8-1989

Greatest Common Subgraphs

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GREATEST COMMON SUBGRAPHS

by

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A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
August 1989

GREATEST COMMON SUBGRAPHS

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A greatest common subgraph of a family $\mathcal{G}$ of graphs, all of the same size, is a graph of maximum size that is a common subgraph of every graph in $\mathcal{G}$. In this dissertation several topics concerning this concept as well as some generalizations, variations and greatest common subgraph parameters are investigated.

Chapter I is an overview of the history of greatest common subgraphs and related topics. It provides also a background for the next chapters.

In Chapter II a greatest common subgraph index is introduced. It divides the set of all graphs into two classes (those of finite index and of infinite index) and the problem of determining which graphs belong to which category is examined. The relationships between the greatest common subgraph index and other graphical parameters are established.

Chapter III is devoted to the study of existence. It is proved that if, for a given graph G, there exist two graphs G_1 and G_2 of equal size such that G is their unique greatest common subgraph, then there exist graphs G_1 and G_2 of size only one greater than size of G, perhaps even when G, G_1 and G_2 are required to have some graphical property. The characterization of such graphs G is presented when the property is that of being connected outerplanar, connected planar, or unicyclic.

In Chapter IV two variations of common substructures are investigated, namely maximal common subgraphs and absorbing common subgraphs. The generalization of greatest common subgraphs for graphs of arbitrary size (not necessarily equal size) is considered. A metric on the set of all graphs is defined in terms of edge deletions and

edge rotations. Bounds on the distance between graphs are given in terms of the size of graphs and size of a greatest common subgraph. Finally, a duality theorem establishes relationships between greatest common subgraphs and least common supergraphs.

Order Number 8923548

Greatest common subgraphs

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Western Michigan University, 1989

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ACKNOWLEDGEMENTS

I wish to thank my advisor Professor Gary Chartrand for his mathematical expertise and assistance; his deep enthusiasm for graph theory has proven to be contagious. I also greatly appreciate his genuine interest in my family's well-being.

Special thanks are due to Professor Ortrud R. Oellermann and Dr. Mark Johnson for their patience and diligence in evaluating and guiding my dissertation. Thanks also to Professor Shashi Kapoor and Professor Kung Wei Yang for serving on my committee and always showing great interest in my research.

I would like to thank the many professors at Western Michigan University who have made the study of higher mathematics such a rewarding experience. In particular, I would like to thank the excellent teachers, Professor Gary Chartrand, Professor Allen Schwenk, Professor Arthur T. White and Professor Kung Wei Yang for their outstanding lectures.

Professor Yousef Alavi and Professor Joseph Buckley provided valuable support and interest in my work and career for which I am grateful.

Grzegorz Kubicki

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CHAPTER I

PRELIMINARIES

In this chapter, we begin with a few preliminary definitions that will be used throughout the dissertation. The second section provides an historical background of the theory of greatest common subgraphs of graphs and its related problems.

1.1 Introduction

All graph-theoretical terms not defined in this dissertation have the meaning as in Chartrand and Lesniak [4].

As usual, $|S|$ denotes the cardinality of a set S. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. The number $|V(G)|$ is called the order of a graph G and $|E(G)|$ is called the size of G.

A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $v \in V(G)$, then $G - v$ denotes the subgraph with vertex set $V(G) - \{v\}$ and whose edges are all those of G not incident with v. If $e \in E(G)$, then $G - e$ is the subgraph having vertex set $V(G)$ and edge set $E(G) - \{e\}$. The graph obtained by the deletion of a set S of vertices or edges, denoted by $G - S$ is defined analogously. If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G induced by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. Similarly, if F is a nonempty subset of $E(G)$, then the subgraph $\langle F \rangle$ induced by F is the graph whose edge set is F and whose vertex set consists of those vertices of G incident with at least one edge of F.

If G_1 and G_2 are two graphs with disjoint vertex sets, then their <u>union</u> $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, whereas their <u>join</u> $G_1 + G_2$ is defined as the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

1.2 Historical Background

The concept of greatest common subgraphs of graphs was introduced in Chartrand, Saba and Zou [6]. A graph G without isolated vertices is called a <u>greatest common subgraph</u> of a set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $n \geq 2$, of graphs of the same size if G is a graph of maximum size that is isomorphic to a subgraph of each graph G_i, $1 \leq i \leq n$. The set of all greatest common subgraphs of $\mathcal{G}$ is denoted by gcs $\mathcal{G}$. For example, if $\mathcal{G} = \{G_1, G_2\}$ for the graphs of Figure 1.1, then gcs $\mathcal{G} = \{H_1, H_2\}$.

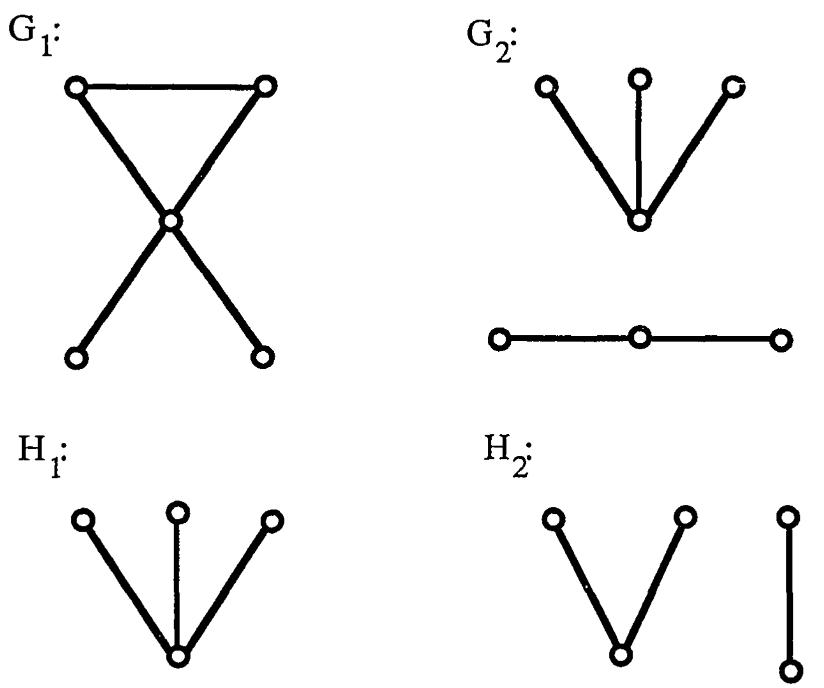


Figure 1.1

It is not unusual that a set $\mathcal{G}$ has more then one greatest common subgraph; in fact, the following result was established in [7].

Theorem 1A For every pair m, n of integers with $n \geq 2$ and $m \geq 1$, there exist n pairwise nonisomorphic graphs $G_1, G_2, \ldots, G_n$ of equal size such that

$$|gcs(G_1, G_2, \ldots, G_n)| = m.$$

However, a more interesting problem is to find, for a given graph G, two nonisomorphic graphs G_1 and G_2 of equal size (or a set $\mathcal{G}$ of graphs of equal size) such that G is the unique greatest common subgraph of G_1 and G_2 (of a set $\mathcal{G}$, respectively). This result was obtained in [7] and we state it for future reference.

Theorem 1B If G is a graph without isolated vertices, then there exist non-isomorphic graphs G_1 and G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$.

In the proof of this result, the graphs G_1 and G_2 constructed have size only one greater than the size of G. The problem of finding, for a given graph G, a family $\mathcal{G}$ of graphs of the same size (but of size large compared to the size of G) such that gcs $\mathcal{G} = \{G\}$ leads to a concept of greatest common subgraph index. The gcs index i(G) of a graph G without isolated vertices is the least positive integer q_0 such that for any integer $q > q_0$ and every set $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $n \geq 2$, of graphs of size q for which $G \in$ gcs $\mathcal{G}$, it follows that $|\text{gcs } \mathcal{G}| > 1$. If no such q_0 exists, then we write $i(G) = \infty$. This concept was introduced in [7], where the values of i(G) for complete graphs, paths and cycles were also established . New results about this topic will be presented in Chapter II.

The problem related to the gcs index is that of determining which graphs G have the property that there are nonisomorphic graphs G_1 and G_2 of equal size such that

$gcs(G_1, G_2) = \{G\}$ and $|E(G_i)| - |E(G)|$ is large, $i = 1, 2$. It was shown in [5] that such graphs exist. This concept motivated the definition of the greatest common subgraph number [5]. In Chapter II, we establish relationships between the gcs index and the gcs number of a graph.

In the proof of Theorem 1B, one of G_1 and G_2 is connected while the other graph is disconnected. However, Chartrand, Johnson and Oellermann [3] proved that if G is connected but not complete, then there are nonisomorphic connected graphs G_1 and G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$. Later a more general class of problems was investigated. Let P be a graphical property. For a given graph G without isolated vertices having property P, we ask whether there exist non-isomorphic graphs G_1 and G_2 of equal size and having property P such that $gcs(G_1, G_2) = \{G\}$. If P is the property of being 2-connected, then the following characterization was given in [5]. For a 2-connected graph G, there exist non-isomorphic 2-connected graphs G_1 and G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$ if and only if $G \not\cong K_n$ $(n \geq 3)$ and $G \not\cong K_n - e$ $(n \geq 4)$. In the same paper, it was shown that for every n-chromatic graph $(n \geq 2)$, we are able to construct non-isomorphic n-chromatic graphs G_1 and G_2 of the same size such that $gcs(G_1, G_2) = \{G\}$. Chartrand and Zou [8] characterized trees that are unique greatest common subgraphs of two suitably chosen nonisomorphic trees of equal size. Let $D(t)$ denote a tree consisting of two stars $K(1, t)$ whose central vertices are connected by a path of length 3. If T is a tree, then $gcs(T_1, T_2) = \{T\}$ for some nonisomorphic trees T_1 and T_2 of equal size if and only if $T \not\cong P_n$, $n = 2, 4, 5, \ldots$ and $T \not\cong D(t)$, $t \geq 2$. In Chapter III we present a solution of this problem when the property P is that of being connected outerplanar, connected planar, or unicyclic.

There are several concepts related to greatest common subgraphs that have been studied. Greatest common induced subgraphs have been considered in [3], [5] and [8] and this concept has proved to be considerably easier to study than the greatest common subgraph concept. Also the related problems for digraphs have been considered in [3]. However, we will not investigate digraphs and induced subgraphs in this dissertation.

Another variation of greatest common substructures, namely maximal common subgraphs of graphs, has been examined by Zou [13]. For a labeled graph G and nonisomorphic subgraphs H and F, we say that H can be extended to F if $V(H) \subseteq V(F)$ and $E(H) \subseteq E(F)$. If G_1 and G_2 are nonempty graphs of equal size, then H is a maximal common subgraph of G_1 and G_2 if H is isomorphic to some subgraph H_1 of G_1 and some subgraph H_2 of G_2, and moreover H_1 and H_2 cannot be extended (in G_1 and G_2, respectively) to any other common subgraph of G_1 and G_2 whose size is one more than that of H. In Chapter IV, we will consider the unlabeled version of maximal common subgraphs.

The concept of greatest common subgraph was introduced in [6] mainly for the purpose of providing an upper bound for a distance between graphs. Chartrand, Saba and Zou [6] defined the distance between graphs of equal order and size in terms of edge rotation. A graph G can be transformed into a graph H by an edge rotation if G contains distinct vertices u, v and w such that $uv \in E(G)$, $uw \notin E(G)$ and $H \cong G - uv + uw$. A graph G_1 can be transformed into a graph G_2, denoted $G_1 \to G_2$, if either (1) $G_1 \cong G_2$, or (2) there exists a sequence

$$G_1 \cong H_0,\ H_1, \ldots, H_n \cong G_2 \ \ (n \geq 1) \tag{1.1}$$

of graphs such that H_i can be transformed into H_{i+1} by an edge rotation or an edge deletion for $i = 0, 1, \ldots, n - 1$.

The edge rotation distance $d(G_1, G_2)$ between graphs G_1 and G_2 of the same order and same size is defined as 0 if $G_1 \cong G_2$ and otherwise as the smallest positive integer n for which there exists a sequence as described in (1.1). Chartrand, Saba and Zou [6] established an upper bound for the distance between graphs.

Theorem 1C If G_1 and G_2 are nonempty graphs of the same order and of size q and if s is the size of a greatest common subgraph of G_1 and G_2, then $d(G_1, G_2) \leq 2(q - s)$.

The concept of distance between graphs will be generalized in Chapter IV for graphs of arbitrary order and size. To give bounds for the distance between two graphs, it will be necessary to define greatest common subgraphs for graphs of arbitrary size.

A concept dual to greatest common subgraph is the concept of a least common supergraph. For a set $\mathcal{G}$ of graphs of equal size, a graph H without isolated vertices is called a least common supergraph of $\mathcal{G}$ if H is a graph of minimum size such that each graph in $\mathcal{G}$ is isomorphic to a subgraph of H. Basic results about least common supergraphs are presented in [2]. We will show in Chapter IV that least common supergraphs are basically the same concept as greatest common subgraphs because of a theorem that gives a relationship between them in terms of a complement operation. With the aid of this idea, many result about greatest common subgraphs can be translated into and expressed for least common supergraphs.

CHAPTER II

GREATEST COMMON SUBGRAPH INDEX

2.1 Introduction

Consider the following existence question: " Is a given graph a unique greatest common subgraph of two suitably chosen nonisomorphic graphs ? ". This question was answered by Chartrand, Saba and Zou [7] where they proved that for every graph G without isolated vertices there exist nonisomorphic graphs G_1 and G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$. In the proof, the size of G_1 (and G_2) was one greater than the size of G.

A natural question arises: For a given graph G, how large can the sizes of graphs G_1 and G_2 be so that $gcs(G_1, G_2) = \{G\}$?

We will explain this idea with an example that was considered in [7]. Let $G \cong K_3$ and let q denote the size of a graph G_i. If $q = 4, 5$ or 6, then graphs G_1 and G_2 both of size q and such that $gcs(G_1, G_2) = \{G\}$ are given in Figure 2.1.

However, if $q > 6$ and each graph G_j contains K_3 as a subgraph, then $K_2 \cup P_3$ is also a subgraph of every G_j. In fact, let v_1, v_2 and v_3 be vertices of a triangle in G_j. If $\deg v_i \geq 4$ for some i $(1 \leq i \leq 3)$, then $K_2 \cup P_3 \subset G_j$ (see Figure 2.2 (a)). On the other hand, if $\deg v_i \leq 3$ for all i, then Gj must contain an edge incident with none of the vertices v_i (as in Figure 2.2 (b)) so that $K_2 \cup P_3 \subset G_j$. Hence K_3 is not the unique greatest common subgraph of G_j, $j = 1, 2$.

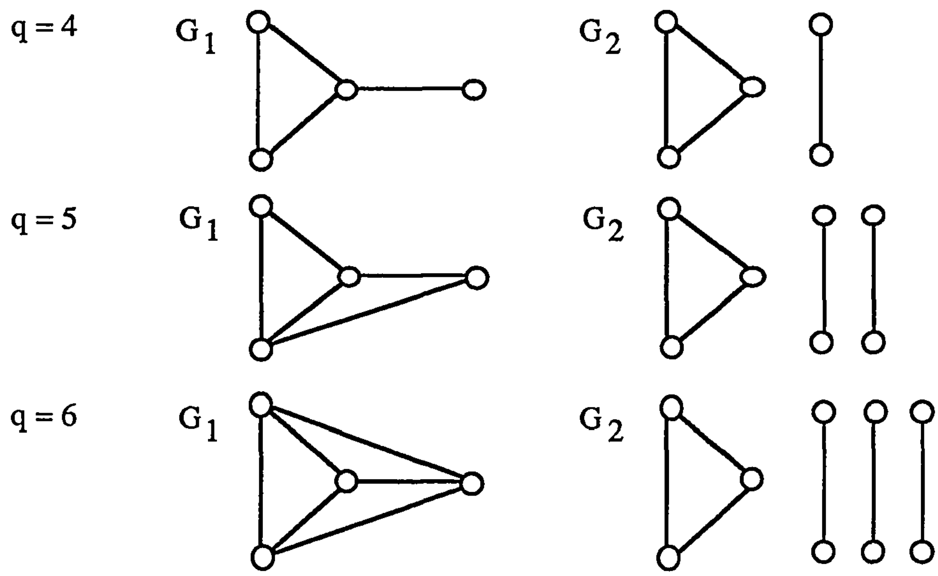


Figure 2.1

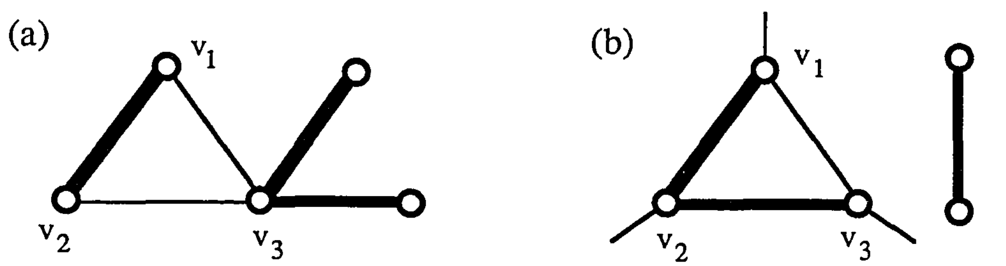


Figure 2.2

Therefore, for the graph $G \cong K_3$ there exist a "breaking point" equal to 6. If $4 \leq q \leq 6$, then we can construct a family $\mathcal{G}$ of graphs all of size q such that gcs $\mathcal{G} = \{G\}$, but when $q > 6$, no such construction is possible. This breaking point will be called the greatest common subgraph index of a graph G and denoted by $i(G)$. In the example presented above we have $i(K_3) = 6$. Before giving a formal definition, we will establish the following fact.

Theorem 2.1 Let G be a graph and let $G_1, G_2, \dots, G_n$, $n \geq 2$, be graphs of equal size for which $gcs(G_1, G_2, \dots, G_n) = \{G\}$. Then for all subsets $E_1 \subset E(G_1) - E(G)$, $E_2 \subset E(G_2) - E(G), \dots, E_n \subset E(G_n) - E(G)$ with $|E_1| = |E_2| = \dots = |E_n| \geq 1$,

$$gcs(G + E_1, G + E_2, \dots, G + E_n) = \{G\}.$$

Proof. Of course, G is a common subgraph of the graphs $G + E_1, G + E_2, \dots, G + E_n$. Suppose, to the contrary, that G is not a unique greatest common subgraph of $G + E_1, G + E_2, \dots, G + E_n$. It means that there exists a common subgraph H of $G + E_1, G + E_2, \dots, G + E_n$, $H \not\cong G$, with $q(H) \geq q(G)$. Of course, H is also a common subgraph of $G_1, G_2, \dots, G_n$, which contradicts the fact that G was the unique greatest common subgraph of $G_1, G_2, \dots, G_n$. ❑

Let G be a given graph of size q. Assume that there exists a family $\mathcal{G}$ of graphs of size q_0, $q_0 > q$, such that $gcs\ \mathcal{G} = \{G\}$. By Theorem 2.1, for every positive integer q' such that $q < q' < q_0$, we are able to construct a family $\mathcal{G}'$ of graphs of size q' such that $gcs\ \mathcal{G}' = \{G\}$.

2.2 Definition of GCS Index and Basic Properties

The formal definition of a gcs index is taken from [7]. For a graph G without isolated vertices, the <u>greatest common subgraph index</u> or <u>gcs index</u> of G, denoted $i(G)$, is the least positive integer q_0 such that for any integer $q > q_0$ and any set

$$\mathcal{G} = \{G_1, G_2, \dots, G_n\},\ n \geq 2,$$

of graphs of size q for which $G \in gcs\ \mathcal{G}$, it follows that $|gcs\ \mathcal{G}| > 1$, i.e., $gcs\ \mathcal{G}$ contains an element different from G. If no such q_0 exists, then we write $i(G) = \infty$.

Immediately from the definition of gcs index and from Theorem 2.1, it follows that if $i(G)$ is finite, then for every positive integer q, $q(G) < q \leq i(G)$, we are able to construct a family $\mathcal{G}$ of graphs of size q such that gcs $\mathcal{G} = \{G\}$, but if $q > i(G)$, then such construction is impossible. Moreover, if $i(G) = \infty$, then for every positive integer q, $q > q(G)$, we can find a family $\mathcal{G}$ of graphs of size q with gcs $\mathcal{G} = \{G\}$.

As the second example we compute the index of the following graph.

Example 2.2 Let $G \cong (K_2 \cup K_1) + K_1$ (see Figure 2.3). We will show that $i(G) = 10$.

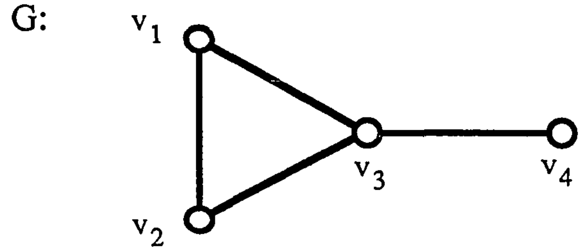


Figure 2.3

To prove that $i(G) \geq 10$, it is enough to find a family $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $n \geq 2$, of graphs of size 10 such that gcs $\mathcal{G} = \{G\}$. Let $\mathcal{G} = \{G_1, G_2, G_3\}$, where $G_1 \cong G \cup 6K_2$, $G_2 \cong (K_2 \cup 7K_1) + K_1$ and $G_3 \cong K_5$ (see Figure 2.4). Then each G_i, $i = 1, 2, 3$, is of size 10, and $\text{gcs}(G_1, G_2, G_3) = \{G\}$.

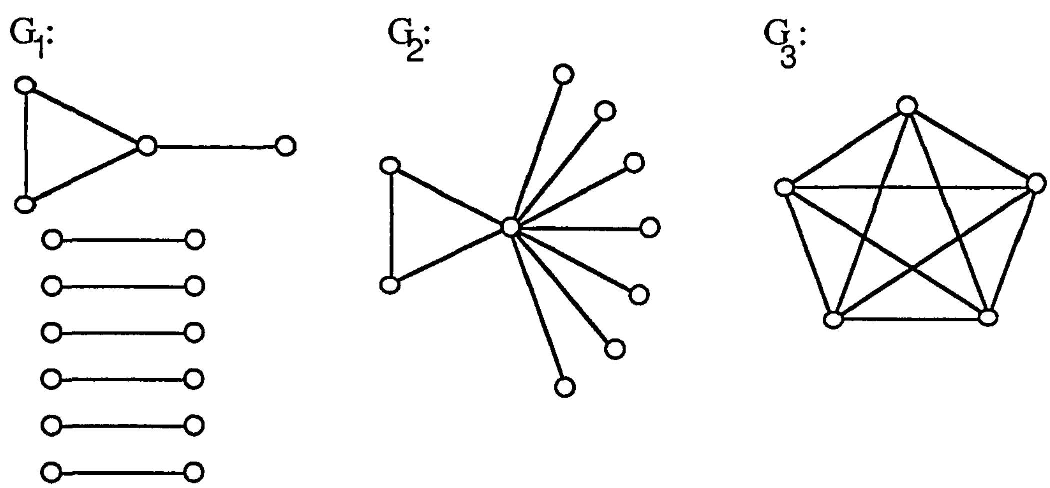


Figure 2.4

To prove the reverse inequality, consider a family $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$, $n \geq 2$, of graphs of size q, $q > 10$, for which $G \in \text{gcs}\,\mathcal{G}$. We will show that $K(1, 3) \cup K_2 \in \text{gcs}\,\mathcal{G}$, so that $|\text{gcs}\,\mathcal{G}| > 1$. Take any G_i, $1 \leq i \leq n$. Because $q(G_i) > 10 = \binom{5}{2}$, it follows that G_i has at least six vertices. Let v_1, v_2, v_3, v_4 be vertices in a copy of G in G_i, and let $x_1, x_2, \dots, x_r$ be other vertices of G_i, $r \geq 2$.

Consider the subgraph of G_i induced by the vertices $x_1, x_2, \dots, x_r$. If it contains an edge, then taking this edge as K_2 and a copy of $K(1, 3)$ from G, we have that $K(1, 3) \cup K_2 \subset G_i$. Otherwise, all edges in G_i are in the graph $\langle\{v_1, v_2, v_3, v_4\}\rangle$ or join a vertex v_l, $1 \leq l \leq 4$, with a vertex x_j, $1 \leq j \leq r$. Because $q(G_i) - q\langle\{v_1, v_2, v_3, v_4\}\rangle \geq 11 - 6 = 5$, we have at least five edges of the second type, so there exists a vertex v_l, $1 \leq l \leq 4$, that is adjacent with at least two vertices from $\{x_1, x_2, \dots, x_r\}$, say v_l is adjacent to x_j and x_k. Then $\langle\{x_j v_l, x_k v_l, v_1 v_2, v_3 v_4\}\rangle \cong K(1, 3) \cup K_2 \subset G_i$, which completes the proof. ❑

The graph G in Example 2.2 was chosen for two reasons. First, it shows that even for relatively simple graphs, finding the gcs index is not trivial. Second, we will return to this example later, when we will discuss the role of the number n (the cardinality of $\mathcal{G}$) in the definition of a gcs index.

Fortunately, it is easier to establish a lower bound for a gcs index.

Theorem 2.3 If G is a noncomplete graph of order p without isolated vertices, then $i(G) \geq \binom{p}{2}$.

Proof. Let us take $G_1 \cong K_p$ and $G_2 \cong G \cup nK_2$, where $n = \binom{p}{2} - q(G)$ (see Figure 2.5). Then $q(G_1) = q(G_2) = \binom{p}{2}$.

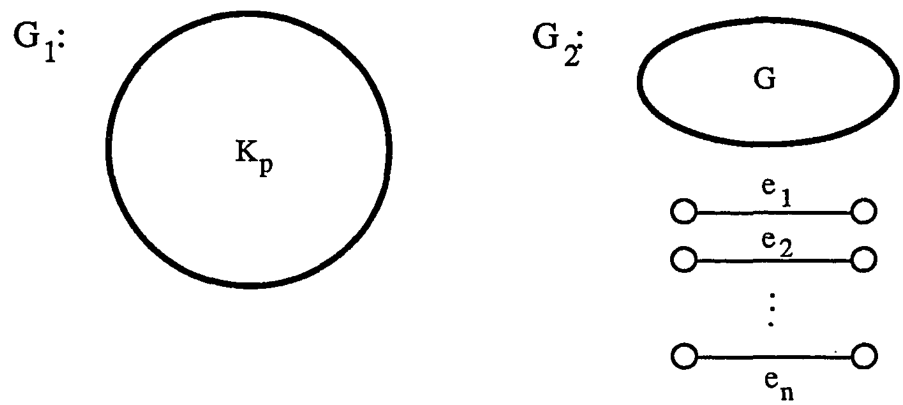


Figure 2.5

We will show that $gcs(G_1, G_2) = \{G\}$. Of course, $G \in gcs(G_1, G_2)$. Assume that there exists a graph H such that $H \ncong G$ and $H \in gcs(G_1, G_2)$. The order of H is at most p. The graph H must use some independent edges from G_2, say it uses r edges among $e_1, e_2, \ldots, e_n$. Then at least $2r$ vertices from a copy of G in G_2 are not present in H. Suppose that $S = \{v_1, v_2, \ldots, v_{2r}\} \subset V(G) - V(H)$. But then the size of the graph $G - S$ is at most $q(G) - r$. Moreover, if the graph induced by S is not isomorphic to rK_2, then this size is strictly less than $q(G) - r$. Because $H \ncong G$, we have the last case, and therefore $q(H) \leq q(G - S) + r < q(G) - r + r = q(G)$. Hence $H \notin gcs(G_1, G_2)$, and the proof is complete. ❑

The construction of the graphs G_1 and G_2 in the proof of Theorem 2.3 gives the following result.

Corollary 2.4 For a noncomplete graph G without isolated vertices, there exist two nonisomorphic graphs G_1 and G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$.

This is (except for the trivial case when $G \cong K_p$) the theorem from [7] mentioned at the beginning of this chapter.

The bound established in Theorem 2.3 is best possible in the sense that there is an infinite family of graphs such that for every graph G of this family, we have $i(G) = \binom{p}{2}$, where $p = p(G)$.

Example 2.5 Let the family consist of graphs kP_4, $k \geq 1$. Let $G \cong kP_4$. Because $p(G) = 4k$, by Theorem 2.3 we have $i(G) \geq \binom{4k}{2}$. To prove the reverse inequality $i(G) \leq \binom{4k}{2}$, it is enough to find for any family $\mathcal{G} = \{G_1, G_2, ..., G_n\}$, $n \geq 2$, of graphs of size $q > \binom{4k}{2}$ such that $G \in \text{gcs } \mathcal{G}$, a graph H belonging to $\mathcal{G}$, where $H \ncong G$. We claim that $H \cong (k-1)P_4 \cup P_3 \cup K_2$ is such a graph. In fact, take any G_i, $1 \leq i \leq n$. Because $G \subset G_i$, the graph G_i contains k copies of P_4. Let us denote them by $H_1, H_2, ..., H_k$. Because $q(G_i) > \binom{4k}{2}$, there is a vertex $v \in V(G_i) - (V(H_1) \cup V(H_2) \cup ... \cup V(H_k))$. If v is adjacent to some vertex of H_j, $1 \leq j \leq k$, then $P_3 \cup K_2 \subset \langle V(H_j) \cup \{v\} \rangle$. Taking $k - 1$ remaining copies of P_4, we have $(k-1)P_4 \cup P_3 \cup K_2 \subset G_i$. On the other hand, if v is not adjacent to a vertex of $H_1 \cup H_2 \cup ... \cup H_k$, then (because v is not an isolated vertex) $K_2 \subset G_i - (V(H_1) \cup V(H_2) \cup ... \cup V(H_k))$, so also $(k-1)P_4 \cup P_3 \cup K_2 \subset G_i$. ❑

If a graph has neither isolated vertices nor end-vertices, then we can improve the lower bound for its gcs index.

Theorem 2.6 Let G be a graph of order p. If $\delta(G) \geq 2$, then $i(G) \geq \binom{p+1}{2}$.

Proof. Let us consider the two graphs $G_1 \cong K_{p+1}$ and $G_2 \cong G \cup nK_2$, where $n = \binom{p+1}{2} - q(G)$ (see Figure 2.6).

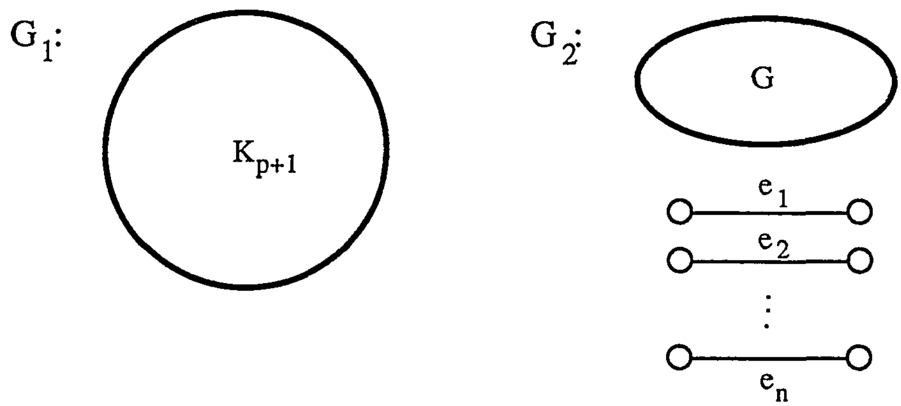


Figure 2.6

Of course, G is a common subgraph of G_1 and G_2. Suppose that there is a common subgraph H of G_1 and G_2 such that $H \not\cong G$ and $q(H) \geq q(G)$. Then H, as a subgraph of G_2, must use some edges among $e_1, e_2, ..., e_n$. If H uses only one edge, then it can use at most p - 1 vertices from a copy of G. Therefore, at least one vertex from this copy is left, and at least two edges are left. Hence, $q(H) \leq 1 + q(G) - 2 = q(G) - 1$ which gives a contradiction. If H uses k edges among $e_1, e_2, ..., e_n$, $k \geq 2$, then at least 2k - 1 vertices of the copy of G are left, and

$$q(H) \leq k + q(G) - \frac{(2k-1)\delta(G)}{2} = q(G) - k + 1 < q(G) \quad \text{for } k \geq 2.$$

This contradiction gives $gcs(G_1, G_2) = \{G\}$, and because $q(G_1) = q(G_2) = \binom{p+1}{2}$, it follows that $i(G) \geq \binom{p+1}{2}$. ❑

In general, the lower bound in Theorem 2.6 cannot be improved. That is, there is an infinite family of graphs such that for every graph G from this family we have $\delta(G) \geq 2$ and $i(G) = \binom{p+1}{2}$.

Example 2.7 Let the family consist of graphs kK_3, $k \geq 1$ and let $G \cong kK_3$. Because $p(G) = 3k$ and $\delta(G) = 2$, by Theorem 2.6 we have $i(G) \geq \binom{3k+1}{2}$. We will prove the reverse inequality. Let $\mathcal{G}$ be a family of graphs of size q, $q > \binom{3k+1}{2}$, such that $kK_3 \in \text{gcs } \mathcal{G}$. We will show that $(k-1)K_3 \cup P_3 \cup K_2 \in \text{gcs } \mathcal{G}$. Let G_i be any graph from the family $\mathcal{G}$. We denote k copies of K_3 in G_i by $H_1, H_2, \ldots, H_k$. Because $q(G_i) > \binom{3k+1}{2}$, the order of G_i is at least $3k+2$. Let $W = V(G_i) - (V(H_1) \cup V(H_2) \cup \ldots \cup V(H_k))$. If there is a vertex $v \in V(H_1) \cup V(H_2) \cup \ldots \cup V(H_k)$ that is adjacent to at least two vertices of W, then $(k-1)K_3 \cup P_3 \cup K_2 \subset G_i$. In fact, without loss of generality, assume that $v \in V(H_1)$ and v is adjacent to x and y, where $x, y \in W$. Then $(H_1 - v) \cup \langle\{vx, vy\}\rangle \cup H_2 \cup \ldots \cup H_k \cong K_2 \cup P_3 \cup (k-1)K_3 \subset G_i$ (see Figure 2.7).

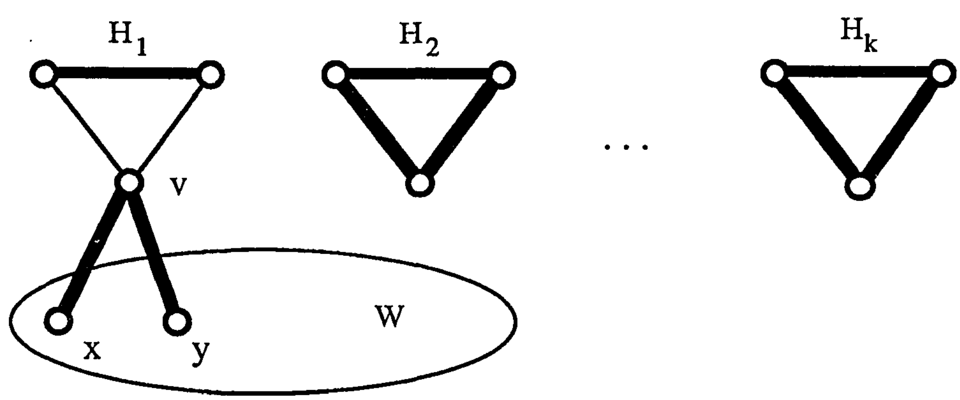


Figure 2.7

Otherwise, if every vertex from $V(H_1) \cup V(H_2) \cup \ldots \cup V(H_k)$ is adjacent to at most one vertex of W, then G_i must contain an edge incident with none of the vertices from $V(H_1) \cup V(H_2) \cup \ldots \cup V(H_k)$. In fact, if G_i contains no such edge, then $q(G_i) \leq \binom{3k}{2} + 3k = \binom{3k+1}{2}$, which produces a contradiction. Therefore, also in this case we have $(k-1)K_3 \cup P_3 \cup K_2 \subset G_i$, which completes the proof. ❑

Using the concept of gcs index, we can divide the set of all graphs into two classes, namely graphs of a finite gcs index and graphs of an infinite gcs index. In the next section we will want to determine which graphs belong to which category.

2.3 Graphs of Infinite Greatest Common Subgraph Index

If a graph G has an infinite gcs index, then we are able to construct, for any positive integer q_0, a family $\mathcal{G}$ of graphs of the same size q, $q > q_0$, such that gcs $\mathcal{G} = \{G\}$.

The next theorem gives a sufficient condition for a graph to have an infinite gcs index.

Theorem 2.8 If G contains a vertex v of maximum degree such that no component of G - v is isomorphic to K_2, then $i(G) = \infty$.

