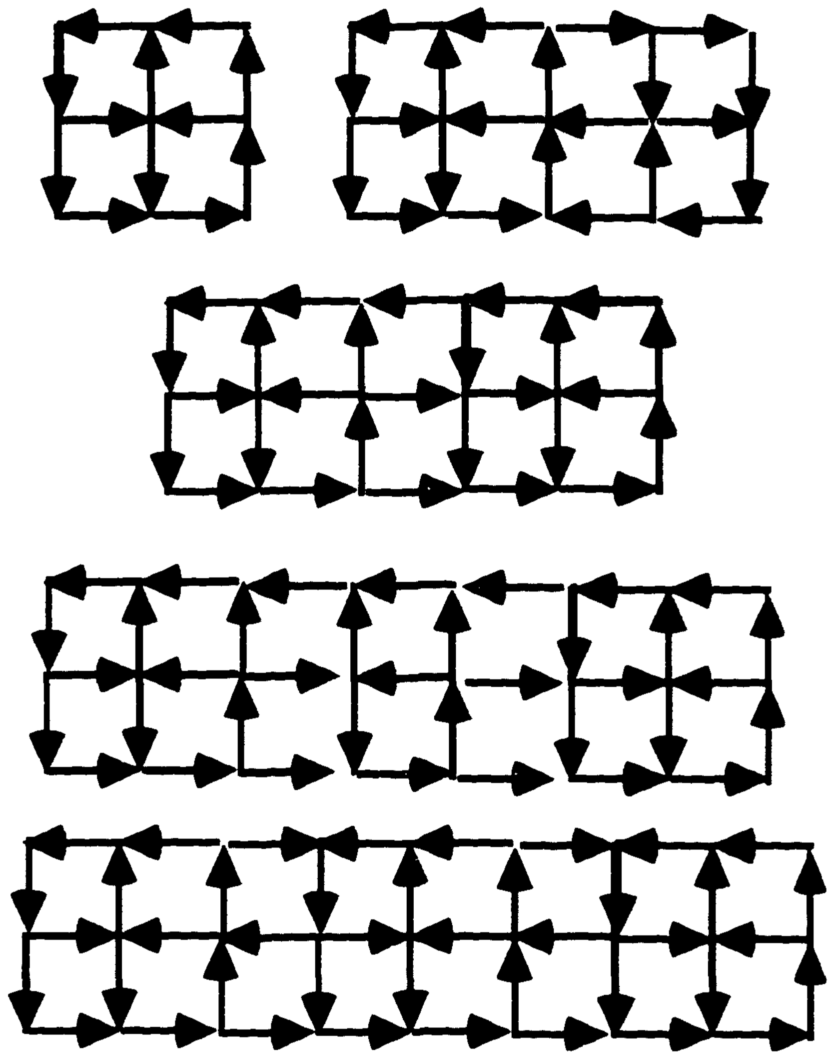


Figure 3.10

For the case $n = 7$, let D be an orientation of $P_3 \times P_7$, and remove the vertices $v_{2,2}$, $v_{2,3}$, $v_{2,5}$, and $v_{2,6}$ from D. If there is a nontrivial strong component left, it must include two degree-2 vertices in the same copy of P_3, without loss of generality $v_{1,1}$ and $v_{3,1}$ (see Figure 3.11(a)). Next, we

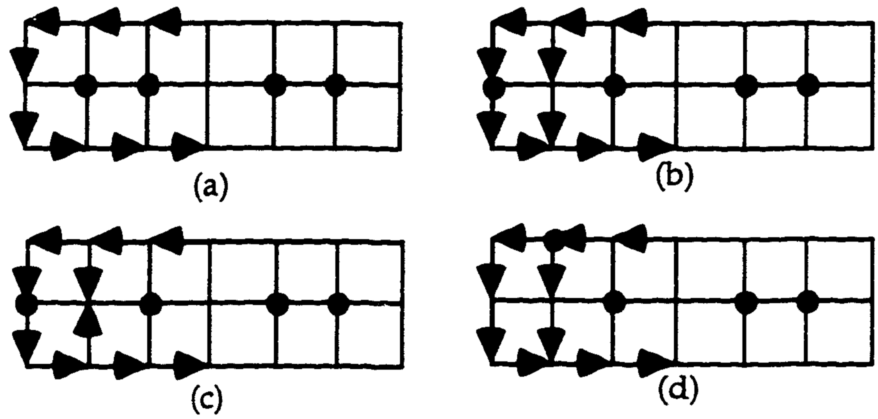


Figure 3.11

remove $v_{2,1}$, $v_{2,3}$, $v_{2,5}$, and $v_{2,6}$ from D, if a nontrivial strong component of what remains contains $v_{2,2}$, then no vertex in V(*,1:3) can lie in a strong component of D - $\{v_{1,2}, v_{2,3}\}$, otherwise no vertex in V(*,1:3) can lie in a strong component of D - $\{v_{2,1}, v_{2,3}\}$. The last two vertices in an I-set will be chosen from V(*,4:7) as in Theorem 3.7.2 for $P_3 \times P_4$. □

In the general case, we mimic a technique of the last theorem, noting that at most $\lfloor m/2 \rfloor$ vertices must be removed from V(*,i) in an orientation of $P_m \times P_n$ to disconnect V(*,1:i) from the rest of the digraph.

