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A Theory of Multiplicity for Multiplicative Filtrations

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A THEORY OF MULTIPLICITY
FOR
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I INTRODUCTION

The Problem

In the classical theory of commutative rings, one important result states that for a 0-dimensional ideal $\mathfrak{a}$ in a noetherian ring A with identity, the Hilbert function $H(n,\mathfrak{a}) = L_A(A/\mathfrak{a}^n)$, that is the A-module length from A to $\mathfrak{a}^n$, is a polynomial with rational coefficients for sufficiently large n. The degree of this polynomial is the altitude s of $\mathfrak{a}$. The multiplicity $\mu(\mathfrak{a})$ of the ideal $\mathfrak{a}$ is $s!$ times the leading coefficient of this polynomial and is frequently expressed by the limit formula of Samuel

$$(1.1) \qquad \mu(\mathfrak{a}) = s! \lim_{n\to\infty} \frac{H(n,\mathfrak{a})}{n^s} .$$

It is always true that $\mu(\mathfrak{a})$ is an integer which, in the geometric situation, can be interpreted as the "multiplicity" or "count" of the intersections of varieties.

The study of sequences of powers of a fixed ideal has been generalized to the study of filtrations where a <u>multiplicative</u> <u>filtration</u> (or just <u>filtration</u>) on a ring A is a sequence $f = \{\mathfrak{a}_n\}$ of ideals of A which satisfies the conditions

$$(1.2) \qquad \begin{aligned} &\text{(i)} \quad \mathfrak{a}_0 = A , \\ &\text{(ii)} \quad \mathfrak{a}_{n+1} \subseteq \mathfrak{a}_n \quad \text{for all} \quad n \geq 0 , \text{ and} \\ &\text{(iii)} \quad \mathfrak{a}_m\mathfrak{a}_n \subseteq \mathfrak{a}_{m+n} \quad \text{for all} \quad m,n \geq 0 . \end{aligned}$$

This is indeed a generalization of the powers of an ideal for, given any ideal $\mathfrak{a}$ of A , the sequence

$$(1.3) \qquad f_{\mathfrak{a}} = \{\mathfrak{a}^n\}$$

forms a filtration on A using the convention that $\mathfrak{a}^0 = A$.

In this dissertation, the concept of multiplicity is extended by the formula analogous to (1.1) to all filtrations for which the limit exists; that is,

$$(1.4) \qquad \mu(f) = s! \lim_{n\to\infty} \frac{L_A(A/\mathfrak{a}_n)}{n^s}$$

provided this limit exists, where $s = \operatorname{alt}(f)$. See (1.13). In general $L_A(A/\mathfrak{a}_n)$ cannot be described by a polynomial as before and multiplicity need not be an integer or even a rational number. This means that the geometric interpretation of multiplicity as counting intersections does not apply even in the case of geometric rings. Some connection does exist, however, and in Chapter 3, a minor strengthening of the theory is developed in order to demonstrate the connection with another generalization of multiplicity. For any filtration of the type considered there, a constant factor of proportionality is obtained which recovers the integer corresponding to the geometric multiplicity. Despite this fact, the general theory developed lies more appropriately in the realm of the arithmetical theory of filtrations with no particular distinction being made as to whether or not the multiplicity is related to some integer in a natural manner. This thesis is to be regarded, then, as one further advance in the broader study of all multiplicative filtrations on a commutative ring, restricted of course to those filtrations for which multiplicity is defined.

Within the text, all statements about multiplicity will be made in terms of submodules of a module instead of ideals in a ring by defining the multiplicity $\mu(f,M)$ of a finitely generated A-module M with respect to a filtration $f = \{a_n\}$ on a noetherian, commutative ring A with identity as

$$(1.5)\qquad \mu(f,M) = s!\lim_{n\to\infty}\frac{L_A(M/a_nM)}{n^s}\ ,\quad s = \mathrm{alt}(f)\ ,$$

whenever this limit exists. If f is <u>approximatable by powers</u> in the sense that for each $j \in \mathbb{N}$, there exists $k_j \in \mathbb{N}$ such that

$$(1.6)\qquad \begin{aligned}&\text{(i)}\quad a_{k_j n} \subseteq a_j^{\,n} \text{ for all } n\ , \text{ and}\\ &\text{(ii)}\quad \frac{k_j}{j} \longrightarrow 1 \text{ as } j \longrightarrow \infty\ .\end{aligned}$$

and if f is 0-dimensional, $\mu(f,M)$ exists for every finitely generated A-module M. The corresponding multiplicity function $\mu(f,\_)$ defined on the category of all finitely generated A-modules has properties very similar to those of the multiplicity function $\mu(a,\_)$ of classical ideal theory; e.g., it is additive and obeys the usual localization and extension formulae. Any filtration of the type $f = f_\alpha$ for some α is trivially approximatable by powers. The class of all filtrations which are approximatable by powers is much wider, however, and even includes many filtrations which arise from nondiscrete rank 1 valuations.

To demonstrate how the theory of Chapter 2 is to be applied, two classes of filtrations on familiar rings are discussed in detail. In fact, all of Chapter 4 is devoted to the description of an unusual but very interesting class of filtrations on $k[X,Y]$, the examination of which led to the key idea of "approximatable by powers".

For Chapters 1, 2, and 4 only the general knowledge of commutative ring theory found in [4] or [5] is assumed. The third chapter quotes without proof and with very little explanation some of the more sophisticated results of [7] and [9].

Preliminary Notions

Throughout this work, whether mentioned explicitly or not, A will denote a commutative ring with identity and any ring homomorphism will be assumed to preserve the identity. A number of common terms of commutative ring theory need to be adapted to simplify statements about filtrations. First recall that the rank of a prime ideal $\mathfrak{p}$ in a ring A is the supremum (possibly $+\infty$) of all n for which there exists a chain of prime ideals

(1.7) $\mathfrak{p}_o \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{n-1} \subsetneq \mathfrak{p}_n = \mathfrak{p}$.

The altitude of an ideal $\mathfrak{a} \neq A$ is defined by

(1.8) $\mathrm{alt}(\mathfrak{a}) = \sup\{\mathrm{rank}(\mathfrak{p}) \mid \mathfrak{p}$ is a minimal prime of $\mathfrak{a}\}$.

The dimension of the ring A is

(1.9) $\mathrm{Dim}(A) = \sup\{\mathrm{rank}(\mathfrak{p}) \mid \mathfrak{p}$ is a maximal ideal of $A\}$.

The dimension of the ideal $\mathfrak{a}$ of A is then

(1.10) $$\dim(\mathfrak{a}) = \mathrm{Dim}(A/\mathfrak{a}) .$$

The radical of an ideal $\mathfrak{a} \neq A$ is given by

(1.11) $\mathrm{rad}(\mathfrak{a}) = \bigcap \{\mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal$\}$.

It is well known that $\mathrm{rad}(\mathfrak{a}) = \{\alpha \in A \mid \alpha^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$.

Let $f = \{\mathfrak{a}_n\}$ be a filtration on a ring A . If $\mathfrak{a}_n = A$ for some $n > 0$, the definition implies that $\mathfrak{a}_n = A$ for all $n \geq 0$; that is, $f = f_A$ in the notation of (1.3). This trivial filtration will ordinarily be excluded from consideration without comment. Conditions (ii) and (iii) of the definition of filtration (1.2) and properties of prime ideals then imply that all the ideals of f except $\mathfrak{a}_o$ have the same radical. Define the radical of f by

(1.12) $\mathrm{rad}(f) = \mathrm{rad}(\mathfrak{a}_n)$ for any $n > 0$.

Since rad(f) is an ideal of A , the altitude and dimension of the filtration f can be defined by

(1.13) $\mathrm{alt}(f) = \mathrm{alt}(\mathrm{rad}(f))$ and

$$\dim(f) = \dim(\mathrm{rad}(f)) .$$

It has been observed [6] that the set of all filtrations F on a ring A, including the trivial ones $f_{(0)}$ and f_A, forms a lattice under the following partial order, meet, and join. For any two elements $f = \{\mathfrak{a}_n\}$ and $g = \{\mathfrak{b}_n\}$ in F,

$$
\begin{aligned}
& f \le g \text{ if } \mathfrak{a}_n \subseteq \mathfrak{b}_n \text{ for all } n \ge 0 , \\
(1.14)\quad & f \wedge g = f \cap g = \{\mathfrak{a}_n \cap \mathfrak{b}_n\} \text{ for all } n \ge 0 \text{, and} \\
& f \vee g = f + g = \{\mathfrak{c}_n\} \text{, where } \mathfrak{c}_n \text{ is the ideal}
\end{aligned}
$$

$$\mathfrak{c}_n = \sum_{i+j=n} \mathfrak{a}_i \mathfrak{b}_j .$$

The set F is also endowed with a join distributive multiplication

$$(1.15) \qquad fg = \{\mathfrak{a}_n \mathfrak{b}_n\} \text{ for all } n \ge 0 .$$

The operations are compatible with the order, associative, and commutative. The two trivial filtrations $f_{(0)}$ and f_A provide, respectively, additive and multiplicative identities.

<u>Note 1.1</u>: It is easily shown that for two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A, $f_{\mathfrak{a}} + f_{\mathfrak{b}} = f_{\mathfrak{a}+\mathfrak{b}}$ and $f_{\mathfrak{a}} f_{\mathfrak{b}} = f_{\mathfrak{a}\mathfrak{b}}$, but for the intersection of $f_{\mathfrak{a}}$ and $f_{\mathfrak{b}}$, in general only $f_{\mathfrak{a}\cap\mathfrak{b}} \le f_{\mathfrak{a}} \cap f_{\mathfrak{b}}$ is true. The question of just "how close" $f_{\mathfrak{a}\cap\mathfrak{b}}$ is to $f_{\mathfrak{a}} \cap f_{\mathfrak{b}}$ is still unanswered.

<u>Note 1.2</u>: The set of all filtrations on a ring with the

same radical form a sublattice of F with the property that for any two elements $f \le g$ of the sublattice, the interval in F determined by f and g is contained in the sublattice.

For any filtration $f = \{\mathfrak{a}_n\}$ and positive integer k, the subsequence determined by the multiples of k forms a filtration $f^{(k)}$; that is,

$$(1.16) \qquad f^{(k)} = \{\mathfrak{a}_{kn}\}_{n=0}^{\infty} .$$

By use of (1.15), f^k is defined and can be expressed $f^k = \{\mathfrak{a}_n{}^k\}$. From (iii) of (1.2), it is immediate that $f^k \le f^{(k)}$.

Let $f = \{\mathfrak{a}_n\}$ be a filtration on a ring A , $g = \{\mathfrak{b}_n\}$ a filtration on a ring B , and $\varphi: A \longrightarrow B$ a ring homomorphism so that by definition, B is an extension of A . The <u>extension</u> of f to B is defined by

$$(1.17) \qquad f^e = \{\mathfrak{a}_n{}^e\} = \{\varphi[\mathfrak{a}_n]B\}$$

and the <u>contraction</u> of g to A by

$$(1.18) \qquad g^c = \{\mathfrak{b}_n{}^c\} = \{\varphi^{-1}[\mathfrak{b}_n]\} .$$

It is easily checked that f^e and g^c are filtrations on B and A respectively and that $f \le f^{ec}$ and $g^{ce} \le g$.

Filtrations which arise as the powers of a fixed ideal will provide a key tool in the subsequent development and the relationship between these and the collection of all filtrations is more than simple containment. The following result demonstrates this fact and provides suspicion that some properties of filtrations of this special type are in fact properties of more general filtrations. In particular, it shows that any statement about a filtration f which is true for every filtration of the type $f_{\mathfrak{a}}$ for some $\mathfrak{a}$ and which is preserved under all contractions, is true for f itself.

Proposition 1.3: For any filtration f on a ring A, there exists a ring extension B and an ideal $\mathfrak{b}$ of B such that f is the contraction of the filtration on B given by the powers of $\mathfrak{b}$; that is, $f_{\mathfrak{b}}{}^{c} = f$.

Proof: For each $\mathfrak{a}_n$ of $f = \{\mathfrak{a}_n\}$, choose a set of generators $\{\alpha_{n,i}\}_{i \in I_n}$. For an indeterminate t over A and the multiplicatively closed set $S = \{t, t^2, \ldots\}$, let $A[t]_S$ be the corresponding ring of quotients. Take B as the subring of $A[t]_S$ generated by t and all quotients of the form $\alpha_{n,i}/t^n$ for each n. That is,

(1.19) $B = A[t, \alpha_{n,i}/t^n$ for $i \in I_n$ and $n = 1,2,\ldots]$.

Then for the ideal $\mathfrak{b} = (t)$ in B,

(1.20) $$\mathfrak{a}_n = \mathfrak{b}^n \cap A \quad \text{for } n \geq 0 .$$

The containment $\mathfrak{a}_n \subseteq \mathfrak{b}^n \cap A$ is immediate since $t^n(\alpha_{n,i}/t^n) = \alpha_{n,i}$ implies that every generator of $\mathfrak{a}_n$ is in $\mathfrak{b}^n \cap A$. To see the reverse inclusion, choose any $\alpha \in \mathfrak{b}^n \cap A$. Then $\alpha = t^n\beta$ for some $\beta \in B$. Since β is in B it is easy to show it has a representation as $\beta = g(t) + \gamma_1/t + \gamma_2/t^2 + \gamma_3/t^3 + \ldots$ where $g(t) \in A[t]$, $\gamma_i \in \mathfrak{a}_i$ for all $i \geq 1$, and all but finitely many $\gamma_i = 0$. Then $t^n\beta \in A$ implies $g(t) = 0$ and $\gamma_i = 0$ for $i \neq n$. Now $\alpha = t^n\beta = \gamma_n \in \mathfrak{a}_n$. Q.E.D.

Remark 1.4: Another, slightly more complicated proof of this result exists showing that B may even be chosen integral over A .

There is an alternate approach to the study of filtrations on a ring which will be employed to develop one aspect of the theory and will be used as a convenient method of presenting particular filtrations.

Let $\bar{R}$ denote the set of nonnegative real numbers together with $+\infty$ and define addition and order on $\bar{R}$ in the usual manner. A pseudo-valuation on a ring A is a function v on A into $\bar{R}$ with the properties

(i) $v(0) = +\infty$

(1.21) (ii) $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ for all $\alpha, \beta \in A$, and

(iii) $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$ for all $\alpha, \beta \in A$.

These conditions define a valuation on A in case (ii) is replaced by

(1.22) (ii)' $v(\alpha\beta) = v(\alpha) + v(\beta)$ for all $\alpha, \beta \in A$.

For any $r \in \bar{R}$, the set $\mathfrak{a}_{v,r} = \{\alpha \in A \mid v(\alpha) \geq r\}$ is an ideal of A and the sequence of ideals

(1.23) $$f_v = \{\mathfrak{a}_{v,n}\}$$

forms a filtration on A which is called the filtration associated with the pseudo-valuation v . Conversely, given a filtration $f = \{\mathfrak{a}_n\}$ on a ring A , there is a class of pseudo-valuations v for which $f_v = f$. Existance of such is obtained by defining v_f on A into $\bar{R}$ via

(1.24) $$v_f(\alpha) = \sup\{n \mid \alpha \in \mathfrak{a}_n\} \text{ for each } \alpha \in A .$$

This function is a pseudo-valuation on A and is called the pseudo-valuation associated with f . Note that $f_{v_f} = f$. The corresponding statement starting with a pseudo-valuation v ; i.e., $v_{f_v} = v$, is in general false since v need not be integrally valued. This situation can be rectified by defining

(1.25) $$[v](\alpha) = [v(\alpha)] , \text{ for each } \alpha \in A ,$$

where $[r]$ denotes the greatest integer less than or equal to the real number r and $[+\infty] = +\infty$. Then $[v]$ is a pseudo-valuation and $v_{f_v} = [v]$.

A graded ring R is a ring with a subsequence of abelian subgroups R^n such that

(1.26)
(i) $R = \bigoplus_{n=0}^{\infty} R^n$ as abelian groups, and
(ii) $R^m R^n \subseteq R^{m+n}$ for all $m,n \geq 0$.

The elements of R^n are said to be homogeneous of degree n and a homogeneous ideal is an ideal of R generated by homogeneous elements. The elements homogeneous of degree 0 form a subring of R. A graded module E over a graded ring $R = \bigoplus R^n$ is an R-module E together with a sequence of subgroups E^n which satisfy

$$(1.27)\qquad \text{(i)}\quad E = \bigoplus_{n=0}^{\infty} E^n \quad \text{as abelian groups, and}$$
$$\text{(ii)}\quad R^m E^n \subseteq E^{m+n} \quad \text{for all } m,n \geq 0 .$$

As before, the elements of E^n are called homogeneous of degree n, and a homogeneous submodule is one which is generated by homogeneous elements.

To every multiplicative filtration $f = \{\mathfrak{a}_n\}$ on a ring A, there corresponds a graded commutative ring with identity, the associated graded ring, defined by

$$(1.28)\qquad G_f(A) = \bigoplus_{n=0}^{\infty} \frac{\mathfrak{a}_n}{\mathfrak{a}_{n+1}}$$

as an abelian group with multiplication being given by the formula

$$(1.29)\qquad \left(\sum_i \alpha_i + \mathfrak{a}_{i+1}\right)\left(\sum_j \beta_j + \mathfrak{a}_{j+1}\right) = \sum_k \left(\sum_{i+j=k} \alpha_i \beta_j\right) + \mathfrak{a}_{k+1} .$$

This multiplication is well defined because f is a multiplicative filtration. That is, if $\alpha' = \alpha + \gamma$ with $\alpha', \alpha \in \mathfrak{a}_i$, $\gamma \in \mathfrak{a}_{i+1}$ and $\beta' = \beta + \delta$ with $\beta, \beta' \in \mathfrak{a}_j$, $\delta \in \mathfrak{a}_{j+1}$, then $\alpha'\beta' = \alpha\beta + \alpha\delta + \gamma\beta + \gamma\delta$ and by the filtration conditions, $\alpha\delta + \gamma\beta + \gamma\delta \in \mathfrak{a}_{i+j+1}$. The identity of $G_f(A)$ is $1_A + \mathfrak{a}_1$. If M is an A-module and $f = \{\mathfrak{a}_n\}$ is a filtration on A , the abelian group

$$(1.30) \qquad G_f(M) = \bigoplus_{n=0}^{\infty} \frac{\mathfrak{a}_n M}{\mathfrak{a}_{n+1} M}$$

has structure as a graded module over the graded ring $G_f(A)$ by defining scalar multiplication as

$$(1.31) \qquad \left(\sum_i \alpha_i + \mathfrak{a}_{i+1}\right)\left(\sum_j m_j + \mathfrak{a}_{j+1} M\right) = \sum_k \left(\sum_{i+j=k} \alpha_i m_j\right) + \mathfrak{a}_{k+1} M \ .$$

The $G_f(A)$-module $G_f(M)$ is called the <u>associated graded module</u> of M with respect to f . In case $f = f_{\mathfrak{a}}$ for some ideal $\mathfrak{a}$, $G_{f_{\mathfrak{a}}}(A)$ and $G_{f_{\mathfrak{a}}}(M)$ will be denoted $G_{\mathfrak{a}}(A)$ and $G_{\mathfrak{a}}(M)$.

