Characterizing and Supporting Change in Algebra Students' Representational Fluency in a CAS/Paper-and-Pencil Environment

Nicole L. Fonger
*Western Michigan University, nicole.m.lanie@wmich.edu*

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CHARACTERIZING AND SUPPORTING CHANGE IN ALGEBRA
STUDENTS’ REPRESENTATIONAL FLUENCY IN A
CAS/PAPER-AND-PENCIL ENVIRONMENT

by

Nicole L. Fonger

A Dissertation
Submitted to the
Faculty of The Graduate College
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requirements for the
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Department of Mathematics
Advisor: Jon D. Davis, Ph.D.

Western Michigan University
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WE HEREBY APPROVE THE DISSERTATION SUBMITTED BY

Nicole L. Fonger

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APPROVED

Jon D. Davis, Ph.D.
Dissertation Review Committee Chair

Steven W. Ziebarth, Ph.D.
Dissertation Review Committee Member

Rose Mary Zbiek, Ph.D.
Dissertation Review Committee Member

Christine A. Browning, Ph.D.
Dissertation Review Committee Member

APPROVED

Susan A. Hagan
Dean of The Graduate College

Date August 2012
CHARACTERIZING AND SUPPORTING CHANGE IN ALGEBRA STUDENTS’ REPRESENTATIONAL FLUENCY IN A CAS/PAPER-AND-PENCIL ENVIRONMENT

Nicole L. Fonger, Ph.D.
Western Michigan University, 2012

Representational fluency (RF) includes an ability to interpret, create, move within and among, and connect tool-based representations of mathematical objects. Taken as an indicator of conceptual understanding, there is a need to better support school algebra students’ RF in learning environments that utilize both computer algebra systems (CAS) and paper-and-pencil. The purpose of this research was to:

(a) characterize change in ninth-grade algebra students’ RF in solving problems involving linear equations, and (b) determine conditions of a CAS and paper-and-pencil learning environment in which those students changed their RF.

Change in RF was measured by comparing results from initial to final semi-structured task-based interviews using a specifically designed framework based on the SOLO taxonomy. Following a design research approach, an instructional theory was used as a lens and object of analysis to determine conditions of the learning environment that supported RF. This theory was posited prior to the study, tested during a five-week collaborative teaching experiment in which a ninth-grade algebra teacher taught all lessons, and revised during ongoing and retrospective analyses.
Each of three student’s performance on linear equation solving tasks posed in the symbolic representation type was initially characterized at prestructural, unistructural, and multistructural levels of RF. Two of the three students demonstrated relational levels of RF in the final characterization based on similar tasks. This change in RF is attributed to a specifically designed instructional intervention based on an instructional theory that includes: (a) an activity structure for representation-specific tasks and techniques, (b) a learning progression that emphasizes a multi-representational approach to equivalence of expressions and solving linear equations, and (c) classroom expectations. A revised activity sequence that incorporates the Cartesian Connection earlier in the progression is proposed.

Results suggest that improving one’s RF may be connected to affect and disposition toward mathematics. Tasks and classroom discourse that were specifically designed to focus on reconciling differences between representations seemed particularly powerful. The use of a task-technique-theory framework might support research and practice efforts aimed at instructional design for learning environments that utilize a combined use of tools for doing and learning mathematics.
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Finally, I am so grateful for my family members who span several households and states. They cease to believe in me as I strive to surpass my own goals and expectations. This work is dedicated to my loving husband, research assistant, and best friend, Judson Wyman Fonger. This work would not be possible without you all.

Nicole L. Fonger
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CHAPTER I

INTRODUCTION

*If you have a problem, there are two paths open to you: either you solve the problem, or you change your view.*

(Chinese Proverb, as cited in Kutzler, 2010)

Each mathematical representation provides a glimpse into a version or phase of a particular mathematical object. When multiple representations (MR) are taken together as complementary aspects of a given algebraic object, a more complete portrayal of the structure of this object can be understood (Kaput, 1989). Assuming a representational perspective on mathematics, activities of representing and connecting representations are key to teaching and learning mathematics with understanding (Hiebert & Carpenter, 1992). From a representational lens, “A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes” (National Research Council [NRC], 2001, p. 119). When conceptual understanding of mathematics is valued, the importance of fluency with MR is made evident.

The goal of developing connections between knowledge structures, and between representations more specifically, permeates current values in school mathematics (e.g., National Council of Teachers of Mathematics [NCTM], 2000; NRC, 2001). Recent curricular reform efforts emphasizing learning for understanding (NCTM, 1989, 2000)
have catalyzed a broadened perspective of algebra in which the verbal, graphic, and tabular counterparts to symbolic representations are central to problem solving and meaning making (Kieran, 2007).