Theorem 3.7.4: For $m \leq n$,

$$\bar{I}(P_m \times P_n) \leq \min\{\nabla(P_m \times P_n) + 1,$$
$$\left\lceil \sqrt{2n} - 2 \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n+1}{\lceil \sqrt{2n} \rceil} \right\rceil, \left\lfloor \sqrt{2n} - 2 \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n+1}{\lfloor \sqrt{2n} \rfloor} \right\rceil\}.$$

Proof: The first term in the minimum comes from Theorem 3.3.2. For the final two terms, we start by recalling that when we remove $\lfloor m/2 \rfloor$ vertices from V(*,i) and disconnect V(*,1:i) from the rest of the digraph a strong component containing V(*,1:i - 1) could still contain the remaining $\lceil m/2 \rceil$ vertices of V(*,i). Hence, the largest of the components could have as many as $m\lceil (n-k)/(k+1) \rceil + \lceil m/2 \rceil$ vertices. Thus,

$$I(D) \le \lfloor m/2 \rfloor k + m\lceil (n-k)/(k+1) \rceil + \lceil m/2 \rceil =$$
$$\lfloor m/2 \rfloor (k-1) + m\lfloor n/(k+1) \rfloor + m \le$$
$$m + m(k-1)/2 + mn/(k+1).$$

We use calculus with fixed m and n, to find that the minimum of the original function occurs when k is either $\lceil \sqrt{2n} - 1 \rceil$ or $\lfloor \sqrt{2n} - 1 \rfloor$. □

Theorem 3.7.5: $\bar{I}(P_m \times C_n) \le$

$$\min\{\nabla(P_m \times C_n) + 1,$$
$$\lfloor \sqrt{2n-2} - 1 \rfloor \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n}{\sqrt{2n-2}} \right\rceil,$$
$$\lceil \sqrt{2n-2} - 1 \rceil \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n}{\sqrt{2n-2}} \right\rceil\}.$$

Proof: The only difference between this and the previous argument is that the second graph in the product is a cycle, so when we put in the k partitions as before we get only k components. Thus the maximum order of a near-uniform strong component in this case will be $m\left\lceil \frac{n-k}{k} \right\rceil + \left\lceil \frac{m}{2} \right\rceil$.

Hence the inequality

$$I(P_m \times C_n) \le \left\lfloor \frac{m}{2} \right\rfloor k + m\left\lceil \frac{n-k}{k} \right\rceil + \left\lceil \frac{m}{2} \right\rceil =$$
$$\left\lfloor \frac{m}{2} \right\rfloor (k-1) + m\left\lfloor \frac{n-1}{k} \right\rfloor + m \le \frac{m(k-1)}{2} + \frac{m(n-1)}{k} + m.$$

Applying calculus we find that the last expression is minimal, with regards to fixed m and n, when k is either $\lfloor\sqrt{2n-2}\rfloor$ or $\lceil\sqrt{2n-2}\rceil$. When these are substituted into the first expression of the inequality, and the result is simplified, we get the desired values. □

Corollary 3.7.6: $\bar{I}(C_m \times C_n) \leq$

$$\min\{\nabla(C_m \times C_n) + 1,$$
$$\left\lfloor\sqrt{2n-2} - 1\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor + m\left\lceil\frac{n}{\sqrt{2n-2}}\right\rceil,$$
$$\left\lceil\sqrt{2n-2} - 1\right\rceil\left\lfloor\frac{m}{2}\right\rfloor + m\left\lceil\frac{n}{\sqrt{2n-2}}\right\rceil\}.$$

Proof: One can see that the extra edges of the cycles will not change the disconnection principle used in the proof of Theorem 3.7.4. The bounds on the orders of the strong components and the calculus aren't any different either, hence we get the same bound if we allow nontrivial strong components. □

One of the most important product graphs is the n-cube. It seems that the behavior of the integrity of an orientation of Q_n depends very much on n. For n = 2 or 3, then $2^{n-2} + 1$ is a sharp upper bound for the integrity of an orientation, but when n is large, the upper bound shown in [3.3] for graphs, and hence for their orientations, of $O(2^n \mathit{log} n/\sqrt{n})$ is better. This is worth further analysis, but appears to be very difficult.

CHAPTER IV

ARC-INTEGRITY OF ORIENTED GRAPHS

4.1 Definitions

In Chapter III we were interested in the effect that removing vertices had on oriented graphs. In this chapter we look at the arc counterpart. The *arc-integrity*, $I'(D)$ of a digraph D is defined as the minimum value of $m(D - S) + |S|$ over all subsets S of the arc set of D. A set S is an *I'-set* of D if $I'(D) = |S| + m(D - S)$.

