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**CAYLEY GRAPH IMBEDDINGS
AND THE ASSOCIATED BLOCK DESIGNS**

by
Brian L. Garman

**A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy**

**Western Michigan University
Kalamazoo, Michigan
August 1976**

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Brian L. Garman

For

Mom and Dad

Linda, Larry, and Jennifer

Larry, Barb, Barclay, and Kyle

Mark and Shellie

Ron and Bill

Jon

the "St."

and

S.J.

all of whom provided incentive
in their own special way.

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CHAPTER 1

PRELIMINARIES IN TOPOLOGICAL GRAPH THEORY

In this chapter we present basic definitions and known results in topological graph theory that we will need throughout our discussion. Other terminology and definitions may be found in Behzad and Chartrand [3], Harary [14], or White [33].

A pseudograph G consists of a pair of finite sets $V(G)$ and $E(G)$, called the vertex set and edge set, respectively. The edge set is made up of subscripted singletons or subscripted doubletons. Singleton edges are called loops and distinct edges (i.e. having distinct subscripts) that are equal as subsets (disregarding subscripts) of $V(G)$ are called multiple-edges. Notationally, we write uv (or vu) for the edge $\{u,v\}$ (usually we suppress edge subscripts). A directed edge consists of an ordered pair (u,v) of elements from $V(G)$. The vertex u is called the initial vertex and v the terminal vertex. We will also write uv for the directed edge (u,v) if the context makes the understanding clear. Two vertices in G are adjacent if they constitute an edge of G ; each edge is incident with a vertex, and conversely, if the edge contains the vertex. A multi-graph is a pseudograph without loops and a graph is a pseudograph without loops

or multiple edges.

A surface is a connected, closed 2-manifold. A surface is orientable if it admits a 2-cell decomposition with coherent orientation (i.e., the boundary of each 2-cell is given an orientation so that each 1-cell portion of the boundary incident with two adjacent 2-cells is oppositely oriented within those two 2-cells); otherwise, it is nonorientable. It is well known that the orientable surfaces are the spheres with n handles S_n ($n = 0, 1, \dots$) and that the nonorientable surfaces are spheres with n crosscaps \tilde{S}_n ($n = 1, 2, \dots$). A pseudo-surface results when finitely many identifications of finitely many points each are made on a given surface; each such identification results in a singular point. The number of points identified for a given singular point is the degree of the singular point. It is immediate that all surfaces are pseudosurfaces. A generalized pseudosurface results when finitely many identifications of finitely many points each are made on a topological space of finitely many components, each of which is a pseudosurface, with a connected topological space resulting. Observe that generalized pseudosurfaces include pseudosurfaces and that a generalized pseudosurface is orientable if and only if each component before identification is an orientable pseudosurface. (A pseudosurface is orientable if and only if its associated surface

is orientable.)

It is well known that any finite 1-complex can be realized in real 3-space R^3 . A pseudograph G is imbedded in a generalized pseudosurface M if the geometric realization of G as a finite 1-complex in R^3 is homeomorphic to a subspace of M . (The image of the homeomorphism is the imbedding.) Each component of $M-G$ is called a region of the imbedding of G in M . Regions homeomorphic to the open unit disk are called 2-cells, and the imbedding is said to be a 2-cell imbedding if all regions are 2-cells. Thus in a 2-cell imbedding, each singular point must be the image of a vertex or a point on the interior of some edge; we will always insist on the former in what follows. A region whose boundary is a closed walk of length n is called an n -sided region; a 3-sided region is called a triangle, and the imbedding is triangular if every region is a triangle. The Euler characteristic $\chi(M)$ of M is a topological invariant and for any 2-cell imbedding of a graph with p vertices, q edges, and r regions $\chi(M) = p - q + r$.

It is widely known that $\chi(S_n) = 2 - 2n$ and $\chi(\tilde{S}_n) = 2 - n$. Petroelje [23] gives $\chi(P)$ for any orientable pseudosurface P . We give a formula for the characteristic of a pseudosurface P regardless of orientation, which is equivalent to Petroelje's when P is orientable. A pseudosurface P may be thought of as consisting

of a surface M of characteristic $\chi(M)$ and a set of singular points. For $i = 1, \dots, t$ suppose there are n_i singular points of degree m_i . Then $\chi(P) = \chi(M) - \sum_{i=1}^t n_i(m_i - 1)$, and P is orientable or nonorientable according to whether M is or is not orientable. For generalized pseudosurfaces we give a similar formula in the form of a theorem.

THEOREM 1.1. Let Q be a generalized pseudosurface composed of c surface components M_1, \dots, M_c and a set of singular points; say there are n_i singular points of degree m_i for each $i = 1, \dots, t$. Then $\chi(Q) = \sum_{i=1}^c \chi(M_i) - \sum_{i=1}^t n_i(m_i - 1)$.

PROOF. Let Q be as given in the hypothesis. Let I be a 2-cell imbedding on Q of a graph with p vertices, q edges, and r regions. Reverse the identification of the singular points to form 2-cell imbeddings on each of the c components M_1, \dots, M_c . The number of edges and regions are unchanged but the number of vertices is increased by $\sum_{i=1}^t n_i(m_i) - \sum_{i=1}^t n_i = \sum_{i=1}^t n_i(m_i - 1)$. Now let p_i, q_i , and r_i be the number of vertices, edges, and regions, respectively, for M_i , $i = 1, \dots, c$. Thus $\sum_{i=1}^c \chi(M_i) = \sum_{i=1}^c p_i - \sum_{i=1}^c q_i + \sum_{i=1}^c r_i = p + \sum_{i=1}^t n_i(m_i - 1) - q + r$. Therefore,

$$\chi(Q) = \sum_{i=1}^c \chi(M_i) - \sum_{i=1}^t n_i (m_i - 1).$$

The genus $\gamma(G)$ or the nonorientable genus $\bar{\gamma}(G)$ of a graph G is the minimum n such that G 2-cell imbeds in S_n or \tilde{S}_n , respectively. For surfaces, minimizing genus is equivalent to maximizing characteristic. It is this latter concept that generalizes to pseudosurfaces and generalized pseudosurfaces, although as we will see some extra care must be taken in formulating the concept for generalized pseudosurfaces. The pseudo-characteristic $\chi'(G)$ or the nonorientable pseudocharacteristic $\tilde{\chi}'(G)$ is the maximum $\chi(M)$ such that G is 2-cell imbeddable in an orientable or nonorientable pseudosurface M , respectively.

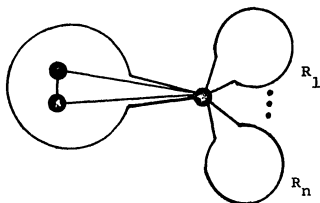
In minimizing genus (or maximizing characteristic) for surfaces for a given graph G , we maximize r , since p and q are fixed. Youngs showed in [36] that an orientable imbedding of a connected graph G is of minimum genus if and only if it has a maximum number of 2-cell regions. For graphs there are no loops or multiple edges so there are never any one or two-sided regions in a given (2-cell) minimal imbedding. In other words, the minimum genus or maximum characteristic can be thought of as maximizing r where only regions having 3 or more sides are allowed. For 2-cell imbeddings on pseudosurfaces no problems arise, for as in surfaces no regions of

length two or less can occur. In generalized pseudosurfaces, however, both 0-sided and 2-sided regions can occur in a 2-cell imbedding (1-sided regions cannot occur if G is a graph). In formulating a definition for generalized pseudocharacteristic, if we allow regions of length two or less our definition would not be desirable--problems arise in the following two ways:

1. The inclusion of 0-sided regions makes the characteristic unbounded since we can attach any arbitrary number of spheres, each having one singular point, to a given vertex. (See Figure 1.1 below.)

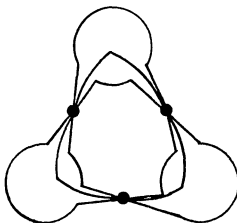
2. Prohibiting 0-sided regions but permitting 2-sided regions allows an imbedding where $p-q+r$ is identically p for any connected graph G . This is obtained by imbedding each edge of G in a sphere as indicated in Figure 1.2.

M:



K_3 imbedded with 2 triangles and n 0-sided regions; $\chi(M)=n+2$

Figure 1.1



K_3 imbedded with 3 2-sided regions
Figure 1.2

Keeping the foregoing discussion in mind, we define the generalized pseudocharacteristic $\chi''(G)$ or the non-orientable generalized pseudocharacteristic $\tilde{\chi}''(G)$ as the maximum $\chi(M)$ such that G is 2-cell imbeddable in an orientable or nonorientable generalized pseudosurface M , respectively, where each 2-cell region has boundary length three or more. Thus $\chi''(G)$ ($\tilde{\chi}''(G)$) is a natural generalization of $\chi'(G)$ and $\chi(G)$ ($\tilde{\chi}'(G)$ and $\tilde{\chi}(G)$).

An n -coloring of a multigraph G is an assignment of n (distinct) colors to the vertices of G so that distinct adjacent vertices are assigned different colors. A multigraph G is n -colorable if there exist an m -coloring of G for some $m \leq n$. The minimum n for which G is n -colorable is called the chromatic number of G and is denoted by $\chi(G)$. The context will make it clear

whether $\chi(G)$ denotes the chromatic number or the characteristic of G .

The dual of a pseudograph G imbedded in a generalized pseudosurface M is a pseudograph G^* whose vertices are the regions of the imbedding; two vertices are adjacent in G^* if the regions they represent share an edge of G in their boundaries. The imbedding of G in M has bichromatic dual if $\chi(G^*) = 2$. (Note that χ stands for chromatic number here.)

The regular complete n -partite graph of order nm is that graph whose vertex set can be partitioned into n partite sets containing m vertices each, such that two vertices are adjacent if and only if they are in different partite sets. We denote this graph by $K_{n(m)}$; it is a simple observation that the complement of $K_{n(m)}$ is n disjoint copies of the complete graph K_m , which we denote by $\overline{K_{n(m)}} = nK_m$. (The complement \overline{G} of a graph G is the graph with same vertex set as that of G and such that two vertices are adjacent in \overline{G} if and only if they are not adjacent in G .)

For a given finite group Γ with a set Δ of generators for Γ , the Cayley color graph $C_\Delta(\Gamma)$ has vertex set Γ , with (g, g') a directed edge--labeled with generator (color) δ_i --if and only if $g' = g\delta_i$. We assume that if $\delta_i \in \Delta$, then $\delta_i^{-1} \notin \Delta$ unless δ_i has order 2. In this latter case, the two directed edges $(g, g\delta_i)$ and

$(g\delta_i, g)$ are represented as a simple undirected edge $[g, g\delta_i]$, labeled with δ_i . For $C_\Delta(\Gamma)$ the associated Cayley graph $G_\Delta(\Gamma)$ is defined by disregarding all the colors and directions associated with the edges. An example of $C_\Delta(\Gamma)$ is given in Figure 1.3 with $\Gamma = \mathbb{Z}_6$ and $\Delta = \{2, 3\}$. A solid edge has group element 2 assigned in the direction of the arrow and a dotted line has group element 3 assigned with no direction specified.

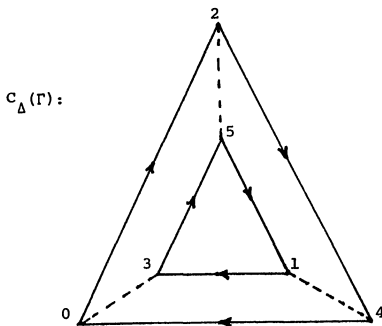


Figure 1.3

CHAPTER 2

COVERING PSEUDOGRAPHS IMBEDDED IN PSEUDOSURFACES

The study of imbedding graphs in generalized pseudo-surfaces, perhaps in connection with the determination of the genus or pseudocharacteristic parameters or as we shall see later in the construction of certain types of block designs, is in general an extremely complicated problem. The voltage graph theory as introduced by Gross gives a powerful method for imbedding certain graphs in orientable surfaces. (This theory is the dual of current graph theory, which was first introduced by Gustin [12], developed by Youngs [37], unified by Jacques [16], and extended by Gross and Alpert [8] and [9,10].) The method of Gross involves imbedding a much simpler related pseudo-graph P into usually a much simpler surface, rather than trying to imbed the given graph G directly. A major result of the theory is that the imbedded G (with its surface) is a (possibly branched) covering space over the imbedded P (with its surface). Stahl [30] (See also Alpert [1] and Ringel [26].) extends the theory of voltage graphs to include both orientable and nonorientable coverings of nonorientable surface imbeddings. In this section we extend the existing theory first to coverings of pseudosurface imbeddings and then to coverings

of generalized pseudosurface imbeddings. To proceed to this end we first review some pertinent ideas in covering space theory and the voltage graph theory of surfaces. It will be convenient to use the current graph theory of surfaces in some of our applications so we conclude the chapter with necessary definitions and theorems even though we will only formulate the extensions for voltage graph theory.