With heightened attention to MR in standards and curriculum, a great deal of research in the domain of algebra has focused on graphic representations of functions (Romberg, Fennema, and Carpenter, 1993) and how graphic representations offer a necessary complement to symbolic representations in coming to understand mathematics in a deeper way (e.g., Knuth, 2000). Moreover, helping to make MR of algebraic objects more accessible, mathematics technology for handheld and computer platforms from graphing calculators to computer algebra systems (CAS) assume a prominent role in multi-representational approaches to school algebra (Chazan & Yerushalmy, 2003; Heid & Blume, 2008; Heid, Choate, Sheets, Zbiek, Schoen, & Teague, 1995).

On the role of technology, NCTM (2000) purports that “students can learn mathematics more deeply with the appropriate use of technology” (p. 25). Housing a variety of representations including symbolic, graphic, numeric, and verbal (written) descriptions, CAS can be considered “representational toolkits” for solving problems and doing mathematics, providing “a means of moving between representations” (Dick & Edwards, 2008, p. 266). Mathematics technology introduces specific representational infrastructures that relate to but are not exactly the same as those available in non-technological environments (Kaput, 1989, 1992; NCTM, 2000). What is valued and expected of learners of mathematics has been ever changed by recognition of the power of technology-based representations. In particular, students’ proficiency with the construction of mathematical representations includes their technical skill in operating
electronic and dynamic representational toolkits. It becomes clear that the representational proficiencies and fluencies expected of students change depending on the available tools.

Students’ representational activities and abilities are arguably significant indicators of students’ proficiency in mathematics. Pea (1987) argued for the now widely acknowledged view that “competency in mathematical problem solving depends partly on one's ability to think in terms of different representational systems during the problem-solving process” (p. 109). Described in terms of students’ flexibility (Greer, 2009; Heinze, Star & Verschaffel, 2009), versatility and adaptability (Huntley, Marcus, Kahan, & Miller, 2007; Sfard & Linchevski, 1994), and representational fluency (Bieda & Nathan, 2009; Sandoval, Bell, Coleman, Enyedy, & Suthers, 2000; Suh & Moyer, 2007; Zbiek, Heid, Blume, & Dick, 2007), competent problem solvers are expected to know what representations have to offer a particular situation, how to construct representations that are needed, and how to choose between appropriate representations.

Representational fluency (RF) has been defined in various ways (e.g., Heid & Blume, 2008; Sandoval et al., 2000) and is integrally connected to affordances of technology-based representations (Heid & Blume, 2008; Zbiek et al., 2007). The bulk of research that has been conducted to better understand algebra students’ RF has employed researcher designed instrumentation in interviews or assessments to measure secondary and post-secondary students’ abilities to translate between MR, coordinate information across MR, or otherwise connect tool-based representations (Adu-Gyamfı, 2007; Bieda & Nathan, 2009; Huntley & Davis, 2008; Huntley et al., 2007; Moschkovich, Schoenfeld, & Arcavi, 1993; Nathan, Alibali, Masarik, Stephens, & Koedinger, 2010; Nathan & Kim,
Some research has been reported on early algebra and RF in elementary grades (e.g., Carraher & Schliemann, 2007; Suh & Moyer, 2007). Few contemporary studies that investigate students’ RF disregard the role of technology (e.g., Filloy, Rojano, & Solares, 2010; Knuth, 2000).

Algebra students’ difficulties with translating and connecting MR have been well documented (e.g., Dreyfus & Eisenburg, 1996; Moschkovich, Arcavi, & Schoenfeld, 1993). In particular, researchers have documented that secondary and post-secondary students’ have persistent difficulties with translating among MR with the availability of graphing calculator technology (Bieda & Nathan, 2009; Herman, 2007; Spitzer, 2008) and without technology (Filloy, Rojano, & Solares, 2010; Knuth, 2000). On the other hand, research has documented that school algebra students can use MR in solving tasks and are successful in translating between MR with the use of technology (e.g., Ruthven, 1990).