4.2 Arc-integrity of Digraphs

We begin with some preliminary results about arc-integrity.

Proposition 4.2.1: If F is a subdigraph of D, then $I'(F) \leq I'(D)$.

Proof: Let S be an I'-set for D. Then $m(F - S) \leq m(D - S)$ since any strong component of F - S must be contained in a strong component of D - S. Hence,

$$I'(D) = m(D - S) + |S| \geq m(F - S) + |S \cap F| \geq I'(F). \quad \square$$

The arc-integrity of a digraph is related to its integrity in the following manner.

Theorem 4.2.2: For every digraph D, $I'(D) \geq I(D)$.

Proof: Let T be an I'-set for D and let S be the set of vertices of out-degree at least 1 in the digraph induced by T. Clearly, $|T| \geq |S|$ and $m(D - T) \geq m(D - S)$. Hence,

$$I'(D) = |T| + m(D\text{-}T) \geq |S| + m(D\text{-}S) \geq I(D). \quad \square$$

This relationship does not imply that all properties of integrity hold for arc-integrity. For example, by Theorem 3.2.3, the integrity of a digraph D is closely related to the integrities of the family of digraphs D - v, where v is a vertex in D. However, there is no corresponding relationship for arc-integrity. The main problem here is that even though we are removing arcs from the digraph, strong components are measured in terms of vertices. In the vertex case, the removal of a single well-selected vertex can often decrease the order of a largest component; while in the arc case we must disconnect the digraph to reduce the order of the largest strong component. The family of circulant digraphs $D(6t + 1,\{1,2,3\})$ where t is a nonnegative integer is an example of this (removal of any single vertex decreases the order of the largest strong component while we must remove at least three arcs to accomplish the same decrease).

Nevertheless, there are some relationships between arc-integrity and strong components. The next proposition gives a sharp upper bound, while those that follow relete arc-integrity to other parameters.

Theorem 4.2.3: For a digraph D, $1 \leq I'(D) \leq m(D) \leq |D|$.

Proof: Let S be the empty set of arcs. Then

$$I'(D) = \min_{T \subseteq E(D)} \{m(D - T) + |T|\} \leq m(D - S) + |S| = m(D) \leq |D|. \quad \square$$

Theorem 4.2.4: For digraph D,

$$I'(D) \geq \max\{\delta^+(D), \delta^-(D)\} + 1.$$

Proof: From Theorem 4.2.2 we know $I'(D) \geq I(D)$ and from Theorem 3.2.8 we have that $I(D) \geq \max\{\delta^+(D), \delta^-(D)\} + 1$, and hence the inequality follows. □

Recall that D' is defined to be the orientation obtained by reversing all the arcs of D. Any strong component in D is also a strong component in D', which gives the following result.

Theorem 4.2.5: For every digraph D, $I'(D') = I'(D)$.

The following characterizations are fairly straight forward but will be helpful.

Theorem 4.2.6: For any digraph D the following are equivalent: (a) $I(D) = 1$, (b) $I'(D) = 1$, and (c) D has no dicycles.

Proof: From Proposition 3.2.6 we know that (a) and (c) are equivalent. If D has no dicycles, then, by definition, $I'(D) = 1$. On the other hand, if $I'(D) = 1$ then the order of a largest strong component must be 1, and so D is acyclic. Hence (b) and (c) are equivalent. □

Theorem 4.2.7: A digraph has arc-integrity 2 if and only if it has at least one dicycle and either some arc is on all of the dicycles or the order of a largest strong component is 2.

Proof: Assume $I'(D) = 2$. Then D must contain a directed cycle. If $m(D) > 2$, then there must be some arc which lies on all dicycles. If not, then for each arc e, $m(D - e) > 1$, so $m(D - S) > 1$, which is a contradiction to the assumption that $I'(D) = 2$.

If D contains a dicycle and the arc e lies on all dicycles, then $m(D - e) = 1$, so $I'(D) \leq 2$. On the other hand if $m(D) = 2$, then $I'(D) \leq 2$; but for any subset S of the arc set $m(D - S) \geq 1$. In either case, $I'(D) = 2$. □

Note that if $I(D) = 2$, then $I'(D)$ can be arbitrarily large (see Figure 4.1 for an example, the 4-flower, with integrity 2 and arc-integrity 4).

