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Maximal and Maximum Independent Sets in Graphs

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MAXIMAL AND MAXIMUM INDEPENDENT SETS IN GRAPHS

by

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MAXIMAL AND MAXIMUM INDEPENDENT SETS IN GRAPHS

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A maximal independent set of a graph G is an independent set which is not contained properly in any other independent set of G. An independent set is called *maximum* if it is of largest cardinality. Denote $i(G)$ to be the number of maximal independent sets of G. These special sets and the parameter $i(G)$ have interested many researchers leading to a number of properties and results. One of these is the determination of the maximum number of maximal independent sets among all graphs of order n, and the extremal graphs. In this investigation,we develop new properties for the number of maximal independent sets $i(G)$ and the number of maximum independent sets $i_m(G)$, as well as determine the largest number of maximal and maximum independent sets possible in a k-connected graph of order n(with n large) and characterize the respective extremal graphs. Finally, we determine the corresponding values for bipartite graphs and connected bipartite graphs, and characterize the extremal graphs.

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To my loving wife Cong
and my son Andrew

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INTRODUCTION

1.1 Definitions and Notations

Let G be a graph. A subset $V' \subseteq V(G)$ is *independent* if no two vertices in V' are adjacent. An independent set V' is *maximal* if $V' \cup \{v\}$ is not independent for any $v \in V(G) - V'$. A *maximum independent set* is an independent set of largest cardinality. The graph G is *well-covered* if every maximal independent set is also maximum (see [12]). A *clique* of G is a complete subgraph which is not contained properly in any other complete subgraph.

Now, for a graph G, let $I(G)$ be the set of maximal independent sets of G and $i(G) = |I(G)|$. Set

$$f(n) = \max\{i(G) : \ G \text{ is a graph of order } n\},$$

$$b(n) = \max\{i(G) : \ G \text{ is a bipartite graph of order } n\},$$

$$f_k(n) = \max\{i(G) : \ G \text{ is a } k\text{-connected graph of order } n\}, \text{ and}$$

$$b_k(n) = \max\{i(G) : \ G \text{ is a } k\text{-connected bipartite graph of order } n\}.$$

Similarly, let $I_m(G)$ be the set of maximum independent sets of G and

$i_m(G) = |I_m(G)|$. Set

$$F(n) = \max\{i_m(G) : \ G \text{ is a graph of order } n\},$$

$$B(n) = \max\{i_m(G) : \ G \text{ is a bipartite graph of order } n\},$$

$$F_k(n) = \max\{i_m(G) : \ G \text{ is a } k\text{-connected graph of order } n\}, \quad \text{and}$$

$$B_k(n) = \max\{i_m(G) : \ G \text{ is a } k\text{-connected bipartite graph of order } n\}.$$

For convenience, we use MIS to represent maximal independent set and kG to denote the disjoint union of k copies of a graph G. Define the graph H_n with n vertices as follows:

$$H_n = \begin{cases} tK_3 & \text{if } n = 3t \\ (t-1)K_3 \cup K_4 \text{ or } (t-1)K_3 \cup 2K_2 & \text{if } n = 3t+1 \\ tK_3 \cup K_2 & \text{if } n = 3t+2. \end{cases}$$

1.2 History and Known Results

In 1960, Miller and Muller [10] solved the problem of finding a family of maximum cardinality which consists of subsets satisfying certain properties. They showed that their subset problem is equivalent to finding the graphs on n vertices which have the greatest number of cliques. Notice that C is a clique of a graph G if and only if $V(C)$ is a MIS of the complement $\overline{G}$ of G. Consequently, the authors solved their original problem by finding the extremal graphs on n vertices which have the maximum number $f(n)$ of MIS's. Independently, in 1965, Moon and Moser [11] obtained the same result in answering the following questions raised by Erdös and Moser: What is the maximum number $c(n)$ of cliques possible in a graph with n vertices and which graphs have so many cliques? Hence they obtained the following two equivalent propositions.

Proposition 1.1. *If* $n \geq 2$, *then*

$$f(n) = \begin{cases} 3^t & \text{if } n = 3t \\ 4 \cdot 3^{t-1} & \text{if } n = 3t + 1 \\ 2 \cdot 3^t & \text{if } n = 3t + 2, \end{cases}$$

and the only graphs of order n *which attain* $f(n)$ *maximal independent sets are the graphs* H_n.

Proposition 1.2. *For* $n \geq 2$, *the maximum number of cliques among all graphs of order* n *is*

$$c(n) = f(n)$$

and the only graphs of order n *with* $c(n)$ *cliques are* $\overline{H}_n$.

Proposition 1.1 has an immediate corollary.

Corollary 1.1. *For a graph* G *of order* n, $i(G) \leq 3^{n/3}$.

Since these results have appeared, many other problems have been formulated. For example, given a class Ψ, one may ask what is the maximum number of cliques (or MIS's) possible in a graph of order n in Ψ and which graphs in Ψ have so many cliques (or MIS's).

For cliques, Hedman [4] found the maximum number in dense graphs. Fan and Liu [1] determined the maximum number of cliques possible in a k-connected graph of order n for each $k \geq 1$.

For MIS's, Wilf [14] found the maximum number among all trees on n vertices (with Sagan [13] later providing a short graph theoretical proof).

Proposition 1.3. *The maximum number of MIS's among all trees on n vertices is*

$$T(n) = \begin{cases} 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 2^{\frac{n-2}{2}} + 1 & \text{if } n \text{ is even} \end{cases}$$

and the only trees of order n which have so many MIS's are T_n shown in Figure 1.1.

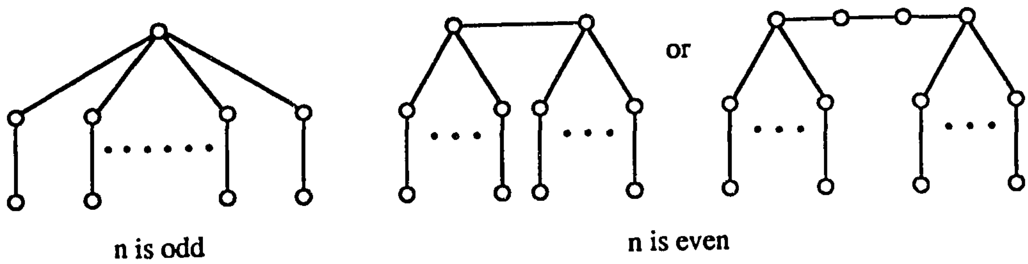


Figure 1.1. The Trees T_n.

In 1987, Fűredi [2] derived the following result.

Proposition 1.4. *Suppose* G *is a graph of order* n*. If* $G \ncong H_n$*, then* $i(G) \leq g(n)$*, where*

$$g(n) = \begin{cases} f(n-1) & \textit{if } n \leq 5 \\ 8 \cdot 3^{t-2} & \textit{if } n = 3t \geq 6 \\ 11 \cdot 3^{t-2} & \textit{if } n = 3t+1 \geq 6 \\ 16 \cdot 3^{t-2} & \textit{if } n = 3t+2 \geq 6\ . \end{cases}$$

In the same paper Fűredi also determined the maximum number $f_1(n)$ of MIS's possible in a connected graph of order n for $n > 50$. Independently, Griggs, Grinstead, and Guichard [3] found $f_1(n)$ for all n, and characterized the connected graphs of order n which have $f_1(n)$ MIS's.

Proposition 1.5. *For* $n \leq 5$, $f_1(n) = n$*; for* $n \geq 6$

$$f_1(n) = \begin{cases} 2 \cdot 3^{t-1} + 2^{t-1} & \textit{if } n = 3t \\ 3^t + 2^{t-1} & \textit{if } n = 3t+1 \\ 4 \cdot 3^{t-1} + 3 \cdot 2^{t-2} & \textit{if } n = 3t+2, \end{cases}$$

and the only connected graph of order $n \geq 6$ *which has* $f_1(n)$ *maximal independent sets, denoted by* $H_{1,n}$*, is shown in Figure 1.2.*

Here, we study some properties for the number, $i(G)$, of MIS's of a graph G. Then we determine the maximum number $f_k(n)$ of MIS's possible in a k-connected graphs of order n and the extremal graphs for large n.

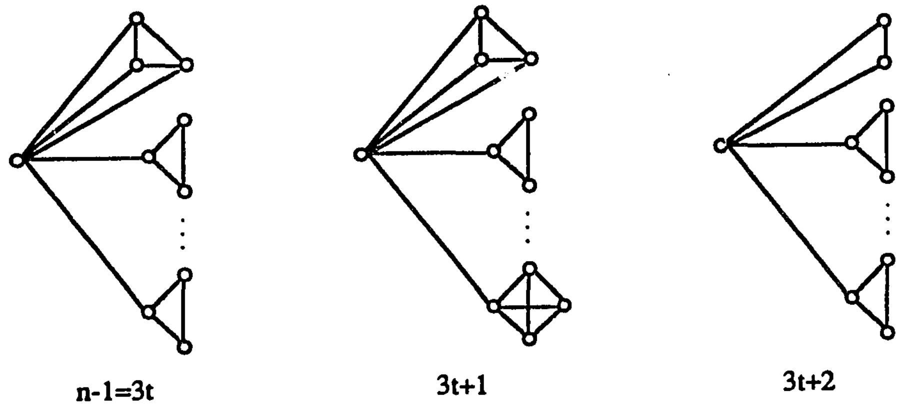


Figure 1.2. The Graph $H_{1,n}$.

We also derive the maximum number of maximum independent sets among all k-connected graphs of order n as well as the extremal graphs. Finally, we develop the corresponding results for bipartite and connected bipartite graphs, respectively.

The maximum number of MIS's among all graphs in a certain class is useful in calculating the complexity of algorithms which cycle through all MIS's. For example, Lawler [7] gave an algorithm for determining the chromatic number of a graph which runs through all MIS's, and it is shown that the running time of that algorithm is $O[nq(1+3^{\frac{1}{3}})^n]$ for graphs of n vertices and q edges. The appearance of $3^{\frac{n}{3}}$ derives from Proposition 1.1.

CHAPTER II

CONSTRAINTS FOR INDEPENDENT SETS

2.1 Constraints for The Number of Maximal Independent Sets of a Graph

Given two functions $f(x)$ and $g(x)$, if $f(x)/g(x) \longrightarrow 0$ as $x \longrightarrow \infty$, then we say $f(x) = o(g(x))$, and if there exist positive constants a, b, c such that $a \leq |\frac{f(x)}{g(x)}| \leq b$ for $|x| \geq c$, then we say $f(x) = O(g(x))$.

Given a family $\Phi = \{G : G$ is a graph with property $P\}$, we define the function $\Delta_\Phi : Z^+ \longrightarrow Z_0^+$ by

$$\Delta_\Phi(n) = \max\{\Delta(G) : G \in \Phi \text{ and } |V(G)| = n\}.$$

Similarly, we define

$$i_\Phi(n) = \max\{i(G) : G \in \Phi \text{ and } |V(G)| = n\},$$

$$i'_\Phi(n) = \max\{i_m(G) : G \in \Phi \text{ and } |V(G)| = n\}, \text{ and}$$

$$\ell_\Phi(n) = \min\{\ell(G) : G \in \Phi \text{ and } |V(G)| = n\},$$

where $\ell(G)$ denotes the length of a longest path of G.

From Proposition 1.5 we see that the maximum number $f_1(n)$ of MIS's possible in a connected graph G of order n satisfies $i(H_{1,n}) = f_1(n) =$

$O(3^{n/3})$ and the extremal graph $H_{1,n}$ shown in Figure 1.2 has maximum degree $\Delta(H_{1,n}) = O(n)$. In fact, this is a necessary condition as the next result indicates.

Theorem 2.1. *For a family* $\Phi = \{G : G \text{ is connected}\}$, *if* $\Delta_\Phi(n) = o(n)$, *then* $i_\Phi(n) = o(3^{n/3})$.

To prove this property, we need the inequality in the proof of Theorem 2.1 in [2], for convenience, we define the number of MIS's for the empty graph as $i(\phi) = 1$.

Lemma 2.1. *For any graph* G *and any* $v \in V(G)$,

$$i(G) \leq i(G - v) + i(G - v - N(v)).$$

The next lemma generalizes Lemma 3.3 in [2], which states:

Proposition 2.1. *Let* G *be a connected graph with* n *vertices and* $\Delta(G) \leq 6$. *Then* $i(G) \leq 3^{\frac{n}{3}} 1.009^{-n+3}$.

Lemma 2.2. *For a given constant* $C \in Z^+$, *there exists a real number* $t > 0$ *such that if* G *is a connected graph of order* $n \geq 3$ *with maximum degree* $\Delta(G) \leq C$, *then* $i(G) \leq 3^{n/3} \cdot (1+t)^{-n+3}$.