II MULTIPLICITY OF FILTRATIONS

The Noetherian Case

A multiplicative filtration $f = \{\mathfrak{a}_n\}$ on a commutative ring A with identity is said to be noetherian in case the associated graded ring $G_f(A)$ is noetherian.

Note 2.1: By elementary properties of graded rings and the Hilbert Basis Theorem, f is noetherian if and only if $R^o = A/\mathfrak{a}_1$ is a noetherian ring and $G_f(A)$ is finitely generated as an algebra over R^o [1, Proposition 10.7, P. 106].

Proposition 2.2: For any ideal $\mathfrak{a}$ in a noetherian ring A , the filtration $f_\mathfrak{a}$ is noetherian and $G_\mathfrak{a}(A)$ is generated as an algebra over $R^o = A/\mathfrak{a}$ by the elements homogeneous of degree 1. If in addition M is a finitely generated A-module, $G_\mathfrak{a}(M)$ is a finitely generated $G_\mathfrak{a}(A)$-module.

Proof: In any ring A , the associated graded ring $G_\mathfrak{a}(A)$ for an ideal $\mathfrak{a}$ is generated as an algebra over R^o by R^1 , the elements homogeneous of degree 1, because each R^n has the form $R^n = \mathfrak{a}^n/\mathfrak{a}^{n+1}$. The added condition that A is noetherian implies R^1 is finitely generated and R^o is a noetherian ring. Note 2.1 now shows that

$G_{\mathfrak{a}}(A)$ is a noetherian ring. The last statement in the proposition is immediate since $R^n E^o = E^n$ for each $E^n = \mathfrak{a}^n M/\mathfrak{a}^{n+1}M$ implying that the image of a set of generators for M in $E_o = M/\mathfrak{a}M$ is a set of generators for $G_{\mathfrak{a}}(M)$ over $G_{\mathfrak{a}}(A)$. Q.E.D.

Let M be an A-module and $\mathfrak{a}$ an ideal of A contained in the annihilator of M . Then M has structure as an $A/\mathfrak{a}$-module with the lattice of submodules of M as an A-module being identically that of M as an $A/\mathfrak{a}$-module. If A is noetherian, $\mathfrak{a}$ is 0-dimensional, and M is finitely generated over A , then M is finitely generated over the ring $A/\mathfrak{a}$. Now $A/\mathfrak{a}$ is both 0-dimensional and noetherian, and therefore Artinian [1, Theorem 8.5, P.90] which in turn implies that M has finite $A/\mathfrak{a}$-length. Since the lattice structure of M as an A-module and as an $A/\mathfrak{a}$-module agree,

$$(2.1) \qquad L_A(M) = L_{A/\mathfrak{a}}(M) < \infty .$$

For any filtration $f = \{\mathfrak{a}_n\}$, $\mathfrak{a}_n$ is contained in $\mathrm{Ann}_A(M/\mathfrak{a}_n M)$ and $\dim(f) = \dim(\mathfrak{a}_n)$ for $n > 0$. The next proposition follows directly from these comments and statement (2.1).

<u>Proposition 2.3</u>: Let A be a noetherian ring, $f = \{\mathfrak{a}_n\}$ a filtration on A with $\dim(f) = 0$, and M a finitely generated A-module. Then for each $n \geq 0$, $L_A(M/\mathfrak{a}_n M) < \infty$.

In the situation of Proposition 2.3, define the (cummulative) <u>Hilbert function</u> by

(2.2) $H_A(n,M,f) = L_A(M/\mathfrak{a}_n M)$ for all $n \geq 0$.

Since no confusion will arise in the following development the A will be suppressed. The symbol $H(n,f)$ denotes $H(n,A,f)$, and $H(n,M,\mathfrak{a})$ denotes $H(n,M,f_{\mathfrak{a}})$ in the classical case of $f = f_{\mathfrak{a}}$ for some ideal $\mathfrak{a}$. The Hilbert function $H(n,M,\mathfrak{a})$ is especially well behaved.

<u>Theorem 2.4</u>: For a 0-dimensional ideal $\mathfrak{a}$ in a noetherian ring A and a finitely generated module M over A , the Hilbert function $H(n,M,\mathfrak{a})$ is described by a polynomial in n for all sufficiently large n .

This is a special case of the following result by letting $R = G_{\mathfrak{a}}(A)$ and $E = G_{\mathfrak{a}}(M)$ and applying Proposition 2.2. In fact Theorem 2.5 extends Theorem 2.4 to the case where f is 0-dimensional, noetherian, and $G_f(A)$ is generated by $\mathfrak{a}_1/\mathfrak{a}_2$ as an algebra over $A/\mathfrak{a}_1$.

<u>Theorem 2.5</u>: Let $R = \bigoplus R^n$ be a noetherian graded ring such that R^0 is 0-dimensional and R is generated as an algebra over R^0 by R^1. If $E = \bigoplus E^n$ is a finitely generated graded R-module, then

$$f(n) = \sum_{i=0}^{n-1} L_{R^0}(E^i)$$

is a polynomial in n for sufficiently large n .

The proof will be omitted since it is readily available from several sources; e.g., [1,Corollary 11.2, P.117] or [4,(20.5), P.68].

<u>Remark 2.6</u>: Usually $H(n,M,\mathfrak{a})$ is regarded as a polynomial and called the Hilbert polynomial.

In the ensuing development it is essential to know the degree of the Hilbert polynomial.

<u>Theorem 2.7</u>: The degree of the polynomial $H(n,M,\mathfrak{a})$ given by Theorem 2.4 is $\operatorname{alt}\left(\frac{\mathfrak{a} + \operatorname{Ann}_A(M)}{\operatorname{Ann}_A(M)}\right)$ taken in the ring $\frac{A}{\operatorname{Ann}_A(M)}$.

For proof see [4, Theorem 22.7, P. 74].

<u>Corollary 2.8</u>: The degree of $H(n,\mathfrak{a})$ is precisely $s = \operatorname{alt}(\mathfrak{a})$ and $\deg(H(n,M,\mathfrak{a}))$ is less than or equal to s for every M in the category of finitely generated A-modules.

<u>Proof</u>: Since $1 \in A$, $\operatorname{Ann}_A(A) = (0)$ so $\operatorname{alt}\left(\frac{\mathfrak{a} + \operatorname{Ann}_A(A)}{\operatorname{Ann}_A(A)}\right) = \operatorname{alt}(\mathfrak{a})$. The second assertion of the corollary is immediate from the fact that the altitude of an ideal can not increase under a ring epimorphism. Q.E.D.

Since Nagata's proof of Theorem 2.7 is rather complicated and since only Corollary 2.8 will be needed in the following work, it should be noted that Corollary 2.8 can be derived in a much more accessible manner from results in [1, Chapter 11] by localization although the explicit result is not stated.

If $G_f(A)$ is not generated as an algebra by elements homogeneous of degree 1, $H(n,f)$ need not be given by a polynomial for large n as the following demonstrates.

Example 2.9: Let $A = k[X]$ where X is an indeterminate over the field k and the filtration $f = \{\mathfrak{a}_n\}$ be given by

$$(2.3) \qquad \mathfrak{a}_n = \begin{cases} (X^{(n+1)/2}) & \text{if } n \text{ is odd, and} \\ (X^{n/2}) & \text{if } n \text{ is even .} \end{cases}$$

More explicitly $f = \{A,(X),(X),(X^2),(X^2),\ldots\}$. Since $L_A((X^n)/(X^{n+1})) = L_A(A/(X)) = 1$ we have the following formula for the Hilbert function of f on A .

$$(2.4) \qquad H(n,f) = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd, and} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Clearly $H(n,f)$ can not be represented by a polynomial for large n .

Definition 2.10: Let A be a noetherian commutative ring with identity, $f = \{\mathfrak{a}_n\}$ a 0-dimensional filtration on A with $\operatorname{alt}(f) = s$, and M a finitely generated A-module. The multiplicity of M with respect to f is defined by

$$\mu_A(f,M) = s!\lim_{n\to\infty}\frac{H(n,M,f)}{n^s} = s!\lim_{n\to\infty}\frac{L_A(M/\mathfrak{a}_nM)}{n^s}$$

whenever this limit exists. As before, $\mu_A(f,M)$ will be denoted $\mu(f,M)$, $\mu_A(f,A)$ as $\mu(f)$, and $\mu_A(f_{\mathfrak{a}},M)$ as $\mu(\mathfrak{a},M)$.

Note 2.11: It remains unanswered whether or not this limit always exists.

Theorem 2.12: If $\mathfrak{a}$ is a 0-dimensional ideal in a noetherian ring A and M is a finitely generated A-module, then $\mu(\mathfrak{a},M)$ exists. Moreover, it is just s! times the leading coefficient of the Hilbert polynomial if $\text{alt}\left(\frac{\mathfrak{a} + \text{Ann}_A(M)}{\text{Ann}_A(M)}\right) = \text{alt}(\mathfrak{a})$ and zero otherwise.

Proof: This is an immediate consequence of Theorems 2.4 and 2.7 and Corollary 2.8. Q.E.D.

Remark 2.13: In this context, the inclusion of s! in Definition 2.10 appears superfluous. Its presence here is for agreement with other definitions of multiplicity in the situation of Theorem 2.12 where it assures that $\mu(\mathfrak{a},M)$ will always be a nonnegative integer [4,(20.8), P.69].

The classical situation having been reviewed, it seems natural to ask about existance of and representations for $\mu(f,M)$ when f is not $f_{\mathfrak{a}}$ for any $\mathfrak{a}$. As a simple example consider the filtration f given in Example 2.9. This filtration is 0-dimensional since (X) is a maximal ideal and $\text{alt}(f) = 1$ since $k[X]$ is a principal ideal domain. Then (2.4) implies

$$(2.5)\qquad \frac{H(n,f)}{n} = \begin{cases} 1/2 + 1/2n & \text{if } n \text{ is odd, and} \\ 1/2 & \text{if } n \text{ is even,} \end{cases}$$

and thus $\mu(f) = 1/2$. The existence of multiplicity in this example is no accident but instead comes from the fact that f is essentially powers of an ideal in the following sense.

<u>Definition 2.14</u>: A filtration $f = \{\mathfrak{a}_n\}$ on a ring is described as being <u>essentially powers of an ideal</u> in case there exists $N \in \mathbb{N}$ such that for all $n \geq 0$

$$\mathfrak{a}_n = \sum_{i=1}^{N} \mathfrak{a}_{n-i}\mathfrak{a}_i \ ,$$

where $\mathfrak{a}_n = A$ for all $n < 0$.

<u>Proposition 2.15</u>: The condition $f = \{\mathfrak{a}_n\}$ with $\mathfrak{a}_n = \sum_{i=1}^{N} \mathfrak{a}_{n-i}\mathfrak{a}_i$ for all $n \geq 0$ is equivalent to saying f is the least filtration $g = \{\mathfrak{b}_n\}$ such that $\mathfrak{b}_i = \mathfrak{a}_i$ for $i = 1,\ldots,N$.

<u>Proof</u>: Directly from the definition, any filtration $g = \{\mathfrak{b}_n\}$ with $\mathfrak{b}_i = \mathfrak{a}_i$ for $i = 1,\ldots,N$ has the property that

$$(2.6)\qquad \mathfrak{c}_n = \sum_{\Sigma i n_i = n} \left(\prod_{i=1}^{N} \mathfrak{a}_i^{\,n_i} \right) \subseteq \mathfrak{b}_n \quad \text{for } n > N \ .$$

Define $h = \{\mathfrak{c}_n\}$ by letting $\mathfrak{c}_n = \mathfrak{a}_n$ for $n \leq N$.

It is straight forward to show h is a filtration so of course the smallest filtration which agrees with a_i for $i = 1,\ldots,N$. Since f is a filtration, this property implies $h \leq f$. To see that $f \leq h$, choose any a_n, $n > N$ and show it is contained in c_n. Examine each product of $a_n = \sum_{i=1}^{N} a_{n-i}a_i$. If $n-i \leq N$, then $a_{n-i}a_i$ is one of the defining products of c_n and $a_{n-i}a_i \subseteq c_n$. If $n-i > N$, then $a_{n-i} = \sum_{j=1}^{N} a_{(n-i)-j}a_j$ and $a_{n-i}a_i = \sum_{j=1}^{N} a_{(n-i)-j}a_j a_i$. Repeat the procedure for each j such that $n-i-j > N$. This process eventually terminates since the $i,j,\ldots$ obtained are all greater than or equal to 1. It now follows that $a_{n-i}a_i \subseteq c_n$ for each $i = 1,\ldots, N$ and therefore $a_n \subseteq c_n$ for each $n > N$. By definition, $f \leq h$, and hence $f = h$. Q.E.D.

Note 2.16: If $f = f_a$ for some a, f is essentially powers with $N = 1$.

As a first step toward the goal of showing that in a noetherian ring multiplicity always exists for any 0-dimensional filtration which is essentially powers, the following is obtained from adapting the argument for a result of Muhly and Sakuma in [3]. In addition to being a necessary step in the theory, this result gives much insight into the nature of all noetherian filtrations and characterizes them in the case of noetherian rings.

Theorem 2.17: If $f = \{\mathfrak{a}_n\}$ is a noetherian filtration on a commutative ring A with identity, there exists $k \in \mathbb{N}$ such that for all $j \geq k$ and all $t \geq 0$,

$$\mathfrak{a}_{k+j} = \mathfrak{a}_k \mathfrak{a}_j + \mathfrak{a}_{k+j+t} .$$

Conversely, if for $f = \{\mathfrak{a}_n\}$ there exists $k \in \mathbb{N}$ such that for all $j \geq k$,

$$\mathfrak{a}_{k+j} = \mathfrak{a}_k \mathfrak{a}_j + \mathfrak{a}_{k+j+1} ,$$

and if A is noetherian, then f is noetherian.

Proof: For each $j \geq 1$ and i with $1 \leq i \leq j$, define

(2.7) $$\mathfrak{b}_{i,j} = \mathfrak{a}_1 \mathfrak{a}_{j-1} + \mathfrak{a}_2 \mathfrak{a}_{j-2} + \cdots + \mathfrak{a}_i \mathfrak{a}_{j-i} + \mathfrak{a}_{j+1} .$$

It follows from the definition of filtration that $\mathfrak{b}_{i,j} \subseteq \mathfrak{a}_j$ for each j and i. By letting $\mathfrak{a}_n = A$ for $n < 0$ the restriction $i \leq j$ may be removed and $\mathfrak{b}_{i,j}$ is defined for all $i,j \in \mathbb{N}$. Thus $\mathfrak{b}_{i,j} = \mathfrak{a}_j$ if $i \geq j$. Fixing i determines an ideal B_i of $G_f(A)$ via

(2.8) $$B_i = \bigoplus_{j=0}^{\infty} \frac{\mathfrak{b}_{i,j}}{\mathfrak{a}_{j+1}} .$$

The sequence $B_1, B_2, \ldots$ is an ascending chain of ideals in $G_f(A)$ which must become constant. That is, there exists $k \in \mathbb{N}$ such that $B_k = B_{k+t}$ for all $t \geq 0$. The direct sum nature of B_i then implies

$$\frac{\mathfrak{b}_{k,j}}{\mathfrak{a}_{j+1}} = \frac{\mathfrak{b}_{k+t,j}}{\mathfrak{a}_{j+1}} \text{ for all } t,j \in \mathbb{N}.$$

Since $\mathfrak{a}_{j+1} \subseteq \mathfrak{b}_{k,j} \subseteq \mathfrak{b}_{k+t,j}$, it follows that

$$\mathfrak{b}_{k,j} = \mathfrak{b}_{k+t,j} \text{ for all } t,j \in \mathbb{N}.$$

Thus all the ideal products used in the definition of $\mathfrak{b}_{k+t,j}$ are contained in $\mathfrak{b}_{k,j}$; in particular, at $i = k + t$ this means that for all j and t

$$\mathfrak{a}_{k+t}\mathfrak{a}_{j-(k+t)} \subseteq \mathfrak{b}_{k,j} = \mathfrak{a}_1\mathfrak{a}_{j-1} +\dots+ \mathfrak{a}_k\mathfrak{a}_{j-k} + \mathfrak{a}_{j+1}.$$

Specifically for $j \geq k$ and $t = j-k$, this becomes

$$\mathfrak{a}_j \subseteq \mathfrak{b}_{k,j}.$$

The other containment always holds, giving the equation

$$(2.9) \quad \mathfrak{a}_j = \mathfrak{a}_1\mathfrak{a}_{j-1} + \mathfrak{a}_2\mathfrak{a}_{j-2} +\dots+ \mathfrak{a}_k\mathfrak{a}_{j-k} + \mathfrak{a}_{j+1}, \quad j \geq k.$$

Replacing j by $j+1$ translates equation (2.9) into

$$(2.10) \quad \mathfrak{a}_{j+1} = \mathfrak{a}_1\mathfrak{a}_j + \mathfrak{a}_2\mathfrak{a}_{j-1} +\dots+ \mathfrak{a}_k\mathfrak{a}_{j-k+1} + \mathfrak{a}_{j+2}, \quad j \geq k-1.$$

Substituting (2.9) into (2.10) and distributing $\mathfrak{a}_1$ leads to

$$\mathfrak{a}_{j+1} = (\mathfrak{a}_1\mathfrak{a}_1\mathfrak{a}_{j-1} + \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_{j-2} +\dots+ \mathfrak{a}_1\mathfrak{a}_k\mathfrak{a}_{j-k} + \mathfrak{a}_1\mathfrak{a}_{j+1})$$
$$+ \mathfrak{a}_2\mathfrak{a}_{j-1} +\dots+ \mathfrak{a}_k\mathfrak{a}_{j-k+1} + \mathfrak{a}_{j+2}, \quad j \geq k,$$

with each ideal product in parentheses contained in one of those which is not. Thus

$$\alpha_{j+1} = \alpha_2\alpha_{j-1} + \alpha_3\alpha_{j-2} + \dots + \alpha_k\alpha_{j-k+1} + \alpha_{j+2}\,, \quad j \geq k\,.$$

Continuing inductively, assume for $2 \leq h < k-1$ that

$$(2.11) \quad \alpha_{j+h} = \alpha_{h+1}\alpha_{j-1} + \alpha_{h+2}\alpha_{j-2} + \dots + \alpha_k\alpha_{j-k+h} + \alpha_{j+h+1}\,,$$

for all $j \geq k$. As before, replace j by $j+1$ to obtain

$$(2.12) \quad \alpha_{j+h+1} = \alpha_{h+1}\alpha_j + \alpha_{h+2}\alpha_{j-1} + \dots + \alpha_k\alpha_{j-k+h+1} + \alpha_{j+h+2}\,, \quad j \geq k-1\,.$$

Using equation (2.9) in (2.12) leads to

$$\alpha_{j+(h+1)} = (\alpha_{h+1}\alpha_1\alpha_{j-1} + \alpha_{h+1}\alpha_2\alpha_{j-2} + \dots + \alpha_{h+1}\alpha_k\alpha_{j-k} + \alpha_{h+1}\alpha_{j+1}) + \alpha_{h+2}\alpha_{j-1} + \dots + \alpha_k\alpha_{j-k+h+1} + \alpha_{j+h+2}\,, \quad j \geq k\,.$$

Again each term in parentheses is contained in one of those which is not, completing the induction argument to give validity of equation (2.11) for $h = k-1$. Explicitly,

$$\alpha_{j+k-1} = \alpha_k\alpha_{j-1} + \alpha_{j+k}\,, \quad j \geq k\,.$$

By replacing j with $j+1$, this implies

$$(2.13) \quad \alpha_{k+j} = \alpha_k\alpha_j + \alpha_{k+j+1}\,, \quad j \geq k\,.$$

Replacing j by $j+1$ in (2.13) gives rise to

$$(2.14) \quad \alpha_{k+j+1} = \alpha_k\alpha_{j+1} + \alpha_{k+j+2}\,, \quad j \geq k-1\,.$$

Substituting (2.14) into (2.13) and using the fact that $\alpha_k\alpha_{j+1} \subseteq \alpha_k\alpha_j$, the equation

$$\alpha_{k+j} = \alpha_k\alpha_j + \alpha_{k+j+2}\,, \quad j \geq k$$

is obtained. Inductively one proceeds to derive

$$a_{k+j} = a_k a_j + a_{k+j+t}\,, \quad j \geq k \text{ and all } t \geq 0\,.$$

Conversely, suppose there exists k such that

(2.15) $$a_{k+j} = a_k a_j + a_{k+j+1} \quad \text{for all } j \geq k\,.$$

For any n, the division algorithm gives $n = mk + r$ with $0 \leq r < k$.

