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Chromatic Partitions

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CHROMATIC PARTITIONS

by

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A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
December 1988

CHROMATIC PARTITIONS

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Western Michigan University, 1988

A proper coloring of a graph G is an assignment of colors, usually positive integers, to the vertices of the graph such that no two adjacent vertices have the same color. The chromatic polynomial of a graph G, denoted $\chi(G; \lambda)$, enumerates the number of distinct colorings of G using λ colors. For each positive integer n, a *partition* of n is a finite non–increasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_p$ such that $\sum \lambda_i = n$. The λ_i are called *parts* of n. We define a *Chromatic Partition* of n on G to be a coloring of G using the parts λ_i of n as colors. The *chromatic partition function*, denoted $\Lambda(G; x_1, x_2, ..., x_p)$, associated with a given graph G with vertex set $V(G)=\{x_1, x_2, ..., x_p\}$, expresses the number of ways of coloring G as a function of the vertex set $x_1, x_2, ..., x_p$.

Chapter I is devoted to the study of the chromatic partition function of a labeled graph G. The chromatic partition function for complete graphs are also investigated. In Chapter II the chromatic partition function for an unlabeled graph is introduced. We also devote the last half of this chapter to the study of determining the ordinary chromatic polynomial from the chromatic partition function.

In Chapter III we investigate various partition generating functions that we can derive from the chromatic partition function. We begin with a study of partitions of an integer into two disjoint sets of specified cardinality such that no repeats occur between the sets. Next, we study labeled chromatic sequences and labeled chromatic cycles and several variations of these types of partitions. Finally, several open problems related to the chromatic partition function are presented in Chapter IV.

Order Number 8909104

Chromatic partitions

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In memory of

Bapa and Nana

ACKNOWLEDGEMENTS

I wish to thank Professor Allen Schwenk, my dissertation advisor, for his outstanding guidance, his never-ending encouragement, his sincere interest towards the completion of this dissertation and for being an outstanding teacher. His enthusiasm for anything mathematical and his unique ability to count and to find the most unusual relations among numbers will always be remembered. I am grateful to Professor Kung Wei Yang for being my second reader and for sharing his philosophical views and mathematical insights with all his students. I am very thankful to Professor Ron C. Read for serving as the outside member on my committee. I am also thankful to both Professor Joseph T. Buckley and Professor Yousef Alavi for serving on my committee despite their busy adminstrative schedules. I am grateful also to Professor Paul Erdos for the inspiration he provides wherever he goes. I also wish to thank Professor Gary Chartrand for sparking my interest in Graph Theory with his outstanding lectures. I am grateful to Professor Shashi Kapoor for his invaluable suggestions for typing this manuscript. Thanks also to all the faculty at Western Michigan University for the many stimulating years. I am also grateful to the Mathematics Department for the assistantships. I am also grateful for the support provided by the Office of Naval Research, Contract N00014-86-K0566. I wish to thank my friends and fellow students for making my stay at Western an enjoyable one. I especially wish to thank both my officemates, Garry Johns and Ghidhewon Abay Asmarom for putting up with my mess in the office! I also wish to thank everyone in the Math office for putting up with all the paper-work I caused with my carelessness with bureaucratic matters.

I am very grateful to my family and friends for their never-ending support, love and understanding. Finally, I would like to thank Shruti for her encouragement, inspiration and patience during the past year, and her family for their continued moral support.

Paresh J. Malde

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CHAPTER I

AN INTRODUCTION TO CHROMATIC PARTITIONS

Section 1.1 Partitions and Coloring

Let G be a graph with vertex set $V(G) = \{x_1, x_2, ..., x_p\}$ and edge set $E(G) = \{e_1, e_2, ..., e_q\}$. In general we will follow the terminology of Chartrand and Lesniak [4]. A *proper coloring*, or to be economical, simply a *coloring* of G is an assignment of colors to the vertices of G so that no two adjacent vertices have the same color. For each positive integer n, a *partition* of n is a finite non–increasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_p$ such that $\sum \lambda_i = n$. The λ_i are called *parts* of n. An ordered partition is called a *composition*. For example, n = 4 has 5 partitions, 4; 3 1; 2 2; 2 1 1; 1 1 1 1, but has 8 compositions, 4; 3 1; 1 3; 2 2; 2 1 1; 1 2 1; 1 1 2; 1 1 1 1. The number of compositions is easily seen to be 2^{n-1}, but no convenient formula for the number of partitions p(n) is known. Instead, generating function identities are usually deemed the best results that one can hope for to determine p(n). Sometimes we wish to limit the number of terms, so we define p(n,k) to be the number of partitions of n using at most k parts. One might also consider partitions with exactly k parts, but this is given by $p(n,k) - p(n,k-1)$, so we do not develop a separate notation to handle this variation. In contrast, the number of compositions with exactly k parts is $\binom{n-1}{k-1}$.

Section 1.2 Chromatic Partition Function

We wish to consider a new concept that combines the features from these two areas. We define a *Chromatic Partition* of n on G to be a coloring of G using the parts λ_i of n as colors. In this context it is not appropriate to require the λ_i to be non–increasing, but in the spirit of proper colorings, we require adjacent parts to have distinct values. Perhaps we should call these chromatic compositions, but we like the sound of chromatic partitions better. The following example illustrates the various partitions of n=6 on $K_{2,1}$.

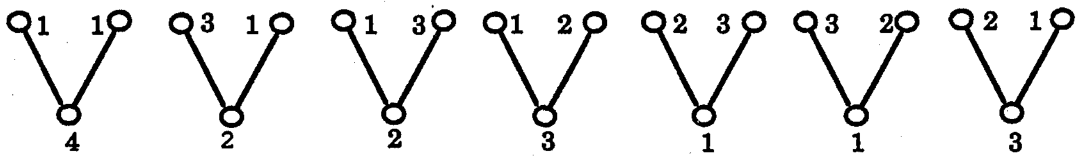


Figure 1.1

It is unreasonable to expect to find a closed formula for the number of chromatic partitions for a given value of n. However, we can develop a generating function whose coefficients are the desired number of possibilities. The *chromatic partition function*, denoted $\Lambda(G; x_1, x_2, ..., x_p)$, associated with a given graph G with vertex set $V(G)=\{x_1, x_2, ..., x_p\}$, expresses the number of ways of coloring G as a function of the vertex set $x_1, x_2, ..., x_p$. Therefore x_i is a vertex as well as a variable. When convenient and obvious from context we will suppress the name of the graph and just write $\Lambda(x_1, x_2, ..., x_p)$. Let $[x_1^{a_1}x_2^{a_2}...x_p^{a_p}]$ denote the coefficient of the $x_1^{a_1}x_2^{a_2}...x_p^{a_p}$ term in $\Lambda(G ; x_1, x_2, ..., x_p)$. Then $[x_1^{a_1}x_2^{a_2}...x_p^{a_p}]$ is 1 if G can be colored with the color assignment a_1 to x_1; a_2 to x_2 ;...; a_p to x_p and is 0 otherwise. For example

$$\Lambda(K_1; x_1) = (x_1 + x_1^2 + x_1^3 + ...) = \frac{x_1}{1 - x_1}$$

$$\Lambda(K_2; x_1,x_2) = \left(\frac{x_1}{1 - x_1}\right)\left(\frac{x_2}{1 - x_2}\right) - \left(\frac{x_1x_2}{1 - x_1x_2}\right).$$

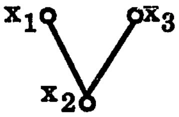
We introduce the notation $X_i = \frac{x_i}{1 - x_i} = (x_i + x_i^2 + x_i^3 + ...)$ to ease the burden of writing sums and products of infinite series. Similarly, it is useful to define $X_{ij} = \frac{x_i x_j}{1 - x_i x_j} = (x_i x_j + x_i^2 x_j^2 + ...)$. The notation is used analogously with additional subscripts, so that $X_{ijk} = \frac{x_i x_j x_k}{1 - x_i x_j x_k}$. For example $\Lambda(K_2) = X_1 X_2 - X_{12}$.

Let G be a graph and let $e = x_i x_j$ be an edge of G. We denote by G^*e the graph formed from G by contracting the edge e and identifying the vertices labeled x_i and x_j to form a new vertex x_{ij} and replacing multiple edges by a single edge. We also denote by $G - e$ the graph formed by deleting the edge e from G. We begin with a fundamental result that allows $\Lambda(G)$ to be determined recursively from the generating functions of graphs of smaller size. This result is analogous to the result of Birkhoff and Lewis [2] for determining the chromatic polynomial of a graph.

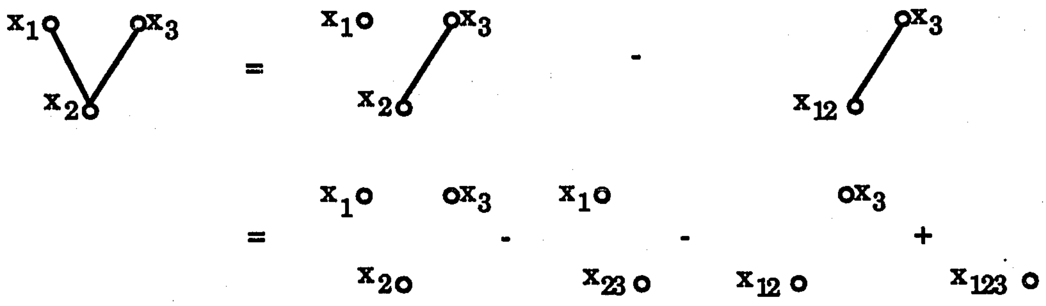
Theorem 1.1 *For any graph G and any arbitrary edge e in G, the partition function $\Lambda(G)$ is given by $\Lambda(G) = \Lambda(G - e) - \Lambda(G^*e)$*

Proof. Let $e = x_i x_j$ be an edge of G. Then $\Lambda(G-e)$ is nearly the same as the counting series for G except it allows the extra possibility of colorings of G where x_i and x_j have been assigned the same color. But $\Lambda(G^*e)$ is precisely the counting series for G in which x_i and x_j have been assigned the same color. Hence, $\Lambda(G) = \Lambda(G-e) - \Lambda(G^*e)$ follows immediately. ❑

We illustrate the use of this theorem with the following example:

Let G: 

Then ,



So that $\Lambda(G) = X_1X_2X_3 - X_1X_{23} - X_{12}X_3 + X_{123}$

Figure 1.2

The following theorem relates the partition function of a graph G to the partition functions of the connected components of G.

Theorem 1.2 *If G has connected components $G_1, G_2, \ldots, G_k$ then* $\Lambda(G) = \Lambda(G_1)\Lambda(G_2)\ldots\Lambda(G_k)$

Proof. Since the components of G are disjoint, the coloring of each is independent of the coloring of the others. Hence, the number of ways of coloring G is simply the product of the number of ways of coloring the separate components. ❑

Section 1.3 The Main Result

For any subgraph H of G we introduce the contraction G*H to be the graph obtained from G by contracting all the edges of H. Observe that G*G has no edges and its order is simply the number of components in G, denoted k(G). Each new vertex has multiple subscripts, as many as the order of the component from which it came. We next present a theorem that evaluates the partition function of a graph G in terms of its spanning subgraphs. We denote by *H(G)* the collection of all spanning subgraphs of G i.e. $H(G) = \{ H \mid G > H, V(G) = V(H) \text{ and } E(G) \supseteq E(H) \}$. The next theorem is effectively an inclusion–exclusion result that evaluates the chromatic partition function of G in terms of improper partitions which use constant values on connected components of H contrary to the rules for chromatic partitions. ❑

Theorem 1.3 *Let G be a graph and let H(G) be the collection of all spanning subgraphs of G. Then*

$$\Lambda(G) = \sum_{H \in H(G)} (-1)^{|E(H)|} \Lambda(H^*H). \tag{1}$$

Proof. We proceed by induction on the size of G. If $|E(G)| = 0$, the only subgraph of G is H = G and H*H = G so that the identity is trivial. Now suppose that the result is true for all graphs of size less than a fixed positive integer m. We must show that the result holds for any graph of size m.

