Distances Associated with Subgraphs and Subdigraphs

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DISTANCES ASSOCIATED WITH SUBGRAPHS AND SUBDIGRAPHS

by

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Steven John Winters, Ph.D.

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The defining properties of several important subgraphs and subdigraphs rely on the concept of distance in graphs and digraphs. In this dissertation, we investigate many of these subgraphs and subdigraphs.

In Chapter I, we present some preliminary definitions and examples. In addition, many known results are recalled. We then introduce several new induced subgraphs and subdigraphs.

In Chapter II, we investigate the general structure of the center and periphery of a graph. We introduce two new induced subgraphs of the center along with a new induced subgraph of the periphery of a graph in order to study these structures.

For every digraph D, there is a corresponding digraph whose vertex set consists of subsets of vertices of D of the same cardinality. In Chapter III, we introduce this multivertex digraph and indicate the motivation for studying these digraphs.

The center and periphery are subgraphs or subdigraphs induced by those vertices of minimum and maximum eccentricity, respectively. In Chapter IV, we introduce two new induced subgraphs and subdigraphs that involve the remaining vertices and investigate their relative location in the graph or digraph.

We continue this investigation in Chapter V by studying the relative location of the median and periphery of a graph or digraph.

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Distances associated with subgraphs and subdigraphs

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To my loving parents

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<div align="right">Steven John Winters</div>

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CHAPTER I

INTRODUCTION

1.1 Definitions and Examples

Distance is one of the most fundamental concepts in the theory of graphs and digraphs. In fact, Buckley and Harary [3] wrote an entire book devoted to the study of distance in graphs. The standard *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G.

The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex furthest from v in G, namely, $e(v) = \max_{u \in V(G)} d(v, u)$. The *radius* rad G of G is the minimum eccentricity among the vertices of G; its *diameter* diam G is the maximum eccentricity. The *center* $C(G)$ of G is the subgraph induced by those vertices of G having minimum eccentricity; the *periphery* $P(G)$ is the subgraph induced by those vertices of G having maximum eccentricity. We say that a vertex v of G is a *central vertex* if v is a vertex in $C(G)$, and v is called a *peripheral vertex* if v is in $P(G)$. We illustrate these concepts by giving an example of a connected graph G along with its center $C(G)$ and periphery $P(G)$ in Figure 1.1. The eccentricity of each vertex of G is also indicated. Furthermore, rad $G = 2$ and diam $G = 4$.

Let v be a central vertex in a connected graph G with rad $G \neq$ diam G. We define the *central distance* $c(v)$ of v as the largest nonnegative integer k such that if $d(v, x) \leq k$, then x is also a central vertex. Let $m = \max\{c(v)\}$ over all central vertices v of G. Then the *ultracenter* $UC(G)$ of G is the subgraph induced by those central vertices v with $c(v) = m$; while the *central fringe* $CF(G)$ of G is the

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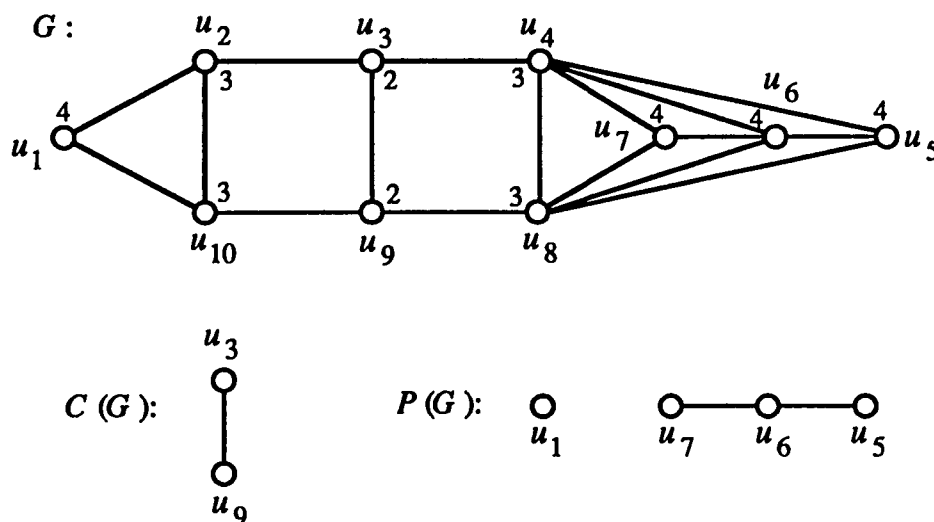
Figure 1.1

subgraph induced by those central vertices v with $c(v) = 0$. Similarly, the *peripheral distance* $p(v)$ of a peripheral vertex v is the largest nonnegative integer k such that if $d(v, x) \le k$, then x is also a peripheral vertex. If $m = \max\{p(v)\}$ over all peripheral vertices v of G, then the *ultraperiphery* $UP(G)$ of G is the subgraph induced by those vertices v with $p(v) = m$. In Figure 1.2, we give an example of a connected graph G along with its ultracenter $UC(G)$, central fringe $CF(G)$, and ultraperiphery $UP(G)$. The eccentricity of each vertex of G is also indicated. We investigate the properties of the ultracenter, central fringe, and ultraperiphery of connected graphs in Chapter II.

A digraph D is *strong* if for every two vertices u and v of D, there is both a u–v (directed) path and a v–u path in D. For vertices u and v in a strong digraph D, the *directed distance* $\overrightarrow{d}(u, v)$ (or $d(u, v)$ if directed distance is clear from context) from u to v is the length of a shortest u–v path in D. We say that a digraph D is *asymmetric* if whenever (u, v) is an arc of D, then (v, u) is not an arc of D.
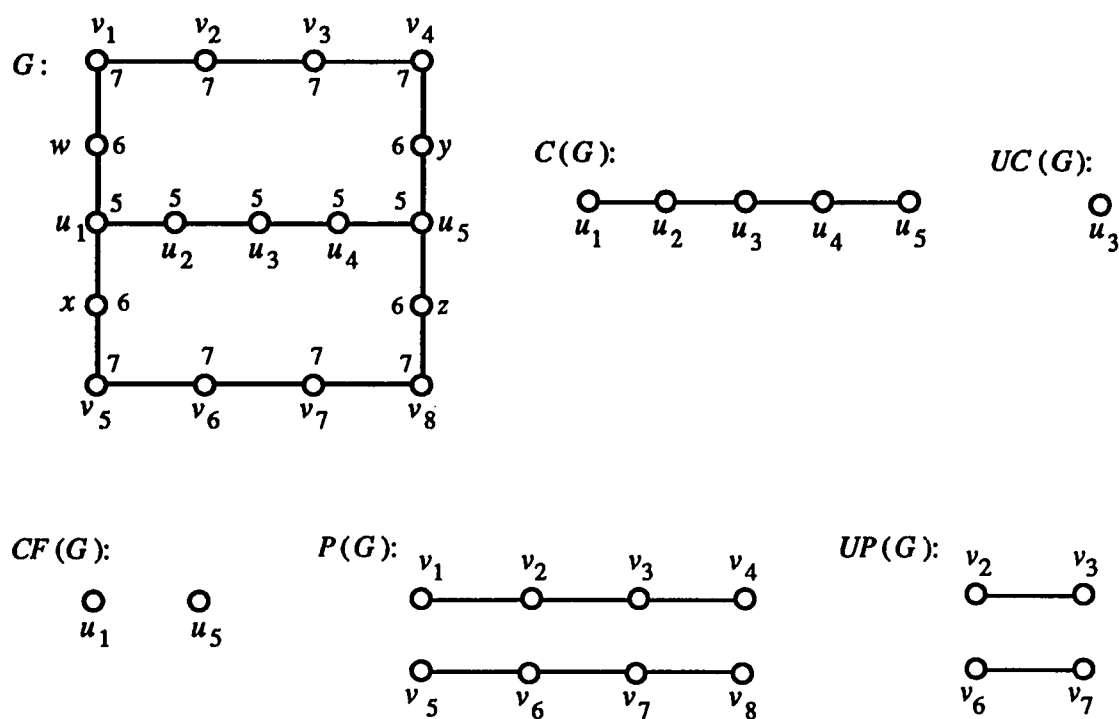
Figure 1.2

For a vertex v in a strong digraph D, the *eccentricity* $e(v)$ of v is the directed distance from v to a vertex furthest from v in D. The *radius* rad D of D is the minimum eccentricity among the vertices of D; while its *diameter* diam D is the maximum eccentricity. The *center* $C(D)$ of D is the subdigraph induced by those vertices of D having minimum eccentricity; while the *periphery* $P(D)$ is the subdigraph induced by those vertices having maximum eccentricity. In Figure 1.3, we give an example of a strong asymmetric digraph D with its center $C(D)$ and periphery $P(D)$. In addition, rad $D = 3$, diam $D = 4$, and the eccentricity of each vertex is indicated.

Let F and H be subgraphs of a connected graph G. Then the standard *distance* $d(F, H)$ between F and H is defined by
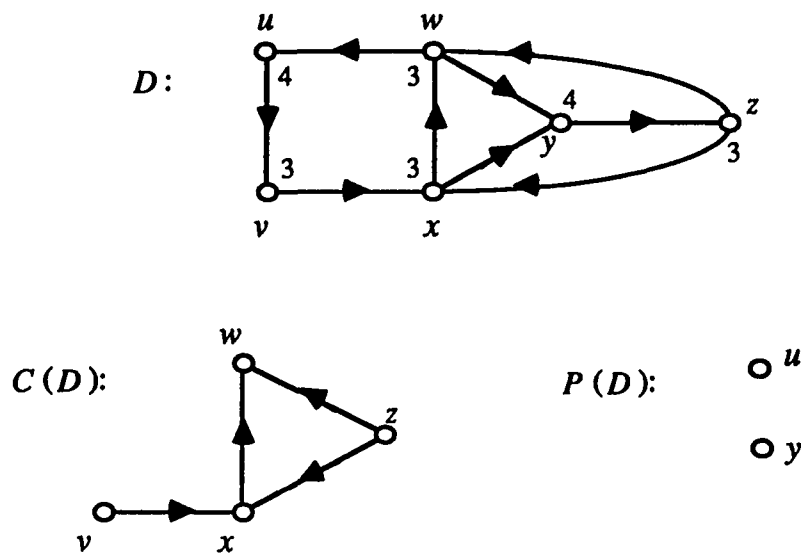
Figure 1.3

$$d(F, H) = \min\{d(u, v) \mid u \in V(F), v \in V(H)\}.$$

For example, if we consider the graph G given in Figure 1.1, then $d(C(G), P(G)) = 2$. Observe that the distance between subgraphs is a generalization of the distance between two vertices; that is, if $V(F) = \{u\}$ and $V(H) = \{v\}$, then $d(F, H) = d(u, v)$.

Similarly, if F and H are subdigraphs of a strong digraph D, then the standard *directed distance* $\overrightarrow{d}(F, H)$ (or $d(F, H)$ if directed distance is clear from context) from F to H is defined by

$$\overrightarrow{d}(F, H) = \min\{\overrightarrow{d}(u, v) \mid u \in V(F), v \in V(H)\}.$$

If D is the digraph given in Figure 1.3 with subdigraphs $F = \langle\{u, v\}\rangle$ and $H = \langle\{z\}\rangle$, then $\overrightarrow{d}(F, H) = 3$, while $\overrightarrow{d}(H, F) = 2$. Clearly, this distance is not a metric, but it is a generalization of the directed distance from one vertex to another.

We now consider another distance introduced by Johns [5] involving subgraphs of the same order. Let G be a connected graph of order p and let n be an integer with $1 \le n \le p$. Furthermore, let F and H be induced subgraphs of G of order n. We define a *pairing* π from the vertex set $V(F) = \{v_1, v_2, \ldots, v_n\}$ to $V(H)$ as a one-to-one mapping from $V(F)$ to $V(H)$. The *subgraph distance* $sd_\pi(F, H)$ *induced by* π between F and H is

$$sd_\pi(F, H) = \sum_{i=1}^{n} d(v_i, \pi(v_i)).$$

The *subgraph distance* $sd(F, H)$ between F and H is defined by

$$sd(F, H) = \min_\pi sd_\pi(F, H).$$

For example, in Figure 1.4, we give a connected graph G along with two induced subgraphs F and H of order 3. We also list all pairings between $V(F)$ and $V(H)$ and compute $sd(F, H)$. In Chapter III, we introduce the corresponding directed distance, and we investigate properties of this directed subdigraph distance that are analogous to properties involving graphs.

The *distance* $d(v)$ of a vertex v in a connected graph G is the sum of the distances from v to the vertices of G; that is, $d(v) = \sum_{u \in V(G)} d(v, u)$. The *median* $M(G)$ of G is the subgraph of G induced by those vertices having minimum distance. In Figure 1.5, we give an example of a connected graph G along with its median $M(G)$. Furthermore, the distance of each vertex is also indicated. Similarly, for a vertex v in a strong digraph D, the *distance* $\vec{d}(v)$ (or $d(v)$ if directed distance is clear from context) of v is the sum of the directed distances from v to the vertices of D, namely, $\vec{d}(v) = \sum_{u \in V(D)} \vec{d}(v, u)$. The *median* $M(D)$ of D is the subdigraph of D induced by those vertices having minimum distance. A strong asymmetric digraph D

Pairing	u_i	v_j	$d(u_i, v_j)$	$sd_{\pi_k}(F, H)$
π_1	u_1	v_1	2	
	u_2	v_2	2	5
	u_3	v_3	1	
π_2	u_1	v_1	2	
	u_2	v_3	2	6
	u_3	v_2	2	
π_3	u_1	v_2	3	
	u_2	v_1	1	5
	u_3	v_3	1	
π_4	u_1	v_2	3	
	u_2	v_3	2	7
	u_3	v_1	2	
π_5	u_1	v_3	3	
	u_2	v_1	1	6
	u_3	v_2	2	
π_6	u_1	v_3	3	
	u_2	v_2	2	7
	u_3	v_1	2	

$$sd(F, H) = 5$$

Figure 1.4

is given in Figure 1.6 along with its median $M(D)$. Again, the distance of each vertex is indicated.
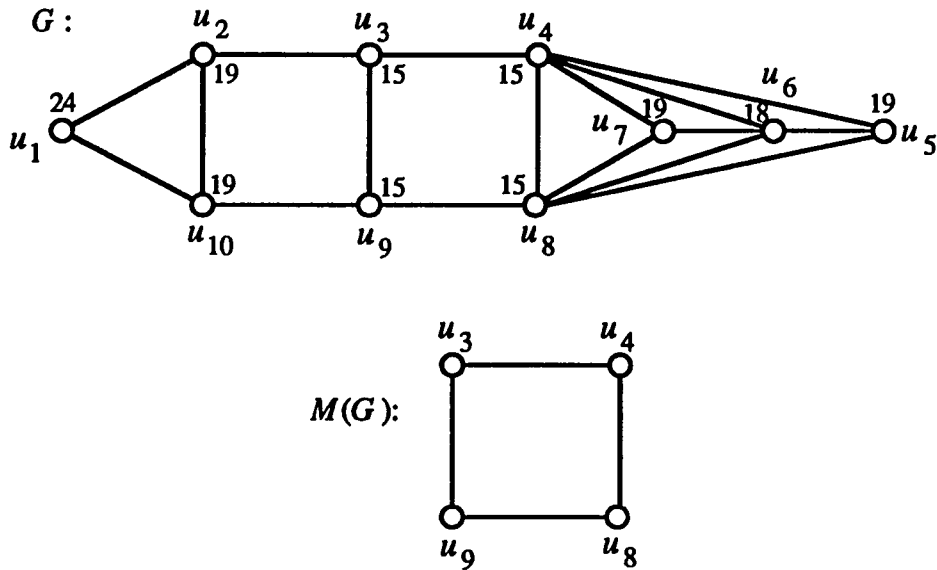
Figure 1.5
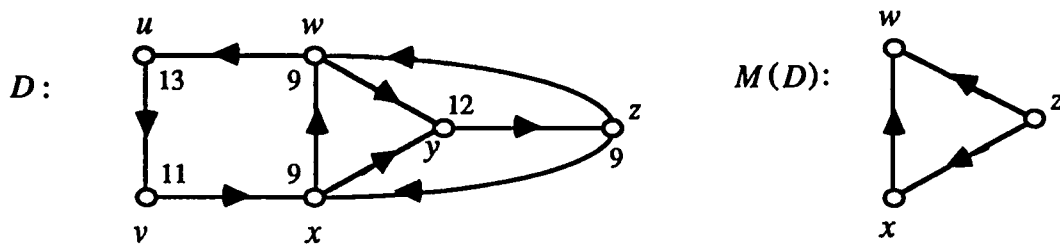
Figure 1.6

1.2 Some Previous Results

Hedetniemi (see [4]) showed that every graph is the center of some connected graph, that is, for every graph G, there exists a connected graph H such that $C(H) \cong G$; while Slater [16] showed that every graph is the median of some connected graph. Since the center and median are two ways of defining the "middle" of a graph, one might expect the center and median of a graph to overlap (have vertices in common) or at least be "close" to each other. Such is not the case, however, as Hendry [11]

proved that for every two graphs F and G, there exists a connected graph H such that $C(H) \cong F$ and $M(H) \cong G$, where $C(H)$ and $M(H)$ are disjoint. Holbert [12] extended this result by showing that for every two graphs F and G and positive integer k, there exists a connected graph H such that $C(H) \cong F$, $M(H) \cong G$, and $d(C(H), M(H)) = k$. Thus, the standard distance between the center and median of a graph can be arbitrarily large. On the other hand, these subgraphs can be arbitrarily close as Novotny and Tian [13] showed when they proved that for any three graphs F, G, and K, where K is isomorphic to an induced subgraph of both F and G, there exists a connected graph H such that $C(H) \cong F$, $M(H) \cong G$, and $C(H) \cap M(H) \cong K$.

Not every graph is the periphery of some graph, however. Bielak and Syslo [2] proved that a graph G is the periphery of some connected graph if and only if $e(x) \neq 1$ for each $x \in V(G)$ or $e(x) = 1$ for each $x \in V(G)$. Chartrand, Johns, and Tian [6] proved that for every asymmetric digraph D, there exists a strong asymmetric digraph H_1 such that $C(H_1) \cong D$ and there exists a strong asymmetric digraph H_2 such that $P(H_2) \cong D$. It was shown by Shaikh [14] that for every two digraphs D_1 and D_2, there exists a strong digraph H such that $C(H) \cong D_1$ and $P(H) \cong D_2$. We now extend this result to asymmetric digraphs.

Theorem 1.1 For every two asymmetric digraphs D_1 and D_2, there exists a strong asymmetric digraph H such that $C(H) \cong D_1$ and $P(H) \cong D_2$.

Proof We define a strong asymmetric digraph H by

$$V(H) = V(D_1) \cup V(D_2) \cup \{z_i \,|\, 1 \leq i \leq 6\}$$

and

$$E(H) = E(D_1) \cup E(D_2) \cup \{(z_i, z_{i+1}) \,|\, 1 \leq i \leq 5\} \cup \{(x, z_1), (z_5, x) \,|\, x \in V(D_2)\}$$
$$\cup \{(x, z_1), (x, z_4), (z_5, x), (z_6, x) \,|\, x \in V(D_1)\}$$

(see Figure 1.7).
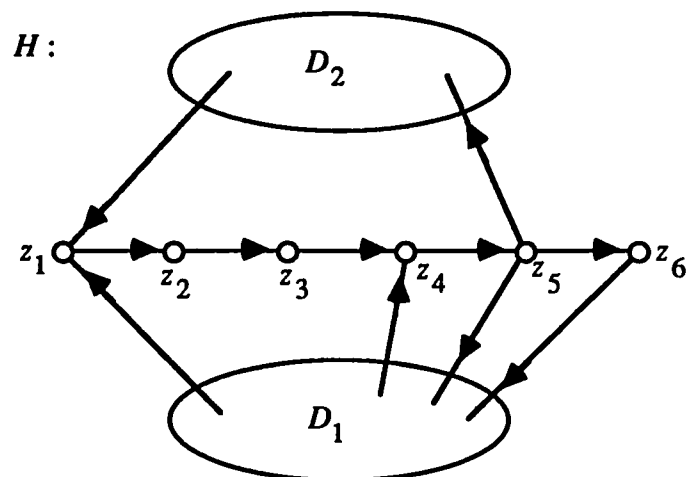
Figure 1.7

From the construction of H, we have

(i) $e(x) = 3$ for $x \in V(D_1)$,

(ii) $e(z_5) = e(z_6) = 4$,

(iii) $e(z_i) = 5$ for $1 \le i \le 4$, and

(iv) $e(x) = 6$ for $x \in V(D_2)$.

Thus, $C(H) \cong D_1$ and $P(H) \cong D_2$. $\square$

The results of Hendry [11], Holbert [12], and Novotny and Tian [13] involving graphs were extended to digraphs in [7]. For every two asymmetric digraphs D_1 and D_2, there exists a strong asymmetric digraph H such that $C(H) \cong D_1$ and $M(H) \cong D_2$, and where the directed distances from $C(H)$ to $M(H)$ and from $M(H)$ to $C(H)$ can be arbitrarily prescribed. Furthermore, if K is a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D_1 and D_2, then there exists a strong asymmetric digraph F such that $C(F) \cong D_1$, $M(F) \cong D_2$, and $C(F) \cap M(F) \cong K$.

In Chapter IV, we introduce the concepts of interior and annulus of a connected graph and of a strong digraph. For a connected graph G with rad $G <$ diam G, we define the *interior* Int(G) of G by

$$\text{Int}(G) = \langle \{v \in V(G) \mid e(v) < \text{diam } G\} \rangle.$$

If rad $G =$ diam G, we define Int(G) $= G$. For a connected graph G with rad $G <$ diam $G - 1$, we define the *annulus* Ann(G) of G by

$$\text{Ann}(G) = \langle \{v \in V(G) \mid \text{rad } G < e(v) < \text{diam } G\} \rangle.$$

If rad $G \geq$ diam $G - 1$, we say that graph G has no annulus. We illustrate these concepts in Figure 1.8 by presenting a graph G along with its interior Int(G) and annulus Ann(G). We also indicate the eccentricity of each vertex of G. Similarly, we define the interior and annulus of a strong digraph in Chapter IV.
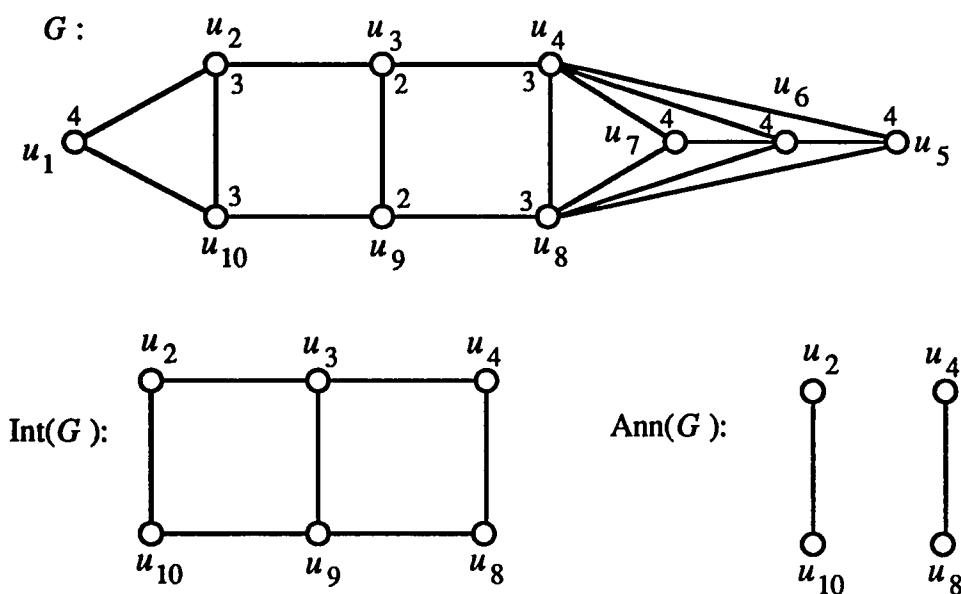
Figure 1.8

Many of the previous results indicate a relationship between the relative location of the center and median or the center and periphery of a graph or digraph. There are similar types of questions involving the relationship between the relative location of other induced subgraphs or subdigraphs. We investigate these results for the interior and annulus of graphs and digraphs in Chapter IV. We continue this investigation in Chapter V by studying the relative location of the median and periphery of a graph or digraph.

CHAPTER II

ULTRACENTERS, CENTRAL FRINGES, AND ULTRAPERIPHERIES OF GRAPHS

2.1 The Ultracenter and Central Fringe of a Graph

The center is among the most studied induced subgraphs of a graph. One reason for this is that the center of a graph has many applications, for example, in facility location problems. Another reason is that the center describes the "middle" of a graph and by examining the center along with other induced subgraphs whose defining property relies on the distance between two vertices, such as the median and periphery, we obtain information about the structure of the graph. But even though the vertices of the center have the same eccentricity, this may not indicate how the center interacts with the rest of the graph. In addition, some of the vertices of the center could be interpreted as being more "central" than others. For example, a central vertex that is adjacent only to central vertices may appear to be more central than one that is adjacent to some noncentral vertex. In Figure 2.1, we illustrate some of these ideas by presenting two graphs of the same order that have isomorphic centers and peripheries but have considerably different structures.

This example motivates us to investigate the structure of the center and periphery of a graph to obtain a better understanding of the overall structure of the graph. We start by introducing two induced subgraphs of the center of a graph.

Let v be a central vertex in a connected graph G with rad $G <$ diam G. The *central distance* $c(v)$ of v is the largest nonnegative integer k such that if $d(v, x) \leq k$, then x is also a central vertex. If $m = \max\{c(v)\}$ over all central vertices v of G,

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$$C(F) \cong C(G):$$

$$P(F) \cong P(G):$$
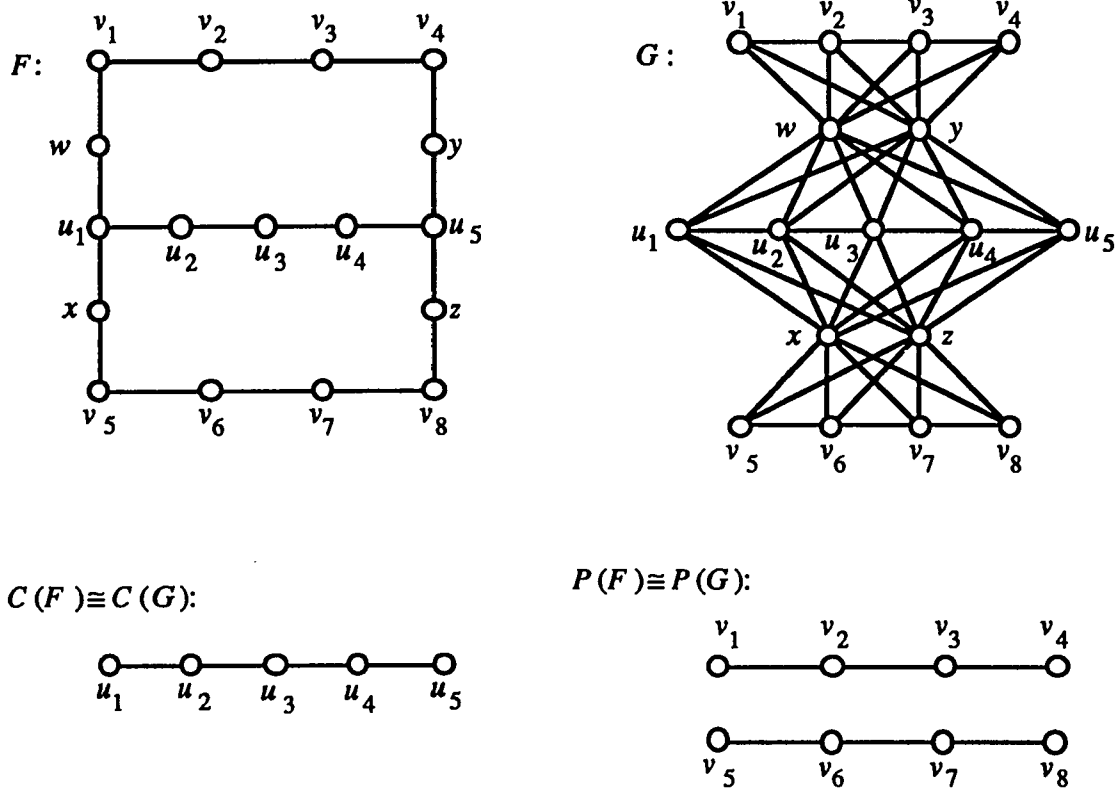
Figure 2.1

then the *ultracenter* $UC(G)$ of G is the subgraph induced by those central vertices v with $c(v) = m$; while the *central fringe* $CF(G)$ of G is the subgraph induced by those central vertices v with $c(v) = 0$. We now investigate which graphs can be the ultracenter (or central fringe) of some connected graph.

Theorem 2.1 Let F and G be graphs and let n be a positive integer. If $\operatorname{diam} F \geq 2n - 1$, then there exists a connected graph H such that $CF(H) \cong F$, $UC(H) \cong G$, and every vertex of $UC(H)$ has central distance n.

Proof Let $v \in V(F)$ such that $e(v) = \operatorname{diam} F$. We partition the vertex set of F as follows:

(i) $A_0 = \{v\}$;

(ii) $A_i = \{x \mid xy \in E(F), y \in A_{i-1}, x \notin \bigcup_{j=0}^{i-1} A_j\}$, $1 \le i \le 2n - 2$; and

(iii) $A_{2n-1} = V(F) - \bigcup_{j=0}^{2n-2} A_j$.

We now construct a connected graph H by

$$V(H) = V(F) \cup V(G) \cup \{v_i, v_i', w_i \mid 1 \le i \le 2n + 2\}$$

$$\cup \{u_i, u_i' \mid 1 \le i \le n - 1\} \cup \{z_i \mid 1 \le i \le 2n - 2\}$$

and

$$E(H) = E(F) \cup E(G) \cup \{u_i u_{i+1}, u_i' u_{i+1}' \mid 1 \le i \le n - 2\}$$

$$\cup \{v_i v_{i+1}, v_i' v_{i+1}', w_i w_{i+1} \mid 1 \le i \le 2n + 1\}$$

$$\cup \{vu_{n-1}, vv_1, vv_1', vw_1, vz_1\} \cup \{xu_1, xu_1' \mid x \in V(G)\}$$

$$\cup \{xu_{n-1}', xv_{2n+2}, xv_{2n+2}', xw_{2n+2}, xz_{2n-2} \mid x \in A_{2n-1}\}$$

$$\cup \{xw_{i+1} \mid x \in A_i, 1 \le i \le n - 1\} \cup \{xw_{i+3} \mid x \in A_i, n \le i \le 2n - 2\}$$

$$\cup \{xz_{i-1}, xz_{i+1} \mid x \in A_i, 2 \le i \le 2n - 3\} \cup \{xz_2 \mid x \in A_1\}$$

$$\cup \{xz_{2n-3}, xz_{2n-2} \mid x \in A_{2n-2}\} \cup \{z_i z_{i+1} \mid 1 \le i \le 2n - 3\}$$

(see Figure 2.2).

Let $B = V(F) \cup V(G) \cup \{u_i, u_i' \mid 1 \le i \le n - 1\} \cup \{z_i \mid 1 \le i \le 2n - 2\}$. The graph H is constructed in such a way that if $x \in B$ and $y \in V(H)$, then x and y are on some cycle (of H) of order at most $4n + 3$. Thus $e_H(x) \le 2n + 1$ for $x \in B$.