Proof. Assume, to the contrary, that the gcs index of G is finite, say $i(G) = q_0$. Let us take $q > q_0$ and define two graphs $G_1 \cong G \cup rK_2$ and $G_2 \cong G + vx_1 + vx_2 + \ldots + vx_r$, where $r = q - q(G)$, as in Figure 2.8. Let the edges in G incident to v be $e_1, e_2, \ldots, e_\Delta$, and let us denote $f_i = vx_i$, $i = 1, 2, \ldots, r$.

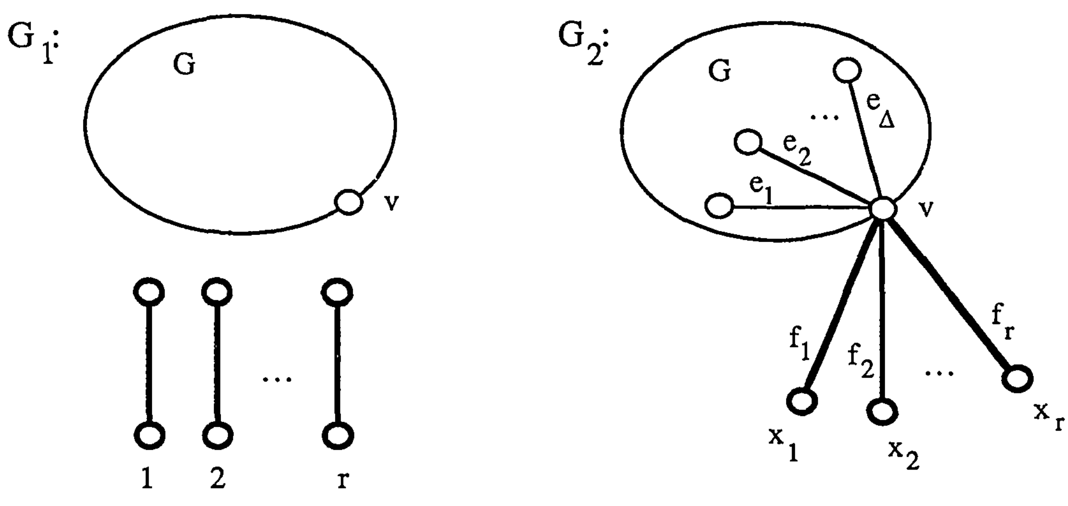


Figure 2.8

Of course, G is a common subgraph of G_1 and G_2. Assume that $H \in gcs(G_1, G_2)$ and $H \not\cong G$. Then at least one component of H is isomorphic to K_2. The graph H, as a subgraph of G_2, must use some of the edges $f_1, f_2, ..., f_r$, say k of them. Then k edges among $e_1, e_2, ..., e_\Delta$ are not in H. Otherwise, the vertex v in H would have degree larger than $\Delta(G)$, but there is no such vertex in G_1. Therefore, G without these k edges has K_2 as a component, as does $G - v$. This contradiction proves that $gcs(G_1, G_2) = \{G\}$, so $i(G) \geq q(G_1) > q_0$, which is impossible. ❑

Using Theorem 2.8, we show that there are some well-known graphs having infinite gcs index.

Corollary 2.9 The following graphs have infinite gcs index:

(a) complete graphs K_n, where $n \neq 3$;

(b) complete bipartite graphs $K(r, s)$, $r, s \geq 1$;

(c) cycles C_n, $n \geq 4$;

(d) paths P_n, $n \neq 4$.

In the exceptional cases, we have $i(K_3) = i(P_4) = 6$. These values were established in [7]. They can also be obtained as special cases of the graphs examined in Examples 2.5 and 2.7, namely for the graphs kK_3 and kP_4 with $k = 1$.

If a graph G is 2 - connected, then $G - v$ is connected for every vertex $v \in V(G)$. Therefore, we have the next corollary.

Corollary 2.10 If G is a 2 - connected graph and $G \not\cong K_3$, then $i(G) = \infty$.

It is well-known (see for example [1], p.131) that for a fixed integer k, almost every graph is k - connected. Using this fact and the previous corollary we have the following result.

Corollary 2.11 Almost every graph has an infinite gcs index.

The condition for a graph to have infinite gcs index given in Theorem 2.8 is sufficient but not necessary.

Example 2.12 There are graphs of an infinite gcs index having the property that removal of a vertex of maximum degree produces a component isomorphic to K_2.

Let G be a graph as in Figure 2.9 obtained by identifying the end-vertex of P_3 and a vertex of K_n, $n \geq 4$. Then G has the unique vertex v of maximum degree, and $G - v \cong K_{n-1} \cup K_2$.

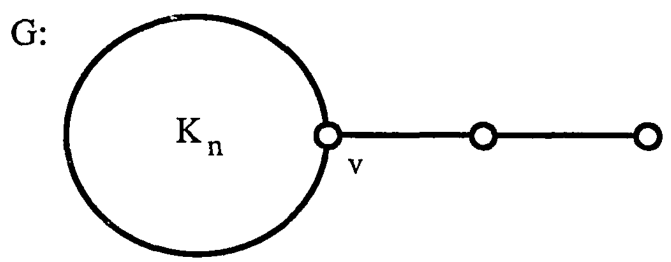


Figure 2.9

To prove that $i(G) = \infty$, we will use a slightly different construction of G_1 and G_2 than that in the proof of Theorem 2.8. Assume, to the contrary, that $i(G) = q_0$. Let us take $q > q_0$ and define two graphs $G_1 \cong G \cup rK_2$ and $G_2 \cong G + wx_1 + wx_2 + \ldots + wx_r$, where $r = q - q(G)$, w is the central vertex of P_3 and $x_1, x_2, \ldots, x_r \notin V(G)$ (see Figure 2.10). Let the two edges in G_2 incident to w be g and h, let $f_i = wx_i$, $i = 1, 2, \ldots, r$, and let $F = \{f_1, f_2, \ldots, f_r, g, h\}$.

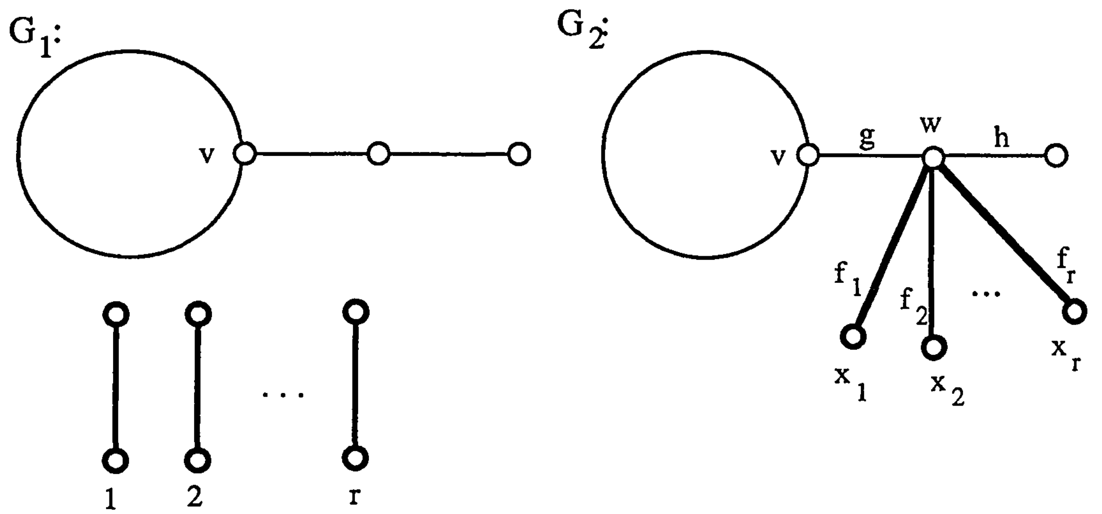


Figure 2.10

We will prove that $gcs(G_1, G_2) = \{G\}$, which gives $i(G) \geq q > q_0$ and produces a contradiction. Of course, G is a common subgraph of G_1 and G_2. Assume that $H \in gcs(G_1, G_2)$ and $H \not\cong G$. Then at least one component of H is isomorphic to K_2. If such a component in G_2 uses an edge belonging to F, then the remaining edges belong to K_n and $q(H) \leq 1 + \binom{n}{2} < q(G)$, which is impossible. Therefore, the component isomorphic to K_2 must be contained in K_n, and only $n - 2$ vertices from K_n are available for $H - K_2$. Because $H \subset G_1$ and $\Delta(G_1) = n + 1$, the graph H can use at most $n + 1$ edges from F (otherwise $\Delta(H) \geq n + 2$, which is impossible). Therefore,

$$q(H) \leq \binom{n-2}{2} + (n+1) + 1 = \binom{n-1}{2} + 4 < \binom{n}{2} + 2 = q(G) \quad \text{for } n \geq 4.$$

Consequently, $i(G) = \infty$. ❑

Because of the last example which shows that in general the converse of Theorem 2.8 is not true, a characterization of graphs of infinite gcs index remains an open problem.

In Examples 2.5 and 2.7 we found that $i(kP_4) = \binom{4k}{2}$ and $i(kK_3) = \binom{3k+1}{2}$. Therefore, the gcs index of a graph can be arbitrarily large (and finite). However, for $k \geq 2$, the graphs in these examples are disconnected. We will prove that there are connected graphs with this property.

Theorem 2.13 The gcs index of connected graphs can be arbitrarily large.

Proof. Let a graph G consist of k triangles with one vertex in common; more formally, $G \cong K_1 + kK_2$ (see Figure 2.11).

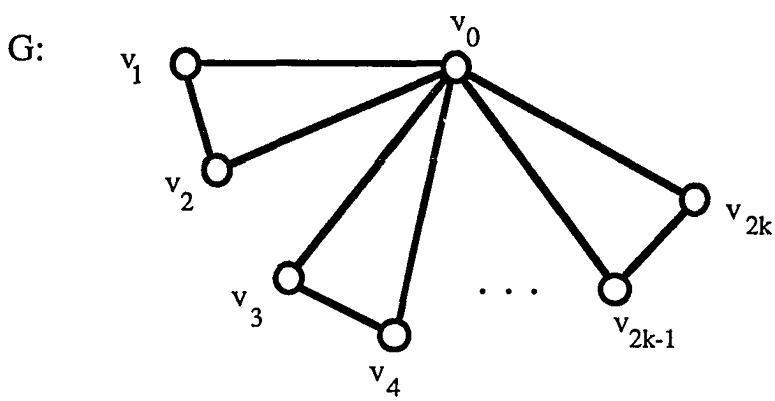


Figure 2.11

Because $\delta(G) = 2$ and $p(G) = 2k + 1$, it follows by Theorem 2.6 that $i(G) \geq \binom{2k+2}{2}$. In fact, we can find a much better lower bound for $i(G)$. Consider the three graphs $G_1 \cong K_{4k-2}$, $G_2 \cong G \cup rK_2$ and $G_3 \cong G + v_1x_1 + v_1x_2 + \ldots + v_1x_r$, where $r = \binom{4k-2}{2} - 3k$ (see Figure 2.12). Then $gcs(G_1, G_2, G_3) = \{G\}$. (The proof of this fact is quite lengthy and we omit it.) Therefore,

$$i(G) \geq \binom{4k-2}{2} = 8k^2 - 10k + 3.$$

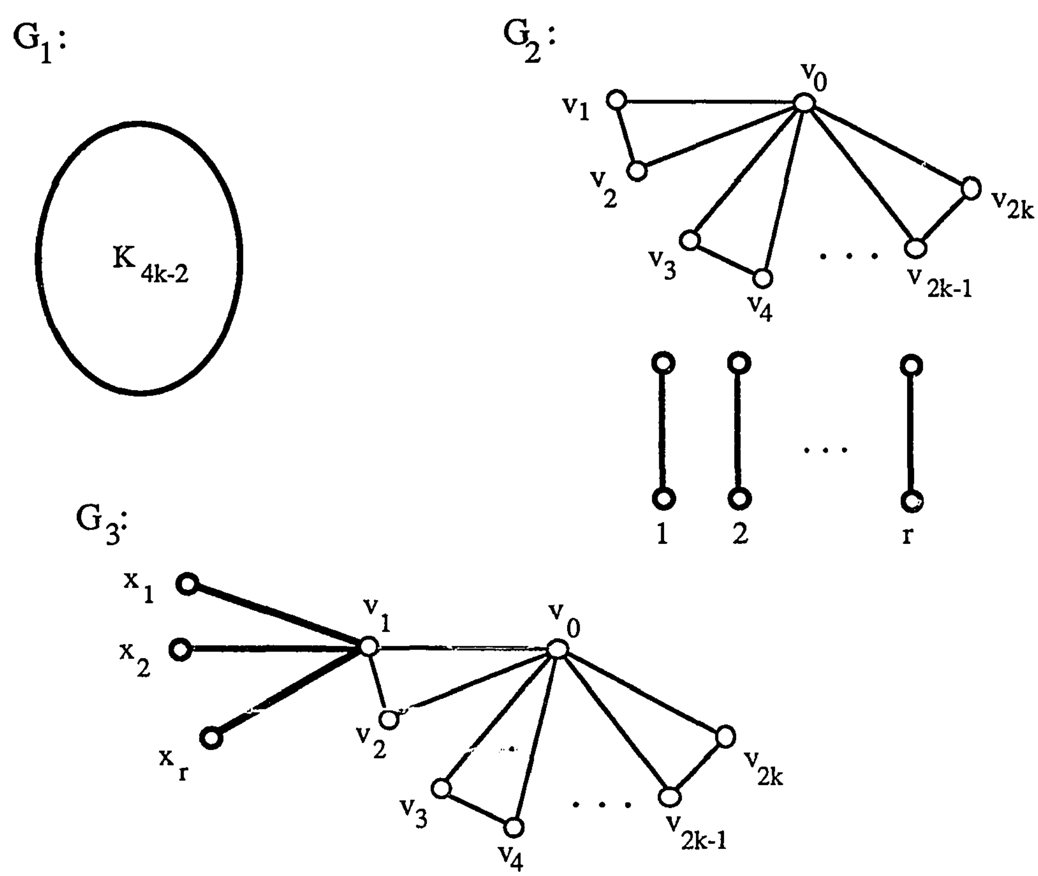


Figure 2.12

To prove that i(G) is finite, we will show that there exists q_0 such that for every integer $q > q_0$ and every set $\mathcal{G} = \{G_1, G_2, ..., G_n\}$, $n \geq 2$, of graphs of size q for which $G \in$ gcs $\mathcal{G}$, we have $|\text{gcs } \mathcal{G}| > 1$. Let us set

$$q_0 = 2k^4 + 4k^3 + 7k^2 - 1,$$

and let $\mathcal{G} = \{G_1, G_2, ... , G_n\}$, $n \geq 2$, be any set of graphs of size $q > q_0$ with $G \in$ gcs $\mathcal{G}$. We will show that

$$K(1, 2k) \cup kK_2 \in \text{ gcs } \mathcal{G}.$$

Let F be a copy of G in G_i. We distinguish two cases.

<u>Case 1</u>: <u>There are more than</u> $(2k + 1)(2k - 1)$ <u>vertices of</u> $G_i - V(F)$ <u>such that each of them is adjacent to some vertex of</u> F. Then, because $p(F) = 2k + 1$, at least one vertex of F is adjacent to at least 2k vertices of $G_i - V(F)$. Let x be such a vertex. By the symmetry of F it suffices to consider two possibilities, namely

(a) $x = v_0$, or

(b) $x = v_i$, for some i, $1 \leq i \leq 2k$.

These two cases are presented in Figure 2.13. The bold edges indicate the subgraph $K(1, 2k) \cup kK_2$ of G_i.

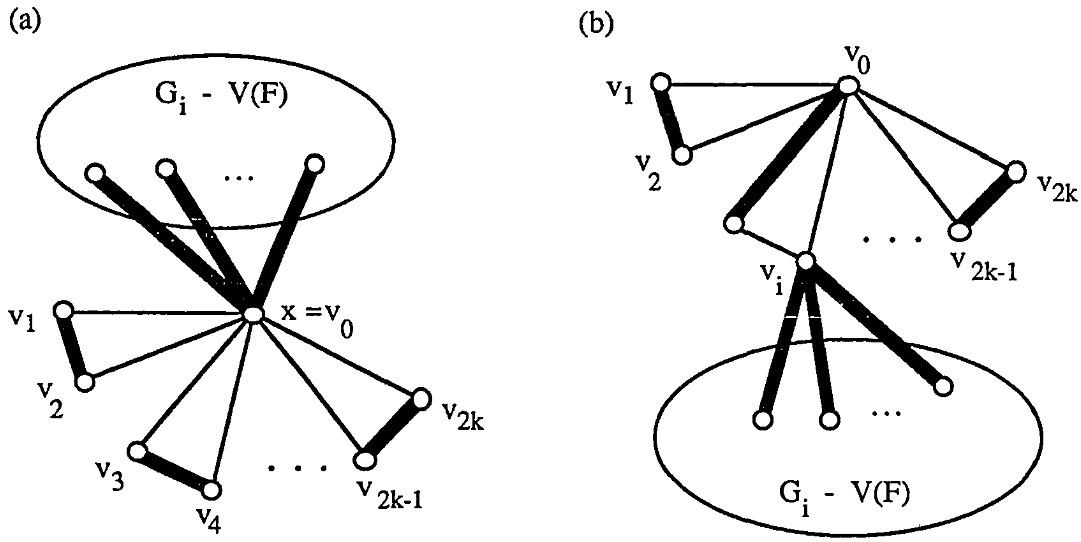


Figure 2.13

<u>Case 2</u>: <u>At most</u> $(2k + 1)(2k - 1)$ <u>vertices of</u> $G_i - V(F)$ <u>are adjacent to some vertex of</u> F. Then we have at most $(2k + 1)(2k + 1)(2k - 1)$ edges between F and $G_i - V(F)$, and, of course, at most $\binom{2k+1}{2}$ edges belong to $\langle V(F) \rangle$, which implies that

$$q(G_i - V(F)) \geq q(G_i) - (2k + 1)^2 (2k - 1) - \binom{2k+1}{2} =$$

$$q(G_i) - (2k + 1)(4k^2 + k - 1) > q_0 - (2k + 1)(4k^2 + k - 1) =$$

$$(2k^4 + 4k^3 + 7k^2 - 1) - (2k + 1)(4k^2 + k - 1) = 2k^4 - 4k^3 + k^2 + k =$$

$$\binom{2k(k-1)}{2} = q(K_{2k(k-1)}).$$

Therefore, the order of $G_i - V(F)$ is $p(G_i - V(F)) > 2k(k - 1)$. If $\Delta(G_i - V(F)) \geq 2k$, then $G_i - V(F)$ contains $K(1, 2k)$ as a subgraph, and taking k independent edges from F, we have that $K(1, 2k) \cup kK_2 \subset G_i$. On the other hand, if $\Delta(G_i - V(F)) < 2k$, then because

$$\beta_1(G_i - V(F)) \geq \frac{p\,(G_i - V(F))}{1 + \Delta(G_i - V(F))}$$

($\beta_1(G)$ is an edge independence number of a graph G) and $1 + \Delta(G_i - V(F)) \leq 2k$, it follows that

$$\beta_1(G_i - V(F)) > \frac{2k(k - 1)}{2k} = k - 1.$$

Therefore, $G_i - V(F)$ contains k independent edges, and taking the star $K(1, 2k)$ as a subgraph of F, we conclude that G_i contains $kK_2 \cup K(1, 2k)$ as a subgraph. ❑

Although we do not know the exact value of the gcs index for the graph G from Theorem 2.13, this graph can also serve as an example to illustrate that the difference between $i(G)$ and the lower bound for the gcs index given by Theorem 2.3 (or Theorem 2.6) can be arbitrarily large.

Next, we will discuss relationships between the gcs index of components of a graph and that of the graph itself.

Theorem 2.14 If $i(G) = \infty$ and $\Delta(G) \geq \Delta(H)$, then $i(G \cup H) = \infty$.

Proof. Since $i(G) = \infty$, for every positive integer q_0 there exist $q > q_0$ and a family $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $n \geq 2$, of graphs of size q such that gcs $\mathcal{G} = \{G\}$.

Let v be a vertex of G of maximum degree and let $r = q - q(G)$. Define a family $\mathcal{H} = \{G_1 \cup H, G_2 \cup H, \ldots, G_n \cup H, H_{n+1}, H_{n+2}\}$, where

$$H_{n+1} \cong G \cup H \cup rK_2 \text{ and}$$
$$H_{n+2} \cong (G + vx_1 + vx_2 + \ldots + vx_r) \cup H,$$

$x_1, x_2, \ldots, x_r \notin V(G \cup H)$.

We will show that gcs $\mathcal{H} = \{G \cup H\}$, so $i(G \cup H) = \infty$. Of course, $G \cup H$ is a common subgraph of $\mathcal{H}$. Suppose, to the contrary, that there exists a graph F, $F \ncong G \cup H$, $q(F) \geq q(G \cup H)$, such that $F \in$ gcs $\mathcal{H}$. Because $F \subset H_{n+1}$ and $\Delta(H_{n+1}) = \Delta(G)$, it follows that $\Delta(F) \leq \Delta(G)$. The graph F as a subgraph of H_{n+2} can use at most $\Delta(G)$ edges incident with the vertex v, and F must use all edges of a copy of H from H_{n+2}. Therefore, $F \cong G' \cup H$, where $q(G') = q(G)$ and $G' \ncong G$ (otherwise, $F \cong G \cup H$). But then G' would be a common subgraph of every graph from $\mathcal{G}$, which is impossible because gcs $\mathcal{G} = \{G\}$. ❑

Corollary 2.15 If $i(G) = i(H) = \infty$, then $i(G \cup H) = \infty$.

Without the assumption $\Delta(G) \geq \Delta(H)$, Theorem 2.14 is not true in general. As an example consider graphs $G \cong K_2$ and $H \cong (K_2 \cup K_1) + K_1$ (H is the graph from Example 2.2). Then $i(G) = \infty$, $i(H) = 10$ but $i(G \cup H)$ is finite. In fact, it is not difficult to show that $i(G \cup H) = 21$.

Theorem 2.14 and Corollary 2.15 can be generalized for graphs with at least three components.

The next example shows that there exist graphs of infinite gcs index such that all components have finite gcs index.

Example 2.16 Let $G \cong K_3 \cup P_4$. To prove that $i(G) = \infty$, we construct for any positive integer q_0, three graphs G_1, G_2, G_3 of size q ($> q_0$) such that $gcs(G_1, G_2, G_3) = \{G\}$. Let $q > \max\{q_0, 6\}$. Define

$$G_1 \cong P_4 \cup [(K_2 \cup (q-6)K_1) + K_1],$$

$$G_2 \cong K_3 \cup [K(1, q-4) + xy],$$

where x is an end-vertex of $K(1, q-4)$, and $y \notin V(K_3) \cup V(K(1, q-4))$ and

$$G_3 \cong K_3 \cup P_4 \cup (q-6)K_2$$

(see Figure 2.14).

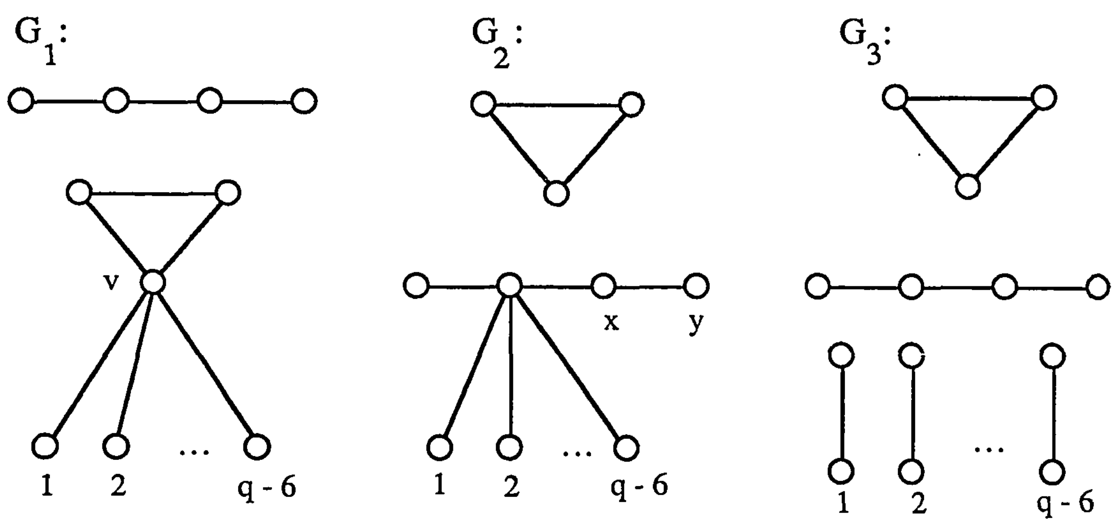


Figure 2.14

We claim that $gcs(G_1, G_2, G_3) = \{G\}$. Of course, G is a common subgraph of G_1, G_2 and G_3. Assume that H is a common subgraph of G_1, G_2 and G_3 and $q(H) \geq q(F)$. Because H is a subgraph of G_3, it follows that $\Delta(H) \leq 2$. Hence H, as a subgraph of G_1, can use at most two edges incident with the vertex $v \in V(G_1)$. Therefore, H can use from the component of G_1 containing the vertex v either

(a) K_3, and then $H \cong G$; or

(b) $K_2 \cup P_3$, and then $H \cong P_4 \cup P_3 \cup K_2 \not\subset G_2$; or

(c) P_4, and then $H \cong 2P_4 \not\subset G_3$.

Therefore, we must have $H \cong G$, which proves that $i(G) = \infty$.

We know that $i(P_4) = i(K_3) = 6$, so the gcs index of every component of G is finite. ❑

However, if all components of a graph are isomorphic, then the fact that the gcs index of a component is finite implies the gcs index of the graph is finite.

Theorem 2.15 Let $G \cong 2F$ and $i(F)$ is finite, say $i(F) = r \leq \binom{s}{2}$. Then $i(G) \leq \binom{2s}{2}$, so G has finite gcs index.

Proof. We prove that if $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a family of graphs of size q, where $q > \binom{2s}{2}$, with $G \in \text{gcs}\ \mathcal{G}$, then $|\text{gcs}\ \mathcal{G}| > 1$.

Let us consider any G_i, $1 \leq i \leq n$. Because $G \cong 2F \subset G_i$, denote two disjoint copies of F in G_i by F_1 and F_2. Because $i(F) \leq \binom{s}{2}$, we have $p(F) = p(F_1) = p(F_2) \leq s$. Assume first that $q(G_i - V(F_1)) \leq \binom{s}{2}$ and $q(G_i - V(F_2)) \leq \binom{s}{2}$. Then

$$q(G_i) \leq q(G_i - V(F_1)) + q(G_i - V(F_2)) + p(F_1)\,p(F_2) \leq \binom{s}{2} + \binom{s}{2} + s^2 = \binom{2s}{2},$$

which gives a contradiction. Therefore, for any i, $1 \leq i \leq n$, we must have $q(G_i - V(F_{k(i)})) > \binom{s}{2}$ for $k(i) = 1$ or $k(i) = 2$. Because $i(F) \leq \binom{s}{2}$, and $F \subset G_i - V(F_{k(i)})$, so F is not the unique greatest common subgraph of $\{G_i - V(F_{k(i)}) \mid i = 1, 2, \ldots, n\} = \mathcal{F}$. There exists F', $(\not\cong F)$, such that $F' \in \text{gcs}\ \mathcal{F}$. Then, $F' \cup F$ is a subgraph of G_i for every $i = 1, 2, \ldots, n$, and $q(F' \cup F) \geq q(2F)$, so $2F$ is not the unique greatest common subgraph of the family $\mathcal{G}$. ❑

Using the same technique as in the proof of Theorem 2.17, we get the following generalization of the above result.

Theorem 2.18 Let $G \cong kF$, where $i(F)$ is finite, say $i(F) = r \leq \binom{s}{2}$. Then G has finite gcs index; in particular, $i(G) \leq \binom{ks}{2}$.

There is reason to believe that if a graph G has finite gcs index, then in a family $\mathcal{G}$ of graphs of maximum size with gcs $\mathcal{G} = \{G\}$, a complete graph is present. Its role is to restrict the order of greatest common subgraphs. Therefore, we believe that the following conjecture is true. Certainly all known examples confirm this hypothesis.

Conjecture. If $i(G)$ is finite, then $i(G) = \binom{k}{2}$ for some integer $k \geq 4$.

In the definition of gcs index, we considered a family $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $n \geq 2$, of graphs of the same size. Now we want to discuss the role of n (the cardinality of $\mathcal{G}$) in this definition. Assuming that $n = 2$ we define an index $i_2(G)$. For a graph G without isolated vertices, $i_2(G)$ is the least positive integer q_0 such that for every integer $q > q_0$ and every set $\mathcal{G} = \{G_1, G_2\}$ of two nonisomorphic graphs of size q for which $G \in$ gcs $\mathcal{G}$, it follows that $|\text{gcs } \mathcal{G}| > 1$. If no such q_0 exists, then we write $i_2(G) = \infty$.

It is immediate from the definition of $i_2(G)$ that $i_2(G) \leq i(G)$ for every graph G (where we allow $i(G)$ and $i_2(G)$ to be infinite).

We show next that $i(G)$ and $i_2(G)$ need not be equal.

Example 2.19 Let $G \cong (K_2 \cup K_1) + K_1$ be the graph considered in Example 2.2. We proved that $i(G) = 10$. We will show that $i_2(G) = 7$.

If we take $G_1 \cong K_2 + 3K_1$ and $G_2 \cong G \cup 3K_2$ (see Figure 2.15), then $q(G_1) = q(G_2) = 7$ and $\text{gcs}(G_1, G_2) = \{G\}$, so $i_2(G) \geq 7$.

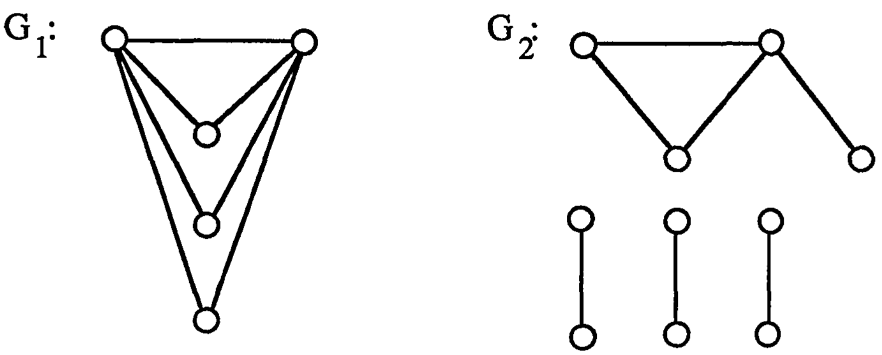


Figure 2.15

Let G_1 and G_2 be graphs of size q (> 7) such that $G \in gcs(G_1, G_2)$. We need to show that $|gcs(G_1, G_2)| > 1$. We consider three cases.

Case 1. Both G_1 and G_2 have at least two components. Then, it is obvious that $K(1, 3) \cup K_2 \subset G_i$, $i = 1, 2$.

Case 2. Only one of G_1 and G_2 is connected, say G_1 is connected. Then we distinguish two subcases according to the order of G_1.

(a) If $p(G_1) \geq 6$, then $K(1, 3) \cup K_2 \subset G_1$ (the proof is the same as in Example 2.2); also $K(1, 3) \cup K_2 \subset G_2$.

(b) If $p(G_1) = 5$, then $K_3 \cup K_2 \subset G_1$; also $K_3 \cup K_2 \subset G_2$.

Case 3. Both G_1 and G_2 are connected. Assuming that $p(G_1) \leq p(G_2)$, we have the following three possibilities:

(a) $p(G_1) = p(G_2) = 5$.

Then $K_3 \cup K_2 \subset G_i$, $i = 1, 2$.

(b) $p(G_1) = 5$ and $p(G_2) \geq 6$.

Denoting by u the unique end-vertex of G, we distinguish three subcases represented in Figure 2.16 (i), (ii) and (iii).

(i) If some vertex $x \in V(G_2) - V(G)$ is adjacent to u, then $K_3 \cup K_2 \subset G_i$, $i = 1, 2$.

(ii) If no $x \in V(G_2) - V(G)$ is adjacent to u but there is an edge e in the graph $G_2 - V(G)$, then also $K_3 \cup K_2 \subset G_i$, $i = 1, 2$.

(iii) If no $x \in V(G_2) - V(G)$ is adjacent to u and no edge is in the graph $G_2 - V(G)$, then there are two vertices $x, y \in V(G_2) - \{v_1, v_2, v_3\}$ adjacent to some vertex among v_1, v_2, v_3 (v_1, v_2 and v_3 are vertices of a triangle in G_2). Then $H \cong (K_2 \cup 2K_1) + K_1$ is a subgraph of both G_1 and G_2. This gives a contradiction, because $q(H) = 5 > q(G)$.

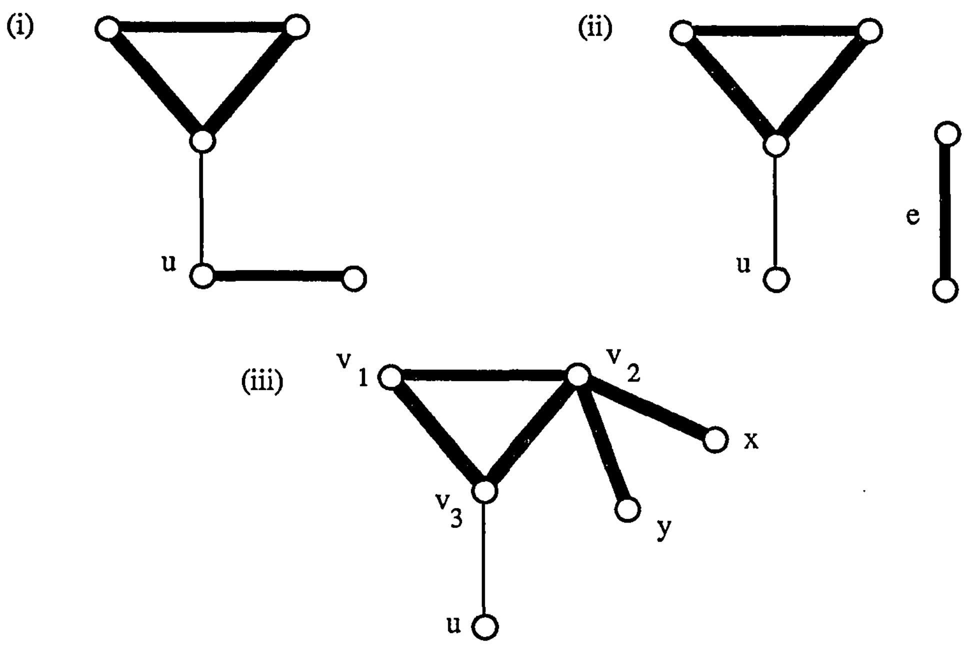


Figure 2.16

(c) $p(G_1) \geq 6$ and $p(G_2) \geq 6$.

Then $K(1, 3) \cup K_2$ is a subgraph of G_i, $i = 1, 2$ (the proof is the same as in Example 2.2). ❑

If we replace $i(G)$ by $i_2(G)$ in Theorems 2.3, 2.6, 2.8 and 2.13 the results remain true. However, the last example shows that the conjecture $i_2(G) = \binom{k}{2}$ for some $k \geq 4$ (if $i_2(G)$ is finite) is false. It remains an open problem whether there exists a graph G with $i_2(G)$ finite and $i(G)$ infinite.

In an analogous way, we can define $i_3(G)$, $i_4(G)$, ... by placing the obvious restriction on the cardinality of a family $\mathcal{G}$ ($|\mathcal{G}| = 3, 4, \ldots$). Then, of course we have

$$i_2(G) \leq i_3(G) \leq i_4(G) \leq \ldots \leq i(G).$$

We do not know whether these inequalities (except the first one) can be strict. In fact, no example of a graph G is known for which

$$i_3(G) < i(G).$$

2.4 Greatest Common Subgraph Number

The greatest common subgraph number is another graphical parameter which measures how large the sizes of G_1 and G_2 can be (in comparison with the size of G), where G_1 and G_2 are nonisomorphic graphs of equal size that satisfy $gcs(G_1, G_2) = \{G\}$. This parameter was defined in [5]. For a graph G without isolated vertices, let $\mathcal{G}(G)$ be the set of all graphs G_1 for which there exists a graph G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$. The set $\mathcal{G}(G)$ is nonempty, because $G \in \mathcal{G}(G)$. We define the <u>gcs number</u> gn(G) of G as

$$gn(G) = \max_{H \in \mathcal{G}(G)} \{ q(H) - q(G) \}$$

if it exists; otherwise $gn(G) = \infty$.

