The Chromatic and Cochromatic Number of a Graph

John Gordon Gimbel
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THE CHROMATIC AND COCHROMATIC NUMBER OF A GRAPH

by

John Gordon Gimbel

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics

Western Michigan University
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THE CHROMATIC AND COCHROMATIC NUMBER OF A GRAPH

John Gordon Gimbel, Ph.D.
Western Michigan University, 1984

Clearly, there are many ways that one can partition the vertex sets of graphs. In the first chapter of this work I examine the problem of determining, for a given graph, the minimum order of a vertex partition having specified properties. In the remaining chapters I concentrate on partitions of two types--those in which each subset induces an empty graph and those in which each subset induces an empty or a complete graph.

The chromatic number of a graph G is the minimum number of subsets into which V(G) can be partitioned so that each subset induces an empty graph. The cochromatic number of G is the minimum number of subsets into which V(G) can be partitioned so that each subset induces a complete or an empty graph. In the second chapter I discuss the relationship between chromatic and cochromatic
numbers of graphs. I also extend known results in the field of cochromatic numbers.

In the third chapter I explore concepts in cochromatic theory which are analogous to well known topics in chromatic theory.

The acochromatic number of a graph G is the maximum order of all vertex partitions of G where each subset induces a complete or an empty graph but the union of any two does neither. I show in Chapter IV that the acochromatic number of a bipartite graph is bounded below by its edge independence number and above by this number plus one.

In the last chapter I discuss switching sets and sequences. I apply knowledge of chromatic and cochromatic theory to this concept.
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In this Chapter we will discuss graphical functions of certain types. For undefined concepts see [BCL1]. We shall say that a graph $G$ is isomorphic to $H$, written $G = H$, if there is a bijection $\gamma$ from the vertex set of $G$ to the vertex set of $H$ with the property that for any pair of distinct vertices $u$ and $v$ of $G$, $u$ is adjacent to $v$ if and only if $\gamma(u)$ is adjacent in $H$ to $\gamma(v)$. That is, $\gamma(u)\gamma(v)$ is an edge of $H$ if and only if $uv$ is an edge of $G$. When we will risk no clarity by doing so we will say that $G$ is the graph $H$ when we mean $G$ is isomorphic to $H$. Given a family $F$ of graphs, we will say $F$ contains a graph $G$, written $G \in F$, if $F$ contains a graph isomorphic to $G$.

By a complete graph on $n$ vertices, denoted $K_n$, we mean a graph in which every pair of vertices are adjacent. Therefore, $K_n$ is the graph on $n$ vertices which is $(n-1)$-regular. It also may be thought of as the graph on $n$ vertices which contains $(n^2-n)/2$
edges. A graph is empty if it has no edges. Any two vertices in an empty graph are nonadjacent.

Given a graph $G$, the complement of $G$, denoted $\overline{G}$, is the graph whose vertex set is the vertex set of $G$, and whose edge set is the set $\{uv | u$ and $v$ are not adjacent in $G\}$. We note that a graph is empty if its complement is a complete graph. In this work, we will be interested in certain properties of graphs which are inherited by their complements.

Given graphs $H$ and $G$, we say $H$ is a subgraph of $G$ if

i) $V(H) \subseteq V(G)$

and ii) $E(H) \subseteq E(G)$.

$H$ is an induced subgraph of $G$ if $H$ is a subgraph of $G$ and $E(H) = \{uv \in E(G) | u, v \in V(H)\}$. Clearly, any complete subgraph is induced. If $H$ is a subgraph of $G$ we write $H \subset G$. If $H$ is an induced subgraph we will write $H < G$. Given a graph $G$ and $V$ a subset of the vertex set of $G$, let $<V>_G$, the graph induced by $V$, be the induced subgraph of $G$ whose vertex set is $V$.

Given a graph $G$, we say $V_1, V_2, ..., V_n$ is a vertex partition of $G$ if $V(G) = \cup V_i$ and the sets $V_1, V_2, ..., V_n$ are mutually disjoint. Now, let $F$ be a family of graphs. We say that $G$ is $F$-partitionable
if there is a vertex partition \( V_1, V_2, \ldots, V_n \) of \( G \) such that \( \langle V_i \rangle \in F \) for each \( i \). Such a partition is called an \( F \)-partition of \( G \). The \( F \)-partition number \( \chi_F(G) \) of an \( F \)-partitionable graph \( G \) is the minimum number of sets in an \( F \)-partition of \( G \). A minimal \( F \)-partition of \( G \) is one whose cardinality is the \( F \)-partition number.

Clearly, for certain \( F \) the function \( \chi_F \) is not defined for all graphs. For example, if \( F \) contains only graphs of even order \( \chi_F(G) \) is undefined for any graph \( G \) of odd order.

**Proposition 1.1**

The function \( \chi_F \) is defined for all graphs if and only if \( K_1 \in F \).

**Proof:**

Clearly, if \( K_1 \notin F \), then the only induced subgraph of \( K_1 \), namely itself, is not a member of \( F \). Thus, \( \chi_F(K_1) \) is not defined.

Conversely, suppose \( K_1 \in F \) and \( G \) is a graph with \( V(G) = \{v_1, v_2, \ldots, v_p\} \). Then the vertex partition \( V_1, V_2, \ldots, V_p \) with \( V_i = \{v_i\} \) has the property that \( \langle V_i \rangle \in G = K_1 \) for each \( i = 1, 2, \ldots, p \). Thus, \( \chi_F(G) \) does indeed exist. \( \square \)

**Proposition 1.2**

Let \( E \) and \( F \) be families of graphs. If \( F \) is a nonempty subset of \( E \) then the domain of \( \chi_F \) is a
subset of the domain of $\chi_E$ and for all graphs $G$ in the domain of $\chi_F$, we have $\chi_E(G) \leq \chi_F(G)$.

**Proof:**

Suppose $\chi_F(G)$ is defined. Let $V_1, V_2, \ldots, V_n$ be an $F$-partition of $G$, with $n = \chi_F(G)$. Since $F \subseteq E$, we see $\langle V_i \rangle \in E$ for $i = 1, 2, \ldots, n$. Hence, $\chi_E(G)$ is defined and $\chi_E(G) \leq n = \chi_F(G)$. 

A family $F$ of graphs is **complete** if for each $G \in F$ and $H \subseteq G$ we have $H \in F$. The family $F$ is **symmetric** if for each $G \in F$, the complement $\overline{G}$ of $G$ is in $F$.

**Proposition 1.3**

Let $G$ be a graph and $H \subseteq G$. If $F$ is a complete family of graphs and $\chi_F(G)$ is defined, then $\chi_F(H)$ is defined and $\chi_F(H) \leq \chi_F(G)$.

**Proof:**

Suppose the hypothesis is satisfied. Let $V_1, V_2, \ldots, V_n$ be a minimal $F$-partition of $G$. Now, let $W_i = V_i \cap V(H)$. Remove the empty sets from the list $W_1, W_2, \ldots, W_n$. The remaining sets form a vertex partition of $H$. Further, for each $W_i$ remaining in the list, $\langle W_i \rangle_H < \langle V_i \rangle_G$; thus $\langle W_i \rangle_H \in F$. Hence, $\chi_F(H)$ is defined and $\chi_F(H) \leq n = \chi_F(G)$. 

**Proposition 1.4**

If $F$ is a symmetric family of graphs, then for
each $G$ for which $\chi_F(G)$ is defined, $\chi_F(\overline{G})$ is defined and equals $\chi_F(G)$.

**Proof:**

Let $V_1, V_2, \ldots, V_n$ be a minimal $F$-partition of $G$. Since $F$ is symmetric $\langle V_i \rangle_G \subseteq F$, for $i = 1, 2, \ldots, n$ and so $\langle V_i \rangle_G \subseteq G$ since $\langle V_i \rangle_G = \overline{\langle V_i \rangle}_G$. Thus, $\chi_F(\overline{G}) \leq n = \chi_F(G)$. Now, replacing $G$ with $\overline{G}$ in the above argument yields $\chi_F(G) = \chi_F(\overline{G}) \leq \chi_F(\overline{G})$.

Hence, $\chi_F(G) = \chi_F(\overline{G})$. $\blacksquare$

We will say that a graph $G$ is a composite of graphs $G_1$ and $G_2$ if
1) $V(G) = V(G_1)\cup V(G_2)$
2) $\langle V(G_1) \rangle_G = G_1$
and 3) $\langle V(G_2) \rangle_G = G_2$.

Two well-known examples of composites are union and join. Let $G_1$ and $G_2$ be vertex disjoint graphs. The union of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is the composite of $G_1$ and $G_2$ with edge set $E(G_1) \cup E(G_2)$. The join of $G_1$ and $G_2$, denoted $G_1 + G_2$, is the graph $\overline{G_1 \cup G_2}$.

**Proposition 1.5**

Given a family $F$ of graphs and two vertex disjoint $F$-partitionable graphs $G_1$ and $G_2$, let $G$ be a composite of $G_1$ and $G_2$. It follows that
\[ x_F(G) \leq x_F(G_1) + x_F(G_2). \]

**Proof:**

Let \( V_1, V_2, \ldots, V_n \) be a minimal \( F \)-partition of \( G \) and \( U_1, U_2, \ldots, U_m \) be a minimal \( F \)-partition of \( G \). Note, then, that \( V_1, V_2, \ldots, V_n, U_1, U_2, \ldots, U_m \) is an \( F \)-partition of \( G \). Hence,

\[ x_F(G) \leq n + m = x_F(G_1) + x_F(G_2). \]

From this and Proposition 1.3 we see that for any family \( F \) of graphs and disjoint \( F \)-partitionable graphs \( G_1 \) and \( G_2 \) that

\[ \max\{x_F(G_1), x_F(G_2)\} \leq x_F(G_1 + G_2) \leq x_F(G_1) + x_F(G_2). \]

and

\[ \max\{x_F(G_1), x_F(G_2)\} \leq x_F(G_1 U G_2) \leq x_F(G_1) + x_F(G_2). \]

**Proposition 1.6**

Let \( F \) be a family of graphs and \( G \) an \( F \)-partitionable graph. If \( k = \max\{|V| : V \subseteq V(G) \text{ and } <V> \in F\} \), then

\[ \frac{p}{k} \leq x_F(G) \]

where \( p \) is the order of \( G \).

**Proof:**

Let \( V_1, V_2, \ldots, V_n \) be a minimal \( F \)-decomposition of \( G \). Note,

\[ p = \sum_{i=1}^{n} |V_i| \leq \sum_{i=1}^{n} k. \]
Hence, $P \leq nk$ and the desired result follows.

Now, suppose $K_1 \in F$. It is readily seen that

$$\frac{P}{k} \leq \chi_F(G) \leq p - k + 1.$$  

Section 1.2

Examples of Vertex Partition Functions

In this section we will consider specific families $F$ of graphs and the resulting functions $\chi_F$.

Perhaps the most used vertex partition function in graph theory is $p(G)$, the order of $G$. Here we let $F = \{K_1\}$ and note that $p(G) = \chi_F(G)$.

A function which is more interesting is $\chi$, the chromatic number. Here we let $F$ be the set of empty graphs. That is,

$$F = \{\overline{K_1}, \overline{K_2}, \overline{K_3}, \ldots\}.$$  

Then $\chi(G)$ is $\chi_F(G)$. We note that $\chi(G)$ is the least number of sets into which the vertex set can be partitioned so that each set induces an empty graph. Often, we think of coloring the vertices of a graph with the fewest number of colors possible so that no two adjacent vertices have the same color assigned. In this sense, the vertex partition is determined by
the coloring: All vertices assigned the same color are placed in the same set.

The independence number \( \beta(G) \) of a graph \( G \) is defined to be

\[
\max \{ |V| : V \subseteq V(G) \text{ and } <V> \text{ is an empty graph} \}.
\]

By Proposition 1.6 and the comment following, we see that

\[
\frac{p(G)}{\beta(G)} \leq \chi(G) \leq p(G) - \beta(G) + 1.
\]

Furthermore, both bounds can be shown to be sharp. Let \( G = K_p \) for \( p \geq 1 \). Now equality holds in this case for both bounds.

A forest is a graph that contains no cycles. Let \( F \) be the set of all forests. The arboricity \( a(G) \) of \( G \) is the minimum number of sets into which the vertex set can be partitioned so that each subset induces a forest. That is, \( a(G) = \chi_F(G) \). Since any empty graph is a forest, Proposition 1.2 implies that \( a(G) \leq \chi(G) \) for each graph \( G \). Since \( K_2 \) and \( K_2 \) are both forests we see that \( \frac{p(G)}{2} \geq a(G) \).

For our last example, let \( \delta(G) \) be the minimum degree of all vertices in the graph \( G \). We say that a graph \( G \) is k-degenerate if

\[
\delta(H) \leq k
\]

for all \( H \subseteq G \). Let \( F(k) = \{ G : G \text{ is a k-degenerate graph} \} \).
The vertex partition number $p_k(G)$ of a graph $G$ is $\chi_{F(k)}(G)$. Notice, $\chi(G) = p_0(G)$ and $a(G) = p_1(G)$ for all graphs $G$. Also, since $K_1 \in F(k)$ for all $k$, $p_k(G)$ is defined for each natural number $k$ and graph $G$. If $n \leq m$ we see that any $n$-degenerate graph is $m$-degenerate. Thus, by Proposition 1.2, for any graph $G$

$$n \leq m \implies p_m(G) \leq p_n(G).$$

The reader interested in this class of functions should consult [LW1], [LW2] and [S2].

In the next chapter we shall turn our attention to functions of the form $\chi_F$, where

$$F = \{ \overline{K_1}, \overline{K_2}, \overline{K_3}, \ldots \} \cup \{ K_1, K_2, K_3, \ldots \}.$$
CHAPTER II

COCHROMATIC NUMBERS

Section 2.1

An Introduction

As we mentioned in Chapter I, the chromatic number of a graph is the fewest number of subsets into which the vertex set can be partitioned so that each subset induces an empty graph. For a graph \( G \), the cochromatic number \( Z(G) \) is the fewest number of subsets into which the vertex set can be partitioned so that each subset induces an empty or a complete graph. These subsets are known as cocolor classes or, when it is clear, as color classes.

We now present without proof several preliminary remarks about cochromatic numbers. Many of these follow as corollaries to propositions in Chapter I.

(a) \( Z(G) \) is defined for all graphs \( G \).

(b) \( Z(G) = 1 \) implies that \( G \) is a complete or an empty graph.

(c) For any graph \( G \), \( Z(G) \leq \chi(G) \).

(d) If \( B \) is a nonempty bipartite graph and \( \nu(B) \geq 3 \) then \( Z(B) = 2 \).

(e) For any even cycle \( C_{2n} \).
with \( n = 2, 3, 4, \ldots \), the cochromatic number is 2.

(f) For any odd cycle other than \( C_3 \), the cochromatic number is 3.

(g) For any graph \( G \), \( Z(G) = Z(\overline{G}) \).

(h) For any graph \( G \), \( Z(G) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor \).

**Theorem 2.1**

If \( G \) is a planar graph then \( Z(G) \leq 4 \).

**Proof:**

By the Four Color Theorem, if \( G \) is planar then \( \chi(G) \leq 4 \). Since \( Z(G) \leq \chi(G) \) the result follows.

This bound can be shown to be sharp. Let \( G = 4K_4 \).

Clearly \( G \) is planar. Later, we shall prove that \( Z(G) = 4 \).

**Section 2.2**

**A Summary of Previous Results**

In this section we will examine several known results in the field of cochromatic numbers.

The subject of cochromatic numbers seems to have arisen from work by Foldes and Hammer (see [FH1]) on polarized and split graphs. A graph \( G \) is **polarized** if its vertex set can be partitioned into \( V_1 \) and \( V_2 \) so that each induces a complete graph in \( G \) or \( \overline{G} \). That is, for \( G \) a nontrivial graph, \( G \) is polarized if and only if \( Z(G) = 2 \) or \( G \) is a complete or an empty graph. A graph \( G \) is a **split** graph
if the above holds and $V_1$ is independent in $G$ and $V_2$ is independent in $\overline{G}$. Foldes and Hammer were able to show that $G$ is a split graph if and only if $G$ and $\overline{G}$ are both chordal. Recall that a graph is chordal if the vertices of any $n$-cycle, $n \geq 4$, induce a graph with at least $n + 1$ edges.

Straight and Lesniak-Foster were the first to use the concept of cochromatic number. In their paper [LS1] on the subject, the following two theorems were proven.

**Theorem 2.2**

If $G$ is a graph of order 3 or more which does not contain $K_3$ as a subgraph, then $Z(G) = \chi(G)$.

We recall that in all cases the cochromatic number of a graph is bounded above by its chromatic number. Theorem 2.2 shows that this bound is sharp for an infinite family of graphs.

**Theorem 2.3**

Let $G = K(p_1, p_2, \ldots, p_n)$ where $p_1 \leq p_2 \leq \ldots \leq p_n$. Then

$$Z(G) = \min \{n, n - j + p_j \mid j = 1, 2, \ldots, n\}.$$ Since $K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n} = K(p_1, p_2, \ldots, p_n)$, we have an immediate corollary.

**Corollary 2.4**

Let $G = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$ where $p_1 \leq p_2 \leq \ldots \leq p_n$.
Then \( Z(G) = \text{Min} \{n, n - j + p_j \mid j = 1, 2, \ldots, n\} \).

The following two results are specific applications of this corollary.