We calculate the eccentricity of each vertex of B by observing the following:

(i) $d(x, v_{n+1}) = 2n + 1$ for each $x \in V(G)$,

(ii) $d(u_i, v_{n+1+i}) = 2n + 1$ for $1 \le i \le n - 1$,

(iii) $d(u_i', v_{n+1-i}) = 2n + 1$ for $1 \le i \le n - 1$,

(iv) $d(v, v_{2n+1}) = 2n + 1$,
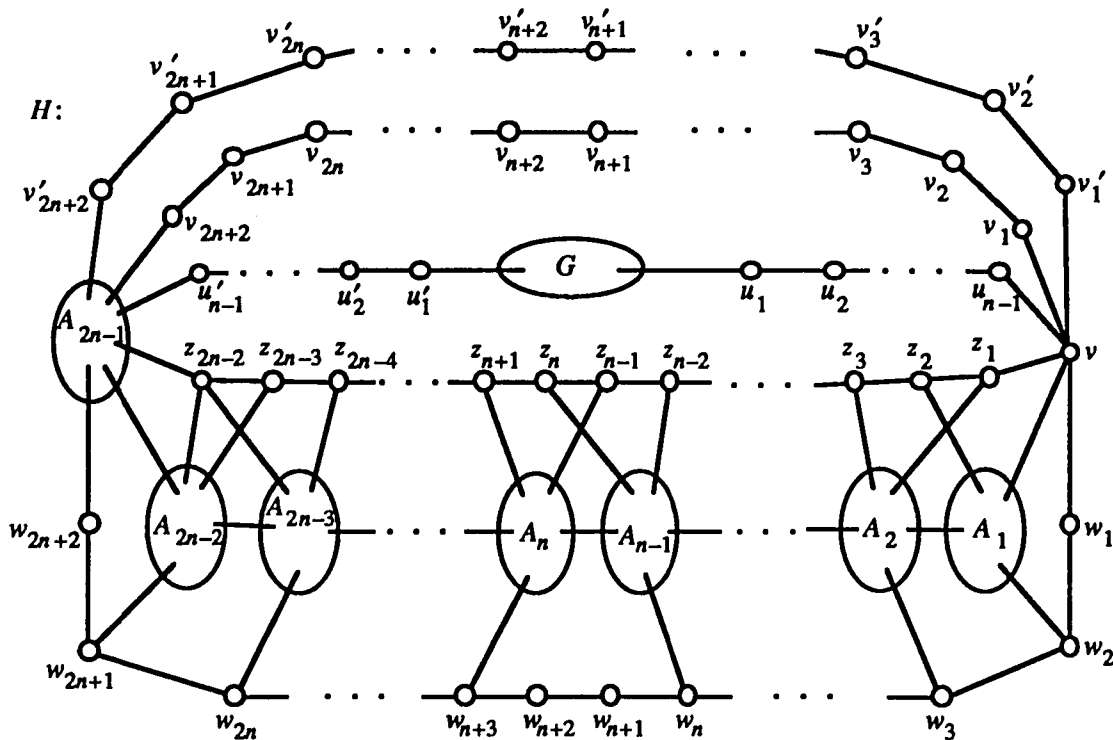
Figure 2.2

(v) $d(z_i, v_{2n+1-i}) = 2n + 1$ for $1 \le i \le 2n - 2$, and

(vi) $d(x, v_{2n+1-i}) = 2n + 1$ for $x \in A_i, 1 \le i \le 2n - 1$.

Therefore $e_H(x) = 2n + 1$ for each $x \in B$. Furthermore, the remaining vertices of H have eccentricities at least $2n + 2$ as illustrated below:

(i) $d(w_i, v_{2n+2-i}) = 2n + 2$ for $1 \le i \le n + 2$,

(ii) $d(w_i, v_{2n+4-i}) = 2n + 2$ for $n + 3 \le i \le 2n + 2$,

(iii) $d(v_i, v'_{2n+2-i}) = 2n + 2$ for $1 \le i \le 2n + 1$,

(iv) $d(v_{2n+2}, v'_2) = 2n + 2$, and

(v) $d(v'_{2n+2}, v_2) = 2n + 2$.

Consequently, $e(x) \geq 2n + 2$ for $x \in V(H) - B$. In addition, from the construction of H, we conclude that $CF(H) \cong F$, $UC(H) \cong G$, and each vertex of $UC(H)$ has central distance n. $\square$

We now state two corollaries which follow immediately from Theorem 2.1.

Corollary 2.2 Let F and G be two graphs. Then there exists a connected graph H such that $CF(H) \cong F$ and $UC(H) \cong G$.

Corollary 2.3 Let G be a graph and n a positive integer. Then there exists a connected graph H such that $UC(H) \cong G$ and every vertex of $UC(H)$ has central distance n.

Now suppose that we are given two graphs F and G such that G is an induced subgraph of F. Is it possible to construct a connected graph H such that $C(H) \cong F$ and $UC(H) \cong G$, where each vertex of $UC(H)$ has central distance $n \geq 1$? This appears to be a difficult problem even for small n, so we will only investigate this question for $n = 1$.

Theorem 2.4 Let F and G be graphs such that G is an induced subgraph of F. Let $V(F) - V(G) = A \cup B$ with $A \cap B = \emptyset$. If for each $x \in V(G)$, there is some $y \in A$ and some $z \in B$ such that $xy, xz \in E(F)$, then there exists a connected graph H such that $UC(H) \cong G$, $C(H) \cong F$, and each vertex of $UC(H)$ has central distance 1.

Proof Let sets A and B be a partition of the vertex set $V(F) - V(G)$. In addition, assume that for each $x \in V(G)$, there is some $y \in A$ and some $z \in B$ such that xy, $xz \in E(F)$. Furthermore, assume that if $x \in V(F) - V(G)$ and $xy \notin E(F)$ for each

$y \in V(G)$, then $x \in B$. Let C be the set of vertices from B that are not adjacent to any vertices of G. We construct a connected graph H by

$$V(H) = V(F) \cup \{v_i, w_i \mid 1 \le i \le 6\} \cup \{u_i \mid 1 \le i \le 6, \ C \ne \varnothing\}$$

and

$$E(H) = E(F) \cup \{v_i v_{i+1}, w_i w_{i+1} \mid 1 \le i \le 5\} \cup \{xv_1, xw_1 \mid x \in B\}$$

$$\cup \{xv_6, xw_6 \mid x \in A\} \cup \{xu_1, xu_4 \mid x \in C, \ C \ne \varnothing\}$$

$$\cup \{u_1 u_2, u_2 u_3, u_3 v_5, u_3 v_6, u_4 u_5, u_5 u_6, u_6 w_5, u_6 w_6 \mid C \ne \varnothing\}$$
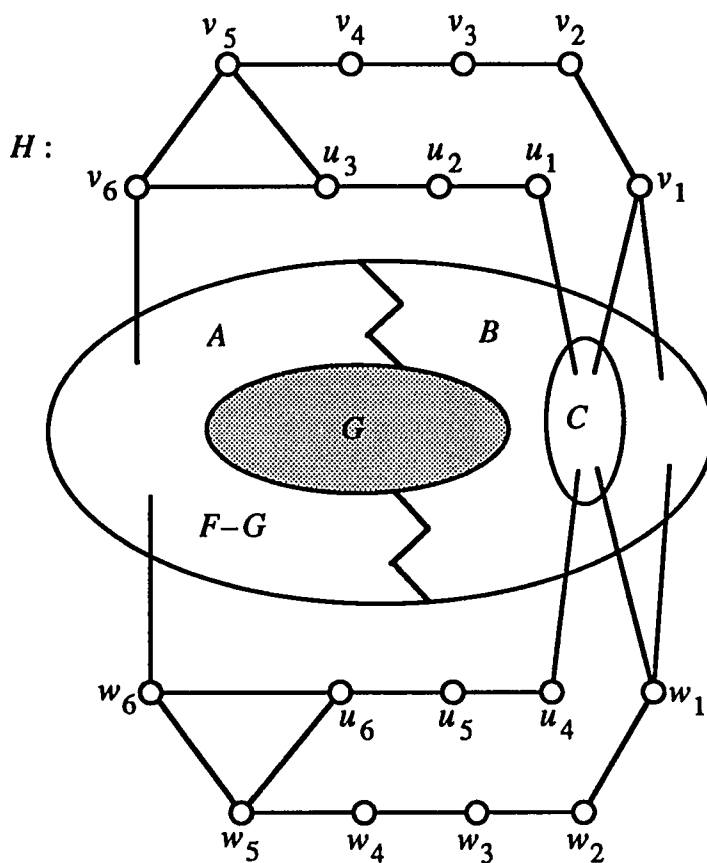
(see Figure 2.3).

Figure 2.3

We compute the eccentricity of each vertex of H as follows:

(i) $e(x) = 4$ for $x \in V(F)$,

(ii) $e(x) = 5$ for $x \in \{u_1, u_4, v_1, v_6, w_1, w_6\}$,

(iii) $e(x) = 6$ for $x \in \{u_2, u_3, u_5, u_6, v_2, v_5, w_2 w_5\}$, and

(iv) $e(x) = 7$ for $x \in \{v_3, v_4, w_3 w_4\}$.

Thus, $C(H) \cong F$. It is clear from the construction of H that $c(x) = 1$ for each $x \in V(G)$ and $c(y) = 0$ for each $y \in V(F) - V(G)$. Therefore, we conclude that $UC(H) \cong G$ and each vertex of $UC(H)$ has central distance 1. $\square$

The next theorem places several restrictions on induced subgraphs of the center that can be the ultracenter of a graph.

Theorem 2.5 If F, G, and H are graphs such that $UC(H) = G$, $C(H) = F$, and each vertex of $UC(H)$ has central distance $n \geq 1$, then

(i) $\deg_F x \geq 2$ for each $x \in V(G)$,

(ii) for each $v \in V(G)$, there exists $x \in V(F) - V(G)$ such that $xv \in E(F)$, and

(iii) $p(C(H)) \geq p(UC(H)) + 2n$.

Proof We will first show that $\deg_F x \geq 2$ for each $x \in V(G)$. Suppose, to the contrary, that there is some vertex z of G such that $\deg_F z \leq 1$. Since $c(z) \geq 1$, we have $\deg_F z = 1$ and $zx \notin E(H)$ for all $x \in V(H) - V(F)$. Therefore, there exists some $y \in V(F)$ such that $zy \in E(H)$. But this means that $e_H(z) = e_H(y) + 1$, which contradicts $C(H) = F$. Thus, $\deg_F x \geq 2$ for each $x \in V(G)$.

To show that condition (ii) must hold, we assume, to the contrary, that there is some vertex z in G that is not adjacent to any vertex of $F - V(G)$. Thus, the vertices adjacent to z are also in G, and they have central distance n. But this implies that

$c(z) = n + 1$, which is impossible. Hence, each vertex of G is adjacent to some vertex of $F - V(G)$.

We prove condition (iii) by showing that $p(F - V(G)) \geq 2n$. It is clear that for $x \in V(UC(H))$, we have $e(x) \geq n + 1$. Now suppose, to the contrary, that $p(F - V(G)) < 2n$. Let $x \in V(UC(H))$ and $y \in V(CF(H))$ such that $d(x, y) = n$. Furthermore, let P: $x = v_0, v_1, \ldots, v_n = y$ be any fixed x-y path of length n. For each $z \in V(CF(H))$, it follows that any x-z path of minimum length must contain a vertex of $\{v_i \mid 1 \leq i \leq n\}$. Clearly, $d(x, v_i) > d(v_1, v_i)$ for $1 \leq i \leq n$. Consequently, for each $z \in V(CF(H))$, we have $d(x, z) = d(x, v_i) + d(v_i, z)$ for some i $(1 \leq i \leq n)$ and

$$d(x, z) = d(x, v_i) + d(v_i, z) > d(v_1, v_i) + d(v_i, z) = d(v_1, z);$$

that is, $e(x) > e(v_1)$, which is a contradiction. Therefore, $p(F - V(G)) \geq 2n$ which implies that $p(C(H)) \geq p(UC(H)) + 2n$. $\square$

2.2 The Ultraperiphery of a Graph

In this section, we investigate the structure of the periphery of a graph. As seen in Figure 2.1, two graphs may have isomorphic peripheries but be quite different in their overall structure. We begin by introducing an induced subgraph of the periphery of a graph.

Let v be a peripheral vertex in a connected graph G with rad $G <$ diam G. Recall that the *peripheral distance* $p(v)$ of v is the largest nonnegative integer k such that if $d(v, x) \leq k$, then x is also a peripheral vertex. Let $m = \max\{p(v)\}$ over all peripheral vertices v of G. Then the *ultraperiphery* $UP(G)$ of G is the subgraph induced by those vertices v with $p(v) = m$. Even though some graphs are not the

periphery of any connected graph, the next result shows that every graph is the ultraperiphery of some connected graph.

Theorem 2.6 Let F be a graph and n a positive integer. Then there exists a connected graph H such that $UP(H) \cong F$ and every vertex of $UP(H)$ has peripheral distance n.

Proof We construct a connected graph H by

$$V(H) = V(F) \cup \{u_i, u_i', v_i', v_i'' \mid 1 \le i \le n\} \cup \{v_i \mid 1 \le i \le n + 4\}$$
$$\cup \{w_i, w_i' \mid 1 \le i \le n + 3\} \cup \{z_i, z_i' \mid 1 \le i \le n - 1\}$$

and

$$E(H) = E(F) \cup \{xu_1, xu_1' \mid x \in V(F)\} \cup \{v_i v_{i+1} \mid 1 \le i \le n + 3\}$$
$$\cup \{u_i u_{i+1}, u_i' u_{i+1}', v_i' v_{i+1}', v_i'' v_{i+1}'' \mid 1 \le i \le n - 1\}$$
$$\cup \{z_i z_{i+1}, z_i' z_{i+1}' \mid 1 \le i \le n - 2\}$$
$$\cup \{w_i w_{i+1}, w_i' w_{i+1}' \mid 1 \le i \le n + 2\}$$
$$\cup \{v_1' v_i, v_1'' v_i \mid 1 \le i \le n\} \cup \{w_n w_i, w_n' w_i' \mid n + 2 \le i \le n + 3\}$$
$$\cup \{v_{n+2} w_1, v_{n+2} w_1', v_{n+4} z_1, v_{n+4} z_1', u_n z_{n-1}, u_n w_{n+3}, u_n' z_{n-1}',$$
$$u_n' w_{n+3}', v_n' w_{n+1}, v_n'' w_{n+1}'\}$$

(see Figure 2.4).

From direct calculations, it follows that $\text{diam}\, H \le 2n + 4$. In addition, we have

(i) $d(x, v_i) = 2n + 4$ for $x \in V(F)$ and $1 \le i \le n$,

(ii) $d(u_i, v_i'') = 2n + 4$ for $1 \le i \le n$, and

(iii) $d(u_i', v_i') = 2n + 4$ for $1 \le i \le n$.
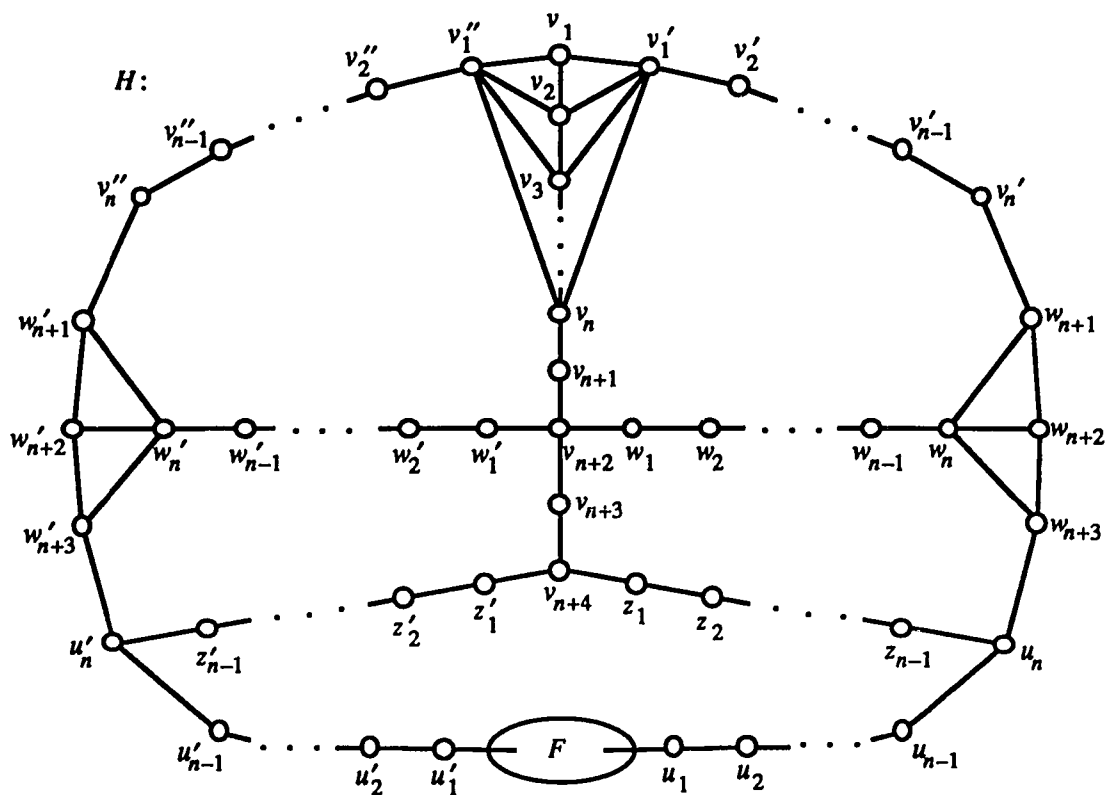
Figure 2.4

Thus, if $A = \{x, u_i, u_i', v_i, v_i', v_i'' \mid x \in V(F), 1 \le i \le n\}$, then $e(y) = 2n + 4$ for each $y \in A$. Furthermore, for each $x \in V(H) - A$, we have $e(x) < 2n + 4$. Consequently, it follows that $UP(H) \cong F$ and each vertex of $UP(H)$ has peripheral distance n. $\square$

We now investigate the case where $n = 0$.

Theorem 2.7 For any graph F, there exists a connected graph H such that $UP(H) \cong F$ and each vertex of $UP(H)$ has peripheral distance 0 if and only if $e_F(x) \ne 1$ for each $x \in V(F)$.

Proof First assume that there exists a connected graph H such that $UP(H) \cong F$ and each vertex of $UP(H)$ has peripheral distance 0. Then $P(H) = UP(H) \cong F$ and, from Bielak and Syslo [2], we have $e_F(x) \neq 1$ for each $x \in V(F)$ or $e_F(x) = 1$ for each $x \in V(F)$. But if F is a complete graph, then $UP(H)$ is not defined, which is a contradiction.

For the converse, we assume that F is a graph with $e_F(x) \neq 1$ for each $x \in V(F)$. We construct a connected graph H by joining a new vertex y to the vertices of F. It follows that $e_H(x) = 2$ for $x \in V(F)$ and $e_H(y) = 1$. Thus $P(H) \cong F$ and $C(H) = \langle \{y\} \rangle$. Furthermore, each vertex of $P(H)$ is adjacent to a vertex in $C(H)$. Therefore, $UP(H) = P(H) \cong F$ and each vertex of $UP(H)$ has peripheral distance 0. $\square$

CHAPTER III

SUBDIGRAPH DISTANCE AND MULTIVERTEX DIGRAPHS

3.1 Subdigraph Distance

For a strong digraph D of order p and an integer n such that $1 \leq n \leq p$, let F and H be induced subdigraphs of D of order n. Following Johns [5], we define a *pairing* π from the set $V(F)$, say $\{v_1, v_2, \dots, v_n\}$, to the set $V(H)$ as a one–to–one correspondence that associates a vertex of F with one of H. The *subdigraph distance induced by* π *from* F *to* H is defined as

$$sd_\pi(F, H) = \sum_{i=1}^{n} d(v_i, \pi(v_i))$$

and the *subdigraph distance from* F *to* H is

$$sd(F, H) = \min_\pi sd_\pi(F, H).$$

Notice that if $V(F) = \{x\}$ and $V(H) = \{y\}$, then $sd(F, H) = d(x, y)$. Thus subdigraph distance is a generalization of directed distance for digraphs. In Figure 3.1, we give a strong digraph D with two induced subdigraphs F and H of order 3. We also list all pairings between $V(F)$ and $V(H)$ and compute $sd(F, H)$ and $sd(H, F)$.

For a strong digraph D and an integer n such that $1 \leq n \leq p$, let F, H, and J be subdigraphs of D of order n. Since $sd(F, H) = sd(H, F)$ is not true in general, subdigraph distance is not a metric. On the other hand, the triangle inequality holds, as we now verify. Suppose that $V(F) = \{v_1, v_2, \dots, v_n\}$. Let π_1 and π_2 be pairings such that $sd(F, H) = sd_{\pi_1}(F, H)$ and $sd(H, J) = sd_{\pi_2}(H, J)$. Then

$$sd(F, J) \leq sd_{\pi_2(\pi_1)}(F, J) = \sum_{i=1}^{n} d(v_i, \pi_2(\pi_1(v_i)))$$

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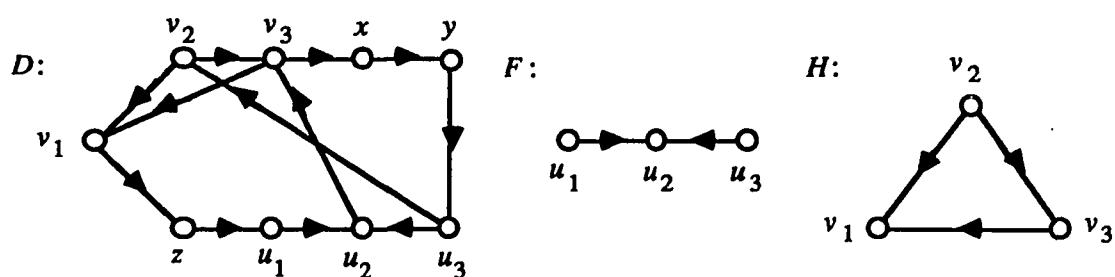
Pairing	u_i	v_j	$d(u_i, v_j)$	$d(v_j, u_i)$	$sd_{\pi_k}(F, H)$	$sd_{\pi_k}(H, F)$
π_1	u_1	v_1	3	2	10	9
	u_2	v_2	5	4		
	u_3	v_3	2	3		
π_2	u_1	v_1	3	2	5	10
	u_2	v_3	1	4		
	u_3	v_2	1	4		
π_3	u_1	v_2	6	3	10	9
	u_2	v_1	2	3		
	u_3	v_3	2	3		
π_4	u_1	v_2	6	3	9	11
	u_2	v_3	1	4		
	u_3	v_1	2	4		
π_5	u_1	v_3	2	3	5	10
	u_2	v_1	2	3		
	u_3	v_2	1	4		
π_6	u_1	v_3	2	4	9	12
	u_2	v_2	5	4		
	u_3	v_1	2	4		

$$sd(F, H) = 5 \text{ and } sd(H, F) = 9$$

Figure 3.1

$$\leq \sum_{i=1}^{n} [d(v_i, \pi_1(v_i)) + d(\pi_1(v_i), \pi_2(\pi_1(v_i)))]$$

$$= sd_{\pi_1}(F, H) + sd_{\pi_2}(H, J)$$

$$= sd(F, H) + sd(H, J),$$

where the second inequality holds since directed distance satisfies the triangle inequality.

There are several results involving distance in subgraphs that have natural analogues in subdigraphs, which we now investigate. The proof of the next theorem and its corollaries are similar to the proofs in [5] corresponding to graphs and are therefore omitted.

Theorem 3.1 For a strong digraph D, let F and H be subdigraphs of D with $p(F) = p(H)$. If $\{u_1, u_2, \ldots, u_k\} \subseteq V(F) \cap V(H)$, then there exists a pairing π from $V(F)$ to $V(H)$ such that $sd(F, H) = sd_\pi(F, H)$ and $\pi(u_i) = u_i$ for $i = 1, 2, \ldots, k$.

For a strong digraph D of order p, let F be an induced subdigraph of D of order n. We define the *subdigraph eccentricity* $e(F)$ of F as

$$e(F) = \max \{sd(F, H) \mid H \text{ is an induced subdigraph of } D \text{ of order } n\},$$

the *n–radius* $\mathrm{rad}_n D$ of D as

$$\mathrm{rad}_n D = \min\{e(F) \mid F \text{ is an induced subdigraph of } D \text{ of order } n\},$$

and the *n–diameter* $\mathrm{diam}_n D$ of D as

$$\mathrm{diam}_n D = \max\{e(F) \mid F \text{ is an induced subdigraph of } D \text{ of order } n\}.$$

The *diameter sequence* of a strong digraph D is defined as the sequence $\text{diam}_1 (D)$, $\text{diam}_2 (D)$, ... , $\text{diam}_{p-1} (D)$. The first corollary shows that the first "half" of the diameter sequence is nondecreasing while the second corollary shows that the diameter sequence is symmetric with respect to its middle term.

Corollary 3.2 Let D be a strong digraph of order p. If n is an integer such that $1 \le n \le \lfloor p / 2 \rfloor - 1$, then $\text{diam}_n (D) \le \text{diam}_{n+1} (D)$.

Corollary 3.3 Let D be a strong digraph of order p. If n is an integer such that $1 \le n \le p - 1$, then $\text{diam}_n (D) = \text{diam}_{p-n} (D)$.

3.2 Multivertex Digraphs

There are numerous results involving distance in graphs and digraphs. Thus, we may gain some insight of subgraph or subdigraph distance if this distance can by represented in terms of distance in graphs or digraphs. This has been done for graphs with several people studying this distance [1], [5], [10], [14], [17]. The next definition illustrates how this is accomplished for digraphs.

Let D be a strong digraph of order p with induced subdigraphs F and H of order n $(1 \le n \le p)$. Then $sd(F, H) = 1$ if and only if $V(F) - V(H) = \{x\}$, $V(H) - V(F) = \{y\}$, and $(x, y) \in E(D)$. We define the *n–multivertex digraph* or *n–digraph* of D as the digraph $M_n(D)$ with

$$V(M_n(D)) = \{v_i \mid S_i \text{ is a set of } n \text{ vertices in } D\}$$

and

$$E(M_n(D)) = \{(v_i, v_j) \mid sd(\langle S_i \rangle, \langle S_j \rangle) = 1\}.$$

Throughout the dissertation, we will assume that $v_i \in V(M_n(D))$ is associated with the set S_i. An example of a digraph and its 2-digraph are given if Figure 3.2 with $S_1 = \{x_1, x_2\}$, $S_2 = \{x_1, x_3\}$, $S_3 = \{x_1, x_4\}$, $S_4 = \{x_2, x_3\}$, $S_5 = \{x_2, x_4\}$, and $S_6 = \{x_3, x_4\}$.
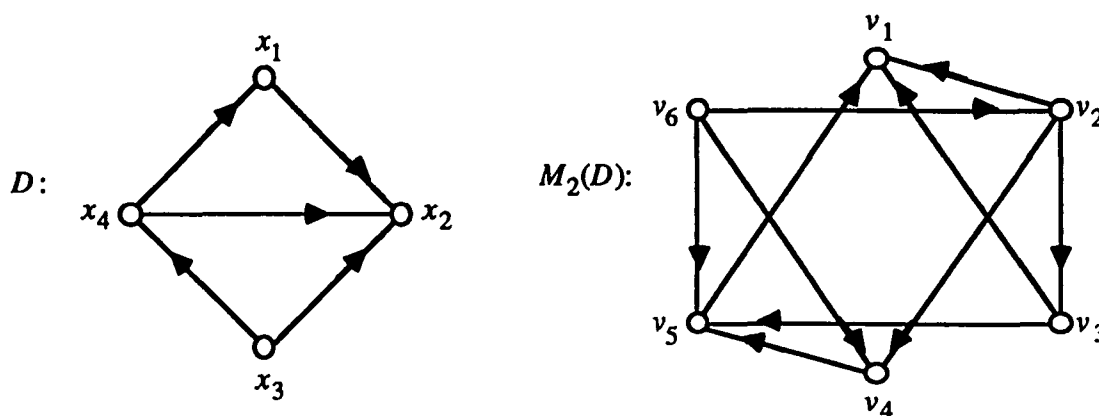
Figure 3.2

The following result shows that subdigraph distance in digraph D can be represented as directed distance in the multivertex digraph $M_n(D)$. Again, the proof of this theorem is similar to a result in [5] corresponding to graphs and is therefore omitted.

Theorem 3.4 Let D be a strong digraph of order p and let n be an integer such that $1 \le n \le p$. If F and H are induced subdigraphs of D of order n with $V(F) = S_\alpha$ and $V(H) = S_\beta$, then

$$sd\ (F, H) = d_{M_n(D)}\ (v_\alpha, v_\beta).$$

We have an immediate consequence of this theorem.

Corollary 3.5 Let D be a strong digraph of order p. If n is an integer such that $1 \le n \le p$, then $\mathrm{diam}_n D = \mathrm{diam}\, M_n(D)$ and $\mathrm{rad}_n D = \mathrm{rad}\, M_n(D)$.

The next result determines the order and size of $M_n(D)$ in terms of the order and size of D.

Theorem 3.6 Let D be a digraph of order p and size q. Then $p(M_n(D)) = \binom{p}{n}$ and $q(M_n(D)) = q\binom{p-2}{n-1}$ for $1 \le n \le p$.

Proof It is clear by the definition of $M_n(D)$ that $p(M_n(D)) = \binom{p}{n}$. For each $(u, v) \in E(D)$, we have $(v_i, v_j) \in E(M_n(D))$ when $S_i = \{x_1, x_2, \ldots, x_{n-1}, u\}$ and $S_j = \{x_1, x_2, \ldots, x_{n-1}, v\}$ where $x_m \ne u$ and $x_m \ne v$, for $1 \le m \le n - 1$. Since there are $\binom{p-2}{n-1}$ ways to choose an $(n - 1)$-element subset of $V(D) - \{u, v\}$, we have $\binom{p-2}{n-1}$ arcs in $M_n(D)$ for each arc in D. Thus, there are $q\binom{p-2}{n-1}$ arcs in $M_n(D)$. ☐

An asymmetric digraph is a *tournament* if its underlying graph is a complete graph. The observation that a digraph D is asymmetric if and only if $M_n(D)$ is asymmetric results in the following corollary.

Corollary 3.7 Let D be an asymmetric digraph of order p. Then D is a tournament if and only if $q(M_n(D)) = \binom{p}{2}\binom{p-2}{n-1}$ for $1 \le n \le p - 1$.

We now present a result concerning the indegrees and outdegrees of the vertices of $M_n(D)$.

Theorem 3.8 Let D be a digraph with $v_i \in V(M_n(D))$. Then

$$id(v_i) = \sum_{x \in S_i} id(x) - q(\langle S_i \rangle)$$

and

$$od(v_i) = \sum_{x \in S_i} od(x) - q(\langle S_i \rangle).$$

Proof Suppose that $S_i \subseteq V(D)$ where $|S_i| = n$, and let $x \in S_i$. Furthermore, suppose that x is adjacent to the vertices $y_1, y_2, \ldots, y_k$. For each $y_m \notin S_i$, $1 \le m \le k$, the vertex v_i is adjacent to v_α in $M_n(D)$, where $S_\alpha = (S_i - \{x\}) \cup \{y_m\}$. Observe that no such arc occurs in $M_n(D)$ when $y_m \in S_i$, which means that $(x, y_m) \in E(\langle S_i \rangle)$. Thus,

$$od(v_i) = \sum_{x \in S_i} od(x) - q(\langle S_i \rangle).$$

A similar argument shows that

$$id(v_i) = \sum_{x \in S_i} id(x) - q(\langle S_i \rangle). \quad \square$$

For a digraph D, the *converse* $\vec{D}$ of D is defined by $V(\vec{D}) = V(D)$ and $E(\vec{D}) = \{(u, v) \mid (v, u) \in E(D)\}$. The following result gives a relationship between the digraphs $M_n(D)$ and $M_{p-n}(\vec{D})$ for a digraph D of order p.

Theorem 3.9 If D is a digraph of order p and n is an integer with $1 \le n \le p - 1$, then $M_n(D) \cong M_{p-n}(\vec{D})$.

