Patterns of Interaction and Mathematical Thinking of High School Students in Classroom Environments that Include Use of JAVA-based, Curriculum-Embedded Software

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PATTERNS OF INTERACTION AND MATHEMATICAL THINKING OF HIGH SCHOOL STUDENTS IN CLASSROOM ENVIRONMENTS THAT INCLUDE USE OF JAVA-BASED, CURRICULUM-EMBEDDED SOFTWARE

by

Karen L. Fonkert

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the requirements for the Degree of Doctor of Philosophy
Department of Mathematics
Advisor: Steven Ziebarth, Ph.D.

Western Michigan University
Kalamazoo, Michigan
April 2012
WE HEREBY APPROVE THE DISSERTATION SUBMITTED BY

Karen L. Fonkert

ENTITLED PATTERNS OF INTERACTION AND MATHEMATICAL THINKING OF HIGH SCHOOL STUDENTS IN CLASSROOM ENVIRONMENTS THAT INCLUDE USE OF JAVA-BASED, CURRICULUM-EMBEDDED SOFTWARE

AS PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

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Mathematics

(Department)

Mathematics Education

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Dean of The Graduate College
This study analyzes the nature of student interaction and discourse in an environment that includes the use of Java-based, curriculum-embedded mathematical software. The software CPMP-Tools was designed as part of the development of the second edition of the Core-Plus Mathematics curriculum. The use of the software on laptop computers in small groups of students, and in whole-class interactive lessons with a single computer at the front of the classroom was explored. Data were collected through observations, interviews, and selected items from the students’ regular assessments. During the observations, classroom discussion was audio-taped and videotaped, and field notes were taken. The interviews of students and teachers were audio and/or videotaped. The analysis of this data revealed that the students engaged in inquiry the majority of the time while they were using CPMP-Tools in small groups. Building on other students’ ideas was the second most frequent interaction pattern in that setting. During the whole-class interactive lessons with a single computer, the two most frequently found interaction patterns were teacher explain and giving new ideas. The most frequently occurring level of mathematical thinking found in both types of classroom environments using CPMP-Tools was the second-highest level in the
framework—Constructing Synthesizing. Therefore, the students were habitually engaged in productive interaction patterns and high levels of mathematical thinking while using the curriculum-embedded software. The dynamic nature and strategic use of colorful visuals used in CPMP-Tools facilitated students’ interactions and high levels of mathematical thinking.
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Karen L. Fonkert
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CHAPTER I
INTRODUCTION

Incremental Changes in Mathematics Instruction

For much of the 20th century, mathematics was predominantly taught to students sitting in rows while the teacher at the blackboard gave an exposé on the new concept(s), method(s), or theorem(s) of the day augmented with worked-out examples of the types of problems they would have for homework (Crabill, 1990; Schoen & Charles, 2003). Such instruction is still the norm in many present day mathematics classrooms. A consequence of this instructional method is often that many students only acquire “a command of a set of facts, procedures, and formulas that they understand in a superficial or disconnected way” (Schoen & Charles, 2003, p. xi).

Among its many recommendations, the National Council of Teachers of Mathematics (NCTM) (1989, 2000) has called for changes to this traditional instructional model, recommending that students be active learners—communicating with each other through discussions that include conjecturing, exploring and justifying claims using tools or manipulatives. Many researchers have shown that students who discuss mathematics with others develop deeper conceptual understanding (Cobb, 2000a, 2000b; Cohen, 1994a; Davidson & Worsham, 1992; Steele, 1999; Vygotsky, 1978, 1981). In addition, NCTM’s Principles and Standards for School Mathematics (2000) specifically emphasizes the use of technology, stating that, “Electronic technologies—calculators and
computers—are essential tools for teaching, learning, and doing mathematics” (p. 24). Incorporating technology and student communication in the classroom can deepen the students’ learning of mathematics (NCTM, 2000).

Mathematics curricula have been developed to address NCTM’s *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and to assist teachers in incorporating them into their classrooms (cf. Coxford, Fey, Hirsch, & Schoen, et al., 2003; Hirsch, 2007; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006; Akers et al., 2004). Most of these mathematics curricula were created with an instructional design that incorporates the use of collaborative problem-based learning, manipulatives, and technology (Hirsch, 2007). With the help of the NSF-funded Standards-based curricula, changes in some mathematics classrooms have taken place as some teachers have changed their teaching practices through sustained professional development and increased the use of technology.

**Change Mechanisms**

Wilson and Lloyd (2000) identified curriculum materials as a catalyst for change for teachers in a study of three high school teachers implementing a Standards-based curriculum that was new to them. The researchers documented the teaching practices of teachers who had begun to share the mathematical authority with their students after beginning to teach the new curriculum. One teacher stood in the back of the classroom providing encouragement to students as they gave the explanations. He had realized that half of his students became disengaged when he did the talking. Another teacher encouraged students to share their thoughts and questions with other students in their
group because he believed the students would have more ownership of the ideas and gain
a deeper understanding through the exposure to different solution methods. Being able to
share the mathematical authority with students is essential when a teacher wishes to have
students learn in groups. This study showed that the curriculum prompted the teachers to
make this change in their teaching.

In another instance of a curriculum effecting change in mathematics instruction,
Breyfogle (2001) documented the changes that occurred when a teacher began teaching a
curriculum that followed the recommendations of NCTM. The teacher in this study
became more open to considering alternate teaching methods. The classroom norms
shifted from being dominated by the teacher talking to the students doing more of the
talking. During whole-class discussions the teacher’s talk shifted from simply eliciting
answers from students, to probing how the students were thinking (Breyfogle, 2001).

The professional development that accompanies some of the NSF-funded
Standards-based curricula also has the potential to promote changes in instructional
practice. Schoen, Finn, Griffin, and Fi (2001) found that the teachers whose students
performed the best had strong professional development that prepared them to teach the
curriculum under study. These teachers used more cooperative learning and less
supplemental traditional worksheets (Schoen, Finn, Griffin, & Fi, 2001). This study
illustrates that a curriculum that enables teachers to achieve the vision put forth by
NCTM may also have a positive effect on student achievement. The study also suggests
that as more teachers experience professional development, cooperative learning may be
utilized more often. In general, cooperative learning has become more prevalent in
mathematics classrooms (Marzano, 1992). This may be a consequence of the
implementation of NSF-funded Standards-based curricula.

**The Potential of Technology to Change Mathematics Teaching**

Despite differential availability and use across schools, technology in mathematics classrooms has also become more common in recent years. The use of graphing calculators to explore topics and expedite tedious calculations is quite commonplace in secondary mathematics classrooms (Alfonso & Long, 2005; Bethell & Miller, 1998). Computers have also been used in mathematics classes in various ways (Alfonso & Long, 2005; Sarama & Clements, 1998; Stevenson, 1998). Some teachers have a computer in the classroom for demonstration purposes, or they may have access to a computer lab to which they will take students. However, with the advent of more affordable laptop computers, the presence of computers in mathematics classrooms is becoming more widespread (Poulsen, 2005).

The use of computers in mathematics classrooms may result in students becoming more active learners. When mathematical software is easily accessible, it can be used to perform the same operations as a graphing calculator and more such as taking on the role of manipulatives to explore mathematical concepts (Bakker & Frederickson, 2005; Clements & Nastasi, 1999). When students are using computer software tools in mathematics class, there is some evidence to suggest that use of them may actually enhance communication among students and increase students’ ability to make conjectures and explore concepts (Clements & Nastasi, 1999; Battista, 2007). Therefore, an increasing potential exists for computers to help teachers make further changes in instructional practice and to achieve the vision of active learning and increased use of
technology recommended by NCTM. This study investigated how the use of computers may accomplish that goal by describing how students interact with each other and with the computer, and what levels of thinking they verbalize while using computer software that was developed in conjunction with the second edition of *Core-Plus Mathematics*.

**Context for This Study**

**The Curriculum**

One of the NCTM-Standards-based mathematics curricula was developed by the *Core-Plus Mathematics Project*, based at Western Michigan University, and funded by the National Science Foundation (NSF). *Core-Plus Mathematics* is a complete high school mathematics curriculum organized around coherent, focused units that consist of problem-based investigations set in real-world contexts that facilitate students’ construction of important and meaningful mathematics. The *Core-Plus Mathematics* pedagogical model emphasizes the use of collaborative learning by using rich contextual tasks that are appropriately challenging to students and by including material to help teachers and students engage in effective group work (Coxford et al., 2003). Since the textbook pages contain progressions of written questions or actions to take, the lessons promote oral and written discourse, and small group work as the primary medium for students’ learning.

In 2002, the *Core-Plus Mathematics Project* was awarded a four-year NSF grant to revise the existing curriculum for a second edition. In addition to a structure that continues to support collaborative learning, this second edition has incorporated curriculum-embedded software to be used by students working at computers. Although
Kaput in 1992 noted that computers had already been used in mathematics classrooms for quite a few years, the second edition of *Core-Plus Mathematics* is the first complete high school mathematics curriculum to include computer software as an integral part of the curriculum materials. All further references to the *Core-Plus Mathematics* curriculum will be to this second edition unless otherwise noted.

**The Technology**

The Java-based software designed for use with the *Core-Plus Mathematics* curriculum is called *CPMP-Tools* (Keller, 2011). *CPMP-Tools* is a suite of both general purpose and custom tools developed for each mathematical strand of the curriculum. The general purpose algebra tools include an electronic spreadsheet and a computer algebra system (CAS), which have the ability to produce linked tables and graphs of functions, to manipulate algebraic expressions, and to solve equations and inequalities. The geometry tools include an interactive drawing tool used for constructing, measuring, manipulating, and transforming geometric figures; an object-oriented programming language used to create animation effects; and a set of custom tools used for studying geometric models of physical mechanisms, and special shapes. The statistics tools include tools for graphic displays of data, simulations of probability experiments, and mathematical modeling of quantitative relationships. The discrete mathematics tools include tools for constructing, manipulating, and analyzing vertex-edge graphs and the matrices that represent them.

The focus of this study primarily involved the geometry and statistics tools.

Some investigations in the *Core-Plus Mathematics* curriculum call for use of custom tools within *CPMP-Tools* designed for specific purposes. For example, a custom tool may have data already entered from a problem in the textbook, or a geometric figure
already drawn and ready to manipulate in prescribed ways. For certain lessons these aspects make the use of CPMP-Tools convenient for both the teacher and students. Another aspect of CPMP-Tools that makes it convenient is that it is accessible to students via the internet. Thus, students may have access to it without having to purchase the software. Also, students might interact with the software in the classroom, at home, in a library, or anywhere that has internet access.

Since Core-Plus Mathematics enables teachers to use collaborative-group learning and technology, classrooms that use this curriculum provide a fitting opportunity to study student interaction in various settings of computer use with different types of CPMP-Tools. Some schools that have implemented the second edition of Core-Plus Mathematics use both a single classroom computer for whole-class interactive lessons, and a computer lab for individual student exploration. The school chosen as a research site for this study had laptop computers for all of its students. Thus, this study provided the opportunity to explore the interactions and mathematical thinking of students while they used the software on laptop computers in small groups, and while a teacher facilitated whole-class discussions with the use of the software at the front of the class with a single classroom computer.

The Mathematical Strands

In addition to different computer environments, observing students using both the statistics tools and the geometry tools provided a wider focus to the study. These two strands were chosen in this study after a careful examination of the entire Core-Plus Mathematics curriculum. Initially, a chart was made of all uses of CPMP-Tools in the curriculum and the number of problems and number of days that each instance would
entail. Investigations were chosen that would make use of *CPMP-Tools* on multiple consecutive days. This allowed for a more detailed storyline of the use of *CPMP-Tools*. While these investigations integrate multiple mathematical topics, they are primarily in the strands of statistics and geometry. Additionally, the research on geometry software seemed to be the most plentiful, but other geometry software was not tied to a particular curriculum. The research on statistics software is just beginning to accumulate. Yet, *CPMP-Tools* contains statistical features that are unique. These considerations made the geometry and statistics strands interesting to study.

A review of the studies related to geometry revealed that many involved similar computer software with a similar focus but within different classroom environments. Clements and Nastasi (1988) investigated how elementary school students learned geometric concepts using Logo. They found that the social and cognitive processes interacted to increase student understanding. Yu (2004) performed a qualitative study to describe the discourse of four pairs of middle school students using *Geometer’s Sketchpad*. He found that the environment created by the activity and the software facilitated conversation between students, and between students and the teacher (Yu, 2004). This study seeks to fill a gap in the literature by researching the social and cognitive processes of high school students using *CPMP-Tools* interactive geometry software.

The statistics strand was chosen to study because statistics education is a growing field (Franklin et al., 2005; Scheaffer, 2000). From the year 2000 to 2011 there have been 70 dissertations published in the area of statistics education (Consortium for the Advancement of Undergraduate Statistics Education, 2011). Counting only U.S.
universities, the number decreases to 45. Only ten of those involve statistical software. Since the inclusion of statistics in high school curricula is relatively new, only five dissertations involved high school students, three involved high school teachers, and only two are similar to this study involving both high school students and technology (Consortium for the Advancement of Undergraduate Statistics Education, 2011). Yet, statistical software is changing the way statistics is taught (Hammerman & Rubin, 2004). This study seeks to extend the findings of Cobb and McClain (2004) who found that statistical software is most effective when a teacher establishes productive classroom norms, carefully plans instructional activities, and manages whole-class discussions well (Cobb & McClain, 2004). Furthermore, Pfannkuch (2005) has called for more research to explore how various statistical tools empower students’ thinking. Additionally, during pilot study interviews with the teachers, there were recurring comments that the investigations that used the CPMP-Tools data analysis software had an impact on students’ learning.

**Statement of the Problem**

Since the second edition of Core-Plus Mathematics was developed, little has been documented about how students collaborate and interact during the investigations that involve CPMP-Tools. We have much to learn about how its use will affect the way students learn mathematics and the dynamics of collaborative learning groups that have access to these tools.

When teachers use CPMP-Tools to facilitate a whole-class discussion with a single computer at the front of the classroom, it may be because they feel this method is
sufficient for the particular purpose. Alternatively, they may not be able to reserve the school’s computer lab, or they do not think that the time spent going to the computer lab is worth the short time needed on the computer for a certain task. In any case, a whole-class discussion using CPMP-Tools may or may not be the ideal situation. “Small-group work presents opportunities for more students to verbalize their questions and thinking than whole-class instruction does” (Grouws, 2003, p.135). During a whole-class teacher-demonstration, the students may not be as actively engaged as they would be if they were operating the software themselves. They may or may not be any more engaged than if they were listening to a teacher presentation. However, the dynamic technological tool may have greater power to engage students’ minds than verbal exposition alone (Sarama & Clements, 1998). Therefore, classroom interactions in this environment need to be investigated.

There seem to be potential disadvantages to the situation where students are using the software on laptops. When computers are present in the midst of students collaborating, the computer may interfere with the students’ discourse. For instance, if each student has their own laptop, they could be tempted to work independently. Or, if pairs of students share a computer, even though they are part of a larger group, the sets of pairs may keep their conversation to themselves instead of with the whole group. Additionally, when the laptop is open, it could form somewhat of a barrier between students. However, these potential disadvantages could be overcome by the characteristics of the software, or by the potential advantages of using laptops given below. Looking for potential advantages and disadvantages was of particular interest to the researcher in this study.
Stevenson (1999) found that most students believed the laptops did not hinder classroom communication, but instead enhanced student interaction. With the use of laptops, students can work together around a table, or with their desks pushed together. The physical proximity can remain the same as it was without computers. Also, the laptops are not stationary. Students can push two laptops next to each other to compare the screens, they can pick the laptop up and show the screen to a student who is not right next to them, or they can push the laptops aside so that they can easily collaborate (Heid, 1997). In this study, the way in which student collaboration occurs with the use of CPMP-Tools on laptop computers was examined.

Other studies have shown that the dynamic nature of software similar to CPMP-Tools (i.e., Fathom or Geometer’s Sketchpad), but not part of a mathematics curriculum, is engaging for students and beneficial for learning mathematics (Hammerman & Rubin, 2004; Jones, 2000; Sarama & Clements, 1998; Tall, 2000). The type of computer environment (whole-class/single computer or small-group/multiple computers) in which the software is used presents another factor to consider. The benefits afforded by the software may be realized more fully in one environment than another, or one environment may be preferable over another under certain circumstances. Analyzing the verbalized mathematical thinking of students in each type of computer environment may shed some light on this issue.

As computer technology becomes a more integral part of mathematics teaching and learning, changes in teaching methods or strategies may be necessary. Specifically, it may be helpful for future users of Core-Plus Mathematics, and other users of curriculum-embedded software, to know what kinds of issues can arise under various conditions.
Student discourse is an important component in learning mathematics (NCTM, 1989, 2000; Vygotsky, 1978). Finding ways to enable productive student discourse, while exploiting the benefits of using CPMP-Tools on computers is a goal worth pursuing.

**Purpose**

Kaput (1992) listed 14 open questions in the area of technology and mathematics education. One of them, “How do social patterns change in mathematics classrooms that are technologically rich?” (p. 550) is consistent with the purpose of this study: to investigate the nature of interactions and discourse of students and teachers, with each other and with the computer, in mathematics classrooms that include the use of curriculum-embedded software on computers. This study examined student collaboration, group dynamics, and discourse while using CPMP-Tools. In addition, the levels of mathematical thinking (Wood, Williams, & McNeal, 2006) revealed in the discourse of students and teachers during particular types of interactions were explored. This study provides a description of the discourse that takes place in two types of computer environments—whole-class/single computer or small-group/multiple computers.

**Research Questions**

Three overarching questions guided this study.

1) *What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools?* Sub-questions that will be used to answer the first question are: a) *What interaction patterns are present in*
mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? and b) What interaction patterns are present in classrooms in which CPMP-Tools is used in whole-class interactive lessons with a single computer?

2) What is the nature of students’ mathematical thinking while using the curriculum-embedded software? Sub-questions that will be used to answer the second question are:
   a) What levels of mathematical thinking are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? and b) What levels of mathematical thinking are present in a classroom in which CPMP-Tools is used in whole-class interactive lessons with a single computer?

3) What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking?

**Significance of the Study**

Three main groups of people may benefit from the findings of this study. First, teachers who are teaching the second edition of *Core-Plus Mathematics* would profit from knowing the kinds of student interactions that may take place during investigations with *CPMP-Tools*. Teachers could use this information to find the best ways to design their lessons to avoid possible pitfalls, or enhance their facilitation of the group work involved in the investigations utilizing the software. It could help them make decisions about the type of computer environment to use for the launch of a lesson, or for an entire investigation. There may be times when a whole-class teacher-led demonstration works well, and times when the students would benefit more from operating the computer themselves.
Second, curriculum developers, including the authors of *Core-Plus Mathematics*, may be interested in a description of how students are pursuing questions and engaging in other types of mathematical thinking while using *CPMP-Tools*, and how students interact with the general purpose tools and the custom tools. They may begin to see what issues arise involving the software or the mathematics surrounding the tools, and whether the use of software affects the types of questions students ask or the way students learn the concepts and how the use of the software affects the collaborative learning of the students, and the students’ responses on assessments.

It may also be interesting to see how the internet access to *CPMP-Tools* allows students to use the tools in more flexible ways. The fact that students can use the software in the library or at home may foster a different relationship between students and *CPMP-Tools*, than that of students and graphing calculators or other kinds of dynamic software. Using the first edition of *Core-Plus Mathematics*, teachers were often hindered in their ability to assign homework that required the use of technology. The fact that some students cannot afford a graphing calculator is still an issue for some teachers who desire to utilize this technology. Most schools cannot provide graphing calculators for all students. Even schools that purchase licenses for software like *Geometer’s Sketchpad* cannot provide access to that software for all students at home. Using the technology is often limited to the classroom, and students must find alternative ways to do problems at home. On the other hand, with the increased availability of *CPMP-Tools*, students are free to explore and solve problems using the technology that they are accustomed to using in class. There is less of a disconnect between the work done at
home and the work done in the classroom. Student interviews may reveal how they use CPMP-Tools outside the classroom, and how that may relate to their in-class work.

Third, the results of this study may be useful for people who provide professional development for Core-Plus Mathematics teachers. The Core-Plus Mathematics Project offers comprehensive professional development and training sessions to teachers who plan to implement, or who are currently implementing Core-Plus Mathematics. The inclusion of CPMP-Tools will likely be a major focus in professional development sessions. The same aspects that are of benefit to future Core-Plus Mathematics teachers may also be of help to professional development planners in identifying issues relevant to teachers who are implementing lessons that include use of computer software.

The results of this study may be of most interest to Core-Plus Mathematics teachers, curriculum developers, and professional developers. However, others who use similar software on computers can gain insight from the results regarding the types of interactions and mathematical thinking present in various computer environments. Additionally, since CPMP-Tools can be obtained via the internet, and is freely available for anyone to use, teachers using different curricula may see like benefits as they attempt to use it with their students.
CHAPTER II
LITERATURE REVIEW

This study is grounded in several bodies of related literature. The students in the study were learning mathematics in a way that is consistent with the social constructivist theory of learning. Thus the literature that explains this theory will be reviewed first. Then an interpretive framework will be reviewed that bridges the gap between social constructivism and cognitive psychology. Collaborative learning is consistent with the social constructivist theory. Studies involving collaborative learning will be examined next, including three key attributes to effective group work. Finally, research concerning the use of computers in mathematics classrooms will be reviewed—including the advantages and disadvantages that have been found, studies that included collaborative learning and computers, studies involving laptop computers, relationships in technology-rich environments, and studies specifically regarding software for the strands of geometry and statistics.

Social Constructivist Theory of Learning Mathematics

The constructivist theory of learning posits that students construct knowledge by making new information fit with what they already know (or believe) about the world (Stiff, Johnson, & Johnson, 1993). Students grow cognitively when they attach new information to existing knowledge. According to this theory, students learn mathematics by taking what they already know about something, using that while exploring something
new, and then adding to, or reconstructing their knowledge (Piaget, 1964). In this way, mathematical facts and rules are incorporated into students’ existing knowledge in a way that makes sense to them.

Social constructivism treats individual people and the social realm as inseparable (Ernest, 1996). According to this theory, learning is a social activity. When students are given opportunities to interact with the teacher and their peers, these are considered opportunities to construct mathematical knowledge (Cobb, Wood, & Yackel, 1990). By providing opportunities for social interaction, small group collaborative learning is a main avenue for students to construct knowledge.

Schoenfeld (1992) makes a connection between learning and social activities when using his framework for categorizing aspects of mathematical thinking. These aspects include the knowledge base, problem-solving strategies, effective use of one’s resources, mathematical beliefs and perspectives, and mathematical practices. He stated that more research is needed on the interactions among these categories, and that the key to understanding these interactions, and the resulting development of mathematical thinking, is to study students in their mathematical community.

People develop their sense of any serious endeavor…from interactions with others. And if we are to understand how people develop their mathematical perspectives, we must look at the issue in terms of the mathematical communities in which students live and the practices that underlie those communities. The role of interactions with others will be central in understanding learning… (Schoenfeld, 1992, p. 363).

Thus Schoenfeld also believes that studying students’ social activity in the mathematics classroom is important for understanding the development of their mathematical thinking.

Consistent with social constructivist theory, language is instrumental to learning (Vygotsky, 1978, 1981). Through discussion, students come to understand a concept by
constructing a shared meaning (Cobb et al., 1990). For example, in Core-Plus Mathematics instruction, definitions of mathematical terms and concepts are agreed upon by the whole class (Hirsch et al., 2008). The definitions and formulas are not explicitly stated in the text in shaded boxes, but are instead discovered and negotiated by the students. Memorizing definitions given in a book does not guarantee that a student has learned the ideas represented by the words. Instead, through communication, ideas are internalized (Steele, 1999). NCTM’s Principles and Standards for School Mathematics (2000) also emphasizes the importance of communication when it states, “Through communication, ideas become objects of reflection, refinement, discussion, and amendment. The communication process also helps build meaning and permanence for ideas and makes them public” (p.60). When ideas are shared publicly, students learn and grow. This kind of instruction allows students to “invent, test, and refine their own ideas rather than to blindly follow procedures given to them by others” (Battista, 1999, p.430). Thus, the communications that occur between students, and between students and teacher, are crucial to their understanding of mathematics.

The term often used to describe the important communication that takes place in mathematics classrooms is discourse. According to Hicks (1995-1996), “The term discourse implies a dialectic of both linguistic form and social communicative practices” (p. 51). Discourse may refer to the oral or written text, or the social interactions that appear in the classroom (Hicks, 1995-1996). Pirie (1998) classified mathematical communication under six headings: ordinary language (everyday vocabulary of a child), mathematical verbal language (spoken or written words), symbolic language (written mathematical symbols), visual representation (diagrams, graphs, tables, etc.), unspoken
but shared assumptions (a means by which students’ mathematical understanding is communicated that is specific to a group of students), and quasi-mathematical language (language that is specific to a group of students that has a mathematical significance not always evident to an outsider). It is important to keep in mind all of these types of communication when interpreting a student’s mathematical understanding (Pirie, 1998).

Other discourse research has resulted in identifying some patterns in the communication that takes place in mathematics classrooms. A long-established pattern of communication in mathematics classrooms is teacher initiation, student response, teacher evaluation (IRE) (Cazden, 1988; Hoetker & Ahlbrandt, 1969; Manouchehri & St. John, 2006; Wood, 1998). A pattern of communication that moves slightly away from traditional discourse is funneling (Wood, 1998). Funneling begins with a student giving an incorrect answer. The teacher then asks a series of leading questions in order to get the student to say the answer he/she wants to hear (Wood, 1998). An alternative pattern is focusing (Wood, 1998). The focusing pattern places the emphasis on a particular student’s thinking by way of the teacher focusing all of the students’ attention on a particular aspect of the original student’s response. The purpose is to clarify the meaning of the student’s response or push the student’s thinking further (Wood, 1998). Another aspect of discourse has been called building (Sherin, Louis, & Mendez, 2000). Building occurs when a student’s verbal response relates to something a classmate has just said, either by agreeing or disagreeing with the previous student’s idea, giving new insight into the previous student’s idea, or drawing on the previous student’s idea to form a conjecture (Sherin, Louis, & Mendez, 2000). The last two interaction patterns are most effective in motivating productive discourse and thus facilitating the development of a
learning community (Manouchehri & St. John, 2006; Wood, Williams, & McNeal, 2006). The patterns of interaction given above, and other lesser known patterns will be used in the analysis framework in this study.

A Bridge Between Social Constructivism and Cognitive Psychology

Through teaching experiments, Cobb and his colleagues discovered that there are three important aspects to the classroom culture. In addition to the more familiar social norms, there are also sociomathematical norms and classroom mathematical practices of which the teacher needs to initiate the negotiation with students (Cobb & Yackel, 1996; Rasmussen, Yackel, & King, 2003; Yackel & Cobb, 1996). Social norms are norms for behavior that might apply in any discipline. Some examples are: explaining answers, justifying solutions, listening to others explanations, and questioning others explanations (Cobb & Yackel, 1996). Sociomathematical norms are those norms that are particular to the discipline of mathematics. Examples of these norms are “what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation” (Cobb, 2000b, p. 70). The third aspect present in the classroom culture is mathematical practices. For instance, when children are just developing a mathematical concept, they are expected to explain and justify the details involving that concept. However, later on in the school year, that concept comes to be accepted by all, and therefore it no longer requires explanation and justification. This illustrates a shift in mathematical practices, which are more specific than sociomathematical norms, and specify the taken-as-shared mathematical content (Cobb, 2000b; Cobb & Yackel, 1996; Yackel & Cobb, 1996).
The three aspects of the “classroom microculture” (Cobb, 2000b, p. 68) that were delineated above are part of the interpretive framework (see Table 1) proposed by Cobb and Yackel (1996), which emerged from nine years of their classroom teaching experiments. The amalgamation of the social perspective (stemming from Vygotskian theory) and the psychological perspective (stemming from cognitive psychology) forms the emergent perspective coined by Cobb and Yackel (Cobb, 2000a; Cobb & Yackel, 1996). The emergent perspective assumes that an individual student’s mathematical activities, and classroom community practices have a reflexive relationship—neither occurs independent of the other, nor does one take precedence over the other (Cobb, 2000a). This perspective bridges the dichotomy between cognitive psychology and sociocultural theory.

Table 1

<table>
<thead>
<tr>
<th>Social Perspective</th>
<th>Psychological Perspective</th>
</tr>
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<tbody>
<tr>
<td>Classroom social norms</td>
<td>Beliefs about our own role, others’ roles, and the general nature of mathematical activity</td>
</tr>
<tr>
<td>Sociomathematical norms</td>
<td>Specifically mathematical beliefs and values</td>
</tr>
<tr>
<td>Classroom mathematical practices</td>
<td>Mathematical conceptions and activity</td>
</tr>
</tbody>
</table>

(Cobb, 2000a, p. 321)

**Cooperative Learning in Mathematics**

Research has shown that cooperative learning is beneficial to students’ learning of mathematics. According to Davidson & Worsham (1992), cooperative learning enhances individual thought by expanding each person’s realm of thinking through shared visions and understandings. In a meta-analysis of 122 studies of the use of cooperative learning
in mathematics classrooms at various grade levels, Johnson, Maruyama, Johnson, & Nelson (1981) found that cooperative learning promotes the use of higher-quality reasoning strategies and the construction of new ideas and solutions. They also found that the mathematical knowledge learned within the group, transferred well to subsequent problems the students did individually. Slavin (1990) reviewed 99 studies that involved cooperative learning in mathematics classes at all grade levels, and concluded that it was effective in improving student achievement. A study that focused only on secondary mathematics students found that students who learned in cooperative learning groups scored increasingly better on three chapter tests than those who learned independently (Whicker, Bol, & Nunnery, 1997). Webb (1982) found that cooperative learning develops higher level thinking skills in mathematics, and Hagman and Hayes (1986) found that it promotes high achievement in mathematics and class attendance. According to Davidson and Kroll (1991), there is evidence that cooperative learning in mathematics “promotes self-esteem, increased efforts to achieve, enhanced psychological health and caring relationships, and ability to take the perspective of another person” (p. 363).

Good, Mulryan, and McCaslin (1992) asserted that simply putting students together in groups will not in and of itself increase student learning, but it is the quality of group work that is important.

A main factor that seems to contribute to the effectiveness of cooperative groups in mathematics classrooms is an explicit structure (Cohen, 1994a; Davidson & Worsham, 1992; Johnson & Johnson, 1990). Cooperative group work is more than just having students work in close proximity (Davidson & Worsham, 1992). It involves communication among students in a way that allows them to develop a shared conception
of the task at hand (Dillenbourg, Baker, Blaye, & O’Malley, 1996). A group structure can facilitate students’ construction of knowledge, and discourse among students (Johnson & Johnson, 1990). It helps to ensure that all students participate equally.

Studies in various disciplines including mathematics, from elementary to post-secondary grade levels, have found that the use of group roles is important for cooperative learning to be effective. A study at East Carolina University (Mennecke & Bradley, 1997) demonstrated the impact of group roles by comparing two sections of an Information Systems course that used the roles with two sections that did not. The researchers found that the teams that used group roles reported a more positive experience with the project, and they also scored significantly higher on the project. In an article about motivating students in mathematics classrooms using cooperative learning, Bernero (2000) cites the assignment of group roles as one of the keys for students to successfully accomplish tasks in elementary mathematics classes. Bernero found that having assigned roles gave the students a sense of responsibility toward the team and the completed work. Giving each person in the group a specific responsibility reduces the likelihood that one person will do most of the work. Another study (Prescott, 1990) involving 51 elementary and secondary classroom teachers in various disciplines, found that clear student roles were the key to successful implementation of cooperative learning. Homan & Poel (1999) found that if group roles were not clearly defined for the group members in a university-level English-as-a Second Language course, the group work was less effective.
Three Key Attributes of Effective Group Work

Cohen conducted or supervised over fifty studies on cooperative learning groups, especially in mathematics classes. The techniques for group work given in this section were a result of those numerous studies. In various settings, “they have proven to be highly effective” (Cohen, 1994a, p.4). “Effective” or “productive” in these studies meant that the students exhibited the following kinds of intellectual and social learning goals: conceptual learning, creative problem solving, increased engagement, active learning, task-oriented behavior, increased talking and working together, decrease in disengagement of a single student in a group, delegation of authority to the students, and students asking questions, explaining, making conjectures, making and analyzing arguments, listening, debating, and making joint decisions (Cohen, 1994a, 1994b, 1996, 1999, 2002).

Cohen found that group roles are the first key attribute of effective group work (1994a, 1994b, 1996, 2002). According to Cohen (1996), the teacher must foster the interaction between students and delegate the authority to the students. She found that the use of group roles transfers the authority to the students. She said that, “unfortunately, the use of roles is rather widely misunderstood as a restriction of intellectual autonomy on the part of the students. The contrary is actually true” (1996, p.13). The roles help the teacher to manage the group’s operations, and therefore the teacher does not have to interfere as much. Two of the logistical practices that she advocated are using a role chart and frequently rotating the groups and the roles of group members (Cohen, Lotan, & Scarloss, 1999). Cohen (1996) has identified the facilitator as a key role because he or she is the person who sees to it that each person gets help if
they need it, and that everyone participates. She mentioned other roles such as reporter, materials manager, and measurer. These roles, with the addition of quality controller, match the roles often used in the *Core-Plus Mathematics* curriculum. (The person with the role of coordinator in *Core-Plus Mathematics* acts as the materials manager). Cohen described how each of the roles fosters group interaction and discourse, avoids problems of nonparticipation and interpersonal difficulty, and facilitates completion of a task (1994a). Each group member is like a different piece to a puzzle; the puzzle cannot be completed without any one of them.

The second key attribute of effective group work in mathematics classrooms is the development of norms of behavior (Cohen, 1994a, 1996). Training students how to engage in communication in a group enhances discourse (Cohen, 1996). Thus students can be taught how to share their ideas, ask questions of each other, make conjectures, justify their ideas, listen to others, build on one another’s ideas, pull ideas together, and come to a consensus (Cohen, 1994a, 1996). Cohen found that these norms of behavior enhanced the communication that occurred in groups in mathematics classrooms and thus facilitated the learning of mathematics.

A third key attribute of effective group work in mathematics classrooms is assigning a “true group task” (Cohen, 1994a, 1994b). Cohen describes a true group task as one that is ill-structured, requires a variety of skills and behaviors, has more than one way to solve it, and is challenging. Correspondingly, (Dillenbourg, Baker, Blaye, & O’Malley, 1996) state that, “It is also clear that the nature of the task influences the results: one cannot observe conceptual change if the task is purely procedural and does not involve much understanding…Some tasks are less ‘shareable’ than others” (p. 199).
Using group work with a mathematics task that does not meet most of Cohen’s criteria will not allow for the realization of the potential for effective group work. Williams (2000) supported the need for challenging tasks in her study of two groups of secondary mathematics students. She found that the combination of the task complexity and the cooperative group learning contributed to sustained student engagement (measured using a psychological concept of flow of consciousness) and increased learning (measured using tests of mathematical understanding administered before and after each instructional period). She defined unfamiliar challenging problems as tasks that: (a) are presented before the relevant mathematical concepts have been “taught”; (b) cannot be solved by the application of algorithmic procedures assumed known to the students; and (c) require students to analyse mathematical representations to connect mathematical ideas and to build concepts new to them (Williams, 2000, p. 657).

**Theoretical Framework for Group Work**

Students in this study collaborated in small groups since group work is an integral part of the *Core-Plus Mathematics* curriculum (Hirsch et al., 1995). The following framework will be used to highlight significant dimensions of, and interactions inherent in, effective group work (see Figure 1). This framework is based primarily on the research and findings of Elizabeth Cohen (1994a).

The outer square represents the task that is the object of the group’s work. Cohen considered a true group task to be a necessary condition for effective and productive group work (1994b). The authors of *Core-Plus Mathematics* strive to create problems that are open-ended and problematic, such that many are not so clear-cut as to have only one right answer. These problems are true group tasks according to Cohen’s criteria (Cohen, 1994a). The completion of the task, and the resulting answer, is enriched by the
work of the group. These kinds of tasks are most successfully completed if each person in the group contributes and plays their part (Cohen, 1994a). Thus the task, represented by the square and its interior, is divided into puzzle pieces signifying the idea that each student has a piece of the puzzle that is needed to complete the task/puzzle.

Figure 1. Framework for effective cooperative group work based on Cohen (1994a, 1994b, 1996).

Each circle represents a student in the group. There are four circles pictured, and therefore four pieces to the puzzle, because this is considered to be the “optimal” number
of students to have in a group (Cohen, 1994a, p. 73). “As the group gets larger there is more of a chance that one or more members will be left out of the interaction almost entirely” (Cohen, 1994a, p. 73). If there are three people in the group, two of them are likely to form a coalition, isolating the third student (Cohen, 1994a). However, due to absences or a total number of students not divisible by four, the group may have to contain a different number of students. In that case, the puzzle can be divided into a different number of pieces. For instance, when there are only three people in the group, the role of quality controller is taken on by all of the members in the group in addition to their other duties.

The labels in the circles correspond to the group roles most commonly used in Core-Plus Mathematics, but those could also be changed to different roles to fit other situations. The important part is that each student has a role (Cohen, 1994a). The bi-directional arrow between each pair of students (circles) signifies the interaction that should occur between them. Thus, each student is connected by bi-directional arrows to the rest of the students because he/she should be interacting with all of them. The nature of that interaction is somewhat determined by the role each student has. For instance, the coordinator makes sure that each student is contributing his or her ideas and, therefore, asks for input from each of his/her group mates. Finally, the activities listed in the middle of the diagram are the expected behaviors of all the students in the group. These norms for behavior are the final key attribute of effective group work (Cohen, 1994a, 1996).
Learning Mathematics with Computers

Another body of research relevant to this study is the research that has been done on the use of computers in mathematics classrooms. The research cited above involves cooperative learning in mathematics classrooms without the use of computers. Following a summary of research below done in mathematics classrooms using computers, there will be a summary of research encompassing cooperative learning with the use of computers. Research specific to laptop computers and other types of mathematical computer software will also be reviewed.

Advantages and Disadvantages of Using Computers in Mathematics Classes

Ruthven and Hennessy (2002) studied computer use in seven different college mathematics departments and identified 10 advantages to computer use in the classrooms: ambience enhanced (break from the routine), restraints alleviated (eliminating the drudgery of some work), tinkering assisted (supported an experimental approach with self-correction), motivation improved, engagement intensified (students more willing to keep trying), routine facilitated (routine components carried out more quickly, easily, and reliably), activity effected (securing and enhancing the pace and productivity of classroom activity), features accentuated (provision of vivid images and striking effects through which mathematical constructs are accentuated), attention raised (avoiding or overcoming the need for attention to subsidiary tasks increases the focus on overarching ideas), and ideas established.

According to Zbiek (2003), using computers in mathematics instruction has the benefits of rapidly generating examples, decreasing computational errors, creating opportunities for students to pose their own problems, and making students’ thinking
more visible to teachers. When technology is used effectively, it allows students to focus on the mathematics by reducing tedious calculations, and engage in problems that interest them by giving freedom to explore. Additionally, technology can be utilized to help students or teachers present, pose, compare, interpret, and reflect on problems and their solutions (Zbiek, 2003). According to Tall (2000), the visual aspect of computers can enhance conceptual thinking processes. Similarly, Peressini and Knuth (2005) found that representations of a problem situation, made on the computer, can be more powerful, can extend beyond the problem, and can help students better understand the mathematics inherent in the task. Clements (1991) found that a computer environment can enhance both figural and verbal creativity in elementary students.

Through multiple representations, computers can help students make connections among mathematical topics and strands. Hollenbeck, Wray, and Fey (2010) assert that this is “one of the most valuable contributions of computers to mathematical work” (p. 273). On a single screen, computer software can produce graphical, numerical, and symbolic representations of a function or other relationship. This visual helps students make connections between concepts. For instance, by viewing the graph, equation, and table all on one screen, they could discover that the y-intercept of a graph is the constant term in the corresponding equation and the value in the table when \( x = 0 \). Making connections in mathematics is beneficial to students’ learning (NCTM, 2000). The development of technology has the ability to link strands like algebra and geometry and thus strengthen the teaching and learning of mathematics (Jones, 2010).