**Corollary 2.5**

\[ Z(K_1 \cup K_2 \cup \ldots \cup K_n) = n. \]

**Proof:**

By Corollary 2.4 we have that

\[ Z(K_1 \cup K_2 \cup \ldots \cup K_n) = \text{Min} \{n, n - j + j \mid j = 1, 2, \ldots, n\} = n. \]

**Corollary 2.6**

\[ Z(mK_n) = \text{Min} \{m, n\}. \]

**Proof:**

Again, by Corollary 2.4,

\[ Z(mK_n) = \text{Min} \{m, m - j + n \mid j = 1, 2, \ldots, m\} = \text{Min} \{m, n\}. \]

We close this section with a summary of known topological results related to cochromatic numbers. By a **surface** we mean a compact 2-manifold. If a graph \( G \) embeds on a surface \( S \), we will write \( G \preceq S \). Otherwise, we will write \( G \not\preceq S \). Given a real-valued graphical function \( f \) and a surface \( S \), we will use the notation \( f(S) \) to mean \( \text{Max} \{f(G) \mid G \not\preceq S\} \). Recall that an orientable surface \( S_n \), of genus \( n \), may be thought of as a sphere with \( n \) "handles" attached.
Theorem 2.7

\[ Z(S) \leq \chi(S) - 1 \] for any surface \( S \), except when \( S \) is the sphere or a nonorientable surface of genus less than 3.

This result is due to Straight and a proof may be found in [S4].

In [S4] and [S5] proofs of the following can be found.

Theorem 2.8

(a) \( Z(N_1) = 4 \)
(b) \( Z(N_2) = 6 \)
(c) \( Z(N_3) = 6 \)
(d) \( Z(N_4) = 7 \)

where \( N_1 \) is the nonorientable surface of genus \( i \).

\( N_1 \) is often referred to as the projective plane and \( N_2 \) is the familiar Klein bottle.

Straight has conjectured that

\[ Z(S) = \text{Max} \{ n | K_1 \cup K_2 \cup \ldots \cup K_n \preceq S \} \]

for any surface \( S \). Specifically, it is not known whether \( Z(S_1) \) is 5.
Section 2.3

Cochromatic and Chromatic Numbers

In this section we examine fundamental relations between cochromatic and chromatic numbers.

Recall that for any graph $G$, the cochromatic number of $G$ is bounded above by the chromatic number. Our first theorem is related to this fact.

**Theorem 2.9**

Given natural numbers $n$ and $m$ with $n \leq m$, there is a graph $G$ with $Z(G) = n$ and $\chi(G) = m$.

**Proof:**

For such an $n$ and $m$, let $G$ be the union of $n$ complete graphs on $m$ vertices. That is, $G = nK_m$. It is not difficult to see that such a graph has a chromatic number of $m$. Using Corollary 2.6 we see that this graph has a cochromatic number of $n$.

Now, if $n = 1$, and $m \geq 2$ there is only one graph, namely $K_m$, with a cochromatic number of $n$ and a chromatic number of $m$. In all other cases where $n < m$, there are an infinite number of blocks with a cochromatic number of $n$ and a chromatic number of $m$. To see this, let $G = (m-1)K_{n-1} + K_r$ where $r \geq n$. Note, $\chi(G) = m$ and $Z(G) = n$.

Now, if $G$ is a graph with a cochromatic...
number of 2 and a chromatic number of 2 it is clearly a bipartite graph other than $K_2$. Foldes and Hammer were able to characterize those graphs with cochromatic number 2 and chromatic number greater than 2.

**Proposition 2.10**

Given a graph $G$, then $Z(G) = 2$ and $\chi(G) > 2$ if and only if $G$ is not $K_1$ or $K_2$ and $G$ does not contain $C_4$, $\overline{C_4}$, or $C_5$ as an induced subgraph.

The proof of this may be found in [FHl]. It was previously known by F. Galvin and follows from work by Gyarfas and Lehel (see [GL1]).

A related question asks if we are given a triple $(m, n, p)$ of natural numbers is there a graph $G$ on $p$ vertices with $Z(G) = m$ and $\chi(G) = n$. Clearly, if there is then $m \leq n \leq p$. This, however, is not sufficient. Using the well-known Nordhaus-Gaddum result [NGl] that for any graph $G$ on $p$ vertices $\chi(G) + \chi(\overline{G}) \leq p + 1$, we see that $Z(G) \leq p + 1 - \chi(G)$. Hence, it must be that $m \leq p + 1 - n$.

This leads us to see that if $p = n + 1$, then $m = 2$ or $m = n = 1$.

The following remark is presented without proof.

**Proposition 2.11**

If $G$ is a graph with $p(G) \geq 5$ and $\chi(G) = p(G) - 3$
then $Z(G) = 2, 3,$ or $4$. Furthermore, if $p(G) = 5$
then $Z(G) = 2$; if $6 \leq p(G) \leq 8$ then $Z(G) = 2$ or
$3$; if $p(G) \geq 9$ then $Z(G)$ can have any of the three
values.

Section 2.4

Bounds on the Cochromatic Number of a Graph

Proposition 2.12

For any graph $G$, $Z(G) \leq \min \{x(G), x(\overline{G})\}$.

Proof:
This follows by recalling that $Z(G) \leq x(G)$ and
that $Z(G) = Z(\overline{G}) \leq x(\overline{G})$.

This bound is sharp in the sense that there are
an infinite number of graphs for which equality holds.
To see this, let $G_n$ be the complete $n$-partite graph
containing $n$ vertices in each partite set. Note that
$G_n$ and $\overline{G}_n$ are both $n$-chromatic. We have already
seen that $Z(G_n) = n$.

The bound is also sharp when $Z(G) = 1$. If,
however, $Z(G) = 2$ we see that $\min \{x(G), x(\overline{G})\}$
is unbounded: let $G_n = K_n \cup \overline{K}_{n-1}$, $n = 2, 3, \ldots$.
Clearly, $Z(G_n) = 2$ and $x(G_n) = x(\overline{G}_n) = n$.

Recall that the symbols $\Delta(G)$ and $\delta(G)$ denote
the maximum and the minimum degrees, respectively,
among the vertices of a graph $G$. The following is
a celebrated theorem which relates chromatic number
and the maximum degree of a graph.

**Theorem 2.13** (Brooks' Theorem)

For any graph $G$, $\chi(G) \leq \Delta(G) + 1$. Furthermore, if $G$ is connected, equality holds if and only
if $G$ is a complete graph or an odd cycle.

The following corollary is stated in [LS1].

**Corollary 2.14**

For any graph $G$ of order $p$, $Z(G) \leq \min \{1 + \Delta(G), p - \delta(G)\}$. Furthermore, if $p \geq 5$ and
both $G$ and $\overline{G}$ are connected, then equality holds
if and only if $G$ or $\overline{G}$ is an odd cycle.

**Proof**

Note that $\Delta(\overline{G}) = p - 1 - \delta(G)$. Now, suppose
$G$ is an odd cycle on $5$ or more vertices. Note also
that $Z(G) = 3$ and $\Delta(G) = 2$ and $p - \Delta(G) > 2$. Thus,
equality holds in this case and also holds when $\overline{G}$ is
an odd cycle on $5$ or more vertices.

Now, suppose $G$ is a graph on $p \geq 5$ vertices,
with both $G$ and $\overline{G}$ connected, and $Z(G) = \min \{1 + \Delta(G), p - \delta(G)\}$. If $Z(G) = 1 + \Delta(G)$ then we conclude
from the fact that $Z(G) \leq \chi(G) \leq 1 + \Delta(G)$ that $\chi(G) = \overline{G}$.
Thus, $G$ is either an odd cycle or a complete graph. If $G$ were a complete graph then $\Delta(G) = p - 1$ and hence $Z(G) = 1 + (p - 1) = p > 5$.
But let us also note that if $G$ were a complete graph then $Z(G) = 1$ and a contradiction would be reached. So, if $Z(G) = 1 + \Delta(G)$ then $G$ must be an odd cycle.

Now, if $Z(G) = p - \delta(G)$ we see that $Z(G) = Z(\bar{G}) \leq \chi(\bar{G}) \leq 1 + \Delta(\bar{G}) = p - (p - 1 - \Delta(\bar{G})) = p - \delta(G)$.
Thus, $\chi(\bar{G}) = 1 + \Delta(\bar{G})$. As above, we must conclude that $\bar{G}$ is an odd cycle. □

**Corollary 2.15**

If $P_4$, the path on four vertices, is not an induced subgraph of a graph $G$ then $Z(G) \leq \min\{\delta(G), \omega(G)\}$

**Proof:**

In [S1] it was shown that if $P_4 \not\subseteq G$ then $\chi(G) = \omega(G)$. Now the complement of $P_4$ is isomorphic to $P_4$. Thus, $P_4 \not\subseteq G \Rightarrow P_4 \cong \bar{G} \Rightarrow \chi(\bar{G}) = \omega(\bar{G}) = \delta(G)$.
It follows from Theorem 2.3 that $P_4 \not\subseteq G \Rightarrow Z(G) \leq \min\{\chi(G), \chi(\bar{G})\} = \min\{\omega(G), \delta(G)\}$.

**Proposition 2.16**

Given $G$, a graph,

$$\min\left\{\frac{P(G)}{\omega(G)}, \frac{P(G)}{\delta(G)}\right\} \leq Z(G).$$

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From Proposition 1.6, we have that

\[ \frac{p(G)}{\max \{ \omega(G), \delta(G) \}} \leq Z(G), \]

from which this follows. 

Other upper bounds on the cochromatic number of a graph may be found by constructing bounds on the chromatic number of the graph. For example, for any graph \( G \) it was shown in [H1] and [SW1] that \( \chi(G) \) is bounded above by

\[ \max \delta(H) + 1, \quad H \prec G \]

from which Brooks' theorem follows. Thus, for any graph \( G \),

i) \( Z(G) \leq \max \delta(H) + 1 \)

\[ H \prec G \]

ii) \( Z(G) \leq \Delta(G) + 1 \).

To see sharpness, select \( n \) and let \( G = nK_n \). Note that equality holds in both of the above.

In [EH1] it was shown that if \( n \geq 3 \) is the length of a longest odd cycle in a graph \( G \) then \( \chi(G) \leq n + 1 \). Hence, if a longest odd cycle in \( G \) has length \( n \) then \( Z(G) \leq n + 1 \). This bound is seen to be sharp by noting that for an odd number \( n \),

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and $m = n + 1$, if $G = mK_m$ then a longest odd cycle in $G$ has length $n$ and $Z(G) = n + 1$.

**Conjecture**

If $B$ is a block with a longest odd cycle of length $n$, then $Z(B) \leq n - 1$.

Recall that earlier we learned that if a graph $G$ contains no $K_3$ as an induced subgraph, then $Z(G) = \chi(B)$. Straight has shown the following in [S3].

**Theorem 2.17**

If $G$ is a graph which does not contain $K_4$ as a subgraph then $\chi(G) \leq Z(G) + 1$.

Hence, for any graph $G$ which does not contain $K_4$, $Z(G) \leq \chi(G) \leq Z(G) + 1$. Straight went on to point out that if a graph $G$ did not contain $K_5$ as a subgraph then $\chi(G)$ was not necessarily bounded by $Z(G) + 2$. His counterexample to this was the graph $H$, whose complement $\overline{H}$ is shown below.

![Fig. 2.1](Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.)
It was not known if this was the only counterexample to this bound. We have found several others. Three of them are now exhibited.

Take the graph $G = C_5 + C_5$. Since $x(C_5) = 3$, we see that $x(G) = 6$. Note also that $Z(G) \leq x(\overline{G}) = x(C_5 + C_5) = 3$. Lastly, notice that $K_5$ is not a subgraph of $G$. If it were, three vertices in one of the two 5-cycles would include a triangle.

If $G_1$ and $G_2$ are the graphs illustrated in Fig. 2.2, then it is straightforward to verify for $i = 1, 2$, that $K_5 \not\subseteq \overline{G}_i$ and $Z(\overline{G}_1) = 3$ and $x(\overline{G}_1) = 6$.

![Fig. 2.2](image-url)
Other graphs can be formed having desired properties by adding any subset of the edge set \{uv_1, uv_2, uv_3, uv_4\} to \( G_1 \), of Figure 2.2, or any subset of the edge set \{uv_i, vu_i \mid i = 1, 2, 3, 4\} to \( G_2 \) of that figure.

**Conjecture**

There are only a finite number of graphs \( G \) with \( Z(G) + 2 < \chi(G) \) which contain no \( K_5 \).

**Conjecture**

If \( G \) is a graph which does not contain \( K_5 \) then \( \chi(G) \leq Z(G) + 3 \).

Straight also asked if it were true that for each natural number \( m \), there was a number \( N_m \) such that for all graphs \( G \)

\[
p(G) \geq m \quad \text{and} \quad K_m \not\subseteq G \Rightarrow \chi(G) \leq Z(G) + N_m.
\]

We see that if there is such an \( N_m \) for a given \( m \), then

\[
N_m \geq \begin{cases} 
\frac{3m - 12}{2} & \text{if } m \text{ is even} \\
\frac{3m - 9}{2} & \text{if } m \text{ is odd.}
\end{cases}
\]

To see this, note the following construction. For \( n = 1, 2, \ldots \) let \( G_n = nC_5 \). Note the following:
1) $\omega(G_n) = 2n$
2) $\chi(G_n) = 3n$
3) $Z(G_n) = 3$.

Now, if $m \geq 4$ is even, pick $n$ so that $m = 2n + 2$.

The graph $G_n$ has order at least $m$ and contains no $K_m$. Furthermore,

$$\chi(G_n) = Z(G_n) + 3n - 3 = Z(G_n) + \frac{3m - 12}{2}.$$ 

Hence, $N_m \geq \frac{3m - 12}{2}$.

If $m \geq 3$ is odd, pick $n$ so that $m = 2n + 1$.

Again, the graph $G_n$ has order at least $m$ and contains no $K_m$. Furthermore,

$$\chi(G_n) = Z(G_n) + \frac{3m - 9}{2}.$$ 

Thus, when $m$ is odd, $N_m \geq \frac{3m - 9}{2}$.

For natural numbers $n$ and $m$, let $r(n,m)$, the Ramsey number for $n$ and $m$, be the least positive integer such that every graph of order $r(n,m)$ contains either $K_m$ or $\overline{K_n}$ as an induced subgraph.

The following theorem answers the question of Straight.

It was proposed by Erdös and uses the well known result that $r(n,n) \leq 4^n$ for all $n \geq 2$. 

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Theorem 2.18

If G is a graph containing no $K_m$ then

$$\chi(G) \leq \Delta(G) + 4^m.$$ 

Proof:

Let G be a graph which contains no $K_m$ as a subgraph. Let $V_1, V_2, \ldots, V_{\Delta(G)}$ be a cocoloring of G with the first $m_1$ classes being those with cardinality at least $m$, provided some exist. If none exist, let $m_1 = 0$. Further, if $m_1 = \Delta(G)$ then $\chi(G) = \Delta(G)$ and we are done. If $m_1 < \Delta(G)$ let $G_1 = \langle V_{m_1+1} \cup V_{m_1+2} \cup \ldots \cup V_{\Delta(G)} \rangle$. If $p(G_1) \leq 4^m$, let $W_1$ be a subset of $V(G_1)$ which induces an empty graph on at least $m$ vertices. Let $G_2 = G_1 - W_1$. Continue in this fashion: if $p(G_i) \geq 4^m$, let $W_i$ be a subset of $V(G_i)$ which induces an empty graph on at least $m$ vertices. Let $G_{i+1} = G_i - W_i$. Now, suppose $p(G_{m_2+1}) < 4^m$. Note that if $i \geq m_1 + 1$ then $|V_i| \leq m - 1$. Hence, $m_2 \leq \Delta(G) - m_1$. Now,

$$\chi(G_{m_2+1}) \leq 4^m,$$

thus $\chi(G) \leq m_1 + m_2 + 4^m$. The desired result now follows. $\square$

From this proof, we see that if G is a graph which does not contain $K_n$ then $\chi(G) \leq \Delta(G) + r(n,n)$. We close this section with several remarks on graphical operations.
Proposition 2.19

Given graphs $G_1$ and $G_2$,

$$Z(G_1 + G_2) \leq Z(G_1) + Z(G_2)$$

and

$$Z(G_1 \cup G_2) \leq Z(G_1) + Z(G_2).$$

Proof:

This follows immediately from Proposition 1.5. $ullet$

To establish sharpness, let $G_1 = G_2 = nK_{2n}$, for some positive integer $n$. By Corollary 2.6, $Z(G_1) = Z(G_2) = n$ and $Z(G_1 \cup G_2) = 2n$. Also, $Z(G_1 + G_2) = 2n$, since $\overline{G_1 \cup G_2} = G_1 + G_2$. Thus, in this example, $Z(G_1 + G_2) = Z(G_1) + Z(G_2)$ and $Z(G_1 \cup G_2) = Z(G_1) + Z(G_2)$.

For graphs $G_1$ and $G_2$, let $G_1 \times G_2$ be the cartesian product of $G_1$ and $G_2$, let $G_1 [G_2]$ be the lexicographic product. We make the following observations for any graphs $G_1$ and $G_2$:

a) $Z(G_1 \times G_2) \leq \min \{ p(G_1)Z(G_2), p(G_2)Z(G_1) \}$

b) $Z(G_1 [G_2]) \leq p(G_1)Z(G_2)$

These bounds can be shown to be sharp by letting $G_1 = \overline{K}_2$ and $G_2 = nK_{2n}$, for some natural number $n$. Now $G_1 \times G_2 = G_1 [G_2] = (2n)K_{2n}$. Notice that

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1) \( p(G_1)Z(G_2) = 2n \)
2) \( p(G_2)Z(G_1) = 2n^2 \)
and 3) \( Z[(2n)K_{2n}] = 2n \).

Hence, sharpness holds in both bounds.

Now, notice that both \( G_1 \) and \( G_2 \) are induced subgraphs of each of the graphs \( G_1UG_2 \) and \( G_1 + G_2 \) and \( G_1 \times G_2 \) and \( G_1 [G_2] \). Hence, the following bound:

\[
\text{Max} \{ Z(G_1), Z(G_2) \} \leq \text{Min} \{ Z(G_1UG_2), Z(G_1 + G_2), Z(G_1 \times G_2), Z(G_1 [G_2]) \}.
\]

If we let \( G_1 = \overline{K}_2 \) and \( G_2 = nK_n \) then equality can be seen to hold.