Let G be a graph of size m, i.e. $|E(G)| = m$ and suppose that $e = x_i x_j$ is an edge of G. Then by Theorem 1.1, $\Lambda(G) = \Lambda(G-e) - \Lambda(G^*e)$. Since $|E(G-e)| = |E(G)|-1 = m-1 < m$, we conclude that

$$\Lambda(G-e) = \sum_{H \in H(G-e)} (-1)^{|E(H)|} \Lambda(H^*H).$$

Similarly, since $|E(G^*e)| < m$ we observe that

$$\Lambda(G^*e) = \sum_{H \in H(G^*e)} (-1)^{|E(H)|} \Lambda(H^*H).$$

Observe that $H(G-e)$ is the collection of all spanning subgraphs of G in which the edge e is not included. We claim that

$$-\sum_{H\in H(G*e)}(-1)^{|E(H)|}\,\Lambda(H*H) = \sum_{H\in H(G)-H(G-e)}(-1)^{|E(H)|}\,\Lambda(H*H)$$

i.e. the second term is the correct count for all spanning subgraphs of G in which the edge e is present. Suppose that on contracting the edge e and identifying the vertices x_i and x_j no multiple edges are formed, then the result is clear since x_i and x_j will be in same component of every spanning subgraph of G containing the edge e. Suppose now that there exists at least one vertex x_k such that x_ix_k and x_jx_k are edges of G. Then $x_ix_jx_k$ forms a triangle and on contracting edge e we obtain a pair of edges joining x_{ij} and x_k. We shall replace the pair by a single edge and show that this does not alter the net count. Notice that the sign of each term in equation (1) depends on the number of edges. Now for each term corresponding to a particular $H < G*e$ in which the edge e is present, there corresponds three subgraphs H_1, H_2, H_3 of G such that $H_i*e = H$. These three subgraphs are

(1) H_1, which contains edges x_ix_j and x_ix_k but not x_jx_k

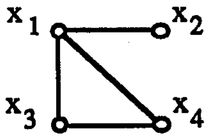
(2) H_2, having edges x_ix_j and x_jx_k present but not x_ix_k

and (3) H_3 in which edges x_ix_j, x_jx_k and x_ix_k are all present.

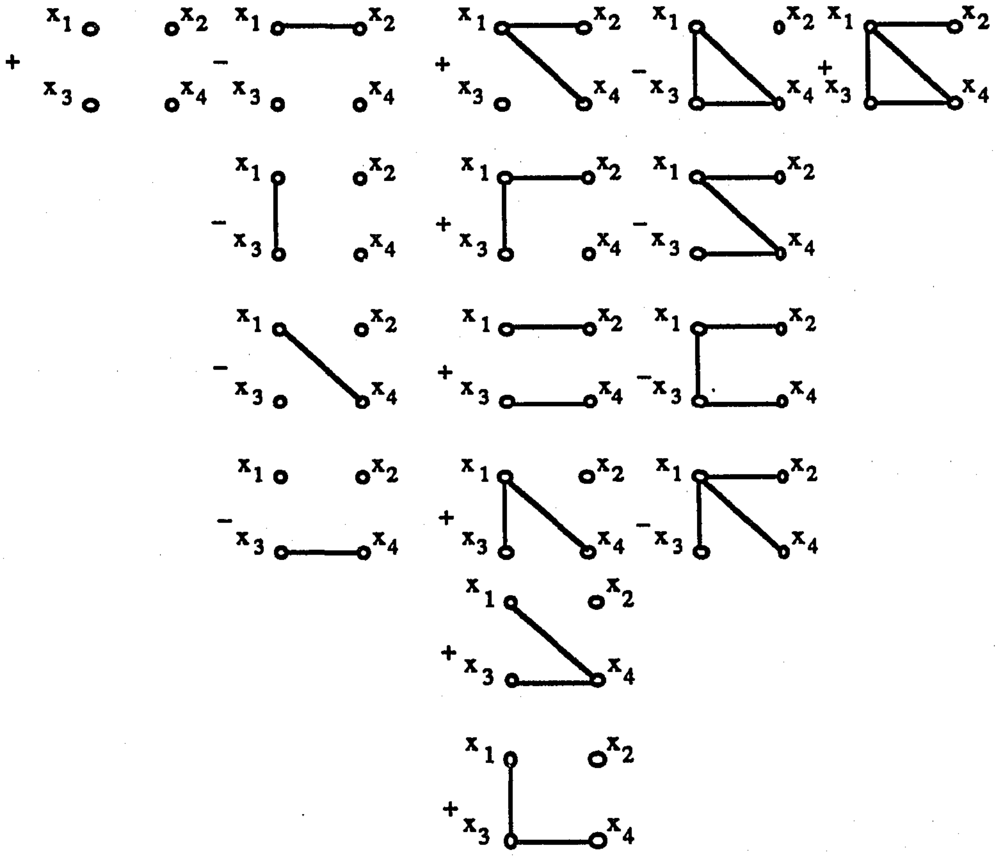
Notice then that $|E(H_1)| = |E(H_2)| = |E(H_3)|-1$ so that the term corresponding to H_2 and H_3 are of opposite sign in (1). Thus both terms may be deleted with no net effect on the sum. The remaining term $(-1)^{|E(H)|}\,\Lambda(H_1*H_1)$ in $\Lambda(G)$ is identical to the $(-1)^{|E(H)|}\Lambda(H*H)$ term in $\Lambda(G*e)$. The extra factor of (-1) accounts for the edge e, and is just what we need to produce the minus sign preceding the summation on the left hand side of (1). Hence we have verified the identity. ❑

We illustrate the usefulness of this result in the example below.

Let G:



Then the subgraphs of G are listed below with the sign indicating odd or even number of edges:



So that

$$
\begin{array}{llllll}
\Lambda(G) = X_1X_2X_3X_4 & - X_{12}X_3X_4 & + X_{124}X_3 & - X_{134}X_2 & + X_{1234} \\
 & - X_{13}X_2X_4 & + X_{123}X_4 & - X_{1234} & \\
 & - X_{14}X_2X_3 & + X_{12}X_{34} & - X_{1234} & \\
 & - X_1X_2X_{34} & + X_{134}X_2 & - X_{1234} & \\
 & & + X_{134}X_2 & & \\
 & & + X_{134}X_2 & &
\end{array}
$$

Figure 1.3

Section 1.4 Complete Graphs

We now apply this theorem to compute the chromatic partition function of a complete graph. Let K_p be the complete graph on p vertices with $V(K_p) = \{x_1, x_2, ..., x_p\}$. We denote by T_p the collection of all partitions of the integer p. Let α be a partition of the set $\{1, 2, ..., p\}$ into disjoint subsets. For each natural number k, $j_k(\alpha)$ is the number of cells of size k in α. Since $\Sigma\, kj_k$ must equal p, each set partition α, having for each k from 1 to p, exactly $j_k(\alpha)$ subsets of cardinality k, can be associated with the unique partition denoted in the vector form $j = (j_1(\alpha), j_2(\alpha), ...)$ in T_p. Let (j) be the collection of all such α associated with j. For example if $p=9$, then $\alpha=\{\{124\},\{3567\},\{8\},\{9\}\}$ has $j_1(\alpha)=2$, $j_2(\alpha)=0$, $j_3(\alpha)=1$ and $j_4(\alpha)=1$, and α is an element of (j) where $j=(2,0,1,1)$ associated with the partition $1^2\, 3^1\, 4^1$. We define $X_\alpha = \prod_{A\in\alpha} X_A$ and then the above example $X_\alpha=X_{124}X_{3567}X_8X_9$ serves as an illustration of this concept.

We are now ready to present the next theorem.

Theorem 1.4 *The chromatic partition function for complete graphs of order p is given by*

$$\Lambda(K_p) = \sum_{j\in T_p}[(-1)^{p-\Sigma j_k}\prod_{k=1}^{p}[(k-1)!]^{j_k}] \sum_{\alpha\varepsilon(j)} X_\alpha \quad .$$

We approach the proof of this theorem via a lemma, interesting in its own right, that compares the number of even versus odd sized connected labeled graphs. Let $G(x,y) = \sum_{p;q}\frac{1}{p!}G_{p,q}x^py^q$ be the counting series for labeled graphs with p vertices and q edges i.e. there are $G_{p,q}$ labeled p,q–graphs. This generating function is exponential in the first variable but ordinary in the second. Let $C(x,y)=\sum_{p;q}\frac{1}{p!}C_{p,q}x^py^q$ be the corresponding counting series for connected labeled p,q–graphs. Let $C_{n,e-o}$ denote

$\sum_q (-1)^q C_{n,q}$, in other words, the difference between the number of graphs of order n with even size and the number with odd size. We find a remarkably simple formula for $C_{n,e-o}$.

Lemma 1.5 *The difference between the numbers of connected labeled graphs of even size and odd size for order n is just*

$$C_{n,e-o} = (-1)^{n-1}(n-1)!.$$

Proof of Lemma 1.5 The C(x,y) and G(x,y) series are related by C(x,y) = ln(1+ G(x,y)) as derived by Riddel [13]. We now replace y by −1 and have C(x,−1) = ln(1+G(x,−1)).

But $$G(x,y) = \sum_{n=1}^{\infty} \frac{x^n}{n!}(1+y)^{\binom{n}{2}}$$

as given by Harary and Palmer [10]. For y=−1 this reduces to

$$C(x,-1) = \ln(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n!} C_{n,e-o}$$

$$= \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n}$$

so that $C_{n,e-o}=(-1)^{n-1}(n-1)!.$ ❑

We are now ready to present the...

Proof of Theorem 1.4 By Theorem 1.3 there is a term in $\Lambda(K_p)$ corresponding to each spanning subgraph of K_p. The sign of this term is determined by the number of edges in the subgraph. Let α be a partition of the set $\{1,2,\ldots,p\}$. Then to each α there is a class of subgraphs of K_p, in which the set of labels of the vertices of each connected component corresponds to the cells of α. If α is an element of $(\mathbf{j})$, we apply Lemma 1.5 to each cell of α to get $\prod_{k=1}^{p}[(k-1)!]^{j_k}$ as the coefficient of X_α since there are exactly $(k-1)!$ terms for each cell of size k in α and there are exactly $j_k(\alpha)$ such parts. Also the sign of each term X_α is

$$\prod_{k=1}^{p}(-1)^{(k-1)j_k} = (-1)^{\Sigma(k-1)j_k} = (-1)^{\Sigma kj_k - j_k} = (-1)^{p-\Sigma j_k}$$

Hence the result follows immediately. ❑

We now use the above theorem to evaluate the chromatic function for complete graphs. In the table below the complete graph K_p has vertex set $V(K_p) = \{x_1, x_2, \ldots, x_p\}$.

Table 1.1

Chromatic Partition Functions Evaluated for the Complete Graphs K_p

$$\Lambda(K_1) = X_1$$

$$\Lambda(K_2) = X_1X_2 - X_{12}$$

$$\Lambda(K_3) = X_1X_2X_3 \begin{array}{l} -X_{12}X_3 \\ -X_{23}X_1 \\ -X_{13}X_2 \end{array} + 2X_{123}$$

$$\Lambda(K_4) = X_1X_2X_3X_4 \begin{array}{l} -X_{12}X_3X_4 \\ -X_{13}X_2X_4 \\ -X_{14}X_2X_3 \\ -X_{23}X_1X_4 \\ -X_{24}X_1X_3 \\ -X_{34}X_1X_2 \end{array} \begin{array}{l} +X_{12}X_{34} \\ +X_{14}X_{23} \\ +X_{13}X_{24} \end{array} \begin{array}{l} +2X_{123}X_4 \\ +2X_{124}X_3 \\ +2X_{134}X_2 \\ +2X_{234}X_1 \end{array} - 6X_{1234}$$

$$\Lambda(K_5) = X_1X_2X_3X_4X_5 \begin{array}{l} -X_{12}X_3X_4X_5 \\ -X_{13}X_2X_4X_5 \\ -X_{14}X_2X_3X_5 \\ -X_{23}X_1X_4X_5 \\ -X_{24}X_1X_3X_5 \\ -X_{34}X_1X_2X_5 \\ -X_{15}X_2X_3X_4 \\ -X_{25}X_1X_3X_4 \\ -X_{35}X_1X_2X_4 \\ -X_{45}X_1X_2X_3 \end{array} \begin{array}{l} +X_{12}X_{34}X_5 \\ +X_{14}X_{23}X_5 \\ +X_{13}X_{24}X_5 \\ +X_{25}X_{34}X_1 \\ +X_{45}X_{23}X_1 \\ +X_{35}X_{24}X_1 \\ +X_{15}X_{34}X_2 \\ +X_{14}X_{35}X_2 \\ +X_{13}X_{45}X_2 \\ +X_{12}X_{35}X_4 \\ +X_{15}X_{23}X_4 \\ +X_{13}X_{25}X_4 \\ +X_{12}X_{45}X_3 \\ +X_{14}X_{25}X_3 \\ +X_{15}X_{24}X_3 \end{array} \begin{array}{l} -2X_{123}X_4X_5 \\ -2X_{124}X_3X_5 \\ -2X_{134}X_2X_5 \\ -2X_{234}X_1X_5 \\ -2X_{145}X_3X_4 \\ -2X_{135}X_2X_4 \\ -2X_{235}X_1X_4 \\ -2X_{145}X_2X_3 \\ -2X_{245}X_1X_3 \\ -2X_{345}X_1X_2 \end{array}$$

$$\begin{array}{l} +2X_{123}X_{45} \\ +2X_{124}X_{35} \\ +2X_{134}X_{25} \\ +2X_{234}X_{15} \\ +2X_{125}X_{34} \\ +2X_{135}X_{24} \\ +2X_{235}X_{14} \\ +2X_{145}X_{23} \\ +2X_{245}X_{13} \\ +2X_{345}X_{21} \end{array} \begin{array}{l} -6X_{1234}X_5 \\ -6X_{2345}X_1 \\ -6X_{1345}X_2 \\ -6X_{1245}X_3 \\ -6X_{1235}X_4 \end{array} + 24X_{12345}$$