However, simply using computers in the classroom does not necessarily guarantee that students will gain a deeper understanding (Hollenbeck, Wray, & Fey, 2010). Much
of the research stated above applies to the effective or appropriate use of technology. It is possible for technology to be used without realizing the potential benefits. For instance, a student could simply use a geometry tool to find measurements without thinking about the reasons for the results or the patterns that exist in the diagram and corresponding measurements. Technology can enhance the curriculum (Hollenbeck, Wray, & Fey, 2010) and teachers play a key role in whether or not this is accomplished. Knowing when to use technology is an important consideration (Kutzler, 2010). Kutzler (2010) used the analogy of knowing when to drive, or walk instead, to another location. If tedious calculations would obscure the structure of the problem then the technology is helpful. For example, when students are trying to grasp the concept of the sum of squared errors, performing all the calculations by hand with a relatively large data set may conceal the true meaning of the statistic—especially when errors are made. However, if learning the calculations are the focus, it may be best to perform them by hand. So when students need to learn how the sum of squared errors is calculated, it may be best to have them perform the calculations by hand—perhaps with a relatively small data set.

Yerushalmy and Chazan (2003) found that when secondary students use computers in algebra classes, they exhibit new strengths that they may not otherwise exhibit. On the other hand, the technology may present new difficulties for the students. For instance, some students found more difficulty using the explicit form of an equation than students from a traditional curriculum because use of the recursive form became more common while using technology (Yerushalmy & Chazan, 2003). In their study, as well as in the Core-Plus Mathematics curriculum, the use of technology changed the
order of some of the key concepts in the curriculum. Thus a major conclusion of the study was that, “…technology is important, but not as something separate, rather as but one component of the curriculum with which students interact” (Yerushalmy & Chazan, 2003, p. 7). Based on observations, analysis of student work and student surveys, McMaster (2005) also found that learning is not necessarily enhanced by the use of computers if it is not accompanied by student-centered, problem-solving activities. Similarly, Fey, Hollenbeck, and Wray (2010) assert that the use of technology is more effective as an integral part of the curriculum. These statements support the choice of studying the computer software developed specifically for use with a particular curriculum such as Core-Plus Mathematics.

Cooperative Learning with Computers

Research in this area is lacking at the high school level. However, some studies that combine both the elements of computers and cooperative learning have been completed in different classroom environments than the ones in the current study. A study was conducted at a middle school that participated in a program called the Computer- and Team-Assisted Mathematics Acceleration (CATAMA) course (Mac Iver, Balfanz, & Plank, 1998). This was a course for middle school students who needed extra help in mathematics and it combined the use of computer-assisted instruction (CAI) and structured cooperative-group learning. Students worked with a partner on a single computer and then two partnerships were combined to form a group. The study reported dramatic gains in the students’ procedural knowledge and skills using the Stanford 9 Achievement Test. It also reported that the interactions between student and teacher, student and student, and student and computer were productive and on-task. The
environment combining computer activities and cooperative learning was engaging and kept even the most restless student focused on mathematics. Based on qualitative observation data, students were talking about mathematics, actively helping each other, and sharing problem-solving strategies (Mac Iver, Balfanz, & Plank, 1998).

Heid (1997) defines technology-intensive instruction as “instruction that assumes constant student access to technology tools” (p. 9). According to Heid (1997), “Cooperative-group work is an instructional strategy that comes naturally in the context of computer-intensive instruction” (p. 37). Student collaboration may even happen unplanned due to the need to share computers, look at each other’s screens, and the interactive nature of computers. The computer can become a “meditational resource which can contribute to create a shared referent between the social partners” (Dillenbourg, Baker, Blaye, & O’Malley, 1996, p. 203). Whether group work is planned or unplanned, the usual roles of teacher and students are shifted. Teachers become technical assistants, collaborators, and facilitators of group work. When using computers, students naturally rely more on other students, communicate more with other students, and take on more responsibility for their own learning (Heid, 1997).

Wood (1992) conducted a college level study with a developmental algebra course. This study used a control group of students who received only lecture and an experimental group that used cooperative learning in a computer laboratory. In the experimental group, 23 students completed the course compared to 15 students in the control group. Using the Fennema-Sherman Mathematics Anxiety and Confidence Scales Test, “the control group showed greater increases in post-course confidence ratings and greater reductions in anxiety ratings than the experimental group” (Wood,
The experimental group was asked to evaluate the group work process they experienced. The comments they gave were overwhelmingly positive about the group work. The experimental group also earned higher grades overall. Of the experimental group, 69% received a grade of A, B, or C, compared to 52% of the control group.

Another study on the integration of computers and cooperative learning occurred in a third-grade mathematics classroom (Xin, 1996). The cooperative learning structure used was the Team-assisted Individualization approach proposed by Slavin (1990). Students were grouped heterogeneously into groups of four with a pair of students sitting at each computer. Three commercially-produced software packages covered third-grade mathematics computation and applications. The results showed that student achievement improved, students liked to work with partners at the computer, and student attitudes about mathematics were more positive than before the program was implemented.

Clements and Nastasi (1999) studied elementary children’s explanations in a Logo computer environment. They found that Logo facilitated peer interaction, which aided metacomponential processing (processes indicative of metacognition) (Clements & Nastasi, 1988). This study also supported an earlier finding by Nastasi, Clements, and Battista (1990) that the resolution of cognitive conflict that arises out of social interchange facilitates the development of higher-level cognitive processes. Additionally, the study furthered social-cognitive theory by determining that the conflict resolution strategies that involve attempts to synthesize viewpoints are particularly important for cognitive growth.

Hennessy, Deane, and Ruthven (2005) reported on a number of small-scale projects undertaken by 15 teacher-researchers. Each project involved developing
materials and pedagogical approaches using various forms of computer-based information and communication technology (ICT) to support teaching and learning in secondary classrooms in six disciplines (English, classics, design technology, geography, history, and science). They found that in 10 of the 17 lessons they observed, all or the majority of the students were engaged in collaborative activity as defined by students discussing the problem, checking suggestions with each other, and sharing technical expertise. The students were purposefully working together at a single computer trying to accomplish a common goal. The teachers in the study reported that there was more peer interaction in the ICT-supported lessons than in other lessons. Some of the reasons cited were the physical ease of students coming together to be close to the computer, the visual stimulation generating ideas, and working on the screen made student thinking visible to others.

Two dissertations were found that focused on topics similar to the current study. Yu (2004) conducted a naturalistic study to describe the discourse of middle school students using Geometer’s Sketchpad (Jackiw, 2001). He found that the students used the linguistic tools of metaphors, metaphorical-action, and literal description. The environment created by the activity and the software facilitated conversation between students and between students and the teacher (Yu, 2004). Taylor (2005) observed and videotaped pairs of middle school students working with mathematical applets designed to explore 7th- and 8th-grade mathematics in accordance with cooperative learning and constructivist principles. A main goal of the study was to see how students interacted with each other while using the computer. She found that some pairs interacted successfully while others did not. A pair was deemed successful if they shared their
thinking about the problems and discussed them until a solution was reached. It was determined qualitatively that pairs were unsuccessful at interacting if they did not construct knowledge together or communicate well (e.g., bickering or not talking to each other). She noted that the students were not used to working together on a regular basis and she attributed the difficulties to different learning styles and ability levels (Taylor, 2005).

The present study investigated whether or not the same kinds of student interactions and attitudes are present in high school mathematics classroom environments that include use of curriculum-embedded software in either small groups of students using laptops or in whole-class teacher-led discussions using a single computer.

**Laptop Computers**

Educational benefits have been found in mathematics, science, and English classes in schools that have provided laptop computers for their students. Benefits included increased student motivation (Gardner, 1994; Rockman, 1998), movement toward student-centeredness (Stevenson, 1998; Rockman, 1998), and better school attendance (Stevenson, 1998). Fisher and Stolarchuk (1998) studied laptop use in middle school science classrooms. They found that laptops had the most positive impact on student learning and attitudes in those classrooms that emphasized the process of inquiry. Rockman (1998) found an increase in cooperative learning and project-based instruction in classrooms that used laptops. Lewis (2005) has taught with laptops for more than eight years and reported advantages such as 1) students have immediate access to the internet without having to go to the computer lab, and 2) students can switch from using the mathematics software to taking notes with the word processor all from their own desk.
Another relevant finding from Stevenson (1999) is that eighty-five percent of the students in his study believed the laptops did not hinder classroom communication but instead enhanced student interaction.

The only major study involving laptop computers in mathematics classrooms found in this review of the literature was done in Canada. Raaflaub and Fraser (2002) surveyed 1173 students in 73 mathematics and science classrooms using laptop computers in Ontario. The survey measured students’ perceptions of actual and preferred classroom learning environments and students’ attitudes toward their classes and towards the use of laptops. The survey was followed by in-depth case studies of selected classrooms. Overall, the researchers found that girls favored the classroom environment with laptops more than boys and students in science classes favored the environment more than the ones in mathematics classes.

**Relationships in a Technology-rich Environment**

In any mathematics classroom there are relationships among, and between, each pair of the following: student, teacher, mathematical activity, and curriculum content. In a technology-rich environment, a fifth element—the technological tool—is added to the mix (Zbiek, Heid, Blume, & Dick, 2007). There are complex relationships between the tool and each of the four original elements in the classroom. Of particular interest in this study is the student-tool relationship. In the same way that student-to-student interactions have the potential to aid in the internalization of ideas, student-to-computer interaction may aid in that internalization (Dillenbourg, Baker, Blaye, & O’Malley, 1996). The term *instrumental genesis* is a construct used to discuss this relationship (Bretscher, 2010; Drijvers, Doorman, Boon, van Gisbergen, 2010; Zbiek, Heid, Blume, & Dick, 2007).
The use of a technological tool has two facets—that of a material artifact and that of a psychological instrument (Artigue, 2002; Zbiek, Heid, Blume, & Dick, 2007). Artigue (2002) described the construct as follows:

Thus an instrument is a mixed entity, part artefact, part cognitive schemes which make it an instrument. For a given individual, the artefact at the outset does not have an instrumental value. It becomes an instrument through a process, called instrumental genesis, involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes (p.250).

Thus instrumental genesis involves a complex relationship between the user of the tool, the tool, and the mathematics inherent in the tool’s use. The value of the tool depends on who is using the tool and how that person is using it. Further, as students develop a relationship with the tool, the way in which they use the tool changes (Heid, 2010). Through instrumental genesis students come to understand the mathematics of the tool (Zbiek, Heid, Blume, & Dick, 2007).

**Dynamic Geometry Software**

Lessons in the Core-Plus Mathematics curriculum offer frequent and rich opportunities to study the use of dynamic geometry software (DGS). Research has shown that DGS can enhance student learning. DGS possesses the unique dragging feature that makes it more powerful than paper-and-pencil methods because students can see many examples in a short time, get immediate feedback, and easily look for properties, special cases, or counterexamples (Marrades & Gutierrez, 2000). Additionally, when an object is dragged, it preserves the properties that were inherent in its construction (Fey, Hollenbeck, & Wray, 2010). If students are only provided with static pictures on paper, misconceptions can arise. For example, when students are only exposed to obtuse angles that have one ray drawn horizontal, they may believe that is the
only way an obtuse angle can appear (Clements & Battista, 1992). However, with the ability to move an angle around on the screen, students can see that an obtuse angle can appear with neither ray in the horizontal position. Battista (2007) found that dragging also provides the advantages of being alluring to students, and making the invariant properties of a figure more apparent. Ruthven, Hennessy, and Deaney (2004) found DGS to be helpful by avoiding the repetition of drawing and measurement and by increasing accuracy. DGS allows students to experiment freely, easily check their intuitions and conjectures, explore before trying to produce a deductive justification, and make meaningful representations of problems (Marrades & Gutierrez, 2000). Students in this study will be using the geometry tools in CPMP-Tools to drag objects and move angles as the research above discusses.

The role of proof in geometry has been a key issue in the debate about using computer software in mathematics classrooms. Laborde (2000) summarizes the results of four studies, all of which concluded that DGS does not threaten the need for proofs in geometry. In fact, use of the software may motivate students to prove why their computer-based construction works, or it may allow students to better understand the justifications that underlie the proof (Laborde, 2000). Marrades and Gutierrez (2000) found that DGS improved the quality of students’ justifications, and helped “secondary school students understand the need for abstract justifications and formal proofs in mathematics” (p. 119). Similarly, Jones (2000) discovered that DGS helped students shift from imprecise ordinary language to more mathematical explanations that facilitates further deductive reasoning.
One of the advantages of DGS previously mentioned is the immediate feedback it provides. This feedback may be in the form of measurements displayed on the screen or in the visual representation of the figures. From a constructivist point of view, the feedback that the DGS provides may be very rich because the visual and the theoretical aspects of geometry are integrated (Laborde, 2001). For instance, as students use the drag mode to test a conjecture, the instant display of measurements update as a figure is manipulated thus providing automatic feedback (Hollebrands, 2007). If the visual result is not as expected, the students will need to alter their thinking. Learning in this way is consistent with constructivist theory. “Evidence supports a constructivist position on how children learn spatial and geometric ideas” (Clements & Battista, 1992, p. 457).

Additionally, since the computer screen allows students’ thinking and actions to be more visible, students may receive more feedback from other students, teachers, or the visual image itself. Visual imagery plays a vital role in mathematical thinking about geometric concepts (Clements & Battista, 1992). According to Hollebrands (2007), the ability to refer to a common shared screen allows students to better communicate their mathematical thinking with each other. Clements and Battista (1992) call for research that addresses the interrelationships between verbal and visual processing to learn more about how students learn geometry with the use of computer software. While Clements and Battista focus on elementary students in their research, the same need for research exists for students of all ages.

Another researcher, Sinclair (2003), had intended to design a study to investigate the benefits and limitations of using pre-constructed, web-based, dynamic geometry sketches in activities at the secondary school level. However, Sinclair was drawn by her
case study data to study the interactions present in the environment, both between the students and between a student and the environment. She noted actions that students took using the software such as: pointing to the screen, dragging the diagram, using the motion button, deducting from the visual image, making erroneous conclusions, checking their understanding, modeling their thinking, reinforcing their ideas, posing inviting questions, and referencing the colors in the diagram. She concluded that the task question and the pre-constructed sketch worked together to create an environment for student exploration. Likewise, while studying how students make use of the affordances provided by DGS, Hollebrands (2007) found that students “used dragging to test a construction, verify a conjecture, observe behaviors of points under the drag mode, and search for invariances” (p. 174). The students used the measurements given by the software to explore relationships, create and verify conjectures, and to check the correctness of a construction.

Laborde (2001) noted that giving students homework that required the use of Cabri (another type of dynamic geometry software) allowed students to experiment with the use of the software at their own pace at home. Students could practice constructing images so that their in-class work could be more efficient. Laborde (2001) found that it was important that students find time to do the manipulations on their own. This is one reason why in the present study the extent to which students use CPMP-Tools at home was explored through student interviews. If Laborde is correct, then students’ use of CPMP-Tools at home could benefit their in-class work.
Statistical Software

Lessons in the Core-Plus Mathematics curriculum also offer frequent and rich opportunities to study the use of statistical software. Statistical software is drastically changing the ways data analyses are performed and statistics is taught (Hammerman & Rubin, 2004). Statisticians use computer-based tools to search for trends and patterns in data. This process is called Exploratory Data Analysis (EDA) and represents a large shift away from the usual way statistics is taught (Pfannkuch, 2005). “There is almost universal agreement among stochastics researchers that computer simulations, computer spreadsheets, and the use of computers to conduct Exploratory Data Analysis (EDA) are the directions in which stochastics education should be headed” (Shaughnessy, 1992, p. 484). According to Cobb (2000b), statistical software can help students reason about data as opposed to trying to recall memorized procedures for manipulating numbers. The software also helps students manage the complexity that is present in some data sets, so that realistic problems can be explored (Hammerman & Rubin, 2004; Konold & Higgins, 2003). Also, with the use of computers, simulations are becoming a more widely used inferential technique (Scheaffer, 2000). A goal of statistics instruction is for students to move beyond the use of textbook formulas and technical procedures and toward understanding statistical analysis and communication of results (Franklin et al., 2005; Scheaffer, 2000). Statistical software can help accomplish this goal.

Using statistical software like CPMP-Tools students can make different plots for the same data set in very little time (Shaughnessy, 1992). The software allows students to explore multiple representations of data so that they can find the most meaningful and convincing plot that answers the question at hand (Bakker & Frederickson, 2005). This
gives students a sense for the shape of the data set that cannot be determined from a table or a single measure of center. Since examining variation is key to statistical analysis, students can observe trends by exploring different models of data (Gould, 2004; Konold & Pollatsek, 2002). Computer software allows students to analyze data sets that are very large—a task that could not be done by making graphs by hand (Bakker & Frederickson, 2005). With software like *Mini-tools* (Cobb, Gravemeijer, Doorman, & Bowers, 1999), even elementary students can make their own statistical plots so that they can have a visual image of data distributions and can more easily learn more sophisticated plots later (Konold & Higgins, 2003). Using *TinkerPlots* (Konold & Miller, 2004) or *Fathom* (Key Curriculum Press, 2000) software enables students to make plots that are familiar and some that are not typically taught in school. These types of software give students the ability to sort and separate data into categories, choose an attribute by which to order or highlight data, and stack and organize a data set in various ways (Hammerman & Rubin, 2004). Students in this study used the software to make multiple statistical plots to explore data sets much like the students did in the studies mentioned above.

Using computers in statistics education presents teachers with some new aspects to consider. Just as with DGS, statistical software can make students’ thinking more visible to others (Hammerman & Rubin, 2004). The plots that students create, and the way they alter them, gives teachers a better idea of the reasoning processes taking place. The visual and exploratory aspects of statistical software are two affordances of statistical software that may be exploited (Shaughnessy, 1992). To aid students’ learning of data analysis, Bakker and Frederickson (2005) found that it was important to engage students in focused reflection on their activities with the software. McClain, Cobb, and
Gravemeijer (2000) also found that students needed to explain and justify the reasoning they went through as they used the software. If students also plot and analyze data by hand in addition to using the software their learning is enhanced (Bakker & Frederickson, 2005, Shaughnessy, 1992). Additionally, Bakker and Frederickson (2005) found that students are very motivated to use data analysis software and that it was rewarding for them. Providing students with classroom experiences in which they use computer software to help them think and reason about data-rich situations may better prepare them for the future (McClain, Cobb, & Gravemeijer, 2000). More research is needed to explore how various statistical tools empower students’ thinking (Pfannkuch, 2005).

According to Cobb and McClain (2004), there are four main things a teacher can do to support students’ learning of statistics: establish productive classroom norms, select suitable computer tools, carefully plan instructional activities, and manage whole-class discussions well. It is important that classroom discussions focus on significant statistical ideas. Cobb & McClain (2004) found that a teacher can accomplish this by listening while students work to conjecture about some of the issues that may arise. One can capitalize on the diverse ways in which students use the tools through whole-class discussions. If certain issues do not surface in the discussion, the teacher can refer to the observations he/she made while the students were at work and refer to those. The four recommendations given above are all relevant to the conditions present in this study.

Since research in statistics education with the use of computer software is relatively new, few dissertations were found on the topic. One dissertation compared the effects of computer manipulatives to concrete manipulatives in teaching elementary probability. The study explains that many teachers do not use concrete manipulatives
(despite the research that supports their use) because of classroom management issues. The results showed that students using the computer manipulatives performed as well as students using the concrete manipulatives. Furthermore, teachers preferred using the computer manipulatives (Phyliss, 2001).

Of the two dissertations found that involved both high school students and technology, one of them focused on the teachers’ implementation of the technology in their high school classrooms rather than on the students’ use of the technology (Shamatha, 2003). The other study was conducted by audio- and video-taping twenty-three students from an Advanced Placement Statistics class. The technology used was graphing calculators not computer software. However, the technology was found to have a significant impact on students’ reasoning about probability (Zimmerman, 2002). Thus, neither dissertation was particularly useful for understanding high school students’ use of statistical computer software.

Possibly the dissertation that was most similar in content to this one was Students’ Conceptual Understanding of Variability (Slauson, 2008). Though she studied college students, the statistical concept in the study was variability. This relates to the ideas of error in prediction, residual, and sum of squared errors studied in this dissertation. She also compared a class of students taught by lecture to a class of students taught using activity-based, hands-on labs designed with a conceptual change framework. Using pre- and post-tests and interviews, Slauson found some improvement of students’ conceptual understanding of variability ideas in the activity-based class. She also found that students’ understanding of the connection between data distributions and measures of variability is important for students to understand standard error.
Claiming that statistics education is still a new and emerging discipline, Garfield and Ben-Zvi (2007) performed a review of the research in statistics education to determine what knowledge had been accumulated, some of which is relevant to this study. They found that “inappropriate reasoning about statistical ideas is widespread and persistent, similar at all age levels (even among some experienced researchers), and quite difficult to change” (Garfield & Ben-Zvi, 2007, p. 4). They also found that “carefully designed sequences of activities using appropriate technological tools can help students improve reasoning and understanding over substantial periods of time” (Garfield & Ben-Zvi, 2007, p. 6). Another finding was that students working in cooperative groups to learn statistics led to higher test grades than students who learned by lecture. Also relevant to the topic in this study, research suggests that “students tend to see and use graphs as illustrations rather than as reasoning tools to learn something about a data set or gain new information about a particular problem or context” (Garfield & Ben-Zvi, 2007, p. 18). Researchers suggest less of an emphasis on drawing graphs and more of an emphasis on using the graph to make sense of the data. The research also suggests that “careful attention be paid to developing these [statistical] concepts first in informal and intuitive ways, leading to more formal notions (Garfield & Ben-Zvi, 2007, p. 25). Core-Plus Mathematics follows the recommendations given in this research review by containing carefully designed sequences of activities, making use of technological tools, using collaborative learning, having students use technology to make graphs so that the emphasis lies on their interpretation, and developing ideas in intuitive ways.
Summary

Research in mathematics education supports the use of cooperative learning groups. NCTM (2000) advocates for discourse as the primary mode of teaching and learning. Cooperative learning enables students to be active participants in their learning, engage in mathematical discourse, and construct an understanding of mathematics. Standards-based mathematics curricula like Core-Plus Mathematics provide the teacher with the rich problems that lend themselves to collaboration and with the teaching strategies and philosophy to support discourse and the construction of mathematical concepts. The Core-Plus Mathematics curriculum now adds to these aspects by incorporating curriculum-embedded software in the classroom and outside the classroom.

The research that has been done on the use of computers in mathematics classrooms suggests that students’ attitudes and learning are positively affected. Other research indicates that computers can help facilitate cooperative learning and that cooperative learning can be enhanced by the use of computers. Some research has been done on the use of dynamic geometry software and statistical software. These studies document advantages of using the software such as efficiency, visual aspects, and immediate feedback. However, more research is needed at the secondary level to discover what actually happens in the classroom when students use such software. According to Good, Mulryan, & McCaslin (1992), “…increased use of small-group learning, better use of computers, better curriculum activities, better whole-class teaching, and more selective use of content units can improve practice” (p. 193), but more information is needed “about how groups can facilitate certain student attitudes and problem-solving abilities, including the knowledge of when to work with others, when to
work alone, and how to vary these two approaches” (p. 193). Dillenbourg, Baker, Blaye, and O’Malley (1996) indicated that studies should be done that perform a detailed analysis of collaborative learning with computers. The present study addresses these issues and takes a closer look at the ways in which secondary students interact in the presence of computers and at the discourse that occurs since it may reveal their mathematical thinking.

*CPMP-Tools* contains many of the features of the computer software discussed in this review of literature, yet it is unique in that it includes “custom tools” developed for use with specific tasks in the *Core-Plus Mathematics* curriculum. Therefore, this study differs from others previously described. The tasks in the curriculum play an integral part in the software use that was observed. The specific meaningful, complex tasks that students worked on during the observations will be described in detail and be important to the analysis of the discourse that occurred.
CHAPTER III
METHODOLOGY

Purpose and Research Questions

The purpose of this study was to investigate the nature of student interaction and the levels of student mathematical thinking in mathematics classrooms that include the use of curriculum-embedded software. The cognitive activity and social interactions of students, as they explore mathematics using CPMP-Tools, are the main focus. Three overarching questions guided this study.

1) What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools? Sub-questions that will be used to answer the first question are: a) What interaction patterns are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? and b) What interaction patterns are present in classrooms in which CPMP-Tools is used in whole-class interactive lessons with a single computer?

2) What is the nature of students’ mathematical thinking while using the curriculum-embedded software? Sub-questions that will be used to answer the second question are: a) What levels of mathematical thinking are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? and b) What levels of mathematical thinking are present in a classroom in which CPMP-Tools is used in whole-class interactive lessons with a single computer?
3) What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking?

Overview of Study Context

This study employed a collective, descriptive case study involving qualitative research methods (Berg, 2004). It is a case study because the focus is on a bounded system—that is the classroom in which CPMP-Tools is utilized (Creswell, 1998). Since more than one classroom was used, the study is collective because it involves several instrumental cases (classrooms) of the use of CPMP-Tools (Berg, 2004). The cases are instrumental because the goal is to understand the phenomenon described in the research questions and not the particular teachers or students (Stake, 1995). The classrooms of four teachers were included in the study. A detailed description of the cases will be provided through an analysis of the themes that emerge (Creswell, 1998). Although the four cases provided a way to organize data for various analyses and comparisons, the general organizing feature for reporting results (see Chapter 4) was by four mathematical concepts where CPMP-Tools played a prominent role in the delivery of the curriculum. Thus, the cases are bound by limitations inherent in working with a fixed set of observation data based on a small subset of lessons within the much larger curriculum from which they were selected.

The interactions and mathematical thinking present in two types of computer environments, single classroom computer and multiple computers, were explored. Then, in order to examine the effect of the curriculum-embedded software, the interactions and
mathematical thinking present in those same classrooms, when the software was not in use, was documented using the same methodology as described below.

At the time of this study, Central Highlands High School was implementing the second edition of the Core-Plus Mathematics program and its curriculum-embedded software CPMP-Tools in a couple different ways with students. When students were working collaboratively, the most common configurations were groups of three or four, with two to four laptops per group. The students involved in this study each had their own laptop, but they did not always remember to bring it to class. Sometimes, a teacher used CPMP-Tools on his/her computer and projected the display onto a screen in front of the classroom. Since the students each had their own laptop computer, some of them individually followed along with the teacher who was working on a single computer at the front of the room.

According to Stake (1995), when selecting cases, “opportunity to learn is of primary importance” (p. 6). Therefore, this study included Central Highlands High School so that cases of student interaction and discourse while using the CPMP-Tools software with small groups of students via laptops and with the use of a single computer used for whole-class interactive lessons could be examined. Also, Core-Plus Mathematics is a Standards-based mathematics curriculum in which “true group tasks” (Cohen, 1994a, 1994b) are central to the pedagogical model. From the beginning of the textbook series (i.e., typically 9th grade), students are taught how to work in groups using the roles of Reader, Experiment, Recorder, and Quality Controller (Hirsch et al., 2008, p. 4) in some problem situations and Reader, Recorder, Quality Controller, and Coordinator (Coxford et al., 2003, p. 8) in others. Therefore, the Core-Plus Mathematics
curriculum contains two of the three key attributes supporting effective group work as identified by Cohen (1994a, 1994b): good group tasks that require discourse and the use of group roles. The third attribute, behavior norms, is the responsibility of the teacher to incorporate. Therefore, Central Highlands High School contained the primary “opportunity” characteristics needed for this study—the use of computers and cooperative learning.

Furthermore, for purposes of contrast it was advantageous to observe the same classrooms at Central Highlands High School when neither the teacher nor the students were using CPMP-Tools. The same teachers and students were observed while they were doing problems that did not lend themselves to the use of technology. Thus, the students’ interaction patterns and levels of mathematical thinking in the non-CPMP-Tools environment were used to establish the levels that normally occurred when students engaged with the CPMP curriculum materials. Comparing the computer environments to the non-computer environments using the same teachers and students was intended to reduce the differences that can occur due to different teachers’ practices, classroom norms, and different students.

Data Collection

Overview

In this study data were collected to provide a detailed and rich description of the phenomenon of students using CPMP-Tools in Core-Plus Mathematics investigations. The nature of students’ collaboration while using the software, including the types of interactions and levels of mathematical thinking during their interactions with the
software, can therefore be described. Additionally, data regarding students’ use of CPMP-Tools both inside and outside of the classroom were gathered through student interviews.

The classroom behaviors and activities of Core-Plus Mathematics teachers and their students were observed and analyzed. Data were collected using audiotape, videotape, and field notes from the class observations. In addition, interviews were conducted with the teachers whose classrooms were observed, and with selected students in those classes. Interviews were recorded and analysis of the responses was conducted by the researcher. Finally, copies of students’ completed test or quiz items related to the content that was aligned with the computer software and observed investigations were collected and those problems relevant to the observed lessons were analyzed. See Appendix F for the HSIRB protocol approval documents.

Observations

Three investigations from the geometry strand and two investigations from the statistics strand of Core-Plus Mathematics were observed and data were collected via audiotape, videotape, and field notes. For every class period in which CPMP-Tools was used in groups, the same class was observed on another day, shortly before or after, when the groups were working without the use of CPMP-Tools. For every class period in which CPMP-Tools was used on the teacher’s computer with a teacher-led demonstration, the same class was observed on another day, shortly before or after, when the teacher was leading another class discussion without the use of CPMP-Tools. A total of 17 class periods were observed without the use of computers, and 20 class periods with the use of computers. The discrepancy in observations occurred because sometimes the
observation without the use of computers took one day whereas the corresponding observation with computers took two days. See Appendix A for a list of the investigations involving the use of CPMP-Tools that were observed. Other observed investigations where CPMP-Tools was not used varied by teacher because each teacher had different plans for the way they implemented investigations.

In addition to audio- and videotaping the investigations, observation forms (see Appendix B) were used to record field notes on the physical characteristics and apparent norms of the classroom—especially any that inhibited or enhanced collaboration. The first page of the form contains space to draw the layout of the room including the placement of the tape recorders, video camera, and any other significant people or objects. The second page contains a table in which the relevant tasks of the lesson and their respective times were recorded. The group-work observation form was modified from one developed by Cohen (1994a) and was used when observing the multiple-computer environment. The teacher-led observation form was adapted from the group work observation form and was used when observing the whole-class discussions when CPMP-Tools was used in an interactive-demo mode on a single computer. Other data collected in field notes contain information on teachers (as well as students) in order to document the ways in which each facilitated whole-class discussion and group work. This was especially prevalent while the teacher was facilitating whole-class discussions. Significant events or quotes from students were also recorded. The purpose of the field notes was to add detail to the description and to aid in the analysis (Stake, 1995).

When students were working in groups, the use or nonuse of group roles was noted as was the level of cooperation among group members or the domination of the
group by any one person. Some of the questions that were answered in the field notes were: How many students are disengaged? Are they all on the same problem? Is there discussion and mathematical debate? The complete list of the specific questions can be found in Appendix B. After each classroom was observed, the researcher’s reactions and reflections were also written in the field notes.

When students were working in groups with laptops, tape recorders were placed at three different groups’ tables. The tape recorders were placed at groups of students that varied according to gender (all girls, all boys, or mixed) or other noticeable characteristics (such as quiet and engaged versus talkative and energetic). The audiotape recorded the details of the groups’ discussions. Additionally, of the three groups that were being tape-recorded, one group at a time was selected to videotape. The videotape added another dimension to the data collected by the audiotape, so that interactions between students could be interpreted more accurately. The video also captured the students’ computer screens so that the actions they took with the software could be viewed.

During teacher-led discussions using the software, the teacher, and students who spoke were videotaped. From time to time, the video captured the whole class from one side of the room to another so as to record the overall classroom atmosphere. Also, the video was used to record images on the classroom projection screen so that the use of the computer software could clearly be seen. The audio from the videotape captured most discussions that took place. However, a tape-recorder was also placed at the front of the room next to the teacher who was operating the computer.
Teacher Interviews

Each of the four teachers in this study was interviewed following multiple observations of their classes. The interviews were one-on-one, semi-structured and approximately 30-minutes in length. See Appendix C for the interview questions. Additional questions were asked based on specific issues that surfaced during the observations, or their responses. The purpose of the interviews was to solicit the teachers’ perspectives on their lessons, their perspective on the use of CPMP-Tools, any additional information that was not available in the observation, or information that supported the data collected in the observation. Since all of these teachers had taught Core-Plus Mathematics without the use of CPMP-Tools, they all could speak to any differences since the implementation of CPMP-Tools. The interviews were recorded on audio or videotape. Immediately after the interview, the researcher’s initial thoughts and reflections were entered into the field notes.

Student Interviews

At least two students from each observed classroom were interviewed for a total of 28 interviews. These students were selected based upon their willingness and availability. The semi-structured interview was video taped and lasted approximately ten to thirty minutes depending on the student’s availability and the information he/she provided. See Appendix D for the pre-planned interview questions. Additional questions were asked to elicit their mathematical thinking during investigation problems where a particular software tool was used. Many questions were asked based on things the students said or the particular software they were using. Using their laptop, the students were asked to reenact the lesson exploration/investigation they performed with the
software and to explain their thinking as they worked. They were asked to demonstrate on the computer how they have been using CPMP-Tools both in and outside of the classroom. The purpose of the student interviews was to get the students’ perspectives on the use of CPMP-Tools, further information on their thinking, and information regarding their use of the software outside the classroom. Immediately after the interview, the researcher’s initial thoughts and reflections were entered into the field notes.

**Assessment Items**

For each of the observed investigations, relevant items from the lesson quiz or the unit test were analyzed to assess the overall student performance where CPMP-Tools was involved. Analysis related to these items is described in the next section. The analysis focused on what students learned with the use of each particular general purpose or custom tool, and how use of the tool may have influenced responses on the assessment item.

**Data Analysis**

**Analysis of Observations**

Selected portions of the audiotapes and videotapes were transcribed for comprehensive analysis. The group discussions were narrowed down by eliminating those that were unclear and those that contained little mathematical discussion. All of the substantive part of a teacher-led videotape was transcribed since that was the only conversation taking place at that time. Pseudonyms for the schools, teachers, and students were used in the transcriptions and reporting.
The method of analysis for the transcribed classroom discussions was based on a quantitative-qualitative research paradigm adapted from Wood, Williams, and McNeal (2006). Two coding schemes were used on the class discussion transcripts—one for analysis of the interaction patterns and the other for students’ mathematical thinking. Each selected transcript was coded twice. The first time it was coded for interaction patterns. It was then coded separately a second time for the level of mathematical thinking. A person with a Ph.D. in mathematics education who is experienced with coding also used the frameworks to code 10% of the transcripts. Initially, there was a 91% agreement in the coding. However, disagreements were resolved through a discussion of the meaning of certain codes. With some clarification, and subsequent changes to the wording of descriptions of some of the codes 100% agreement was achieved to show that the results were replicable.

The coding scheme for interaction patterns is given in Figure 2. Most of the patterns listed are the same ones used in the study by Wood, Williams, and McNeal (2006, pp. 253-255). A majority of the additions and adaptations made to the interaction patterns were based upon the results of two pilot studies. The relevant details of those pilot studies can be found in Appendix E. As a result of continued review of the literature, the construct of instrumental genesis was also added to the analysis (Bretscher, 2010; Drijvers, Doorman, Boon, van Gisbergen, 2010; Zbiek, Heid, Blume, & Dick, 2007). Using this new code, the data were analyzed for instances in which the computer tool became the means by which a student constructed a personal schema, or instances in which the student changed how he/she used the tool due to a newfound understanding of
the mathematics inherent in the tool. This use of the term may be different than the way some researchers have referred to the process of instrumental genesis.

<table>
<thead>
<tr>
<th>Labels and Description of Types of Interaction Patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Collect Answers.</strong> (ca) The purpose of the interaction pattern is to make public the answer(s) to a problem. The pattern consists of the teacher collecting from students an answer or answers for a problem.</td>
</tr>
<tr>
<td><strong>IRE</strong> (Cazden, 1988; Hoetker &amp; Ahlbrandt, 1969). (ire) The purpose in the interaction pattern is for the teacher to check whether students know what the teacher expects them to know by asking low-level test questions. Students’ answers are constrained to short answers which the teacher evaluates.</td>
</tr>
<tr>
<td><strong>Give Expected Information.</strong> (gei) The purpose in the interaction pattern is for students to give information that has been taught, and that students are expected to know in order to evaluate their knowledge. This pattern is more open than the IRE in that students’ responses are not constrained to a short answer. However, this does not involve true open-ended questions.</td>
</tr>
<tr>
<td><strong>Funnel</strong> (Bauersfeld, 1980). (fn) This interaction pattern serves the purpose of allowing the teacher to correct a students’ incorrect answer instead of telling the student the correct answer. Through a series of test questions (the answer to which is either right or wrong) or fill-in-the-blank questions (teacher asks a question or makes a statement but leaves a blank at the end for the student to fill in), it leads student(s) to the answer.</td>
</tr>
<tr>
<td><strong>Teacher Explain.</strong> (tex) In this interaction pattern, the teacher gives explanations for the mathematical ideas/concepts of the lesson or situation under consideration. The purpose is to tell students the information they are expected to learn and know. The pattern is often distinguished by long segments of teacher talk.</td>
</tr>
<tr>
<td><strong>Hint to Solution.</strong> (hs) The purpose of this pattern of interaction is for the teacher to ensure that students can solve a non-routine or open problem and get the correct answer without struggling, being confused, or taking a long time. The teacher gives a hint to the solution that takes the mathematical challenge of the non-routine problem away for the students, often reducing the problem to a simple calculation. This also assures the teacher that the students will get the correct answer.</td>
</tr>
<tr>
<td><strong>Exploring Methods.</strong> (em) This function of the interaction pattern is for students to give explanations for how they solved the problem or arrived at answers to a problem. The goal is for students to give several different strategies.</td>
</tr>
<tr>
<td><strong>Teacher Elaborate.</strong> (tel) A teacher uses this form of interaction to expand on, elaborate, or add information to a student’s explanation as a way of providing information that the teacher believes is lacking in the students’ comments for those listening as well as the student explaining.</td>
</tr>
<tr>
<td><strong>Focus</strong> (Wood, 1994). (fc) The teacher uses this pattern of interaction to orient students to a critical aspect of a problem that they need to solve or resolve. The teacher gives a summary of what the group knows and understands and then...</td>
</tr>
</tbody>
</table>
asks a high-level question that focuses attention on a critical aspect of the problem that they still need to solve or resolve and then turns the discussion to student(s) to solve/resolve.

Inquiry. (iq) The basic idea that underlies this pattern of interaction is understanding and exploring ideas, thoughts, and/or methods. Students or teacher ask questions or make statements that indicate they do not understand a student’s explanation, method, or idea; and ask for clarification of the student’s meaning, or explore the idea further.

Inquiry Using the Software. (iqs) This type of interaction is reserved for times when the students are using the software to explore an idea or test a conjecture. This interaction covers DeVilliers (1997) idea of proof as verification in that it is used for the discovery or invention of new results.

Answering Another Student’s Question. (asq) In this interaction, one or more students are answering a question posed by another student. The student(s) explaining is/are helping another student understand.

Giving New Idea/Making a Conjecture. (gi) In this instance, a student gives an idea about which he/she is not yet convinced. This could be in the form of a question.

Argument. (ar) The basic idea that underlies this pattern is “disagreement” with answers, strategies (methods), ideas, or thoughts reported by others. Typically, this interaction pattern begins with a challenge from a student listener. Students participate in the resolution of their disagreement through turn-taking discourse. The final resolution results in a change of answers, strategies (methods), ideas, or thoughts on the part of one or more students.

Proof of Answer by Manipulation of the Software. In this form of interaction, the software is used to solve the problem. Here the software becomes the means to the correct answer. Following DeVilliers’ (1997) categorization of proof, two subcategories were used. Proof used in the sense of explanation (pse) is when the software is used to provide the insight that explains why a conjecture is true. Proof used in the sense of systematization (pss) is when the software provides the means for organizing the results into a theorem.

Resolution of Conceptual Issue Using the Software. (rs) In this pattern, the purpose is to use the software as a tool to think with and as a way to help students’ understanding after they have attempted but have not resolved conflicting answers or methods themselves through other representations. Here the software provides insight into a conceptual issue and/or acts as a mediator for understanding a symbolic representation.