It is not difficult to establish a relationship between the gcs number and the gcs index of a graph.

Theorem 2.20 Let G be a graph without isolated vertices that has finite index $i_2(G)$. Then $gn(G) = i_2(G) - q(G)$.

Proof. Let $i_2(G) = q_0$. This implies that there are two graphs G_1 and G_2 of size q_0 such that $gcs(G_1, G_2) = \{G\}$. Hence $G_1 \in \mathcal{G}(G)$ and

$$gn(G) = \max_{H \in \mathcal{G}(G)} \{ q(H) - q(G) \} \geq q(G_1) - q(G), \text{ or}$$

$$gn(G) \geq i_2(G) - q(G) \qquad (2.1)$$

On the other hand, for any integer $q > q_0$ and any pair G_1, G_2 of graphs of size q for which $G \in gcs(G_1, G_2)$, we have $|gcs(G_1, G_2)| > 1$. This means that if G_1 has size $q > q_0$, then $G_1 \notin \mathcal{G}(G)$ and therefore

$$gn(G) < q - q(G) \text{ for any } q > q_0, \text{ or}$$

$$gn(G) \leq q_0 - q(G) = i_2(G) - q(G). \qquad (2.2)$$

The inequalities (2.1) and (2.2) give the desired formula. ❑

Theorem 2.21 If $i_2(G) = \infty$, then $gn(G) = \infty$.

Proof. If $i_2(G) = \infty$, then for every positive integer q_0 there exist two graphs G_1 and G_2 such that $q(G_1) = q(G_2) > q_0$ and $gcs(G_1, G_2) = \{G\}$. Hence $\mathcal{G}(G)$ contains graphs of arbitrarily large size, and therefore $gn(G) = \infty$. ❑

Using Theorems 2.20 and 2.21 and results about $i_2(G)$ mentioned after Example 2.19, we can list the following facts concerning $gn(G)$.

Corollary 2.22

(i) For any graph G without isolated vertices $gn(G) \geq \binom{p(G)}{2} - q(G)$.

(ii) For any graph G with $\delta(G) \geq 2$, $gn(G) \geq \binom{p(G)+1}{2} - q(G)$.

(iii) If G contains a vertex v of maximum degree such that no component of $G - v$ is isomorphic to K_2, then $gn(G) = \infty$.

(iv) For every positive integer k, there is a connected graph G for which $gn(G)$ is finite and $gn(G) > k$.

By Theorem 2.20 and the inequality $i_2(G) \leq i(G)$, we have the following result.

Corollary 2.23 If $i(G)$ is finite, then $gn(G)$ is finite, and

$$gn(G) \leq i(G) - q(G).$$

The last inequality may be strict as the graph G from Example 2.2 shows. Namely, if $G \cong (K_2 \cup K_1) + K_1$, then $gn(G) = i_2(G) - q(G) = 7 - 4 = 3$, but $i(G) - q(G) = 10 - 4 = 6$.

Finally, by Theorems 2.20 and 2.21 and Corollary 2.23, we have the following result.

Corollary 2.24 Let G be a graph without isolated vertices.

(i) The gcs number $gn(G)$ is infinite if and only if the index $i_2(G)$ is infinite.

(ii) If the gcs number $gn(G)$ is infinite, then the gcs index $i(G)$ is infinite.

Whether the converse of (ii) is true is unknown.

CHAPTER III

GREATEST COMMON SUBGRAPHS OF GRAPHS WITH SPECIFIED PROPERTIES

3.1 Greatest Common Subgraphs and Hereditary Properties.

Assume that $\{G\} = \text{gcs}(G_1, G_2)$ for some graphs G, G_1 and G_2, where G_1 and G_2 are nonisomorphic graphs of the same size. In this section we want to show that we can choose G_1 and G_2 such that their sizes are only one greater than the size of the graph G. Such a choice may even be possible if the graphs G, G_1 and G_2 are required to have some specified property.

As a special case of Theorem 2.1, we have the following result.

Theorem 3.1 Let G be a graph. If G_1 and G_2 are nonisomorphic graphs of equal size for which $\text{gcs}(G_1, G_2) = \{G\}$, then for $e \in E(G_1) - E(G)$ and $f \in E(G_2) - E(G)$,

$$\text{gcs}(G + e, G + f) = \{G\}.$$

Let P be a graphical property. We are interested in the problem of determining, for a given graph G with property P, the existence of two nonisomorphic graphs G_1 and G_2 with property P and of equal size, such that G is the unique greatest common subgraph of G_1 and G_2.

A graphical property P is <u>hereditary</u> if, whenever a graph G has the property P, then every subgraph of G also has property P. For example, planarity, outerplanarity, being acyclic, and being n-colorable are hereditary properties, whereas connectedness is not.

Theorem 3.1 has an immediate counterpart if we consider hereditary properties.

Theorem 3.2 Let G_1 and G_2 be nonisomorphic graphs of equal size for which $gcs(G_1, G_2) = \{G\}$, where all three graphs G, G_1 and G_2 have a hereditary property P. Then for every $e \in E(G_1) - E(G)$ and $f \in E(G_2) - E(G)$, it follows $gcs(G + e, G + f) = \{G\}$, where both $G + e$ and $G + f$ are graphs with property P.

If P is not a hereditary property, it may happen that $G + e$ does not have the property P even when both G and G_1 do. However, if a property P is any of the following:

(1) being connected,

(2) being outerplanar and connected,

(3) being planar and connected,

(4) being unicyclic,

then we have the next result and its corollary.

Theorem 3.3 Let G_1 and G_2 be nonisomorphic graphs of equal size such that $gcs(G_1, G_2) = \{G\}$, where all three graphs G, G_1 and G_2 have property P. Then there exist edges $e \in E(G_1) - E(G)$ and $f \in E(G_2) - E(G)$ such that $gcs(G+ e, G+ f) = \{G\}$ and both $G + e$ and $G + f$ are graphs with property P.

Corollary 3.4 Assume that for a given graph G there exist nonisomorphic graphs G_1 and G_2 of equal size such that $gcs(G_1, G_2) = \{G\}$, where all three graphs G, G_1 and G_2 have property P. Then we can choose G_1 and G_2 so that $q(G_1) = q(G_2) = q(G) + 1$.

3.2 Outerplanar Graphs

Recall that a graph G is <u>outerplanar</u> if G can be embedded in the plane in such a way that every vertex of G lies in the boundary of the exterior region. In this section we want to determine all connected outerplanar graphs G for which there exist two nonisomorphic connected outerplanar graphs G_1 and G_2 of the same size with $gcs(G_1, G_2) = \{G\}$. If we remove the assumption about the connecttedness of G, G_1 and G_2, then the answer is easy. Namely, all outerplanar graphs G have the above property and a construction of graphs G_1 and G_2 can be the same as in the proof of Proposition 2 [7].

Therefore, we assume that graphs G, G_1 and G_2 are outerplanar and connected. Let us first define a special family of outerplanar graphs: $F_n \cong K_1 + P_n$, where $n \geq 1$. The first four graphs of this family, namely the graphs $F_1 \cong K_2$, $F_2 \cong K_3$, F_3 and F_4 are represented in Figure 3.1.

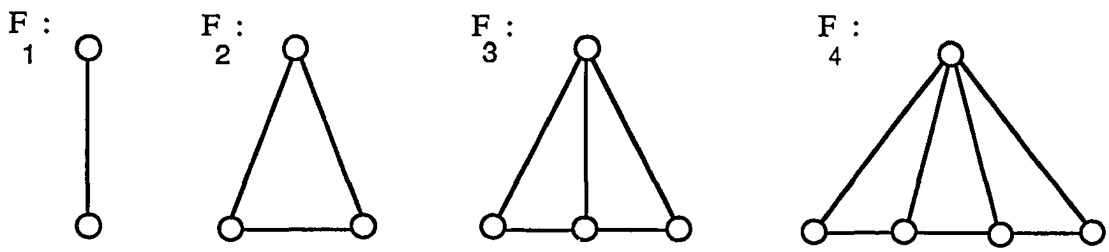


Figure 3.1

Of course, if $G \cong K_2$ or $G \cong K_3$, then there are no connected nonisomorphic graphs G_1 and G_2 of the same size with $gcs(G_1, G_2) = \{G\}$. For example, if $G_1 \supset K_2$, $G_2 \supset K_2$ and G_1, G_2 are nonisomorphic connected graphs of the same size, then P_3 is a common subgraph of both G_1 and G_2.

Let $G \cong F_3$ and assume that G_1 and G_2 are nonisomorphic connected outerplanar graphs with $gcs(G_1, G_2) = \{G\}$. By Corollary 3.4 we can assume that $q(G_1) = q(G_2) = q(F_3) + 1 = 6$. Using the symmetry of the graph F_3 we can assume without loss of generality that $G_1 \cong F_3 + e$ and $G_2 \cong F_3 + f$ (see Figure 3.2). But then not only F_3, but also the graph H indicated by bold lines in

Figure 3.2 is a common subgraph of both G_1 and G_2, hence $gcs(G_1, G_2) = \{F_3, H\}$, which gives a contradiction.

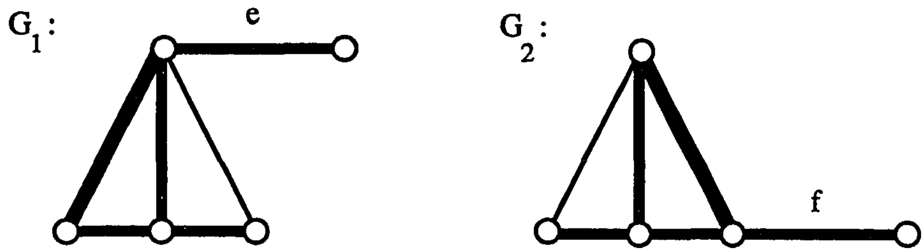


Figure 3.2

Finally, assume that $G \cong F_4$. By the symmetry of the graph F_4, there are only three connected outerplanar graphs G_1, G_2 and G_3 such that $G_i \supset F_4$ and $q(G_i) = q(F_4) + 1 = 8$, $i = 1, 2, 3$. But then not only F_4 but also the graph H (marked by bold lines in Figure 3.3) is a subgraph of G_i, $i = 1, 2, 3$. Therefore, it is impossible to find two connected outerplanar graphs G_1 and G_2 with $gcs(G_1, G_2) = \{F_4\}$.

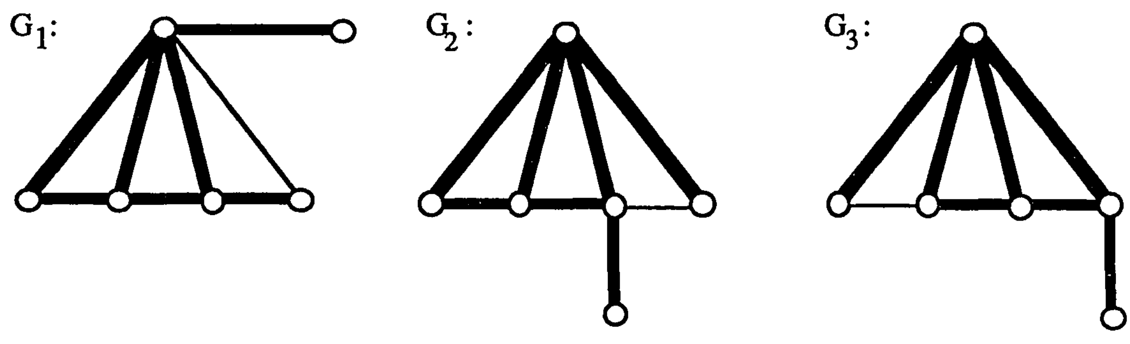


Figure 3.3

Therefore, we proved that if $G \cong F_n$, $n = 1, 2, 3$ or 4, then we are not able to construct two nonisomorphic connected outerplanar graphs G_1 and G_2 of the same size such that G is the unique greatest subgraph of G_1 and G_2. In the main theorem of this section we will show that these four graphs F_n, $n = 1, 2, 3$ or 4, are the only exceptions. In the proof of the theorem it will be convenient to use the following concept. A <u>branch</u> of a graph G at a vertex v is a maximal connected subgraph of G containing v as a non-cut-vertex.

Theorem 3.5 Let G be a connected outerplanar graph such that $G \ncong F_n$, n = 1, 2, 3, 4. Then there exist two nonisomorphic connected outerplanar graphs G_1 and G_2 of the same size for which $gcs(G_1, G_2) = \{G\}$.

Proof. We consider the following cases.

<u>Case 1. The graph G has an end-vertex.</u>

<u>Subcase 1.1. There are two vertices</u> $x, y \in V(G)$ <u>such that</u> $\deg x < \Delta(G)$, $\deg y < \Delta(G)$, $xy \notin E(G)$ <u>and</u> $G + xy$ <u>is outerplanar</u>. Construct two graphs $G_1 \cong G + xy$ and $G_2 \cong G + vw$, where v is a vertex of maximum degree and $w \notin V(G)$ (see Figure 3.4).

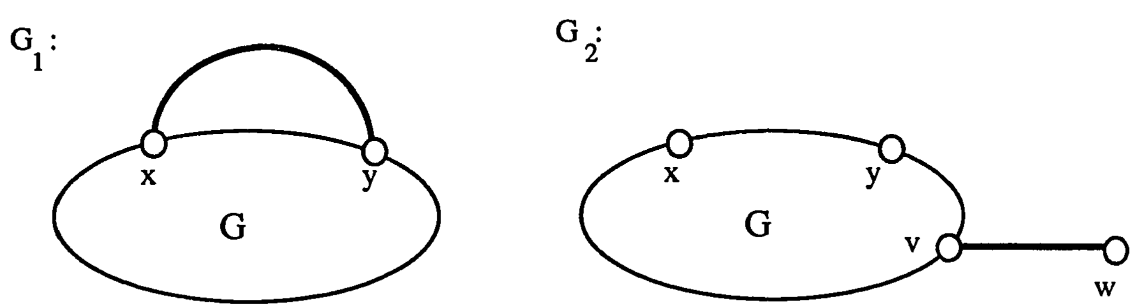


Figure 3.4

Of course, $G_1 \ncong G_2$, so $G \in gcs(G_1, G_2)$. Let $H \in gcs(G_1, G_2)$. To obtain H we must remove one edge from G_1 and one edge from G_2. But the graph G_1 has more regions than G_2 does, so we have to remove a cycle edge, say e, from G_1 (G_1 - e will be connected), and a bridge, say f, from G_2. But $\deg_{G_2} v > \Delta(G_1)$, so the edge f must be incident with v. If f is a terminal bridge, then $H \cong G_2 - f \cong G$. Otherwise, the graph $G_2 - f$ is disconnected (has two nontrivial components), but $G_1 - e$ is connected, which produces a contradiction. Therefore, $H \cong G$ and $gcs(G_1, G_2) = \{G\}$.

Subcase 1.2. For any two vertices x, y with $\deg x < \Delta(G)$, $\deg y < \Delta(G)$, $xy \notin E(G)$ the graph $G + xy$ is not outerplanar. We can assume that $\Delta(G) \geq 3$. If $\Delta(G) \leq 2$, then $G \cong P_n$, $n \geq 3$, so $G + xy \cong C_n$ and $G + xy$ is outerplanar. Therefore, the conditions of Subcase 1.1 hold.

Observe that:

(1) there is no vertex in G with two (or more) branches that are trees (otherwise, we could join two end-vertices from two trees and the resulting graph would still be outerplanar);

(2) if a branch at a vertex v is a tree T, then $T \cong K_2$ or $T \cong P_3$ (otherwise, we could join two end-vertices, or if $T \cong P_n$, $n > 3$, say P_n: $v = u_1, u_2, \ldots, u_n$ we could join u_n and u_{n-2} ($\deg u_n = 1$, $\deg u_{n-2} = 2 < \Delta(G)$), in both cases producing an outerplanar graph, so the conditions of Subcase 1.1 would hold).

Therefore, at any vertex v of G if a branch at v is a tree, then the branch is either K_2 or P_3 (and only one such branch at v is present). We consider two subcases.

Subcase 1.2.1. Each terminal edge is incident with a vertex of degree 2. Let v be a vertex with a branch isomorphic to P_3, say v, u, w (see Figure 3.5). In some fixed embedding of G in the plane, let x be a neighbor of v such that vx is an edge following vu as we proceed in counterclockwise direction about v. Then $\deg x = \Delta(G)$; otherwise $G + wx$ would be outerplanar and the conditions of Subcase 1.1 would hold.

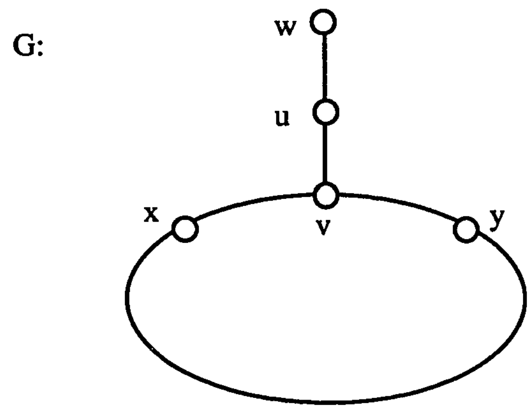


Figure 3.5

Define $G_1 \cong G + wx$ and $G_2 \cong G + uz$, where $z \notin V(G)$ (see Figure 3.6). Because $\deg_{G_1} x > \Delta(G_2)$ and G_1 has more regions than G_2 does, to obtain $H \in gcs(G_1, G_2)$ we must remove a cycle edge from G_1 and this cycle edge must be incident with x. In this way we can produce at most one additional end-vertex. Therefore, we must reduce the number of end-vertices in G_2. But we have to remove a bridge, say f, from G_2, so either $f = uw$ or $f = uz$ and $H \cong G_2 - f \cong G$.

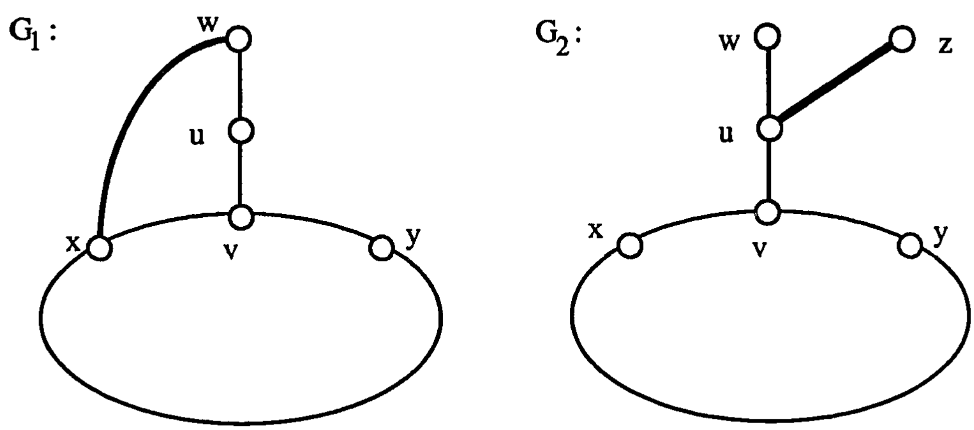


Figure 3.6

<u>Subcase 1.2.2. There is a terminal edge that is incident with a vertex of degree at least</u> 3. Let v be a vertex of maximum degree among vertices that are adjacent to end-vertices. Define $G_1 \cong G + wx$ and $G_2 \cong G + vz$, where w is an end-vertex

adjacent to v, vertex x is adjacent to v and vx is the next edge after vw, as we proceed in counterclockwise direction about v, and $z \notin V(G)$ (see Figure 3.7).

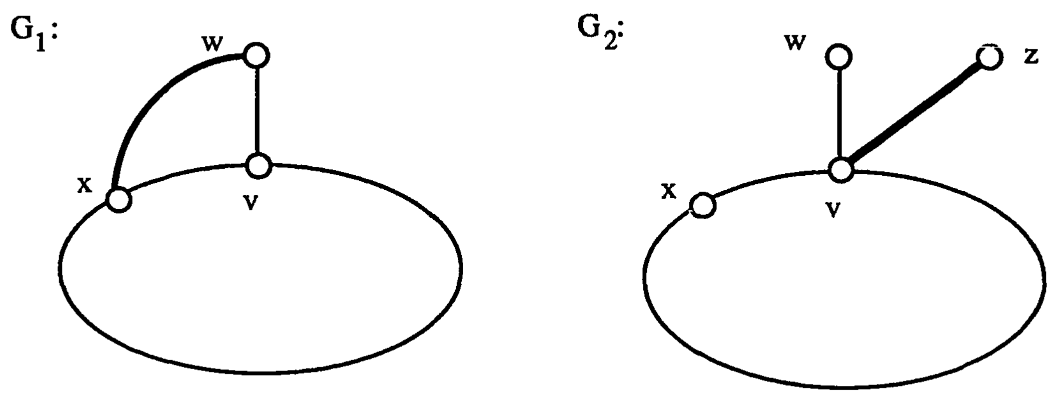


Figure 3.7

Of course, $G \in gcs(G_1, G_2)$. To obtain $H \in gcs(G_1, G_2)$ we must remove a cycle edge from G_1 and a bridge f from G_2. If this bridge is vw or vz, then $G_2 - f \cong G$. Otherwise, in G_2 the vertex v is adjacent to two end-vertices w and z. To produce two such end-vertices in G_1 a cycle edge e must be removed from a block, say v', w', z' isomorphic to K_3 (we cannot use v', if it is adjacent to an end-vertex, because $\deg_{G_1} v' < \deg_{G_2} v$). But then the graph $G_1 - e$ (see Figure 3.8) is not isomorphic to $G_2 - f$, since they have a different number of blocks isomorphic to K_3, which is impossible.

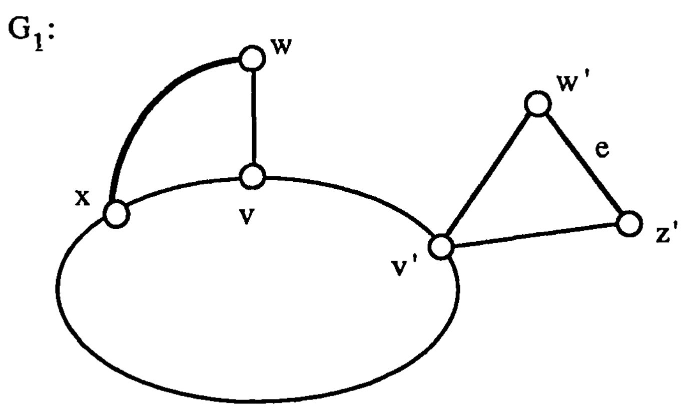


Figure 3.8

Case 2. The graph G has no end-vertices, but G is not maximal outerplanar.

There are two vertices x and y of G such that $xy \notin E(G)$ and $G + xy$ is outerplanar. Define two graphs $G_1 \cong G + xy$ and $G_2 \cong G + xu$, where $u \notin V(G)$ (see Figure 3.9).

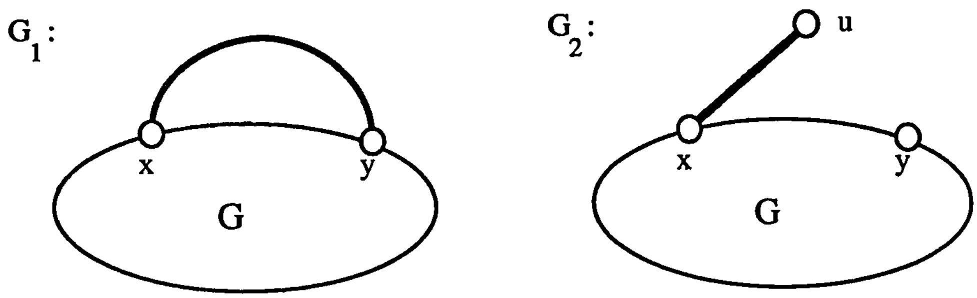


Figure 3.9

Then $G \in gcs(G_1, G_2)$. To produce $H \in gcs(G_1, G_2)$ we must remove a cycle edge e from G_1 ($G_1 - e$ will be connected) and a bridge f from G_2. If f is not a terminal bridge, then $G_2 - f$ is disconnected which is impossible. If f is a terminal bridge, then $f = xu$ and $H \cong G_2 - f \cong G$.

Case 3. The graph G is maximal outerplanar.

Since $G \ncong F_n$, $n = 1, 2, 3, 4$, it follows that $\Delta(G) \geq 4$.

Subcase 3.1. There is a vertex v of maximum degree that is not adjacent to a vertex of degree 2. Let x be a vertex of degree 2. Define $G_1 \cong G + vw$ and $G_2 \cong G + xy$, where $w, y \notin V(G)$ (see Figure 3.10).

Of course, $G \in gcs(G_1, G_2)$. Let $H \in gcs(G_1, G_2)$. To obtain H as a subgraph of G_1, we must remove an edge incident with v from G_1. If vw is removed, then $H \cong G$. If any other edge, say e, is removed from G_1, then in $G_1 - e$ we have only one end-vertex (namely w) that is adjacent to a vertex of degree at least 4. But in G_2, the end-vertex y is adjacent to a vertex of degree at most 3, so we have to remove the edge xy from G_2 to produce H. Therefore, $H \cong G$.

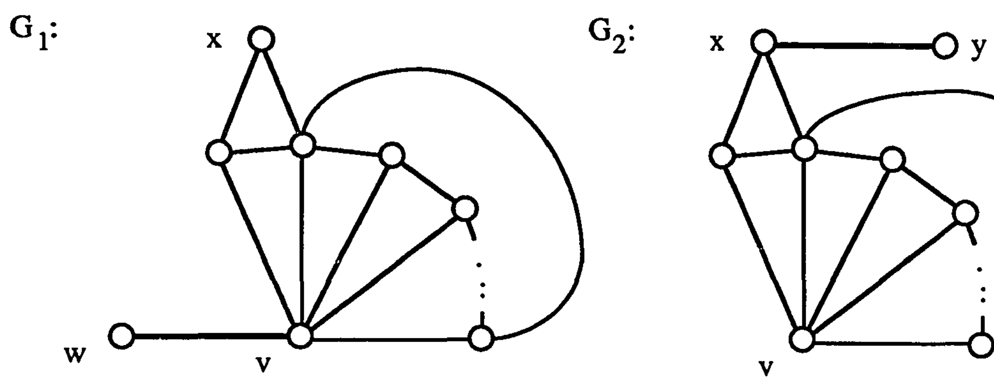


Figure 3.10

Subcase 3.2. There is a vertex v of maximum degree that is adjacent to exactly one vertex, say x, of degree 2. Define $G_1 \cong G + v_1w$ and $G_2 \cong G + x_2y$, where $w, y \notin V(G)$ and where v_i and x_i $(i = 1, 2)$ correspond to v and x in G (see Figure 3.11).

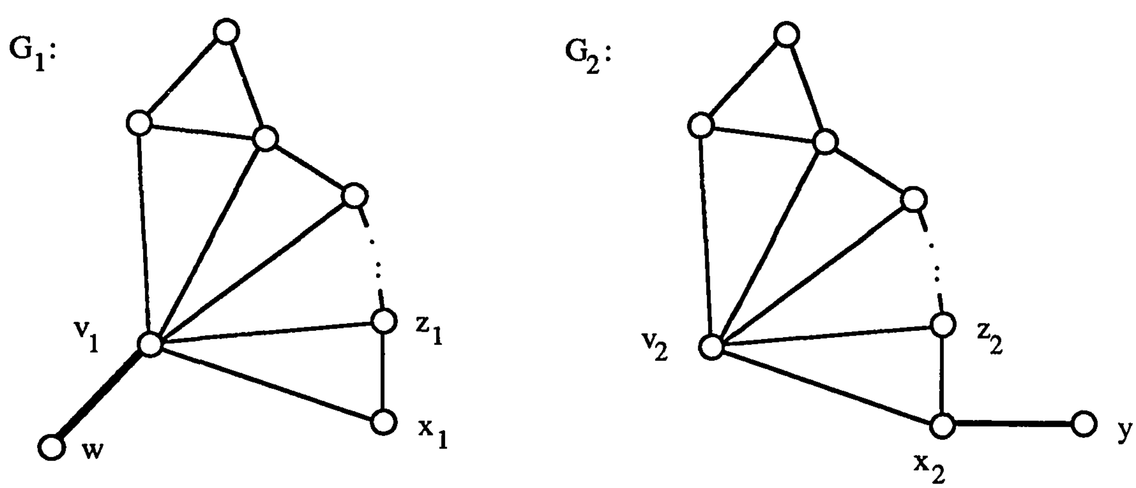


Figure 3.11

Of course, $G \in gcs(G_1, G_2)$. To obtain $H \in gcs(G_1, G_2)$ we must remove an edge f from G_2. If $f = x_2y$, then $H \cong G_2 - f \cong G$. Otherwise, $G_2 - f$ has an end-vertex, namely y, adjacent to the vertex x_2 of degree 2 or 3. To produce such an end-vertex in G_1, it is necessary to remove an edge e incident with v_1. The only possibility is $e = v_1x_1$. Then in $G_1 - e$ the end-vertex x_1 is adjacent to the vertex z_1. Therefore, $\deg_{G_1-e} z_1 = \deg_{G_2-f} z_2 = 3$. In $G_1 - e$ the neighbors z_1 and v_1 of two end-vertices are adjacent, and $\deg_{G_1-e} v_1 \geq 4$. The same must occur in $G_2 - f$. Thus, the second end-vertex (one is y) must be adjacent to v_2 (because $\deg_{G_2-f} z_2 \leq 3$). But v_2 in G_2 is not adjacent to any vertex of degree 2, so removing f from G_2 (f incident with v_2) does not produce an additional end-vertex.

Subcase 3.3. Every vertex of maximum degree is adjacent to two vertices of degree 2. Consider a plane embedding of G where all the vertices of G lie on the exterior region. Let v be a vertex of maximum degree n and suppose that $u_1, u_2, \ldots, u_n$ be the vertices adjacent to v as they appear in clockwise order about v in this embedding (see Figure 3.12). Then $\deg u_1 = 2$ and $\deg u_n = 2$.

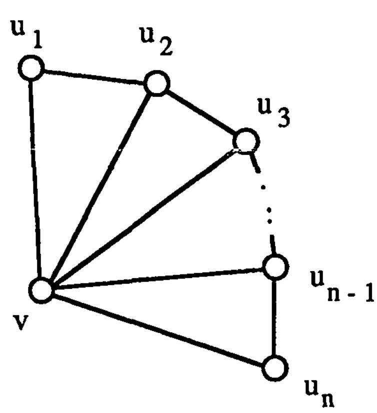


Figure 3.12

Consider the graph $G - v$.

Subcase 3.3.1. Suppose that $G - v \cong P_n$. Then $G \cong K_1 + P_n$ and $n \geq 5$, because for $n \leq 4$, we would have $G \cong F_n$. Define $G_1 \cong G + vw$ and $G_2 \cong G + u_3y$, where $w, y \notin V(G)$ (see Figure 3.13).

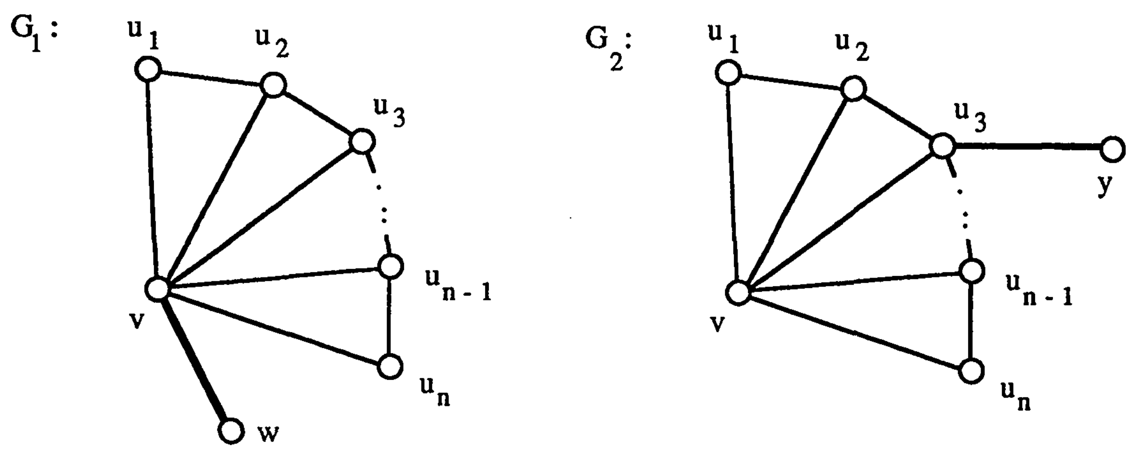


Figure 3.13

Of course, $G \in gcs(G_1, G_2)$. If $H \in gcs(G_1, G_2)$, then we must remove from G_1 an edge e incident with v. If $e = vw$, then $H \cong G$. Otherwise, in $G_1 - e$ there is an end-vertex, namely w, adjacent to the vertex v of degree n, $n \geq 5$. To produce such an end-vertex in G_2, we must remove either the edge u_1u_2 or the edge $u_{n-1}u_n$. In both cases, $G_2 - f$ has an end-vertex, namely y, adjacent to the vertex of degree 4. But we cannot produce such an end-vertex by removing e from G_1.

Subcase 3.3.2. Suppose that $G - v \ncong P_n$.

Let us define

$$m = \min\{ i \mid 2 \leq i \leq n-1, \ \deg_{G-v} u_i \geq 3\}$$

and

$$M = \max\{ i \mid 2 \leq i \leq n-1, \ \deg_{G-v} u_i \geq 3\}.$$

If $m - 1 \geq n - M$, define x to be u_1. Otherwise, define x to be u_n. Without loss of generality, assume that $x = u_1$. Define $G_1 \cong G + vw$ and $G_2 \cong G + u_1y$, where $w, y \notin V(G)$ (see Figure 3.14).

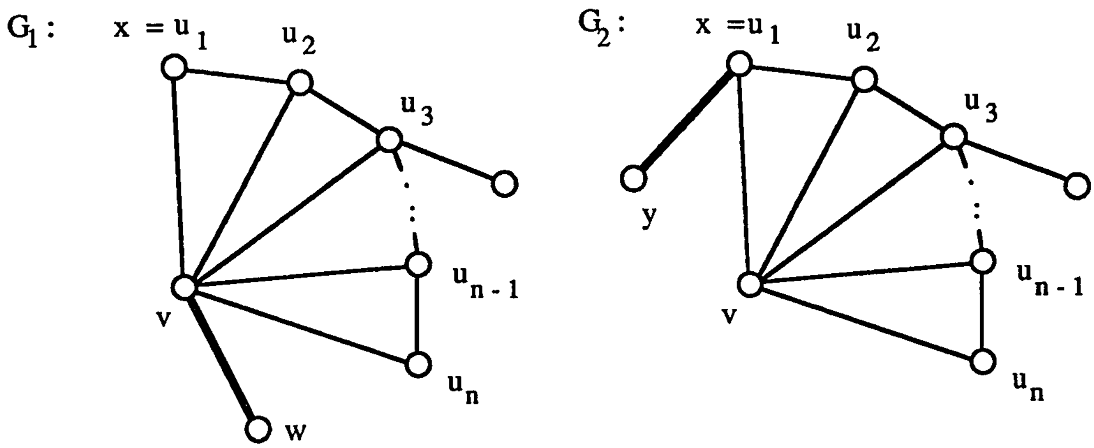


Figure 3.14

Of course, $G \in gcs(G_1, G_2)$. If $H \in gcs(G_1, G_2)$, then we must remove an edge f from G_2. If $f = u_1y$, then $H \equiv G$. Otherwise, in $G_2 - f$ there is an end-vertex, namely y, adjacent to the vertex u_1 of degree 2 or 3. To produce such an end-vertex, we must remove from G_1 either the edge vu_1 (and necessarily $\deg_G u_2 = 3$) or the edge vu_n ($\deg_G u_{n-1} = 3$). In both cases, the neighbors of end-vertices in $G_1 - e$ are adjacent. Therefore, the neighbors of end-vertices in $G_2 - f$ must be adjacent. Because u_1 is the neighbor of the end-vertex y, the second neighbor is either u_2 or v. But $\deg_{G_2} u_2 = 3$, so the second neighbor must be v. This is only possible if $f = u_{n-1}u_n$, but then $G_1 - e \not\equiv G_2 - f$. To prove this fact, consider the longest paths that use only the vertex v and vertices of degree 3 in $G_1 - e$ and in $G_2 - f$ between end-vertices. In $G_1 - e$ the path is either

$$x, u_2, \dots, u_{m-1}, v, w \qquad \text{or}$$

$$x, u_{n-1}, \dots, u_{n-M}, v, w$$

and in both cases is shorter than the path $y, x, u_2, \dots, u_{m-1}, v, u_n$ in $G_2 - f$. ❑

3.3 Planar Graphs

In this section we determine all connected planar graphs G for which there exist two nonisomorphic connected planar graphs G_1 and G_2 of the same size with $gcs(G_1, G_2) = \{G\}$. If we remove the requirement that G, G_1 and G_2 be connected, then the answer is immediate. Namely, all planar graphs G have the above property and a construction of graphs G_1 and G_2 can be the same as in the proof of Proposition 2 [7].