Section 1.5 Expressing The Partition Function in Terms of Subgraphs

Keeping in mind the complexity involved in computing the chromatic partition function, $\Lambda(G)$, of a graph G, we present a result that allows us to compute $\Lambda(G)$ in terms of the partition function of its subgraphs. We first define a special product of the partition functions of graphs. Let α be a partition of the set $Z = \{1,2,\ldots,p\}$. Define K_α to be the graph vertex set $V(K_\alpha) = Z$ and the components of K_α are the complete graphs on the cells of α. We say that $x,y \in Z$ are related under α if they are in the same cell of α or equivalently if they are connected in K_α. Now suppose that β is another partition of Z. Then we define $G_{\alpha\cup\beta}$ to be the graph $K_\alpha \cup K_\beta$, where the vertices with the same labels are identified and multiple edges are replaced by a single edge. We say that $x,y \in Z$ are related under $\alpha \cup \beta$ if they are connected by a path in $G_{\alpha\cup\beta}$. Therefore $\gamma = \alpha\cup\beta$ is the partition of Z obtained from α and β. We illustrate this concept below:

$$\text{Let } \alpha = \{\{135\},\{78\},\{26\},\{4\}\}$$

$$\text{and } \beta = \{\{142\},\{63\},\{8\},\{7\},\{5\}\}$$

$$\text{then } \alpha\cup\beta = \{\{123456\},\{78\}\}$$

We are now ready to define the special product of the partition functions:

$$X_\alpha \cup X_\beta = X_{\alpha\cup\beta}$$

so that in the above example

$$X_{135}X_{78}X_{26}X_4 \cup X_{142}X_{63}X_8X_7X_5 = X_{123456}X_{78}\,.$$

This product is defined to be distributive over addition and subtraction and the associative property holds i.e.

$$(X_\alpha + X_\beta) \cup X_\gamma = X_\alpha \cup X_\gamma + X_\beta \cup X_\gamma$$

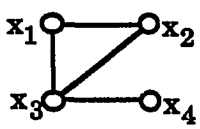
and

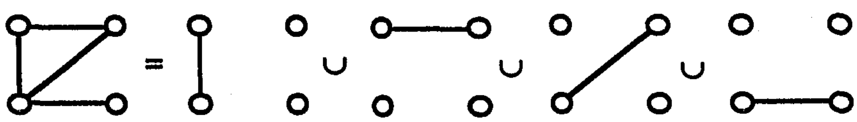
$$(X_\alpha \cup X_\beta) \cup X_\gamma = X_\alpha \cup (X_\beta \cup X_\gamma)\,.$$

We need some more definitions. Let G be a graph with vertex set $V(G) = \{x_1, x_2, ..., x_p\}$ and let $e = x_ix_j$ be an edge of G. We define G_e to be the subgraph of G with vertex set $V(G_e) = V(G) = \{x_1, x_2, ..., x_p\}$, but the edge set $E(G_e) = \{e\}$, i.e. G_e is the spanning subgraph of G containing only the edge e. Now observe that $\Lambda(G_e) = X_1X_2...X_i..X_j..X_p - X_1X_2..X_{i-1}X_{ij}X_{i+1}..X_{j-1}X_{j+1}...X_p$ by theorem 1. We are now ready to present the next theorem.

Theorem 1.6 *Let G be a graph with vertex set* $V(G) = \{x_1, x_2, ..., x_p\}$ *and edge set* $E(G) = \{e_1, e_2, ..., e_q\}$. *Then* $\Lambda(G) = \Lambda(G_{e_1}) \cup \Lambda(G_{e_2}) \cup ... \cup \Lambda(G_{e_q})$.

Before presenting the proof of this theorem we shall illustrate its use.

Let G: 

Then G: 

So that

$$\Lambda(G) = (X_1X_2X_3X_4 - X_{13}X_2X_4) \cup (X_1X_2X_3X_4 - X_{12}X_3X_4) \cup$$
$$(X_1X_2X_3X_4 - X_{23}X_1X_4) \cup (X_1X_2X_3X_4 - X_{34}X_1X_2)$$

Figure 1.4

Proof. By Theorem 1.3, every term of $\Lambda(G)$ corresponds to some spanning subgraph of G. But each such spanning subgraph also corresponds to exactly one term above i.e. the first term of $\Lambda(G_{e_i})$ corresponds to the case when the edge e_i is not present in the subgraph and the second term corresponds to the case when it is present. The result follows immediately. ❑

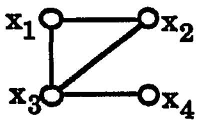
This yields the next result.

Corollary. *Let* G *be a graph with vertex set* $V(G) = \{x_1, x_2, ..., x_p\}$ *and let* G_1 *and* G_2 *be two spanning subgraphs of* G *such that* $E(G_1) \cap E(G_2) = \varnothing$. *Then* $\Lambda(G) = \Lambda(G_1) \cup \Lambda(G_1)$.

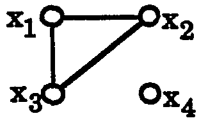
Proof. The operation just defined is commutative so that we can group the respective edges to form G_1 and G_2 so that the result follows immediately. ❑

We will illustrate the usefulness of this corollary next.

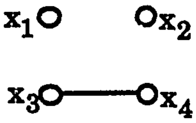
Let G:



Let G_1:



and G_2:



Then $\Lambda(G) = \Lambda(G_1) \cup \Lambda(G_1)$.

$$
\begin{aligned}
&= (X_1X_2X_3X_4 - X_{12}X_3X_4 + 2X_{123}X_4) \cup (X_1X_2X_3X_4 - X_{34}X_1X_2) \\
&\qquad - X_{23}X_1X_4 \\
&\qquad - X_{13}X_2X_4 \\
&= (X_1X_2X_3X_4 - X_{12}X_3X_4 + 2X_{123}X_4 - 2X_{1234}) \\
&\qquad - X_{23}X_1X_4 + X_{12}X_{34} \\
&\qquad - X_{13}X_2X_4 + X_{234}X_1 \\
&\qquad - X_1X_2X_{34} + X_{134}X_2
\end{aligned}
$$

Figure 1.5

In the next chapter we will consider the unlabeled chromatic partition function.

CHAPTER II

THE UNLABELED CHROMATIC PARTITION FUNCTION

Section 2.1 Group Theoretic Preliminaries

We proceed to address the problem of determining the chromatic partition function of an unlabeled graph. Let G be a graph with vertex set $V(G) = \{x_1, x_2, ..., x_p\}$ and let $\Gamma(G)$ denote the automorphism group of G. We denote by $\Gamma_I(G)$ the subset of $\Gamma(G)$ in which every orbit consists solely of independent vertices. We denote by $\Lambda(G; \Gamma(G))$ the chromatic partition function for the unlabeled graph underlying the labeled graph G. Then the following is an immediate consequence of the restricted form of Burnside's Lemma, which we state below. We shall adopt the notation of Harary and Palmer [10]. Let A be a permutation group with object set $X = \{1, 2, 3, ..., n\}$. Then $x,y \in X$ are called A–equivalent, or similar, if there is a permutation $\alpha \in A$ such that $\alpha x = y$. This is easily seen to be an equivalence relation on the set X and the equivalence classes are called the orbits of A. Let $N(A)$ denote the number of orbits of A. We often restrict A to a subset Y of X where Y is a union of orbits of A, and we denote by $A|Y$ the set of permutations on Y obtained by restricting those of A to Y. We denote by $j_1(\alpha)$, $\alpha \in A$, the number of elements of X fixed by α. Similarly for $\beta \in A|Y$, we denote by $j_1(\beta|Y)$ the number of elements of Y fixed by β. We now state

Burnside's lemma. *If A is a permutation group acting on some object set X, then the number of orbits, $N(A)$ of A is given by* $N(A) = \frac{1}{|A|}\sum_{\alpha\in A} j_1(\alpha)$. *Similarly, if Y is a subset of X consisting of the union of certain orbits of A, then* $N(A/Y) = \frac{1}{|A|}\sum_{\alpha\in A} j_1(\alpha/Y)$.

Section 2.2 The Unlabeled Chromatic Partition Function

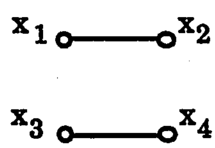
We are now ready to state the next result which allows us to compute the unlabeled chromatic partition function from the labeled case.

Theorem 2.1 *Let G be graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_p\}$. Then*

$$\Lambda(G;\Gamma(G)) = \frac{1}{|\Gamma(G)|}\sum_{\alpha\in\Gamma_I(G)} \Lambda(G)\cup X_\alpha .$$

Before presenting the proof of the above result we shall illustrate it with an example.

Let G:



Then $\Gamma(G) = \{(1)(2)(3)(4), (12)(3)(4), (1)(2)(34), (23)(14), (13)(24), (12)(34), (1324), (1423)\}$

and $\Gamma_I(G) = \{(1)(2)(3)(4), (23)(14), (13)(24)\}$

$$\Lambda(G) = X_1X_2X_3X_4 - X_1X_2X_{34} - X_{12}X_3X_4 + X_{12}X_{34}$$

$$\Lambda(G;\Gamma(G)) = \frac{1}{8}(\Lambda(G) + \Lambda(G)\cup X_{23}X_{14} + \Lambda(G)\cup X_{13}X_{24})$$

$$= \frac{1}{8}(X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{12}X_{34} - 2X_{1234} - X_1X_2X_{34} + X_{23}X_{14} + X_{13}X_{24})$$

Figure 2.1

We now present the proof of the theorem.

Proof. Let τ be a proper assignment of colors to the vertices of G, where the colors are natural numbers. We denote the resulting colored graph by G_τ. Let **G** be the collection of all such colored graphs. We let $\Gamma(G)$ act on **G** as follows: if $\alpha \in \Gamma(G)$ then $\alpha(G_\tau) = G_\gamma$ where γ is the coloring of G obtained by letting α permute the vertices of G but fixing the colors. We now apply Burnside's Lemma to count the number of equivalent classes of **G** under $\Gamma(G)$. If $\alpha \in \Gamma(G) - \Gamma_I(G)$, however, then improper colorings of G may result under the action of α, so we restrict $\Gamma(G)$ to $\Gamma_I(G)$ and apply the restricted form of Burnside's Lemma. If $\alpha \in \Gamma_I(G)$, then we count the fixed elements of **G** by observing that $\alpha(G_\tau) = G_\tau$ if and only if τ assigns colors which are constant on the vertex orbits of α. Therefore we see that $j_1(\alpha)$ can be replaced by $\Lambda(G) \cup X_\alpha$. Hence applying Burnside's Lemma we have

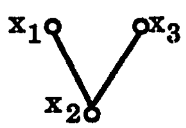
$$\Lambda(G;\Gamma(G)) = \frac{1}{|\Gamma(G)|} \sum_{\alpha \in \Gamma_I(G)} \Lambda(G) \cup X_\alpha . \quad \square$$

Section 2.3 The Ordinary Chromatic Polynomial Derived From the Chromatic Partition Function

We will now see how this new concept relates to the two areas from which it was derived, namely the chromatic theory of graphs and the theory of partitions in number theory. Let G be a graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_p\}$. We denote by $\chi(G;\lambda)$, the chromatic polynomial of a graph G, where $\chi(G,\lambda)$ is the number of of distinct proper colorings of the labeled graph G from a palette of λ colors. The chromatic polynomial is related to the chromatic partition function in the following manner: if $\Lambda(G;x_1,x_2,\ldots,x_p) = \sum_\alpha X_\alpha$ then $\chi(G;\lambda) = \sum_\alpha \lambda^{|\alpha|}$ where

$|\alpha|$ is the number of cells in α. For example $\Lambda(K_2) = X_1X_2 - X_{12}$ and so we have $\chi(K_2;\lambda) = \lambda^2 - \lambda$. We can easily verify this transformation since each α corresponds to some spanning subgraph H of G and $|\alpha|$ corresponds to the number of connected components of H and we assign the λ colors freely to these connected components and apply the inclusion–exclusion principle. We will denote by $X_i^n = x_i + x_i^2 + x_i^3 + ...+x_i^n = \frac{x_i - x_i^{n+1}}{1-x_i}$ and $X_{ij}^n = x_i x_j + ...+x_i^n x_j^n$ and so on. If we substitute each X_i by X_i^λ, X_{ij} by X_{ij}^λ and so on, in $\Lambda(G)$, then we obtain the $\chi(G;\lambda)$ proper colorings of G with λ colors. We illustrate these concepts in the following example