Proof. Given a constant $C \in Z^+$, let t be a real number such that

$$0 < t < \min\{0.009, [3^{1/3} \cdot 5/7]^{\frac{1}{3C-2}} - 1\}.$$

We now prove $i(G) \le 3^{n/3} \cdot (1+t)^{-n+3}$ by induction on n. For $n \le 3$, it is easy to check the result. Assume the result for any connected graph of order less than n (with $n \ge 4$) which has maximum degree at most C. Let G be a connected graph of order n with $\Delta(G) \le C$. If $\Delta(G) \le 3$, then it follows from Proposition 2.1 that

$$i(G) \le 3^{n/3}(1.009)^{-n+3} \le 3^{n/3}(1+t)^{-n+3}.$$

So we assume $\Delta = \Delta(G) \ge 4$. Let $v \in V(G)$ such that $\deg(v) = \Delta$. By Lemma 2.1, we have

$$i(G) \le i(G-v) + i(G-v-N(v)).$$

Suppose $G_1, G_2, \cdots, G_k$ are the components of $G-v$ and $p_i = |V(G_i)|$ for $1 \le i \le k$, and let $H_1, H_2, \cdots, H_r$ be the components of $G-v-N(v)$ with $h_j = |V(H_j)|$ for each j. Then it follows from the induction hypothesis that

$$\begin{aligned} i(G) &\le \prod_{j=1}^{k} i(G_j) + \prod_{j=1}^{r} i(H_j) \\ &\le \prod_{j=1}^{k} 3^{p_j/3}(1+t)^{-p_j+3} + \prod_{j=1}^{r} 3^{h_j/3}(1+t)^{-h_j+3} \\ &= 3^{(n-1)/3}(1+t)^{-(n-1)+3k} + 3^{(n-1-\Delta)/3}(1+t)^{-(n-1-\Delta)+3r}. \end{aligned}$$

Since $k \leq \Delta$ and $r \leq \Delta(\Delta-1)$, we have

$$i(G) \leq 3^{(n-1)/3}(1+t)^{-(n-1)+3\Delta} + 3^{(n-1-\Delta)/3}(1+t)^{-(n-1-\Delta)+3\Delta(\Delta-1)}$$
$$\leq 3^{n/3}(1+t)^{-n+3} \cdot [(1+t)^{3\Delta-2} \cdot 3^{-1/3} + (1+t)^{3\Delta^2-2\Delta} \cdot 3^{-(\Delta+1)/3}].$$

From the choice of t and the assumption $4 \leq \Delta \leq C$ it follows that

$$(1+t)^{3\Delta-2} \cdot 3^{-1/3} \leq (1+t)^{3C-2} \cdot 3^{-1/3} < 5/7.$$

Now raise the inequality to the Δ power and multiply by 3 to get

$$(1+t)^{(3\Delta-2)\cdot\Delta} \cdot 3^{-(\Delta-3)/3} < 3 \cdot (5/7)^{\Delta} < 1.$$

Thus

$$i(G) \leq 3^{n/3}(1+t)^{-n+3} \cdot (5/7 + 3^{-4/3}) < 3^{n/3} \cdot (1+t)^{-n+3}$$

as required. This completes the proof of the lemma. □

Lemma 2.3. *There exist a positive integer M and another positive integer C dependent on M such that if G is a connected graph of order n with maximum degree $\Delta \geq C$, then $i(G) \leq 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3}$.*

Proof. We first choose a positive integer M such that $a = e^{3/M} \cdot 3^{-1/3} < 1$, then choose a positive integer C such that $a + a^C \cdot 3^{-1/3} \leq 1$, $\frac{3^{\frac{C}{3}}}{CM} \geq 1$, and $\frac{1}{CM} < A = \min\{0.009, (3^{1/3} \cdot 5/7)^{\frac{1}{3C-2}} - 1\}$. Now we use induction to prove $i(G) \leq 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3}$ for any connected graph G of order n with maximum degree $\Delta \geq C$. For $n = C + 1$, any graph G of order $C + 1$ with maximum degree C must satisfy $i(G) \leq i(G - v) + 1$, where $\deg(v) = \Delta(G) = C$. From Corollary 1.1 we obtain $i(G) \leq 3^{(n-1)/3} + 1$. Since $(1 + 1/x)^x \leq e$ for any integer $x > 0$ and $e^{1/M} < 3^{1/3}$, it follows that

$$3^{1/3} \cdot (1 + \frac{1}{CM})^{-C} = 3^{1/3} \cdot (1 + \frac{1}{CM})^{-CM\cdot(1/M)}$$
$$\geq 3^{1/3} \cdot e^{-1/M} > 1,$$

which implies

$$i(G) \leq 3^{\frac{C}{3}} + 1 \leq 3^{\frac{C}{3}}(1 + \frac{1}{CM}) \leq 3^{\frac{C+1}{3}} \cdot (1 + \frac{1}{CM})^{-(C+1)+3}$$

as required. Assume the result for any connected graph of order less than n (with $n \geq C + 2$) which has maximum degree at least C. Now suppose G is a connected graph of order n such that $\Delta = \Delta(G) \geq C$. Note that from Lemma 2.2, for any t such that $\frac{1}{CM} < t < A$, if R is a connected graph of order $r \geq 3$ with maximum degree at most C, then

$$i(R) \leq 3^{r/3} \cdot (1 + t)^{-r+3} \leq 3^{r/3} \cdot (1 + \frac{1}{CM})^{-r+3} \leq 3^{r/3} \cdot (1 + \frac{1}{M\Delta})^{-r+3}.$$

Also note that if a connected graph R has order $r \leq 2$, then $i(R) \leq 3^{r/3} \leq 3^{r/3} \cdot (1 + \frac{1}{M\Delta})^{-r+3}$. Together with the induction hypothesis, similar to the proof of Lemma 2.2, we obtain the inequality

$$\begin{aligned}
i(G) &\le 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3} \\
&\quad \cdot [(1 + \frac{1}{M\Delta})^{3\Delta-2} \cdot 3^{-1/3} + (1 + \frac{1}{M\Delta})^{(3\Delta-2)\Delta} \cdot 3^{-(\Delta+1)/3}] \\
&\le 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3} \\
&\quad \cdot [(1 + \frac{1}{M\Delta})^{3\Delta} \cdot 3^{-1/3} + (1 + \frac{1}{M\Delta})^{3\Delta^2} \cdot 3^{-(\Delta+1)/3}] \\
&\le 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3} \cdot [e^{3/M} \cdot 3^{-1/3} + (e^{3/M} \cdot 3^{-1/3})^{\Delta} \cdot 3^{-1/3}] \\
&\le 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3} \cdot (a + a^C \cdot 3^{-1/3}) \le 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3}
\end{aligned}$$

as required. Therefore the Lemma follows. □

Proof of Theorem 2.1. Let G be a connected graph of order n in Φ such that $i(G) = i_\Phi(n)$. By Lemma 2.3, there exist two positive integers C and M such that if $\Delta = \Delta(G) \ge C$, then

$$i(G) \le 3^{n/3} \cdot (1 + \frac{1}{M\Delta})^{-n+3}$$

which implies $i(G) = o(3^{n/3})$ since $\Delta(G) \le \Delta_\Phi(n) = o(n)$ and $(1 + \frac{1}{M\Delta})^{-M\Delta} \le (1 + \frac{1}{MC})^{-MC}$. But if $\Delta(G) \le C$, then Lemma 2.2 implies $i(G) = o(3^{n/3})$. Therefore the theorem holds. □

The following example shows that Theorem 2.1 is best possible.

Example 2.1. *For any constant integer* $k > 0$, *let* $n = (3n_1 + 1)k$ *and let* G *be the connected graph of order* n *shown in Figure 2.1. Then it is* *easy to see that* G *has maximum degree* $n_1 + 2 = (n-k)/(3k) + 2$ *and* $i(G) = O(3^{n/3})$.

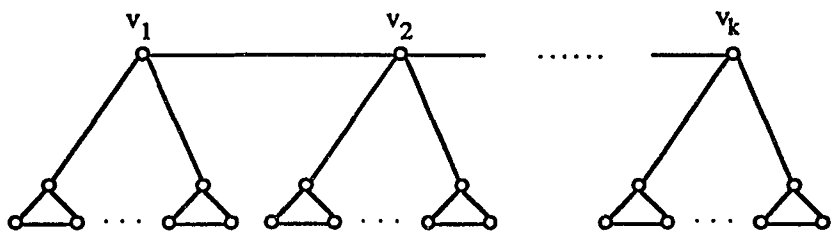


Figure 2.1. A Graph G of Order n With $i(G) = O(3^{\frac{n}{3}})$.

Notice that the extremal graph H_n for $f(n)$ has no path of length greater than 3 and the extremal graph $H_{1,n}$ for $f_1(n)$ shown in Figure 1.2 has no path of length greater than 7. This led Erdős to suggest the following property.

Theorem 2.2. *If* Φ *is a family of graphs such that* $\ell_\Phi(n) \longrightarrow \infty$ *as* $n \longrightarrow \infty$, *then* $i_\Phi(n) = o(3^{n/3})$.

We will prove the following stronger theorem.

Theorem 2.3. *There exists a real number* $t > 0$ *such that if* G *is a graph of order* n, *then*

$$i(G) \leq 3^{n/3} \cdot (1+t)^{-l+3},$$

where l *is the length of a longest path of* G.

To prove Theorem 2.3, we need the following identity in the proof of Theorem 1 in [11]. For our purpose, we have complemented the clique version to the MIS version.

Proposition 2.2. *For a graph G and $x, y \in V(G)$, if x and y are adjacent, then*

$$i(G(x;y)) = i(G) + i(y) - i(x) + \beta(x),$$

where $G(x;y)$ denotes the graph obtained by removing the edges incident with x except the edge xy and replacing them by edges joining x to each vertex of $N(y) - \{x\}$, $i(x)$ denotes the number of MIS's of G containing x, and $\beta(x)$ denotes the number of MIS's in $G - x$ which are contained entirely in $G - x - N(x)$.

We also need a basic fact for $i(G)$.

Fact 2.1. *If H is an induced subgraph of a graph G, then $i(H) \leq i(G)$.*

Proof of Theorem 2.3. Choose $t > 0$ such that $(1+t)^2 \cdot g(n) < 3^{n/3}$ and $(1+t)^5 \cdot [3^{-1/3} + (1+t) \cdot 3^{-4/3}] < 1$. We proceed by induction on n. For $n \leq 5$, the result is easy to check. Assume the result for all graphs of order less than n (with $n \geq 6$). Now let G be a graph of order n and let $P = v_1 v_2 \cdots v_l v_{l+1}$ be a longest path of G of length l. For $l \leq 5$, it follows directly from Corollary 1.1 and Proposition 1.4 that

$$i(G) \leq 3^{n/3} \cdot (1+t)^{-l+3}$$

since $(1+t)^2 \cdot g(n) < 3^{n/3}$. Suppose $l \geq 6$. By Proposition 2.2, we have

$$i(G(x;y)) = i(G) + i(y) - i(x) + \beta(x).$$

Now, for $i(v_3) \leq i(v_4)$, we let $x = v_3$ and $y = v_4$; for $i(v_3) > i(v_4)$, we let $x = v_4$ and $y = v_3$. then $i(G) \leq i(G(x;y))$. Assume $i(v_3) \leq i(v_4)$. Then v_5 has at least three neighbors v_3, v_4, and v_6 in $G(x;y)$, $G_1 = G(x;y) - v_5$ has a path of length $l-5$, and $G_2 = G(x;y) - \{v_3, v_4, v_5, v_6\}$ has a path of length $l-6$. By Lemma 2.1 and Fact 2.1, we have

$$i(G(x;y)) \leq i(G_1) + i(G(x;y) - v_5 - N(v_5)) \leq i(G_1) + i(G_2).$$

Then it follows from the induction hypothesis that

$$\begin{aligned} i(G) \leq i(G(x;y)) &\leq 3^{(n-1)/3} \cdot (1+t)^{-l_1+3} + 3^{(n-4)/3} \cdot (1+t)^{-l_2+3} \\ &\leq 3^{(n-1)/3} \cdot (1+t)^{-l+5+3} + 3^{(n-4)/3} \cdot (1+t)^{-l+6+3} \\ &= 3^{n/3} \cdot (1+t)^{-l+3} \cdot [(1+t)^5 \cdot 3^{-1/3} + (1+t)^6 \cdot 3^{-4/3}] \\ &\leq 3^{n/3} \cdot (1+t)^{-l+3}, \end{aligned}$$

where l_j is the length of a longest path of G_j for $j = 1$ and 2. For the case $i(v_3) > i(v_4)$, it follows that v_2 has at least three neighbors v_1, v_3, and v_4 in $G(x;y)$, $G_1 = G(x;y) - v_2$ has a path of length $l-2$, and $G_2 = G(x;y) - \{v_1, v_2, v_3, v_4\}$ has a path of length $l-3$. Similar to the above, we obtain

$$i(G) \leq 3^{n/3} \cdot (1+t)^{-\ell+3}.$$

Therefore, the theorem follows. □

Given any constant C, let $G = K_{C+1} \cup H_{n-C-1}$. Then G has a longest path of length C and $i(G) = O(3^{n/3})$. This shows that Theorem 2.2 is best possible.

2.2 Constraints for the Number of Maximum Independent Sets

First of all, there is a similar inequality to Lemma 2.1 here, where we define $i_m(\phi) = 1$.

Theorem 2.4. *For any graph G and any $v \in V(G)$,*

$$i_m(G) \leq i_m(G - v) + i_m(G - v - N(v)).$$

Proof. For any maximum independent set V' of G, if $v \notin V'$, then V' is a maximum independent set of $G - v$; if $v \in V'$, then $V' - v$ is a maximum independent set in $G - v - N(v)$ unless $G - v - N(v) = \phi$. Therefore the theorem holds. □

Notice that for any graph G, $i_m(G) \leq i(G)$. From Theorems 2.1 and 2.2 the following two theorems follow directly.

Theorem 2.5. *For a family $\Phi = \{G : G \text{ is connected}\}$, if $\Delta_\Phi(n) = o(n)$, then $i'_\Phi(n) = o(3^{n/3})$.*

Theorem 2.6. *If* Φ *is a family of graphs such that* $\ell_\Phi(n) \longrightarrow \infty$ *as* $n \longrightarrow \infty$, *then* $i'_\Phi(n) = o(3^{n/3})$.

CHAPTER III

MAXIMAL INDEPENDENT SETS

3.1 Maximal Independent Sets in k-Connected Graphs

We begin this section by defining the k-connected graphs $H_{k,n}$ on n vertices and the function $m_k(n)$ such that $i(H_{k,n}) = m_k(n)$ for each $k \geq 2$ and $n-k \geq 16$.

Let

$$m_2(n) = \begin{cases} 3^t + 2^t + 1 & \text{if } n-2 = 3t \\ 4 \cdot 3^{t-1} + 3 \cdot 2^{t-1} + 2 & \text{if } n-2 = 3t+1 \\ 2 \cdot 3^t + 2^{t+1} & \text{if } n-2 = 3t+2, \end{cases}$$

and define $H_{2,n}$ to be the graph shown in Figure 3.1.