Proof Consider a mapping ϕ from $V(M_n(D))$ onto $V(M_{p-n}(\vec{D}))$ such that for each $v_i \in V(M_n(D))$, we have $\phi(v_i) = v_i'$, where $S_i' = V(D) - S_i$. Observe that if (v_i, v_j)

$\in E(M_n(D))$, then $S_i - S_j = \{x\}$, $S_j - S_i = \{y\}$, and $(x, y) \in E(D)$. So it follows that $S'_j - S'_i = \{x\}$, $S'_i - S'_j = \{y\}$, and $(v'_j, v'_i) \in E(M_{p-n}(\vec{D}))$. Similarly, if $(v'_j, v'_i) \in E(M_{p-n}(\vec{D}))$, we conclude that $(v_i, v_j) \in E(M_n(D))$. Thus, ϕ is an isomorphism from $V(M_n(D))$ onto $V(M_{p-n}(\vec{D}))$, completing the proof. $\square$

A tournament D is transitive if, whenever (u, v) and (v, w) are arcs of D, then (u, w) is also an arc of D. We now determine those n-digraphs of D that are (transitive) tournaments.

Theorem 3.10 Let D be a digraph of order p. Then

(i) $M_n(D)$ is not a tournament for $2 \leq n \leq p - 2$, and

(ii) $M_{p-1}(D)$ is a (transitive) tournament if and only if D is a (transitive) tournament.

Proof Statement (i) is obvious for $p \leq 3$, so assume that $p \geq 4$. Let S_i be any set of n distinct vertices of $V(D)$ for $2 \leq n \leq p - 2$. Since $|V(D) - S_i| \geq 2$, there exist $u, v \in V(D)$ such that $u, v \notin S_i$. Now choose S_j such that $u, v \in S_j$, which means there is no arc between v_i and v_j in $M_n(D)$.

The statement (ii) follows directly from Theorem 3.9. $\square$

The next result shows that each induced subdigraph of order n of a digraph D is isomorphic to some induced subdigraph of $M_n(D)$.

Theorem 3.11 Let D be a digraph of order p, and let n be an integer such that $1 \leq n \leq p - 1$. Then $D - \{x_1, x_2, \ldots, x_{n-1}\} \cong \langle\{v_i \in V(M_n(D)): |S_i - \{x_1, x_2, \ldots, x_{n-1}\}| = 1\}\rangle$ for distinct vertices $x_1, x_2, \ldots, x_{n-1}$.

Proof Let ϕ be a mapping from $V(D) - \{x_1, x_2, \dots, x_{n-1}\}$ onto $T \subseteq V(M_n(D))$, for $1 \le n \le p - 1$, where $T = \{v_i \in V(M_n(D)) : |S_i - \{x_1, x_2, \dots, x_{n-1}\}| = 1\}$. For $u_i \in V(D) - \{x_1, x_2, \dots, x_{n-1}\}$, define $\phi(u_i) = v_i$ where $S_i = \{x_1, x_2, \dots, x_{n-1}, u_i\}$. Since $(v_i, v_j) \in E(\langle T \rangle)$ if and only if $(u_i, u_j) \in E(D - \{x_1, x_2, \dots, x_{n-1}\})$, it follows that ϕ is an isomorphism from $V(D) - \{x_1, x_2, \dots, x_{n-1}\}$ onto T. $\square$

We now present upper and lower bounds for the n–radius and n–diameter of a digraph.

Theorem 3.12 Let D be a strong digraph of order p, and let n be an integer such that $1 \le n \le \lfloor p / 2 \rfloor$. Then

$$n \le \mathrm{rad}_n D \le \mathrm{diam}_n D \le n \,\mathrm{diam}\, D,$$

and there exists a digraph D such that

$$n = \mathrm{rad}_n D = \mathrm{diam}_n D = n \,\mathrm{diam}\, D.$$

Proof Let $S_1 = \{x_1, x_2, \dots, x_n\}$ be any set of n distinct vertices of digraph D. Since $n \le \lfloor p / 2 \rfloor$, there exists a set S_2 of n distinct vertices contained in $V(D) - S_1$. Let $F_1 \cong \langle S_1 \rangle$ and $F_2 \cong \langle S_2 \rangle$. Then for a pairing π with $sd_\pi(F_1, F_2) = sd(F_1, F_2)$, we have

$$e(v_1) = e(F_1) \ge sd(F_1, F_2) = sd_\pi(F_1, F_2) = \sum_{i=1}^{n} d(x_i, \pi(x_i)) \ge \sum_{i=1}^{n} 1 = n.$$

Thus, $n \le \mathrm{rad}_n D$. It is clear that $\mathrm{rad}_n D \le \mathrm{diam}_n D$.

Now consider a set S_3 of n distinct vertices of digraph D such that $e(v_1) = d_{M_n(D)}(v_1, v_3)$. Let $F_3 \cong \langle S_3 \rangle$. Then for a pairing π with $sd_\pi(F_1, F_3) = sd(F_1, F_3)$, we have

$$e(v_1) = d_{M_n(D)}(v_1, v_3) = sd(F_1, F_3) = sd_\pi(F_1, F_3) = \sum_{i=1}^{n} d(x_i, \pi(x_i)) \leq n \text{ diam } D.$$

Therefore, $\text{diam}_n D \leq n \text{ diam } D$.

For a digraph $D \cong K_{2n}^*$, we have

$$n = \text{rad}_n D = \text{diam}_n D = n \text{ diam } D. \quad \Box$$

The next corollary follows from Corollary 3.5 and Theorem 3.12.

Corollary 3.13 Let D be a strong digraph of order p, and let n be an integer such that $1 \leq n \leq \lfloor p / 2 \rfloor$. Then

$$n \leq \text{rad } M_n(D) \leq \text{diam } M_n(D) \leq n \text{ diam } D,$$

and there exists a digraph D such that

$$n = \text{rad } M_n(D) = \text{diam } M_n(D) = n \text{ diam } D.$$

3.3 Relationships Between Digraphs and Multivertex Digraphs

In general, the multivertex digraphs of a digraph D are more complicated then D itself. Thus, it would be desirable to determine properties of multivertex digraphs of a digraph D by properties possessed by D. Another way of saying this is to ask: What properties of multivertex digraphs of a digraph D are inherited from D? In addition, if we are given some multivertex digraph $M_n(D)$, can we determine any properties of the digraph D? In this section, we investigate these questions with respect to distance concepts. We start by showing that each path in the multivertex digraph $M_n(D)$ corresponds to n paths in the digraph D.

Lemma 3.14 Let D be a digraph of order p. For any $v_0 - v$ path in $M_n(D)$ $(1 \le n \le p)$ with $S_0 = \{x_1, x_2, \ldots, x_n\}$, there is some pairing π from S_0 to S such that there is an $x_i - \pi(x_i)$ path in D for $i = 1, 2, \ldots, n$. Furthermore, if $S_0 - S = \{x\}$ and $S - S_0 = \{y\}$, then there is an $x - y$ path in D.

Proof Let $P: v_0, v_1, \ldots, v_m = v$ be a $v_0 - v$ path in $M_n(D)$. Since $(v_j, v_{j+1}) \in E(M_n(D))$ for $0 \le j \le m - 1$ if and only if $S_j - S_{j+1} = \{x\}$, $S_{j+1} - S_j = \{y\}$, and $(x, y) \in E(D)$, we can construct n directed trails in digraph D by algorithm TRAIL (see Figure 3.3).

Algorithm TRAIL:

> For $i = 1$ to n
>
>> Let $k \leftarrow 0$, $t_0 \leftarrow x_i$, and $T_i \leftarrow \{x_i\}$.
>>
>> For $j = 0$ to $m - 1$
>>
>>> If $S_j - S_{j+1} = \{t_k\}$, then
>>>
>>>> Let $t_{k+1} \leftarrow y$ where $S_{j+1} - S_j = \{y\}$.
>>>>
>>>> Let $T_i \leftarrow T_i \cup \{t_{k+1}\}$.
>>>>
>>>> Let $k \leftarrow k + 1$.

Figure 3.3

For each $i = 1, 2, \ldots, n$, we construct the trail T_i: $x_i = t_0, t_1, \ldots, t_k$ where $t_k \in S$. If we construct the n trails simultaneously by algorithm TRAIL, then for any j $(0 \le j \le m - 1)$, the last vertex in the trail T_i $(1 \le i \le n)$ constructed thus far by the algorithm corresponds to a vertex of D in S_{j+1}, and the set of these n vertices of D is S_{j+1}. Since each set S_j $(0 \le j \le m)$ consists of n distinct vertices of D, there

must be some pairing π from S_0 to S such that for $i = 1, 2, \dots, n$, there is an $x_i - \pi(x_i)$ path in D.

Now assume that S_0 and S are two sets of n distinct vertices of D with $S_0 - S = \{x\}$ and $S - S_0 = \{y\}$. We show that there is an $x - y$ path in digraph D. Suppose that $S_0 = \{x_1, x_2, \dots, x_{n-1}, x\}$ and $S = \{x_1, x_2, \dots, x_{n-1}, y\}$. By the previous discussion, there is some pairing π from S_0 to S such that there is an $x_i - \pi(x_i)$ path in D for each $i = 1, 2, \dots, n - 1$ and an $x - \pi(x)$ path in D. If $\pi(x) = y$, then the proof is complete. So assume that $\pi(x) = x_i$ for some i ($1 \le i \le n - 1$). Since a pairing π is a one–to–one correspondence between S_0 and S, we have $\pi(x_i) \in S - \{x_i\}$. If $\pi(x_i) = y$, then we have an $x - x_i$ path and an $x_i - y$ path in D. This means that there is an $x - y$ path in digraph D. So assume that $\pi(x_i) = x_j$ for some $j \ne i$ ($1 \le j \le n - 1$). Then we have $\pi(x_j) \in S - \{x_i, x_j\}$. If $\pi(x_j) = y$, then we have an $x - x_i$ path, an $x_i - x_j$ path, and an $x_j - y$ path in D and, thus, an $x - y$ path in D. Continuing in this fashion, we eventually reach y since S and S_0 have the same (finite) cardinality. Therefore, it follows that there is an $x - y$ path in digraph D. $\square$

A digraph D is *unilateral* if for each pair x, y of vertices of D, there is an $x - y$ path or a $y - x$ path in D. One of the consequences of the preceding lemma is the following result.

Corollary 3.15 If D is a digraph of order p such that for some integer n ($1 \le n \le p - 1$) the digraph $M_n(D)$ is unilateral, then D is unilateral.

Proof Let $x, y \in V(D)$ and suppose that $S_1 = \{x_1, x_2, \dots, x_{n-1}, x\}$ and $S_2 = \{x_1, x_2, \dots, x_{n-1}, y\}$ are two sets of n distinct vertices of D. Since $M_n(D)$ is

unilateral, there is a $v_1 - v_2$ path or a $v_2 - v_1$ path in $M_n(D)$. Now, by Lemma 3.14, there is an $x - y$ path or a $y - x$ path in D. ◻

The converse of Corollary 3.15 is not true in general. To see this, consider the digraphs D and $M_2(D)$ shown in Figure 3.4. Let $S_1 = \{x_1, x_2\}$, $S_2 = \{x_1, x_3\}$, $S_3 = \{x_1, x_4\}$, $S_4 = \{x_2, x_3\}$, $S_5 = \{x_2, x_4\}$, and $S_6 = \{x_3, x_4\}$. Observe that even though D is unilateral, there is neither a $v_3 - v_4$ path nor a $v_4 - v_3$ path in $M_2(D)$.
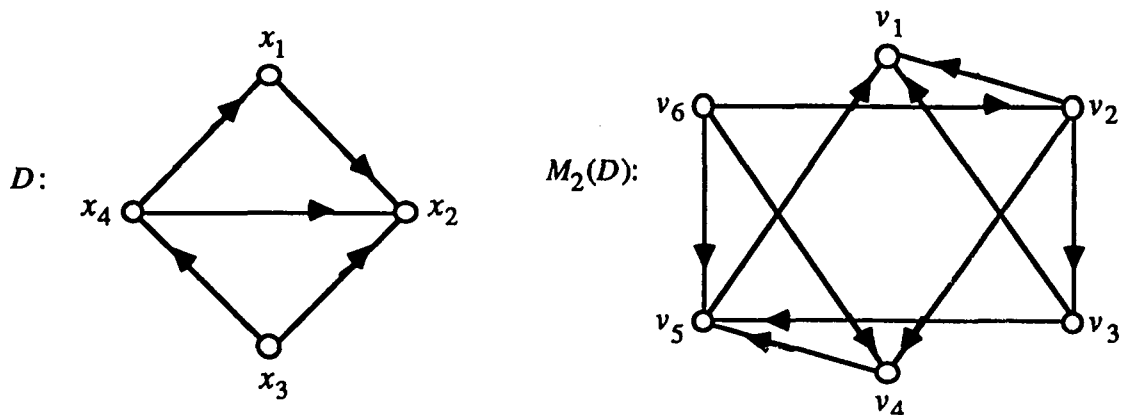

Figure 3.4

Observe that a digraph D is strong if and only if its converse $\overleftarrow{D}$ is strong. Thus, by Theorem 3.9, a digraph D of order p is strong if and only if $M_{p-1}(D)$ is strong. In fact, we can even say more.

Corollary 3.16 Let D be a digraph of order p. If $M_n(D)$ is strong for some integer n $(1 \le n \le p - 1)$, then D is strong.

Proof Let $x, y \in V(D)$ and suppose that $S_1 = \{x_1, x_2, \dots, x_{n-1}, x\}$ and $S_2 = \{x_1, x_2, \dots, x_{n-1}, y\}$ are two sets of n distinct vertices of D. Since $M_n(D)$ is strong, there is a $v_1 - v_2$ path and a $v_2 - v_1$ path in $M_n(D)$. Then by Lemma 3.14, we have an $x - y$ path and a $y - x$ path in D. ◻

The next result shows that if D is a strong digraph, then all n-digraphs of D are also strong.

Theorem 3.17 If D is a strong digraph of order p, then for each integer n $(1 \leq n \leq p)$, the digraph $M_n(D)$ is strong.

Proof Assume that the statement is false. Then there exists an integer n $(1 \leq n \leq p)$ and vertices v_i and v_j in $M_n(D)$ such that there is no $v_i - v_j$ path in $M_n(D)$. In addition, assume that $A = \{S_k \mid$ there is a $v_i - v_k$ path in $M_n(D)\}$ and let $S_t \in A$ such that

$$|S_t \cap S_j| = \max\{|S_k \cap S_j| : S_k \in A\} < n.$$

Thus, $M_n(D)$ contains a $v_i - v_t$ path. Let $x \in S_t - S_j$ and $y \in S_j - S_t$. Since D is strong, there exists an $x - y$ path in D. Suppose that P: $x, w_1, w_2, \ldots, w_s, y$ is a shortest $x - y$ path in D. Furthermore, assume that

$$\{w_1, w_2, \ldots, w_s\} \cap S_t = \{z_1, z_2, \ldots, z_r\}.$$

Without loss of generality, we can rewrite

$$P: x, w_1, w_2, \ldots, z_1, w_{1,1}, w_{1,2}, \ldots, z_2, w_{2,1}, w_{2,2}, \ldots, z_r, w_{r,1}, w_{r,2}, \ldots, y.$$

Observe that the following sequence of sets of vertices in D beginning with S_t and ending with $S_m = (S_t - \{x\}) \cup \{y\}$ corresponds to a sequence of vertices in $M_n(D)$ that forms a $v_t - v_m$ path in $M_n(D)$.

$$S_t, (S_t - \{z_r\}) \cup \{w_{r,1}\}, (S_t - \{z_r\}) \cup \{w_{r,2}\}, \ldots, (S_t - \{z_r\}) \cup \{y\},$$

$$(S_t - \{z_r, z_{r-1}\}) \cup \{y, w_{r-1,1}\}, (S_t - \{z_r, z_{r-1}\}) \cup \{y, w_{r-1,2}\}, \ldots,$$

$$(S_t - \{z_r, z_{r-1}\}) \cup \{y, z_r\} = (S_t - \{z_{r-1}\}) \cup \{y\}, (S_t - \{z_{r-1}, z_{r-2}\}) \cup \{y, w_{r-2,1}\},$$

$$(S_t - \{z_{r-1}, z_{r-2}\}) \cup \{y, w_{r-2,2}\}, \dots, (S_t - \{z_{r-1}, z_{r-2}\}) \cup \{y, z_{r-1}\}$$

$$= (S_t - \{z_{r-2}\}) \cup \{y\}, \dots, (S_t - \{x\}) \cup \{y\} = S_m.$$

Thus, there is a $v_t - v_m$ path and, consequently, a $v_i - v_m$ path in $M_n(D)$ and $S_m \in A$. But $|S_m \cap S_j| = |S_t \cap S_j| + 1$, which is a contradiction. $\square$

The next result is a consequence of Corollary 3.16 and Theorem 3.17.

Corollary 3.18 Let D be a digraph of order p. If $M_n(D)$ is strong for some integer n $(1 \le n \le p - 1)$, then the digraph $M_m(D)$ is strong for each integer m $(1 \le m \le p - 1)$.

A digraph D is said to be *weak* if its underlying graph is connected; that is, D is weak if for $u, v \in V(D)$, there is a $u - v$ semipath in D. There is a result similar to Lemma 3.14 that involves semipaths.

Lemma 3.19 Let D be a digraph of order p. For any $v_0 - v$ semipath in $M_n(D)$ $(1 \le n \le p)$ with $S_0 = \{x_1, x_2, \dots, x_n\}$, there is some pairing π from S_0 to S such that there is an $x_i - \pi(x_i)$ semipath in D for $i = 1, 2, \dots, n$. In addition, if $S_0 - S = \{x\}$ and $S - S_0 = \{y\}$, then there is an $x - y$ semipath in D.

This lemma can be proved by imitating the proof of Lemma 3.14 with "path" replaced by "semipath". From the "proof technique" presented in Theorem 3.17 and from Lemma 3.19, we have two corollaries, which we state without proof.

Corollary 3.20 If D is a weak digraph of order p, then for each integer n $(1 \le n \le p)$, the digraph $M_n(D)$ is weak.

Corollary 3.21 If D is a digraph of order p such that for some integer n $(1 \le n \le p - 1)$, the digraph $M_n(D)$ is weak, then D is weak. Furthermore, the digraph $M_m(D)$ is weak for each $m = 1, 2, \ldots, p$.

An *eulerian circuit* of a strong digraph D is a directed circuit containing all the arcs of D. A digraph possessing an eulerian circuit is called an *eulerian digraph*. From [9], a strong digraph D is eulerian if and only if $\text{id}(v) = \text{od}(v)$ for every vertex v of D.

Theorem 3.22 Let D be a digraph of order $p \ge 3$ and let n be an integer with $2 \le n \le p - 1$. Then the digraph $M_n(D)$ is eulerian if and only if D is eulerian.

Proof Assume that D is eulerian. Then D is strong and for each $u \in V(D)$, we have $\text{id}(u) = \text{od}(u)$. From Theorem 3.17, it follows that $M_n(D)$ is also strong. In addition, by Theorem 3.8, we have

$$\text{id}(v_i) = \sum_{x \in S_i} \text{id}(x) - q(\langle S_i \rangle)$$

and

$$\text{od}(v_i) = \sum_{x \in S_i} \text{od}(x) - q(\langle S_i \rangle).$$

Since $\text{id}(u) = \text{od}(u)$ for each $u \in V(D)$, we conclude that $\text{id}(v_i) = \text{od}(v_i)$ for each $v_i \in V(M_n(D))$, that is, $M_n(D)$ is eulerian.

Now suppose that $M_n(D)$ is an eulerian digraph. Then $M_n(D)$ is strong and by Corollary 3.16, the digraph D is also strong. Furthermore, for $v_i \in V(M_n(D))$, we have $id(v_i) = od(v_i)$. From this and Theorem 3.8, it follows that

$$id(v_i) = \sum_{x \in S_i} id(x) - q(\langle S_i \rangle) = \sum_{x \in S_i} od(x) - q(\langle S_i \rangle) = od(v_i)$$

and

$$\sum_{x \in S_i} id(x) = \sum_{x \in S_i} od(x).$$

Suppose that there exists $w \in V(D)$ such that $id(w) < od(w)$. Since

$$\sum_{x \in V(D)} id(x) = \sum_{x \in V(D)} od(x),$$

there exists $y \in V(D)$ such that $id(y) > od(y)$. Let $S \subseteq V(D)$ be a set of n distinct vertices such that $w, y \in S$, and let $z \in V(D)$ such that $z \notin S$. If $id(z) \geq od(z)$, then for $S_1 = (S - \{w\}) \cup \{z\}$, we have

$$\sum_{x \in S_1} id(x) = \sum_{x \in S} id(x) - id(w) + id(z)$$

$$= \sum_{x \in S} od(x) - id(w) + id(z)$$

$$> \sum_{x \in S} od(x) - od(w) + od(z) = \sum_{x \in S_1} od(x),$$

which is a contradiction. So $id(z) < od(z)$. Let $S_2 = (S - \{y\}) \cup \{z\}$. Then

$$\sum_{x \in S_2} id(x) = \sum_{x \in S} id(x) - id(y) + id(z)$$

$$= \sum_{x \in S} od(x) - id(y) + id(z)$$

$$< \sum_{x \in S} od(x) - od(y) + od(z) = \sum_{x \in S_2} od(x).$$

Again, we have a contradiction. Therefore, $id(x) = od(x)$ for all $x \in V(D)$. Since the digraph D is strong, it follows that D is eulerian. $\square$

A digraph D is called *regular of degree* r or *r-regular* if $id(v) = od(v) = r$ for every vertex v of D. For a connected graph, Wright [17] stated necessary and sufficient conditions for the n-multivertex graph to be regular. For an asymmetric digraph D, we present necessary and sufficient conditions for the multivertex graph $M_n(D)$ to be regular.

Theorem 3.23 Let D be an asymmetric digraph of order p and let n be an integer with $2 \le n \le p - 1$. Then the digraph $M_n(D)$ is regular if and only if (1) D is regular of order $n + 1$, (2) D is an m-regular tournament, $m \ge 1$ (of order $2m + 1$), (3) $D \cong H \cup K_1$, where H is an $(n - 1)$-regular tournament (of order $2n - 1$), or (4) $q(D) = 0$.

Proof If D is regular of order $n + 1$, then $M_n(D) \cong \vec{D}$ by Theorem 3.9. Thus, $M_n(D)$ is regular.

Suppose next that D is an m-regular tournament. Thus, if S is a set of n distinct vertices of D with $2 \le n \le p - 1$, then $q(\langle S \rangle) = \binom{n}{2}$. So, for $v_i, v_j \in V(M_n(D))$ with $1 \le i, j \le \binom{p}{n}$, it follows from Theorem 3.8 that

$$id(v_i) = \sum_{x \in S_i} id(x) - q(\langle S_i \rangle) = \sum_{x \in S_j} id(x) - q(\langle S_i \rangle)$$

$$= \sum_{x \in S_j} id(x) - q(\langle S_j \rangle) = id(v_j)$$

and

$$\mathrm{id}(v_i) = \sum_{x \in S_i} \mathrm{id}(x) - q(\langle S_i \rangle) = \sum_{x \in S_i} \mathrm{od}(x) - q(\langle S_i \rangle) = \mathrm{od}(v_i).$$

Therefore, $M_n(D)$ is regular.

Assume now that $D \cong H \cup K_1$, where H is an $(n-1)$-regular tournament. Let y be the isolated vertex of D and let S be any set of n distinct vertices of D. Then

$$\mathrm{id}(v) = \sum_{x \in S} \mathrm{id}(x) - q(\langle S \rangle) = \sum_{x \in S} \mathrm{od}(x) - q(\langle S \rangle) = \mathrm{od}(v).$$

Now let S_1 and S_2 be any two sets of n distinct vertices of D such that $y \in S_1$ and $y \notin S_2$. Observe that $q(\langle S_1 \rangle) = \binom{n-1}{2}$ and $q(\langle S_2 \rangle) = \binom{n}{2}$. Also, for $w \in S_2$ and $T = S_2 - \{w\}$, we have

$$\sum_{x \in T} \mathrm{id}(x) = \sum_{x \in S_1} \mathrm{id}(x).$$

Thus, for $w \in S_2$, it follows that

$$\mathrm{id}(v_2) = \sum_{x \in S_2} \mathrm{id}(x) - q(\langle S_2 \rangle) = \sum_{x \in S_2} \mathrm{id}(x) - \binom{n}{2}$$

$$= \sum_{x \in S_2} \mathrm{id}(x) - \left[\binom{n-1}{2} + (n-1) \right] = \sum_{x \in S_2} \mathrm{id}(x) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in T} \mathrm{id}(x) + \mathrm{id}(w) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in S_1} \mathrm{id}(x) + \mathrm{id}(w) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in S_1} \mathrm{id}(x) + (n-1) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in S_1} \mathrm{id}(x) - q(\langle S_1 \rangle) = \mathrm{id}(v_1).$$

From this, we conclude that $M_n(D)$ is regular.

If D is a digraph of size 0 and order p, it is clear that $M_n(D)$ has size 0 for $2 \le n \le p - 1$, that is, $M_n(D)$ is regular.

For the converse, suppose that $M_n(D)$ is regular for some integer n, where $2 \le n \le \lfloor p / 2 \rfloor$. We first show that $\mathrm{id}(x) = \mathrm{od}(x)$ for each $x \in V(D)$. For $v_i \in V(M_n(D))$, we have $\mathrm{id}(v_i) = \mathrm{od}(v_i)$. Thus, from Theorem 3.8,

$$\sum_{x \in S_i} \mathrm{id}(x) - q(\langle S_i \rangle) = \sum_{x \in S_i} \mathrm{od}(x) - q(\langle S_i \rangle),$$

that is,

$$\sum_{x \in S_i} \mathrm{id}(x) = \sum_{x \in S_i} \mathrm{od}(x).$$

Suppose that there is some $u \in V(D)$ such that $\mathrm{id}(u) < \mathrm{od}(u)$. Since

$$\sum_{x \in V(D)} \mathrm{id}(x) = \sum_{x \in V(D)} \mathrm{od}(x),$$

there exists $y \in V(D)$ such that $\mathrm{id}(y) > \mathrm{od}(y)$. Let $S \subseteq V(D)$ be a set of n distinct vertices such that $u, y \in S$, and let $z \in V(D) - S$. We consider two cases.

Case 1 *Assume that* $\mathrm{id}(z) \ge \mathrm{od}(z)$. Let $S' = (S - \{u\}) \cup \{z\}$. Since $\mathrm{id}(z) \ge \mathrm{od}(z)$ and $\mathrm{id}(u) < \mathrm{od}(u)$, we have

$$\sum_{x \in S'} \mathrm{id}(x) = \sum_{x \in S} \mathrm{id}(x) - \mathrm{id}(u) + \mathrm{id}(z) = \sum_{x \in S} \mathrm{od}(x) - \mathrm{id}(u) + \mathrm{id}(z)$$

$$> \sum_{x \in S} \mathrm{od}(x) - \mathrm{od}(u) + \mathrm{od}(z) = \sum_{x \in S'} \mathrm{od}(x),$$

which is a contradiction.

Case 2   *Assume that* $\mathrm{id}(z) < \mathrm{od}(z)$. In this case, we let $S' = (S - \{y\}) \cup \{z\}$ and by a calculation similar to Case 1, we have the contradiction

$$\sum_{x \in S'} \mathrm{id}(x) < \sum_{x \in S'} \mathrm{od}(x).$$

Therefore, $\mathrm{id}(x) = \mathrm{od}(x)$ for each $x \in V(D)$.

We now show that D is regular or $D \cong H \cup K_1$, where H is regular. If D is not regular, then

(i)   there exist $x, y \in V(D)$ such that $1 \leq \mathrm{id}(x) < \mathrm{id}(y)$, or

(ii)   there exist $x, z \in V(D)$ such that $\mathrm{id}(x) = 0$, $\mathrm{id}(z) \geq 1$, and for each $y \in V(D) - \{x, z\}$, we have $\mathrm{id}(y) = 0$ or $\mathrm{id}(y) = \mathrm{id}(z)$.

We consider both of these cases.

Case 1   *Assume that there exist* $x, y \in V(D)$ *such that* $1 \leq \mathrm{id}(x) < \mathrm{id}(y)$. Since $\mathrm{id}(y) \geq 2$ and $\mathrm{id}(y) = \mathrm{od}(y)$, we only consider $p \geq 5$. Let $N^-(x) = \{u \mid (u, x) \in E(D)\}$, $N^+(x) = \{u \mid (x, u) \in E(D)\}$, and $N(x) = N^-(x) \cup N^+(x)$. For $A \subseteq V(D)$, let $\mathrm{id}_A(x)$ be the number of vertices in A that are adjacent to x, and let $\mathrm{od}_A(x)$ be the number of vertices in A that are adjacent from x. For any set $A \subseteq V(D) - \{x, y\}$ of $n - 1$ distinct vertices, let $S_1 = A \cup \{x\}$, $S_2 = A \cup \{y\}$, and $T = V(D) - A$. Then

$$\text{id}(v_1) = \sum_{z \in A} \text{id}_T(z) + \text{id}_T(x) - \text{od}_A(x)$$

and

$$\text{id}(v_2) = \sum_{z \in A} \text{id}_T(z) + \text{id}_T(y) - \text{od}_A(y).$$

Since $M_n(D)$ is regular, it is clear that $\text{id}(v_1) = \text{id}(v_2)$. From this, we must have

$$\text{id}_T(x) - \text{od}_A(x) = \text{id}_T(y) - \text{od}_A(y). \tag{3.1}$$

In a similar fashion, if we consider the corresponding equations for $\text{od}(v_1)$ and $\text{od}(v_2)$, it follows that

$$\text{id}_A(x) - \text{od}_T(x) = \text{id}_A(y) - \text{od}_T(y). \tag{3.2}$$

Subcase 1.1   *Assume that* $N(x) \cap N(y) \neq \emptyset$. Suppose that $z \in N(x) \cap N(y)$ and $w \in N(y)$ such that $w \notin N(x) \cup \{x\}$. Let $A \subseteq V(D) - \{x, y, w\}$ be a set of $n - 1$ distinct vertices such that $z \in A$, and let $T = V(D) - A$. Without loss of generality, assume that $z \in N^+(x)$.

Subcase 1.1.1   *Assume that* $z \in N^+(y)$. If $B = (A - \{z\}) \cup \{w\}$ and $U = V(D) - B$, then

$$\text{id}_T(x) - \text{od}_A(x) = \text{id}_U(x) - \text{od}_B(x) - 1 = \text{id}_U(y) - \text{od}_B(y) - 1$$
$$= \text{id}_T(y) - \text{od}_A(y) - 1,$$

where the second equality follows from (3.1) and we consider both cases of $w \in N^-(y)$ or $w \in N^+(y)$ for the third equality. This is a contradiction.

Subcase 1.1.2   Assume that $z \in N^-(y)$. Using (3.2) and a calculation similar to Subcase 1.1.1 with $B = (A - \{z\}) \cup \{w\}$ and $U = V(D) - B$, we have

$$id_A(x) - od_T(x) = id_B(x) - od_U(x) + 1 = id_B(y) - od_U(y) + 1$$
$$= id_A(y) - od_T(y) + 1.$$

Again, we have another contradiction.

Subcase 1.2   Assume that $N(x) \cap N(y) = \varnothing$. Let $A \subseteq V(D) - \{x, y, w\}$ be a set of $n - 1$ distinct vertices such that $u \in A$, where $u \in N(x)$ and $w \in N(y)$. If $T = V(D) - A$, $B = (A - \{u\}) \cup \{w\}$, and $U = V(D) - B$, then a calculation similar to the above results in

$$id_T(x) - od_A(x) = id_U(x) - od_B(x) - 1 = id_U(y) - od_B(y) - 1$$
$$= id_T(y) - od_A(y) - 2.$$

From this contradiction, we conclude that Case 1 is impossible.

Case 2   Assume that there exist $x, z \in V(D)$ *such that* $id(x) = 0$, $id(z) \geq 1$, *and for each* $y \in V(D) - \{x, z\}$, *we have* $id(y) = 0$ *or* $id(y) = id(z)$. Suppose that there are two vertices $x, y \in V(D)$ such that $id(x) = id(y) = 0$. For some $(u, w) \in E(D)$, let $S \subseteq V(D) - \{u, w\}$ be a set of n distinct vertices such that $x, y \in S$, and let $T = V(D) - S$. We consider three subcases.