Let us consider first regular maximal planar graphs. Because every planar graph contains a vertex of degree at most 5, the degree of regularity is at most 5. Therefore, if we denote by T(r) an r-regular maximal planar graph, then $1 \le r \le 5$ and $T(1) \cong K_2$, $T(2) \cong K_3$, $T(3) \cong K_4$, $T(4) \cong K_{2,2,2}$ is the graph of the octahedron, and T(5) is the graph of the icosahedron (see Figure 3.15).

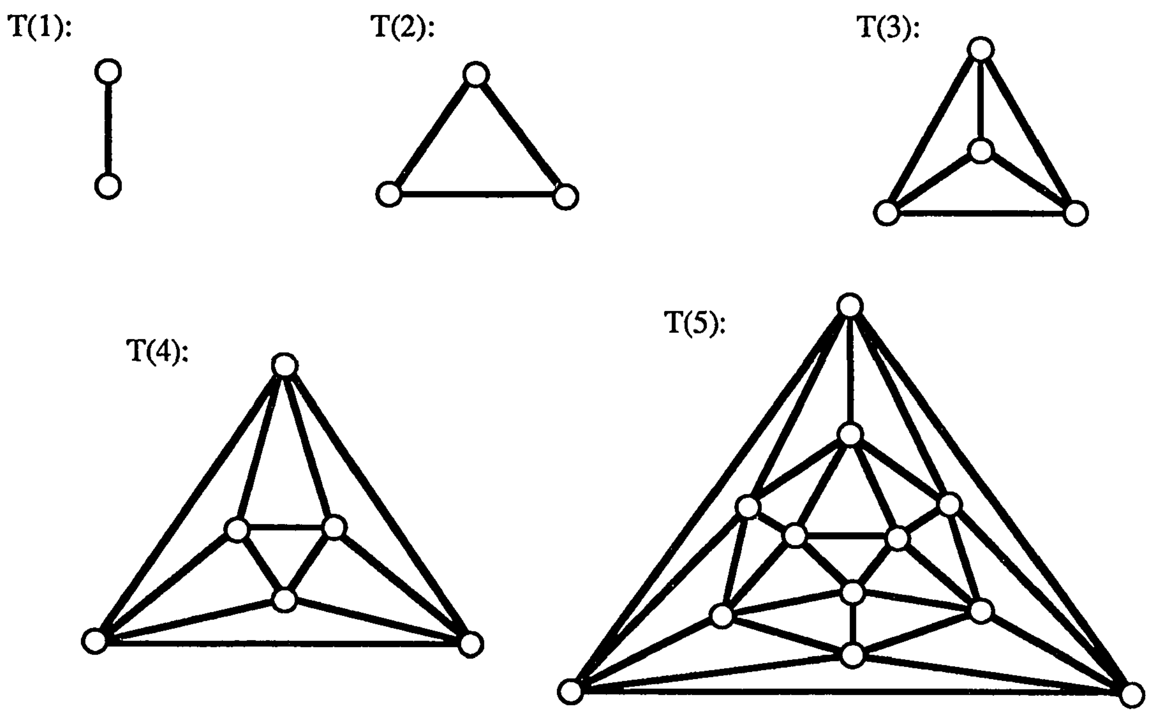


Figure 3.15. Five regular maximal planar graphs.

Let us note that if $G \cong T(r)$ for some r, $1 \le r \le 5$, then there are no connected nonisomorphic graphs G_1 and G_2 of the same size with $gcs(G_1, G_2) = \{G\}$. In fact, by Corollary 3.4 we can assume that $G_1 \cong G + e$ and $G_2 \cong G + f$, where both graphs $G + e$ and $G + f$ are planar and connected. Therefore, the edge e (as well as the edge f) must be incident with one vertex of G. But if G is a regular maximal planar graph then $G + e \cong G + f$, so $G_1 \cong G_2$, which is impossible.

We state the following two lemmas without proof.

Lemma 3.6 If G is a maximal planar graph with degree set $\mathcal{D}(G) = \{3, 4\}$, then G is isomorphic to the graph T(3, 4) given in Figure 3.16.

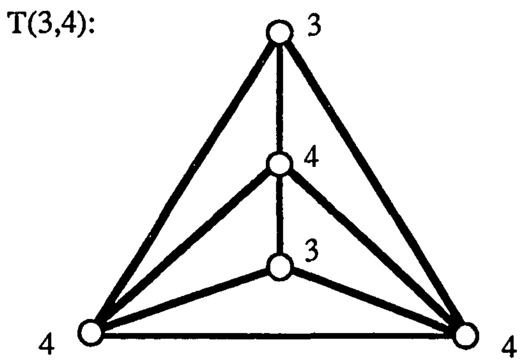


Figure 3.16

Lemma 3.7 There are exactly four nonisomorphic maximal planar graphs G with degree set $\mathcal{D}(G) = \{4, 5\}$, namely the four graphs given in Figure 3.17.

The fact that there is no a maximal planar graph G of order 11 and $\mathcal{D}(G) = \{4, 5\}$ (or equivalently, with ten vertices of degree 5 and one vertex of degree 4) follows from a theorem of Grünbaum ([10], pp. 272-275).

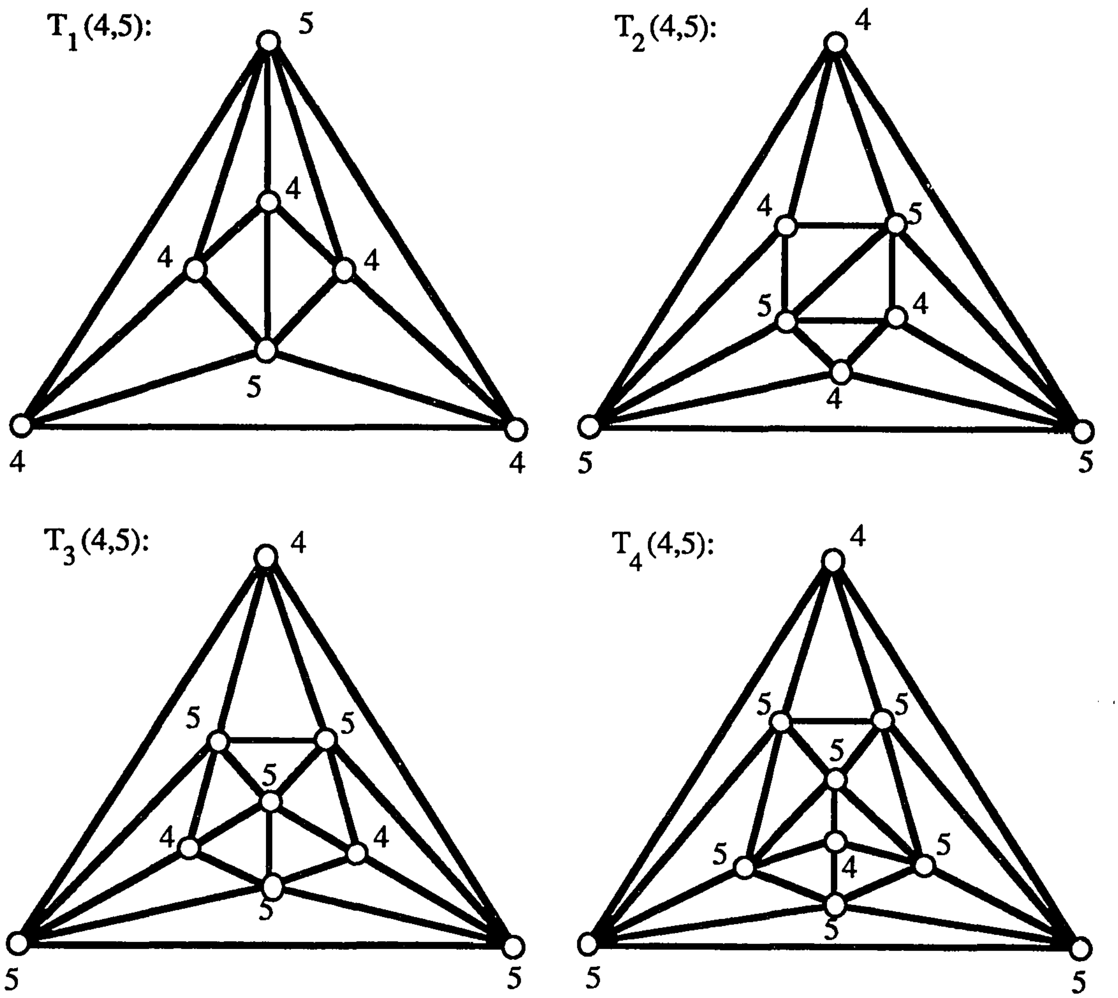


Figure 3.17

Theorem 3A Let G(k, t) denote a maximal planar graph in which the vertex degrees are multiples of k with t exceptions. Then:

(i) There does not exist a G(k, 1) for k = 2, 3, 4, 5.

(ii) There does not exist a G(k, 2) in which the two exceptional vertices are adjacent for k = 2, 3, 4, 5.

If G is a maximal planar graph with degree set $\mathcal{D}(G) = \{5, 6\}$, then G has exactly twelve vertices of degree 5. By Grünbaum's theorem, there is no maximal planar graph with $\mathcal{D}(G) = \{5, 6\}$ and with only one vertex of degree 6. However, Grünbaum and Motzkin [11] constructed a maximal planar graph with twelve vertices of degree 5 and n vertices of degree 6 for every $n \geq 2$. A different construction of such graphs was given by Etourneau [9], together with a proof that all such graphs (maximal planar with degree set $\{5, 6\}$) are 5-connected. In the proof of the main theorem we will use the following well-known fact observed first by Whitney [12].

Theorem 3B If a planar graph G is 3-connected, then G is uniquely embeddable on the sphere.

Let us consider first "exceptional " connected planar graphs.

Theorem 3.8 If $G \cong T(3, 4)$ from Figure 3.16 or $G \cong T_i(4, 5)$, where $i = 2$ or 3, from Figure 3.17, then there do not exist nonisomorphic connected planar graphs G_1 and G_2 of equal size with $gcs(G_1, G_2) = \{G\}$.

Proof. By Corollary 3.4, if there exist two nonisomorphic connected planar graphs G_1 and G_2 of the same size such that $gcs(G_1, G_2) = \{G\}$, then we can assume that $G_1 \cong G + e$ and $G_2 \cong G + f$ for some edges e and f.

If $G \cong T(3, 4)$, then G_1 and G_2 are as shown in Figure 3.18, and $gcs(G_1, G_2) = \{G, G - g + e\}$, so $gcs(G_1, G_2)$ is not unique.

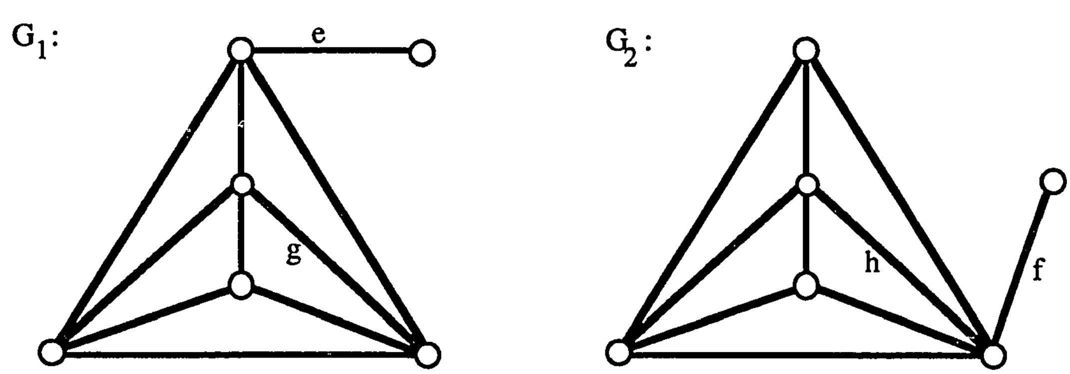


Figure 3.18

We have the same situation for the graph $T_2(4, 5)$ or the graph $T_3(4, 5)$. For example, if $G \cong T_2(4, 5)$, then G_1 and G_2 are as shown in Figure 3.19, where $G_1 - g \cong G_2 - h$. Therefore, $\{G, G - g + e\} \subseteq gcs(G_1, G_2)$ and $gcs(G_1, G_2)$ is not unique.❑

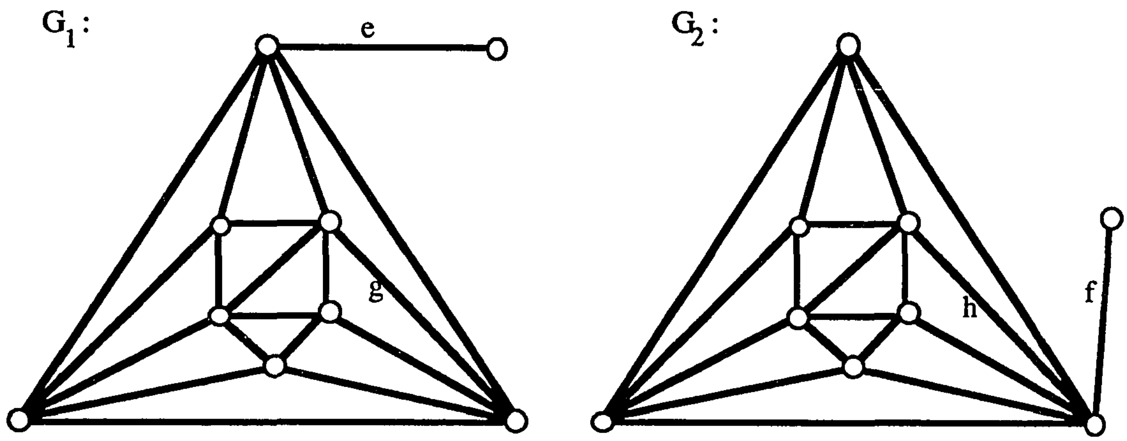


Figure 3.19

Therefore, we have proved that if $G \cong T(r)$, $1 \le r \le 5$, or G is isomorphic to $T(3, 4)$, $T_2(4, 5)$ or $T_3(4, 5)$, then we are not able to construct two nonisomorphic connected planar graphs G_1 and G_2 of the same size such that G is the unique greatest common subgraph of G_1 and G_2. In the main theorem of this section we will show that these eight graphs are the only exceptions. In the proof of the theorem it will be convenient to use the following notation. If a graph G is embedded in the plane and a boundary of a region is an r - cycle ($r \ge 3$), then we will call this region an <u>r - region.</u> A vertex of degree n is called an <u>n - vertex</u>.

Theorem 3.9 Let G be a connected planar graph such that G is not isomorphic to any of the graphs: $T(r)$ $(1 \le r \le 5)$, $T(3, 4)$, $T_2(4, 5)$ and $T_3(4, 5)$. Then there exist two nonisomorphic connected planar graphs G_1 and G_2 of the same size for which $gcs(G_1, G_2) = \{G\}$.

Proof. We consider the following cases.

Case 1. The graph G has an end-vertex.

Subcase 1.1. There are two vertices $x, y \in V(G)$ such that $\deg x < \Delta(G)$, $\deg y < \Delta(G)$, $xy \notin E(G)$ and $G + xy$ is planar. Construct two graphs $G_1 \cong G + xy$ and $G_2 \cong G + vw$, where v is a vertex of maximum degree and $w \notin V(G)$. Then, in the same manner as in the proof of Theorem 3.5, we can show that $gcs(G_1, G_2) = \{G\}$.

Subcase 1.2. For any two vertices x, y with $\deg x < \Delta(G)$, $\deg y < \Delta(G)$, $xy \notin E(G)$ the graph $G + xy$ is not planar. We can assume that $\Delta(G) \geq 3$. If $\Delta(G) \leq 2$, then $G \cong P_n$, $n \geq 3$, so $G + xy \cong C_n$ and $G + xy$ is planar. Therefore, the conditions of Subcase 1.1 hold. Using the similar arguments as in the proof of Theorem 3.5, we can construct graphs G_1 and G_2 with $gcs(G_1, G_2) = \{G\}$.

Case 2. The graph G has no end-vertices, but G is not maximal planar.

There are two vertices x and y of G such that $xy \notin E(G)$ and $G + xy$ is planar. Define two graphs $G_1 \cong G + xy$ and $G_2 \cong G + xu$, where $u \notin V(G)$. Then $gcs(G_1, G_2) = \{G\}$ and the proof of this fact is the same as in Theorem 3.5.

Case 3. The graph G is maximal planar.

Subcase 3.1. Assume G contains two vertices u and v such that $\deg u - \deg v \geq 2$ (or $\Delta(G) - \delta(G) \geq 2$). Let u be a vertex of maximum degree and v a vertex of minimum degree. Consider two graphs $G_1 \cong G + ux$ and $G_2 \cong G + vy$, where $x, y \notin V(G)$. Of course, $G \in gcs(G_1, G_2)$. To obtain $H \in gcs(G_1, G_2)$ we must remove one edge, say e, from G_1 and one edge from G_2. If $e = ux$, then $H \cong G_1 - e \cong G$. Otherwise, in $G_1 - e$ there is the end-vertex x adjacent to the vertex of degree $\Delta(G)$. We cannot produce such an end-vertex by removing one edge from G_2. Therefore, $H \cong G$ and $gcs(G_1, G_2) = \{G\}$.

<u>Subcase 3.2</u>. <u>Assume</u> $\Delta(G) - \delta(G) \leq 1$. Because G is not regular, $\Delta(G) - \delta(G) = 1$, or $\mathcal{D}(G) = \{d, d + 1\}$, where $3 \leq d \leq 5$ ($d \geq 6$ is impossible because every planar graph has a vertex of degree at most 5). By Lemma 3.6, if $d = 3$ then $G \cong T(3, 4)$, but we assumed that G is not isomorphic to T(3, 4). Therefore, the only two possibilities are $\mathcal{D}(G) = \{4, 5\}$ or $\mathcal{D}(G) = \{5, 6\}$, and we consider them in two subcases.

<u>Subcase 3.2.1</u>. <u>Assume that</u> $\mathcal{D}(G) = \{4, 5\}$. By Lemma 3.7, there are exactly four such graphs, namely the graphs $T_i(4, 5)$, $1 \leq i \leq 4$, but the graphs $T_2(4, 5)$ and $T_3(4, 5)$ were excluded in the assumption of the theorem.

If $G \cong T_1(4, 5)$, then we define two graphs $G_1 \cong G + ux$ and $G_2 \cong G + vy$ as in Figure 3.20, where $x, y \notin V(G)$. The numbers in Figure 3.20 denote degrees of vertices.

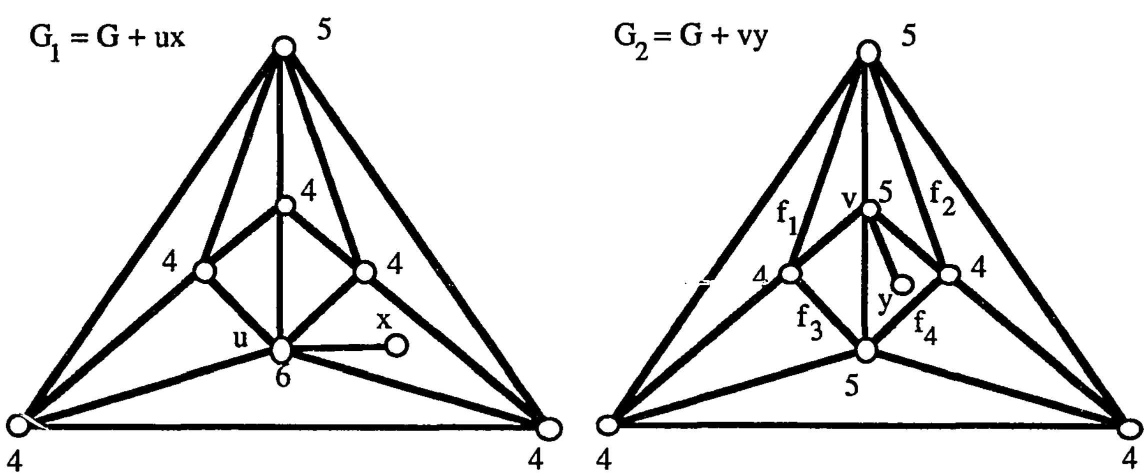


Figure 3.20

Of course, $G \in gcs(G_1, G_2)$. To obtain $H \in gcs(G_1, G_2)$, we have to remove from G_1 an edge e that is incident with the vertex u. If $e = ux$, then $H \cong G$.

Otherwise, in $G_1 - e$ the unique end-vertex x is adjacent to the vertex u that lies on the boundary of the 4-region of the form A represented in Figure 3.21 (numbers denote degrees of vertices). Because the graph $G - e$ is 3-connected (and planar), it follows from Theorem 3B that there is a unique (up to the orientation) embedding of the graph $G - e$ in the plane. Therefore, if we neglect the orientation and different possibilities of placing the end-vertex x in the plane, there is only one embedding of the graph $G_1 - e$ in the plane. The same is true for the graph $G_2 - f$. If $G_1 - e \cong G_2 - f$, then a 4-region with a vertex on its boundary that is adjacent to the end-vertex y must be present in $G_2 - f$. Therefore, the removed edge f must be one among f_1, f_2, f_3, f_4. But then the 4-region in $G_2 - f$ is of the form B represented in Figure 3.21 and $G_1 - e \not\cong G_2 - f$.

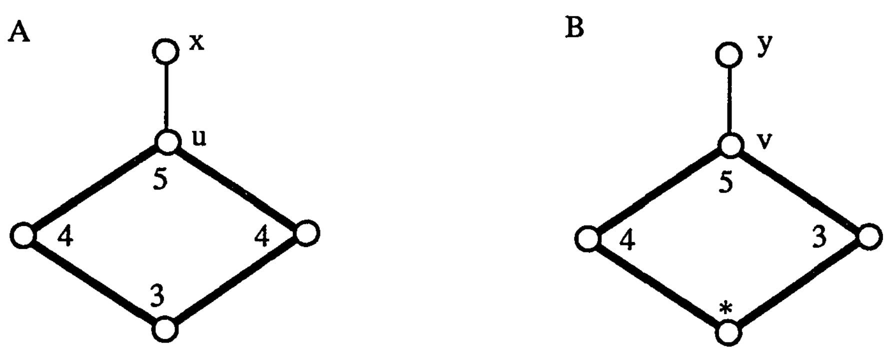


Figure 3.21

If $G \cong T_4(4, 5)$, then we construct two graphs $G_1 \cong G + ux$ and $G_2 \cong G + vy$, where u is a vertex of degree 5, v is a vertex of degree 6, and $x, y \notin V(G)$. The neighborhoods N(u) and N(v) of vertices u and v in G_1 and G_2, respectively, are as in Figure 3.22.

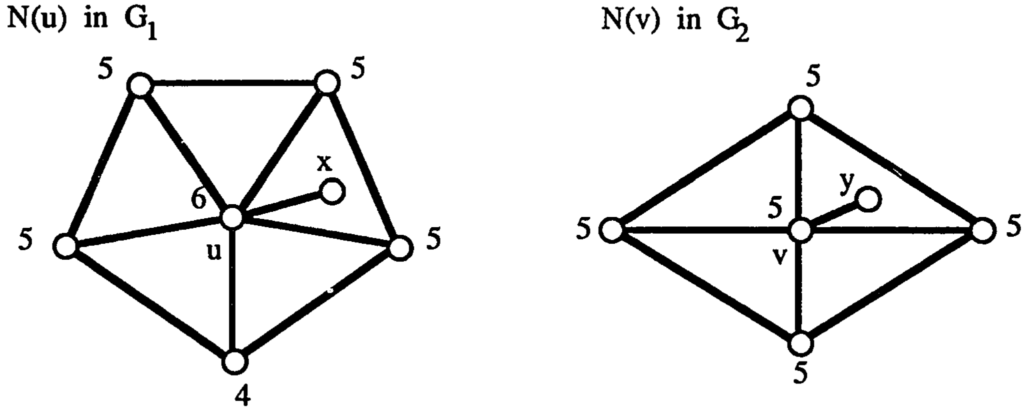


Figure 3.22

By the same reason as mentioned above, for every $e \in E(G_1)$ and every $f \in E(G_2)$ the graphs $G_1 - e$ and $G_2 - f$ are uniquely embeddable in the plane. If we remove the edge e that is incident with the vertex u (except e = ux), then the 4-region in $G_1 - e$ is of the form A represented in Figure 3.23. But the only possible 4-region in $G_2 - f$ is of the form B (see Figure 3.23), so $G_1 - e \not\cong G_2 - f$.

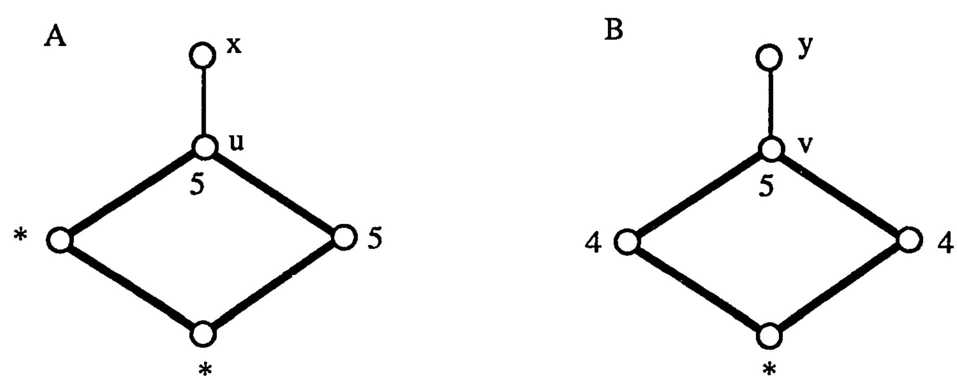


Figure 3.23

<u>Subcase 3.2.2</u>. <u>Assume that</u> $\mathcal{D}(G) = \{5, 6\}$. In the proof of the theorem in this case it will occasionally be more convenient to work with the dual graph. Let us note that if G is a maximal planar graph with $\mathcal{D}(G) = \{5, 6\}$, then its dual graph G* is a planar

cubic graph every region of which is a pentagon (5 - cycle) or a hexagon (6 - cycle). We will also use the following notation. If a vertex v has degree n, we will denote it by v(n). Let G be a plane graph. If the vertices adjacent to a vertex v(n) are $v_1, v_2, \ldots, v_n$, and deg $v_i = d_i$, $i = 1, 2, \ldots, n$, then the (ordered) neighborhood of the vertex v(n) will be denoted by $N(v) = (v_1(d_1), v_2(d_2), \ldots, v_n(d_n))$ or, more simply, by $N(v) = (d_1 d_2 \ldots d_n)$ if the names of vertices are not important.

Let u(6) and v(5) be vertices of G (of degree 6 and 5, respectively). We define two graphs $G_1 \cong G + ux$ and $G_2 \cong G + vy$, where $x, y \notin V(G)$. Of course, G_1 and G_2 are nonisomorphic connected planar graphs of equal size, and $G \in gcs(G_1, G_2)$. Suppose that $H \in gcs(G_1, G_2)$ and $H \ncong G$. To obtain H we must remove one edge, say e, from G_1. The edge e must be incident to the vertex u, but $e \neq ux$. Therefore, in the graph $G_1 - e$, the unique terminal bridge ux is incident to the vertex u that lies on the boundary of a 4 - region (see Figure 3.24). The same configuration must occur in the graph $G_2 - f$. Therefore, the removed edge f must joint two consecutive vertices from the neighborhood of v (see Figure 3.24).

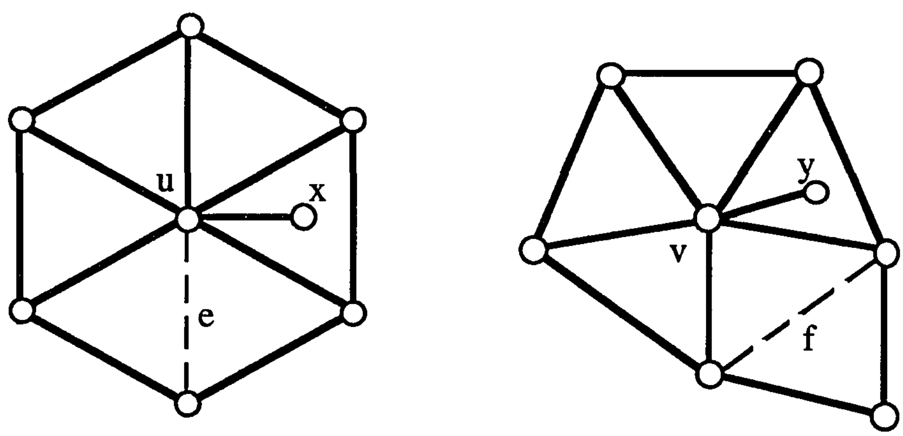


Figure 3.24

Consider the unique 4 - region u, u_1, u_2, u_3 in G_1 - e and the unique 4 - region v, v_1, v_2, v_3 in G_2 - f (see Figure 3.25). Because the graphs G - e and G - f are 3 - connected (and planar), using Theorem 3B, we conclude that they are uniquely (up to the orientation) embeddable in the plane. But x and y are the only end-vertices in G_1 - e and G_2 - f, respectively. Therefore, we must have the following correspondence between vertices in G_1 - e and G_2 - f:

$x \leftrightarrow y$, $u \leftrightarrow v$, $u_2 \leftrightarrow v_2$,

$u_1 \leftrightarrow v_1$ (and then $u_3 \leftrightarrow v_3$) or

$u_1 \leftrightarrow v_3$ (and then $u_3 \leftrightarrow v_1$).

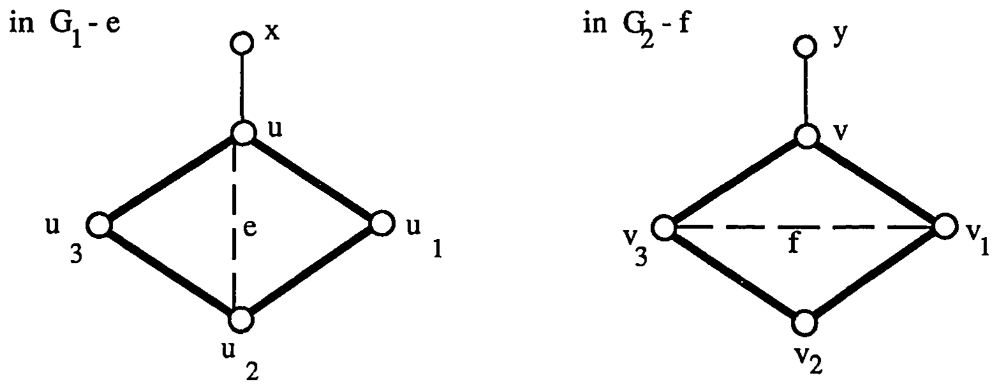


Figure 3.25

Now we can make the following observations.

Observation 1. In the neighborhood of the vertex v(5) in G, there are two consecutive vertices of degree 6.

Otherwise, in the graph G_2 - f at least one of the vertices v_1 and v_3 (see Figure 3.25) has degree 4, but both vertices u_1 and u_3 have degree at least 5.

Observation 2. The edge f removed from G_2 joins two vertices of degree 6. Moreover, the second vertex (one is v) that is adjacent to both of these vertices, namely v_2, must have degree 5.

In fact, the degree of v_2 is 5 or 6. On the other hand, the vertex v_2 in G_2 - f corresponds to the vertex u_2 in G_1 - e whose degree is 4 or 5. Hence, its degree must be 5.

Observation 3. In the neighborhood of the vertex u(6) in G the configuration (565) must be present.

In fact, consider the vertices u_1, u_2, u_3 adjacent to u. The vertices u_1 and u_3 correspond to the vertices v_1 and v_3 in G_2 - f, so their degrees are 5. The vertex u_2 in G has degree 6 because it has degree 5 in G_1 - e.

Observation 4. If the vertex u(6) in G is adjacent to s vertices of degree 6, then the vertex v(5) in G is adjacent to s + 1 vertices of degree 6.

In fact, the vertices u in G_1 - e and v in G_2 - f must correspond to each other, so the neighborhood of u must correspond to the neighborhood of v. By removing the edge e from G_1 we reduced the number of 6 - vertices adjacent to u by 1. But removing the edge f from G_2 reduces the number of 6 - vertices adjacent to v by 2.

In the construction of graphs G_1 and G_2, vertices u(6) and v(5) can be chosen arbitrarily. Therefore, by Observation 4, we have the following.

Observation 5. Every vertex u of degree 6 in G must be adjacent to the same number, say s, of 6 - vertices. Then every vertex v of degree 5 in G must be adjacent to s + 1 vertices of degree 6.

By Observation 3, it follows that $s \geq 1$. Of course, s + 1 does not exceed the degree of v(5), so $s + 1 \leq 5$. Finally, $1 \leq s \leq 4$. We will distinguish four cases according to the value of s, the number of 6 - vertices adjacent to a vertex u of degree 6 in G.

Assume that s = 1. In the dual graph G* we must have the configuration of Figure 3.26, where the numbers denote the degrees of vertices in G. But by Observation 1, the vertex v must be adjacent to two consecutive vertices of degree 6, or in G*, two adjacent 6 - regions. This implies that v is adjacent to three 6 - vertices, which gives a contradiction.

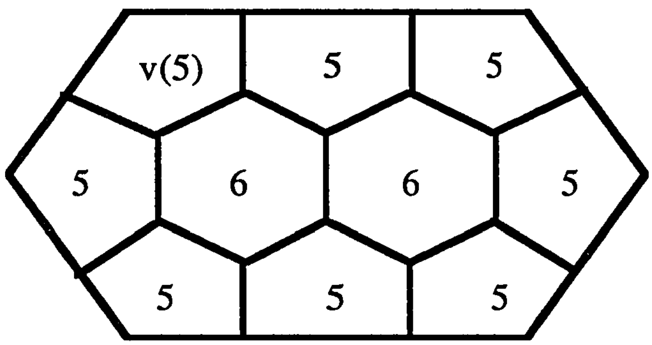


Figure 3.26

Assume that s = 2. Suppose first that there is a vertex of degree 5 in G, say the vertex v, with a neighborhood N(v) = (6^2565). Then in the dual graph G* we have the configuration of Figure 3.27. We use the labeled vertices u(6) and v(5) for the construction of the graphs G_1 and G_2.

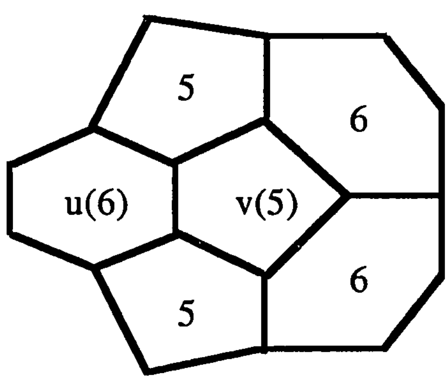


Figure 3.27

The neighborhood of the vertex v in G_2 - f is of the form $(5^2 6 5^2)$, where we list vertices starting and ending on the boundary of 4 - region and we do not consider the end-vertex x (see Figure 3.28). But in the neighborhood of the vertex u(6) in G_1 - e we have three consecutive vertices of degree 5, which gives a contradiction.

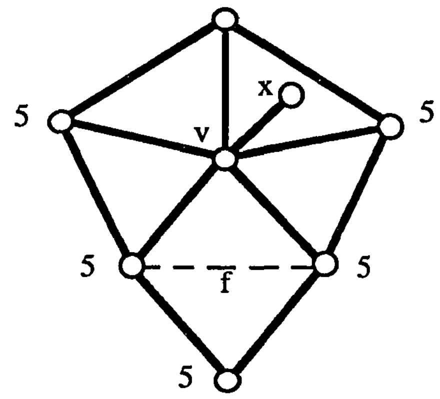


Figure 3.28

In the other case, the neighborhood of every vertex v is $N(v) = (6^3 5^2)$. Then in G* the configuration shown in Figure 3.29 is forced. Therefore, there exists a vertex of degree 5, namely the vertex v', with the neighborhood $N(v') = (6^2 565)$ and in fact this case cannot happen.