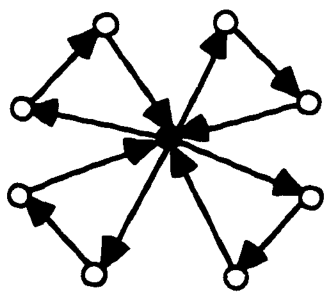


Figure 4.1

4.3 Orientations of Graphs

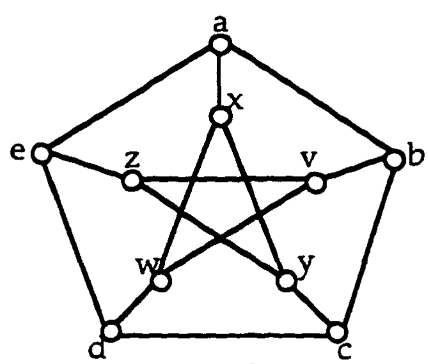
Similar to what we did in Chapter III, we note that when D is an orientation of a graph G and S' is a subset of the arc set of D which corresponds to a subset S of the edge set of G, a strong component of D - S' must be contained in a connected component of G - S. This gives the following lemma.

Lemma 4.3.1: If D is an orientation of the graph G, then $I'(D) \leq I'(G)$. □

Thus a natural upper bound for the arc-integrity of a digraph is the edge-integrity of the underlying graph. This bound will be more helpful for arc-integrity than the corresponding bound for the integrity.

We now consider the arc-integrity of the classes of oriented graphs considered in Chapter III. As with integrity, the arc-integrity for different orientations of the same graph will generally not be the same. As noted before, every graph has an acyclic orientation; so the sharp lower bound for the arc-integrity of an orientation of a graph is always 1. Also, it follows from Proposition 4.2.3, that each orientation D of a graph G on p vertices has $I'(D) \leq p$. The *maximum arc-integrity for the class of orientations of G*, denoted $\bar{I}'(G)$, is the maximum arc-integrity attained by any orientation of G.

As in the two preceding chapters, our first nontrivial example will be the Petersen graph (Figure 4.2). If D is an orientation of P in which each



Petersen Graph
Figure 4.2

of the sets {a, b, c, d, e} and {v, w, x, y, z} induce a 5-cycle, then Theorems 3.2.7 and 4.2.2 imply that $I'(D) \geq 3$, and hence so is $\bar{I}'(P)$. As was demonstrated in Chapter III, for each orientation D of P there are two vertices, say x and y, for which $m(D - \{x,y\}) = 1$. Each of these vertices has either indegree or outdegree at most 1. If the two arcs associated with these degrees are removed, then the resulting digraph is acyclic, and so $\bar{I}'(P) \leq 3$. We can combine the inequalities to give $\bar{I}'(P) = 3$.

Notice that for the Petersen graph, $\bar{I}(P) = \bar{I}'(P)$. It is not generally the case that $\bar{I}(G) = \bar{I}'(G)$. We know that this statement is true for forests, and the following lemma gives another sufficient condition for this to be the case.

Theorem 4.3.2: For any graph G with $\Delta(G) \leq 3$, $\bar{I}(G)=\bar{I}'(G)$.

Proof: Assume that $\Delta(G) \leq 3$. From Theorem 4.2.2, we have $I'(D) \geq I(D)$, and hence $\bar{I}(G) \leq \bar{I}'(G)$. Assume D is an orientation of G for which $\bar{I}'(G) = I'(D) > \bar{I}(G)$ (hence $I'(D) > I(D)$), and let S be an I-set of D. For each vertex v in S, either $deg^+(v)$ or $deg^-(v)$ is 1. Thus, there is a single arc e incident with v such that if e is removed from D, then v constitutes a nontrivial strong component of D - e. Let T be a collection of such arcs, one incident with each vertex v in S. Clearly $|S| \geq |T|$ and $m(D - S) \geq m(D - T)$; so

$$I(D) = |S| + m(D - S) \geq |T| + m(D - T) \geq I'(D),$$

which is a contradiction. Therefore the equality holds. □

From Proposition 4.2.7 and Theorem 4.2.8 we derive the following result.

Theorem 4.3.3: For every integer $n \geq 3$, $\bar{I}'(C_n) = 2$.

4.4 Bipartite Tournaments

Consider the complete bipartite graph $K_{n,m}$, with partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. Recall that the smallest order a nontrivial strong component can have is 4.

Theorem 4.4.1: For every integer $m \geq 2$, $\bar{I}'(K_{2,m}) = \lfloor m/2 \rfloor + 1$.

Proof: Every 4-cycle must contain both of the vertices x_1 and x_2 in the first partite set. Clearly, $\min\{deg^-(x_1), deg^+(x_1), deg^-(x_2), deg^+(x_2)\} \leq \lfloor m/2 \rfloor$. Without loss of generality assume the minimum is $deg^-(x_1)$. Then the removal of all arcs incident to x_1 will destroy all dicycles, and hence $\bar{I}'(K_{2,m}) \leq \lfloor m/2 \rfloor + 1$. Let D be the orientation of $K_{2,m}$ in which the only arcs from X to Y are (x_1, y_j) where $j = 1, 2, \ldots, \lfloor m/2 \rfloor$, or (x_2, y_j) where $j = \lfloor m/2 \rfloor + 1, \lfloor m/2 \rfloor + 2, \ldots, m$ (see Figure 4.3). Then $I'(D) = \lfloor m/2 \rfloor + 1$, and thus the upper bound is achieved. □