Let G:



then $\Lambda(G;\lambda) = X_1X_2X_3 - X_{12}X_3 + X_{123} - X_{23}X_1$

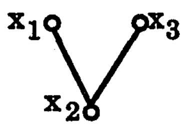
and $\chi(G;\lambda) = \lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda-1)^2$

so that for $\lambda=3$, $\chi(G;3) = 12$ and we obtain the 12 proper colorings of G in

$$X_1^3X_2^3X_3^3 - X_{12}^3X_3^3 + X_{123}^3 - X_{23}^3X_1^3 = x_1^3x_2^2x_3^3 + x_1^3x_2^2x_3 + x_1^3x_2x_3^3$$
$$x_1^3x_2x_3^2 + x_1^2x_2^3x_3^2 + x_1^2x_2^3x_3$$
$$x_1^2x_2x_3^3 + x_1^2x_2x_3^2 + x_1x_2^3x_3^2$$
$$x_1x_2^2x_3^3 + x_1x_2^3x_3 + x_1x_2^2x_3.$$

Figure 2.2

Let $\chi(G;\Gamma(G);\lambda)$ denote the chromatic polynomial for the unlabeled graph underlying the labeled graph G. Then we obtain $\chi(G;\Gamma(G);\lambda)$ from $\Lambda(G;\Gamma(G))$ by replacing each X_α by $\lambda^{|\alpha|}$ as in the labeled case. For example

Let G: 

$$\text{then } \Lambda(G;\Gamma(G)) = \tfrac{1}{2}(X_1X_2X_3 - X_{12}X_3 + X_{13}X_2) - X_{23}X_1$$

$$\text{so that } \chi(G;\Gamma(G);\lambda) = \tfrac{1}{2}(\lambda^3 - \lambda^2)$$

Figure 2.3

The theory of partitions has been an extensively studied field of number theory. The chromatic partition function allows us to compute the generating function for partitions with specified properties. We will now investigate the chromatic partition functions for various families of graphs and interpret the resultant partition generating function obtained by simply replacing each x_i with an x. For example if $\Lambda(\bar{K}_p;\Gamma(\bar{K}_p);x,x,\ldots,x) = a(x) = \sum a_nx^n$ then a_n is the number of partitions of n into exactly p parts. On the other hand, if $\Lambda(K_p;\Gamma(K_p);x,x,\ldots,x) = b(x) = \sum b_nx^n$ then b_n is the number of partitions of n into exactly p distinct parts. These two families of graphs are the extremes and in each case it would be easier to obtain the partition generating function directly. Specifically,

$$a(x) = \frac{x^p}{\prod_{n=1}^{p}(1-x^n)} \quad \text{and} \quad b(x) = \frac{x^{p(p+1)/2}}{\prod_{n=1}^{p}(1-x^n)}\,.$$

However, if we now consider the intermediate families of graphs we get partition functions with some interesting properties. For example if $\Lambda(P_m; x, x, \ldots, x) = c(x) = \sum c_nx^n$, then c_n is the number of sequences of length m and weight n with no 'levels'. If $\Lambda(K_{m,n}; \Gamma(K_{m,n}); x, x, ..., x) = e(x) = \sum e_q x^q$, then $e(x)$ is the generating function for the number of partitions into two sets of size m and n with no repeats between the sets. We will derive general partition identities in the next chapter.

Section 2.4 Specified Minimum Color Difference

We will conclude this chapter with a discussion of chromatic partitions where the color difference between adjacent vertices is at least some positive integer k. The next example illustrates the various partitions of n=7 on $K_{2,1}$ with k=2, i.e. color difference at least 2.

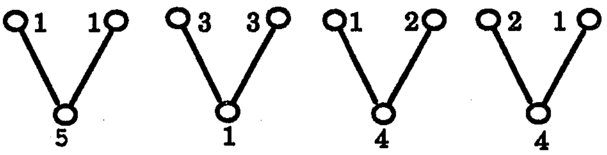


Figure 2.4

We define $\chi^k(G)$ to be the minimum number of colors required to properly color the graph G with color difference k between adjacent vertices. We now present a theorem that relates $\chi^k(G)$ to $\chi(G)$, the ordinary chromatic number of a graph. This result has been shown in Cozzens and Roberts [5].

Theorem 2.2 $\chi^k(G) = 1 + k(\chi(G) - 1)$.

Proof. It is clear that any graph G can be k–colored with at most $1 + k(\chi(G) - 1)$ colors since for any coloring of G one can obtain a proper k–coloring by replacing color i with color $1+k(i-1)$. Therefore suppose G is a graph with $\chi^k(G) \leq k(\chi(G) - 1)$. Let τ be a proper k–coloring of G with $\chi^k(G)$ colors. Now any color t_1, where $1 < t_1 < k+1$ can be replaced by the color 1 since any vertex colored t_1 in G is not adjacent to any vertex colored 1. Proceeding to the next interval, any color t_2 where $k+1 < t_2 < 2k+1$ can be altered to the color $k+1$. Continuing in this manner we obtain a k–coloring of G with at most $\chi(G) - 1$ colors $1, k+1, \ldots, k(\chi(G) - 2) + 1$. We now obtain a proper coloring of G with at most $\chi(G) - 1$ colors by replacing the color $tk+1$ with $t+1$, which is obviously a contradiction. ❑

CHAPTER III

PARTITION IDENTITIES

Section 3.1 Introduction

In this chapter we will investigate the partition identities obtained from the chromatic partition function. We do this by replacing each x_i with an x in $\Lambda(G; x_1, x_2, ..., x_p)$ or $\Lambda(G; \Gamma(G); x_1, x_2, ..., x_p)$ to obtain a generating function for partitions whose properties depend upon the structure of the graph. For example $\Lambda(K_p; \Gamma(K_p); x, x, ..., x)$ is the generating function for partitions with exactly p distinct parts. We will investigate various families of graphs.

Section 3.2 Bipartitions

We begin with the partitions obtained from the family of complete bipartite graphs. Let $P_{m,n}(x) = \Lambda(K_{m,n}; \Gamma(K_{m,n}); x, x, ..., x)$ denote the generating function for partitions of an integer into $m+n$ parts with the parts separated into two sets S_m, S_n such that $|S_m| = m$ and $|S_n| = n$ and further more $S_m \cap S_n = \emptyset$, i.e. repeats are allowed within each set S_m, S_n, but no part is repeated between the sets. We shall call these bipartite partitions or, to be brief, ***bipartitions***. For example, in the analysis of $P_{4,5}(x)$, 1 3 4 4 : 2 2 5 5 8 is a legal bipartition but 1 2 4 4 : 2 3 5 5 8 is *not* legal because of the repeated 2. The generating function for $P_{m,n}(x)$ reduces to a remarkably simple formula.

Theorem 3.1 *Let m and n be positive integers. Then*

for $m \neq n$,
$$P_{m,n}(x) = \frac{(x^{m+2n}+x^{n+2m})\prod_{i=1}^{m+n-1}(1+x^i)}{\prod_{i=1}^{m}(1-x^{2i})\prod_{i=1}^{n}(1-x^{2i})}$$

and
$$P_{m,m}(x) = \frac{x^{3m}\prod_{i=1}^{2m-1}(1+x^i)}{\prod_{i=1}^{m}(1-x^{2i})^2}$$

Proof. We will prove this result by demonstrating a one-to-one correspondence between bipartitions and the terms in the above formula. We shall consider two cases.

CASE 1. $m \neq n$.

Let $S_{m,n}$ be the set of all bipartitions with m,n parts. We now prescribe an algorithm for obtaining the unique term in the expansion of the formula that corresponds to an arbitrary element of $\alpha = [a_1\ a_2\ \dots\ a_m : b_1\ b_2 \dots b_n] \in S_{m,n}$ where $1 \leq a_1 \leq a_2 \dots \leq a_m$ and $1 \leq b_1 \leq b_2 \dots \leq b_n$ and no $a_i = b_j$. Each step of the algorithm contributes a term T_i within the formula.

Algorithm :

Step 1 :

If $a_1 < b_1$ and $a_1 \equiv 0 \bmod 2$, then set $T_0 = x^{2m+n}$.
If $a_1 < b_1$ and $a_1 \equiv 1 \bmod 2$, then set $T_0 = x^{m+2n}$.
If $a_1 > b_1$ and $b_1 \equiv 0 \bmod 2$, then set $T_0 = x^{m+2n}$.
If $a_1 > b_1$ and $b_1 \equiv 1 \bmod 2$, then set $T_0 = x^{2m+n}$.

We modify sequence α as follows:

$$\alpha = [a_1\text{-}2,\ a_2\text{-}2,\ \ldots,\ a_m\text{-}2 : b_1\text{-}1,\ b_2\text{-}1, \ldots,\ b_n\text{-}1] \text{ if } T_0 = x^{2m+n}\ ,$$

or
$$\alpha = [a_1\text{-}1,\ a_2\text{-}1,\ \ldots,\ a_m\text{-}1 : b_1\text{-}2,\ b_2\text{-}2, \ldots,\ b_n\text{-}2] \text{ if } T_0 = x^{m+2n}\ .$$

Of course α may now have some equal terms on opposite sides.

Step 2 :

For $i = 1$ to $m+n-1$, examine the $i+1^{st}$ smallest part in α (ignoring the a:b distinction). If it is an even integer leave α unchanged and set $T_{m+n-i} = 1$. But if it is odd, decrease it and every larger part by 1 and set $T_{m+n-i} = x^{m+n-i}$. That is T_{m+n-i} counts the effect of reducing the $m+n-i$ largest parts by one each.

Step 3 :

Now α has only even parts that correspond to a pair of even partitions F_m and F_n.

Let us illustrate this algorithm with a sequence in $S_{8,9}$. We only list T_i's that are not equal to 1.

	m = 8								:	n = 9								
For $\alpha =$	[2	2	9	9	10	11	12	14	:	7	7	8	8	13	15	16	17	17]
$T_0 = x^{2m+n}$	0	0	7	7	8	9	10	12	:	6	6	7	7	12	14	15	16	16
$T_{13} = x^{13}$			6	6	7	8	9	11	:			6	6	11	13	14	15	15
$T_9 = x^9$					6	7	9	10	:					10	12	13	14	14
$T_8 = x^8$						6	7	9	:					9	11	12	13	13
$T_7 = x^7$							6	8	:					8	10	11	12	12
$T_3 = x^3$									:							10	11	11
$T_2 = x^2$									:								10	10
	0	0	6	6	6	6	6	8	:	6	6	6	6	8	10	10	10	10

F_m:F_n $\qquad (x^6)^5\ (x^8)^1\ :\ (x^6)^4\ (x^8)^1\ (x^{10})^4$

We reverse the procedure to obtain the original sequence. The algorithm for the reversal process is as follows :

Algorithm:

Step 1:

Start with an even sequence (0's permitted) in non-decreasing order in each cell. If $T_0 = x^{2m+n}$, add 2 to the m-cell and 1 to the n-cell, or vice-versa if $T_0 = x^{m+2n}$.

Step 2:

For $i = n+m-1$ to 1, if $T_i = x^i$ add 1 to the i largest parts to obtain a new sequence. If $T_i = 1$ there is no change, proceed to the next smaller value of i.

We illustrate this idea below with an example.

					m				:					n				
Start with	0	0	6	6	6	6	6	8	:	6	6	6	6	8	10	10	10	10
$T_0 = x^{2m+n}$	2	2	8	8	8	8	8	10	:	7	7	7	7	9	11	11	11	11
$T_{13} = x^{13}$			9	9	9	9	9	11	:			8	8	10	12	12	12	12
$T_9 = x^9$					10	10	10	12	:					11	13	13	13	13
$T_8 = x^8$						11	11	13	:					12	14	14	14	14
$T_7 = x^7$							12	14	:					13	15	15	15	15
$T_3 = x^3$									:							16	16	16
$T_2 = x^2$									:								17	17

To give $\alpha = [2\ 2\ 9\ 9\ 10\ 11\ 12\ 14 : 7\ 7\ 8\ 8\ 13\ 15\ 16\ 17\ 17]$

Observe that, for each bipartition we start with, we obtain a set of terms unique to that sequence. Conversely, starting with a set of terms from the formula we construct a unique $S_m{:}S_n$ bipartition. This establishes a one-to-one correspondence. It remains to produce the formula in the theorem.

The factor $(x^{m+2n} + x^{2m+n})$ in the formula allows for the possible choices of T_0. The factor $\prod_{i=1}^{m+n-1} 1+x^i$ accounts for the sequence of terms $T_1, T_2, \ldots, T_{n+m-i}$. Each T_i represents the selection of either x^i or 1 from the i^{th} factor in the product. Finally, F_m refers to the set of m even parts and is counted by $\prod_{i=1}^{m}(1-x^{2i})^{-1}$. Similarly F_n is counted by $\prod_{i=1}^{n}(1-x^{2i})^{-1}$. This completes the proof of the case 1 in which $m \neq n$.