If $n \geq k$, $n = (m-1)k + (k+r)$ with $k \leq k + r < 2k$.

Then

$$\begin{aligned} a_n &= a_{(m-1)k+(k+r)} \\ &= a_{k+(m-2)k+(k+r)} \\ &= a_k a_{(m-2)k+(k+r)} + a_{n+1} \quad \text{(by (2.15))} \\ &= a_k\left(a_k a_{(m-3)k+(k+r)} + a_{n-k+1}\right) + a_{n+1} \\ &= a_k^2 a_{(m-3)k+(k+r)} + a_k a_{n-k+1} + a_{n+1} \\ &= a_k^2 a_{(m-3)k+(k+r)} + a_{n+1} \\ &\;\;\vdots \\ &= a_k^{m-1} a_{k+r} + a_{n+1}\,. \end{aligned}$$

Now

$$\frac{a_n}{a_{n+1}} = \begin{cases} \dfrac{a_n}{a_{n+1}} & \text{for } n < k\text{, and} \\ \dfrac{a_k^{m-1} a_{k+r}}{a_{n+1}} = \left(\dfrac{a_k}{a_{k+1}}\right)^{m-1}\left(\dfrac{a_{k+r}}{a_{k+r+1}}\right) & \text{for } n \geq k\,. \end{cases}$$

Thus $G_f(A)$ is generated as an algebra over $A_o = A/a_1$ by a_n/a_{n+1}, $n = 1,\ldots,2k-1$. Now the assumption that A is noetherian implies that A_o is noetherian and

$G_f(A)$ is finitely generated as an algebra over A_o. By Note 2.1, f is noetherian. Q.E.D.

Proposition 2.18: If $f = \{\mathfrak{a}_n\}$ is a noetherian filtration on a noetherian commutative ring A and if f has the property that for each n there exists $\varphi(n)$ such that $\mathfrak{a}_{\varphi(n)} \subseteq (\mathrm{rad}(f))^n$, then there exists k such that for all $j \geq k$

$$\mathfrak{a}_{k+j} = \mathfrak{a}_k \mathfrak{a}_j .$$

Note 2.19: In the language of filtration topologies, the condition $\mathfrak{a}_{\varphi(n)} \subseteq (\mathrm{rad}(f))^n$ is equivalent to saying that f (by its ideals) and $\mathrm{rad}(f)$ (by its powers) generate the same ring topology on A.

Proof: Let k be given by Theorem 2.17. Then for all $j \geq k$ and all $t \geq 1$,

$$\mathfrak{a}_{k+j} = \mathfrak{a}_k \mathfrak{a}_j + \mathfrak{a}_{k+j+t} . \tag{2.16}$$

Since in a noetherian ring every ideal contains a power of its radical, there exists $n, m \in \mathbb{N}$ such that $(\mathrm{rad}(f))^n \subseteq \mathfrak{a}_k$ and $(\mathrm{rad}(f))^m \subseteq \mathfrak{a}_j$. It follows that

$$\mathfrak{a}_{\varphi(n+m)} \subseteq (\mathrm{rad}(f))^{n+m} = (\mathrm{rad}(f))^n (\mathrm{rad}(f))^m \subseteq \mathfrak{a}_k \mathfrak{a}_j .$$

The desired result now follows by letting $t = \varphi(n + m)$ in equation (2.16). Q.E.D.

Theorem 2.20: A filtration $f = \{\mathfrak{a}_n\}$ on a noetherian ring A is essentially powers of an ideal if and only if f is noetherian and has the property that for each n there exists $\varphi(n)$ such that $\mathfrak{a}_{\varphi(n)} \subseteq (\mathrm{rad}(f))^n$.

<u>Proof</u>: If f is essentially powers of an ideal there is by definition an N such that $\mathfrak{a}_n = \sum_{i=1}^{N} \mathfrak{a}_{n-i}\mathfrak{a}_i$ for all n and, by (2.6) in the proof of Proposition 2.15, $\mathfrak{a}_n$ can be represented as

$$\mathfrak{a}_n = \sum_{\Sigma i n_i = n} \prod_{i=1}^{N} \mathfrak{a}_i^{n_i} . \tag{2.17}$$

Then $G_f(A)$ is generated by $\mathfrak{a}_i/\mathfrak{a}_{i+1}$, $i = 1,\ldots,N$, as an algebra over $A/\mathfrak{a}_1$. Each of these is finitely generated and from Note 2.1 f is noetherian. For the function φ , the linear function $\varphi(n) = Nn$ can be used because for each power product $\prod \mathfrak{a}_i^{n_i}$ with $\sum i n_i = Nn$ in (2.17),

$$\prod \mathfrak{a}_i^{n_i} \subseteq \prod \mathfrak{a}_1^{n_i} = \mathfrak{a}_1^{\Sigma n_i} \subseteq (\mathrm{rad}(f))^{\Sigma n_i} \subseteq (\mathrm{rad}(f))^n ,$$

with the final containment coming from the fact that each i is no greater than N so that $n = \sum \frac{i n_i}{N} \leq \sum n_i$.

Conversely, use the k of Proposition 2.18 to write

$$\mathfrak{a}_n = \mathfrak{a}_k^{m-1}\mathfrak{a}_{k+r} \subseteq \mathfrak{a}_{n-(k+r)}\mathfrak{a}_{k+r}$$

for each $n \geq k$, where $n = mk + r$, $0 \leq r < k$.

Then for all n , $\mathfrak{a}_n \subseteq \sum_{i=1}^{2k-1} \mathfrak{a}_{n-i}\mathfrak{a}_i$, with the opposite inclusion always being true. Hence f is essentially powers of an ideal. Q.E.D.

Corollary 2.21: If $f = \{\mathfrak{a}_n\}$ is a filtration on a noetherian ring A which is essentially powers of an ideal, there exists $k \in \mathbb{N}$ such that

$$\mathfrak{a}_{kn} = \mathfrak{a}_k^{\,n} \quad \text{for all } n \geq 0 .$$

Proof: By Theorem 2.20, the conditions of Proposition 2.18 are satisfied. Let k be given by Proposition 2.18. Then for all $n > 0$,

$$\mathfrak{a}_{kn} = \mathfrak{a}_{k+k(n-1)} = \mathfrak{a}_k\mathfrak{a}_{k(n-1)} = \cdots = \mathfrak{a}_k^{\,n-1}\mathfrak{a}_k = \mathfrak{a}_k^{\,n} .$$ Q.E.D.

Extracting the property assured by Corollary 2.21, a filtration $f = \{\mathfrak{a}_n\}$ will be said to possess a regular subsequence of powers of an ideal in case there exists $k \in \mathbb{N}$ with

$$f^{(k)} = f_{\mathfrak{a}_k} \; ; \tag{2.18}$$

that is, $\mathfrak{a}_{nk} = \mathfrak{a}_k^{\,n}$ for all $n \geq 0$.
The connection with multiplicity is summarized in the following theorem.

Theorem 2.22: Let A be a noetherian ring, M a finitely generated A-module, $f = \{\mathfrak{a}_n\}$ a 0-dimensional filtration of altitude s . If f possesses a regular subsequence $f^{(k)}$ of powers of an ideal, then $\mu(f,M)$ exists and $\mu(f,M) = \frac{1}{k^s}\,\mu(\mathfrak{a}_k,M)$.

Proof: For each n , one has by the division algorithm $n = q_n k + r_n$, $0 \le r_n < k$. Since $q_n k \le n < (q_n+1)k$, it follows that $\mathfrak{a}_{(q_n+1)k} \subseteq \mathfrak{a}_n \subseteq \mathfrak{a}_{q_n k}$, which in turn implies

$$L_A\left(\frac{M}{\mathfrak{a}_{q_n k}M}\right) \le L_A\left(\frac{M}{\mathfrak{a}_n M}\right) \le L_A\left(\frac{M}{\mathfrak{a}_{(q_n+1)k}M}\right) .$$

But $\mathfrak{a}_{q_n k} = \mathfrak{a}_k^{q_n}$ and $\mathfrak{a}_{(q_n+1)k} = \mathfrak{a}_k^{q_n+1}$ so this may be expressed in terms of Hilbert functions as

$$H(q_n,M,\mathfrak{a}_k) \le H(n,M,f) \le H(q_n+1,M,\mathfrak{a}_k) .$$

Multiplying by the positive constant $\frac{s!}{n^s}$ and adjusting the terms leads to

$$\frac{s!}{k^s}\left(\frac{q_n k}{n}\right)^s \frac{H(q_n,M,\mathfrak{a}_k)}{(q_n)^s} \le s!\,\frac{H(n,M,f)}{n^s}$$

$$\le \frac{s!}{k^s}\left(\frac{(q_n+1)k}{n}\right)^s \frac{H(q_n+1,M,\mathfrak{a}_k)}{(q_n+1)^s} .$$

Passing to the limit and using Theorem 2.12 to give the same limit for the first and last expressions provides the desired result

$$s! \lim_{n\to\infty} \frac{H(n,M,f)}{n^s} = \frac{1}{k^s}\,\mu(\mathfrak{a}_k,M) . \qquad \text{Q.E.D.}$$

Corollary 2.23: If f is a 0-dimensional filtration on a noetherian ring A , which is essentially powers of an ideal, then for any finitely generated A-module M , $\mu(f,M)$ exists and is a rational number.

Proof: This is immediate from Corollary 2.21, Theorem 2.22, and the fact that $\mu(\mathfrak{a},M)$ is an integer.

The General Case

It has not yet been shown that there exist filtrations on noetherian rings which are not noetherian. Trivial ones abound as can be seen by the following.

Example 2.24: Let $A = \mathbb{Z}$, the integers, and $f = \{\mathbb{Z}, 2\mathbb{Z}, 2^2\mathbb{Z}, 2^2\mathbb{Z}, 2^3\mathbb{Z}, 2^3\mathbb{Z}, 2^3\mathbb{Z}, \ldots\}$. Then f is a filtration on A which is not noetherian but its multiplicity does exist and is zero. In fact, multiplicity exists for any filtration on $\mathbb{Z}$. See Corollary 2.31.

A far more interesting situation is given below.

Example 2.25: Let $A = k[X] = k[X_1,\ldots,X_n]$ with the X_i being indeterminates over the field k. Let τ_i, $i = 1,\ldots,n$ be nonnegative real numbers. For any $\sum \alpha_{(i)}X^{(i)} = \sum \alpha_{i_1,\ldots,i_n} X_1^{i_1}\ldots X_n^{i_n} \in A$ define

$$(2.19) \qquad v_\tau\left(\sum \alpha_{(i)}X^{(i)}\right) = \min\{i_1\tau_1 + \ldots + i_n\tau_n \mid \alpha_{(i)} \neq 0\}.$$

It can be verified that v_τ defines a valuation on A which by (1.23) gives rise to the filtration $f_\tau = \{\mathfrak{a}_m\}$ where $\mathfrak{a}_m = \{\varphi \in A \mid v_\tau(\varphi) \geq m\}$ for each m. If at least one τ_i is irrational, for definiteness say τ_j,

f_τ is not noetherian. This is because the "φ-condition" of Theorem 2.20 is satisfied by taking $\varphi(n)$ to be the linear function Nn where N is any integer greater than or equal to $\max_i\{\tau_i\}$. If f_τ was noetherian, Theorem 2.20 would imply that it was essentially powers and Corollary 2.21 would give a regular subsequence $f^{(k)}$ of powers of an ideal; that is, $\mathfrak{a}_{km} = \mathfrak{a}_k^{\,m}$ for all m. That this is false, may be seen as follows. Choose a set of generators for $\mathfrak{a}_k$ of the form $X^{(k)}$ which is possible since $\sum \alpha_{(i)} X^{(i)}$ is in $\mathfrak{a}_k$ if and only if $X^{(i)}$ is also, for each $\alpha_{(i)} \neq 0$. Then one of the generators needed is $X_j^{k_j}$ where k_j is the smallest positive integer q for which $q\tau_j \geq k$. Since $k_j\tau_j > k$ there exist positive integers s and t such that $k_j\tau_j > \frac{s}{t}\,\tau_j > k$. From this inequality both $k_j t > s$ and $s\tau_j > kt$ are derived. The first implies $X_j^{\,s} \notin \mathfrak{a}_k^{\,t}$ because the smallest p for which $X_j^{\,p} \in \mathfrak{a}_k^{\,t}$ is $k_j t$, and the second implies $X_j^{\,s} \in \mathfrak{a}_{kt}$. Hence for this choice of t, $\mathfrak{a}_{kt} \neq \mathfrak{a}_k^{\,t}$.

In the case where all τ_i are rational and greater than zero, the question of existance of multiplicity has already been settled affirmatively by Corollary 2.23 because f_τ is essentially powers. Since later remarks will depend on this fact, a proof is included for completeness. Let $\tau = (\tau_i)$ with $\tau_i = a_i/b_i$ for each $i = 1,\ldots,n$ where $a_i, b_i \in \mathbb{N}$ and let $a = \prod_{i=1}^{n} a_i$.

It will be shown that for all m ,

(2.20) $$\mathfrak{a}_m = \sum_{k=1}^{na} \mathfrak{a}_{m-k}\mathfrak{a}_k \quad .$$

For any generator $X^{(i)}$ of $\mathfrak{a}_m$ use the division algorithm to express $i_j = c_j m_j + r_j$, $0 \le r_j < c_j$ where $c_j = a_1 \cdots a_{j-1} b_j a_{j+1} \cdots a_n$ for each j . Then $X^{(i)}$ may be factored

$$X^{(i)} = X_1^{i_1} \ldots X_n^{i_n} = \prod X_j^{c_j m_j} \prod X_j^{r_j} \text{ , } j = 1,\ldots,n.$$

For each j , $v_T(X_j^{c_j}) = c_j a_j/b_j = a$ implying that $\prod X_j^{c_j m_j} \in \mathfrak{a}_a^{\Sigma m_j}$. Since $v_T(\prod X_j^{r_j}) = \sum r_j a_j/b_j$, it follows that $\prod X_j^{r_j} \in \mathfrak{a}_{[\Sigma r_j a_j/b_j]}$, where $[r]$ denotes the greatest integer less than or equal to the real number r . Thus

(2.21) $$X^{(i)} \in \mathfrak{a}_a^{\Sigma m_j} \mathfrak{a}_{[\Sigma r_j a_j/b_j]} \quad .$$

Since $v_T(X^{(i)}) = \sum i_j a_j/b_j$ and $X^{(i)} \in \mathfrak{a}_m$,

$$m \le \sum i_j a_j/b_j = \sum c_j m_j a_j/b_j + \sum r_j a_j/b_j$$
$$= a \sum m_j + \sum r_j a_j/b_j \quad .$$

From the fact that $m - a \sum m_j$ is an integer,

$$m - a \sum m_j \le \left[\sum r_j a_j/b_j\right] \quad .$$

Since $r_j a_j/b_j < c_j a_j/b_j = a$, one has $\left[\sum r_j a_j/b_j\right] < na$, so a fortiori $k = m - a \sum m_j < na$.

Because $\mathfrak{a}_{[\Sigma r_j a_j/b_j]} \subseteq \mathfrak{a}_k$ and $\mathfrak{a}_a^{\Sigma m_j} \subseteq \mathfrak{a}_{a\Sigma m_j}$, it follows from (2.21) that

$$X^{(i)} \in \mathfrak{a}_{a\Sigma m_j}\mathfrak{a}_k = \mathfrak{a}_{m-k}\mathfrak{a}_k .$$

The last ideal product is one of those appearing in the right side of (2.20) and $X^{(i)}$ was arbitrarily chosen implying $\mathfrak{a}_m \subseteq \sum_{k=1}^{na} \mathfrak{a}_{m-k}\mathfrak{a}_k$. The reverse containment always holds completing the argument. Q.E.D.

Existance of multiplicity for f_τ in the irrational case will be derived from results in this section.

As a first step in the theory of multiplicity for filtrations which need not be noetherian, the following approximation formula is derived.

<u>Proposition 2.26</u>: Let $f = \{\mathfrak{a}_n\}$ be a 0-dimensional filtration on a noetherian ring A with alt(f) = s and let M be a finitely generated A-module. Then

$$\lim_{k\to\infty} \frac{1}{k^s}\,\mu(\mathfrak{a}_k, M)$$

exists and provides an upper bound for

$$s!\,\limsup \left\{\frac{H(n,M,f)}{n^s}\right\} .$$

<u>Proof</u>: Fix k and choose any $n \in \mathbb{N}$. By the division algorithm express $n = q_n k + r_n$, $0 \le r_n < k$. Then for all $m \in \mathbb{N}$,

$$\frac{s!\ L_A\left(\frac{M}{\alpha_n^{\,m}M}\right)}{m^s} = \frac{s!\ L_A\left(\frac{M}{(\alpha_{q_n k+r_n})^m M}\right)}{m^s} \le \frac{s!\ L_A\left(\frac{M}{(\alpha_{k(q_n+1)})^m M}\right)}{m^s}$$

$$\le \frac{(q_n+1)^s\ s!\ L_A\left(\frac{M}{\alpha_k^{(q_n+1)m} M}\right)}{((q_n+1)m)^s} .$$

In terms of Hilbert polynomials this becomes

$$\frac{s!\ H(m,M,\alpha_n)}{m^s} \le \frac{(q_n+1)^s\ s!\ H((q_n+1)m,M,\alpha_k)}{((q_n+1)m)^s} .$$

Taking limits with respect to m implies

$$\mu(\alpha_n,M) \le (q_n+1)^s\ \mu(\alpha_k,M) .$$

Dividing by $((q_n+1)k)^s$, one obtains

$$\left(\frac{1}{(q_n+1)k}\right)^s \mu(\alpha_n,M) \le \frac{1}{k^s}\ \mu(\alpha_k,M) ,$$

for all n, so that

$$\limsup \left\{\frac{1}{n^s}\ \mu(\alpha_n,M)\right\} \le \frac{1}{k^s}\ \mu(\alpha_k,M) .$$

Since k was chosen arbitrarily and all the terms are bounded below by zero, the desired limit exists.