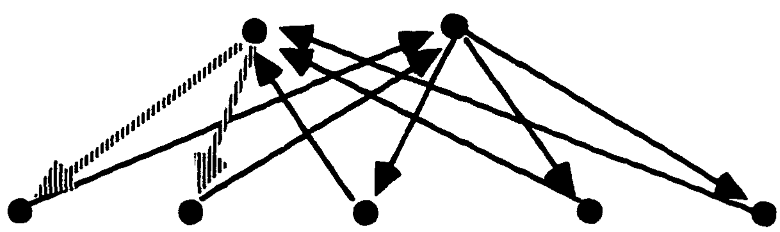


Figure 4.3

Recall that in Chapter III we used the decycling number of a graph G in order to derive an upper bound for $\bar{I}(G)$. We use a similar approach to find upper bounds for $\bar{I}'(K_{3,m})$.

For a digraph D, a subset T of the arc set of D is called a *feedback arc set* if every directed cycle in D contains some arc of T (or $m(D - T) = 1$).

The *feedback arc index* $\nabla'(D)$ of a digraph D is the minimum cardinality of a feedback arc set. Note that if D is an oriented graph, then $I'(D) \leq \nabla'(D) + 1$.

Lemma 4.4.2: If D is an orientation of $K_{3,m}$, then $\nabla'(D) \leq \lceil 2m/3 \rceil$.

Proof: Without loss of generality we may assume that D is strongly connected. For each vertex y_i in Y, either $deg^+(y_i)$ or $deg^-(y_i)$ equals 1. Label the arc associated with degree 1 incident with y_i as e_i. Let E be the collection of the arcs e_i. Note that $|E| \leq m$, and that every directed cycle of length 2k must contain exactly k arcs of E. We complete the proof by induction.

When m = 3, then we must remove at most two arcs of E in order for the remaining digraph to have no directed cycles. Suppose k is the smallest value of m for which some orientation D of $K_{3,m}$ has $\nabla'(D) > \lceil 2m/3 \rceil$. We can assume that $|E| = k$, for otherwise D is not strongly connected. If all of the arcs of E are removed from D, then $m(D - E) = 1$. For distinct vertices x_i and x_j in X, let Y^+ be the set of outneighbors of x_i by arcs of E, let Y^- be the set of inneighbors of x_j by arcs of E, and let $Y^* = Y - (Y^+ \cup Y^-)$. All the vertices in $Y^+ \cup Y^-$ are inneighbors to x_j and outneighbors from x_i; hence any directed cycle in D must contain a vertex of Y^*, the order of the largest strong component of the remaining graph is 1. All we need now show is that for some pair x_i, x_j, the cardinality of the set of arcs of E either incident to x_i or incident from x_j is at least $\lfloor k/3 \rfloor$. If this is not true, then for all (i,j) = (1,2), (1,3), (2,3), (2,1), (3,1), (3,2) the cardinality of the set of arcs of E either incident to x_i or incident from x_j is strictly less than

$\lfloor k/3 \rfloor$. However, if we sum the cardinalities of these sets we should get $2|E|$, which equals 2k; hence we get a contradiction, and the theorem holds. □

Theorem 4.4.3: For $m \geq 3$, $\bar{I}'(K_{3,m}) = \lceil 2m/3 \rceil + 1$.

Proof: The upper bound for $\bar{I}'(K_{3,m})$ was demonstrated in Lemma 4.5.2; while the orientation of $K_{3,m}$ for which the set E is $\{(x_i, y_j : i \equiv j \pmod 6)\}$ $\{(y_j, x_i) : i + 3 \equiv j \pmod 6)\}$ has arc-integrity $\lceil 2m/3 \rceil + 1$. □

4.5 Tournaments

When we remove a vertex from a tournament, the result is also a tournament. This simple fact was used in proving some results on the integrity of tournaments in the previous chapter. However, the same cannot be said for arc removal; so we will have to be a litt more resourceful. Nonetheless, an argument similar to that used for tournaments in Chapter III will be helpful. For a given positive integer n, let g(n) denote the maximum number k of arcs for which every n-tournament contains a set of k arcs that generate no cycles. Then an upper bound for $\bar{I}'(K_n)$ is $|E(K_n)| - g(n) + 1$. If n is small, then this bound is better than the bound from Section 1.2 of $I'(K_n) \leq n$.

The maximum arc-integrity of tournaments of order n is given in the following theorem.

Theorem 4.5.1: $\bar{I}'(T_n) = \begin{cases} 1 \text{ if } n = 1, 2 \\ 2 \text{ if } n = 3, 4 \\ 4 \text{ if } n = 5 \\ n \text{ if } n \geq 6 \end{cases}$

Proof: If $n \leq 6$, the upper bound for $\bar{I}'(T_n)$ follows from Theorem 4.5.2 due to Reid [4.6].