Subcase 2.1   Assume that $id_T(w) \neq od_S(w)$. If $S_1 = (S - \{x\}) \cup \{w\}$, then $id(v_1) = id(v) - od_S(w) + id_T(w) \neq id(v)$, which contradicts $M_n(D)$ being regular.

Subcase 2.2   Assume that $id_T(u) \neq od_S(u)$. For $S_1 = (S - \{x\}) \cup \{u\}$, we have $id(v_1) = id(v) - od_S(u) + id_T(u) \neq id(v)$. Again, this is a contradiction.

Subcase 2.3   Assume that $\text{id}_T(w) = \text{od}_S(w)$ *and* $\text{id}_T(u) = \text{od}_S(u)$. *If* $S_1 = (S - \{x, y\}) \cup \{u, w\}$, then $\text{id}(v_1) = \text{id}(v) + \text{id}_T(u) + (\text{id}_T(w) - 1) - \text{od}_S(u) - \text{od}_S(w) \neq \text{id}(v)$, where the -1 occurs from $(u, w) \in E(D)$. This is a contradiction.

Thus, if $M_n(D)$ is regular, with $2 \leq n \leq \lfloor p/2 \rfloor$, then D is regular or $D \cong H \cup K_1$, where H is m–regular with $m \geq 1$.

Assume that D is m–regular with $m \geq 1$ and $p(D) \neq 2m + 1$. Since D is asymmetric, we have $p(D) \geq 2m + 2$. Also, for each $v_i, v_j \in V(M_n(D))$, it follows that

$$\text{id}(v_i) = \sum_{x \in S_i} \text{id}(x) - q(\langle S_i \rangle) = \sum_{x \in S_j} \text{id}(x) - q(\langle S_j \rangle) = \text{id}(v_j).$$

Since D is regular, we conclude that

$$q(\langle S_i \rangle) = q(\langle S_j \rangle).$$

Let $x, y \in V(D)$ such that $y \notin N(x) \cup \{x\}$. Let $A \subseteq V(D) - \{x, y\}$ be any set of $n - 1$ distinct vertices and let $S_1 = A \cup \{x\}$ and $S_2 = A \cup \{y\}$. Using $q(\langle S_1 \rangle) = q(\langle S_2 \rangle)$, we have $|N(x) \cap A| = |N(y) \cap A|$. Since this is true for each $A \subseteq V(D) - \{x, y\}$, we conclude that $N(x) = N(y)$. Furthermore, if $B = V(D) - (N(x) \cup \{x\})$, then $N(x) = N(z)$ for each $z \in B$. From this, it follows that $2m = N(u) \geq |B| + 1$ for $u \in N(x)$. Thus,

$$p = 1 + |N(x)| + |B| = 1 + 2m + |B|$$
$$\leq 2m + 2m = 4m$$

and

$$n \leq \lfloor p/2 \rfloor \leq 2m = |N(x)|.$$

For some $u \in N(x)$, suppose that $w \in V(D)$ such that $w \notin N(u) \cup \{u\}$. Let $A \subseteq N(x) - \{u, w\}$ be a set of $n-2$ distinct vertices where $2 \leq n \leq \lfloor p/2 \rfloor \leq 2m$. If $S_1 = A \cup \{u, x\}$ and $S_2 = A \cup \{u, w\}$, observe that $n - 1 = |N(x) \cap S_1| > |N(w) \cap S_2|$. From this, it follows that $q(\langle S_1 \rangle) > q(\langle S_2 \rangle)$, which is a contradiction. Thus, we must have $p(D) = 2m + 1$.

Now suppose that $D \cong H \cup K_1$, where H is m–regular with $m \geq 1$. Let $y \in V(D)$ such that $\text{id}(y) = \text{od}(y) = 0$. Let S_1 and S_2 be any two sets of n distinct vertices of D such that $y \in S_1$ and $y \notin S_2$. Observe that v_1 is not adjacent to or from v_2. Thus, $\langle \{v_i \mid y \in S_i\} \rangle$ and $\langle \{v_j \mid y \notin S_j\} \rangle$ are the two components of $M_n(D)$. For $3 \leq n \leq \lfloor p/2 \rfloor$, if we consider all sets of n distinct vertices of D that contain vertex y, we conclude by the previous discussion that $p(H) = 2m + 1$. Similarly, if $n = 2$, $p(H) \geq 4$, and we consider all sets of two distinct vertices of D that do not contain vertex y, then $p(H) = 2m + 1$. The only case that remains is $n = 2$ with $p(H) = 3$. Since H is m–regular with $m \geq 1$, it follows that $m = 1$. Therefore, $p(H) = 2m + 1$. If $T = S_2 - \{w\}$ for some $w \in S_2$, then we have

$$\text{id}(v_2) = \sum_{x \in S_2} \text{id}(x) - q(\langle S_2 \rangle) = \sum_{x \in S_2} \text{id}(x) - \binom{n}{2}$$

$$= \sum_{x \in S_2} \text{id}(x) - \left[\binom{n-1}{2} + (n-1) \right] = \sum_{x \in S_2} \text{id}(x) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in T} \text{id}(x) + \text{id}(w) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in S_1} \text{id}(x) + \text{id}(w) - \left[q(\langle S_1 \rangle) + (n-1) \right]$$

$$= \sum_{x \in S_1} \text{id}(x) + m - \left[q(\langle S_1 \rangle) + (n-1) \right] = \text{id}(v_1).$$

From this, we must have $m = n - 1$. Thus, H is an $(n - 1)$-regular tournament (of order $2n - 1$).

Now suppose that $M_n(D)$ is regular for some integer n, where $\lfloor p / 2 \rfloor < n \leq p - 1$. If $n = p - 1$, then by Theorem 3.9, we have $M_n(D) \cong \vec{D}$. From this, it follows that D is regular of order $n + 1$. So assume that $\lfloor p / 2 \rfloor < n < p - 1$. Using the isomorphism from Theorem 3.9, we have $M_n(D) \cong M_{p-n}(\vec{D})$. Notice that $2 \leq p - n \leq \lfloor p / 2 \rfloor$, so we can apply our previous discussion with $M_{p-n}(\vec{D})$ to determine $\vec{D}$. Thus $\vec{D}$ satisfies condition (2), (3), or (4) from the statement of this theorem. Observing that D also satisfies condition (2), (3), or (4) completes the proof. □

3.4 Multivertex Digraphs With Prescribed Center and Periphery

Recall that the *center* $C(D)$ of a strong digraph D is the subdigraph induced by those vertices v of D with $e(v) = \mathrm{rad}\, D$; while the *periphery* $P(D)$ is the subdigraph induced by those vertices v of D with $e(v) = \mathrm{diam}\, D$. We have a similar definition for the center and periphery of the n–digraph $M_n(D)$ of D. We define the n–*center* $C(M_n(D))$ of D as

$$C(M_n(D)) = \langle \{ v_i \in V(M_n(D)) \mid e(\langle S_i \rangle) = \mathrm{rad}_n D \} \rangle$$

and the n–*periphery* $P(M_n(D))$ of D as

$$P(M_n(D)) = \langle \{ v_i \in V(M_n(D)) \mid e(\langle S_i \rangle) = \mathrm{diam}_n D \} \rangle.$$

Since $M_1(D) \cong D$, we have $C(M_1(D)) \cong C(D)$ and $P(M_1(D)) \cong P(D)$. Thus, the n–center and n–periphery of D are a generalization of the center and periphery of D. In [6], it is shown that every digraph can be the center (or the periphery) of some digraph. The next theorem proves the corresponding results with n–digraphs.

Theorem 3.24 For every two asymmetric digraphs F_1 and F_2 and integer $n \geq 2$, there exists a strong asymmetric digraph D such that $C(M_n(D)) \cong F_1$ and $P(M_n(D)) \cong F_2$.

Proof Let $H_0 \cong H_1 \cong \overline{K_{n-1}}$ and let $H_2 \cong H_3 \cong \overline{K_{2n}}$. Assume that $V(F_1) = \{x_1, x_2, \ldots, x_m\}$ and $V(F_2) = \{y_1, y_2, \ldots, y_r\}$. We define a strong asymmetric digraph D by

$$V(D) = V(H_0) \cup V(H_1) \cup V(H_2) \cup V(H_3)$$
$$\cup V(F_1) \cup V(F_2) \cup \{w_i \mid 1 \leq i \leq 8\}$$

and

$$E(D) = E(F_1) \cup E(F_2) \cup \{(x,y), (y, w_5) \mid x \in V(H_1), y \in V(F_2)\}$$
$$\cup \{(x,y), (z,x), (x, w_1), (x, w_3), (x, w_6), (x, w_8), (w_8, z) \mid x \in V(H_0),$$
$$y \in V(H_1), z \in V(F_1)\}$$
$$\cup \{(w_2, x), (w_4, y), (x, w_8), (y, w_8) \mid x \in V(H_2), y \in V(H_3)\}$$
$$\cup \{(w_1, w_2), (w_3, w_4), (w_5, w_6), (w_6, w_7), (w_7, w_8)\}$$

(see Figure 3.5).

We compute the eccentricity of each vertex of D. Observe that for $x \in V(D) - \{w_3, w_4\}$ and $y \in V(H_3)$ with $y \neq x$, we have $e(x) = d(x, y)$. Also, for $y \in V(H_2)$, it is clear that $e(w_3) = d(w_3, y)$ and $e(w_4) = d(w_4, y)$. From this, it follows that

(i) $e(x) = 3$ for $x \in V(H_0)$;

(ii) $e(x) = 4$ for $x \in V(F_1)$;

(iii) $e(x) = 6$ for $x \in V(H_2) \cup V(H_3)$;
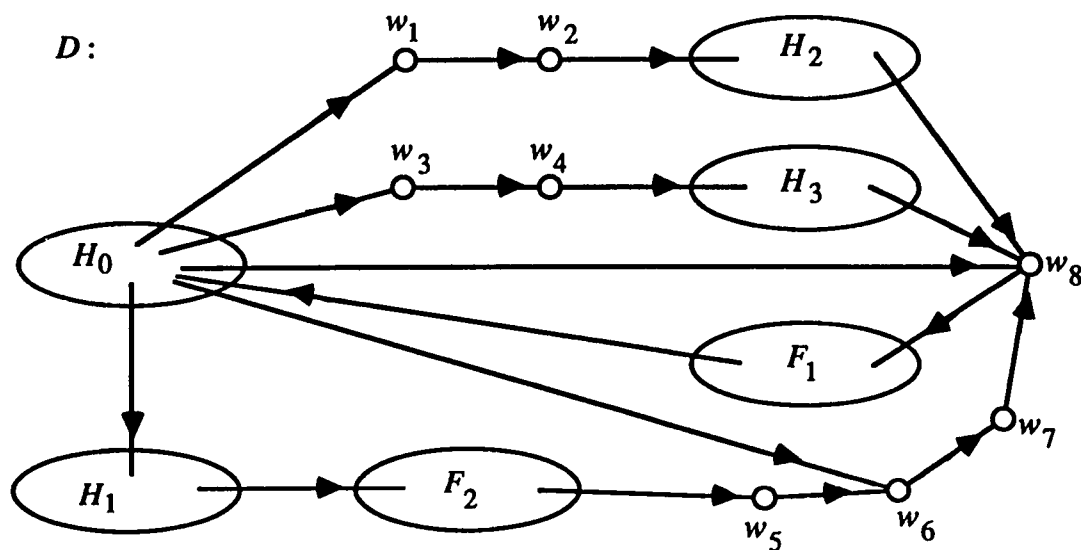
Figure 3.5

(iv) $e(x) = 9$ for $x \in V(F_2)$;

(v) $e(x) = 10$ for $x \in V(H_1)$; and

(vi) $e(w_8) = 5$, $e(w_7) = 6$, $e(w_2) = e(w_4) = e(w_6) = 7$, $e(w_1) = e(w_3) =$

$e(w_5) = 8$.

Assume throughout this proof that $S_i = V(H_0) \cup \{x_i\}$ for $1 \leq i \leq m$. Let S be any set of n distinct vertices of D. Recall that S_i is associated with the vertex v_i of $M_n(D)$, the set S with the vertex v, and S' with the vertex v'. Since $e(x) = 3$ for $x \in V(H_0)$ and $e(x_i) = 4$ for $1 \leq i \leq m$, it follows that $e(v_i) \leq 3(n-1) + 4 = 3n + 1$. If $S \subseteq V(H_3)$, then we calculate $d(v_i, v) = 3n + 1$. Thus $e(v_i) = 3n + 1$ for $1 \leq i \leq m$.

We now show that the remaining vertices of $M_n(D)$ have eccentricity exceeding $3n + 1$. Let $S \subseteq V(D) - \{w_1, w_2\}$ be a set of n distinct vertices such that $S \neq S_i$ for $1 \leq i \leq m$. It is clear that for each $x \in S$ and $y \in V(H_2) - S$, we have

$d(x, y) \geq 3$. Furthermore, there exist two vertices $u_1, u_2 \in S$ such that $d(u_1, y) \geq 4$ and $d(u_2, y) \geq 4$ or $d(u_1, y) \geq 5$ and $d(u_2, y) \geq 3$ for each $y \in V(H_2) - S$. From this, it follows that if $S' \subseteq V(H_2) - S$ is a set of n distinct vertices, then $d(v, v') \geq 8 + 3(n - 2) = 3n + 2$. Similarly, if $S \subseteq V(D) - \{w_3, w_4\}$ is a set of n distinct vertices with $S \neq S_i$ ($1 \leq i \leq m$) and $S' \subseteq V(H_3) - S$, then $d(v, v') \geq 3n + 2$. Thus, if $S \subseteq V(D) - \{w_1, w_2\}$ or $S \subseteq V(D) - \{w_3, w_4\}$, then $e(v) \geq 3n + 2$.

Suppose that $A = \{w_1, w_2\}$ and $B = \{w_3, w_4\}$. Let S be any set of n distinct vertices of D. We consider three cases.

Case 1 Assume that $|S \cap A| = 1$ and $|S \cap B| \geq 1$. Let $S' \subseteq V(H_2) - S$ be a set of n distinct vertices. For $y \in V(H_2)$ and $x \in S - \{w_1, w_2, w_3, w_4\}$, we have $d(w_1, y) > d(w_2, y) = 1$, $d(w_3, y) > d(w_4, y) = 7$, and $d(x, y) \geq 3$. Thus $e(v) \geq d(v, v') \geq 1 + 7 + 3(n - 2) = 3n + 2$.

Case 2 Assume that $|S \cap A| = 2$ and $|S \cap B| = 1$. Let $S' \subseteq V(H_3) - S$ be a set of n distinct vertices. For $y \in V(H_3)$ and $x \in S - \{w_1, w_2, w_3, w_4\}$, we have $d(w_1, y) = 6$, $d(w_2, y) = 7$, $d(w_3, y) > d(w_4, y) = 1$, and $d(x, y) \geq 3$. Again we compute $e(v) \geq d(v, v') \geq 6 + 7 + 1 + 3(n - 3) = 3n + 5$.

Case 3 Assume that $|S \cap A| = |S \cap B| = 2$. Let $S' \subseteq V(H_2) - S$ be a set of n distinct vertices. Then using the same technique as the previous two cases, we have $e(v) \geq d(v, v') \geq 1 + 2 + 6 + 7 + 3(n - 4) = 3n + 4$.

From this, we conclude that if $S \subseteq V(D)$ is a set of n distinct vertices such that $S \neq S_i$ for $1 \leq i \leq m$, then $e(v) \geq 3n + 2$. Therefore, $C(M_n(D)) \cong \langle\{v_i \mid 1 \leq i \leq m\}\rangle$ and $(v_i, v_j) \in E(M_n(D))$ if and only if $(x_i, x_j) \in E(F_1)$, that is, $C(M_n(D)) \cong F_1$.

We now show that $P(M_n(D)) \cong F_2$. Assume that $S'_i = V(H_1) \cup \{y_i\}$ for $1 \leq i \leq r$, and let S be any set of n distinct vertices of D. Since $e(x) = 10$ for $x \in V(H_1)$ and $e(y_i) = 9$ for $1 \leq i \leq r$, we have $e(v'_i) \leq 10(n - 1) + 9 = 10n - 1$. When $S \subseteq V(H_3)$, we compute $d(v'_i, v) = 10n - 1$. Thus $e(v'_i) = 10n - 1$ for $1 \leq i \leq r$.

To complete the proof, we show that if $v \in V(M_n(D)) - \{v'_i \mid 1 \leq i \leq r\}$, then $e(v) \leq 10n - 2$. Let $S \subseteq V(D)$ be a set of n distinct vertices with $S \neq S'_i$ for $1 \leq i \leq r$. Then there exist two vertices $x, y \in S$ such that $e(x) \leq 9$ and $e(y) \leq 9$ or $e(x) \leq 8$ and $e(y) \leq 10$. Therefore, if $S' \subseteq V(D)$ is any set of n distinct vertices, then $d(v, v') \leq 18 + 10(n - 2) = 10n - 2$. Thus, $P(M_n(D)) \cong \langle \{v'_i \mid 1 \leq i \leq r\} \rangle$ and $(v'_i, v'_j) \in E(M_n(D))$ if and only if $(y_i, y_j) \in E(F_2)$, so that $P(M_n(D)) \cong F_2$. $\square$

CHAPTER IV

DIGRAPHS AND GRAPHS WITH PRESCRIBED INTERIOR AND ANNULUS

4.1 Interiors and Annuli of Digraphs

In this section, we investigate the topological concepts of interior and annulus for strong digraphs. For a strong digraph D with $\text{rad}\,D < \text{diam}\,D$, the *interior* $\text{Int}(D)$ of D is defined by

$$\text{Int}(D) = \langle\{v \in V(D) \mid e(v) < \text{diam}\,D\}\rangle.$$

If $\text{rad}\,D = \text{diam}\,D$, we define

$$\text{Int}(D) = D.$$

The *annulus* $\text{Ann}(D)$ of a strong digraph D is defined only when $\text{rad}\,D < \text{diam}\,D - 1$ and is defined by

$$\text{Ann}(D) = \langle\{v \in V(D) \mid \text{rad}\,D < e(v) < \text{diam}\,D\}\rangle.$$

Otherwise, we say that D has no annulus. A strong digraph D is shown in Figure 4.1 with its interior $\text{Int}(D)$ and annulus $\text{Ann}(D)$. The eccentricity of each vertex of D is also indicated.

In Chapter I, it is shown that if D and F are asymmetric digraphs, then there exists a strong asymmetric digraph H such that $C(H) \cong D$ and $P(H) \cong F$. It is a natural question to ask if there are similar results involving other pairs of induced subdigraphs of H. In this chapter, we investigate strong asymmetric digraphs with a

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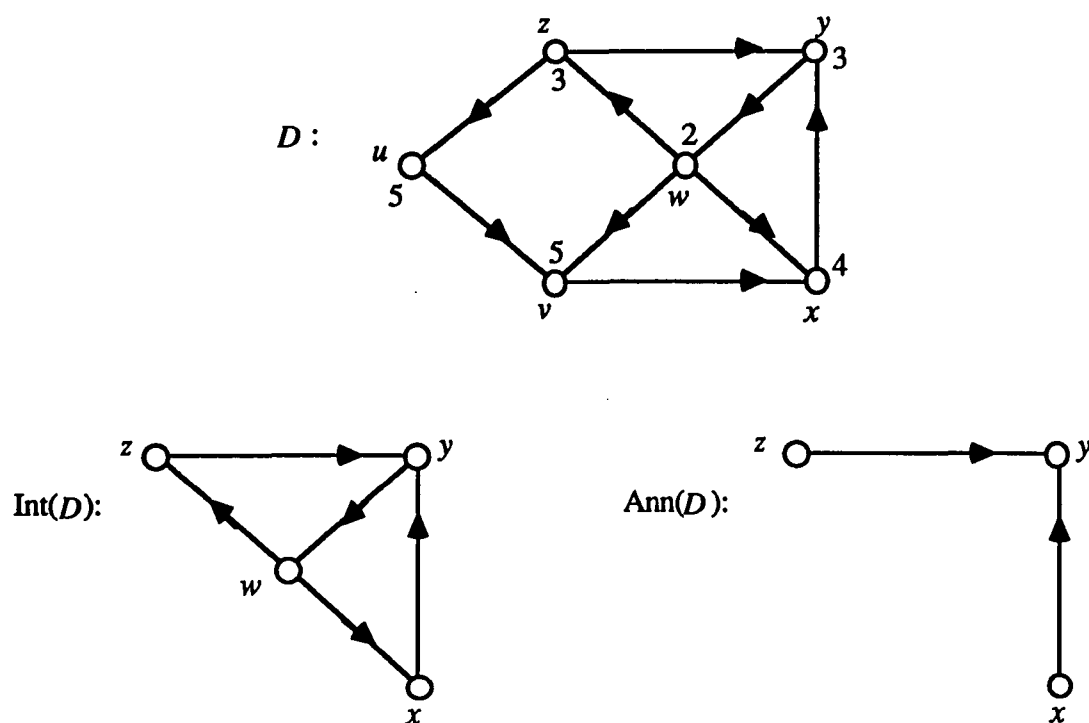
Figure 4.1   The interior and annulus of a strong digraph

pair of prescribed induced subdigraphs, where the pair is chosen from the center, interior, annulus, and periphery.

4.2 Strong Asymmetric Digraphs With Prescribed Center and Interior

From the definition of interior, it is clear that the center of a strong digraph H is an induced subdigraph of the interior of H. In addition, the interior of H is isomorphic to the center of H if and only if $\text{rad } H \geq \text{diam } H - 1$. For any subdigraph F of an asymmetric digraph D, our first result states precisely when D can be embedded in some strong asymmetric digraph H such that the interior and center of H are D and F, respectively.

Theorem 4.1 Let D be an asymmetric digraph and let F be an induced subdigraph of D. Then there exists a strong asymmetric digraph H containing D as an induced subdigraph such that $\text{Int}(H) = D$ and $C(H) = F$ if and only if $F = D$ or for each $y \in V(F)$, there exists $x \in V(D) - V(F)$ such that there is an $x - y$ path in D.

Proof Assume that $F \neq D$ and assume further that for each $y \in V(F)$, there is an $x - y$ path in D for some $x \in V(D) - V(F)$. Let $S = \{x \in V(D) - V(F) \mid d(\langle\{x\}\rangle, F) = 1\}$ and let $m = \max_{y \in V(F)} d(\langle S \rangle, \langle\{y\}\rangle)$. We define a strong asymmetric digraph H by

$$V(H) = V(D) \cup \{v_i, w_i \mid 1 \le i \le m + 4\}$$

and

$$E(H) = E(D) \cup \{(x, v_1), (v_2, x), (v_{m+4}, x), (x, w_1), (w_{m+4}, x) \mid x \in V(D) - V(F)\}$$

$$\cup \{(y, v_2), (y, w_2) \mid y \in V(F)\} \cup \{(v_{m+4}, w_1), (w_{m+4}, v_1)\}$$

$$\cup \{(v_i, v_{i+1}), (v_j, v_{m+4}), (w_i, w_{i+1}), (w_j, w_{m+4}) \mid 1 \le i \le m + 2,$$

$$1 \le j \le m + 3\}$$

(see Figure 4.2).

For each $y \in V(F)$, observe that

(i) $d(y, x) \le 2$ for $x \in V(D) - V(F)$;

(ii) $d(y, v_i) = d(y, w_i) = i - 1$ for $2 \le i \le m + 3$; and

(iii) $d(y, v_1) \le 3$, $d(y, w_1) \le 3$, and $d(y, v_{m+4}) = d(y, w_{m+4}) = 2$.

If $y, y_1 \in V(F)$, then $d(y, y_1) \le d(y, x) + d(x, y_1) \le 2 + m$ for some $x \in S$. Since $d(y, v_{m+3}) = m + 2$, we have $e(y) = m + 2$ for each $y \in V(F)$.

For $x \in V(D) - V(F)$, if follows that
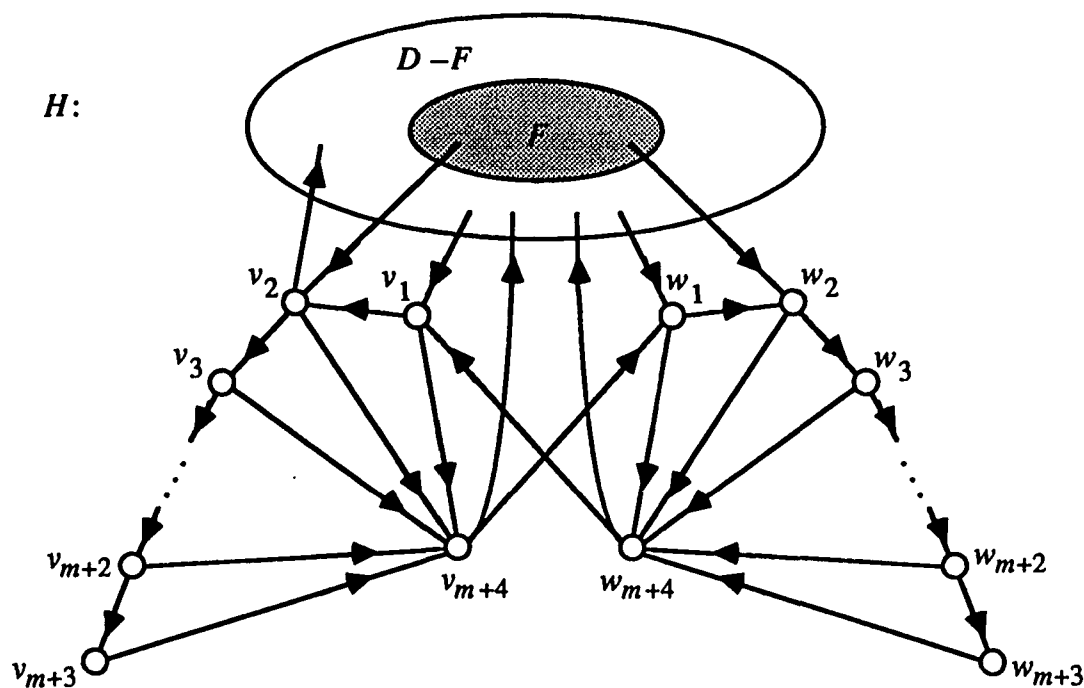
Figure 4.2

(i) $d(x, x_1) \leq 3$ for $x_1 \in V(D) - V(F)$;

(ii) $d(x, v_i) = d(x, w_i) = i$ for $1 \leq i \leq m + 3$; and

(iii) $d(x, v_{m+4}) = d(x, w_{m+4}) = 2$.

For $y \in V(F)$, we have $d(x, y) \leq d(x, x_1) + d(x_1, y) \leq 3 + m$ for some $x_1 \in S$. Thus, $e(x) = m + 3$ for each $x \in V(D) - V(F)$.

We now show that $e(v_i) = e(w_i) = m + 4$ for $1 \leq i \leq m + 4$. By the construction of H, for $1 \leq i \leq m + 3$, we have

(i) $d(v_i, x) \leq 2$ and $d(w_i, x) = 2$ for $x \in V(D) - V(F)$;

(ii) $d(v_i, y) \leq d(v_i, x_1) + d(x_1, y) \leq 2 + m$ for $y \in V(F)$ and some $x_1 \in S$;

(iii) $d(w_i, y) \leq d(w_i, x_1) + d(x_1, y) \leq 2 + m$ for $y \in V(F)$ and some $x_1 \in S$;

(iv) $d(v_i, w_j) = d(w_i, v_j) = j + 1$ for $1 \leq j \leq m + 3$;

(v) $d(v_i, w_{m+4}) = d(w_i, v_{m+4}) = 3$; and

(vi) $d(v_i, v_j) \leq m + 4$ and $d(w_i, w_j) \leq m + 4$ for $1 \leq j \leq m + 4$.

Since $d(v_i, w_{m+3}) = d(w_i, v_{m+3}) = m + 4$ for $1 \leq i \leq m + 3$, it follows that $e(v_i) = e(w_i) = m + 4$.

To complete this part of the proof, we need to show that $e(v_{m+4}) = e(w_{m+4}) = m + 4$. Using Figure 4.2, we observe that

(i) $d(v_{m+4}, x) = d(w_{m+4}, x) = 1$ for $x \in V(D) - V(F)$;

(ii) $d(v_{m+4}, y) \leq d(v_{m+4}, x_1) + d(x_1, y) \leq 1 + m$ for $y \in V(F)$ and some $x_1 \in S$;

(iii) $d(w_{m+4}, y) \leq d(w_{m+4}, x_1) + d(x_1, y) \leq 1 + m$ for $y \in V(F)$ and some $x_1 \in S$;

(iv) $d(v_{m+4}, w_j) = d(w_{m+4}, v_j) = j$ for $1 \leq j \leq m + 3$;

(v) $d(v_{m+4}, w_{m+4}) = d(w_{m+4}, v_{m+4}) = 2$; and

(vi) $d(v_{m+4}, v_j) = d(w_{m+4}, w_j) = j + 1$ for $1 \leq j \leq m + 3$.

Thus, $e(v_{m+4}) = e(w_{m+4}) = m + 4$, and we conclude that $\text{Int}(H) = D$ and $C(H) = F$.

Now suppose that $F = D$. We define a strong asymmetric digraph H by

$$V(H) = V(D) \cup \{v_i, w_i \mid 1 \leq i \leq 4\}$$

and

$$E(H) = E(D) \cup \{(x, v_1), (x, w_1), (v_4, x), (w_4, x) \mid x \in V(D)\}$$

$$\cup \{(v_i, v_4), (w_i, w_4) \mid 1 \leq i \leq 3\} \cup \{(v_4, w_1), (w_4, v_1)\}$$

$$\cup \{(v_i, v_{i+1}), (w_i, w_{i+1}) \mid 1 \leq i \leq 2\}$$

(see Figure 4.3).

H:
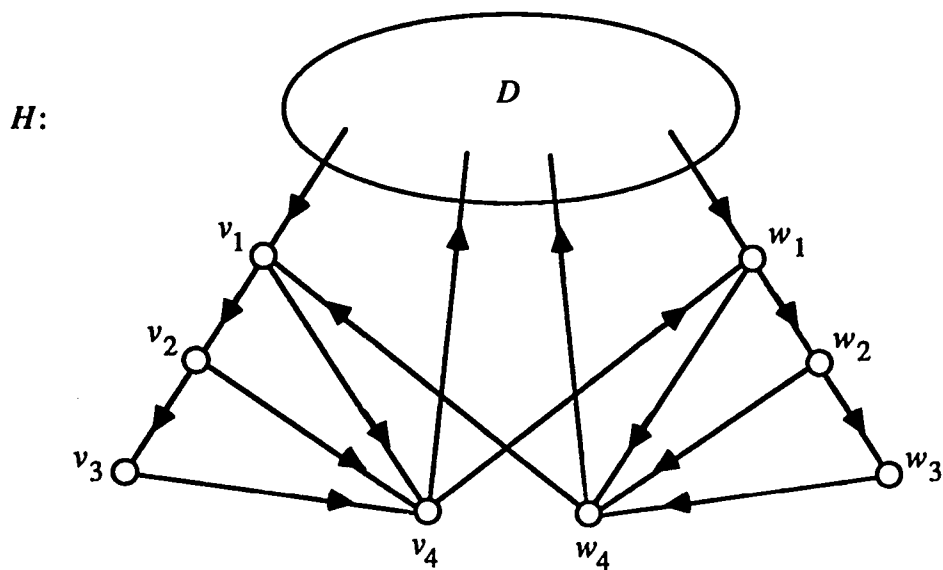
Figure 4.3

For $x, x_1 \in V(D)$ and $1 \le i \le 4$, we have $d(x, x_1) \le 3$ and $d(x, v_i) = d(x, w_i) \le 3$. Since $d(x, v_3) = d(x, w_3) = 3$, it follows that $e(x) = 3$ for $x \in V(D)$. For $z \in V(H) - V(D)$ and $1 \le i \le 4$, we have from the construction of H that $d(v_i, z) \le 4$ and $d(w_i, z) \le 4$. It is clear that $d(v_i, x) \le 2$ and $d(w_i, x) \le 2$ for $x \in V(D)$ and $1 \le i \le 4$. Since

$$d(v_i, w_3) = d(v_4, v_3) = d(w_i, v_3) = d(w_4, w_3) = 4$$

for $1 \le i \le 3$, we conclude that $e(v_i) = e(w_i) = 4$ $(1 \le i \le 4)$. Thus, $\text{Int}(H) = C(H) = D = F$.