For the second statement, it is now sufficient to show

$$s!\ \limsup \left\{\frac{H(n,M,f)}{n^s}\right\} \le \frac{1}{k^s}\ \mu(\alpha_k,M) \quad \text{for each } k .$$

The argument is very similar. Let n be arbitrary and express $n = q_n k + r_n$, $0 \le r_n < k$ as before. Then

$$\frac{s!\ L_A\left(\frac{M}{\mathfrak{a}_n M}\right)}{n^s} \le \frac{s!\ L_A\left(\frac{M}{\mathfrak{a}_{k(q_n+1)}M}\right)}{n^s} \le \left(\frac{q_n+1}{n}\right)^s \frac{s!\ L_A\left(\frac{M}{(\mathfrak{a}_k)^{q_n+1}M}\right)}{(q_n+1)^s} .$$

By use of Hilbert functions, this becomes

$$\frac{s!\ H(n,M,f)}{n^s} \le \left(\frac{q_n+1}{n}\right)^s \frac{s!\ H(q_n+1,M,\mathfrak{a}_k)}{(q_n+1)^s} .$$

The limit of the right side is known giving

$$s!\ \limsup\left\{\frac{H(n,M,f)}{n^s}\right\} \le \frac{1}{k^s}\,\mu(\mathfrak{a}_k,M) . \qquad \text{Q.E.D.}$$

<u>Definition 2.27</u>: The limit given by Proposition 2.26 will be called the <u>natural</u> <u>upper</u> <u>bound</u> for the multiplicity of M with respect to f (even though the multiplicity itself may not be defined).

<u>Note 2.28</u>: Trivial examples show that this natural upper bound need not be attained even when $\mu(f,M)$ exists. (See Example 2.33). In fact, the following example shows that the ring and filtration may be chosen in such a manner that the two disagree by a preassigned positive integer multiple.

<u>Example 2.29</u>: Let $A = \mathbb{Z}[X]/(X^k) = \mathbb{Z}[x]$ and $f = \{\mathfrak{a}_n\}$ where $\mathfrak{a}_n = 2^n A + xA = (2^n, x)$. Then f is a 0-dimensional filtration on A with rank(f) = 1 and $\mu(f) = 1$ but $\lim_{n\to\infty} \frac{1}{n}\,\mu(\mathfrak{a}_n) = k$.

Proof: Since $(x) \subseteq \mathrm{Ann}_A(A/\mathfrak{a}_n)$ and since $A/\mathfrak{a}_n \approx \mathbb{Z}/2^n\mathbb{Z}$, $L_A(A/\mathfrak{a}_n) = L_{\mathbb{Z}}(\mathbb{Z}/2^n\mathbb{Z}) = n$. Thus $\mu(f) = 1! \lim_{n\to\infty} \frac{L_A(A/\mathfrak{a}_n)}{n} = 1$. To compute $\lim_{n\to\infty} \frac{1}{n} \mu(\mathfrak{a}_n)$, first compute $\mu(\mathfrak{a}_n)$ for each n. The following is a composition series from A to $(\mathfrak{a}_n)^m$:

$$A \supseteq (2,x) \supseteq (2^2,x) \supseteq \ldots \supseteq (2^{nm},x) \supseteq (2^{nm},2x,x^2) \supseteq \ldots$$
$$\supseteq (2^{nm},2^{n(m-1)}x,x^2) \supseteq (2^{nm},2^{n(m-1)}x,2x^2,x^3) \supseteq \ldots$$
$$\supseteq (2^{nm},2^{n(m-1)}x,\ldots,2^{n(m-(k-1))}x^{k-1}) = (\mathfrak{a}_n)^m .$$

Hence $L_A(A/(\mathfrak{a}_n)^m) = nm + n(m-1)+\ldots+n(m-k+1) = n(km - k(k-1)/2)$ and thus $\mu(\mathfrak{a}_n) = nk$. Therefore $\frac{1}{n}\mu(\mathfrak{a}_n) = k$ for each n. Q.E.D.

Theorem 2.30: In the situation of Proposition 2.26, if all but finitely many of the ideals $\mathfrak{a}_n$ can be generated by sets of s elements, $\mu(f,M)$ exists and is the natural upper bound.

Proof: For any $\epsilon > 0$ choose a sequence $\{n_i\}$ such that

$$(2.22) \qquad s!\,\frac{H(n_i,M,f)}{(n_i)^s} \le s! \liminf \left\{\frac{H(n,M,f)}{n^s}\right\} + \epsilon$$

which exists since 0 is a lower bound. Note that $H(n_i,M,f) = H(1,M,\mathfrak{a}_{n_i}) = L_A(M/\mathfrak{a}_{n_i}M)$ and by hypothesis $\mathfrak{a}_{n_i}$ can be generated by a set of s elements. In this situation the multiplicity defined in this work agrees with that in Northcott [5] (see Theorem 13, P.329) and in his development, Theorem 6 (p.308) gives the inequality

$$\mu(\mathfrak{a}_{n_i}, M) \leq L_A(M/\mathfrak{a}_{n_i}M) \leq s!\, L_A(M/\mathfrak{a}_{n_i}M) \quad .$$

Substituting back into (2.22) gives the inequality

$$\frac{1}{n_i^{\,s}}\, \mu(\mathfrak{a}_{n_i}, M) \leq s!\, \lim\inf \left\{ \frac{H(n,M,f)}{n^s} \right\} + \epsilon \quad ,$$

which together with Proposition 2.26, proves the result.

Corollary 2.31: Multiplicity exists and is the natural upper bound for any nontrivial filtration and finitely generated module over a principal ideal domain.

Proof: Except for the two trivial filtrations $f_{(0)}$ and f_A, every filtration is 0-dimensional and has altitude 1. Q.E.D.

In the case of a filtration for which the multiplicity exists and is the natural upper bound, several important theorems follow from the corresponding results for ideals using simply the fact that the limit of a sum is the sum of the limits. Theorems 2.32, 2.34, 2.35 and 2.36 are of this type with Theorems 2.35 and 2.36 being proven in more generality in that only existance of multiplicity is assumed. It will subsequently be shown that any filtration which is approximatable by powers, mentioned in Chapter I (1.6), does satisfy the condition that for itself and for each of its localizations multiplicity exists and is the natural upper bound. By Theorem 2.22 and properties of localization, it can already be seen that the hypotheses of these theorems are satisfied if the filtration involved possesses a regular subsequence of powers.

<u>Theorem 2.32</u> (<u>Additivity</u>): Let $f = \{\mathfrak{a}_n\}$ be a 0-dimensional filtration on a noetherian ring A. If f is such that $\mu(f,M)$ exists and is the natural upper bound for every finitely generated A-module M, then the real valued function $\mu(f,\_)$ is additive on the category of finitely generated A-modules. That is, if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{2.23}$$

is a short exact sequence of finitely generated A-modules, then

$$\mu(f,M) = \mu(f,M') + \mu(f,M'') \ .$$

<u>Proof</u>: By hypothesis $\mu(f,M) = \lim_{n\to\infty} \frac{1}{n^s} \mu(\mathfrak{a}_n,M)$. Since $\mu(\mathfrak{a}_n,\_)$ is additive [4, Theorem (23.3), P.76] applying it to (2.23) implies

$$\begin{aligned} \mu(f,M) &= \lim_{n\to\infty} \frac{1}{n^s} (\mu(\mathfrak{a}_n,M') + \mu(\mathfrak{a}_n,M'')) \\ &= \lim_{n\to\infty} \frac{1}{n^s} \mu(\mathfrak{a}_n,M') + \lim_{n\to\infty} \frac{1}{n^s} \mu(\mathfrak{a}_n,M'') \\ &= \mu(f,M') + \mu(f,M'') \ . \end{aligned}$$

Q.E.D.

To see that the multiplicity function for a 0-dimensional filtration f need not be additive even in case f is noetherian, consider the following.

Example 2.33: Let $A = \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, $f = \{\mathbb{Z}_4, 2\mathbb{Z}_4, 2\mathbb{Z}_4, \ldots\}$ and take the short exact sequence

$$(2.24) \qquad 0 \longrightarrow 2\mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_4 \xrightarrow{c} \mathbb{Z}_2 \longrightarrow 0$$

where i represents the injection of $2\mathbb{Z}_4$ into $\mathbb{Z}_4$ and c the canonical epimorphism $\mathbb{Z}_4 \longrightarrow \mathbb{Z}_4/2\mathbb{Z}_4 \approx \mathbb{Z}_2$. For this choice of filtration, $\text{alt}(f) = \dim(f) = 0$ and multiplicity trivially exists for every finitely generated $\mathbb{Z}_4$-module M ; explicitly,

$$\mu(f,M) = 0! \lim_{n\to\infty} \frac{L_A(M/2M)}{n^0} = L_A(M/2M) = L_{\mathbb{Z}}(M/2M) .$$

Then, with respect to the short exact sequence (2.24),

$$\mu(f,2\mathbb{Z}_4) = L_{\mathbb{Z}}(2\mathbb{Z}_4/2(2\mathbb{Z}_4)) = L_{\mathbb{Z}}(2\mathbb{Z}_4/(0)) = L_{\mathbb{Z}}(\mathbb{Z}_2) = 1,$$

$$\mu(f,\mathbb{Z}_4) = L_{\mathbb{Z}}(\mathbb{Z}_4/2\mathbb{Z}_4) = L_{\mathbb{Z}}(\mathbb{Z}_2) = 1 \text{ , and}$$

$$\mu(f,\mathbb{Z}_2) = L_{\mathbb{Z}}(\mathbb{Z}_2/2\mathbb{Z}_2) = L_{\mathbb{Z}}(\mathbb{Z}_2) = 1 .$$

It is now clear that

$$\mu(f,\mathbb{Z}_4) \neq \mu(f,2\mathbb{Z}_4) + \mu(f,\mathbb{Z}_2) .$$

Theorem 2.34: Let $f = \{\mathfrak{a}_n\}$ be a 0-dimensional filtration on a noetherian ring A with the property that for any finitely generated A-module M, $\mu_A(f,M)$ exists and is the natural upper bound. Then

$$\mu_A(f,M) = \sum_{\mathfrak{p}} L_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\mu_A(f,A/\mathfrak{p}) ,$$

where $\mathfrak{p}$ runs over all minimal primes of A such that $\dim(\mathfrak{p}) = \mathrm{alt}(f)$.

Proof: By hypothesis,

$$\mu_A(f,M) = \lim_{n\to\infty} \frac{1}{n^s} \mu(\mathfrak{a}_n, M) .$$

This implies

$$\mu_A(f,M) = \lim_{n\to\infty} \frac{1}{n^s} \sum_{\mathfrak{p}} L_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\, \mu_A(\mathfrak{a}_n, A/\mathfrak{p})$$

where $\mathfrak{p}$ runs over all minimal primes of A such that $\dim(\mathfrak{p}) = \mathrm{alt}(f)$ by [7,P.V-3]. Then

$$\mu_A(f,M) = \sum_{\mathfrak{p}} L_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \lim_{n\to\infty} \frac{1}{n^s} \mu_A(\mathfrak{a}_n, A/\mathfrak{p})$$

since there are only finitely many such $\mathfrak{p}$. Thus

$$\mu_A(f,M) = \sum_{\mathfrak{p}} L_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\, \mu_A(f, A/\mathfrak{p}) . \quad \text{Q.E.D.}$$

For a filtration f on a ring A and a prime ideal $\mathfrak{p}$ of A, let $f_{\mathfrak{p}}$ denote the extension of f in the local ring $A_{\mathfrak{p}}$.

Theorem 2.35 (Localization formula): Let f be a 0-dimensional filtration on a noetherian ring A and let M be a finitely generated A-module. If $\mu_{A_{\mathfrak{p}}}(f_{\mathfrak{p}}, M_{\mathfrak{p}})$ exists for each prime ideal $\mathfrak{p}$ which contains $\mathrm{rad}(f)$ and for which $\mathrm{alt}(\mathfrak{p}) = \mathrm{alt}(f)$, then $\mu_A(f,M)$ exists and

$$\mu_A(f,M) = \sum_{\mathfrak{p}} \mu_{A_{\mathfrak{p}}}(f_{\mathfrak{p}},M_{\mathfrak{p}}) \quad ,$$

where $\mathfrak{p}$ runs over all such primes.

Proof: This can be done directly using the localization formula for lengths [5, Theorem 12, P.166]. However by taking $A = A'$ in the following theorem, the direct approach is unnecessary. Q.E.D.

Theorem 2.36 (Extension Formula): Let f be a 0-dimensional filtration on a noetherian ring A which is a finite integral extension of a subring A' and let M be a finitely generated A-module. If $\mu_{A_{\mathfrak{p}}}(f_{\mathfrak{p}},M_{\mathfrak{p}})$ exists for each prime ideal $\mathfrak{p}$ which contains $\mathrm{rad}(f)$ and for which $\mathrm{alt}(\mathfrak{p}) = \mathrm{alt}(f)$, then $\mu_{A'}(f,M)$ exists and

$$\mu_{A'}(f,M) = \sum_{\mathfrak{p}} \mu_{A_{\mathfrak{p}}}(f_{\mathfrak{p}},M_{\mathfrak{p}}) \; [A/\mathfrak{p}:A'/\mathfrak{p}^c]$$

where $\mathfrak{p}$ runs over all such primes.

Note 2.37: The expression $\mu_{A'}(f,M)$ has not been defined since f is a filtration on A , not A' . The definition is, however, entirely analogous to Definition 2.10 only $L_A(M/\mathfrak{a}_n M)$ is replaced by $L_{A'}(M/\mathfrak{a}_n M)$ for each n .

Remark 2.38: The only reason finiteness of A over A' is assumed is to assure that each $[A/\mathfrak{p}:A'/\mathfrak{p}^c] < \infty$. If this condition is imposed separately the finiteness may be dropped and the same proof gives the result. In fact, by requiring the properties of Nagatas's [4,Theorem 21.2, P.70] on the rings A and A' and letting rad(f) be the zero dimensional ideal given there one can even drop the condition that A be integral over A' .

Proof: From the fact that $\mu_{A_p}(f_p,M_p)$ exists for each such p ,

$$\sum_p \mu_{A_p}(f_p,M_p)\ [A/p:A'/p^c] = \sum_p s!\lim_{n\to\infty}\frac{L_{A_p}\left(\frac{M_p}{(a_n)_pM_p}\right)}{n^s}[A/p:A'/p^c]$$

where $s = \mathrm{alt}(f)$. Since $L_{A_q}\left(\frac{M_q}{(a_n)_qM_q}\right) = 0$ for those maximal ideals which do not contain rad(f) and since

$$\lim_{n\to\infty}\frac{L_{A_q}\left(\frac{M_q}{(a_n)_qM_q}\right)}{n^s} = 0$$ for those which contain rad(f)

but for which $\mathrm{alt}(q) < s$, this equation may be rewritten as

$$\sum_p \mu_{A_p}(f_p,M_p)\ [A/p:A'/p^c] = \sum_q s!\lim_{n\to\infty}\frac{L_{A_q}\left(\frac{M_q}{(a_n)_qM_q}\right)[A/q:A'/q^c]}{n^s}$$

where q runs over all maximal ideals of A . All but finitely many $L_{A_q}\left(\frac{M_q}{(a_n)_qM_q}\right) = 0$ so the sum and the limit may be interchanged to obtain

$$\sum_p \mu_{A_p}(f_p,M_p)\ [A/p:A'/p^c] = s!\lim_{n\to\infty}\frac{\sum_q L_{A_q}\left(\frac{M_q}{(a_n)_qM_q}\right)[A/q:A'/q^c]}{n^s}\ .$$

By properties of localization $\frac{M_q}{(\alpha_n)_q M_q} \approx \left(\frac{M}{\alpha_n M}\right)_q$, and the extension formula for lengths [5, Theorem 13, P.168] may be applied. This implies

$$\sum_{\mathfrak{p}} \mu_{A_{\mathfrak{p}}}(f_{\mathfrak{p}}, M_{\mathfrak{p}}) \; [A/\mathfrak{p} : A'/\mathfrak{p}^c] \;=\; s! \lim_{n\to\infty} \frac{L_{A'}\left(\frac{M}{\alpha_n M}\right)}{n^s}$$

and the last expression is the definition of $\mu_{A'}(f,M)$.

Q.E.D.

Corollary 2.39: Multiplicity exists and is the natural upper bound for any nontrivial filtration and any finitely generated module over a Dedekind domain.

Proof: Every localization is a discrete valuation ring and consequently a principal ideal domain [1, Theorem 9.3, P.95]. By Corollary 2.31 and the localization formula the result is immediate. Q.E.D.

The remainder of this chapter will be devoted to the study of two equivalent forms of the condition "approximatable by powers" introduced in Chapter I. The first form considered appears to be the more natural generalization of the "essentially powers" situation and lends itself more easily to the proof of Proposition 2.45. The second is formulated in such a way that until one final condition is imposed the results are proved more generally and, if the filtration is given by a pseudo-valuation, it is somewhat easier to apply.

Suppose for the filtration $f = \{\mathfrak{a}_n\}$ the condition

(2.25) there exists $k \in \mathbb{N}$ such that $\mathfrak{a}_{kn} \subseteq (\mathrm{rad}(f))^n$ for all n

is satisfied. This is a strengthening of the "φ-condition" used in Theorem 2.20. However, an inspection of the proof of Theorem 2.20 shows that whenever f is essentially powers, condition (2.25) is satisfied. Thus on a noetherian ring, a filtration $f = \{\mathfrak{a}_n\}$ is essentially powers if and only if f is noetherian and condition (2.25) is satisfied. In the sequel the consequences of condition (2.25) will be investigated in the absence of the requirement that f be noetherian.

In the case that A is noetherian, every ideal contains a power of its radical and consequently it is easy to verify that if for the filtration $f = \{\mathfrak{a}_n\}$, condition (2.25) is satisfied, then for each $j \in \mathbb{N}$ there exists some $k_j \in \mathbb{N}$ such that

(2.26) $$\mathfrak{a}_{k_j n} \subseteq \mathfrak{a}_j^n \quad \text{for all } n \in \mathbb{N}.$$

Lemma 2.40: Let $f = \{\mathfrak{a}_n\}$ be a filtration with the property that for each j there exist positive integers k_j which satisfy (2.26) and let t_j be the least such integer. Then the sequence of ratios $\{t_j/j\}$ converges.

Proof: To show that the sequence is bounded, fix j and consider any m. Use the division algorithm to express $m = jq_m + r_m$, $0 \le r_m < j$, which with (2.26) implies

$$\alpha_{t_j(q_m+1)n} \subseteq \alpha_j^{(q_m+1)n} \subseteq (\alpha_{j(q_m+1)})^n \subseteq (\alpha_{jq_m+r_m})^n = \alpha_m^{\,n}$$

for all n. Since t_m is the least k_m for which $\alpha_{k_m n} \subseteq \alpha_m^{\,n}$ for all n, it follows that $t_m \le t_j(q_m + 1)$.