A covering space of a topological space X is a pair (\tilde{X}, p) consisting of a topological space \tilde{X} and a continuous map $p: \tilde{X} \rightarrow X$ such that each point $x \in X$ has a neighborhood U for which p maps each component of $p^{-1}(U)$ homeomorphically onto U . The map p is called a covering projection. In all of our applications \tilde{X} and X will be generalized pseudosurfaces. If Y is a subset of X and \tilde{Y} is a subset of \tilde{X} such that p maps \tilde{Y} homeomorphically onto Y , we say that Y lifts to \tilde{Y} or that \tilde{Y} is a lift of Y . For a given topological space, a path f is a continuous map from the closed unit interval $[0,1]$ into the given space. Although a path is a map by definition, we will find it convenient to identify the path f with its image. The initial point of path f is $f(0)$. The following theorem essentially states that paths in the space X are lifted to unique paths in \tilde{X} .

THEOREM 2.1. Let (\tilde{X}, p) be a covering space of X , $\tilde{x} \in \tilde{X}$, and $x = p(\tilde{x})$; let f be a path in X with initial point x . Then there exists a unique path \tilde{f} in \tilde{X} which is a lift of f and such that \tilde{x} is the initial point of \tilde{f} .

PROOF. See [22] Chapter 5 Lemma 3.1.

THEOREM 2.2. If (\tilde{X}, p) is a covering space of X , then the sets $p^{-1}(x)$ for all $x \in X$ have the same cardinality.

PROOF. See [22] Chapter 5 Lemma 3.4.

If n is the common cardinality of the sets $p^{-1}(x)$, we say that the covering is n -fold.

For our purposes here, it is sufficient to say that (\tilde{X}, p) is a branched covering space of a topological space X , if there exists a discrete set B of points of X such that $(\tilde{X} - p^{-1}(B), p)$ is a covering space of $X - B$. The points of B are the branch points. If b is a branch point, then for some sufficiently small open neighborhood U of b , the restricted map $p : \tilde{U} \rightarrow U - \{b\}$ is n -fold, where n is a positive integer and \tilde{U} is a component of $p^{-1}(U - \{b\})$ in \tilde{X} . We call n the degree of branching at b . An example of each of these types of spaces may be helpful. Define a map p_n from the unit circle S^1 onto itself by the equation $p_n(1, \theta) = (1, n\theta)$, where (r, θ) denotes polar coordinates

in the plane R^2 . The map p_n wraps the circle around itself n times. For n a positive integer, it is readily seen that the pair (S^1, p_n) is an n -fold covering space of S^1 . The map $q_n(1, \phi, \theta) = (1, \phi, n\theta)$ (n a positive integer and here we use spherical coordinates) that wraps the sphere around itself n times leaving the north and south poles fixed defines a branched covering projection. The north and south poles are the only branch points, each having degree of branching n . For more details the reader may refer to Fox [5] or Massey [22].

We now give the definitions and main construction of voltage graph theory according to Gross [8]. (See also Gross and Alpert [9], [10], and Stahl [30].) A reduced voltage pseudograph is a pair (G, β) consisting of a graph G and a function β from the directed edges of G to a group Γ (usually finite) such that the value of β on a directed edge is the inverse in Γ of its value on the oppositely directed edge; that is, if k is a directed edge and k^{-1} the oppositely directed edge, $\beta(k^{-1}) = [\beta(k)]^{-1}$. The values of β are called voltages and the function β is called a voltage assignment in Γ for G . To a given reduced voltage pseudograph (G, β) there is associated a derived graph G^β whose vertex set is the cartesian product $V \times \Gamma$ of the set V of vertices of G and the voltage group Γ . Given two vertices (u, g) and (v, h) in G^β , there is a directed

edge from (u,g) to (v,h) in G if and only if there is a directed edge from u to v in G and $h = g\beta(uv)$ in Γ . A voltage pseudograph is a triple $(G,\beta,c:G \rightarrow M)$ such that the pair (G,β) is a reduced voltage graph and $c:G \rightarrow M$ gives a 2-cell imbedding $(c(G))$ of G on a surface M . For a given region f in an imbedding c , the cyclic product of voltages around the boundary of f consistent with the directed edges is called an excess voltage and such products are unique up to conjugacy. The order of an excess voltage around a given region f is thus unique, and if the order is 1 we say Kirchoff's Voltage Law (KVL) holds around f .

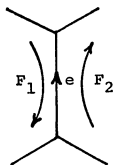
It follows almost immediately from the construction of the derived graph G^β that each region boundary in G with n edges lifts to a set of region boundaries in G^β each with $n \cdot e$ edges, where e is the order of an excess voltage about the given region in the imbedding of G . The derived surface M^β and the derived imbedding $c^\beta:G^\beta \rightarrow M^\beta$ are obtained by first identifying each component of a lifted region boundary with the sides of a 2-cell (unique to that component) and then performing the standard identification of edges from surface topology. If we regard pseudographs as topological spaces with edges and loops homeomorphic to closed intervals and circles, respectively, then G^β is an $|\Gamma|$ -fold covering space of G .

It is often important to know the number of components of G^β . This information is given in the following theorem; but first, we give necessary preliminary definitions. Given a reduced voltage pseudograph (G, β) and a closed walk $w: e_1, \dots, e_n$ (note that each e_i is a directed edge in (G, β)) at a vertex v of G , define $\beta(w) = \prod_{i=1}^n \beta(e_i)$. The local group at v , denoted by Γ_v , is defined for all $v \in V(G)$ as $\Gamma_v = \{ \beta(w) \mid w \text{ is a closed walk at } v \}$. It is simple to verify that Γ_v is a subgroup of Γ and that if u and v are two vertices belonging to the same component of G , then Γ_u and Γ_v are conjugate subgroups of Γ . Thus the index of Γ_v in Γ is independent of v if the pseudograph G is connected.

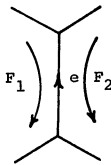
THEOREM 2.3. (Stahl[30]) Given a connected reduced voltage pseudograph (G, β) with group Γ , the number of components of the covering graph G^β equals the index of Γ_v in Γ for any v of G .

It is well known that only orientable surfaces can cover orientable surfaces, whereas both nonorientable and orientable surfaces may cover nonorientable surfaces. A second theorem by Stahl gives a method for determining the type of orientability of the covering space when the base space is nonorientable. Once again preliminary notation and definitions are needed.

Given a voltage pseudograph $(G, \eta, c: G \rightarrow M)$ with voltages in Γ , we say that the closed walk w of G is η -trivial if $\eta(w)$ is the identity element of Γ . Let G be a pseudograph 2-cell imbedded and Ω a set of orientations for the regions of this imbedding. An edge e bounding regions F_1 and F_2 (F_1 and F_2 not necessarily distinct) is coherently oriented if e occurs in the oriented boundary of one region and e^{-1} occurs in the oriented boundary of the other region. Otherwise, edge e is noncoherently oriented. (See Figure 2.1.) It is essentially by definition that an imbedding is orientable if and only if there exists a set of region orientations for which each edge is coherently oriented.



A Coherently Oriented Edge



A Noncoherently Oriented edge

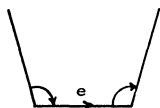
Figure 2.1

Stahl's orientability-criterion theorem uses generalized imbedding schemes (P, λ) to describe imbeddings algebraically. Here P is a set of vertex rotations and λ is a voltage assignment in the group Z_2 . (A vertex rotation at v is a cyclic permutation of the directed edges terminating at v .) For our purposes it is not necessary to go into a full discussion of such schemes. We are primarily interested in applying his orientability criterion to imbeddings of voltage pseudographs given in a particular type of planar polygonal form. To do this it suffices to know how the values of λ are assigned to the edges of the voltage pseudograph. The interested reader may see [30] for the precise nature of λ and complete discussion of generalized imbedding schemes. First we state Stahl's theorem and then describe how to apply this theorem for our special needs.

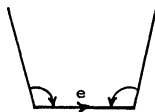
THEOREM 2.4 (Stahl[30]). Let (G, β) be a reduced voltage pseudograph with generalized imbedding scheme (P, λ) (describing an imbedding of (G, β)). Then the derived surface M^β is orientable if and only if every β -trivial closed walk in G is also λ -trivial.

The imbeddings of (G, β) to which we will apply Theorem 2.4 will always be given in a planar polygonal form in which all the vertices of G are located on the periphery. (See Examples 1 and 2 following.) Stahl

shows (See [30].), in this event, that an edge in G is assigned $\lambda = 1$ if and only if its dual edge (once a set of region orientations for the dual G^* has been chosen) is noncoherently oriented. For convenience in application we may circumvent passing to the dual by assigning orientations to the vertices of G instead of the regions of the imbedded G^* . Those edges uv in G for which the local orientations at u and v agree (both clockwise or both counterclockwise) correspond to coherently oriented edges in G^* and those edges uv in G for which the local orientations disagree correspond to noncoherently oriented edges in G^* . (See Figure 2.2.) Thus edges in G for which local vertex orientations disagree are precisely the edges that are assigned $\lambda = 1$.



Agreeing local vertex
orientations $\lambda = 0$



Disagreeing local vertex
orientations $\lambda = 1$

Figure 2.2

The following examples will, perhaps, elucidate the foregoing discussion.

Example 1. Let (G, β) be the reduced voltage pseudo-graph imbedded in planar polygonal form as described in Figure 2.3. The voltages are in $\Gamma = \mathbb{Z}_{12}$. The pseudo-graph G is actually a bouquet of five loops imbedded in the Klein bottle, $\tilde{\mathbb{S}}_2$. (The two edges labeled "2" are to be identified, in the direction given by the arrows, as are the two edges labeled "3".)

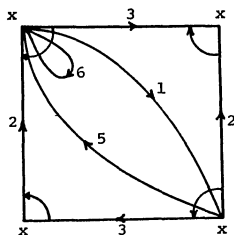


Figure 2.3

The vertex orientations as assigned in Figure 2.3 dictate that $\lambda = 1$ on precisely those edges whose voltages are 1, 2, and 5. The closed walk w consisting of following the loop with voltage 6 and passing 3 times around the loop with voltage 2 is β -trivial but

not λ -trivial; therefore, the derived surface is non-orientable. In fact, the derived surface is \tilde{S}_8 . Furthermore, it is easily shown that the derived graph is $K_4(3)$ (after identifying the sides in each digon lying over the loop with voltage 6).

Example 2. This example is very similar to Example 1 but the planar polygonal form (See Figure 2.4.) is slightly altered from that given in Figure 2.3, producing a voltage pseudograph with 2 vertices, imbedded in the projective plane, \tilde{S}_1 . (Again the two edges labeled "2" are to be identified, in the direction given by the arrows, as are the two edges labeled "3".) After we identify these two vertices, the reduced voltage pseudograph is again a bouquet of five loops but this time imbedded in a (nonorientable) pseudosurface with 1 singular point of degree 2.