Proof of Answer by Student Explanation. (pe) The function of this interaction pattern is for the teacher to ensure that the class has been told a correct strategy, idea, or concept. This is accomplished by selecting a student(s) to explain who the teacher knows has a correct strategy, idea, or concept. After the student has presented, the teacher may “review” the strategy of the student(s) and then check with the class for agreement.

Building Consensus. (bc) In this interaction pattern, either the teacher or a student participates as a facilitator attempting to move the class or group to agreement on a major mathematical idea or concept through questioning and
through students giving explanations. Students participate by offering their ideas about the concept or idea as they contribute to the consensus about an idea or concept. This pattern of interaction happens after much debate, and is used to establish common ground among the students.

**Checking for Consensus.** (cc) The teacher or student participates by checking with the students and listening to find out if they have any questions or comments about an idea, strategy, or concept that a student explained. The student who explained may be asked further questions or to re-explain by the listening students. In some cases, listeners give another different strategy for solving the problem or offer further explanation. The outcome is public agreement on the validity of an idea or concept given by the student explaining.

**Develop Conceptual Understanding.** (dcu) The teacher facilitates students’ conceptual understanding by posing an open-ended question that addresses a specific mathematical idea or concept. Students participate by giving their ideas in response to the teacher’s question(s). This is different from focusing in that it is not in response to a student’s answer.

**Pupil Self-Nominate.** (psn) In this interaction, a student volunteers a mathematical idea, problem, or insight that goes beyond the topic of discussion and then explains and/or justifies his/her idea. This pattern represents a high level of student autonomy as a participant.

**Building on Other Students’ Ideas** (Sherin, Louis, & Mendez. 2000). (bsi) In this interaction, a student uses something that another student has said, and adds something to it to further the idea.

**Making Observations.** (mo) This designation is reserved for an interaction done in the presence of the software used on the computer. Student(s) make statements based on what they see on the computer screen. The statements are simply observations that are not final answers, and are not prompted by teacher questioning.

**Technical Software Statement.** (tec) This interaction is used for any statements that refer to the logistics of using the software rather than the mathematical thinking involved.

**Off-Task.** (off) This code will be used for any statements that do not relate to the mathematics or the task at hand.

**Instrumental Genesis** (ig) This code refers to the instances when the computer tool becomes the means by which a student constructs a personal schema, or instances in which the student changes how he/she is using the tool due to the newfound understanding of the mathematics inherent in the tool. The tool becomes a psychological instrument for the student to use in building mathematical knowledge.

**Figure 2.** Interaction patterns adapted from Wood, Williams, and McNeal (2006).

Below is an example of the coding used on a segment of a transcript. Each line receives one of the interaction codes indicating the interaction pattern that occurred. For example, *ire* is the traditional *Initiation, Response, Evaluation* interaction pattern.
(Cazden, 1988; Hoetker & Ahlbrandt, 1969). The teacher asks a low-level question, the student answers, the teacher then confirms or denies the answer. This interaction typically takes three lines of text. Each line receives the _ire_ code. The other interaction code in the example below is _gei_ which refers to the students giving expected information—information that has been previously taught.

(tex) Mr. L.: …So, here’s what we’re going to do, we got angle A. Is angle A an

(ire) acute, a right, or an obtuse angle right now?

(ire/C) Amber: Acute.

(ire) Mr. L: It’s acute, right? Why, because it’s—

(gei/C) Several students: Smaller than 90.

(ire) Mr. L: What is it right now?

(ire/C) Katelyn: 36.

(ire) Mr. L: 36 degree angle, ok? How long is AB?

(ire/C) Boy and Girl: 18.

(ire) Mr. L: 18 units, right? How long is BC?

(ire/C) Students: 12.

(ire) Mr. L: 12, ok.

After trying various potential units of analysis for the transcripts, it was decided that each line should receive a code in order to give a measure of the amount of words (and therefore approximate time) devoted to each interaction. For instance, _tex_ is the _teacher explain_ interaction pattern. The example above only contains one line of the teacher explaining. However, there were many times when this interaction occurred for 5, 10, or even 15 lines of text in a row. Therefore, each line would get counted as having
that interaction to give it the appropriate weight. When deciding on a unit of analysis, Fey (1970) came to a similar conclusion. The term “interaction” is similar to the term “move” that Fey (1970) used when he was studying patterns of verbal communication in mathematics classrooms. He used the term move to refer to a piece of the conversation that structured the discourse. He considered using a time interval such as five seconds. However, he determined that not only was this logistically difficult to use, it also did not make as much sense given the nature of verbal speech. People may pause, or say a lot in a short amount of time. Discourse does not occur on regular time intervals. A sentence does not work because verbal utterances are rarely sentences. Therefore he settled on a half-line of text as the unit, and counted the number of half-lines of text that each move/interaction spanned. In this study, each person’s statement—whether it was a sentence, fragment, or series of ideas strung together—was usually contained on one line. Further, in all but approximately 10 of 2,301 lines of transcript analyzed, the line contained a single type of interaction. In those 10 instances, the line was broken into two and counted as two different codes.

The interaction codes are presented in lower-case letters and each line of text is coded with one of these. However, only student statements receive a code for the level of mathematical thinking. Thus, there are fewer codes given for mathematical thinking than for interactions. The codes for the level of mathematical thinking are in capital letters. In the example above, all of the students’ statements are at the Recognizing comprehending level as designated by the letter C. The teachers’ statements are an integral part of the interactions in the classroom, but it was not essential to code their
mathematical thinking because only the students’ mathematical thinking was of interest in this study. Therefore, only student statements have capital letter codes next to them.

The types of interaction patterns found in each type of classroom environment were counted and recorded in a table similar to Table 2. These data served to answer the first research question, *What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools?*

Table 2

<table>
<thead>
<tr>
<th>Example of Table Showing Type and Number of Occurrences of Interaction Patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type of Classroom Environment</strong></td>
</tr>
<tr>
<td>Interaction Pattern</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

According to Wood, Williams, and McNeal (2006), the interaction patterns that are consistent with traditional instruction are: *IRE, funnel, give expected information, teacher explain,* and *hint to solution.* The interaction patterns they labeled as consistent with reform (standards-based) mathematics instruction and constructivist theory are: *exploring methods, teacher elaborate, focus, inquiry, inquiry using the software,* *answering another student’s question, giving new idea/making a conjecture, argument,* *proof of answer by manipulation of the software, resolution of conceptual issue using the software, proof of answer by student explanation,* *building consensus, checking for consensus, develop conceptual understanding, pupil self-nominate,* *building on other students’ ideas, instrumental genesis,* and *making observations.* The interaction patterns found in all types of instruction are: *collect answers, technical software statements,* and
Thus, the interaction patterns present in a particular type of classroom environment aided in describing the nature of that environment.

The coding scheme for mathematical thinking is given in Table 3. The first two columns containing categories of mathematical thinking and corresponding examples of cognitive activity are the same as those given in Wood, Williams, and McNeal (2006). The information in the third column, which contains examples of mathematical thinking revealed in class discussion, was taken from transcripts of recorded conversations of students working with CPMP-Tools during the first pilot study. Examples, different from those shown in the table created by Wood, Williams, and McNeal (2006), were needed because their study was performed with elementary school children. However, Williams performed an earlier study in 2000 with secondary mathematics students, using this same coding scheme to describe the students’ levels of mathematical thinking. Therefore, this hierarchy of mathematical thinking, based on the widely accepted Bloom’s Taxonomy (Noble, 2004), has been applied to all ages. Yet, the hierarchy was repeatedly refined (Wood, 2000; Wood, Williams, & McNeal, 2006) resulting in the finer distinctions shown in Table 3. The categories represent levels of a hierarchy of mathematical thinking with the least demanding listed first. While a student’s statement may fit under more than one of the descriptions, the highest level that applied was always chosen as the code for that statement because the goal was to find the highest level of mathematical thinking in each interaction.

Table 3

<table>
<thead>
<tr>
<th>Levels of Mathematical Thinking</th>
<th>Labels and Description of Levels of Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical Thinking</td>
<td>Examples of Cognitive Activity</td>
</tr>
</tbody>
</table>

65
<table>
<thead>
<tr>
<th>Level</th>
<th>Thinking Revealed in Class Discussion</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizing comprehending</td>
<td>Understand concepts behind taught idea or known strategy.</td>
<td>Brandt: Is it a right triangle? Kelli: Is ABC a right angle? Brandt: Oh, yes. Kelli: Then it is a right triangle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The students have used the Geometry Tools to construct a triangle, and are using the measurements on the screen to determine that it is a right triangle.</td>
</tr>
<tr>
<td>Recognizing applying</td>
<td>Know when to use a known mathematical idea.</td>
<td>Kaeden: Ah, it's a trigonometric function. We know angle A and the length of AB. So, the tangent of A equals BC over AB.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The students are using the Explore SSA Geometry Tool, and are figuring out whether or not the triangle they constructed is the only possible triangle.</td>
</tr>
<tr>
<td>Building-with analyzing</td>
<td>Apply known mathematical procedures in a new context. Solve using a problem with a slight twist.</td>
<td>Bianca: When an inscribed angle intercepts a semi-circle, what is the measure of the angle? Emily: So, that's like a half circle, right? Brooke: Yeah, well, an intercepted angle is half the measure of the arc, so I guess it would be 90 degrees.</td>
</tr>
<tr>
<td></td>
<td>Familiarize self with problem using specific numerical examples. Systematize the numerical results and search for patterns.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>The students are using the Geometry tool to explore what happens when in inscribed angle intercepts a semi-circle.</td>
</tr>
<tr>
<td>Building-with synthetic-analyzing</td>
<td>Contrast and comparison of two methods for the difference. Interconnect various representations, operations, and assumptions. Use or recognize more than one pathway to solve a problem. Produce an independent generalization—“small discovery.” Analyze one case, or form a guiding principle to formulate a new rule.</td>
<td>[Referring to the example from recognizing applying] Tashawn: We got 10.4 too, but we used angle C instead of angle A. The tangent of angle C equals AB over BC. It gives you the same thing.</td>
</tr>
<tr>
<td>Building with evaluative-analyzing</td>
<td>Interconnect solution pathways for the purpose of identifying flaws and strengthening arguments. Pull together ideas for making a judgment. Evaluate whether a method or result is reasonable, efficient, or elegant.</td>
<td></td>
</tr>
<tr>
<td>-----------------------------------</td>
<td>------------------------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Jalen:</td>
<td>We’ll use sine.</td>
<td></td>
</tr>
<tr>
<td>Tamra:</td>
<td>But we don’t have a ratio, so we can’t use sine.</td>
<td></td>
</tr>
<tr>
<td>Jalen:</td>
<td>What do you mean we don’t have a ratio?</td>
<td></td>
</tr>
<tr>
<td>Tamra:</td>
<td>The sine of A would be BC over AC, but we don’t know AC.</td>
<td></td>
</tr>
<tr>
<td>This is another group of students using the Geometry Tool Explore SSA, and trying to determine whether or not the triangle they constructed using the given angle A, and sides AB and BC, is the only possible triangle.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constructing synthesizing</td>
<td>Formulate mathematical arguments to explain discovered patterns. Explore the problem from many perspectives rather than just work toward a solution. Integrate concepts to create new thought or ideas (new insight). Could vary in: Number of concepts involved. Diversity of the domains concepts were drawn from. Size of the conceptual leap. Spontaneity with which the process is undertaken. Progressively explore the problem to continually develop new insights.</td>
<td></td>
</tr>
<tr>
<td>Kaliana:</td>
<td>If the radiuses [sic] are the same then it’s an isosceles triangle.</td>
<td></td>
</tr>
<tr>
<td>Taea:</td>
<td>So, then the angles have to be the same.</td>
<td></td>
</tr>
<tr>
<td>Kaliana:</td>
<td>Yeah, because then you know angle one, and then you have an isosceles triangle.</td>
<td></td>
</tr>
<tr>
<td>Taea:</td>
<td>So the two other angles are the same.</td>
<td></td>
</tr>
<tr>
<td>Kaliana:</td>
<td>And they all have to add up to 180.</td>
<td></td>
</tr>
<tr>
<td>Taea:</td>
<td>So, we can take 180 minus angle one, and divide that by two.</td>
<td></td>
</tr>
<tr>
<td>The students are examining a triangle that they have inscribed in a circle using the Geometry tools, and reasoning about the angles.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constructing evaluating</td>
<td>Progressively reflect on the situation as a whole for the purpose of recognizing inconsistent information and/or finding a more elegant solution pathway. Reflect upon the process or problem solution for the purpose or recognizing its limitations and its application to other contexts. Reflect upon the solution pathway developed and its possible contribution to generic mathematical processes to employ in the future.</td>
<td></td>
</tr>
<tr>
<td>Jadyn:</td>
<td>[regarding finding the measure of an inscribed angle] Wouldn’t it be 112?</td>
<td></td>
</tr>
<tr>
<td>Marcus:</td>
<td>The central angle would be 112, but they want the inscribed angle.</td>
<td></td>
</tr>
<tr>
<td>Shealy:</td>
<td>Yeah, and look at it, it’s not greater than 90.</td>
<td></td>
</tr>
<tr>
<td>Jadyn:</td>
<td>Oh, so it must be the 56. It’s going to be half.</td>
<td></td>
</tr>
<tr>
<td>The students have constructed an inscribed angle in a circle using Geometry Tools. There are four measurements given on the screen. They are trying to figure out which one is the measure of the inscribed angle.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Adapted from Wood, Williams, and McNeal (2006) based on pilot study data of Explore Angles in Circles & Explore SSA (Hirsch et al., 2008, Keller, 2011).
Each line of transcript where a student’s mathematical thinking was determined to be present was coded with the corresponding level of mathematical thinking. The levels of mathematical thinking found in each type of classroom environment were counted and recorded in a table such as the one in Table 4. These data served to answer the second research question: What is the nature of students’ mathematical thinking while using the curriculum-embedded software?

Table 4

<table>
<thead>
<tr>
<th>Type of Classroom Environment</th>
<th>Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole-Class/Single Computer</td>
<td>Recognizing</td>
</tr>
<tr>
<td>Whole-Class/No Computers</td>
<td>Comprehending (C)</td>
</tr>
<tr>
<td>Small-Group/Multiple Computers</td>
<td>Recognizing</td>
</tr>
<tr>
<td>Small-Group/No Computer</td>
<td>Applying (A)</td>
</tr>
<tr>
<td>Whole-Class/Single Computer</td>
<td>Building-with</td>
</tr>
<tr>
<td>Whole-Class/No Computer</td>
<td>Analyzing (AN)</td>
</tr>
<tr>
<td>Small-Group/Multiple Computers</td>
<td>Building-with (SA)</td>
</tr>
<tr>
<td></td>
<td>Synthetic-Analyzing</td>
</tr>
<tr>
<td>Small-Group/No Computer</td>
<td>Building-with (EA)</td>
</tr>
<tr>
<td></td>
<td>Evaluative-Analyzing</td>
</tr>
<tr>
<td>Whole-Class/Single Computer</td>
<td>Constructing</td>
</tr>
<tr>
<td>Whole-Class/No Computer</td>
<td>Synthesizing (SN)</td>
</tr>
<tr>
<td>Small-Group/Multiple Computers</td>
<td>Constructing</td>
</tr>
<tr>
<td>Small-Group/No Computer</td>
<td>Evaluating (E)</td>
</tr>
</tbody>
</table>

Furthermore, the number of occurrences of each kind of mathematical thinking found in each type of interaction pattern in each type of classroom environment was tabulated as in Table 5. For each line of student text that received a code for mathematical thinking, a tally was placed in the corresponding interaction pattern row for the interaction code that line of text received. A similar table was created for the whole-class and small-group environments without computer use. The abbreviations for each level of mathematical thinking, shown in Table 4, were used in the third row of Table 5.
Table 5

*Example of Table Showing Students’ Levels of Mathematical Thinking within Interaction Patterns*

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Levels of Mathematical Thinking</th>
<th>Whole-Class/Single Computer</th>
<th>Small-Group/Multiple Computers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>A</td>
<td>AN</td>
</tr>
<tr>
<td>Collect Answers</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IRE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Give Expected</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Information</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Funnel</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher Explain</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hint to Solution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exploring Methods</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher Elaborate</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Focus</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inquiry</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inquiry Using the</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Software</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Answering another</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>student’s question</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Giving new idea/making a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>conjecture</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Argument</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proof of answer by</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>manipulation of</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>software</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Resolution of</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>conceptual issue</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>using software</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proof of answer by</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>student explanation</td>
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<tr>
<td>Building consensus</td>
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<tr>
<td>Checking for</td>
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<td></td>
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<tr>
<td>consensus</td>
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<tr>
<td>Develop conceptual</td>
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<tr>
<td>understanding</td>
<td></td>
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<tr>
<td>Pupil self-</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>nominate</td>
<td></td>
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<td></td>
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<tr>
<td>Building on other</td>
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<td>students' ideas</td>
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<tr>
<td>Making observations</td>
<td></td>
<td></td>
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<tr>
<td>Instrumental</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>genesis</td>
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<td>statement</td>
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<td></td>
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<tr>
<td>Off-task</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
These data help to answer the third research question: *What is the relationship between the patterns of interaction that exist in the classrooms and level of students’ expressed mathematical thinking?*

**Analysis of the Interviews**

The transcripts of the interviews and the notes written afterwards were reviewed, entries were highlighted, and notes were written in the margins of the text. From this initial analysis, the categories that emerged were tabulated in a process called “categorical aggregation” (Creswell, 1998). Themes that emerged repeatedly were noted and direct interpretations of specific instances during the interviews were used (Creswell, 1998). Tables consisting of the themes that emerged in the teacher and student interviews were created. The interview data were compared with the observation data to look for evidence that supported or refuted ideas stated in the interview. These data were used to help describe the nature of classroom environments that include curriculum-embedded software on computers.

**Analysis of the Assessment Items**

A copy was made of the quiz or test items (those related to mathematical ideas contained in the observed investigations) for each of the students in the observed classrooms. A form of categorical aggregation was used to analyze all the students’ responses. A first categorical division consisted of correct answers versus not completely correct answers. For each of those two categories, a second round of analyses occurred that formed subcategories from the themes that emerged in the students’ responses. The students’ approaches to the problems and types of errors they made served as the basis for the categories that were formed. Evidence of student thinking influenced by *CPMP-*
Tools was of particular interest. This analysis was used to help describe the students’ mathematical thinking regarding the content developed with use of the computer software. A person with a Ph.D. in mathematics education and an emphasis in statistics analyzed 50 of the students’ tests to check the reliability of the analysis. There was 100% agreement on the interpretation of the responses.

**Overall Analysis**

Before drawing final conclusions, all data were analyzed by reviewing the field notes, audiotape and videotape transcriptions. Methodological triangulation was accomplished through comparing the data collected via the various methods used (Stake, 1995). The tables that were created from the student discourse data, the teacher interview data, and the student interview data were reviewed in order to identify themes, make connections between categories, and notice patterns. Patterns and correspondence between two or more types of data were sought. The methods of categorical aggregation and direct interpretation (Creswell, 1998) were applied to the analysis of all of the data described in the previous sections. In using categorical aggregation, categories were formed from constructs that appeared repeatedly in the data. Categories that were supported by multiple forms of the data were determined. From these categories, the meanings emerged. Therefore, conclusions supported by multiple sources could be drawn about the nature of students’ interactions and mathematical thinking. See Table 6 below for the forms of data that were used to answer each research question. Additionally, using direct interpretation, a specific occurrence was analyzed and meaning was drawn from it. In other words, some explanations came from one significant or striking data source—one environment type, one classroom, or one student. The data
from the *No Computers* classroom environments were compared to the data from the other two classroom environments, and any differences were noted. Data source triangulation was accomplished when the same patterns emerged from different classes of students in the same computer environment (Stake, 1995). Through this analysis, the nature of student interaction while utilizing the curriculum-embedded software in the second edition of *Core-Plus Mathematics* was described.

Table 6

*The Correlation between the Data Collection Methods and the Answers to each Research Question*

<table>
<thead>
<tr>
<th>Research Question</th>
<th>Transcripts from Audio/Video Tape</th>
<th>Field Notes</th>
<th>Interviews</th>
<th>Assessment Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Interaction coding</td>
<td>X*</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Mathematical thinking coding</td>
<td>X</td>
<td>X</td>
<td>X (especially student interviews)</td>
</tr>
<tr>
<td>3</td>
<td>Two-dimensional pairing of interaction codes and mathematical thinking codes</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

An X indicates that the data source was used to answer the corresponding research question.

**Triangulating Data and Reporting the Findings**

The findings are reported by adhering to case study methodology. According to Stake (1995), the “best story needs to be found” (p. 121). However, Stake (1995) goes on to say that, “Once in a while, it will be useful to present the report in story form and, much more often, it will be useful to tell a few stories or vignettes to illustrate what is going on, but case study reporting generally is not storytelling” (p. 127). As an alternative to storytelling, one format suggested by Stake (1995) is to give a “description
one by one of several major components of the case” (p. 127). This is the format used in this dissertation. A description of each observed investigation is presented first, followed by illustrative examples from each teacher’s (case) data that constitute—“briefly described episodes [that] illustrate an aspect of the case, perhaps one of the issues” (Stake, 1995, p. 129). Data from other sources (e.g., interviews and assessments) are then triangulated within each investigation “story” to help address the main research questions with special attention to comparing differences in each of the teacher’s classrooms, including mathematical contexts, the computer environments, and small group and whole-class interactions.
CHAPTER IV

RESULTS

In this chapter the results are given from all data collection. The chapter is organized around each of the four sets of observations that involved the use of CPMP-Tools for a particular mathematical concept. The data are presented chronologically as they occurred in the set of observations. First, the investigation as it was designed in the textbook is described, followed by examples of key interactions that highlight each teacher’s implementation of that investigation. The interaction patterns (RQ1) and levels of mathematical thinking (RQ2) found in each teacher’s classroom are shown in tables disaggregated in ways that allow for comparisons of various investigations. Relevant data from the student and teacher interviews are woven within each lesson narrative to provide a richer description of the classroom environments of the observed teachers. Finally, the results of those assessment items, where they are relevant to the observed investigations, are discussed.

Five CPMP investigations using CPMP-Tools were observed, each involving multiple class periods, multiple days, multiple problems, and in some cases multiple teachers (see Appendix A). There were three investigations from the geometry strand that came from three different units, and two investigations from the statistics strand that came from the same unit. The two statistics investigations are reported in the same section since they occurred on consecutive days, the assessment item relates to both investigations, and they refer to the same mathematical concept. Additionally, for each
investigation using computers, the same class was observed on another day close to that
time when the students were not using computers. Each class period was 45 minutes
long. Four teachers’ classrooms were observed. Mr. Louiselle and Mr. Foster taught
Course 2 of Core-Plus Mathematics, and Mr. Nelson and Mr. Kirkwood taught Course 1
of Core-Plus Mathematics.

In the last section of this chapter, the data are combined from all of the classroom
observations in order to answer each research question. The presentation and resulting
analysis of the data coincide with the methodology used by Wood, Williams, and McNeal
(2006). Table 19 combines the data on interaction patterns (RQ1) found in all four
mathematical topics studied. Additionally, the data from the observations of the same
classes when they were not using CPMP-Tools are presented in this and subsequent
tables. Table 20 combines the data on levels of mathematical thinking (RQ2) found in all
four mathematical topics studied, as well as the observations when they were not using
CPMP-Tools. Tables 21 and 22 contain data relevant to the third research question.
These tables contain information relevant to the relationship between the interaction
patterns and the levels of mathematical thinking for all of the classroom observations.

**Mathematical Concept One: Tessellations**

The two Course 1 teachers’ (Mr. Kirkwood and Mr. Nelson) classes were
observed while their students studied tilings using the geometry custom tools shown in
Figures 5-10. Each teacher had two Course 1 classes, so a total of four classes were
observed using this tool. In all four classes, the students worked in small groups using
their laptops to investigate tessellations. Thus, this investigation was categorized under the Small-Group/Multiple Computers environment.

**The Investigation**

In *Course 1, Unit 6, Lesson 2, Investigation 3 (Patterns with Polygons)*, students explore “*Which polygons or combinations of polygons will tile the plane?*” (Hirsch et al., 2008, p. 407). In this investigation, students discover why some polygons will tile the plane, and others will not. The investigation begins with a picture of a real-life tiling from the Center for Mathematics and Computing at Carleton College in Northfield, Minnesota (see Figure 3). Then, in the first problem students are given two regular tessellations and asked to find the symmetry in them (see Figure 4). The students are also asked to find the sum of the measures of the angles at a common vertex in the pictures, and the measure of each angle at that vertex. These ideas lay the foundation for the students to discover that transformations can be used on a shape to create a tessellation, and that a tessellation can be created when the sum of the angle measures around each vertex is 360 degrees. The custom tool in *CPMP-Tools* is designed to highlight these ideas.

The second problem in this investigation asks students to cut out a triangle and trace it multiple times to create a tessellation. Ideally, each person in the group would use a different shaped triangle so that they can explore whether or not any triangle can tile the plane. After using the paper triangles, the students are to explore more triangles using the custom tool *Tilings with Triangles or Quadrilaterals* (Keller, 2011) shown in Figure 5. The lighter shaded triangle in Figure 5 is the starting triangle—it appears
**Check Your Understanding**

Being able to recognize traffic signs by their shape and color is important when driving and is often tested on exams for a driver’s license. Examine the school crossing sign at the right.

a. Identify the shape of the sign and describe the symmetries of this shape.

b. Use the design specifications shown and symmetry to find the lengths of the remaining sides of the sign.

c. Find the measures of the remaining interior angles.

d. On a copy of the shape, extend the sides to form an exterior angle at each vertex. Find the measure of each exterior angle.

**Investigation 3 Patterns with Polygons**

One of the most common and interesting applications of polygon shapes is their use as tiles for floors and walls. The photo below shows portions of two different tilings at the Center for Mathematics and Computing at Carleton College in Northfield, Minnesota. The portion of the tiling in the center of the photo is based on special tiles and a procedure for placing them created by Sir Roger Penrose, a British mathematician at the University of Oxford. Can you identify the types of polygons used for the Penrose tiles?

Later in this lesson you will have the opportunity to explore the variety of patterns that can be created with these tiles. Researchers have recently discovered that certain chemicals naturally organize themselves in similar patterns, some of which are used to make nonstick coating for pots and pans.

As you work on the following problems, make notes that will help you answer this question:

*Which polygons or combinations of polygons will tile the plane?*

*Figure 3.* The beginning of the tessellation investigation (Hirsch et al., 2008, p. 407).
Figure 4. Problems 1 and 2 of the tessellation investigation (Hirsch et al., 2008, p. 408).
automatically when the tool is selected. However, the shape of this triangle can be altered by clicking and dragging on one of the vertices. Each successive triangle is created by clicking on the midpoint of one of the edges of an adjacent triangle. When a midpoint is selected, an image of the triangle is created by a half-turn around the selected midpoint. Students can continue this process until they fill the screen, creating a tessellation of triangles. The shape of the original triangle (and therefore its images) can also be changed after the images have been created.

Figure 5. Tilings with Triangles or Quadrilaterals custom tool from CPMP-Tools.

The third problem in the investigation is similar to the second problem except that the students are to use quadrilaterals instead of triangles. Again, the students are to cut a shape out of poster paper and try to use it to tile the plane. Then, after making a conjecture about which quadrilaterals will tile the plane, they are to use the software to explore the problem further, and determine the underlying reasons for why it works. The same custom tool is shown in Figure 6 as was shown in Figure 5, but this time under
“Shape,” the quadrilateral was chosen. The lighter shaded quadrilateral appears and can be manipulated and copied in the same way that the triangle was manipulated and copied in Figure 5. The angles in each quadrilateral are numbered and color-coded to help students see why any quadrilateral can be used to tile the plane. As long as rotation is used and each of the four angle measures of the original quadrilateral come together at each vertex, there will be no gaps or overlaps because the measures of the angles of a quadrilateral sum to 360 degrees. In all classes, most students were working on, or just finishing, problem number three when the class ended.

Figure 6. Tilings with Triangles or Quadrilaterals custom tool from CPMP-Tools.

Mr. Kirkwood’s Classes

Mr. Kirkwood’s 1<sup>st</sup>-period class spent the least amount of time in discussion before they began to work in groups. Mr. Kirkwood briefly introduced the software and gave the students basic instructions on how to use it, then had them work in groups. By the time he got to his 8<sup>th</sup> period (his second of two Course 1 classes), he realized that the students needed more of an introduction to the investigation in order to give them a
purpose for exploring with the tool. The 1st-hour students had spent a lot of time enthusiastically playing with the tool and making fun and interesting shapes with the triangles and quadrilaterals, but Mr. Kirkwood constantly had to push them to figure out why the shapes tiled the plane.

In 8th period, Mr. Kirkwood first asked his students what it meant to “tile the plane.” They discussed this until they appeared to have a shared understanding of that idea. Then Mr. Kirkwood asked what polygons the students saw in the picture shown in the beginning of the investigation in the book (see Figure 3). Next, he briefly led the students through the first problem of the investigation. Finally, he displayed the CPMP-Tool Tilings with Triangles or Quadrilaterals on the screen and gave brief instructions about its use. He launched them into group work with the following questions: “Does it have any gaps? Does it have any overlaps? If it doesn’t, the big question is in 2c—why doesn’t it?” The groups spent the rest of the class period exploring Problems 2 and 3 using CPMP-Tools. Below is a segment of transcript illustrating a typical set of interactions including inquiry using the software and proof of answer by manipulation of the software of from one group in this class. Using the tool shown in Figure 5, the group is trying to justify the fact that any triangle will tile the plane.

Kylie: And now, if we go like this, see it’s going to be the same no matter what.

Abby: Yeah.

Kylie: Because it’s always going to make one [tiling].

Abby: So you can tile the plane, can’t you?!

Kylie: Yeah—ooh that one’s cool! Yes it will because it will always be connecting with the same size side.
Tyler: Oh, you made a new one.

Kylie: Yes, I did! To prove my theory!

In this segment, Kylie is reasoning that when a triangle rotates to create another one, the sides that come together are congruent. The side on which the midpoint is selected rotates such that the image of that side coincides with the original side. She is using this idea to justify the fact that there will be no gaps or overlaps. She then creates another tiling to test and verify her conjecture.

Mr. Kirkwood usually had his students do parts a and b of the second and third problems where the students use cut-out shapes and trace them on paper. He was accustomed to doing this activity since this was the only way the students investigated this topic in the first edition of *Core-Plus Mathematics*. However, because of a shortened class period the day his class was observed, he decided to skip that step and go right to using *CPMP-Tools* to tile the plane with triangles. He ended up thinking that the investigation went better than usual. In the interview he said that the computer tool that tiles the plane with the shape that is chosen saves so much time and is much more accurate. He felt that skipping the tedious and inaccurate cutting and tracing and retracing kept the flow of students’ thinking moving smoothly, so that the main point was not lost along the way. The transcripts verified that the students were able to start creating triangles with a click of the mouse and immediately start exploring and making conjectures about why the triangles were tiling the plane.

**Mr. Nelson’s Classes**

Having already talked to Mr. Kirkwood by 5th period, Mr. Nelson used that information by spending nearly ten minutes introducing the investigation before the
students began to work in groups. He asked the students to look at the picture at the beginning of the investigation and name the shapes shown. In the 6th period, he also pushed them to find convex and non-convex polygons as foreshadowing for the third problem in the investigation when they should explore both types of quadrilaterals. He then prompted them, “What’s interesting about those is they—” and a student responded “all come together.” He referred to the beehives that they had examined a few days earlier, and the way that those hexagons had fit together. He explained that the purpose of this investigation is to figure out what polygons will “fit together” and connected this informal terminology to the more formal “tile the plane.” He also used the square tiles on the floor to make the point that we know that some quadrilaterals will tile the plane, but queried “Will all quadrilaterals tile the plane?” He instructed them to skip problem number one, start with problem number two, go to problem number three in the investigation, and then gave them brief instructions on how to use the software. By 6th period (his second teaching of two Course 1 classes), he tried to streamline the investigation and focus their attention on the key questions so he gave the instructions as follows: “Do number 2—do triangles tile the plane? Explain why. Number 3—do quadrilaterals tile the plane? Explain why.” The students appeared to be motivated to explore the topic. He seemed to have piqued their curiosity with the introduction, and they went to work with the computer software. He pointed out that one student was already starting to explore with the software before he had finished his launch. Many groups in his classes successfully justified the fact that any triangle will tile the plane. The group discussion that follows illustrates this line of reasoning.

Erika: Oh my word, I got it!
Delanie: 180 right?

Kristin: Yeah.

Erika: No, it makes 360.

Delanie: If you go all the way around?

Erika: Ahh! That was fun!

Delanie: Super, super fun! Ok, these all work because…

Erika: That angle would like equal 360.

Delanie: This angle?!

Erika: No, like any angle. Around the vertex.

Kristin: Like this 1, 2, 3, right?

Erika: Yeah, around the vertex, when it rotates.

In this segment, Erika discovers that around each vertex in the tessellation, there are two copies of each angle 1, 2, and 3 of the original triangle (see Figure 5). Delanie had noticed in the beginning that the angles 1, 2, and 3 come together to make 180 degrees either because they form a straight line, or because the three angles in any triangle add up to 180 degrees. But Erika pushes the others to see that if you go all the way around any vertex, the two sets of the angles of the triangle will always add up to 360. Therefore, there will be no gaps or overlaps and so any triangle will tile the plane. The group had justified their discovery using the visual provided by CPMP-Tools.

**Interactions Displayed During the Tessellation Investigation**

Table 7 below shows the patterns of interaction (including the two examples above) that were found while the students worked in small groups on the tessellation investigation. The counts represent the interactions found in a sample of the groups in
each classroom. The “Combined” column refers to the total interactions found together in both teachers’ classrooms.

The interaction that occurred most frequently was inquiry. The interactions that coincide with the behavior norms of effective group work also have high frequencies: answering another student’s question, giving new idea/making a conjecture, building on other students’ ideas. The teacher explain interaction occurred when the teacher interacted with a group and explained ideas to them. The teacher explained such things as how to manipulate the software, the question they were trying to answer, and the pattern of angles around each vertex. The focus interaction pattern also occurred when the teacher was interacting with the group. There were more instances of the teacher asking focusing questions (16) than there were of the teacher explaining (11).

Inquiry using the software also occurred quite often. While analyzing these transcripts, the researcher had noted that much of the time the students were in inquiry mode—exploring, and making and testing conjectures. There were 42 instances of instrumental genesis found in this investigation. At these moments, the tool became an extension of the student’s mind. For example, a student said, “look, I tiled the plane!” He did not say that the computer tiled the plane, or CPMP-Tools tiled the plane, but that he had tiled the plane. This interaction also occurred frequently when the tool was used to create an idea that the student had spoken such as “I want to reflect the midpoint.” The student’s thoughts were played out on the computer screen. Only 14 interactions (4%) were associated with traditional mathematics instruction (Wood, Williams, & McNeal, 2006).
Table 7

*The Frequencies of each Type of Interaction Found during Tessellation Group Work*

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Type of Classroom Environment</th>
<th>Small-Group/ Multiple Computers</th>
<th>Teacher Nelson</th>
<th>Kirkwood</th>
<th>Combined</th>
<th>Only #2 Nelson</th>
<th>Only #2 Kirkwood</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collect answers</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>IRE</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>Give expected information</td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>2</td>
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<td>0</td>
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<tr>
<td>Teacher explain</td>
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<td>10</td>
<td>1</td>
<td>11</td>
<td>8</td>
<td>1</td>
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<td>Hint to solution</td>
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<tr>
<td>Exploring methods</td>
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<td>0</td>
<td>0</td>
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<td>Teacher elaborate</td>
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<td>Focus</td>
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<td>10</td>
<td>6</td>
<td>16</td>
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<td>6</td>
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<td>Inquiry</td>
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<td>20</td>
<td>40</td>
<td>60</td>
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<td>28</td>
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<td>Inquiry using the software</td>
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<td>14</td>
<td>15</td>
<td>29</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>Answering another student’s question</td>
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<td>3</td>
<td>27</td>
<td>30</td>
<td>3</td>
<td>16</td>
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<tr>
<td>Giving new idea/making a conjecture</td>
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<td></td>
<td>18</td>
<td>21</td>
<td>39</td>
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<td>13</td>
<td>15</td>
<td>2</td>
<td>7</td>
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<tr>
<td>Proof of answer by manipulation of software</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
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<tr>
<td>Resolution of conceptual issue using software</td>
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<td>2</td>
<td>0</td>
<td>1</td>
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<tr>
<td>Proof of answer by student explanation</td>
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<td>0</td>
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<tr>
<td>Building consensus</td>
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<td>Develop conceptual understanding</td>
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<td></td>
<td></td>
<td>117</td>
<td>222</td>
<td>339</td>
<td>101</td>
<td>161</td>
</tr>
</tbody>
</table>

Examining Mr. Nelson’s classroom only, *inquiry* was still the most commonly found interaction pattern. For example, one of his students said, “Wait, hold on, how do I do a third one here?” This student had rotated one triangle around the midpoint of the original triangle, and was wondering how to continue to make images of the original triangle. The second most frequently occurring interaction pattern was *giving new idea/making a conjecture*. This occurred in response to the question “Do you think any triangle could be used to tile the plane?” and a student said, “Yes, because they [the angles of a triangle] will always add up to 180.” *Instrumental genesis* and *inquiry using the software* occurred the third and fourth most frequently. The *teacher explain* interaction pattern occurred much more frequently in Mr. Nelson’s classroom than it did in Mr. Kirkwood’s classroom. In Mr. Nelson’s classroom, there were long periods of
teacher talk even when he was working with a group. On the other hand, Mr. Kirkwood tended to ask the group a probing question and move on, leaving the group to struggle with the question.

In Mr. Kirkwood’s classroom, *inquiry* was also the predominant interaction pattern. *Building on other students’ ideas* and *answering another student’s question* were the next two most frequently occurring interaction patterns. These interaction patterns are based on student-to-student communication and were not in Mr. Nelson’s top four most frequently occurring interaction patterns. An example of *building on other students’ ideas* occurs in the following segment that refers to the students testing their conjecture that any triangle will tile the plane: Barb: I already tested it with this one, and it works. John: And I tested it with that one. The next most common interaction pattern in Mr. Kirkwood’s classes was *instrumental genesis*. Like *inquiry*, this interaction pattern was also commonly found in Mr. Nelson’s classroom. A notable difference between the two classrooms is the fact that the interaction patterns *argument* and *making observations* occurred much more often in Mr. Kirkwood’s classes than they did in Mr. Nelson’s classes. The *argument* interaction pattern is another one that emphasizes student-to-student interaction and debate. For instance, one of Mr. Kirkwood’s students said, “Well I don’t think any one [triangle] would work because if I made this one differently it might not have worked.” The student is arguing with the group about whether or not a triangle will always tile the plane. The *making observations* interaction pattern involves statements made based on the visual image on the computer screen such as, “Whoa, it makes it look like the lines are crooked.”
The last two columns in Table 7 show the interaction patterns that occurred only during the groups’ work on problem number two in the investigation. This problem was a focus of the analysis because it was the largest common segment of lesson material covered by both teachers during the observations. The frequencies of answering another student’s question, and building on other students’ ideas are considerably higher for Mr. Kirkwood’s classroom than they are for Mr. Nelson’s classroom. Meanwhile, teacher explain occurred more frequently in Mr. Nelson’s classroom. This indicated that Mr. Nelson spent more time than Mr. Kirkwood explaining the ideas to the students. Conversely, students in Mr. Kirkwood’s classes spent more time productively interacting with each other than students in Mr. Nelson’s classes.

**Levels of Mathematical Thinking Displayed During the Tessellation Investigation**

Table 8 below shows the levels of mathematical thinking verbalized as students worked in small groups on the tessellation investigation. Transcripts from the same sample of groups as those used for the interaction patterns were analyzed. The frequencies and percentages are listed in each teacher’s classroom since the total number of codes varies greatly from one classroom to the other. In both classrooms, a majority of the students’ discussion was at the Constructing Synthesizing level of mathematical thinking. That means that a majority of the time students were exploring the problem from multiple perspectives, formulating arguments to explain discovered patterns, or integrating concepts to create new insights. An example of students exploring the problem and creating a new insight is shown in the following excerpt from a group’s discussion while using the tool shown in Figure 6.

Trevor: Move it down a little bit.
Gabby: I think if we grab it we can move it.

Andrew: Trevor.

Trevor: What?

Andrew: Look, it fits together perfectly!