Then for $m \ge j$,

$$(2.27) \quad \frac{t_m}{m} \le \frac{t_j(q_m + 1)}{m} = \frac{t_j(q_m + 1)}{jq_m + r_m} = \frac{t_j}{j}\left(\frac{1 + 1/q_m}{1 + r_m/jq_m}\right) \le \frac{2t_j}{j}.$$

Thus $\{t_m/m\}$ is bounded by $\max_p\{t_p/p, 2t_j/j\}$ where $p = 1,\ldots,j-1$. Let $L = \lim\inf\{t_m/m\}$. For any $\epsilon > 0$ choose i such that $t_i/i < L + \epsilon$. From statement (2.27), one has

$$\frac{t_m}{m} \le \frac{t_i}{i}\left(\frac{1 + 1/q_m}{1 + r_m/iq_m}\right) \text{ for all } m \ge i .$$

As $m \longrightarrow \infty$, $\left(\frac{1 + 1/q_m}{1 + r_m/iq_m}\right) \longrightarrow 1$ which implies that $t_m/m < L + \epsilon$ for all sufficiently large m. Hence L is actually the limit of the sequence $\{t_m/m\}$. Q.E.D.

In the ensuing development the actual value of $\lim_{j\to\infty}(t_j/j)$ in Lemma 2.40 does not play the important role but rather whether or not this limit is less than or equal to one.

Definition 2.41: A filtration $f = \{\mathfrak{a}_n\}$ is approximatable by powers in case there exists a sequence of positive integers $\{k_j\}$ with $k_j \geq j$ for each j such that $\mathfrak{a}_{k_j n} \subseteq \mathfrak{a}_j^n$ for all n and $(k_j/j) \longrightarrow 1$.

Note 2.42: Let $f = \{\mathfrak{a}_n\}$ be as in Lemma 2.40. If for some j , t_j is less than or equal to j , then the containments

$$\mathfrak{a}_{jn} \subseteq \mathfrak{a}_{t_j n} \subseteq \mathfrak{a}_j^n \subseteq \mathfrak{a}_{jn}$$

imply that f has a regular subsequence of powers $f^{(j)}$. The following remark then implies f is approximatable by powers.

Remark 2.43: If $f = \{\mathfrak{a}_n\}$ has a regular subsequence of powers $f^{(k)}$, then it is approximatable by powers. In particular, if f is essentially powers on a noetherian ring, by Corollary 2.21 and this remark, f is approximatable by powers. To prove Remark 2.43 use the division algorithm to express each integer j as $j = kq_j + r_j$, $0 \leq r_j < k$. Then for all n ,

$$\mathfrak{a}_{k(q_j+1)n} = \mathfrak{a}_k^{(q_j+1)n} = (\mathfrak{a}_{k(q_j+1)})^n \subseteq \mathfrak{a}_j^n .$$

The sequence obtained by taking $k_j = k(q_j + 1)$ provides the conclusion.

Theorem 2.44: Let $f = \{\mathfrak{a}_n\}$ be a 0-dimensional filtration on a noetherian ring A and let M be a finitely generated A-module. If f is approximatable by powers, then $\mu(f,M)$ exists and is the natural upper bound.

Proof: Let $s = \mathrm{alt}(f)$. By Proposition 2.26 it suffices to show

$$\lim_{n\to\infty} \frac{1}{n^s}\,\mu(\mathfrak{a}_n,M) \le s!\,\liminf \left\{\frac{H(n,M,f)}{n^s}\right\}.$$

As usual, the argument depends on the division algorithm. For each n, let k_n be given by Definition 2.41 and one has $k_n q_m \le m = k_n q_m + r_m$, $0 \le r_m < k_n$ for each m. From $\mathfrak{a}_{k_n q_m} \subseteq \mathfrak{a}_n^{q_m}$, it follows that

$$L_A\left(\frac{M}{\mathfrak{a}_n^{q_m}M}\right) \le L_A\left(\frac{M}{\mathfrak{a}_{k_n q_m}M}\right) \le L_A\left(\frac{M}{\mathfrak{a}_m M}\right).$$

Multiplying by $\frac{s!}{m^s}$ and changing to Hilbert functions implies that

$$\left(\frac{q_m}{m}\right)^s \frac{s!\,H(q_m,M,\mathfrak{a}_n)}{(q_m)^s} \le \frac{s!\,H(m,M,f)}{m^s}.$$

By passing to the limit with respect to m, one obtains

$$\frac{1}{k_n^s}\,\mu(\mathfrak{a}_n,M) \le s!\,\liminf\left\{\frac{H(m,M,f)}{m^s}\right\} \quad \text{for each } n.$$

Since $\frac{1}{k_n^s}\,\mu(\mathfrak{a}_n,M) = \left(\frac{n}{k_n}\right)^s \frac{1}{n^s}\,\mu(\mathfrak{a}_n,M)$ and $\frac{n}{k_n} \longrightarrow 1$, this implies

$$\lim_{n\to\infty} \frac{1}{n^s}\, \mu(\mathfrak{a}_n, M) \le s!\ \liminf \left\{\frac{H(m,M,f)}{m^s}\right\}. \quad \text{Q.E.D.}$$

The next proposition summarizes some of the preservation properties of the condition "approximatable by powers".

<u>Proposition 2.45</u>: Let $f = \{\mathfrak{a}_n\}$, $g = \{\mathfrak{b}_n\}$ be two filtrations on a commutative ring A which are approximatable by powers and B a commutative ring extension of A . Then

(i) $f + g$,

(ii) fg , and

(iii) f^e are approximatable by powers .

<u>Proof</u>: Let $\{k'_j\}$ be given by the fact that f is approximatable by powers and similarly $\{k''_j\}$ for g .

(i) For each j , let $k_j = \max\{k'_j, k''_j\}$. Then $k_j \ge j$ for every j and $(k_j/j) \longrightarrow 1$. Letting $\mathfrak{c}_i$ denote the i^{th} ideal of $f + g$ it is claimed that $\mathfrak{c}_{k_j n} \subseteq \mathfrak{c}_j^{\,n-2}$. By definition of $f + g$, $\mathfrak{c}_{k_j n}$ is generated by all products of the form $\mathfrak{a}_s\mathfrak{b}_t$ where $s + t = k_j n$ so it suffices to show that each of these ideal products are in $\mathfrak{c}_j^{\,n-2}$. The division algorithm gives the equations $s = k_j q_s + r_s$, $0 \le r_s < k_j$ and $t = k_j q_t + r_t$, $0 \le r_t < k_j$. Then

$$\mathfrak{a}_s\mathfrak{b}_t = \mathfrak{a}_{k_j q_s + r_s}\mathfrak{b}_{k_j q_t + r_t} \subseteq \mathfrak{a}_{k_j q_s}\mathfrak{b}_{k_j q_t} \subseteq \mathfrak{a}_j^{\,q_s}\mathfrak{b}_j^{\,q_t} .$$

The ideals $\mathfrak{a}_j$ and $\mathfrak{b}_j$ are in $\mathfrak{c}_j$ so this containment chain may be extended giving

$$\mathfrak{a}_s \mathfrak{b}_t \subseteq \mathfrak{c}_j^{\,q_s+q_t} .$$

The expressions for s and t above give $k_j n = s + t \le k_j(q_s + 1) + k_j(q_t + 1)$ from which it follows that $n - 2 \le q_s + q_t$. Thus

$$\mathfrak{c}_{k_j n} \subseteq \mathfrak{c}_j^{\,n-2} \quad \text{for all } n .$$

This condition and the fact that $(k_j/j) \longrightarrow 1$ imply $f + g$ is approximatable by the following argument. Let $\{t_j\}$ be the sequence of smallest integers m such that

$$\mathfrak{c}_{mn} \subseteq \mathfrak{c}_j^{\,n} \quad \text{for all } n .$$

This sequence exists because for $m = 3k_j$, the statement is true; explicitly,

$$\mathfrak{c}_{3k_j n} \subseteq \mathfrak{c}_j^{\,3n-2} \subseteq \mathfrak{c}_j^{\,n} \quad \text{for all } n .$$

From the fact that

$$\mathfrak{c}_{k_j mn} \subseteq \mathfrak{c}_j^{\,mn-2} \subseteq \mathfrak{c}_j^{\,mn-2n} \subseteq (\mathfrak{c}_{j(m-2)})^n ,$$

it is seen that $t_{j(m-2)} \le k_j m$.

Then $\frac{t_{j(m-2)}}{j(m-2)} \le \frac{k_j m}{j(m-2)}$ for all j and $m > 2$.

Since $\frac{k_j}{j} \longrightarrow 1$, $\lim_{j\to\infty} \frac{t_j}{j} \le \frac{m}{m-2}$ for all $m > 2$.

Therefore $\{t_j/j\}$ converges to some limit L with $0 \le L \le 1$. If $L = 1$, the proof is complete. If $L < 1$, replace t_j by j whenever $t_j < j$. The resulting sequence then has the desired properties.

(ii) Using $k_j = \max\{k'_j, k''_j\}$ for each j, one obtains the chain

$$\mathfrak{a}_{k_j n}\mathfrak{b}_{k_j n} \subseteq \mathfrak{a}_{k'_j n}\mathfrak{b}_{k''_j n} \subseteq \mathfrak{a}_j^{\,n}\mathfrak{b}_j^{\,n} = (\mathfrak{a}_j\mathfrak{b}_j)^n$$

which implies fg is approximatable by powers.

(iii) Let $\varphi : A \longrightarrow B$ be the ring homomorphism which makes B an extension of A. The extended filtration f^e is approximatable by powers with the same sequence $\{k'_i\}$ as for f since

$$(\mathfrak{a}_{k'_j n})^e = \varphi[\mathfrak{a}_{k'_j n}]B \subseteq \varphi[\mathfrak{a}_j^{\,n}]B = (\varphi[\mathfrak{a}_j]B)^n = (\mathfrak{a}_j^{\,e})^n .$$

Q.E.D.

<u>Remark 2.46</u>: The property that a filtration be approximatable by powers need not be preserved under ring contrations and as yet the answer is unknown for the intersection of two such filtrations. The first statement is manifested in the fact that any filtration is the contraction of a filtration given by the powers of a fixed ideal in some ring extension. See Proposition 1.3.

<u>Theorem 2.47</u>: If f is a 0-dimensional filtration on a noetherian ring A which is approximatable by powers, the function $\mu(f,\_)$ is an additive function on the category of finitely generated A-modules and satisfies the localization formula. If in addition, A is a finite

integral extension of the subring A', $\mu_{A'}(f,\_)$ satisfies the extension formula as well.

Proof: Since localization is a form of ring extension, Theorem 2.44 and Proposition 2.45 imply that the hypotheses of Theorems 2.32 and 2.34-2.36 are satisfied. Q.E.D.

The second form of the "approximatable by powers" condition will be stated in terms of the product of a filtration with a real number. In preparation for this definition, recall that for a pseudo-valuation v on a ring A and a real number $\lambda \geq 0$ the function λv defined by

$$\lambda v(\alpha) = \lambda(v(\alpha)) \quad \text{for all } \alpha \in A \tag{2.28}$$

is a pseudo-valuation on A.

Definition 2.48: For a filtration $f = \{\mathfrak{a}_n\}$ on a commutative ring A and a positive real number λ, define λf via

$$\lambda f = f_{\lambda v_f} .$$

See (1.23) and (1.24). That is, for each n,

$$\mathfrak{a}_{\lambda f,n} = \{\alpha \in A \mid \lambda v_f(\alpha) \geq n\} \quad \text{where} \quad v_f(\alpha) = \sup\{m \mid \alpha \in \mathfrak{a}_m\} .$$

Proposition 2.49: Let $f = \{\mathfrak{a}_n\}$ be a filtration on a noetherian ring A, let M be an A-module, and let

λ be a positive real number. Then $\mu(f,M)$ exists if and only if $\mu(\lambda f,M)$ exists in which case

$$\mu(\lambda f,M) = \frac{1}{\lambda^s}\mu(f,M) \quad \text{where} \quad s = \text{alt}(f) .$$

<u>Proof</u>: Since the two multiplicities refer to different filtrations it must be verified that $\text{alt}(\lambda f) = \text{alt}(f)$. Although this is not difficult to show directly, it follows from Lemma 2.53 that $\text{rad}(\lambda f) = \text{rad}(f)$, thus $\text{alt}(\lambda f) = \text{alt}(f)$.

By considering rational sequences converging to λ from above and below and the continuity of $\frac{1}{\lambda^s}\mu(f,M)$ as a function of λ , it suffices to show Proposition 2.49 for λ rational, say $\lambda = a/b$, $a,b \in \mathbb{N}$. First note that for any filtration $g = \{\mathfrak{b}_n\}$ both the existence and the value of $\mu(g,M)$ can be found from any regular subsequence $g^{(k)}$. To prove this fact, use the division algorithm to express $n = kq_n + r_n$, $0 \le r_n < k$. Then $n - k < kq_n \le n < k(q_n + 1)$ and

$$L_A(M/\mathfrak{b}_{n-k}M) \le L_A(M/\mathfrak{b}_{kq_n}M) \le L_A(M/\mathfrak{b}_nM) \le L_A(M/\mathfrak{b}_{k(q_n+1)}M) .$$

By multiplying by $\frac{s!}{n^s}$ and changing to Hilbert functions this becomes

$$\left(\frac{n-k}{n}\right)^s \frac{s!\ H(n-k,M,g)}{(n-k)^s} \le \left(\frac{q_n}{n}\right)^s \frac{s!\ H(q_n,M,g^{(k)})}{(q_n)^s}$$

$$\le \frac{s!\ H(n,M,g)}{n^s} \le \left(\frac{q_n+1}{n}\right)^s \frac{s!\ H(q_n+1,M,g^{(k)})}{(q_n+1)^s}$$

and passing to the limit with respect to n, existance of either limit implies existance of the other in which case

$$(2.29) \qquad \frac{1}{k^s}\,\mu(g^{(k)},M) = \mu(g,M) \quad .$$

Since $\mathfrak{a}_{\frac{a}{b}f,na} = \{\alpha \mid \frac{a}{b}v_f(\alpha) \geq na\} = \{\alpha \mid v_f(\alpha) \geq nb\} = \mathfrak{a}_{nb}$, it follows that $\frac{a}{b}f^{(a)} = f^{(b)}$ and therefore

$$\mu(\tfrac{a}{b}f^{(a)},M) = \mu(f^{(b)},M) \quad .$$

From (2.29) one obtains

$$a^s\mu(\tfrac{a}{b}f,M) = \mu(\tfrac{a}{b}f^{(a)},M) = \mu(f^{(b)},M) = b^s\mu(f,M) \ .$$

Q.E.D.

<u>Theorem 2.50</u>: Let $f = \{\mathfrak{a}_n\}$ be a 0-dimensional filtration on a noetherian ring A and let M be an A-module. If there exists a sequence $\{f_n\}$ of filtrations on A and a sequence $\{\lambda_n\}$ of positive real numbers satisfying

(i) $f_n \leq f \leq \lambda_n f_n$ for each n,
(ii) $\mu(f_n,M)$ exists for each n,
(iii) the sequence $\{\mu(f_n,M)\}$ converges to L, and
(iv) the sequence $\{\lambda_n\}$ converges to 1,

then $\mu(f,M)$ exists and $\mu(f,M) = L$.

<u>Proof</u>: Let $s = \operatorname{alt}(f_n)$ for some n. Condition (i) implies $\operatorname{alt}(f) = \operatorname{alt}(f_m) = \operatorname{alt}(\lambda_m f_m) = s$ for all m.

From the ordering of the filtrations,

$$\mu(\lambda_n f_n, M) \leq s!\ \lim\inf \left\{ \frac{L_A\left(\frac{M}{\mathfrak{a}_m M}\right)}{m^s} \right\}$$

$$\leq s!\ \lim\sup \left\{ \frac{L_A\left(\frac{M}{\mathfrak{a}_m M}\right)}{m^s} \right\} \leq \mu(f_n, M) \quad .$$

Proposition 2.49 implies $\mu(\lambda_n f_n, M) = \frac{1}{\lambda_n^s}\mu(f_n, M)$ for each n . Since by hypothesis $1/\lambda_n^s \longrightarrow 1$, the result follows by taking limits with respect to n . Q.E.D.

It is straight forward to show that condition (i) of Theorem 2.50 on a filtration $f = \{\mathfrak{a}_n\}$ is preserved under all ring extensions, since $(\lambda_n f_n)^e \leq \lambda_n (f_n^e)$, so criteria may be obtained from this theorem to establish theorems about additivity, localization and extension as before. The difficulty in applying Theorem 2.50 is that one requires considerable knowledge about the approximating filtrations f_n . Conditions (ii) and (iii) of Theorem 2.50 are automatically satisfied if it is required that each f_n be essentially powers (Corollary 2.23) and that the sequence $\{f_n\}$ be monotone increasing. However, the following characterization of filtrations which are approximatable by powers shows that under this added hypothesis nothing new is being considered.

<u>Theorem 2.51</u>: Let $f = \{\mathfrak{a}_m\}$ be a filtration on a noetherian ring A . There exists a sequence of filtrations

$\{f_n\}$ each of which is essentially powers and a sequence of positive real numbers $\{\lambda_n\}$ which converges to 1 such that for each n

$$f_n \le f \le \lambda_n f_n ,$$

if and only if f is approximatable by powers.

Remark 2.52: The proof of Theorem 2.51 will imply that each of the filtrations f_n may be chosen to be the least filtration such that the first N_n ideals of f_n agree with those of f for some $N_n \in \mathbb{N}$. Furthermore the N_n may be taken to be monotone increasing so that the resulting f_n are monotone increasing as well.

The following computational lemma is derived in order to facilitate the proof of Theorem 2.51. As usual, $\{r\}$ denotes the least integer greater than or equal to the real number r .

Lemma 2.53: Let $f = \{\mathfrak{a}_n\}$ be a filtration on a ring A and let λ be a positive real number. Then

(i) $\frac{1}{\lambda}(\lambda f) \le f$, and

(ii) $f^{(\{\lambda k\})} \le \left(\frac{1}{\lambda}f\right)^{(k)}$ for any $k \in \mathbb{N}$.