Section 2.5

Cochromatic Number and the Order of a Graph

Given a positive integer \( n \), we define \( Z(n) \) to be

\[
\text{Max} \{ Z(G) \mid p(G) = n \}.
\]

**Theorem 2.20**

For \( n = 1, 2, \ldots \)

\[
Z(n) \geq \begin{cases} 
  m & \text{if } n - m^2 < m \\
  m + 1 & \text{otherwise}
\end{cases}
\]

where \( m = \lfloor \sqrt{n} \rfloor \).

**Proof:**

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If $n$ is a perfect square, let $G = mK_m$. Since $Z(G) = m$ and $G$ has $n$ vertices we know that $Z(n) \geq m$. If $n$ is not a perfect square, let $G = mK_m \cup K_{n-m^2}$. Notice, $G$ has $n$ vertices. By Corollary 2.4 we have

$$Z(G) = \text{Min } \{ m + 1, m + 1 - j + p_j \}$$

where

$$p_j = \begin{cases} 
  m & \text{if } j < m + 1 \text{ and } n - m^2 \geq m \\
  n-m^2 & \text{if } j = m + 1 \text{ and } n - m^2 \geq m \\
  n-m^2 & \text{if } j = 1 \text{ and } n - m^2 < m \\
  m & \text{if } j > 1 \text{ and } n - m^2 < m
\end{cases}$$

A case by case analysis provides the desired result. \(\Box\)

**Corollary 2.21**

For any positive integer $n$, the inequality $Z(n) \geq \lceil \sqrt{n} \rceil$ holds.

**Proof:**

This follows by examining both cases in the preceding bound. \(\Box\)

Lesniak-Foster and Straight showed in [LS1]
that $Z(8) = 3$. It is not as difficult to show the following:

$$
Z(1) = 1 \\
Z(2) = 1 \\
Z(3) = 2 \\
Z(4) = 2 \\
Z(5) = 3 \\
Z(6) = 3 \\
Z(7) = 3 \\
Z(9) = 4 \\
Z(10) = 4 \\
Z(11) = 4
$$

**Proposition 2.22**

For $n = 1, 2, ...$

$$
Z(n) \leq \left( \frac{n + 1}{3} \right) + r(n)
$$

where

$$
r(n) = \begin{cases} 
1 & \text{if } n - 3 \left\{ \frac{n-5}{3} \right\} = 5 \\
0 & \text{otherwise}
\end{cases}
$$

**Proof:**

If $n = 1, 2, 3, 4,$ or $5$ the inequality clearly holds. So, suppose $G$ is a graph on $n \geq 6$ vertices. We wish to show that $Z(G) \leq \left( \frac{n+1}{3} \right) + r(n)$. Now, recall that any graph on $6$ or more vertices contains three vertices which induce either a complete or an
empty graph. Successively remove from $V(G)$ groups of three vertices with this property until five or fewer vertices remain. Note, $\left\lfloor \frac{n-5}{3} \right\rfloor$ such classes have been formed. Now, if five vertices remain, the induced graph will have a cochromatic number of 3 or less. In which case $Z(G) \leq \left\lfloor \frac{n-5}{3} \right\rfloor + 3 = \left\lfloor \frac{n+1}{3} \right\rfloor + 1$.

However, 5 vertices will remain only if

$$n - 3 \cdot \left\lfloor \frac{n-5}{3} \right\rfloor = 5$$

If the remaining graph has 4 or fewer vertices, it induces a graph with a cochromatic number of 1 or 2. In which case $Z(G) \leq \left\lfloor \frac{n-5}{3} \right\rfloor + 2 = \left\lfloor \frac{n+1}{3} \right\rfloor$.

Thus, the desired bound is established. 

**Corollary 2.23**

$$\lim_{n \to \infty} \frac{1}{n} = 1$$

**Proof:**

This follows from the fact that

$$\lim_{n \to \infty} \frac{1}{n} \leq \lim_{n \to \infty} \left\lfloor \frac{n+1}{3} \right\rfloor + 2}^{1/n}$$

$$\leq \lim_{n \to \infty} \left( \frac{n}{3} + n \right)^{1/n}$$

$$= 1$$
Corollary 2.24

\[ Z(17) = 6. \]

**Proof:**

We first show that \( Z(17) \geq 6 \). Note the graph \( G \) in Fig. 2.3. Also note, \( \omega(G) = \delta(G) = 3 \). By Proposition 1.6 we know that \( Z(G) \geq \{17/3\} \). Hence, \( Z(G) \geq 6 \). Since \( G \) has 17 vertices, it follows that \( Z(17) \geq 6 \).

Fig. 2.3

It is not difficult to show that \( Z(G) \) actually equals 6.

It was shown in \([K1]\) that the graph in Fig. 2.3
is the only graph on 17 vertices which contains no $K_4$ or $\overline{K_4}$ as an induced subgraph. So, suppose $G'$ is some graph on 17 vertices other than $G$. Remove four vertices from $G'$ which induce either a complete or an empty graph. The remaining graph has 13 vertices. According to Proposition 2.14, the remaining graph can have a cochromatic number of no more than 5. Hence, $Z(G') \leq 6$. 

Corollary 2.25

For any $n \geq 17$, $Z(n) \leq \left\lfloor \frac{n + 7}{4} \right\rfloor$.

Proof:

By the preceding corollary, this bound holds if $n = 17$. So, assume $G$ is a graph on $n \geq 18$ vertices. Now, it is well known that any graph on 18 vertices, or more, contains $K_4$ or $\overline{K_4}$ as an induced subgraph. Thus, we may successively remove groups of four vertices from $G$, each which induces a complete or an empty graph, until a graph on 17 or fewer vertices remains. This remaining graph has a cochromatic number of no more than 6. Hence

\[ Z(G) \leq \left\lfloor \frac{n - 17}{4} \right\rfloor + 6 \]

\[ = \left\lfloor \frac{n + 7}{4} \right\rfloor \].
Theorem 2.26

For any natural numbers \( m \) and \( n \) with \( n \geq r(m,m) \),

\[
Z(n) \leq \left\{ \frac{1 + n - r(m,m)}{m} \right\} + Z(r(m,m) - 1)
\]

Proof:

The proof follows in a similar fashion to the proof of the preceding corollary: remove sets of vertices of order \( m \) from any graph on \( n \) vertices, so that each set induces a complete or an empty graph. Do this until a graph on fewer than \( r(m,m) \) vertices is formed. This graph has a cochromatic number of no more than \( Z(r(m,m) - 1) \). Thus, the bound is established. ■

From Theorem 2.26 we see for a fixed \( m \) and any \( n \) that

\[
Z(n) \leq \left\{ \frac{1 + n - r(m,m)}{m} \right\} + Z(r(m,m) - 1) + r(m,m)
\]

\[
\leq \left\{ \frac{1 + n}{m} \right\} + Z(r(m,m) - 1) + r(m,m)
\]

\[
\leq \frac{1 + n}{m} + Z(r(m,m) - 1) + r(m,m) + 1
\]

\[
= \frac{n + (1 + m + m Z(r(m,m) - 1) + mr(m,m))}{m}
\]

We will use this and the following observation in the proof of the next theorem.

Lemma 2.27

For any \( n = 1, 2, \ldots \) the following inequality
holds
\[ Z(n + 1) \leq Z(n) + 1. \]

**Proof:**

Let \( G \) be a graph on \( n + 1 \) vertices with \( Z(G) = Z(n + 1) \). Select \( v \), a vertex of \( G \). Now, \( Z(G-v) \geq Z(G) - 1 \). Otherwise, we could cocolor \( G \) with less than \( Z(G) \) colors, by taking a minimal cocoloring of \( G - v \) and adding an extra color class, if necessary, for \( v \). Since \( G - v \) has \( n \) vertices

\[ Z(n) \geq Z(G - v) \geq Z(G) - 1 \]
\[ = Z(n + 1) - 1. \]

The desired result now follows.

**Theorem 2.28**

The function \( Z : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) is a monotone increasing surjection which is constant on arbitrarily large classes of integers.

**Proof:**

We first begin by showing that for all \( n = 1, 2, \ldots \) we have that

\[ Z(n) \leq Z(n + 1). \]

Let \( G \) be a graph on \( n \) vertices with \( Z(G) = Z(n) \). Let \( G' = G \cup K_1 \). Since \( G \) is an induced subgraph of \( G' \), \( Z(G) \leq Z(G') \). The graph \( G' \) has \( n + 1 \) vertices,
Thus

\[ Z(n) = Z(G) \]
\[ \leq Z(G') \]
\[ \leq Z(n + 1) . \]

Hence, \( Z \) is monotone increasing.

Recall that \( Z(n) \geq \lfloor \sqrt{n} \rfloor \). Hence, \( Z \) is unbounded. This, together with the fact that \( Z \) increases by at most one between successive integers shows that \( Z \) is a surjection.

Now, we wish to show that \( Z \) is constant on intervals of arbitrarily large size. So, suppose there is an \( L > 1 \) with the property that

\[ Z(n) < Z(n + L) , \]

for all \( n \geq 1 \). Since \( Z(1) = Z(2) \), we see \( L \geq 2 \).

Choose \( M > 1 \) so that

\[ Z(n) \leq \frac{n + M}{2L} \]

for all \( n = 1, 2, \ldots \).

Now, since \( Z(1) = 1 \),

\[ K < Z(1 + KL) , \]

for any \( K \geq 1 \).

So \( M < Z(1 + LM) \)

\[ \leq \frac{1 + LM + M}{2L} \]
\[ \leq \frac{2M + LM}{2L} . \]

Hence, \( 1 < \frac{2 + L}{2L} \leq \frac{L + L}{2L} = 1 . \)
Hence, an absurd conclusion is reached. Thus, for some $n$, $Z(n) = Z(n + L)$ and $Z$ is seen to be constant on an interval of length $L$. 

Conjecture

$Z(12) = 4$.

Since $Z(11) = 4$ it follows that $Z(12) \geq 4$.

From Proposition 2.22 we know that $Z(12) \leq 5$.

Any graph $G$ on 12 vertices which has a chromatic number of 5 would necessarily have the following properties:

(a) $\omega(G) = \omega(\overline{G}) = 3$
(b) $\beta(G) = \beta(\overline{G}) = 3$
(c) $P_4 \not\subseteq G, \overline{G}$
(d) $\chi(G), \chi(\overline{G}) \geq 5$

Thus, $G$ cannot contain $K_4$ and yet it must have a chromatic number of at least five. This would also apply to its complement. Indeed, it seems unlikely that a graph with all of these properties exists.
CHAPTER III

FURTHER RESULTS ON THE COCHROMATIC NUMBER

Section 3.1

Critically Cochromatic Graphs

A graph is said to be $n$-cochromatic if it has a cochromatic number of $n$. A cocoloring of a graph is an $n$-cocoloring if it has cardinality $n$. A graph $G$ is critically cochromatic if $Z(G-v) < Z(G)$ for each vertex $v$ of $G$. A graph is critically $n$-cochromatic if it is critically cochromatic and $n$-cochromatic. Clearly, if $G$ is critically $n$-cochromatic then $Z(G-v) = n-1$ for all vertices $v$ in $G$. Several examples of critically cochromatic graphs are odd cycles of order 5 or more, graphs of the form $K_1 U K_2 U \ldots U K_n$ and the graph shown in Fig. 3.1.

Fig. 3.1

Notice that the graph shown in Figure 3.1 is not critically chromatic. Examples can also be given of critically chromatic graphs which are not critically

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cochromatic. All nontrivial complete graphs are of this type.

**Conjecture**

If $G$ is both a critically chromatic graph and critically cochromatic then $G = C_{2n + 1}$ for some $n \geq 2$.

There clearly are critically cochromatic graphs of orders 3, 5, 6, 7, 9, 10 and 11. To see this, note that the graphs $P_3$, $C_5$, $K_{1, UK_2 UK_3}$, $C_7$, $C_9$, $K_{1, UK_2 UK_3 UK_4}$ and $C_{11}$ are critically cochromatic graphs with these orders. Clearly, there are no critically cochromatic graphs of orders 1 and 2. It is straightforward to verify that there is no critically cochromatic graph of order 4. It is not known whether there are critically cochromatic graphs of orders 8 and 12.

**Proposition 3.1**

For each $n \geq 13$ there is a critically cochromatic graph of order $n$.

**Proof:**

We will examine two cases depending on the parity of $n$.

**Case 1:** Suppose $n$ is odd. We have already noted that all odd cycles are critically cochromatic.

**Case 2:** Suppose $n$ is even. Let $m = n - 9$. Notice,
m is odd and $m \geq 5$. Let $G = K_{4}UK_{5}UC_{m}$. This graph has order $n$. Furthermore, since $Z(C_{m})=3$, and since $Z(K_{4}UK_{5})=2$, we see that $Z(G) \leq 5$. Now, $Z(G) > 4$.

For if the vertices of $G$ were partitioned into four cocolor classes, at least two of the vertices in the 5-clique would be placed in a single class. Removing the vertices in this class from $G$ produces a graph with cochromatic number 3 and which contains $K_{4}UC_{m}$ as an induced subgraph. Repeating this argument, we may remove the vertices of the 4-clique and conclude that the remaining graph, the $m$-cycle, has cochromatic number 2. Since we know this to be false, we see that $Z(G) > 4$ and hence $Z(G) = 5$.

If we remove a vertex from $G$ we are left with one of the graphs $K_{3}UK_{5}UC_{m}$, or $K_{4}UK_{4}UC_{m-1}$, or $K_{4}UK_{4}UC_{m}$. Since each of these graphs has cochromatic number 4, we may conclude that $G$ is a critically cochromatic graph of order $n$.

Suppose that $v$ is a vertex in a critically $n$-cochromatic graph and $V_{1}, V_{2}, \ldots, V_{n-1}$ is a cocoloring of $G-v$. Note, no $V_{i}$, $i=1, 2, \ldots, n-1$ contains exactly one vertex. Otherwise, it could be combined with $v$ and $G$ would be cocolored with $n-1$ colors. So, $|V_{i}| \geq 2$, $i=1, 2, \ldots, n-1$; thus

$$p(G) \geq 1 + 2(n-1) = 2n-1.$$
For \( n=2 \) and \( n=3 \) this cannot be improved. Consider the graphs \( P_3 \) and \( C_5 \). However, for \( n \geq 4 \) we have an improved bound.

**Theorem 3.2**

If a graph \( G \) is critically \( n \)-cochromatic then

(i) \( p(G) \geq 3n-4 \) (when \( n \geq 3 \))

and (ii) \( p(G) \geq 4n-12 \) (when \( n \geq 9 \)).

**Proof:**

To see (i), suppose that vertex \( v \) is selected from \( G \). Let \( V_1, V_2, \ldots, V_{n-1} \) be a cocoloring of \( G - v \). Suppose also that \( |V_1| = |V_2| = |V_3| = 2 \).

Then, \( \{v\} UV_1UV_2UV_3 \) has seven vertices. Since \( \chi(7) = 3 \), we can cocolor \( G \) with \( n-1 \) colors and hence, a contradiction. So, at most two color classes have cardinality 2. Hence

\[
p(G) \geq 1 + 2 \cdot 2 + 3(n-3) = 3n-4.
\]

To prove (ii) let \( \{v\} \) and \( V_1, V_2, \ldots, V_{n-1} \) be as above with \( n \geq 9 \). Suppose, that \( |V_i| \leq 3 \), \( i = 1, 2, \ldots, 8 \). Then \( \{v\} UV_1UV_2U...UV_8 \) has no more than 25 vertices and a cochromatic number of 9. But, by Corollary 2.12, \( \chi(25) \leq 8 \). Thus, a contradiction. So, at most seven of these classes have a cardinality 3 or less. From our preceding work we know that at most
two have a cardinality of two. Thus,
\[ p(G) \geq 1 + 2 \cdot 2 + 3 \cdot 5 + 4 \cdot (n-8) = 4n - 12. \]

The first bound is sharp; consider \( C_5 \). It is not known whether (ii) is sharp.

Clearly, these bounds can be generalized. The concept is, however, strongly linked with the function \( Z : Z^+ \rightarrow Z^+ \) in the following sense. Let \( C(m) \) be
\[ \operatorname{Min} \{ n \mid Z(n) = m \}, \text{ for } m \text{ a natural number. Given } m, \ Z(C(m)) = m \text{ and } Z(C(m)-1) = m - 1. \]

Now, any graph which is critically \( n \)-cochromatic must have an order of at least \( C(n) \). Also, any graph on \( C(n) \) vertices with a cochromatic number of \( n \) is critically \( n \)-cochromatic. Accordingly, as we learn more about the function \( Z : Z^+ \rightarrow Z^+ \), more will be known about critically cochromatic graphs.

The proof of the preceding bounds led us to observe that for any vertex \( v \) in a critically \( n \)-cochromatic graph \( G \), there was an \( n \)-coloring of \( G \) which assigned \( v \) to a class by itself. Thus, there are at least as many different \( n \)-colorings as there are vertices.

**Proposition 3.3**

Given two vertices \( u \) and \( v \) in a critically \( n \)-cochromatic graph \( G \), there is an \( n \)-coloring of
G which contains \{u,v\} as a cocolor class.

**Proof:**

Let G be a critically n-cochromatic graph with vertices u and v. Remove u and v from G. Let us cocolor G-u-v with n-1 colors. Add to this cocoloring the class \{u,v\}. Since \{u,v\} induces either a complete or an empty graph, this is an n-cocoloring of G.

Suppose that G is a graph with \(Z(G) = n\). Let \(V_1, V_2, ..., V_n\) be a cocoloring of G. One can see that \(G - V_n\) has cochromatic number n-1; the partition \(V_1, V_2, ..., V_{n-1}\) is a cocoloring of \(G - V_n\). Thus, \(Z(G - V_n) < n-1\). But if \(Z(G - V_n) = m < n-1\) then we could add \(V_n\) to an m-cocoloring of \(G - V_n\) and cocolor G with less than n colors. We will use this observation several times in this chapter.