CASE 2. m=n

We obtain the claimed formula by observing that every sequence of $S_{m,m}$ is counted twice in Case 1 if m=n. Therefore We simply set $n = m$ and divide the result by 2 to obtain the correct formula. ❑

Section 3.3 Labeled Paths

Let P_m denote the path with m vertices, having $\Lambda(P_m; x_1, x_2, ..., x_m)$ as its chromatic partition function. Let $P_m(x)$ denote the polynomial obtained from the chromatic partition function by replacing each x_i with an x, i.e. $P_m(x) = \Lambda(P_m; x, x, ..., x)$. Suppose $P_m(x) = \sum a_k x^k$, then a_k enumerates the number of ways to partition the integer k into sequences of length m of positive integers with no repeats allowed between adjacent elements. For example if m=3 then $a_7 = 9$ since there are exactly nine sequences 151; 124; 142; 214; 241; 412; 421; 232; 313. In this section we obtain a surprising formula for $P_m(x)$. We need a few definitions first...

Let m be a positive integer, then define A_m to be the set of ordered partitions of m in which there is at most one part of size 1, and furthermore if 1 does appear, then it must be placed at the beginning of the ordered partition. For example if m=6 then $A_6 = \{6, 15, 24, 42, 33, 123, 132, 222\}$. Observe that 51, 213, 312, 231, 321 are all forbidden by the leading 1 condition and 141, 114 (and many others) are forbidden because they repeat the part 1. If k is a positive integer then we denote by Π_k the polynomial $(1-x)(1-x^2)(1-x^3)...(1-x^k)$ i.e.

$$\Pi_1 = (1-x)$$

$$\Pi_2 = (1-x)(1-x^2)$$

$$\Pi_3 = (1-x)(1-x^2)(1-x^3)$$

$$\vdots$$

For each $\alpha \in A_m$ we define

$$\Pi_\alpha = \prod_{t \in \alpha} (\Pi_t) .$$

For example for $\alpha \in A_6$ given by $\alpha = 132$ we find

$$\Pi_\alpha = \Pi_1\Pi_3\Pi_2 = (1-x)(1-x)(1-x^2)(1-x^3)(1-x)(1-x^2) .$$

For each $\alpha \in A_m$, let α_f denote the first element of α and let α_r denote the rest of α. In the example above, $\alpha_f = 1$ and $\alpha_r = 32$.

We denote by
$$t(\alpha) = \begin{cases} \alpha_f & \text{if } \alpha_r = \varnothing \\ \alpha_f \cdot \prod_{k \in \alpha_r}(k-1) & \text{otherwise} \end{cases}$$

For example $t(132) = 1{\cdot}2{\cdot}1 = 2$

and $t(3622) = 3{\cdot}5{\cdot}1{\cdot}1 = 15.$

Define $d(\alpha)$ to be the degree of the polynomial Π_α. Therefore

$$\begin{aligned} d(\alpha) &= \sum_{k \in \alpha} d(\Pi_\alpha) \\ &= \sum_{k \in \alpha} (1+2+\ldots+k) \\ &= \sum_{k \in \alpha} \binom{k+1}{2} . \end{aligned}$$

Thus $d(132) = 1+6+3 = 10$ and $d(3622) = 6+21+3+3 = 33$.

We are now ready to present the next result which provides a surprisingly compact form for $P_m(x)$.

Theorem 3.2 *Let m be a positive integer, then*

$$P_m(x) = \sum_{\alpha \in A_m} \frac{t(\alpha)\, x^{d(\alpha)}}{\Pi_\alpha}. \tag{2}$$

Before presenting the proof we will illustrate with an example. This will be instructive since the proof is a construction of a 1:1 correspondence between each "chromatic sequence" and each term in the above expression.

As we saw earlier, $A_6 = \{6, 15, 24, 42, 33, 123, 132, 222\}$.

Then according to the above expression we obtain...

$$P_6(x) = \frac{6x^{21}}{\Pi_6} + \frac{4x^{16}}{\Pi_1\Pi_5} + \frac{6x^{13}}{\Pi_2\Pi_4} + \frac{4x^{13}}{\Pi_4\Pi_2} + \frac{6x^{12}}{\Pi_3\Pi_3} + \frac{2x^{10}}{\Pi_1\Pi_3\Pi_2} + \frac{2x^{10}}{\Pi_1\Pi_2\Pi_3} + \frac{2x^9}{\Pi_2\Pi_2\Pi_2}$$

$$= \frac{6x^{21}}{\Pi_6} + \frac{4x^{16}}{\Pi_1\Pi_5} + \frac{10x^{13}}{\Pi_2\Pi_4} + \frac{6x^{12}}{\Pi_3\Pi_3} + \frac{4x^{10}}{\Pi_1\Pi_2\Pi_3} + \frac{2x^9}{\Pi_2\Pi_2\Pi_2}$$

In the second line we have collected terms with common denominator, as one would surely do before actually computing the series.

Proof. We establish a one-to-one correspondence between each term in the formula and each sequence, in the manner described below.

Let $\alpha \in A_m$, then corresponding to α there is a "basic partition" composed of cells of decreasing sequences, one cell $k, k-1, k-2, \ldots, 2, 1$ for each $k \in \alpha$, and these cells preserve the order given by α. For example if α=323, the basic sequence is 321 21 321. Observe that the 'weight' of the basic sequence is just the degree $d(\alpha)$ of Π_α i.e. $d(\alpha)$ is the sum of the elements in the basic sequence of α. Also corresponding to each α, is a product of infinite series disguised in the denominator as Π_α. For example

$$\frac{1}{\Pi_3\Pi_2\Pi_3} = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x)(1-x^2)(1-x)(1-x^2)(1-x^3)}$$

$$= (1+x+...)(1+x^2+...)(1+x^3+...)(1+x+...)(1+x^2+..)$$

$$(1+x+...)(1+x^2+..)(1+x^3+..)$$

Now in the expansion of this product, corresponding to each term we will obtain an "augmented basic partition" by piling onto each cell of basic partition the amount specified by that term in a manner that maintains the decreasing pattern on each cell. For example if we select the term $(x^1)^2(x^2)^3(x^3)^1$; $(x^1)^4(x^2)^1$; $(x^1)^3(x^2)^0(x^3)^1$, from the expansion above, then to the basic partition

		3	2	1		2	1		3	2	1
we add	$(x^3)^1$	1	1	1				$(x^3)^1$	1	1	1
	$(x^2)^3$	3	3		$(x^2)^1$	1	1	$(x^2)^0$	0	0	
	$(x^1)^2$	2			$(x^1)^4$	4		$(x^1)^3$	3		
to obtain		9	6	2		7	2		7	3	2

as the augmented basic partition.

To each augmented partition obtained from α, there corresponds exactly $t(\alpha)$ chromatic sequences. We obtain these $t(\alpha)$ sequences by rotating each cell of the augmented basic partition allowing the first cell complete freedom to be rotated to any position, but all subsequent cells have one forbidden position. For the k^{th} cell $(k>1)$ we shall associate a sequence of half open intervals. For example the cell 8753 has four intervals : [3,5); [5,7); [7,8); and [8,3). Here the unique improper interval is viewed as the complement $N-[3,8)$ so that the full set of four, partitions the natural numbers into intervals. In this example, the four intervals are just {3,4}, {5,6}, {7}, and everything else. As we assemble all possible partitions that can be obtained from a single augmented sequence, we rotate each cell subject to the constraint that the k^{th} cell can be rotated to any position except one, namely the one in which the last entry of the k-1st cell is at least as large as the 1st entry in the rotated kth cell, but also smaller than the last entry. For example, if the k-1st cell ends with a 5 or a 6, and the kth cell is obtained from 8753, we permit rotations giving 8753, 7538, 3875 but we forbid 5387. In effect, the forbidden pattern would either create a repeated value on neighbors or else produce ambiguity in the cell size. After all 55387 is forbidden, and 65387 could be a cell of size 5. Similarly if 2 or 9 both lie in the interval [8,3) of the sequence 8753, we avoid that interval to obtain 2,7538; 2,5387; and 2,3875 or 9,7538; 9,5387; and 9,3875. Therefore, for each augmented basic pattern, we can multiply the number of permitted rotations on each cell to obtain $t(\alpha)$ as the number of sequences that can be generated from that particular augmented sequence.

In our previous example with 962 72 732 as the augmented basic partition obtained from $\alpha = 323$, we compute $t(\alpha)=3\cdot1\cdot2=6$ sequences, namely:

962 72 732
962 72 327
296 72 732
296 72 327
629 27 327
629 27 273.

To show that this construction is one-to-one and onto we consider an arbitrary sequence of length m and determine its unique augmented basic partition and hence the α from which it must be derived. Let $\gamma = a_1, a_2,\ldots,a_m$ with $a_i \neq a_{i+1}$ (for all i). Partition the sequence γ into decreasing segments as we move from left to right. An example will serve to illustrate here. Let γ = 4 12 10 3 12 10 4 2 8 5 4 3. Decomposing γ into decreasing segments we obtain the four cells 4 12 10 3 12 10 4 2 8 5 4 3.

The right-most cell will include all of the last segment in γ and perhaps a portion of the penultimate segment. How much?. Identify the right-most element a_j in the penultimate segment that satisfies $a_j \geq a_m$ (the last element in γ). Then the right-most cell contains a_{j+1} through a_m. If it happens that $a_j \geq a_m$ for all a_j in the penultimate segment then the rightmost cell is comprised of the final two segments. Remove the final cell and repeat the procedure to find the next rightmost cell, and so on, until γ is exhausted. We illustrate this procedure with an example below.

Let $\gamma = \underline{2}\ \ \underline{10\ 9\ 7\ 2\ 1}\ \ \underline{4\ 3\ 2}\ \ \underline{5\ 4}\ \ \underline{10\ 7\ 6}$ be a chromatic sequence of length fourteen. Then the rightmost cell of γ is 5 4 10 7 6 to obtain a reduced sequence $\gamma' = \underline{2}\ \ \underline{10\ 9\ 7\ 2\ 1}\ \ \underline{4\ 3\ 2}$. The rightmost cell of γ' is 1 4 3 2. To obtain $\gamma'' = \underline{2}\ \ \underline{10\ 9\ 7\ 2}$. Then the rightmost cell of γ'' is 10 9 7 2. Therefore the cells of γ are (2) (10 9 7 2) (1 4 3 2) (5 4 10 7 6) and the augmented basic partition corresponding to γ is (2) (10 9 7 2) (4 3 2 1) (10 7 6 5 4) obtained from $\alpha = 1445$.

To demonstrate that the procedure is one-to-one, we must establish that γ can be obtained from a unique $\alpha \in A_m$. Suppose to the contrary that γ can be obtained from some other $\beta \in A_m$. Then the cell structure of β must differ from that of α in at least one position. Then at that position the sequence γ must violate the forbidden rotation rule. For example for γ above one can ask why couldn't the cell structure of γ be (2)(10 9 7 2 1)(4 3 2)(5 4 10 7 6) ?. On closer inspection, we notice that the 2nd cell ends in a 1, and the only allowed rotations are 243 and 324, thus 432 is a forbidden rotation of the 3rd cell if the second ends in a 1. Therefore we have a one-to-one correspondence between each term in (2) and each "chromatic sequence". This completes the proof. ❑

We present in the following page a table of these partition generating functions and in Appendix A a table of values computed to x^{45} for paths.

Table 3.1

Partition Functions for Labeled Paths

P2: $\frac{2x^3}{\Pi_2}$

P3: $\frac{3x^6}{\Pi_3} + \frac{x^4}{\Pi_1\Pi_2}$

P4: $\frac{4x^{10}}{\Pi_4} + \frac{2x^7}{\Pi_1\Pi_3} + \frac{2x^6}{\Pi_2\Pi_2}$

P5: $\frac{5x^{15}}{\Pi_5} + \frac{3x^{11}}{\Pi_1\Pi_4} + \frac{7x^9}{\Pi_2\Pi_3} + \frac{x^7}{\Pi_1\Pi_2\Pi_2}$

P6: $\frac{6x^{21}}{\Pi_6} + \frac{4x^{16}}{\Pi_1\Pi_5} + \frac{10x^{13}}{\Pi_2\Pi_4} + \frac{6x^{12}}{\Pi_3\Pi_3} + \frac{4x^{10}}{\Pi_1\Pi_2\Pi_3} + \frac{2x^9}{\Pi_2\Pi_2\Pi_2}$

P7: $\frac{7x^{28}}{\Pi_7} + \frac{5x^{22}}{\Pi_1\Pi_6} + \frac{13x^{18}}{\Pi_2\Pi_5} + \frac{17x^{16}}{\Pi_3\Pi_4} + \frac{6x^{14}}{\Pi_1\Pi_2\Pi_4} + \frac{11x^{12}}{\Pi_2\Pi_2\Pi_3}$

$+ \frac{4x^{13}}{\Pi_1\Pi_3\Pi_3} + \frac{x^{10}}{\Pi_1\Pi_2\Pi_2\Pi_2}$

P8: $\frac{8x^{36}}{\Pi_8} + \frac{6x^{29}}{\Pi_1\Pi_7} + \frac{16x^{24}}{\Pi_2\Pi_6} + \frac{22x^{21}}{\Pi_3\Pi_5} + \frac{12x^{20}}{\Pi_4\Pi_4} + \frac{8x^{19}}{\Pi_1\Pi_2\Pi_5}$

$+ \frac{12x^{17}}{\Pi_1\Pi_3\Pi_4} + \frac{16x^{16}}{\Pi_2\Pi_2\Pi_4} + \frac{20x^{15}}{\Pi_2\Pi_3\Pi_3} + \frac{6x^{13}}{\Pi_1\Pi_2\Pi_2\Pi_3} + \frac{2x^{12}}{\Pi_2\Pi_2\Pi_2\Pi_2}$

Remarks.