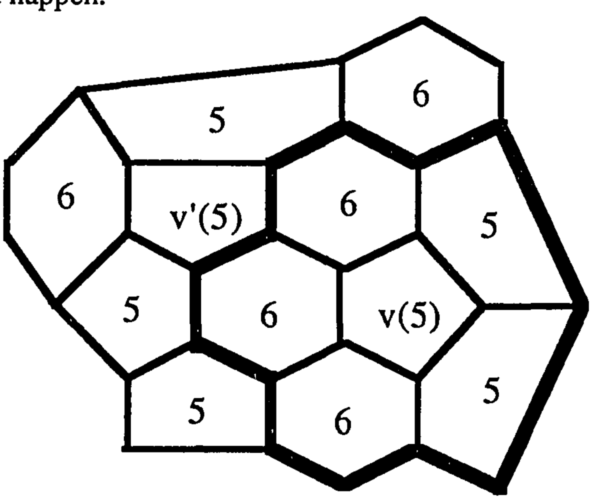


Figure 3.29

Assume that s = 3. Then every neighborhood of v(5) is $N(v) = (6^4 5)$ and in the graph G* the configuration of Figure 3.30 is forced.

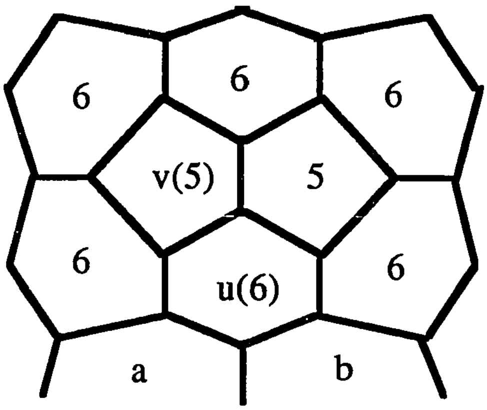


Figure 3.30

Because not both a and b (see Figure 3.30) can be 5 - regions (6 - regions), we can assume by symmetry that a is 5 - region and b is 6 - region. With this assumption the graph G* must have the subgraph shown in Figure 3.31.

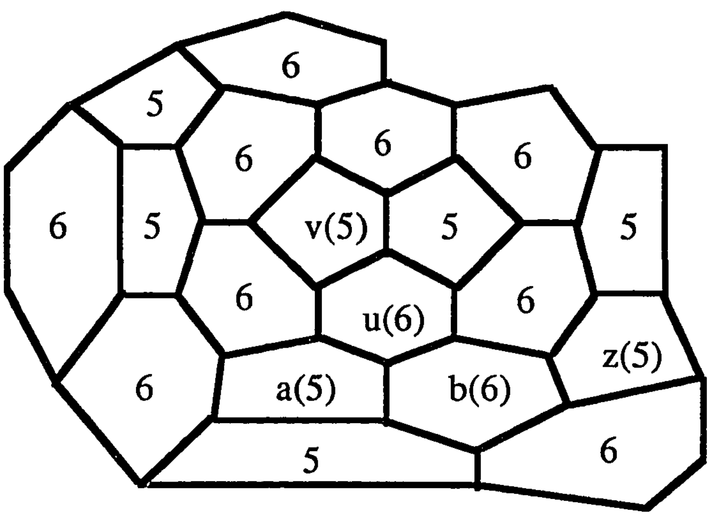


Figure 3.31

If we use the two labeled vertices u(6) and v(5) for the construction of G_1 and G_2, then the neighborhood of u in $G_1 - e$ is $N(u) = (5^2 6^2 5)$, as shown in Figure 3.32, and the vertex z (adjacent to two vertices of degree 6 from this neighborhood) has degree 5.

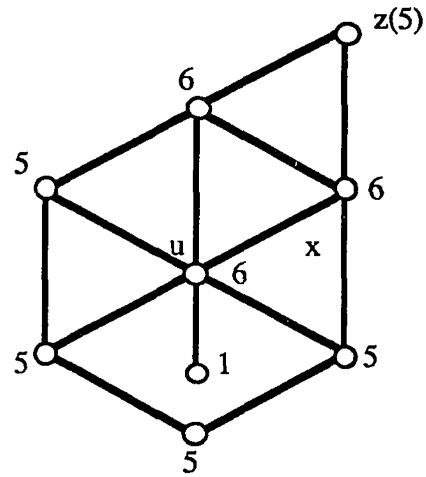


Figure 3.32

But by removing an edge f from G_2, we can obtain the neighborhood of v of one of the two types shown in Figure 3.33.

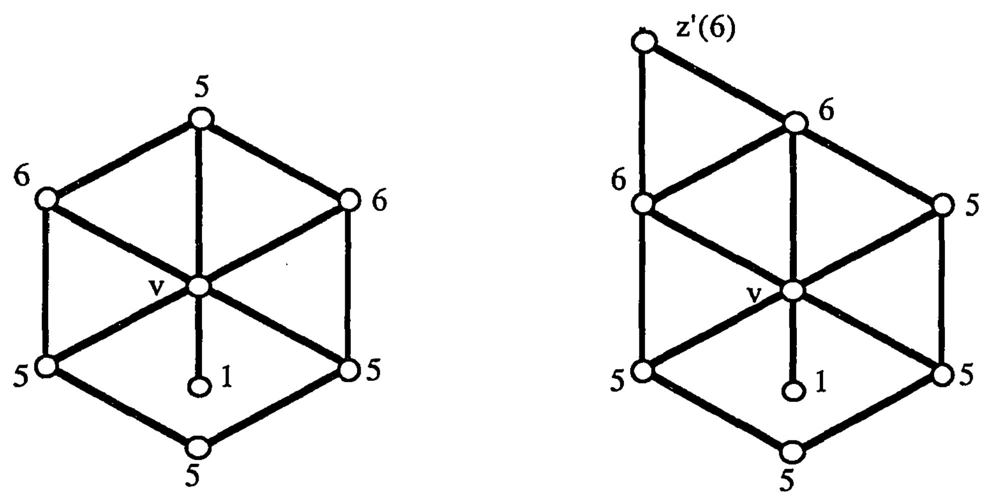


Figure 3.33

Either N(v) = (56565), so <N(v)> $\not\cong$ <N(u)>, or N(v) = (56^25^2). But the vertex z' (adjacent to two vertices of degree 6 from the neighborhood) has degree 6. This contradiction completes the proof in the case when s = 3.

Assume finally that s = 4. Then in G* we have the situation shown in Figure 3.34 and the neighborhood of the labeled vertex u is N(u) = (6^256^25). But by Observation 3 the configuration (565) must be present in N(u), which produces a contradiction. ❑

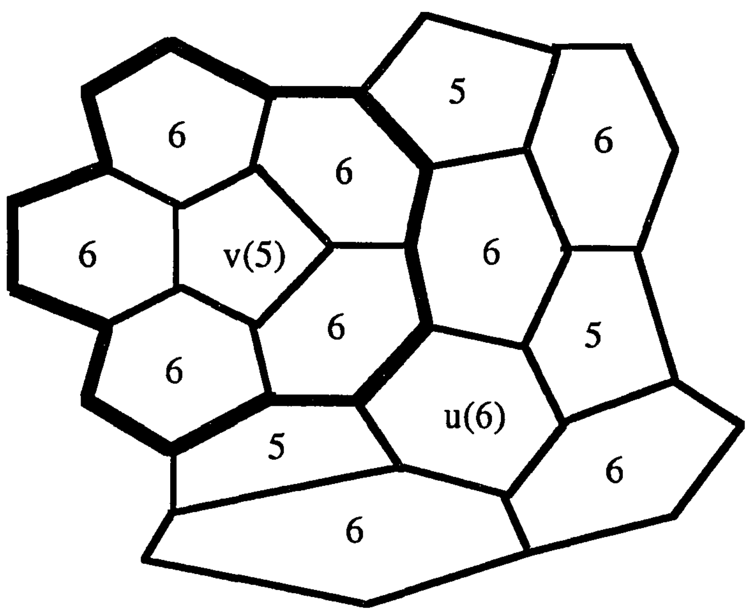


Figure 3.34

3.4 Unicyclic Graphs

Let us recall that a graph G is unicyclic if G is connected and contains exactly one cycle. In this section we will determine all unicyclic graphs for which there exist two nonisomorphic unicyclic graphs G_1 and G_2 of the same size with gcs(G_1, G_2) = {G}.

Let us first define two special families of unicyclic graphs $\{C(3k, n) \mid k \geq 1,\ n \geq 1\}$ and $\{D(4k, n) \mid k \geq 1,\ n \geq 1\}$. The graph $C(3k, n)$ consists of the cycle of length $3k$, every third vertex of which is the central vertex of a star $K(1, n)$ none of whose edges lie on the cycle. The graphs $C(3, 4)$ and $C(9, 1)$ are shown in Figure 3.35.

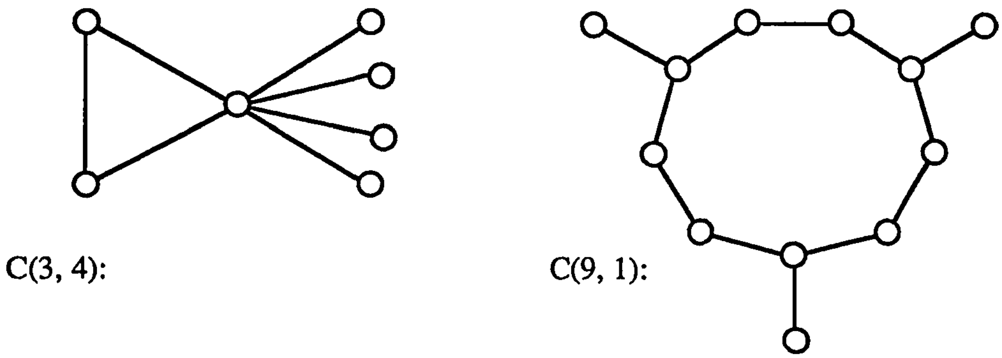


Figure 3.35

The graph $D(4k, n)$ consists of a cycle $v_1, v_2, \ldots, v_{4k}, v_1$ of length $4k$ such that every vertex v_i with $i \equiv 1$ or $2 \pmod 4$ is a central vertex of a star $K(1, n)$ none of whose edges lie on the cycle. The graphs $D(4, 3)$ and $D(8, 2)$ are given in Figure 3.36.

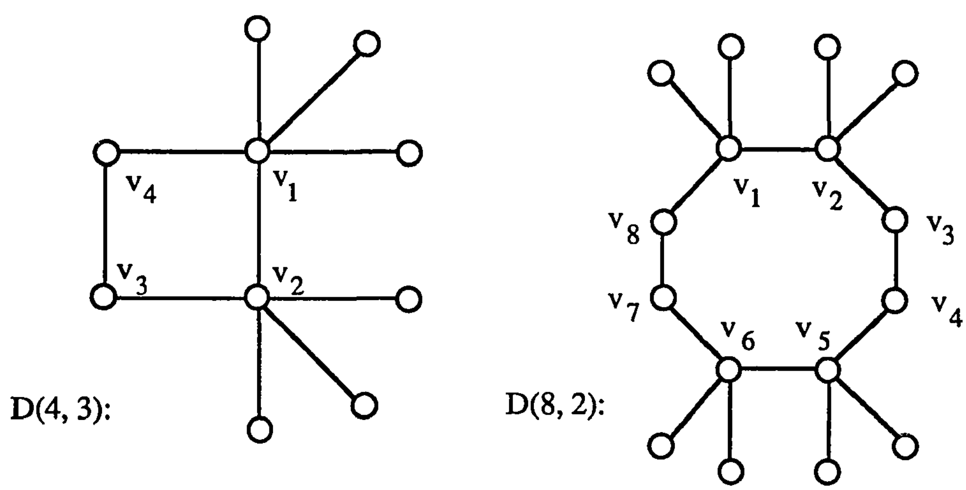


Figure 3.36

First we will show that graphs from these two families as well as cycles are "exceptional" unicyclic graphs.

Theorem 3.10 If $G \cong C_k$, $k \geq 3$, $G \cong C(3k, n)$ or $G \cong D(4k, n)$, where $k \geq 1$ and $n \geq 1$, then there do not exist nonisomorphic unicyclic graphs G_1 and G_2 of equal size with $gcs(G_1, G_2) = \{G\}$.

Proof. Assume, to the contrary, that we are able to construct such graphs G_1 and G_2. By Corollary 3.4, we may assume that $q(G_1) = q(G_2) = q(G) + 1$. Because graphs G_1 and G_2 are unicyclic, G_1 (and G_2) is obtained by adding exactly one terminal edge to the graph G.

If $G \cong C_k$, $k \geq 3$, then $G_1 \cong G_2 \cong G + vx$, where $v \in V(C_k)$ and $x \notin V(C_k)$, which gives a contradiction.

If $G \cong C(3k, n)$, $k \geq 1$, $n \geq 1$, then by the symmetry of the graph $C(3k, n)$ there are only three unicyclic graphs G_1, G_2 and G_3 such that $q(G_i) = q(G) + 1$ and $G \subset G_i$ $(i = 1, 2, 3)$. But then not only $C(3k, n)$ but also the caterpillar T of diameter $3k$ whose $3k + 1$ vertices on the longest path have degrees $1, 2, n + 2, 2, 2, n + 2, \ldots, 2, 2, n + 2, 1$ is a subgraph of G_i, $i = 1, 2, 3$. Therefore, for every pair $i, j \in \{1, 2, 3\}$, $i \neq j$, we have that $\{G, T\} \subset gcs(G_i, G_j)$ which contradicts the fact that a greatest common subgraph is unique.

To illustrate this fact consider $G \cong C(3, 4)$. Then G_1, G_2 and G_3 are represented in Figure 3.37 where the caterpillar T is marked by bold edges.

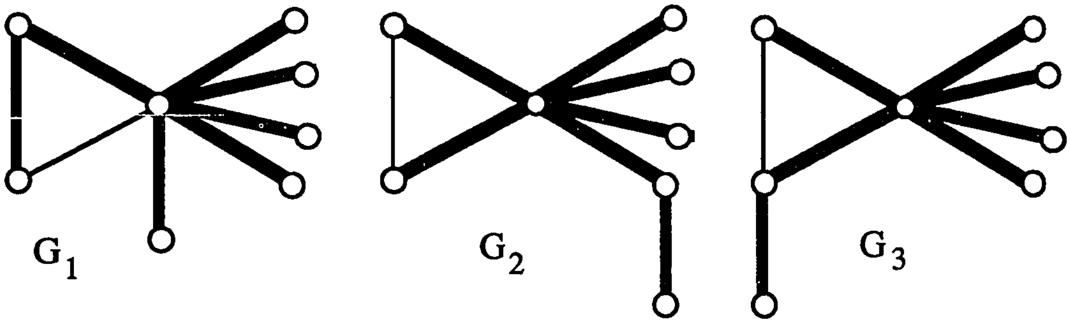


Figure 3.37

Finally, if $G \cong D(4k, n)$, $k \geq 1$, $n \geq 1$, then by the symmetry of $D(4k, n)$ there are three possibilities to construct unicyclic graphs G_i such that $G \subset G_i$ and $q(G_i) = q(G) + 1$. But then the caterpillar T of diameter $4k$ whose $4k + 1$ vertices on the longest path have degrees $1, 2, n + 2, n + 2, 2, 2, n + 2, n + 2, \ldots, 2, 2, n + 2, n + 2, 1$ is a subgraph of G_i, $i = 1, 2, 3$. Therefore, $\{G, T\} \subset gcs(G_i, G_j)$ for every $i, j \in \{1, 2, 3\}$, $i \neq j$, which produces a contradiction.

As an illustration consider $G \cong D(8, 2)$. Then G_1, G_2 and G_3 are given in Figure 3.38 where T is marked by bold edges. ❑

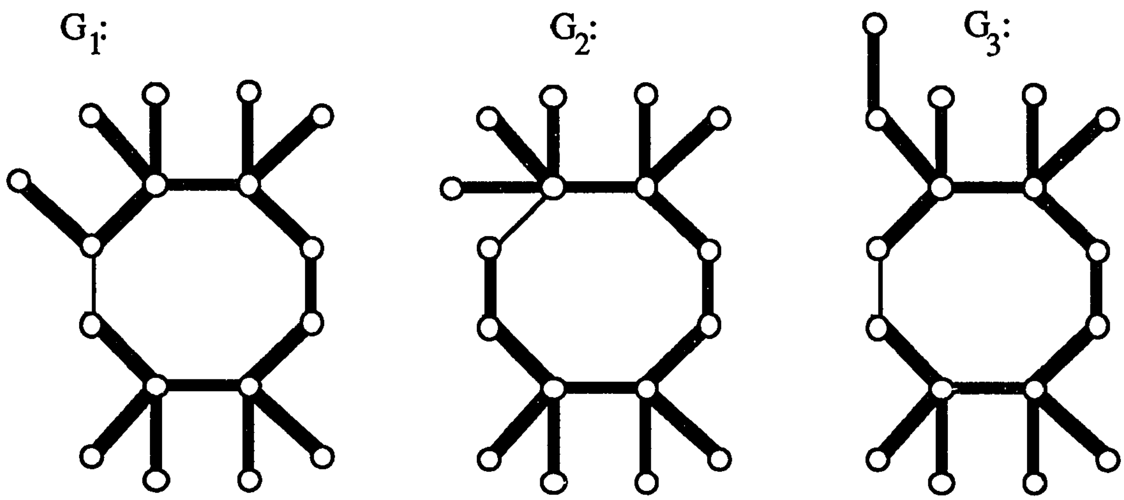


Figure 3.38

The characterization of unicyclic graphs will be completed if we show that the graphs from the three families described in Theorem 3.10 are the only exceptions. In the proof of the main theorem of this section we will use the following notation. If G is an unicyclic graph and C is its subgraph that is the cycle, then for every vertex $v \in V(G)$ the distance $d(v, C)$ from v to the cycle is the length of the shortest v-u path, where $u \in V(C)$. If $e = xy \in E(G)$, $d(x, C) = d - 1$ and $d(y, C) = d$, then a level of the edge e is defined to be d.

Theorem 3.11 Let G be a unicyclic graph such that G is not a cycle and G is not isomorphic to any of the graphs $C(3k, n)$ or $D(4k, n)$, $k \geq 1$, $n \geq 1$. Then there exist two nonisomorphic unicyclic graphs G_1 and G_2 of the same size for which $gcs(G_1, G_2) = \{G\}$.

Proof. By Corollary 3.4, if a construction of G_1 and G_2 is possible, then we can assume that $q(G_1) = q(G_2) = q(G) + 1$. Therefore, a graph G_1 (as well as a graph G_2) is obtained from the graph G by adding a terminal edge (together with its end-vertex). We will denote vertices of a copy of G in G_1 by u, v, w, ... , whereas the corresponding vertices of a copy of G in G_2 will be denoted by u', v', w', ...

We distinguish several cases.

Case 1. There is a vertex of maximum degree that does not lie on the cycle.

Among the vertices of maximum degree, let v be a vertex such that:

(a) the distance from v to the cycle is a maximum;

(b) if there is more than one vertex satisfying (a), choose v such that a tree branch of G - vw has maximum order (w is the vertex adjacent to v that lies on the path to the cycle);

(c) if there are at least two vertices of maximum degree that satisfy (a) and (b), choose among them a vertex adjacent to a maximum number of end-vertices.

If we choose such a vertex v, let d be its distance from the cycle, let T be a tree branch of v of maximum order, and $u \in V(T)$ be a vertex in that branch adjacent to v. Note that deg u < deg v. Let t be the number of end-vertices adjacent to v.

Subcase 1.1. Assume deg u $\leq$ deg v - 2. We define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$, as in Figure 3.39.

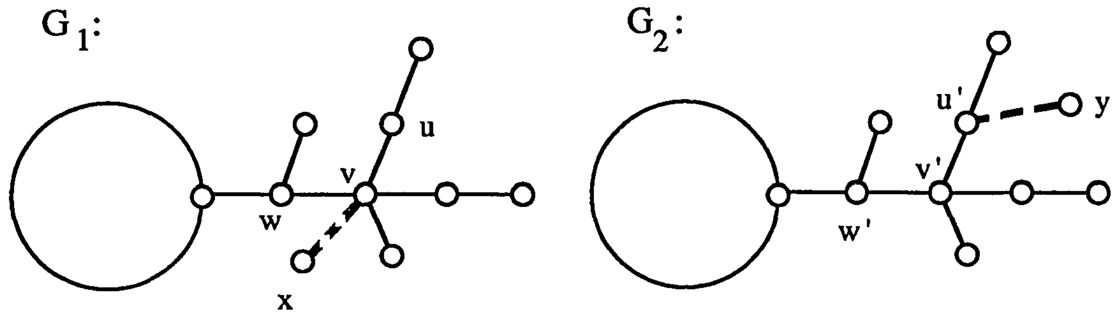


Figure 3.39

Of course, $G \in gcs(G_1, G_2)$. To produce $H \in gcs(G_1, G_2)$, we must remove one edge, say e, from G_1 and the edge e must be incident to the vertex v. If e = wv, then the tree component of G_1 - e has a vertex (namely v) of degree $\Delta(G)$. But then the edge f we must remove from G_2 to produce H is necessarily f = w'v' and the tree component of G_2 - f has no vertex of degree $\Delta(G)$, which gives a contradiction. Therefore, the edge e must be on the level d + 1. Then from G_2 we have to remove an edge on the level d + 2, and this edge f must be incident to the vertex u' (otherwise, the vertex v' in G_2 - f would have a tree branch of order greater than the order of any tree branch in G_1 - e). If e is a terminal edge, then G_1 -

$e \cong H \cong G$; otherwise, the vertex v is adjacent to $t + 1$ end-vertices in $G_1 - e$. But by removing f from G_2 we cannot get such a vertex, which again gives a contradiction.

Subcase 1.2. Assume that $\deg u = \deg v - 1$ but the tree component of $G - wv$ is not a bicentral symmetrical tree with the center $\{v, u\}$. Define G_1 and G_2 as in Subcase 1.1. The proof is exactly the same as above where the additional assumption about the tree component is needed to exclude the possibility of removing $e = wv$ from G_1 and $f = w'v'$ from G_2.

Subcase 1.3. If $\deg u = \deg v - 1$ and the tree component of $G - wv$ is a bicentral symmetrical tree with the center $\{v, u\}$. Let S be a tree branch of the vertex v of the second greatest order and let z be a vertex of S that is adjacent to v. Define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + z'y$, where $x, y \notin V(G)$ (see Figure 3.40).

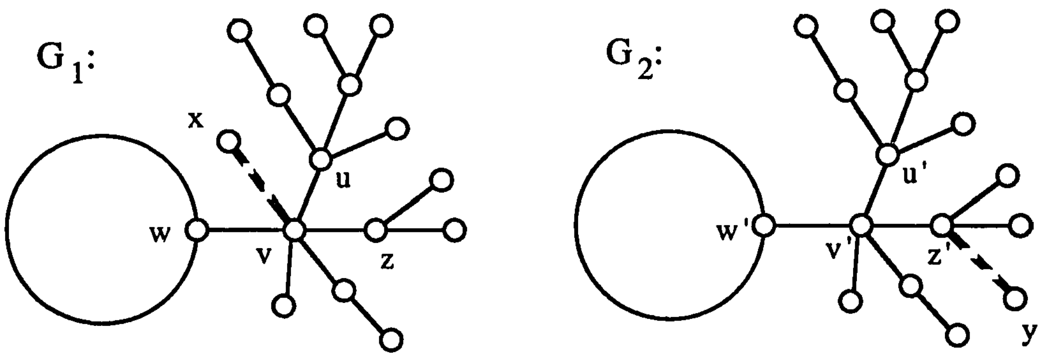


Figure 3.40

Then, of course, $G \in gcs(G_1, G_2)$. To produce $H \in gcs(G_1, G_2)$, we must remove from G_1 one edge e that is incident to v. We cannot remove $e = wv$, because then $f = w'v'$ must be removed from G_2 but the tree components of $G_1 - e$ and $G_2 - f$ are different. If e is a terminal bridge, then $G_1 - e \cong G$. Otherwise, the graph $G_1 - e$ has an additional terminal bridge incident to the vertex v. We must produce such a terminal bridge in $G_2 - f$ and at the same time remove an edge incident to the vertex z'. Because this is impossible, the contradiction shows that $gcs(G_1, G_2) = \{G\}$.

Case 2. All vertices of maximum degree are on the cycle, but there is a vertex of maximum degree that has a tree branch nonisomorphic to K_2.

We will distinguish several subcases.

Subcase 2.1. There is the unique vertex of maximum degree, say the vertex v, and deg v = 3. Let u be the vertex adjacent to v that lies outside the cycle. If we define $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$, then it is easy to check that $gcs(G_1, G_2) = \{G\}$.

Subcase 2.2. There is the unique vertex of maximum degree, say the vertex v, with deg v $\geq$ 4.

Subcase 2.2.1. Assume first that two vertices adjacent to the vertex v that lie on the cycle have degree 2. Let T be a tree branch of v of maximum order, and let u be a vertex of the tree T that satisfies deg u < deg v - 1 and the distance between u and v is minimum. Then we define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$ (see Figure 3.41).

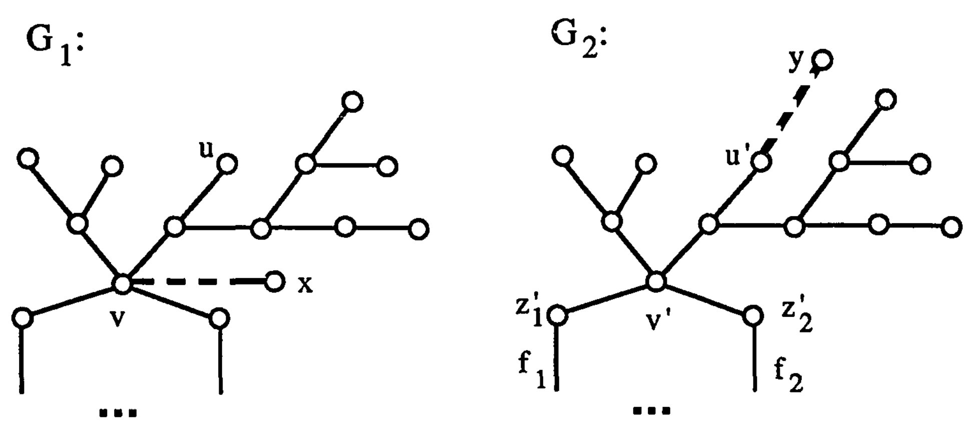


Figure 3.41

Of course, $G \in gcs(G_1, G_2)$. To get $H \in gcs(G_1, G_2)$, we must remove from G_1 an edge e that is incident to the vertex v. If e is a terminal bridge, then $H \cong G$. If e is a non-terminal bridge, then we have an additional end-vertex adjacent to v in $G_1 - e$. To get such vertex in $G_2 - f$, the removed edge f must be from level 2 and from the branch T. This is impossible, because in G_2 the degree of the vertex from T adjacent to v' is at least 3. Therefore, the edge e must be a cycle edge (incident to v), and also f must be a cycle edge (either f_1 or f_2 to produce one extra end-vertex adjacent to v'). Then one of the branches of v in $G_1 - e$ is T. There is no such branch of v' in $G_2 - f$, because $\deg_{G_2 - f} z'_1 \leq 2$ and $\deg_{G_2 - f} z'_2 \leq 2$. This gives a contradiction and proves that $gcs(G_1, G_2) = \{G\}$.

Subcase 2.2.2. Only one vertex, say u, on the cycle that is adjacent to v has degree 2. Define $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$. Then $gcs(G_1,G_2) = \{G\}$. In fact, we cannot remove a bridge from G_1. Removing a cycle edge e produces an additional end-vertex adjacent to v in $G_1 - e$. We cannot produce such an end-vertex in $G_2 - f$.

Subcase 2.2.3. Both vertices on the cycle that are adjacent to v have degree at least 3. We define G_1 and G_2 as in Subcase 2.2.1 and use the same arguments in the proof.

Subcase 2.3. There are at least two vertices of maximum degree. Among the vertices of maximum degree let v be a vertex with the following properties:

(a) v is adjacent to a maximum number of end-vertices (say, to t end-vertices);

(b) if there are at least two vertices that satisfy (a), choose among them a vertex that has a tree branch of maximum size (let T be this tree branch and let $u \in V(T)$ be adjacent to v);

(c) if there are at least two vertices that satisfy (a) and (b), choose a vertex v such that deg u is maximum (of course, $\deg u \leq \Delta(G) - 1$).

We consider two possibilities.

Subcase 2.3.1. Assume $T \not\cong K_2$. Define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$, as in Figure 3.42.

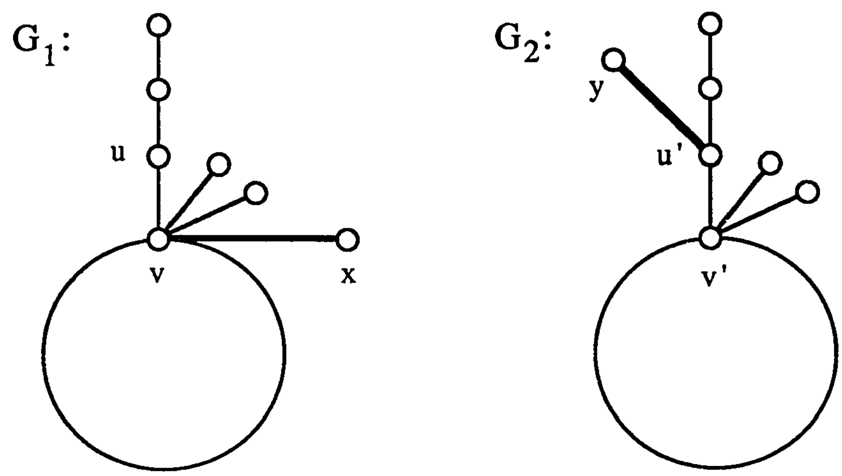


Figure 3.42

Of course, $G \in gcs(G_1, G_2)$. Assume that $H \in gcs(G_1, G_2)$ and $H \cong G_1 - e \cong G_2 - f \not\cong G$. The edge e removed from G_1 must be a cycle edge incident with the vertex v. Therefore, f is also a cycle edge. Assume first that f is incident with v'. The removal of the edge f produces in $G_2 - f$ a vertex corresponding to v; let w'_1 be this vertex. The edge f must be incident with a neighbor z'_1 of w'_1 (see Figure 3.43). Let d be the number of $\Delta(G)$ - vertices in G. If $\deg_{G_2} u' < \Delta(G)$, then in $G_2 - f$ there are $d - 1$ vertices of maximum degree. Therefore, the edge e must join the vertex v with the vertex w of degree $\Delta(G)$.

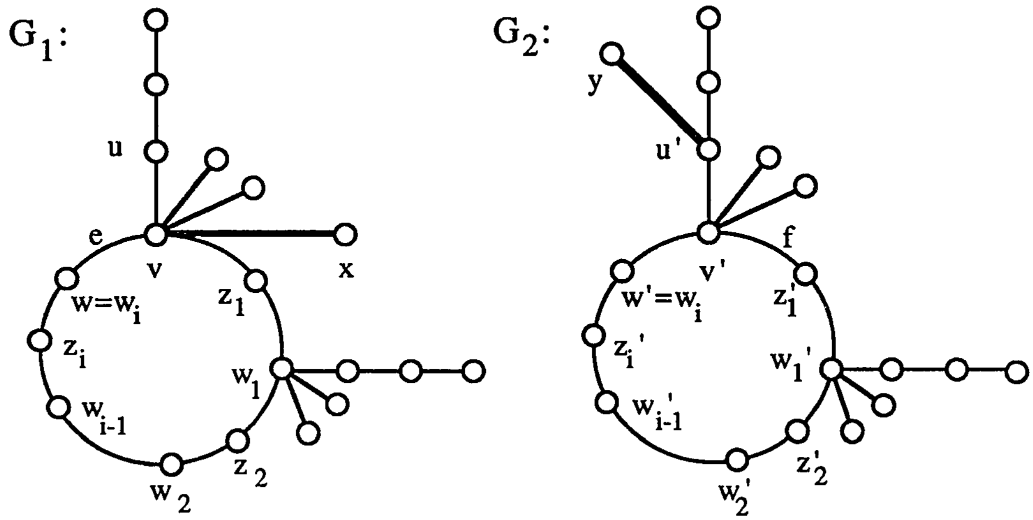


Figure 3.43

Then we must have the following correspondence between vertices in $G_1 - e$ and $G_2 - f$:

$$\begin{aligned} v &\leftrightarrow w_1' \\ z_1 &\leftrightarrow z_2' \\ w_1 &\leftrightarrow w_2' \\ &\cdots \\ z_{i-1} &\leftrightarrow z_i' \\ w_{i-1} &\leftrightarrow w_i' = w' \\ z_i &\leftrightarrow v' \end{aligned}$$

which produces a contradiction because $\deg_{G_1-e} z_i = 2 < \deg_{G_2-f} v' = \Delta(G) - 1$. On the other hand, if $\deg_{G_2} u' = \Delta(G)$, then in $G_2 - f$ there are two vertices of maximum degree, namely w_1' and u', such that $d(w_1', u') = q(C) - 1$. In $G_1 - e$, this is only possible if $e = wv$ and $\deg_{G_1-e} w = \Delta(G)$. But then $\deg_{G_1} w = \Delta(G) + 1$, which gives a contradiction.

Therefore, the edge f is not incident with v'. Then $\deg_{G_2-f} v' = \Delta(G)$, v' is adjacent to t end-vertices and v' has a tree branch $T' \cong T + u'y$ of order greater than the order of T. Therefore, v' must correspond to some vertex w in $G_1 - e$ whose tree branch T' is obtained by the removal of the cycle edge e. Moreover, w has a tree branch, say B, such that B is produced by removing f from G_2 (see Figure 3.44). Also by removing f, we must produce in $G_2 - f$ a vertex w'_1 that corresponds to v. The edge f must be incident with a neighbor of w'_1.

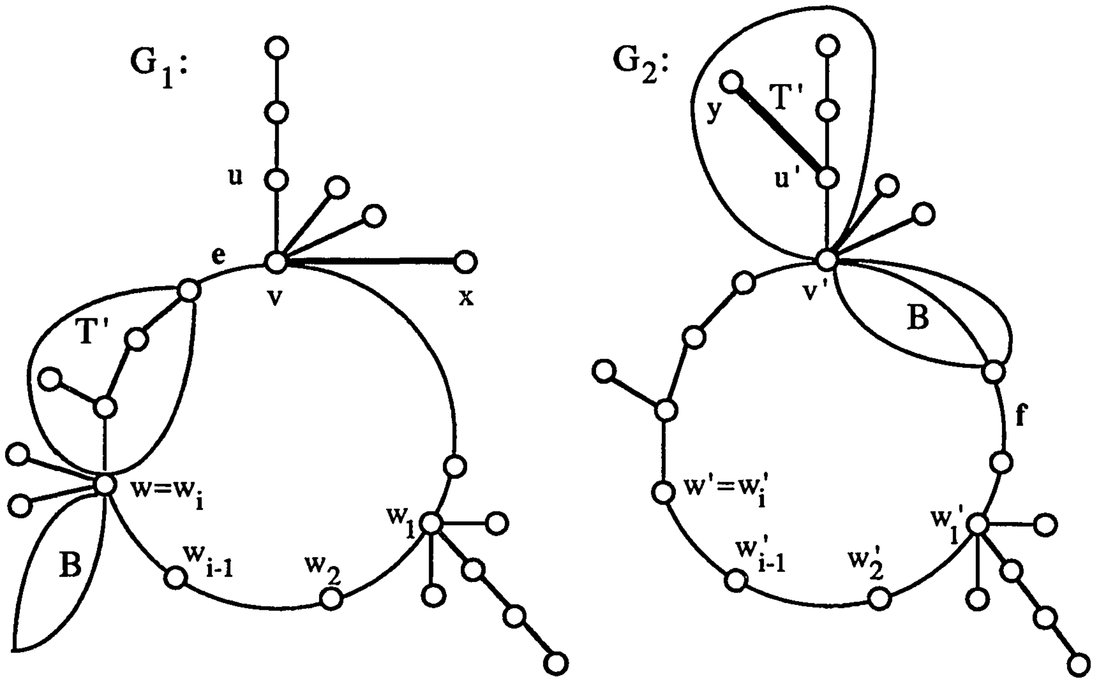


Figure 3.44

Therefore, we have the following correspondence between vertices of $G_1 - e$ and $G_2 - f$:

$$v \leftrightarrow w'_1$$
$$w_1 \leftrightarrow w'_2$$
$$\ldots$$
$$w_{i-1} \leftrightarrow w'_i$$

and the segment S between vertices v' and w'_1 (containing the edge f) must be repeated between w'_1 and w'_2, ... , w'_{i-1} and w'_i. Finally, we must have $w_i = w'$, so the branch B must be isomorphic to T.