Theorem 4.5.2: For $n \leq 8$, the values of $g(n)$ are given in Table 5.

Table 5

Maximum Number of Arcs g(n) Which Generate
No Cycles in Any Orientation of K_n

n	g(n)
2	1
3	2
4	5
5	7
6	10
7	13
8	20

For $n \leq 5$, the regular or near-regular tournament on n vertices has the arc-integrity stated in the theorem.

When $n = 6$, the near-regular tournaments all have arc-integrity of at least 5, and if T is the tournament of Figure 4.4, then $I'(T) = 6$.

For the case $n \geq 7$, then each regular or near-regular n-tournament T has arc-integrity n. Suppose this is not so, and let k be the minimum value for which it does not hold. Let T_k be a regular or near-regular tournament on k vertices for which $I'(T_k) < k$, and let S an I'-set for T_k. Let

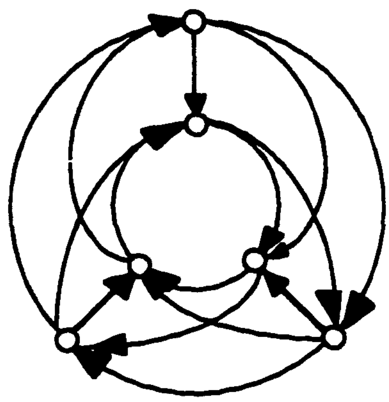


Figure 4.4

r be the order of the dominating strong component R of T_k - S. Note that we can assume $r \leq \lfloor k/2 \rfloor$; otherwise we look at the converse of T_k and remove the arcs associated with S. We must look at two cases depending upon the parity of k.

Case 1. The integer $k = 2t + 1$ is odd. Here T_k is regular. The set S must contain at least $rt - \binom{r}{2}$ arcs of R; so $k = 2t + 1 > I'(T_k)\ \ rt - \binom{r}{2} + r$, which is equivalent to the statement $r^2 - (2t + 2)r + 2(2t + 1) > 0$, which can be factored to give$(r - k)(r - 2) > 0$. Since $r \leq k$, we know that $r < 2$ for the inequality to be true. However, if $r = 1$, then T_k - R is near-regular of order $k - 1$; so $I'(T_k) = |S| + m(T_k - S) = t + |S\ \ (T_k - R)| + m((T_k - R) - S)$, which is at least $t + I'(T_k - R) \geq k$. This is a contradiction to our assumption.

Case 2. The integer $k = 2t$ is even. Here T_k is near-regular. This means that S contains at least $(t - 1)r - \binom{r}{2}$ arcs into vertices of R; so

$$k = 2t > I'(T_k) \geq (t-1)r - \binom{r}{2} + r,$$

which when simplified and factored is$(r - (k - 2))(r - 2) > 0$. As in the previous case r must be at most 2; hence $r = 1$. Now S must contain at least the arcs into R, so $|S| \geq t - 1$. This implies that $m(T_k - S) \leq t + 1$, hence we can use an argument similar to the one which demonstrated $|R| = 1$ to argue that the order of R^*, the strong component of T_k - S dominated by all other components, is 1 as well. Therefore S must contain all arcs into R and all arcs out of R^*, so $|S| \geq (t-1) + (t-1) - 1 = 2t - 3$. We know that $I'(T_k) < 2t$, and hence $m(T_k - S) \leq 2$. This implies that $m(T_k - S) = 1$, therefore $|S|$ will be at least k - 1 since at least two more arcs must be removed for this to be achieved. This contradicts the assumption on $I'(T_k)$, and the theorem is proved. □

4.6 Unions of Graphs

As in the previous chapter, we get some general bounds for the maximum arc-integrity for orientations of three of the most widely used binary operations on graphs, unions, joins, and cartesian products.

Theorem 4.6.1: $\max_i\{\bar{I}'(G_i)\} \leq \bar{I}'(\bigcup_{i=1}^{n} G_i) \leq \sum_{i=1}^{n} \bar{I}'(G_i) - n + 1.$

Proof: The first inequality is a corollary of Theorem 4.2.1. The second inequality is derived from Lemma 4.3.1 and the theorem of [4.1] which states that for graphs $G_1, G_2, \ldots, G_n$ $I'(\bigcup_{i=1}^{n} G_i) \leq \sum_{i=1}^{n} I'(Gi) - n + 1$. □

These bounds are shown to be sharp by $C_n \cup (n - 1)K_2$ and nC_{n+1}.

4.7 Products of Paths and Cycles

The maximum arc-integrity for $P_2 \times P_n$ comes as a corollary to Theorems 3.7.1 and 4.3.2.