We now make some interesting observations of $P_m(x)$ given by the formula in Theorem 3.2.

Observation 3.1 *The sum of the coefficients,*

$$\sum_{\alpha \in A_m} t(\alpha) = 2^{m-1} .$$

Proof. We first note that there are exactly 2^{m-1} ways to choose an odd number of cells from a row of length m. (Choose an arbitrary number from the first m-1 cells. If that is even then we include that last cell to make the total odd. Otherwise exclude the last cell.) We must show how the individual terms $t(\alpha)$ relate to these odd subsets. For each subset, let the even selection mark the beginning of a new cell while each odd choice provides a pointer within that cell. We illustrate this correspondence below, with m=5. The $\sum_{\alpha \in A_m} t(\alpha) = 2^4 = 16$ terms obtained are

(o _ _ _ _)	Π_5
(_ o _ _ _)	Π_5
(_ _ o _ _)	Π_5
(_ _ _ o _)	Π_5
(_ _ _ _ o)	Π_5
(o)(e _ _ o)	$\Pi_1\Pi_4$
(o _)(e _ o)	$\Pi_2\Pi_3$
(o _ _)(e o)	$\Pi_3\Pi_2$
(_ o)(e _ o)	$\Pi_2\Pi_3$
(_ o _)(e o)	$\Pi_3\Pi_2$
(_ _ o)(e o)	$\Pi_3\Pi_2$
(o)(e o _ _)	$\Pi_1\Pi_4$
(o)(e _ o _)	$\Pi_1\Pi_4$
(o _)(e o _)	$\Pi_2\Pi_3$
(_ o)(e o _)	$\Pi_2\Pi_3$
(o)(e o)(e o)	$\Pi_1\Pi_2\Pi_2$

We note that this procedure allows complete freedom for the pointer in the first cell, but each subsequent cell loses one position, and that is exactly how $t(\alpha)$ was derived!.

❑

A second approach to analyzing these 2^{m-1} terms is to make use of generating functions. We observe that the first cell has complete freedom and if it is a cell of size k then there are exactly k rotations allowed for the first cell. Therefore

$$q(x) = x + 2x^2 + 3x^3 + \ldots = \frac{x}{(1-x)^2}$$

counts the number of ways to specify the first cell. For subsequent cells we have $k-1$ choices for each cell of size k. Therefore

$$p(x) = x^2 + 2x^3 + 3x^4 + .. = \frac{x^2}{(1-x)^2}$$

enumerates the ways to specify a single cell. Since there may be 0,1,2,... subsequent cells, we find that all possibilities are counted by

$$q(x)[\,1 + p(x) + p^2(x) + \ldots.] = \frac{q(x)}{1-p(x)} = \frac{x}{1-2x} = x + 2x^2 + 4x^3 + \ldots = \sum_m 2^{m-1}x^m$$

Thus $\sum_{\alpha \in A_m} t(\alpha) = 2^{m-1}$. ❑

We make a second observation about the number of terms that appear in the formula. Let F_m denote the m^{th} Fibonacci number i.e. $F_1=1$; $F_2=1$; $F_3=2$; $F_4=3$; etc. The number of terms in Theorem 3.2 is given by this sequence. We state this as...

Observation 3.2 *The number of terms,* $|A_m| = F_m$.

Proof. We show that $|A_{m+2}| = |A_{m+1}| + |A_m|$ and that $|A_1|=1$, $|A_2|=1$. Since $A_1=\{1\}$ and $A_2=\{2\}$, it follows that $|A_1|=1$, $|A_2|=1$. To obtain A_{m+2} from A_{m+1} and A_m we add to each element $\alpha \in A_m$, one to the first member and attach a one to the beginning and to each $\beta \in A_{m+1}$ we just add one to the first element and take the union of the two new sets. In general,

$\alpha \in A_m$, $\alpha=\alpha_f\alpha_r$ becomes $1,(\alpha_f+1),\alpha_r$

and $\beta \in A_{m+1}$, $\beta=\beta_f\beta_r$ becomes $(\beta_f+1),\beta_r$.

In this manner we obtain the elements of A_{m+2}, with the terms beginning with a 1 derived from A_m and the rest derived from A_{m+1}. This is exactly how the Fibonacci numbers are defined. ❑

Let A^*_m denote the set of unordered partitions of n in which the part 1 appears at most once. Thus $|A^*_m|$ is the number of terms obtained after collecting terms with common denominators of the formula in Theorem 3.2. The size of A^*_m can be expressed in terms of the classical partition numbers p(n) already introduced in Chapter 1.

Observation 3.3 $|A^*_m| = p(m) - p(m-2)$.

Proof. Recall that $|A^*_m|$ is the number of partitions of m with the 1 appearing at most once. Then

$$\sum_{m=1}^{\infty} |A^*_m| x^m = (1+x)\prod_{i=2}^{m}(1+x^i+x^{2i}+\ldots)$$
$$= \frac{(1+x)(1-x)}{\Pi_m}$$
$$= \frac{1}{\Pi_m} - \frac{x^2}{\Pi_m}$$
$$= \sum_{m=1}^{\infty} p(m)x^m - \sum_{m=1}^{\infty} p(m)x^{m+2}$$
$$= \sum_{m=1}^{\infty} (p(m)-p(m-2))x^m.$$

So it follows immediately that $|A^*_m| = p(m) - p(m-2)$. ❑

Section 3.4 Unlabeled Paths

Let P_m denote a path with m vertices, as in the last section. Then $\Lambda(P_m ; \Gamma(P_m); x)$ denotes the unlabeled chromatic partition function for the path. For brevity, let $P^*_m(x) = \Lambda(P_m ; \Gamma(P_m); x) = \sum a^*_k x^k$. Then a^*_k enumerates the number of ways to partition the integer k into unlabeled sequences of length m, unlabeled in the sense that two sequences are equivalent if one can be written in the reverse order of the other. For example 37143 is equivalent to 34173 and is therefore included only once in the count for unlabeled sequences of length 5 and weight 18. Thus, for m=5, it contributes only 1 to a^*_{18} but 2 to a_{18}. We will now present a result that provides us with a formula for $P^*_m(x)$ in terms of $P_k(x)$ for k<m. We shall denote the empty graph [10] as P_0 and take $P_0(x)$ to be identically equal to 1. This result is a direct application of Theorem 1.1 and Theorem 1.6.

Theorem 3.3 *The unlabeled path partition function is given by,*

for m even: $P^*_m(x) = \frac{1}{2}P_m(x)$

for m odd: $P^*_m(x) = \frac{1}{2}\{ P_m(x) + \sum_{k=0}^{r=(m-1)/2} [(-1)^k P_{r-k}(x^2)\frac{x^{2k+1}}{1-x^{2k+1}}] \}$

Proof. From Theorem 1.6, we obtain

$$P^*_m(x) = \Lambda(P_m ; \Gamma(P_m); x) = \frac{1}{|\Gamma(P_m)|} \sum_{\alpha\in\Gamma_1(P_m)} \Lambda(P_m)UX_\alpha$$

where we recall that $\Gamma(P_m)$ is the symmetry group of P_m and $\Gamma_1(P_m)$ is the subset of $\Gamma(P_m)$ in which every orbit consists of independent vertices. Therefore we consider two cases:

CASE 1. m is even.

Then $|\Gamma(P_m)| = 2$ and $\Gamma_1(P_m)$ is just the set containing the identity element, e. Therefore we immediately obtain

$$P^*_m(x) = \frac{1}{2}\Lambda(P_m)UX_e = \frac{1}{2}\Lambda(P_m) = \frac{1}{2}P_m(x).$$

CASE 2. m is odd.

Then $|\Gamma(P_m)| = |\Gamma_1(P_m)| = 2$.

$$\text{Therefore } P^*_m(x) = \frac{1}{2} \sum_{\alpha\in\Gamma_1(P_m)} \Lambda(P_m)UX_\alpha$$
$$= \frac{1}{2}(\Lambda(P_m)UX_e + \Lambda(P_m)UX_\beta)$$

where β consists of 1 fixed element and $\frac{m-1}{2}$ transpositions. We will now obtain a simple expression for $\Lambda(P_m)UX_\beta$. Let $P_{k,r}$ denote the path of length r where the first vertex has weight k and the remaining r vertices have weight 2. It is easy to see that if a vertex x_i has weight s then in the chromatic partition function one replaces x_i by x_i^s. Now observe that $\Lambda(P_m)UX_\beta = \Lambda(P_{1,(m-1)/2})$. Also it is an easy application of Theorem 1.1 to see that

$$\Lambda(P_{k,r}) = \frac{x^k}{1-x^k} P_r(x^2) - \Lambda(P_{k+2,r-1}). \tag{3}$$

For example

$$\begin{aligned}\Lambda(P_{1,4}) &= \frac{x}{1-x} P_4(x^2) - \Lambda(P_{3,3}) \\ &= \frac{x}{1-x} P_4(x^2) - \frac{x^3}{1-x^3} P_3(x^2) + \Lambda(P_{5,2}).\end{aligned}$$

After several applications of (3) we observe that

$$\Lambda(P_{1,4}) = \sum_{k=0}^{4} (-1)^k \frac{x^{2k+1}}{1-x^{2k+1}} P_{4-k}(x^2).$$

In the general case we see that

$$\Lambda(P_{1,r}) = \sum_{k=0}^{r} (-1)^k \frac{x^{2k+1}}{1-x^{2k+1}} P_{r-k}(x^2). \quad \square$$

This result is effectively an inclusion-exclusion result and is therefore not very computationally efficient. We now present a second expression for $P^*_m(x)$ when m is odd. This result is obtained by manipulating the generating function for $P^*_{(m+1)/2}(x)$. We need to define two new polynomials. Let k be a positive integer, then we define $\Pi^*_k := \prod_{i=1}^{k}(1-x^{2i})$ and for $k = 0$, Π^*_0 is identically equal to 1. Also for k a positive integer we define

$$w_k = \sum_{p=1}^{k} \left(\frac{\prod_{q=p}^{k}(1-x^{2q-1})^{-1}}{x^{k-p+1}\Pi^*_{p-1}} \right).$$

For example

$$\Pi^*_4 = (1-x^2)(1-x^4)(1-x^6)(1-x^8)$$

and $$w_4 = \frac{1}{x^4(1-x)(1-x^3)(1-x^5)(1-x^7)} + \frac{1}{x^3(1-x^2)(1-x^3)(1-x^5)(1-x^7)} + \frac{1}{x^2(1-x^2)(1-x^4)(1-x^5)(1-x^7)} + \frac{1}{x(1-x^2)(1-x^4)(1-x^6)(1-x^7)} .$$

We are now ready to present the next theorem.