Only in this special case, we can have $G_1 - e \cong G_2 - f$. But then the graph G is "cyclically" symmetrical, i.e., it is of the form as in Figure 3.45. It contains the cycle $a_0, a_1, \ldots a_{3i+2}, a_0$ of length 3(i+1). The tree branch T is present at $a_0, a_3, a_6, \ldots, a_{3i}$, and the remaining tree branches of these vertices are denoted by F. The tree branch R that is present at the vertices $a_1, a_4, \ldots, a_{3i+1}$ satisfies $R + a_0a_1 \cong T$.

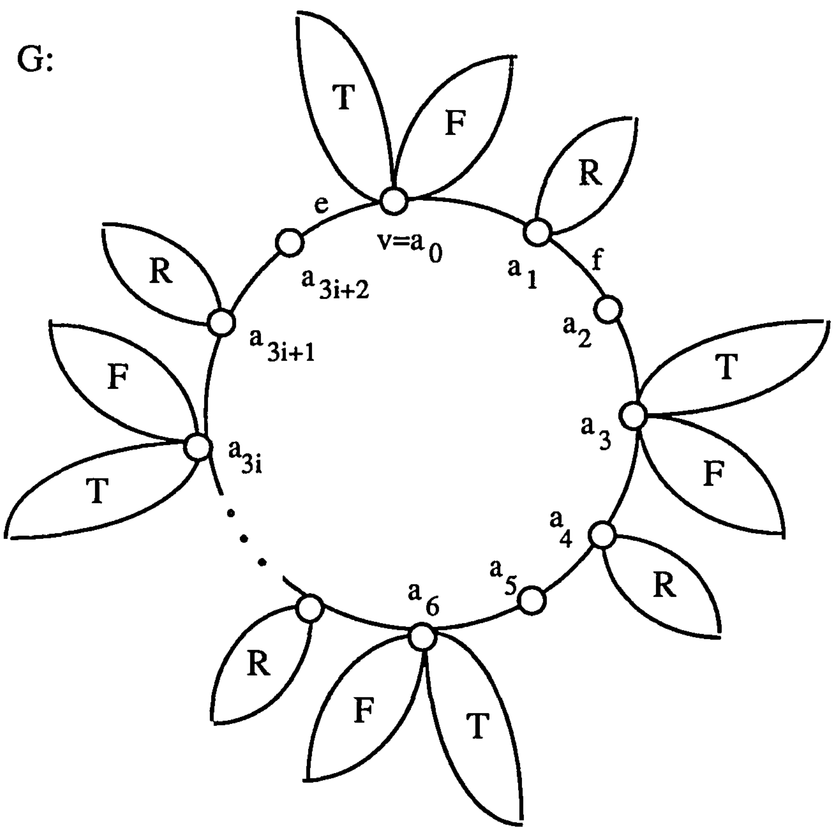


Figure 3.45

Then in fact $G_1 - e \cong G_2 - f$, where $e = a_{3i+2}a_0$ and $f = a_1a_2$. But in this special case we can construct G_1 and G_2 in a different way. Let $G_1 \cong G + vx$ and $G_2 \cong$

$G + a_2y$, where $x, y \notin V(G)$ (note that $\deg a_2 = 2$). Then it is easy to check that $gcs(G_1, G_2) = \{G\}$.

Subcase 2.3.2. Assume that $T \cong K_2$. Let v be a vertex of maximum degree all of whose t tree branches are isomorphic to K_2. Let w be a vertex of maximum degree that has a tree branch of maximum order. If there is more than one such pair (v, w), let us choose v and w whose distance $d(v, w)$ is minimum. Let B be a tree branch of w of maximum order (by assumption, $B \ncong K_2$) and let $u \in V(B)$ be an end-vertex from this branch. Define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$ (see Figure 3.46).

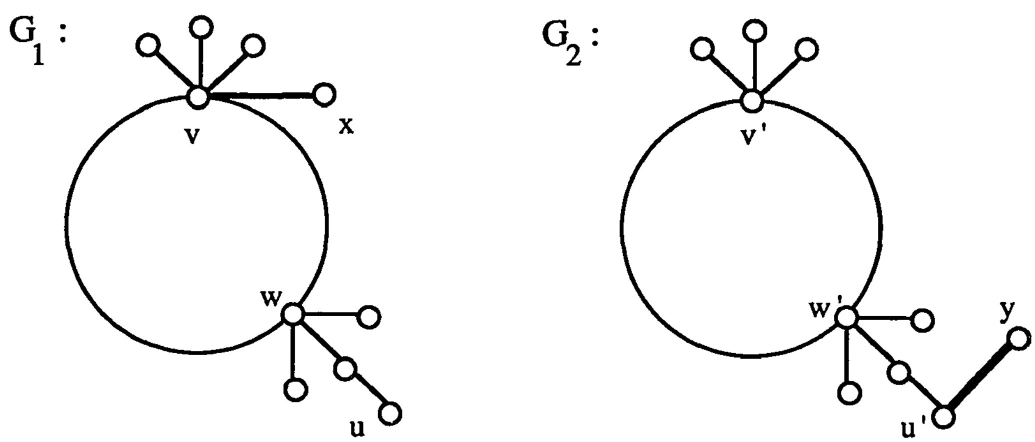


Figure 3.46

Of course, $G \in gcs(G_1, G_2)$. Assume that $H \in gcs(G_1, G_2)$ and $H \cong G_1 - e \cong G_2 - f \ncong G$. The edge e must be a cycle edge incident with the vertex v, so f is also a cycle edge. Suppose first that f is incident with the vertex w'. By the removal of f, we must produce in $G_2 - f$ a vertex corresponding to v, so f is incident with a neighbor of a $\Delta(G)$-vertex. Therefore, $d(v, w) \leq 2$. If $d(v, w) = 1$, we have the situation shown in Figure 3.47, where the vertices of the cycle are denoted by $v_0 = v$,

$v_1 = w, v_2, v_3, \dots, v_{n-1}, v_n = v_0$. The edge e is incident with the vertex v_0, so either $e = v_0v_1$ or $e = v_{n-1}v_0$.

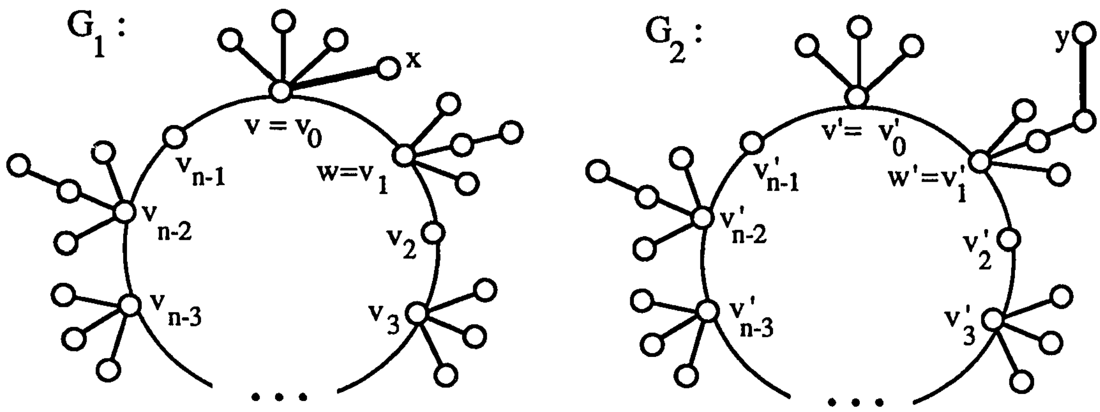


Figure 3.47

If $e = v_0v_1$, then we have the following correspondence between vertices in G_1 - e and G_2 - f:

$v_0 \leftrightarrow v'_3$

$v_{n-1} \leftrightarrow v'_4$

...

$v_4 \leftrightarrow v'_{n-1}$

$v_3 \leftrightarrow v'_0$

$v_2 \leftrightarrow w'_1$,

which gives a contradiction, because $\deg v_2 = 2$ and $\deg v'_1 = \Delta(G) - 1 \geq 3$.

If $e = v_{n-1}v_0$, then we have the following correspondence between vertices in G_1 - e and G_2 - f:

$v_0 \leftrightarrow v'_3$

$v_1 \leftrightarrow v'_4$

...

$v_i \leftrightarrow v'_{i+3}$

...

$v_{n-3} \leftrightarrow v'_n = v'_0$

$v_{n-2} \leftrightarrow v'_1,$

which gives a contradiction because $\deg v_{n-2} = \Delta(G) > \deg v'_1 = \Delta(G) - 1$.

If $d(v, w) = 2$, then, without loss of generality, we can assume that $v = v_0$ and $w = v_2$. The edge e is either $v_{n-1}v_0$ or v_0v_1, whereas $f = v'_1v'_2$ or $f = v'_2v'_3$. Consider the case when $e = v_{n-1}v_0$ and $f = v'_1v'_2$ (see Figure 3.48). The other three cases can be treated in a similar way.

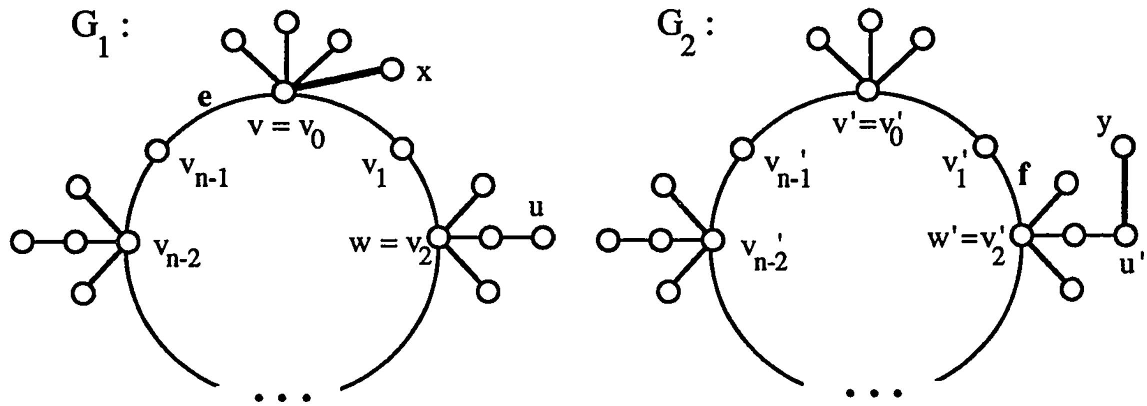


Figure 3.48

We have the following correspondence between the vertices of $G_1 - e$ and $G_2 - f$:

$v_0 \leftrightarrow v'_0$

$v_1 \leftrightarrow v'_{n-1}$

$v_2 \leftrightarrow v'_{n-2}$ $(\deg v_2 = \Delta(G) \Rightarrow \deg v'_{n-2} = \Delta(G)-1)$

...

$v_{n-2} \leftrightarrow v'_2,$

which gives a contradiction because $\deg v_{n-2} = \Delta(G) > \deg v'_2 = \Delta(G) - 1$.

Therefore, the edge f is not incident with w'. Since $\deg_{G_2 - f} w' = \Delta(G)$ and w' has a tree branch $B' \cong B + u'y$, it follows that a vertex z in $G_1 - e$ corresponding to w' must have a tree branch B' that is produced by removing the edge e. Moreover, z has a tree branch, say T, such that T is produced by the removal of f from G_2. Also by removing f, we must produce in G_2 - f a vertex corresponding to v. Assuming that $v = v_0$ and $w = v_i$, we have two possibilities. The first is $f = v'_1v'_2$. Then we have the situation as in Figure 3.49 and the following correspondence between the vertices of G_1 - e and G_2 - f must hold:

$v_0 \leftrightarrow v'_0$

$v_1 \leftrightarrow v'_{n-1}$ (so $\deg v'_{n-1} = \deg v_{n-1} = 2$)

$v_2 \leftrightarrow v'_{n-2}$

...

$v_i \leftrightarrow v'_{n-i}$ (which implies that $T \cong B$)

...

$v_k \leftrightarrow v'_{n-k}$.

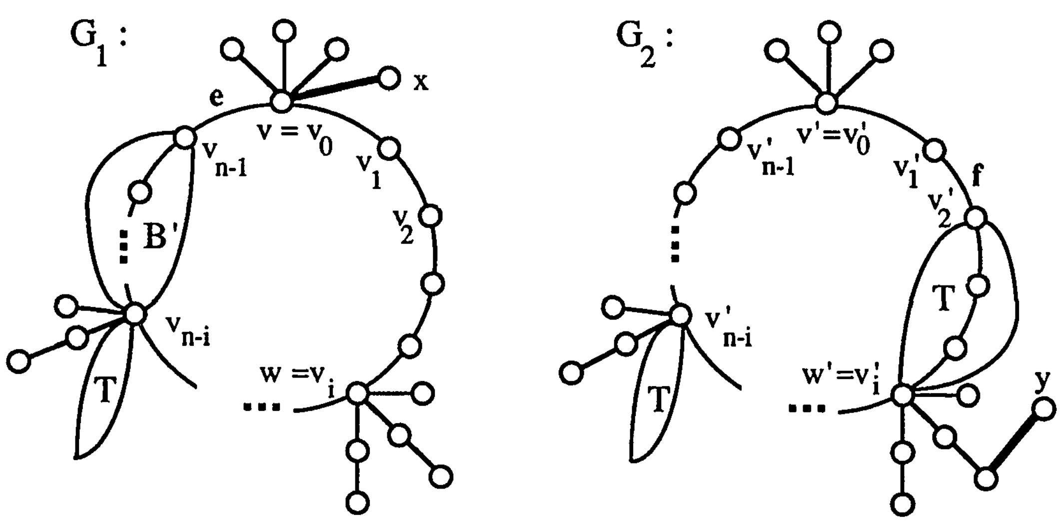


Figure 3.49

Therefore, the graph G is symmetrical, i.e. the vertex v_k is similar to the vertex v_{n-k}, where n is the length of the cycle of G and $k = 0, 1, 2, \dots, n$. But then we can use a different construction for G_1 and G_2. If we define $G_1 \equiv G + vx$ and $G_2 \equiv G + u'y$, where u is an end-vertex adjacent to v and $x,y \notin V(G)$, then it is easy to check that $gcs(G_1, G_2) = \{G\}$.

It remains to consider the case when $f \neq v'_1v'_2$. Then , because z corresponds to $w' = v_i'$ and v corresponds to a vertex different than v', say v'_{i+d} we must have $f = v'_{i+d-2}\, v'_{i+d-1}$ (see Figure 3.50).

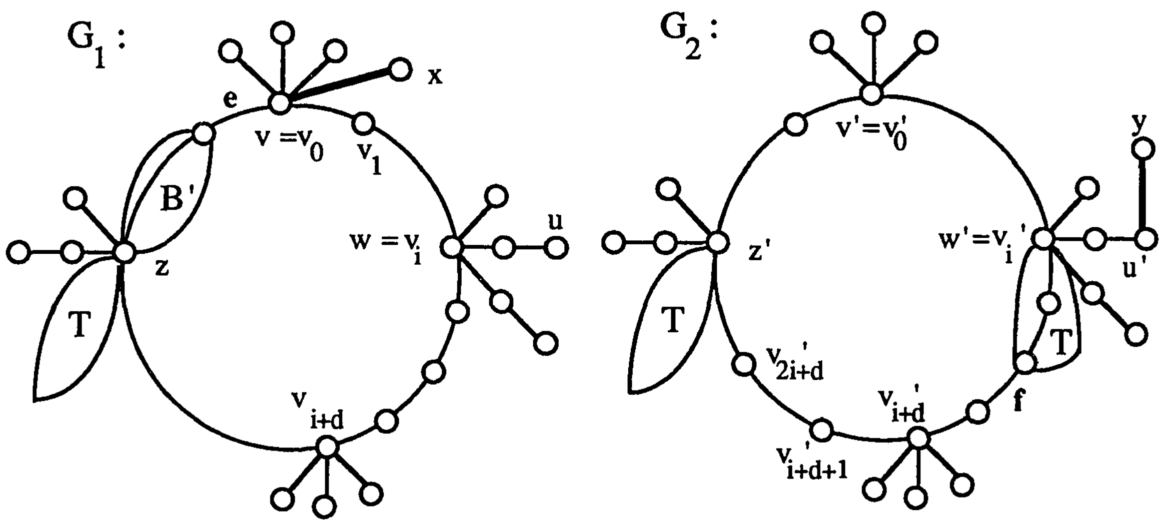


Figure 3.50

Then the following correspondence between the vertices of $G_1 - e$ and $G_2 - f$ is forced:

$$v_0 \leftrightarrow v'_{i+d}$$

$$v_1 \leftrightarrow v'_{i+d+1}$$

$$\dots$$

$$v_i \leftrightarrow v'_{2i+d}$$

$$\dots$$

$$v_{i+d} \leftrightarrow v'_{2(i+d)}.$$

Since $i \leq d$ (the vertices v and w were chosen in such a way that the distance $d(v,w)$ was a minimum), it follows that there exists a positive integer k such that

$$v_{(k-1)(i+d)+i} \leftrightarrow v'_{k(i+d)+i} = z'.$$

Then also

$$v_{k(i+d)} \leftrightarrow v'_0$$

$$v_{k(i+d)+i} = z \leftrightarrow v'_i.$$

This implies that the graph G is "cyclically" symmetrical, i.e. the segments

$$v_0 - v_1 - \dots - v_i - v_{i+d},$$

$$v_{i+d} - v_{i+d+1} - \dots - v_{2i+d} - \dots - v_{2(i+d)},$$

$$\dots$$

$$v_{k(i+d)} - v_{k(i+d)+1} - \dots - v_{(k+1)(i+d)}$$

are isomorphic (it may happen that there is only one segment). But in this special case we can construct G_1 and G_2 in a different way. Let us note that $\deg v_{n-1} = \deg v_{i+d-1} = 2$. It is easy to check that if $G_1 \cong G + v_0x$ and $G_2 \cong G + v'_{n-1}y$, where $x, y \notin V(G)$, then $gcs(G_1, G_2) = \{G\}$.

Case 3. All vertices of maximum degree lie on the cycle and all their tree branches are isomorphic to K_2.

Subcase 3.1. There is exactly one vertex of maximum degree with all tree branches isomorphic to K_2. Let v be such a vertex, $\deg v = \Delta(G)$, and let u be an end-vertex adjacent to v. Define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where $x, y \notin V(G)$. Of course, $G \in gcs(G_1, G_2)$. To produce $H \in gcs(G_1, G_2)$, we must remove from G a cycle edge e that is incident with v. But then in $G_1 - e$ the vertex v is adjacent to $\Delta(G) - 1$ end-vertices. To produce such a vertex in $G_2 - f$, it is necessary that the cycle is a triangle and both vertices on the cycle that are adjacent to v' have degree 2. Therefore, we have $G \cong C(3, n)$ but we assumed that it is not the case.

Subcase 3.2. There are at least two vertices of maximum degree on the cycle and all of their tree branches are isomorphic to K_2. Let us denote by $T(n)$ a tree consisting of two stars $K(1, n)$ whose central vertices are connected by a path of length 3. We will distinguish several subcases.

Subcase 3.2.1. There is a vertex v of maximum degree such that v is not a vertex of a segment isomorphic to $T(\Delta(G) - 2) = T$. Define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where u is an end-vertex adjacent to v and $x, y \notin V(G)$ (see Figure 3.51).

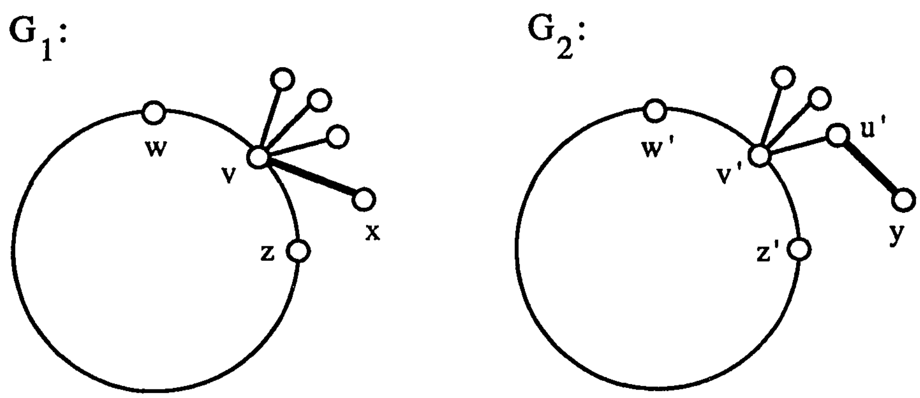


Figure 3.51

Of course, $G \in gcs(G_1, G_2)$. Assume that $H \in gcs(G_1, G_2)$ and $H \cong G_1 - e \cong G_2 - f \ncong G$. The edge e removed from G_1 must be a cycle edge incident with the vertex v, so either $e = wv$ or $e = vz$. If $f \neq w'v'$ and $f \neq v'z'$, then $\deg_{G_2 - f} v' = \Delta(G)$. To produce in $G_1 - e$ a vertex corresponding to v', it is necessary that v is a vertex from a segment isomorphic to T, which gives a contradiction. Therefore, the edge f is either $w'v'$ or $v'z'$. Assume without loss of generality that $f = v'z'$. Since we must produce an additional end-vertex in $G_2 - f$, it follows that the other vertex adjacent to z' has degree $\Delta(G)$; let v'_1 be this vertex (see Figure 3.52).

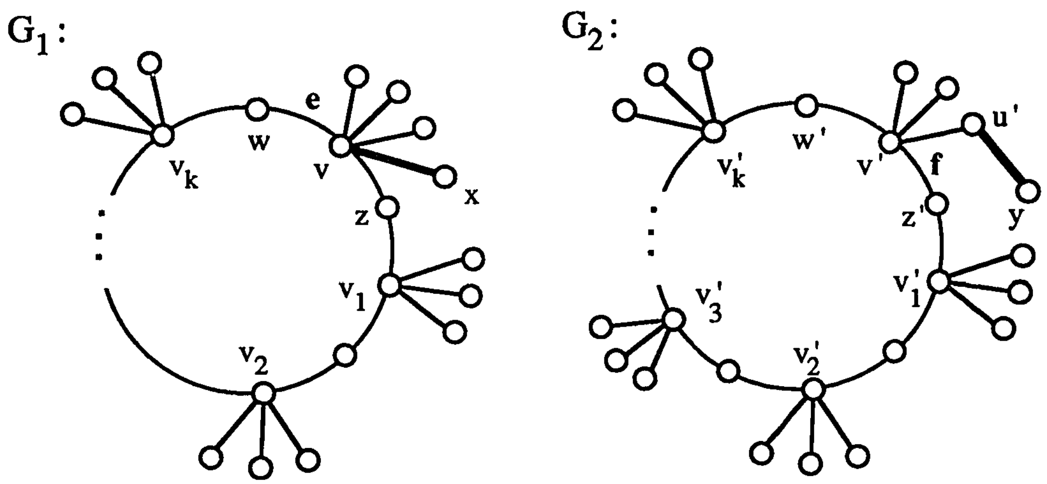


Figure 3.52

Then there exists an integer $k \geq 1$ such that the following correspondence between vertices of $G_1 - e$ and $G_2 - f$ holds:

$$\begin{array}{lcl} v & \leftrightarrow & v'_1 \\ v_1 & \leftrightarrow & v'_2 \\ v_2 & \leftrightarrow & v'_3 \\ \dots & & \\ v_k & \leftrightarrow & v' \end{array}$$

which gives a contradiction because v' has a tree branch isomorphic to P_3 whereas v_k does not.

Subcase 3.2.2. Every vertex v of maximum degree occurs in a segment isomorphic to $T(\Delta(G) - 2) = T$. The segment isomorphic to T will be called a T - segment. Let v be a vertex that is present in exactly one T - segment. If $G \not\cong C(3k, n)$, then such a vertex v exists. We define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y$, where u is an end-vertex adjacent to v and $x, y \notin V(G)$ (see Figure 3.53).

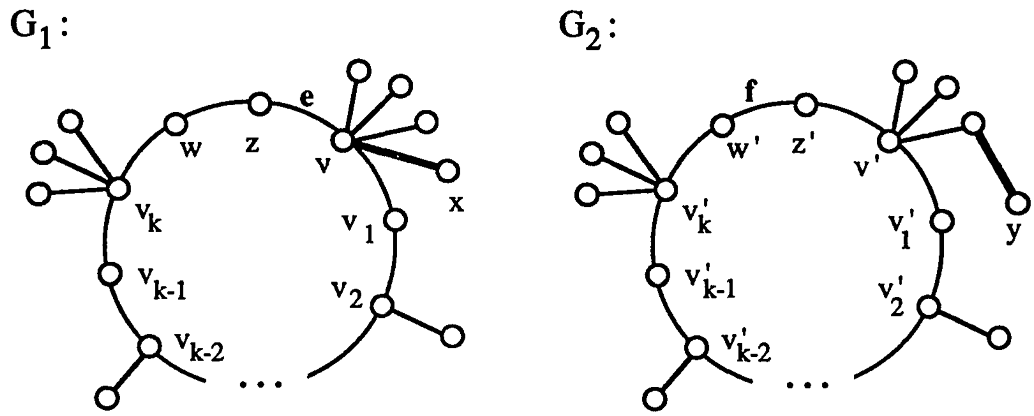


Figure 3.53

Of course, $G \in gcs(G_1, G_2)$. Assume that $H \in gcs(G_1, G_2)$ and $H \cong G_1 - e \cong G_2 - f \not\cong G$. By the observations made in the proof of Subcase 3.2.1, we must have $e = zv$ and $f = w'z'$. Then, if $v_1, v_2, \dots, v_k$ are the remaining vertices on the cycle, we must have the following correspondence between vertices of $G_1 - e$ and $G_2 - f$:

$$v \leftrightarrow v'_k$$

$$v_1 \leftrightarrow v'_{k-1}$$

$$\dots$$

$$v_{k-1} \leftrightarrow v'_1$$

$$v_k \leftrightarrow v'.$$

Therefore, the graph G must be symmetrical, i.e. the vertices v_i and v_{k-i} are similar, where $0 \le i \le k$ and $v_0 = v$; also the vertex w is similar to the vertex z. Otherwise, for the two graphs G_1 and G_2 constructed above we would have $gcs(G_1, G_2) = \{G\}$. Because of this symmetry and the fact that v occurs in exactly one T - segment, every vertex of maximum degree is present in exactly one such segment. In other words, vertices of maximum degree occur in pairs, two in a T - segment. Since $G \not\cong C(4k, n)$, it follows that there is a vertex on the cycle that does not belong to a T - segment. Among all such vertices, let u be a vertex adjacent to a vertex from a T - segment, say v, and u is of highest degree. Then $2 \leq \deg u < \Delta(G)$. We define two graphs $G_1 \cong G + vx$ and $G_2 \cong G + u'y'$, where $x, y \notin V(G)$, as in Figure 3.54.

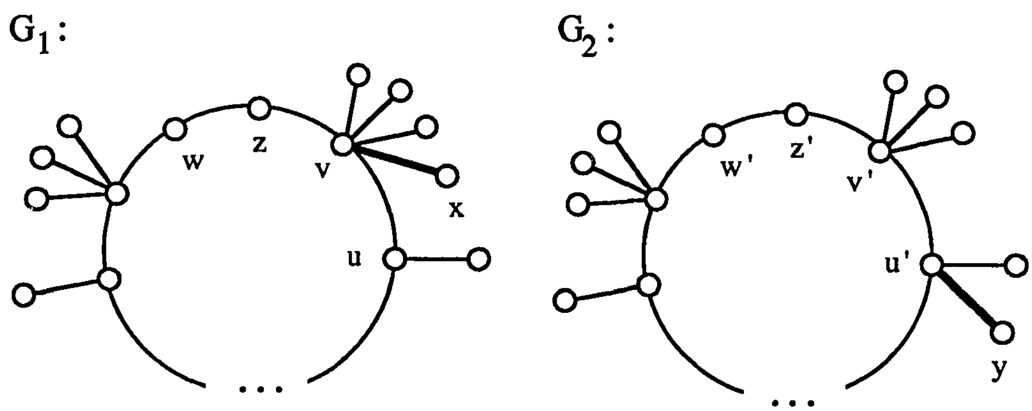


Figure 3.54

Of course, $G \in gcs(G_1, G_2)$. Assume that $H \in gcs(G_1, G_2)$ and $H \cong G_1 - e \cong G_2 - f \not\cong G$. The edge e is either e = zv or e = vu. If e = zv, then in $G_1 - e$ we reduced the number of T - segments. Therefore, we have to reduce that number in $G_2 - f$, and at the same time produce an additional end-vertex. This is only possible if f is the middle edge of the segment T, but then we produce two vertices each incident to n + 1 terminal bridges. This gives a contradiction, because there is no such pair of vertices in $G_2 - f$. Therefore, we must have e = vu. Then the graph $G_1 - e$ has one extra end-vertex (compared to G), namely the vertex x, but the graph $G_2 - f$ has at least two extra end-vertices (y' and a vertex corresponding to x which had to be produced). This contradiction shows that $gcs(G_1, G_2) = \{G\}$. ❑

CHAPTER IV

VARIATIONS OF GREATEST COMMON SUBGRAPHS

4.1 Maximal Common Subgraphs

Let G_1 and G_2 be nonisomorphic graphs of equal size. The set of all common subgraphs of G_1 and G_2 can be considered as a set partially ordered by a "being a subgraph" relation. Maximal common subgraphs are the maximal elements in this partially ordered set. Therefore, the following definition is well justified. A graph H without isolated vertices is a maximal common subgraph of G_1 and G_2 if $H \subset G_i$, $i = 1, 2$, and there is no F such that $H \subsetneq F \subset G_i$, $i = 1, 2$. The set of all maximal common subgraphs of G_1 and G_2 is denoted by $\mathrm{mcs}(G_1, G_2)$.

For instance, if $G_1 \cong K(3, 3)$ and $G_2 \cong K(1, 3) \cup K_4$, then $\mathrm{mcs}(G_1, G_2) = \{H_1, H_2, H_3\}$, where $H_1 \cong K(1,3)$, $H_2 \cong 2K(1, 2)$ and $H_3 \cong C_4 \cup K_2$ (see Figure 4.1).

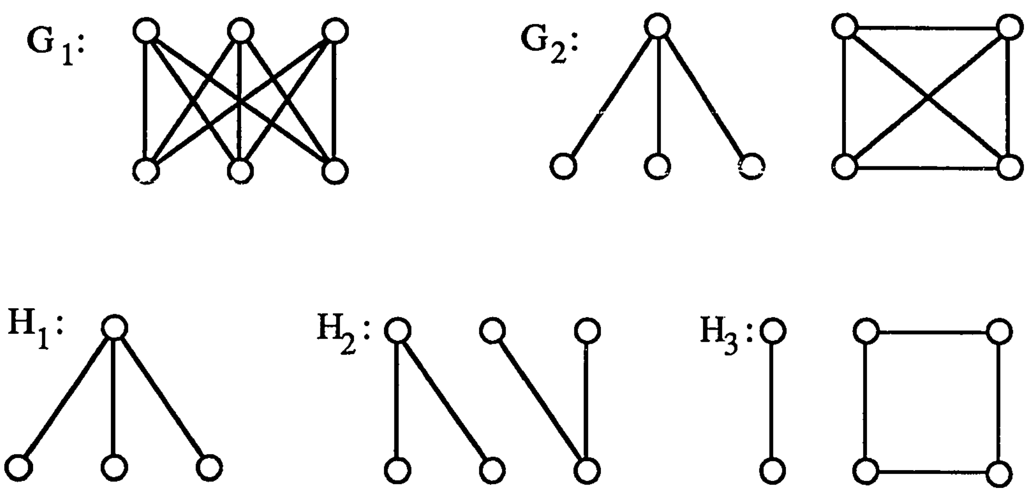


Figure 4.1

This example shows that this concept is different than greatest common subgraphs. Clearly, a greatest common subgraph of two graphs G_1 and G_2 is also a maximal common subgraph of G_1 and G_2, so $gcs(G_1, G_2) \subset mcs(G_1, G_2)$. For the above example, we have $gcs(G_1, G_2) = \{C_4 \cup K_2\}$, so the reverse inclusion does not hold in general.

Let us note in the above example that the graphs H_1, H_2 and H_3, belonging to $mcs(G_1, G_2)$, have different sizes (3, 4 and 5, respectively). Of course, if H is a maximal common subgraph of G_1 and G_2 having maximum size, then H is a greatest common subgraph of G_1 and G_2, and vice versa.

The first result shows that the difference between the sizes of a greatest common subgraph and a maximal common subgraph can be arbitrarily large.

Theorem 4.1 For every positive integer M, there exist graphs G_1 and G_2 of equal size and graphs $G \in gcs(G_1,G_2)$ and $H \in mcs(G_1, G_2)$ with $q(G) - q(H) > M$.

Proof. Let $G_1 \cong K_{n-1} \cup C_n$ and $G_2 \cong (K_{n-1} \cup K_1) + K_1$ be graphs indicated in Figure 4.2.

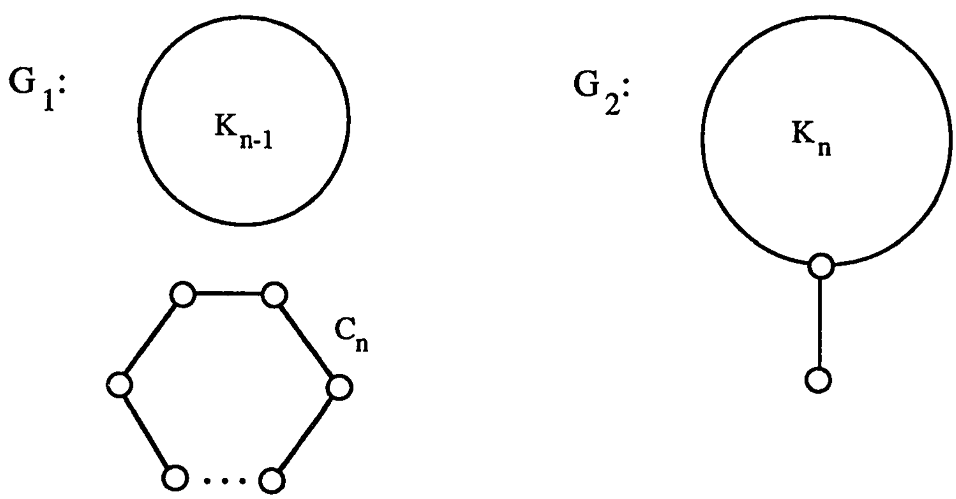


Figure 4.2

Then $q(G_1) = q(G_2) = \frac{1}{2}(n^2 - n + 2)$,

$gcs(G_1, G_2) = \{K_{n-1} \cup K_2\}$, and

$mcs(G_1, G_2) \supseteq \{C_n, K_{n-1} \cup K_2\}$.

If we let $G \cong K_{n-1} \cup K_2$ and $H \cong C_n$, then $q(G) - q(H) = \frac{(n-1)(n-2)}{2} + 1 - n = \frac{1}{2}(n^2 - 5n + 4) > M$ for sufficiently large n. ❒

The set of maximal common subgraphs can have arbitrarily large cardinality. Moreover, we can have a wide range of sizes of maximal common subgraphs.

Theorem 4.2 For every positive integer N, there exist graphs G_1 and G_2 of equal size and N graphs $H_1, H_2, \dots, H_N$ with $q(H_i) \neq q(H_j)$ for $1 \leq i < j \leq N$ such that

$$\{H_1, H_2, \dots, H_N\} \subseteq mcs(G_1, G_2).$$

Proof. Let N be given. First we define a family of graphs H_i, $1 \leq i \leq N$, as follows: $H_1 \cong C_{N!}$. For $2 \leq i \leq N$ let H_i be the graph obtained from i cycles $C_{N!/i}$ and a path P_i by identifying an end-vertex of P_i and one vertex from each of the i cycles. Let us define two graphs $G_1 \cong K_{N!}$ and $G_2 \cong H_1 \cup H_2 \cup \dots \cup H_i \cup \dots \cup H_N \cup K(1, r)$, where $r = \binom{N!}{2} - N \times N! - \binom{N}{2}$ (see Figure 4.3). Then $q(G_1) = \binom{N!}{2} = q(G_2)$. Since $p(H_i) = N! = p(G_1)$ for every i, $1 \leq i \leq N$, and H_i contains a cycle $C_{N!/i}$ that is not contained in a component different from H_i, it follows that $H_i \in mcs(G_1, G_2)$, $i = 1, 2, \dots, N$. We have also $q(H_i) = N! + (i - 1)$, so $q(H_i) \neq q(H_j)$ for $1 \leq i < j \leq N$. ❒

In the next result, we will characterize those graphs G that are maximal common subgraphs for some suitably chosen graphs G_1 and G_2 of equal size but are not greatest common subgraphs of G_1 and G_2.