Emily: Nice!...Yeah, now it’s a parallelogram!

Table 8

*The Frequencies and Percentages of each Level of Mathematical Thinking Found in each Teacher’s Classroom during Tessellation Group Work*

<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
<th>Type of Classroom Environment</th>
<th>Small-Group/Multiple Computers</th>
<th>Nelson</th>
<th>Kirkwood</th>
<th>Combine</th>
<th>#2 Nelson</th>
<th>#2 Kirkwood</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizing Comprehending (C)</td>
<td></td>
<td></td>
<td>5</td>
<td>3</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Recognizing Applying (A)</td>
<td></td>
<td></td>
<td>3</td>
<td>4%</td>
<td>0</td>
<td>0%</td>
<td>0</td>
</tr>
<tr>
<td>Building-with Analyzing (AN)</td>
<td></td>
<td></td>
<td>8</td>
<td>12%</td>
<td>11</td>
<td>7%</td>
<td>19 8</td>
</tr>
<tr>
<td>Building-with Synthetic-Analyzing (SA)</td>
<td></td>
<td></td>
<td>0</td>
<td>0%</td>
<td>8</td>
<td>5%</td>
<td>8 0 0%</td>
</tr>
<tr>
<td>Building-with Evaluative-Analyzing (EA)</td>
<td></td>
<td></td>
<td>4</td>
<td>6%</td>
<td>48</td>
<td>29%</td>
<td>52 3 5%</td>
</tr>
<tr>
<td>Constructing Synthesizing (SN)</td>
<td></td>
<td></td>
<td>49</td>
<td>71%</td>
<td>84</td>
<td>51%</td>
<td>133 45 71%</td>
</tr>
<tr>
<td>Constructing Evaluating (E)</td>
<td></td>
<td></td>
<td>0</td>
<td>0%</td>
<td>10</td>
<td>6%</td>
<td>10 0 0%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>69</td>
<td>100%</td>
<td>164</td>
<td>100%</td>
<td>233 63 100%</td>
</tr>
</tbody>
</table>

Note: The last column does not have a total of 100% due to rounding the individual percents to the nearest whole percent.

The level of thinking that occurred with the second highest frequency in Mr. Kirkwood’s classroom was *Building-with Evaluative-Analyzing* that involves analyzing and evaluating ideas or arguments. As an example, this happened during this lesson each time a group answered the question, “Do you think any triangle could be used to tile a plane?” (Hirsch et al., 2008, p. 408), and after exploring, came down to the conclusion—“yes.”

The level of thinking that occurred with the second highest frequency in Mr. Nelson’s classes was *Building-with Analyzing*. This level of mathematical thinking involves applying known procedures in a new context, or systematizing the results and searching for patterns. For example, a student in Mr. Nelson’s class made the following
statement, “Two of the numbers are the same. This and this both have a 1, and they’re both the same.” Here the student is noticing a pattern that at each vertex there are two angle ones, and presumably two of every angle. This level of mathematical thinking was the third most frequently found level in Mr. Kirkwood’s classes. While the *Building-with Analyzing* level of thinking is the second most common level found in Mr. Nelson’s classroom, the frequencies are all very close in number for all the levels except the most common—*Constructing Synthesizing*.

There were also ten instances of the highest level of mathematical thinking during this investigation—all found in Mr. Kirkwood’s classroom. This occurred when the students made an evaluation but did so with reflection on the situation as a whole, such as when Kylie (in a transcript given previously) had said, “Yes it will because it will always be connecting with the same size side.” She is answering the same question given above about whether or not any triangle will tile the plane, but she has justified it in general. She is speaking about all the triangle tessellations the group had created on the computer screen, and all possible tessellations of triangles that could be made. Mr. Kirkwood’s classes did have more time to work in groups than Mr. Nelson’s classes had. This could have impacted the level of mathematical thinking that the groups were able to attain.

The last two columns of Table 8 again show the levels of mathematical thinking while students worked on number two of the investigation, the common unit of in-class lesson coverage across both teachers. Proportionally, there were more instances of lower-level mathematical thinking found in Mr. Nelson’s classes. The relative frequency for *Building-with Evaluative-Analyzing* was much higher in Mr. Kirkwood’s classes than it was in Mr. Nelson’s classes. Yet, the next highest level of mathematical thinking,
Constructing Synthesizing, occurs proportionally more in Mr. Nelson’s classes than in Mr. Kirkwood’s classes. Mr. Nelson’s students were spending a lot of the time exploring the problem from many perspectives as well as some of the lower-level activities like recognizing concepts or applying known procedures, while Mr. Kirkwood’s students spent almost the whole time exploring the problem from many perspectives and pulling together ideas to make a judgment.

Types of Interaction Patterns Found in Each Level of Mathematical Thinking

Table 9 quantifies all of the types of interactions within each level of mathematical thinking that occurred during the collaborative learning of tessellations. In the combined classrooms, the highest frequencies of mathematical thinking occurred at the Constructing Synthesizing (SN) and Building-with Evaluative-Analyzing (EA) levels. So the highest numbers in Table 9 occur in these two columns. It is noteworthy that 40 instances of instrumental genesis occurred while students were exploring tessellations. There has been much recent research relating to this construct. Some of this research infers that instrumental genesis is an ideal interaction pattern during computer use (Zbiek, Heid, Blume, & Dick, 2007). Other types of interaction patterns in the two columns with the highest frequencies are inquiry, inquiry using the software, answering another student’s question, giving new idea/making a conjecture, building on other students’ ideas, and making observations. Two of these interaction patterns are technology-dependent—inquiry using the software and making observations. Three of these interactions are behaviors indicative of effective group work—answering another student’s question, giving new idea/making a conjecture and building on other students’ ideas (Cohen, 1994a, 1996).
The Rest of the Investigation

Since most students completed up to problem number three in both classes, the rest of the investigation was assigned for homework. The teachers had no reservations assigning the rest of this investigation as part of their homework since all of their students had their own laptops, and had access to the internet in order to use CPMP-Tools. The next day

Table 9

The Number of each Type of Interaction Found in each Level of Mathematical Thinking During Tessellation Group Work Across Both Teachers

<table>
<thead>
<tr>
<th>Small-Group/Multiple Computers</th>
<th>Levels of Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Interaction Pattern</strong></td>
<td><strong>C</strong></td>
</tr>
<tr>
<td>Collect answers</td>
<td>C</td>
</tr>
<tr>
<td>IRE</td>
<td></td>
</tr>
<tr>
<td>Give expected information</td>
<td>1</td>
</tr>
<tr>
<td>Funnel</td>
<td></td>
</tr>
<tr>
<td>Teacher explain</td>
<td></td>
</tr>
<tr>
<td>Hint to solution</td>
<td></td>
</tr>
<tr>
<td>Exploring methods</td>
<td></td>
</tr>
<tr>
<td>Teacher elaborate</td>
<td></td>
</tr>
<tr>
<td>Focus</td>
<td>1</td>
</tr>
<tr>
<td>Inquiry</td>
<td>1</td>
</tr>
<tr>
<td>Inquiry using the software</td>
<td></td>
</tr>
<tr>
<td>Answering another student’s question</td>
<td>1</td>
</tr>
<tr>
<td>Giving new idea/making a conjecture</td>
<td>2</td>
</tr>
<tr>
<td>Argument</td>
<td>1</td>
</tr>
<tr>
<td>Proof of answer by manipulation of software</td>
<td>2</td>
</tr>
<tr>
<td>Resolution of conceptual issue using software</td>
<td>1</td>
</tr>
<tr>
<td>Proof of answer by student explanation</td>
<td></td>
</tr>
<tr>
<td>Building consensus</td>
<td></td>
</tr>
<tr>
<td>Checking for consensus</td>
<td></td>
</tr>
<tr>
<td>Develop conceptual understanding</td>
<td></td>
</tr>
<tr>
<td>Pupil self-nominate</td>
<td></td>
</tr>
<tr>
<td>Building on other students’ ideas</td>
<td>2</td>
</tr>
<tr>
<td>Making observations</td>
<td>2</td>
</tr>
<tr>
<td>Instrumental genesis</td>
<td></td>
</tr>
<tr>
<td>Technical software statement</td>
<td></td>
</tr>
<tr>
<td>Off-task</td>
<td></td>
</tr>
</tbody>
</table>

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students shared in class the tessellations they had found. By observing this, it was clear that most students had done the homework using *CPMP-Tools*.

**Follow-up Interview Results**

Four students from each of Mr. Kirkwood’s and Mr. Nelson’s classes were interviewed the day after the classroom observation. Every one of the eight students could explain why all triangles and quadrilaterals will tile the plane, as illustrated in the transcript below.

Researcher: Can you show me what you would do to see if you can tile the plane with a triangle?

[Student creates an image similar to Figure 5.]

Researcher: Is there a reason that you chose to make kind of a hexagon out of the first few?

Student: Because it makes the 360 degree angle right there.

[Student points to the vertex in the center of the six triangles.]

Researcher: How do you know that’s 360 degrees?

Student: Because it makes a circle.

Researcher: Ok, so what angles come together to make that 360 degrees?

Student: 180 right here—that straight line—1, 2 and 3, and then another 180.

Researcher: If you had a different triangle, would you still end up with the 1, 2, and 3 and another 1, 2, and 3?

Student: Yes [shows it by clicking and dragging the figure to make the triangles all different shapes.]

Researcher: So are you convinced that any triangle will tile the plane?
Student: Yes.

Researcher: Can you explain why?

Student: Well like whatever you do you have the 1, 2, and 3 angles, and that makes a straight line, so then you have another set on the other side and that makes another straight line, and $180 + 180$ is 360.

Researcher: Does this program help you see that?

Student: Yes.

Additionally, the researcher asked the students to show what they had done with CPMP-Tools for homework. The fourth problem in the investigation asks students to explore other polygons to see whether or not they will tile the plane. See Figure 7 for an example of using the tool Tilings with Regular Polygons (Keller, 2011) to attempt to tile the plane with pentagons. A student had created this image on their computer during an

Figure 7. Using CPMP-Tools to try to tile the plane with pentagons.
interview the day after the classroom observation. Using this custom tool, the student chooses the number of sides that they want the polygon to have, then the polygon appears and the students can make as many copies of it as they want. The student explained using the image shown in Figure 7 that a regular pentagon will not tile the plane. There is no way to fill in the gap in the middle with another pentagon.

Later in the investigation, the students are shown an example of a semiregular tessellation, and are then asked to explore whether or not one can be made with a regular hexagon, two squares and an equilateral triangle at each vertex. See Figure 8 for an example of how the same tool used in Figure 7 can be used to create semiregular tessellations by selecting different numbers of sides to get the different regular polygons. Clearly, the suggested semiregular tessellation does work as a student had shown in an interview using the image shown below.

Figure 8. A semiregular tessellation with a hexagon, two squares, and an equilateral triangle at each vertex made with CPMP-Tools.
The succeeding questions ask students to explore some other semiregular tessellations and to find one of their own. During the interviews, students recreated the tessellations that they had found. A segment of one of the interview transcripts follows.

Researcher: So go ahead and show me what you were doing with this *Tilings with Regular Polygons* custom tool?

[Student makes drawing shown in Figure 9 below.]

*Figure 9.* A tessellation created by a student who was interviewed.

Researcher: Oh, how many sides is the big one?

Student: 12.

Researcher: 12 sides? And then you have a triangle and a square?

Student: Yep.

Researcher: So do you know what angles you have coming together?

Student: This is 60, 60, 90, and 150.

Researcher: Interesting. Did you find some other ones to work?
The student is working with the software.

Student: There’s this one.

[The student put 7 hexagons together.]

Student: Or you could do this.

[The student made a hexagon with squares and triangles around it as shown in Figure 8.]

Researcher: Is this what you did for homework?

Student: Yeah.

Researcher: So you were playing around at home finding all these different ones to work?

Student: Yep.

Researcher: So how do you know when they work?

Student: If the vertex of all the angles add up to 360.

Researcher: So do you want to show me one that doesn’t work?

Student: Ok.

[Draws the pentagons shown in Figure 7.]

Student: Five doesn’t work because you can’t put anything in between there. So it all doesn’t add up—it’s not tiled.

Researcher: So, does this program allow you to quickly see whether or not a shape is going to work and tile the plane?

Student: Yeah.

Researcher: Have you found that helpful to learn what you need to learn in this investigation?

Student: Yes! [enthusiastically]

The interview above was typical of those conducted during this investigation.
Additionally, all of the students interviewed had an understanding of why the shapes tile the plane, or why they will not tile the plane. They were all able to point to a vertex and show that the angles around any particular vertex would add up to 360 degrees if the tiling worked. The interview transcript above, and an additional one below illustrate this. Figure 10 below shows the tessellation that the student in the interview below had created.

![Tessellation Image]

_Figure 10._ Student-created tessellation consisting of two hexagons and two triangles at each vertex.

Student: You can click on the size polygon that you want—I’ll just do 6—and just click anywhere and it puts it automatically in there. And then to add more you just switch the number of sides and keep clicking to put them in. So like there’s two more, and there’s another shape that we are talking about right now.

Researcher: So would that arrangement tile the plane?

Student: Yes.
Researcher: How do you know that?

Student: Because of the angles that come together, they add up to 360.

Researcher: Does having this picture on the computer help you to see that?

Student: Yeah, definitely it’s easier to—when you can put them together then it makes it easier. It’s harder on paper—they’re loose.

Using the visual aid provided by the software, the students were able to see how the shapes fit together with no gaps or overlaps. Extending the idea that they learned using all triangles or all quadrilaterals, the students observed the sum of the angles at each vertex and determined that they added up to 360 degrees if the combination of shapes tiled the plane. Students seemed to have fun exploring with this custom tool, and were able to use it quite easily. They liked that they did not have to cut out all the shapes and trace them over and over. Similar to the results found during the classroom observations, the students would get excited about what they had created on their screen. Statements such as, “Cool!” or “Look at this!” can be found in the audio of both the interviews and classroom observations.

**Relevant Assessment Results**

Both teachers gave the same assessment. The assessment item that relates to the investigation observed is as follows: *Can two squares and three equilateral triangles be used to make a semiregular tessellation of the plane? Explain your reasoning.* Of the 133 students’ assessments that were analyzed, 74 (56%) were correct and 59 were incorrect. (These results seem slightly skewed by one sixth-hour class that had 6 correct and 16 wrong which accounts for 27% of the incorrect answers.) Among the 74 correct answers, 34 of them (46%) drew a picture that resembled the *CPMP-Tools* custom tool.
shown in Figure 8 above (with triangles in the middle instead of the hexagon). This finding suggests that use of the software influenced some students’ thinking in obtaining the correct answer. Other correct answers simply had the numbers written for each angle that would surround a vertex and added them up such as: $90 + 90 + 60 + 60 + 60 = 360$, as the student had stated in the interview above regarding Figure 9. There were 35 responses like that. The other 5 correct answers did not have clear justification. Two simply stated that the sum of the angles was not over 360 degrees. One of the students had answered “yes” correctly, but then said that the reason was because “it doesn’t tile the plane.” This student and six others in the incorrect category seemed to be confused about what it meant to “tile the plane.”

Seven students in the incorrect category seemed to be confused about what an equilateral triangle looked like. Some drew, and others stated, that it was like a 45-45-90 triangle. One student stated that he/she is “not sure what semiregular means.” Sixteen incorrect responses either used the wrong numbers when adding or added incorrectly. Eighteen students said that the shapes just would not fit together—that there would be gaps or overlaps. Another student drew the correct shapes, but with the lack of accuracy in a paper-and-pencil drawing, the shapes had a gap and he/she concluded that it did not work. That student may be assuming that a drawing by hand will be as accurate as the drawings that the software provided because they determined in class whether or not a shape would tile the plane by putting images of the shapes together on the computer screen. Another student said that the angles added up to 360 degrees but that the shapes overlapped. This may have also resulted from an attempt to draw them by hand. Three students did not think it would work with more than one type of shape. They may not
have used CPMP-Tools to tile the plane with more than one shape, even though the students that were interviewed had all tiled the plane with more than one shape. The other incorrect answers consisted of four blank, one with no explanation, and two with reasons that did not have anything related to the correct reason.

Conclusions

While students were investigating tessellations in this study the interaction pattern they engaged in the most was inquiry. This interaction pattern elicited the top five levels of mathematical thinking, with the second-highest level occurring the most. Students were making conjectures about whether or not a shape would tile the plane and using the software to confirm or deny their conjectures. Students also spent a lot of time giving their ideas, building on other students’ ideas, and answering each other’s questions. These interactions tended to elicit the higher levels of thinking as well. Using the software gave the students much to discuss. Additionally, instrumental genesis was found more in this investigation than the others. Therefore, students were using the software as an extension of their own thinking. The visual appeal of this software may have contributed to this result.

When the investigation interaction patterns and levels of thinking are linked to the student assessments, some of the desired high-level outcomes seem tempered since only 56% of the students in Mr. Nelson’s and Mr. Kirkwood’s classes got the related assessment item completely correct. Many of them drew pictures that resembled Figure 8 or 9 to show that the shapes would fit together. However, some of them relied too heavily on their pencil and paper drawing. If they could not draw the correct shapes accurately, then they assumed it would not work. Conversely, if they drew the wrong
shapes (such as a 45-45-90 triangle instead of an equilateral triangle), then they either assumed it would not work because their drawing did not work, or they made the drawing work even with an incorrect shape. In either case, students were making rather large assumptions about the ability of their own drawing to prove their answer. This was somewhat understandable given the fact that they used the visual display in CPMP-Tools to confirm whether or not the shapes would tile the plane. Therefore, as students use this tool, it should be stressed that the computer can draw with an accuracy that we do not have on our own. Using the software to check whether or not a conjecture is likely to be true is much different than using a paper and pencil free-hand sketch.

Mathematical Concept Two: Regression Lines

The two Course 2 teachers’ (Mr. Louiselle’s and Mr. Foster’s) classes were observed while students were performing two investigations in the data analysis and statistics strand. There were both whole-class and small-group uses of CPMP-Tools during these investigations. However, as was noted in the particular field notes for this investigation, the researcher determined that Mr. Foster’s treatment of group work was contrived, thus rendering comparisons on that dimension unreliable. Therefore, the comparison of teachers’ data for these investigations will focus on each of the teachers’ facilitation of whole-class discussion with the use of CPMP-Tools on the screen at the front of the room. Each teacher had two Course 2 classes, so a total of four classes were observed using the data analysis tools. These data were categorized under the Whole-Class/Single Computer classroom environment.

The Investigations
In *Course 2*, Unit 4, Lesson 2, Investigation 1 (*How Good Is the Fit?*) and Investigation 2 (*Behavior of the Regression Line*), students use the data analysis tools in *CPMP-Tools* to make scatterplots, find the regression equation, graph the regression line, display the residuals on the graph, calculate the sum of squared errors (SSE), display the centroid, and determine the effect of an influential point (see Figure 11). Each problem is set in a real-life context. Some of these contexts are: the curb weight and highway mpg of a set of compact cars, index of radioactive waste and cancer deaths in a set of communities in Oregon, height and hip angle of a set of horses while running, and the season batting average and World Series batting average for a set of Chicago White Sox players. As students explore these contexts with the data analysis software, they are asked questions in the investigation that compel them to interpret things like the meaning of the slope of the regression line and the effect of an influential point.

In Investigation 1, students learn that a residual is the difference between the observed value (actual data) and the predicted value obtained from a best-fit line. They also learn how to calculate the sum of squared errors (SSE) by hand using a data set containing three ordered pairs, and discover that the least squares regression line has a smaller SSE than any other line.

Both teachers facilitated a whole-class discussion using *CPMP-Tools* on the second investigation problems number one and two (see Figures 11 and 12). In problem number one, students discover that the centroid \((\bar{x}, \bar{y})\) will always lie on the least squares regression line. In problem number two, students create a scatterplot using *CPMP-Tools* of the data from 16 different kinds of compact cars’ *Curb Weight* and *Highway mpg*. The compact car data are already stored in *CPMP-Tools*. Students have to select the data
from the list of stored data sets, and choose scatterplot in the graph menu of the software. Part a of problem number two asks students to create a line on the scatterplot by pressing the button that looks like a pencil drawing a line (see the top of the computer screen shot in Figure 12) and clicking and dragging the line until they think it best fits the data. Part b of this problem asks students to compare the line they created on the screen to the least squares regression line. The regression line will automatically appear when they press the second button from the left on the scatterplot screen, the button that looks like a line on a scatterplot. Figure 12 shows a screen shot of the software displaying both the regression line and the line created by the computer user. Part c asks students to determine the effect on the regression line of removing a point from the data set.

Figure 11. The beginning of Investigation 2 of the regression lines lesson (Hirsch et al., 2008, p. 286).
Now, use data analysis software to examine the idea of finding the line on a scatterplot that minimizes the sum of squared errors (residuals), or SSE.

Figure 12. The second problem of Investigation 2 from the regression lines lesson (Hirsch et al., 2008, p. 287).

**Figure 12.** The second problem of Investigation 2 from the regression lines lesson (Hirsch et al., 2008, p. 287).

a. Using the moveable line capability of the software, visually find a line that you think best fits the compact car (curb weight in hundreds of pounds, highway mpg) data.

b. Compare the line you found visually and its equation with the regression line and its equation.

c. The Honda Civic Hybrid is an outlier and so may have a large effect on the location of the regression line. To investigate the effect of this point, first remove from the plot the lines you found in Part a.

i. Delete the point for the Honda Civic Hybrid from the data set. How do the regression line and equation change?

ii. Replace the point for the Honda Civic Hybrid and then delete the point for the Mercedes-Benz C280, which is the second heaviest car. How do the regression line and equation change in this case?

iii. Does the Honda Civic Hybrid or the Mercedes-Benz C280 have more influence on the regression line and equation?

An *influential point* is a special type of outlier. It strongly influences the equation of the regression line or the correlation. When such a point is removed from the data set, the slope or y-intercept of the regression line changes quite a bit. The interpretation of “quite a bit” depends on the real-life situation. You will further examine the idea of an influential point in the next two problems.
When the teachers implemented this problem with the students they expanded on part \( b \) by examining the sum of squared errors for both the regression line and the line created by the computer user. Figure 13 shows the regression line for the compact car data with its squared errors. When the fourth button from the left at the top of the screen is clicked, the software draws the squares corresponding to the squared error, drawn on each residual, for each data point. This gives students a visual of the geometric interpretation of “squared errors.”

![Figure 13](image)

*Figure 13. An example of a regression line and its corresponding sum of squared errors drawn by the data analysis tools in CPMP-Tools.*

The classes also experimented with some other features of CPMP-Tools. With both the regression line and the other line displayed, the sum of squared errors can be shown for both lines. The squares of the regression line appear in blue on the computer screen (the darker squares in Figure 14), and the squares of the other line appear in
yellow (the lighter squares in Figure 14). Additionally, there is an *error thermometer* feature also shown in Figure 14 that gives the sum of the areas of the squares (sum of squared errors). Students can click on the line they created and drag it around to try to make their SSE smaller. The error thermometer will rise and fall accordingly as the areas of the squares change. This feature will also demonstrate that the least squares regression line has the smallest SSE. In other words, a line cannot be created that causes the error thermometer to go below the level of the regression line.

*Figure 14.* The least squares regression line and a possible line of best fit created by a student and the corresponding squared errors for each line created with the data analysis tools in *CPMP-Tools.*
Mr. Foster’s Classes

As Mr. Foster’s students entered the room, he handed them a number. This number was intended to indicate the group that each student would belong to that day. Students were supposed to find their group mates and put their desks together. However, before having students work in groups, Mr. Foster spent the first part of the class period leading the class in a discussion about number two of Investigation 2. Mr. Foster brought up *CPMP-Tools* on the screen in front of the room. He instructed students to go to *Course 2* in *CPMP-Tools*. Students had many questions at this step. Mr. Foster and some of the students helped other students get to this step. Then Mr. Foster told the students that they were to just watch the front screen, and not work on their own computers. In fact, when a student asked a question, he said, “Just watch. If you wouldn’t be playing instead of watching, you’d be good.”

Beginning to work on problem number two, Mr. Foster selected the compact car data and created the scatterplot with *CPMP-Tools*. The students were impressed. One said, “That’s crazy!” and another said, “How do they know [the data]?” Mr. Foster said, “They just do. They made it for you to make your life easier.” Then Mr. Foster went on to explain how to create a line that best fits the data. Here he asked for students’ input as to where to put the line. As a class, they adjusted the slope and the vertical position of the line until the class agreed on a line. Next, Mr. Foster explained how to make the regression line appear on the plot. Again, students were excited about how easily this was accomplished with *CPMP-Tools*. One student asked why one would bother to make his/her own line if he/she could just click the regression line button. Mr. Foster replied that it was “good to be able to eye-ball it.” Then Mr. Foster clicked on the residuals
button (third button from the left in the screen shot in Figure 12), and asked students “what did it just draw in there?” A couple students called out, “residuals.” Then he said, “Now watch what this does,” and he clicked on the squared errors button. The first response from the students was “Whoa!” The subsequent discussion is as follows:

Mr. Foster: What did that do?

Troy: I don’t know.

Jon: It made a box.

Mr. Foster: It made a box. Why did it make a box?

Troy: Cuz.

Jon: Mine looks different.

Mr. Foster: What is the area of those boxes?

Several students: The residuals squared! [emphatically]

In this exchange Mr. Foster is asking the students to interpret the results of displaying the squared errors on the best-fit line by asking three focusing questions. Later it will be shown that without the use of CPMP-Tools, Mr. Foster did not typically ask that many focusing questions during a whole-class discussion. The questions in the exchange above led students to discover that the squares they saw on the screen were actually representing the quantity of the squared residuals. Mr. Foster finished the whole-class discussion by displaying a visual similar to Figure 14 and comparing the sum of squared errors of the regression line to that of the line the class had agreed upon. The students were then supposed to finish the rest of the investigation in groups (yet the groups never fully formed) using the features of CPMP-Tools that Mr. Foster had just used in the whole-class discussion. However, many students could be heard on the audio
saying that they were lost and confused. The students who were interviewed right after this class did not know what a residual was and did not know how to operate the software. After comparing the data from Mr. Louiselle’s students below, it could be speculated that this was because he did not let them operate their computers along with him.

**Mr. Louiselle’s Classes**

Mr. Louiselle was observed facilitating a whole-class discussion on the same part of Investigation 2 using the same tools on the screen at the front of the room as Mr. Foster used as described above. One of the main differences between the two teachers’ classes was that Mr. Louiselle’s students had already attempted to use the functions of *CPMP-Tools* on the problems for homework the night before. So, while many of Mr. Foster’s students were lost in the first step, Mr. Louiselle’s students had already gotten past that and had at least tried to perform the computer operations so they had more of a frame of reference for what was displayed on the screen. Also, Mr. Louiselle’s questions requested students to give the next steps. For example, in the following discussion, it is the students that give the information that is supposed to be learned in problem number one of Investigation 2. Since the centroid is always on the regression line, any line of best fit will necessarily go through that point.

Mr. Louiselle: What’s the first thing that you want to know before you draw in your own line?

Tracy: How many…

Mr. Louiselle: You can’t use the regression line right now, you have to come up with your own line. What piece of information would be very helpful in knowing?
Tracy: Outliers.

Megan: The centroid.

Mr. Louiselle: Hey, did you hear that? Say it loud again please.

Megan: The centroid.

Mr. Louiselle: The centroid. Wouldn’t it be nice to know the centroid?

Brian: Sure.

After a discussion about the centroid, where Mr. Louiselle prompted students to explain to the rest of the class what the centroid was and why we would want to know it, he continued with problem number two as follows: “Now, you’re supposed to draw in your own line. How do we do that with CPMP-Tools?” So, the students tell him how to click on the first button on the left at the top of the screen and move it around. Throughout this discussion, Mr. Louiselle feigns as if he doesn’t know how to operate the software, and entices students to explain to him and the rest of the class how to perform the operations. When they get to the part where the error thermometer is used, Mr. Louiselle simply puts it up on the screen and asks, “What is this telling me right now?” One student says, “Not very close” and another one says, “A lot of error.” The students were immediately interpreting the meaning of the visual image on the screen. He asked them if the slope should be positive or negative by looking at the points, and by considering the context. He asked, “So would it make sense that as the weight goes up, what would you expect with the miles per gallon?” Students were thinking critically to determine that the slope should be negative.

There was a time when a student asked Mr. Louiselle to just tell her how to do something with the software. He replied, “If I just give you food, you’re fed for a day. If
I show you how to fish, …”. This was understood to mean that Mr. Louiselle did not just tell students what buttons to push and exactly what to do. He led them to discover how to do what they needed to do with an understanding of why they were doing it.

Mr. Louiselle also led the students through part c of problem number two where they determine the effect of removing an outlier from the data set. For this part, Mr. Louiselle also asked the students to think critically as follows.

Mr. Louiselle: There’s a point that seems to be an outlier right? And what was that point in our data?
Ashlynn: The hybrid.
Mr. Louiselle: The hybrid, and now like we said before, why would you compare a hybrid with…
Ashlynn: Regular cars.

Mr. Louiselle: Regular gasoline cars. And so they said, well what if we take that car out. If you remember, we select the point we want. If you want to get rid of it, I call it the ghostbuster symbol, but it’s basically the big ol’ no button right—the universal symbol for no. Ok, you click on that, and now what happens to the regression line?
Ashlynn: It changes.
Bailey: It goes down.

Mr. Louiselle was drawing their attention to the fact that the outlier was a different type of car and that it would make sense for that car to not fit the trend of the rest of the data. Without this point, the slope of the regression line changed, as did the SSE.

Observing Mr. Louiselle’s students revealed that they generally knew how to operate the data analysis tools, and could interpret the meaning of the graph and the SSE.
Below is an excerpt from one of the discussions with two of Mr. Louiselle’s students. This discussion occurred in his classroom during lunch. Students frequently come in to his classroom during lunch if they have questions. It was a good opportunity to hear the students’ thinking as their own teacher, whom the students are comfortable with, asked them questions. Jessica is controlling the computer, and her friend Chloe is following along. Figure 14 shows the features of the data analysis software that Jessica was using during this discussion.

[Jessica forms a line of best fit on a scatterplot using Stat-Tools.]

Jessica: Then do you want me to like, get the thermometer to see how…

Mr. Louiselle: You do what you need to do—you tell me what you’re doing. And Chloe, you step in anytime and help her out.

Jessica: Well, like this [the thermometer], this is telling me how accurate my line is, that I’m making.

Mr. Louiselle: Can you tell me what you mean by accurate?

Jessica: Like how close it is to the linear regression line.

Mr. Louiselle: And why do you say that?

Jessica: Because the linear regression line is the best line.

Mr. Louiselle: Why is that?

Jessica: Because it has the…

Chloe: The best fit.

Jessica: Yeah, because it has the smallest residuals.

Mr. Louiselle: Can you show me what the residuals look like on this?

Chloe: I don’t get what the residual is.
Jessica: It’s like the distance from here to here [using the pointer to point on the screen].

Mr. Louiselle: Ok, and what do you click up there to show that visually?

[clicks the residual button]

Mr. Louiselle: And what is the residual?

Jessica: The distance from the point to the line that I made [pointing on the screen].

Mr. Louiselle: And you told me earlier that the regression line was the line of best fit, how come? You can put the regression line in. [Clicks the regression line button] How did you do?

Chloe: Pretty good.

Mr. Louiselle: Yeah, that’s not bad. You had 405, and the regression line has 387. And what is this number telling us?

Jessica: Oh, is it the sum of all the residuals?

Mr. Louiselle: Close.

Jessica: Sum of the squared errors.

Mr. Louiselle: And when you clicked that box, [the squared errors box] what did that show us?

Jessica: The squares of the residuals.

Mr. Louiselle: So, what did that number show us?

Jessica: Oh, it’s the sum of all the squares.

In this excerpt, Jessica has shown a good understanding of both using the software and the concepts behind it. She knew what to click on to make the error thermometer appear, to make the regression line, to show the residuals, and to show the squared errors. She also knew that the regression line would be the line that fit the data the best, and that
when she created her first line she knew she had to approximate the regression line. She also knew the meaning of a residual—she could point to it on the screen. At first she said that the best-fit line had the smallest residuals. But after the teacher questioned her further, she realized that what the error thermometer was measuring was the sum of the squared residuals. Although the teacher said that Chloe could step in and help Jessica out, it seemed to be the other way around. Jessica explained what a residual was to Chloe.

During the interviews of Mr. Louiselle’s students, when asked the meaning of a residual, the students would open up their computer to show a residual and a visual for the sum of squared errors using the data analysis tools in CPMP-Tools. These students said that they liked the immediate feedback given by the error thermometer, and the many things that are already set up like having the data already entered. When asked to describe a time that CPMP-Tools helped them learn math, the students mentioned the animations in Course 2, Unit 3, and the regression line lessons.

When Mr. Louiselle was interviewed he said the following about the data analysis tools: “Students will say, ‘I don’t want to take the time to put the stuff in my calc and calculate the regression line’. The computer software saves time and gives students instant feedback. The feature of clicking and seeing all the squares, it’s a visual connection. On the calculator they had to visualize in their head. And even if you drew it on the board it was a still picture. Now you can see what happens instantly when the regression line is changed. It gives them a better sense of the correlation coefficient by looking at the graph. They are a lot more confident. The lesson has gone extremely well—it’s fun. With the calculator, they didn’t have as much of a sense of what the line
would look like. The computer allows them to make and test conjectures about regression lines and many other things with ease. The challenge then is first make sure they can do it with paper and pencil, and understand it, but now we give them the freedom to let the computer do it. And now the focus is on interpreting what the computer gives them. That visual I think is extremely important.”

**Interactions Displayed During the Regression Line Investigation**

Table 10 shows the patterns of interaction that were found while the *Course 2* teachers led interactive class discussions using the data analysis tools for problem number two of Investigation 2. Since this investigation was conducted as a *whole-class* discussion, the frequencies represent the spoken words of all students in the classrooms. The “Combined” column refers to the total interactions found together in both teachers’ classrooms. The columns for Foster and Louiselle contain a column of frequencies and a column of percentages of the total interactions in that teacher’s classroom.

The *teacher explain* interaction had the highest frequency for each teacher. This is somewhat understandable given that it is a teacher-led discussion. However, without the use of computer software there are teachers who lead class discussions without *explaining* to the students. Yet, when the teacher needs to explain how to use the software, explaining may be unavoidable. For example, Mr. Louiselle said, “Just make sure that under ‘course’ you have two marked off and that gives you access to everything from *Course 2*. We go to *statistics* again, *data analysis.*” While the *teacher explain* interaction occurred the most in both classrooms, it only made up 22% of the interactions in Mr. Louiselle’s classroom compared to 45% of the interactions in Mr. Foster’s
Table 10

Frequencies and Percentages of each Type of Interaction Found in each Classroom during a Whole-class Discussion about the Regression Line

<table>
<thead>
<tr>
<th>Type of Classroom Environment</th>
<th>Interaction Pattern</th>
<th>Whole-Class/Single Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher</td>
<td>Foster</td>
<td>Louiselle</td>
</tr>
<tr>
<td>Collect answers</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>IRE</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>Give expected information</td>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td>Funnel</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Teacher explain</td>
<td>35</td>
<td>45</td>
</tr>
<tr>
<td>Hint to solution</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Exploring methods</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Teacher elaborate</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Focus</td>
<td>8</td>
<td>10%</td>
</tr>
<tr>
<td>Inquiry</td>
<td>8</td>
<td>10%</td>
</tr>
<tr>
<td>Inquiry using the software</td>
<td>4</td>
<td>5%</td>
</tr>
<tr>
<td>Answering another student’s question</td>
<td>1</td>
<td>1%</td>
</tr>
<tr>
<td>Giving new idea/making a conjecture</td>
<td>14</td>
<td>18%</td>
</tr>
<tr>
<td>Argument</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Proof of answer by manipulation of software</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Resolution of conceptual issue using software</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Proof of answer by student explanation</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Building consensus</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Checking for consensus</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Develop conceptual understanding</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Pupil self-nominate</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>Building on other students’ ideas</td>
<td>0</td>
<td>6%</td>
</tr>
<tr>
<td>Making observations</td>
<td>0</td>
<td>2%</td>
</tr>
<tr>
<td>Instrumental genesis</td>
<td>0</td>
<td>4%</td>
</tr>
<tr>
<td>Technical software statement</td>
<td>4</td>
<td>5%</td>
</tr>
<tr>
<td>Off-task</td>
<td>0</td>
<td>2%</td>
</tr>
<tr>
<td>Total</td>
<td>78</td>
<td>99%</td>
</tr>
</tbody>
</table>

classroom. Also, the interaction with the second highest frequency in Mr. Louiselle’s classroom was focus which occurred almost as often as teacher explain in his classes.

Focus was also the second highest frequency in the combined classrooms. Here are some examples of the focusing questions asked by the teachers: “What are we comparing here?” “What do we need that equation for?” “Does that look like a good regression line to you guys or not?” “What did it just draw in there?” The focusing questions engage the students more in the thinking that is involved than when the teacher just explains. The third highest frequency in the combined classrooms was inquiry. Here are some examples of the inquiry questions asked by the students: “What is the centroid?” “How did you get that?” and “Why didn’t you just cheat [use the regression line created by the
software] instead of doing your own?” Inquiry using the software did not occur as often in this investigation as it did when students were working in groups because when the teacher is leading the exploration in a particular direction, the students have less room to explore their own conjectures with the software. Yet, it did occur when the students were exploring what would happen to the regression line and SSE when a point was deleted from the data set.

Levels of Mathematical Thinking Displayed During the Regression Line Investigation

Table 11 shows the levels of mathematical thinking verbalized during the whole-class discussions about the regression line. The frequencies and percentages are listed in each teacher’s classroom since the total number of codes is not equal in both classrooms. In both classrooms combined, Building-with Analyzing was the level of mathematical thinking that was displayed most often, and Building-with Evaluative-Analyzing occurred nearly as often. However, when examining each teacher’s classrooms, there are different results. In Mr. Foster’s classroom Recognizing Comprehending occurred as often as

Table 11

<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
<th>Type of Classroom Environment</th>
<th>Whole-Class/Single Computer</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Foster</td>
<td>Louiselle</td>
<td>Combined</td>
</tr>
<tr>
<td>Recognizing Comprehending (C)</td>
<td>6</td>
<td>9</td>
<td>18%</td>
</tr>
<tr>
<td>Recognizing Applying (A)</td>
<td>1</td>
<td>4%</td>
<td>7</td>
</tr>
<tr>
<td>Building-with Analyzing (AN)</td>
<td>6</td>
<td>26%</td>
<td>13</td>
</tr>
<tr>
<td>Building-with Synthetic-Analyzing (SA)</td>
<td>3</td>
<td>13%</td>
<td>0</td>
</tr>
<tr>
<td>Building-with Evaluative-Analyzing (EA)</td>
<td>5</td>
<td>22%</td>
<td>13</td>
</tr>
<tr>
<td>Constructing Synthesizing (SN)</td>
<td>2</td>
<td>9%</td>
<td>7</td>
</tr>
<tr>
<td>Constructing Evaluating (E)</td>
<td>0</td>
<td>0%</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>100%</td>
<td>49</td>
</tr>
</tbody>
</table>
**Building-with Analyzing**, while in Mr. Louiselle’s classroom **Building-with Evaluative-Analyzing** occurred as often as **Building-with Analyzing**.

**Building-with Analyzing** involves applying a known procedure in a new context, familiarizing oneself with the problem, or searching for patterns and an example from this lesson is as follows. All of the student statements in this excerpt were coded AN.

Mr. Louiselle: Ok again, one of the things we started looking at is how you describe a scatterplot. Strength was one of the features right? Is it linear, or is it curved? What is this?

Several students: Linear.

Mr. Louiselle: Yeah, it’s linear and is it positive or negative?

Several students: Negative.

Mr. Louiselle: Now, if you’re not sure, if you have a hard time looking at points, consider the context of the problem. What are we comparing here?

Taylor: Curb weight and miles per gallon.

Ian: Weight.

Mr. Louiselle: Weight of a car with?

Taylor: Miles per gallon.

Mr. Louiselle: Miles per gallon right? So would it make sense that as the weight goes up, what would you expect with the miles per gallon?

Taylor: Go down.

In the exchange above, the students were looking for patterns in the data and familiarizing themselves with the problem.