Corollary 4.7.1: $\bar{I}'(P_2 \times P_n) =$

$$\min\{\left\lfloor \frac{n}{2} + 1 \right\rfloor, \left\lceil \sqrt{2n} \right\rceil + 2\left\lceil \frac{n-1}{\left\lceil \sqrt{2n} \right\rceil} \right\rceil - 3, \left\lfloor \sqrt{2n} \right\rfloor + 2\left\lceil \frac{n-1}{\left\lfloor \sqrt{2n} \right\rfloor} \right\rceil - 3\}.$$

The argument for the arc-integrity of $P_3 \times P_n$ is similar to that used in Theorem 3.7.2 on the integrity of $P_3 \times P_n$.

Theorem 4.7.2: $\bar{I}'(P_3 \times P_n) =$

$$\min\{n, \left\lceil \sqrt{3n-3} \right\rceil + 3\left\lceil \frac{n}{\left\lceil \sqrt{3n-3} \right\rceil} \right\rceil - 1, \left\lfloor \sqrt{3n-3} \right\rfloor + 3\left\lceil \frac{n}{\left\lfloor \sqrt{3n-3} \right\rfloor} \right\rceil - 1\}.$$

Proof: Let D be an orientaion of $P_3 \times P_n$. Then as in Theorem 3.7.2, there is some single arc between V(*,i) and V(*,i+1) whose removal leaves V(*,1:i) disconnected from V(*,i + 1:n) (see Figure 4.5). This means that there is a set S of k arcs for which D - S has at least k + 1 strong components.

The digraph derived by extending the orientation of $P_3 \times P_5$ in Figure 4.6 to the general case $P_3 \times P_n$ (orient the arcs of the 4-cycle $C_{i,j}$ clockwise if i + j is even and counter-clockwise if the sum is odd) attains

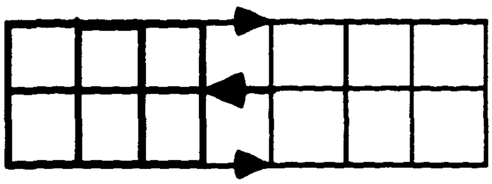


Figure 4.5

the lower bound for the theorem. In this orientation, removal of an arc contained in V(*,2) increases the number of strong components by at least

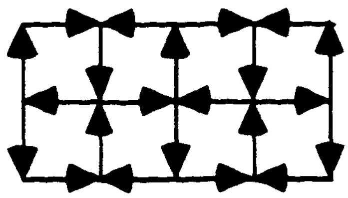


Figure 4.6

1. These components are either of the form V(*,i:j) (of order 3(j - i + 1)) or are trivial. Let T be an I'-set for D of order k. As in Theorem 3.7.2 the orders of the components of D - T should be nearly equal. Therefore, m(D - T) will either be 1 or $3\left\lceil \frac{n}{k+1} \right\rceil$.

Case 1: m(D - T) = 1. There are n - 1 pairwise arc-disjoint 4-cycles (the $C_{i,j}$ where i + j is even), so we need to remove at least n - 1 arcs. However, if the arcs of V(2,*) are removed the remaining digraph has no dicycles.

Case 2: m(D - T) = $3\left\lceil \frac{n}{k+1} \right\rceil$. In a manner similar to the previous chapter, calculus is used to find the arc-integrity. Here

$$I'(D) = k + 3\lceil n/(k+1) \rceil = k + 3\lfloor (n+k)/(k+1) \rfloor \leq$$
$$k + 3(n+k)/(k+1).$$

The last expression is minimized when $k = \sqrt{3n - 3} - 1$. Hence, the first expression (arc-integrity) is minimized when k is either $\lceil \sqrt{3n-3} - 1 \rceil$ or $\lfloor \sqrt{3n-3} - 1 \rfloor$. When these values are substituted into the arc-integrity expression we get the desired values.

Let $r = \lceil \sqrt{3n-3} \rceil$ and $s = \lfloor \sqrt{3n-3} \rfloor$, and let $(v_{t,i}, v_{t,i+1})$ be an arc whose removal leaves V(*,1:i) disconnected from V(*,i + 1:n). Then the upper bound is demonstrated when D is an orientation of $P_3 \times P_n$ and T is one of the following sets:

$$T_1 = \{(v_{t,i}, v_{t,i+1}) \mid i = 1,\dots,n-1\}$$
$$T_2 = \{(v_{t,i}, v_{t,i+1}) \mid i = j(\lceil n/r \rceil + 1), \text{ and } j = 1,\dots,r\}$$
$$T_3 = \{(v_{t,i}, v_{t,i+1}) \mid i = j(\lceil n/s \rceil + 1), \text{ and } j = 1,\dots,s\}. \quad \square$$

Theorem 4.7.3: When $m \le n$, then $\bar{I}'(P_m \times P_n) \le$

$$\min\Big\{(n-1)\lfloor m/2 \rfloor + 1,$$
$$\left\lfloor \frac{m}{2} \right\rfloor\left(\left\lceil \sqrt{2n-2} \right\rceil - 1\right) + m\left\lceil \frac{n}{\left\lceil \sqrt{2n-2} \right\rceil} \right\rceil,$$
$$\left\lfloor \frac{m}{2} \right\rfloor\left(\left\lfloor \sqrt{2n-2} \right\rfloor - 1\right) + m\left\lceil \frac{n}{\left\lfloor \sqrt{2n-2} \right\rfloor} \right\rceil\Big\}.$$

Proof: Let D be an orientaion of $P_m \times P_n$. As in Theorem 3.7.3, there is some set of $\lfloor m/2 \rfloor$ arcs between V(*,i) and V(*,i+1) whose removal leaves V(*,1:i) disconnected from V(*,i + 1:n). This means that there is a set S of $\lfloor m/2 \rfloor k$ arcs for which D - S has at least k + 1 strong components.