Substantial research has accumulated on students’ problem solving performances on specifically crafted tasks that begin in a variety of initial representations that might require or allow access to technology. Through interview methodology, researchers have analyzed students’ flexible use of MR and have built models or frameworks that have contributed to our ability to categorize students’ RF (and disfluency) (e.g., Izsák, 2003; Bieda & Nathan, 2009). Others have relied on data collection methods that involve researcher designed written instrumentation or specific tasks (e.g., Nathan et al., 2010; Keller & Hirsch, 1998; Ruthven 1990). This work is important because it highlights specific areas in which students’ cognitions can be developed via richer connections between MR, ultimately bolstering their conceptual understanding of mathematics.
In sum, we know a great deal about students’ successes and difficulties in using MR to solve problems in both technology and non-technology settings. However, in many research reports that tie students’ understanding to instructional situations that emphasized MR, the acts of teaching are treated as a black box (Adu-Gyamfi, 2002). In other words, we lack sufficient information on particular facets of technology-rich classroom practice that may help to determine how students’ understanding of the relationships among MR can be strengthened (Heinze, Star, & Verschaffel, 2009).

Gaps in the Related Research

Determining means of support for students’ development of RF with CAS technology is a targeted area of interest for both researchers and practitioners alike (Arbaugh, Herbel-Eisenmann, Ramirez, Knuth, Kranendonk, & Quander, 2010; Heinze, Star, & Verschaffel, 2009; Zbiek, 2003). Central to a research-guiding question of NCTM’s *Linking Research and Practice* report (Arbaugh et al., 2010) is a general inquiry into “how technology use relates to students developing mathematical ways of thinking (e.g., proof and reasoning) and mathematical understandings that apply across many mathematical topics (e.g., symbol sense and use of representations)” (p. 20). Of particular interest to the present study is the practitioner question, “How do we help students to be facile in moving among representations, including those created by technology?” (p. 21). Facility in translating between representations is a key building block of students developing RF.

An overwhelming majority of the research on RF that utilizes technology involves non-CAS graphing calculator technology (e.g., Bieda & Nathan; Herman, 2007; Huntley
& Davis, 2008; Huntley et al., 2007). With the aforementioned practitioner question as a basic building block, exploring the means of support for students’ development of RF in combined CAS and paper-and-pencil environments is addressing a significant gap in the literature. Research on teaching and learning school algebra with technology-based representations has heretofore documented the learning environments in which students develop strengthened conceptions of the relationships between representations, and the ability to interpret and employ representations in doing mathematics (e.g., Heid & Blume, 2008).

Within this niche, Heid (2010) and Kieran and Yerushalmy (2004) strongly recommend the development of collaborative relationships between researchers and practitioners with respect to effective use of CAS in classrooms. In a similar vein, Kieran (2007) recommended teaching experiment methodology for bridging the gap between research on teaching and learning that could be specific to the design of instruction and instructional tasks that incorporate the use of mathematics technology for the teaching and learning of algebra. Kieran and Yerushalmy (2004) go further in highlighting the need for such research with average or lower ability algebra students.

Research that targets students’ multi-representational activity with CAS and/or graphing calculators has typically targeted higher ability students (e.g., Huntley & Davis, 2008; Knuth, 2000). In their extensive review of MR and technology, Kieran and Yerushalmy (2004) recount that “Research on the long-term learning of algebra with technology shows that it is not only high-performing students who can excel with the aid of technology; students who tend to be low-performers in algebra do better when they can move part of the responsibility to the tool (e.g., if they solve equations using an
intersection on a graph) or when they can work on manipulations in context” (p. 143). Kieran and Yerushalmy (2004) also identify gaps in the literature on explaining how students use technology in coming to understand algebra. When focused on CAS technology in particular, relatively few classroom-based research studies have been conducted with this particular mathematics technology and population of average ability algebra students (Pierce, Stacey, Wander, & Ball, 2011).

Research Questions and Purpose

This study employed methodology consistent with a design research approach (Gravemeijer & Cobb, 2006) to investigate the following research questions:

1. How does students’ RF in solving problems involving linear equations change as a result of learning how to solve linear equation problems within a CAS and paper-and-pencil environment?

2. Under what conditions does a group of ninth-grade algebra students change their RF in solving problems involving linear equations within a CAS and paper-and-pencil environment?

The theoretical orientation that I assume on teaching and learning mathematics is consistent with Cobb’s emergent perspective (cf. Cobb & Yackel, 1996). From this lens, both psychological aspects of individual students’ learning, and social aspects of the situation or community of that individual are of significant concern.

Situated within the context of a teaching experiment conducted in collaboration with a classroom teacher (Cobb, 2000), pre-test data was used to purposefully select a stratified sample of three students within a group of ninth-grade algebra students of
which to study in more depth. To answer the first research questions, case studies (Stake, 1995) of these three select students incorporate performance data on pre- and post-assessments, initial and final semi-structured task-based interviews, and artifact collection in order to document students’ development of RF throughout the duration of the teaching experiment.