The **Recognizing Comprehending** level of mathematical thinking in Mr. Foster’s classroom all occurred in the beginning of the lesson when they were trying to understand
the first steps in the problem—the data they were using and how to make a scatterplot of those data. The *Building-with Evaluative-Analyzing* occurred in both classes when they were trying to find a line of best fit as in the following excerpt from Mr. Louiselle’s class.

Mr. Louiselle:  Now, is that a good fit right there, my line?  
Alli:  Yes-ish.  Maybe.  
Several students:  No.

When answering this question, the students are pulling together ideas to make a judgment or evaluating whether a result is reasonable. This level of mathematical thinking also occurred in Mr. Louiselle’s class when the students evaluated the effect of removing an outlier as follows:  Mr. Louiselle:  Did it change the slope a lot?;  Megan:  A little bit.

The *Constructing Synthesizing* level of mathematical thinking did not occur in these whole-class discussions as much as it occurred in the group-work of the tessellation problems. This is perhaps because the discussion was led by the teacher so the students may not have had as much of an opportunity to explore the problem from many perspectives or progressively explore the problem to develop new insights. When it did occur in this lesson, it was often when the student was clearly following along on his/her own computer such as Mr. Foster’s student who said, “How do I make it [the slope] negative?”

**Types of Interaction Patterns Found in Each Level of Mathematical Thinking**

In the combined classrooms, the highest level of mathematical thinking that was observed (SN) occurred during the interactions *focus, inquiry, inquiry using the software, giving new idea/making a conjecture, building on other students’ ideas* and making
observations. The interactions give expected information, focus, inquiry and giving new idea/making a conjecture had the most occurrences spanning levels of mathematical thinking from Recognizing Comprehending (C) to Constructing Synthesizing (SN). All of these interaction patterns except for give expected information are consistent with constructivist theory. There are generally lower frequencies in this table than in the group work table since only student statements are coded for mathematical thinking and

Table 12

The Number of each Type of Interaction Found in each Level of Mathematical Thinking during Whole-class Discussions about the Regression Line across Both Teachers

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Levels of Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>Collect answers</td>
<td></td>
</tr>
<tr>
<td>IRE</td>
<td>2</td>
</tr>
<tr>
<td>Give expected info</td>
<td>3</td>
</tr>
<tr>
<td>Funnel</td>
<td></td>
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<tr>
<td>Teacher explain</td>
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<tr>
<td>Hint to solution</td>
<td></td>
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<tr>
<td>Exploring methods</td>
<td></td>
</tr>
<tr>
<td>Teacher elaborate</td>
<td></td>
</tr>
<tr>
<td>Focus</td>
<td>2</td>
</tr>
<tr>
<td>Inquiry</td>
<td></td>
</tr>
<tr>
<td>Inquiry using the software</td>
<td>1</td>
</tr>
<tr>
<td>Answering another student’s question</td>
<td>2</td>
</tr>
<tr>
<td>Giving new idea/making a conjecture</td>
<td>2</td>
</tr>
<tr>
<td>Argument</td>
<td></td>
</tr>
<tr>
<td>Proof of answer by manipulation of software</td>
<td></td>
</tr>
<tr>
<td>Resolution of conceptual issue using software</td>
<td></td>
</tr>
<tr>
<td>Proof of answer by student explanation</td>
<td></td>
</tr>
<tr>
<td>Building consensus</td>
<td></td>
</tr>
<tr>
<td>Checking for consensus</td>
<td></td>
</tr>
<tr>
<td>Develop conceptual understanding</td>
<td></td>
</tr>
<tr>
<td>Pupil self-nominate</td>
<td></td>
</tr>
<tr>
<td>Building on other students’ ideas</td>
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<tr>
<td>Making observations</td>
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<tr>
<td>Instrumental genesis</td>
<td></td>
</tr>
<tr>
<td>Technical software statement</td>
<td></td>
</tr>
<tr>
<td>Off-task</td>
<td></td>
</tr>
</tbody>
</table>

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in a teacher-led discussion, many lines of text are that of the teacher’s spoken words. There are fewer opportunities to capture students’ mathematical thinking. The table at the end of this chapter containing all of the whole-class/single computer data has more revealing patterns.

**Relevant Assessment Results**

In this case, both teachers did not give the same assessment. The context shown in item number 1 in Figure 15 was used by both teachers. However, the questions asked by each teacher varied. Mr. Foster’s test required many numerical calculations, and did not assess the conceptual understanding that can be gained from using *CPMP-Tools* as much as what Mr. Louiselle’s test assessed. One item that was common on both teachers’ assessments is as follows: *Using the equation in Part a/b [Part a or b depended on which teacher’s test it was. The equation was the regression line.], find the sum of the squared errors for the regression line.* On this item, only 8 (17%) out of 47 students in Mr. Foster’s classes got the correct answer of 921. Of the 39 students who got it wrong, five seemed to be doing the correct procedure but must have made some mistakes in the many calculations that are required. Many did not show their work. Three students gave the correlation coefficient. One student answered “Mexico,” and one did cubic regression and gave that equation as an answer. Another student said, “I have no idea,” and another said, “nothing seems to be going right.”

On the same item, 14 (28%) out of 50 of Mr. Louiselle’s students got it completely correct. Of the 36 students who got it wrong, 13 of them showed the correct steps. These students had a good understanding of the problem but just made a simple arithmetic error. Two students left it blank, one gave the correlation coefficient squared,
and one said “Mexico.” The students in both classrooms that answered “Mexico” presumably thought the question was asking which country had the greatest residual. The other 19 students attempted to find an answer but used an incorrect procedure. However, 7 of these 19 students had procedures that closely resembled the correct procedure.

Based on Mr. Foster’s students’ comments on the test, lack of work shown, or work that was far from the correct procedure, Mr. Louiselle’s students appeared to have more of the right idea about this problem than Mr. Foster’s students did on this item. Additionally, on Mr. Louiselle’s test papers, 24/50 of the students drew the residuals in on the graph compared to 9/47 of Mr. Foster’s students. Mr. Foster’s students drew no other markings on their graph. Meanwhile, three of Mr. Louiselle’s students drew in an additional line of best fit for two regression lines much like the two lines shown on the screen in Figure 14, and three others drew the squares (one of those students’ papers is shown in Figure 15). These students may have created a mental picture of the image they saw on the computer screen during the investigation and were re-creating it on paper.
There are two other items on Mr. Louiselle’s test that are related to concepts students could learn from using the data analysis tools in CPMP-Tools. These items were not on Mr. Foster’s test. One of them is as follows: *Find the point with the largest positive residual. Which country does it represent? Find the value of the residual.* Out of the 50 students in Mr. Louiselle’s classes, 19 (38%) answered this question completely correct. Three students gave the correct residual, but the wrong country. Ten other students had the right country but gave the wrong residual. One student left the residual blank, and one squared the correct residual. Seven students answered that Mexico has the largest absolute value of all the residuals. The other nine students gave other countries. Most of these errors do not appear to be big misconceptions.

The other test item of interest was as follows: *On the scatterplot, circle the point that is an outlier. Then describe how the slope of the regression line and the correlation coefficient would change if that point were deleted from the data set.* On this item, 30 (60%) of the 50 students answered correctly that the slope would decrease. Three students said the slope would stay the same, four gave no answer for the slope, and seven said that the slope increased. Six students answered that the line would go up because the outlier pulled it down. This answer could represent a correct understanding without precisely stating that the slope decreases. These students are probably picturing what happens to the left side of the graph shown in Figure 15 when the lowest point on the left is removed. That end of the graph goes up when the outlier is removed because the outlier was pulling it down. This answer may reflect a visual image that the students gained from using CPMP-Tools.
The second part of the same test item asks about the effect on the correlation coefficient. On this part, six (12%) students answered correctly—three of them saying it would go down and three of them giving the exact value to which it decreased. Four students were partly correct in saying that the correlation coefficient would change very little (in fact it goes from .87 to .78). Eighteen students neglected to respond about the correlation coefficient. The other 22 students said that the correlation coefficient would increase and/or get closer to 1. This answer is actually understandable since it is often the case that when an outlier is removed the correlation coefficient gets stronger. However, in this case the set of points loses some of its linear quality and appears more cloud-like without the outlier.

Based on the interviews of students from both classrooms, it was clear that the students from Mr. Foster’s class had had less experience with using the data analysis tools in CPMP-Tools. Mr. Foster’s students struggled to show, or were reluctant to show what they could do with the data analysis tools. It did not seem that they had used the software for homework, and not much productive use of the software by the students was observed in class. These students also were less likely to know the meaning of residuals or the sum of squared errors. When asked about how CPMP-Tools is used in their group work, one of Mr. Foster’s students said, “I think we talk about what buttons to push but not as much about what we’re really doing.”

On the contrary, there was a lot of evidence of the use of the data analysis software in Mr. Louiselle’s class. The students seemed to follow along with him—some actually following along on their own laptops on their desk—when he facilitated a whole-class discussion using CPMP-Tools. They asked questions such as, “when I tried to do
that at home last night, I couldn’t figure out how to …,” that suggested that they had used the software on their homework. Every student of Mr. Louiselle’s that was interviewed used their computer to show what they could do with the software. This may have contributed to their slightly better performance on these assessment items.

**Conclusions**

Mr. Foster’s classroom discussion could be characterized by the teacher doing most of the explaining and the students giving some ideas. Mr. Louiselle’s classroom discussion contained a greater variety of interaction patterns. There was still a large number of teacher explain interactions, but there was also a substantial amount of focusing and inquiry occurring. Mr. Louiselle used students input in the discussion which resulted in higher levels of mathematical thinking displayed by the students. There were more technical software statements during this investigation. This may be because the statistical tools have more options available and more complex operations, than the custom geometry tools used in this study. In the combined teachers’ classrooms, the students gave ideas that spanned all levels of mathematical thinking except the highest level. However, when students built on other students’ ideas, the result was mainly in the second and third highest levels of mathematical thinking. As students used someone else’s idea to go further in the line of reasoning, the thinking tended to become deeper.

In the discussions that Mr. Louiselle led in the statistics unit, he was frequently asking students questions to make them think critically about the topic. For example, in a discussion about the Chicago White Sox context he guided students to see that the data revealed a scandal. All students were engaged in this discussion and many participated to try to analyze the patterns in the data. As it turns out, even though baseball player Nemo
Leibold had a World Series batting average that was the furthest from his season batting average, and his data point was the furthest away from the regression line, he was not accused of throwing the game while other players were accused. Students were able to make that discovery as a result of this rich discussion that was facilitated by the use of CPMP-Tools.

The assessment for the regression lines investigations looked different for each teacher. Mr. Foster’s questions required more procedures than Mr. Louiselle’s, and Mr. Louiselle’s questions required more of the type of interpretation that students had been doing with the use of CPMP-Tools. On the item that both tests had in common, students had to find the sum of squared errors (without the use of CPMP-Tools). Even though, Mr. Louiselle’s students had used the software to find the SSE for the most part, a larger percentage of his students (28%) got the question completely correct. Eight students (17%) got the question right in Mr. Foster’s classes.

Additionally, the assessments collected from Mr. Louiselle’s classes showed more evidence of the influence of using CPMP-Tools in this lesson. This finding is consistent with those of the observations and interviews, that Mr. Foster’s students did not have many opportunities to learn the material with the use of CPMP-Tools. During the observation, the students were told not to use their own computers but just watch Mr. Foster. The students’ questions also revealed that they had not used the tool themselves. Meanwhile, Mr. Louiselle’s students had already used the tool for their homework the night before the observation, and they were allowed to follow along on their own computer as Mr. Louiselle worked on the computer at the front of the room.

Additionally, while Mr. Foster was telling the students how to use the software,
and what to click on, Mr. Louiselle’s students were telling him how to use the software and what to click on. If a student asked how to do something with the software, Mr. Louiselle did not just tell him/her how to do it, he led them to figure it out on their own. Therefore, Mr. Louiselle’s students demonstrated greater competency in using CPMP-Tools during class and in the interviews. Overall, familiarity with and frequent use of the tools seems to have a positive effect on understanding of some regression concepts.

**Mathematical Concept Three: Side-Side-Angle Triangle Condition**

Mr. Louiselle’s two Course 2 classes were also observed while they studied the side-side-angle (SSA) condition for triangles. Mr. Louiselle facilitated an interactive whole-class discussion during this investigation, so these data were categorized under the Whole-Class/Single Computer classroom environment.

**The Investigation**

In Course 2, Unit 7, Lesson 2, Investigation 3 (*Triangle Models—Two, One, or None*?), students explore the SSA condition for triangles, and the conditions under which there will be zero, one, or two possible triangles with two side lengths and a non-included angle given. The custom tool designed for this investigation is entitled *Explore SSA*. The investigation begins with a real-life example of a triangle found in the mechanism for a cold frame, which is a box used in a flower nursery to help the flowers grow (see the top picture in Figure 16). In this mechanism, two sides of the triangle are a fixed length. One of these sides is along the top and this side can be raised or lowered by the other fixed side of the triangle, the piece that props up the top. When the top is opened in some positions, the prop can either be placed closer to the vertex of the non-included angle (as
shown in the lower picture in Figure 16), or farther away from it (similar to the top picture in Figure 16). For example, it is possible to have the same given measurements as shown in the lower picture in Figure 16, but with the prop placed close to the left end of the base. Therefore, this real-life situation is a case of being able to form two possible non-congruent triangles with two pairs of corresponding sides and one pair of non-included angles congruent.

Figure 16. Diagrams of the cold frame (Hirsch et al., 2008, p. 500).

Students are introduced to this context in number one of the investigation. In this problem, they are just familiarizing themselves with the context—making a sketch of the triangle formed in the cold frame and making some conjectures about the effects of moving the prop in the cold frame mechanism. In number two (see Figure 17), the students are asked how many triangles can be formed with the measures of two sides and
a non-included angle fixed. They may use the *Explore SSA* geometry custom tool shown in Figures 18, 19 and 20 to explore the situation. In these figures, side BC can be thought of as the prop, side AB is the top, and angle A is at the hinge. Using *CPMP-Tools*, the students experiment with different lengths of BC when the measure of angle A, and the length of AB are fixed. Figures 18, 19, and 20 show what happens when the length of BC is less than the value it would be if triangle ABC were a right triangle, when the length of BC is exactly the length needed to create right triangle ABC, and when the length of BC is greater than the value that makes triangle ABC a right triangle. This exploration allows students to discover the conditions under which one, two, or no triangles are possible with two adjacent sides and the non-included angle given.

Problem number three is similar to number two except in part a the students are to use an obtuse angle for angle A, and in part b a right angle for angle A. After they explore all the possibilities, students come back to the cold frame context in problem number four as shown in Figure 16. This problem asks students if they think there could be another position for the prop to be placed such that the angle remains the same. The prop stays the same length, as does the length on the top from the hinge to the point where the prop is attached. Therefore, this is a situation just like they had explored with *CPMP-Tools*. 
You will return to Problem 1 after first exploring how many differently shaped triangles can be formed given two side lengths (say, $AB$ and $BC$) and the measure of an angle ($\angle A$) not included between them. In Problem 2, you will explore the side-side-angle (SSA) condition where $\angle A$ is acute. In Problem 3, you will explore the case where $\angle A$ is not an acute angle.

2. Use a compass and ruler or software like the interactive geometry “Explore SSA” custom tool to conduct the following experiment.

a. Draw an acute angle, $\angle A$. Fix point $B$ on one side of $\angle A$ by marking off length $AB$. Suppose $BC$ is very short, as in the figure on the right. How many triangles $ABC$ can be formed with the three given parts: $\angle A$, $AB$, and $BC$? Explain.

b. With the same $\angle A$ and length $AB$, try making $BC$ longer. There is a minimum length of $BC$ for which a triangle with the three given parts is formed, as shown in the next figure.

i. In this case, what kind of triangle is $ABC$?

ii. Is the triangle shown the only possible triangle determined by $\angle A$, $AB$, and $BC$?

iii. Write the length $BC$ in terms of length $AB$ and a trigonometric function of $\angle A$.

c. With the same $\angle A$ and length $AB$, continue to make $BC$ longer. Try to find lengths $BC$ for which two noncongruent triangles can be formed with the same three given parts.

i. Draw a figure like those above that shows an example of this situation.

ii. Over what interval of lengths $BC$ will two noncongruent triangles be formed?

d. If $BC$ is greater than or equal to all values in the interval you found in Part cii, how many triangles can be formed with the three given parts? Draw a figure like those above that shows an example of this situation.

e. In Parts a–d, you explored the SSA condition when $\angle A$ is acute. Summarize your findings by describing the lengths, or intervals of lengths, $BC$ for which the given side-side-angle parts determine:

i. no triangle.

ii. exactly one triangle.

iii. two noncongruent triangles.

Figure 17. Problem number two of Investigation 3: Triangle Models—Two, One, or None? (Hirsch et al., 2008, p. 499).
Figure 18. Explore SSA custom tool with no triangle formed.

Figure 19. Explore SSA custom tool with a right triangle formed.
Mr. Louiselle’s Classes

During this investigation, Mr. Louiselle facilitated a whole-class discussion using the Explore SSA custom tool on his computer and projected it on the screen at the front of the room. A segment of the transcript from this class discussion is given below. Gwen and Claire were two students who were operating the teacher’s computer. Figures 18, 19, and 20 show the kind of images that the students were looking at during the discussion.

Mr. Louiselle: What happened there?

Gwen: Our edges don’t touch.

Mr. Louiselle: Why did it—it doesn’t touch—why did the triangle go away?

Gwen: Because it’s too short.

Mr. Louiselle: Ok, keep going just very slowly until we get something.

Gwen: Like this way?

Mr. Louiselle: There you go, go back a little bit, keep going.
Jorden: Back.

Mr. Louiselle: There you go, whoops.

Gwen: Go back?

Claire: No, you have to get the triangle back.

Mr. Louiselle: A little bit smaller, perfect. What do you notice about this?

Chloe: It’s a right triangle.

Mr. Louiselle: A right triangle? Does it appear to be a right triangle?

Students: Yeah.

Micah: Yes, sir.

Mr. Louiselle: Ok. When that got smaller, what happened? When Gwen made BC smaller than that—go a little bit smaller than 12 please.

Hannah: Oh, the line went away.

Mr. Louiselle: The line went away.

Gwen: Do you want me to make it smaller?

Mr. Louiselle: Right after that. It’s too short to touch. Now, make it longer please.

What’s happening here?

Hannah: There are two possibilities.

Mr. Louiselle: Why are there two possibilities?

Hannah: Because there are two BC’s. It can be on the left or on the right.

Mr. Louiselle: Do you see that there’s two triangles in here, guys?

Students: Yeah.

This excerpt from the class discussion was the typical line of reasoning that occurred in this investigation. Students first noticed that when side BC of the triangle
was too short, a triangle would not be formed. When BC was lengthened just long enough to form a triangle, a right triangle appeared. When BC was lengthened even further, two triangles appeared. These two triangles contained the same angle A and side AB, and a side BC (or BC’) that was the same length in both triangles (since BC and BC’ are both radii of the same circle), but the triangles clearly were not congruent. Thus, students discovered that the SSA condition does not guarantee the congruence of triangles.

This excerpt also illustrates some common characteristics of the whole-class/single computer environment. Statements such as “go back a little bit, keep going” or “make it smaller” indicate the action taking place in the exploration. Additionally, more students participated in the discussions that involved CPMP-Tools. The observation forms that were part of the field notes contained tallies of the number of students that participated in each class discussion. In this short segment of the discussion, there were at least six students actively participating. However, there is only evidence that Hannah understands the underlying concepts that are supposed to be discovered. As with any lesson, that can be a disadvantage to facilitating the discussion as a whole-class activity as opposed to having each group of students explore the situation on their own.

In this segment, there are also some examples of statements made which are observations (making observation interaction pattern) about the occurrences on the computer screen—“Our edges don’t touch,” “Because it’s too short,” “Oh, the line went away,” and “It’s a right triangle.” The observations are based on the image on the screen and are beneficial for building students’ understanding of the underlying concepts. They
need to see that when segment BC is too short, there will be no triangle, and when segment BC is a particular length it will form a right triangle. These statements combined with the statements made by students in the interviews—“It is helpful to see the concept being discussed” and “It helps you show your thinking”—are an indication of the visual benefit of using *CPMP-Tools*.

Next in the lesson, Mr. Louiselle created another image with the software by changing the length of AB from 17 to 18, and asked the students if they could find the shortest length for BC that would create a triangle. Through an interactive discussion, the students came to the conclusion that they could set up an equation using the sine function. The sine of angle A equals BC divided by AB. After they calculated the answer for BC, he asked them to predict what would happen when BC is shorter or longer than this length.

Mr. Louiselle: So in this case would somebody find out what the shortest distance is going to be? What does it have to be? Who’s got their calculator handy? Sam, you’ve always got yours ready.

Sam: 12.72.

Mr. Louiselle: 12.72. So if I pick anything smaller than 12.72 what’s the problem?

Kyle: It won’t touch.

Mr. Louiselle: It won’t touch. So guess what, how many possibilities are there?

Anything less than 12.7, guess what?

Ashley: No triangle.

Mr. Louiselle: No triangle. So if I do anything larger than 12.7, what do you think is going to happen?
After further discussion, the students realized that there would be two possible triangles when BC was longer than 12.72. They verified it using the software by showing that when BC equaled 14, two triangles appeared on the screen. The students were able to make connections between the value they obtained with an equation, and the image on the screen created by the software. In interviews with Mr. Louiselle’s students, they said that they liked the fact that there was immediate feedback using CPMP-Tools and that many things were already set up like the geometric drawing that was already created.

When Mr. Louiselle was interviewed about the SSA investigation he said: “I remember doing that on the chalkboard with meter sticks and needing three hands, and this is so helpful. The Explore SSA geometry tool allows student to clearly see the conditions under which one, two, or no triangles will exist. CPMP-Tools facilitates student ownership of their learning. The software allowed students to feel more ownership of the mathematical ideas. The students decide what angle measures or lengths of sides they wish to explore. Since students don’t have to do the drawing and the measuring with a ruler, protractor and compass; measurement error is avoided and time is saved. It takes care of students’ inability to measure accurately which results in them missing the patterns.”

**Interactions Displayed During the SSA Investigation**

Table 13 shows the interaction patterns that were found during the observations of the investigation on the side-side-angle condition for triangles. These data contributed to the total of the whole-class/single computer classroom environment interaction patterns. Since these observations were of whole-class discussions as opposed to small-group
discussions, all students’ spoken words were transcribed and included in the counts in this table.

The interaction patterns that are typical of traditional math instruction (*IRE*, *give expected information*, *funnel*, *teacher explain*, and *hint to solution*) occurred 15.3% of the time, while the rest of the interaction patterns that are more consistent with standards-based math instruction (Wood, Williams, & McNeal, 2006) occurred 84.7% of the time. Therefore, this investigation performed with the use of *CPMP-Tools* was implemented according to standards-based math instructional strategies a majority of the time.

The *giving new idea* interaction occurred most frequently (19.4%). The SSA investigation evoked many ideas from the students. For instance, “There might be two different answers” “It [the third side of a triangle] can’t be longer than both sides added together” or “It’s a right triangle so you can get the third angle.” The second most frequent interaction pattern was *develop conceptual understanding* (12.8%). For example, Mr. Louiselle was trying to develop conceptual understanding when he asked, “When would we have no solutions?” or “It turns into one—why does it turn into one?” Another example of *developing conceptual understanding* was when Mr. Louiselle said, “This is good because your intuition says there has to be some boundaries. Why do there have to be some boundaries?” The third most frequent interaction pattern was *teacher explain* (10.1%). Given the fact the classroom environment for this investigation was a whole-class, teacher-led interactive discussion, this is not surprising. When a teacher is leading the discussion, he/she may spend a significant amount of time explaining to the class.
Table 13

Frequencies and Percentages of each Type of Interaction Found in Mr. Louiselle’s Classroom during Whole-class Discussion about the SSA Triangle Condition

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Whole-Class/Single Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency</td>
</tr>
<tr>
<td>Collect answers</td>
<td>0</td>
</tr>
<tr>
<td>IRE</td>
<td>9</td>
</tr>
<tr>
<td>Give expected info</td>
<td>5</td>
</tr>
<tr>
<td>Funnel</td>
<td>1</td>
</tr>
<tr>
<td>Teacher explain</td>
<td>49</td>
</tr>
<tr>
<td>Hint to solution</td>
<td>10</td>
</tr>
<tr>
<td>Exploring methods</td>
<td>0</td>
</tr>
<tr>
<td>Teacher elaborate</td>
<td>22</td>
</tr>
<tr>
<td>Focus</td>
<td>35</td>
</tr>
<tr>
<td>Inquiry</td>
<td>24</td>
</tr>
<tr>
<td>Inquiry using software</td>
<td>47</td>
</tr>
<tr>
<td>Answering another student’s question</td>
<td>0</td>
</tr>
<tr>
<td>Giving new idea/making a conjecture</td>
<td>94</td>
</tr>
<tr>
<td>Argument</td>
<td>10</td>
</tr>
<tr>
<td>Proof of answer by manipulation of software</td>
<td>0</td>
</tr>
<tr>
<td>Resolution of conceptual issue using software</td>
<td>0</td>
</tr>
<tr>
<td>Proof of answer by student explanation</td>
<td>0</td>
</tr>
<tr>
<td>Building consensus</td>
<td>27</td>
</tr>
<tr>
<td>Checking for consensus</td>
<td>6</td>
</tr>
<tr>
<td>Develop conceptual understanding</td>
<td>62</td>
</tr>
<tr>
<td>Pupil self-nominate</td>
<td>0</td>
</tr>
<tr>
<td>Building on other students’ ideas</td>
<td>12</td>
</tr>
<tr>
<td>Making observations</td>
<td>43</td>
</tr>
<tr>
<td>Instrumental genesis</td>
<td>6</td>
</tr>
<tr>
<td>Technical software statement</td>
<td>22</td>
</tr>
<tr>
<td>Off-task</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>484</td>
</tr>
</tbody>
</table>

Some computer-related interactions occurred frequently in Mr. Louiselle’s class. *Making observations* occurred 43 times (8.9%). The following are some examples of statements students made which were observations about the occurrences on the computer screen: “Our edges don’t touch,” “Because it’s too short,” “Oh, the line went away,” and “It’s a right triangle.” Students were referring to these kinds of observations when they said in the interviews that “It is helpful to *see* the concept being discussed” and “It helps you *show* your thinking.” *Inquiry using the software* is another computer-related interaction that occurred frequently (9.7%) during this investigation. This interaction repeatedly occurred when the students were using the software to change the length of side BC of the triangle in order to explore what would happen to the whole
shape of the diagram. *Inquiry* and *inquiry using the software* interactions occurred a combined 14.7% of the time indicating that students spent a significant amount of time in inquiry mode. It is also notable that there were zero instances of off-task interactions in Mr. Louiselle’s class during this investigation.

**Levels of Mathematical Thinking Displayed During the SSA Investigation**

Table 14 shows the levels of mathematical thinking verbalized by all students during the SSA investigation. A majority of the frequencies are contained in the highest three levels of mathematical thinking (which are the bottom three rows of Table 14). Furthermore, the lowest two levels of mathematical thinking (top two rows) have very low frequencies.

Seventy percent of the student’s statements were at the *Constructing Synthesizing* level. This level of mathematical thinking happens when students are exploring the problem from many perspectives or exploring the problem to continually develop new insights. When students are using *CPMP-Tools* they are often exploring the problem. Though it occurred much less frequently, *Building-with Evaluative Analyzing* was the Table 14

*The Frequencies and Percentages of each Level of Mathematical Thinking Found in Mr. Louiselle’s Classroom during Whole-class Discussion about the SSA Triangle Condition*

<table>
<thead>
<tr>
<th>Type of Classroom Environment</th>
<th>Whole-Class/Single Computer</th>
<th>Mathematical Thinking</th>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizing Comprehending (C)</td>
<td></td>
<td>6</td>
<td>3%</td>
<td></td>
</tr>
<tr>
<td>Recognizing Applying (A)</td>
<td></td>
<td>5</td>
<td>2%</td>
<td></td>
</tr>
<tr>
<td>Building-with Analyzing (AN)</td>
<td></td>
<td>15</td>
<td>7%</td>
<td></td>
</tr>
<tr>
<td>Building-with Synthetic-Analyzing (SA)</td>
<td></td>
<td>2</td>
<td>1%</td>
<td></td>
</tr>
<tr>
<td>Building-with Evaluative-Analyzing (EA)</td>
<td></td>
<td>23</td>
<td>11%</td>
<td></td>
</tr>
<tr>
<td>Constructing Synthesizing (SN)</td>
<td></td>
<td>144</td>
<td>70%</td>
<td></td>
</tr>
<tr>
<td>Constructing Evaluating (E)</td>
<td></td>
<td>11</td>
<td>5%</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>206</td>
<td>99%</td>
<td></td>
</tr>
</tbody>
</table>
second most frequently found level of mathematical thinking (11%). This level of mathematical thinking often involves pulling together ideas for the purpose of making a judgment. Of the lowest four levels of mathematical thinking, *Building-with Analyzing* is the only level with a frequency in the double-digits. In these instances the students were applying a known procedure in a new context or familiarizing themselves with the problem using specific numerical examples.

A good illustration of the higher levels of mathematical thinking was the piece of transcript previously given in the section that described Mr. Louiselle’s lesson on SSA. In that segment of transcript, many instances of higher-level mathematical thinking occurred. Using the *Explore SSA* custom tool, the computer controller (Gwen) lengthened the segment BC so that the diagram went from showing no triangle, to one right triangle, to two triangles. When BC was shorter than 12 in this case, a triangle could not be formed, when it was longer than 12 two triangles were formed. The first instance of *Building-with Evaluative-Analyzing* (EA) occurred when Gwen evaluated the result shown on the computer screen and decided that it was not sufficient because it did not form a triangle—“Our edges don’t touch.” Thus, Gwen evaluated the result during the process of exploring/building, as opposed to the evaluating that occurred when the ideas were well formed—*Constructing Evaluating*. So, this was at the 3rd highest level of mathematical thinking. Then, when she said “Because it’s too short,” Gwen formulated an argument to explain the discovered pattern. From that point on, the students progressively explored the problem to continually develop new insights. Thus, a majority of the discussion in this transcript was at the *Constructing Synthesizing* level of mathematical thinking, which is the 2nd highest level of mathematical thinking.
Types of Interaction Patterns Found in Each Level of Mathematical Thinking

Table 15 contains data from Mr. Louiselle’s classes during the SSA investigation. The highest frequencies are in the highest three levels of mathematical thinking. The frequencies in the lower four levels of mathematical thinking are all single digit numbers—mostly ones and twos. Therefore, most of the interactions in this investigation took place at the higher levels of mathematical thinking. The largest concentration of Table 15

*The Number of each Type of Interaction Found in each Level of Mathematical Thinking during Mr. Louiselle’s Whole-class Discussions about the SSA Triangle Condition*

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>C</th>
<th>A</th>
<th>AN</th>
<th>SA</th>
<th>EA</th>
<th>SN</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collect answers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IRE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Give expected information</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
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frequencies occurs in the *Constructing Synthesizing* (SN) column. As students work with *CPMP-Tools* they are often exploring the problem, which is characteristic of the *Constructing Synthesizing* level of mathematical thinking.

A row that has a concentration of frequencies is the *giving new idea/making a conjecture* interaction pattern. This interaction occurs at all levels of mathematical thinking except for the lowest level. Thus, sometimes students suggested ideas in class that demonstrated lower levels of mathematical thinking and sometimes the ideas demonstrated higher levels of mathematical thinking. The highest frequencies in the table are in the *Constructing Synthesizing* (SN) column, *giving new idea/making a conjecture* and *making observations* rows. These combinations of interactions and level of mathematical thinking characterize much of the discourse in this investigation. There were also six instances of *instrumental genesis*.

**Relevant Assessment Results**

The assessment item that relates to the SSA condition is as follows: *Determine the number of triangles possible given the following information. Provide a sketch(s) and explain your answer.* \( m\angle A = 40^\circ, a = 6, c = 8 \). Out of the 22 assessments collected from Mr. Louiselle’s students, 12 (55%) of them were completely correct with the same reason shown in Figure 21 below—“6 is greater than the height of the right triangle and it is less than the 8 (other side length).” This is the discovery the students should have made when using the *Explore SSA* software. A correct answer consisted of the number two in the blank and a correct explanation.
Four of Mr. Louiselle’s students answered that there was one possible triangle. Of these students, one had side $a$ and side $c$ switched accidentally, two tried to use the inequalities stated in Figure 21 but did not have it quite right, and one said “b/c there is only one way you can make this triangle work.” Five of Mr. Louiselle’s students answered that there were zero possible triangles. Two of these students said that $a$ was smaller than $c$ so “it wouldn’t reach the bottom,” one said that it wasn’t “long enough to reach,” one said “because it is a right triangle and it doesn’t work,” and one said “you need three sides.” And the last student of Mr. Louiselle’s gave no numerical answer, just drew a picture of the given information.

The solution method shown in Figure 21 appears to be motivated by the use of CPMP-Tools. This student did not calculate two possible lengths for side $c$. The student has drawn the triangle in the same orientation as the triangle that was drawn with the
software in Figure 20. He or she was reasoning that there are two triangles because 6 (side BC) is greater than the height of the triangle—just like when they used the software to make segment BC longer than when it was perpendicular to segment AC. This previously given segment of transcript illustrates how students develop the idea that if side \(a\) is greater than the height of a right triangle that has the same length as side \(c\) and the same measure as angle \(A\), then there are two possible triangles. The segment of transcript below illustrates how they develop the understanding that side \(a\) also has to be less than side \(c\) in order for there to be two possible triangles. See Figures 22 and 23 for the images the students were viewing on the screen.

Mr. Louiselle: There you go. How many triangles?

Brennan: Two.

Mr. Louiselle: Ok, two. Drag it a little longer.

Aliyah: Three, there’s going to be three.

Mr. Louiselle: You think there will be three possibilities?

Aliyah: Just kidding.

Wes: It goes back to one.

Mr. Louiselle: Whoa, what happened there? When she started going further, ok, there’s none.

Aliyah: It started going back.

Jarrod: Keep going.

Mr. Louiselle: Keep going, keep going.

Brennan: Go, go, go, go.

Aliyah: It turns into one.
Mr. Louiselle: It turns into one, why does it turn into one?

Brennan: Aah.

Emerson: Because BC is longer than AB.

Figure 22. Explore SSA custom tool with two triangles shown.

Figure 23. Explore SSA custom tool with BC is longer than AB.

As the student controlling the computer makes side BC longer and longer past the point where two triangles are formed as in Figure 22 they see the point at which it goes back to only forming one triangle as in Figure 23. This occurs once side BC becomes
longer than side $AB$ because $BC$ can no longer intercept the circle at two points. This explains the second part of the answer in Figure 21—that 6 is less than 8. Therefore, there will still be two triangles.

All of the students in Mr. Louiselle’s classes that had an answer of two got the problem completely correct. There were not many opportunities to make a mistake using the solution method shown in Figure 21. If the student had a complete conceptual understanding of the situation, then they got the answer completely correct. Ten of Mr. Louiselle’s students (45%) gave the wrong answers, zero, one, and no answer. Most of these students had a correct visual representation of the situation, and displayed some conceptual understanding of the problem, but still had a misconception somewhere.

This solution method in a geometry context is in line with Mr. Louiselle’s stated philosophy on data analysis. One day after a class in the data analysis strand he said, “What’s important is that they can understand the graph, data, etc. It’s not as much—can they perform all these calculations by hand—it’s, can they interpret what’s really going on.” Here in this geometry context, his students showed that they could interpret what was going on in the triangle, not by using formulas and calculations, but by using the relationships between the sides and angles inherent in the figure. The assessments and student discourse suggest that with the use of CPMP-Tools, his goal was accomplished.

Conclusions

The fact that giving new idea was the most frequently occurring interaction is evidence that the discussions were interactive with students providing many of the ideas for discussion. Mr. Louiselle asked many focusing questions allowing the students to focus on the important aspects of the problem being discussed. Much of the discussion
time was also spent developing conceptual understanding. Beyond just making
observations based on the action taking place on the screen, the students were pressed to
look for the reasons for those actions and relationships. There were also many building
consensus interactions in Mr. Louiselle’s class discussion. Therefore, ideas were
exchanged until the class agreed on conclusions.

Seventy percent of the lines of text in this investigation contained the
*Constructing Synthesizing* (SN) level of mathematical thinking. Thus, a majority of the
interaction patterns fell in the SN column. Only the *giving idea* interaction pattern had a
significant amount of instances at lower levels of mathematical thinking. Yet, in this
investigation many more of the students’ ideas were at higher levels of mathematical
thinking than at lower levels.

Qualitative differences were found in discussion with the use of *CPMP-Tools*
compared to discussion without the use of *CPMP-Tools*. With the use of *CPMP-Tools*,
there was more active language—words like “stop,” or “right there,” or “go back” were
often found. This type of language was typical when *CPMP-Tools* was used by the
students because there was often an action taking place on the screen. This also helps to
explain why there were many making observations interaction patterns in this
investigation.

The responses on Mr. Louiselle’s related assessment item revealed which students
understood the dynamic relationships among the lengths of the sides of the triangles.
None of his students mindlessly set up an equation to solve for two possible lengths of
one of the sides of the triangle. Instead, they often made a drawing that resembled the
visual provided by the software, and provided a reasoned argument that explained the
relationships between the sides of the triangle. Many students demonstrated a good conceptual understanding of the problem situation.

Mathematical Concept Four: Transformations/Computer Animations

Observations included Mr. Louiselle’s teaching of Course 2, Unit 3, Coordinate Methods. These observations were only performed in Mr. Louiselle’s classes because he was the only Course 2 teacher that was using CPMP-Tools to do the computer animations in Unit 3. Two classes were observed for three consecutive days while they were using CPMP-Tools. A fourth day of observations recorded the presentations of the students’ computer animations—the culmination of Unit 3. During these investigations, the students worked in small groups using CPMP-Tools on their laptops. Thus, these investigations were categorized under the Small-Group/Multiple Computers classroom environment.

The Investigations

In Course 2, Unit 3, Lesson 2 (Coordinate Models of Transformations) students discovered how to represent transformations with coordinate rules of the form \((x, y) \rightarrow (\_\_, \_\_)\). For example, the coordinate rule for a reflection over the line \(y = x\) is \((x, y) \rightarrow (y, x)\). Then, in Lesson 3 (Transformations, Matrices, and Animation) of the same unit, students discover how to represent those same transformations (translations, reflections, rotations, and size transformations) with a matrix. For example, the transformation matrix for a reflection over the line \(y = x\) is \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]. When this matrix is multiplied by any coordinate pair \[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\] the result is \[
\begin{bmatrix}
y \\
x
\end{bmatrix}
\]. The coordinates for all the
points that make up a figure can be put into a matrix, and that coordinate matrix can be multiplied by the transformation matrix in order to perform the transformation on the whole figure.

Most of the observed computer use was in Lesson 3, Investigation 2 (*Building and Using Size Transformation Matrices*). In the first three problems of the investigation students explore ways to animate a simple drawing of a space shuttle (see Figure 24). The first problem asks students to find the coordinate rules to perform a size transformation and translations. The second and third problems ask students to find the matrices that could be multiplied by the coordinate matrix of the space shuttle in order to perform various size transformations. Beginning with the fourth problem of the investigation (see Figure 25), the students learn some simple programming language that can be used to write an animation program that will draw the space shuttle and cause it to grow larger (using size transformations) and move across the screen (using translations). By the end of this investigation, students can write their own program in the programming window of *CPMP-Tools*, using matrices to perform the transformations that animate the figure.

**Mr. Louiselle’s Computer Animation Project**

When the class finished Lesson 3 of Unit 3, they began working on their animation project. For this project, they had to create their own figure, decide on what transformations to use on that figure, and create a computer program that would perform those transformations in order to animate the figure. The students were observed for two days while they worked in groups creating their animation program. Mr. Louiselle completely relinquished the authority to the groups during these observations. He just let
Begin by examining the space shuttle model shown below.

a. What is the coordinate matrix for a similar shuttle model (in the same position) whose sides are twice the length of those in the given model? Half the length?

b. Write two coordinate rules $(x, y) \rightarrow (__, __)$ that would resize the shuttle model as described in Part a.

c. What is the coordinate matrix for the image of the given shuttle model when translated:
   i. 5 units to the right?
   ii. 3 units down?
   iii. 5 units to the right and 3 units down?

d. Write coordinate rules for these three transformations in the form $(x, y) \rightarrow (__, __)$.