**Proposition 3.4**

If G is a critically n-cochromatic graph then \(Z(G-u-v) = n-1\) for all u and v in V(G).

**Proof:**

By Proposition 3.3 there is an n-cocoloring of G which has \{u,v\} as a cocolor class. By the preceding remark the above statement follows.

From this, we see that if G is a critically n-
cochromatic graph then $G-v$ is stable in the sense that the removal of any other vertex will not affect the cochromatic number.

Every critically $n$-cochromatic graph has many different $n$-cocolorings; at least as many as the order of the graph. In the next section, we will examine graphs which have only one minimal cocoloring.

Section 3.2

Uniquely Cochromatic Graphs

Suppose $G$ is a graph with chromatic number $n$. The graph $G$ is uniquely chromatic if given the color classes for two colorings of $G$, say $\{S_1, S_2, \ldots, S_n\}$ and $\{T_1, T_2, \ldots, T_n\}$, $n = \chi(G)$, then $\{S_1, S_2, \ldots, S_n\} = \{T_1, T_2, \ldots, T_n\}$. In this case, we will also say that $G$ is uniquely $n$-chromatic. All complete graphs and all even cycles are uniquely chromatic graphs. Likewise, a graph is uniquely cochromatic if any two $Z(G)$-cocolorings of $G$ induce the same vertex partition of $G$. A graph is uniquely $n$-cochromatic if it is uniquely cochromatic and has cochromatic number $n$. All $n$-cycles, where $n$ is even and greater than 3, are uniquely cochromatic as are graphs of the form $mK_p$ where $p > m$. Also, the complement of any uniquely cochromatic is uniquely cochromatic.

Proposition 3.4
No uniquely cochromatic graph is critically cochromatic.

**Proof:**

We have observed that any critically cochromatic graph $G$ has as many $Z(G)$-cocolorings as vertices. Thus, every critically cochromatic graph has at least two $Z(G)$-cocolorings.

Given a graph $G$, the **connectivity** of $G$, denoted $\kappa(G)$, is the fewest number of vertices whose removal from $G$ produces a disconnected or trivial graph. We say that $G$ is $m$-connected if $\kappa(G) \geq m$. We shall see that critically cochromatic graphs and connectivity are related. But first, we need two preliminary remarks.

**Lemma 3.5**

If $G$ is an $n$-connected graph and $H$ is formed by adding a vertex to $G$ and at least $n$ edges incident with the added vertex, then $H$ is $n$-connected.

This observation was made in [BCL1]. Its proof is not difficult.

**Lemma 3.6** (Chartrand-Geller)

If $G$ is a uniquely $n$-chromatic graph then $G$ is $(n-1)$-connected.

A proof of this may be found in [CG1].
Theorem 3.7

If a graph $G$ is uniquely $n$-cochromatic then $G$ or $\overline{G}$ is $\left\lceil \frac{n-2}{2} \right\rceil$-connected.

Proof:

Let $G$ be a uniquely $n$-cochromatic graph. Set $m$ equal to $\left\lceil \frac{n}{2} \right\rceil$. Let us label the cocolor classes of $G$ with $V_1, V_2, \ldots, V_n$. Now at least $m$ of these classes induce empty graphs in $G$ or $\overline{G}$. If $m$ of these classes induce empty graphs in $G$, then we will show that $G$ is $(m-1)$-connected. Otherwise, we will see that $\overline{G}$ is $(m-1)$-connected.

For the first case, let us suppose, without loss of generality, that the classes $V_1, V_2, \ldots, V_m$ induce empty graphs in $G$. Let $H$ be the graph $\langle V_1 U V_2 U \ldots U V_m \rangle$. The chromatic number of $H$ can be seen to be $m$: since $V_1, V_2, \ldots, V_m$ induce empty graphs, $\chi(H) \leq m$. If $\chi(H)$ were less than $m$, we could partition $V(H)$ into less than $m$ color classes, then add to this the sets $V_{m+1}, \ldots, V_n$. This, however, is a coloring of $G$ of cardinality less than $n$, and hence a contradiction. Further, we see that $H$ is uniquely $m$-chromatic. For, if there were an $m$-coloring of $H$ different from $V_1, V_2, \ldots, V_m$ we could take this second coloring, add to it the sets $V_{m+1}, V_{m+2}, \ldots, V_n$ and this would produce an $n$-cocoloring of $G$ different.
from our original. But, since $G$ was uniquely cochromatic, this is impossible. We conclude that $H$ is uniquely $m$-chromatic and by Lemma 3.6 $H$ must be $(m-1)$-connected. Now, suppose $v$ is a vertex of $G-V(H)$. Given $i$ between 1 and $m$, the vertex $v$ must be adjacent to some vertex of $V_i$. For if it were adjacent to no vertices in $V_i$, we could partition the vertices of $G$ as we did before, except that $v$ would be placed in $V_i$ instead of its original cocolor class. This would produce a second $n$-cocoloring of $G$ and contradict the fact that $G$ is uniquely $n$-cocolorable. Thus, $<V_1UV_2U\ldots UV_mU\{v\}>_G$ is the graph $H$, with an added vertex $v$, together with at least $m$ edges incident to $v$. Thus, $<V_1UV_2U\ldots UV_mU\{v\}>_G$ is $(m-1)$-connected. Continue to add vertices of $G-V(H)$ and incident edges. Repeating the above argument, we see that with each addition, an $(m-1)$-connected graph is formed. Completing this process with $G$, we see that $G$ is $(m-1)$-connected. Since

$$m - 1 = \left\lceil \frac{n}{2} \right\rceil - 1$$

$$= \left\lceil \frac{n-2}{2} \right\rceil$$

We see that in this case, $G$ is $\left\lceil \frac{n-2}{2} \right\rceil$-connected.

For the second case, suppose $V_1', V_2', \ldots, V_m$ induce empty graphs in $G$. Repeat the entire argument.
in the first case, with $\bar{G}$ in place of $G$. One concludes that $\bar{G}$ is $\{\frac{n-2}{2}\}$-connected.

**Corollary 3.8**

If $G$ is a uniquely $n$-c chromatic graph, then

$$\text{Max}\{\delta(G), \delta(\bar{G})\} \geq \left\lfloor \frac{n-2}{2}\right\rfloor .$$

**Proof:**

We need merely note that if a graph $G$ is $m$-connected, then $\delta(G) \geq m$. Since under the hypothesis $G$ or $\bar{G}$ is $\{\frac{n-2}{2}\}$-connected, we see that the conclusion holds.

### Section 3.3

Comaximal and Minimally Cochromatic Graphs

In this section we shall consider the removal of edges from a graph and the effect that this has on the cochromatic number of the graph. Our first remark shows that such an operation alters the cochromatic number by at most one.

**Proposition 3.9**

Given a graph $G$ and an edge $e$ of $G$ then

$$Z(G) - 1 \leq Z(G-e) \leq Z(G) + 1 .$$

**Proof:**

Suppose that edge $e$ is incident with the vertex $u$. Let $V_1, V_2, \ldots, V_n$ be a cocoloring of $G-e$. 

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where \( n = Z(G-e) \). Without loss of generality suppose that \( u \in V_1 \). Now, if \( V_1 \) is a singleton, then \( V_1, V_2, \ldots, V_n \) induce complete or empty graphs in \( G \). If \( V_1 \) is not a singleton, then \( V_1-u, V_2, V_3, \ldots, V_n, \{u\} \) each induce complete or empty graphs in \( G \). Hence, \( G \) can be cocolored with \( Z(G-e) + 1 \) colors.

Similarly, \( Z(G-e) \leq Z(G) + 1 \).

We will say that a graph \( G \) is minimally cochromatic if the removal of any edge from \( G \) reduces the cochromatic number of \( G \). A graph \( G \) is minimally \( n \)-cochromatic if \( Z(G) = n \) and \( G \) is minimally cochromatic. A graph \( G \) is comaximal if the removal of any edge increases the cochromatic number of \( G \).

As examples, one can see that any graph of the form \( K_1 \cup K_2 \cup \ldots \cup K_n, n \geq 2 \), is minimally \( n \)-cochromatic; any complete graph on three or more vertices is comaximal. Empty graphs are vacuously comaximal as they are minimally cochromatic. If a graph \( G \) contains an edge, its removal will not both increase and decrease the cochromatic number of \( G \). Hence, empty graphs are the only graphs which are both comaximal and minimally cochromatic.

There are graphs which are critically cochromatic but not minimally cochromatic. It is easy to verify that the graph in Figure 3.1 has this property.
However, as our next theorem indicates, minimally cochromatic graphs are, in general, critically cochromatic.

**Theorem 3.10**

If $G$ is a minimally cochromatic graph containing no isolated vertices then $G$ is critically cochromatic.

**Proof:**

Assume $G$ is a graph satisfying the hypothesis of this theorem. Select $v$ a vertex of $G$ and $e$ an edge incident to $v$. We wish to show that $Z(G-v) < Z(G)$. Note that $G-v$ is an induced subgraph of $G-e$. Thus $Z(G-v) < Z(G-e) < Z(G)$.

Since $v$ was arbitrarily chosen, we conclude that $G$ is critically cochromatic.

We now turn our attention to the concept of comaximal graphs.

**Theorem 3.11**

If $G$ is a comaximal graph, then $G$ is uniquely cochromatic.

**Proof:**

As we have noted, any empty graph or any $K_p$, $p \geq 3$, is comaximal. But clearly, any empty or complete graph is uniquely cochromatic.

So, suppose $G$ is a nonempty noncomplete comaximal graph. Let $V_1, V_2, \ldots, V_n$ be a cocoloring of $G$,
where \( n = Z(G)^2 \). Now, there is no edge \( uv \) of \( G \) with \( u \) in one class and \( v \) in another. Otherwise, we could cocolor \( G - uv \) with the same classes \( V_1, V_2, \ldots, V_n \). Now, we observe that each class \( V_i \) induces a complete graph. If this were not the case, some set, say \( V_1 \), would induce an empty graph. The set \( V_2 \) does not induce an empty graph, for otherwise \( V_1 U V_2, V_3, V_4, \ldots, V_n \) would be an \((n-1)\)-cocoloring of \( G \). Now select vertex \( v \) from \( V_2 \). Remove any edge \( e \) incident with \( v \). Cocolor the resulting graph with \( V_1 U \{v\}, V_2 - \{v\}, V_3, V_4, \ldots, V_n \). This contradicts the fact that \( G - e \) must have cochromatic number greater than \( n \).

Now, suppose \( G \) is not uniquely cochromatic. Let \( U_1, U_2, \ldots, U_n \) be an \( n \)-cocoloring of \( G \) different from our original. Since it is different, two vertices, say \( x \) and \( y \), which were in the same cocolor class originally must now be in separate classes. Since this original class induced a complete graph, \( x \) and \( y \) are adjacent in \( G \). But \( U_1, U_2, \ldots, U_n \) is a cocoloring of \( G - xy \) contradicting the fact that \( G \) is comaximal. Hence, there is only one minimal cocoloring of \( G \). Thus, \( G \) is uniquely cochromatic. \( \blacksquare \)

**Corollary 3.12**

Any comaximal graph is not critically cochromatic.
A nonempty graph $G$ is comaximal if and only if $G = K_{n_1} U K_{n_2} U \ldots U K_{n_m}$ where

1. $3 \leq n_1 \leq n_2 \leq \ldots \leq n_m$,
2. $i < n_i$ for $i = 1, 2, \ldots, m$,
3. $n_i = i + 1 \Rightarrow n_{i-1} = i + 1$.

Proof:

Suppose $G$ is a nonempty comaximal graph with $Z(G) = m$. Then it follows from the proof of Theorem 3.12 that $G = K_{n_1} U K_{n_2} U \ldots U K_{n_m}$, where $n_1 \leq n_2 \leq \ldots \leq n_m$. We also saw that in a comaximal graph such as this,
We will first show that \( 3 \leq n_1 \). Clearly, \( n_1 \) and \( n_2 \) are not both 1. Suppose that \( n_1 = 1 \). Let \( u \) be the vertex in \( K_{n_1} \). Let \( v \) be a vertex in \( K_{n_2} \). Let \( V_1 = \{u, v\} \). Let \( V_2 = V(K_{n_2} - v) \). Cocolor \( G \) with \( V_1, V_2, \ldots, V_m \) with \( V_i = V(K_{n_i}) \), \( i = 3, 4, \ldots, m \). This contradicts the fact that \( G \) is uniquely cochromatic. So, suppose that \( n_1 = 2 \). The removal of the only edge in \( K_{n_1} \) will clearly not increase its cochromatic number. We conclude that \( 3 \leq n_1 \).

We next show that \( i \leq n_i \) for \( i = 1, 2, \ldots, m \). To see this suppose that there is an \( i \) between 1 and \( m \) with the property that \( n_i < i \). Again, there are two ways to \( m \)-cocolor \( G \). We may cocolor \( G \) as we did above with \( V(K_{n_1}), \ldots, V(K_{n_m}) \) as cocolor classes or we may partition the vertices of \( K_{n_1} \cup \ldots \cup K_{n_m} \) into \( i \) sets, each of which induces an empty graph and put the vertices of \( K_{n_1} \cup \ldots \cup K_{n_{i+1}} \) into \( m - i \) sets, each of which induces a complete graph. Thus, if \( n_i < i \) for some \( i \), the graph \( G \) is not uniquely cochromatic and thus, not comaximal.

Now, suppose that there is a \( k \) with \( n_k = k + 1 \) but \( n_{k-1} \neq k+1 \). Since \( k-1 < n_{k-1} \) we see that \( n_{k-1} = k \). Let \( e = uv \) be an edge of \( K_{n_k} \). Let \( H = G - e \). We will see that \( Z(H) \leq m \). But this will be a contradiction;
since $G$ is comaximal, $Z(H) > m$. Cocomber $H$ so that the cliques $K_{n_{k+1}}$, $K_{n_{k+2}}$, ..., $K_{n_m}$ are each assigned to distinct classes. Select a vertex $v_i$ from each $K_{n_i}$ where $i = 1, 2, \ldots, k-1$. Let \( \{v_1', v_2', \ldots, v_{k-1}', u, v\} \) be a cocomber class. Note that $K_{n_1} U K_{n_2} \ldots U K_{n_k} = \{v_1', v_2', \ldots, v_{k-1}', u, v\} = K_{n_1-1} U K_{n_2-1} \ldots U K_{n_{(k-1)-1}} U K_{n_k-2}$ where $n_1-1 \leq n_2 \leq \ldots \leq n_k-1 - 1 \leq n_k - 2 = k - 1$. This graph can be cocombered with $k-1$ colors, where each color class induces an empty graph. Hence, $H$ can be cocombered with $(m-k) + 1 + (k-1) = m$ colors. We conclude that if $n_k = k + 1$ for some $k$, then $n_{k-1} = k + 1$.

We are now prepared to prove that the converse statement holds. Suppose that $G = K_{n_1} U K_{n_2} \ldots U K_{n_m}$ where $n_1$, $n_2$, ..., $n_m$ satisfy the three conditions given in the theorem. We will first show that $Z(G) = m$ and then that for any edge $e$ of $G$ we have $Z(G-e) \geq m+1$.

Since $G$ is the union of $m$ cliques, $Z(G) \leq m$. Also, $G$ contains $K_{n_1} U K_{n_2} \ldots U K_{n_m}$ as an induced subgraph. This follows from the fact that $n_i > i$ for $i = 1, 2, \ldots, m$. Thus $Z(G) \geq Z(K_{n_1} U K_{n_2} \ldots U K_{n_m}) = m$. It follows that $Z(G) = m$.

Now, suppose that we are given an edge $e = uv$. Say that $e \in K_{n_i}$. We would like to show that $Z(G-e) \geq m + 1$. We will do this by examining the following
three cases.

Case 1: Suppose that \( i \geq 2 \) and that \( n_i \geq i + 2 \).

Let \( H \) be the graph

\[
K_3 UK_4 UK_5 U \ldots UK_i UK_{i+2} - eUK_{i+2}
\]

\[
UK_{i+3} U \ldots UK_{m+1}.
\]

Note that \( H \) is an induced subgraph of \( G-e \). We will show that \( Z(H) \geq m + 1 \) and thus, conclude that \( Z(G-e) \geq m + 1 \). Let us cocolor \( H \) with as few colors as possible.

Suppose that \( u \) and \( v \) are in the same cocolor class. Call this class \( V \). Now, \( H - V \) contains

\[
K_1 UK_2 UK_3 U \ldots UK_{i-1} UK_i UK_{i+1} U \ldots UK_m.
\]

as an induced subgraph. This follows from the fact that \( V \) can contain at most one vertex from each of the cliques which do not contain \( e \). But the above graph has cochromatic number of \( m \). Thus, \( Z(H) \geq m + 1 \) and our desired result is attained.

So, let us suppose instead that \( u \) and \( v \) are placed in different cocolor classes. Without loss of generality say that \( u \in V_1 \) and \( v \in V_2 \), where \( V_1 \) and \( V_2 \) are cocolor classes. Now, either \( V_1 \) and \( V_2 \) both induce complete graphs or at least one of them does not.
If both $V_1$ and $V_2$ induce complete graphs, then both come from the same component $K_{i+2}$. Thus, $H - (V_1 \cup V_2)$ contains $K_3 \cup K_3 \cup \ldots \cup K_{i+2} \cup K_{i+3} \cup \ldots \cup K_{m+1}$ as an induced subgraph. But let us note in turn that this graph contains $K_1 \cup K_2 \cup \ldots \cup K_{m-1}$ as an induced subgraph. Hence, $Z(H - (V_1 \cup V_2)) \geq m - 1$ and $Z(H) \geq m + 1$.