We claim that these are precisely the conditions needed for the existence of a strong asymmetric digraph H with $\text{Int}(H) = D$ and $C(H) = F$. That is, if $F \ne D$ and if there exists some $y \in V(F)$ such that for each $x \in V(D) - V(F)$, there is no $x - y$ path in D, then there does not exist a strong asymmetric digraph H with $\text{Int}(H) = D$

and $C(H) = F$. Suppose, to the contrary, that there is some strong asymmetric digraph H with $\text{Int}(H) = D$ and $C(H) = F$. Assume that the vertex y_1 of F has the property that for each $x \in V(D) - V(F)$, there is no $x - y_1$ path in D. Since $F \neq D$, we have $P(H) = \langle V(H) - V(D) \rangle$. Observe that if $e_H(y) = m$ for $y \in V(F)$, then $e_H(z) \geq m + 2$ for $z \in V(P(H))$. For $y \in V(F)$ and $x \in V(H)$ with $(x, y) \in E(H)$, it follows that $e_H(x) \leq 1 + e_H(y) = 1 + m$. Thus, we must have $x \in V(D)$. But this says that for each $x \in V(H) - V(F)$, there is no $x - y_1$ path in H, which contradicts that H is strong. □

Corollary 4.2 Let D be an asymmetric digraph and let F be a proper induced subdigraph of D. Then there exists a strong asymmetric digraph H containing D as an induced subdigraph such that $\text{Int}(H) = D$ and $\text{Ann}(H) = F$ if and only if for each $y \in V(D) - V(F)$, there exists $x \in V(F)$ such that there is an $x - y$ path in D.

The proof of Corollary 4.2 follows directly from Theorem 4.1, which states that there exists a strong asymmetric digraph H with $\text{Int}(H) = D$ and $C(H) = D - V(F)$ if and only if $D - V(F) = D$ or for each $y \in V(D) - V(F)$, there exists an $x \in V(F)$ such that there is an $x - y$ path in D. Since $D \neq F$ and $V(F) \neq \emptyset$ in Corollary 4.2, we have $\text{Ann}(H) = \text{Int}(H) - V(C(H)) = F$.

We state the following two corollaries without proof.

Corollary 4.3 Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that $\text{Int}(H) = D_1 \cong D$ and $C(H) = F_1 \cong F$ if and only if $F \cong D$ or there is some induced subdigraph of $D_1 \cong D$, say $F_1 \cong F$, with the property that for each $y \in V(F_1)$, there exists $x \in V(D_1) - V(F_1)$ such that there is an $x - y$ path in D_1.

Corollary 4.4 Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that $\text{Int}(H) = D_1 \cong D$ and $\text{Ann}(H) = F_1 \cong F$ if and only if there is some induced subdigraph of $D_1 \cong D$, say $F_1 \cong F$, with the property that for each $y \in V(D_1) - V(F_1)$, there exists $x \in V(F_1)$ such that there is an $x - y$ path in D_1.

4.3 Strong Asymmetric Digraphs With Prescribed Annulus and Periphery

The next result shows that for any two asymmetric digraphs D and F, there is some strong asymmetric digraph H such that the annulus and periphery of H are D and F, respectively. Furthermore, the distance from the annulus to the periphery of H can be arbitrarily large.

Theorem 4.5 Let D and F be asymmetric digraphs and let $n \geq 2$ be an integer. Then there exists a strong asymmetric digraph H such that $P(H) \cong F$ and $\text{Ann}(H) \cong D$ with $d(\text{Ann}(H), P(H)) = n$. In addition, if $p(D) \geq 2$, then there exists a strong asymmetric digraph H such that $P(H) \cong F$ and $\text{Ann}(H) \cong D$ with $d(\text{Ann}(H), P(H)) = 1$.

Proof Let $t = \max\{3, n\}$. For $n = 2$, we define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{v_i \mid 1 \leq i \leq 3\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$$
$$\cup \{(y, x), (x, v_1), (v_2, x), (v_i, y) \mid x \in V(D), y \in V(F), 1 \leq i \leq 3\}$$

(see Figure 4.4(a)).

For $n \geq 3$, we define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{v_i \mid 1 \leq i \leq t\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(v_i, v_{i+1}) \mid 1 \leq i \leq t-1\}$$
$$\cup \{(y, x), (x, v_1), (v_{t-1}, x) \mid x \in V(D), y \in V(F)\}$$
$$\cup \{(v_{t-1}, y), (v_t, y), (v_t, v_1) \mid y \in V(F)\}$$

(see Figure 4.4(b)).

For $x \in V(D)$, it follows from Figure 4.4 that

(i) $d(x, v_i) = i$ for $1 \leq i \leq t$;

(ii) $d(x, x_1) \leq d(x, v_{t-1}) + d(v_{t-1}, x_1) = (t-1) + 1 = t$ for $x_1 \in V(D)$;

(iii) $d(x, y) \leq d(x, v_{t-1}) + d(v_{t-1}, y) = (t-1) + 1 = t = n$ for $y \in V(F)$ and $n \geq 3$;

and

(iv) $d(x, y) = 2$ for $y \in V(F)$ and $n = 2$.

Thus, $e(x) = t$ for $x \in V(D)$.

Observe that for $y \in V(F)$ and $x \in V(D)$, we have $e(y) \leq e(x) + 1 = t + 1$ since $d(y, x) = 1$. Because $d(y, v_t) = t + 1$, it follows that $e(y) = t + 1$ for each $y \in V(F)$.

It is clear from the construction of H that for $1 \leq i \leq t$,

(i) $d(v_i, x) \leq t - 1$ for $x \in V(D)$;

(ii) $d(v_i, y) \leq t - 1$ for $y \in V(F)$; and

(iii) $d(v_i, v_j) \leq t - 1$ for $1 \leq j \leq t$.

H :
(n = 2)
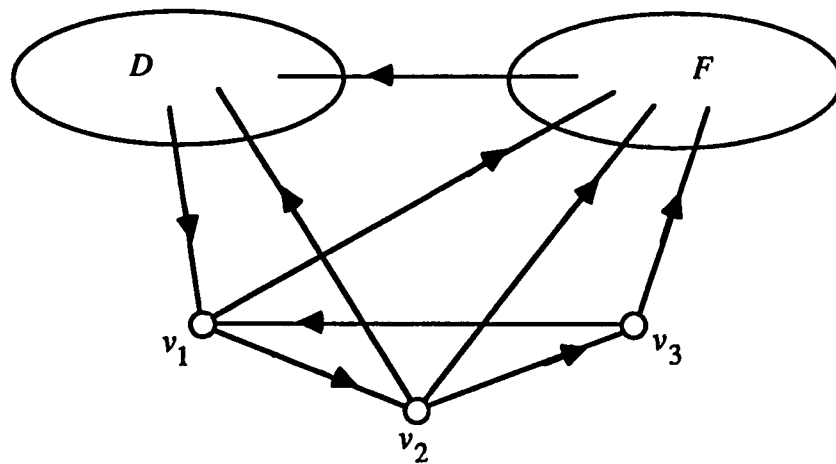
(a)

H :
($n \geq 3$)
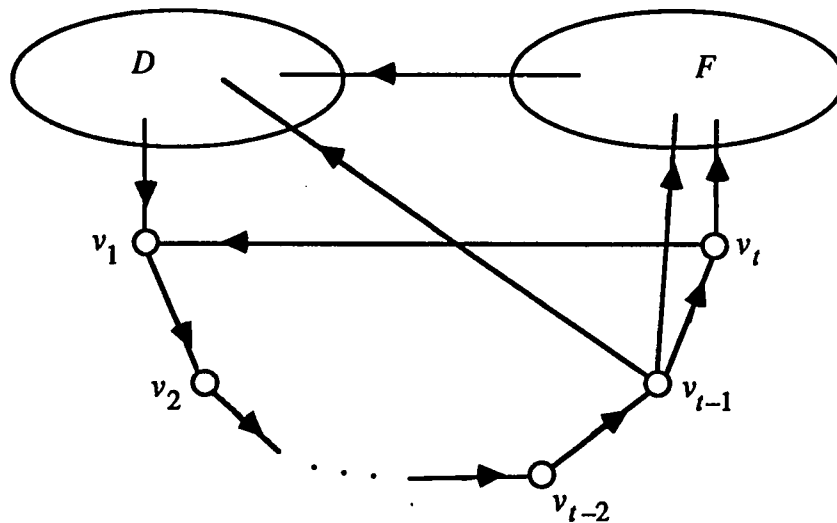
(b)

Figure 4.4

Since $d(v_i, v_{i-1}) = d(v_1, x) = t - 1$ for $2 \leq i \leq t$ and $x \in V(D)$, it follows that $e(v_i) =$ $t - 1$ $(1 \leq i \leq t)$. Thus, $P(H) \cong F$ and $\text{Ann}(H) \cong D$. For $x \in V(D)$ and $y \in V(F)$, we have

$$d(x, y) = d(x, v_{n-1}) + d(v_{n-1}, y) = (n - 1) + 1 = n.$$

Consequently, $d(\text{Ann}(H), P(H)) = n$.

Now suppose that $p(D) \geq 2$. We will show that there exists a strong asymmetric digraph H such that $P(H) \cong F$ and $\text{Ann}(H) \cong D$ with $d(\text{Ann}(H), P(H)) = 1$. For some $x_1 \in V(D)$, let $S = V(D) - \{x_1\}$. Define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{v_1, v_2, v_3\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(y, x), (x_1, y) \mid x \in S, y \in V(F)\}$$

$$\cup \{(x, v_1), (v_2, x), (v_3, x), (v_i, y) \mid x \in V(D), y \in V(F), 1 \leq i \leq 3\}$$

$$\cup \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$$
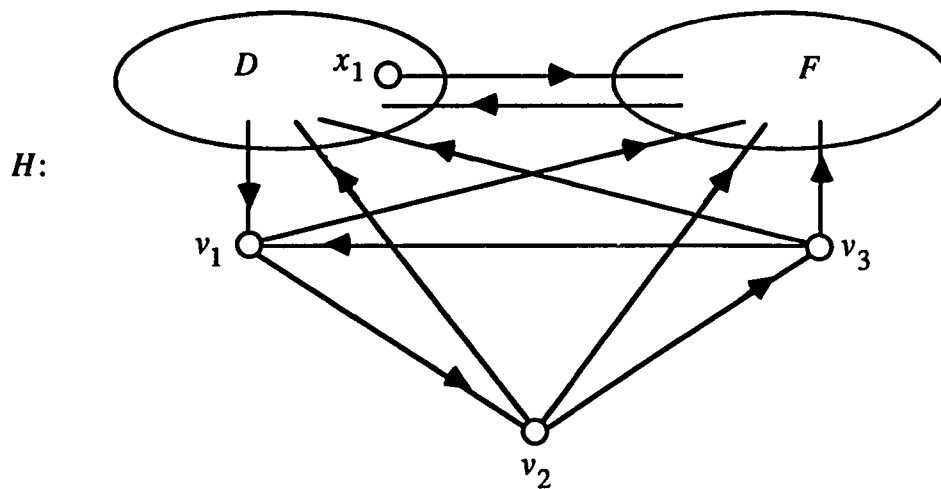
(see Figure 4.5).

Figure 4.5

It follows from the construction of H that for $x \in V(D)$,

(i) $d(x, v_i) = i$ for $1 \le i \le 3$;

(ii) $d(x, y) \le 2$ for $y \in V(F)$; and

(iii) $d(x, x_2) \le 3$ for $x_2 \in V(D)$.

Thus, $e(x) = 3$ for $x \in V(D)$. Observe that for $y \in V(F)$ and $x \in S \subset V(D)$, we have $d(y, x) = 1$. This means that $e(y) \le e(x) + 1 = 4$. Since $d(y, v_3) = 4$, it follows that $e(y) = 4$ for $y \in V(F)$. It is clear that $e(v_i) = 2$ for $1 \le i \le 3$. Thus, we conclude that $P(H) \cong F$ and $Ann(H) \cong D$ with $d(Ann(H), P(H)) = 1$. $\square$

Corollary 4.6 Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that $P(H) \cong F$ and $Ann(H) \cong D$.

From the previous theorem, if the annulus of a strong asymmetric digraph H is defined, then the distance from the annulus to the periphery of H may be arbitrarily large. On the other hand, our next result shows that the distance from the periphery to the annulus and the distance from the annulus to the center of H must be 1.

Theorem 4.7 Let H be a strong asymmetric digraph containing an annulus. Then

$$d(P(H), Ann(H)) = d(Ann(H), C(H)) = 1.$$

Proof Assume that $\mathrm{rad}\, H = k$ and $\mathrm{diam}\, H = m$. Since H has an annulus, we have $m - 2 \ge k$. Observe that if $(y, z) \in E(H)$ for $y \in V(P(H))$ and $z \in V(H)$, then $e(z) \ge m - 1 > k$; that is, $z \in V(P(H)) \cup (Ann(H))$. Thus, every arc that leaves $P(H)$ must be incident to a vertex of $Ann(H)$. Since H is strong, we have $d(P(H), Ann(H)) = 1$.

Now it follows from H being strong that there exists some $x \in V(H) - V(C(H))$ and $y \in V(C(H))$ such that $(x, y) \in E(H)$. But for each such x, we must have $e(x) \le k + 1 < m$. Thus, $x \in V(\text{Ann}(H))$ and we conclude that $d(\text{Ann}(H), C(H)) = 1$. $\square$

4.4 Strong Asymmetric Digraphs With Prescribed Center and Annulus

We now present a result similar to Theorem 4.5 with the periphery of H replaced with the center of H.

Theorem 4.8 Let D and F be asymmetric digraphs and let n be a positive integer. Then there exists a strong asymmetric digraph H such that $\text{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \text{Ann}(H)) = n$ if and only if $n \ge 2$, $p(D) \ge 2$ or $q(F) \ge 1$.

Proof We consider four cases.

Case 1 Assume that $n \ge 2$. For $n = 2$, we define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \le i \le 3\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(u_1, u_2), (u_2, u_3), (v_1, v_2), (v_2, v_3)\}$$
$$\cup \{(u_i, x), (v_i, x), (x, y), (y, u_1), (y, v_1) \mid x \in V(D), y \in V(F), 1 \le i \le 3\}$$

(see Figure 4.6(a)).

For $n \ge 3$, define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \le i \le n + 1\} \cup \{w_i \mid 1 \le i \le 7\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \le i \le n\}$$

$$\cup \{(u_i, x), (v_i, x), (x, y), (y, u_1), (y, v_1) \mid x \in V(D), y \in V(F),$$

$$n - 1 \le i \le n + 1\}$$

$$\cup \{u_i, w_6), (v_i, w_5) \mid 1 \le i \le n - 2\}$$

$$\cup \{(w_i, w_{i+1}) \mid 1 \le i \le 6\} \cup \{(y, w_2) \mid y \in V(F)\}$$

$$\cup \{(w_1, w_3), (w_2, u_1), (w_3, w_5), (w_4, v_1), (w_5, w_2), (w_6, w_1), (w_7, w_5),$$

$$(w_7, v_1)\}$$

(see Figure 4.6(b)).

For $y \in V(F)$, observe that

(i) $d(y, u_i) = d(y, v_i) = i$ for $1 \le i \le n + 1$;

(ii) $d(y, w_i) \le 3$ for $1 \le i \le 7, n \ge 3$;

(iii) $d(y, x) = d(y, v_{n-1}) + d(v_{n-1}, x) = (n - 1) + 1 = n$ for $x \in V(D)$; and

(iv) $d(y, y_1) \le d(y, x) + d(x, y_1) = n + 1$ for $y_1 \in V(F), x \in V(D)$.

Since $d(y, v_{n+1}) = n + 1$, we conclude that $e(y) = n + 1$ for each $y \in V(F)$.

For each $x \in V(D)$ and $y \in V(F)$, we have $d(x, y) = 1$. Since $e(y) = n + 1$ for $y \in V(F)$, it follows that $e(x) \le n + 2$ for $x \in V(D)$. But $d(x, u_{n+1}) = n + 2$ implies that $e(x) = n + 2$ for each $x \in V(D)$.

It is clear that $d(u_i, x) = d(v_i, x) = 1$ for $x \in V(D)$ and $n - 1 \le i \le n + 1$. Thus, in a fashion similar to above, we have $e(u_i) \le n + 3$ and $e(v_i) \le n + 3$ $(n - 1 \le i \le n + 1)$. Observing that $d(u_i, v_{n+1}) = d(v_i, u_{n+1}) = n + 3$ for $n - 1 \le i \le n + 1$, we conclude that $e(u_i) = e(v_i) = n + 3$ $(n - 1 \le i \le n + 1)$.

(a)
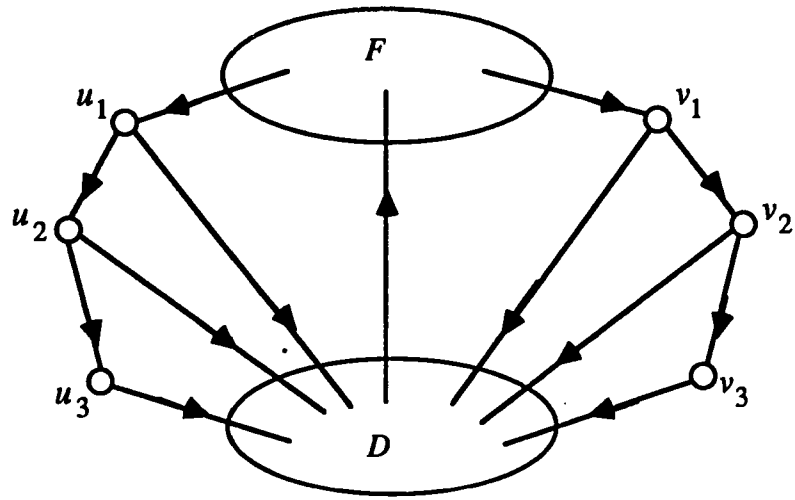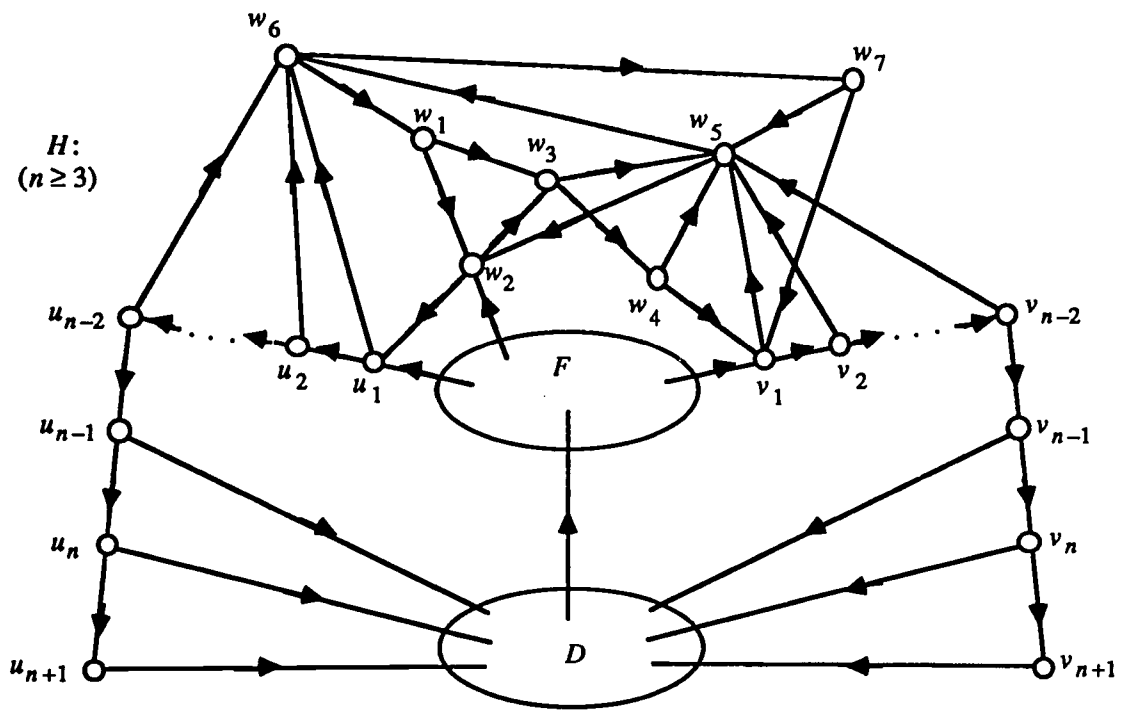
(b)

Figure 4.6

By the construction of H, it follows that $d(u_i, v_1) = d(v_i, u_1) = 3$ for $1 \le i \le n - 2$. With this fact, it can be seen for $z \in V(H)$ that $d(u_i, z) \le n + 3$, $d(v_i, z) \le n + 3$, and $d(u_i, v_{n+1}) = d(v_i, u_{n+1}) = n + 3$ $(1 \le i \le n - 2)$. This means that $e(u_i) = e(v_i) = n + 3$ for $1 \le i \le n - 2$.

To show that $e(w_i) = n + 3$ $(1 \le i \le 7)$, we make the observation that for $i \in \{1, 2, 5\}$ and $j \in \{3, 4, 6, 7\}$, we have $d(w_i, v_1) = d(w_j, u_1) = 3$, $d(w_i, u_1) \le 2$, and $d(w_j, v_1) \le 2$. Also, $d(w_i, w_j) \le 4$ for $1 \le i \ne j \le 7$. From this, it follows that $e(w_i) \le n + 3$ $(1 \le i \le 7)$. Since $d(w_i, v_{n+1}) = d(w_j, u_{n+1}) = n + 3$ $(i \in \{1, 2, 5\}$ and $j \in \{3, 4, 6, 7\})$, we conclude that $e(w_i) = n + 3$ $(1 \le i \le 7)$. Thus, $\text{Ann}(H) \cong D$ and $C(H) \cong F$. From the construction of H, it is clear that $d(C(H), \text{Ann}(H)) = n$.

Case 2 *Assume that* $n = 1, p(D) \ge 2, and\ p(F) \ge 2$. For some $x_1 \in V(D)$ and some $y_1 \in V(F)$, let $S = V(D) - \{x_1\}$ and $T = V(F) - \{y_1\}$. We define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \le i \le 3\}$$

and

$$E(H) = E(D) \cup E(F) \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \le i \le 2\}$$
$$\cup \{(u_i, x), (v_i, x), (y, u_1), (y, v_1) \mid x \in V(D), y \in V(F), 1 \le i \le 3\}$$
$$\cup \{(y_1, x_1)\} \cup \{(x_1, y) \mid y \in T\} \cup \{(x, y) \mid x \in S, y \in V(F)\}$$

(see Figure 4.7).

For $y \in V(F)$, it is clear from the construction of H that $e(y) = 3$. Using a technique similar to the preceding case, we can show that $e(x) = 4$ and $e(u_i) = e(v_i) = 5$ for $x \in V(D)$ and $1 \le i \le 3$. Thus, $\text{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \text{Ann}(H)) = 1$.
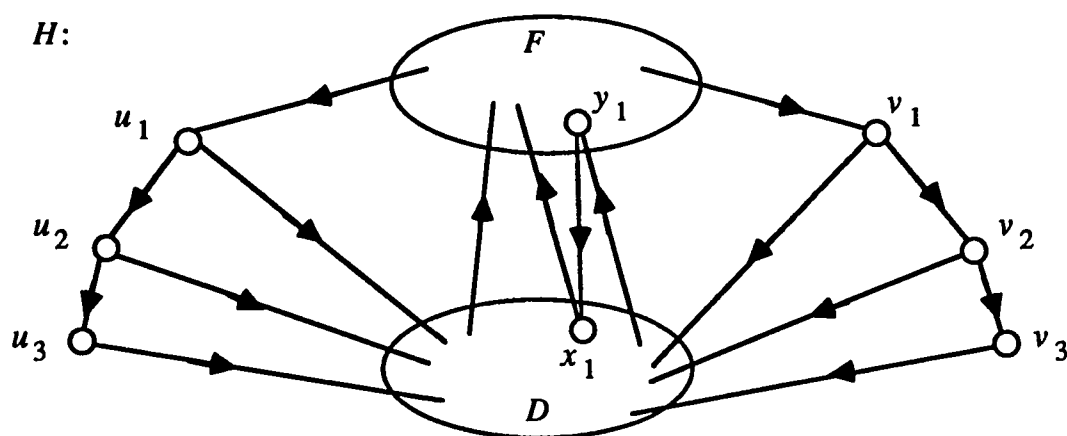
H:

Figure 4.7

Case 3   *Assume that* $n = 1$, $p(D) \geq 2$, *and* $p(F) = 1$. Let $V(F) = \{y\}$ and $S = V(D) - \{x_1\}$ for some $x_1 \in V(D)$. We define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leq i \leq 3\}$$

and

$$E(H) = E(D) \cup \{(y, x_1), (y, u_1), (y, v_1), (y, u_2), (y, v_2), (x_1, u_1), (x_1, v_1)\}$$

$$\cup \{(x, y), (u_1, x), (v_1, x) \mid x \in S\} \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \leq i \leq 2\}$$

$$\cup \{(u_i, x), (v_i, x) \mid x \in V(D), 2 \leq i \leq 3\}$$

(see Figure 4.8).

It is clear from the construction of H that $e(y) = 2$. For $x \in S$, we have $d(x, y) = 1$ and, thus, $e(x) \leq 3$. Since $d(x, v_3) = 3$, it follows that $e(x) = 3$ for each $x \in S$. Observe that for $x \in S$, we have $d(x_1, x) \leq 2$, $d(x_1, y) \leq 3$, and $d(x_1, u_i) = d(x_1, v_i) = i$ $(1 \leq i \leq 3)$. Thus, $e(x_1) = 3$. Clearly, $d(u_i, x) = d(v_i, x) = 1$ for $x \in S$ and $1 \leq i \leq 3$. Consequently, $e(u_i) \leq 4$ and $e(v_i) \leq 4$ $(1 \leq i \leq 3)$. Since $d(u_i, v_3) =$

H:
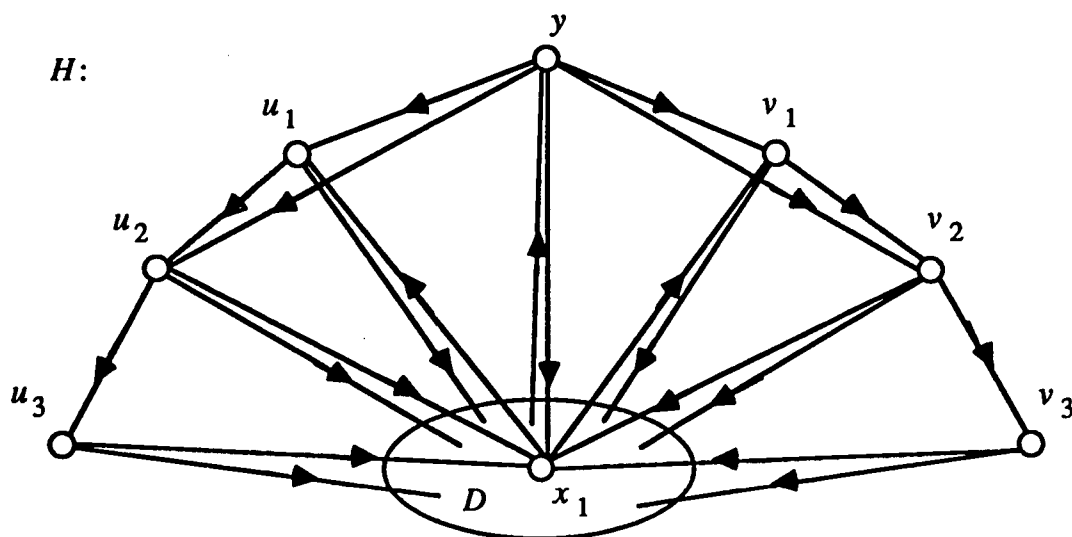
Figure 4.8

$d(v_i, u_3) = 4$ $(1 \leq i \leq 3)$, it follows that $e(u_i) = e(v_i) = 4$. Therefore, $\text{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \text{Ann}(H)) = 1$.

Case 4 *Assume that* $n = 1, p(D) = 1, and$ $q(F) \geq 1$. Let $y_1 \in V(F)$ such that the indegree of y_1 in F is at least 1. Suppose that $V(D) = \{x\}$ and $S = V(F) - \{y_1\}$. We define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F) \cup \{u_i, v_i \mid 1 \leq i \leq 4\}$$

and

$$E(H) = E(F) \cup \{(u_i, x), (v_i, x), (y, u_1), (y, v_1) \mid y \in V(F), 1 \leq i \leq 4\}$$
$$\cup \{(x, y) \mid y \in S\} \cup \{(y_1, x)\}$$

(see Figure 4.9).

H:
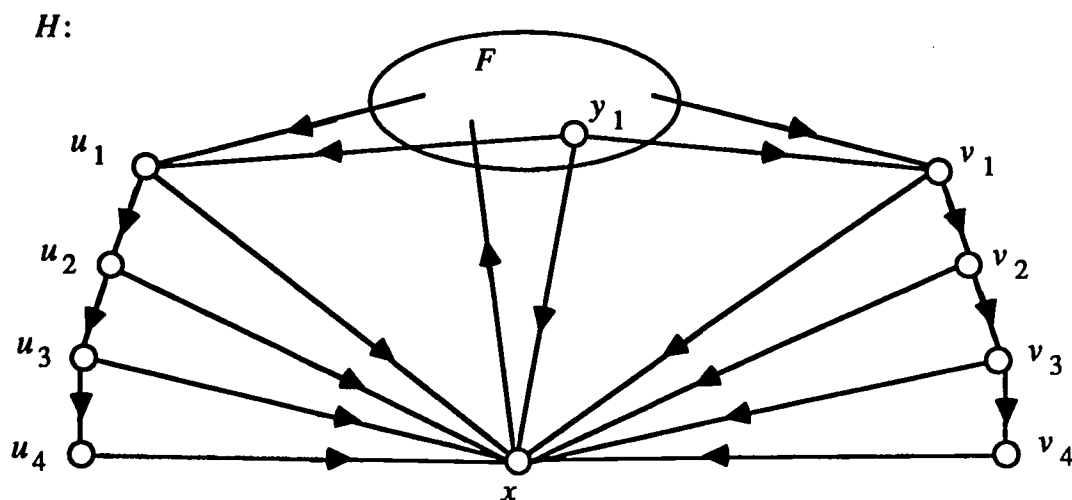
Figure 4.9

Using the methods described in Cases 1 and 2, we find that $e(y) = 4$, $e(x) =$ 5, and $e(u_i) = e(v_i) = 6$ for $y \in V(F)$ and $1 \le i \le 4$. Thus, $\text{Ann}(H) \cong D$ and $C(H)$ $\cong F$ with $d(C(H), \text{Ann}(H)) = 1$.

Conversely, assume that $n = 1, p(D) = 1$ and $q(F) = 0$. Suppose, to the contrary, that there is some strong asymmetric digraph H with the property that $\text{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \text{Ann}(H)) = 1$. Then there is some vertex y of $C(H)$ such that $(y, x) \in E(H)$ for some $x \in V(\text{Ann}(H))$. Since H is an asymmetric digraph with $p(\text{Ann}(H)) = 1$ and $q(C(H)) = 0$, we have $(z, y) \notin E(H)$ for each $z \in [V(C(H)) \cup V(\text{Ann}(H))] - \{y\}$. If there is some vertex v of H such that v is adjacent to a vertex in $C(H)$, then $e(v) \le \text{rad } H + 1$. Thus, v must be a vertex in $C(H)$ or $\text{Ann}(H)$. This means that y has indegree 0, which contradicts the fact that H is strong. Therefore, it follows, for asymmetric digraphs D and F with $p(D) = 1$ and $q(F) = 0$, that there does not exist a strong asymmetric digraph H such that $\text{Ann}(H) \cong D$ and $C(H) \cong F$ with $d(C(H), \text{Ann}(H)) = 1$. $\square$

Corollary 4.9 Let D and F be asymmetric digraphs. Then there exists a strong asymmetric digraph H such that $\text{Ann}(H) \cong D$ and $C(H) \cong F$.