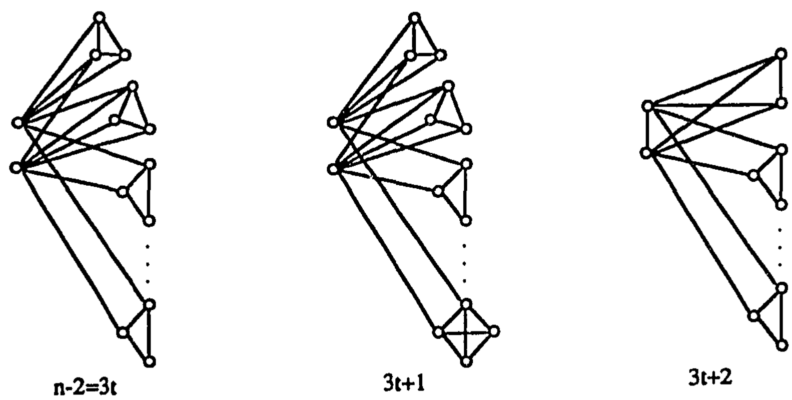
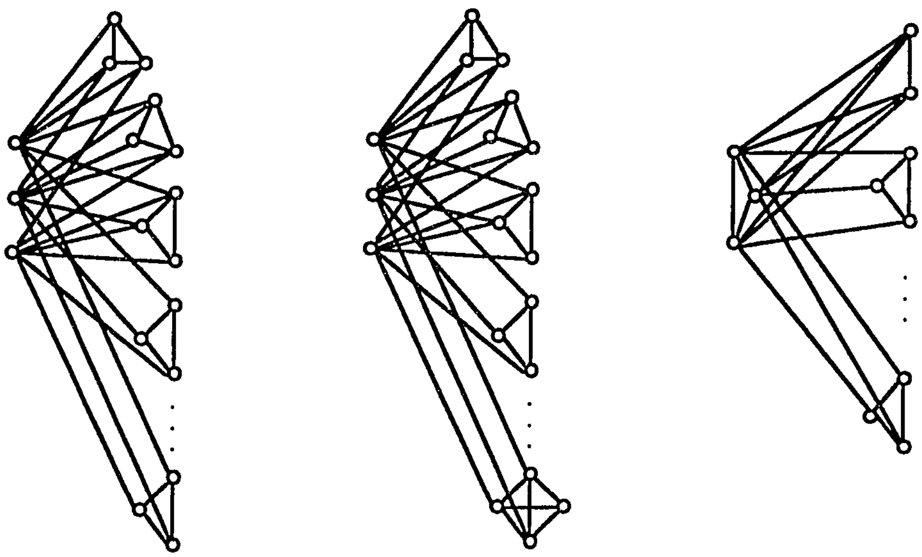


Figure 3.1. The Graph $H_{2,n}$.

Let

$$m_3(n) = \begin{cases} 3^t + 3\cdot 2^{t-1} + 4 & \text{if } n-3 = 3t \\ 4\cdot 3^{t-1} + 9\cdot 2^{t-2} + 7 & \text{if } n-3 = 3t+1 \\ 2\cdot 3^t + 3\cdot 2^t & \text{if } n-3 = 3t+2, \end{cases}$$

and define $H_{3,n}$ to be the graph shown in Figure 3.2.



n-3=3t   3t+1   3t+2

Figure 3.2. The Graph $H_{3,n}$.

Let

$$m_4(n) = \begin{cases} 3^t + 3\cdot 2^{t-1} + 15 & \text{if } n-4 = 3t \\ 4\cdot 3^{t-1} + 9\cdot 2^{t-2} + 41 & \text{if } n-4 = 3t+1 \\ 2\cdot 3^t + 3\cdot 2^t + 4 & \text{if } n-4 = 3t+2, \end{cases}$$

and define $H_{4,n}$ to be the graphs shown in Figure 3.3.

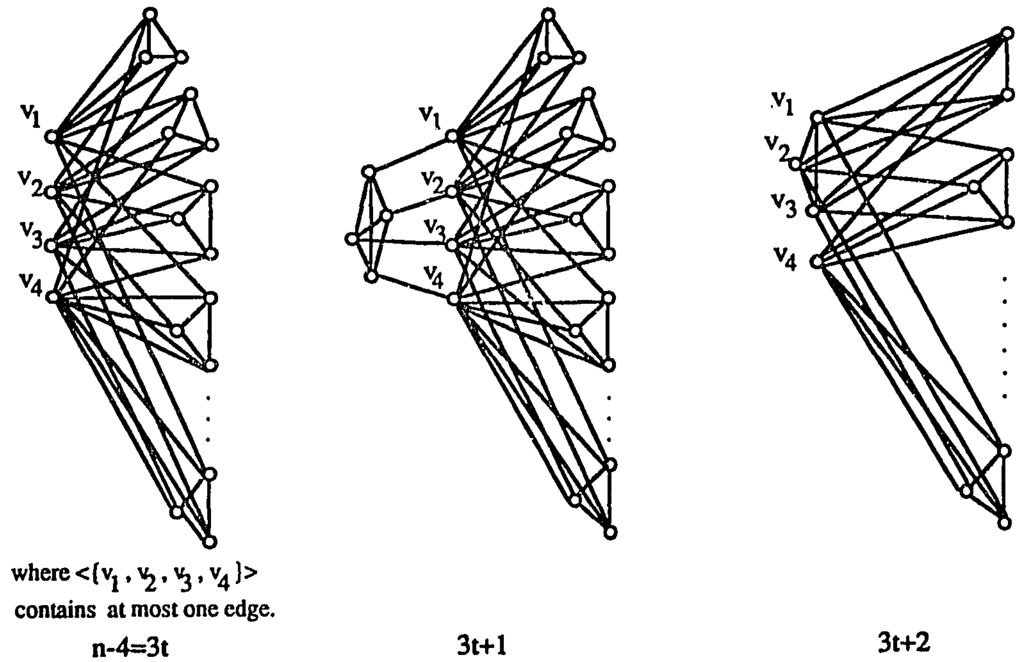


Figure 3.3. The Graphs $H_{4,n}$.

For $k \geq 5$, $n-k \geq 16$, and $n-k \not\equiv 2 \bmod 3$, let

$$m_k(n) = \begin{cases} m_4(n-k+4)+k-4 & \text{if } k \leq 9 \\ f(n-k)+f(k)+3\cdot 2^{t-1}+14 & \text{if } k \geq 10 \text{ and } n-k=3t \\ f(n-k)+f(k)+9\cdot 2^{t-2}+40 & \text{if } k \geq 10 \text{ and } n-k=3t+1 \end{cases}$$

and define $H_{k,n}$ to be the graphs obtained from $H_{4,n-k+4}$ by adding $k-4$ vertices $v_5, v_6, \cdots, v_k$ and for $k \leq 9$, joining each new vertex to every other vertex; for $k \geq 10$, joining each new vertex to every vertex in $H_{4,n-k+4} - \{v_1, v_2, v_3, v_4\}$, then adding some edges between v_i and v_j such that the induced subgraph on $A = \{v_1, \cdots, v_k\}$ is isomorphic to H_k and $\{v_1, v_2, v_3, v_4\}$ is independent.

For $k \geq 5$, $n-k \geq 16$, and $n-k = 3t+2$, let

$$m_k(n) = f(n-k) + f(k) + 3 \cdot 2^t + 2$$

and define $H_{k,n}$ to be the graphs obtained from $H_{5,n-k+5}$ shown in Figure 3.4 by adding $k-5$ vertices $v_6, v_7, \cdots, v_k$ and joining each new vertex to every vertex in $H_{5,n-k+5} - \{v_1, \cdots, v_5\}$, then adding some edges between v_i and v_j so that the induced subgraph on $A = \{v_1, v_2, \cdots, v_k\}$ is isomorphic to H_k.

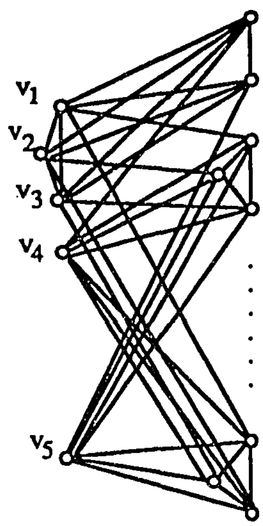


Figure 3.4. The Graph $H_{5,3r+1}$.

Then it is easy to check that $H_{k,n}$ is k-connected and $i(H_{k,n}) = m_k(n)$ for each $k \geq 2$. Hence the next lemma follows immediately.

Lemma 3.1. *For each* $k \geq 2$ *and* $n-k \geq 16$, $f_k(n) \geq m_k(n) > f(n-k) = O(3^{\frac{n}{3}})$.

We will prove that $f_k(n) = m_k(n)$ for $2 \le k \le 4$ and $k \ge 10$. To do that, we first develop two structure properties for the extremal graphs for $f_k(n)$.

Theorem 3.1. *Given an integer $k \ge 1$, there exist positive constants n_0 and C such that for $n \ge n_0$, if G is a k-connected graph of order n and $i(G) = f_k(n)$, then G has k vertices of degree at least $C \cdot n$.*

Proof. Suppose, to the contrary, that the theorem is not true. Then there exist an integer $k \ge 1$ and a k-connected graph G of order $n \ge n_0$ such that $i(G) = f_k(n)$ and for any constant $C > 0$ there are at most $k-1$ vertices which have degree at least $C \cdot n$. Now, choose a constant C_0 so that the number t of vertices with degree at least $C_0 \cdot n$ is maximized. Of course, we have $t \le k-1$. Let $v_1, v_2, \cdots, v_t$ be the vertices of G such that $\deg(v_j) \ge C_0 \cdot n$ for each j. Then from the choice of C_0 we conclude that $\Delta(G - \{v_1, v_2, \cdots, v_t\}) = o(n-t)$. By applying Lemma 2.1 t times, we obtain

$$\begin{aligned}
i(G) &\le i(G - v_1) + i(G - v_1 - N(v_1)) \\
&\le i(G - v_1 - v_2) + i(G - \{v_1, v_2\} - N(v_2)) + i(G - v_1 - N(v_1)) \\
&\le i(G - \{v_1, v_2, \cdots, v_t\}) + i(G - v_1 - N(v_1)) + i(G - \{v_1, v_2\} - N(v_2)) \\
&\qquad + \cdots + i(G - \{v_1, v_2, \cdots, v_t\} - N(v_t)).
\end{aligned}$$

Notice that each $i(G - \{v_1, v_2, \cdots, v_j\} - N(v_j)) \le f(\lceil n - C_0 \cdot n \rceil)$. Consequently, we have

$$i(G) \le i(G - \{v_1, v_2, \cdots, v_t\}) + t \cdot f(\lceil n - C_0 \cdot n \rceil).$$

Since $t \le k-1$ and G is k-connected, $G' = G - \{v_1, v_2, \cdots, v_t\}$ is connected. Recall that $\Delta(G') = o(n-t)$, it follows from Theorem 2.1 that $i(G') = o(3^{(n-t)/3})$. Thus

$$\begin{aligned} i(G) \cdot 3^{-n/3} &\le i(G') \cdot 3^{-n/3} + t \cdot f(\lceil n - C_0 \cdot n \rceil) \cdot 3^{-n/3} \\ &\le i(G') \cdot 3^{-3/n} + t \cdot 3^{\lceil n - C_0 \cdot n \rceil /3} \cdot 3^{-n/3} \longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned}$$

which contradicts Lemma 3.1. Therefore the theorem follows. □

From Theorem 3.1 our second structure property of the extremal graphs for $f_k(n)$ follows.

Theorem 3.2. *Given an integer $k \ge 1$, there exists a positive integer n_0 such that if G is a k-connected graph of order $n \ge n_0$ with $i(G) = f_k(n)$, then $G - \{v_1, v_2, \cdots, v_k\} \cong H_{n-k}$, where $\{v_1, v_2, \cdots, v_k\} \subseteq V(G)$.*

Proof. First, by using Theorem 3.1, we obtain a constant $C > 0$ and k vertices $v_1, v_2, \cdots, v_k$ in $V(G)$ such that $\deg(v_j) \ge C \cdot n$ for each j. Then, applying Lemma 2.1 k times similar to the proof of Theorem 3.1, we have

$$i(G) \le i(G - \{v_1, v_2, \cdots, v_k\}) + k \cdot f(\lceil n - C \cdot n \rceil).$$

If $G' = G - \{v_1, v_2, \cdots, v_k\} \not\cong H_{n-k}$, then it follows from Proposition 1.4 that

$$\begin{aligned} i(G) &\le g(n-k) + k \cdot f(\lceil n - C \cdot n \rceil) \\ &< f(n-k) < f_k(n) \text{ when } n \ge n_0 \text{ for some } n_0 \in Z^+, \end{aligned}$$

contradicting $i(G) = f_k(n)$. Therefore $G - \{v_1, v_2, \cdots, v_k\} \cong H_{n-k}$. □

In the following discussions of this section, we assume G is a k-connected graph of order n with $i(G) = f_k(n)$. Then Theorem 3.2 tells us that there exists $n_0' \in Z^+$ such that for $n \geq n_0'$

$$G' = G - A \cong H_{n-k},$$

where $A = \{v_1, v_2, \cdots, v_k\} \subseteq V(G)$. That is, if we let $G_1, G_2, \cdots, G_r$ be the components of G' with $|V(G_1)| \leq |V(G_2)| \leq \cdots \leq |V(G_r)|$, then for $n-k = 3t$, $r = t$ and each G_j is a triangle; for $n-k = 3k+1$, either $r = t$, $G_r = K_4$, and the other G_j's are triangles or $r = t+1$, $G_1 = G_2 = K_2$, and the other G_j's are triangles; for $n-k = 3t+2$, $r = t+1$, $G_1 = K_2$, and the other G_j's are triangles.