The upper bound for the theorem is shown as follows. For fixed j there is a set of $\lfloor m/2 \rfloor$ arcs of the form $(v_{i,j}, v_{i,j+1})$ whose removal disconnects V(*,1:j) from V(*,i + 1:n). Let T be a set of k such disconnecting sets, chosen in a way that the orders of the components of D - T are approximately uniform, as in Theorem 3.7.3. Here, m(D - T) will either be 1 or $m\lceil n/(k+1)\rceil$.

Case 1: Let k = n - 1. Then m(D - T) = 1 and hence, the minimum occurs at the first expression.

Case 2: For k < n - 1, choose k to minimize $\lfloor m/2\rfloor k + m\lceil n/(k+1)\rceil$. Recall that $I(D) \le \lfloor m/2\rfloor k + m\lceil n/(k+1)\rceil = \lfloor m/2\rfloor k + m\lfloor (n+k)/(k+1)\rfloor \le (m/2)k + m(n/(k+1)$. The final expression has a minimum when $k = \sqrt{2n-2} - 1$, and hence the first expression attains its minimum when k is either $\lceil \sqrt{2n-2} - 1\rceil$ or $\lfloor \sqrt{2n-2} - 1\rfloor$. When these values are substituted into the the first expression we get the last two terms in the minimum of the theorem. □

Theorem 4.7.4: $\bar{I}'(P_m \times C_n) \le$

$$\min\{(n+1)\lfloor m/2\rfloor + 1,$$
$$\left\lceil \sqrt{2n-2} + 1\right\rceil \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n}{\left\lceil \sqrt{2n-2}\right\rceil} \right\rceil,$$
$$\left\lfloor \sqrt{2n-2} + 1\right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n}{\left\lfloor \sqrt{2n-2}\right\rfloor} \right\rceil\}$$

Proof: Let D be an orientation of $P_m \times C_n$. Between V(i,*) and V(i + 1,*) (for i = 1,2) there are at least $\lceil m/2\rceil$ arcs oriented from one of the P_m to the other, hence at most $\lfloor m/2\rfloor$ oriented in the other direction. Remove k > 1 of these sets of $\lfloor m/2\rfloor$ arcs so that the sets are as evenly spaced as possible. If all the sets are oriented in the "same direction" (see Figure 4.7 where the dotted arcs are the arcs which are removed and the dotted line traces a directed cycle which remains), then the digraph might not be disconnected. If one of the k sets is replaced by the $2\lfloor m/2 \rfloor$ arcs which are

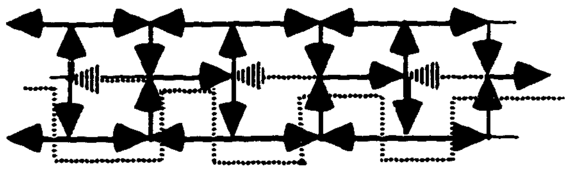


Figure 4.7

oriented in the opposite direction, then removal of this new set T separates D into at least k components.

If k = n, then m(D - T) = 1. Therefore $I'(D) \leq (n + 1)\lfloor m/2 \rfloor + 1$. Assume k < n, then we want to minimize $(k+1)\lfloor m/2 \rfloor + m\lceil n/k \rceil$ (which is at least I'(D)) with respect to k. Using calculus as before, we find the minimum occurs when k is either $\lceil \sqrt{2n-2} \rceil$ or $\lfloor \sqrt{2n-2} \rfloor$, and hence, the theorem follows. □

Theorem 4.7.5: For $m \leq n$,

$$\bar{I}'(C_m \times C_n) \leq \min\{(n+1)\lfloor m/2 \rfloor + m,$$
$$\left\lceil \sqrt{2n-2} + 1 \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n}{\lceil \sqrt{2n-2} \rceil} \right\rceil,$$
$$\left\lfloor \sqrt{2n-2} + 1 \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + m\left\lceil \frac{n}{\lfloor \sqrt{2n-2} \rfloor} \right\rceil\}.$$

Proof: The addition of the extra arcs does not change the disconnection principle used in Theorem 4.7.4, nor the calculus involved in finding the last two bounds. The only difference between the arguments is that when k=n, the number m(D-T) may be m. Hence, we get the first expression. □

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