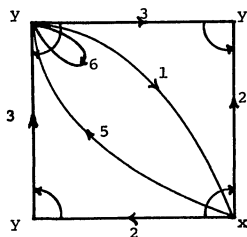


Figure 2.4

Now the vertex orientations as assigned in Figure 2.4 indicate that $\lambda = 1$ on precisely those edges whose voltages are 2 and 3. It can be argued quite easily that every β -trivial closed walk beginning at either vertex x or vertex y is also λ -trivial; consequently, the derived surface (derived from the \tilde{S}_1 imbedding prior to identifying x and y) is orientable. Here $\Gamma_y = \{0, 3, 6, 9\}$ so $|\Gamma|/|\Gamma_y| = 3$ and in fact each of the 3 components before identification is a sphere. Once again the derived graph (after identification of proper vertices--see Theorem 2.6) is $K_4(3)$.

We next give the main theorem of voltage graph theory due to Gross [8] and then extend this to pseudosurfaces. Our interest in producing triangular imbeddings and block designs lies primarily in the pseudosurface extension but for completeness, we further expand the theory to generalized pseudosurfaces.

THEOREM 2.5 (Gross). Let the voltage pseudograph $(G, \beta, c: G \rightarrow M)$ with voltages in group Γ have regions R_1, \dots, R_r , having number of sides s_1, \dots, s_r , and carrying excess voltages of orders e_1, \dots, e_r , respectively. Then the derived imbedding $c^\beta: G^\beta \rightarrow M^\beta$ is a (possibly branched) covering of the imbedding $c: G \rightarrow M$. For $i = 1, \dots, r$, there are $|\Gamma|/e_i$ regions lying over region R_i , each with $e_i \cdot s_i$ sides and (for $e_i \neq 1$) a branch point of degree e_i in its interior.

As we have stated before, Stahl has extended this theorem in the nonorientable case through what he calls a generalized embedding scheme. For our purposes here it is not necessary to go into his general development but the interested reader may see [30].

THEOREM 2.6. Let (G, β) be a reduced voltage pseudo-graph with group Γ 2-cell imbedded in a pseudosurface $M(G)$. Let regions R_1, \dots, R_r have number of sides s_1, \dots, s_r and carry excess voltages of orders e_1, \dots, e_r , respectively. Then the 2-cell imbedding (in a generalized pseudosurface) of the derived graph G^β is a (possibly branched) covering over the 2-cell imbedding of (G, β) . For $i = 1, \dots, p$, there are $|\Gamma|/e_i$ regions lying over region R_i , each with $e_i \cdot s_i$ sides and (for $e_i \neq 1$) a branch point of degree e_i in its interior.

PROOF. Let (G, β) be a reduced voltage pseudograph with group Γ , 2-cell imbedded in a pseudosurface $M(G)$ having singular points v_1, \dots, v_k . Form a new pseudo-graph $(G, \beta)'$ from (G, β) by reversing the identifications comprising the singular points and label the new vertices associated with v_i by $v_{i1}, v_{i2}, \dots, v_{in_i}$ for each $i = 1, \dots, k$. Observe that this gives a 2-cell surface imbedding of $(G, \beta)'$ whose regions are precisely the same as the regions of the imbedding of (G, β) . Now invoke the surface version of this theorem, getting a

2-cell imbedding of $(G^\beta)'$ as a (possibly branched) covering over the imbedding of $(G, \beta)'$. Let $\Gamma = \{g_1, \dots, g_t\}$ and for $j = 1, 2, \dots, t$ identify the set of vertices $\{(v_{i1}, g_j), \dots, (v_{in_i}, g_j)\}$ where $i = 1, \dots, k$; this gives a 2-cell generalized pseudosurface imbedding of G that is a (possibly branched) covering space of the imbedding of (G, β) . The singular point v_i with degree n_i in $M(G)$ lifts to $|\Gamma| = t$ singular points each with degree n_i in the derived imbedding of G ($1 \leq i \leq k$). Since the regions of G^β are the same as those of $(G^\beta)'$ the statements in the last sentence of the theorem are immediate.

The generalized pseudosurface version of the above theorem now follows by applying the pseudosurface extension componentwise (after reversing proper singular point identifications).

THEOREM 2.7. Let (G, β) be a reduced voltage pseudo-graph with group Γ 2-cell imbedded in a generalized pseudosurface $M(G)$. Let regions R_1, \dots, R_r have number of sides s_1, \dots, s_r and carry excess voltages of orders e_1, \dots, e_r , respectively. Then the 2-cell imbedding of the derived graph G^β is a (possibly branched) covering over the 2-cell imbedding of (G, β) . For $i = 1, \dots, r$, there are $|\Gamma|/e_i$ regions lying over region R_i , each with $e_i \cdot s_i$ sides and (for $e_i \neq 1$) a branch point of

degree e_i in its interior.

PROOF. Let (G, β) be a reduced voltage pseudograph with group Γ 2-cell imbedded in a generalized pseudo-surface $M(G)$. Let d_1, \dots, d_k be precisely those singular vertices which have been formed by identifying at least two vertices from different components (that is, those singular points which do not consist entirely of vertices from the same component). Form a new pseudograph $(G, \beta)'$ from (G, β) by reversing the identifications of precisely the singular points d_1, \dots, d_k and label the new vertices associated with d_i by d_{i1}, \dots, d_{ia_i} for each $i = 1, \dots, k$. Observe that this gives a 2-cell pseudosurface imbedding in each component of $(G, \beta)'$ whose regions are precisely the same as those in the imbedding of (G, β) . Now for each component, invoke Theorem 2.6, getting a 2-cell imbedding of $(G^\beta)'$ as a (possibly branched) covering over the imbedding of $(G, \beta)'$. Let $\Gamma = \{g_1, \dots, g_t\}$ and for $j = 1, \dots, t$ identify the set of vertices $\{(d_{i1}, g_j), \dots, (d_{ia_i}, g_j)\}$ where $i = 1, 2, \dots, k$; this gives a 2-cell generalized pseudo-surface imbedding of G^β that is a (possibly branched) covering space of the imbedding of (G, β) . The singular point d_i with degree a_i in $M(G)$ lifts to $|\Gamma|$ singular points each with degree a_i in the derived imbedding of G^β ($1 \leq i \leq k$). The remaining claims of the theorem follow as before.

The following corollary relates the characteristic between base space and covering space when no branching occurs.

COROLLARY. Let (G, β) be a reduced voltage pseudograph with group Γ 2-cell imbedded in a generalized pseudo-surface $M(G)$ and suppose the KVL holds around every region. Then the characteristic of the derived surface M^β is given by $\chi(M^\beta) = |\Gamma| \chi(M)$.

PROOF. By construction of the derived graph, there is a $|\Gamma|$ -fold lifting of vertices and edges. Since the KVL holds on each region in $M(G)$, by the theorem there is an $|\Gamma|$ -fold covering of each region in $M(G)$. Thus

$$\begin{aligned}\chi(M^\beta) &= |\Gamma|p - |\Gamma|q + |\Gamma|r \\ &= |\Gamma|(p - q + r) \\ &= |\Gamma|\chi(M).\end{aligned}$$

We now give a brief discussion of current graph theory for the orientable surface case. The basic concepts are dual to those of voltage graph theory; however, one difference is that we describe the imbeddings of the current graphs by rotation systems. It is because of this that we choose to give a separate development of current graph theory rather than a simultaneous one. The reader may see [10] for a more detailed study of current graphs than we give below.

A current pseudograph (G, ϕ, α) is a triple consisting of a pseudograph G , a rotation system ϕ that assigns to each vertex v of G a cyclic permutation ϕ_v of the oriented edges with initial point v , and a function α from the directed edges of G to a group Γ such that the value of α on a directed edge is the inverse in Γ of its value on the oppositely directed edge. The values of α are called currents and the function α is called a current assignment in Γ for G . The rotation system on the graph G describes combinatorially a 2-cell imbedding of G in a surface $M(G, \phi)$. This rotation system gives rise to a permutation ϕ' on the set of oriented edges of G , and the elements of the orbit set C of ϕ' are called circuits of (G, ϕ) and correspond to region boundaries in $M(G, \phi)$. To a given current graph (G, ϕ, α) there is associated a derived graph G^α and a derived rotational system ϕ^α ; the vertex set of G^α is the cartesian product $C \times \Gamma$ of the set C of circuits of (G, ϕ) and the current group Γ . For every oriented edge $k = (u, v)$ in circuit c and every $g \in \Gamma$, there is an edge $[(c, g), (d, h)]_k$ where $h = g\alpha(k)$ and d is the circuit containing the edge $s = (v, u)$ oppositely oriented to that of k ((v, u) is the opposite of (u, v)). If the currents on circuit c are cyclically ordered as a_0, \dots, a_{n-1} and if the opposites of the oriented edges k_0, \dots, k_{n-1} carrying those currents lie in circuits

c_0, \dots, c_{n-1} , respectively, then the rotation at vertex (c, g) of (G^α, ϕ^α) is the cyclic permutation that carries the oriented edge $[(c, g), (c_i, ga_i)]_{k_i}$ to the oriented edge $[(c, g), (c_{i+1}, ga_{i+1})]_{k_{i+1}}$ for $i = 0, \dots, n-1 \pmod{n}$.

The rotation system ϕ^α on G^α then gives a 2-cell imbedding of G^α in a surface $M(G^\alpha, \phi^\alpha)$. The product of the inflowing currents at v , taken in the cyclic order given by ϕ_v , is called the excess current at v and is unique up to conjugacy. If the order of the excess current at v is 1, we say Kirchoff's Current Law (KCL) holds at vertex v .

THEOREM 2.8 (Gross and Alpert). Let the current graph (G, ϕ, α) with currents in group Γ have vertices v_1, \dots, v_p with degrees k_1, \dots, k_p and carrying excess currents of orders e_1, \dots, e_p , respectively. Then the 2-cell imbedding (G^α, ϕ^α) in $M(G^\alpha, \phi^\alpha)$ is a (possibly branched) covering of the 2-cell imbedding of G^* in $M(G, \phi)$ dual to (G, ϕ) in $M(G, \phi)$. For $i = 1, \dots, p$ there are $|\Gamma|/e_i$ regions lying over region v_i^* (dual to v_i in G), each with $k_i \cdot e_i$ sides and (for $e_i \neq 1$) a branch point of degree e_i in its interior.

CHAPTER 3

BLOCK DESIGNS

Block designs are widely used by statisticians in the design of experiments. There are many types of such designs but we will only be interested in two: a so called balanced incomplete block design (BIBD) and a generalization of this, a partially balanced incomplete block design (PBIBD).

It is not our interest in this thesis to design statistical experiments that would use PBIB designs, but rather, it is our purpose to increase the known number of PBIB designs by constructing new such designs. Nevertheless, we motivate the statistical use of block designs by the following two examples.

Example 1. Consider the following wine tasting problem. Suppose we have 4 wines to test and 4 wine tasters. In designing our experiment, we might like, ideally, that each wine taster sample all 4 wines, but experience has shown that a person can no longer distinguish the qualities of wine after having sampled 3 wines. Thus since we cannot have any one person tasting all 4 wines, we do the next best thing and require that each taster test exactly 3 wines. It also seems only natural

that we would want each wine tasted the same number of times in order to help "balance" the experiment; let's say 3 times for this experiment. Finally, the fact that two wines tasted together may have an influence on their respective tastes (for example tasting a sweet and dry wine may elicit a different opinion from the taster than testing two dry wines together) leads us to insist that every two wines be tasted together the same number of times, say twice, so as to help reduce extraneous influences. Table A in Figure 3.1 gives a solution to this problem. The integers 1,2,3, and 4 represent the 4 wines and the rows (blocks), the wine tasters. Each row then indicates exactly which three wines will be tried by that taster. This then is an example of a balanced incomplete block design. (see below for the precise definition.)