To determine G, we need to explore the structure in between $< A >$ and G'. For that purpose, we divide the set $I(G)$ of MIS's of G into three disjoint subsets: $I(G) = I_1 \cup I_2 \cup I_3$, where $I_1 = \{S : S \in I(G) \text{ and } S \subseteq V(G')\}$, $I_2 = \{S : S \in I(G) \text{ and } S \subsetneq A\}$, and $I_3 = I(G) - (I_1 \cup I_2)$, so $i(G) = |I_1| + |I_2| + |I_3|$. By Proposition 1.1, we have $|I_1| \leq f(n-k)$ and $|I_2| \leq f(k)$. Our goal is to develop an upper bound for $|I_3|$. Then we prove that for $k \geq 10$, $i(G)$ is maximized only if each $|I_j|$ attains its upper bound.

We first list some basic facts and properties. Since G is k-connected, we have an immediate fact.

Fact 3.1. *Let p_n be the number of non-adjacent pairs with one vertex in A and the other one in $V(G')$. Then*

$$p_n \leq \begin{cases} 6t & \text{if } n-k=3t \\ 6t+6 & \text{if } n-k=3t+1 \\ 6t+2 & \text{if } n-k=3t+2. \end{cases}$$

Property 3.1. *Given $S \subseteq A$, then for each j, either $N(S) \cap V(G_j) = V(G_j)$ or $|N(S) \cap V(G_j)| \geq |S|$.*

Proof. Suppose, to the contrary, that there exists j between 1 and r such that

$$N(S) \cap V(G_j) \neq V(G_j) \text{ and } |N(S) \cap V(G_j)| < |S|.$$

Then $A' = (A-S) \cup (N(S) \cap V(G_j))$ is a cut set of G with $|A'| < |A| = k$, contradicting the assumption that G is k-connected. Therefore the property holds. □

From this property, it is easy to see that each v_i in A has at least one neighbor in each component G_j. Notice that $|V(G_j)| \leq 3$ for each j when $n-k \not\equiv 1 \bmod 3$ and $|V(G_j)| \leq 4$ for each j when $n-k \equiv 1 \bmod 3$. Property 3.1 has the following two immediate consequences.

Property 3.2. *For $n-k \not\equiv 1 \bmod 3$, if $S \in I(G)$ and $|S \cap A| \geq 3$, then $S \in I_2$. Furthermore, for each pair v_i and v_j in A, there is at most one MIS in I_3 which contains v_i and v_j.*

Property 3.3. *For $n-k \equiv 1 \bmod 3$, if $S \in I(G)$ and $|S \cap A| \geq 4$, then $S \in I_2$. Furthermore, for each set $X \subseteq A$, if $|X| = 2$, then there are at most two MIS's in I_3 which contain X; if $|X| = 3$, then there exists at most one MIS in I_3 which contains X.*

Denote $I_3(v_j) = \{S : S \in I_3 \text{ and } S \cap A = \{v_j\}\}$ and $i_3(v_j) = |I_3(v_j)|$. Then from Properties 3.2 and 3.3 the following property follows directly.

Property 3.4. *For $k \geq 2$, we have*

$$|I_3| \leq \begin{cases} \sum_{j=1}^{k} i_3(v_j) + \binom{k}{2} & \text{if } n-k \not\equiv 1 \bmod 3 \\ \sum_{j=1}^{k} i_3(v_j) + 2 \cdot \binom{k}{2} + \binom{k}{3} & \text{if } n-k \equiv 1 \bmod 3. \end{cases}$$

Now, for each $v_i \in A$, let $n_j(v_i)$ be the number of components of G' of order at least 3 which have exactly j neighbors of v_i. The following fact is easy to see.

Fact 3.2. *For each $v_j \in A$,*

$$i_3(v_j) \leq \begin{cases} 2^{n_1(v_j)+1} & \text{if } n-k = 3t+1,\ G_r = K_4, \\ & \quad \text{and } |N(v_j) \cap V(G_r)| = 2 \\ 3 \cdot 2^{n_1(v_j)-1} & \text{if } n-k = 3t+1, G_r = K_4, \\ & \quad \text{and } |N(v_j) \cap V(G_r)| = 1 \\ 2^{n_1(v_j)} & \text{otherwise.} \end{cases}$$

Property 3.5. *For* $k \geq 1$, *we have*

$$\sum_{i=1}^{k} n_1(v_i) \leq \begin{cases} n-k & \text{if } n-k \not\equiv 2 \bmod 3 \\ n-k-2 & \text{if } n-k \equiv 2 \bmod 3. \end{cases}$$

Proof. To prove this property, it suffices to prove that for each component G_j, there are at most $|V(G_j)|$ vertices in A which have exactly one neighbor in G_j. Suppose, to the contrary, that there exists a component G_j such that there are at least $|V(G_j)|+1$ vertices in A which have exactly one neighbor in G_j. Then there are two vertices in A, say v_1 and v_2, such that

$$|N(\{v_1, v_2\}) \cap V(G_j)| = 1 < |\{v_1, v_2\}|,$$

violating Property 3.1. Therefore, the property holds. □

Property 3.6. *There exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$ *and* $n-k = 3t+1$, *then* $G' = G - A$ *must be* $(t-1)K_3 \cup K_4$.

Proof. Let $n-k = 3t+1$. Suppose G' is not isomorphic to $(t-1)K_3 \cup K_4$, then $G' \cong H_{n-k}$ implies G' must be $(t-1)K_3 \cup 2K_2$. For $k = 2$, it follows from Fact 3.2 that $i_3(v_1) + v_3(v_2) \leq 2^{t-1} + 2^{t-1}$, which implies

$$\begin{aligned} i(G) &= |I_1| + |I_2| + |I_3| \leq f(n-2) + 2 + (i_3(v_1) + i_3(v_2) + 1) \\ &\leq f(n-2) + 2^t + 3 < m_2(n) \leq f_2(n), \end{aligned}$$

contradicting the choice of G. For $k \geq 3$, let $D_k = \{(x_1, \cdots, x_k) \in Z^k : \sum_{j=1}^{k} x_j \leq 3t+1 \text{ and } 0 \leq x_j \leq t-1\}$. Then from Fact 3.2 and Properties 3.4 and 3.5 it follows that

$$\begin{aligned}
&\sum_{j=1}^{k} i_3(v_j) \\
&\qquad \leq \max\{2^{x_1} + 2^{x_2} + \cdots + 2^{x_k} : (x_1, \cdots, x_k) \in D_k\} \\
&\qquad \leq 3 \cdot 2^{t-1} + 2^4 + k - 4 \text{ and} \\
i(G) &\leq f(n-k) + f(k) + \left(\sum_{j=1}^{k} i_3(v_j) + 2 \cdot \binom{k}{2} + \binom{k}{3}\right) \\
&\leq f(n-k) + f(k) + 3 \cdot 2^{t-1} + k + 12 + 2 \cdot \binom{k}{2} + \binom{k}{3},
\end{aligned}$$

this implies

$$i(G) < m_k(n) \leq f_k(n) \text{ when } n \geq n_0 \text{ for some } n_0 \in Z^+,$$

contradicting the choice of G. Therefore, the property holds. □

From now on, we assume n is large enough so that $G' = G - A \cong H_{n-k}$ and $G' = (t-1)K_3 \cup K_4$ for $n-k = 3t+1$. In order to establish further properties of G, we need the following remark. For convenience, we call a vertex v_i in A a bad-vertex if $|N(v_i) \cap V(G_j)| < |V(G_j)|$ for each $1 \leq j \leq r$. Then, for each bad-vertex v_j, $S_j = \{X - \{v_j\} : X \in I_3(v_j)\} \subseteq I(G')$ and $|S_j| = i_3(v_j)$. Furthermore, for any two bad-vertices v_i and v_j, $|S_i \cap S_j| \leq 2$. Hence we have the following remark.

Remark 3.1. *The following claims are true:*

(i) If A has a bad-vertex v_i, then $|I_1| \le f(n-k) - i_3(v_i)$.

(ii) If A has two bad-vertices, say v_1 and v_2, then

$$|I_1| \le f(n-k) - [i_3(v_1) + i_3(v_2) - 2].$$

Next, we first establish the equality $f_2(n) = m_2(n)$ when n is large enough.

Property 3.7. *For* $k = 2$, A *has no bad-vertex and*

$$i_3(v_1) + i_3(v_2) \le \begin{cases} 2^t & \text{if } n-2 = 3t \\ 3 \cdot 2^{t-1} & \text{if } n-2 = 3t+1 \\ 2^{t+1} & \text{if } n-2 = 3t+2. \end{cases}$$

Furthermore, for $n-k \equiv 2 \bmod 3$, *the equality holds only if* $G \cong H_{2,n}$. *For* $n-k \not\equiv 2 \bmod 3$, *the equality holds only if either* $G \cong H_{2,n}$, *or* v_1 *is adjacent to* v_2, $n_1(v_1) = n_1(v_2) = t-1$, *and for each* $j = 1$ *or 2 there exists a triangle component* G_{ℓ_j} *such that* $N(v_j) \supsetneqq V(G_{\ell_j})$.

Proof. To prove this property, it suffices to prove that A has no bad-vertex. If both v_1 and v_2 are bad-vertices, then it follows from Remark 3.1 that

$$\begin{aligned} i(G) &= |I_1| + |I_2| + |I_3| \\ &\le f(n-2) - [i_3(v_1) + i_3(v_2) - 2] + 2 + [i_3(v_1) + i_3(v_2) + 1] \\ &= f(n-2) + 5 < m_2(n) \le f_2(n), \end{aligned}$$

contradicting $i(G) = f_2(n)$. Suppose there is exactly one of v_1 and v_2 which is a bad-vertex, say v_1. Then, by Fact 3.2 and Remark 3.1, it follows that $|I_1| \leq f(n-2) - i_3(v_1)$ and

$$i_3(v_2) \leq \begin{cases} 2^{t-1} & \text{if } n-2 = 3t \\ 3 \cdot 2^{t-2} & \text{if } n-2 = 3t+1 \\ 2^t & \text{if } n-2 = 3t+2. \end{cases}$$

Hence

$$\begin{aligned} i(G) &\leq f(n-2) - i_3(v_1) + 2 + [i_3(v_1) + i_3(v_2) + 1] \\ &= f(n-2) + 3 + i_3(v_2) < m_2(n) \leq f_2(n), \end{aligned}$$

contradicting the choice of G again. Therefore A has no bad-vertex and the property holds. □

From Property 3.7, the next theorem follows easily.

Theorem 3.3. *There exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$, *then*

$$f_2(n) = m_2(n)$$

and the only 2-connected graph of order n *which has* $f_2(n)$ *maximal independent sets is* $H_{2,n}$.

We now assume $k \geq 3$. In order to determine $f_k(n)$, we need more properties of G.

Property 3.8. *For* $n - k = 3t$, *there exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$, *then each* $n_1(v_j) \leq t-1$ *for* $1 \leq j \leq k$.

Proof. Without loss of generality, assume $n_1(v_1) \geq n_1(v_2) \geq \cdots \geq n_1(v_k)$. If $n_1(v_1) = n_1(v_2) = t$, then v_1 and v_2 are bad-vertices , so it follows from Remark 3.1 that

$$|I_1| \leq f(n-k) - [i_3(v_1) + i_3(v_2) - 2].$$

Applying Fact 3.2 and Property 3.5, we obtain

$$\begin{aligned}
&\sum_{j=3}^{k} i_3(v_j) \\
&\quad \leq \max\{2^{x_3} + \cdots + 2^{x_k} : \text{each } x_j \in Z,\ 0 \leq x_j \leq t \text{ and } \sum_{j=3}^{k} x_j \leq t\} \\
&\quad \leq 2^t + k - 3.
\end{aligned}$$

Thus, by Property 3.4,

$$\begin{aligned}
i(G) &= |I_1| + |I_2| + |I_3| \\
&\leq f(n-k) - [i_3(v_1) + i_3(v_2) - 2] + f(k) + \sum_{j=1}^{k} i_3(v_j) + \binom{k}{2} \\
&\leq f(n-k) + f(k) + 2^t + k - 1 + \binom{k}{2} \\
&< m_k(n) \leq f_k(n) \text{ for } n \geq n_0 \in Z^+,
\end{aligned}$$

contradicting the choice of G. If $n_1(v_1) = t$ and $n_1(v_2) < t$, then it follows from Remark 3.1 that

$$|I_1| \leq f(n-k) - i_3(v_1).$$

Similarly, we have

$$\sum_{j=2}^{k} i_3(v_j) \leq 2 \cdot 2^{t-1} + 2^2 + k - 4 \text{ and}$$
$$i(G) < m_k(n) \leq f_k(n) \text{ for } n \geq n_0 \in Z^+,$$

contradicting the choice of G again. Therefore the property holds. □

Property 3.9. *For* $n - k = 3t + 1$, *there exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$, *then* $n_1(v_i) \leq t - 1$ *for each* v_i *with only one neighbor in* $G_r(= K_4)$ *and* $n_1(v_i) \leq t - 2$ *for each* v_i *with exactly two neighbors in* G_r.