4.5 Strong Asymmetric Digraphs With Prescribed Interior and Periphery

We now present a sufficient condition for two asymmetric digraphs D and F to be isomorphic to the periphery and interior, respectively, of some strong asymmetric digraph.

Theorem 4.10 Let D and F be asymmetric digraphs. If F is nontrivial and strong, then there exists a strong asymmetric digraph H such that $P(H) \cong D$ and $\text{Int}(H) \cong F$.

Proof Assume that $\text{diam } F = m \geq 2$, and let $x, y \in V(F)$ such that $d(x, y) = m$. We define a strong asymmetric digraph H by

$$V(H) = V(D) \cup V(F)$$

and

$$E(H) = E(D) \cup E(F) \cup \{(z, x) \mid z \in V(D)\}$$
$$\cup \{(v, z) \mid v \in V(F) - \{x\}, z \in V(D)\}$$

(see Figure 4.10).

For $z \in V(D)$ and $v \in V(F)$, it is clear, from the construction of H, that $e(z) = m + 1$ and $e(v) \leq m$. Thus, $P(H) \cong D$ and $\text{Int}(H) \cong F$. $\square$

If D and F are asymmetric digraphs such that F is not strong, then even when D is strong, we can draw no conclusion about the existence of a strong
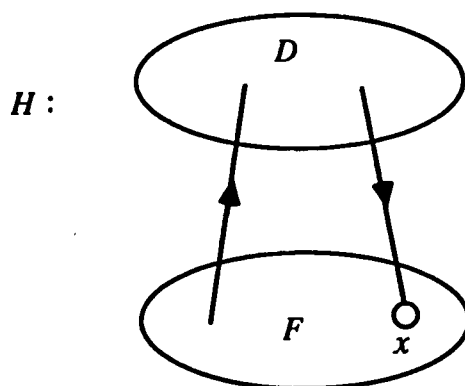
Figure 4.10

asymmetric digraph H such that $P(H) \cong D$ and $\text{Int}(H) \cong F$. For example, define D by $V(D) = \{x_1, x_2, x_3\}$ and $E(D) = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$, and define F by $V(F) = \{y_1, y_2\}$ and $E(F) = \{(y_1, y_2)\}$. We claim that no strong asymmetric digraph H exists with $P(H) \cong D$ and $\text{Int}(H) \cong F$. Suppose, to the contrary, that there is some strong asymmetric digraph H with this property. It is clear that $V(H) = V(D) \cup V(F)$. Since H is strong and the indegree of vertex y_1 in F is 0, there must be some arc from D to y_1. Assume that $(x_1, y_1) \in E(H)$. From this, it follows that $e(x_1) = 2$. Since H is asymmetric, we have $(y_2, y_1) \notin E(H)$. Thus, $e(y_2) \geq 2$, which contradicts the fact that $\text{Int}(H) \cong F$.

On the other hand, if we define D by $V(D) = \{x_1, x_2, x_3, x_4\}$ and $E(D) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)\}$, and define F by $V(F) = \{y_1, y_2\}$ and $E(F) = \{(y_1, y_2)\}$, then there exists a strong asymmetric digraph H such that $P(H) \cong D$ and $\text{Int}(H) \cong F$. We define H by

$$V(H) = V(D) \cup V(F)$$

and

$$E(H) = E(D) \cup E(F) \cup \{(x_1, y_1), (x_3, y_1)\}$$

$$\cup \{(y_2, x) \mid x \in V(D)\}$$
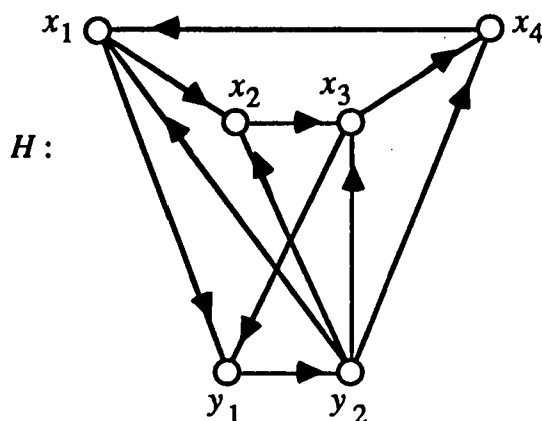
(see Figure 4.11).

H :

Figure 4.11

It is clear from the construction of H that $e(x) = 3$ and $e(y) = 2$ for $x \in$ $V(D)$ and $y \in V(F)$. Thus, $P(H) \cong D$ and $\text{Int}(H) \cong F$.

4.6 Connected Graphs With Prescribed Interior and Annulus

For a connected graph G with rad $G <$ diam G, the *interior* $\text{Int}(G)$ of G is defined as the subgraph induced by those vertices v with $e(v) <$ diam G. Otherwise, if rad $G =$ diam G, we say $\text{Int}(G) = G$. When rad $G <$ diam $G - 1$, the *annulus* $\text{Ann}(G)$ of a connected graph G is defined as the subgraph induced by those vertices v with rad $G < e(v) <$ diam G. We say that G has no annulus if rad $G \geq$ diam $G -$ 1. A connected graph G is shown in Figure 4.12 with its interior $\text{Int}(G)$ and annulus $\text{Ann}(G)$.
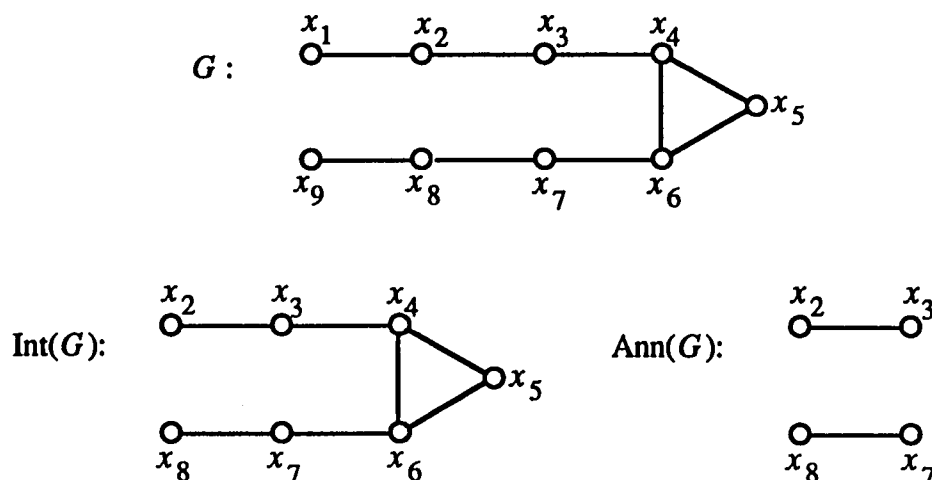
Figure 4.12

It was proved in [8] that every graph is isomorphic to the interior of some connected graph. Furthermore, every connected graph G is isomorphic to the interior, but not the center, of some connected graph if and only if G is not complete. We now present a sufficient condition for two graphs to be isomorphic to the annulus and center of some connected graph.

Theorem 4.11 Let F and G be graphs such that F has no vertices of eccentricity 1. Then there exists a connected graph H such that $\mathrm{Ann}(H) \cong F$ and $C(H) \cong G$.

Proof First assume that F is a connected graph with diam $F \geq 3$. Then there exist $u, v \in V(F)$ such that $d(u, v) = 3$. We define a connected graph H by

$$V(H) = V(F) \cup V(G) \cup \{w_1, w_2\}$$

and

$$E(H) = E(F) \cup E(G) \cup \{w_1 u, w_2 v\}$$

$$\cup \ \{xy \mid x \in V(F), y \in V(G)\}$$
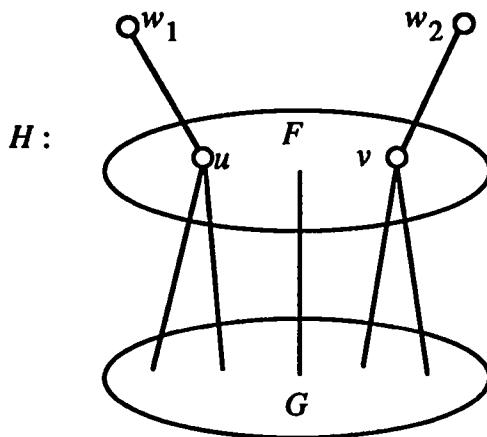
(see Figure 4.13).

Figure 4.13

From the construction of H, we have $e(x) = 2$ for each $x \in V(G)$. Since each vertex of F is adjacent to a vertex of G, it follows that $e(y) \leq 3$ for each $y \in V(F)$. Observe that for each $y \in V(F)$, we have $d(y, u) \geq 2$ or $d(y, v) \geq 2$, from which it follows that $d(y, w_1) \geq 3$ or $d(y, w_2) \geq 3$. Thus, $e(y) = 3$ for $y \in V(F)$. We also have $e(w_1) = e(w_2) = 4$ since vertices w_1 and w_2 are each adjacent to a vertex of F and $d(w_1, w_2) = 4$. Therefore, $\text{Ann}(H) \cong F$ and $C(H) \cong G$.

Now assume that F is a graph of order p with $e(x) = 2$ for each $x \in V(F)$. Also assume that $V(F) = \{x_1, x_2, \ldots, x_p\}$. We define a connected graph H by

$$V(H) = V(F) \cup V(G) \cup \{w_i \mid 1 \leq i \leq p\}$$

and

$$E(H) = E(F) \cup E(G) \cup \{x_i w_i \mid 1 \leq i \leq p\}$$

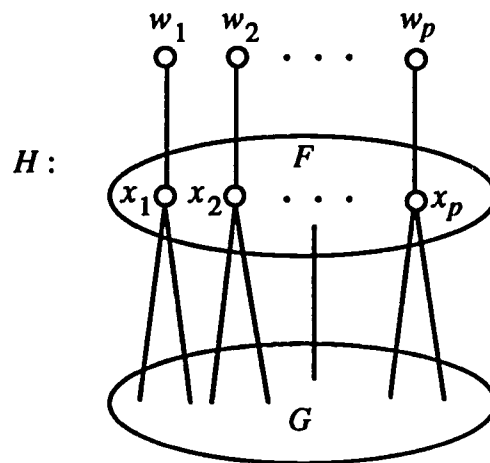$$\cup \{xy \mid x \in V(F), y \in V(G)\}$$

· (see Figure 4.14).

Figure 4.14

It is clear by the construction of H that $e(y) = 2$ for each $y \in V(G)$. Since each vertex of F is adjacent to a vertex of G, we have $e(x_i) \leq 3$ for $1 \leq i \leq p$. Similarly, $e(w_i) \leq 4$ for $1 \leq i \leq p$. Also, for each $x_i \in V(F)$, there exists $x_j \in V(F)$ such that $d(x_i, x_j) = 2$. From this, we have $d(x_i, w_j) = 3$ and $d(w_i, w_j) = 4$. Thus, we conclude that $e(x_i) = 3$ and $e(w_i) = 4$ for $1 \leq i \leq p$; that is, $\text{Ann}(H) \cong F$ and $C(H) \cong G$.

Assume next that F is not connected. Let $u, v \in V(F)$ such that u and v belong to different components of F. We define a connected graph H by

$$V(H) = V(F) \cup V(G) \cup \{w_1, w_2\}$$

and

$$E(H) = E(F) \cup E(G) \cup \{w_1 u, w_2 v\}$$

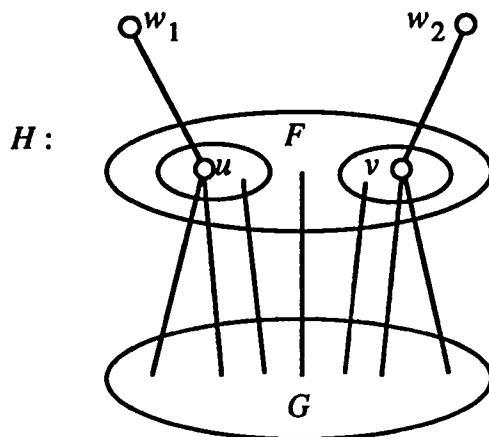$$\cup \ \{xy \mid x \in V(F), y \in V(G)\}$$

- (see Figure 4.15).

Figure 4.15

By the construction of H, we have $e(y) = 2$ for $y \in V(G)$. It follows that $e(x) \le 3$ and $e(w_i) \le 4$ for $x \in V(F)$ and $1 \le i \le 2$, since each vertex of F is adjacent to a vertex of G and w_1 and w_2 are both adjacent to a vertex of F. Observing that $d(x, u) \ge 2$ or $d(x, v) \ge 2$ for each $x \in V(F)$, it follows that $d(x, w_1)$ ≥ 3 or $d(x, w_2) \ge 3$. Also, $d(w_1, w_2) \ge 4$, from which we conclude that $e(x) = 3$ and $e(w_1) = e(w_2) = 4$ for $x \in V(F)$. Therefore, $\text{Ann}(H) \cong F$ and $C(H) \cong G$. $\quad\square$

In Theorem 4.11, we presented a sufficient condition for a graph to be isomorphic to the annulus of some connected graph. The next theorem shows that this condition is also necessary.

Theorem 4.12 Let F and H be graphs such that H is connected and $\text{Ann}(H) \cong$ F. Then F has no vertices of eccentricity 1.

Proof Assume that there exists $u \in V(F)$ such that $e(u) = 1$. Suppose, to the contrary, that there is some connected graph H such that $\text{Ann}(H) \cong F$. Assume that $\text{diam } H = n$ and $\text{rad } H = m$. Observe that for each $w_1 \in V(H) - V(\text{Ann}(H))$ with w_1 adjacent to a vertex of $\text{Ann}(H)$ and $e(w_1) = n$, there exists $w_2 \in V(H) - V(\text{Ann}(H))$ such that $d(w_1, w_2) = n$. There must also be some vertex $y \in V(H) - V(\text{Ann}(H))$ such that $e(y) = m$. Since $\text{Ann}(H)$ is defined, $\text{diam } H - \text{rad } H \geq 2$ and vertex y is not adjacent to any vertex of eccentricity n. Therefore, $m \geq 2$. Observe that

$$d(y, w_2) = d(y, z) + d(z, w_2) \leq m$$

for some $z \in V(\text{Ann}(H))$. Thus,

$$d(w_1, w_2) \leq d(w_1, z) + d(z, w_2) \leq 3 + (m - 1) = m + 2,$$

that is, $\text{diam } H - \text{rad } H \leq 2$. If $w_1 u \in E(H)$ with $e(u) = 1$ in $\text{Ann}(H)$, then

$$d(w_1, w_2) \leq d(w_1, z) + d(z, w_2) \leq 2 + (m - 1) = m + 1,$$

which is impossible. Thus, each vertex of $\text{Ann}(H)$ has eccentricity $m + 1$ and u is not adjacent to a vertex of eccentricity $m + 2$. Assume that $d(u, w) = m + 1$ for some $w \in V(H)$. Then $e(w) = m + 2$ and $d(u, w) = d(u, z) + d(z, w)$ for some $z \in V(\text{Ann}(H))$. But, from this, we have $d(y, w) = d(y, z) + d(z, w) \geq d(u, w)$ for some $z \in V(\text{Ann}(H))$. This contradiction completes the proof. $\square$

From the proof of Theorems 4.11 and 4.12, we have the following corollary.

Corollary 4.13 For every graph F, there exists a connected graph H such that $\text{Ann}(H) \cong F$ if and only if F has no vertices of eccentricity 1.

CHAPTER V

MEDIANS AND PERIPHERIES OF GRAPHS AND DIGRAPHS

5.1 Strong Asymmetric Digraphs With Prescribed Median and Periphery

For a vertex v in a strong digraph D, the *distance* $d_D(v)$ of v in D is the sum of the directed distances from v to the vertices of D; that is, $dD(v) = \sum_{u \in V(D)} d(v, u)$. The *median* $M(D)$ of D is the subdigraph of D induced by those vertices having minimum distance. In Figure 5.1, a strong digraph D is shown with its median $M(D)$. The distance of each vertex is also indicated.
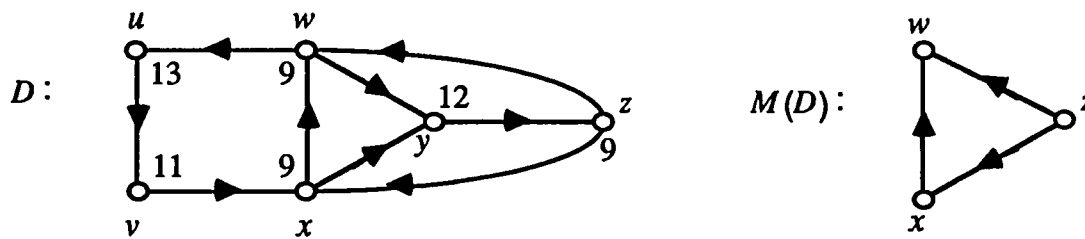
Figure 5.1

In [6] it was shown that for every two asymmetric digraphs D_1 and D_2, there exists a strong asymmetric digraph H such that $C(H) \cong D_1$ and $M(H) \cong D_2$, and where the directed distances from $C(H)$ to $M(H)$ and from $M(H)$ to $C(H)$ can be arbitrarily prescribed. In addition, if K is a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D_1 and D_2, then there exists a strong asymmetric digraph F such that $C(F) \cong D_1$, $M(F) \cong D_2$, and $C(F) \cap M(F) \cong K$. In the next two sections, we present similar results involving the median and periphery of a graph.

We begin by recalling two lemmas from [7].

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Lemma 5.1 Let D be a strong asymmetric digraph and let F be an induced subdigraph of D with $d_D(u, v) \le 3$ for all $u, v \in V(F)$. Then there exists a strong asymmetric digraph H containing D as an induced subdigraph such that

(i) $d_H(u) = d_H(v)$ for all $u, v \in V(F)$, and

(ii) if $V(H) \ne V(D)$, then $\max\{d(u, v) \mid u \in V(F), v \in V(H) - V(D)\} = 2$.

Lemma 5.2 Let D be a strong asymmetric digraph and let F be an induced subdigraph of D such that $d_D(u, v) \le 3$ and $d_D(u) = d_D(v)$ for all $u, v \in V(F)$. Then there exists a strong asymmetric digraph H containing D as an induced subdigraph such that

(i) $M(H) \cong F$, and

(ii) if $V(H) \ne V(D)$, then $\max\{d(u, v) \mid u \in V(F), v \in V(H) - V(D)\} = 2$.

We now illustrate the construction of the digraph H from Lemma 5.2. Suppose that $d_D(v) = k$ for all $v \in V(F)$ and let $n = \left\lceil \frac{k-p(F)}{2} \right\rceil + 2$. We construct a strong asymmetric digraph H by adding $2n$ new vertices u_i and v_i $(1 \le i \le n)$ to D, the arcs (u_i, v_i) for $1 \le i \le n$, together with the arcs joining all vertices of F to u_i and the arcs joining all vertices of F from v_i for $1 \le i \le n$ (see Figure 5.2).

Let D and F be asymmetric digraphs. With the aid of Lemmas 5.1 and 5.2, we can show that there exists a strong asymmetric digraph H such that $M(H) \cong D$ and $P(H) \cong F$, where $M(H)$ and $P(H)$ are disjoint. Furthermore, since the center and median are two ways of defining the "middle" of a digraph and the periphery defines the "exterior" of a digraph, it is not suprising that the distances from the median to the periphery and from the periphery to the median can be arbitrarily prescribed. What may
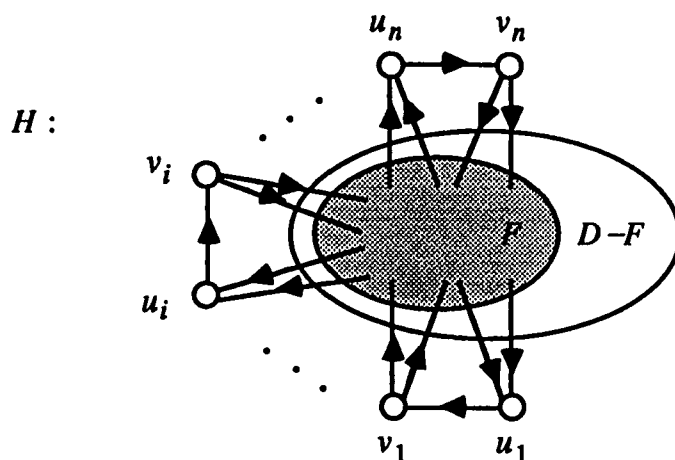
H :

Figure 5.2

be suprising is that the median and periphery can intersect in any common induced subdigraph.

Theorem 5.3 Let D and F be asymmetric digraphs and let m and n be positive integers. Then there exists a strong asymmetric digraph H such that $M(H) \cong D$ and $P(H) \cong F$ with $d(P(H), M(H)) = m$ and $d(M(H), P(H)) = n$ if and only if (1) $m + n \geq 3$, (2) $p(D) \geq 2$, or (3) $p(F) \geq 2$.

Proof We construct a strong asymmetric digraph H_0 from D by adding two new vertices u_0 and v_0, the arc (u_0, v_0), together with the arcs joining all vertices of D to u_0 and the arcs joining all vertices of D from v_0. By applying Lemma 5.1 to H_0, we construct a strong asymmetric digraph H_1 with

(1) $d_{H_1}(u) = d_{H_1}(v)$ for all $u, v \in V(D)$ and

(2) if $V(H_1) \neq V(H_0)$, then $\max\{d(u, v) \mid u \in V(D), \ v \in V(H_1) - V(H_0)\} = 2$.

We consider three cases.

Case 1 Assume that $m + n \geq 3$. We define a strong asymmetric digraph H_2 by

$$V(H_2) = V(H_1) \cup V(F) \cup \{u_i, v_j \mid 1 \leq i \leq n - 1, \ 1 \leq j \leq m + 2\}$$

and

$$E(H_2) = E(H_1) \cup E(F) \cup \{(x, v_1), (u_{n-1}, y) \mid x \in V(H_1), y \in V(D)\}$$

$$\cup \{(v_i, x), (x, u_1) \mid m - 1 \leq i \leq m + 2, x \in V(F)\}$$

$$\cup \{(u_i, u_{i+1}), (v_j, v_{j+1}), (v_{m+2}, v_m) \mid 1 \leq i \leq n - 2, 1 \leq j \leq m + 1\}$$

$$\cup \{(x, y) \mid m = 1, x \in V(D), y \in V(F)\}$$

$$\cup \{(y, x) \mid n = 1, y \in V(F), x \in V(D)\}$$
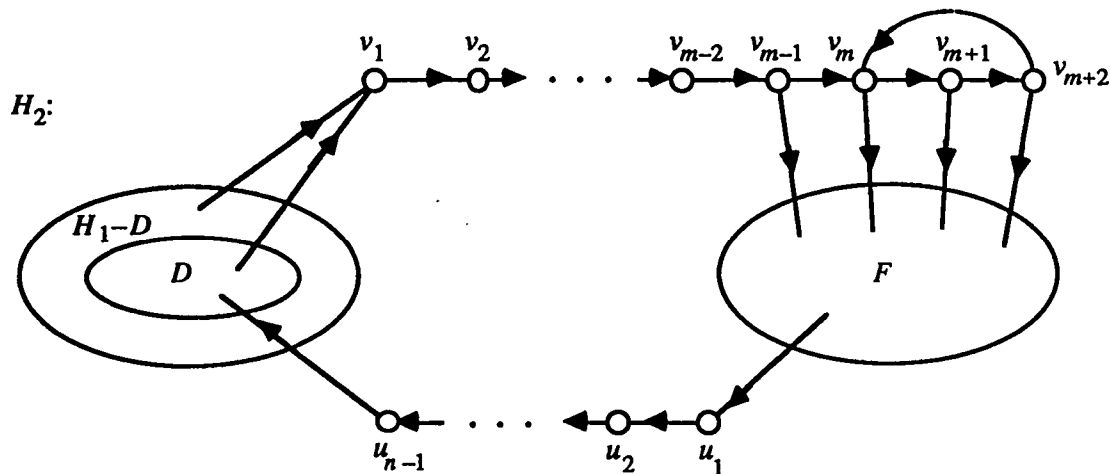
(see Figure 5.3).

Figure 5.3

We now use Lemma 5.2 to construct a strong asymmetric digraph H_3 with

(1) $M(H_3) \cong D$ and

(2) if $V(H_3) \neq V(H_2)$, then $\max\{d(u, v) \mid u \in V(D), v \in V(H_3) - V(H_2)\} = 2$.

The strong asymmetric digraph H is constructed by joining all vertices of $H_3 - H_2$ to v_1. Observe that for each $x \in V(H_2)$, it follows that $d_{H_3}(x) = d_H(x)$. In particular, we have $d_{H_3}(u_0) = d_H(u_0)$ and $d_{H_3}(v_0) = d_H(v_0)$. In addition, from the construction of H, it follows that for each $x \in V(H_3) - V(H_2)$, we have $d_H(x) = d_H(u_0)$ or $d_H(x) = d_H(v_0)$. Thus $M(H) \cong D$. We calculate the eccentricity of each vertex of H as

(1) $e(x) = m + n + 2$ for $x \in V(F)$ and

(2) $e(x) \le m + n + 1$ for $x \in V(H) - V(F)$.

Therefore $P(H) \cong F$.

Case 2 Assume that $m = n = 1$ and $p(F) \ge 2$. Let y be a vertex of F and let w be a vertex in D. We define a strong asymmetric digraph H_2 by

$$V(H_2) = V(H_1) \cup V(F) \cup \{v_i \mid 1 \le i \le 5\}$$

and

$$E(H_2) = E(H_1) \cup E(F) \cup \{(x, v_1), (x, v_2) \mid x \in V(H_1)\}$$
$$\cup \{(x, v_1) \mid x \in V(F)\} \cup \{(v_i, v_{i+1}) \mid 1 \le i \le 4\}$$
$$\cup \{(v_5, v_3), (w, y), (y, u_0)\} \cup \{(v, x) \mid v \in V(F) - \{y\}, x \in V(D)\}$$
$$\cup \{(v_4, x), (v_5, x) \mid x \in V(F)\}$$

(see Figure 5.4).

We construct strong asymmetric digraphs H_3, by applying Lemma 5.2, and H by joining $V(H_3) - V(H_2)$ to the vertices v_1 and v_2. Using calculations similar to those in Case 1, we conclude that H has the desired properties.
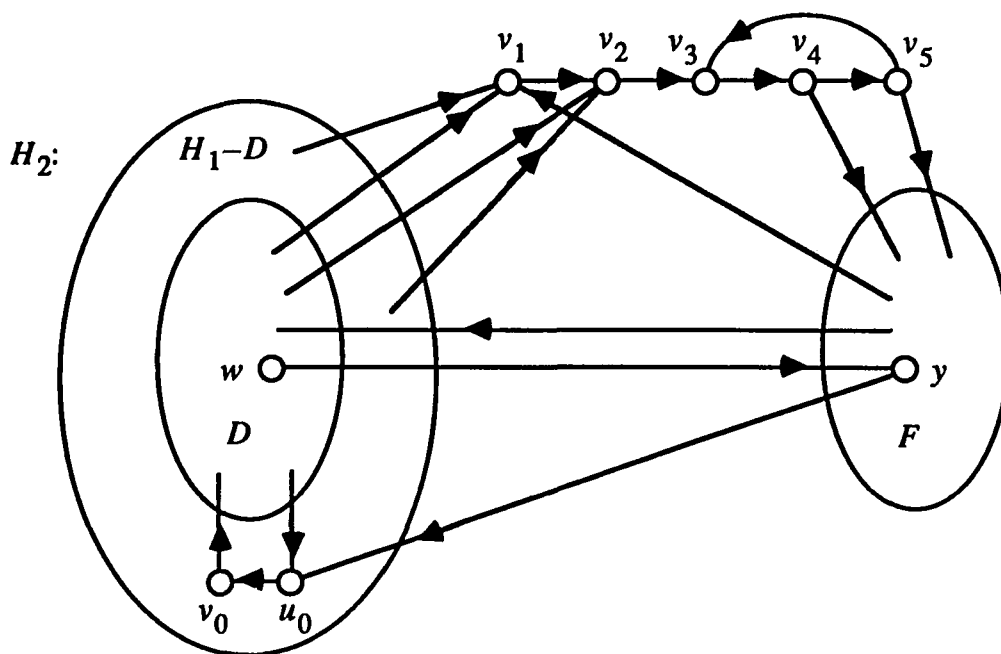
Figure 5.4

Case 3   *Assume that* $m = n = 1$, $p(F) = 1$, *and* $p(D) \geq 2$. Let y be a vertex of D and assume that $V(F) = \{w\}$. We define a strong asymmetric digraph H_2 by

$$V(H_2) = V(H_1) \cup V(F) \cup \{v_i \mid 1 \leq i \leq 4\}$$

and

$$E(H_2) = E(H_1) \cup \{(x, v_1) \mid x \in V(H_1)\} \cup \{(y, w), (v_4, v_2), (v_3, w), (v_4, w)\}$$
$$\cup \{(v_i, v_{i+1}) \mid 1 \leq i \leq 3\} \cup \{(w, x) \mid x \in V(D) - \{y\}\}$$

(see Figure 5.5).

We construct strong asymmetric digraphs H_3 and H as in Case 1. Again, H has the desired properties.
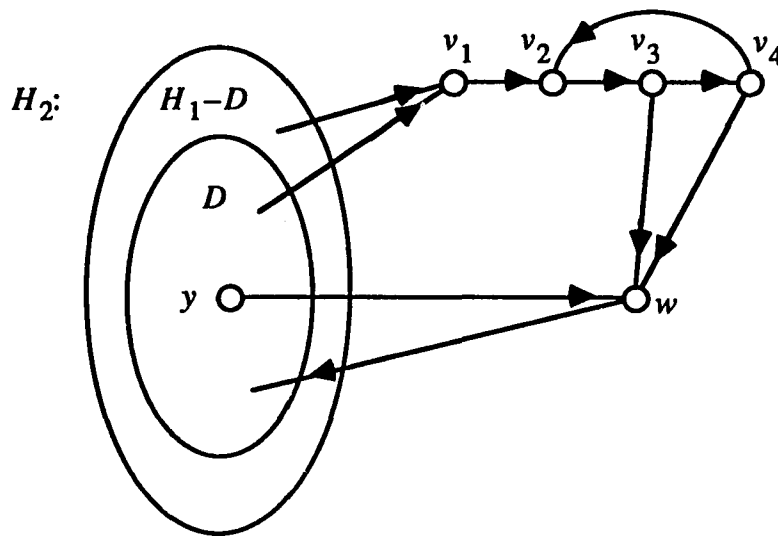
Figure 5.5

The only case that remains is when $m = n = 1$ and $p(D) = p(F) = 1$. Clearly, there is no digraph H with the desired properties since H is asymmetric. ◻

5.2 Strong Asymmetric Digraphs With Intersecting Median and Periphery

Recall that the center and median are two ways of describing the middle of a digraph, while the periphery describes the exterior of a digraph. The next theorem shows that not only can the periphery and median intersect in any common induced subdigraph, but, in addition, the distances from the center to the median and from the median to the center can be arbitrarily large.

Theorem 5.4 Let D_1, D_2, and F be asymmetric digraphs, and let m and n be positive integers such that $m + n \geq 3$. In addition, let K be a nonempty asymmetric digraph isomorphic to an induced subdigraph of D_1 and a proper induced subdigraph of D_2. Then there exists a strong asymmetric digraph H such that $P(H) \cong D_1$,

$M(H) \cong D_2$, $C(H) \cong F$, and $P(H) \cap M(H) \cong K$ with $d(M(H), C(H)) = m$ and $d(C(H), M(H)) = n$.