Theorem 3.4 *For m a positive odd integer,*

$$P^*_m(x) = \frac{1}{2}\left\{ P_m(x) + \sum_{\alpha \in A_{(m+1)/2}} \frac{w_{\alpha_f}\, t(\alpha)\, x^{2d(\alpha)}}{\alpha_f \Pi^*_{\alpha_r}} \right\},$$

where we recall that A_r *is the set of ordered partitions of* r *defined in Section 3.2,* $d(\alpha)$ *is the degree of* Π_α*,* $t(\alpha)$ *is the special product of* α*, and* $\Pi^*_{\alpha_f}$ *is defined similar to* Π_{α_f} *in Section 3.2.*

Sketch of Proof. We will illustrate the proof with an example. Let us compute $P^*_9(x)$. By Theorem 1.6, $P^*_9(x) = \frac{1}{2}(P_9(x) + \Lambda(P_{1,4}))$. As in the last theorem, we obtain an expression for $\Lambda(P_{1,4})$ by manipulating the result for $P_5(x)$. As we saw in the previous section:

$$P_5(x) = \frac{5x^{15}}{\Pi_5} + \frac{3x^{11}}{\Pi_1\Pi_4} + \frac{4x^9}{\Pi_2\Pi_3} + \frac{3x^9}{\Pi_3\Pi_2} + \frac{x^7}{\Pi_1\Pi_2\Pi_2} .$$

We recall that the coefficient of each term was obtained by rotating the decreasing cycles in a prescribed manner, with the first cell having complete freedom of rotation. We note that $P_{1,4}$ is just P_5 with the first vertex having weight 1 and the remaining having weight 2. To account for the first element of the initial cell having single weight, we replace x by x^2 everywhere except for the first rotated position of the initial cell. We obtain α_f terms for each $\alpha \in A_{(m+1)/2}$ since each initial cell can be rotated to α_f positions. For example the term $\frac{3x^9}{\Pi_3\Pi_2}$ in $P_5(x)$ gives rise to three terms in $P_{1,4}(x)$ namely :

$$\frac{x^{18}}{x^3(1-x)(1-x^3)(1-x^5)\Pi_2^*} + \frac{x^{18}}{x^2(1-x^2)(1-x^3)(1-x^5)\Pi_2^*} + \frac{x^{18}}{x(1-x^2)(1-x^4)(1-x^5)\Pi_2^*}.$$

Observe that the leading factor of x^k in the denominator corresponds to the rotation of the initial cell which places the element of size k in the first position. Of course this means that any piling that involves this element need not not be doubled since the first element of the sequence has weight 1. ❑

The last result provides us with an expression that allows us to compute the result in an efficient manner, even though the terms get complicated. We illustrate the generating function for unlabeled chromatic sequences of length 9.

$$P_9^*(x) = \frac{1}{2}\Big\{P_9(x) + \frac{x^{30}}{x^5(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)}$$

$$+ \frac{x^{30}}{x^4(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^9)} + \frac{x^{30}}{x^3(1-x^2)(1-x^4)(1-x^5)(1-x^7)(1-x^9)}$$

$$+ \frac{x^{30}}{x^2(1-x^2)(1-x^4)(1-x^6)(1-x^7)(1-x^9)} + \frac{x^{30}}{x(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^9)}$$

$$+ \frac{3x^{22}}{x(1-x)\Pi_4^*}$$

$$+ \frac{2x^{18}}{x^2(1-x)(1-x^3)\Pi_3^*} + \frac{2x^{18}}{x(1-x^2)(1-x^3)\Pi_3^*}$$

$$+ \frac{x^{18}}{x^3(1-x)(1-x^3)(1-x^5)\Pi_2^*} + \frac{x^{18}}{x^2(1-x^2)(1-x^3)(1-x^5)\Pi_2^*} + \frac{x^{18}}{x(1-x^2)(1-x^4)(1-x^5)\Pi_2^*}$$

$$+ \frac{x^{14}}{x(1-x)\Pi_2^*\Pi_2^*} \}$$

Section 3.5 Weighted Paths

We now wish to enumerate sequences in which the first two elements are equal and the remaining sequence is chromatic. Let P_m^k denote the path on m vertices with the first vertex having weight k. Then $\Lambda(P_m^2) = P_{m+1}(x) - \frac{x}{1-x}P_m(x)$ follows immediately from Theorem 1.1. We now present a result that is again a modification of the generating function of Theorem 3.2. We need to define a new polynomial. For k a positve integer, let $v_k = \sum_{i=1}^{k} \frac{(1-x)^i x^{k-i+1}}{\Pi_{k+1}}$.

For example
$$v_4 = \frac{x^4}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)} + \frac{x^3}{(1-x)(1-x^3)(1-x^4)(1-x^5)} + \frac{x^2}{(1-x)(1-x^2)(1-x^4)(1-x^5)} + \frac{x}{(1-x)(1-x^2)(1-x^3)(1-x^5)}.$$

We close this section with a theorem which we present without proof. The idea is very similar to the previous result in that instead of avoiding the first position of the rotated initial cell we need to add an extra piling.

Theorem 3.5 *For a path P_m with a single endpoint of weight 2 and the remaining vertices of weight 1,*

$$\Lambda(P_m^2) = \sum_{\alpha \in A_m} \frac{v_{\alpha_f}\, t(\alpha)\, x^{d(\alpha)}}{\alpha_f \, \Pi_{\alpha_r}}.$$

Section 3.6 Labeled Cycles

We will now consider the chromatic partition function for cycles. In this section we will obtain an expression for $\Lambda(C_n)$ in terms of $P_m(x)$ for $m \leq n$. Let $C_n(x) = \Lambda(C_n ; x)$ and let $C_{k,n}$ denote the cycle of length n in which one of the vertices has weight k and the rest have weight 1. Let $C_{k,n}(x) = \Lambda(C_{k,n} ; x)$ and note that $C_n(x) = C_{1,n}(x)$. We first obtain an expression for $C_{k,n}(x)$ in terms of $P_m(x)$ for $m \leq n$.

Theo

$$C_{k,n}(x) = \sum_{j=1}^{n-1} (-1)^{j-1} \frac{x^{k+j-1}}{1-x^{k+j-1}} jP_{n-j}(x) \quad + \quad (-1)^{n-1}(n-1)\frac{x^{k+n-1}}{1-x^{k+n-1}} \quad .$$

$$C_{k,n}(x) = \sum_{j=1}^{n-1} (-1)^{j-1} \frac{x^{k+j-1}}{1-x^{k+j-1}} jP_{n-j}(x) \quad + \quad (-1)^{n-1}(n-1)\frac{x^{k+n-1}}{1-x^{k+n-1}} \quad .$$

Proof. The proof proceeds by induction on n and an application of Theorem 1.1. We recall the definition of P^k_n to be the path of length n where the first vertex has weight k and the rest have weight 1. Then an easy application of Theorem 1.1 gives us a formula for $P^k_n(x)$,

$$P^k_n(x) = \sum_{j=1}^{n} (-1)^{j-1} \left(\frac{x^{k+j-1}}{1-x^{k+j-1}}\right) P_{n-j}(x)$$

It can also easily be shown that for all positive integers k,

$$C_{k,3}(x) = \sum_{j=1}^{2} (-1)^{j-1} \left(\frac{x^{k+j-1}}{1-x^{k+j-1}}\right) \; jP_{3-j}(x) \quad + \quad 2\frac{x^{k+2}}{1-x^{k+2}} \quad .$$

Now suppose that the result holds for $C_{k,n}(x)$, we show that the result holds for $C_{k,n+1}(x)$. By Theorem 1.1, $C_{k,n+1}(x) = P^k_{n+1}(x) - C_{k+1,n}(x)$. Therefore by the induction hypothesis

$$C_{k,n+1}(x) = \sum_{j=1}^{n+1}(-1)^{j-1}\left(\frac{x^{k+j-1}}{1-x^{k+j-1}}\right) P_{n-j+1}(x)$$
$$- \sum_{j=1}^{n-1}(-1)^{j-1}\left(\frac{x^{k+j}}{1-x^{k+j}}\right) jP_{n-j}(x) - (-1)^{n-1}(n-1)\frac{x^{k+n}}{1-x^{k+n}} .$$

We combine the j=n+1 term in the first summation with the final term and shift the indices in the second summation to get,

$$= \sum_{j=1}^{n}(-1)^{j-1}\left(\frac{x^{k+j-1}}{1-x^{k+j-1}}\right) P_{n-j+1}(x)$$
$$+ \sum_{j=2}^{n}(-1)^{j-1}\left(\frac{x^{k+j-1}}{1-x^{k+j-1}}\right)(j-1)P_{n-j+1}(x) + (-1)^{n}(n)\frac{x^{k+n}}{1x^{k+n}} .$$

The two summations can now be combined to form

$$= \sum_{j=1}^{n}(-1)^{j-1}\left(\frac{x^{k+j-1}}{1-x^{k+j-1}}\right)jP_{n-j+1}(x) + (-1)^{n}(n)\frac{x^{k+n}}{1-x^{k+n}} ,$$

which is exactly what we want. Therefore the result is true. ❑

Setting k=1 in the theorem produces the main result of this section.

Corollary. *The partition function for a labeled cycle C_n with $n \geq 3$ is*

$$C_n(x) = \sum^{n-1}(-1)^{j-1}\left(\frac{x^j}{1-x^j}\right) jP_{n-j}(x) + (-1)^{n-1}(n-1)\frac{x^n}{1-x^n} .$$

$$C_n(x) = \sum_{j=1}^{n-1}(-1)^{j-1}\left(\frac{x^j}{1-x^j}\right) jP_{n-j}(x) + (-1)^{n-1}(n-1)\frac{x^n}{1-x^n}$$

This last result is again an application of inclusion-exclusion and is therefore not very computationally desirable. For example, suppose $C_n(x) = \sum c_k x^k$, then to compute the coefficients c_k one would need to evaluate at least $p(n-1) + p(n-2)$ terms, where we recall that $p(n)$ is the number of partitions of n and we saw in Section 3.2 that each $P_m(x)$ had $p(m) - p(m-2)$ terms. Of course we would like an expression for $C_n(x)$ similar to $P_m(x)$ which is easier to evaluate and which has a combinatorial explanation. After simplifying the expression for $C_n(x)$ for various values of n, we noticed a similar pattern for which we have no combinatorial explanation. We believe there is an explanation similar to the proof of $P_m(x)$ and will present this as a conjecture. We need a few definitions to present the conjecture. Let B_m denote the set of all ordered partitions of m in which each part is of size at least 2. For example $B_6 = \{ 6, 42, 24, 33, 222 \}$. For $\beta \in B_m$, let β_f denote the first element, and β_r be the remaining elements. Let $s(\beta)$ denote the special product of the elements of β described below.

$$s(\beta) = \beta_f \cdot \prod_{k \in \beta_r} (k-1)$$

For example $s(42) = 4{\cdot}3{\cdot}1 = 12$. We are now ready to present the conjecture.

Conjecture. *For m a positive integer* ≥ 3,

$$C_m(x) = \sum_{\beta \in B_m} \frac{s(\beta)\, x^{d(\beta)}}{\Pi\beta}$$

The table below shows these partition functions verified up to the case $m = 7$.

Table 3.1

Partition Functions for Labeled Cycles

C3: $\frac{6x^{6}}{\Pi 3}$

C4: $\frac{12x^{10}}{\Pi 4} + \frac{2x^{6}}{\Pi 2\Pi 2}$

C5: $\frac{20x^{15}}{\Pi 5} + \frac{4x^{9}}{\Pi 2\Pi 3} + \frac{6x^{9}}{\Pi 3\Pi 2}$

C6: $\frac{30x^{21}}{\Pi 6} + \frac{6x^{13}}{\Pi 2\Pi 4} + \frac{12x^{13}}{\Pi 4\Pi 2} + \frac{12x^{12}}{\Pi 3\Pi 3} + \frac{2x^{9}}{\Pi 2\Pi 2\Pi 2}$

C7: $\frac{42x^{28}}{\Pi 7} + \frac{8x^{18}}{\Pi 2\Pi 5} + \frac{20x^{18}}{\Pi 5\Pi 2} + \frac{18x^{16}}{\Pi 3\Pi 4} + \frac{24x^{16}}{\Pi 4\Pi 3} + \frac{4x^{12}}{\Pi 2\Pi 2\Pi 3}$

$+ \frac{4x^{12}}{\Pi 2\Pi 3\Pi 2} + \frac{6x^{12}}{\Pi 3\Pi 2\Pi 2}$

CHAPTER IV

OPEN PROBLEMS

Section 4.1 Introduction

In this final chapter we discuss various topics (related to chromatic partitions) that we studied during the course of this dissertation. We also mention a few open problems at the end of each section. We believe that this new concept has many areas open for further study and we hope to motivate those ideas here.

Section 4.2 Graphs with Identical Partition Functions

It is well known that there are non-isomorphic graphs which share the same chromatic polynomial. A very natural question to ask is the following:

Can two non-isomorphic graphs have the same chromatic partition function?

In other words do there exist two graphs G,H with $G \not\cong H$, such that $\Lambda(G;x) = \Lambda(H;x)$? The graphs in Figure 4.1 provide an affirmative answer. This pair first appears in Chalcraft [3] where they provided examples for other graph polynomials. Their critical property is their similarity under the deletion and contraction of certain edges.

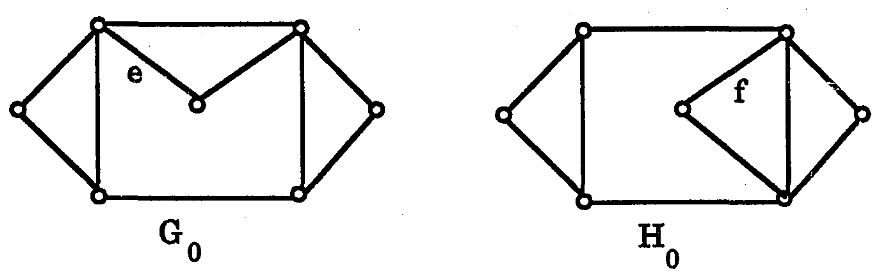


Figure 4.1

Theorem 4.1 *The graphs G_0 and H_0 in Figure 4.1 have the same partition function, that is $G_0 \not\cong H_0$ and $\Lambda(G_0;x) = \Lambda(H_0;x)$.*

Proof. It is easy to see that $G_0 \not\cong H_0$ since H_0 has an edge that lies in two triangles but G_0 has no such edge. To prove that $\Lambda(G_0;x) = \Lambda(H_0;x)$ we simply observe that $G_0 - e \cong H_0 - f$ and that $G_0 * e \cong H_0 * f$ so that by Theorem 1.1,

$$\begin{aligned}\Lambda(G_0;x) &= \Lambda(G_0 - e\,;x) - \Lambda(G_0 * e\,;x)\\ &= \Lambda(H_0 - f\,;x) - \Lambda(H_0 * f\,;x)\\ &= \Lambda(H_0;x). \quad \square\end{aligned}$$

In fact we can construct graphs with identical partition functions for any order greater than 7 by using the join operation.