We are left with the possibility that $V_1$ or $V_2$ induces an empty graph.

Let us suppose, without loss of generality, that $V_1$ induces an empty graph. Now, $H - V_1$ contains $K_3 \cup K_3 \cup \ldots \cup K_{i-1} \cup K_{i} \cup K_{i+1} \cup \ldots \cup K_{m}$ as an induced subgraph.

This is because each component of $H$ can contain at most one vertex of $V_1$. Note, this graph contains $K_1 \cup K_2 \cup \ldots \cup K_m$ as an induced subgraph. But again, the cochromatic number of this graph is $m$. Hence, $Z(H) \geq m + 1$.

We conclude, that in this case, $Z(G - e) \geq m + 1$.

**Case 2:** Suppose that $i \geq 2$ and $n_i = i + 1$. Note, that in this case, $n_{i-1} = i + 1$. Again, we will construct $H$, an induced subgraph of $G - e$, and show that $Z(H) \geq m + 1$.

Let $H$ be the graph $K_3 \cup K_4 \cup \ldots \cup K_{i-1} \cup K_{i+1} \cup \ldots \cup K_{m+1}$. Notice, $H$ is an induced

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subgraph of $G-e$. Let us cocolor $H$ with the fewest number of colors possible. We will see that $Z(H) \geq m + 1$.

Suppose that $u$ and $v$ are both in the cocolor class $V$. Then $V$ must induce an empty graph. No other vertices in the component of $H$ to which $u$ and $v$ belong are in $V$. This is because all other vertices in this component are adjacent to $u$ and $v$. Also, at most one vertex from each of the other components is in $V$. Thus, $H-V$ contains

$$K_2 \cup K_2 \cup \ldots \cup K_{i-2} \cup K_1 \cup K_{i-1} \cup K_{i+1} \cup K_{i+2} \cup \ldots \cup K_m$$

as an induced subgraph. But it is clear that this graph contains $K_1 \cup K_2 \cup K_3 \cup \ldots \cup K_m$ as an induced subgraph. Thus, $Z(H-V) \geq m$ and $Z(H) \geq m + 1$.

So, suppose that $u$ is in cocolor class $V_1$ and that $v$ is in $V_2$. If $V_1$ and $V_2$ are both cliques then both are subgraphs of the same component. So $H-(V_1V_2)$ contains

$$K_3 \cup K_3 \cup K_4 \cup \ldots \cup K_{i-1} \cup K_{i+1} \cup K_{i+2} \cup \ldots \cup K_{m+1}$$

as an induced subgraph. But this graph contains $K_1 \cup K_2 \cup \ldots \cup K_{m-1}$ as an induced subgraph. We conclude that $Z(G) \geq m + 1$.

So, suppose that one of $V_1$ or $V_2$ induces an empty graph. Without loss of generality say that $V_1$...
is an empty graph. Now, at most one vertex from each component of \( H \) is in \( V_1 \). So \( H-V_1 \) contains

\[
K_2 \cup K_3 \cup \ldots \cup K_{i-2} \cup K_{i-1} \cup K_i \cup K_{i+1} \cup \ldots \cup K_m
\]
as an induced subgraph. But this graph has cochromatic number \( m \). It too contains \( K_1 \cup K_2 \cup \ldots \cup K_m \) as an induced subgraph. Hence, \( Z(H) > m + 1 \).

We conclude that in this second case, \( Z(G-e) > m + 1 \).

**Case 3:** Suppose that \( i = 1 \). Let \( H = K_3 - e \cup K_4 \cup K_5 \cup \ldots \cup K_{m+1} \). Now, cocolor \( H \) with as few colors as possible. Let \( V \) be the cocolor class containing \( u \). Now \( V \) does not contain all vertices in the first component. This is because the first component is neither a complete nor an empty graph. Further, at most one vertex from each of the other components is in \( V \). For if two vertices, say \( x \) and \( y \), were in \( V \) and both in some component other than the first one, the graph induced by \( u, x, y \) would be \( K_1 \cup K_2 \). Since \( V \) is complete or empty this is impossible. We conclude that \( H-V \) contains an induced subgraph isomorphic to \( K_1 \cup K_2 \cup \ldots \cup K_m \). Since this graph has a cochromatic number of \( m \) we conclude that \( Z(H) > m + 1 \).

Thus, \( Z(G-e) > m + 1 \) and our third and final case is complete.

The converse is now established.
its cochromatic number. For example, if an edge is added to \( P_3 \) it becomes \( K_3 \) and its cochromatic number subsequently drops. If an edge is added to \( K(3,3) \) its cochromatic number is increased. Note that \( \overline{G + e} = \overline{G} - e \), where \( e \) is any edge not in \( G \). This observation yields two corollaries.

**Corollary 3.14**

Given a graph \( G \) and an edge \( e \) in \( \overline{G} \) then

\[
Z(G) - 1 \leq Z(G + e) \leq Z(G) + 1.
\]

**Proof:**

We first observe that \( Z(G + e) = Z(\overline{G + e}) = Z(\overline{G} - e) \).

From Proposition 3.9 we have \( Z(\overline{G}) - 1 \leq Z(\overline{G} - e) \leq Z(\overline{G}) + 1 \).

Since \( Z(\overline{G}) = Z(G) \) the desired result follows. \( \square \)

In a similar manner, the following can be established.

**Corollary 3.15**

A graph \( G \) has the property that the addition of any edge increases its cochromatic number if and only if \( G \) is complete or if it is of the form \( K(n_1, n_2, \ldots, n_m) \) where

1) \( 3 \leq n_1 \leq n_2 \leq \cdots \leq n_m \)
2) \( i < n_i \) for \( i = 1, 2, \ldots, m \)
3) \( i + 1 = n_i \Rightarrow i + 1 = n_{i-1} \).

This completes the work of this chapter. In the next chapter, we will be concerned with a concept closely related to cochromatic theory.
CHAPTER IV

THE ACOCHROMATIC NUMBER OF A GRAPH

Given a graph $G$, let a vertex partition $V_1, V_2, \ldots, V_n$ be a complete coloring if the following holds:

$$i=j \iff <V_i, V_j>$$

is an empty graph.

That is, each subset in this partition induces an empty graph but the union of any two sets does not. We can then define $\chi(G)$ by

$$\chi(G) = \min \{n \mid V_1, V_2, \ldots, V_n \text{ is a complete coloring of } G \}.$$ 

The achromatic number, $a\chi(G)$, of $G$ is defined by

$$a\chi(G) = \max \{n \mid V_1, V_2, \ldots, V_n \text{ is a complete coloring of } G \}.$$ 

More can be found on this subject in [BH1], [Gl], [HH1], [HHPl], and [HML].

Likewise, let us say that a vertex partition $V_1, V_2, \ldots, V_n$ of a graph $G$ is a complete cocoloring if the following holds: $i=j \iff <V_i, V_j>$ is a complete or empty graph. We can then define $Z(G)$ by

$$Z(G) = \min \{n \mid V_1, V_2, \ldots, V_n \text{ is a complete cocoloring of } G \}.$$ 

The acochromatic number, $aZ(G)$, of $G$ is defined by

$$aZ(G) = \max \{n \mid V_1, V_2, \ldots, V_n \text{ is a complete cocoloring of } G \}.$$ 

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Proposition 4.1

For a graph $G$, the acocromatic number of $G$ is bounded above by $\binom{p}{2}$, where $p$ is the order of $G$.

Proof:

Let $V_1, V_2, \ldots, V_n$ be a complete cocoloring of $G$ with $n = aZ(G)$. At most one class is a singleton. For suppose $V_i$ and $V_j$ were singletons with $i \neq j$. Then $\langle V_i, U V_j \rangle$ would be complete or empty, contradicting the definition of complete cocolorings. So, $p = \sum_{i=1}^{n} |V_i| \geq 1 + 2(n-1)$. Thus, $\frac{p+1}{2} \geq n = aZ(G)$. But since $aZ(G)$ is an integer, $\{\frac{n}{2}\} = \lceil \frac{p+1}{2} \rceil \geq aZ(G)$.

We now compute the acocromatic number of several graphs.

First, $aZ(K_p) = 1$ for all $p$. Suppose the vertex set of a complete graph is decomposed into two or more subsets. The union of any two subsets induces a complete graph. Hence, such a decomposition is not a complete cocoloring.

For all $n \geq 4$ we have $aZ(C_n) = \{\frac{n}{2}\}$. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle with $v_i$ adjacent to $v_{i+1}$ where $i = 1, 2, \ldots, n-1$ and $v_1$ is adjacent to $v_n$. Let us examine the cases where $n$ is even and odd. We will show that in each $aZ(C_n) \geq \{\frac{n}{2}\}$.

Case 1: Suppose that $n$ is even. Decompose the vertex set into classes $\{v_1, v_2\}, \{v_3, v_4\}, \ldots$. 

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Clearly, each class induces a complete graph. Further, the union any two distinct classes does not induce a complete graph, for this would imply that the cycle contained $K_4$. So, in this case $AZ(G) \geq \frac{n}{2} = \{\frac{n}{2}\}$.

**Case 2:** Suppose that $n$ is odd. Decompose the vertex set into $\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_n\}$. This decomposition has cardinality $\{\frac{n}{2}\}$. But as in Case 1, no two sets of cardinality 2 may be combined to induce $K_4$. Also, $\langle \{v_j, v_{j+1}\} \cup \{v_n\} \rangle$ is either $P_3$ or $K_2UK_1$, depending on whether $j = 1, n-2$. Thus, in this case $Z(C_n) \geq \{\frac{n}{2}\}$.

But from the preceding remark, $Z(C_n) \leq \{\frac{n}{2}\}$. Thus, equality holds in each case.

By a similar argument one sees that for $P_n$, a path on $n$ vertices, $AZ(P_n) = \{\frac{n}{2}\}$.

**Remark 4.2** Given $G$ a graph, $AZ(G) = AZ(\overline{G})$.

**Proof:**

This follows from noting that $\langle V_1UV_j \rangle_G$ is a complete graph if and only if $\langle V_1UV_j \rangle_{\overline{G}}$ is an empty graph. Also, $\langle V_1UV_j \rangle_G$ is empty if and only if $\langle V_1UV_j \rangle_{\overline{G}}$ is complete.

It follows that $V_1, V_2, \ldots, V_n$ is a complete cocoloring of $G$ if and only if it is a complete cocoloring of $\overline{G}$. 

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It follows that the acochromatic number of an empty graph is 1.

We recall that \( Z(G) \leq \chi(G) \) for any graph \( G \). It is natural to ask what relationship exists between the functions \( aZ \) and \( a\chi \). We will show that in general, neither \( aZ(G) - a\chi(G) \) nor \( a\chi(G) - aZ(G) \) is bounded above. This is easily seen from the following constructions.

Let \( G_n = nK_2 \). We see that by placing each pair of adjacent vertices into distinct classes, a complete co-coloring is formed: each class induces \( K_2 \) and the union of any two induces \( 2K_2 \). Thus, \( aZ(G_n) \geq n \). From Remark 4.1, we conclude that \( aZ(G_n) = n \).

We will now compute \( a\chi(G_n) \) to show that \( aZ(G) - a\chi(G) \) is not bounded above. Suppose that \( r = a\chi(G_n) \). Now, given any two distinct classes from a complete coloring of \( G_n \), there must be at least one edge incident with a vertex in both classes. Hence, \( \left( \frac{n}{2} \right) \geq n \). Thus, \( r \leq \frac{1 + \sqrt{1 + 8n}}{2} \). Further, if \( k \) is an integer greater than 1 and \( \left( \frac{k}{2} \right) \leq n \) we see that there is a complete coloring of \( G_n \) of cardinality \( k \). Label the vertices of \( G_n \) with \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \) so that the edge set of \( G_n \) is \( \{ u_i v_i \mid i = 1, 2, \ldots, n \} \). Set \( m \) equal to \( \left( \frac{k}{2} \right) \) and let \( P_1, P_2, \ldots, P_m \) be the combinations of elements taken two at a time from \( \{ 1, 2, \ldots, n \} \).
k}. For i=1, 2, ..., m denote \( P_i \) as \( \{ r_i, t_i \} \) where \( r_i < t_i \). Form a complete coloring of \( G_n \) with the sets \( S_1, S_2, ..., S_k \) where vertex \( u_i \) is in \( S_{r_i} \) and \( v_i \) is in \( S_{t_i} \), \( i = 1, 2, ..., m \). If \( m < n \) complete the coloring by placing \( u_i \) is \( S_1 \) and \( v_i \) in \( S_2 \) where \( i = m+1, m+2, ..., n \).

Accordingly, \( ax(G^n) \geq \text{Max} \{ m \mid \binom{m}{2} \leq n \} \). Thus, we conclude \( ax(G^n) = \left[ 1 + \frac{\sqrt{1+8n}}{2} \right] \). Also, \( aZ(G^n) - ax(G) = \left[ \frac{2n-1-\sqrt{1+8n}}{2} \right] \). Hence, \( aZ(G) - ax(G) \) is not bounded above.

We now note that \( ax(G) - aZ(G) \) is also unbounded above. Recall that the acocromatic number of a complete graph is 1. Clearly, the achromatic number of the complete graph \( K_p \) is \( p \). Thus \( ax(K_p) - aZ(K_p) = p - 1 \).

We may think of the achromatic and the acocromatic number as being unrelated in the sense that the achromatic number of a graph may be arbitrarily larger than the acocromatic number or may be arbitrarily smaller. We here conjecture that for all natural numbers \( n \) and \( m \), if \( m \geq 2 \) then there is a graph \( G \) with \( aZ(G)=n \) and \( ax(G)=m \). We also conjecture that for integers \( m \) and \( n \) with \( 2 \leq m \leq n \) there are graphs \( G_1 \) and \( G_2 \) such that \( \chi(G_1)=m \).
\[ ax(G_1) = n \]
\[ Z(G_2) = m \]
\[ aZ(G_2) = n . \]

We will say that a set of edges is independent in a graph \( G \) if no two edges in the set are adjacent. The edge independence number of a graph \( G \) is the maximum cardinality of all independent sets of edges from \( G \). The edge independence number of \( G \) will be denoted \( \beta_1(G) \). An independent set of edges is maximum if it contains \( \beta_1(G) \) edges. A set \( S \) of independent edges in a graph \( G \) is a \( 1 \)-factor if every vertex of \( G \) is incident with an edge of \( S \).

Before we present Theorem 4.4 let us make a useful observation.

**Lemma 4.3**

If \( B \) is a bipartite graph then \( ax(B) \leq \beta_1(B) + 1 \).

**Proof:**

It was shown by König [Kl] that \( \alpha_0(B) = \beta_1(B) \) for a bipartite graph \( B \), where \( \alpha_0(B) \) is the smallest number of vertices which cover the edges of \( B \). Further, in [HH1] it was shown that \( ax(G) \leq \alpha_0(G) + 1 \), for any graph \( G \). The desired result now follows. \( \square \)

Let us recall the well known result that a graph is bipartite if and only if it contains no odd cycles. Specifically, a bipartite graph contains no triangles.
We will make use of this and the fact that a subgraph of a bipartite graph is bipartite.

**Theorem 4.4**

For a bipartite graph, $\beta_1(B) \leq \alpha_2(B) = \beta_1(B) + 1$,

where the lower bound holds if and only if $B$ contains a 1-factor.

**Proof:**

Set $m = \beta_1(B)$. Let $\{c_1, c_2, \ldots, c_m\}$ be a set of independent edges in $B$. For $i = 1, 2, \ldots, m$ let $V_i$ be the set of vertices incident with $c_i$.

Let $V_{m+1} = V(B) - \bigcup V_i$.

If $V_{m+1} = \emptyset$, we see $V_1, V_2, \ldots, V_m$ is a complete cocoloring of $B$. Each set induces a complete graph on two vertices. Since $B$ contains no triangles, the union of any two distinct sets, $V_i$ and $V_j$, does not induce a complete graph. Since each set induces an edge the union of any two does not induce an empty graph.

If $V_{m+1} \neq \emptyset$, we see that $V_1, V_2, \ldots, V_{m+1}$ is a complete cocoloring of $B$. Clearly, $\langle V_{m+1} \rangle$ is empty, for if it contained an edge, $\{c_1, c_2, \ldots, c_m\}$ would not be maximum. Note, for $i=1, 2, \ldots, m$ the graph $\langle V_i U V_{m+1} \rangle$ is not complete. If it were complete, $B$ would contain a triangle, since $V_i U V_{m+1}$ has
cardinality at least 3. Since \( V_i \) induces an edge, 
\( <V_iUV_{m+1}> \) is not empty. As we noted in the case where 
\( V_{m+1} \) is empty, the union of any other two classes does 
not induce a complete or an empty graph. Since 
a\( \leq \) is the maximum cardinality of all complete co-
colorings, \( \beta_1(B)=m \leq a\leq(B) \).

Now, suppose we have chosen a complete cocoloring 
\( V_1, V_2, \ldots, V_n \) with \( n=a\leq(B) \). Say \( V_1, V_2, \ldots, V_m \) 
are the only sets which induce nontrivial complete 
graphs. Once again, since \( B \) contains no triangles, 
each of the sets \( V_1, V_2, \ldots, V_m \) induce complete 
graphs on two vertices. Let \( c_i \) be the edge induced 
by \( V_i \), where \( i=1, 2, \ldots, m \). Note, \( \{c_1, c_2, \ldots, 
\}
\) is a set of independent edges.

If \( m=n \) then \( a\leq(B)=m \leq \beta_1(B) \leq \beta_1(B)+1 \).