The second research question was addressed though analyses that assumed a social lens on classroom practices and discourse patterns documented during instruction, in addition to data from regular debriefing sessions between the researcher and classroom teacher. The research methodology targeted at answering the second question included the development, experimentation, and refinement of an instructional theory that targeted students’ RF in the mathematical context of linear expressions, equivalence of expressions, linear equations, and solving linear equations.

Results of this study are directed at addressing the two main purposes of this research. First, three students’ RF was characterized according to an analytic framework for RF that was derived out of pilot study research (Fonger, 2011). Findings are suggestive of two students’ change in RF from prestructural, unistructural, and multistructural levels to relational levels with a greater number of unistructural and multistructural levels in their final characterization. One student persisted at the prestructural level of RF, not demonstrating change in RF over the course of the teaching experiment. Second, conditions of the learning environment are explained through several components of an instructional theory. Three aspects of this theory to highlight here are: (a) an activity structure that describes the nature of representation-specific activities in the classroom, (b) elements of a learning progression that provided the
theoretical foundation for the design of instructional tasks based on specific CAS and P&P techniques, and (c) classroom expectations that were communicated as normative aspects of the learning environment.

Overview

In the next chapter, a review of relevant literature covers both psychological and social perspectives on using tools to study change in RF. The definition of RF is further elaborated, research results on students’ mathematical activity and use of tool-based representations in doing and communicating mathematics are explored, and classroom mathematical activity is examined from a representational lens, including attention to curriculum and instruction.

In light of the research purpose and research questions, the third chapter details the research design and methodology that were employed. Then the findings are articulated with respect to both the set of case studies and the teaching experiment. In the final chapter a discussion of the study includes attention to the limitations and ideas for future research.
CHAPTER II

LITERATURE REVIEW

Recent curricular reform efforts emphasizing learning for understanding (NCTM, 1989, 2000) have catalyzed a broadened perspective of algebra in which the verbal, graphic, and numeric counterparts to symbolic representations are central to problem solving and meaning making (Kieran, 2007). Helping to make multiple representations (MR) of algebraic objects more accessible, mathematics technology for handheld and computer platforms from a graphing calculator to a computer algebra system (CAS) assume a prominent role in multi-representational approaches to school algebra (Heid & Blume, 2008; Heid, Choate, Sheets, & Zbiek, 1995; Kaput, 1992; Zbiek, Heid, Blume & Dick, 2007) and their use in this context is supported by research (Chazan & Yerushalmy, 2003; Kieran, 2007).

Algebra, conceptualized broadly as a representational system, is a topic that was historically taught at the collegiate level (Osborne & Crosswhite, 1970), and is now taught as a high school course, with the exception of the fact that approximately 30% of students in the United States take algebra in the eighth-grade (Perie, Moran, & Luktas, 2005 as cited in Walston & McCarroll, 2010). Reflected in the most recent policy adopted in 45 states and territories of the United States, the recommendations of the Common Core State Standards Initiative (CCSSI, 2010) are not too dissimilar from the aforementioned recommendations of other national groups (yet take an arguably more
conservative stance) in terms of what is valued in school algebra with respect to MR and technology. Specifically, the CCSS for high school mathematics include relatively few standards that make explicit mention of mathematics technology—three (of 27) in algebra, four (of 28) in functions, three (of 43) in geometry, and two (of 31) in statistics and probability.

Despite the paucity of explicit reference to the role of technology within the content standards, the mathematical practices are clearer in articulating a role for technology. The Common Core State Standards call for students to “Use appropriate tools strategically,” and to gain skill in converting between symbolic, graphic, numeric, and verbal representations of equations, inequalities, expressions, and functions in coming to understand high school algebra. The role of technology in learning mathematics is also elaborated as part of the mathematical practice of “Make Sense of Problems and Persevere in Solving Them”:

Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends.

In light of the educational context for this research, one of the goals of the ensuing review of literature is to synthesize the research base on what is known with respect to students’ understanding of MR and the connections between them, particularly when students have access to and learn mathematics with technology. The construct of representational fluency (RF), defined next, is used to organize the literature review around three main sections. First, the definitions for representations and RF are given to establish the meaning of this terminology for purposes of this study. The latter two
sections coincide with the research questions that guided this study—both the psychological and situated aspects of teaching and learning are addressed. Extant literature on students’ mathematical activity as directly or indirectly related to RF is reviewed followed by a summary of literature on RF and classroom practices, including curriculum and instruction.