Matrix representations for size transformations with center at the origin can be found using the same method you used to find rotation matrices.

a. Determine the entries $a$, $b$, $c$, and $d$ of the matrix for a size transformation with magnitude 2.

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
2x \\
2y
\end{bmatrix}
\]

b. What should be the image of the point $F(-2, 4)$ under this transformation? Multiply the transformation matrix by the one-column matrix for point $F$ and check to see if you get the correct image point.

c. Multiply the size transformation matrix you found in Part a by the matrix for $\triangle A\!E\!H\!$, where $A$, $E$, and $H$ are the wing tips of the shuttle model.
   i. Compare the coordinates of the image $\triangle A\!E\!H\!'$ with those found using the appropriate coordinate rule for a size transformation with magnitude 2.
   ii. Compare the lengths of $EH$ and $E'H$. Why does that relationship make sense?

d. Find the matrix for a size transformation with magnitude $\frac{1}{2}$. With magnitude 5.

How could you use the idea of multiplying a matrix by a real number to find the image of a point or a polygon under a size transformation of magnitude 3 with center at the origin? Of magnitude $\frac{1}{4}$? Compare methods with others and resolve any differences.

Figure 24. Problem number 1 - 3 of Investigation 2: Building and Using Size Transformation Matrices (Hirsch et al., 2008, p. 238).
Use the Roll Over Algorithm on page 235 and the following questions to help you develop an algorithm for a program that will repeatedly scale the space shuttle model by a factor of 1.5 using a size transformation with center at the origin.

- What information would you need to input?
- What processing would the program need to complete?
- What information should the calculator or computer output?

### Resizing Algorithm

**Step 1.** Set up the coordinate matrix representing the shuttle. (input)

**Step 2.**

**Step 3.**

The animation described in Problem 4 can be created using commands such as:

```plaintext
Resizing Program
let shuttle = \[[8,0][2,0][2,4][0,4][0,8][0,4][-2,4][-2,0][-8,0][8,0]\]
let sizematrix = \[[1.5,0][0,1.5]\]
draw shuttle
repeat 4 [draw [let shuttle = [sizematrix*shuttle]] pause 500]
```

a. Test the program by entering it in the Command window of your software.

b. Suppose the last three lines of the program were replaced by these two lines:

```plaintext
draw shuttle
repeat 4 [draw [let shuttle = [scale shuttle 1.5]] pause 500]
```

What animation do you think will be produced by the modified program? Check your conjecture by running the modified program.

c. Write a series of commands that could be used to create an animation that repeatedly scales the shuttle by a factor of 4 with center at the origin.

d. Test your program by entering it in the Command window. Revise commands as necessary.

Unlike in the cases of rotations and size transformations, translations cannot be represented with $2 \times 2$ matrices. (See Extensions Task 18.) Instead of using matrix multiplication, the coordinate rule form is frequently used to describe translations. For example, you can translate the shuttle and display the image with the commands shown below.

```plaintext
let shuttle = \[[8,0][2,0][2,4][0,4][0,8][0,4][-2,4][-2,0][-8,0][8,0]\]
let shuttle = [translate shuttle [5,-3]]
draw shuttle
```

*Figure 25. Problem number 4 – 6 of Investigation 2: Building and Using Size Transformation Matrices* (Hirsch et al., 2008, p. 239).
them start working on their animations and walked around and helped. He was very encouraging to the students throughout the project. During one class period he said, “You’re doing great guys—you really are! I love looking around the room and seeing stuff like this—people are doing the hand pump [makes the gesture]—you know, ‘yes!’ because they got it to draw the way they wanted it to. Or, ‘yes, the mouth turned out ok!’ or ‘man, check out my cool mouth, it’s distorted!’ I mean that’s what I want—I want you to be having fun with this!” Later on, he said, “I know you can do this. I have faith in you!” Even though Mr. Louiselle was not leading the class in any whole-class activities during these observations, he was actively facilitating the groups’ work and encouraging the students to keep trying.

The students overcame many struggles while working on their animation project. One of the mathematical challenges the students experienced was how to create their initial figure using the coordinates of the points that made up the figure. They had to decide whether they could put all of their points in one matrix or use multiple matrices to represent various pieces of the object. For example, in the truck program which will be given later, the wheels were represented in a separate matrix so that they could rotate separately. If they were not in a separate matrix, the whole truck would rotate when the coordinates were multiplied by the rotation matrix. The students also had to figure out the correct order of the points because consecutive points in the coordinate matrix would be connected. The following exchange was about these issues.

Victoria: Why don’t you just put that in this matrix?

Amy: Then wouldn’t it connect all the points?

Victoria: Then it would be all together.
Some students also had to figure out how to draw certain objects. For instance, one student drew a fish but she was trying to figure out what features to animate.

Heather: I don’t know what to do with it, I mean, I want to have a bigger fish come out, or I want to do bubbles. So how should I do that?

Mr. Louiselle: What does a bubble look like?

Heather: A circle.

Mr. Louiselle: Ok, so maybe draw some circles.

Heather: Draw circle?

Mr. Louiselle continued by referring Heather to a sheet that contained programming notes, and led her to realize that the dimension that was needed was the radius.

Another kind of struggle was getting the right transformations to accomplish the desired motions. The students had to know the numbers that go in the matrix for each type of transformation, and they had to know how to multiply the transformation matrix by the coordinates matrix and make the program draw it. The following episode shows a student struggling to get a rotation to work.

Addison: It’s not going. I have everything. I have LET ROTATE 90 and the matrix, and it’s not going.

Mr. Louiselle: Did you put the right matrix for the rotation?

Addison: Um hum.

Mr. Louiselle: And then you drew it?

Addison: I did this times thing.

Mr. Louiselle: Yeah, but did you draw it afterwards?

Addison: Yeah. It said it doesn’t know what rotation is.
Jennifer: I don’t get it. Wait, I don’t get why you said 0, 1, 1, 0.

Addison: It’s 0, 1, -1, 0.

Brianna: Next to your rotation matrix, you just said draw—it would be draw rotation 90, not draw crayon.

Addison finally got the rotation to work after a classmate helped her find the error in the program.

Even though the project was an individual one, there was much evidence of student collaboration during the work. Students were often showing each other their animations, and asking each other for help. Below is an example of this type of discourse where two students collaborate on aspects of one of their animation projects:

Moria: Oh my gosh, I don’t know what to do now.

Alaina: Make a fin.

Moria: How can I just put little fins on top of him?

Alaina: Why not?

Moria: It will look like…

Alaina: Have two fins coming off—two triangles coming off the side and then one…that would look so good!

Moria: Are you serious?

Alaina: Yeah!

As this excerpt illustrates, the students were asking each other many questions and answering each other’s questions.

According to the field notes, every student worked on their animation project continually the whole class period. The students were so engaged that one day it was
noted in the field notes that some students stayed five minutes after class to keep working on their animations. Though it was challenging, the student persevered.

Another interesting observation involved the students who were struggling with the project at first. Those who struggled to come up with an idea to draw and animate, or those who struggled with the programming code, expressed even more of a sense of accomplishment when their animation was finally complete. Here are a few quotes from these students: “Now I even understand transformations!” and “I was frustrated at first because I couldn’t get it to do what I wanted it to. But now that it works, I am so excited! You get a sense of accomplishment when you’re done. And you know if you ever go into computer business, you know how to make designs.” “I finished my animation and it’s beautiful—maybe not beautiful—but I just think it’s pretty cool!” There was much excitement in the room on the last day of the unit when the teacher projected each of the students’ animation projects on the screen in front of the class. One student had struggled quite a bit to get her snail to move across the screen. She was so proud of herself when she got it to work. The class cheered for her when her snail moved across the screen projected at the front of the room.

Every student interviewed had used *CPMP-Tools* at home for homework at least once. The students who were currently working on their animation projects were especially spending quite a bit of time using *CPMP-Tools* at home. All of the students interviewed had access to the internet, and were able to use *CPMP-Tools* at home.

**Interactions Displayed During the Transformations Investigations**

Table 16 shows the patterns of interaction that were found while students worked in small groups on their laptop computers during the transformations investigations. The
data in Table 16 are all from Mr. Louiselle’s classroom because his class was the only one observed for this mathematical concept. The counts represent a sample of the group discussions.

The interaction that occurred most frequently is inquiry using the software. Thus, students spent 40.3% of the time using the software to explore ideas and concepts. This is reasonable given the fact that students were using CPMP-Tools the entire time that they were working on their animation projects. The second most frequent interaction pattern was making observations. This interaction pattern also involves the use of CPMP-Tools. During these interactions students were making statements based on what they saw on the computer screen such as, “Whoa! Look right here!” or “Look, it

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<tr>
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<td>Proof of answer by manipulation of software</td>
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157
worked!” The interaction of interest in much current research on computer use, *instrumental genesis*, was also found. This code is used when the computer becomes an extension of the student’s own mind. For example, when working on her animation project, a student said, “Well, *I* was thinking more like this…” She was using the computer software to show her group her thinking.

Three interactions were found that are evidence of students collaborating with each other—*answering another student’s question, argument, and building on other students’ ideas*. All of these interactions involved an exchange between students as opposed to the computer or the teacher. As was illustrated in previous segments of transcript, students were asking and answering questions with each other, debating ideas, and building on each other’s ideas. These types of interactions are shown in the following segment of transcript:

Lauren: Here how does this mouth look?

Ashlynn: You need an opening in the middle.

Mark: Draw 9, -1,…

Ashlynn: I was going to say like you could have two little circles on top.

First, Lauren asks other students a question. Ashlynn answers the question. Mark builds on this answer. Then Ashlynn argues with Marks answer.

The interactions *IRE, teacher explain, and focus* are the result of the teacher working with the groups. When the teacher asks a low-level question like, “What does a bubble look like?” it receives an *IRE* code. When the teacher asks a higher-level question that gets the student to focus their thinking on an important aspect of the problem like, “How do you normally do a size transformation?” it receives a *focus* code.
When the teacher just explains something to the student such as, “But you can’t call it the same—remember what we talked about—you had to call that I one and I two. If you rename it the same thing, something that you did before is going to disappear,” that receives a *teacher explain* code. The sum of the teacher-related codes (18) is the same as the sum of the student-interaction codes (18). This could be an indication that the teacher is helping the students as much as the students are helping each other.

**Levels of Mathematical Thinking Displayed During the Transformations Investigations**

Table 17 shows the levels of mathematical thinking verbalized as students worked in small groups on the transformation/animations investigations. The level of mathematical thinking that occurred most frequently was *Constructing Synthesizing*. This was also true in the other *Small-group/ Multiple-computer* classroom environment (see mathematical concept one), the tessellation investigation. When students are using *CPMP-Tools*, especially in small groups, they spend a majority of the time exploring the problem from many perspectives or integrating concepts to create new insights. Most lines of the previously stated segments of transcript have been at the *Constructing Synthesizing* level of mathematical thinking.

The levels of mathematical thinking just above and below *Constructing Synthesizing* both occurred the second most frequently. The highest level of mathematical thinking *Constructing Evaluating*, occurred when a student reflected on the process they had just completed and evaluated why it did not work—“Oh, I know why it didn’t work, I had 75 instead of 90.” An example of the *Building-with Evaluative-Analyzing* level of mathematical thinking was when a student said, “I have way too many
points for mine. Like it’s going on forever.” Here the student was making a judgment about their work while they were still in the middle of the process, not at the end like the

Constructing Evaluating level of mathematical thinking. Overall, a majority of the discourse as students explored transformations took place using the three highest levels of mathematical thinking.

Types of Interaction Patterns Found in Each Level of Mathematical Thinking

Table 18 shows the number of each type of interaction found in each level of mathematical thinking as students explored transformations. The highest interaction pattern/level of mathematical thinking pair is inquiry using the software/Constructing Synthesizing (SN). This pair really characterizes the discourse that took place during these investigations. Students were continuously exploring and engaging in inquiry.

A majority of the interactions were at the higher levels of mathematical thinking. Since Constructing Synthesizing (SN) had the highest frequency in Table 17, it is reasonable that this column would have the highest frequencies in Table 18. It is interesting that the only interactions in the lowest three levels of mathematical thinking took place while the
Table 18

*The Number of each Type of Interaction Found in each Level of Mathematical Thinking during Transformations/Computer Animations Group Work*

<table>
<thead>
<tr>
<th>Small-Group/Multiple Computers</th>
<th>Levels of Mathematical Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interaction Pattern</td>
<td>C</td>
</tr>
<tr>
<td>Collect answers</td>
<td></td>
</tr>
<tr>
<td>IRE</td>
<td></td>
</tr>
<tr>
<td>Give expected information</td>
<td></td>
</tr>
<tr>
<td>Funnel</td>
<td></td>
</tr>
<tr>
<td>Teacher explain</td>
<td></td>
</tr>
<tr>
<td>Hint to solution</td>
<td></td>
</tr>
<tr>
<td>Exploring methods</td>
<td></td>
</tr>
<tr>
<td>Teacher elaborate</td>
<td></td>
</tr>
<tr>
<td>Focus</td>
<td>1</td>
</tr>
<tr>
<td>Inquiry</td>
<td>2</td>
</tr>
<tr>
<td>Inquiry using the software</td>
<td></td>
</tr>
<tr>
<td>Answering another student’s question</td>
<td>8</td>
</tr>
<tr>
<td>Giving new idea/making a conjecture</td>
<td>1</td>
</tr>
<tr>
<td>Argument</td>
<td></td>
</tr>
<tr>
<td>Proof of answer by manipulation of software</td>
<td>2</td>
</tr>
<tr>
<td>Resolution of conceptual issue using software</td>
<td></td>
</tr>
<tr>
<td>Proof of answer by student explanation</td>
<td></td>
</tr>
<tr>
<td>Building consensus</td>
<td></td>
</tr>
<tr>
<td>Checking for consensus</td>
<td></td>
</tr>
<tr>
<td>Develop conceptual understanding</td>
<td></td>
</tr>
<tr>
<td>Pupil self-nominate</td>
<td></td>
</tr>
<tr>
<td>Building on other students’ ideas</td>
<td>3</td>
</tr>
<tr>
<td>Making observations</td>
<td>1</td>
</tr>
<tr>
<td>Instrumental genesis</td>
<td></td>
</tr>
<tr>
<td>Technical software statement</td>
<td></td>
</tr>
<tr>
<td>Off-task</td>
<td></td>
</tr>
</tbody>
</table>

teacher was interacting with the group (though higher levels of mathematical thinking were also present). Generally, when the group was working without the aid of the teacher, their discourse remained at high levels of thinking.

**Assessment Results**

The assessment for the transformations/computer animations investigations that were observed was the animation project. Each student had to develop their own
animation by creating a program to be interpreted and executed by CPMP-Tools that would draw an object and put it in motion. They used matrices to store the coordinates of their drawings, and to perform the transformations. On the final day of this unit, each student’s program was run on the screen in the front of the class for all to see. Each student’s animation was recorded on video as it was displayed on the screen. Every student successfully completed this project.

CPMP-Tools includes a window in which computer programs can be written. There are some basic commands that students learn that work in CPMP-Tools. Below is an example of a student’s program code used in CPMP-Tools to create their animation of a truck. The let statements define the points that are drawn and connected with segments to create the objects. The let translatetruck statement further down defines the translation that will be used and repeated to make the truck move across the screen. Below that is a let rotmatrix statement that defines a rotation matrix. Then, this rotation matrix is multiplied by the wheel matrix one and two in the complicated last statement indicating that the wheels are going to rotate.

clearscreen

pause 1000

let truck = [[30,2],[30,3],[31,3],[31,4],[35,4],[36,3],[36,2],[30,2]]

style truck filled on fillcolor 0 100 200

draw truck

let wheel1 = [circle [[32,1],[33,1]]]

style wheel1 filled on fillcolor 0 0 0

draw wheel1
let wheel2 = [circle [[35,1][36,1]]]

style wheel2 filled on fillcolor 0 0 0
draw wheel2

let land = [[30,0][3.5,0][0,3][0,0][3.5,0][0,0][0,-8.5][-13,-8.5]
[-13,0][-27,0][-27,11]]
draw land

let flag = [[-27,6][-25,5][-27,4]]

style flag filled on fillcolor 255 0 0
draw flag

pause 50

let translatetruck = [[-.25,0][-25,0][-.25,0][-.25,0][-.25,0][-.25,0]
[-.25,0][-25,0]]

let translatewheel = [[-.25,0][-25,0]]

repeat 110 [draw [let truck = [translatetruck+truck] let wheel1 =
[translatewheel+wheel1] let wheel2 = [translatewheel+wheel2] pause 15]]

let rotmatrix = [[.9961,-.0871][.0871,.9961]]

let translatetruck1 = [[-1,1][-1,1][-1,1][-1,1][-1,1][-1,1][-1,1][-1,1]]

let translatewheel1 = [[-1,1][-1,1]]

repeat 8 [draw [let truck = [rotmatrix*truck] let wheel1 =
[rotmatrix*wheel1] let wheel2 = [rotmatrix*wheel2] let truck =
[translatetruck1+truck] let wheel1 = [translatewheel1+wheel1] let
wheel2 = [translatewheel1+wheel2] pause 5]]
Some students had programs that were simpler than this, but all students were able to draw a figure and make it move. For instance, one student-created animation had two trees drawn and a bird that flew across the screen. Another was a snail that moved slowly across the screen. Yet another was a fish with bubbles coming up around him. Many students impressed both the teacher and the researcher with the creative and complex animations they were able to create. Some were extremely elaborate given that these were 10th-grade students with little to no computer programming experience.

**Conclusions**

There were four main interaction patterns found during the observations of this unit: *focus, inquiry, inquiry using the software, and making observations*. The *focus* interaction occurred when Mr. Louiselle was helping students with their program and asking focusing questions to allow them to find their mistakes, or figure out what to do next. *Inquiry* and *inquiry using the software* made up almost half of the interactions during these investigations. This is reasonable given that students spent most of the time trying to figure out how to create their animation and trying different transformations on their objects. Students were usually making observations when they ran their program and examined the results of the commands that they had written. Related to these conclusions is the fact that this set of observations contained the highest percentage (82.1%) of interactions at the *Constructing Synthesizing* level of mathematical thinking. Students were constantly exploring using the software to create their own animation. Therefore, virtually all of the interaction patterns that contained levels of mathematical thinking were categorized at the *Constructing Synthesizing* level.
The observations that took place during the transformation investigations show a difference between students’ interactions and behavior before they used CPMP-Tools and while they were using CPMP-Tools. The teacher was surprised by the students who stood out as leaders during these investigations. Also, students who were previously disengaged became very engaged. All students successfully completed the animation project and had their program run and projected on the screen in front of the room at the end of the unit. All animations used a drawing generated by matrices and made use of transformations generated by matrices.

**Combined Results of Classroom Observations**

In this section, the results from all the classroom observations are combined into one table to provide information relevant to each research question. The data are disaggregated according to the type of classroom environment. However, the data from the four teachers and multiple lessons are pooled because it is the classroom environment that is of interest in this study. The teacher is a variable that will change with each classroom implementation of Core-Plus Mathematics. The lessons chosen for this study were examples of the use of CPMP-Tools, and represent a subset of the many other lessons in the curriculum that make use of the software. Additionally, there were observations of the same classes when they were not using CPMP-Tools. The results of these observations are also given in the following section.

**Interaction Patterns**

Four types of classroom environments were observed—whole-class/single computer, small-group/multiple computers, whole-class/no computers, and small-
group/no computer. The whole-class/single computer environment was a teacher-led interactive discussion in which the teacher used CPMP-Tools on his computer and displayed the screen at the front of the class for all students to see. All student and teacher discussion was transcribed and analyzed for this environment. The small-group/multiple computers environment consisted of students working in groups and using CPMP-Tools on their own laptop computers. Due to the large number of groups involved, a sample of the groups’ discussions was transcribed and analyzed for this environment. The whole-class/no computers environment was a teacher-led class discussion during which CPMP-Tools was not utilized. All student and teacher discussions were transcribed and analyzed for this environment. Finally, the small-group/no computer classroom environment consisted of students working in groups without the use of CPMP-Tools. A sample of the groups’ discussions was transcribed and analyzed for this environment. Data for this classroom environment comprised the smallest percent of total collected. Due to a prevailing lack of group structure, when CPMP-Tools was not needed, students were less likely to truly engage in collaborative learning.

The data in Table 19 are used to answer the first research question: What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools? Sub-questions: a) What types of interactions are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? and b) What types of interactions are present in classrooms in which CPMP-Tools is used in whole-class interactive lessons with a single computer? After the data were analyzed, the number of each type of interaction pattern found in each
type of classroom environment was counted and entered into the table below. Therefore the table reveals the types of interactions that are present in each type of classroom environment.

When CPMP-Tools was utilized in a whole-class discussion, 16.2% of the interactions were of the teacher explain type. While this was still the most common form of interaction in this whole-class/single computer environment, less of the discussion consisted of the teacher explaining the material than in the whole-class/no computers classroom environment (30.4%). The second most frequently occurring interaction in the whole-class/single computer environment was giving new idea/making a conjecture (14.3%). The teachers in this study all asked for student input during their whole-class discussions so it is reasonable that this was the second most frequently occurring interaction pattern. Although, the percentage was higher in the whole-class/ single computer environment (14.3%) than in the whole-class/no computer environment (8.9%). The focus interaction type occurred the third most frequently in the whole-class/single computer environment making up 10.9% of the interactions compared to 6.5% in the whole-class/no computer environment. Focusing is a desirable interaction as it consists of the teacher asking higher-level questions that enable the students to think critically about the topic at hand (Wood, 1994). Instrumental genesis occurred ten times (1.3%) in the whole-class discussions when the students would tell the teacher (or student computer-controller) what to do with the software to model their thinking.

Again, the most common form of interaction found in the whole-class/no computer classroom environment was the teacher explain pattern with 263 lines of text out of 865 (30.4%) containing explanations by the teacher. This was the highest
Table 19

The Number and Percentage of each Type of Interaction Found in each Type of Classroom Environment

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Whole-Class/ Single Computer</th>
<th>Whole-Class/ No Computers</th>
<th>Whole-Class/ Multiple Computers</th>
<th>Small-Group/ No Computer</th>
<th>Small-Group/ Multiple Computers</th>
<th>Small-Group/ Single Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collect Answers</td>
<td>0</td>
<td>14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>IRE</td>
<td>23</td>
<td>152</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Give Expected Information</td>
<td>31</td>
<td>106</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Funnel</td>
<td>4</td>
<td>25</td>
<td>0</td>
<td>21</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Teacher Explain</td>
<td>129</td>
<td>263</td>
<td>14</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Hint to Solution</td>
<td>12</td>
<td>7</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Exploring Methods</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Teacher Elaborate</td>
<td>33</td>
<td>33</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Focus</td>
<td>87</td>
<td>106</td>
<td>29</td>
<td>22</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>Inquiry</td>
<td>55</td>
<td>65</td>
<td>70</td>
<td>38</td>
<td>38</td>
<td>38</td>
</tr>
<tr>
<td>Inquiry Using the Software</td>
<td>71</td>
<td>83</td>
<td>17.8%</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Answering another student question</td>
<td>2</td>
<td>3</td>
<td>8.1%</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Giving new idea/making a conjecture</td>
<td>114</td>
<td>77</td>
<td>9.9%</td>
<td>29</td>
<td>29</td>
<td>29</td>
</tr>
<tr>
<td>Argument</td>
<td>10</td>
<td>13</td>
<td>22</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Proof of answer by manipulation of software</td>
<td>0</td>
<td>0</td>
<td>0.6%</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Resolution of conceptual issue using software</td>
<td>0</td>
<td>0</td>
<td>1.3%</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Proof of answer by student explanation</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Building consensus</td>
<td>27</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Checking for consensus</td>
<td>6</td>
<td>4</td>
<td>0.5%</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Develop conceptual understanding</td>
<td>62</td>
<td>0</td>
<td>7.8%</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Pupil self-nominate</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Building on other students' ideas</td>
<td>19</td>
<td>17</td>
<td>2.4%</td>
<td>10.5%</td>
<td>10.5%</td>
<td>10.5%</td>
</tr>
<tr>
<td>Making observations</td>
<td>50</td>
<td>0</td>
<td>6.3%</td>
<td>7.9%</td>
<td>7.9%</td>
<td>7.9%</td>
</tr>
<tr>
<td>Instrumental Genesis</td>
<td>10</td>
<td>0</td>
<td>1.3%</td>
<td>9.6%</td>
<td>9.6%</td>
<td>9.6%</td>
</tr>
<tr>
<td>Technical software statement</td>
<td>49</td>
<td>0</td>
<td>6.2%</td>
<td>2.4%</td>
<td>2.4%</td>
<td>2.4%</td>
</tr>
<tr>
<td>Off-task</td>
<td>2</td>
<td>2</td>
<td>0.3%</td>
<td>1.1%</td>
<td>1.1%</td>
<td>1.1%</td>
</tr>
<tr>
<td>Total</td>
<td>796</td>
<td>865</td>
<td>467</td>
<td>173</td>
<td>173</td>
<td>173</td>
</tr>
</tbody>
</table>

frequency found in the table. The second most common interaction pattern in the whole-class-no computer classroom environment was IRE (initiation-response-evaluation)
(Cazden, 1988; Hoetker & Ahlbrandt, 1969) with 17.6%, compared to 2.9% in the whole-class/single computer environment. The third most common interaction pattern was the give expected information pattern with 12.3%, compared to 3.9% in the whole-class/single computer environment.

An example of the common IRE interaction pattern taken from a whole-class discussion during a statistics investigation is as follows:

Mr. Foster: What is x bar again?

Blake: Mean.

Ashley: Two.

Mr. Foster: Two, it’s the mean. So, if I take 1, my first value, minus the mean, what am I going to get? 1 minus 2 is negative 1. [This is also an example of Mr. Foster asking and answering his own question]. If I take my next x value minus the mean, I’m going to get?

Scott: Two.

Mr. Foster: Two minus two is?

Many students: Zero.

In this exchange, Mr. Foster is asking low level questions that only require a one-word answer from the students. Consequently, the level of mathematical thinking for all of the student lines of text except for Ashley’s, is at the lowest, Recognizing Comprehending, level. The line for Ashley received the code for the next highest level of mathematical thinking, Recognizing Applying, because she had to find the mean to give the answer. This is the kind of discussion that characterized much of the whole-class discussion without the use of CPMP-Tools.
With the use of CPMP-Tools, the discussion became deeper. The teacher explained that interaction code was most frequently found (16.2%) since the discussion was led by the teacher. However, the giving new idea/making a conjecture interaction type occurred almost as often with 14.3% of the total interactions in that environment. Below is an example that is also from the statistics unit:

Mr. Foster: For our regression line. Does that look like a good regression line to you guys or not?

A few students: Yeah.

Mr. Foster: You sure? Ok, then what we can do is this blue line is the actual regression line. Ok, so if I click this—oh, we didn’t do so good, we have not so good of a slope.

Bryan: Why didn’t you just cheat instead of doing your own?

Mr. Foster: Because it’s good to be able to eye-ball it. Watch this you guys. I can turn it back off by clicking it again. If I draw this, what did it just draw in there?

A couple students: Residuals.

Mr. Foster: My residuals. It just drew in all my residuals. Now watch what this does.

Ron: Whoa!!

Mr. Foster: What did that do?

Ron: I don’t know.

Seth: It made a box.

Mr. Foster: It made a box, why did it make a box?

Ron: Cuz.

Seth: Mine looks different.

Mr. Foster: What is the area of those boxes?
Several students: the residuals squared! [emphatically]

In this exchange, there are some lines of statements made by Mr. Foster that were labeled teacher explain. However, all of the questions Mr. Foster asked were coded as focusing. These questions are asking the students to think critically about the situation and make sense of it. The students were asked to judge whether or not it was a good regression line and to analyze the meaning of aspects of the visual display on the screen. Most of these student lines of text were labeled with giving new idea/making a conjecture and with the Building-with Analyzing and Building-with Synthetic-Analyzing codes for mathematical thinking because the students were applying known procedures in a new context and making small discoveries. This productive type of discussion happened more often when CPMP-Tools was used in the whole-class setting than when it was not used.

In the small-group classroom environments, some interactions occurred that are usually found in whole-class discussions. Interactions such as teacher explain, funnel, and focus require participation by the teacher. These interactions were found in the small-group discussions because the teacher was working with the groups, as they should from time to time. The most common interaction pattern found in the small-group/multiple computers environment was inquiry using the software (17.8%). This means that 17.8% of the time the students were using the computer software to answer their questions and explore the mathematics contained in the investigation. Of course this interaction can only be found in a computer environment. So, it is interesting to compare interactions that can occur in either small-group classroom environment. For instance, the interaction type answering another student’s question made up 8.1% of the interactions in the computer environment, while the corresponding amount in the non-
computer environment is 6.4%. The inquiry interaction was the most common in the small-group/no computer environment at 22%. The corresponding percentage in the computer environment was 15%. However, much of the inquiry in the computer small-group environment was labeled with the inquiry using the software or instrumental genesis interaction codes. If the number of inquiry interactions is combined with the inquiry using the software and instrumental genesis interactions in the small-group/multiple computers environment, then the percentage is 42.4%. Another productive interaction that can be found in either small-group environment is building on other students’ ideas (Sherin, Louis, & Mendez, 2000). In this interaction, a student makes a statement that builds on what another student has said. When CPMP-Tools is in use, this interaction occurred 10.5% of the time, compared to 8.1% when CPMP-Tools is not in use. Giving Ideas also occurred 9.9% of the time in the small-group/multiple computers environment. Thus, student-to-student interactions were occurring even when students were working individually on their own laptop computer. In summary, students were asking (inquiry) and answering each other’s questions, and building on each other’s ideas more often while using CPMP-Tools than without using CPMP-Tools.

One of the teachers had regularly structured groups established. These students worked in pre-determined groups on a regular basis. However, the data show that even then, there was less group discussion and student interaction in this situation than when CPMP-Tools was being utilized. There were more off-task interactions in the small-group/no computer environment (9.8%) than the small-group/multiple computer environment (1.1%). The usual way in which group work occurred during the observations of the other two teacher’s classrooms whose classroom data were included
in the small-group categories was when the teacher said something like, “work with someone around you on number…” Therefore, there were many legitimate groups formed, but still some pairs of students, or single students worked alone. But when students were using CPMP-Tools, there was more student-to-student interaction. Students were getting up to look at someone else’s computer, and they were turning their screens around to show each other what they were doing.

Levels of Mathematical Thinking

In the four types of classroom environments, the levels of verbalized mathematical thinking were also determined and counted. This was done in order to collect information relevant to the second research question: What is the nature of students’ mathematical thinking while using the curriculum-embedded software? Sub-questions: a) What levels of mathematical thinking are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? and b) What levels of mathematical thinking are present in a classroom in which CPMP-Tools is used in whole-class interactive lessons with a single computer? Table 20 contains the number of instances of each level of mathematical thinking found in each type of classroom environment. There are fewer total codes for mathematical thinking than interaction types because lines of teacher text did not receive a code for mathematical thinking. Additionally, some other interactions did not contain mathematical thinking, especially those that were off-task, or statements such as “where do you want to start?” While this question represents a student interaction promoting effective group work, it does not contain mathematical thinking.
Table 20

The Number and Percentage of each Level of Mathematical Thinking Codes Found in each Type of Classroom Environment

<table>
<thead>
<tr>
<th>Type of Classroom Environment</th>
<th>Mathematical Thinking</th>
<th>Whole-Class/Single Computer</th>
<th>Small-Group/Multiple Computers</th>
<th>Whole-Class/No Computers</th>
<th>Small-Group/No Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizing Comprehending (C)</td>
<td>21</td>
<td>7.2%</td>
<td>9</td>
<td>2.9%</td>
<td>87</td>
</tr>
<tr>
<td>Recognizing Applying (A)</td>
<td>13</td>
<td>4.5%</td>
<td>3</td>
<td>1.0%</td>
<td>57</td>
</tr>
<tr>
<td>Building-with Analyzing (AN)</td>
<td>37</td>
<td>12.7%</td>
<td>22</td>
<td>7.1%</td>
<td>40</td>
</tr>
<tr>
<td>Building-with (SA) Synthetic-Analyzing</td>
<td>5</td>
<td>1.7%</td>
<td>10</td>
<td>3.2%</td>
<td>27</td>
</tr>
<tr>
<td>Building-with (EA) Evaluative-Analyzing</td>
<td>42</td>
<td>14.4%</td>
<td>56</td>
<td>18.0%</td>
<td>26</td>
</tr>
<tr>
<td>Constructing Synthesizing (SN)</td>
<td>163</td>
<td>55.8%</td>
<td>197</td>
<td>63.3%</td>
<td>5</td>
</tr>
<tr>
<td>Constructing Evaluating (E)</td>
<td>11</td>
<td>3.8%</td>
<td>14</td>
<td>4.5%</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>292</td>
<td>100%</td>
<td>311</td>
<td>100%</td>
<td>242</td>
</tr>
</tbody>
</table>

The total number of mathematical thinking codes in each type of classroom environment is noteworthy. Almost every line of transcribed text was coded with an interaction pattern. Therefore the totals in the last row of Table 19, the table containing student interactions, represent the total lines of text that were analyzed for each classroom environment. The same text was analyzed twice—once for interaction patterns and once for levels of mathematical thinking. Examining the percents of the total number of interactions that contain a level of mathematical thinking results in the following:

Whole-Class/Single Computer, 37%; Small-Group/Multiple Computers, 67%; Whole-Class/No Computers, 28%; and Small-Group/No Computer, 50%. Whole class discussion recorded in this study contains the smallest ratios of mathematical thinking to interactions (lines of coded text) because the teachers in this study did a majority of the talking during whole-class discussion. Nonetheless, the small-group/multiple computer
environment results are notable. This environment contains the highest percentage of mathematical thinking (67%), and is considerably higher than that of the small-group/no computer environment (50%). In other words, for every 100 codes (lines) of student interactions, 67 represented mathematical thinking in the small-group/multiple computers environment compared to 50 lines in the small-group/no computer environment. Additionally, the small-group/multiple computer environment contained the greatest percentages of the two highest levels of mathematical thinking, Constructing Synthesizing and Constructing Evaluating.

The levels of mathematical thinking found in the whole-class/single computer classroom environment were widely distributed. However, the majority of the levels of mathematical thinking were found at the Constructing Synthesizing (55.8%) level of mathematical thinking. There were fewer instances of Building-with Synthetic-Analyzing found in this first column because this level of thinking involves the comparison of multiple solution methods. In the investigations that were observed with CPMP-Tools used on a single computer at the front of the class, students were not presenting multiple ways of doing the problems. The teacher basically guided the students through a single solution method. Yet, the levels of mathematical thinking above and below this level (Building-with Evaluative-Analyzing and Building-with Analyzing) occurred in relatively equal amounts (14.4% and 12.7% respectively).

The data show that the highest level of mathematical thinking, Constructing Evaluating, only occurred in the two computer environments. The first four rows of these columns of Table 20 contain relatively low percentages of codes for those lower levels of mathematical thinking. However, the 5th and 6th levels of mathematical thinking
(Building-with Evaluative-Analyzing and Constructing Synthesizing) contain the bulk of the codes found in the computer environments. The Constructing Synthesizing category contains the highest percentages of mathematical thinking codes (55.8% and 63.3%) in both of the computer environments. Thus, students were spending a majority of their time exploring the problem from many perspectives, formulating mathematical arguments to explain discoveries, or developing new insights.

Conversely, the lowest levels of mathematical thinking were frequently found in the whole-class/no computer environment. The majority of the codes found in that environment are in the lowest three levels of mathematical thinking (the first three rows of Table 20). Both of the whole-class environments had higher percentages at the Recognizing Comprehending level (7.2% and 36.0%) than their corresponding small-group environments (2.9% and 8.0%). Yet, the whole-class/single computer environment had larger percentages at the higher levels of thinking than the whole-class/no computer environment. The highest percentage of codes was found at the Constructing Synthesizing level of mathematical thinking in the single computer environment just like in the small-group/multiple computer environment. However, in the whole-class/no computer environment, the lowest percentages of mathematical thinking were found at the highest two levels of mathematical thinking Constructing Synthesizing (2.1%) and Constructing Evaluating (0.0%).

In the small-group/no computer classroom environment, Building-with Analyzing (43.7%) was the most common level of mathematical thinking found. Thus, students in this setting were often applying known procedures in a new context, familiarizing themselves with a problem using specific numerical examples, or searching for patterns
in the numerical results. There were no instances of the highest level of mathematical thinking (*Constructing Evaluating*) found in the small group/no computer classroom environment. The non-computer classroom environments were similar in that there were fewer instances of the higher levels mathematical thinking found, and more instances of the lower levels of mathematical thinking than the computer environments.

**Relationship Between Types of Interactions and Levels of Mathematical Thinking**

The third research question is: *What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking?* Tables 21 and 22 provide information relevant to this question. For each line of text that was coded with a level of mathematical thinking, the interaction code that was assigned to that line of text has been tallied in these two tables. Thus the relationship between the levels of mathematical thinking and the interaction patterns can be examined. The abbreviations shown for the levels of mathematical thinking correspond to those shown in Table 20. Table 21 contains the data for the two computer environments and Table 22 contains the data for the non-computer environments. Note that these tables will not contain all of the interactions that were included in Table 19 because only lines of text coded with a level of mathematical thinking were examined.

Again, the small-group/multiple computers environment stands out with higher frequencies concentrated in the higher levels of thinking/productive interactions positions in the table. The higher levels of mathematical thinking are furthest to the right in each classroom environment section of the table. The highest frequencies are found in the *Constructing Synthesizing* column in the small-group/multiple computer environment. The corresponding rows for these high numbers are student interactions such as *inquiry,*
Table 21

The Number of each Type of Interaction Found in each Level of Mathematical Thinking for each of the Two Computer Classroom Environments

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Whole-Class/Single Computer</th>
<th>Small-Group/Multiple Computers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>Collect Answers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IRE</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Give Expected</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Information</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Funnel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher Explain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hint to Solution</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Exploring Methods</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher Elaborate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Focus</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Inquiry</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Inquiry Using the</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Software</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Answering another</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>student’s question</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Giving new idea/making a</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>conjecture</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Argument</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Proof of answer by</td>
<td></td>
<td></td>
</tr>
<tr>
<td>manipulation of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>software</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Resolution of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>conceptual issue</td>
<td></td>
<td></td>
</tr>
<tr>
<td>using software</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proof of answer by</td>
<td></td>
<td></td>
</tr>
<tr>
<td>student explanation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building consensus</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Checking for</td>
<td></td>
<td></td>
</tr>
<tr>
<td>consensus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Develop conceptual</td>
<td></td>
<td></td>
</tr>
<tr>
<td>understanding</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pupil self-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>nominate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building on other</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>students’ ideas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Making observations</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Instrumental</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>genesis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Technical software</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>statement</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Off-task</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
inquiry using the software, answering another student’s question, giving new idea/making a conjecture, argument, building on other students’ ideas, making observations and instrumental genesis. This shows that when students are working in groups with the use of CPMP-Tools, they are engaged in productive interactions and high levels of thinking.