So, suppose that \( m<n \). Let \( H=<V_{m+1}UV_{m+2} \ldots
UV_n> \). Observe that \( H \) is a bipartite graph and 
\( V_m, V_{m+1}, V_{m+2}, \ldots, V_n \) is a complete coloring of \( H \).
Thus, \( n-m \leq \chi(H) \). From the preceding lemma it fol-
lows that \( n-m \leq \beta_1(H)+1 \). Let \( \{f_1, f_2, \ldots, f_k\} \) 
be a set of independent edges in \( H \) with \( k=\beta_1(H) \).
Since the vertices in \( V_1UV_2U \ldots UV_m \) are disjoint from 
the vertices of \( H \) we see that \( \{c_1, c_2, \ldots, c_m, 
\}
\) is independent in \( B \). Thus, \( m+k \leq 
\beta_1(B) \), that is \( m+\beta_1(H) \leq \beta_1(B) \). The following is now
established:
\[ a^2(B) = n = m + (n - m) \]
\[ \leq m + \beta_1(H) + 1 \]
\[ \leq \beta_1(B) + 1. \]

Now, if \( B \) contains a 1-factor, then \( \beta_1(B) = \left\lfloor \frac{\beta_1(B)}{2} \right\rfloor \). Since \( a^2(B) \leq \left\lfloor \frac{\beta_1(B)}{2} \right\rfloor \) we conclude that \( a^2(B) = \beta_1(B) \).

Lastly, suppose that \( \beta_1(B) = a^2(B) \) but that \( B \)
does not contain a 1-factor. Let \( \{c_1, c_2, \ldots, c_m\} \) be a maximum independent set of edges. For \( i = 1, 2, \ldots, m \) let \( V_i \) be the vertices incident with \( c_i \). Let \( V_{m+1} = V(G) - \bigcup V_i \). Since \( B \) does not contain a 1-factor, \( V_{m+1} \) is nonempty. However, since \( \{c_1, c_2, \ldots, c_m\} \) is a maximum set of independent edges \( < V_{m+1} \) is empty. As we noted earlier, \( V_1, V_2, \ldots, V_{m+1} \) is a complete cocoloring of \( B \) having cardinality \( \beta_1(B) + 1 \). This contradicts the fact that \( a^2(B) = \beta_1(B) \). We conclude that if \( a^2(B) = \beta_1(B) \) then \( B \) contains a 1-factor.

**Corollary 4.5**

For \( 1 \leq n \leq m \), \( a^2(K(n,m)) = \begin{cases} n, & \text{if } n=m \\ n+1, & \text{otherwise.} \end{cases} \)

**Proof:**

This follows by noting \( \beta_1(K(n,m)) = n \) and \( K(n,m) \) has a 1-factor if and only if \( n=m \).

**Theorem 4.6**
Given natural numbers $r$, $m$ and $n$ with $r \leq m \leq n$
we have that $\alpha(Z(K(r,m,n))) = 1$ if $n = 1$; otherwise

$$\alpha(Z(K(r,m,n))) = \begin{cases} 
  r+m+1 & \text{if } r+m < n \\
  \lfloor \frac{r+m+n}{2} \rfloor & \text{if } n \leq r+m \leq n+3 \\
  \lceil \frac{r+m+n}{2} \rceil & \text{if } n+3 < r+m 
\end{cases}$$

**Proof:**

If $n = 1$ then $K(r,m,n)$ is a triangle and has an
acochromatic number of 1. So, suppose that $n \geq 2$.
Let $H_1 = K_r$ and $H_2 = K_m$ and $H_3 = K_n$. Let $G = H_1 + H_2 + H_3$.

Note, $K(r,m,n) = H_1 + H_2 + H_3$. We examine the following
cases.

**Case 1.** Suppose $r+m < n$. Pair each vertex in
$H_1$ and $H_2$ with a vertex in $H_3$. Now, consider each
such pair to be a color class. Place all unpaired
vertices in a single class, say $V$. Note, $V$ induces
an empty graph and each of the other classes induces a
complete graph. If $V$ is a singleton, it is also
complete. But, if so, its union with any other class
would induce $K_1 U K_2$. It is easy to verify that the
union of any other two classes does not induce a com-
plete graph. So, such a partition is a complete co-
coloring of $G$. This has cardinality $r+m+1$.

Now, to see that $\alpha(Z(G)) \leq r+m+1$, suppose that
the vertices of $G$ are partitioned into $r+m+2$ or more sets. Since $H_1+H_2$ has $r+m$ vertices, two sets are contained entirely in $H_3$. Since $H_3$ is empty, the union of these two sets induces an empty graph. Thus, such a decomposition could not be a complete cocoloring.

**Case 2.** Suppose $r+m > n$ and $r+m+n$ is even. If $r < m$, pair $m-r$ vertices in $H_2$ with vertices in $H_3$ and remove these vertices from consideration. This leaves $r$ vertices in $H_1$ and $r$ vertices in $H_2$ and $n-(m-r)$ vertices in $H_3$. Note, $n-(m-r) = n-m+r-2m$. This has even parity. Let

$$k = \frac{n-(m-r)}{2}.$$  

Pair $k$ vertices of $H_1$ with $k$ vertices in $H_3$. Likewise, pair $k$ unpaired vertices in $H_2$ and $H_3$. Remove all paired vertices from consideration. There remain $r-k$ vertices in $H_1$ and $r-k$ vertices in $H_2$. Match these vertices together, two to a class; one from $H_1$, one from $H_2$.

As in Case 1, each set of vertices in this partition induces $K_2$ but the union of any two will not induce a complete graph. Thus, this is a complete cocoloring. Since this partition has cardinality $\frac{\beta(G)}{2}$ and since $\alpha_2(G) \leq \{\frac{\beta(G)}{2}\}$ we see that equality holds. That is

$$\alpha_2(G) = \frac{r+m+n}{2}.$$
Case 3. Suppose \( r+m+n \) is odd and \( n \leq r+m \leq n+3 \).

There are two subcases to consider: when \( r+m=n+1 \) and when \( r+m=n+3 \). In both cases we will show that \( \alpha Z(G) \geq \frac{r+m+n}{2} \). Since \( \alpha Z(G) \leq \left\{ \frac{\delta(G)}{2} \right\} = \left\{ \frac{r+m+n}{2} \right\} \), equality will be established.

Suppose \( r+m=n+1 \). Since \( n \geq 2 \), we see \( m \geq 2 \).

Let \( V_1 \) be a set of two vertices chosen from \( H_2 \).

Let \( V_2 \) be a vertex chosen from \( H_3 \). Note, \( H_1+H_2+H_3-(V_1UV_2)\sim K(r, m-2, n-1) \). Also note, \( r+(m-2) = n-1 \).

Color the vertices of \( G-(V_1UV_2) \) as we did in Case 2.

Add these color classes to \( V_1, V_2 \). This is a complete cocoloring of \( G \) using \( 2+r+(m-2) \) colors. But, \( r+m = \left\{ \frac{r+m+n}{2} \right\} \). Hence, \( \alpha Z(G) \geq \left\{ \frac{r+m+n}{2} \right\} \).

Suppose \( r+m = n+3 \). Note that in this case \( r \geq 2 \). For if \( r=1 \) then \( n+3 = m+1 \) and \( m > n \).

Let \( V_1 \) be two vertices chosen from \( H_1 \) and \( V_2 \) be two vertices chosen from \( H_2 \) and \( V_3 \) be a vertex chosen from \( H_3 \). Note, \( H_1+H_2+H_3-(V_1UV_2UV_3)=K(r-2, m-2, n-1) \).

But \( (r-2)+(m-2) = n-1 \). Again, color \( G-(V_1UV_2UV_3) \) as in Case 2. Add the color classes to \( V_1, V_2, V_3 \). This is a complete cocoloring of \( G \) of cardinality \( 3+(r-2)+(m-2) = r+m-1 \). But \( r+m-1 = \left\{ \frac{r+m+n}{2} \right\} \). So again, equality is established.

Case 4. Suppose \( r+m+n \) is odd and \( r+m > n+3 \).

We first show that \( \alpha Z(G) \neq \left\{ \frac{\delta(G)}{2} \right\} \). And thus, \( \alpha Z(G) \)
Suppose \( aZ(G) = \{ \frac{p(G)}{2} \} \). Let \( V_1, V_2, \ldots, V_q \) be a complete cocoloring with \( q = \{ \frac{p(G)}{2} \} \). Since \( p(G) \) is odd, some class, say \( V_1 \), contains a single vertex and all others contain exactly two vertices. Suppose the vertex in \( V_1 \) is contained in \( H_i \). Let \( j, k \) be two different integers between 1 and 3 other than \( i \). Now, no \( V_t \) contains two vertices, one from \( H_j \) the other from \( H_k \). Otherwise, \( V_1 \cup V_t \) would induce a complete graph. Also, no more than one of these cocolor classes is contained in \( V(H_j) \) and no more than one is in \( V(H_k) \). Otherwise, the union of two sets would induce an empty graph. Hence, \( q - 3 \) or more of these classes contain exactly one vertex from \( H_i \) and exactly one vertex from \( H_j \cup H_k \). Thus, \( |H_i| - 1 \geq |H_j| + |H_k| - 4 \). But \( |H_i| - 1 \leq n - 1 \) and \( |H_j| + |H_k| - 4 \geq x + m - 4 \). We conclude that \( n - 1 \geq x + m - 4 \) and thus \( n + 3 > x + m \), a contradiction.

Now, let \( V \) be a set containing three vertices, one taken from each of the graphs \( H_1, H_2 \) and \( H_3 \). Note, \( G - V \) is a graph with an even number of vertices. Color it as we did in Case 2. Add the new classes to \( V \). This produces a complete cocoloring of order

\[
1 + \frac{r + m + n - 3}{2} = \left\lceil \frac{r + m + n}{2} \right\rceil.
\]

One quickly sees that the results from each of
these cases implies the statement of the theorem.

Suppose \( 1 \leq n_1 \leq n_2 \leq \ldots \leq n_m \) now if \( n_1 + n_2 + \ldots + n_{m-1} < n_m \) then, as in Case 1 of the preceding proof,

\[
a_2(K(n_1, n_2, \ldots, n_m)) = 1 + \sum_{i=1}^{m-1} n_i.
\]

This chapter ends with the statement of an unsolved problem in this field. It is known that for a graph \( G \) and an integer \( k \) with \( \chi(G) \leq k \leq \Delta(G) \) there is a complete coloring of \( G \) which uses exactly \( k \) colors. Does a similar statement hold for complete cocolorings. That is, for any graph \( G \), if \( \Delta(G) \leq k \leq a_2(G) \) is there a complete cocoloring of \( G \) of cardinality \( k \)?
We began this work with a discussion of the broad framework within which coloring theory lies. In this last chapter we will examine what may be thought of as an application. That is, we will see specific relations between coloring theory and another branch of graph theory. This branch is known as switching theory.

Let G be a graph and H a subset of V(G). The switched graph on the set H, denoted by $S_H(G)$, is the graph with vertex set V(G) and edge set

$$E(S_H(G)) = E(H) \cup E(G-H) \cup \{ uv \in E(G) | u \in H \text{ and } v \notin H \}.$$  

If $H = \emptyset$, define $S_H(G)$ to be $G$. We will also refer to $S_H(G)$ as the graph $G$ switched on H or G switched on H. We will refer to H as the switching set.

We note that if V is the vertex set of G then G switched on V is $\overline{G}$. So, in a sense, $S_H(G)$ is intermediate to $G$ and $\overline{G}$.

The reader interested in this concept should
Suppose we are given a graph $G$. Label the vertices $v_1, v_2, \ldots, v_p$. A labeling such as this not only provides identification of each vertex but also provides an order on the vertices. Let $V_n = \{ v_1, v_2, \ldots, v_n \}$, $n = 1, 2, \ldots, p$. We can then form a sequence of graphs $G_0, G_1, \ldots, G_p$ which is defined by

$$G_i = \begin{cases} G & \text{if } i = 0 \\ S_{V_i}(G) & \text{otherwise.} \end{cases}$$

Note that $G_{p-1} = G_p = \overline{G}$.

Let the chromatic sequence $\chi_0, \chi_1, \ldots, \chi_p$ associated with $G$ and the labeling $v_1, v_2, \ldots, v_p$ be defined by $\chi_n = \chi(G_n)$, $n = 0, 1, \ldots, p$.

Likewise, let the cochromatic sequence $Z_0, Z_1, \ldots, Z_p$ be defined by $Z_n = Z(G_n)$, $n = 0, 1, \ldots, p$.

If $\chi_0$ equals 1 then $G$ is the empty graph and $G_n = K_n + \overline{K}_{p-n}$, $n = 1, 2, \ldots, p-1$. Also $G_p = K_p$.

Thus, if $\chi_0$ equals 1 then $\chi_n$ must be $n + 1$, for $n \leq p-1$. Likewise, if $\chi_0 = p$ then $\chi_0, \chi_1, \ldots, \chi_p$ must be the sequence $p, p-1, \ldots, 2, 1, 1$.

If $Z_0$ equals 1 then $G$ is empty or complete. If $G$ is empty then $G_n$ was determined above. If $G$ is complete then $G_n = K_{p-n} \cup \overline{K}_n$, where $1 \leq n \leq p-1$. Also, $G_p = \overline{K}_p$. Thus,
\[ Z_n = \begin{cases} 
1 & \text{if } n = 0, \text{ or } p-1 \\
2 & \text{otherwise.} 
\end{cases} \]

Several properties characteristic of chromatic and cochromatic sequences are apparent. By the Nordhaus-Gaddum result \([NG1]\), \(2\sqrt{p} \leq \chi_0' + \chi_p' \leq p + 1\) and \(\chi_0' \leq \chi_p' \leq \left(\frac{p + 1}{2}\right)^2\). Note that \(Z_i \leq Z_i\) for \(0 \leq i \leq p\). Thus, \(Z_0 Z_p \leq \left(\frac{p + 1}{2}\right)^2\). Also, since \(Z(G) = Z(\overline{G})\), we have \(Z_0 = Z_p\). Lastly, since \(G_{p-1} = G_p\) we conclude \(\chi_{p-1} = \chi_p\) and \(Z_{p-1} = Z_p\).

Section 5.2

Chromatic and Cochromatic Sets

A set \(S\) is chromatic if there is a chromatic sequence \(\chi_0', \chi_1', \ldots, \chi_p'\) associated with a graph \(G\) and a labeling of \(V(G)\) such that \(S = \{\chi_0', \chi_1', \ldots, \chi_p'\}\). In general, it seems rather difficult determining which sequences are chromatic. Such is not the case with sets. We shall characterize such sets after the following remark is established.

**Proposition 5.1**

Given a chromatic sequence \(\chi_0', \chi_1', \ldots, \chi_p'\) associated with a graph \(G\) and a labeling \(v_1, v_2, \ldots, v_p\) on the vertices, then

\[ \chi_{i-1} - 1 \leq \chi_i \leq \chi_{i-1} + 1 \]
for $1 \leq i \leq p$. Furthermore, $x_1 - 1 \leq x_0 \leq x_1 + 1$.

**Proof:**

Let $1 \leq i \leq p$. Partition $V(G_{i-1})$ into $x_{i-1}$ sets, each of which induces an empty graph. If ${v_i}$ is a color class in this coloring then we have a coloring of $G_i$. Otherwise, we obtain a coloring of $G_i$ by placing $v_i$ in a new color class. In any case $x_i \leq x_{i-1} + 1$.

Similarly, given a $x_i$-coloring of $G_i$ we can produce a coloring of $G_{i-1}$ using at most $x_i + 1$ colors. Thus, $x_{i-1} - 1 \leq x_i$.

In a like manner, it can be shown that $x_1 - 1 \leq x_0 \leq x_1 + 1$. ■

It follows from the proposition that every chromatic set $S$ has an "intermediate value" property. That is, for any $i$ and $j$ in $S$, if there is an integer $m$ between $i$ and $j$, then $m$ is in $S$. As we shall see, this property characterizes chromatic sets.

**Theorem 5.2**

A set $S$ is chromatic if and only if $S = \{m, m + 1, m + 2, \ldots, n\}$ for some natural numbers $m$ and $n$.

**Proof:**

By the preceding remark, we need only show that
if $S = \{m, m+1, \ldots, n\}$ for some natural numbers $m$ and $n$, then $S$ is a chromatic set. If $n = 1$ the case is trivial. So suppose $n > 1$.

Let $G = \overline{K}_m + K_{n-1}$. Label the vertices of $G$ as $v_1, v_2, \ldots, v_{m+n-1}$ so that

$$<v_1, v_2, \ldots, v_m> = \overline{K}_m$$

and

$$<v_{m+1}, \ldots, v_{m+n-1}> = K_{n-1}.$$ 

We see that $G_0 = \overline{K}_m + K_{n-1}$. Hence, $\chi_0 = n$. Also, $G_m = K_m \cup K_{n-1}$. Thus, $\chi_m = \text{Max} \{m, n-1\}$. Note, for $1 \leq i \leq m$, the graph $G_i$ is $\overline{K}_{m-i} + (K_{n-1} \cup K_i)$. This has a chromatic number of $n$. For $m < i < n + m - 1$, the graph $G_i$ is $K_m \cup K_n - (i-m) \cup \overline{K}_{i-m}$. So $\chi_i = \text{Max} \{m, n-1-i+m\}$. In this case, we notice that $n \geq \chi_i \geq m$. Lastly, note $G_{n+m-1} = K_m \cup \overline{K}_{n-1}$. Here, $\chi_{n+m-1} = m$. So, $\{x_0, x_1, \ldots, x_{n+m-1}\} \subseteq \{m, m+1, \ldots, n\}$. But, since $x_0 = n$ and $x_{m+n-1} = m$, it follows by the preceding proposition that $\{m, m+1, \ldots, n\} \subseteq \{x_0, x_1, \ldots, x_{n+m-1}\}$. Hence, equality between the two sets holds and the proof is established.

We shall discuss a similar concept for cochromatic numbers momentarily.