The first section is heavily oriented toward theoretical perspectives thus little empirical research is cited therein. The importance of this extended discussion lies in the fact that these perspectives directly informed the design of the study. Moreover, both the theoretical orientations and the empirical results discussed in the latter two sections are directly tied to the instructional theory that was conjectured, tested, and revised as part of the teaching experiment.

Representations and Representational Fluency

An introduction of the constructs of representations and RF unfolds through a theoretical discussion regarding issues of representation. Various forms and manifestations of representations are discussed including internal representations, external representations, MR, and technology-based representations.

Representation Defined

In a rudimentary description, a representation is "something which stands for something else from someone's point of view" (Morgan, Mariotti, & Maffei, 2009, p. 242). Goldin and Kaput (1996) identify a useful theme that is paramount in addressing the construct of representation, that of the relationship between thought and language, the
signified and the signifier, respectively. Connecting the proposed definition to Goldin and Kaput’s perspective, the “something” is the language or signifier, and the “something else” is the mental construct, thought, or signified. Highlighting “someone’s point of view” in defining representation is particularly significant in the distinction between internal and external representations.

**Internal and External Representations**

De Corte, Greer and Verschaffel (1996) posit the popular view that representations involve “internal processes as well as external embodiments” (p. 535). An internal representation is personal to an individual and derives from one’s cognition and experiences, where as an external representation is social in that it has the potential to be shared or communicated via various media (e.g., printed in textbooks, written by hand, conveyed using technology). Two somewhat conflicting perspectives on the relationship between internal and external representations appear in the literature.

Attention to both internal and external representations is a theme that permeates research on student knowledge from the cognitive line of research (e.g., Goldin & Kaput, 1996). For example, drawing broadly from insights gained from cognitive science research (e.g., Gardner, 1985), Hiebert and Carpenter (1992) assert the perspective that a student’s internal representations, part of their web of internal mental structures, can be accessed through their use of external representations (p. 65-66). This is a feasible perspective for it allows one to theoretically have access to aspects of students’ internal

---

1 Thanks to a committee member’s comment, it is noted that the process-object difference here is different than the internal-external representation distinction being made.
representations and cognition. Offering a counter perspective, Sandoval, Bell, Coleman, Enyedy, and Suthers (2000) contend that, “It may or may not be the case that [students’ epistemic] representations correspond to representations or inscriptions common to scientific practice” (p. 6). So it should not necessarily be assumed that students’ internal representations are the same as external representations used in conventional practice.

In light of these theoretical orientations, it is assumed that the creation or selection of external representations, made by hand or with technology (i.e., tool-based) is not sufficient in and of itself to infer deep insight into a student’s cognition. Instead, it is the interpretation and communication about these representations that can lead one to infer about students’ thinking and understanding of representations. For instance, consistent with the position of Sandoval and colleagues (2000), external representations in and of themselves do not carry particular meanings; such meanings arise from the context of their use, such as in language (cf. Wittgenstein, 1958/1994 as cited in Roth & McGinn, 1998).

In regards to representations of algebraic objects in particular, Sfard and Linchevski (1994) posit that the same representation of the same object can be seen differently depending on the problem to be solved and the flexibility of the learner to perceive and apply sometimes both process and object conceptions of an idea. For the current study, it is assumed that a representation does not hold its own meaning, but instead, that meaning is constructed by each individual and is situated with respect to the context and conventions of the community. To broaden the perspective beyond various meanings of single representations, the ensuing discussion considers the intricacies of multiple representations and representation types of the same mathematical object.
Multiple Representations

In mathematics, several different representations may serve to signify a single mathematical object. Multiple external representations of a mathematical object bring the theory-laden structure of this object into the tangible world of existence for scrutiny, making it easier to communicate and document its properties and to characterize and utilize its form. When MR are taken together simultaneously a more complete picture of the object can be understood (e.g., Kaput, 1989). In algebra in particular, prominent representation types of linear equations include graphic, numeric, symbolic and verbal representations.