Row 1	1 3 2
Row 2	2 4 3
Row 3	3 1 4
Row 4	4 2 1

Table A

Figure 3.1

Example 2. We are planning a wine tasting party with

8 couples, i.e., 16 tasters and 8 wines. As before we know that no one can discriminately judge after tasting 3 wines so we again require each person to sample exactly 3 wines. We still want each wine to be tested the same number of times, say 6 times. However, there is a new stipulation in this experiment: there are four brands of wine each having two types. (For example one brand may be Gallo with the two types being Burgundy and Hearty Burgundy.) Interest lies in the influence among different brands but not between types of the same brand name. Therefore, we require that wines having different brand names be tasted together the same number of times, say twice, but wines of the same brand should not be sampled together. Table B, Figure 3.2 gives a solution to the wine party problem where the integers 1 to 8 represent wines, integers equivalent modulo 4, brand names, and rows, wine tasters. This is an example of a partially balanced incomplete block design. (The exact definition of a PBIBD follows below.)

Row 1	1	3	8	Row 9	1	4	3
Row 2	2	4	1	Row 10	2	5	4
Row 3	3	5	2	Row 11	3	6	5
Row 4	4	6	3	Row 12	4	7	6
Row 5	5	7	4	Row 13	5	8	7
Row 6	6	8	5	Row 14	6	1	8
Row 7	7	1	6	Row 15	7	2	1
Row 8	8	2	7	Row 16	8	3	2

Table B

Figure 3.2

In the next chapter we shall see the connection between certain imbedded graphs and block designs. First, however, we give precise definitions for both BIB and PBIB designs.

A (v,b,r,k,λ) -balanced incomplete block design (BIBD) is an arrangement of v objects into b blocks with:
 i) each object appearing in exactly r blocks; ii) each block containing exactly k ($k < v$) objects; and iii) each pair of distinct objects appearing together in exactly λ blocks. Simple counting arguments establish the following well known result.

THEOREM 3.1. If a (v,b,r,k,λ) -BIBD exists, then

- (i) $vr = bk$
- (ii) $\lambda(v-1) = r(k-1)$.

These conditions have also been shown to be sufficient for $k = 3,4,5$ (and for some of the cases $k = 6,7$) by Hanani [13] and for fixed k and λ with v large enough by Wilson [35].

The condition that λ be an integer is quite restrictive; this prompted Bose and Nair to introduce a related design with multiple lambdas (association classes). We concern ourselves with the case of two association classes. It will prove useful to give a definition of these designs in terms of a strongly regular graph, which we now define.

Let x and y be distinct vertices in a graph G

and denote their nonadjacency by $h = 1$ and adjacency by $h = 2$. Let $P_{ij}^h(x, y)$ be the number of vertices adjacent to both x and y ($i = j = 2$), adjacent to x but not to y ($i = 2, j = 1$), adjacent to y but not to x ($i = 1, j = 2$), and adjacent to neither x nor y ($i = j = 1$), respectively. A graph G ($G \neq K_n, \overline{K_n}$) is strongly regular if the eight integers $P_{ij}^h(x, y)$ are independent of x and y .

Bose and Clatworthy [4] show the eight conditions can be reduced to the two conditions that $P_{22}^1(x, y)$ and $P_{22}^2(x, y)$ be independent of x and y . A $(v, b, r, k, \lambda_1, \lambda_2)$ -partially balanced incomplete block design (PBIBD) with two association classes can be described as a strongly regular graph whose v vertices are arranged into b blocks with: i) each vertex appearing in exactly r blocks, ii) each block containing exactly k vertices, iii) each nonadjacent pair of vertices (first associates) appearing together in exactly λ_1 blocks, and iv) each adjacent pair of vertices (second associates) appearing together in exactly λ_2 blocks. We will be concerned entirely with designs which are based on the regular complete n -partite graphs $K_{n(m)}$. These graphs are readily seen to be strongly regular since $P_{22}^1(x, y) = m(n-1)$ and $P_{22}^2(x, y) = m(n-2)$ are independent of x and y . A BIBD or PBIBD is resolvable if the b blocks can be partitioned into exactly bk/v sets with each such set containing

each of the v objects exactly once. Similarly, a BIBD or PBIBD is n -partially resolvable if there are n (for some n between 1 and bk/v) pairwise disjoint sets of blocks with each such set containing each of the v objects exactly once. Thus a bk/v - partially resolvable design is resolvable.

Two block designs D_1 and D_2 are isomorphic if there exists a one-to-one correspondence $\Psi: \Omega_1 \rightarrow \Omega_2$ between their object sets such that $\{x_1, \dots, x_k\}$ is a block in D_1 if and only if $\{\Psi(x_1), \dots, \Psi(x_k)\}$ is a block in D_2 .

CHAPTER 4

BLOCK DESIGNS AND IMBEDDED GRAPHS

In 1897 L. Heffter [15] first explored imbedded graphs as a means of constructing block designs. Not many graph imbeddings were known then, so very few new block designs evolved. Ringel and Youngs' solution to the Heawood map color problem has recently effected a flurry of research on imbedding graphs. This new research has provided an impetus for both Alpert and White to reexamine Heffter's idea of creating block designs by imbedding graphs. Alpert shows [2] that there is a one-to-one correspondence between $(v,b,r,3,2)$ -BIB designs and triangular imbeddings of K_v in generalized pseudosurfaces, and White [34] shows that each $(v,b,r,3,0,2)$ -PBIB design corresponds to a triangular imbedding of a strongly regular graph in a generalized pseudosurface, and conversely. We will be concerned primarily with triangular imbeddings of regular complete n -partite graphs.

In generating new PBIB designs by graph imbeddings a natural question arises: namely, given a strongly regular graph G triangularly imbedded in nonhomeomorphic generalized pseudosurfaces, are the corresponding block designs isomorphic? This question is answered in the following theorem [34].

THEOREM 4.1. Let G be a strongly regular graph tri-
angularly imbedded in nonhomeomorphic generalized pseudo-
surfaces T_1 and T_2 . Then the block designs determined
by the two imbeddings are not isomorphic.

In [7] Garman, Ringeisen, and White introduce the
strong tensor product $G_1 \otimes G_2$ of graphs G_1 and G_2 .
The vertex set of G_1 strong tensor product with G_2 is
the cartesian product of $V(G_1)$ and $V(G_2)$, and the edge
set is $\{ (u_1, u_2)(v_1, v_2) \mid [u_1 = v_1 \text{ and } u_2 v_2 \in E(G_2)] \text{ or } [u_i v_i \in E(G_i) \text{ for } i = 1 \text{ and } 2] \}$. It will be helpful
to think of the strong tensor product as a combination of
two other graphical operations: the sum and the tensor
product. Given two graphs G_1 and G_2 with $V(G_1) = V(G_2)$
and $E(G_1) \cap E(G_2) = \emptyset$, the sum $G_1 + G_2$ of G_1 and G_2
is that graph whose vertex set equals $V(G_1)$ and edge
set equals $E(G_1) \cup E(G_2)$. The tensor product of graphs
 G_1 and G_2 is denoted by $G_1 \otimes G_2$; the vertex set of
 G_1 tensor product with G_2 is the cartesian product of
 $V(G_1)$ and $V(G_2)$ and the edge set is $\{ (u_1, u_2)(v_1, v_2) \mid u_i v_i \in E(G_i), i = 1 \text{ and } 2 \}$. It is an easy exercise
to see that $K_2 \otimes G = 2G + K_2 \otimes G$ where, as before, $2G$
denotes two disjoint copies of G . The following two
theorems are special cases of two theorems given (among
many others) in [7].

THEOREM 4.2. $K_2 \otimes K_{n(m)} = K_{n(2m)}$.

THEOREM 4.3. Let G be a triangularly imbedded graph in an orientable surface, with bichromatic dual. Then $K_2 \otimes G$ has an orientable (surface) triangular imbedding, with bichromatic dual.

We next extend Theorem 4.3 to generalized pseudosurfaces (nonorientable as well as orientable).

THEOREM 4.4. Let G be a triangularly imbedded graph in a generalized pseudosurface, with bichromatic dual. Then $K_2 \otimes G$ has a triangular imbedding in a generalized pseudosurface, with bichromatic dual. Furthermore, the imbedding of $K_2 \otimes G$ is orientable if and only if the imbedding of G is orientable.

PROOF. We first consider the case when G is triangularly imbedded in a nonorientable surface \tilde{S}_n . We regard $K_2 \otimes G$ as $2G + K_2 \otimes G$. Transform the triangular imbedding of G on \tilde{S}_n into a planar polygonal form P_1 and form a second copy of this imbedding by taking a "mirror image" of the planar polygonal form P_1 calling this new form P_2 . The planar polygonal form P_1 can easily be constructed by pasting the triangles of the imbedding together in the plane along common edges. Since the imbedding is triangular each edge appears in exactly two triangles so that the polygon formed will have the property that each perimeter edge appears twice along the perimeter.

Note that many different forms can be constructed; however, each such form will yield the same surface after the perimeter edges are identified according to standard topological procedure from surface topology. Because G is triangularly imbedded the number of regions r is even, say $r = 2t$. (This follows from the equation $2q = 3r$.) Since G has bichromatic dual, half of the regions must be colored with one color whereas the remaining half must be colored with the other color. Let regions R_1, \dots, R_t be colored black and regions R_{t+1}, \dots, R_{2t} be colored white in P_1 in accordance with $\chi(G^*) = 2$. For $i = 1, \dots, 2t$, let R_i' be the region in P_2 corresponding to R_i in P_1 , that is, R_i' is the mirror image of R_i . Color regions R_1', \dots, R_t' white and regions R_{t+1}', \dots, R_{2t}' black. We must add the tensor product edges to form $K_2 \otimes G$ from $2G$. This will be accomplished by attaching cylinders between the two imbeddings represented by P_1 and P_2 , so as to form \tilde{S}_m for some m (after identification of respective perimeter edges of the polygons) and then triangulating the cylinders with these edges. We will see that the resulting triangular imbedding also has bichromatic dual.

Consider R_1 in P_1 and its mirror image R_1' in P_2 . Next, run a topological cylinder T_1 between R_1 and R_1' attaching one base, B_1 , of T_1 in the interior of R_1 and the other base, B_2 , of T_1 in the interior

of R_1' . The edges xy' , xz' , yz' , yx' , zx' , and zy' can now be imbedded along T_1 as shown in Figure 4.1. Note that six triangular regions are formed along T_1 and that these can be 2-colored consistently with the 2-colorings of P_1 and P_2 . Now repeat this process, joining region R_i in P_1 with R_i' in P_2 by cylinder T_i , $i = 1, 2, \dots, t$, and adding the six required tensor product edges along each cylinder. At this stage we have added precisely $6t = 2q$ ($2q =$ twice the number of edges in G) edges of $K_2 \otimes G$ so that $K_2 \otimes G$ is triangularly imbedded on the nonorientable surface represented by the union of the two surfaces described by P_1 and P_2 (after identification of respective perimeter edges) and as altered by the cylinder attachments. Moreover, the imbedding of $K_2 \otimes G$ has bichromatic dual.

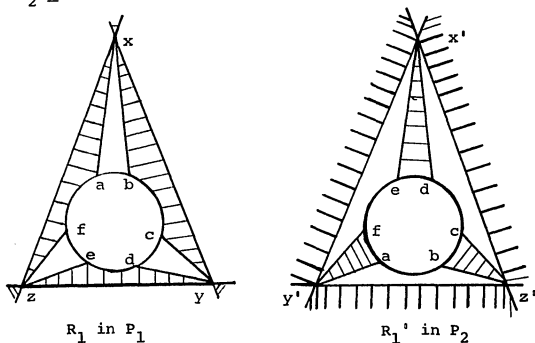


Figure 4.1

We now consider the case when G is triangularly imbedded with bichromatic dual in a generalized pseudo-surface. Let G' denote the graph obtained from G by reversing each singular point identification. For the given imbedded G , we form G' and thereby obtain a triangular imbedding (since the regions are unchanged) of G' in a possibly disconnected surface M . We next apply either Theorem 4.3 or the first case of the present theorem (as appropriate) to each component of M , giving a triangular imbedding with bichromatic dual of $K_2 \otimes G'$. Finally, we obtain a triangular imbedding with bichromatic dual of $K_2 \otimes G$ from $K_2 \otimes G'$ by reidentifying to obtain the singular points in each copy of G' . It is clear from the foregoing proof that the imbedding of $K_2 \otimes G$ is orientable if and only if the imbedding of G is orientable.