Proof. Without loss of generality, assume that $v_1, \cdots, v_x$ are vertices in A which have exactly one neighbor in G_r and $v_{x+1}, \cdots, v_{x+y}$ are vertices which have exactly two neighbors in G_r. Then $n_1(v_i) \leq t$ for $i \leq x$ and $n_1(v_j) \leq t - 1$ for $x + 1 \leq j \leq x + y$. Let h be the number of vertices in $\{v_1, \cdots, v_{x+y}\}$ on which the values of function n_1 attain their upper bounds. Similar to the proof of Property 3.8, we obtain $h = 0$ by obtaining a contradiction in each of the cases $h = 1$ and $h \geq 2$ for $n \geq n_0$. Therefore the property holds. □

Property 3.10. *There exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$, *then there are three vertices in* A, *say* v_1, v_2, v_3, *satisfying:*

(1) if $n-k \not\equiv 2 \bmod 3$, *then there are components* G_{ℓ_1}, G_{ℓ_2}, G_{ℓ_3} *of* G' *such that for each* $1 \leq j \leq 3$, $|N(v_j) \cap V(G_{\ell_j})| = 3 = |V(G_{\ell_j})|$ *and* v_j *is adjacent to exactly one vertex in each of the other components;*

(2) if $n-k \equiv 2 \bmod 3$, *then for each* $1 \leq j \leq 3$, $|N(v_j) \cap V(G_1)| = 2 = |V(G_1)|$ *and* v_j *is adjacent to exactly one vertex in each of the other components.*

Proof. First, we claim that there are three vertices in A, say v_1, v_2, v_3, such that

(i) if $n-k = 3t$, then $n_1(v_j) = t-1$ for $1 \leq j \leq 3$;

(ii) if $n-k = 3t+1$, then for each $1 \leq j \leq 3$, $n_1(v_j) = t-1$ and v_j is adjacent to exactly one vertex in $G_r = K_4$;

(iii) if $n-k = 3t+2$, then for each $1 \leq j \leq 3$, $n_1(v_j) = t$, namely, v_j is adjacent to exactly one vertex in each triangle component of G'.

Suppose, to the contrary, that the claim is not true. Then it follows from Fact 3.2 and Properties 3.4, 3.5, 3.8, and 3.9 that

$$|I_3| \leq \begin{cases} \sum_{j=1}^{k} i_3(v_j) + \binom{k}{2} & \text{if } n-k \not\equiv 1 \mod 3 \\ \sum_{j=1}^{k} i_3(v_j) + 2\cdot\binom{k}{2} + \binom{k}{3} & \text{if } n-k \equiv 1 \mod 3 \end{cases}$$

$$\leq \begin{cases} \sum_{j=1}^{k} 2^{x_j} + \binom{k}{2} & \text{if } n-k \not\equiv 1 \mod 3 \\ 3\cdot(2^{x_1-1} + 2^{x_2-1} + 2^{x_3-1} + 2^{x_4-1}) & \\ +2^{x_5} + \cdots + 2^{x_k} + 2\cdot\binom{k}{2} + \binom{k}{3} & \text{if } n-k \equiv 1 \mod 3, \end{cases}$$

where $x_j = n_1(v_j)$ for $1 \leq j \leq k$ such that if $n-k=3t$, then $0 \leq x_j \leq t-1$ for $j=1$ or 2, $0 \leq x_j \leq t-2$ for $j \geq 3$, and $\sum_{j=1}^{k} x_j \leq 3t$; if $n-k=3t+1$, then $0 \leq x_j \leq t-1$ for $j \neq 3$ and 4, $0 \leq x_j \leq t-2$ for $j=3$ or 4, and $\sum_{j=1}^{k} x_j \leq 3t+1$; if $n-k=3t+2$, then $0 \leq x_j \leq t$ for $j=1$ or 2, $0 \leq x_j \leq t-1$ for $j \geq 3$, and $\sum_{j=1}^{k} x_j \leq 3t$. Thus

$$
\begin{aligned}
i(G) &= |I_1| + |I_2| + |I_3| \le f(n-k) + f(k) + |I_3| \\
&\le \begin{cases}
f(n-k) + f(k) + 2^t + 2^{t-2} + 2^4 \\
\quad +k - 4 + \binom{k}{2} & \text{if } n-k = 3t \\
f(n-k) + f(k) + 2\cdot(3\cdot 2^{t-2}) + 3\cdot 2^2 \\
\quad +3 + 2^{t-1} + k - 5 + 2\cdot\binom{k}{2} + \binom{k}{3} & \text{if } n-k = 3t+1 \\
f(n-k) + f(k) + 2^{t+1} + 2^{t-1} + 2 \\
\quad +k - 4 + \binom{k}{2} & \text{if } n-k = 3t+2
\end{cases} \\
&< m_k(n) \le f_k(n) \text{ when } n \ge n_0 \in Z^+,
\end{aligned}
$$

contradicting the choice of G. Now, by Remark 3.1 and similar to the proof of Property 3.8, we conclude that v_1, v_2, v_3 are not bad-vertices. Therefore Property 3.10 follows. □

Notice from Property 3.1 that if $v_{j_1}, v_{j_2}, v_{j_3} \in A$ and G_ℓ is a triangle component of G' such that $|N(v_{j_i}) \cap V(G_\ell)| = 1$ for $1 \le i \le 3$, then v_{j_1}, v_{j_2}, and v_{j_3} are adjacent to distinct vertices of G_ℓ. We call such a component a <u>good-component</u>. Since G is k-connected, we have an immediate consequence.

Property 3.11. *For each good component* G_ℓ *of* G'*, suppose* v_1, v_2, v_3 *are three vertices in* A *which have exactly one neighbor in* G_ℓ*, than each* v_j *in* $A - \{v_1, v_2, v_3\}$ *is adjacent to all vertices of* G_ℓ*.*

Theorem 3.4. *For* $k \geq 3$ *and* $n-k \equiv 2 \bmod 3$*, there exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$*, then*

$$f_k(n) = m_k(n)$$

and the only k-connected graphs of order n *which have* $f_k(n)$ *maximal independent sets are the graphs* $H_{k,n}$*.*

Proof. Let $k \geq 3$ and $n-k = 3t+2 \geq 16$. From Lemma 3.1 it follows that $f_k(n) \geq m_k(n)$. On the other hand, let G be a k-connected graph of order n such that $i(G) = f_k(n)$. From the above discussions it follows that there exists $n_0 \in Z^+$ such that if $n \geq n_0$, then $G' = G - A \cong H_{n-k}$ for some subset $A = \{v_1, \cdots, v_k\} \subseteq V(G)$ and G has the above properties. By Property 3.10, we obtain three vertices in A, say v_1, v_2, and v_3, such that for each $1 \leq i \leq 3$,

$$N(v_i) \cap V(G_1) = V(G_1) \text{ and } |N(v_i) \cap V(G_\ell)| = 1 \text{ for } 2 \leq \ell \leq t+1.$$

Then Property 3.11 implies that for each $v_j \in A - \{v_1, v_2, v_3\}$,

$$N(v_j) \cap V(G_\ell) = V(G_\ell) \text{ for } \ell \geq 2.$$

Since G is k-connected, we have $\delta(G) \geq k$, which implies that each vertex in G_1 is adjacent to at least $k-1$ vertices in A. Without loss of generality, assume that one vertex in G_1 may be not adjacent to v_4 and the other vertex in G_1 may be not adjacent to v_5. Then it is easy to see that $i(G) \leq i(H_{k,n})$ and the equality holds only if $G \cong H_{k,n}$. This concludes the proof. □

Theorem 3.5. *For* $n-k=3t$ *or* $3t+1$ *with* $3 \leq k \leq 4$ *or* $k \geq 10$, *there exists* $n_0 \in Z^+$ *such that if* $n \geq n_0$, *then*

$$f_k(n) = m_k(n)$$

and the only k*-connected graphs of order* n *which have* $f_k(n)$ *maximal independent sets are the graphs* $H_{k,n}$.

Proof. Assume $n-k = 3t$ or $3t+1$ with $3 \leq k \leq 4$ or $k \geq 10$ and $n-k \geq 16$. From Lemma 3.1 it follows that $f_k(n) \geq m_k(n)$. On the other hand, let G be a k-connected graph of order n such that $i(G) = f_k(n)$. Then there exists $n_0 \in Z^+$ such that if $n \geq n_0$, then $G' = G-A \cong H_{n-k}$ for some $A = \{v_1, \cdots, v_k\} \subseteq V(G)$ and G has the above properties. Suppose v_1, v_2, v_3 are three vertices in A satisfying 1) of Property 3.10 and G_{ℓ_1}, G_{ℓ_2}, G_{ℓ_3} are the corresponding triangle components of G'. By Property 3.11, we know that each vertex in $A - \{v_1, v_2, v_3\}$ is adjacent to all vertices in each triangle of G' differing from G_{ℓ_j} for $1 \leq j \leq 3$. Since G is

k-connected, there are at most 6 non-adjacent pairs of vertices with one vertex in $A - \{v_1, v_2, v_3\}$ and the other one in G_{ℓ_j} for $1 \leq j \leq 3$, and for $n-k = 3t+1$, there are at most 3 additional non-adjacent pairs of vertices with one vertex in $A - \{v_1, v_2, v_3\}$ and the other one in $G_r = K_4$. Since each vertex in a triangle component of G' is not adjacent to at most two vertices in A, we have an immediate claim:

Claim 1. For each $v_j \in A$, if $|N(v_j) \cap V(G')| = |V(G')| - i$ and the vertices of G' which are not adjacent to v_j are in triangle components, then v_j is contained in at most $x(i)$ maximal independent sets in I_3, where $x(i) = 1$ if $i = 1$, and 3 if $i = 2$.

Furthermore, we have the following claim (for the proof, see Appendix A).

Claim 2. $G_{\ell_1}, G_{\ell_2}, G_{\ell_3}$ are all distinct.

Now, for $k = 3$ or 4, it is easy to see the result. For $k \geq 10$, from Claims 1 and 2 it is easy to check that

$$|I_3| \leq \begin{cases} 3 \cdot 2^{t-1} + 2^3 + \binom{4}{2} & \text{if } n-k = 3t \\ 9 \cdot 2^{t-2} + 3 \cdot 2^3 + 2 \cdot \binom{4}{2} + \binom{4}{3} & \text{if } n-k = 3t+1 \end{cases}$$

and the equality holds only if there is a vertex in $A - \{v_1, v_2, v_3\}$, say v_4, such that $G - \{v_5, \cdots, v_k\}$ is isomorphic to the graph $H_{4,n-k+4}$ where $\{v_1, v_2, v_3, v_4\}$ is independent. Recall that $|I_1| \leq f(n-k)$ and $|I_2| \leq f(k)$.

It follows that

$$i(G) = |I_1| + |I_2| + |I_3| \leq m_k(n) = i(H_{k,n})$$

with the equality only if each $|I_j|$ attains its upper bound, which implies $G \cong H_{k,n}$. Therefore the theorem follows. □

Theorem 3.5 gave the asymptotic value of $f_k(n)$ for $n - k \not\equiv 2 \bmod 3$ with $3 \leq k \leq 4$ or $k \geq 10$. Similarly, one can also determine the asymptotic values of $f_k(n)$ for $n - k \not\equiv 2 \bmod 3$ with $5 \leq k \leq 9$. Here we omit the detail discussions.

3.2. Maximal Independent Sets in Bipartite Graphs

For $n \geq 2$ and $0 \leq r \leq \lfloor n/2 \rfloor$, define

$$B'_{n,r} = \begin{cases} tK_2 & \text{if } n = 2t \\ rK_2 \cup T_{2(t-r)+1} & \text{if } n = 2t+1. \end{cases}$$

Theorem 3.6. *The maximum number of MIS's among all bipartite graphs of order n is*

$$b(n) = 2^{\lfloor n/2 \rfloor}$$

and the only bipartite graphs of order n which have so many MIS's are $B'_{n,r}$.

Proof. We proceed by induction on n. For $1 \leq n \leq 3$, it is easy to see the result. Assume the result for $n \leq k$. Now let $n = k + 1 \geq 4$. Since

$i(B'_{n,r}) = 2^{\lfloor n/2 \rfloor}$, it follows that $b(n) \geq 2^{\lfloor n/2 \rfloor}$. On the other hand, let G be a bipartite graph of order n such that $i(G) = b(n)$, then $i(G) \geq 2^{\lfloor n/2 \rfloor}$. We claim that $G \cong B'_{n,r}$ for some r between 0 and $\lfloor n/2 \rfloor$. Suppose, to the contrary, that $G \ncong B'_{n,r}$ for any r. If G is disconnected, then it is easy to see $i(G) < 2^{\lfloor n/2 \rfloor}$ from the induction hypothesis since the number of MIS's of G is the product of the corresponding numbers of its components, contradicting the choice of G. Hence we assume G is connected, which implies $\Delta(G) \geq 2$. Let $v \in V(G)$ such that $\deg(v) = \Delta(G) = \Delta$. We consider three cases.

Case 1. n is even. By Lemma 2.1 and the induction hypothesis, it follows that

$$i(G) \leq i(G - v) + i(G - v - N(v)) \leq 2^{\lfloor (n-1)/2 \rfloor} + 2^{\lfloor (n-\Delta-1)/2 \rfloor} < 2^{\lfloor n/2 \rfloor},$$

contradicting the choice of G.

Case 2. n is odd and $\Delta(G) \geq 3$. If $\Delta(G - v) = 1$, then G must be a graph with the structure shown in Figure 3.5.

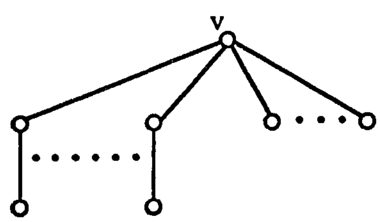


Figure 3.5. Graphs G With $\Delta(G - v) = 1$.

Since $G \not\cong B'_{n,r}$ for any r, $G - v \not\cong [(n-1)/2]K_2$, and so $i(G) < 2^{\lfloor n/2 \rfloor}$, contradicting the choice of G. For $\Delta(G - v) \geq 2$, let $G_1 = G - v$ and let $u \in V(G_1)$ such that u has maximum degree in G_1. Applying Lemma 2.1 twice and using the induction hypothesis, we obtain

$$i(G) \leq i(G - v - u) + i(G - v - u - N(u)) + i(G - v - N(v))$$
$$\leq 2^{\lfloor (n-2)/2 \rfloor} + 2^{\lfloor (n-4)/2 \rfloor} + 2^{\lfloor (n-4)/2 \rfloor} = 2^{\lfloor n/2 \rfloor}.$$

Furthermore, $i(G) = 2^{\lfloor n/2 \rfloor}$ implies that $G - v - u \cong B'_{n-2,r}$, $G - v - u - N(u) \cong B'_{n-4,r}$ or $B'_{n-5,r}$, and $G - v - N(v) \cong B'_{n-4,r}$ or $B'_{n-5,r}$, but there is no such bipartite graph of order n. Hence $i(G) < 2^{\lfloor \frac{n}{2} \rfloor}$, contradicting the choice of G.