Proof of (i): For any n , $\alpha \in \mathfrak{a}_{\frac{1}{\lambda}(\lambda f),n}$ implies $\frac{1}{\lambda}v_{\lambda f}(\alpha) \ge n \Rightarrow v_{\lambda f}(\alpha) \ge \lambda n \Rightarrow v_{\lambda f}(\alpha) \ge \{\lambda n\}$ since $v_{\lambda f}$ is integral valued. Thus $\alpha \in \mathfrak{a}_{\lambda f,\{\lambda n\}}$ which in turn implies $\lambda v_f(\alpha) \geq \{\lambda n\} \Rightarrow v_f(\alpha) \geq \frac{\{\lambda n\}}{\lambda}$

$$\Rightarrow v_f(\alpha) \geq \left\{\frac{\{\lambda n\}}{\lambda}\right\} \geq \left\{\frac{\lambda n}{\lambda}\right\} = n \Rightarrow \alpha \in \mathfrak{a}_n .$$

<u>Proof of (ii)</u>: For any n , $\alpha \in \mathfrak{a}_{f(\{\lambda k\}),n} = \mathfrak{a}_{\{\lambda k\}n}$ implies $v_f(\alpha) \geq \{\lambda k\}n \geq \lambda kn \Rightarrow \frac{1}{\lambda}v_f(\alpha) \geq kn$

$$\Rightarrow \alpha \in \mathfrak{a}_{\frac{1}{\lambda}f,kn} = \mathfrak{a}_{\left(\frac{1}{\lambda}f\right)(k),n} .$$

<u>Proof of Theorem 2.51</u>: For each n , f_n is essentially powers on a noetherian ring so by Corollary 2.21 there exists $k_n \in \mathbb{N}$ such that $\mathfrak{a}_{f_n,k_nm} = (\mathfrak{a}_{f_n,k_n})^m$ for all m . From Lemma 2.53 and the fact that $f \leq \lambda_n f_n$ it follows that for all m ,

$$\mathfrak{a}_{\{\lambda_n k_n\}m} \subseteq \mathfrak{a}_{\lambda_n f_n,\{\lambda_n k_n\}m} \subseteq \mathfrak{a}_{\frac{1}{\lambda_n}(\lambda_n f_n),k_nm}$$

(2.30)

$$\subseteq \mathfrak{a}_{f_n,k_nm} = (\mathfrak{a}_{f_n,k_n})^m \subseteq (\mathfrak{a}_{k_n})^m .$$

Since $\mathfrak{a}_{k_n} \subseteq \operatorname{rad}(f)$, the condition (2.25) and hence condition (2.26) are satisfied for the filtration f . Let t_j denote the smallest positive integer for which $\mathfrak{a}_{t_jm} \subseteq \mathfrak{a}_j{}^m$ for all m . By Lemma 2.40, the sequence $\{t_j/j\}$ converges to some limit L . If $L < 1$, Note 2.42 together with Remark 2.43 imply f is approximatable by powers so it suffices to show $L \leq 1$. By the choice of k_n above $a_{f_n, k_n jm} = (a_{f_n, k_n j})^m$ so that (2.30) remains valid when k_n is replaced by $k_n j$ for any j. this implies

$$a_{\{\lambda_n k_n j\}m} \subseteq (a_{k_n j})^m \quad \text{for} \quad j,m \in \mathbb{N} .$$

By definition of $t_{k_n j}$, $t_{k_n j} \leq \{\lambda_n k_n j\}$. Thus

$$L = \lim_{j\to\infty} \frac{t_j}{j} = \lim_{j\to\infty} \frac{t_{k_n j}}{k_n j} \leq \lim_{j\to\infty} \frac{\{\lambda_n k_n j\}}{k_n j} = \lambda_n .$$

Since $\lambda_n \longrightarrow 1$ and $L \leq \lambda_n$ for every n, f is approximatable by powers.

Conversely, let $f = \{a_n\}$ be such that for each j there exists $k_j \geq j$ such that $a_{k_j n} \subseteq a_j^{\,n}$ for all n and $\{k_j/j\}$ converges to 1. For each j, let $N_j \in \mathbb{N}$ be defined inductively by $N_1 = k_1^{\,2}$ and $N_j = \max\{N_{j-1}, k_j^{\,2}\}$ for $j > 1$. Define $f_j = \{a_{f_j, n}\}$ by

$$a_{f_j, n} = \sum_{i=1}^{N_j} a_{n-i} a_i .$$

Letting $\lambda_j = \frac{k_j + 1}{j}$ for each j, it is claimed that

(2.31) $\qquad f_j \leq f \leq \lambda_j f_j$ for each j.

By construction $f_j \leq f$ for each j. To show $f \leq \lambda_j f_j$, it is required to show that $a_n \subseteq a_{\lambda_j f_j, n}$ for each j and all n. Since each $\lambda_j > 1$, clearly $a_{f_j, n} \subseteq a_{\lambda_j f_j, n}$. Hence for $n \leq N_j$, $a_n = a_{f_j, n} \subseteq a_{\lambda_j f_j, n}$.

For $n > N_j$, the division algorithm implies $n = k_jq_n+r_n$, $0 \le r_n < k_j$, and therefore

$$\mathfrak{a}_n = \mathfrak{a}_{k_jq_n+r_n} \subseteq \mathfrak{a}_{k_jq_n} \subseteq \mathfrak{a}_j^{q_n} .$$

Since $j \le k_j \le k_j^2 \le N_j$, it is immediate that $\mathfrak{a}_j = \mathfrak{a}_{f_j,j}$ so that $\mathfrak{a}_j^m \subseteq \mathfrak{a}_{f_j,jm}$ for all m . Then this chain of containments can extend to imply

$$\mathfrak{a}_n \subseteq \mathfrak{a}_{f_j,jq_n} .$$

Thus $\alpha \in \mathfrak{a}_n$ implies $v_{f_j}(\alpha) \ge jq_n$. Now multiply by $\lambda_j = \frac{k_j + 1}{j}$ to obtain

$$\lambda_j v_{f_j}(\alpha) \ge (k_j + 1)q_n = k_jq_n + q_n .$$

By choice of N_j , $k_j^2 \le N_j < n$ so that $k_j < \frac{n}{k_j}$. But q_n is the greatest integer less than or equal to n/k_j and therefore $k_j \le q_n$. One now has

$$\lambda_j v_{f_j}(\alpha) \ge k_jq_n + k_j > k_jq_n + r_n = n .$$

Hence $\alpha \in \mathfrak{a}_n$ implies $\alpha \in \mathfrak{a}_{\lambda_j f_j,n}$ for every n so by definition $f \le \lambda_j f_j$. Statement (2.31) has now been verified and the fact that $\lambda_j \longrightarrow 1$ is trivial. Q.E.D.

As an example of the results just considered, return to Example 2.25, the filtration f_τ on $k[X] = k[X_1,\dots,X_n]$ with $\tau = (\tau_j)$, $j = 1,\dots,n$ and each $\tau_j > 0$. It was shown at that time that if each τ_j is rational, f_τ is essentially powers. If some τ_j are not rational, choose a sequence of vectors $\tau^m = (\tau_j^m)$ with τ_j^m rational and such that $0 \le \tau_j - \tau_j^m \le \frac{1}{m}$. Let N_m be given by the fact that each f_{τ^m} is essentially powers and Definition 2.14. Nothing is lost by requiring $N_m < N_{m+1}$ so that $N_m \to \infty$ as $m \to \infty$. Let f_m be the least filtration which agrees with f_τ for α_i, $i = 1,\dots,N_m$. It is claimed that $f_m \le f_\tau \le \lambda_m f_m$ where $\lambda_m = \left(\frac{N_m+1}{N_m}\right)\lambda_m'$ and $\lambda_m' = \max\{\tau_j/\tau_j^m\}$. The second inequality is the only one for which there is any difficulty so it is sufficient to show that $[v_\tau] \le \lambda_m v_{f_m}$. First note that $\lambda_m' v_{\tau^m} \ge v_\tau$ since

$$\lambda_m' v_{\tau^m}\left(\sum \alpha_{(i)} X^{(i)}\right) = \min\left\{\sum i_j \lambda_m' \tau_j^m \;\middle|\; \alpha_{(i)} \ne 0\right\}$$
$$\ge \min\left\{\sum i_j \tau_j \;\middle|\; \alpha_{(i)} \ne 0\right\} = v_\tau\left(\sum \alpha_{(i)} X^{(i)}\right).$$

Let $\alpha \in A$. If $v_{\tau^m}(\alpha) \le N_m$, $[v_\tau](\alpha) > v_{f_m}(\alpha)$ only if

$$N_m \le v_{f_m}(\alpha) < v_\tau(\alpha) \le \lambda_m' v_{\tau^m}(\alpha) \le \lambda_m' N_m \le \lambda_m' v_{f_m}(\alpha) \le \lambda_m v_{f_m}(\alpha).$$

For $v_{\tau^m}(\alpha) > N_m$, $\frac{N_m+1}{N_m} \ge \frac{v_{\tau^m}(\alpha)}{[v_{\tau^m}](\alpha)}$ which implies that

$$\lambda_m v_{f_m}(\alpha) \ge \lambda_m'\left(\frac{N_m+1}{N_m}\right)[v_{\tau^m}](\alpha) \ge \lambda_m' v_{\tau^m}(\alpha) \ge v_\tau(\alpha).$$

Since $\lambda_m \to 1$, f_τ is approximatable by powers. Q.E.D.

In Chapter III it will be required to know more about $\mu(f_\tau, M)$ than can be found just from the fact that

f_τ is approximatable by powers. The next proposition provides this information as well as giving an alternate proof of the existance of $\mu(f_\tau, M)$.

<u>Proposition 2.54</u>: For $f_\tau = \{\mathfrak{a}_n\}$ on $A = k[X_1,\ldots,X_n]$ with $\tau = (\tau_i)$, $\tau_i > 0$, $i = 1,\ldots,n$ and any finitely generated A-module M ,

$$\mu(f_\tau, M) = \frac{1}{\tau_1\cdots\tau_n}\mu((X),M) ,$$

where $(X) = (X_1,\ldots,X_n)$, the maximal ideal generated by the X_i .

<u>Proof</u>: First observe that the restriction $\tau_i > 0$ for all i implies some power of each X_i is in $\mathfrak{a}_1$ so $\text{rad}(f_\tau) = (X)$ and $\text{alt}(f_\tau) = n$ [5, Theorem 3, P.281]. By continuity of $\frac{1}{\tau_1\cdots\tau_n}\mu((X),M)$ as a function of $\tau = (\tau_1,\ldots,\tau_n)$, it suffices to restrict the proof to the case where all τ_i are rational numbers. Assume that $\tau_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathbb{N}$. The method of proof is to approximate the filtration f_τ by use of the powers of the ideal $\mathfrak{a} = (X_1^{c_1},\ldots,X_n^{c_n})$ where $c_j = a_1\cdots a_{j-1}b_j a_{j+1}\cdots a_n$. Note that $\mathfrak{a} \subseteq \mathfrak{a}_a$, where $a = \prod a_i$, and therefore $\mathfrak{a}^m \subseteq \mathfrak{a}_{am}$ for all m . For any m , let $m = aq_m + r_m$, $0 \le r_m < a$. Then

$$\mathfrak{a}^{q_m+1} \subseteq \mathfrak{a}_{a(q_m+1)} \subseteq \mathfrak{a}_m$$

which implies

$$n! \lim_{m\to\infty} \frac{L_A(M/\alpha_m M)}{m^n} = n! \lim_{m\to\infty} \frac{L_A(M/\alpha^{q_m+1} M) - L_A(\alpha_m M/\alpha^{q_m+1} M)}{m^n}$$

(2.32)

$$= n! \lim_{m\to\infty} \frac{L_A(M/\alpha^{q_m+1} M)}{m^n} - n! \lim_{m\to\infty} \frac{L_A(\alpha_m M/\alpha^{q_m+1} M)}{m^n} ,$$

provided both limits exist. These limits will now be computed.

$$n! \lim_{m\to\infty} \frac{L_A(M/\alpha^{q_m+1} M)}{m^n} = n! \lim_{m\to\infty} \left(\frac{q_m+1}{m}\right)^n \frac{L_A(M/\alpha^{q_m+1} M)}{(q_m+1)^n}$$

$$= \frac{1}{a^n} \mu(\alpha, M)$$

$$= \frac{1}{a^n} \mu((X_1^{c_1}, \ldots, X_n^{c_n}), M) .$$

In this situation μ is the same as Northcott's e_R so by [5, Corollary 1 P.311],

$$\frac{1}{a^n} \mu((X_1^{c_1}, \ldots, X_n^{c_n}), M) = \frac{c_1 \cdots c_n}{a^n} \mu((X_1, \ldots, X_n), M)$$

(2.33)

$$= \frac{c_1 \cdots c_n}{(c_1 a_1/b_1) \cdots (c_n a_n/b_n)} \mu((X), M)$$

$$= \frac{1}{\tau_1 \cdots \tau_n} \mu((X), M) .$$

It remains to be shown that $n! \lim_{m\to\infty} \frac{L_A(\alpha_m M/\alpha^{q_m+1} M)}{m^n} = 0.$

By equation (2.20) f_τ was shown to be essentially powers with $\alpha_m = \sum_{i=1}^{na} \alpha_{m-i} \alpha_i$. From the argument given there, more information can be derived; namely,

$$\alpha_m = \sum_{i=1}^{na} \alpha^{m_i} \alpha_i \quad , \tag{2.34}$$

where m_i is the least integer such that $am_i + i \geq m$.

This fact follows from noting that in statement (2.21), α_a may be replaced by α . Since $am_i + i \geq m$ and $i \leq na$ one obtains $m_i \geq \frac{m - na}{a}$ and the fact that m_i is an integer implies

$$m_i \geq \left\{\frac{m - na}{a}\right\} \text{ for each } m_i \text{ of (2.34),}$$

where $\{r\}$ denotes the least integer greater than or equal to r . Hence the right side of equation (2.34) can be factored as

$$\alpha_m = \alpha^{\{\frac{m-na}{a}\}} \sum_{i=1}^{na} \alpha^{m_i - \{\frac{m-na}{a}\}} \alpha_i \quad \text{for all } m \geq na \text{ ,}$$

leading to the approximation

$$\alpha_m \subseteq \alpha^{\{\frac{m-na}{a}\}} \quad \text{for all } m \geq na \text{ .}$$

From $m = aq_m + r_m$, subtraction of an and division by a shows that

$$\left\{\frac{m - na}{a}\right\} \geq \frac{m - na}{a} = q_m - n + \frac{r_m}{a} \geq q_m - n$$

which implies the even rougher approximation

$$\alpha_m \subseteq \alpha^{q_m - n} \quad .$$

Applying this to lengths one has

$$n!\ \limsup \left\{ \frac{L_A\left(\dfrac{\alpha_m M}{\alpha^{q_m+1} M}\right)}{m^n} \right\} \le n!\ \lim_{m\to\infty} \frac{L_A\left(\dfrac{\alpha^{q_m-n} M}{\alpha^{q_m+1} M}\right)}{m^n}$$

$$= n!\ \lim_{m\to\infty} \sum_{i=0}^{n} \frac{L_A\left(\dfrac{\alpha^{q_m-n+i} M}{\alpha^{q_m-n+i+1} M}\right)}{m^n}$$

$$= n!\ \lim_{m\to\infty} \sum_{i=0}^{n} \frac{H(q_m-n+i+1,M,\alpha) - H(q_m-n+i,M,\alpha)}{m^n}$$

$$= \sum_{i=0}^{n} \left(n!\ \lim_{m\to\infty} \left(\frac{q_m-n+i+1}{m}\right)^n \frac{H(q_m-n+i+1,M,\alpha)}{(q_m-n+i+1)^n} \right.$$

$$\left. - n!\ \lim_{m\to\infty} \left(\frac{q_m-n+i}{m}\right)^n \frac{H(q_m-n+i,M,\alpha)}{(q_m-n+i)^n} \right)$$

$$= \sum_{i=0}^{n} \left(\frac{1}{a^n}\ \mu(\alpha,M) - \frac{1}{a^n}\ \mu(\alpha,M) \right)$$

$$= 0 \quad .$$

Thus $n!\ \lim_{m\to\infty} \dfrac{L_A(\alpha_m M/\alpha^{q_m+1} M)}{m^n}$ exists and is 0 .

From (2.32) and (2.33) the proof is complete. Q.E.D.

III FILTERED MODULES OVER FILTERED RINGS

Let M be a module over A, a commutative ring with identity.

Definition 3.1: A filtration $g = \{M_n\}_{n=0}^{\infty}$ on M is a sequence of A-submodules of M which satisfies

(i) $M_o = M$

(ii) $M_{n+1} \subseteq M_n$ for all n.

If the filtration $g = \{M_n\}$ on M has the property that each M/M_n has finite A-length, a Hilbert function for g may be defined as

(3.1) $$H(n,g) = H_A(n,g) = L_A(M/M_n) \text{ for all } n.$$

Definition 3.2: Let $g = \{M_n\}$ be a filtration on M, an A-module, with $L_A(M/M_n) < \infty$ for each n, and s a natural number. The multiplicity of g with respect to s is

$$\mu(s,g) = s! \lim_{n\to\infty} \frac{H(n,g)}{n^s}$$

whenever this limit exists.

Example 3.3: For a simple example showing that this limit need not always exist even if the sequence of ratios $\left\{\frac{H(n,g)}{n^s}\right\}$ is bounded, consider the following filtration on $\mathbb{Z}$ over $\mathbb{Z}$. Let $g = \{a_n\}$ with $a_n = (2^{2^{k_n}})$ where

$n = 2^{k_n} + r_n$ with $0 \le r_n < 2^{k_n}$ for $n \ge 1$. That is,

$$g = \{\mathbb{Z},(2),(2^2),(2^2),(2^4),(2^4),(2^4),(2^4),(2^8),\ldots\} .$$

Then the sequence of ratios $\left\{\frac{H(n,g)}{n^1}\right\}$ is bounded but contains subsequences converging to different limits. For instance, the subsequence of terms determined by $n_i = 2^i$ has value 1 for each i but the subsequence of terms determined by $n_i = 2^i - 1$ has value $\frac{2^{i-1}}{2^i - 1}$ for each i and therefore converges to $\frac{1}{2}$.

Definition 3.4: Let $f = \{\mathfrak{a}_n\}$ be a (multiplicative) filtration on a ring A and $g = \{M_n\}$ a filtration on an A-module M. Then g is called an f-filtration in case

$$\mathfrak{a}_m M_n \subseteq M_{m+n} \quad \text{for all } m,n \in \mathbb{N} .$$

The filtration g is a stable f-filtration if it is an f-filtration and there exists an N such that

$$M_n = \sum_{i=0}^{N} \mathfrak{a}_{n-i} M_i \quad \text{for all } n ,$$

where $\mathfrak{a}_j = A$ if $j < 0$.

Note 3.5: For any filtration $f = \{\mathfrak{a}_n\}$ on a ring A and A-module M, the sequence $g = \{M_n\}$ with $M_n = \mathfrak{a}_n M$ forms a stable f-filtration on M since

$$\mathfrak{a}_m M_n = \mathfrak{a}_m(\mathfrak{a}_n M) = (\mathfrak{a}_m \mathfrak{a}_n)M \subseteq \mathfrak{a}_{m+n} M = M_{m+n} \quad \text{and}$$

$$M_n = \sum_{i=0}^{0} \mathfrak{a}_{n-i} M_i = \mathfrak{a}_n M_o = \mathfrak{a}_n M .$$

This was the situation encountered throughout Chapter II.

Remark 3.6: In case f is the powers of a fixed ideal, say $f = \{\mathfrak{a}^n\}$, the conditions for a filtration $g = \{M_n\}$ on an A-module to be a stable f-filtration reduce to

(3.2) $\mathfrak{a}M_n \subseteq M_{n+1}$ for all n, and

(3.3) $\mathfrak{a}M_n = M_{n+1}$ for all sufficiently large n.

To verify this remark, note that condition (3.2) is equivalent to g being an f-filtration, since for any m and n, (3.2) implies

$$\mathfrak{a}^m M_n = \mathfrak{a}^{m-1}\mathfrak{a}M_n \subseteq \mathfrak{a}^{m-1}M_{n+1} \subseteq \ldots \subseteq M_{n+m},$$

and the converse is immediate by taking $m = 1$. If (3.3) is satisfied, that is, if there exists $N \in \mathbb{N}$ such that $\mathfrak{a}M_n = M_{n+1}$ for all $n \geq N$, then for any k,

$$M_{N+k} = \mathfrak{a}M_{N+k-1} = \ldots = \mathfrak{a}^k M_N \subseteq \sum_{i=0}^{N} \mathfrak{a}^{N+k-i}M_i \subseteq M_{N+k}$$

which implies g is f-stable. Conversely, if g is f-stable there is N such that $M_n = \sum_{i=0}^{N} \mathfrak{a}^{n-i}M_i$ for all n. Then for $n \geq N$,

$$M_n = \sum_{i=0}^{N} \mathfrak{a}^{n-N}\mathfrak{a}^{N-i}M_i \subseteq \mathfrak{a}^{n-N}M_N \subseteq M_n$$

from which it follows that $M_n = \mathfrak{a}^{n-N}M_N$ for all $n \geq N$. Then for any $n \geq N$,

$$\mathfrak{a}M_n = \mathfrak{a}\mathfrak{a}^{n-N}M_N = \mathfrak{a}^{n+1-N}M_N = M_{n+1}$$

and condition (3.3) is satisfied.