As we have seen, each triangular imbedding in a generalized pseudosurface of a strongly regular graph determines a PBIBD having $k = 3$, $\lambda_1 = 0$, and $\lambda_2 = 2$. By applying Theorem 4.4 to those imbeddings having bichromatic dual, we generate a "doubled" PBIBD having $k = 3$, $\lambda_1 = 0$, and $\lambda_2 = 2$, that is, v , the number of objects, and r , the number of blocks each object appears in, are doubled. Furthermore, there is a simple recipe to give the objects and blocks (regions) of the doubled PBIBD in terms of the objects and blocks stemming from the

t

original PBIBD. Thus if $\bigcup_{i=1}^t \{[a_{i1}, a_{i2}, a_{i3}]\}$ are the blocks (regions) colored white and $\bigcup_{i=1}^t \{[b_{i1}, b_{i2}, b_{i3}]\}$ are the blocks (regions) colored black in the initial design (where $b = 2t$), then the union of the following 4 expressions are the blocks colored white

1. $\bigcup_{i=1}^t \{[a_{i1}, a_{i2}, a_{i3}]\}$
2. $\bigcup_{i=1}^t \{[b_{i1}, b'_{i2}, b'_{i3}]\}$
3. $\bigcup_{i=1}^t \{[b'_{i1}, b'_{i2}, b_{i3}]\}$
4. $\bigcup_{i=1}^t \{[b'_{i1}, b_{i2}, b'_{i3}]\}$,

and the union of the next 4 expressions are the blocks colored black in the doubled design (the number of blocks here is $8t = 4b$)

1. $\bigcup_{i=1}^t \{[a'_{i1}, a'_{i2}, a'_{i3}]\}$
2. $\bigcup_{i=1}^t \{[b'_{i1}, b_{i2}, b_{i3}]\}$
3. $\bigcup_{i=1}^t \{[b_{i1}, b_{i2}, b'_{i3}]\}$
4. $\bigcup_{i=1}^t \{[b_{i1}, b'_{i2}, b'_{i3}]\}$.

In this connection, we see that triangular imbeddings of strongly regular graphs with bichromatic duals are particularly valuable. The next theorem gives a sufficient condition to impose on an imbedded reduced

voltage pseudograph (G, β) so that the derived imbedding of G will have bichromatic dual.

THEOREM 4.5. Suppose an imbedding of a reduced voltage pseudograph (G, β) in a generalized pseudosurface has bichromatic dual. Then the derived imbedding of the derived graph G^β also has bichromatic dual.

PROOF. We note that in the construction of G^β each region boundary f in G with n edges lifts to a set of region boundaries in G^β ; each region boundary in this set has $n \cdot e$ edges where e is the order of the excess voltage around f . Since the imbedding of G has bichromatic dual its regions can be colored with exactly 2 colors. This induces a coloring of the regions of G^β by assigning to any region R in the imbedding of G^β the same color as the unique region onto which R projects. It is immediate that this gives a 2-coloring of the regions of G since two regions in G^β sharing a common edge k project onto two regions in G sharing the unique edge that lifts to k .

We illustrate Theorem 4.5 with the following voltage pseudograph. (See Figure 4.2) In this example the imbedding is in the nonorientable surface \tilde{S}_6 . As indicated by Figure 4.2, the imbedding has bichromatic dual. Theorem 4.5 implies that the derived imbedding also has

CHAPTER 5

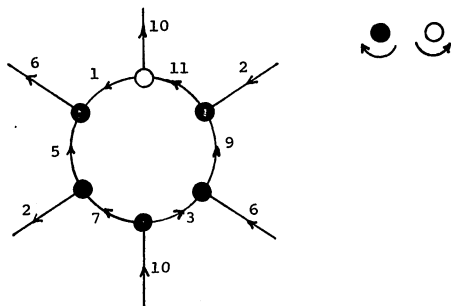
GENERATING NEW BLOCK DESIGNS

In Chapter 4 we saw that every triangular imbedding of a strongly regular graph in a generalized pseudo-surface yields a partially balanced incomplete block design. In this chapter we use the theories of both voltage and current graphs to generate triangular imbeddings for $K_{4(n)}$, the regular complete 4-partite graphs of order $4n$. We show for n even that each imbedding has bichromatic dual and how it splits to give two related designs with the parameters b, r, λ_1 , and λ_2 each one-half of its original value. Finally for $n \equiv 6 \pmod{12}$, we show that $K_{4(n)}$ is n -partially resolvable and how this can be used to produce triangular imbeddings of $K_{5(n)}$ for the same values of n . The block designs that we give in this chapter can be reinterpreted to yield results for the genus, pseudocharacteristic, and generalised pseudocharacteristic parameters. We do this in Chapter 6.

We begin by showing that $K_{4(n)}$ for $n \equiv 2 \pmod{4}$ can be triangularly imbedded with bichromatic dual in an orientable surface. We do this by employing current graph theory but we must first verify several preliminary properties and results.

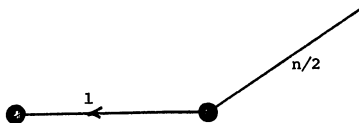
For the graph in Figure 5.3, we understand that a solid dot has its incident edges ordered clockwise; a hollow dot, counterclockwise. Next we define the graph G_n in Figure 5.3 and describe the assignment of the labels (currents). Let $\Gamma = \mathbb{Z}_{4n}$ and $\Delta = \{1, 2, 3, 4, \dots, 8s+4\} - \{4, 8, 12, \dots, 8s+4\}$, where $n = 4s + 2$ and $s = 0, 1, 2, \dots$. The order of G_n is $n = 4s + 2$, and the cardinality of the edge-set of G_n is $3n/2 = 6s + 3$. The vertices of G_n may be pictured as equally spaced around the circumference of a circle. Pick any vertex and represent it with a hollow dot; represent all other vertices with a solid dot. Starting with the hollow dot, order the vertices consecutively, in a clockwise direction, as 1st through $(4s+2)nd$. Every two diametrically opposite vertices are adjacent. Note that this gives a schematic representation of G_n in the plane; the actual imbedding of G_n is on a surface of genus $1+s$ --we shall prove this shortly. For the following discussion it will be convenient to make two definitions: an edge joining two diametrically opposite vertices will be called a diameter-edge; an edge joining two consecutive vertices on the circle will be referred to as a circle-edge. We define two functions: D and C , where $D(k)$ gives the (undirected) label of the diameter-edge incident with the k th vertex, and $C(k)$ and $C(k+1)$ give the labels of the two circle-edges incident with the k th vertex.

(Note that the 1st circle-edge is immediately to the left of the 1st vertex.) The two functions are described by the following recipes:



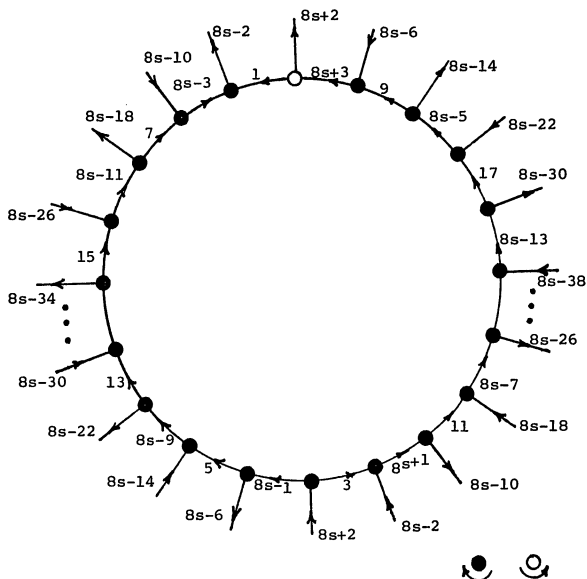
Case $n = 6$ A Current Graph for $K_4(6)$

Figure 5.1



A Current Graph for the Graph G_n

Figure 5.2



A Current Graph for $K_{4(n)}$ when $n \equiv 2 \pmod{4}$

Figure 5.3

$$D(k) = \begin{cases} (8s+2) - 8(k-1) & \text{for } 1 \leq k \leq (s+1) \\ 6 + 8[k - (s+2)] & \text{for } (s+2) \leq k \leq (2s+1) \\ (8s+2) - 8[k - (2s+2)] & \text{for } (2s+2) \leq k \leq (3s+2) \\ 6 + 8[k - (3s+3)] & \text{for } (3s+3) \leq k \leq (4s+2) \end{cases}$$

and

$$C(k) = \begin{cases} 1 + 4(k-1) & \text{for } 1 \leq k \leq (2s+1), k \text{ odd} \\ (8s+3) - 4(k-2) & \text{for } 2 \leq k \leq (2s+2), k \text{ even} \\ (8s-1) - 4[k - (2s+3)] & \text{for } (2s+3) \leq k \leq (4s+1), k \text{ odd} \\ 5 + 4[k - (2s+4)] & \text{for } (2s+4) \leq k \leq (4s+2), k \text{ even.} \end{cases}$$

The diameter-edges labeled with integers congruent to 2 (mod 8) have their arrows directed away from the center of the circle if k is odd, toward the center if k is even. Those diameter-edges labeled with integers congruent to 6 (mod 8) have the arrow directed toward the center if k is odd, away from the center if even. The k th circle-edge has its arrow pointing counter-clockwise for $1 \leq k \leq (2s+2)$ and clockwise for $(2s+3) \leq k \leq (4s+2)$. By appealing to the equivalence classes (mod 8), it is easy to see that the function C has exactly $4s + 2$ distinct values--one for each of the $4s + 2$ circle-edges. Similarly, the function D has $4s + 2$ values, but only $2s + 1$ are distinct since for each k , $1 \leq k \leq (2s + 1)$, the k th and $(k + 2s + 1)$ st vertices (diametrically opposite vertices) are joined by a common diameter-edge. Thus each of the $2s + 1$

diameter-edges has exactly one of the $2s + 1$ distinct values of D assigned to it. Moreover, (again by appealing to equivalence classes $(\text{mod } 8)$) no values of C and D are equal so that Δ is composed precisely of the $6s + 3$ distinct values of C and D .

We have stated that Figure 5.3 represents an imbedding of G_n on a surface of genus $1 + s$. This imbedding, in fact, has exactly one region. To prove this, our plan of attack will be to give a current graph whose derived graph is G_n imbedded with three regions and to apply a lemma of Ringeisen [24] that will reduce the 3-region imbedding to a 1-region imbedding of G . The lemma of Ringeisen is as follows.

LEMMA 5.1. (The Edge-Adding Technique). Let G be a connected graph and v and w denote nonadjacent vertices of G . Let T be a 2-cell imbedding of G which has vertex v on the boundary of region R_v and vertex w on the boundary of region R_w . Let G' be the graph G with edge $[v, w]$ added. Then if $R_v \neq R_w$, G' has a 2-cell imbedding with one less region than T .