Case 3. n is odd and $\Delta(G) = 2$. Then G must be a path of order $n \geq 7$ since G is a connected bipartite graph not isomorphic to $B'_{n,r}$. Let $u \in V(G)$ such that $\deg(u) = 2$ and u is adjacent to an end vertex w. It follows from Lemma 2.1 and the induction hypothesis that

$$i(G) \leq i(G - v) + i(G - u - N(u)) = i(G - u - w) + i(G - u - N(u))$$
$$< 2^{\lfloor (n-2)/2 \rfloor} + 2^{\lfloor (n-3)/2 \rfloor} = 2^{\lfloor n/2 \rfloor},$$

contradicting the choice of G. Hence $G \cong B'_{n,r}$ for some r and the result is true for $n = k + 1$, and so the theorem follows. □

3.3. Maximal Independent Sets in Connected Bipartite Graphs

In this section, we will develop two properties for the number of MIS's of a bipartite graph. As a consequence, the maximum number $b_1(n)$ of MIS's among all connected bipartite graphs of order n is obtained.

For $n \geq 2$ and $0 \leq r \leq \lfloor n/2 \rfloor$, define

$$B_{n,r} = \begin{cases} 2P_4 \cup [(n-8)/2]K_2 \text{ or } T_{n-2r} \cup rK_2 & \text{if } n \text{ is even} \\ T_{n-2r} \cup rK_2 & \text{if } n \text{ is odd.} \end{cases}$$

Then $i(B_{n,r}) = 2^r T(n-2r)$ or $9 \cdot 2^{(n-8)/2}$, where $T(0) = 1$. Note that $B_{n,r} = B'_{n,r}$ if n is odd.

Theorem 3.7. *If G is a forest of order n and $G \ncong B_{n,r}$, then $i(G) < T(n)$.*

Proof. If G is connected, then G is a tree and Proposition 1.3 implies $i(G) < T(n)$. Hence we assume G is disconnected. Let $G_1, G_2, \cdots, G_k$(and $k \geq 2$) be the components of G. Then

$$i(G) = \prod_{j=1}^{k} i(G_j).$$

For n odd, it follows directly from Theorem 3.6 and Proposition 1.3 that $i(G) < T(n)$. For n even, without loss of generality, let $|V(G_i)| = n_i$ such that $n_1 \geq n_2 \geq \cdots \geq n_k$. We consider two cases.

Case 1. One of $n_1, \cdots, n_k$ is odd. Suppose n_t is odd, then $n - n_t$ is also odd. Let $H = G - G_t$. It follows from Theorem 3.6 that

$$i(G) = i(G_t) \cdot i(H) \leq 2^{(n_t-1)/2} \cdot 2^{(n-n_t-1)/2} = 2^{(n-2)/2} < T(n).$$

Case 2. $n_1, \cdots, n_k$ are all even. Let h be the integer between 0 and k such that $n_j > 2$ for $1 \le j \le h$ and $n_j = 2$ for $h+1 \le j \le k$. Since $G \ncong B_{n,r}$, we have $h \ge 1$ and $h = 1$ implies $G_1 \ncong T_{n-2(k-1)}$. If $h = 1$, then from Proposition 1.3 it follows that

$$i(G) = i(G_1) \cdot i(G - G_1) \le 2^{(n_1-2)/2} \cdot 2^{(n-n_1)/2} < T(n).$$

If $h = 2$, then either $n_1 \ge 6$ or $n_1 = n_2 = 4$ but not both G_1 and G_2 are P_4. It follows that

$$\begin{aligned} i(G) &= i(G_1) \cdot i(G_2) \cdot i(G - G_1 - G_2) \\ &\le \begin{cases} 6 \cdot 2^{(n-8)/2} & \text{if } n_1 = n_2 = 4 \\ (2^{(n_1-2)/2} + 1)(2^{(n_2-2)/2} + 1) \cdot 2^{(n-n_1-n_2)/2} & \text{if } n_1 \ge 6, \end{cases} \end{aligned}$$

and so $i(G) < T(n)$. For $h \ge 3$, similarly, we have $i(G) < T(n)$. Therefore the theorem holds. □

Corollary 3.1. *If G is a forest of even order n such that $G \ncong (n/2)K_2$, then*

$$i(G) \le 2^{(n-2)/2} + 2^{(n-4)/2}.$$

Proof. If $G \ncong B_{n,r}$, then the result follows directly from Theorem 3.7. If $G \cong B_{n,r}$, then either $G = 2P_4 \cup [(n-8)/2]K_2$ or $r \le (n-4)/2$, so either $i(G) = 9 \cdot 2^{(n-8)/2} < 2^{(n-2)/2} + 2^{(n-4)/2}$ or

$$i(G) \le \max\{2^r(2^{(n-2r-2)/2} + 1) : 0 \le r \le (n-4)/2\} \le 2^{(n-2)/2} + 2^{(n-4)/2}. \square$$

For n even and $3 \leq r \leq n/2$, define $D_{n,r} = B_{2r} \cup (n/2 - r)K_2$, where B_{2r} has the structure shown in Figure 3.6.

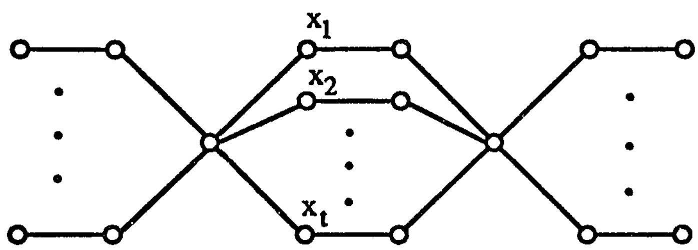


Figure 3.6. The Graphs B_{2r}, Where $r \geq 3$ and $t \geq 2$.

Then $i(D_{n,r}) = 2^{n/2-r} \cdot i(B_{2r}) = 2^{n/2-r}(2^{r-1}+1) \leq 2^{(n-2)/2} + 2^{(n-6)/2}$.

Theorem 3.8. *If G is a bipartite graph of order $n \geq 4$ which contains cycles, then*

$$i(G) \leq z(n) = \begin{cases} 3 \cdot 2^{(n-5)/2} & \text{if } n \text{ is odd} \\ 2^{(n-2)/2} & \text{if } n \text{ is even} \end{cases}$$

with the only exception that n is even and $G \cong D_{n,r}$.

Remark: The graphs with the structures shown in Figure 3.7 attain the upper bounds in Theorem 3.8.

Figure 3.7. Graphs of Order n With $z(n)$ MIS's.

To prove this theorem, we first develop several lemmas. In the following discussion, we denote S to be the set of all bipartite graphs with cycles which are not isomorphic to $D_{n,r}$ and we say property $P(m)$ holds if $i(H) \leq z(p)$ for any graph $H \in S$ of order $p < m$.

Lemma 3.2. *Suppose* $n \geq 7$ *is odd and property* $P(n)$ *holds. If* G *is a connected graph in* S *of order* n *such that* $\delta(G) \geq 2$ *and each vertex of degree* ≥ 4 *is in all cycles of* G, *then* $i(G) \leq 3 \cdot 2^{(n-5)/2}$.

Proof. We first assume G has a vertex v of degree 3 such that $G - v$ has cycles. For $G - v \not\cong D_{n-1,r}$, it follows from Lemma 2.1, property $P(n)$ and Theorem 3.6 that

$$i(G) \leq i(G-v) + i(G-v-N(v)) \leq 2^{(n-1-2)/2} + 2^{\lfloor (n-4)/2 \rfloor} = 3 \cdot 2^{(n-5)/2}.$$

For $G-v \cong D_{n-1,r}$, since $\delta(G) \geq 2$, $G-v$ must be connected, so $G-v \cong B_{n-1}$ and $i(G-v) = 2^{(n-1-2)/2} + 1$. Furthermore, $G_1 = G - v - N(v)$ is either an acyclic graph not isomorphic to $B_{n-4,r}$ or a graph having cycles. It follows from Theorem 3.7 or property $P(n)$ that

$$i(G_1) \leq \max\{2^{(n-4-1)/2} - 1,\ \ 3 \cdot 2^{(n-4-5)/2}\} = 2^{(n-5)/2} - 1.$$

Hence

$$i(G) \leq i(G-v) + i(G-v-N(v)) \leq 2^{(n-1-2)/2} + 1 + 2^{(n-5)/2} - 1 = 3 \cdot 2^{(n-5)/2}.$$

Now, we consider the case that each vertex of degree ≥ 3 is in all cycles of G. Then G must be a graph with the structures shown in Figure 3.8.

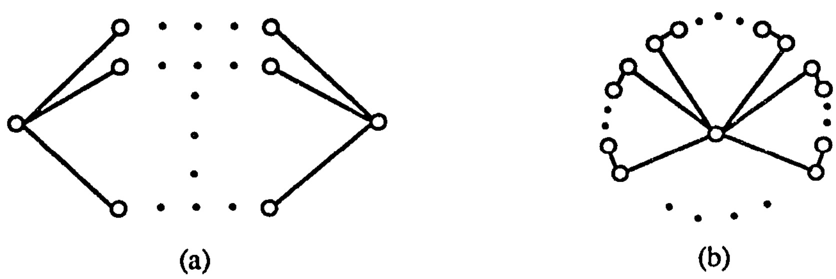


Figure 3.8. The Graphs With Each Vertex of Degree ≥ 3 in All Cycles.

Let $u \in V(G)$ which is in all cycles of G. Then $\deg(u) \geq 3$ and $G - u \not\cong B_{n-1,r}$ since n is odd, $\delta(G) \geq 2$, and G is bipartite. By Theorem 3.7, we have $i(G-u) \leq 2^{(n-1-2)/2}$ which gives

$$i(G) \leq i(G-u) + i(G-u-N(u)) \leq 2^{(n-3)/2} + 2^{\lfloor (n-4)/2 \rfloor} = 3 \cdot 2^{(n-5)/2}.$$

Therefore the lemma holds. □

Lemma 3.3. *Suppose* $n \geq 6$ *is even and Property* $P(n)$ *holds. If* G *is a graph in* S *of order* n *and* G *has a path* $P_5 = u_1u_2u_3u_4u_5$ *such that* $\deg(u_3) = 2$, $G - \{u_1, u_2, u_3, u_4, u_5\}$ *has cycles, and* $G - \{u_2, u_3, u_4, u_5\} \not\cong D_{n-4,r}$, *then* $i(G) \leq 2^{(n-2)/2}$.

Proof. From the assumption it follows that $G_1 = G - \{u_1, u_2, u_3\} \in S$, $G_2 = G - \{u_2, u_3, u_4\} \in S$, and $G_3 = G - \{u_2, u_3, u_4, u_5\} \in S$. By applying Lemma 2.1 twice, and using Fact 2.1 and property $P(n)$, we obtain

$$\begin{aligned} i(G) &\leq i(G - u_2 - u_4) + i(G - u_2 - u_4 - N(u_4)) + i(G - u_2 - N(u_2)) \\ &\leq i(G_2) + i(G_3) + i(G_1) \\ &\leq 3 \cdot 2^{(n-3-5)/2} + 2^{(n-4-2)/2} + 3 \cdot 2^{(n-3-5)/2} = 2^{(n-2)/2}. \end{aligned}$$ □

Lemma 3.4. *Suppose* $n \geq 6$ *is even and property* $P(n)$ *holds. If* G *is a connected graph in* S *of order* n *such that* $\delta(G) \geq 2$ *and each vertex of degree* ≥ 4 *is in all cycles of* G, *then* $i(G) \leq 2^{(n-2)/2}$.

Proof. We say a vertex v is good if $\deg(v) = 3$ and $G - v$ has cycles. Since $\delta(G) \geq 2$ and G is bipartite, $G - v - N(v) \not\cong B_{n-4,r}$ for any good vertex v.

We first consider the case that G has good vertices. If G has a good vertex v such that either $G_1 = G - v - N(v)$ is acyclic or G_1 has cycles but $G_1 \not\cong D_{n-4,r}$, then it follows from Theorem 3.7 or property $P(n)$ that

$i(G_1) \le 2^{(n-4-2)/2}$ and by Lemma 2.1,

$$i(G) \le i(G-v) + i(G_1) \le 3 \cdot 2^{(n-1-5)/2} + 2^{(n-6)/2} = 2^{(n-2)/2}.$$

For the case that $G_1 \cong D_{n-4,r}$ for every good vertex v, since $\delta(G) \ge 2$ and G is bipartite, G_1 must be connected, that is $G_1 \cong B_{n-4}$. Furthermore, G_1 must be a graph with the structure shown in Figure 3.9.

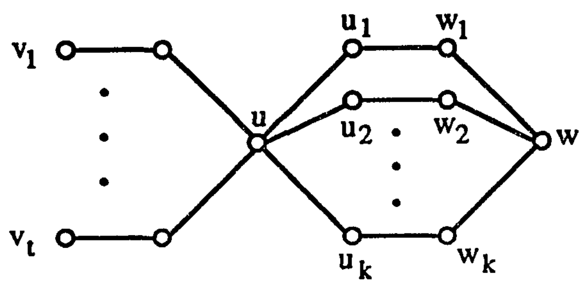


Figure 3.9. The Graphs B^*_{2r}.