We next consider the question of the existence of graphs $G$ with the property that the cochromatic sequence associated with $G$ and a labeling of the vertices is unaffected by relabeling of the vertices.
That is, if \( v_1, v_2, \ldots, v_p \) and \( v_1', v_2', \ldots, v_p' \) are labelings of \( V(G) \) and the corresponding cochromatic sequences are \( Z_1, Z_2, \ldots, Z_p \) and \( Z_1', Z_2', \ldots, Z_p' \), respectively, then \( Z_i = Z_i' \) for \( i = 1, 2, \ldots, p \).

Clearly, all complete and empty graphs have this property, although \( Z_i \) in any case is either one or two.

We shall present a somewhat stronger example by showing that for each \( N \geq 1 \) there is a graph \( G_N \) whose cochromatic sequence is constantly \( N \), regardless of the labeling of the vertices of \( G_N \). In order to present this result we will need the following lemma.

Lemma 5.3

Let \( A = \bigcup S_i \), where \( S_1, S_2, \ldots, S_n \) is a sequence of pairwise disjoint sets with \( |S_i| = n \) for each \( i \). Let \( B = \{B_1, B_2, \ldots, B_m\} \) be a partition of \( A \), where \( m < n \). Then, there is a set \( B_k \) containing distinct elements \( w, x, y, \) and \( z \) such that

\[
\{w, x\} \subseteq S_i, \\
\{y, z\} \subseteq S_j, \\
\text{and } i \neq j.
\]

Proof:

Since \( n > m \) each \( S_i \) must have at least two elements in some \( B_j \). Use the axiom of choice to
define a function $f$ with the property that $S_i$ has at least two elements in $B_f(i)$, for $i = 1, 2, \ldots, n$. By the pigeonhole principle, there exist $i$ and $j$ such that $1 \leq i \neq j \leq n$ and $f(i) = f(j)$.

Now, $S_i$ has two elements in $B_f(i)$ as does $S_j$.

**Theorem 5.4**

For each positive integer $N$, there is a graph $G_N$ with the property that for any $H \subseteq V(G_N)$ the cochromatic number of $S_H(G_N)$ is $N$.

**Proof:**

Let $G_N = K_N \times K_N = K_{N(N)}$. Let $H$ be a subset of the vertex set of $G_N$. We first show that $Z(S_H(G_N)) \leq N$. Let $S_1, S_2, \ldots, S_N$ denote the partite sets of $G_N$. Now, if $S_i \cap H$ is nonempty label the vertices of $S_i \cap H$ as $u_1^i, u_2^i, \ldots, u_{p_i}^i$ if $i = 1$; as $u_1^i, u_2^i, \ldots, u_{p_i}^i$ if $i > 1$ and $|S_i \cap H| < i$ and as $u_1^i, u_2^i, \ldots, u_{i-1}^i, u_{i+1}^i, \ldots, u_{p_i}^i$ in all other cases. Note, in any case, $p_i \leq N+1$.

We now describe a three step process for partitioning $V(S_H(G_N))$ into $N$ cocolor classes $V_1, \ldots, V_N$. 

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**Step 1**

For $i = 1, 2, \ldots, N$ let $V_i$ initially be defined to equal $S_i \cap H$.

**Step 2**

If $V_i$ is the empty set, add $u_i^j$ to $V_i$. Note that in such a case $S_i \subseteq H$. Thus, $p_i = N + 1$.

Also note that at this point, each $V_i$ is non-empty and induces an empty graph in $G_N$ as well as $S_H(G_N)$.

**Step 3**

For each $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$, if $u_i^j$ exists, add this vertex to $V_j$. Since $u_i^j$ is in $H$, it is not adjacent in $S_H(G_N)$ to any elements in $S_k$, $k \neq i$. In particular, $u_i^j$ is not adjacent to any $u_j^k$ nor to any vertex in $S_j$. Thus, each $V_j$ induces an empty graph in $S_H(G_N)$.

Since each vertex is a member of some $V_i$, it follows that $Z(S_H(G_N)) \leq N$.

We now use the preceding lemma to show that $N \leq Z(S_H(G_N))$. Suppose, to the contrary, that $Z(S_H(G_N)) < N$. Let $C_1, C_2, \ldots, C_m$ be a coloring of $S_H(G_N)$, where $m < N$. By the lemma, there are distinct vertices $u, v, w, x$ and some integer $k$ such that
\{u, v, w, x\} \subseteq C_k$, and there are sets $S_i$ and $S_j$, $i \neq j$, such that $\{u, v\} \subseteq S_i$ and $\{w, x\} \subseteq S_j$.

We now produce a contradiction by showing that $C_k$ induces neither a complete nor an empty graph in $S_H(G_N)$. Without loss of generality, there are two cases to be considered. We will denote $V(G_N)$ by $V$.

**Case 1**

Suppose $\{u, v\} \subseteq V-H$. Since $\{u, v\} \subseteq S_i-H$, it follows that $u$ and $v$ are not adjacent in $S_H(G_N)$. Hence, $C_k$ must induce an empty graph in $S_H(G_N)$. If $w$ or $x$ is in $H$, then $w$ is adjacent with $x$ in $S_H(G_N)$, contradicting the fact that $C_k$ induces an empty graph in $S_H(G_N)$. So neither $w$ nor $x$ is in $H$. Since $v$ and $w$ are adjacent in $G_N$ they are adjacent in $S_H(G)$. Thus, $C_k$ does not induce an empty graph in $S_H(G_N)$ and a contradiction is reached.

**Case 2**

Suppose that $u \in H$. Since $u$ and $v$ are non-adjacent in $G_N$ they are adjacent in $S_H(G_N)$. But $u$ and $w$ are adjacent in $G_N$ so they are not adjacent in $S_H(G_N)$. So $C_k$ does not induce either a complete or an empty graph in $S_H(G)$. Again, a
contradiction is reached.

We conclude that \( N \leq Z(S_H(G_N)) \) and hence that \( N = Z(S_H(G_N)) \).

Thus, for any natural number \( m \), if \( H \subseteq V(K_m(m)) \) then \( Z(S_H(K_m(m))) = m \). By a similar proof, a corresponding statement holds for \( mK_m \).

Further, any labeling \( v_1, v_2, \ldots, v_N \) of the vertices of \( G_N \) produces a corresponding cochromatic sequence which is constantly \( N \). One sees by a similar argument that a corresponding statement holds for chromatic sequences.

A set \( S \) is cochromatic if there is a graph \( G \) and a labeling \( v_1, v_2, \ldots, v_p \) of \( V(G) \) with corresponding sequence \( G_0, G_1, \ldots, G_p \) such that \( S = \{Z(G_i) | i = 0, 1, 2, \ldots, p\} \).

**Theorem 5.5**

A set \( S \) is cochromatic if and only if \( S \) is of the form \( \{m, m+1, \ldots, n\} \) where \( m \) and \( n \) are integers satisfying \( 1 \leq n \leq 2m \).

**Proof:**

Suppose \( S \) is a cochromatic set. As in the case of chromatic sets we must have the "intermediate value" property. The proof of this follows exactly...
like the proof in the chromatic set case. So, $S$ has the form \{m, m+1, m+2, \ldots, n\} for some natural numbers $m$ and $n$.

To see that $n \leq 2m$, let $G$ be a graph together with a labeling $v_1, v_2, \ldots, v_p$ on $V(G)$ such that $m = \min \{z_0, z_1, \ldots, z_p\}$ where $z_0, z_1, \ldots, z_p$ is the cochromatic sequence corresponding to $G$ and the given labeling.

We consider two cases.

**Case 1**

Suppose $z_0 = m$.

Let $S_1, S_2, \ldots, S_m$ be a cocoloring of $G$ and let $H = \{v_1, v_2, \ldots, v_k\}$, where $1 \leq k \leq p$. Let $T_i = S_i \cap H$ and $T_i' = S_i - H$, for $i = 1, 2, \ldots, m$. Then the nonempty sets among $T_1, T_2, \ldots, T_m, T_1', T_2', \ldots, T_m'$ define an $r$-cocoloring of $S_H(G)$, where $r \leq 2m$. Hence, in this case, $z_j \leq 2m$ for $j = 1, 2, \ldots, p$.

**Case 2**

Suppose $z_0 > m$. Let $k$ be chosen so that $z_k = m$. We show that $z_i \leq 2m$ in the case where $k<i$. The case where $i<k$ follows with a similar proof.
Let $H = \{v_1, v_2, \ldots, v_k\}$. Let $S_1, S_2, \ldots, S_m$ be a cocoloring of $S_H(G)$. Fix $i > k$. For $j = 1, 2, \ldots, m$ let $T_j = S_j \cap \{v_{k+1}, v_{k+2}, \ldots, v_i\}$ and $T'_j = S_j \setminus T_j$. Then the non-empty sets among $T_1, T_2, \ldots, T_m, T'_1, T'_2, \ldots, T'_m$ form a cocoloring of $G_i$. Hence, $Z_i \leq 2m$.

We now conclude that $\text{Max}\{Z_0, Z_1, \ldots, Z_p\} \leq 2m$. This concludes the first half of the proof.

Now, suppose $S = \{m, m+1, \ldots, n\}$, where $m$ and $n$ are integers with $1 \leq m \leq 2m$. We wish to show that such a set is cochromatic. We do this by constructing a graph $G$ and a labeling $v_1, v_2, \ldots, v_p$ of $V(G)$ so that the corresponding cochromatic set $\{Z_0, Z_1, \ldots, Z_p\}$ is in fact equal to the set $S$.

If $S$ is a singleton then by Theorem 5.4 the set $S$ is cochromatic. So, assume $m < n$. Thus $n \geq 2$.

Let $k = n - m$. Also, let $A_i$ be a set of $n$ vertices for each $i = 1, 2, \ldots, k$. Define each $A_i$ so that $A_1, A_2, \ldots, A_k$ are mutually disjoint. Also, let $B_1, B_2, \ldots, B_m$ be pairwise disjoint sets of new vertices, each set with cardinality $n$. Now define $S_i$ so that

$$S_i = \begin{cases} A_i \cup B_i, & 1 \leq i \leq k \\ B_i, & k < i \end{cases}$$
for each $i = 1, 2, \ldots, m$. Let $G$ be the complete $m$-partite graph with partite sets $S_1, S_2, \ldots, S_m$. Note, $p(G) = n^2$.

Label the elements of $A_1 U A_2 U \ldots U A_k$ with $v_1, v_2, \ldots, v_{kn}$ so that

$$A_1 = \{v_1, v_2, \ldots, v_n\}$$
$$A_2 = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$$

and so forth. Label the elements of $B_1 U B_2 U \ldots U B_m$ with $v_{kn+1}, v_{kn+2}, \ldots, v_n$, where

$$B_1 = \{v_{kn+1}, v_{kn+2}, \ldots, v_{kn+n}\}$$
$$B_2 = \{v_{kn+n+1}, v_{kn+n+2}, \ldots, v_{kn+2n}\}$$

and so forth.

We show (i) $Z(G) = m$; (ii) $Z(S_i \{v_1, v_2, \ldots, v_{kn}\}) = n$; and (iii) for any $r = 1, 2, \ldots, n^2$, $m \leq Z(S_i \{v_1, v_2, \ldots, v_r\}) \leq n$.

To verify (i) we note that $G$ contains $K_m(m)$ as an induced subgraph; thus $m = Z(K_m(m)) \leq Z(G)$.

However, $G$ is an $m$-partite graph; thus $Z(G) \leq m$.

We conclude that $Z(G) = m$.

To prove (ii), let $H = A_1 U A_2 U \ldots U A_k$. Now note, $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_m$ is a cocoloring of $S_H(G)$. Thus, $Z(S_H(G)) \leq m + k = n$. Suppose then that $Z(S_H(G)) < n$. Note that $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_m$ is a collection of $n$ pairwise disjoint sets.
of cardinality $n$. By Lemma 5.3 any cocoloring of $S_H(G)$ with less than $n$ colors produces a cocolor class containing four distinct vertices, say $u$, $v$, $w$, and $x$ such that for some $i$ and $j$ one of the following three cases occurs:

1. $\{u, v\} \subseteq A_i$
   $\{w, x\} \subseteq A_j$, $i \neq j$

2. $\{u, v\} \subseteq A_i$
   $\{w, x\} \subseteq B_j$

3. $\{u, v\} \subseteq B_i$
   $\{w, x\} \subseteq B_j$, $i \neq j$.

Each case, however, leads to a contradiction. To see this, we will examine Case 1; the other two cases follow similarly.

Note that $\{u, v, w, x\} \subseteq H$. Since $u$ and $v$ are not adjacent in $G$, they are adjacent in $S_H(G)$. So $\{u, v, w, x\}$ does not induce either a complete or an empty graph in $S_H(G)$. Hence, $\{u, v, w, x\}$ does not lie in a single cocolor class and a contradiction is reached. Thus, $n \leq Z(S_H(G))$. We have now verified (ii).

To see that $m \leq Z(S_H(G))$, where $H = \{v_1, v_2, \ldots, v_r\}$ for any $r = 1, 2, \ldots, n^2$ let $B = B_1 \cup B_2 \cup \ldots \cup B_m$ and let $H' = H \cap B$. We note that $K_m(m) \leq B_G$ and
\(<B>_{S_H(G)} = S'_{H}(<B>_G)\). Also, by the remark following
Theorem 5.4, \(Z(<B>_{S_H(G)}) \geq m\). Thus, \(m \leq Z(S_H(G))\).

We now conclude the proof by showing that for any
\(H = \{v_1, v_2, \ldots, v_r\}\) with \(r \leq n^2\), the cochromatic
number of \(S_H(G)\) is no more than \(n\). There are
three cases to consider.

Case 1

Suppose \(r \leq kn\), that is, \(H \subseteq A_1 U A_2 U \ldots U A_k\).
Choose \(i\) so that \(v_r \in A_i\). Let \(T = A_i \cap H\) and \(T' = B_i U (A_i - H)\).
Then \(A_1, A_2, \ldots, A_{i-1}, T, T', B_1, B_2, \ldots, B_{i-1}, S_{i+1}, S_{i+2}, \ldots, S_m\) defines a \(t\)-cocoloring of
\(S_H(G)\), where \(t = i + m\). Since \(i \leq k\), we have
\(Z(S_H(G)) \leq k + m = n\).

Case 2

Suppose \(v_r \in B_1 U B_2 U \ldots U B_k\). Choose \(i\) so that
\(v_r \in B_i\). In this case, let \(T = A_i U (B_i \cap H)\) and \(T' = B_i - H\).
Then a \(t\)-cocoloring of \(S_H(G)\) is defined
by \(S_1, S_2, \ldots, S_{i-1}, T, T', A_{i+1}, A_{i+2}, \ldots, A_k, B_{i+1}, B_{i+2}, \ldots, B_m\), where \(t = 1 + k - i + m\). Now since \(i \geq 1\),
we see that \(1 + k - i + m \leq k + m = n\). Thus, in this case,
\(Z(S_H(G)) \leq n\).

Case 3

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Suppose \( v^r \in B_{k+1} U B_{k+2} U \ldots U B_m \). Choose \( i \) so that \( v^r \in B_i \). Let \( T = B_i \setminus H \) and \( T' = B_i \setminus H \). Now, we can define a cocoloring on \( S_h(G) \) with the decomposition \( S_1, S_2, \ldots, S_{i-1}, T, T', S_{i+1}, \ldots, S_m \). Thus, \( Z(S_h(G)) \leq m+1 \). Since \( m < n \), in this final case \( Z(S_h(G)) \leq n \). □

It was shown in the proof of Theorem 5.5 that if \( S \) is a cochromatic set whose largest element is \( n \) then \( S \) is a cochromatic set for a graph of order \( n^2 \). As our next theorem indicates, unless \( S = \{1\} \), such graphs exist of all orders of at least \( n^2 \).

**Theorem 5.6**

Given a set of positive integers \( S = \{m, m+1, m+2, \ldots, n\} \), with \( n \leq m \) then for any \( k \geq n^2 \) there is a graph \( G \) on \( k \) vertices and a labeling of \( V(G) \) so that the corresponding cochromatic set equals \( S \), provided \( S \neq \{1\} \).

**Proof:**

Clearly, if \( \{Z_i \mid i = 0, 1, \ldots, p\} = \{1\} \), then \( Z(G_0) = 1 \). Hence, \( G_0 \) is either a complete or an empty graph. If \( G_0 \) has more than two vertices then \( Z(G_1) = 2 \), so assume \( S \neq \{1\} \).

We note that in the construction in the last
Theorem: If more vertices are added to a graph, we may produce graphs of arbitrarily large order with the same cochromatic sets.

Thus, we will concentrate on the singletons $S = \{n\}$, $n \geq 2$. Given $n \geq 2$ and $k \geq 0$, we will produce a graph on $n^2 + k$ vertices with a labeling on the vertex set corresponding to a cochromatic set $\{n\}$. Let $H_1$ be the empty graph on $n+k$ vertices. Let $H_i = K_n$, $i = 2, 3, \ldots, n$. Label the vertices of $H_i$ with $v_1, v_2, \ldots, v_{n+k}$. For $i = 2, 3, \ldots, n$, label the vertices of $H_i$ with $u_{ni+1}, u_{ni+2}, \ldots, u_{n(i+1)}$. Let $G = H_1 U H_2 U \ldots U H_n$. Let the switching sequence $G_0, G_1, G_2, \ldots, G_{n^2+k}$ be defined by $G$ and the sequence $v_1, v_2, \ldots, v_{n+k-1}, u_{2n+1}, u_{2n+2}, \ldots, u_{3n}, u_{3n+1}, u_{3n+2}, \ldots, u_{2n+1}, u_{2n+2}, \ldots, u_{n^2+n}, v_{n+k}$. Thus, our sequence begins by listing all but one element of $H_1$, then the elements of $H_2, H_3, \ldots, H_n$, consecutively, and ending with $v_{n+k}$, the previously unlisted vertex in $H_1$. Clearly, $Z(G_0) = n$ and thus $Z(G_{n^2+k}) = n$. So, suppose $1 \leq i \leq n^2+k$. We would like to show that $Z(G_i) = n$.