Conditions (3.2) and (3.3) on a filtration $g = \{M_n\}$ have been studied and have been described by saying g is <u>$\mathfrak{a}$-stable</u>, [See 1, P.105].

<u>Remark 3.7</u>: Although the fact will not be used it should be noted that when $g = \{M_n\}$ is an f-filtration on an A-module M, g has an <u>associated graded module</u> via

$$G_g(M) = \bigoplus_{n=0}^{\infty} \frac{M_n}{M_{n+1}}$$

which has a module stucture over $G_f(A)$. If g is f-stable and each M_n is finitely generated, $G_g(M)$ is a finitely generated $G_f(A)$-module. These considerations are not helpful here, however, since $G_f(A)$ is ordinarily not a noetherian ring.

It was shown by Example 3.3 that even in very simple cases the multiplicity of a filtration on a module need not exist. The situation is much better for filtrations which are stable with respect to a multiplicative filtration.

<u>Theorem 3.8</u>: Let $f = \{\mathfrak{a}_n\}$ be a 0-dimensional filtration on a noetherian ring A and let $g = \{M_n\}$ be a stable f-filtration on an A-module M. Then for $s = \operatorname{alt}(f)$, $\mu(s,g)$ exists if and only if $\mu(f,M)$ exists, in which case they are equal.

Proof: Let N be such that $M_n = \sum_{i=0}^{N} \mathfrak{a}_{n-i} M_i$ for all n. Then for $n \geq N$, $\mathfrak{a}_{n-i} \subseteq \mathfrak{a}_{n-N}$ and of course $M_i \subseteq M$ so

$$\mathfrak{a}_n M \subseteq M_n \subseteq \mathfrak{a}_{n-N} M \subseteq M_{n-N} . \tag{3.4}$$

The first three members of this chain imply

$$\left(\frac{n-N}{n}\right)^s \frac{s!\ H(n-N,M,f)}{(n-N)^s} \leq \frac{s!\ H(n,g)}{n^s} \leq \frac{s!\ H(n,M,f)}{n^s}$$

and passing to the limit, existance of $\mu(f,M)$ forces $\mu(s,g)$ to exist and equal $\mu(f,M)$. The converse follows from a similar argument using the last three members of the chain (3.4). Q.E.D.

In [8] and [9], Smoke applied a definition of multiplicity due to Serre [7] and Fraser [2] to finitely generated graded algebras over a field k and to finitely generated graded modules over such algebras. The multiplicity function defined there ordinarily takes values in $\mathbb{Z}[[t]]$. In this work the emphasis has been in a different direction more closely related to the classical formulation of multiplicity as an integer derived from the leading coefficient of some polynomial. The multiplicity defined here is always a real number. It is encouraging to see that when these two types of multiplicity are comparable, there is a strong connection between their values. To restrict consideration to those finitely generated graded algebras for which his multiplicity is a real number (so in fact an integer) for every finitely generated graded module, Smoke imposed the condition of regularity; i.e., the algebra has finite global dimension. As he proved [9,Theorem 7.5 p.38] the condition of regularity is equivalent to requiring that the algebra be a finitely generated graded polynomial algebra over the field k . It is in this situation that the connection between the two types of multiplicity can be given for all finitely generated modules by multiplication with a fixed positive integer which depends only on the algebra.

Let $R = \bigoplus_{p=0}^{\infty} R^p$ be a noetherian graded ring, $R^0 = k$, a field. Any finitely generated graded R-module $M = \bigoplus_{p=0}^{\infty} M^p$ has the property that for some N ,

$M = R\left(\bigoplus_{p=0}^{N} M^p\right)$. On checking degrees, it follows that

$M^p = \sum_{i=0}^{N} R^{p-i}M^i$ for $p \geq N$ and thus

$$\bigoplus_{p\geq q} M^p = \sum_{i=0}^{N} \left(\bigoplus_{s\geq q-i} R^s\right)\left(\bigoplus_{t\geq i} M^t\right) \tag{3.5}$$

for each $q \geq N$.

This situation may be interpreted in terms of filtrations as follows. Let

$$f = \{a_n\} \text{ where } a_n = \bigoplus_{p\geq n} R^p \text{ for each } n \geq 0 . \tag{3.6}$$

By taking a set of homogeneous generators $\{r_1,\ldots,r_n\}$ of R over k, R is seen to be the homomorphic image of $A = k[X_1,\ldots,X_n]$. Let $\tau = (d_i)$ where $d_i = \deg(r_i)$ for each i; that is, $r_i \in R^{d_i}$. Define f_τ on A via Example 2.25. Since the d_i are positive integers, hence rational numbers, f_τ is essentially powers. Observe that under the extension from A to R, $f_\tau{}^e = f$ so that f is essentially powers as well. Since $\frac{R}{\mathfrak{a}_1} \approx k$, f is 0-dimensional and by Corollary 2.23, $\mu(f,M)$ exists for every finitely generated R-module M.

The finitely generated graded R-module $M = \bigoplus_{p=0}^{\infty} M^p$ is filtered by defining

(3.7) $g = \{M_n\}$ where $M_n = \bigoplus_{p\geq n} M^p$ for each n.

Equation (3.5) implies that g is f-stable. By Theorem 3.8 and the fact that $\mu(f,M)$ exists, it follows that $\mu(s,g)$ exists with $\mu(s,g) = \mu(f,M)$ where $s = \operatorname{alt}(f)$. Furthermore $\mu(s,\_)$ is an additive function on the category of finitely generated graded R-modules. Incidentally, Corollary 2.23 implies $\mu(s,g)$ is a rational number for each finitely generated R-module.

At this point, Smoke's multiplicity $e_R(M)$ is an element in $\mathbb{Z}[[t]]$ and as such is not comparable with $\mu(s,g)$. If, however, R is regular, Smoke showed that the polynomial ring A above can be chosen isomorphic to

R and he can compose from $\mathbb{Z}[[t]]$ to $\mathbb{Z}$ and recover the formulation of multiplicity given by Serre[7] in this situation; that is

$$(3.8) \qquad e_R(M) = \sum_{i=1}^{n} (-1)^i \dim_k \mathrm{Tor}_i^R(k,M) .$$

The relationship which exists between these two types of multiplicity is stated in the following theorem.

Theorem 3.9: In the situation described above, identify $A = R$. Then $d_1 \ldots d_n \mu_A(n,g) = e_A(M)$.

Proof: Since Theorem 3.8 implies $\mu_A(n,g) = \mu_A(f_T, M)$, it follows from Proposition 2.54 that

$$\mu_A(n,g) = \frac{1}{d_1 \ldots d_n} \mu_A((X),M) ,$$

where $(X) = (X_1, \ldots, X_n)$ is the maximal ideal generated by $\{X_1, \ldots, X_n\}$. Thus to complete the proof one needs only to show that

$$(3.9) \qquad \mu_A((X),M) = e_A(M) .$$

That is, $e_A(M)$ is just the multiplicity of M with respect to the maximal ideal (X) . To prove (3.9) note first that $\mathrm{alt}(X) = n$. Hence by definition

$$\mu_A((X),M) = n! \lim_{m \to \infty} \frac{L_A(M/(X)^m M)}{m^n} .$$

The limit formula of Samuel implies [5, Theorem 13, P.329] that this multiplicity is the same as Northcott's

$e_A(X_1,\ldots,X_n,M)$ [5, P.299]. Then by [5, Theorem 5, P.370] $\mu_A((X),M)$ is the Euler-Poincaré characteristic of the homology complex of the Koszul complex $K(X;M)$ of M with respect to $X_1,\ldots,X_n$. That is,

$$\mu_A((X),M) = \sum_{i=1}^{n} (-1)^i L_A(H_i K(X;M)) .$$

Since (X) annihilates each homology module, [5, Theorem 3, P.364], $L_A(H_i K(X;M)) = L_k(H_i K(X;M)) = \dim_k(H_i K(X;M))$ and the equation above may be rewritten as

$$(3.10) \qquad \mu_A((X),M) = \sum_{i=1}^{n} (-1)^i \dim_k H_i K(X;M) .$$

Now $X_1,\ldots,X_n$ form an A-sequence (i.e., $((X_1,\ldots,X_i):X_{i+1}) = (X_1,\ldots,X_i)$ for each $i = 0,\ldots,n-1$) so [7, Proposition 2, P.IV-4] the Koszul complex $K(X)$ of A with respect to $X_1,\ldots,X_n$ provides a projective resolution of $k \approx A/(X)$ with the augmentation map $K(X) \longrightarrow A/(X)$ being just the canonical map $K_1^A(X) \approx A \longrightarrow A/(X)$. The Koszul complex of M with respect to $X_1,\ldots,X_n$ is just

$$K(X;M) = K(X) \otimes M .$$

Thus

$$H_i K(X;M) = H_i(K(X) \otimes M) = \mathrm{Tor}_i^A(A/(X),M) = \mathrm{Tor}_i^A(k,M)$$

which together with (3.10) implies

$$(3.11) \qquad \mu_A((X),M) = \sum_{i=1}^{n} (-1)^i \dim_k \mathrm{Tor}_i^A(k,M) .$$

Since $A = R$, (3.8) and (3.11) prove (3.9). Q.E.D.

IV AN EXAMPLE

In this chapter, another class of pseudo-valuations on $k[X,Y]$, k a field, is considered. The main result, Proposition 4.4, establishes for each pseudo-valuation in the class necessary and sufficient conditions on the parameters for the corresponding filtration to be approximatable by powers.

Let $A = k[X,Y]$ and let $\tau = \{\tau_n\}$ be a sequence of positive integers such that

$$(4.1) \quad \begin{aligned} &\text{(i)} \quad \tau_{n+2} > \tau_{n+1}\tau_n \quad \text{for each } n \in \mathbb{N} \text{ and} \\ &\text{(ii)} \quad \gcd(\tau_n, \tau_{n+1}) = 1 \quad \text{for each } n \in \mathbb{N} . \end{aligned}$$

Using this sequence of integers, inductively define the following sequence of polynomials in A,

$$(4.2) \quad \begin{aligned} &\text{(i)} \quad \alpha_1 = X \\ &\text{(ii)} \quad \alpha_2 = Y , \text{ and} \\ &\text{(iii)} \quad \alpha_n = \alpha_{n-1}^{\tau_{n-2}} + \alpha_{n-2}^{\tau_{n-1}}, \text{ for all } n > 2 . \end{aligned}$$

<u>Proposition 4.1</u>: For any $\beta \in A$, there exists a unique representation for β as

$$\beta = \sum a_{(k)}\alpha^{(k)} = \sum a_{k_1,\ldots,k_n}\alpha_1^{k_1}\cdots\alpha_n^{k_n}, \quad 0 \le k_i < \tau_{i+1}$$

where $(k) = (k_1,\ldots,k_n,0,0,\ldots)$ for some $n \in \mathbb{N}$ and $a_{(k)} \in k$. This representation will be called the <u>standard</u> <u>form</u> for β.

Proof: Existance of such a representation will be known provided the result is shown for each X^nY^m, $n,m \in \mathbb{N} \cup \{0\}$. The proof is by induction on $n + m$.

If $n + m = 1$, then X^nY^m is either X or Y and the statement is true. Assume validity for $n + m = k > 1$. The argument now depends on two very similar cases; namely $n > 0$ or $m > 0$. If $n > 0$, $X^nY^m = X(X^{n-1}Y^m)$ and by the inductive hypothesis

$$X^nY^m = X\left(\sum a_{(k)}\alpha^{(k)}\right) = \sum a_{k_1,\ldots,k_n}\alpha_1^{k_1+1}\alpha_2^{k_2}\ldots\alpha_n^{k_n},$$

$0 \le k_i < \tau_{i+1}$. The other case is almost the same,

$$X^nY^m = Y\left(\sum a_{(k)}\alpha^{(k)}\right) = \sum a_{k_1,\ldots,k_n}\alpha_1^{k_1}\alpha_2^{k_2+1}\alpha_3^{k_3}\ldots\alpha_n^{k_n},$$

$0 \le k_i < \tau_{i+1}$. These expressions need not be in the proper form but they both can be so rewritten by establishing the statement: Any monomial $\alpha^{(k)} = \alpha_1^{k_1}\ldots\alpha_n^{k_n}$, $k_j < \tau_{j+1}$ for $j \ne i$ and $k_i < 2\tau_{i+1}$, has a representation $\alpha^{(k)} = \sum a_{(k')}\alpha^{(k')}$, $0 \le k_j' < \tau_{j+1}$ for all j. To establish this assume $\tau_{i+1} \le k_i < 2\tau_{i+1}$ (otherwise it is already correctly expressed) and use $\alpha_i^{\tau_{i+1}} = \alpha_{i+2} - \alpha_{i+1}^{\tau_i}$ to express

$$\alpha^{(k)} = \alpha_1^{k_1}\ldots\alpha_i^{k_i-\tau_{i+1}}\alpha_{i+1}^{k_{i+1}}\alpha_{i+2}^{k_{i+2}+1}\alpha_{i+3}^{k_{i+3}}\ldots\alpha_n^{k_n}$$
$$- \alpha_1^{k_1}\ldots\alpha_i^{k_i-\tau_{i+1}}\alpha_{i+1}^{k_{i+1}+\tau_i}\alpha_{i+2}^{k_{i+2}}\ldots\alpha_n^{k_n}.$$

Since $0 \le k_i - \tau_{i+1} < \tau_{i+1}$, $k_{i+2} + 1 < 2\tau_{i+3}$, and $k_{i+1} + \tau_i < 2\tau_{i+2}$, the problem exponent has been transferred to higher indices. Continuing if necessary to the index n , the process terminates.

Proof of uniqueness of the representation is much more difficult and because one aspect of the proof is rather cumbersome only an incomplete proof will be given.

First note that uniqueness of the representation is equivalent to k-linear independence of B , the set of all power products $\alpha^{(k)}$, $0 \le k_i < \tau_{i+1}$ for all i . To this end, fix n and consider B_n the set of all $\alpha^{(k)}$ such that $k_i = 0$ for $i > n$. Since $B_n \subseteq B_{n+1}$ for each n and $\bigcup B_n = B$, it suffices to show B_n is linearly independent over k for each n . This is proved by induction on $i = 1,\ldots,n-1$ of the statement: $B_{n,i} = \{\alpha^{(k)} \in B_n \mid k_j = 0 \text{ for } j < n-i\}$ is a linearly independent set. It is essential to the proof that each consecutive pair α_n, α_{n+1} be algebraically independent. This follows at once from noting that α_{n-1} is integral over $k[\alpha_n, \alpha_{n+1}]$ and continuing inductively $k[X,Y] = k[\alpha_1, \alpha_2]$ is integral over $k[\alpha_n, \alpha_{n+1}]$. Then the assumption that α_n and α_{n+1} be algebraically dependent leads to a contradiction.

It is easily seen that B_1 is linearly independent so assume $n > 1$. Suppose $i = 1$ and

$$0 = \sum a_{(k)}\alpha^{(k)} \quad \text{with} \quad \alpha^{(k)} \in B_{n,1} \quad \text{for each } (k).$$

$$= \sum a_{k_{n-1},k_n}\alpha_{n-1}^{k_{n-1}}\alpha_n^{k_n}, \quad 0 \le k_i < \tau_{i+1} .$$

From the independence of α_{n-1} and α_n map to $k[T]$, T a new indeterminate, via $\alpha_{n-1} \mapsto T^{\tau_{n-1}}$, $\alpha_n \mapsto T^{\tau_n}$. Then

$$0 = \sum a_{(k)} T^{\tau_{n-1}k_{n-1} + \tau_n k_n} .$$

Cancellation within the sum requires that

$\tau_{n-1}k_{n-1} + \tau_n k_n = \tau_{n-1}k'_{n-1} + \tau_n k'_n \Rightarrow \tau_{n-1}(k_{n-1} - k'_{n-1})$

$= \tau_n(k'_n - k_n)$. But $|k_{n-1} - k'_{n-1}| < \tau_n$ and $(\tau_{n-1},\tau_n) = 1$;

thus $k_{n-1} = k'_{n-1}$, $k_n = k'_n$ and no cancelling can occur. Thus $a_{(k)} = 0$ for all (k).

Assume, then, that $B_{n,i}$ has been shown to be linearly independent for $i \ge 2$. The proof that $B_{n,i+1}$ is also linearly independent follows from knowing that each polynomial of the form

(4.3) $$Z^{\tau_{n-i}} - (\alpha_{n-i+1} - \alpha_{n-i}^{\tau_{n-i-1}})$$

is irreducible over $k[\alpha_{n-i},\alpha_{n-i+1}] = k[\alpha_{n-i},\dots,\alpha_n]$, and hence over $k(\alpha_{n-i},\alpha_{n-i+1})$ (since $k[\alpha_{n-i},\alpha_{n-i+1}]$ is isomorphic to $k[S,T]$, a unique factorization domain). Then $\{1,\alpha_{n-i-1},\alpha_{n-i-1}^2,\dots,\alpha_{n-i-1}^{\tau_{n-1}-1}\}$ is a basis for $k(\alpha_{n-i-1},\dots,\alpha_n)$ as a vector space over $k(\alpha_{n-i},\dots,\alpha_n)$. It now follows that $B_{n,i+1}$ is linearly independent over

k as well. The proof that the polynomial (4.3) is irreducible will be omitted but is outlined as follows. First translate the problem to new indeterminates S,T, using the fact that α_{n-i} and α_{n-i+1} are algebraically independent; then show that $Z^m - (S - T^n)$ is irreducible over $k[S,T,Z]$ for any relatively prime $m,n \in \mathbb{N}$. Although tedious, the argument is straight forward; i.e., assume a factorization in $k[S,T,Z]$,

$$fg = Z^m - (S - T^n) ,$$

and derive that f or g must be in k. It is helpful in this endeavor to first show that $Z^m \pm T^n$ is irreducible in $k[Z,T]$. Q.E.D.

Using the unique representation given by Proposition 4.1, define for any $\beta = \sum a_{(k)}\alpha^{(k)} \in A$

$$(4.4) \quad v_\tau(\beta) = v(\beta) = \min\{\Sigma k_i\tau_i \mid a_{(k)} \neq 0\} .$$

Proposition 4.2: The function v defined by (4.4) is a pseudo-valuation on A.