Clearly, if $t \ge 1$, then $N(w) \cap N(v) = \phi$ and if $t = 0$, then either $N(u) \cap N(v) = \phi$ or $N(w) \cap N(v) = \phi$. Hence, without loss of generality, assume $N(w) \cap N(v) = \phi$. Now $\deg(v) = 3$ and $\delta(G) \ge 2$ imply $G - w$ has cycles. Thus, by the assumption that each vertex of degree ≥ 4 is in all cycles of G, it follows that $2 \le \deg(x) \le 3$ for $x = w$ or $x \in N(v)$ and if $\deg(u) \ge 4$, then G has at most one edge between $N(v)$

and $A = \{u_j : 1 \le j \le k\} \cup \{w_j : 1 \le j \le k\}$. Notice that either $N_1 = N(N(v)) \cap \{u_j : 1 \le j \le k\} = \phi$ or $N_2 = N(N(v)) \cap \{w_j : 1 \le j \le k\} = \phi$. Consequently, we have $\deg(w) = 2$ for otherwise $G - x - N(x) \ncong D_{n-4,r}$ for a good vertex x, where $x = w$ if $N_1 = \phi$ and $x = u$ otherwise, contradicting the assumption that $G - v - N(v) \cong D_{n-4,r}$ for every good vertex v. Similarly, we conclude that each vertex in $N(v)$ has degree 2. Now, for $\deg(u) \ge 4$, by Lemma 3.3, we have $i(G) \le 2^{(n-2)/2}$. For $\deg(u) \le 3$, since $\delta(G) \ge 2$ and $G - v - N(v) \cong D_{n-4,r}$ for every good vertex v, G must be the graph Q shown in Figure 3.10 and $i(G) \le 2^{(n-2)/2}$.

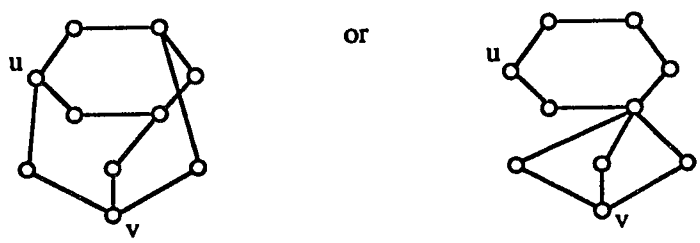


Figure 3.10. The Graphs Q.

Hence we assume G has no good vertex, that is each vertex of degree ≥ 3 must be contained in all cycles of G. This implies G must be a graph with the structures shown in Figure 3.8. If G is a cycle, then it is easy to see $i(G) \le 2^{(n-2)/2}$. Suppose G is not a cycle. If G is a graph of the structure in Figure 3.8 (b), then G has only one vertex v of degree ≥ 3. Since G is a bipartite graph of even order, $\deg(v) \ge 6$ and $G - v$ consists of $k \ge 3$

disjoint paths of even length, say $P_{n_1}, P_{n_2}, \cdots, P_{n_k}$. It follows that

$$i(G-v) = \prod_{j=1}^{k} i(P_{n_j}) \le \prod_{j=1}^{k} 2^{(n_j-1)/2} \le 2^{(n-4)/2},$$

and so

$$i(G) \le i(G-v) + i(G-v-N(v)) \le 2^{(n-4)/2} + 2^{\lfloor (n-7)/2 \rfloor} < 2^{(n-2)/2}.$$

Suppose G is a graph of the structure in Figure 3.8 (a), then G has exactly two vertices of degree ≥ 3, say v_1 and v_2. It follows that $\deg(v_1) = \deg(v_2) = d \ge 3$ and $G - \{v_1, v_2\}$ consists of k disjoint paths, say $P_{n_1}, P_{n_2}, \cdots, P_{n_k}$ with $n_1 \ge n_2 \ge \cdots \ge n_k$, where $k = d$ if v_1 and v_2 are not adjacent and $k = d-1$ otherwise. Since G is bipartite, either all n_j's are odd or all n_j's are even. For the former case, we have $k = d \ge 4$ and

$$i(G - \{v_1, v_2\}) = \prod_{j=1}^{k} i(P_{n_j}) \le \prod_{j=1}^{k} 2^{(n_j-1)/2} \le 2^{(n-6)/2}$$

which gives

$$\begin{aligned} i(G) &\le i(G - \{v_1, v_2\}) + i(G - v_1 - v_2 - N(v_2)) + i(G - v_1 - N(v_1)) \\ &\le 2^{(n-6)/2} + 2^{\lfloor (n-5)/2 \rfloor} + 2^{\lfloor (n-5)/2 \rfloor} < 2^{(n-2)/2}. \end{aligned}$$

For the latter case, let $P_{n_1} = u_1 u_2 \cdots u_{n_1}$. By Lemma 3.3, we assume $n_1 \le 4$. If $n_1 = 4$, then both $L_1 = G - \{u_1, u_2, u_3\}$ and $L_2 = G - \{v_2, u_3, u_4\}$ have cycles and $L_3 = G - \{u_1, u_2, u_3, v_1\}$ is a tree. Furthermore, $L_3 \ncong T_{n-4}$ unless G is one of the graphs shown in Figure 3.11.

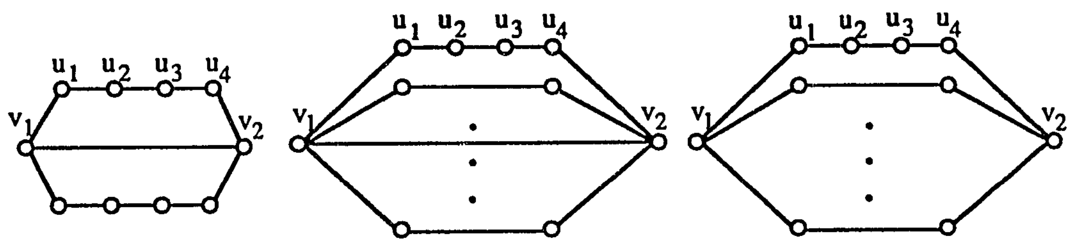


Figure 3.11. Graphs G With $L_3 \cong T_{n-4}$.

If $L_3 \ncong T_{n-4}$, then it follows that

$$\begin{aligned} i(G) &\le i(G - \{u_1, u_3\}) + i(G - u_1 - u_3 - N(u_1)) + i(G - u_3 - N(u_3)) \\ &= i(L_1) + i(L_3) + i(L_2) \\ &\le 3 \cdot 2^{(n-3-5)/2} + 2^{(n-4-2)/2} + 3 \cdot 2^{(n-3-5)/2} = 2^{(n-2)/2}. \end{aligned}$$

If $L_3 \cong T_{n-4}$, then it is easy to check $i(G) \le 2^{(n-2)/2}$. Hence we assume $n_1 \le 2$ which implies $n_1 = n_2 = \cdots = n_k = 2$. Since $G \ncong B_n$, we conclude that v_1 must be adjacent to v_2, and so $i(G) = 2^{(n-2)/2}$. Therefore, we have shown that $i(G) \le 2^{(n-2)/2}$ in each case and the lemma holds. □

Proof of Theorem 3.8. We proceed by induction on n. For $4 \le n \le 5$, the result is easy to check. Assume the result for $n \le k$. Then for any non-acyclic bipartite graph H of even order $p \leq k$,

$$i(H) \leq \begin{cases} 2^{(p-2)/2} + 1 & \text{if } H \text{ is connected} \\ 2^{(p-2)/2} + 2^{(p-6)/2} & \text{if } H \text{ is disconnected.} \end{cases}$$

Now, consider $n = k + 1 \geq 6$. Let $G \in S$ be a graph of order n. If G is disconnected, then the result follows directly from Theorem 3.6 and the induction hypothesis. Hence we assume G is connected. We consider two cases.

Case 1. G has a vertex v of degree ≥ 4 such that $G - v$ has cycles. In this case, if n is even, then the result follows easily from Lemma 2.1, Theorem 3.6, and the induction hypothesis. If n is odd, then one of the followings must be true.

(i) $G - v \not\cong D_{n-1,r}$. Then, similarly, it follows easily that $i(G) \leq 3 \cdot 2^{(n-5)/2}$.

(ii) $G - v \cong D_{n-1,r}$. In this case, if $\deg(v) = 4$, then $G' = G - v - N(v)$ is either an acyclic graph not isomorphic to $B_{n-5,r}$ or an non-acyclic graph not isomorphic to $D_{n-5,r}$. Hence it follows from Theorem 3.6 or Theorem 3.7 or the induction hypothesis that $i(G') \leq 2^{(n-5-2)/2}$ which gives

$$\begin{aligned} i(G) &\leq i(G - v) + i(G') \\ &\leq (2^{(n-1-2)/2} + 2^{(n-1-6)/2}) + 2^{(n-7)/2} = 3 \cdot 2^{(n-5)/2}. \end{aligned}$$

Case 2. Every vertex of degree ≥ 4 is in all cycles of G. We first assume

$\delta(G) = 1$. Let v be an end vertex of G and u be the neighbor of v. Let $G_1 = G - \{u, v\}$ and $G_2 = G - u - N(u)$. Then one of the following holds.

(1) n is odd. In this case, if $\deg(u) \geq 3$, then it follows from Lemma 2.1 and Theorem 3.6 that

$$i(G) \leq i(G_1) + i(G_2) \leq 2^{\lfloor (n-2)/2 \rfloor} + 2^{\lfloor (n-4)/2 \rfloor} = 3 \cdot 2^{(n-5)/2}.$$

If $\deg(u) = 2$, then G_1 has cycles and $G_2 = G - \{u, v, w\} \not\cong [(n-3)/2]K_2$ since G is a bipartite graph with cycles, where w is the second neighbor of u. By Corollary 3.1 or the induction hypothesis, we have

$$\begin{aligned} i(G_2) &\leq \max\{2^{(n-3-2)/2} + 2^{(n-3-4)/2}, 2^{(n-3-2)/2} + 2^{(n-3-6)/2}\} \\ &= 2^{(n-3-2)/2} + 2^{(n-3-4)/2} \end{aligned}$$

which gives

$$i(G) \leq i(G_1) + i(G_2) \leq 3 \cdot 2^{(n-2-5)/2} + 2^{(n-5)/2} + 2^{(n-7)/2} = 3 \cdot 2^{(n-5)/2}.$$

(2) n is even. For $\deg(u) \geq 4$, similarly, we conclude that $G_1 \not\cong [(n-2)/2]K_2$,

$$\begin{aligned} i(G_1) &\leq \max\{2^{(n-2-2)/2} + 2^{(n-2-4)/2},\ 2^{(n-2-2)/2} + 2^{(n-2-6)/2}\} \\ &= 2^{(n-2-2)/2} + 2^{(n-2-4)/2}, \end{aligned}$$

and $i(G) \leq i(G_1)+i(G_2) \leq 2^{(n-2)/2}$. If $\deg(u) = 2$, then G_1 is a connected bipartite graph with cycles, and so $i(G_1) \leq 2^{(n-2-2)/2}+1$. Since $G \not\cong D_{n,r}$, G_2 is either an acyclic bipartite graph not isomorphic to $B_{n-3,r}$ or non-acyclic. It follows from Theorem 3.7 or induction hypothesis that $i(G_2) \leq \max\{2^{(n-4)/2} - 1,\ 3\cdot 2^{(n-3-5)/2}\} = 2^{(n-4)/2} - 1$ which gives $i(G) \leq i(G_1)+i(G_2) \leq 2^{(n-2)/2}$. Hence we assume $\deg(u) = 3$. Now, if G_1 is disconnected, then G_1 must have cycles since G has cycles, and so $G_2 \not\cong [(n-4)/2]K_2$. It follows that $i(G_1) \leq 2^{(n-2-2)/2} + 2^{(n-2-6)/2}$ and

$$\begin{aligned} i(G_2) &\leq \max\{2^{(n-4-2)/2} + 2^{(n-4-4)/2},\ 2^{(n-4-2)/2} + 2^{(n-4-6)/2}\} \\ &= 2^{(n-4-2)/2} + 2^{(n-4-4)/2}. \end{aligned}$$

Hence

$$\begin{aligned} i(G) &\leq i(G_1) + i(G_2) \\ &\leq 2^{(n-2-2)/2} + 2^{(n-2-6)/2} + 2^{(n-4-2)/2} + 2^{(n-4-4)/2} = 2^{(n-2)/2}. \end{aligned}$$

If G_1 is connected, then $i(G_1) \leq 2^{(n-2-2)/2} + 1$ and $G_2 \not\cong [(n-4)/2]K_2$ unless G is a graph with the structure shown in Figure 3.12 where G_1 is a tree not isomorphic to T_{n-2}. It follows from Theorem 3.6 or Proposition 1.3 that either $i(G_2) \leq 2^{(n-4)/2} - 1$ or $i(G_2) = 2^{(n-4)/2}$ and $i(G_1) \leq 2^{(n-4)/2}$ which gives

$$i(G) \leq i(G_1) + i(G_2) \leq 2^{(n-4)/2} + 2^{(n-4)/2} + 1 - 1 = 2^{(n-2)/2}.$$

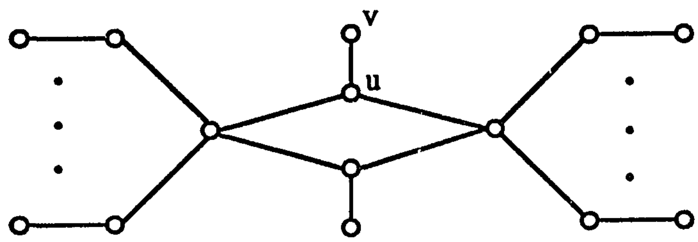


Figure 3.12. Graphs G With $G - u - N(u) = \frac{n-4}{2}K_2$.

Now, assume $\delta(G) \geq 2$. Then the inequality follows directly from Lemmas 3.2 and 3.4. Therefore we have shown $i(G) \leq z(n)$ for $n = k + 1$ and the theorem holds. □

From Theorems 3.7 and 3.8, the following two results follow immediately.