Let $\Gamma_1 = Z_n = \{0, 1, 2, \dots, n-1\}$ and $\Delta_1 = \{1, n/2\}$, where $n = 4s + 2$ and $s = 0, 1, 2, \dots$. It is easily seen that $G_{\Delta_1}(\Gamma_1) = G_n$. Moreover, the current graph imbedded in the sphere has as its derived graph G_n 2-cell imbedded with three n -sided regions. (See Figure 5.2.) It

is clear that the order of the excess current at the vertex of degree 1 is n , so this vertex contributes $|\Gamma_1|/e_1 = n/n = 1$ region of length $k_1 e_1 = 1 \cdot n = n$ to the imbedding $G_{\Delta_1}(\Gamma_1)$. The sum of the currents directed away from the vertex of degree two is $1 + n/2$; we give a number theoretic argument to show that the order of this element is $n/2$ and, hence, the order of the excess current at this vertex is $n/2$. Once this is established, the vertex of degree two will account for $|\Gamma_1|/e_2 = n/(n/2) = 2$ other regions of length $k_2 e_2 = 2(n/2) = n$; hence there are exactly three regions each having n sides. Since $n = 4s + 2$, $1 + n/2 = 2s + 2$. From number theory, $[2s + 2, 4s + 2] = (2s + 2)(4s + 2)/(2s + 2, 4s + 2)$, where $[2s + 2, 4s + 2]$ and $(2s + 2, 4s + 2)$ denote the least common multiple and greatest common divisor of $2s + 2$ and $4s + 2$, respectively. The Euclidean algorithm shows that $(2s + 2, 4s + 2) = 2$; thus $[2s + 2, 4s + 2] = (2s + 2)(2s + 1)$. It follows that $(n/2) + 1 = 2s + 2$ has order $n/2 = 2s + 1$. In fact, we can write down the three regions immediately by successively adding the out-flowing currents around the vertices of the current graph in Figure 5.2. As noted above, one region (region A below) is derived from the vertex of degree 1 and two regions (regions B and C below) are derived from the other vertex. Furthermore, it is easily argued that each of the n -sided regions A, B, and C is, in fact, an

n -cycle (and thus a Hamiltonian cycle for G_n). The three regions are:

region A: $0, n-1, n-2, \dots, 3, 2, 1$

region B: $0, 1, \frac{n}{2} + 1, \frac{n}{2} + 2, 2, 3, \frac{n}{2} + 3, \frac{n}{2} + 4,$
 $\dots, -2, -1, \frac{n}{2} - 1, \frac{n}{2}$

region C: $-1, 0, \frac{n}{2}, \frac{n}{2} + 1, 1, 2, \frac{n}{2} + 2, \frac{n}{2} + 3$
 $-3, -2, \frac{n}{2} - 2, \frac{n}{2} - 1$.

Next we verify that the derived graph (from G_n) is the Cayley graph $G_\Delta(\Gamma)$ and that $G_\Delta(\Gamma) = K_{4(n)}$ where $\Gamma = Z_{4n}$ and $\Delta = \{1, 2, 3, 4, \dots, 2n\} - \{4, 8, \dots, 2n\}$ for $n = 4s + 2$ and $s = 0, 1, 2, \dots$. Once we have established that the imbedding of G_n has one region (we will do this shortly), it will follow immediately that the derived graph is $G_\Delta(\Gamma)$. To see the second part, we set $\bar{\Delta} = \{4, 8, \dots, 2n\}$ and observe that $\bar{\Delta}$ partitions the vertex set of $G_\Delta(\Gamma)$ into four sets:

$$A_0 = \{0, 4, 8, \dots, 4n-4\} ,$$

$$A_1 = \{1, 5, 9, \dots, 4n-3\} ,$$

$$A_2 = \{2, 6, 10, \dots, 4n-2\} ,$$

$$\text{and } A_3 = \{3, 7, 11, \dots, 4n-1\} .$$

The generator $2n$ of $\bar{\Delta}$ has order 2 in Z_{4n} and therefore contributes 1 to the degree of each vertex

in $G_{\bar{\Delta}}(\Gamma)$; each of the other $(n/2) - 1$ generators of $\bar{\Delta}$ has order ≥ 3 and thus contributes 2 to the degree of each vertex in $G_{\bar{\Delta}}(\Gamma)$. It is thus easy to see that each vertex in the four disjoint sets A_0, A_1, A_2 , and A_3 has degree $2[(n/2) - 1] + 1 \cdot 1 = n - 1$. Since the order of each of these sets is n and each set gives rise to a component of $G_{\bar{\Delta}}(\Gamma)$, it is clear that each set determines one copy of K_n ; and so, $G_{\bar{\Delta}}(\Gamma) = 4K_n$ (four copies of K_n). Now we observe that

$$G_{\Delta}(\Gamma) = \overline{G_{\bar{\Delta}}(\Gamma)} = \overline{4K_n} = K_{4(n)}.$$

Figure 5.3 actually represents G_n imbedded in a surface of genus $1 + s$ with one $3n/2 = 6s + 3$ sided region. To verify this, let $G' = G_n = G_{\Delta_1}(\Gamma_1)$, where $\Gamma_1 = Z_n$ and $\Delta_1 = \{1, n/2\}$, as given before. Let T be the imbedding of G' given by the current graph in Figure 5.2 (this imbedding is guaranteed to be a 2-cell imbedding). We have seen that this imbedding has exactly 3 regions (each region boundary is an n -cycle). Let v and w be any two adjacent vertices of G' and form $G = G' - [v, w]$ (note that one of v and w becomes the hollow vertex in Figure 5.3). This reduces the number of regions in T to two. Since each of v and w lies in all three region boundaries of G' (recall each region boundary is an n -cycle), each lies in the remaining two region boundaries of $G = G' - [v, w]$. Call the two distinct

regions R_v and R_w . Now apply Ringeisen's lemma to get a 2-cell imbedding of $G' = G_n$ with one region; hence there is exactly one $3n/2 = (6s+3)$ -sided region. Moreover, with $p = n = 4s + 2$, $q = 6s + 3$, and $r = 1$, the Euler formula gives $\gamma = 1 + s$. Finally we verify that KCL holds at the k th vertex of G_n ($1 \leq k \leq 4s + 2$) by exhibiting an equation (for each k) in which both sides represent the sum of the currents (directed away) at the k th vertex. Since one side of the equation will always be zero, it will be immediate that the KCL holds. We single out two unusual vertices, the $(2s + 2)$ nd and the $(4s + 2)$ nd, and divide the remaining $4s$ vertices into four major cases:

- 1) $1 \leq k \leq s + 1$
- 2) $s + 2 \leq k \leq 2s + 1$
- 3) $2s + 3 \leq k \leq 3s + 2$
- 4) $3s + 3 \leq k \leq 4s + 1$

That the KCL holds for vertex k , $k = 2s + 2$ or $4s + 2$, follows from the equations:

$$\begin{aligned} k = 2s + 2 : \quad & -[8s + 2] + 3 + [8s - 1] = 0, \text{ and} \\ k = 4s + 2 : \quad & [8s - 2] - [8s - 3] - 1 = 0, \end{aligned}$$

where the three terms of each equation represent the directed current on the k th diameter-edge, k th circle-edge, and $(k+1)$ st circle-edge, respectively.

Case 1. $1 \leq k \leq s + 1$

First, we consider k even. By using the functions C and D defined before, it is easy to see that the three edges incident with the k th vertex v are labeled and directed as follows:

- a) the k th diameter-edge--labeled $D(k) = 8s + 2 - 8(k-1)$
and directed toward v .
- b) the k th circle-edge--labeled $C(k) = 8s + 3 - 4(k-2)$
and directed away from v .
- c) the $(k+1)$ st circle-edge--labeled $C(k+1) = 1 + 4(k+1-1)$
and directed toward v .

The corresponding equation is then

$$-[8s + 2 - 8(k-1)] + [8s + 3 - 4(k-2)] - [1 + 4k] = 0.$$

Therefore the KCL holds at the k th vertex for k even and $1 \leq k \leq s + 1$. Second, we suppose that k is odd. Again by using the functions C and D , we write the label and direction of each of the three edges incident with the k th vertex v . Namely,

- a) the k th diameter-edge--labeled $D(k) = 8s+2-8(k-1)$
and directed away from v .
- b) the k th circle-edge--labeled $C(k) = 1 + 4(k-1)$
and directed away from v .
- c) the $(k+1)$ st circle-edge--labeled $c(k+1)=8s+3-4(k+1-2)$
and directed toward v .

The sum of the currents at v is then

$$[8s + 2 - 8(k-1)] + [1 + 4(k-1)] - [8s + 3 - 4(k-1)] = 0.$$

Therefore the KCL holds for case 1.

In cases 2 through 4 we only list the equations representing the sum of the currents (directed away) at the k th vertex v . In each equation the first term represents the current directed away from v of the k th diameter-edge; the second term, the k th circle-edge; and the third term, the $(k+1)$ st circle-edge. The cases are:

Case 2. $s + 2 \leq k \leq 2s + 1$

k even:

$$[6 + 8(k - (s + 2))] + [8s + 3 - 4(k-2)] - [1 + 4k] = 0$$

k odd:

$$-[6 + 8(k - (s + 2))] + [1 + 4(k-1)] - [8s + 3 - 4(k-1)] = 0$$

Case 3. $2s + 3 \leq k \leq 3s + 2$

k even:

$$-[8s+2-8(k-(2s+2))] - [5+4(k-(2s+4))] + [(8s-1)-4(k+1-(2s+3))] = 0$$

k odd:

$$[8s+2-8(k-(2s+3))] - [8s-1-4(k-(2s+3))] + [5+4(k+1-(2s+4))] = 0$$

Case 4. $3s + 3 \leq k \leq 4s + 1$

k even:

$$[6+8(k-(3s+3))]-[5+4(k-(2s+4))]+[8s-1-4(k+1-(2s+3))] = 0$$

k odd:

$$-[6+8(k-(3s+3))]-[8s-1-4(k-(2s+3))]+[5+4(k+1-(2s+4))] = 0.$$

Hence the KCL holds at each vertex of G_n . Thus since G_n is regular of degree 3 every region in the derived graph $G_\Delta(\Gamma) = K_{4(n)}$ is triangular.

Next we apply Theorem 4.5 to prove that the imbeddings for $K_{4(n)}$, $n \equiv 2 \pmod{4}$, given above have bichromatic dual and then apply Theorem 4.4 to obtain triangular imbeddings having bichromatic dual for $K_{4(n)}$, $n \equiv 0 \pmod{4}$, and thus we will have obtained triangular imbeddings of $K_{4(n)}$ for all even values of n .

The current graph G_n given in Figure 5.3 is bichromatic. (Simply start at any given vertex and then proceed cyclically, alternately coloring the vertices black and white. Diametrically opposite vertices are adjacent but since $n \equiv 2 \pmod{4}$ such vertices will be assigned different colors.) Temporarily, we translate from current theory to voltage theory by dualizing the imbedded current graph G_n to obtain the imbedded dual G_n^* . It is now obvious that this imbedding of G_n^* has bichromatic dual, and it is clear that Theorem 4.5 can

be invoked to prove that the derived imbedding of the derived graph $K_{4(n)}$ has bichromatic dual for $n \equiv 2 \pmod{4}$. In the other even cases, namely $n \equiv 0 \pmod{4}$, we write $n = 2^s \cdot 2m$ uniquely, where m is odd and $s \geq 1$. Since $2m \equiv 2 \pmod{4}$, we apply Theorem 4.4 s times to our imbedding of $G = K_{4(2m)}$ and thus produce a triangular imbedding with bichromatic dual for $K_{4(n)}$. All together, then, we have found a triangular imbedding of $K_{4(n)}$ with bichromatic dual for each positive even integer n . For each of these imbeddings, as we have seen, we get a $(4n, 4n^2, 3n, 3, 0, 2)$ -PBIBD. Theorem 4.1 shows these designs are different from those found in [13], and direct comparison with those found in [17] shows that the above designs are new.

We determine $(4n, 4n^2, 3n, 3, 0, 2)$ -PBIB designs for n odd by triangularly imbedding $K_{4(n)}$ again but this time by appealing to voltage graph theory. We consider the two cases $n \equiv 1 \pmod{4}$ and $n \equiv 3 \pmod{4}$. In each case we exhibit a voltage pseudograph satisfying requisite properties sufficient to insure that the derived imbedding of $K_{4(n)}$ is triangular.