Proof: The fact that $v(\beta + \gamma) \geq \min\{v(\beta),v(\gamma)\}$ is immediate from using the standard form representations of each and then adding. The result is then in standard form with possibly more zero coefficients. To show $v(\beta\gamma) \geq v(\beta) + v(\gamma)$, it suffices to show the result where β and γ are defining monomials $\beta = \alpha^{(j)}$,

$\gamma = \alpha^{(k)}$. Even more is true in this case; namely,

$$(4.5) \qquad v(\alpha^{(j)}\alpha^{(k)}) = v(\alpha^{(j)}) + v(\alpha^{(k)}) .$$

Since $\alpha^{(j)}\alpha^{(k)} = \alpha^{(j)+(k)}$, statement (4.5) is implied by the statement

$$(4.6) \qquad v(\alpha^{(m)}) = \Sigma m_i \tau_i , \quad 0 \le m_i , i = 1,\ldots,n , n \in \mathbb{N} .$$

If each $m_i < \tau_{i+1}$, there is nothing to prove. Choose, then, the smallest index i for which $m_i \ge \tau_{i+1}$ and using $\alpha_{i+2} = \alpha_{i+1}^{\tau_i} + \alpha_i^{\tau_{i+1}}$ express

$$\alpha_1^{m_1}\ldots\alpha_i^{m_i}\ldots\alpha_n^{m_n} = \alpha_1^{m_1}\ldots\alpha_i^{m_i-\tau_{i+1}}(\alpha_{i+2} - \alpha_{i+1}^{\tau_i})\alpha_{i+1}^{m_{i+1}}\ldots\alpha_n^{m_n}$$

$$= \alpha_1^{m_1}\ldots\alpha_i^{m_i-\tau_{i+1}}\alpha_{i+1}^{m_{i+1}}\alpha_{i+2}^{m_{i+2}+1}\alpha_{i+3}^{m_{i+3}}\ldots\alpha_n^{m_n}$$

$$- \alpha_1^{m_1}\ldots\alpha_i^{m_i-\tau_{i+1}}\alpha_{i+1}^{m_{i+1}+\tau_i}\alpha_{i+2}^{m_{i+2}}\ldots\alpha_n^{m_n} .$$

If $m_i - \tau_{i+1} \ge \tau_{i+1}$, repeat the process again at i obtaining two more terms for each of those in the expression. If $m_i - \tau_{i+1} < \tau_{i+1}$, move, in each of these terms, to the next index for which the exponent is "too large". Eventually, $\alpha_1^{m_1}\ldots\alpha_n^{m_n}$ will be reduced to a sum of terms of the form $\pm\alpha_1^{k_1}\ldots\alpha_p^{k_p}$ with $k_i < \tau_{i+1}$ for all $i = 1,\ldots,p$. Inspection of this procedure shows that precisely one of the terms has exactly the value $\Sigma m_i \tau_i$ and all the others have higher value since

$$(4.7)\quad \begin{aligned} v(\alpha_{i+2}) &= \tau_{i+2} > \tau_i\tau_{i+1} = v(\alpha_{i+1}^{\tau_i}) \text{ , and} \\ v(\alpha_{i+1}^{\tau_i}) &= v(\alpha_i^{\tau_{i+1}}) \end{aligned}$$

at every i where a substitution is made. Then collecting terms gives the standard form representation for $\alpha_1^{m_1}\ldots\alpha_n^{m_n}$ with precisely one term of value $\Sigma m_i\tau_i$ and all others greater. By definition,

$v(\alpha_1^{m_1}\ldots\alpha_n^{m_n}) = \Sigma m_i\tau_i$. Q.E.D.

Remark 4.3: Equation (4.5) is motivation for suspecting that $v = v_\tau$ is always a valuation. It is not; for suppose $\tau = \{\tau_i\}$ is defined by $\tau_1 = 1$, $\tau_2 = 2$, and $\tau_{n+1} = \tau_n\tau_{n-1} + 1$ for $n \geq 2$. Let $\beta = \alpha_1\alpha_2 + \alpha_3$ and $\gamma = \alpha_1\alpha_2 - \alpha_3$. Then

$$v(\beta\gamma) = v(\alpha_1^2\alpha_2^2 - \alpha_3^2) = v(\alpha_2^2\alpha_3 - \alpha_4) = 7 \text{ , but}$$

$$v(\beta) + v(\gamma) = 3 + 3 = 6 \text{ .}$$

By taking char(k) = 2 , this example shows that v need not be homogeneous; i.e., there are some $\beta \in A$ and $n \in \mathbb{N}$ for which $v(\beta^n) \neq nv(\beta)$.

Proposition 4.4: For any sequence $\tau = \{\tau_n\}$ which satisfies (4.1), the filtration $f_\tau = \{a_n\}$ which corresponds to the pseudo-valuation v_τ is approximatable by powers if and only if the infinite product $\prod_{i=1}^{\infty} \frac{\tau_{i+2}}{\tau_i\tau_{i+1}}$ converges to a finite limit.

Proof: The proof will be based on the equivalent formulation of "approximatable by powers" given by Theorem 2.51. By Remark 2.52 it is seen that attention may be restricted to approximating filtrations of the form

(4.8) $f_n = \{a_{n,j}\}$ where $a_{n,j} = \sum_{i=1}^{N_n} a_{j-i}a_i$, $N_n \in \mathbb{N}$.

Lemma 4.5: Let f_n be given by (4.8) with $N_n = \tau_n\tau_{n-1}$ and $v_n = v_{f_n}$. Then for $n \geq 4$

(i) $v_n(\alpha_j) = \tau_j$ for $j \leq n$,

(ii) $v_n(\alpha_{n+1}) = \tau_{n-1}\tau_n$,

(iii) $v_n(\alpha_{n+2}) = \tau_{n-1}\tau_n^2$, and

(iv) $v_n(\alpha_{n+k}) = \tau_{n-1}\tau_n^2\tau_{n+1}\cdots\tau_{n+k-2}$ for $k \geq 3$.

Proof: By the definition of f_n , $a_{n,i} = a_i$ for $i \leq \tau_n\tau_{n-1}$ and therefore $v_n(\alpha_j) = v(\alpha_j)$ for $j \leq n$. For $j = n + k$, $k \in \mathbb{N}$, the argument is by induction on k with the difficult point being for $k = 1$. To show $v_n(\alpha_{n+1}) = \tau_n\tau_{n-1}$ replace the τ_j sequence by τ'_j where $\tau'_j = \tau_j$ for $j \leq n$ and $\tau'_j = \tau'_{j-2}\tau'_{j-1} + 1$ for $j > n$. This sequence satisfies (4.1) and the resulting sequence $\{\alpha'_n\}$ of (4.2) has the property that $\alpha'_i = \alpha_i$ for $i = 1,\ldots,n$ and $\alpha'_i = \alpha'^{\tau'_{i-2}}_{i-1} + \alpha'^{\tau'_{i-1}}_{i-2}$ for all $i > n$. The definitions of α_{n+1} and α'_{n+1} agree so that $\alpha_{n+1} = \alpha'_{n+1}$ but its value has been replaced by

$\tau_n\tau_{n-1} + 1$. For the rest of the proof of the case $k = 1$, the primes will be omitted since only small indices occur and v_n is the same whether obtained from the orginal filtration or from the new one. This is because v_n is defined entirely by the ideals $\mathfrak{a}_i$, $i = 1,\ldots,\tau_n\tau_{n-1}$ which agree for f_τ and $f_{\tau'}$, since the smallest index j for which $v_\tau(\alpha_j)$ might differ from $v_{\tau'}(\alpha_j)$ is $n + 1$ and then each value exceeds $\tau_n\tau_{n-1}$. In fact, the only reason for introducing the primes at all is to assure that the value of α_{n+1} may be assumed to be $\tau_n\tau_{n-1} + 1$ in the argument.

First it will be shown that

$$(4.9) \qquad v_n(\alpha_{n+1}) < v(\alpha_{n+1}) = \tau_n\tau_{n-1} + 1 .$$

Suppose this assertion is false. Then

$$\alpha_{n+1} \in \mathfrak{a}_{n,\tau_n\tau_{n-1}+1} = \sum_{i=1}^{\tau_n\tau_{n-1}} \mathfrak{a}_{\tau_n\tau_{n-1}+1-i}\mathfrak{a}_i .$$

It then follows that α_{n+1} can be represented as

$$\alpha_{n+1} = \Sigma(\Sigma\beta\gamma) , \ \beta \in \mathfrak{a}_{\tau_n\tau_{n-1}+1-i} , \ \gamma \in \mathfrak{a}_i .$$

Represent each pair β and γ in their standard forms,

$$\beta = \sum a_{(s)}\alpha^{(s)} \quad \text{such that} \quad \min\{\Sigma s_j\tau_j \mid a_{(s)} \neq 0\} = \tau_n\tau_{n-1}+1-i,$$

$$\gamma = \sum b_{(t)}\alpha^{(t)} \quad \text{such that} \quad \min\{\Sigma t_j\tau_j \mid b_{(t)} \neq 0\} = i .$$

Expanding the right side of the above, it follows that

$$\alpha_{n+1} = \sum c_{(m)}\alpha^{(m)} \quad \text{where} \quad m_j < 2\tau_{j+1} \quad \text{for all} \quad j .$$

The unique representation for $\sum c_{(m)}\alpha^{(m)}$ can be obtained by reducing each $c_{(m)}\alpha^{(m)}$ to its unique representation and then combining similar terms. Of course, this must be α_{n+1} since it is already in the unique form. Hence reduction of some $\alpha^{(m)}$ with $m_j < 2\tau_{j+1}$ for all j, must yield a term which is a non-zero multiple of α_{n+1}. Since reduction of $\alpha^{(m)}$ yields only terms of value greater than or equal to $v(\alpha^{(m)})$ and since the value of each term of $\beta\gamma$ is greater than or equal to $\tau_n\tau_{n-1}+1$ it must be true that this $\alpha^{(m)}$ has value exactly $\tau_n\tau_{n-1}+1$. Thus $v(\alpha^{(m)}) = v(\alpha_1^{m_1}\cdots\alpha_q^{m_q}) = \Sigma m_i\tau_i = \tau_n\tau_{n-1}+1$. Since $q > n$ and $m_q > 0$ are only possible in this expression if one of the indices is $n+1$ and the corresponding $m_{n+1} = 1$ and all other $m_i = 0$ and since this situation contradicts the choice of $\alpha^{(m)}$ as a product of lower terms, it may be assumed that $q \le n$. It is claimed that the assumption of α_{n+1} appearing as a term in the reduction of $\alpha^{(m)}$ to standard form is false. Let $\alpha_1^{p_1}\cdots\alpha_{n-1}^{p_{n-1}}\alpha_n^{p_n}$ be from any term in the second to the last stage of the reduction of $\alpha^{(m)}$ to standard form. That is $p_j < \tau_{j+1}$ for $j = 1,\ldots,n-2$ and $\Sigma p_i\tau_i \ge \tau_n\tau_{n-1}+1$. If $\Sigma p_i\tau_i > \tau_n\tau_{n-1}+1$, it cannot yield a term of α_{n+1} so assume equality. If $p_n > 0$, each term of the reduction has a factor of

α_{n+2} , $\alpha_{n+1}^{\tau_n}$, or α_n (depending on whether or not $p_n \geq \tau_{n+1}$) and in any case, the term α_{n+1} cannot be obtained. Thus $p_n = 0$. Now $p_1\tau_1 + \cdots + p_{n-1}\tau_{n-1} = \tau_n\tau_{n-1} + 1$. If $p_{n-1} > \tau_n$, the left side of this equation is greater than $\tau_n\tau_{n-1} + 1$ and we have a contradiction. If $p_{n-1} = \tau_n$, cancel to obtain $p_1\tau_1 + \dots + p_{n-2}\tau_{n-2} = 1$. If $\tau_1 > 1$, this is impossible; otherwise, $p_1 = 1$, $p_j = 0$ for $j = 2,\dots,n-2$ and α_1 will appear as a factor in the reduction, again a contradiction. If $p_{n-1} < \tau_n$, $\alpha_1^{p_1}\dots\alpha_{n-1}^{p_{n-1}}$ is already in standard form and cannot be reduced to yield the term α_{n+1} . Thus $\alpha_{n+1} \notin a_{n,\tau_n\tau_{n-1}+1}$ which implies $v_n(\alpha_{n+1}) < \tau_n\tau_{n-1} + 1$. On the other hand, since $\alpha_{n+1} = \alpha_n^{\tau_{n-1}} + \alpha_{n-1}^{\tau_n}$ it follows that

$$v_n(\alpha_{n+1}) \geq \min\{\tau_{n-1}v_n(\alpha_n), \tau_n v_n(\alpha_{n-1})\}$$

$$= \min\{\tau_{n-1}v(\alpha_n), \tau_n v(\alpha_{n-1})\} = \tau_n\tau_{n-1} .$$

Hence $v_n(\alpha_{n+1}) = \tau_n\tau_{n-1}$.

For the remainder of the argument, the τ sequence is the original one. For $k = 2$, the fact that $\alpha_{n+2} = \alpha_{n+1}^{\tau_n} + \alpha_n^{\tau_{n+1}}$ implies

$$v_n(\alpha_{n+2}) \geq \min\{\tau_n v_n(\alpha_{n+1}), \tau_{n+1} v_n(\alpha_n)\} ,$$

with equality holding in case one of these has strictly smaller value than the other. Now

$\tau_n v_n(\alpha_{n+1}) = \tau_n \tau_n \tau_{n-1} = \tau_{n-1}\tau_n^2$ and $\tau_{n+1} v_n(\alpha_n) = \tau_n \tau_{n+1}$. Since $\tau_{n+1} \geq \tau_n \tau_{n-1} + 1 > \tau_n \tau_{n-1}$, the first expression has minimum value. Thus $v_n(\alpha_{n+2}) = \tau_{n-1}\tau_n^2$. Similarly $v_n(\alpha_{n+3}) = \tau_{n-1}\tau_n^2\tau_{n+1}$, and in general

$$v_n(\alpha_{n+k}) = \tau_{n-1}\tau_n^2\tau_{n+1}\cdots\tau_{n+k-2} .$$

The proof of Lemma 4.5 is now complete.

Returning now to the proof of Proposition 4.4, suppose $f = f_\tau$ is approximatable by powers. Then there is a sequence of filtrations f_n' of the type (4.8) and a sequence of real numbers $\lambda_n \longrightarrow 1$ such that

$$f_n' \leq f \leq \lambda_n f_n' \quad \text{for all } n .$$

Choose any n , for simplicity $n = 1$ and let N_1 be given by (4.8). Since $\tau_m \tau_{m-1} \longrightarrow \infty$ as $m \longrightarrow \infty$, an m can be chosen with $N_1 \leq \tau_m \tau_{m-1}$. Let f_m be given by (4.8) with $N_m = \tau_m \tau_{m-1}$. Then

$$(4.10) \qquad f_1' \leq f_m \leq f \leq \lambda_1 f_1' \leq \lambda_1 f_m ,$$

and consequently $\lambda_1 v_m \geq v$ where $v_m = v_{f_m}$. In particular, $\lambda_1 v_m(\alpha_{m+k}) \geq v(\alpha_{m+k})$ for all $k \in \mathbb{N}$. By Lemma 4.5, it follows that

$$\lambda_1 \tau_{m-1}\tau_m^2\tau_{m+1}\cdots\tau_{m+k-2} \geq \tau_{m+k} ,$$

or equivalently

$$\lambda_1 \geq \frac{\tau_{m+k}}{\tau_{m-1}\tau_m^2\tau_{m+1}\cdots\tau_{m+k-2}} = \prod_{i=1}^{k} \frac{\tau_{m+i}}{\tau_{m+i-2}\tau_{m+i-1}} .$$

This inequality is true for all k so the infinite product converges to a limit no greater than λ_1. Multiplying by the first finitely many products

$\frac{\tau_{j+2}}{\tau_j \tau_{j+1}}$, $j = 1,\ldots,m-2$, one has

$$\prod_{i=1}^{\infty} \frac{\tau_{i+2}}{\tau_i \tau_{i+1}} < \infty . \tag{4.11}$$

Conversely, suppose $\prod_{i=1}^{\infty} \frac{\tau_{i+2}}{\tau_i \tau_{i+1}}$ converges. Let f_n be as in Lemma 4.5 and let $\lambda_n = \prod_{i=n-1}^{\infty} \frac{\tau_{i+2}}{\tau_i \tau_{i+1}}$. It is claimed that for $n \geq 4$

$$f_n \leq f \leq \lambda_n f_n , \quad \lambda_n \longrightarrow 1 \text{ as } n \longrightarrow \infty . \tag{4.12}$$

The inequality $f_n \leq f$ is trivial and the fact that $\lambda_n \longrightarrow 1$ as $n \longrightarrow \infty$ is standard from the theory of convergent products. To prove $f \leq \lambda_n f_n$, recall that the value of any element $\beta \in k[X,Y]$ is found by expressing β in standard form,

$$\beta = \sum a_{(k)} \alpha^{(k)} ,$$

and letting $v(\beta) = \min\{\Sigma k_i \tau_i \mid a_{(k)} \neq 0\}$. Since $\lambda_n v_n$ is a pseudo-valuation,

$$\lambda_n v_n(\beta) \geq \min\{\Sigma k_i \lambda_n v_n(\alpha_i) \mid a_{(k)} \neq 0\} .$$

Thus it suffices to show $\lambda_n v_n(\alpha_j) \geq v(\alpha_j) = \tau_j$ for all j. If $j \leq n$, $v_n(\alpha_j) = v(\alpha_j)$, so assume $j = n + k$, $k \in \mathbb{N}$. Using Lemma 4.5, one computes

$$\lambda_n v_n(\alpha_{n+k}) = \lambda_n \tau_{n-1}\tau_n{}^2\tau_{n+1}\cdots\tau_{n+k-2}$$

$$= \prod_{i=n-1}^{\infty} \frac{\tau_{i+2}}{\tau_i\tau_{i+1}} \tau_{n-1}\tau_n{}^2\tau_{n+1}\cdots\tau_{n+k-2}$$

$$= \prod_{i=n-1}^{n+k-2} \frac{\tau_{i+2}}{\tau_i\tau_{i+1}} \tau_{n-1}\tau_n{}^2\tau_{n+1}\cdots\tau_{n+k-2} \prod_{i=n+k-1}^{\infty} \frac{\tau_{i+2}}{\tau_i\tau_{i+1}}$$

$$= \tau_{n+k} \prod_{i=n+k-1}^{\infty} \frac{\tau_{i+2}}{\tau_i\tau_{i+1}}$$

$$> \tau_{n+k} = v(\alpha_{n+k}) \ .$$

Thus (4.12) is established. Q.E.D.

Remark 4.6: In the proof of the statement, "f is approximatable by powers implies $\prod_{i=1}^{\infty} \frac{\tau_{i+2}}{\tau_i\tau_{i+1}} < \infty$", the fact that $\lambda_n \longrightarrow 1$ was not used in any way. The mere fact that there exists some λ and some filtration f_1 which is essentially powers with

(4.13) $$f_1 \le f \le \lambda f_1$$

forces the product to converge and hence f to be approximatable by powers. Since f_1 is essentially powers, f_1 has a regular subsequence of powers $f_1{}^{(k)}$. Then statement (2.30) implies

$$\mathfrak{a}_{\{\lambda k\}m} \subseteq (\mathrm{rad}(f))^m \text{ for all } m \ ;$$

that is, condition (2.25) is satisfied. The converse is also true; i.e. if condition (2.25) is satisfied, (2.31) implies that some essentially powers filtration f' and some λ can be found such that (4.13) is satisfied. Thus a filtration of the type considered in this chapter is approximatable by powers if and only if condition (2.25) is satisfied. It is unknown whether or not this is the case for every filtration on a noetherian ring.

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