Corollary 3.2. *If G is a bipartite graph of order n such that $G \not\cong B_{n,r}$ and $D_{n,r}$, then $i(G) < T(n)$.*

Theorem 3.9. *The maximum number of MIS's among all connected bipartite graphs of order n is*

$$b_1(n) = T(n)$$

and the only connected bipartite graphs of order n with $b_1(n)$ MIS's are T_n and B_n shown in Figures 1.1 and 3.6.

CHAPTER IV

MAXIMUM INDEPENDENT SETS AND OPEN PROBLEMS

4.1. Maximum Independent Sets in k-Connected Graphs

Note that the graphs H_n, $H_{1,n}$, and $H_{2,n}$ are well-covered, it follows that $F(n) = f(n)$ and $F_k(n) = f_k(n)$ for $k \leq 2$. In this section, we derive $F_k(n)$ for $k \geq 3$ in a similar way as in Section 3.1.

Given two positive integers k and n such that $k \leq n$, define

$$D_n = \begin{cases} K_k + tK_3 & \text{if } n = 3t + k \\ K_k + [(t-1)K_3 \cup K_4] & \text{if } n = 3t + k + 1 \\ K_k + (tK_3 \cup K_2) & \text{if } n = 3t + k + 2. \end{cases}$$

Since the graph D_n is k-connected and $i_m(D_n) \geq f(n-k)$, the next lemma follows immediately.

Lemma 4.1. *For each* $k \in Z^+$, $F_k(n) \geq f(n-k) = O(3^{\frac{n}{3}})$.

By applying Lemma 4.1 and Theorems 2.4 and 2.5, we obtain two structure properties of the extremal graphs for $F_k(n)$ in a way similar to Theorems 3.1 and 3.2.

Theorem 4.1. *Given an integer* $k \geq 1$, *there exist positive constants* n_0

and C *such that for* $n \geq n_0$, *if* G *is a* k*-connected graph of order* n *and* $i_m(G) = F_k(n)$, *then* G *has* k *vertices of degree* $\geq C \cdot n$.

Theorem 4.2. *Given an integer* $k \geq 1$, *there exists a positive integer* n_0 *such that if* G *is a* k*-connected graph of order* $n \geq n_0$ *with* $i_m(G) = F_k(n)$, *then*

$$G - \{v_1, \cdots, v_k\} \cong H_{n-k}, \text{ where } \{v_1, \cdots, v_k\} \subseteq V(G).$$

Now, let G be a k-connected graph of order $n \geq n_0$ such that $i_m(G) = F_k(n)$ and let $A = \{v_1, v_2, \cdots v_k\} \subseteq V(G)$ such that $G' = G - A \cong H_{n-k}$. Similar to Section 3.1, we divide the set $I_m(G)$ of maximum independent sets of G into three disjoint subsets:

$$\begin{aligned}
I_m(G) &= I_1^* \cup I_2^* \cup I_3^*, \text{ where} \\
I_1^* &= \{S : S \in I_m(G) \text{ and } S \subseteq V(G')\}, \\
I_2^* &= \{S : S \in I_m(G) \text{ and } S \subseteq A\}, \text{ and} \\
I_3^* &= I_m(G) - (I_1^* \cup I_2^*),
\end{aligned}$$

so $i_m(G) = |I_1^*| + |I_2^*| + |I_3^*|$.

From now on, we assume $n \geq 4k + 8$. Then $|I_2^*| = 0$ since each maximum independent set contains at least $k + 3$ vertices. Clearly, we have $|I_1^*| \leq f(n-k)$. Also, parallel to the discussions for $f_k(n)$ in Section 3.1, we can derive that for $k \geq 3$,

$$|I_3^*| \leq r(n,k) = \begin{cases} 3\cdot 2^{t-1} + \binom{3}{2} & \text{if } n-k = 3t \\ 9\cdot 2^{t-2} + 2\cdot\binom{3}{2} & \text{if } n-k = 3t+1 \\ 3\cdot 2^{t} & \text{if } n-k = 3t+2 \end{cases}$$

and both $|I_1^*| = f(n-k)$ and $|I_3^*| = r(n,k)$ hold only if G is one of the graphs $H'_{k,n}$ which are k-connected and have an induced subgraph isomorphic to $H_{3,n-k+3}$.

Therefore, by setting

$$M_k(n) = f(n-k) + r(n,k)$$

$$= \begin{cases} 3^t + 3\cdot 2^{t-1} + 3 & \text{if } n-k = 3t \\ 4\cdot 3^{t-1} + 9\cdot 2^{t-2} + 6 & \text{if } n-k = 3t+1 \\ 2\cdot 3^t + 3\cdot 2^t & \text{if } n-k = 3t+2, \end{cases}$$

then we have the following theorem.

Theorem 4.3. *For any* $k \geq 3$, *there is a positive integer* n_0 *such that if* $n \geq n_0$, *then*

$$F_k(n) = M_k(n)$$

and the only k-connected graphs of order n *which have* $F_k(n)$ *maximum independent sets are* $H'_{k,n}$.

4.2. Open Problems

In Section 3.1, we determined $f_k(n)$ for n large enough. We have the following open problem.

PROBLEM 1. Determine $f_k(n)$ and the extremal graphs for small n.

I conjecture that for small n the extremal graphs for $f_k(n)$ will have similar structure to $H_{k,n}$.

Similarly, there is a problem for maximum independent sets.

PROBLEM 2. Determine $F_k(n)$ and the extremal graphs for small n.

In Sections 3.2 and 3.3, we determined $b(n)$ and $b_1(n)$. It will be nice to solve the following questions.

PROBLEM 3. What is $b_k(n)$ for $k \geq 2$ and which k-connected bipartite graphs of order n have so many MIS's?

PROBLEM 4. What is $B_k(n)$ for $k \geq 1$ and which k-connected bipartite graphs of order n have so many maximum independent sets?

Appendix A

The Proof of Claim 2

Suppose, to the contrary, that $G_{\ell_1}, G_{\ell_2}, G_{\ell_3}$ are not distinct, say $G_{\ell_1} = G_{\ell_2} = G_1$. For $k = 3$ or 4, it is easy to check $i(G) < m_k(n)$, contradicting the choice of G. So we assume $k \geq 10$. First we consider $n - k = 3t$. Since $i(G) = |I_1| + |I_2| + |I_3| \geq m_k(n) = f(n-k) + f(k) + 3 \cdot 2^{t-1} + 14$, it follows that

$$|I_3| \geq 3 \cdot 2^{t-1} + 14.$$

From Property 3.2 it follows that

$$|\{X : X \in I_3 \text{ and } X \cap A \subseteq \{v_1, v_2, v_3\}\}| \leq 3 \cdot 2^{t-1} + 2.$$

Without loss of generality, let $v_4, \cdots, v_{3+t_1+t_2}$ be vertices such that each v_j is not adjacent to $x_j \geq 2$ vertices in G' for $4 \leq j \leq 3 + t_1$ with $x_4 \geq x_5 \geq \cdots \geq x_{3+t_1}$ and each v_j is not adjacent to one vertex in G' for $4 + t_1 \leq j \leq 3+t_1+t_2$. Recall that each v_i in $A - \{v_1, v_2, v_3\}$ is adjacent to all vertices of each component in G' differing from G_{ℓ_j} for $1 \leq j \leq 3$ and there are at most 6 non-adjacent pairs with one vertex in $A - \{v_1, v_2, v_3\}$ and the other vertex in G_{ℓ_j} for $1 \leq j \leq 3$. It follows that $\sum_{j=1}^{t_1} x_j + t_2 \leq 6$, so $t_1 \leq 3$ and $t_1 = 3$ implies $t_2 = 0$ and $x_4 = x_5 = x_6 = 2$. Furthermore, for each $4 \leq j \leq t_1 + 3$, $i_3(v_j) \leq x_j \leq 4$, and $i_3(v_j) \leq 2$ if $x_j = 3$. Let $I_3' = \{X : X \in I_3$ and $X \cap \{v_4, \cdots, v_{3+t_1+t_2}\} \neq \phi\}$. Then $|I_3| \leq 3 \cdot 2^{t-1} + 2 + |I_3'|$. Now, by applying Claim 1 it follows that $y_1 = |\{X : X \in I_3'$ and $X \cap \{x_{4+t_1}, \cdots, v_{3+t_1+t_2}\} \neq \phi\}| \leq t_2$ and for each vertex v_j with $4 \leq j \leq 3 + t_1$, if $x_j = 2$, then there are at most 3 MIS's in I'_3 which contain v_j. Hence, by Property 3.2, we have

$$y_2 = |\{X : X \in I'_3 \text{ and } X \cap A \subseteq \{v_1, \cdots, v_{3+t_1}\}\}| \leq \begin{cases} 4+3 & \text{if } t_1 = 1 \\ 2 \cdot (2+3) + 1 & \text{if } t_1 = 2 \text{ and } x_1 = x_2 = 3 \\ (4+3) + 3 & \text{if } t_1 = 2 \text{ and } x_1 \neq x_2 \\ 3 \cdot 3 & \text{if } t_1 = 3. \end{cases}$$

It follows that $|I'_3| = y_1 + y_2 \leq 11$ which gives

$$|I_3| \leq 3 \cdot 2^{t-1} + 13 < 3 \cdot 2^{t-1} + 14 \leq |I_3|,$$

a contradiction. Therefore the claim is true in this case.

Next, we consider $n - k = 3t + 1$. Since

$$i(G) = |I_1| + |I_2| + |I_3| \geq m_k(n) = f(n-k) + f(k) + 9 \cdot 2^{t-2} + 40,$$

it follows that

$$|I_3| \geq 9 \cdot 2^{t-2} + 40.$$

On the other hand, it is easy to see from Property 3.3 that

$$\begin{aligned} z_1 &= |\{X : X \in I_3 \text{ and } X \cap A \subseteq \{v_1, v_2, v_2\}\}| \\ &= \sum_{j=1}^{3} i_3(v_j) + |\{X : X \in I_3 \text{ and } |X \cap \{v_1, v_2, v_3\}| = 2\}| \\ &\quad + |\{X : X \in I_3 \text{ and } X \cap A = \{v_1, v_2, v_3\}\}| \\ &\leq 3 \cdot (3 \cdot 2^{t-2}) + 2 \cdot \binom{3}{2} + \binom{3}{3} - 2 = 9 \cdot 2^{t-2} + 5. \end{aligned}$$

From the discussion for $n - k = 3t$ it follows that

$$z_2 = |\{X : X \in I_3, X \cap \{v_4, \cdots, v_k\} \neq \phi, \text{ and } X \cap V(G_r) = \phi\}| \leq 11.$$

Now let

$$I_3' = \{X : X \in I_3, X \cap \{v_4, \cdots, v_k\} \neq \phi, \text{ and } X \cap V(G_r) \neq \phi\}$$

and $z_3 = |I_3'|$. Then $|I_3| = z_1 + z_2 + z_3$. Without loss of generality, let t_1 be the integer such that

$$y_j = |V(G_r)| - |N(v_j) \cap V(G_r)| > 0 \text{ for } 4 \leq j \leq 3 + t_1$$

and $N(v_j) \supseteq V(G_r)$ for $j \geq 4 + t_1$. Then, for each $X \in I_3'$, $X \cap \{v_{4+t_1}, \cdots, v_k\} = \phi$. Since each vertex in G_r is not adjacent to at most 3 vertices in A, each X in I_3' contains exactly one vertex in $\{v_4, \cdots, v_{3+t_1}\}$.

For $4 \le j \le 3 + t_1$, let x_j be the number of vertices in $\bigcup_{i=1}^{3} G_{\ell_i}$ which are not adjacent to v_j. Recall that each vertex in $A - \{v_1, v_2, v_3\}$ is adjacent to all vertices of each triangle component in G' differing from G_{ℓ_j} for $1 \le j \le 3$, there are at most 6 non-adjacent pairs of vertices with one vertex in $A - \{v_1, v_2, v_3\}$ and the other one in G_{ℓ_j} for $1 \le j \le 3$ and 3 non-adjacent pairs of vertices with one vertex in $A - \{v_1, v_2, v_3\}$ and the other one in G_r. It follows that $\sum_{j=4}^{3+t_1} y_j \le 3$, $\sum_{j=4}^{3+t_1} x_j \le 6$, each $x_j \le 4$ and $i_3(v_j) \le x_j y_j$. Hence, for $t_1 = 1$, it follows from Property 3.3 that

$$z_3 \le i_3(v_4) + 2 \cdot \binom{3}{1} + \binom{3}{2} \le x_4 \cdot y_4 + 6 + 3 \le 4 \cdot 3 + 9 = 21.$$

For $t_1 = 2$, since $y_4 + y_5 \le 3$, we may assume $y_4 \le 2$ and $y_5 = 1$. It follows that

$$\begin{aligned} z_3 &\le \left[i_3(v_4) + 2 \cdot \binom{3}{1} + \binom{3}{2} \right] + \left[i_3(v_5) + \binom{2}{1} + 1 \right] \\ &= [i_3(v_4) + i_3(v_5)] + 12 \le [x_4 y_4 + x_5] + 12 \le 22. \end{aligned}$$

For $t_1 = 3$, we have $y_4 = y_5 = y_6 = 1$, and so

$$\begin{aligned} z_3 &\le \left[i_3(v_4) + \binom{2}{1} + 1\right] + \left[i_3(v_5) + \binom{2}{1} + 1\right] + \left[i_3(v_6) + \binom{2}{1} + 1\right] \\ &\le (x_4 + x_5 + x_6) + 9 \le 6 + 9 = 15. \end{aligned}$$

Hence we have shown that

$$|I_3| = z_1 + z_2 + z_3 \le (9 \cdot 2^{t-2} + 5) + 11 + 22 = 9 \cdot 2^{t-2} + 38,$$

contradicting $|I_3| \geq 9 \cdot 2^{t-2} + 40$. Therefore the claim is also true for $n - k = 3t + 1$. This concludes the proof. □

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