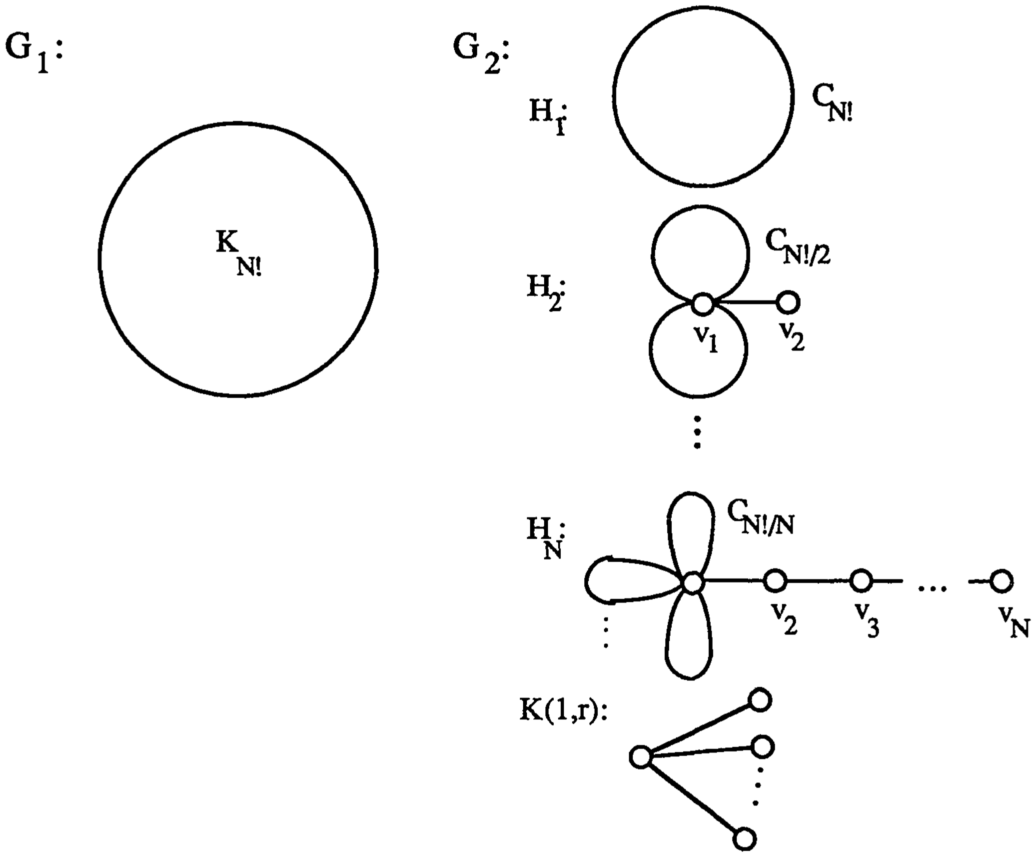


Figure 4.3

Theorem 4.3 Let G be a graph without isolated vertices such that $G \not\cong K(1, r)$, $r = 1, 2$. Then there exist two nonisomorphic graphs G_1 and G_2 of equal size such that $G \in \mathrm{mcs}(G_1, G_2)$ but $G \notin \mathrm{gcs}(G_1, G_2)$.

Proof. Assume that the size of G is q. First, suppose that G contains a component other than a star. Let H be such a component of maximum size and let $v \in V(H)$. Consider two graphs G_1 and G_2 as in Figure 4.4. The graph G_1 is obtained from G and $K(1, q+1)$ by identifying v and the vertex of maximum degree in $K(1, q+1)$. The graph $G_2 \cong G \cup K(1, q+1)$.

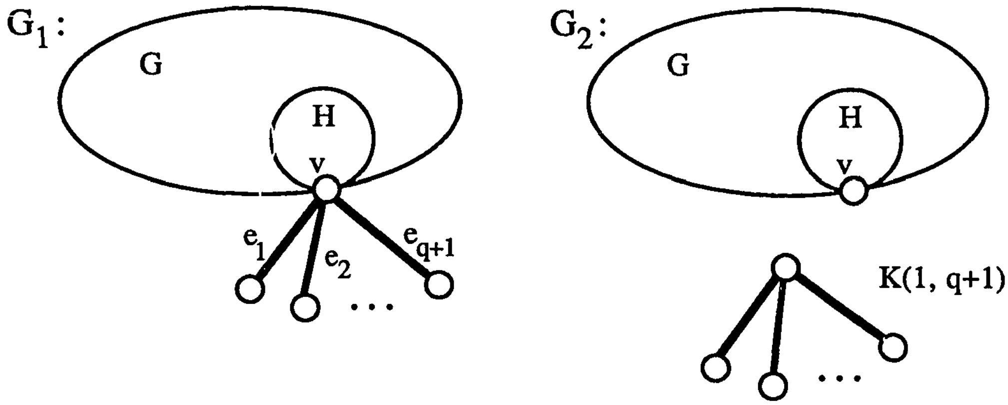


Figure 4.4

Then $G \notin gcs(G_1, G_2)$, because $K(1, q+1)$ is a common subgraph of G_1 and G_2 of size $q + 1$. We will show that $G \in mcs(G_1, G_2)$. Suppose that G' is a common subgraph of G_1 and G_2 with $q(G') > q(G)$ and such that $G' \supseteq G$. Then G' must contain some of the edges $e_1, e_2, \ldots, e_{q+1}$ (the edges from the star $K(1,q+1)$ of G_1). If H' is the component of G' with these edges, then there is no component in G_2 corresponding to H'. This contradiction shows that $G \in mcs(G_1, G_2)$.

Suppose next that all components of G are stars and that there is a component with at least four edges. Let H be such a component of maximum size, say s. Consider two graphs G_1 and G_2 as in Figure 4.5.

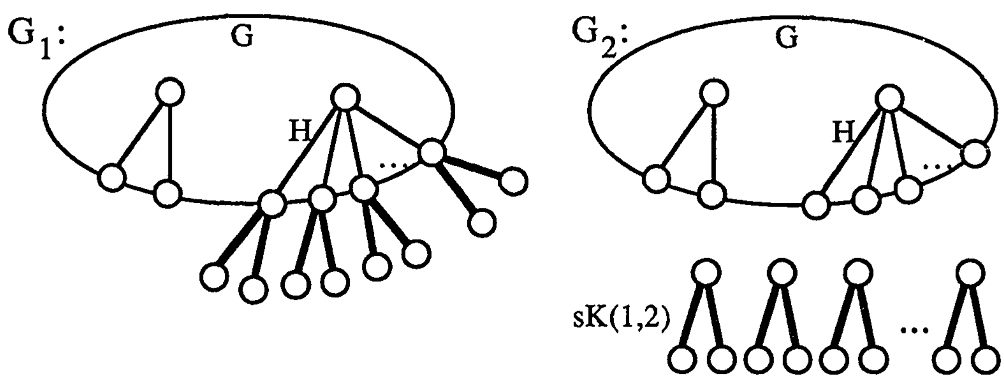


Figure 4.5

Then $G \in mcs(G_1, G_2)$, but $G \notin gcs(G_1, G_2)$ because $F \cong (G - H) \cup sK(1, 2)$ is a subgraph of both G_1 and G_2, and the size of F is $q(F) = q(G) - k + 2k = q(G) + k > q(G)$.

Next, assume that all components of G are stars, each with at most three edges, but there is a component, say H, isomorphic to $K(1, 3)$. We define two graphs $G_1 \cong G - V(H) \cup [(K_3 \cup K_1) + K_1]$ and $G_2 \cong G \cup K_3 \cup K_2$ (see Figure 4.6).

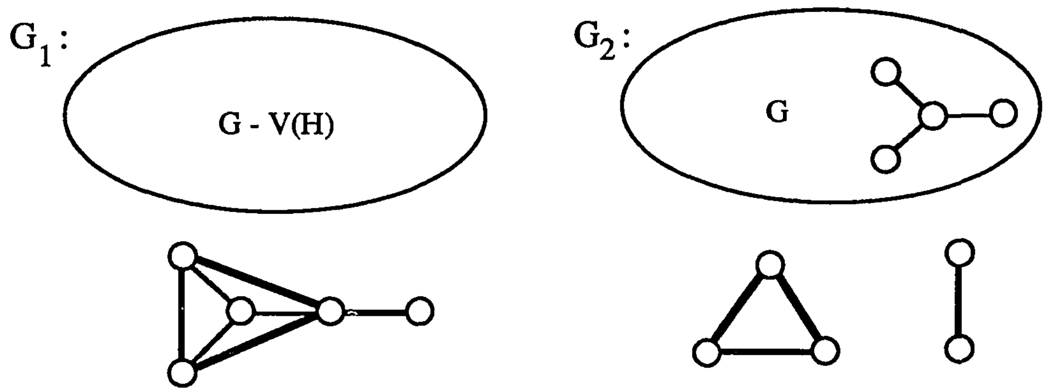


Figure 4.6

Then $G \in mcs(G_1, G_2)$, but $G \notin gcs(G_1, G_2)$ because $G' \cong (G - V(H)) \cup K_3 \cup K_2$ is a common subgraph of both G_1 and G_2 with $q(G') = q(G) + 1$.

The remaining case is if G consists of components isomorphic to K_2 or $K(1,2)$. If G has at least two components isomorphic to $K(1,2)$, then we denote two of them by H_1 and H_2. Define

$$G_1 \cong (G - V(H_1 \cup H_2)) \cup K_6,$$

$$G_2 \cong (G - V(H_1)) \cup K(1, 13).$$

Then $G \in mcs(G_1, G_2)$, but $G \notin gcs(G_1, G_2)$ because $G' \cong (G - V(H_1 \cup H_2)) \cup K(1, 5)$ is a common subgraph of both G_1 and G_2 with $q(G') = q(G) + 1$.

If G has at least two components isomorphic to K_2, denote two of them by H_1 and H_2. Define

$$G_1 \cong (G - V(H_1 \cup H_2)) \cup K_4,$$

$$G_2 \cong (G - V(H_1)) \cup K(1, 5).$$

Then $G \in mcs(G_1, G_2)$, but $G \notin gcs(G_1, G_2)$ because $G' \cong (G - V(H_1 \cup H_2)) \cup K(1, 3)$ is a common subgraph of both G_1 and G_2 with $q(G') = q(G) + 1$.

If G contains a component, say H_1, isomorphic to K_2, and a component, say H_2, isomorphic to $K(1, 2)$, then define

$$G_1 \cong (G - V(H_1 \cup H_2)) \cup K_5,$$

$$G_2 \cong (G - V(H_1)) \cup K(1, 9).$$

Then $G \in mcs(G_1, G_2)$, but $G \notin gcs(G_1, G_2)$ because $G' \cong (G - V(H_1 \cup H_2)) \cup K(1, 4)$ is a common subgraph of both G_1 and G_2 with $q(G') = q(G) + 1$. ❐

Let us note that if $G \cong K_2$ or $G \cong K(1, 2)$, then for any graphs G_1 and G_2 we have: $G \in mcs(G_1, G_2)$ implies that $G \in gcs(G_1, G_2)$. In fact, suppose that there are two graphs G_1 and G_2 such that $K_2 \in mcs(G_1, G_2)$ and $K_2 \notin gcs(G_1, G_2)$. Taking $H \in gcs(G_1, G_2)$, we have that $q(H) \geq 2$, so $H \supseteq K_2$, which contradicts the fact that $K_2 \in mcs(G_1, G_2)$. Next, let $G \cong K(1, 2)$ and suppose that there are two graphs G_1 and G_2 such that $G \in mcs(G_1, G_2)$ and $G \notin gcs(G_1, G_2)$. If $H \in gcs(G_1, G_2)$, then $q(H) \geq 3$. If $H \supseteq K(1, 2)$, we have a contradiction. Therefore, $H \supseteq 3K_2$. We conclude that $G_i \supseteq 3K_2$ and $G_i \supseteq K(1, 2)$, $i = 1, 2$. Therefore, $G_i \supseteq K(1, 2) \cup K_2$ for $i = 1, 2$, and $G \cong K(1, 2)$ is not a maximal common subgraph of G_1 and G_2.

4.2 Absorbing Common Subgraphs

An absorbing common subgraph of G_1 and G_2 is a common subgraph of both G_1 and G_2 that "absorbs" every other common subgraph of G_1 and G_2. More formally, let G_1 and G_2 be graphs of the same size. A graph G without isolated vertices is an absorbing common subgraph of G_1 and G_2 if $G \subset G_1$, $G \subset G_2$ and for a graph H (without isolated vertices) such that $H \subset G_1$ and $H \subset G_2$, it follows that $H \subset G$. If an absorbing common subgraph of G_1 and G_2 exists, then it is unique and we denote it by $\mathrm{acs}(G_1, G_2)$.

Unlike greatest common subgraphs and maximal common subgraphs, it may happen (and in fact it is quite typical) that an absorbing common subgraph does not exist. Clearly, a necessary condition for two nonisomorphic graphs G_1 and G_2 of equal size to have an absorbing common subgraph is that $|\mathrm{gcs}(G_1, G_2)| = 1$. As we shall see next, this is not a sufficient condition. Before presenting a theorem that gives a necessary and sufficient condition for two graphs to have an absorbing common subgraph, we need the following lemma.

Lemma 4.4 Every common subgraph of two nonisomorphic graphs G_1 and G_2 of equal size is contained in some maximal common subgraph of G_1 and G_2.

The proof is straightforward and will be omitted. It is also a consequence of a general theory of partially ordered sets. Let us note that for a common subgraph H, a maximal common subgraph of G_1 and G_2 containing H, whose existence is guaranteed by Lemma 4.4, need not be unique.

Theorem 4.5 For every pair of nonisomorphic graphs G_1 and G_2 of equal size, $\mathrm{acs}(G_1, G_2)$ exists if and only if $|\mathrm{mcs}(G_1, G_2)| = 1$.

Proof. Suppose that $G = acs(G_1, G_2)$ exists and that $|mcs(G_1, G_2)| \geq 2$. Then there exists $H \in mcs(G_1, G_2)$ such that $H \not\cong G$. Since H is a maximal common subgraph of G_1 and G_2, and G is a common subgraph of G_1 and G_2, it follows that $H \not\subset G$. This contradicts the fact that $G = acs(G_1, G_2)$.

For the converse, suppose that $mcs(G_1, G_2) = \{G\}$. We will show that $G = acs(G_1, G_2)$. In fact, let H be a common subgraph of G_1 and G_2. By Lemma 4.4, H is contained in some maximal common subgraph of G_1 and G_2, but since G is the unique maximal common subgraph of G_1 and G_2, it follows that $H \subset G$. ☐

Example 4.6 Consider two graphs $G_1 \cong K_n \cup K_2$ and $G_2 \cong (K_{n-1} \cup K_1) + K_1$ as in Figure 4.7.

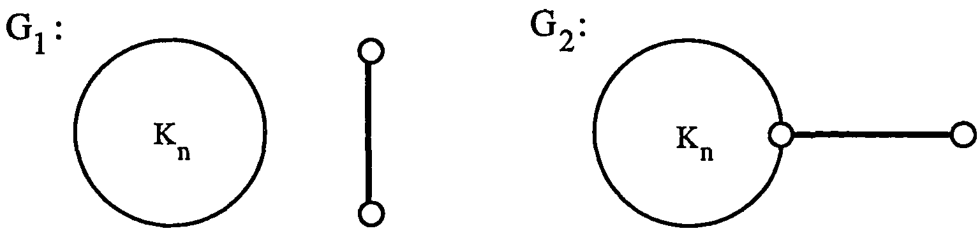


Figure 4.7

It is easy to check that $mcs(G_1, G_2) = \{K_n, K_{n-1} \cup K_2\}$, so by Theorem 4.5 $acs(G_1, G_2)$ does not exist. Note that in this example $acs(G_1, G_2)$ does not exist even if G_1 and G_2 have a unique greatest common subgraph (in fact, $gcs(G_1, G_2) = \{K_n\}$).

In the remainder of this section we will discuss the following existence problem. Let a graph G (without isolated vertices) be given. Do there exist nonisomorphic graphs G_1 and G_2 of equal size such that $G = acs(G_1, G_2)$?

Let us first make the following observation. If for a given graph G, there are nonisomorphic graphs G_1 and G_2 of equal size such that $G = acs(G_1, G_2)$, one can find such G_1 and G_2 of size $q(G_1) = q(G_2) = q(G) + 1$. In fact, if G_1 and G_2 are graphs such that $G = acs(G_1, G_2)$ and $q(G_1) = q(G_2) > q(G) + 1$, then take any $e \in E(G_1) - E(G)$ and any $f \in E(G_2) - E(G)$ and define $G_1' \cong G_1 + e$ and $G_2' \cong G_2 + f$. Then G is a common subgraph of G_1' and G_2' and any common subgraph H of G_1' and G_2' is a subgraph of G (because H is also a common subgraph of G_1 and G_2, and $G = acs(G_1, G_2)$).

Unlike the cases for greatest common subgraphs and maximal common subgraphs, not every graph G is an absorbing common subgraph of two suitably chosen graphs G_1 and G_2. Consider, for example, a complete graph K_n, where $n \geq 2$. Suppose that $K_n = acs(G_1, G_2)$. By the above observation, we can assume that $q(G_1) = q(G_2) = q(K_n) + 1$ and, because the graphs G_1 and G_2 are nonisomorphic, G_1 and G_2 must be the graphs of Example 4.6. But then $acs(G_1, G_2)$ does not exist, so the contradiction is produced. As we will see next, the complete graphs are not the only exceptions.

The characterization of complete bipartite graphs that are the absorbing common subgraphs is given in the following theorem.

Theorem 4.7 Let G be a complete bipartite graph, $G \cong K(m, n)$ where $m \leq n$. There exist nonisomorphic graphs G_1 and G_2 of equal size such that $G = acs(G_1, G_2)$ if and only if $m = 1$, $m = 2$ or $n = m + 1$.

Proof. If $G \cong K(1, n)$, then $G = acs(G_1, G_2)$ for $G_1 \cong K(1, n + 1)$ and $G_2 \cong K(1, n) \cup K_2$.

Assume that $G \cong K(2, n)$. Consider the two graphs G_1 and G_2 given in Figure 4.8.

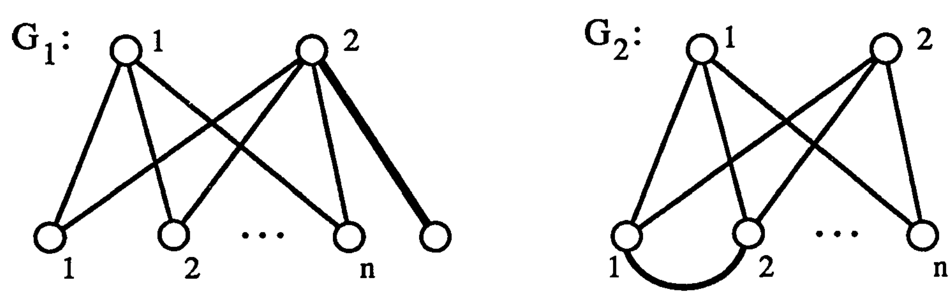


Figure 4.8

Then $G = acs(G_1, G_2)$. In fact, take any H such that $H \subset G_1$ and $H \subset G_2$. Because $H \subset G_1$, it follows that $H \subset K(2, n + 1)$. But $p(H) \leq p(G_2) = n + 2$, so $H \subset K(2,n) \cong G$, in which case the result follows, or $H \subset K(1, n + 1)$. In the latter case, $H \subset K(1, n)$ because $\Delta(G_2) \leq n$, so here too $H \subset G$.

Assume next that $G \cong K(m, m + 1)$, where $m \geq 1$. Define two graphs G_1 and G_2 as in Figure 4.9.

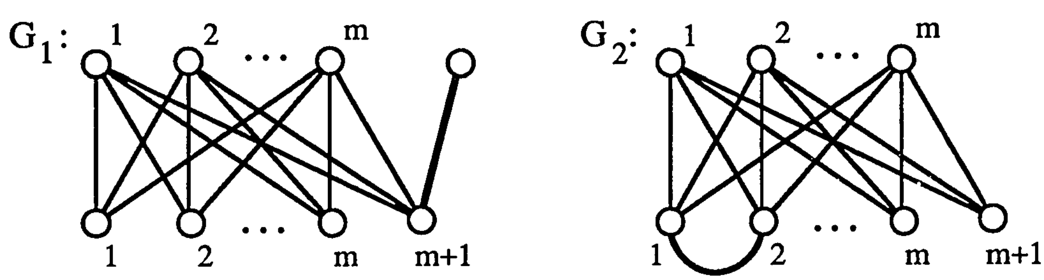


Figure 4.9

We claim that $G = acs(G_1, G_2)$. In fact, take any H such that $H \subset G_1$ and $H \subset G_2$. Because $H \subset G_1$, we have $H \subset K(m + 1, m + 1)$, but $p(H) \leq 2m + 1$, so $H \subset K(m, m + 1)$.

For the converse we consider two cases.

Case 1. Suppose $G \cong K(n, n)$, where $n \geq 3$. Then $K(1, n) \cup K_2 \subset G + e$ for every $e \notin E(G)$ but $K(1, n) \cup K_2 \not\subset G$. Therefore, there are no graphs G_1 and G_2 such that $G = acs(G_1, G_2)$.

Case 2. Suppose $G \cong K(m, m + r)$, where $m \geq 3$ and $r \geq 2$. We can construct a graph G_i with $q(G_i) = q(G) + 1$ and $G \subset G_i$ in five ways. These possibilities are shown in Figure 4.10.

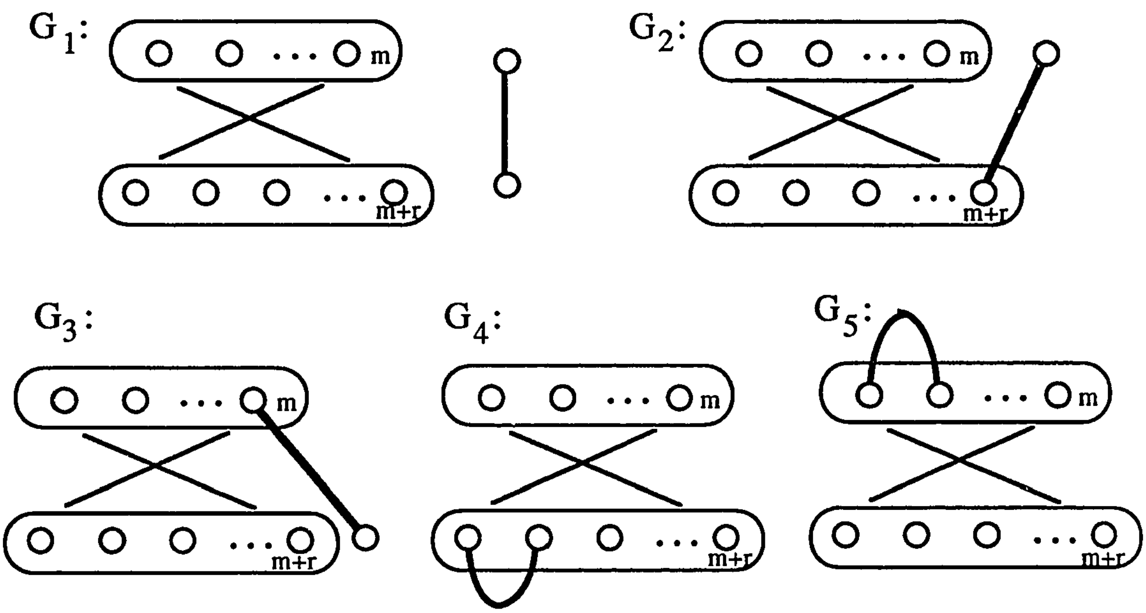


Figure 4.10

There are ten possibilities to select a pair (G_i, G_j), $i \neq j$, and in Table 4.11 we give a graph H such that $H \subset G_i$ and $H \subset G_j$, but $H \not\subset G$. ❐

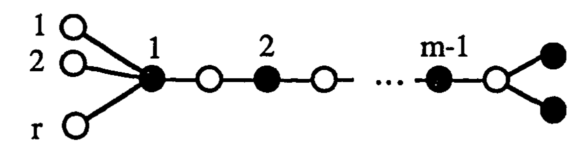
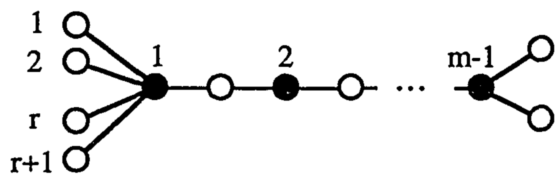
i	j	H
1	2	$K(m, m + r - 1) \cup K_2$
1	3	$K(m - 1, m + r) \cup K_2$
1	4	$K(m, m + r - 2) \cup K_2$
1	5	$K(m - 2, m + r) \cup K_2$
2	3	$K(1, r + 1) \cup P_{2m - 1}$
2	4	$K(m, m + r - 2) \cup K_2$
2	5	
3	4	
3	5	$K(m - 2, m + r) \cup K_2$
4	5	K_3

Table 4.11

Not even all trees are absorbing common subgraphs of two nonisomorphic graphs G_1 and G_2 of equal size. We will present two infinite families of such trees. Let the tree $T_1(k)$, $k \geq 3$, be obtained from the star $K(1, k)$ by joining each end-vertex to $k - 2$ new vertices so that these new vertices each have degree 1. For example, the tree $T_1(5)$ is given in Figure 4.12.

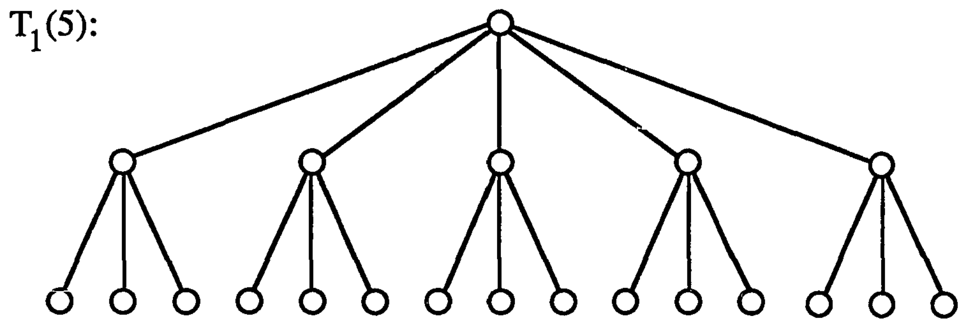


Figure 4.12

Then, $K(1, k) \cup K_2 \not\subset T_1(k)$, but $K(1, k) \cup K_2 \subset T_1(k) + e$ for every edge $e \notin E(T_1(k))$. This proves that $T_1(k) \neq acs(G_1, G_2)$ for any nonisomorphic graphs G_1 and G_2 of equal size.

For the second example, consider a family $T_2(k)$, $k \geq 1$. For a given $k \geq 1$, the tree $T_2(k)$ is obtained by identifying one end-vertex from each of three paths P_{2k+1} (see Figure 4.13).

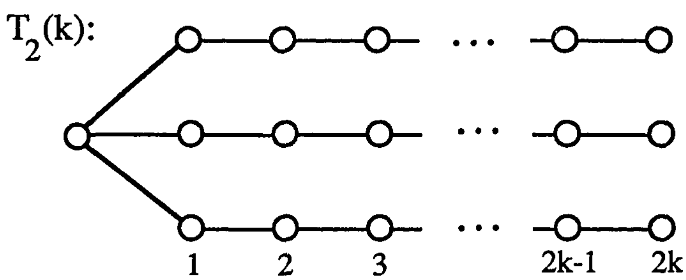


Figure 4.13

Then, $H \cong K(1, 3) \cup [3(k - 1) + 1]K_2 \not\subset T_2(K)$, but $H \subset T_2(k) + e$ for every $e \notin E(T_2(k))$.

With the aid of the next theorem, we will be able to construct infinite families of graphs that are absorbing common subgraphs.

Theorem 4.8 Assume that $G \cong H_{n_1} \cup H_{n_2} \cup \ldots \cup H_{n_k}$, where $k \geq 2$ and the graphs H_n satisfy the following conditions:

(1) $p(H_n) = n \geq 3$;

(2) $H_n - u - v + e \subset H_n$ for every $u, v \in V(H_n)$ and $e \notin E(H_n - u - v)$;

(3) $H_{n+1} - u \subset H_n$ for every $u \in V(H_{n+1})$.

If there exist n_i and n_j such that $n_j = n_i + 1$, then G is an absorbing common subgraph for some nonisomorphic graphs G_1 and G_2 of equal size.

Proof. First, let us note that using condition (3) twice, we have $H_{n+1} - u - v \subset H_{n-1}$ for every $u, v \in V(H_{n-1})$, which implies that $H_{n-1} + e \subset H_{n+1}$.

We can assume without loss of generality that $n_2 = n_1 + 1$. Define two graphs $G_1 \cong G \cup K_2$ and $G_2 \cong G + uv$, where $u \in V(H_{n_1})$ and $v \in V(H_{n_2})$ as in Figure 4.14.

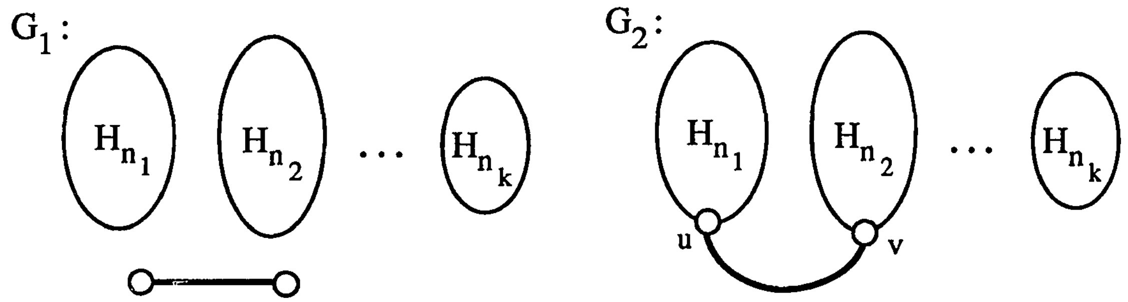


Figure 4.14

Then $G = acs(G_1, G_2)$. To prove this fact, consider any graph H that is a common subgraph of G_1 and G_2. If H does not have K_2 as a component, then $H \subset G$. If H has K_2 as a component, then at least two vertices from $V(H_{n_1}) \cup V(H_{n_2}) \cup \ldots \cup V(H_{n_k})$ are not in H. Assume that $x, y \notin V(H)$ but $x, y \in V(G)$. If $x, y \in V(H_{n_i})$ for some i, $1 \leq i \leq k$, then, by condition (2), $H \subset G$. If $x \in V(H_{n_i})$ and $y \in V(H_{n_j})$, where $n_i \neq n_1$ or $n_j \neq n_2$, then $H \subset G_2$ implies $H \subset G$

(we can place H inside G_2 in such a way that the edge uv is not used). Finally, if $x \in V(H_{n_i})$ and $y \in V(H_{n_j})$, where $n_i = n_1$ and $n_j = n_2$ (or vice versa), then

$$H \subset K_2 \cup H_{n_1 - 1} \cup H_{n_2 - 1} \cup H_{n_3} \cup \ldots \cup H_{n_k} \equiv$$
$$K_2 \cup H_{n_1 - 1} \cup H_{n_1} \cup H_{n_3} \cup \ldots \cup H_{n_k} \subset G,$$

because from condition (3), $K_2 \cup H_{n_1 - 1} \subset H_{n_2}$. □

Before presenting families of graphs that satisfy conditions (1) - (3) of Theorem 4.8, let us recall the concept of the r^{th} power of a graph. For a graph G, the r^{th} power of a graph G, denoted G^r, is a graph whose vertex set is $V(G^r) = V(G)$ and whose edge set $E(G^r) = \{uv \mid u, v \in V(G) \text{ and } d_G(u, v) \leq r\}$.

Corollary 4.9 A graph whose components H_n are of the form:

(a) $H_n \equiv K_n$,

(b) $H_n \equiv C_n$,

(c) $H_n \equiv P_n$,

(d) $H_n \equiv P_n^r$, where $2 \leq r \leq n - 2$

satisfies the conditions of Theorem 4.8, and therefore, if it contains two components of consecutive orders, then it is an absorbing common subgraph for some graphs G_1 and G_2.

If $G \equiv K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k}$ and there are no n_i, n_j such that $n_j = n_i + 1$, then there are no G_1 and G_2 for which $G = acs(G_1, G_2)$. In fact, taking $H \equiv K_2 \cup K_{n_1 - 1} \cup K_{n_2 - 1} \cup \ldots \cup K_{n_k - 1}$, we have $H \subset G_i$ for every G_i such that $q(G_i) > q(G)$ and $G \subset G_i$, but $H \not\subset G$.

Using this observation and Corollary 4.9, we have the following result.

Corollary 4.10 Let $G \cong K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k}$. Then there exist nonisomorphic graphs G_1 and G_2 of equal size such that $G = acs(G_1, G_2)$ if and only if there are integers n_i and n_j such that $n_i = n_j + 1$.

Theorem 4.11 If $G \cong K_n + (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k})$, where $k \geq 2$, then there exist nonisomorphic graphs G_1 and G_2 of the same size with $G = acs(G_1, G_2)$.

Proof. Define $G_1 \cong G + xy$, where $x \in V(K_{n_1})$, $y \in V(K_{n_2})$, and $G_2 \cong G + uv$, where $u \in V(K_n)$ and $v \notin V(G)$ (see Figure 4.15).

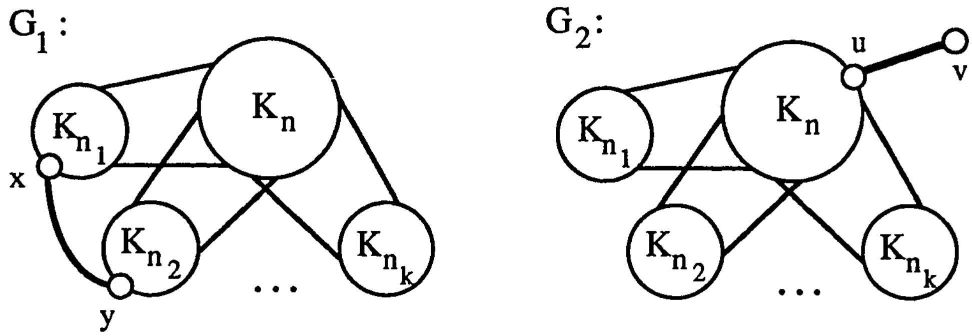


Figure 4.15

Then $G = acs(G_1, G_2)$. In fact, if $H \subset G_1$, then $p(H) \leq n + n_1 + n_2 + \ldots + n_k$. So at least one vertex, say z, from $V(G_2)$ is not in H. Since $G_2 - z \subset G$ for every z, it follows that $H \subset G$. ❒

Corollary 4.12 Let H be a complete k - partite graph of order p, where $k \geq 2$. Then for every n with $n > p$, the graph $G \cong K_n - E(H)$ is an absorbing common subgraph for some graphs G_1 and G_2. In particular, $G \cong K_n - e$ is an absorbing common subgraph.

This result follows from the fact that such a graph G is of the form described in Theorem 4.11.

4.3 Greatest Common Subgraphs for Graphs of Arbitrary Size and Distance Between Graphs

The concept of greatest common subgraphs can be generalized for graphs of arbitrary size (not necessary equal). A graph G without isolated vertices is called a greatest common subgraph of a set $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$, $n \geq 2$, of graphs if G is a graph of maximum size that is isomorphic to a subgraph of each graph G_i, $1 \leq i \leq n$. The set of all greatest common subgraphs of G is denoted by gcs $\mathcal{G}$ and the size of every graph belonging to gcs $\mathcal{G}$ by qgcs $\mathcal{G}$. For example if $\mathcal{G} = \{G_1, G_2\}$ for the graphs of Figure 4.16, then gcs $\mathcal{G} = \{H_1, H_2, H_3\}$ and qgcs $\mathcal{G} = 5$.