Corollary *For every graph G, $G + G_0 \not\cong G + H_0$, but $\Lambda(G + G_0\,;x) = \Lambda(G + H_0\,;x)$.*

We can now ask the same question for trees. Are there trees T_1, T_2 such that $T_1 \not\cong T_2$ but $\Lambda(T_1;x) = \Lambda(T_2;x)$? It is well known that any two trees of the same order share the same chromatic polynomial. However here we believe that the answer to the above question is in the negative. We propose the following

Conjecture 4.1 *Given any two trees T_1 and T_2 such that $T_1 \not\cong T_2$ then $\Lambda(T_1;x) \neq \Lambda(T_2;x)$.*

In fact we also believe that the following stronger assertion is true!

Conjecture 4.2 *Given any two forests F_1 and F_2 such that $F_1 \doteq F_2$ then $\Lambda(F_1;x) \neq \Lambda(F_2;x)$.*

In the next section we will consider the sum of a coloring of a graph G.

Section 4.3 The Chromatic Sum of a Graph

For a given graph G, the chromatic sum ΣG, is defined in Kubicka's dissertation [11] as the minimal possible total that can occur among all proper colorings of G using natural numbers for the colors. From the perspective of chromatic partitions, it is the smallest power of x that appears in $\Lambda(G;x)$. We show in [14] that the chromatic sum for a connected graph G with e edges is tightly bound by $\lceil \sqrt{8e} \rceil \leq \Sigma G \leq \left\lfloor \frac{3(e+1)}{2} \right\rfloor$. It is also known that the chromatic sum is not always achieved by using only the chromatic number of distinct colors. That is, there are graphs for which extra colors must be used to attain the chromatic sum [11]. It appears that the upper bound of $\left\lfloor \frac{3(e+1)}{2} \right\rfloor$ applies not only to the coloring attaining the minimum sum, but also to any Grundy coloring [7] achieving the chromatic number. Specifically,

Conjecture 4.3 *Let G be a graph and let τ be a color assignment with color classes $\Phi_1, \ldots, \Phi_{\chi(G)}$ such that $|\Phi_i| \geq |\Phi_j|$ for all $i \leq j$ and that given $u \in \Phi_j$ there exists $v \in \Phi_i$ such that $uv \in E(G)$ for all $i = 1, 2, \ldots, j-1$. Then*

$$\sum_{i=1}^{\chi(G)} i|\Phi_i| \leq \left\lfloor \frac{3(e+1)}{2} \right\rfloor$$

We finally end this dissertation with some...

Section 4.4 Some Other Open Problems

1. *Complete n-partite graphs (unlabeled).*

In Section 3.1 we obtained an expression for the unlabeled chromatic partition function for the complete bipartite graphs. The very 'nice' combinatorial explanation for that result leads us to suspect that the complete k–partite case may have a similar general expression. Therefore we pose the following question; Is there a 'nice' expression for $\Lambda(K_{n_1,n_2,\ldots,n_k}); \Gamma(K_{n_1,n_2,\ldots,n_k}); x)$ The jump from the case $k=2$ to $k=3$ seems to be significant in solving this problem.

2. *Cycles.*

The labeled cycles problem has already been mentioned as a conjecture in Section 3.5. If the proof of this conjecture has a nice combinatorial explanation then the unlabeled cycle case may be approachable.

3. *Trees.*

In Section 3.2 we obtained a nice expression for $\Lambda(P_m ; x)$. The partition function for a tree in general with maximum degree ≥ 3 seems to be difficult. It will certainly be easier to search for identities for various families of trees for example caterpillars.

APPENDIX A

Partition Functions Evaluated for Paths.

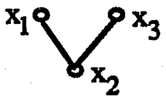
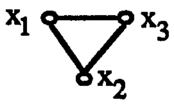
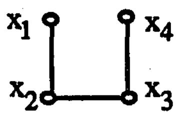
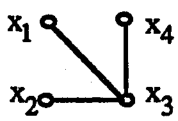
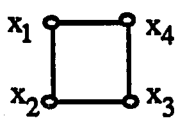
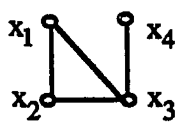
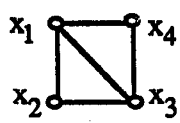
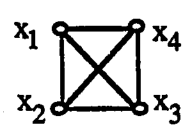
Partition Function Evaluated for Paths.

x^n	P_2	P_3	P_4	P_5	P_6	P_7
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	2	0	0	0	0	0
4	2	1	0	0	0	0
5	4	2	0	0	0	0
6	4	7	2	0	0	0
7	6	9	6	1	0	0
8	6	15	14	3	0	0
9	8	21	24	15	2	0
10	8	28	46	30	10	1
11	10	35	66	68	30	4
12	10	46	100	119	76	24
13	12	54	138	204	168	69
14	12	66	192	316	320	188
15	14	78	246	489	580	410
16	14	91	324	696	968	858
17	16	104	402	987	1558	1586
18	16	121	506	1340	2380	2837
19	18	135	612	1801	3540	4739
20	18	153	746	2348	5078	7672
21	20	171	882	3035	7160	11868
22	20	190	1054	3833	9804	17951
23	22	209	1224	4812	13238	26270
24	22	232	1432	5935	17510	37718
25	24	252	1644	7273	22884	52878
26	24	276	1896	8792	29418	72962
27	26	300	2148	10576	37462	98813
28	26	325	2448	12576	47054	132115
29	28	350	2748	14887	58638	173973
30	28	379	3098	17465	72272	226606
31	30	405	3450	20401	88454	291575
32	30	435	3854	23651	107262	371649
33	32	465	4260	27319	129312	468880
34	32	496	4726	31349	154644	586814
35	34	527	5190	35861	183994	728017
36	34	562	5716	40791	217442	896860
37	36	594	6246	46260	255782	1096660
38	36	630	6840	52212	299114	1332675
39	38	666	7434	58776	348386	1608997
40	38	703	8100	65881	403652	1932042
41	40	740	8766	73667	466012	2306716
42	40	781	9506	82068	535550	2740661
43	42	819	10248	91225	613442	3239897
44	42	861	11066	101067	699812	3813319
45	44	903	11886	111748	796012	4468113

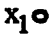
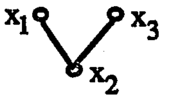
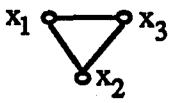
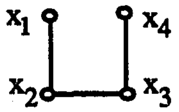
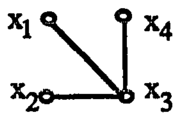
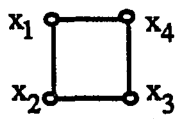
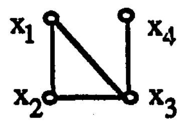
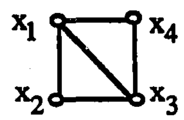
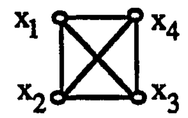
APPENDIX B

The Graphs of Order up to 4.

The Labeled Chromatic Partition Function and Ordinary Chromatic Polynomial.

G	$\Lambda(G)$	$\chi(G)$
	X_1	λ
	$X_1X_2 - X_{12}$	$\lambda^2 - \lambda$
	$X_1X_2X_3 - X_{12}X_3 + X_{123} - X_{23}X_1$	$\lambda(\lambda - 1)^2$
	$X_1X_2X_3 - X_{12}X_3 + 2X_{123} - X_{23}X_1 - X_{13}X_2$	$\lambda^3 - 3\lambda^2 + 2\lambda$
	$X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{123}X_4 - X_{1234} - X_1X_{23}X_4 + X_1X_{234} - X_1X_2X_{34} + X_{12}X_{34}$	$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$
	$X_1X_2X_3X_4 - X_1X_{23}X_4 + X_{123}X_4 - X_{1234} - X_{13}X_2X_4 + X_2X_{134} - X_1X_2X_{34} + X_1X_{234}$	$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$
	$X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{123}X_4 - 3X_{1234} - X_1X_{23}X_4 + X_1X_{234} - X_1X_2X_{34} + X_{12}X_{34} - X_{14}X_2X_3 + X_{14}X_{23} + X_{124}X_3 + X_{134}X_2$	$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$
	$X_1X_2X_3X_4 - X_{12}X_3X_4 + 2X_{123}X_4 - 2X_{1234} - X_1X_{23}X_4 + X_1X_{234} - X_1X_2X_{34} + X_{12}X_{34} - X_{13}X_2X_4 + X_{134}X_2$	$\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$
	$X_1X_2X_3X_4 - X_{12}X_3X_4 + 2X_{123}X_4 - 4X_{1234} - X_1X_{23}X_4 + 2X_{134}X_2 - X_1X_2X_{34} + X_{124}X_3 - X_{13}X_2X_4 + X_{234}X_1 - X_{14}X_2X_3 + X_{12}X_{34} + X_{14}X_{23}$	$\lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda$
	$X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{12}X_{34} + 2X_{123}X_4 - 6X_{1234} - X_{13}X_2X_4 + X_{14}X_{23} + 2X_{124}X_3 - X_{14}X_2X_3 + X_{13}X_{24} + 2X_{134}X_2 - X_{23}X_1X_4 + 2X_{234}X_1 - X_{24}X_1X_3 - X_{34}X_1X_2$	$\lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda$

The Unlabeled Chromatic Partition Function and the Unlabeled Chromatic Polynomial.

G	$\Lambda(G; \Gamma(G))$	$\chi(G; \Gamma(G); \lambda)$
	X_1	λ
	$\frac{1}{2}(X_1X_2 - X_{12})$	$\frac{1}{2}(\lambda^2 - \lambda)$
	$\frac{1}{2}(X_1X_2X_3 - X_{12}X_3 + X_{13}X_2 - X_{23}X_1)$	$\frac{1}{2}(\lambda^3 - \lambda^2)$
	$\frac{1}{6}(X_1X_2X_3 - X_{12}X_3 + 2X_{123} - X_{23}X_1 - X_{13}X_2)$	$\frac{1}{6}(\lambda^3 - 3\lambda^2 + 2\lambda)$
	$\frac{1}{2}(X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{123}X_4 - X_{1234} - X_1X_{23}X_4 + X_1X_{234} - X_1X_2X_{34} + X_{12}X_{34})$	$\frac{1}{2}(\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda)$
	$\frac{1}{6}(X_1X_2X_3X_4 - X_1X_{23}X_4 + X_{14}X_2X_3 + X_{1234} - X_{13}X_2X_4 + X_{24}X_1X_3 - X_1X_2X_{34} + X_{12}X_3X_4 - X_{12}X_{34} - X_{12}X_{24} - X_{14}X_{23})$	$\frac{1}{6}(\lambda^4 - \lambda^2)$
	$\frac{1}{8}(X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{12}X_{34} - 2X_{1234} - X_1X_{23}X_4 + X_{14}X_{23} - X_1X_2X_{34} + X_{13}X_{24} - X_{14}X_2X_3 + X_1X_3X_{24} + X_{13}X_2X_4)$	$\frac{1}{8}(\lambda^4 - 2\lambda^3 + 3\lambda^2 - 2\lambda)$
	$\frac{1}{2}(X_1X_2X_3X_4 - X_{12}X_3X_4 + 2X_{123}X_4 - 2X_{1234} - X_1X_{23}X_4 + X_1X_{234} - X_1X_2X_{34} + X_{12}X_{34} - X_{13}X_2X_4 + X_{134}X_2)$	$\frac{1}{2}(\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda)$
	$\frac{1}{4}(X_1X_2X_3X_4 - X_{12}X_3X_4 + 2X_{123}X_4 - 2X_{1234} - X_1X_{23}X_4 + 2X_{134}X_2 - X_1X_2X_{34} + X_1X_3X_{24} - X_{13}X_2X_4 + X_{234}X_1 - X_{14}X_2X_3 + X_{12}X_{34} - X_{13}X_{24})$	$\frac{1}{4}(\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda)$
	$\frac{1}{24}(X_1X_2X_3X_4 - X_{12}X_3X_4 + X_{12}X_{34} + 2X_{123}X_4 - 6X_{1234} - X_{13}X_2X_4 + X_{14}X_{23} + 2X_{124}X_3 - X_{14}X_2X_3 + X_{13}X_{24} + 2X_{134}X_2 - X_{23}X_1X_4 + 2X_{234}X_1 - X_{24}X_1X_3 - X_{34}X_1X_2)$	$\frac{1}{24}(\lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda)$

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