The structure given in Figure 5.5 is a voltage pseudograph for the case $n \equiv 1 \pmod{4}$. (See Figure 5.4 for the special case $n = 5$.) We now precisely describe this structure and voltage assignment. Let $n = 4s + 1$ with

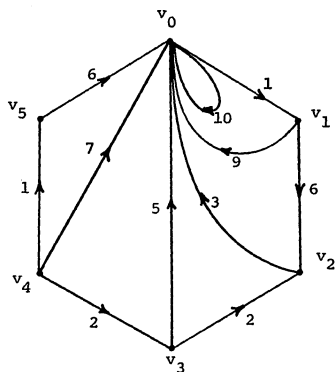
$s = 1, 2, \dots$ and $\Gamma = \mathbb{Z}_{4n}$ and begin with a $(4s + 2)$ -gon imbedded in the plane. Distinguish a vertex v_0 on the polygon and label the remaining vertices as $v_1, v_2, \dots, v_{4s+1}$ cyclically with a clockwise rotation. Construct exactly one chordal edge between v_0 and v_i for each $i = 2, 3, 4, \dots, 4s$. Finally, add a second edge between v_0 and v_1 that is contained within triangle $v_0 v_1 v_2$ and then a loop at v_0 inside the region bounded by the two edges between v_0 and v_1 . This produces a monogon and $4s + 1$ triangular regions. The voltages around the perimeter of the polygon are assigned as follows: Assign 1 to the edge between v_0 and v_1 . Next assign $8s - 2$ ($\equiv 2 \pmod{4}$) to the edge between v_1 and v_2 ; then proceed cyclically, in a clockwise sense, by assigning a voltage that is 4 less than that of the immediately preceding assigned voltage until voltage 2 is reached. At this point we are half-way around the polygon. Continue by labeling the next edge 2 and each successive edge with a voltage 4 more than that assigned to the edge immediately before it until $8s - 6$ is reached. The last two edges are assigned 1 and $8s - 2$, respectively. Observe that each of the voltages assigned to perimeter edges appears exactly twice. The chords of the polygon have the following voltage assignment: $(8s+1) - 4(\frac{k-1}{2})$ (each $\equiv 1 \pmod{4}$) to the edge between v_0 and v_k for $k = 1, 3, 5, \dots, 4s - 1$ and $3 + 4(\frac{k-2}{2})$ (each $\equiv 3 \pmod{4}$)

to the edge between v_0 and v_k for $k = 2, 4, \dots, 4s$. Assign $8s + 2$ to the loop at v_0 . Lastly we designate a direction to each of the edges. The perimeter edges assigned with voltage 1 or a voltage congruent to 6 modulo 8 are directed clockwise whereas those with voltage congruent to 2 modulo 8 are directed counterclockwise. All chordal edges are directed toward v_0 and the loop at v_0 is directed clockwise.

The polygonal structure in Figure 5.5 determines a unique surface (by identifying, in the standard topological manner from surface topology, the pairs of edges on the perimeter having the same voltage assigned). It is not difficult to show that there are exactly two vertices (v_1 and v_{4s+1} represent one vertex whereas the remaining vertices of the polygon represent the other vertex (call these vertices x and y , respectively)) in the imbedded pseudograph formed from this process. It is relatively simple to verify that after identifying x and y we have a bouquet of $6s + 2$ loops based at a single vertex imbedded in a nonorientable pseudosurface of characteristic $1 - 2s$ having exactly one singular point of degree 2. The derived graph is the Cayley graph $G_{\Delta}(\Gamma) = K_{4(n)}$; the covering space is seen to be a pseudosurface: since $\Gamma_y = \Gamma$ (Observe that the two loops $8s-1$ and 5 based at y may be combined to show

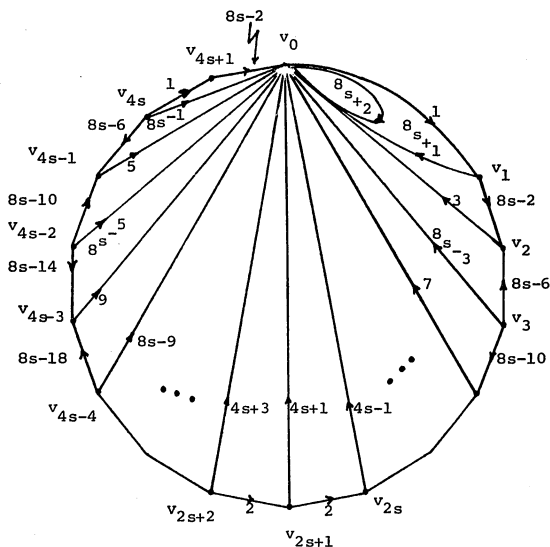
that $16s + 3$, which is relatively prime to $|\Gamma|$, is in Γ_y). Theorem 2.3 implies there is one component in the covering space before x and y are identified. Furthermore, Stahl's orientation theorem, Theorem 2.4, shows that the covering space is nonorientable: the rotations around the vertices x and y may be chosen so that the perimeter voltage 2 has $\lambda = 1$ assigned to it, and the loop voltage $8s + 2$, $\lambda = 0$. Then traversing $(4s+1)$ loops of 2 followed by 1 loop of $8s + 2$ gives a β -trivial walk at y , which is not λ -trivial. Finally, appealing to congruency classes modulo 8 , we see that the KVL holds on each triangular region, which thus lifts to triangular regions in the derived graph. The only other region is a monogon, which lifts to $(8s+2)$ digons. After identifying the opposite edges in each of these digons, we are left with a triangular imbedding of the derived graph $K_{4(n)}$ for $n \equiv 1 \pmod{4}$. That the block designs determined by these imbeddings are new follows from Theorem 4.1, since previously known designs on the parameters are imbedded in surfaces or strictly generalized pseudosurfaces (i.e., not a pseudosurface). (See [13] and [17].)

Triangular imbeddings of $K_{4(n)}$ for the case $n \equiv 3 \pmod{4}$ can be obtained from the voltage pseudograph shown in Figure 5.7. (See Figure 5.6 for the case $n = 7$.) We note that this structure is quite similar to the one



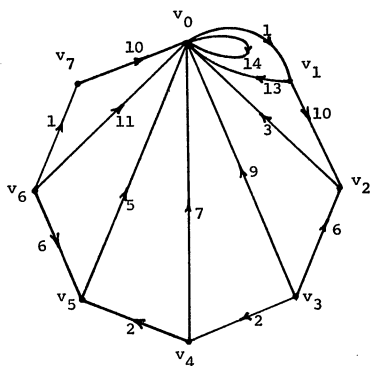
Case $n = 5$ A Voltage Pseudograph for $K_4(5)$

Figure 5.4



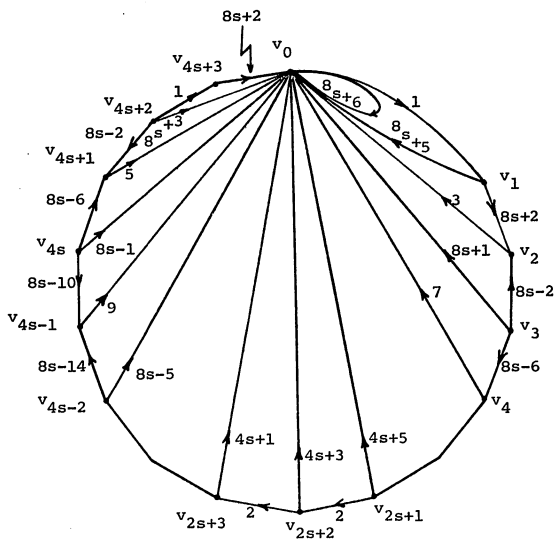
Case $n \equiv 1 \pmod{4}$ A Voltage Pseudograph for $K_4(n)$

Figure 5.5



Case $n = 7$ A Voltage Pseudograph for $K_4(7)$

Figure 5.6



Case $n \equiv 3 \pmod{4}$ A Voltage Pseudograph for $K_4(n)$

Figure 5.7

used in the case $n \equiv 1 \pmod{4}$ so we give a minimum of details since the verification of such details is analogous. This time we begin with a $(4s + 4)$ -gon imbedded in the plane, adding chordal edges and the loop as before. There are $n = 4s + 4$ vertices and the voltage assignment is as given in Figure 5.7 (See Figure 5.6 for special case $n = 7$). Once again there are two distinct vertices x and y ; v_1 and v_{4s+3} represent x and the remaining polygonal vertices represent y . Loops with voltages $8s + 3$ and 5 can be combined to show that $(16s + 11) \in \Gamma_y$; hence, $\Gamma = \Gamma_y$. As in the case $n \equiv 1 \pmod{4}$ rotations can be assigned giving $\lambda = 1$ on perimeter voltage 2 and $\lambda = 0$ on loop voltage $8s + 3$. Traversing $(4s+3)$ loops of voltage 2 followed by 1 loop of voltage $8s + 3$ gives a β -trivial walk at y that is not λ -trivial. Thus by Theorem 2.4 the covering space is nonorientable. The KVL holds on each triangular region and the monogon lifts to $8s + 6$ digons. Performing the same operation on the digons as we did for the case $n \equiv 1 \pmod{4}$, we again obtain a triangular imbedding of $K_{4(n)}$. Finally, since the covering space is a nonorientable pseudosurface, we know the associated block designs are new (by Theorem 4.1). We note in passing that in these two cases ($n \equiv 1$ or $n \equiv 3 \pmod{4}$) neither dual is bichromatic (since no dual can be bichromatic if even one vertex of the original graph has odd degree), so that we are unable to extend

these imbeddings to other designs by applying Theorem 4.4.

The foregoing proves the following theorem.

THEOREM 5.1 New $(4n, 4n^2, 3n, 3, 0, 2)$ -PBIB designs exist (and can be constructed) for each integer $n \geq 2$.

Designs based on imbeddings with bichromatic dual have a particular fruitful feature--they split naturally into two "half sized" designs, that is, the split designs have half as many blocks. The blocks of one of these designs consist of all the regions of one color class; the blocks of the second, all the regions of the other color class. We observe that if the original design is a $(v, 2b, 2r, 3, 0, 2)$ -PBIBD, then each split design is a $(v, b, r, 3, 0, 1)$ -PBIBD. We illustrate the split designs in Table 5.1 below by using the imbedding of $K_4(2)$ employed in the proof of Theorem 5.1. The design based on $K_4(2)$ is an $(8, 16, 6, 3, 0, 2)$ -PBIBD and so each of the split designs is an $(8, 8, 3, 3, 0, 1)$ -PBIBD.

White colored regions

1 3 0
2 4 1
3 5 2
4 6 3
5 7 4
6 0 5
7 1 6
0 2 7

Black colored regions

1 4 3
2 5 4
3 6 5
4 7 6
5 0 7
6 1 0
7 2 1
0 3 2

Table 5.1

The imbeddings of $K_{4(n)}$ for n even used in Theorem 5.1 have bichromatic dual; we give in Theorem 5.2 the split designs that are, consequently, produced.

THEOREM 5.2. For each even value of n , two $(4n, 2n^2, 3n/2, 3, 0, 1)$ -PBIB designs exist (and can be constructed).

LEMMA 5.2. For each $n \equiv 6 \pmod{12}$ there exists a triangular orientable surface imbedding of $K_{4(n)}$ for which the corresponding PBIB design is n -partially resolvable.