In the whole-class/single computer environment, the inquiry, giving new idea/making a conjecture, and making observations rows contain the most frequencies; as well as the Constructing Synthesizing (SN) column. Thus during whole-class discussion with the use of CPMP-Tools, students were exploring the mathematical topic by volunteering their ideas and making observations based on the action taking place on the computer screen at the front of the classroom. The interaction code inquiry using the software frequently was paired with the Constructing Synthesizing level of thinking. By asking questions and using the software to answer those questions, the students were engaging in inquiry with the use of CPMP-Tools/progressively exploring the problem to develop new insights. These are instances of inquiry using the software. However, some of these codes were given to a teacher line of text. Therefore, they are not examined for a level of mathematical thinking. This facet gives some insight into the patterns in Table 21 shown in the inquiry using the software row. Most inquiry using the software interactions are at the Constructing Synthesizing level of mathematical thinking. However, this combination likely occurred more in the small-group/multiple computers environment (46 occurrences compared to 9) because the students were asking almost all of the questions. Still, inquiry was occurring in this whole-class discussion with the assistance of the visual aspect of CPMP-Tools engaging the students’ thinking.
Table 22

*The Number of each Type of Interaction Found in each Level of Mathematical Thinking for each of the Two Non-computer Classroom Environments*

<table>
<thead>
<tr>
<th>Interaction Pattern</th>
<th>Whole-Class/No Computers</th>
<th>Small-Group/No Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>Collect Answers</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>IRE</td>
<td>30</td>
<td>24</td>
</tr>
<tr>
<td>Give Expected</td>
<td>22</td>
<td>13</td>
</tr>
<tr>
<td>Information</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Funnel</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Teacher Explain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hint to Solution</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Exploring Methods</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher Elaborate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Focus</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Inquiry</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Inquiry Using the</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Software</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Answering another</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>student's question</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Giving new idea/making a</td>
<td>19</td>
<td>12</td>
</tr>
<tr>
<td>conjecture</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Argument</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Proof of answer by</td>
<td></td>
<td></td>
</tr>
<tr>
<td>manipulation of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>software</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Resolution of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>conceptual issue</td>
<td></td>
<td></td>
</tr>
<tr>
<td>using software</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proof of answer by</td>
<td></td>
<td></td>
</tr>
<tr>
<td>student explanation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building consensus</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Checking for</td>
<td></td>
<td></td>
</tr>
<tr>
<td>consensus</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Develop conceptual</td>
<td></td>
<td></td>
</tr>
<tr>
<td>understanding</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pupil self-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>nominate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building on other</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>students’ ideas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Making observations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Instrumental</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Genesis</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Technical software</td>
<td></td>
<td></td>
</tr>
<tr>
<td>statement</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Off-task</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Both non-computer classroom environments have higher frequencies toward the left side of their section of Table 22 at the lower levels of mathematical thinking. The whole-class/no computer environment has a large concentration of codes in the square that contains the two rows IRE and give expected information, and the Recognizing Comprehending (C) and Recognizing Applying (A) columns for mathematical thinking. This would indicate that there were many instances of low-level questions being asked by the teacher, and short answers given by the students. Not surprisingly, most of the IRE interactions resulted in the lowest levels of mathematical thinking. The whole-class/no computer environment also contains frequencies of the focus, inquiry, giving new idea/making a conjecture, argument, and building on other students’ ideas interactions at many levels of mathematical thinking. These interactions occurred frequently in this environment but also varied in the level of mathematical thinking. In other words, the questions asked and ideas given could be at any level of mathematical thinking.

In the small-group/no computer environment, there are clusters of codes in the following rows: inquiry, answering another student’s question, giving new idea/making a conjecture, and building on other students’ ideas. So while productive group interactions were occurring, they mainly involved the lower to mid-levels of mathematical thinking. This observation is in contrast to the inquiry, answering another student’s question, giving new idea/making a conjecture and building on other students’ ideas rows in the small-group/multiple computers environment, where the levels of mathematical thinking are mostly in the highest three levels of mathematical thinking for these interactions. The highest frequencies in the small-group/no computer environment are in the Building-With Analyzing (AN) column.
According to Wood, Williams, and McNeal (2006), the interaction patterns that are consistent with traditional mathematics instruction are *IRE, give expected information, funnel, teacher explain, and hint to solution*. Collect answers, off-task, and *technical software statement* interactions are neutral with respect to traditional versus constructivist classroom environments. The rest of the interactions were typical of the classrooms that Wood, Williams, and McNeal (2006) had labeled as reform (standards-based) classrooms. In their study, they established that the interactions found in the traditional classrooms contained lower levels of mathematical thinking. In this study, in all of the classroom environments, a majority of the frequencies in the rows that represent traditional interactions fall in the lowest three levels of mathematical thinking. In each classroom environment section of Table 21 and Table 22, these results are in a 5 x 3 rectangle spanning the rows from *IRE* to *hint to solution* and the columns from *C* to *AN* levels of mathematical thinking. Most of the frequencies in these five rows are contained those rectangles, suggesting that these results align with the results of Wood, Williams, and McNeal (2006).

During whole-class discussions without the use of *CPMP-Tools*, 64% of the interactions found were consistent with traditional mathematics instruction, and 32% of the interactions were consistent with standards-based mathematics instruction and constructivist theory. With the use of *CPMP-Tools*, 25% of the interactions found were consistent with traditional mathematics instruction, and 69% of the interactions were consistent with standards-based mathematics instruction and constructivist theory. Additionally, a greater incidence of higher levels of mathematical thinking were found in the whole-class/single computer environment than in the whole-class/no computer
environment. Perhaps a more notable observation is that fewer incidents of lower levels of mathematical thinking were observed when CPMP-Tools was used in the class discussion.

In the small-group/multiple computer environment, 92% of the interactions were consistent with standards-based mathematics instruction and constructivist theory, and only 4% of the interactions found were consistent with traditional mathematics instruction. Whereas, in the small-group/no computer environment, 71% of the interactions were consistent with standards-based mathematics instruction and constructivist theory, and 19% of the interactions found were consistent with traditional mathematics instruction. The number of inquiry using the software interactions combined with the inquiry interactions in the small-group/multiple computers environment, made up 32.8% of the interactions in this environment. In the small-group/no computer environment, the inquiry interaction percentage was 22%. When the students were using CPMP-Tools, they spent more time inquiring, asking and answering each other’s questions, and building on each other’s ideas. Higher levels of mathematical thinking were also found when the students were using CPMP-Tools in their groups than when they were not. Students were constantly in the exploring mode resulting in mostly inquiry interaction codes and Constructing Synthesizing level of mathematical thinking.

Many of the typically standards-based interaction patterns have been shown to be characteristics of effective group work and to enhance student learning (Cobb & Yackel, 1996; Cohen, 1994a, 1994b, 1996; Hicks, 1995-1996; Manouchehri & St. John, 2006; Pirie, 1998, Sherin, Louis, & Mendez, 2000; Wilson & Lloyd, 2000; Wood, 1994, 1998; Wood, Williams, and McNeal, 2006; Yackel & Cobb, 1996). When the students were
using *CPMP-Tools*, these interaction patterns and characteristics of effective group work were found to be present. Therefore, the computer classroom environments contained the conditions that were conducive to productive collaborative learning and effective teaching and learning.

**Summary**

This chapter described the data that came from all of the observations of computer use in the four teachers’ mathematics classrooms. The tessellations and transformations investigations were conducted with a small-group/multiple computers classroom environment. The regression line and SSA investigations were conducted with a whole-class/single computer classroom environment. The mathematics involved in each type of investigation, and the patterns of interaction and levels of mathematical thinking for each was examined. All of the data were then combined to show the overall trends in each type of classroom environment. Additionally, the data from the observations of the no computer classroom environments were included in the overall tables as an indication of the interaction patterns and levels of mathematical thinking that were normally present in the same classrooms without the use of the computer software *CPMP-Tools*.

In general, the computer classroom environments contained higher levels of mathematical thinking and more of the standards-based type of interaction patterns than the non-computer classroom environments. The use of *CPMP-Tools* seemed to elicit the *Constructing Synthesizing* level of mathematical thinking because students were constantly exploring the situation. Many of the interaction patterns associated with effective group work were present in both of the small-group classroom environments.
However, the small-group/multiple computer classroom environment contained these interactions at higher levels of mathematical thinking than the corresponding non-computer environment. The use of *CPMP-Tools* seemed to moderate the effects of not having group roles or even assigned groups. When students were in small-groups using *CPMP-Tools* they interacted with each other more and made more statements that contained mathematical thinking.

During whole-class interactive discussions, the teacher spent much of the time explaining in both the computer and non-computer environments, though there was less of this in the computer environment. Instead, when *CPMP-Tools* was being used in the discussion, students contributed more of their ideas and made observations based on what they saw on the computer screen. The teacher also asked more focusing questions and questions that would help develop a conceptual understanding of the topic. All of these types of interactions were occurring at higher levels of thinking when *CPMP-Tools* was used than when it was not.
CHAPTER V
SUMMARY, DISCUSSION, AND RECOMMENDATIONS

In this chapter, the results of the analysis of the whole-class and small-group classroom discussions, of the field notes, of the student and teacher interviews, and of the students’ assessment items are interpreted in order to fully answer each of the research questions. Data representing the analysis of the classroom discourse that was in Chapter IV are interpreted and expanded upon and the themes and patterns found in the data are used to describe the nature of students’ interaction and mathematical thinking in classroom environments that utilize CPMP-Tools and to give a rich description of the collective cases.

The chapter is organized in five main sections. The first section describes the setting, including characteristics of the high school. In the four sections that follow, all of the data are used to interpret and to describe the effect of CPMP-Tools on small-group collaboration, whole-class discussion, and students’ thinking related to each of the four mathematical topics studied. The description of each of the cases (the closed, bounded settings of the teachers’ classrooms) of CPMP-Tools being utilized—the socio-mathematical culture and the teachers’ teaching methods—will be integrated throughout the four sections. Finally, limitations, implications for further study, and overall conclusions are stated.

The Setting

The cases in this study consisted of four teachers’ classrooms at one high school that was using CPMP-Tools in their implementation of the second edition of Core-Plus Mathematics. Prior to implementing the second edition, this school had been using the
first edition of the *Core-Plus Mathematics* curriculum materials for seven years. As a result of these experiences, the teachers were familiar with the philosophy of the program and its associated teaching practices. This was a private high school with typically high-achieving students. There were approximately 1000 students in the school with a student-teacher ratio of 18 to 1. The students were 98% Caucasian. The particular classes observed were *CPMP Courses 1* and *2*. *Course 1* students are typically in 9th grade. *Course 2* students are typically in 10th grade.

The teachers whose students were included in this study were Mr. Louiselle, Mr. Kirkwood, Mr. Nelson, and Mr. Foster. While the teachers collaborated regularly to plan instruction, there were definite differences in how each enacted the curriculum. The socio-mathematical norms that each teacher created in his classroom had a large impact on the interaction patterns and resulting levels of mathematical thinking. Therefore, the teachers who taught each of the lessons will be described in the following sections corresponding to each of the content topics.

This high school embraced technology. All of the classrooms observed for this study had a smartboard which, among other things, served as the screen in the front of the classroom used to project the image of the teacher’s computer. In at least one classroom, the image was also projected on a screen in the back of the room. During the observations, the teachers used the smartboard to display class notes from a previous class, to draw on, and to project images from the internet. Additionally, students used their laptops outside of the mathematics classrooms to do science experiments, to research on the internet, and to write papers for other classes. Although laptops were also
used for social purposes such as listening to music, showing each other their pictures, and playing games, they seemed to be used mostly for academic purposes.

Mathematical Concept One: Tessellations

In *Course I*, Unit 6, Lesson 2, Investigation 3 (*Patterns with Polygons*), students explore “Which polygons or combinations of polygons will tile the plane?” (Hirsch et al., 2008, p. 407). Students are given two regular tessellations and asked to find the symmetry in them. The students are also asked to find the sum of the measures of the angles at a common vertex in the diagrams, and the measure of each angle at that vertex. The students used *CPMP-Tools* to explore whether or not any triangle, any quadrilateral, or any pentagon will tile the plane. They discovered that the sum of the measures of the angles around each vertex equals 360 degrees. Therefore, any regular tessellation must be made up of a shape whose angles are factors of 360. Later, the students explored semiregular tessellations by discovering which combinations of regular polygons will tile the plane.

The Teachers’ Socio-mathematical Norms

This mathematical concept was investigated by students in two classrooms in which the teachers exhibited two slightly different management styles. Thus, this set of observations allowed for some direct comparisons across common material taught by the two teachers. This is relevant since an individual student’s mathematical activities, and classroom community practices have a reflexive relationship—neither occurs independent of the other, nor does one take precedence over the other (Cobb, 2000a). Mr. Kirkwood’s and Mr. Nelson’s classes were observed during the tessellation
investigations. Mr. Kirkwood was skillful at classroom management and had good rapport with his students. He started a 1st-hour class by asking what was going on in their lives, and how the various sporting events went the night before. When he called for their attention, the students responded immediately. When students misbehaved or got off-task, he dealt with it immediately. Even on a very warm day during 8th period, his students stayed attentive throughout the whole lesson. Generally, about half of the students in his class contributed to whole-class discussions. Mr. Kirkwood asked many good questions, but often answered them himself too quickly. His classes were also in different rooms with various desk arrangements. However, he consistently had students sitting in structured groups.

Mr. Kirkwood, like the other teachers in this study, did not have a tangible set of group work rules or roles. However, it was evident from the field notes and transcriptions of the students at-work, that most groups had a student performing the role of reader, and there was a tacit expectation that all students participate and work together. It was evident that they had initially learned to work in groups with the use of group roles. He did not have to say anything to students about getting to work. While students were at work in groups, Mr. Kirkwood usually relinquished the authority to the students. He did not tell students how to solve the problems, but instead asked probing questions to encourage them to think about the problems in productive ways. Typically, when all groups were working, only one student’s hand was raised at a time. This suggests that the role of coordinator was being utilized since only the coordinator in the group should raise their hand, and only when no one in the group can answer the question.
Additionally, only one student was observed not working with his group over the course of two class periods for two days.

Mr. Nelson was a bit more lax about classroom management. As he was introducing the investigation, one student was observed playing a game on the computer. In another class, two students were observed off task during most of the whole-class discussion. However, he was effective in not telling students the mathematics they were supposed to discover. He asked questions, and frequently called on particular students to answer. Even though students sat in an orderly fashion at the beginning of class, they were allowed to move wherever they wanted for the small-group work. This arrangement left at least one student to work alone on both days in each of the two class periods that were observed, and friends working together, some of whom had difficulty resisting the temptation to socialize throughout the group work time. There were no prominent discipline issues. However, since the group organization was less structured in his classroom, it was harder for any one student’s behavior to stand out.

Mr. Nelson’s facilitation of group work was rather relaxed. By his own admission, he was not a group “guru.” However, he professed to be in favor of group work, and told students to work together. Yet, the students were not in structured groups. When they were supposed to get into groups, students got up from their usual seats and sat by their friends to work. According to the field notes and data collected using the observation form, many students were relying on the teacher (many hands up waiting for the teacher), and more single students were dominating a group than in Mr. Kirkwood’s class. The students who worked alone often monopolized much of the teacher’s time because they did not have anyone else with whom to discuss the problems. Due to this,
and an overall lack of structure, the teacher did not make it around to all of the groups when they had their hands up. Sometimes, this resulted in a group spending the rest of the hour socializing. Therefore, this study concurred with others that a main factor that seems to contribute to the effectiveness of cooperative groups in mathematics classrooms is an explicit structure (Cohen, 1994a; Davidson & Worsham, 1992; Johnson & Johnson, 1990). Group roles are needed (Cohen, 1994a, 1994b, 1996, 2002) as well as specific groups formed by the teacher.

### The First Research Question

*What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools?* The tessellations investigation was used to help answer the first sub-question to this investigation: a) *What types of interactions are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers?* While students were exploring tessellations the interaction pattern they engaged in the most was *inquiry*. Students were making conjectures about whether or not a shape would tile the plane and using the software to confirm or deny their conjectures. Students also spent a combined 34% of the time giving their ideas, building on other students’ ideas, and answering each other’s questions. They were collectively trying to form a rationale for why all triangles and/or all quadrilaterals would tile the plane. Also, *instrumental genesis* which has been suggested to be an ideal interaction pattern for student learning (Bretscher, 2010; Drijvers, Doorman, Boon, van Gisbergen, 2010; Zbiek, Heid, Blume, & Dick, 2007) was found more in this investigation than the others. Therefore, students were using the
software as an extension of their own thinking. The visual appeal of this software may have contributed to this result.

The interaction patterns that coincide with the behavior norms of effective group work also had the highest frequencies: answering another student’s question (9%), giving new idea/making a conjecture (12%), building on other students’ ideas (14%). Yet, these interactions were found more often in Mr. Kirkwood’s class (37% instances compared to 27%) since he had structured groups. Students knew whom they were supposed to be working with, and that it was their responsibility to work together. In Mr. Nelson’s classroom, there were long periods of teacher talk even when he was working with a group. Consequently, there were more teacher explain interaction patterns found (8.5% compared to 0.5%). On the other hand, Mr. Kirkwood tended to ask the group a probing question and move on, leaving the group to struggle with the question. This may be the reason that the interaction patterns argument and making observations occurred much more often in Mr. Kirkwood’s classes (14%) than they did in Mr. Nelson’s classes (6%). Interactions such as these that are typical of effective collaborative learning and inquiry-based learning may occur more frequently when structured groups are formed.

The Second Research Question

What is the nature of students’ mathematical thinking while using the curriculum-embedded software? Sub-question: What levels of mathematical thinking are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? A majority of the students’ discussion was at the Constructing Synthesizing level of mathematical thinking in both teachers’ classrooms (Nelson, 71%; Kirkwood, 51%). So a majority of the time students were exploring the problem from multiple
perspectives by trying differently shaped polygons using CPMP-Tools, formulating arguments to explain why a polygon did or did not tile the plane on the computer screen, and integrating concepts to create new insights. The level of thinking that occurred with the second highest frequency in Mr. Kirkwood’s classroom was Building-with Evaluative-Analyzing (29%). His students were analyzing and evaluating each other’s ideas or arguments as they tried to explain why a shape would tile the plane on the computer screen. The level of thinking that occurred with the second highest frequency in Mr. Nelson’s classes was Building-with Analyzing (12%). His students were often applying known properties in a new context (such as the sum of the measures of the angles of the triangles is 180 degrees), or systematizing the results and searching for patterns (such as the fact that there were two of each numbered angle around each vertex on the screen). Some students did not go beyond this lower level of thinking to explain and justify the patterns that they were observing. In Mr. Kirkwood’s classroom, 10 instances of the highest level of mathematical thinking occurred during this investigation. This occurred when the students made an evaluation about whether or not the triangles or quadrilaterals they had created with CPMP-Tools would tile the plane but did so with reflection on the situation as a whole. They had already used the visuals provided by CPMP-Tools to come up with the conjecture that the sum of the measures of the angles around each vertex had to be 360 degrees for it to work, and they were creating more images on the screen to confirm and explain that concept. Proportionally, there were more instances of lower-level mathematical thinking found in Mr. Nelson’s classes (23% vs. 13%). Again, it may be that if the groups are not well-defined, their discussion is less likely to reach the higher levels of mathematical thinking.
On the related assessment item, many of the students drew pictures that resembled Figure 8 or 9 (see Chapter IV) to show that the shapes would fit together. However, some of them relied too heavily on their pencil and paper drawing. If they could not draw the correct shapes accurately, then they assumed it would not work. Conversely, if they drew the wrong shapes then they either assumed it would not work because their drawing did not work, or they made the drawing work even with an incorrect shape. In either case, students were making some unwarranted assumptions about the ability of their own drawing to justify their answer. Therefore, as students use this tool, it should be stressed that the computer can draw with an accuracy that we do not have on our own. The idea of using the software to prove whether or not a conjecture is true is much different than using a paper and pencil free-hand sketch.

During the tessellations investigation, the teachers discovered that they needed to highlight the underlying mathematical ideas behind the software tool. The students were having fun and enjoying the visual display of the geometry tool, but the curriculum intended for students to focus their attention on whether the shapes would tile the plane when the sum of the measures of the angles around each vertex add up to 360 degrees. By the time these students were interviewed, all of them could explain why any triangle or quadrilateral would tile the plane.

Another issue that students had when answering this assessment item was not knowing the definitions of the associated words. Misunderstanding the word “tiling” and in one case, “semiregular”, caused them to get the question wrong. Some said yes because it does not tile the plane, and some said no because it does tile the plane, when the answer should be yes because it does tile the plane. Most of these students had the
right idea that the shapes fit together with no gaps or overlaps, but did not know that that meant the shapes tiled the plane. So, while students are using this type of software that appeared to be fun and engaging, it may help to emphasize the meaning of the words that are involved with this concept.

**The Third Research Question**

*What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking?* The highest frequencies of interaction patterns occurred at the Constructing Synthesizing and Building-with Evaluative-Analyzing levels. Inquiry elicited the top five levels of mathematical thinking, with the second-highest level Constructing Synthesizing (SN) occurring the most. The exploring that the students did with the tessellations raised their level of mathematical thinking to that of Constructing Synthesizing (SN). The students explored by using CPMP-Tools to create many different triangles and quadrilaterals to see whether or not any triangle or any quadrilateral would tile the plane. With the angles color-coded and numbered, and the ability to click on a vertex and change the shape of all of the shapes in the tiling at once, it became clear that even though the shapes changed, the same angles would always surround each vertex, and the measures of those angles would always sum to 360 degrees. While exploring, students also spent a lot of time giving their ideas, building on other students’ ideas, and answering each other’s questions. These interactions tended to elicit the higher levels of thinking as well. This result aligned with that of (Sherin, Louis, & Mendez, 2000). When students build on other students’ ideas, their thinking becomes deeper. The visuals provided by CPMP-Tools gave the students much to discuss and therefore much to think about. All but two
of the 42 instrumental genesis interactions took place at the second highest level of mathematical thinking (SN). All of these interactions were facilitated by the use of the software with small groups of students.

Mathematical Concept Two: Regression Lines

In Course 2, Unit 4, Lesson 2, Investigation 1 (How Good Is the Fit?) and Investigation 2 (Behavior of the Regression Line), students used the Data Analysis tools in CPMP-Tools to make scatterplots, find the regression equation, graph the regression line, display the residuals on the graph, calculate the sum of squared errors (SSE), display the centroid, and determine the effect of an influential point. Each problem is set in a real-life context. Some of these contexts are: the curb weight and highway mpg of a sample of cars, index of radioactive waste and cancer deaths in a sample of communities in Oregon, height and hip angle while running for a sample of horses, and the season batting average and World Series batting average for Chicago White Sox players. As students explored these contexts with the data analysis software, they were asked questions in the investigation that compel them to interpret things like the meaning of the slope of the regression line and the effect of an influential point. The software provides a dynamic modeling line, a visual representation of the residuals and of the sum of squared errors.

The Teachers’ Socio-mathematical Norms

This mathematical concept was taught by Mr. Louiselle and Mr. Foster, each of whom had a different teaching philosophy and disposition toward group work and technology. Mr. Louiselle’s classroom management skills were quite good as evidenced
by the minimal misbehavior noted in the field notes. He had a very good rapport with the
students, and they listened well to him. He also was very comfortable with technology.

Since he was one of many teachers who did not have their own classroom, his
classes took place in rooms with various arrangements of tables or desks. Some
arrangements facilitated collaboration better than others—generally tables are more
conducive to group work. Yet, it was clear that he had developed a culture of students
helping each other. He consistently relinquished authority to the groups of students by
allowing them to do the thinking and discover the mathematical concepts themselves. He
did little telling, often answered questions with other questions, and expressed his high
expectations to the students that they could all complete their work successfully. He was
also the only teacher that had students present in front of class during these observations.

Over the course of two pilot studies and the observations performed for this
dissertation, some of the teachers had been observed during four different school years.
Over these four years, Mr. Louiselle grew the most in his facilitation of group work. This
finding is consistent with the findings of Wilson and Lloyd (2000) who found that
standards-based curriculum materials can effect change in teachers’ instructional
strategies. Earlier, he was not much in favor of using group work. Most of his class time
was spent in whole-class discussion. By the fourth year, most of his class periods
included group work, and he was an avid supporter of this instructional method.
However, he still did not structure the groups. Most of the time, he would tell them to
work on the investigation with someone else, and most students would find someone to
work with. However, there were always students left who worked on their own.
Consequently, these students often needed his help because they had no one to talk to,
and no one to ask questions. On average, five students simultaneously spent over five minutes with their hand up. When they gave up, they went up to the teacher’s edition to read the teacher’s notes for the problem. This seemed to be permitted by the teacher because it was observed frequently. In one of his Course 2 classrooms, there were only tables with two students per table. In this class, most of the pairs of students worked together. Yet, some did not because it was not required. He asked good probing and extending questions when he was at each group, staying with the group until they came to a resolution. Due to the length of time he stayed with each group, some groups received no attention during a particular class period. One class period, he spent almost half of the time working with one group.

Mr. Foster attempted to maintain control of the behavior in the classroom, however he had the most discipline issues. This could be due to his instructional methods since he asked questions during whole-class discussion, but he often answered them immediately himself. He led students through many of the investigation questions and this approach hindered the students’ discovery of the concepts and associated conceptual understanding. A few students in each class asked questions or made comments, but there was no evidence that the rest of the class understood during class. Mr. Foster was the only teacher out of those observed that had his own classroom. In all observations, the desks remained separated in rows.

Group work was not a regular part of Mr. Foster’s instruction so group behavior norms had not been developed. During one observation only half of the students were engaged in the task, but even the engaged half did not have much discussion. During another observation on another day, Mr. Foster told the students to “work with someone
next to them.” Three groups of three students and one group of two students were
formed. Fourteen students worked individually as they were most accustomed to doing.
Since Mr. Foster did not engage students in meaningful group work his “group” work
data were not included in the overall data analysis.

The First Research Question

What is the nature of the interactions present among students, and between the
teacher and students, in classroom utilization of CPMP-Tools? The sub-question
addressed by the regression lines investigation is: What types of interactions are present
in classrooms in which CPMP-Tools is used in whole-class interactive lessons with a
single computer? The teacher explain interaction occurred the most in both classrooms.
However, it occurred over twice as often in Mr. Foster’s classes as it did in Mr.
Louiselle’s classes (45% vs. 22%). Mr. Foster’s classroom discussion could be
characterized by the teacher doing most of the explaining and the students giving some
ideas.

Mr. Louiselle’s classroom discussion, on the other hand, contained a greater
variety of interaction patterns. There was still a large number of teacher explain
interactions, but there was also a substantial amount of focusing (19%) and inquiry (11%)
occurring. Focusing was the second highest frequency in both Mr. Louiselle’s and the
combined classrooms (17%). When examining tables and graphs created with CPMP-
Tools, he often focused the students’ attention on the aspect of the data that was key to
making inferences. The third highest frequency in the combined classrooms was inquiry
(11%). Therefore, even though this was a teacher-led discussion, the students were
engaging in inquiry-based learning. The real-world contexts brought out students’
natural curiosity. *Inquiry using the software* did not occur as often in this investigation as it did when students were working in groups because when the teacher is leading the exploration in a particular direction, the students have less room to explore their own conjectures with the software. Yet, it did occur when the students were exploring what would happen to the regression line and SSE when a value was deleted from the data set.

Also, there were more technical software statements during this investigation. This may be because the statistical tools have more options available and more complex operations, than the custom geometry tools used in this study. Students needed to figure out how to make a plot, make their own line of best fit, have the software plot the regression line, show the residuals, show the squared errors, and show the sum of squared errors. The software is quite user friendly, but there are just more features involved that tend to generate more statements regarding its use.

**The Second Research Question**

*What is the nature of students’ mathematical thinking while using the curriculum-embedded software?* Sub-question: *What levels of mathematical thinking are present in a classroom in which CPMP-Tools is used in whole-class interactive lessons with a single computer?* The levels of mathematical thinking that occurred most often during the regression line investigation were different in each teacher’s classroom. In Mr. Foster’s classroom *Recognizing Comprehending* occurred as often as *Building-with Analyzing* (26%), while in Mr. Louiselle’s classroom *Building-with Evaluative-Analyzing* occurred as often as *Building-with Analyzing* (27%).

*Building-with Analyzing* involves applying a known procedure in a new context, familiarizing oneself with the problem, or searching for patterns. This occurred in both
teachers’ classes when the class first looked at the data displayed by CPMP-Tools and tried to find trends. The Recognizing Comprehending level is the lowest level of mathematical thinking and occurred in Mr. Foster’s classroom mainly in the beginning of the lesson when they were trying to understand the first steps in the problem, the data they were using and how to make a scatterplot of that data. Mr. Foster spent more time on this than Mr. Louiselle because Mr. Louiselle’s students already knew how to make a scatterplot with CPMP-Tools. The Building-with Evaluative-Analyzing occurred in both classes when they were trying to find a line of best fit. Mr. Louiselle spent more time on this than Mr. Foster. CPMP-Tools allows students to experiment with creating their own line of best fit by clicking and dragging, and then compare it to the least squares regression line. The Constructing Synthesizing level of mathematical thinking did not occur in these whole-class discussions as much as it occurred in the group-work of the tessellation problems (13% vs. 57%). This is perhaps because the discussion was led by the teacher, so the students may not have had as much of an opportunity to explore the problem themselves with CPMP-Tools to develop new insights.

Mr. Louiselle used students’ input in the discussion which resulted in higher levels of mathematical thinking displayed by the students. As in the statistics investigation where Mr. Louiselle guided students in a discussion about the Chicago White Sox players wherein students discovered that the data revealed a scandal, he was frequently asking students questions to make them think critically about the topic. He also made the students tell him each step to perform with CPMP-Tools. During class and in the interviews, his students displayed proficiency with using CPMP-Tools.
The assessment for the regression lines investigations looked different for each teacher. Mr. Foster’s questions required more procedures than Mr. Louiselle’s, and Mr. Louiselle’s questions required more of the type of interpretation that students had been doing with the use of CPMP-Tools. One item that both tests had in common, asked the students to find the sum of squared errors (without the use of CPMP-Tools). Even though Mr. Louiselle’s students had used the software to find the SSE for the most part, a slightly a larger percentage of his students got the question completely correct (28% vs. 17%) and more got it partially correct. The students who got it wrong in Mr. Foster’s class had answers that were much smaller or larger in magnitude than the correct answer of 921 such as, 0.11, or -384450.933, or 40,743.65, or “Mexico,” or the cubic regression equation. It seems that many of them never learned the meaning of SSE even after the interviews that were conducted with them. Most of the students in Mr. Louiselle’s classes had answers that were relatively close in magnitude to the correct answer. Based on interviews and classroom discussion, Mr. Louiselle’s students at least had an understanding of the meaning of SSE, and therefore were more likely to be on the right track (except for one of his students who also answered “Mexico”). This understanding was facilitated by the software’s data analysis tools which create actual squares on the graph that represent the square of each residual. The sum of the areas of the squares represents the SSE. In the interviews, these students could create a representation of the SSE using CPMP-Tools and describe the concept. Knowing the meaning of SSE allowed the students to recognize when an answer was reasonable.

Additionally, the graphs of the assessments collected from Mr. Louiselle’s classes showed more evidence of the influence of using CPMP-Tools in this lesson. Half of his
students drew the residuals in on the graph compared to 9 (19%) of Mr. Foster’s students. While three of Mr. Louiselle’s students drew in two regression lines much like the two regression lines shown on the screen in Figure 14, and three others drew the squares representing the squared errors, Mr. Foster’s students drew no other markings on their graph. This finding is consistent with those of the observations and interviews, that Mr. Foster’s students did not have many opportunities to learn the material with the use of CPMP-Tools. Mr. Louiselle’s students demonstrated that they knew the geometric meaning of residuals and squared errors.

Additionally, Mr. Louiselle’s students were given more of an opportunity to show the knowledge that they had gained from the use of CPMP-Tools. The following question from the assessment allowed students to use the visual image that they may have created in their mind of the residuals on the graph after seeing many examples with the residual tab using CPMP-Tools. Find the point with the largest positive residual. Which country does it represent? Find the value of the residual. As reported in Chapter IV, 38% of his students answered the question completely correctly. However, many more were at least partially correct, and no serious misconceptions were evident. Another question from Mr. Louiselle’s assessment was as follows: On the scatterplot, circle the point that is an outlier. Then describe how the slope of the regression line and the correlation coefficient would change if that point were deleted from the data set. Again, this question allowed students to recall a visual image of the regression line changing after they deleted a point from previous data sets. Sixty percent of the students were completely correct about the slope in this question. However, if the other six students who answered that “the line would go up because the outlier pulled it down” were added
to the number of correct students, then the percentage rises to 72. These other six students essentially had the right idea, and were visualizing the way the line would move. As for the correlation coefficient, 12% answered correctly by giving the correct value 0.87. However, their error may be linked to a common observation that a correlation coefficient will often increase when an outlier is removed. Yet, this question asked students to think more critically about the shape of the data with the outlier removed. Overall, the students’ assessments from Mr. Louiselle’s class demonstrated more evidence of the influence of the use of CPMP-Tools in their responses. The interviews of Mr. Louiselle’s students also revealed that they had a good understanding of meanings of the graph, the SSE, the residuals, and the correlation coefficients. However, as was previously noted, Mr. Foster’s students did not have as much of an opportunity to show this kind of understanding with the questions that they were given. It seemed Mr. Louiselle’s enthusiasm for the technology may have rubbed off on his students, whereas, Mr. Foster’s apparent uncertainty with the use of the software may have negatively influenced his students.

**The Third Research Question**

*What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking?* In the combined classrooms, the highest level of mathematical thinking that was observed *Constructing Synthesizing* (SN) occurred during the interactions focus, inquiry, inquiry using the software, giving new idea/making a conjecture, building on other students’ ideas and making observations. The data analysis with the use of CPMP-Tools involved in this investigation brought out many of these interactions and high levels of mathematical
thinking. Students used the visual display of the data to make predictions, observations, and draw conclusions. The interactions give expected information, focus, inquiry and giving new idea/making a conjecture had the most occurrences spanning levels of mathematical thinking from Recognizing Comprehending (C) to Constructing Synthesizing (SN). These interactions are more likely because this was a teacher-led discussion where the teacher asked focusing questions that enabled students to share their ideas and engage in inquiry.

Mathematical Concept Three: Side-Side-Angle Triangle Condition

In Course 2, Unit 7, Lesson 2, Investigation 3 (Triangle Models—Two, One, or None?), students explore the SSA condition for triangles, and the conditions under which there will be zero, one, or two possible triangles with two consecutive side lengths and the non-included angle given. The investigation begins with a real-life example of a triangle found in the mechanism for a cold frame, which is a box used in a flower nursery to help seedlings get started (see Chapter IV, Figure 16). In this mechanism, two consecutive sides of the triangle are a fixed length. One of these sides is along the top and this side can be raised or lowered by the other fixed side of the triangle, the piece that props up the top. When the top is opened in some positions, the prop can either be placed closer to the vertex of the non-included angle, or farther away from it, creating two possible triangles with the same measures for two sides and the non-included angle. Students use the Explore SSA geometry custom tool to explore the possibilities that arise under these conditions and to discover when there can be two triangles, one triangle, or
no triangles formed. They could try different lengths for the triangle in seconds with immediate feedback.

Since Mr. Louiselle was the only Course 2 teacher to use CPMP-Tools for this investigation, only observations of his classroom are included here. This set of observations presented an opportunity to study a whole-class interactive discussion with the use of a geometry software tool. The other observations involving the geometry tools were mostly of small-group work.

**The First Research Question**

*What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools?* Sub-question: *What types of interactions are present in classrooms in which CPMP-Tools is used in whole-class interactive lessons with a single computer?* During the SSA investigation, giving *new idea* was the most frequently occurring interaction pattern (19.4%). This indicates that the discussions were interactive with students providing many of the ideas for discussion. The dynamic visual provided by CPMP-Tools elicited many of these ideas. Mr. Louiselle asked many *focusing* questions (7.2%) allowing the students to focus on the important aspects of the problem being discussed such as the critical lengths of the changing side of the triangle. As one side of the triangle changed by clicking and dragging the slider, the whole diagram changed accordingly. This allowed students to easily check their conjectures about when there would be one, two, or no triangles formed.

Much of the discussion time was also spent *developing conceptual understanding* (12.8%). Mr. Louiselle continually asked the students why there was one triangle, or two
triangles, or no triangles. Beyond just making observations of the changing triangle(s), the students were pressed to look for the reasons for those changes and relationships between the sides of the triangle. CPMP-Tools allowed students to see the effect on the rest of the triangle when the length of one side was changed. When two triangles were formed, the use of the circle in the construction of the diagram gave proof that two corresponding sides of the two triangles (the radii of the circle) were congruent. With the other corresponding pair of sides and angles being identical to themselves, the visual representation proved that the SSA condition does not guarantee congruence. These aspects of the diagram on the screen helped students reason about the SSA condition.

There were also a number of building consensus interactions (5.6%) in Mr. Louiselle’s class discussion. The class exchanged ideas about when there would be one, two, or no triangles until the class agreed. Since this was a whole-class, teacher-led interactive discussion, it is not surprising that teacher explain was the third most frequent interaction pattern (10.1%). Inquiry using the software occurred (9.7%) almost as often as the teacher explain interaction pattern. So even though it was a whole-class discussion, the students were still engaged in the inquiry as the software was utilized. The visual provided by the software was engaging for the students and provoked students’ participation.

The Second Research Question

What is the nature of students’ mathematical thinking while using the curriculum-embedded software? Sub-question: What levels of mathematical thinking are present in a classroom in which CPMP-Tools is used in whole-class interactive lessons with a single computer? Seventy percent of the lines of text in this investigation contained the
Constructing Synthesizing level of mathematical thinking. As students changed the length of one side of the triangle, they were exploring the situation and the relationships present in the drawing on the screen.

The mathematical thinking data for this investigation closely resembled that of the tessellation investigation as the frequencies at each level of mathematical thinking were very similar. The common feature in both of these investigations is that the software required clicking and dragging. As students performed those operations with the software, it was usually coded as SN because they were exploring the problem to develop new insights. So, even though the tessellation investigation was a small-group/multiple computer classroom environment and the SSA investigation was a whole-class/single computer classroom environment, the majority of lines of text for both investigations were coded as Constructing Synthesizing.

The responses on Mr. Louiselle’s related assessment item revealed which students understood the dynamic relationships among the lengths of the sides of the triangles. None of his students set up an equation to solve for two possible lengths of one of the sides of the triangle. Instead, they often made a drawing that resembled the visual provided by the software, and explained the relationships between the sides of the triangle. They typically reasoned that there are two triangles because the length of side BC is greater than the height of the triangle—just like when they used the software to make segment BC longer than when it was perpendicular to segment AC. Additionally, they used the fact that side BC also has to be less than side AB in order for there to be two possible triangles—another relationship they discovered using the software. They
demonstrated an understanding of the relationships between the lengths of the sides of the triangle that was facilitated by the use of CPMP-Tools.

**The Third Research Question**

*What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking?* The highest frequencies of interaction patterns are in the highest three levels of mathematical thinking. The frequencies in the lower four levels of mathematical thinking are all single digit numbers—mostly ones and twos. The *Constructing Synthesizing* column contains most of the data, with the largest quantities in the *giving new idea/making a conjecture* (59) and *making observations* (38) rows. These combinations of interactions and level of mathematical thinking characterize much of the discourse in this investigation. As students were exploring the problem, they were sharing ideas and making conjectures about why there were two triangles, and making observations about the changing visual representation shown on the screen. *CPMP-Tools* allowed them to make and test many conjectures in a short amount of time because with a click and drag the whole diagram would change and display the result. They tested acute triangles, obtuse triangles and right triangles. They did not have to create the desired angles or lengths of segments with a ruler and protractor, and repeatedly draw the diagram. Doing this with manual devices may not have allowed them to examine as many different possibilities.

The ideas that students gave were sometimes at lower levels of mathematical thinking, and sometimes at higher levels of mathematical thinking. Yet, in this investigation many more of the students’ ideas were at higher levels of mathematical thinking than at lower levels (81 instances vs. 11 instances). They were explaining the
complex relationships between the sides of the triangle(s). The triangle drawn in the circle in CPMP-Tools allowed students to explain why two triangles were possible with certain lengths. The dynamic constructed diagram in CPMP-Tools facilitated students’ thinking and reasoning.

Mathematical Concept Four: Transformations/Computer Animations

Another set of observations was performed in Mr. Louiselle’s Course 2 classroom during Unit 3, Coordinate Methods. The four days of computer use observed during this unit spanned the largest number of days studying one topic in this study, and afforded an opportunity to do more extensive research on some geometry lessons. In this unit, students discovered coordinate rules for transforming the coordinates of a figure in the coordinate plane using translations, reflections, rotations, and size transformation. For example, the rule for rotating a point 180 degrees about the origin is \((x, y) \rightarrow (-x, -y)\). Later, students accomplished the same transformations through the use of matrices. By the end of this unit, students could create their own figure using coordinates stored in a matrix, and some simple draw commands in the programming window of the software. Then, they chose transformations to perform with matrices, and created an animation using their figure.

The First Research Question

*What is the nature of the interactions present among students, and between the teacher and students, in classroom utilization of CPMP-Tools?* Sub-question: *What types of interactions are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers?* There were four main interaction
patterns found during the observations of this unit: focus, inquiry, inquiry using the software, and making observations. The interaction that occurred most frequently is inquiry using the software (40.3%). This is reasonable given the fact that students were using CPMP-Tools the entire time that they were working on their animation projects. They were testing lines of programming code by running the program to see if it did what they wanted it to do. They were trying to figure out how to create their animation and trying different transformations on their objects.