We do this by first noting that $n \leq Z(G_i)$. To see this, suppose $Z(G_i) < n$. Let $S_1, S_2, \ldots, S_r$, $r < n$, be a cocoloring of $G_i$, where, without loss of generality, $v_{n+k} \in S_r$. We see that for any $H_j$,
$2 \leq j \leq n$, at most one vertex of $H_j$ is in $S_x$. If this were not the case, there would be vertices $u$ and $v$ in some $H_j$, $2 \leq j \leq n$, with $(u, v) \subseteq S_x$. Now $u$ or $v$ is in the switching set. If not, $(u, v, v_{n+k})$ would induce neither a complete or an empty graph in $G_i$. So, say $u$ is in the switching set. This would imply $uv \notin E(G_i)$ and $uv_{n+k} \notin E(G_i)$. Once again, $S_x$ could not induce in $G_i$ a complete or an empty graph.

Note now that $G_i - S_x$ has a cochromatic number less than $n-1$. Also, $G - S_x$ contains $(H_2 U H_3 U \ldots U H_n) - S_x$ as an induced subgraph. Thus $(n-1)k_{n-1} \leq G$. By the note following Theorem 5.4 $G_i - S_x$ has a cochromatic number of at least $n-1$. Hence, a contradiction.

We now show that $Z(G_i) \leq n$. We will consider two cases; first, where the switching set is contained entirely in $H_1$; secondly, where it is not.

Assume that the switching set $H$ is contained in $H_1$. Then $H \cup H_2$ induces a complete graph in $S_{H}(G)$. So $V(H_1) - H, V(H_2) U H, V(H_3), V(H_4), \ldots, V(H_n)$ is an $n$-coring of $S_{H}(G)$.

Now suppose $H$ is not contained in $H_1$. Then $H_1 - H$ is simply the vertex $v_{n+k}$. Form sets $J_1, J_2, \ldots, J_n$. Do this by placing $v_{n+k}$ in $J_1$ and all the vertices of $H \cap H_1$ into $J_2$. If $u_{n+k}, 2 \leq k \leq n$, $1 \leq j \leq n$, is in $H$, place it in set $J_k$, if not, place it
in $J_i$. Note that each $J_i$ induces a complete graph in $S_{H}(G)$. Thus $Z(S_{H}(G)) \leq n$.

Thus, in either case the cochromatic number of $S_{H}(G)$ is less than or equal to $n$, and equality is established. Since this graph has order $(n+k)+(n-l)n = n^2+k$ the proof of the theorem is established. ■

In general, if we are given a set $S = \{m, m+1, \ldots, n\}$ it seems that it would be difficult to determine a minimum $p$ for which there exists a graph $G$ on $p$ vertices with a vertex labeling which corresponds to a cochromatic set equal to $S$. Certainly $Z(p) \geq n$. By the construction in the proof of Theorem 5.4 we know that such a minimum $p$ would be no greater than $n^2$. This would correspond with the fact that $Z(p) \geq \sqrt{n} \brace$.

In the case of chromatic sets, a formula for such a $p$ is known.

For natural numbers $m$ and $n$ with $m \leq n$, let $S(m, n)$ signify

$$\text{Min \{p| \text{there is a graph } G \text{ on } p \text{ vertices with a vertex labeling which corresponds to a chromatic set equal to } \{m, m+1, \ldots, n\}}\}.$$

**Theorem 5.7**

For all natural numbers $m$ and $n$, with $m \leq n$,
$S(m, n) = \begin{cases} 
n & \text{if } m \leq \frac{n+1}{2} \\
2m-1 & \text{if } m > \frac{n+1}{2} 
\end{cases}$

that is, $S(m, n) = \max \{n, 2m-1\}$.

**Proof:**

Note first that $\max \{n, 2m-1\} \leq S(m, n)$. To see this, suppose that $G$ is a graph and $v_1, v_2, \ldots, v_p$ a labeling of $V(G)$ with a corresponding switching sequence $G_0, G_1, \ldots, G_p$ with the property that $
\{x(G_i) | i = 0, 1, \ldots, p\} = \{m, m+1, \ldots, n\}$.
n Select $k$ so that $x(G_k) = n$. Note, $G_k$ has at least $n$ vertices. Hence, $p(G) = p(G_k) \geq n$. Also, $x(G_0) \geq m$ and $x(G_p) \geq m$. Recall that $G_0 = G_p$. By the Nordhause-Gaddum theorem [NG1] it follows that $2m \leq x(G_0) + x(G_p) \leq 2p(G) + 1$. Thus $2m-1 \leq p(G)$. We conclude that $\max \{n, 2m-1\} \leq S(m, n)$.

Now, if $n = m$, the statement of the theorem is easy to prove. Let $H_1 = \overline{K_m}$ and $H_2 = K_{m-1}$. Let $G = H_1 + H_2$. Label the vertices of $H_1$ with $v_1, v_2, \ldots, v_m$. Label the vertices of $H_2$ with $v_{m+1}, v_{m+2}, \ldots, v_{2m-1}$. Let $x_0, x_1, \ldots, x_{2m-1}$ be the corresponding chromatic sequence. Clearly, $x(G_0) = m$. If $1 \leq i \leq m$, $G_i$ contains a vertex in $H_1$ not in the switching set $\{v_1, v_2, \ldots, v_i\}$. Call one such vertex
Now $G_i$ contains the subgraph $v+H_2=K_m$. Thus, $\chi(G_i) \leq m$. Color the vertices of $\{v_1, v_2', \ldots, v_i\}$ with the colors $1, 2, \ldots, i$. Color the vertices $\{v_{i+1}, v_{i+2}', \ldots, v_m\}$ with the color $m$. Color the vertices of $H_2$ with the colors $1, 2, \ldots, m-1$. This is an $m$-coloring of $G_i$. Thus, $\chi(G_i) \leq m$ and we conclude that $\chi_i = m$. Note that $G_m = K_m \cup K_{m-1}$. Thus, $\chi_m = m$.

Suppose $2m-1 > i > m$. Let $j = i-m$. Note that $G_i = K_m \cup K_{m-j-1} \cup K_j$. In this case, $\chi_i = m$.

In the last case to be considered, suppose $i = 2m-1$. Then $G_i = K_m \cup K_{m-1}$. Thus, $\chi_{2m-1} = m$. So, we conclude that $\{\chi_0, \chi_1, \ldots, \chi_{2m-1}\} = \{m\}$. Hence $S(m, n) = 2m-1$.

So let us suppose that $m < n$. Here, we note that $n = \text{Max} \{n, 2m-1\}$ if and only if $m < \frac{n+1}{2}$. We consider the following two cases.

Case 1

Suppose that $m < \frac{n+1}{2}$. Let $H_1 = K_{n-m}$ and $H_2 = K_m$. Let $G = H_1 \cup H_2$. Label the vertices of $H_1$ with $v_1, v_2', \ldots, v_{n-m}$ and the vertices of $H_2$ with $v_{n-m+1}, v_{n-m+2}, \ldots, v_n$. Note that $\chi(G) = m$. Also, if $1 < i < n-m$, the graph $G_i$ contains a clique on $m+i$ vertices. This is induced by $\{v_1, v_2', \ldots, v_i\} \cup V(H_2)$. Color these vertices with $1, 2, \ldots, m+i$. Color the remaining vertices of $H_1$ with a color used in color-
ing the vertices of $H_2$. This shows us that $m \leq \chi(G_i) = m + i \leq m + (n - m) = n$.

Note, if $i = n - m$, the graph $G_i$ is $K_n$. In which case $\chi(G_i) = n$.

If $n - m < i < n$ let $t = i - (n - m)$. Note that $G_i = K_{n-m} + \overline{K_t \cup K_{m-t}}$. Thus, $\chi(G_i) = (n-m) + (m-t) = n - t$.

Since $1 \leq t < m$, we have that
\[ \frac{n+1}{2} = n - \frac{n+1}{2} + 1 \leq n - m + 1 \leq \chi(G_i) \leq n. \]

Lastly, note that $G_n = K_{n-m} + \overline{K_m}$. Thus $\chi(G_n) = n - m + 1$. So, as before, $m \leq \chi(G_n) \leq n$.

We may conclude that $\{\chi(G_0), \chi(G_1), \ldots, \chi(G_n)\} = \{m, m+1, \ldots, n\}$. Hence, in this case, $S(m, n) \leq n$.

**Case 2**

Suppose that $\frac{n+1}{2} < m$. That is, $n < 2m - 1$. Let $H_1 = \overline{K_{n-m}}$. Let $H_2 = \overline{K_{2m-n}}$ and let $H_3 = K_{m-1}$.

Notice, since $n < 2m - 1$, that $1 \leq 2m - n - 1 \leq m - 1$ and that $H_3$ has at least one vertex. Let $G$ be the graph $H_1 \cup (H_2 + H_3)$. Label the vertices of $G$ in the following manner. Let the vertices of $H_1$ be labeled $v_1, v_2, \ldots, v_{n-m}$. Label the vertices of $H_2$ with $v_{n-m+1}, v_{n-m+2}, \ldots, v_m$. Lastly, label the vertices of $H_3$ with $v_{m+1}, v_{m+2}, \ldots, v_{2m-1}$. We make the following three claims.

**Claim 1**

$\chi(G_0) = m$. 

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Claim 2
\[ x(G_{n-m}) = n. \]

Claim 3
\[ m \leq x(G_i) \leq n \quad \text{for} \quad i = 0, 1, \ldots, 2m-1. \]

Proof of Claim 1:

Color the vertices in \( H_3 \) with the colors 1, 2, \ldots, \( m-1 \). Color the vertices of \( H_1 \cup H_2 \) with the color \( m \). Since \( H_1 \cup H_2 \) is an empty graph and since \( H_3 \) has exactly \( m-1 \) vertices, this must form a proper coloring of \( G \). Thus, \( x(G) \leq m \).

Now, select a vertex \( v \) from \( H_2 \). Since \( v + H_3 = K_m \) is a subgraph of \( G \), we see \( x(G) \geq m \).

Proof of Claim 2:

Notice, \( G_{n-m} \) is the graph \( G \) switched on \( V(H_1) \). Thus, \( V(H_1) \cup V(H_3) \) induces a complete graph on \( n-1 \) vertices in \( G_{n-m} \). But each vertex in \( H_2 \) is adjacent to all vertices in this clique. It is now clear that since the vertices of \( H_2 \) induce an empty graph in \( G_{n-m} \), the chromatic number of \( G_{n-m} \) is \( n \).

Proof of Claim 3:

This claim has been established in the case where \( i = 0 \). We did this by showing that \( x(G) = m \).
Suppose then that $1 \leq i < n - m$. Let us note that $G_i = [K_{n-m-1} \cup (K_{2m-n} + K_{m-1})] + K_i$. Hence, $\chi(G_i) = m + i$. Thus, $m \leq \chi(G_i) \leq m + (n - m) = n$.

In the proof of Claim 2 we showed that $\chi(G_{n-m}) = n$. Suppose then that $n - m < i < m$. Here, $G_i = K_{n-m} + [K_{m-i} + (K_{i-n+m} \cup K_{m-1})]$. Hence, $\chi(G_i) = (n-m) + 1 + \max\{i-n+m, m-1\} = \max\{1+i, n\}$. But $1+i < 1+m \leq n$. So $\chi(G_i) = n$.

Note that $G_m = K_{n-m} + (K_{2m-n} \cup K_{m-1})$. Thus, $\chi(G_m) = \max\{m, n-1\}$. Clearly, $m \leq \chi(G_m) \leq n-1$.

Now, if $m \leq i < 2m-1$, then $G_i = K_{n-m} + (K_{2m-n} \cup K_{i-m} \cup K_{2m-1-i})$. Thus, $\chi(G_i) = n-m + \max\{2m-n, 2m-1-i\} = \max\{m, n-m-1-i\}$. But, since $n-m-1-i \leq n-1$, we see that $m \leq \chi(G_i) \leq n-1$.

If $i = 2m-1$, then $G_i = \bar{G} = K_{n-m} + (K_{2m-n} \cup K_{m-1})$. In which case, it is clear that $\chi(G_i) = m$. This concludes the proof of Claim 3.

We have constructed a graph on $2m-1$ vertices with a corresponding chromatic set $\{m, m+1, \ldots, n\}$. Thus, $S(m, n) \leq 2m-1$.

Equality now holds in both cases and the proof of the theorem is established.

Suppose that $G$ is a graph, with a vertex
labeling of $v_1, v_2, \ldots, v_p$ which corresponds to a switching sequence of $G_0, G_1, \ldots, G_p$. Suppose
$
\{x(G_i)|i=0, 1, \ldots, p\} = \{m, m+1, \ldots, n\}.
$
We know that there is a lower bound on $p$. Is there a maximum value which $p$ can attain for fixed values of $m$ and $n$? The answer is affirmative although exact bounds are not known in all cases. To see that such a bound exists, note that

$$2\sqrt{x(G)+x(G)} = x(G_0)+x(G_p) \leq 2n.$$ 

Thus, $p \leq n^2$.

For natural numbers $m$ and $n$ with $m \leq n$ let $p(m, n)$ signify

Max $\{p(G) | \text{there is a switching sequence $G_0, G_1, \ldots, G_p$ associated with a graph $G$ and a vertex labeling of $G$ so that $\{x(G_i) | i = 0, 1, \ldots, p\} = \{m, m+1, \ldots, n\}$} \}$.

**Proposition 5.8**

For natural numbers $n$ and $m$ with $n \leq m$,

$p(m, n) \leq m(2n-1)$.

**Proof:**

Let $v_1, v_2, \ldots, v_p$ be a labeling of the vertices of a graph $G$. Let $G_0, G_1, \ldots, G_p$ be the
corresponding switching sequence. Suppose \( \{ \chi(G_i) | i = 0, 1, \ldots, p \} = \{m, m+1, \ldots, n\} \). Let \( k \) be chosen so that \( \chi(G_k) = m \). If \( k = 0 \), let \( H = \emptyset \). If not, let \( H = \{v_1, v_2, \ldots, v_k\} \). Let \( C_1, C_2, \ldots, C_m \) be the color classes of a coloring of \( S_H(G) \). Now, let \( D_i = C_i \cap H \) and \( E_i = C_i - H \), for \( i = 1, 2, \ldots, m \).

We make two claims.

**Claim 1**

For \( i = 1, 2, \ldots, m \), the cardinality of \( D_i \) is no more than \( n \). Furthermore, if for some \( i \) it is the case that \( |D| = n \) then \( |E_i| = 0 \).

**Claim 2**

For \( i = 1, 2, \ldots, m \) the cardinality of \( E_i \) is no more than \( n \).

Once both claims are established we will conclude that \( |C_i| = |D_i \cup E_i| \leq 2n - 1 \), \( i = 1, 2, \ldots, m \).

Hence, \( p(G) = |C_1 \cup C_2 \cup \ldots \cup C_m| \leq m(2n - 1) \).

**Proof of Claim 1:**

Select \( i \) from between 1 and \( m \). If \( |D_i| > n \) then \( D_i \) would induce in \( G_0 \) a complete graph on more than \( n \) vertices. Hence, \( \chi(G_0) > n \), a contraction. So it is established that \( |D_i| \leq n \).
Now, suppose that $|D_i| = n$ and that $E_i \neq \emptyset$.

Select a vertex $v$ from $E_i$. Since $(v)UD_i$ induces an empty graph on $n+1$ vertices in $S_H(G)$, the graph $D_i^v = K_{n+1}$ will be a subgraph of $G_0$. Thus, $\chi(G_0) \geq n+1$. Again, a contradiction is reached.

This concludes the proof of Claim 1.

Proof of Claim 2:

Suppose there is an $i$ between 1 and $m$ such that $|E_i| \geq n+1$. Now, since $E_i$ and $H$ are disjoint and since $E_i$ induces an empty graph in $S_H(G)$, we see $E_i$ must induce an empty graph in $G$, since $G = G_p$, the graph $G_p$ contains a complete graph on $n+1$ vertices. So $\chi(G_p) \geq n+1$. This contradicts the fact that $\chi(G_p) \leq n$. It now follows that $|E_i| \leq n$, $i = 1, 2, \ldots, m$ and the claim and proof are established.

We see that the bound from the above remark is sharp in the following sense. For each $n$, a natural number, $p(1, n) = 2n-1$. If $n = 1$, this is clearly true. So suppose that $n \geq 2$. Let $H = K_{n-1}$ and $H' = K_n$. Let $G = H + H'$. Label the vertices of $H$ with $v_1, v_2, \ldots, v_{n-1}$. Label the vertices in the $H'$ with $v_n, v_{n+1}, \ldots, v_{2n-1}$. Clearly, $\chi(G_0) = n$. Also, if $1 \leq i < n-1$ then $G_i = K_{n-1} \cup (K_{n-1-i} + K_n)$. In which case $\chi(G_i) = n-i$. Now, if $i = n-1$ then $G_i$ is an empty
graph. Hence \( \chi(G_i) = 1 \). If \( n \leq i < 2n-1 \) then \( G_i = K_{n-1} \cup [K_{n-(i-(n-1))} + K_{i-(n-1)}] \), in which case \( \chi(G_i) = i-n+2 < (2n-1)-n+2 \). Thus, \( \chi(G_i) \leq n \). Also, \( G_{2n-1} = K_n \cup K_{n-1} \). Hence, \( \chi(G_{2n-1}) = n \).

Thus, \( \{\chi(G_i) | i = 0, 1, \ldots, p\} = \{1, 2, \ldots, n\} \).

To summarize, chromatic sets are unrestricted in the sense that any set of natural numbers having the intermediate value property is a chromatic set. Cochromatic sets are restricted, as we have seen. However, if we are given a chromatic set, the order of a graph associated with this set is bounded. There are no upper bounds on the orders of graphs associated with cochromatic sets.
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