PROOF. Referring to the current graph for $K_{4(n)}$ in Figure 5.3, we distinguish $4t + 2$ vertices, where $n = 12 + 6$ for $t = 0, 1, 2, \dots$. We will see that each such vertex will generate 3 distinct sets of $16t + 8$ regions (blocks) each in the derived imbedding (which we have seen before to be a triangular orientable surface imbedding) with the union of the blocks in each such set containing precisely the $4n$ vertices of $K_{4(n)}$. (See Figure 5.8 for the case $n = 18$.) All together there are $3(4t + 2)$ or n sets of the required type (pairwise disjoint). For notational convenience let the triple (a, b, c) represent the outflowing currents at any given fixed vertex of the current graph G_n in Figure 5.3, where the first and third coordinates represent circle-edge currents and the second coordinate represents the

diameter-edge current. The vinculum designates the negative (in Z_{4n}) of the current listed. The $4t + 2$ distinguished vertices can then be conveniently represented as below where $s = 3t + 1$:

$$a_u: (12u + 1, 8s + 2 - 24u, \overline{8s + 3 - 12u}) \text{ for } 0 \leq u \leq t$$

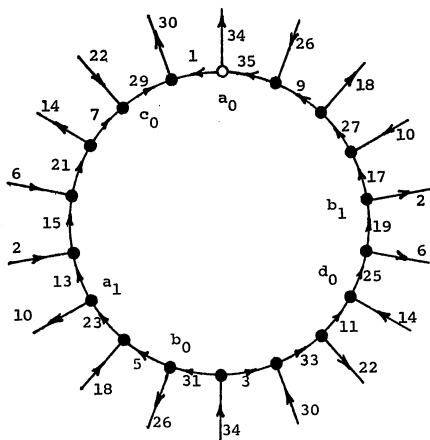
$$b_u: (12u + 5, 8s - 6 - 24u, \overline{8s - 1 - 12u}) \text{ for } 0 \leq u \leq t$$

$$c_u: (\overline{12u + 7}, \overline{8s - 10 - 24u}, 8s - 3 - 12u) \text{ for } 0 \leq u \leq t-1$$

$$d_u: (\overline{12u + 11}, \overline{8s - 18 - 24u}, 8s - 7 - 12u) \text{ for } 0 \leq u \leq t-1.$$

We observe that each such vertex gives rise to an initial triangular region of the form $0--a--a+b$ (for convenience in terminology we will call the first symbol in this triple an initial vertex) where $\{0, a, a+b\}$ constitutes a complete set of residues modulo 3 and that this initial region in turn generates $4n - 1$ other regions, namely, $1--(a+1)--(a+b+1)$, $2--(a+2)--(a+b+2)$, ..., $(4n-1)--(a+4n-1)--(a+b+4n-1)$. Appealing to congruency classes modulo 3, we see that by choosing those regions whose initial vertices are congruent to either 0, 1, or 2, respectively gives 3 sets of regions with each such set consisting of precisely all $4n$ vertices of $K_{4(n)}$. This produces the required n sets of blocks and so the corresponding designs are n -partially resolvable.

THEOREM 5.3. New $(5n, 20n^2/3, 4n, 3, 0, 2)$ -PBIB designs exist (and can be constructed) for $n \equiv 6 \pmod{12}$.



Regions Generated
by Vertex a_0

1	35	0
2	36	1
3	37	2
4	38	3
5	39	4
.	.	.
71	33	70
0	34	71

Corresponding 3 Sets of Regions

1	35	0	2	36	1	3	37	2
4	38	3	5	39	4	6	40	5
7	41	6	8	42	7	9	43	8
.
70	32	69	71	33	70	0	34	71

Case $n = 18$ in Lemma 5.2

Figure 5.8

PROOF. By lemma 5.2 and its proof we know that orientable triangular surface imbeddings of $K_{4(n)}$ for $n \equiv 6 \pmod{12}$ can be constructed whose corresponding PBIB designs are n -partially resolvable. Each of the n sets determined by the n -partial resolvability consists of $4n/3$ regions (blocks) that partition the vertex set of $K_{4(n)}$. For each such set, we place a vertex v in the interior of each of the $4n/3$ triangular regions and then connect v with an edge to each of the three distinct vertices in the boundary enclosing v . Finally, for each given set, we identify the newly placed $4n/3$ vertices as one. This process creates n new vertices of which each is adjacent to the original $4n$ vertices of $K_{4(n)}$ but none is adjacent to any other new vertex. We further observe that a total of $4n^2/3$ triangular regions were destroyed and $4n^2$ created, for a net gain of $8n^2/3$ triangular regions or a total of $20n^2/3$ such regions. This, then, gives a triangular pseudosurface imbedding of $K_{5(n)}$ as well as new $(5n, 20n^2/3, 4n, 3, 0, 2)$ -PBIB designs for $n \equiv 6 \pmod{12}$. Known designs (See [13].) on these parameters are in strictly generalized pseudosurfaces.

By applying Theorem 4.4 repeatedly to the voltage pseudograph in Figure 4.2 triangular imbeddings for the 7-partite graphs $K_{7(5 \cdot 2^k)}$ ($k = 0, 1, 2, \dots$) are obtained. Theorem 5.4 gives the new PBIB designs thus determined.

THEOREM 5.4. New $(7n, 14n^2, 6n, 3, 0, 2)$ -PBIB designs exist (and can be constructed) where $n = 5 \cdot 2^k$ and $k = 0, 1, \dots$.

The imbeddings used in Theorem 5.4 have bichromatic dual and thus determine a pair of split designs for each value of n . We summarize this in the next theorem.

THEOREM 5.5. Two $(7n, 7n^2, 3n, 3, 0, 1)$ -PBIB designs exist (and can be constructed) where $n = 5 \cdot 2^k$ and $k = 0, 1, \dots$.

FROM GENUS TO GENERALIZED PSEUDOCHARACTERISTICS

Historically the genus parameter for graphs is difficult to determine. Usually only ad hoc arguments were used in finding $\gamma(G)$, depending upon the graphs G considered. Current graphs were employed by Ringel and Youngs in their solution to the Heawood problem, where they found the genus of the complete graph. Their solution and techniques not only revived interest in the genus parameter, but also lead researchers to develop more sophisticated imbedding techniques. In the present, the voltage graph theory, introduced by Gross, is becoming widely known. Gross (together with Youngs) has also unified and reorganized current graph theory. Theories of voltage and current graphs for orientable surfaces are equivalent since they are dual to each other, but in particular applications, one method may be more convenient to apply than the other.

The nonorientable genus parameter has often been shunned, perhaps since nonorientable surfaces cannot be realized in real 3-space, making intuition for imbeddings difficult. The extension of voltage graph theory to nonorientable surfaces by Stahl [30] and the

of "cascade" current graphs by Youngs [38] and Jungerman [18] have given not only a firm foundation for nonorientable imbeddings but a new dignity to them.

Like nonorientable imbeddings, pseudosurface and generalized pseudosurface imbeddings have been avoided in the past. Creating block designs via these types of imbeddings has gained acceptance among topological graph theorists. In this connection, we address ourselves in this chapter not only to the genus parameters but also to the pseudocharacteristic and generalized pseudocharacteristic parameters.

The class of graphs $K_{4(n)}$ has been extremely fruitful. Ringel conjectured in 1969 that the genus of $K_{4(n)}$ is given by $(n-1)^2$ or equivalently the orientable characteristic is $2n(2-n)$. Gross and Alpert verified this for the cases $n \equiv 1$ or $5 \pmod{6}$ [11] and Garman [6] showed this for $n \equiv 2 \pmod{4}$. (See also Chapter 5 of this thesis.) Recently Jungerman [17] showed that Ringel's conjecture is true for every $n \geq 1$ except $n = 3$. The exceptional case $K_{4(3)}$ should triangulate the quadruple torus (a sphere with 4 handles) or at least the Euler formula allows for this, but Jungerman has shown, by an exhaustive computer search, that it does not. The actual genus is 5 or equivalently the orientable characteristic is -8. The voltage pseudograph in Figure 2.3 (Chapter 2) shows that the generalized pseudocharacteristic is -6.

Petroelje [23] shows that the pseudocharacteristic of $K_{4(n)}$ is $2n(2 - n)$ for all n . In particular for $n = 3$, $\chi'(K_{4(3)}) = -6$.

In the nonorientable case Jungerman [19] has also recently shown that $\bar{\gamma}(K_{4(n)}) = 2(n - 1)^2$ for all $n \geq 3$. The voltage graphs in Figures 5.5 and 5.7 were constructed independently from Jungerman's work; we note that interchanging the voltages on the edges $v_{4s}v_{4s+1}$ and $v_{4s+1}v_0$ in Figure 5.5 and on edges $v_{4s+2}v_{4s+3}$ and $v_{4s+3}v_0$ in Figure 5.7 produces nonorientable surface imbeddings. (They were nonorientable pseudosurface imbeddings.) These yield nonorientable triangular surface covers that give $\bar{\gamma}(K_{4(n)})$ for n odd, in agreement with Jungerman's result. We note in passing that by appropriately adjusting the triangular regions in Figures 5.5 and 5.7, we can change the covering surfaces to orientable generalized pseudosurfaces and thus produce generalized pseudosurface imbeddings yielding $\chi''(K_{4(n)})$ for n odd. Rather than pursuing this we give two theorems that follow immediately from Theorem 5.1 and its proof and essentially are reinterpretations of those results. A third theorem follows in a similar manner from Theorem 5.3 and its proof.

THEOREM 6.1. $\gamma(K_{4(n)}) = (n - 1)^2$ for n even.

THEOREM 6.2. $\tilde{\chi}'(K_{4(n)}) = 2n(2 - n)$ for n odd.

THEOREM 6.3. $\chi'(K_{5(n)}) = 5n(3-2n)/3$ for $n \equiv 6 \pmod{12}$.

The nonorientable genus of $K_{7(n)}$ for $n = 5 \cdot 2^k$ ($k = 0, 1, \dots$) follows immediately from Example 4.2 and Theorem 5.4. We state these results in Theorem 6.4.

THEOREM 6.4. $\bar{\gamma}(K_{7(n)}) = 2 + 7n(n-1)$ where $n = 5 \cdot 2^k$ and $k = 0, 1, \dots$

We conclude the chapter by summarizing the known genus formulas for the complete n -partite graphs K_{p_1, \dots, p_n} . We note that $K_{n(m)}$ is a subclass of this family and, in particular, the complete graphs belong to this class since $K_n = K_{n(1)}$. Braces are used to denote the least integer function.

$$1. [28] \quad \gamma(K_n) = \left\{ \frac{(n-3)(n-4)}{12} \right\}, \quad n \geq 3.$$

$$2. [25] \quad \bar{\gamma}(K_n) = \left\{ \frac{(n-3)(n-4)}{6} \right\}, \quad \begin{matrix} n \geq 3. \\ n \neq 7 \end{matrix}$$

$$3. [27] \quad \gamma(K_{n,m}) = \left\{ \frac{(m-2)(n-2)}{4} \right\}, \quad m, n \geq 2.$$

$$4. [32] \quad \gamma(K_{mn,n,n}) = \frac{(mn-1)(n-1)}{2}, \quad m, n \geq 1.$$

$$5. [29] \quad \gamma(K_{n,n,n}) = \frac{(n-1)(n-2)}{2}, \quad n \geq 1.$$

$$6. [32] \quad \gamma(K_{p,q,r}) = \left\{ \frac{(p-2)(q+r-2)}{4} \right\}, \quad p \geq q \geq r$$

and $q + r \leq 6$.

$$7. [31] \quad \gamma(K_{n,n,n-2}) = \frac{(n-2)^2}{2}, \quad n \text{ even and } n \geq 2.$$

$$8. [31] \quad \bar{\gamma}(K_{n,n,n-2}) = (n-2)^2, \quad n \geq 3.$$

$$9. [31] \quad \gamma(K_{2n,2n,n}) = \frac{(3n-2)(n-1)}{2}, \quad n \geq 1.$$

$$10. [31] \quad \gamma(K_{n,n,n-4}) = (n-2)(n-3), \quad n \text{ even and } n \geq 4.$$

$$11. [18], [6], (\text{and Theorem 6.1})$$

$$\gamma(K_{n,n,n,n}) = \begin{cases} (n-1)^2, & n \neq 3 \\ 5, & n = 3. \end{cases}$$

$$12. [19] \quad \bar{\gamma}(K_{n,n,n,n}) = 2(n-1)^2, \quad n \geq 3.$$

$$13. [21], [9] \quad \gamma(K_{n(2)}) = \left\{ \frac{(n-3)(n-1)}{3} \right\}, \quad n \neq 11.$$

$$14. [20] \quad \bar{\gamma}(K_{n(2)}) = 2 \left\{ \frac{(n-3)(n-1)}{3} \right\}, \quad n \geq 5.$$

$$15. [7] \quad \gamma(K_7(2^k)) = 1 + 7 \cdot 2^{k-1} (2^k - 1), \quad k \geq 0.$$

$$16. [7] \quad \gamma(K_{n(m)}) = \frac{(mn-3)(mn-4)}{12} - \frac{mn(m-1)}{12}$$

for $n \equiv 3 \pmod{12}$ and $m = 2^k, k \geq 0$.

$$17. (\text{Theorem 6.4})$$

$$\bar{\gamma}(K_7(5 \cdot 2^k)) = 2 + 35 \cdot 2^k (5 \cdot 2^k - 1) \quad k \geq 0.$$

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