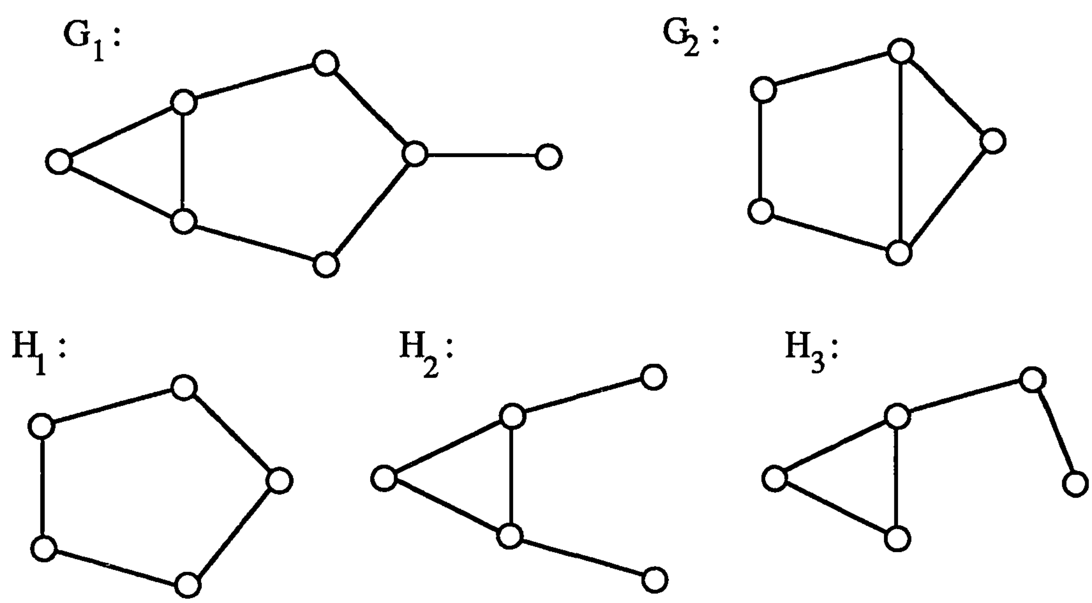


Figure 4.16

We did not need this concept in the previous chapters, because there is no reasonable definition of the index of a graph that would be expressed in terms of graphs of different sizes. Also in Chapter 3, we could restrict our attention to the graphs of equal size, because by Corollary 3.4, for a given graph G the constructed graphs G_1 and G_2 satisfied $q(G_1) = q(G_2) = q(G) + 1$.

Consider the following problem:

For a given graph G without isolated vertices do there exist nonisomorphic graphs G_1 and G_2 such that $gcs(G_1, G_2) = \{G\}$?

Without any restriction on the size of G_1 and G_2, this question has an affirmative answer since we can take $G_1 \cong G$ and G_2 can be any supergraph of G. If G_1 and G_2 are assumed to have the same size, then the positive answer follows from Proposition 2 in [7], where $q(G_1) = q(G_2) = q(G) + 1$.

Suppose that we require graphs G, G_1 and G_2 to have different sizes. To show that such G_1 and G_2 exist for a given graph G, we introduce the following lemma.

Lemma 4.13 Let $gcs(G_1, G_2) = \{G\}$. Then for every $E_1 \subset E(G_1) - E(G)$ and $E_2 \subset E(G_2) - E(G)$, we have

$$gcs(G_1 - E_1, G_2 - E_2) = \{G\}.$$

The proof is similar to the proof of Theorem 2.1 and is thus omitted.

Theorem 4.14 For every graph G without isolated vertices, there exist graphs G_1 and G_2 such that G_1, G_2 and G have distinct sizes and $gcs(G_1, G_2) = \{G\}$.

Proof. Assume first that G is a graph of order p such that $G \not\cong K_p$ and $G \not\cong K_p - e$. By Theorem 2.3, $i(G) \geq \binom{p}{2}$, and, moreover, from the proof of Theorem 2.3, it follows that there exist nonisomorphic graphs H_1 and H_2 of size $\binom{p}{2}$ such that $gcs(H_1, H_2) = \{G\}$. Since G is non-complete and $G \not\cong K_p - e$, it follows that $q(H_1) = q(H_2) = \binom{p}{2} \geq q(G) + 2$. Taking any $f \in E(H_2) - E(G)$ and defining $G_1 \cong H_1$ and $G_2 \cong H_2 - f$, from Lemma 4.13, we have that $gcs(G_1, G_2) = \{G\}$. Also, $q(G_1) > q(G_2) = q(G_1) - 1 > q(G)$.

If $G \cong K_p$, $p \neq 3$, or $G \cong K_p - e$, $p \geq 3$, then by Theorem 2.8, $i(G) = \infty$ and, moreover, for every positive integer q_0, there exist graphs H_1 and H_2 of size q_0 such that $gcs(H_1, H_2) = \{G\}$. If $q_0 \geq q(G) + 2$ and G_1, G_2 are as above, then we have $q(G_1) > q(G_2) > q(G)$ and $gcs(G_1, G_2) = \{G\}$.

Finally if $G \cong K_3$, then $gcs(G_1, G_2) = \{G\}$, where $G_1 \cong K_3 \cup 2K_2$ and $G_2 \cong (K_1 \cup K_2) + K_1$. So $q(G_1) = 5 > q(G_2) = 4 > q(G) = 3$. ❐

From Lemma 4.13 and the proof of Theorem 4.14, we have the following.

Corollary 4.15 For every graph G without isolated vertices, there exist graphs G_1 and G_2 such that $q(G_1) = q(G) + 2$, $q(G_2) = q(G) + 1$ and $gcs(G_1, G_2) = \{G\}$.

Now we turn our attention to the concept of distance between graphs of arbitrary size. In this section we consider graphs to be equivalent if they differ only by isolated vertices. For example the three graphs of Figure 4.17 are equivalent.

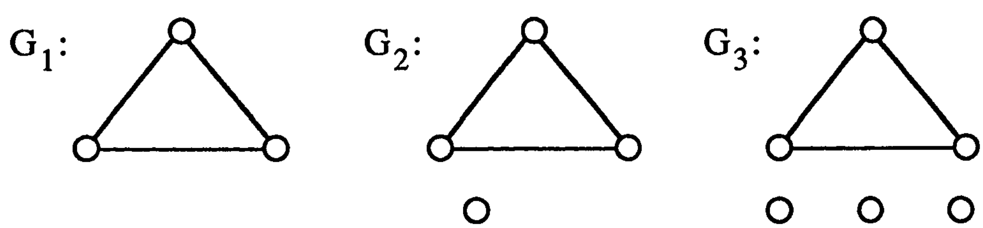


Figure 4.17

More formally, a graph is an equivalence class of the relation $\oplus$:

$G \oplus H$ if and only if there exists an integer $k \geq 0$ such that $G \cong H \cup kK_1$ or $H \cong G \cup kK_1$.

The concept of distance between graphs of the same size was introduced in [6] and the definition of edge rotation is adopted from this paper. A graph G can be transformed into a graph H by an edge rotation if G contains distinct vertices u, v and w such that $uv \in E(G)$, $uw \notin E(G)$ and $H \cong G - uv + uw$. A graph G can be transformed into a graph H by an edge deletion if G contains an edge e such that $H \cong G - e$.

A graph G_1 can be transformed into a graph G_2, written $G_1 \rightarrow G_2$, if either

(1) $G_1 \cong G_2$, or

(2) there exists a sequence $G_1 \cong H_0, H_1, \ldots, H_n \cong G_2$ $(n \geq 1)$ of graphs such that H_i can be transformed into H_{i+1} by an edge rotation or an edge deletion for $i = 0, 1, \ldots, n - 1$.

We note that $G_1 \rightarrow G_2$ for every pair of graphs G_1 and G_2 with $q(G_1) \geq q(G_2)$. To see this note that there is a subgraph H_1 of G_1 such that $q(H_1) = q(G_2)$ and $G_1 \rightarrow H_1$ by edge deletions. Then, by Proposition 1 [6], $H_1 \rightarrow G_2$ by edge rotations.

Let G_1 and G_2 be arbitrary graphs. We define the distance $d(G_1, G_2)$ between G_1 and G_2 as 0 if $G_1 \cong G_2$ and, otherwise, as the smallest positive integer n for which there exists a sequence $H_0, H_1, \ldots, H_n$ of graphs such that $G_1 \cong H_0$, $G_2 \cong H_n$ (or $G_2 \cong H_0$, $G_1 \cong H_n$) and H_i can be transformed into H_{i+1} by an edge rotation or an edge deletion for $i = 0, 1, \ldots, n - 1$.

To show that the function d is a metric on the set of all graphs, we need a preliminary lemma.

Lemma 4.16 If $d(G_1, G_2) = n$ and $q(G_1) - q(G_2) > 0$, then we can choose the sequence $H_0, H_1, \ldots, H_n$ in such a way that the first $q(G_1) - q(G_2)$ transformations of H_i into H_{i+1} are edge deletions.

Proof. Assume that there are graphs H_i, H_{i+1}, H_{i+2} such that $H_i \rightarrow H_{i+1}$ by an edge rotation (say $H_{i+1} \cong H_i - uv + uw$) and $H_{i+1} \rightarrow H_{i+2}$ by an edge deletion (say $H_{i+2} \cong H_{i+1} - e$). Then if $e \neq uw$, we can take H_i, $H'_{i+1} \cong H_i - e$, H_{i+2} and perform the following transformations: $H_i \longrightarrow H'_{i+1}$ by an edge deletion, $H'_{i+1} \longrightarrow H_{i+2}$ by an edge rotation ($H_{i+2} \cong H'_{i+1} - uv + uw$), so we can have an edge deletion first and an edge rotation second. It is not possible that $e = uw$; for otherwise the sequence H_i, H_{i+1}, H_{i+2} could be replaced with H_i, H_{i+2} thus performing a single edge deletion ($H_i \longrightarrow H_{i+2}$ because $H_{i+2} \cong H_i - uv$) and saving one transformation.

Repeating the argument above we can have all edge deletions come first. ❒

Corollary 4.17 If $q(G_1) - q(G_2) = s > 0$, then there exists a subgraph F_1 of G_1 such that $q(F_1) = q(G_2)$ and $d(G_1, G_2) = s + d(F_1, G_2)$.

In fact, if $d(G_1, G_2) = n$, then by Lemma 4.16, we can choose the sequence $G_1 \cong H_0, H_1, \ldots, H_n \cong G_2$ in such a way that first s transformations of H_i into H_{i+1} are edge deletions. Therefore, $F_1 \cong H_s$ satisfies the conditions: $q(F_1) = q(G_2)$ and $d(G_1, G_2) = s + d(F_1, G_2)$.

Theorem 4.18 The function d is a metric on the set of all graphs.

Proof. Certainly, $d(G_1, G_2) \geq 0$ for every two graphs and $d(G_1, G_2) = 0$ if and only if $G_1 \oplus G_2$. Moreover, d is symmetric. Thus, we need only verify the triangle inequality, that is, $d(G_1, G_2) + d(G_2, G_3) \geq d(G_1, G_3)$ for any graphs G_1, G_2 and G_3.

Assume first that $q(G_1) \geq q(G_2) \geq q(G_3)$. If $d(G_1, G_2) = n$ and $d(G_2, G_3) = m$, then there exist sequences $H_0, H_1, \ldots, H_n$ and $H'_0, H'_1, \ldots, H'_m$ such that $H_0 \cong G_1$, $H_n \cong G_2$, $H'_0 \cong G_2$, $H'_m \cong G_3$ and H_i (H'_i) can be transformed into H_{i+1} (H'_{i+1}) by an edge deletion or edge rotation. Taking the sequence $G_1 \cong H_0, H_1, \ldots, H_n \cong G_2 \cong H'_0, H'_1, \ldots, H'_m \cong G_3$, we transform G_1 into G_3 using $n + m$ deletions and rotations. Therefore, $d(G_1, G_3) \leq n + m$. A similar argument can be employed if $q(G_1) \leq q(G_2) \leq q(G_3)$.

Assume next that $q(G_2) > \max(q(G_1), q(G_3))$. By Lemma 4.16, $d(G_1, G_2) = 1 + d(G_1, G_2 - e)$ for some edge $e \in E(G_2)$ and $d(G_2, G_3) = 1 + d(G_2 - f, G_3)$ for some $f \in E(G_2)$. But

$$d(G_2 - e, G_2 - f) = \begin{cases} 0 & \text{if } e = f \\ \leq 1 & \text{if } e \text{ is adjacent to } f \\ \leq 2 & \text{otherwise} \end{cases}$$

To see the last inequality, assume that $e = x_1x_2$ and $f = y_1y_2$, where the four vertices x_1, x_2, y_1 and y_2 are distinct. If the four edges x_1y_1, x_1y_2, x_2y_1 and x_2y_2 belong to $E(G_2)$, then we can use two edge rotations to transform $G_2 - e$ into $G_2 - f$. For example, rotate x_1y_1 into e first and f into x_1y_1 next. Otherwise, if at least one edge among these four is missing, say $x_1y_1 \notin E(G_2)$, we transform $G_2 - e$ into $G_2 - f$ by rotating f into x_1y_1 and then x_1y_1 into e. Therefore, $d(G_2 - e, G_2 - f) \leq 2$. We have

$$\begin{aligned} d(G_1, G_2) + d(G_2, G_3) &= 1 + d(G_1, G_2 - e) + 1 + d(G_2 - f, G_3) = \\ & d(G_1, G_2 - e) + 2 + d(G_2 - f, G_3) \geq \\ & d(G_1, G_2 - e) + d(G_2 - e, G_2 - f) + d(G_2 - f, G_3) \geq \\ & d(G_1, G_2 - f) + d(G_2 - f, G_3). \end{aligned}$$

The last inequality follows from the triangle inequality for the graphs $G_1, G_2 - e$ and $G_2 - f$ since they satisfy the monotonity condition on the number of edges in the first part of the proof.

Consequently, we have proved that

$$d(G_1, G_2) + d(G_2, G_3) \geq d(G_1, G_2 - f) + d(G_2 - f, G_3)$$

and the graph $G_2' \equiv G_2 - f$ has size one less than that of G_2. Repeating the above argument (if necessary), perhaps several times we eventually find G_2' such that $q(G_2') \leq \max(q(G_1), q(G_3))$ and $d(G_1, G_2) + d(G_2, G_3) \geq d(G_1, G_2') + d(G_2', G_3)$.

Again from the first part of the proof $d(G_1, G_2') + d(G_2', G_3) \geq d(G_1, G_3)$, so the triangle inequality holds for G_1, G_2 and G_3.

The proof in the case when $q(G_2) < \min(q(G_1), q(G_3))$ is similar and therefore omitted. ❐

To illustrate this concept, we will find the distance between two graphs G_1 and G_2 considered at the beginning of this section. In Figure 4.18 the graph G_1 is transformed into G_2 by three transformations, namely:

$G_1 \cong H_0 \longrightarrow H_1$ by deleting the edge e,

$H_1 \longrightarrow H_2$ by deleting the edge f, and

$H_2 \longrightarrow H_3 \cong G_2$ by rotation the edge uv into the edge uw.

Therefore, $d(G_1, G_2) \leq 3$.

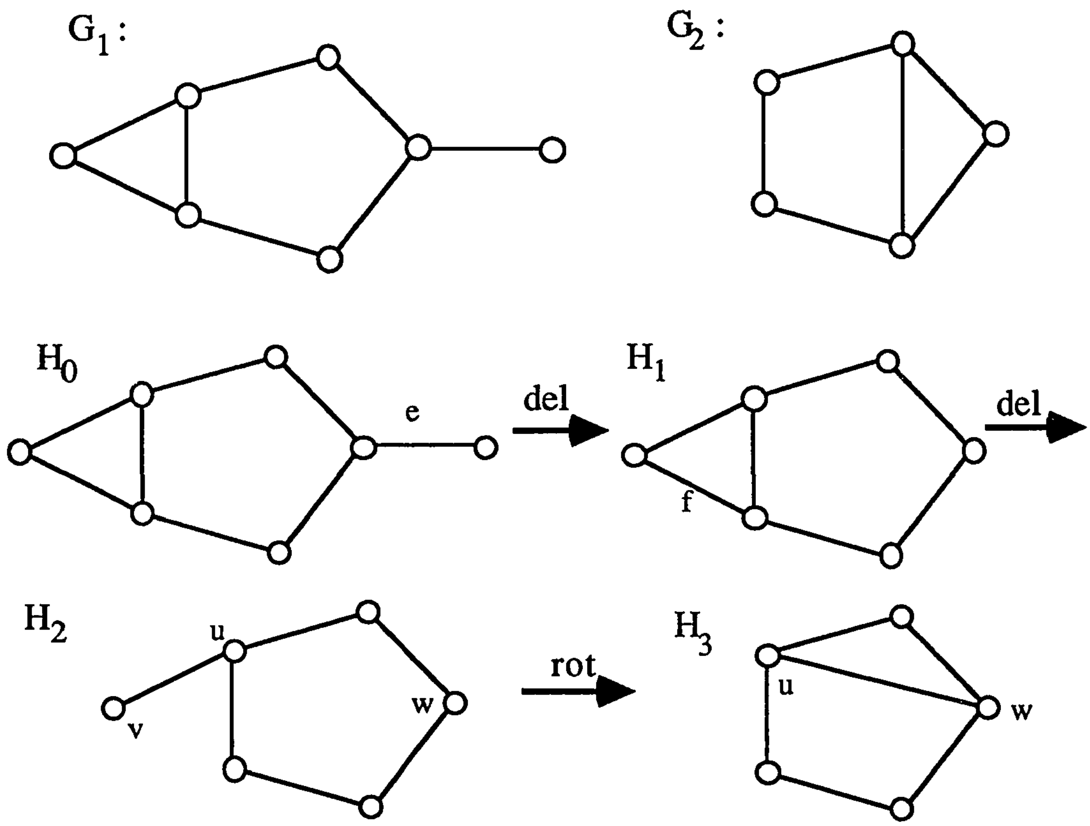


Figure 4.18

Since $q(G_1) = 8$ and $q(G_2) = 6$, it follows that at least two transformations (edge deletions) are necessary to transform G_1 into G_2. Suppose, that we can transform G_1 into G_2 using exactly two edge deletions. Then G_2 is a subgraph of G_1 but this is not the case since G_2 contains a 4-cycle, whereas G_1 does not. Therefore, $d(G_1, G_2) = 3$.

In order to present upper and lower bounds for the distance between graphs, we give the following two lemmas.

Lemma 4.19 Let G_1 and G_2 be two graphs. If G_1 is transformed into G_1' by an edge rotation, then $|\, qgcs(G_1, G_2) - qgcs(G_1', G_2)\,| \leq 1$.

The proof of the lemma is straightforward and is just omitted.

Lemma 4.20 If G_1 and G_2 are graphs such that $q(G_1) \geq q(G_2)$ and $qgcs(G_1, G_2) = c$, then there exists a subgraph G_1' of G_1 such that $q(G_1') = q(G_2)$ and $qgcs(G_1', G_2) = c$.

Proof. Let $H \in gcs(G_1, G_2)$, where $q(H) = c$. Therefore, $H \subset G_1$. Let $E = E(G_1) - E(H)$. Taking arbitrary edges $e_1, e_2, \dots, e_k \in E$, we have $H \in gcs(G_1 - e_1 - e_2 - \dots - e_k, G_2)$. If $k = q(G_1) - q(G_2)$, then we set $G_1' \cong G_1 - e_1 - e_2 - \dots - e_k$. □

Theorem 4.21 Let G_1 and G_2 be graphs of size q_1 and q_2, respectively, and let $qgcs(G_1, G_2) = c$. Then

$$\max\{q_1, q_2\} - c \;\leq\; d(G_1, G_2) \;\leq\; q_1 + q_2 - 2c.$$

Proof. We can assume, without loss of generality, that $q_1 \geq q_2$, say $q_1 - q_2 = s$. For the proof of the lower bound, consider a sequence $G_1 \cong H_0, H_1, \dots, H_n \cong G_2$ of graphs, where $n = d(G_1, G_2)$, H_i is transformed into H_{i+1} by an edge deletion for $i = 0, 1, \dots, s - 1$, and H_i is transformed into H_{i+1} by an edge rotation for $i =$

$s, s+1, \dots, n-1$. Therefore, we have $q(H_s) = q(G_2)$ and $qgcs(H_s, G_2) \le qgcs(G_1, G_2) = c$. Consider the following $n - s + 1$ positive integers:

$$qgcs(H_s, G_2),\ qgcs(H_{s+1}, G_2), \dots, qgcs(H_{n-1}, G_2),\ qgcs(H_n, G_2).$$

The difference between any two consecutive terms in this sequence is at most 1 (by Lemma 4.19), the first integer does not exceed c, and the last integer is equal to q_2. Thus we have

$$n - s \ge q_2 - c,$$

or since $s = q_1 - q_2$ and $n = d(G_1, G_2)$, it follows that

$$d(G_1, G_2) \ge q_1 - c.$$

To verify the upper bound, let G_1' be a subgraph of G_1 such that $q(G_1') = q(G_2) = q_2$ and $qgcs(G_1', G_2) = qgcs(G_1, G_2) = c$ (such a subgraph exists by Lemma 4.20). We have

$$d(G_1, G_2) \le q_1 - q_2 + d(G_1', G_2) \le q_1 - q_2 + 2(q_2 - c) = q_1 + q_2 - 2c,$$

where the second inequality follows by Proposition 4 [6]. □

We now show that the bounds given in Theorem 4.21 are sharp. For the lower bound, let $G_1 \cong K(1, q_1)$ and $G_2 \cong q_2K_2$, where $q_1 \ge q_2$. Then we have $q_1 - q_2$ edge deletions and $q_2 - 1$ edge rotations to transform G_1 into G_2, so

$$d(G_1, G_2) = (q_1 - q_2) + q_2 - 1 = q_1 - 1$$

which is the same as the value of the lower bound because $qgcs(G_1, G_2) = q(K_2) = 1$.

For the upper bound, let $G_1 \cong q_1K_2$ and $G_2 \cong K_{2n}$, where $q_1 \ge \binom{2n}{2} = n(2n - 1) = q_2$. Then $qgcs(G_1, G_2) = q(nK_2) = n$. We have also

$$d(G_1, G_2) = (q_2 - q_1) + d(q_2K_2, G_2).$$

Let us note that by single edge rotation we can increase the degree of one vertex by 1. Since the graph q_2K_2 has all vertices of degree 1 and the graph $G_2 \cong K_{2n}$ has 2n vertices of degree $2n - 1$ we need $2n(2n - 2)$ edge rotations to transform q_2K_2 into K_{2n}. Therefore,

$$d(G_1, G_2) = q_2 - q_1 + 2n(2n - 2) = q_2 - n(2n - 1) + 4n^2 - 4n = q_2 + (2n^2 - n) - 2n =$$
$$= q_2 + q_1 - 2c. \square$$

If $G_2 \subset G_1$ and $q(G_1) - q_1 \geq q_2 = q(G_2)$, then $qgcs(G_1, G_2) = q(G_2) = q_2$. The lower bound given in Theorem 4.21 is $q_1 - q_2$, while the upper bound is $q_1 + q_2 - 2q_2 = q_1 - q_2$, so $d(G_1, G_2) = q_1 - q_2$. This is true of course because we can transform G_1 into G_2 by deleting $q_1 - q_2$ edges.

If in the definition of graph transformations the following operations are permitted:

(1) edge rotation and edge addition, or

(2) edge rotation, edge deletion and edge addition,

then a distance between graphs defined in terms of these transformations is the same as the distance introduced before.

4.4 Least Common Supergraphs

Let $\mathcal{G}$ be a set of graphs all having the same size. A graph G without isolated vertices is <u>a least common supergraph</u> of $\mathcal{G}$ (see [2]) if G is a graph of minimum size that is isomorphic to some supergraph of each graph in $\mathcal{G}$. The set of all least common supergraphs of $\mathcal{G}$ is denoted by lcs $\mathcal{G}$ or lcs $(G_1, G_2, \ldots, G_n)$ if $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$.

For the graphs G_1 and G_2 of Figure 4.19(a), lcs $(G_1, G_2) = \{H_1, H_2, H_3\}$, where these graphs are shown in Figure 4.19(b).

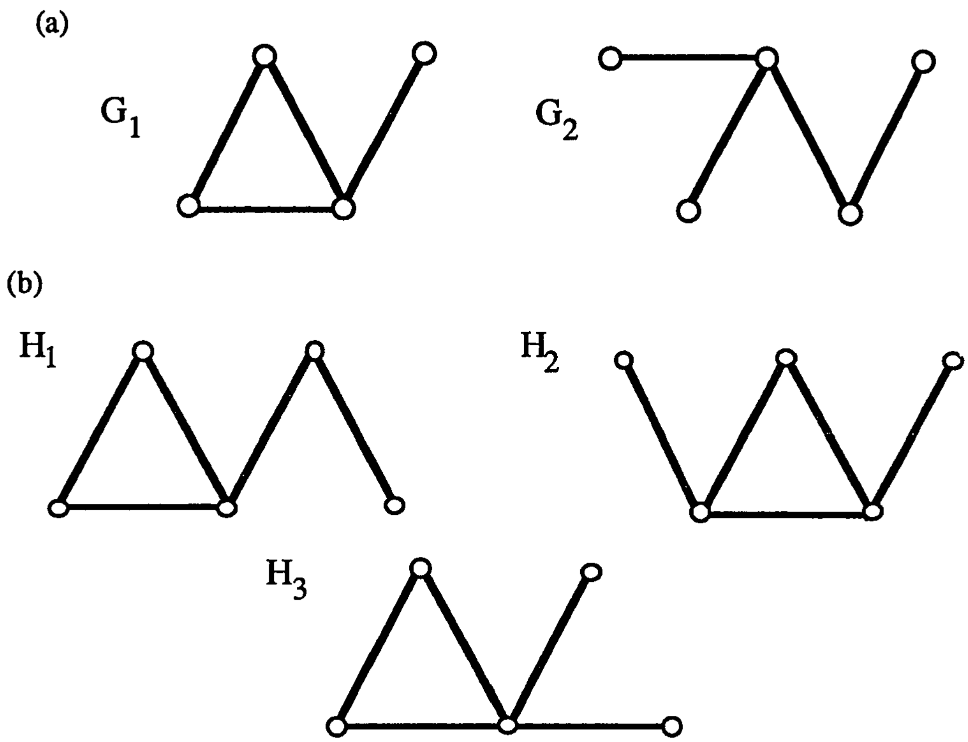


Figure 4.19

We will describe a relationship between least common supergraphs and greatest common subgraphs in terms of a complement operation.

Let $\mathcal{G}$ be a set of graphs, all of the same size, where $|\mathcal{G}| \geq 2$. We describe how to determine lcs $\mathcal{G}$ by finding gcs $\mathcal{G}'$ for a related set $\mathcal{G}'$ of graphs.

Let G be a graph and p an integer with $p \geq p(G)$. The graph G(p) is defined by

$$G(p) \cong G \cup [\, p - p(G) \,] K_1,$$

that is, G(p) is obtained by adding p - p(G) isolated vertices to G.

In what follows, least common supergraphs and greatest common subgraphs are permitted to have isolated vertices.

Theorem 4.22 Let $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ be a family of graphs of the same size and let $p = \max\{p(H) \mid H \in \mathrm{lcs}\,\mathcal{G}$ and H has no isolated vertices$\}$. Then $H \in \mathrm{lcs}(G_1, G_2, \dots, G_n)$ if and only if $\overline{H(p)} \in \mathrm{gcs}(\overline{G_1(p)}, \overline{G_2(p)}, \dots, \overline{G_n(p)})$.

Proof: First, let us note that the integer

$$p = \max\{p(H) \mid H \in \mathrm{lcs}\,\mathcal{G} \text{ and } H \text{ has no isolated vertices}\}$$

is well-defined. In fact, for any family $\mathcal{G}$, $\mathrm{lcs}\,\mathcal{G}$ consists of graphs of the same size. There are only finitely many such graphs without isolated vertices.

Suppose that $H \in \mathrm{lcs}(G_1, G_2, \dots, G_n)$. Hence $G_i \subset H$ for every $i = 1, 2, \dots, n$ and H is a graph of smallest size with this property. Because $p(H) \le p$, it follows that $\overline{H(p)} \subset \overline{G_i(p)}$ for $i = 1, 2, \dots, n$. Therefore $\overline{H(p)}$ is a common subgraph of $\overline{G_i(p)}$, $i = 1, 2, \dots, n$. Suppose that $\overline{H(p)} \notin \mathrm{gcs}\{\overline{G_i(p)} \mid i = 1, 2, \dots, n\}$. Then there is a greatest common subgraph F of $\overline{G_1(p)}, \overline{G_2(p)}, \dots, \overline{G_n(p)}$ such that $q(F) > q(\overline{H(p)})$. Because $F \subset \overline{G_i(p)}$ for $i = 1, 2, \dots, n$, if we consider $G \cong \overline{F(p)}$, then $G_i(p) \subset G$, $i = 1, 2, \dots, n$, and $q(G) < q(H)$. Therefore, G is a common supergraph of $\mathcal{G}$ of smaller size than that of H, which contradicts the fact that $H \in \mathrm{lcs}\,\mathcal{G}$.

For the converse, assume that $\overline{H(p)} \in \mathrm{gcs}(\overline{G_1(p)}, \overline{G_2(p)}, \dots, \overline{G_n(p)})$. Thus $\overline{H(p)} \subset \overline{G_i(p)}$, $i = 1, 2, \dots, n$, and $\overline{H(p)}$ is of the largest size among all such graphs with this property. Since $G_i(p) \subset H(p)$, it follows that $G_i \subset H$ for $i = 1, 2, \dots, n$. Suppose that there exists a least common supergraph G of $G_1, G_2, \dots, G_n$ such that $q(G) < q(H)$. Then $G_i \subset G$, so $\overline{G(p)} \subset \overline{G_i(p)}$ for $i = 1, 2, \dots, n$. Also $q(\overline{G(p)}) > q(\overline{H(p)})$, which contradicts the fact that $\overline{H(p)} \in \mathrm{gcs}(\overline{G_1(p)}, \overline{G_2(p)}, \dots, \overline{G_n(p)})$. ❐

An illustration of Theorem 4.22 is presented in Figure 4.20.

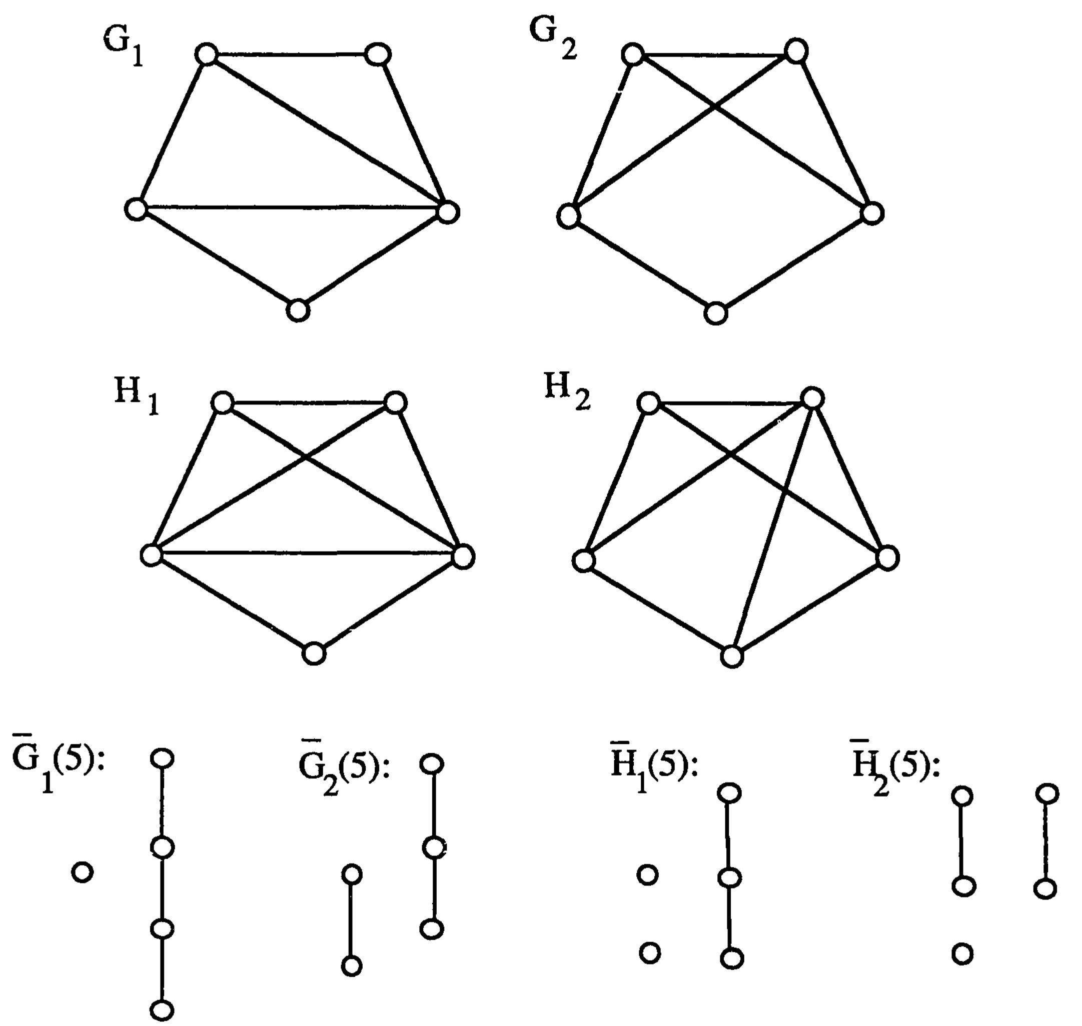


Figure 4.20

If we put $\mathcal{G} = \{G_1, G_2\}$, then lcs $\mathcal{G} = \{H_1, H_2\}$ and, moreover,

$$\max \{p(H) \mid H \in \text{lcs } \mathcal{G} \text{ and } H \text{ has no isolated vertices}\} = 5.$$

Therefore, by Theorem 4.22 we have:

$$H_i \in \text{lcs } \mathcal{G} \quad \text{if and only if} \quad \overline{H_i(5)} \in \text{gcs}(\overline{G_1(5)}, \overline{G_2(5)}), \quad \text{for } i = 1, 2.$$

The second part of this equivalence is much easier to verified because the graphs $\overline{G_1(5)}$ and $\overline{G_2(5)}$ are sparse.

We note that Theorem 4.22 is true for any $p \geq \max\{p(H) \mid H \in \text{lcs}\ \mathcal{G}\}$. Further, if G_1 and G_2 are graphs of size q, then $4q - 2 \geq \max\{p(H) \mid H \in \text{lcs}(G_1, G_2)\}$. To see this let $G \in \text{gcs}(G_1, G_2)$ and $H \in \text{lcs}(G_1, G_2)$. Then it follows from the fact that $q(G) \geq 1$ and since $q(G) + q(H) = 2q$, that $q(H) \leq 2q - 1$. So, since H has no isolated vertices $p(H) \leq 2q(H) \leq 2(2q - 1) = 4q - 2$.

The difference between the orders of a graph $H \in \text{lcs}(G_1, G_2)$ and $\max\{p(G_1), p(G_2)\}$ can be arbitrarily large. For example, let $G_1 \cong k P_3$ and $G_2 \cong 2k K_2$, so $p(G_1) = 3k$, $p(G_2) = 4k$ and $q(G_1) = q(G_2) = 2k$. Then $H \cong k P_3 \cup k K_2 \in \text{lcs}(G_1, G_2)$. Thus $p(H) = 5k$ and $p(H) - \max\{p(G_1), p(G_2)\} = 5k - 4k = k$.

Let $\mathcal{G} = \{G_1, G_2, ..., G_n\}$ be a family of graphs of the same size q, and suppose we know how to determine $\text{gcs}\ \mathcal{G}'$ of a set $\mathcal{G}'$ of graphs of the same size. We describe how to find $\text{lcs}\ \mathcal{G}$. We proceed as follows:

(1) Find an integer p such that $p \geq \max\{p(H) \mid H \in \text{lcs}\ \mathcal{G}\}$. For example, we can take $p = 2\sum_{i=1}^{n} q(G_i) = 2nq$; or if $\mathcal{G} = \{G_1, G_2\}$ we can choose $p = 4q - 2$.

(2) Construct the family $\overline{\mathcal{G}(p)} = \{\overline{G_1(p)}, \overline{G_2(p)}, ..., \overline{G_n(p)}\}$.

(3) Find $\text{gcs}\ \overline{\mathcal{G}(p)}$.

(4) Determine the complement of each graph in $\text{gcs}\ \overline{\mathcal{G}(p)}$.
Then $\text{lcs}\ \mathcal{G} = \{\overline{G(p)} \mid G \in \text{gcs}\ \overline{\mathcal{G}(p)}\}$.

(5) If we want to have graphs without isolated vertices, delete all isolated vertices from graphs $H \in \text{lcs}\ \mathcal{G}$.

Because of Theorem 4.22, we can consider least common supergraphs as a "dual variation" of greatest common subgraphs. Therefore, many result about greatest common subgraphs can be translated into and expressed for least common supergraphs. However, this topic will not be explored in this dissertation.

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