The second most frequent interaction pattern was making observations (11.3%). This interaction pattern also involved the use of CPMP-Tools. During these interactions students were making statements based on what was happening with the animation on which they were working. Students were usually making observations when they ran their program and examined the results of the commands that they had programmed.

Another computer-related interaction that occurred was instrumental genesis (2.4%). The programming code used with the computer becomes an extension of the student’s own mind. Students used the computer software to show each other their thinking when they wrote their programming code and/or ran their program.

The focus interaction (10.5%) occurred when Mr. Louiselle was helping students with their program and asking focusing questions to allow them to find their mistakes, or figure out what to do next. Interactions were also found that are evidence of students collaborating with each other—answering another student’s question (6.5%), argument (5.6%), and building on other students’ ideas (2.4%). These interactions all involve an exchange between students. Students were asking and answering questions with each other, debating ideas, and building on each other’s ideas when they were helping each
other with their programs. Even though each student was working on their own program that was different from anyone else’s, they were still collaborating with each other. Viewing each other’s programs seemed to be engaging for the students.

As the students were learning the transformations, and the coordinate rules for them, there were days when their computers were not in use. The students were observed working in their groups on these days, as well as the days when they were working on their computer animations. The biggest transformation of student interactions occurred during the observations of students working on the computer animations. There was a large difference between students’ interactions and behavior before they used CPMP-Tools and while they were using CPMP-Tools. Quiet students sometimes became more outgoing, and students who were previously disengaged often became very engaged.

There was a marked increase in the activity and energy level in the classroom when the students were working on their computer animations. Students were helping each other with their programming code, and showing each other the drawings they had made. Students displayed creativity in their drawings and ideas for animation, and new strengths surfaced from many students. Some students, who were very quiet before, became more vocal. Some students seemed to exhibit more confidence. Students who were technologically savvy, and those who caught on quickly to the programming code sometimes became new leaders. Other students would walk across the room to ask them questions, and seek their input. Mr. Louiselle said that he “is seeing different kids shine” and, “the kids are asking other kids for help whom they normally wouldn’t ask.” During the observations just prior to these, the students remained at their desks, and only talked to people immediately next to them, if they talked to anyone at all.
The use of *CPMP-Tools* seemed to increase student engagement. The number of disengaged students in the classroom went from an average of nine to none in this unit. There were two boys in Mr. Louiselle’s along the side of the room who would always have their laptops open even when the lesson did not require *CPMP-Tools*. They were usually looking up other things on the internet, and not engaged in the lesson. When the animations began, they were engaged in the work of creating their animation the entire class period. Additionally, a girl who sat in the middle of the room was very quiet during all of the non-computer observations. She was often one left to work alone during “group” work. However, she had a good understanding of the content, and was good at using *CPMP-Tools* and figuring out how to create programs. Once the students started working on their animations, she was observed getting up out of her seat and helping other students in the class while others were asking and answering different types of questions and interacting constantly.

**The Second Research Question**

What is the nature of students’ mathematical thinking while using the curriculum-embedded software? Sub-question: What levels of mathematical thinking are present in mathematics classrooms where small groups of students use CPMP-Tools on laptop computers? The observations of the computer animation project contained the highest percentage of interactions at the *Constructing Synthesizing* (SN) level of mathematical thinking (82.1%). Students were constantly exploring using the software to create their own animation. Therefore, virtually all of the interaction patterns that contained levels of mathematical thinking were categorized at the *Constructing Synthesizing* level. They would type lines of programming code in the programming window which is located
right under the window where the animation will display, and immediately see the results of those lines of code right above. So, they could make a change in the programming code, and see how the figure changed as a result on the same screen. The levels of mathematical thinking just above and below Constructing Synthesizing both occurred the second most frequently. These highest levels of mathematical thinking occurred also because students were creating a product by applying knowledge in a new setting.

During the transformations/animations lessons, it was found that CPMP-Tools helped to enable students to understand transformations. When one of the students said, “Now I even understand transformations” it was clear that working on the animations had deepened the knowledge of transformations. Having to use the correct transformation matrices to make the object on the screen move in the desired way caused the students to develop a deeper understanding of the patterns of coordinates and how the coordinates will change under each transformation. As the students displayed their finished animations in front of the whole class on the last day of this unit, they all could identify the transformations that were appearing on the screen. Additionally, the quality of the animation programs that the students created showed that this activity was beneficial to their learning. All students successfully completed the animation project. All animations used a drawing generated by matrices and made use of transformations that were also generated by matrices.

The Third Research Question

What is the relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking? The highest interaction pattern/level of mathematical thinking pair is inquiry using the software/Constructing
Synthesizing (27 of 78). This pair of interactions characterizes much of the discourse that took place during these investigations. Students were continuously exploring and engaging in inquiry as they worked on their animation project. They would have to write some of the program and then run it to test it. Then they would go back to the program to write some more or edit parts that did not work correctly, and then run it again.

It is interesting that the four interactions in the lowest three levels of mathematical thinking took place while the teacher was interacting with the group. There were also higher levels of mathematical thinking that took place while the teacher was working with the group; however, when the group was working without the aid of the teacher, their discourse remained at high levels of thinking. This may not always be the case for any investigation. But this particular investigation lent itself to being student-led. First, this may have been because many students were “technology-savvy.” These students are often more comfortable with the technology than their teachers. Second, the students were creating their own programs, which often requires different skills than when they worked together to discover a predetermined concept. There was less of a need for the teacher to correct their thinking.

Limitations

There may be some limitations to this study. This school had a higher than average socio-economic status. It is possible that in a school with lower socio-economic status that some of the investigations would not have resulted in students using the tool successfully or displaying the higher levels of mathematical thinking. For example, the computer animations project may have been more difficult for students who have less
opportunity to interact with technology. On the other hand, the interactions associated with standards-based (Wood, Williams, & McNeal, 2006) mathematics instruction which were prevalent in this study may have been more beneficial to students with lower socio-economic status than any other methods of instruction. Since the findings of Boaler (2002, 2004) suggest that this is the case, conducting a similar study with students with lower socio-economic status could produce more dramatic results.

There are also other aspects of this school that may have affected the results. The teachers in this study were all experienced with the curriculum. The school was supportive of the use of the curriculum and of the innovative characteristics of it such as CPMP-Tools. Most of the teachers were also willing to use the technology. There may be different results when the teachers are resistant to using technology. Furthermore, the teachers in this study collaborated quite frequently, giving each other support and helpful tips.

Also, almost all of the students in this school have access to the internet, and therefore CPMP-Tools, at home. There was evidence that the students in Mr. Nelson’s, Mr. Kirkwood’s, and Mr. Louiselle’s classes used CPMP-Tools at home frequently. The results of a study with schools where many students do not have Internet access at home may have been very different from those of this study. The effect of CPMP-Tools on students’ learning could be hindered if the students do not practice using the tools at home and/or do not use it to do their homework. This is suggested by some of the data presented from Mr. Foster for the Regression Lines topic.

Another limitation of this study is the differential ability of teachers to facilitate group work. Three of the teachers in this study were fairly skilled at facilitating effective
group work, but the groups were not as structured as they could have been. If all of the groups were formed by the teachers, and the teachers regularly expected students to use group roles, the non-computer group work would likely have shown more interaction than it did in this study. It would be interesting to observe a teacher who is highly skilled at facilitating group work, and view his/her classes with and without the use of \textit{CPMP-Tools}. Would improved facilitation of group work improve the non-computer environments as much as the computer environments? Or, would the performance in all of the environments appear similar? In this study the use of the computer software enhanced the group collaboration when the teacher’s facilitation of that collaboration was lacking.

This study was also limited to one school with four teachers. The results could be different with different teachers or more teachers. However, for an exploratory study such as this, four teachers were sufficient. There are many studies of students using computer software that used only one teacher \cite{i.e., Taylor, 2005}, and/or as few as eight students \cite{i.e., Yu, 2004}. These studies examined each student’s thinking in more depth, while the present study sought to make broader assertions about students’ interactions and mathematical thinking with multiple uses of computer software. Additionally, the content studied was only from the geometry and data analysis strands. \textit{CPMP-Tools} also contains software for use in the discrete math and algebra strands. Perhaps different results would be discovered from studying the \textit{CPMP-Tools} use in the other two strands.

Further, the \textit{Core-Plus Mathematics} curriculum contains many more investigations within the geometry and statistics strands that make use of \textit{CPMP-Tools}. This study included just a sample of those investigations. Yet, the number of class
periods observed required considerable time and effort. Still, a more in-depth focus on the existing lessons or on specific subgroups of students would likely produce more definitive results.

The framework used in this study for interaction patterns and levels of mathematical thinking was quite extensive. It is possible that other researchers would have coded some lines of text with a different code. As with most qualitative research, individual interpretation can vary. There were times when more than one code could apply. In this case, the researcher chose the most specific code or the one that best fit the interaction present. Some of the coding was also checked by a second researcher. Furthermore, with a total of 2,301 interactions coded, a few differences in coding individual lines would likely not change the overall patterns.

Implications for Further Study

While three of the four teachers in this study would be considered to be in favor of standards-based mathematics curricula and instruction, it would be interesting to repeat the study with another teacher who was highly skilled at facilitating student-centered whole-class discussion as well as one who is highly skilled at facilitating group work. Since an individual student’s mathematical activities, and classroom community practices have a reflexive relationship—neither occurs independent of the other (Cobb, 2000a), the socio-mathematical norms developed by the teacher can have an impact on students’ mathematical thinking. Some of the interaction codes listed in Figure 2 were not found in this study. However, those interactions do occur as the researcher has observed them before this study. One of those interaction patterns is exploring methods. This
interaction is common in standards-based mathematics instruction after students have explored a problem in their groups. Multiple students share their method for solving the problem. In the observations in this study, the students would work in groups until the end of the class period. For the most part, there was no summary, or a reporting-out of their solutions. It is possible that there were other days when the students were given the opportunity to share various solution methods.

Other interaction patterns that were not observed in this study, or that were observed with very low frequencies, are proof of answer by manipulation of software, proof of answer by student explanation, checking for consensus, and pupil self-nominate. Most of these interactions will all usually occur during whole-class discussions when teachers continue to ask focusing questions until students start to form conclusions. At this point in the discussion, students have more of an opportunity to prove their conjectures or volunteer to share a major discovery (pupil self-nominate). Once a student has shared a major mathematical idea in the discussion, then the teacher asks if other students agree (checking for consensus). The computer can be used to fulfill the roles of explanation and discovery (proof of answer by manipulation of software and proof of answer by student explanation) (De Villiers, 1998). These interactions may not have been found often in this study because many of the whole-class discussions took place at the beginning of the investigation, when the students were still exploring, and not coming to conclusions yet. However, repeating this study with a teacher who regularly facilitates these kinds of whole-class discussions at the end of investigations would be interesting. There could be even higher levels of mathematical thinking found with the use of CPMP-Tools. Moreover, another similar study may show the robustness of the classification of
interactions used in this study. The coding could be improved and refined with further use.

It could also prove fruitful to repeat the study with the use of CPMP-Tools in the algebra and discrete math strands. There are very different uses of the tools in these areas. The algebra tools include features similar to those of a graphing calculator, especially those like the TI-89. So, it would be interesting to see what differences can be found in the students’ use of the computer software compared to the use of graphing calculators. In the discrete math strand, the software can be used to create vertex-edge graphs and find minimal spanning trees or Hamiltonian circuits. These functions have not been used before with any technology of which the researcher is aware. So, it would be a novel area to study.

Furthermore, a similar study could be done with either more or fewer students. This study contained about 350 students from four teachers’ classrooms. One possible variation of the study could use more teachers and/or more students. This version could help to verify the patterns of interaction and levels of mathematical thinking found in this study. However, this would be a significant undertaking. There was already abundant data collected in this study. Another variation would use fewer students and likely follow them through more consecutive days of study. It would be interesting for the researcher to sit near a select group of students for an extended time, watching them work with the software, and listening to all of their conversation. That configuration differed from this study because more sampling of multiple groups and multiple topics was used. A more focused study could reveal more about how CPMP-Tools affects the students’ thinking.
For instance, a study could examine how students strategically select tools to help them explore and discover ideas and mathematical concepts.

Other researchers may want to explore the effect of CPMP-Tools on student outcomes. Although students’ assessments were collected in this study, it was more to learn about students’ thinking than it was to create a causal relationship between the use of CPMP-Tools and student achievement. Such a link is difficult to establish given the number of variables involved. Perhaps more of the variables could be controlled in a future study.

**Overall Conclusions**

Several things were learned from the observations of each investigation. As Schoenfeld, 1992 stated, “The role of interactions with others will be central in understanding learning…” (p. 363). By examining the interactions and discourse during each of the investigations, some things were learned about how students learned the mathematical concepts.

The Patterns with Polygons (tessellations) investigation showed how engaging and productive it can be for students to use CPMP-Tools. However, for some students, the mathematical concepts got lost in the amusement. Some students were enjoying creating unique tilings with various shaped triangles and quadrilaterals, but were not focusing on the reason that these tilings could be made. Therefore, the teacher had to go around to each group and focus their attention on the angles surrounding each vertex. Highlighting the mathematical ideas and reinforcing the mathematical language that is associated with the topic of this geometry tool may prevent students from missing the
mathematical concept (Jones, 2000; Laborde, 2000; Marrades & Gutierrez, 2000). These data also suggest that the quality of small-group facilitation could have an impact on the interaction patterns and levels of mathematical thinking displayed. This idea is supported by social constructivist theory which posits that learning occurs through social activity (Cobb, Wood, & Yackel, 1990; Ernest, 1996; Vygotsky, 1978, 1981). So, if students are not interacting as much they have fewer opportunities to learn. Finally, there were aspects of the geometry software that aided students’ understanding of the concepts. The color-coded and numbered angles in the triangles and quadrilaterals allowed students to focus on the fact that two copies of each angle of the triangle were always surrounding each vertex. This observation facilitated the discovery that the angles around each vertex sum to 360. Also, the ability to click and drag the original figure which simultaneously changes all the other copies of the figure in the tiling allowed students to see that this discovery remained true for many different shapes of triangles and quadrilaterals. They could conjecture that it would work for any triangle or quadrilateral because clicking and dragging the shape did not change the angles that surrounded each vertex. When an object is dragged, it preserves the properties that were inherent in its construction (Fey, Hollenbeck, & Wray, 2010). Having students perform these operations themselves while working in groups was most beneficial.

The data analysis (regression line) observations revealed some differences in student understanding based on the extent of use of CPMP-Tools. The students in Mr. Foster’s class used CPMP-Tools in these lessons, but not as much as the students in Mr. Louiselle’s class. Most of Mr. Foster’s students only watched the teacher use the software. Mr. Foster manipulated the software on the computer at the front of the room,
asking for little student input. Students had many questions and said that they were confused. According to the interview data, fewer students in his class had used the software themselves. Mr. Foster’s lessons stressed the procedural knowledge more than the conceptual understanding that CPMP-Tools can facilitate. The students may have benefited more if they had manipulated the software themselves in order to develop a conceptual understanding of the concepts.

According to Cobb and McClain (2004), there are four main things a teacher can do to support students’ learning of statistics: establish productive classroom norms, select suitable computer tools, carefully plan instructional activities, and manage whole-class discussions well. Mr. Louiselle followed all of these suggestions. The students in Mr. Louiselle’s class were all expected to use CPMP-Tools for their homework in this section, and it was clear that most did. Mr. Louiselle’s students demonstrated greater conceptual understanding of the data analysis topics than Mr. Foster’s students on the assessments and in the interviews. This understanding was facilitated by the representation of the SSE created by CPMP-Tools. Statistical software makes students’ thinking more visible to others (Hammerman & Rubin, 2004). They could display their own line that indicated the trend they saw in the data. The squares are shown on the graph for each residual, the sum of their areas is shown in the “error thermometer,” and the squares and error thermometer will both change accordingly as the lines is dragged to a different position. If two lines were being compared, the squares (squared errors) would be a different color for each to allow for easier comparison of the size of the squares. The error thermometer was also color-coded accordingly to clearly see which line had a greater sum of squared errors. The squared errors and the sum of squared
errors were a different color for each line so that students could distinguish between them. This dynamic visual representation appears to be advantageous to students’ understanding of the SSE, residuals, and regression lines. The multiple representations provided by the graph, table, algebraic equation, and error thermometer (essentially a bar graph) helped students make connections among mathematical concepts of residuals, squared errors, and regression lines (Bakker & Frederickson, 2005; Hollenbeck, Wray, & Fey, 2010). Since Mr. Louiselle’s students had used the software themselves, their verbalized statements suggested more of a conceptual understanding of the concepts. Even when Mr. Louiselle was demonstrating the use of the software at the front of the class, the students could make sense of what he was showing them because they had previously done the same operations themselves.

The investigation on the topic of the SSA condition for triangles allowed for an examination of the use of CPMP-Tools using an interactive whole-class discussion. These students approached a familiar topic in a novel way. Since they learned the topic with CPMP-Tools, they used a very visual method to determine the conditions under which two triangles that had the same measures for two sides and the non-included angle were possible. CPMP-Tools facilitated students’ understanding of the SSA triangle condition through the constructed diagram that includes a circle along with the triangle(s). When two triangles are present, two corresponding sides of the triangles are radii of the circle which clearly shows that the segments are congruent. Using that and the given information, students are convinced that the SSA condition does not guarantee congruence. They also discover that these two triangles appear whenever the moveable side is greater than the height of the triangle, and when it is less than the height of the
triangle no triangle is formed. They find out that the minimum length of the moveable side is the length that forms a right triangle because when they make it shorter than that, the triangle immediately disappears from the screen. The visual representation is quite helpful to students. Dynamic geometry software possesses the unique dragging feature that makes it more powerful than paper-and-pencil methods because students can see many examples in a short time, get immediate feedback, and easily look for properties, special cases, or counterexamples (Marrades & Gutierrez, 2000). The sides of the triangle were also different colors which corresponded to the colors of the segments used as sliders to change the length of the sides. Additionally, when two triangles were formed, the side BC and BC’ were both the same color to highlight the fact that they were the same length and yet were contained in two different-shaped triangles. The geometry software used in this study appeared to facilitate students’ understanding because the students displayed a good conceptual understanding of the situation during class discussion, in the interviews and on the assessment. Using this particular software was effective as a whole-class discussion because the teacher was able to focus the students’ thinking on the underlying mathematical ideas.

The unit involving computer animations/transformations illustrated the potential of CPMP-Tools to push students to achieve things they perhaps did not think they could achieve, or to illuminate the talents of those students whose abilities are not always apparent. There were two unforeseen results. The students who were creative, or who were adept at technology, and some who were unengaged by traditional mathematics topics, excelled at this endeavor. Many of these students gave a completely different impression in the observations when comparing their performance with and without the
use of *CPMP-Tools* in the lesson. Additionally, students who were intimidated by the technology or the programming proved that with perseverance they could successfully complete difficult mathematical tasks. The observations of the computer animation project contained the highest percentage of interactions at the *Constructing Synthesizing* level of mathematical thinking. The combination of the *Constructing Synthesizing* level of mathematical thinking and *inquiry using the software* interaction pattern was the most prevalent during investigations of this mathematical topic. *CPMP-Tools* aided students’ thinking during these investigations by having the programming window immediately below the animation that was created on the screen. Therefore, the result of a change to the program could instantly be seen. The feedback that this geometry software provided may be very rich because the visual and the theoretical aspects of geometry are integrated (Laborde, 2001). This aspect may have helped the students to persevere until their program accomplished the desired animation. This use of *CPMP-Tools* was effective in small groups because the students were creating their own animation, and because they could help each other.

A revealing observation was that many of the productive student interactions (such as *inquiry, answering another student’s question, giving new idea, argument, building on other students’ ideas, and instrumental genesis*) that occurred in all of the teachers’ classrooms seemed to be associated with the use of *CPMP-Tools* on the laptops. Students often turned their laptop around to show other students in the group what they had on their screen. Students would get up out of their seat to point at the screen in front of other students. Heid (1997) found that students pushed two laptops next to each other to compare the screens, picked the laptop up to show the screen to a student who was not
right next to them, or pushed the laptops aside so that they could easily collaborate. The same things were observed in this study. Additionally, students tended to be closer in proximity to one another when they were using CPMP-Tools. The findings from the audiotape of student work lend further evidence to the observation that the use of CPMP-Tools on the laptops acted as a magnet to draw the students together. The number of interactions in a given period of time was greater when students were using CPMP-Tools in their groups, than when they were working in groups without CPMP-Tools. When students were working in groups without the computer, there were often long pauses with no discussion. There was clearly more collaboration during the animation project than before the students starting using computers in that lesson. As was noted in the field notes, use of CPMP-Tools on the laptops seemed to help overcome some of the teachers’ lack of group structure. As Dillenbourg, Baker, Blaye, and O’Malley (1996) discovered, student collaboration may even happen unplanned due to the need to share computers, look at each other’s screens, and the interactive nature of computers. The overall CPMP environment as a whole tended to motivate students to collaborate and the use of CPMP-Tools seemed to be a good focal point for structured group activities. As Stevenson (1999) had found in other high school subjects, and Clements and Nastasi (1999) found with elementary students, the use of the computer seemed to enhance student interaction.

When CPMP-Tools was utilized in the classroom, the types of student interactions found most prevalently were focus, inquiry, inquiry using the software, answering another student’s question, giving new idea/making a conjecture, argument, building consensus, developing conceptual understanding, building on other students’ ideas, making observations, instrumental genesis, and teacher explain. These terms define the
nature of the interactions present among students, and between the teacher and students, while \textit{CPMP-Tools} is being utilized. According to Wood, Williams, and McNeal (2006), these interactions are more aligned with standards-based mathematics instruction. The dynamic nature of the software and the visuals it created on the screen facilitated the above interactions. It gave the students a focal point for discussion and more purpose in interacting with each other. This observation was confirmed by the fact that the same classroom was observed not using the software, then using the software, then later not using the software again. Thus, the increase in standard-based interactions appeared to be linked to the use of the software. Use of the software \textit{CPMP-Tools} prompted and enhanced the quality of student interactions.

The nature of students’ mathematical thinking while using the curriculum-embedded software can be characterized by the 2\textsuperscript{nd} and 3\textsuperscript{rd}-highest levels of mathematical thinking—\textit{Building-with Evaluative-Analyzing} and \textit{Constructing Synthesizing}. The greatest frequencies of levels of mathematical thinking were found at these two levels. Again, the dynamic and visual characteristics of \textit{CPMP-Tools} allowed the students to remain in the exploring phase and consequently to pull together ideas to come to conclusions. As Zbiek (2003) had found, using the computers in mathematics instruction had the benefits of rapidly generating examples, decreased computational errors, opportunities for students to pose their own problems, and made students’ thinking more visible to teachers. The effective use of color in \textit{CPMP-Tools} allows students to focus on important aspects and relationships. The representations of a problem situation made on the computer can be more powerful, can extend beyond the problem, and can help students better understand the mathematics inherent in the task (Peressini & Knuth,
The more the students explored, the more ideas they generated. These ideas built upon one another until the intended discovery was made. Conversely, when CPMP-Tools was not being utilized, the greatest frequencies of levels of mathematical thinking were found at the lowest three levels of mathematical thinking—Recognizing, Comprehending, Recognizing Applying, and Building-with Analyzing. However, a direct causal relationship here is complicated by the nature of the task. It is possible that the tasks that require the use of CPMP-Tools are inherently more mentally engaging and could be a variable in the effect on students’ level of mathematical thinking.

The relationship between the patterns of interaction that exist in the classrooms and levels of students’ expressed mathematical thinking is complicated. However, most notable is that the traditional teacher initiation, student response, teacher evaluation (IRE) and give expected information (gei) interaction patterns typically result in low levels of mathematical thinking, while the inquiry and inquiry using the software interactions result in high levels of mathematical thinking. The types of interactions found most frequently in the computer classroom environments have been shown to be beneficial to students’ learning, and/or to lend themselves to effective group work. When students were using CPMP-Tools in groups there were more productive and desirable interactions and higher levels of mathematical thinking than in any other classroom environment configurations. This finding is consistent with that of Webb (1982) who found that cooperative learning develops higher level thinking skills in mathematics. This may suggest that students should be allowed to use CPMP-Tools to explore the mathematics in their small groups whenever it is feasible.
The dynamic nature of software like CPMP-Tools is engaging for students, and has the power to illustrate the mathematical concepts clearly. Other advantages are the ease of manipulating the objects on the computer screen (clicking and dragging), the accuracy of measurements, the precision of movement of the objects, and the colorful visuals. Additionally, most students today are comfortable with using computers. They compose their papers on the computer, and use the internet for many other school assignments. In this study, students were observed using their computer in math class to copy and paste the screen shots from CPMP-Tools into their homework papers, and save their work. These observations suggest that the use of computers and mathematical software such as CPMP-Tools may actually enhance communication among students, and increase students’ ability to make conjectures and explore concepts. Through communication, ideas are internalized (NCTM, 2000; Steele, 1999). So, increased communication can lead to increased learning. Therefore, the use of computer software like CPMP-Tools can help teachers make further changes to instructional practice and to achieve the vision put forth in NCTM’s Principles and Standards for School Mathematics (2000) and Focus in High School Mathematics: Reasoning and Sense Making (2009).
REFERENCES


Appendix A

Investigations Observed That Used CPMP-Tools
Appendix A

Investigations Observed That Used CPMP-Tools

<table>
<thead>
<tr>
<th>Course</th>
<th>Unit</th>
<th>Lesson</th>
<th>Investigation</th>
<th>CPMP-Tool</th>
<th>Extent of Use</th>
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<tbody>
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<td></td>
<td>Geometry Strand</td>
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<tr>
<td>1</td>
<td>6</td>
<td>2</td>
<td>3: Patterns with Polygons</td>
<td>Tessellations</td>
<td>4 multi-part tasks/ 1 class period/ Mr. Kirkwood &amp; Mr. Nelson/ 2 classes per teacher for a total of 4 class periods</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2: Building and Using Size Transformation Matrices</td>
<td>Programming—animation project</td>
<td>5 multi-part tasks/ 4 class periods/ Mr. Louiselle/ 2 classes per teacher for a total of 8 class periods</td>
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<td>7</td>
<td>2</td>
<td>3: Triangle Models—Two, One, or None?</td>
<td>Explore SSA</td>
<td>2 multi-part tasks/ 1 class period/ Mr. Louiselle/ 2 classes per teacher for a total of 2 class periods</td>
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<td>Statistics Strand</td>
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<td>2</td>
<td>4</td>
<td>2</td>
<td>1: How Good Is the Fit?</td>
<td>Data Analysis</td>
<td>4 multi-part tasks/ 1 class period/ 2 of Mr. Louiselle’s classes &amp; 1 of Mr. Foster’s classes/ a total of 3 class periods</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2: Behavior of the Regression Line</td>
<td>Data Analysis</td>
<td>4 multi-part tasks/ 1 class period/ 2 of Mr. Louiselle’s classes &amp; 1 of Mr. Foster’s classes/ a total of 3 class periods</td>
</tr>
</tbody>
</table>

Note: 17 additional class periods were observed that did not involve the use of CPMP-Tools
Appendix B

Observation Forms
Appendix B

Observation Forms

General Information for All Class Periods

A. School/Type of Computer Environment ________________________________

B. No. of Students _______ Girls _______ Boys _______ Length of Period _______

C. Sketch of Room Layout

D. Physical Characteristics of Classroom

<table>
<thead>
<tr>
<th>1. Are students regularly sitting in groups?</th>
<th>No</th>
<th>Yes</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Evidence of group role designations?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Evidence of group work artifacts?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Lesson outline

<table>
<thead>
<tr>
<th>Time</th>
<th>Activity</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Group Work Observation Form

#### A. Introduction of the task

<table>
<thead>
<tr>
<th></th>
<th>Rating</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Are the instructions clear?</td>
<td>VC SC SNC NC</td>
<td></td>
</tr>
<tr>
<td>2. Does the teacher make use of visual aids or CPMP-Tools?</td>
<td>No Yes</td>
<td></td>
</tr>
<tr>
<td>3. Does the teacher engage students in a discussion?</td>
<td>No Yes</td>
<td></td>
</tr>
<tr>
<td>4. Are the students attentive to the introduction?</td>
<td>VA SA SNA NA</td>
<td></td>
</tr>
<tr>
<td>5. Is the assignment of group roles clear?</td>
<td>VC SC SNC NC</td>
<td></td>
</tr>
</tbody>
</table>

#### B. Students at work in groups

<table>
<thead>
<tr>
<th></th>
<th>None</th>
<th>Number</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. How many students are waiting for teacher assistance?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. How many groups are not engaged in the task?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. How many groups have students who are working individually?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. How many students are contributing?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>For each group</strong></td>
<td>No Yes</td>
<td>Notes</td>
<td></td>
</tr>
<tr>
<td>5. Are there any students who are dominating the group?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Are students listening to each other?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Are the groups using group roles?  No _______ Yes _______

If yes, for each role, put a check next to the group role behaviors observed. If no, put a check next to the group role behaviors that any person in the group is observed performing.

<table>
<thead>
<tr>
<th>Reader</th>
<th>Coordinator</th>
<th>Recorder</th>
<th>Quality Controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reads the assigned work aloud for the group</td>
<td>Keeps the group on task</td>
<td>Writes a summary of the group’s decisions and ideas</td>
<td>Monitors the group’s results</td>
</tr>
<tr>
<td>Uses appropriate volume and tone of voice</td>
<td>Makes sure everyone is participating</td>
<td>Reads back answers to the group to ensure agreement and accuracy</td>
<td>Makes sure the group’s work is well done</td>
</tr>
<tr>
<td>Explains the reading when any group member needs clarification</td>
<td>Keeps track of time</td>
<td>Records data when necessary</td>
<td>Makes suggestions for improvements of answers</td>
</tr>
<tr>
<td>Seeks assistance with explanations from other group members</td>
<td>Makes sure everyone understands before moving on</td>
<td>Shares the group’s work when necessary, orally, or on the board</td>
<td>Makes sure all parts of the question are answered</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Asks the teacher for help if everyone in the group is stumped</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Gets materials for the group</td>
<td></td>
</tr>
</tbody>
</table>
### C. Teacher during group work

<table>
<thead>
<tr>
<th>Question</th>
<th>No</th>
<th>Yes</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Is the teacher relinquishing authority to the groups?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Is the teacher spending most of the time getting students back on task?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Is the teacher telling students how to do the problems?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Is the teacher asking probing questions?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Is the teacher asking focusing questions?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Is the teacher asking extending questions to groups that need them?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. When a group has a question, does the teacher speak to the whole group?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Does the teacher assign competence to low-status members?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Does the teacher make explicit verbal note of positive group work behaviors?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Does the teacher ask groups to solve their own group problems?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Is the teacher holding students accountable to performing their roles?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### D. After group work

<table>
<thead>
<tr>
<th>Question</th>
<th>No</th>
<th>Yes</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Do students produce a group product to present?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Do students present answers to particular questions orally?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Do students present up in front of the class?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Does every group contribute to the summary?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Is there any kind of evaluation of the groups’ performance?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Number</td>
<td>Notes</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>--------</td>
<td>-------</td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>How many students are disengaged?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>How many students are contributing to the conversation?</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>No</th>
<th>Yes</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.</td>
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<td>4.</td>
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<td>5.</td>
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<td>6.</td>
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<td>7.</td>
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<td>8.</td>
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<td>9.</td>
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<tr>
<td>10.</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>11.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. The observation forms were adapted from Cohen, 1994a, pp. 140-141.
Appendix C

Teacher Interview Protocol
Appendix C

Teacher Interview Protocol

1. Think about what this investigation would have looked like without use of the ______ custom tool in CPMP-Tools. What advantages does the tool provide?

   [Further probe, if needed] For instance, do students use the objects on the screen to make more conjectures, debate, or justify?

2. How would you describe the nature of students’ interaction while using CPMP-Tools? What types of interactions have you observed among students? If you facilitate a whole-class discussion using CPMP-Tools, how do they interact with you or each other? How do you think students’ use of the software affects student interaction?

3. In what ways have you seen the software facilitate the students’ mathematical thinking and reasoning?

4. Do you assign homework that requires students to use CPMP-Tools? If yes, are most students able to access the software outside of class? What are the advantages to having that access?

5. Is there anything else you would like to share regarding the students’ use of CPMP-Tools?
Appendix D

Student Interview Protocol
Appendix D

Student Interview Protocol

1. When you use CPMP-Tools in class, do you usually operate the software, or does a classmate, or both? When you operate the software, how do the other students contribute to the activity? When someone else in your group is operating the software, how do you contribute to the activity?

2. How has CPMP-Tools helped you learn math? [If needed, Does it help you picture the problem better? Does it make it easier to make and test conjectures?] Describe a time when CPMP-Tools has helped you learn math.

3. How does use of the software on the computers help or hinder you in collaborating with other students?

4. Do you use CPMP-Tools outside the classroom? If yes, where? For what reason (homework, or other)? How does this ease of access help you with your homework and with learning math?

5. When you were doing number __________, how were you thinking about ______________? [Have student show on the computer what they did, and talk about their thinking]
Appendix E

Framework Development and Evolution
Appendix E

Framework Development and Evolution

One of the purposes of the two pilot studies conducted prior to this dissertation was to develop and refine the frameworks for interaction patterns and mathematical thinking that were used to analyze the data. This appendix contains the information from the pilot studies relevant to that development.

The research conducted in the first pilot study was intended to determine the nature of a collaborative environment that includes *CPMP-Tools* on laptop computers. The methodology in this study involved emergent coding such that the behaviors found were determined by forming categories from the data. Table 23 shows the percentage that each type of student behavior occurred out of the 749 verbalized and transcribed ideas given by students working in groups. The top four student behaviors exhibited by students working in groups in the presence of laptop computers were all norms of behavior that have been identified as desirable for effective groups (Cohen, 1994a, 1996). This finding suggests that, overall, even though the students were working on laptops, they were engaging in effective group work.

Table 23

<table>
<thead>
<tr>
<th>Student Behavior</th>
<th>Percent that Behavior Occurred Out of the Total Recorded Behaviors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Building on other’s ideas</td>
<td>25.6%</td>
</tr>
<tr>
<td>Asking questions</td>
<td>25.1%</td>
</tr>
<tr>
<td>Giving ideas</td>
<td>16.3%</td>
</tr>
<tr>
<td>Answering other’s questions</td>
<td>10.0%</td>
</tr>
<tr>
<td>Reader reading text</td>
<td>7.5%</td>
</tr>
<tr>
<td>Off task</td>
<td>5.3%</td>
</tr>
<tr>
<td>Making observations</td>
<td>4.7%</td>
</tr>
<tr>
<td>Justifying ideas</td>
<td>3.5%</td>
</tr>
<tr>
<td>Technical software statements</td>
<td>2.0%</td>
</tr>
</tbody>
</table>
In the second pilot study, the types of student behaviors were broken down to provide further detail. The framework from Wood, Williams, and McNeal (2006) provided that detail. However, behaviors found in the first pilot study, that were not covered by a behavior listed in the framework, were added to the framework and were used both in the second pilot study and in the dissertation study. These included *Answering Another Student’s Question, Giving New Idea/Making a Conjecture, Building on Other Students’ Ideas, Making Observations, Off Task, and Technical Software Statements*. These interaction patterns emerged in the first pilot study as students were talking with each other during group work. However, in the second pilot study, they all occurred at least once in whole-class discussion. The new code *Making Observations* emerged also due to the use of *CPMP-Tools*. This code is specifically used when students are making an observation based on something they view on the computer screen.

During the second pilot study, it became clear that the framework would also need to be modified based on the use of computer software. A new computer-related code emerged during the second pilot study—*Inquiry using the Software (iqs)*. This type of interaction is reserved for times when the students are using the software to explore an idea or test a conjecture. The following example contains two questions of inquiry (labeled with the code *iqs*) that were explored with the *Explore SSA* custom tool. The rest of the coding will be explained below. In this segment, the students are in a whole-class setting experimenting with varying lengths of BC to see what happens to the triangle(s). See Figures 26 and 27 for examples of the pictures they were viewing on the screen.
(iqs) Mr. L: Now, make me see larger than 10.5

(gi/SN) Claire: We’re going to have two triangles

(ire) Mr. L: We’re going to have two triangles, does that make sense?

(ire/C) Claire: Yes.

(iqs) Mr. L: Now Gwen, would you do me a favor, put your
pointers on C, now drag that out a little bit.

*Figure 26. Explore SSA custom tool with BC smaller than 10.5.*
Additionally, two of the codes were adapted to fit the computer software environment as opposed to other types of manipulatives. The study performed by Wood, Williams, and McNeal (2006) involved students working with cubes to help them solve the problems. Thus, two of the interaction patterns used in that study were called *Proof of Answer by Cubes* and *Resolution of Conceptual Issue with Cubes*. For the purposes of this study, those codes were relabeled *Proof of Answer by Manipulation of the Software* and *Resolution of Conceptual Issue using the Software*. The descriptions of these codes could be altered by changing the words “material (cubes)” (Wood, Williams, & McNeal, 2006, p. 254) to “software.” With only this slight modification, the codes made sense in the case of students using *CPMP-Tools*. Additionally, another graduate student also coded transcripts from the second pilot study, and after the coding was compared, any differences that appeared were resolved. This resulted in slight modifications to the wording of the coding scheme in order to be clearer about the meaning of each code.
The final change to the framework for interaction patterns occurred after the proposal defense. It was suggested that the work of De Villiers (1997, 1998, 1999) be reviewed in order to further refine the code for *Proof of Answer by Manipulation of the Software*. Proof has traditionally served the purpose of verifying the correctness of mathematical statements (De Villiers, 1997). However, proof can take on roles that are even more important than that of verification. These roles are: explanation, discovery, communication, intellectual challenge, and systemization (De Villiers, 1997, 1998, 1999). Also, in reality, verification does not only come from a formal proof. Dynamic geometry software has taken on the role of verifying mathematical conjectures in some cases. Therefore, the idea of proof in a computer environment needs to be expanded to include especially the roles of explanation and discovery (De Villiers, 1998). The idea of *proof as discovery* is equivalent to the existing code *Inquiry Using the Software* in which the students are using the software to explore an idea or test a conjecture. The idea of *proof as communication*, where proof is the verbal negotiation of meaning using acceptable explanations and arguments (De Villiers, 1998), is covered by the code *Proof of Answer by Student Explanation*. The last two—proof taking on the role of *intellectual challenge* and *systemization*—take place in advanced courses in geometry and will not be found in this study. The main impact of De Villiers work on the coding scheme is in the description of the code for *Proof of Answer by Manipulation of the Software*. The description of this code was modified to embody the idea of *proof as explanation* in which the software provides the insight into why a statement is true.
Appendix F

HSIRB Protocol Approval Documents
Date: April 19, 2007

To: Steven Ziebarth, Principal Investigator
   Kareem Fonkert, Student Investigator for dissertation

From: Amy Naigle, Ph.D., Chair

Re: HSIRB Project Number: 07-04-13

This letter will serve as confirmation that your research project entitled "Patterns of Interaction and Mathematical Thinking of Secondary Students in a Classroom Environment that Includes Use of Java-based Curriculum-embedded Software" has been approved under the expedited category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may only conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: April 19, 2008
Western Michigan University

Date: May 17, 2010

To: Steven Ziebarth, Principal Investigator
    Karen Fonkert, Student Investigator for dissertation

From: Amy Naugle, Ph.D., Chair

Re: HSIRB Project Number: 10-05-09

This letter will serve as confirmation that your research project titled "The Analysis of Patterns of Interaction and Mathematical Thinking of High School Students in Classroom Environments that Include Use of Java-based, Curriculum-embedded Software" has been approved under the exempt category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may only conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition, if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: May 17, 2011
Appendix G

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Date: March 22, 2012

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    Summerville, SC 29485
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ISBN: 0078615216
Description of material: Figures 3, pg. 77; 4, pg. 78; 11, pg. 104; 12, pg. 286; 16, pg. 287; 17, pg. 131; 24, pg. 499; 25, pg. 151 (ONLY)

Fee: WAIVED [Thesis]

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Title: Assistant Professor of Mathematics Education

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