Proof Assume that $V(D_1) = \{s_1, s_2, \dots, s_{p_1}\}$, $V(D_2) = \{t_1, t_2, \dots, t_{p_2}\}$, and $p(K) = k$. Without loss of generality, assume that $\langle \{s_1, s_2, \dots, s_k\} \rangle \cong \langle \{t_{i_1}, t_{i_2}, \dots, t_{i_k}\} \rangle \cong K$ and $s_j \to t_{i_j}$ is an isomorphism between $\langle \{s_1, s_2, \dots, s_k\} \rangle$ and $\langle \{t_{i_1}, t_{i_2}, \dots, t_{i_k}\} \rangle$ for $1 \leq j \leq k$. We first construct an asymmetric digraph H_0 by identifying s_j and t_{i_j} and labeling the resulting vertex again by s_j for $1 \leq j \leq k$. We now define a strong asymmetric digraph H_1 by

$$V(H_1) = V(H_0) \cup \{u_0, v_0\} \cup \{x_i \mid 1 \leq i \leq m - 1\}$$
$$\cup \{y_i \mid 1 \leq i \leq n - 1\} \cup \{z_i \mid 1 \leq i \leq n + 2\}$$

and

$$E(H_1) = E(H_0) \cup \{(x, u_0), (v_0, x), (y_{n-1}, x), (z_{n+2}, x) \mid x \in V(H_0)\}$$
$$\cup \{(u_0, v_0), (u_0, x_1), (v_0, x_1)\} \cup \{(x, x_1) \mid x \in V(D_2) - V(D_1)\}$$
$$\cup \{(x, x_i), (x, y_1), (x, z_1), (x_{m-1}, x) \mid x \in V(F), 1 \leq i \leq m - 2\}$$
$$\cup \{(x_i, x_{i+1}) \mid 1 \leq i \leq m - 2\} \cup \{(y_i, y_{i+1}) \mid 1 \leq i \leq n - 2\}$$
$$\cup \{(z_i, z_{i+1}) \mid 1 \leq i \leq n + 1\} \cup \{(x, y) \mid x \in V(F), y \in V(H_0), n = 1\}$$
$$\cup \{(x, y), (u_0, y), (v_0, y) \mid x \in V(D_2) - V(D_1), y \in V(F), m = 1\}$$

(see Figure 5.6).

We calculate the eccentricity of each vertex of H_1; namely,

(i) $e(x) = m + n + 3$ for $x \in V(D_1)$,

(ii) $n + 3 \leq e(x) \leq m + n + 2$ for $x \in V(H_1) - (V(D_1) \cup V(F))$, and
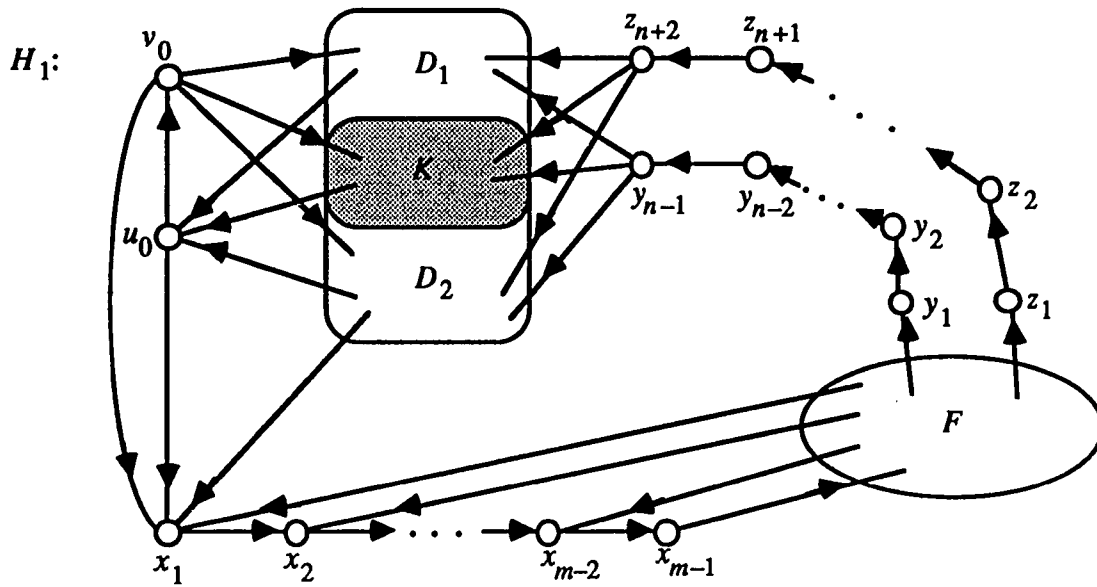
(iii) $e(x) = n + 2$ for $x \in V(F)$.

Figure 5.6

Since H_1 is strong and $d_{H_1}(x, y) \le 3$ for all $x, y \in V(D_2)$, we can apply Lemma 5.1 to construct a strong asymmetric digraph H_2 containing H_1 as an induced subdigraph such that

(i) $d_{H_2}(x) = d_{H_2}(y)$ for all $x, y \in V(D_2)$, and

(ii) if $V(H_2) \ne V(H_1)$, then $\max\{d(x, y) \mid x \in V(D_2), y \in V(H_2) - V(H_1)\} = 2$.

Now define a strong asymmetric digraph H_3 from H_2 by joining all vertices from $V(H_2) - V(H_1)$ to vertex x_1 (if $m = 1$, then join these vertices to the vertices of F). Observe that adding these arcs does not change the distance of the vertices of D_2; that is, $d_{H_3}(x) = d_{H_2}(x)$ for all $x \in V(D_2)$. Assume that $d_{H_3}(x) = b$ for all $x \in V(D_2)$ and let $c = \left\lceil \dfrac{b - p(H_3)}{2} \right\rceil + 2$. Using Lemma 5.2, we construct a strong asymmetric digraph H_4 such that $M(H_4) \cong D_2$ by adding $2c$ new vertices u_i and v_i ($1 \le i \le c$)

to H_3, the arcs (u_i, v_i) for $1 \le i \le c$, together with the arcs joining all vertices of D_2 to u_i and the arcs joining all vertices of D_2 from v_i for $1 \le i \le c$. Furthermore, if $V(H_4) \ne V(H_3)$, then

$$\max\{d(x, y) \mid x \in V(D_2),\ y \in V(H_4) - V(H_3)\} = 2.$$

We define a strong asymmetric digraph H from H_4 by joining all vertices from $V(H_4) - V(H_3)$ to x_1 (again, if $m = 1$, then join these vertices to the vertices of F). We compute the eccentricity of each vertex of H as follows:

(i) $e(x) = m + n + 3$ for $x \in V(D_1)$,

(ii) $n + 3 \le e(x) \le m + n + 2$ for $x \in V(H) - (V(D_1) \cup V(F))$, and

(iii) $e(x) = n + 2$ for $x \in V(F)$.

Thus, $P(H) \cong D_1$ and $C(H) \cong F$.

We now calculate the distance of each vertex of H. Assume that $d_{H_3}(x) = b$ for all $x \in V(D_2)$ and let $c = \left\lceil \dfrac{b - p(H_3)}{2} \right\rceil + 2$. By the construction of H, we have $d_H(v, x) = d_{H_3}(v, x)$ for $v \in V(D_2)$ and $x \in V(H_3)$. Therefore, for $v \in V(D_2)$,

$$d_H(v) = \sum_{i=1}^{c} (d(v, u_i) + d(v, v_i)) + \sum_{x \in V(H_3)} d_H(v, x)$$

$$= 3c + \sum_{x \in V(H_3)} d_H(v, x) = 3c + \sum_{x \in V(H_3)} d_{H_3}(v, x)$$

$$= 3c + d_{H_3}(v) = 3c + b.$$

If $v \in V(H_3) - V(D_2)$, then

$$d_H(v) = \sum_{i=1}^{c} (d(v, u_i) + d(v, v_i)) + \sum_{x \in V(H_3)} d_H(v, x)$$

$$\geq 5c + \sum_{x \in V(H_3)} d_H(v, x) \geq 5c + p(H_3) - 1.$$

In addition, by considering the vertices adjacent from u_i and v_i for $0 \leq i \leq c$, we see that $d_H(u_i) > d_H(u_0)$ and $d_H(v_i) > d_H(v_0)$ for $1 \leq i \leq c$. Since $c = \left\lceil \dfrac{b-p(H_3)}{2} \right\rceil + 2$ and $p(H_3) \geq 7$, it follows that $5c + p(H_3) - 1 > 3c + b$.

Consequently, we conclude that $M(H) \cong D_2$. $\square$

We now consider the case where $m = n = 1$.

Theorem 5.5 Let $D_1, D_2,$ and F be asymmetric digraphs. Let K be a nonempty asymmetric digraph isomorphic to an induced subdigraph of D_1 and a proper induced subdigraph of D_2. Then there exists a strong asymmetric digraph H such that $P(H) \cong D_1$, $M(H) \cong D_2$, $C(H) \cong F$, and $P(H) \cap M(H) \cong K$ with $d(M(H), C(H)) = d(C(H), M(H)) = 1$.

Proof We first construct the digraph H_0 in the same way that H_0 was constructed in Theorem 5.4. Now define a strong asymmetric digraph H_1 by

$$V(H_1) = V(H_0) \cup V(F) \cup \{u_0, v_0, z_1, z_2, z_3\}$$

and

$$E(H_1) = E(H_0) \cup E(F) \cup \{(x, u_0), (v_0, x), (z_3, x), (z_1, x) \mid x \in V(H_0)\}$$

$$\cup \{(u_0, v_0), (z_1, z_2), (z_2, z_3)\}$$

$$\cup \{(x, z_1), (x, y), (z, x), (u_0, x), (v_0, x) \mid x \in V(F), y \in V(K),$$

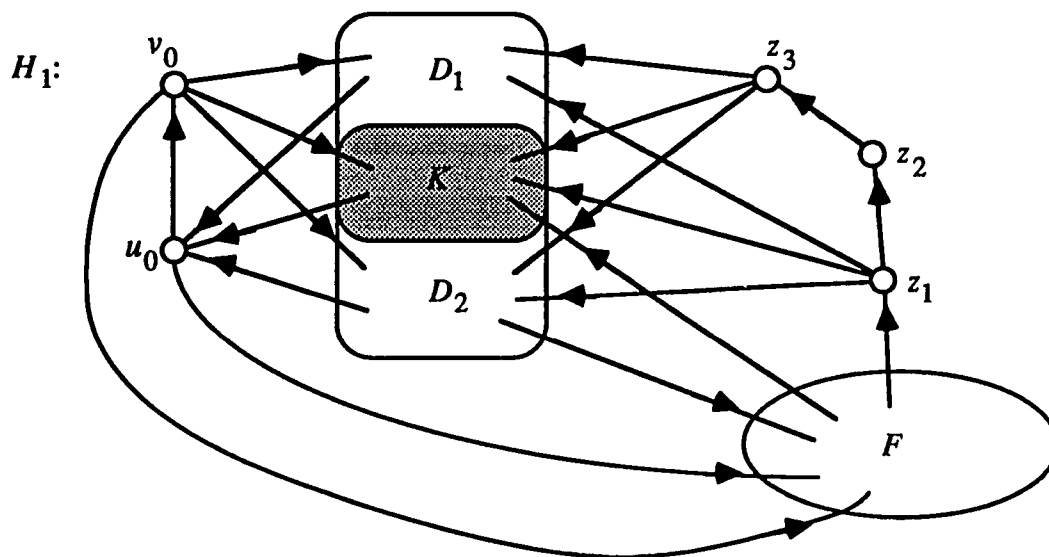$$z \in V(D_2) - V(K)\}$$

(see Figure 5.7).

Figure 5.7

We construct the digraph H from H_1 with the same construction presented in Theorem 5.4. All the calculations are the same as those in Theorem 5.4. Thus, we conclude that $P(H) \cong D_1$, $M(H) \cong D_2$, $C(H) \cong F$, and $P(H) \cap M(H) \cong K$ with $d(M(H), C(H)) = d(C(H), M(H)) = 1$. $\square$

Observe that if $K \cong D_2$ in Theorem 5.4, then $d(M(H), C(H)) = m + 1$. Thus, with this modification and by joining vertex z_{n+2} to vertex v_0 in H_1 in the proof of Theorem 5.4, we have the following corollary.

Corollary 5.6 Let D_1, D_2, and F be asymmetric digraphs, and let m and n be positive integers such that $m \geq 2$. In addition, let K be a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D_1 and D_2. Then there exists a

strong asymmetric digraph H such that $P(H) \cong D_1$, $M(H) \cong D_2$, $C(H) \cong F$, and $P(H) \cap M(H) \cong K$ with $d(M(H), C(H)) = m$ and $d(C(H), M(H)) = n$.

The only case that remains is when $m = 1$ and $K \cong D_2$. It turns out that there may or may not be a strong asymmetric digraph H with the desired properties. For example, say digraphs D_1, $D_2 \cong K$, F, and H are given as in Figure 5.8 with $p \geq 4$. Then $d(v) = d(x) = p + 10$ and since $p \geq 4$, we have $d(t) > p + 10$ for each $t \in V(H) - \{v, x\}$. In addition, $e(u) = 2$ and $e(t) = 3$ for $t \in V(H) - \{u\}$. Therefore, $P(H) \cong D_1$ $M(H) \cong D_2$, $C(H) \cong F$, and $P(H) \cap M(H) \cong K$ with $d(M(H), C(H)) = d(C(H), M(H)) = 1$. Thus, we have an infinite class of digraphs with this property. For most choices of the digraphs D_1, $D_2 \cong K$, and F, there does not exist a strong asymmetric digraph H with the appropriate properties. An example that illustrates this point is $D_1 \cong D_2 \cong K \cong \overline{K_2}$ and $F \cong K_1$. Observe that if a strong asymmetric digraph H exists with the desired properties, then H has order 3. Since H is strong and asymmetric, it follows that H is a directed triangle, which is a contradiction.

Corollary 5.7 Let D_1 and D_2 be asymmetric digraphs, and let K be a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D_1 and D_2. Then there exists a strong asymmetric digraph H such that $P(H) \cong D_1$, $M(H) \cong D_2$, and $P(H) \cap M(H) \cong K$.

5.3 Connected Graphs With Distant Median and Periphery

The definition of the median of a connected graph G is analogous to the definition of the median of a strong digraph. The *distance* $d_G(v)$ of a vertex v in G is the sum of the distances from v to the vertices of G; that is, $d_G(v) = \sum_{u \in V(G)} d(v, u)$.
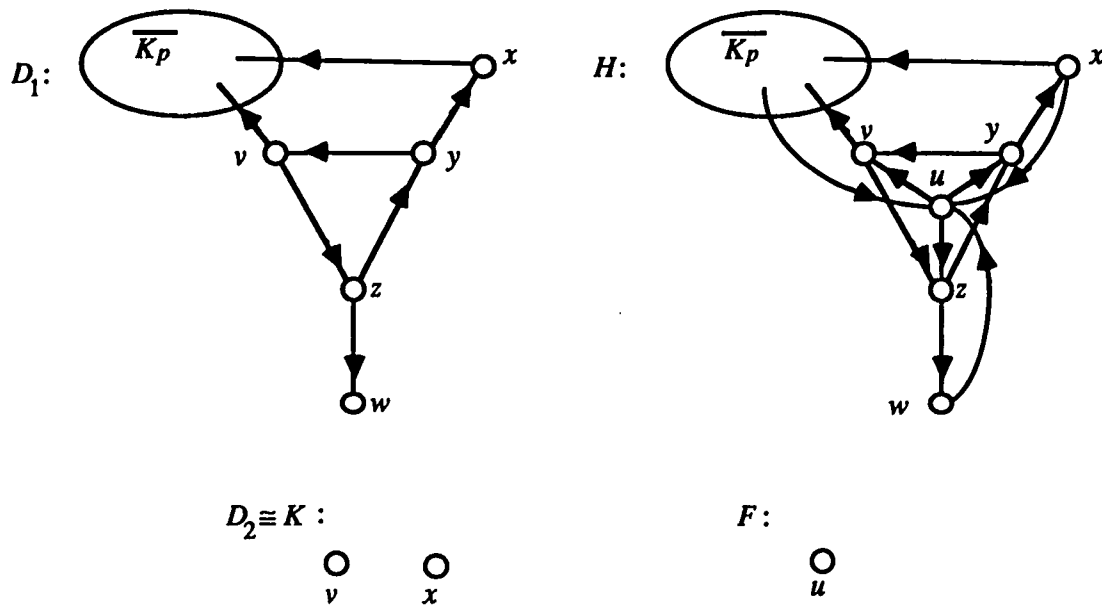
Figure 5.8

The *median* $M(G)$ of G is the subgraph induced by those vertices having minimum distance.

In Sections 5.1 and 5.2, we proved that the distances from the median to the periphery and from the periphery to the median of a strong asymmetric digraph can be arbitrarily large. In addition, the median and periphery can intersect in any common induced subdigraph. It is a natural question to ask if there is a similar relationship between the median and the periphery of a connected graph. Before we answer this question, we present two lemmas.

For any graph F, Lemma 5.8 says that we can construct some connected graph H that contains F as an induced subgraph such that each vertex of F has the same distance in H. Furthermore, H has the property that the distance between each pair of vertices in H is at most 2, and each vertex of H that is not in F is adjacent to some vertex of F. We will use the simplified notation $d(u, F)$ to represent the subgraph distance $d(\langle\{u\}\rangle, F)$.

Lemma 5.8 For any graph F, there exists a connected graph H that contains F as

an induced subgraph such that

(1) $d_H(u) = d_H(v)$ for all $u, v \in V(F)$,

(2) $d_H(u, v) \leq 2$ for all $u, v \in V(H)$, and

(3) $d(u, F) = 1$ for each $u \in V(H) - V(F)$.

Proof Define a connected graph H_0 by joining a new vertex w to F. Let $m_\Delta(H_0) = \max\{d_{H_0}(x) \mid x \in V(F)\}$, $m_\delta(H_0) = \min\{d_{H_0}(x) \mid x \in V(F)\}$, and $n = m_\Delta(H_0) - m_\delta(H_0)$. If $n = 0$, then $d_{H_0}(x) = d_{H_0}(y)$ for all $x, y \in V(F)$, and choosing $H = H_0$ gives us the desired result. For $n \geq 1$, let $S_\Delta(H_0) = \{x \in V(F) \mid d_{H_0}(x) = m_\Delta(H_0)\}$. We define a connected graph H_1 by

$$V(H_1) = V(H_0) \cup \{x_1\}$$

and

$$E(H_1) = E(H_0) \cup \{x_1 w\} \cup \{z x_1 \mid z \in S_\Delta(H_0)\}$$
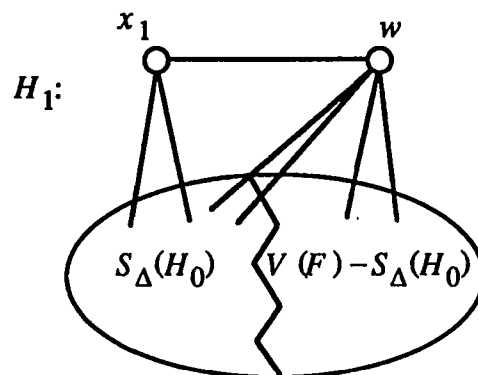
(see Figure 5.9).

Figure 5.9

From the construction of H_1, we have $e_{H_1}(w) = 1$ and, thus, $d_{H_1}(u, v) \le 2$ for

all $u, v \in V(H_1)$. It also follows that for $z \in S_\Delta(H_0)$,

$$d_{H_1}(z) = d_{H_1}(z, x_1) + \sum_{t \in V(H_0)} d_{H_1}(z, t)$$

$$= 1 + \sum_{t \in V(H_0)} d_{H_0}(z, t) = 1 + d_{H_0}(z) = m_\Delta(H_0) + 1.$$

Similarly, for $z \in V(F) - S_\Delta(H_0)$,

$$d_{H_1}(z) = d_{H_0}(z) + 2 \ge m_\delta(H_0) + 2,$$

and there exists some vertex $z_1 \in V(F) - S_\Delta(H_0)$ such that $d_{H_1}(z_1) = m_\delta(H_0) + 2$.
Define $m_\Delta(H_1) = \max\{d_{H_1}(x) \mid x \in V(F)\}$ and $m_\delta(H_1) = \min\{d_{H_1}(x) \mid x \in V(F)\}$.
Then $m_\Delta(H_1) = m_\Delta(H_0) + 1$ and $m_\delta(H_1) = m_\delta(H_0) + 2$, from which it follows that
$m_\Delta(H_1) - m_\delta(H_1) = n - 1$. Let $S_\Delta(H_1) = \{x \in V(F) \mid d_{H_1}(x) = m_\Delta(H_1)\}$. Observe

that

$$S_\Delta(H_1) = S_\Delta(H_0) \cup \{x \in V(F) - S_\Delta(H_0) \mid d_{H_0}(x) = m_\Delta(H_0) - 1\}.$$

Now define a connected graph H_2 by

$$V(H_2) = V(H_1) \cup \{x_2\}$$

and

$$E(H_2) = E(H_1) \cup \{x_2 w\} \cup \{z x_2 \mid z \in S_\Delta(H_1)\}.$$

By a similar argument, it follows that $m_\Delta(H_2) - m_\delta(H_2) = n - 2$. By repeating this
process $n - 2$ times and letting $H = H_n$, we conclude that $m_\Delta(H) = m_\delta(H)$. Thus,

$d_H(u) = d_H(v)$ for all $u, v \in V(F)$. Furthermore, from the construction of H, it follows that $d_H(u, v) \le 2$ for all $u, v \in V(H)$ and $d(z, F) = 1$ for each $z \in V(H) - V(F)$. ❑

The next lemma states that if we are given any graph F and apply Lemma 5.8, then there exists a connected graph H such that the median of H is F; that is, any graph is the median of some connected graph. In addition, the distance between any two vertices of H is at most 2.

Lemma 5.9 Let G be a connected graph and let F be an induced subgraph of G such that

(1) $d_G(u) = d_G(v)$ for all $u, v \in V(F)$,

(2) $d_G(u, v) \le 2$ for all $u, v \in V(G)$, and

(3) $d_G(z, F) = 1$ for each $z \in V(G) - V(F)$.

Then there exists a connected graph H that contains G as an induced subgraph such that $M(H) \cong F$ and $d_H(u, v) \le 2$ for all $u, v \in V(H)$.

Proof Suppose that $d_G(u) = k$ for all $u \in V(F)$, and let $m = k + 2$. We construct a connected graph H from G by joining m new vertices v_i $(1 \le i \le m)$ to F (see Figure 5.10).

Since $d_G(u, v) \le 2$ for all $u, v \in V(F)$, it follows that $d_G(u, v) = d_H(u, v)$. So, for $u \in V(F)$, we have

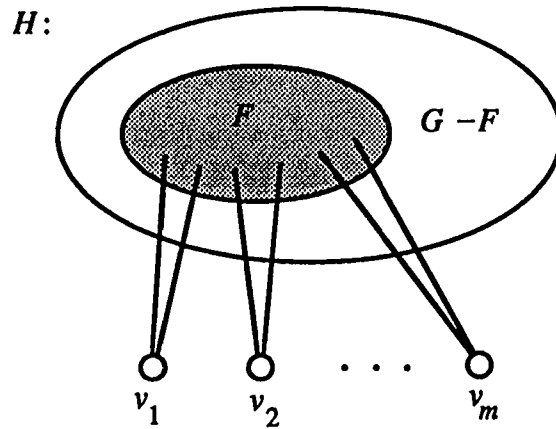$$d_H(u) = \sum_{i=1}^{m} d(u, v_i) + \sum_{x \in V(G)} d_H(u, x) = m + \sum_{x \in V(G)} d_H(u, x)$$

Figure 5.10

$$= m + \sum_{x \in V(G)} d_G(u, v) = m + d_G(u) = m + k = 2k + 2.$$

For $1 \le i \le m$, it follows that

$$d_H(v_i) = \sum_{1 \le j \ne i \le m} d(v_i, v_j) + \sum_{x \in V(F)} d(v_i, x) + \sum_{x \in V(G)-V(F)} d(v_i, x)$$

$$= 2(m-1) + p(F) + \sum_{x \in V(G)-V(F)} d(v_i, x) \ge 2m - 1 = 2k + 3.$$

If $u \in V(G) - V(F)$, then

$$d_H(u) = \sum_{i=1}^{m} d(u, v_i) + \sum_{x \in V(G)} d(u, x) = 2m + \sum_{x \in V(G)} d(u, x)$$

$$\ge 2m + p(G) - 1 \ge 2m + 1 = 2k + 5.$$

Therefore, $M(H) \cong F$. To complete the proof, we must show that $d_H(u, v) \le 2$ for all $u, v \in V(H)$. Since $d_G(x, y) \le 2$ for all $x, y \in V(G)$, it follows that $d_H(x, y) \le 2$. It is clear from the construction of H that for $x \in V(F)$, we have $d(v_i, x) = 1$ $(1 \le i \le$

m) and $d(v_i, v_j) = 2$ $(1 \le i < j \le m)$. Since $d_G(z, F) = 1$ for each $z \in V(G) - V(F)$, we compute $d(v_i, z) = 2$ for $1 \le i \le m$. Thus $d_H(u, v) \le 2$ for all $u, v \in V(H)$. $\Box$

We are now prepared to determine the distance between the median and the periphery of a connected graph. Recall that the distances from the median to the periphery and from the periphery to the median in a strong asymmetric digraph can be arbitrarily prescribed. One might expect a similar result for graphs, but this is not the case.

Theorem 5.10 Let F and G be any two graphs, and let m be a positive integer. In addition, let $\delta = \min\{e(x) \mid x \in V(G)\}$. Then there exists a connected graph H such that $M(H) \cong F$ and $P(H) \cong G$ with $d(M(H), P(H)) = m$ if and only if (1) $\delta \ge 3$ and $m < \delta$ or (2) $\delta = 2$, $m = 1$, and F is a complete graph.

Proof First assume that $\delta \ge 3$ and $m < \delta$. From Lemma 5.8, there exists a connected graph H_1 that contains F as an induced subgraph such that (1) $d_{H_1}(u) = d_{H_1}(v)$ for all $u, v \in V(F)$, (2) $d_{H_1}(u, v) \le 2$ for all $u, v \in V(H_1)$, and (3) $d(u, F) = 1$ for each $u \in V(H_1) - V(F)$. Now by Lemma 5.9, there exists a connected graph H_2 that contains H_1 as an induced subgraph such that $M(H_2) \cong F$ and $d_{H_2}(u, v) \le 2$ for all $u, v \in V(H_2)$. Assume that $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let $r = \left\lfloor \frac{\delta-1}{2} \right\rfloor$ and $t = \left\lfloor \frac{\delta-4}{2} \right\rfloor$. For $\delta \ge 4$, we define a connected graph H_3 by

$$V(H_3) = V(H_2) \cup V(G) \cup \{v_{j,i} \mid 1 \le j \le p, \ 1 \le i \le r\}$$
$$\cup \{w_1, w_2\} \cup \{v_{1,r+i} \mid 1 \le i \le t\}$$

and

$$E(H_3) = E(H_2) \cup E(G) \cup \{v_{j,i}v_{j,i+1} \mid 1 \le j \le p, \ 1 \le i \le r-1\}$$

$$\cup \; \{v_i v_{i,1} \mid 1 \le i \le p\} \cup \{v_{i,r} v_{1,r+1} \mid 1 \le i \le p, \; t \ge 1\}$$

$$\cup \; \{v_{i,r} w_j \mid 1 \le i \le p, \; 1 \le j \le 2, \; t = 0\}$$

$$\cup \; \{v_{1,r+t} w_i \mid 1 \le i \le 2, \; t \ge 1\}$$

$$\cup \; \{v_{1,i} v_{1,i+1} \mid r+1 \le i \le r+t-1\} \cup \{w_i y \mid 1 \le i \le 2, \; y \in V(H_2)\}$$

$$\cup \; \{v_{1,i} y \mid i = m-1, \; y \in V(H_2), \; 2 \le m \le \delta - 2\}$$

$$\cup \; \{v_1 y \mid y \in V(H_2), \; m = 1\} \cup \{v_{i,r} v_{j,r} \mid 1 \le i < j \le p, \; \delta \text{ odd}\}$$

(see Figure 5.11 (a) and (b)). For $\delta = 3$, we define H_3 by

$$V(H_3) = V(H_2) \cup V(G) \cup \{v_{i,1} \mid 1 \le i \le p\}$$

and

$$E(H_3) = E(H_2) \cup E(G) \cup \{v_i v_{i,1} \mid 1 \le i \le p\} \cup \{v_1 y \mid y \in V(H_2), m = 1\}$$

$$\cup \; \{v_{i,1} v_{j,1} \mid 1 \le i < j \le p\} \cup \{v_{i,1} x \mid 1 \le i \le p, \; x \in V(H_2)\}$$
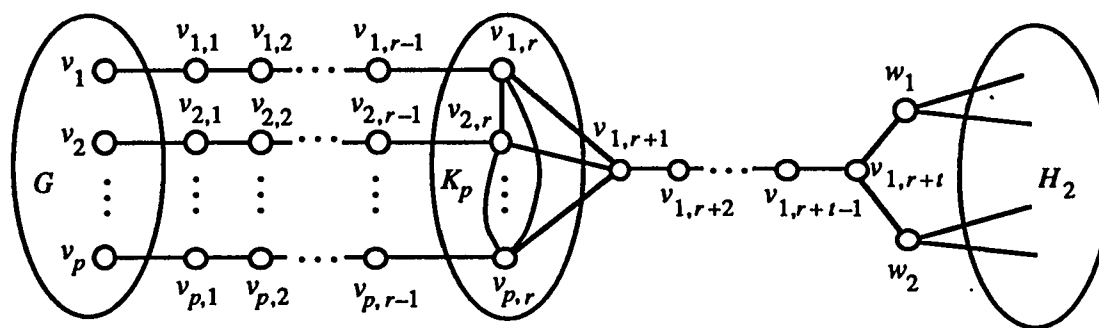
(see Figure 5.11 (c)).

Since $d_{H_2}(x, y) \le 2$ for all $x, y \in V(H_2)$ and w_i $(i = 1, 2)$ is joined to all vertices of H_2, it follows that $d_{H_3}(x, y) = d_{H_2}(x, y)$ and $d_{H_3}(x, z) = d_{H_3}(y, z)$ for all $x, y \in V(H_2)$ and $z \in V(H_3) - V(H_2)$. Thus, for $u, v \in V(F)$, we have
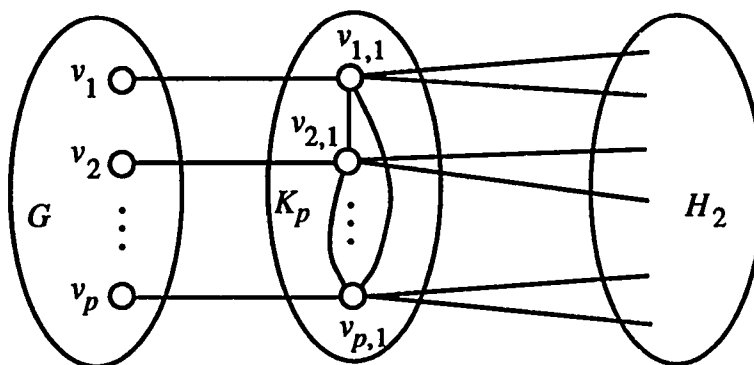
$$d_{H_3}(u) = \sum_{x \in V(H_2)} d_{H_3}(u, x) + \sum_{x \in V(H_3) - V(H_2)} d_{H_3}(u, x)$$

$$= \sum_{x \in V(H_2)} d_{H_2}(u, x) + \sum_{x \in V(H_3) - V(H_2)} d_{H_3}(u, x)$$

$$= d_{H_2}(u) + \sum_{x \in V(H_3) - V(H_2)} d_{H_3}(u, x) = d_{H_2}(v) + \sum_{x \in V(H_3) - V(H_2)} d_{H_3}(u, x)$$
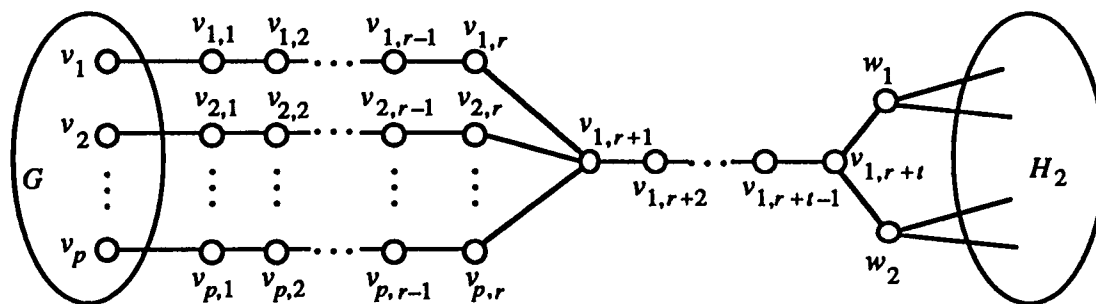
(a) The graph H_3 with $\delta \geq 6$, δ even, and $m = \delta - 1$.

(b) The graph H_3 with $\delta \geq 7$, δ odd, and $m = \delta - 1$.

(c) The graph H_3 with $\delta = 3$ and $m = 2$.

Figure 5.11

$$= d_{H_2}(v) + \sum_{x \in V(H_3)-V(H_2)} d_{H_3}(v, x)$$

$$= \sum_{x \in V(H_2)} d_{H_2}(v, x) + \sum_{x \in V(H_3)-V(H_2)} d_{H_3}(v, x)$$

$$= \sum_{x \in V(H_2)} d_{H_3}(v, x) + \sum_{x \in V(H_3)-V(H_2)} d_{H_3}(v, x) = d_{H_3}(v).$$

In addition, for $u \in V(F)$ and $v \in V(H_2) - V(F)$, it follows that

$$d_{H_3}(u) = d_{H_2}(u) + \sum_{x \in V(H_3)-V(H_2)} d_{H_3}(u, x) = d_{H_2}(u) + \sum_{x \in V(H_3)-V(H_2)} d_{H_3}(v, x)$$

$$< d_{H_2}(v) + \sum_{x \in V(H_3)-V(H_2)} d_{H_3}(v, x) = d_{H_3}(v).$$

Assume that $d_{H_3}(x) = k$ for all $x \in V(F)$. Let $n = 2k$. Now define a connected graph H by

$$V(H) = V(H_3) \cup \{u_i \mid 1 \le i \le 2n\}$$

and

$$E(H) = E(H_3) \cup \{u_i x \mid 1 \le i \le 2n, \ x \in V(H_2)\}$$

$$\cup \{u_{2i-1}w_1, u_{2i}w_2 \mid 1 \le i \le n, \ \delta \ge 4\}$$

$$\cup \{u_{2i-1}v_{1,1}, u_{2i}v_{2,1} \mid 1 \le i \le n, \ \delta = 3\}.$$

From the construction of H, it follows that $d_H(x, y) = d_{H_3}(x, y)$ for all $x, y \in V(H_3)$. For $x \in V(H_2)$, we have

$$d_H(x) = d_{H_3}(x) + \sum_{i=1}^{2n} d(x, u_i) = d_{H_3}(x) + 2n.$$

So, for $x \in V(F)$ and $y \in V(H_2) - V(F)$, we conclude that $d_H(x) = k + 2n = 5k$ and $d_H(y) > 5k$. For $z \in V(H_3) - V(H_2)$, it follows that

$$d_H(z) = \sum_{x \in V(H_3)} d(z, x) + \sum_{i=1}^{2n} d(z, u_i) \geq p(H_3) + 3n > 3n = 6k.$$

For $1 \leq i \leq 2n$, we have

$$d_H(u_i) = \sum_{1 \leq j \neq i \leq 2n} d(u_i, u_j) + \sum_{x \in V(H_3)} d(u_i, x) = 2(2n - 1) + \sum_{x \in V(H_3)} d(u_i, x)$$

$$< 2(2n - 1) + 2 = 4n = 8k.$$

Thus, $M(H) \cong F$.

We now show that $P(H) \cong G$. Observe that for each $v_i \in V(G)$ there exists $v_j \in V(G)$ such that $d_G(v_i, v_j) \geq \delta$. But,

$$d_H(v_i, v_j) = d(v_i, v_{i,r}) + d(v_{i,r}, v_{j,r}) + d(v_{j,r}, v_j) = \delta.$$

Thus, for each $v_i \in V(G)$, we have $e_H(v_i) = \delta$. Also, it follows from the construction of H that $e_H(x) < \delta$ for $x \in V(H) - V(G)$. Therefore, $P(H) \cong G$. Furthermore, it is clear from the construction of H that $d(M(H), P(H)) = m$.

Now assume that $\delta = 2$, $m = 1$, and F is a complete graph. We define a connected graph H by joining all vertices of F to the vertices of G. It is clear from

the construction of H that for $x \in V(G)$ and $y \in V(F)$, we have $e(x) = 2$ and $e(y) = 1$. From this, we conclude that $M(H) \cong F$, $P(H) \cong G$, and $d(M(H), P(H)) = 1$.

For the converse, assume that there exists a connected graph H such that $M(H) \cong F$, $P(H) \cong G$, and $d(M(H), P(H)) = m \geq 1$. Observe that for $x \in V(P(H))$, we have $e_H(x) \leq \delta$. Thus, for each $x \in V(P(H))$ and $y \in V(H) - V(P(H))$, it follows that $d(x, y) < \delta$, namely, $d(P(H), M(H)) = m < \delta$. From this, it is clear that $\delta \neq 1$. If $\delta = 2$, then $e_H(x) < \delta$ for each $x \in V(H) - V(P(H))$. Thus, F is a complete graph. $\square$

5.4 Connected Graphs With Intersecting Median and Periphery

Let F_1, F_2, and K be graphs with K isomorphic to an induced subgraph of both F_1 and F_2. We define the *supergraph set* $S(F_1, F_2; K)$ by

$$S(F_1, F_2; K) = \{G \mid H_1 \text{ and } H_2 \text{ are induced subgraphs of } G, H_1 \cong F_1 \text{ and } H_2 \cong F_2,$$
$$\langle V(H_1) \cap V(H_2) \rangle \cong K, \text{ and } V(G) = V(H_1) \cup V(H_2)\}.$$

In Figure 5.12, we are given graphs F_1, F_2, and K along with G_1, G_2, and G_3, which are the three possible ways of overlapping F_1 and F_2 with intersection K; that is, $S(F_1, F_2; K) = \{G_1, G_2, G_3\}$. We now give necessary and sufficient conditions for the median and periphery of a graph to intersect.

Theorem 5.11 Let F_1, F_2, and K be graphs where K is isomorphic to an induced subgraph of both F_1 and F_2. Then there exists a connected graph H such that $M(H) \cong F_1$, $P(H) \cong F_2$, and $M(H) \cap P(H) \cong K$ if and only if

(1) there is some graph $G \in S(F_1, F_2; K)$ such that for each $x \in V(F_2)$, there exists $y \in V(F_2)$ with $d_G(x, y) \geq 3$,
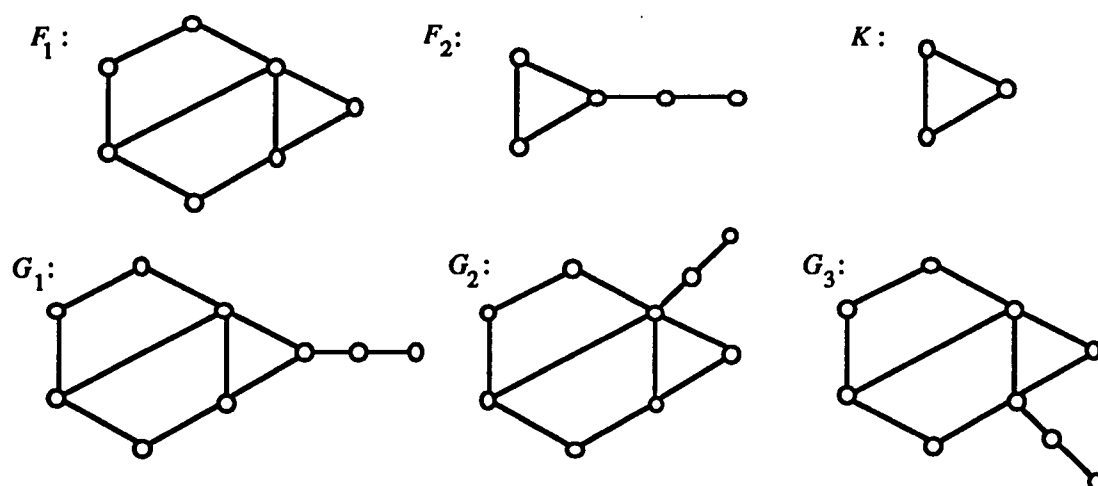
Figure 5.12

(2) $e(x) = 2$ for each $x \in V(F_2)$ and $M(F_2) \cong F_1 \cong K$, or

(3) $e(x) = 1$ for each $x \in V(F_2)$ and $F_2 \cong F_1 \cong K$.

Proof Assume that condition (1) holds and $V(F_2) = \{u_1, u_2, \ldots, u_r\}$. We show that there exists a connected graph H such that $M(H) \cong F_1$, $P(H) \cong F_2$, and $M(H) \cap P(H) \cong K$. We start by constructing sets B_1 and B_2 by Algorithm Partition Vertices with the property that $B_1 \cup B_2 = V(F_1)$, $B_1 \cap B_2 = \varnothing$, and for each $x \in B_i \cap K$, there exists $y \in V(F_2) - B_i$ such that $d_G(x, y) \geq 3$ for $1 \leq i \leq 2$ (see Figure 5.13).

We define a connected graph H_0 by

$$V(H_0) = V(G) \cup \{u_i' \mid 1 \leq i \leq 4r\}$$

and

$$E(H_0) = E(G) \cup \{u_i' u_j' \mid 1 \leq i < j \leq 4r\}$$

$$\cup \{u_i' x \mid 1 \leq i \leq 4r, x \in V(F_1) - V(K)\}$$

Algorithm Partition Vertices:

If $F_1 \cong K$, then

Let $B_1 = \{v_1\}$, and let $B_2 = \varnothing$.

If $F_1 \not\cong K$, then

If $v_1 \in V(K)$, then

Let $B_1 = \varnothing$, and let $B_2 = \{v_1\}$.

If $v_1 \in V(F_1) - V(K)$, then

Let $B_1 = \{v_1\}$, and let $B_2 = \varnothing$.

For $i = 2$ to p,

If $v_i \in V(F_1) - V(K)$, then

Let $B_1 \leftarrow B_1 \cup \{v_i\}$.

If $v_i \in V(K)$, then

If there exists some $x \in V(F_2) - V(K)$ such that $d_{H_0}(v_i, x) \geq 3$, then

Let $B_2 \leftarrow B_2 \cup \{v_i\}$.

If $d_{H_0}(v_i, x) \leq 2$ for all $x \in V(F_2) - V(K)$, then

If $d_{H_0}(v_i, x) \leq 2$ for all $x \in B_2 \cap V(K)$, then

Let $B_2 \leftarrow B_2 \cup \{v_i\}$.

If $d_{H_0}(v_i, x) \geq 3$ for some $x \in B_2 \cap V(K)$, then

Let $B_1 \leftarrow B_1 \cup \{v_i\}$.

Figure 5.13

$$\cup \{u_i\, u'_{4i-j} \mid 1 \leq i \leq r,\ 0 \leq j \leq 3\}$$

$$\cup \{u_i\, u'_{4j} \mid d_G(u_i, u_j) \geq 3, i > j, u_i, u_j \in B_k, k = 1, 2\}$$
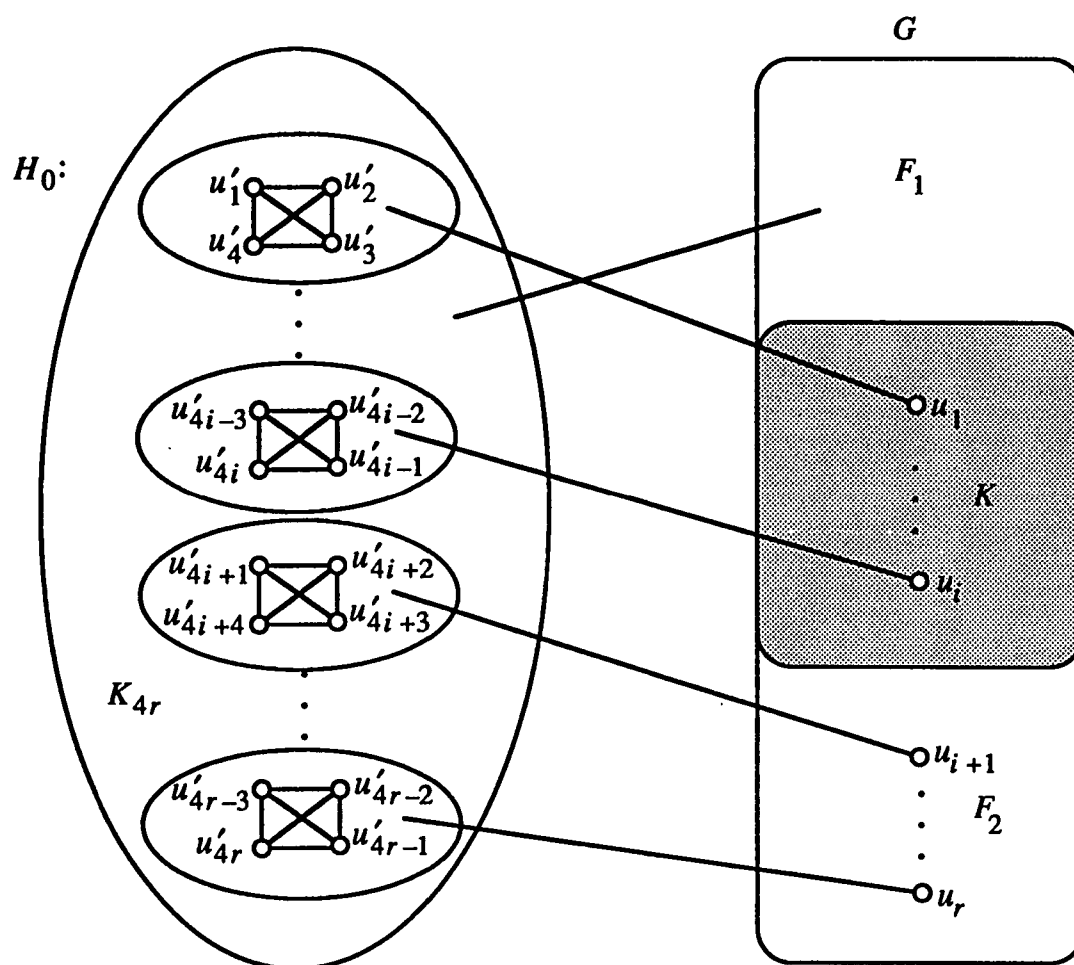
(see Figure 5.14).

Figure 5.14

Observe that $p(F_2) \geq 2$ and, thus, $p(H_0) \geq 8$. We consider two cases for the order of F_1.

Case 1 Assume that $p(F_1) = p \geq 2$. In addition, suppose that $V(F_1) = \{v_1, v_2, \dots, v_p\}$ and $d_{H_0}(v_i) = n_i$ for $1 \leq i \leq p$. Let $n = \min\{n_i \mid 1 \leq i \leq p\}$, and let A_i be a set of $n_i - n + 1$ new vertices, where $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq p$. Let $t = \sum_{i=1}^{p} |A_i|$.

We define a connected graph H_1 by

$$V(H_1) = V(H_0) \cup \left(\bigcup_{i=1}^{p} A_i \right)$$

and

$$E(H_1) = E(H_0) \cup \{v_i x \mid 1 \le i \le p, x \in A_i\}$$

$$\cup \{xy \mid x \in A_i, \; y \in V(H_1) - V(G), \; 1 \le i \le p\}$$
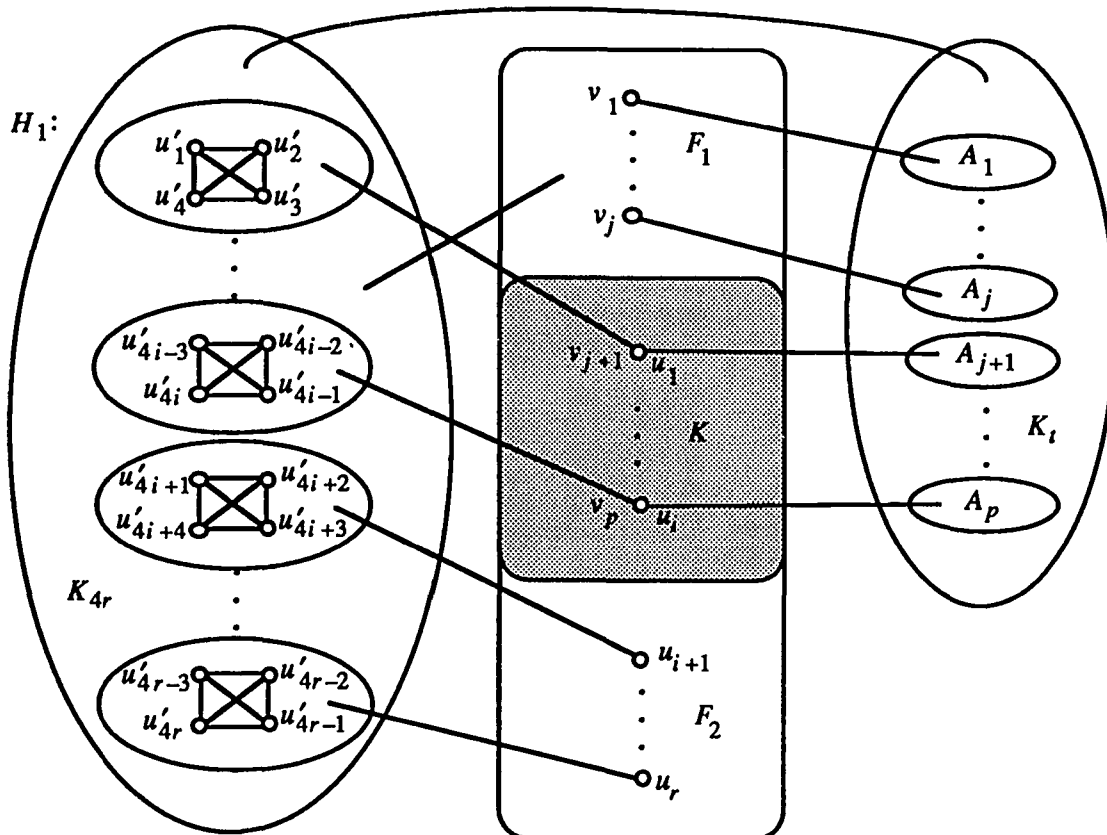
(see Figure 5.15).

Figure 5.15

For $1 \le i < k \le p$, we have

$$d_{H_1}(v_i) = \sum_{x \in V(H_0)} d_{H_1}(v_i, x) + \sum_{x \in V(H_1)-V(H_0)} d(v_i, x)$$

$$= \sum_{x \in V(H_0)} d_{H_0}(v_i, x) + \sum_{x \in V(H_1)-V(H_0)} d(v_i, x)$$

$$= d_{H_0}(v_i) + \sum_{x \in V(H_1)-V(H_0)} d(v_i, x) = n_i + \sum_{x \in V(H_1)-V(H_0)} d(v_i, x)$$

$$= n_i + \sum_{x \in A_i} d(v_i, x) + \sum_{\substack{x \in A_j \\ j \neq i}} d(v_i, x)$$

$$= n_i + (n_i - n + 1) + 2\sum_{j \neq i} (n_j - n + 1)$$

$$= n - 1 + 2(n_i - n + 1) + 2\sum_{j \neq i} (n_j - n + 1)$$

$$= n - 1 + 2\sum_{j=1}^{p} (n_j - n + 1)$$

$$= n - 1 + 2(n_k - n + 1) + 2\sum_{j \neq k} (n_j - n + 1)$$

$$= n_k + (n_k - n + 1) + 2\sum_{j \neq k} (n_j - n + 1)$$

$$= n_k + \sum_{x \in A_k} d(v_k, x) + \sum_{\substack{x \in A_j \\ j \neq k}} d(v_k, x)$$

$$= n_k + \sum_{x \in V(H_1)-V(H_0)} d(v_k, x) = d_{H_0}(v_k) + \sum_{x \in V(H_1)-V(H_0)} d(v_k, x)$$

$$= \sum_{x \in V(H_0)} d_{H_0}(v_k, x) + \sum_{x \in V(H_1)-V(H_0)} d(v_k, x)$$

$$= \sum_{x \in V(H_0)} d_{H_1}(v_k, x) + \sum_{x \in V(H_1)-V(H_0)} d(v_k, x) = d_{H_1}(v_k).$$

Suppose that $d_{H_1}(v_i) = m$ for $1 \leq i \leq p$. We now define a connected graph H

by

$$V(H) = V(H_1) \cup \{w_i, x_i, y_i, z_i \mid 1 \leq i \leq m\}$$

and

$$E(H) = E(H_1) \cup \{w_i x_i, y_i z_i, w_i z_i, x_i y_i \mid 1 \leq i, j \leq m\}$$

$$\cup \{xw_i, xz_i \mid x \in B_1, 1 \leq i \leq m\} \cup \{xx_i, xy_i \mid x \in B_2, 1 \leq i \leq m\}$$

$$\cup \{u'_{4i-3} w_j, u'_{4i-2} x_j, u'_{4i-1} y_j, u'_{4i} z_j \mid 1 \leq i \leq r, 1 \leq j \leq m\}$$

(see Figure 5.16).

For $1 \leq i \leq p$, we compute

$$d_H(v_i) = \sum_{x \in V(H_1)} d_H(v_i, x) + \sum_{j=1}^{m} [d(v_i, w_j) + d(v_i, x_j) + d(v_i, y_j) + d(v_i, z_j)]$$

$$= \sum_{x \in V(H_1)} d_H(v_i, x) + 6m = \sum_{x \in V(H_1)} d_{H_1}(v_i, x) + 6m$$

$$= d_{H_1}(v_i) + 6m = m + 6m = 7m.$$

For $y \in V(H_1) - V(F_1)$, we have

$$d_H(y) = \sum_{x \in V(H_1)} d(y, x) + \sum_{j=1}^{m} [d(y, w_j) + d(y, x_j) + d(y, y_j) + d(y, z_j)]$$

$$\geq \sum_{x \in V(H_1)} d(y, x) + 7m \geq p(H_1) - 1 + 7m > 7m,$$

where $p(H_1) > p(H_0) \geq 8$. If $1 \leq j \leq m$, then

$$d(w_j) = \sum_{1 \leq i \neq j \leq m} [d(w_j, w_i) + d(w_j, x_i) + d(w_j, y_i) + d(w_j, z_i)] + d(w_j, x_j)$$
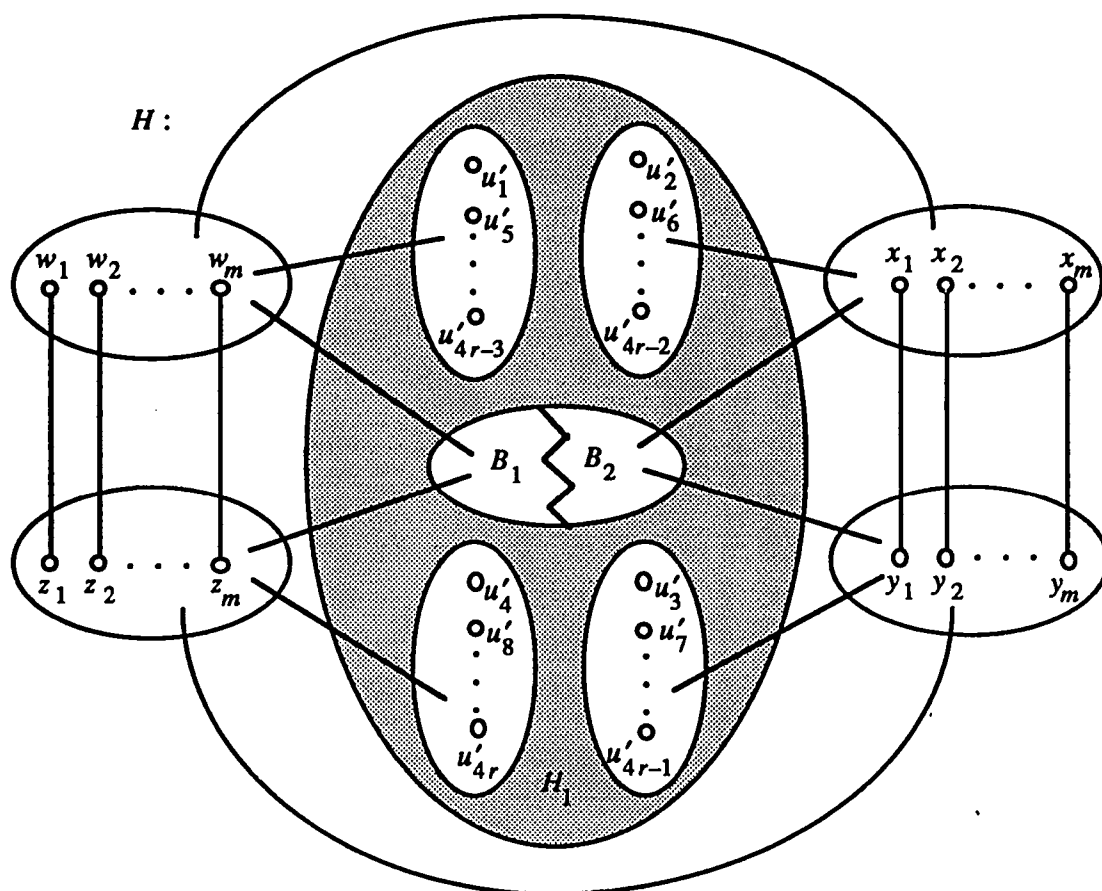
Figure 5.16

$$+ d(w_j, y_j) + d(w_j, z_j) + \sum_{x \in V(H_1)} d(w_j, x)$$

$$= 7(m-1) + 1 + 2 + 1 + \sum_{x \in V(H_1)} d(w_j, x)$$

$$\geq 7m - 3 + p(H_1) > 7m.$$

By a similar calculation, it follows that $d(x_j) > 7m$, $d(y_j) > 7m$, and $d(z_j) > 7m$ for $1 \leq j \leq m$. Thus, $M(H) \cong F_1$. The digraph H was constructed in such a way that

$e_H(x) = 3$ and $e_H(y) = 2$ for $x \in V(F_2)$ and $y \in V(H) - V(F_2)$. Therefore, $P(H) \cong F_2$.

Case 2   *Assume that* $p(F_1) = 1$. Suppose that $d_{H_0}(v_1) = m$. We define a connected graph H by

$$V(H) = V(H_0) \cup \{y_i, z_i \mid 1 \le i \le m\}$$

and

$$E(H) = E(H_0) \cup \{v_1 y_i, v_1 z_i \mid 1 \le i \le m\}$$

$$\cup \{\{u'_{4i-3}y_j, u'_{4i-2}z_j, u'_{4i-1}z_j, u'_{4i}y_j \mid 1 \le i \le r, 1 \le j \le m\}$$

(see Figure 5.17).

By calculations similar to those in Case 1, we find that

$$d_H(v_1) = 3m$$

and

$$d_H(x) > 3m \text{ for } x \in V(H) - \{v_1\}.$$

Thus, $M(H) \cong F_1$. Again, by the construction of H, we have $e_H(x) = 3$ and $e_H(y) = 2$ for $x \in V(F_2)$ and $y \in V(H) - V(F_2)$; that is, $P(H) \cong F_2$.

If conditions (2) or (3) hold, then it is clear that the graph $H \cong F_2$ has the desired properties.

We now prove the converse. Suppose that there exists a connected graph H such that $M(H) \cong F_1$, $P(H) \cong F_2$, and $M(H) \cap P(H) \cong K$. If there exists $x \in V(F_2)$ with $e_{F_2}(x) = 1$, then all vertices in $P(H)$ have eccentricity 1; that is, $P(H) \cong H \cong F_2$.
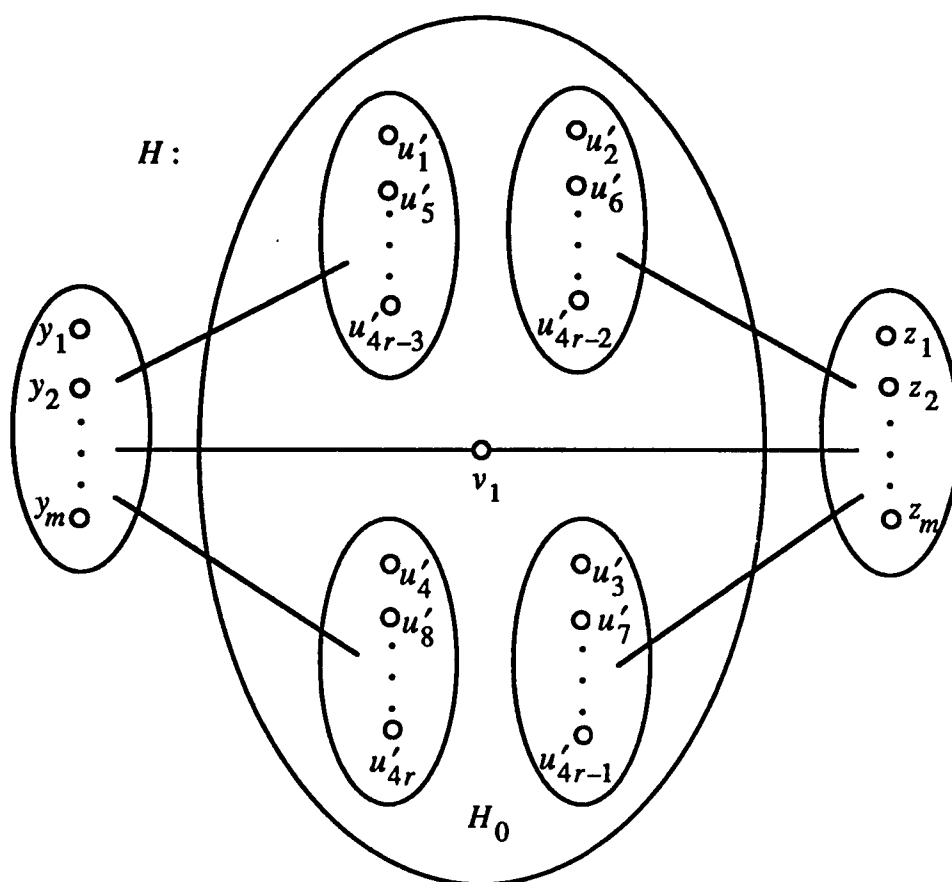
Figure 5.17

In addition, each vertex of H has the same distance. Thus, $M(H) \cong H \cong F_1$ and statement (3) holds.

In the next case, we assume that F_2 has no vertices of eccentricity 1. Furthermore, suppose that for each $G \in S(F_1, F_2; K)$, there exists $x \in V(F_2)$ such that for each $y \in V(F_2)$, we have $d_G(x, y) \leq 2$. Since

$$\text{diam } H \leq \max\{d_G(x, y) \mid x, y \in V(F_2), G \in S(F_1, F_2; K)\},$$

it follows that $e(x) = 2$ for all $x \in V(P(H))$. In addition, for each $y \in V(H) - V(P(H))$, we have $e(y) = 1$ and $y \in V(M(H))$. So all vertices of $M(H)$ have

eccentricity 1 or no vertices of $M(H)$ have eccentricity 1. Since K is an induced subgraph of F_2, it has no vertices of eccentricity 1, which implies that all vertices of $M(H)$ have eccentricity 2. Thus, H has no vertices of eccentricity 1 and $P(H) \cong H \cong G \cong F_2$. Consequently, $F_1 \cong K$ and $M(F_2) \cong M(H) \cong F_1$, which completes the proof. $\square$

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