In promoting the use of MR in the study of algebra in secondary school, Friedlander and Tabach (2001) outline several advantages and disadvantages to each verbal, numeric, graphic, and algebraic (symbolic) representation types. For example, numeric representations are often an intuitive and natural entry point into learning algebra but by their discrete nature, when taken alone they may not convey the entire problem situation well enough or be useful as a tool to generalize or successfully solve all types of tasks that require exact solutions. On the other hand, symbolic representations are best for generalizing patterns and relationships in a concise and accurate way and may be necessary to use in proving and justifying generalizations but when used in isolation of other representations often obscure access to a meaningful understanding of the object they represent making it difficult to interpret. Still yet, graphic representations are best as a visual tool for conveying relationships between real-valued functions or patterns.
however only portions of their entire domain and range can be viewed at once, and like numeric representations, may not allow for exact solutions.

In sum, each representation of a mathematical entity only tells part of the story of the meaning of the object in question; certain nuances of a linear pattern or relationship are best understood when MR are taken together. While representations give us access to external embodiments of mathematical ideas, the existence of MR is not sufficient for learning. Instead, the interpretation of and actions taken on representations are needed. Essentially, the individual must engage in interactions with the representations. For example, connections between external representations can be taken as indicators for cognitive connections (Hiebert & Carpenter, 1992). Another type of interaction on representations is reflection (cf. Simon, Tzur, Heinz, & Kinsel, 2004).

Tool-Based Representations

The notion of a “tool” is commonly traced to Vygotsky or others who have built on Vygotsky’s work such as Pea. According to Pea (1987), a cognitive technology is a tool of the intellect provided by the culture. More specifically, "A cognitive technology is any medium that helps transcend the limitations of the mind" and that “make external the intermediate products of thinking” (p. 91). In the present study, the primary cognitive technologies or tools of focus are paper-and-pencil, or P&P for short, and CAS. Significant to this study, both P&P and CAS provide means to create external representations. Other tools such as hands-on manipulatives (e.g., blocks) will not be prominent aspects of the learning experiences of the targeted population of students, thus will not be considered as primary to this discussion.
The choice to focus on the use of P&P and CAS is multifaceted. First, the use of P&P as tools is still valued as an appropriate means to do and communicate about mathematics (e.g., CCSSI, 2010). Second, CAS afford capabilities to create symbolic, graphic, numeric, geometric, and verbal (written) representation types, and can be considered “representational toolkits” for solving problems and doing mathematics, providing “a means of moving between representations” (Dick & Edwards, 2008, p. 266). CAS technology extends graphing calculator technology in one important way: the ability to manipulate symbolic representations and display results in both exact and approximate forms. Contemporary CAS are heralded for their dynamic capabilities in which MR can be viewed simultaneously and manipulated with clicking and dragging features so that changes in one representation (e.g., a graph of a function) are reflected in another (e.g., a function table). Thus CAS provide access to all representation types significant to an algebra student’s experiences, and facilitate movement between representation types.

A goal of using cognitive technologies can be to capture the processes of students' mathematical thinking (Pea, 1987), so when multiple tools (e.g., CAS and P&P) are being used in students’ mathematical experiences, different facets of students’ mathematical thinking may be inferable from students’ use and interpretation of the external representations they create. Beyond the creation of representations, the use of CAS require interpretation of the results, especially in cases where the forms created by CAS are unexpected (cf. Zbiek et al., 2007, p. 1195). Thus beyond the dynamic linking capabilities of CAS, the use of CAS necessarily involves a certain level of interpretation. We will see in the next section how this particular aspect of tool use contributes an important aspect of one’s RF.
To summarize, MR offer complementary perspectives of a mathematical object that can help to reveal its structure (Kaput, 1989). Fluency in using and developing connections between representations is an important component of conceptual understanding (NRC, 2001). National organizations for mathematics teachers and educators value MR, and the use of technology such as CAS in school mathematics (e.g., NCTM, 2000), which is becoming a prevalent representational toolkit and is incorporated in some contemporary high school textbooks (Davis & Fonger, 2010). To better support students’ conceptual understanding of algebra in the information age, we need to come to a better understanding of how students interpret the links between MR using CAS as a representational toolkit (Arbaugh et al., 2010). The construct of RF, defined next, is one avenue to approaching this research problem in the current educational climate.

**Representational Fluency Defined**

Students’ representational activities and abilities are arguably significant indicators of students’ proficiency in mathematics (e.g., NRC, 2001). The call for a multi-representational approach to algebra necessitates common language to describe mathematical practices associated with one’s ability to represent mathematical ideas. Albeit inconsistently defined, one construct that is beginning to permeate the literature related to these aims is called RF. In brief, this construct conveys the sense of facile interaction with representations. The versatility of this construct is made evident in the final two sections of the literature review on students’ mathematical activity and cognition, and classroom practices including curriculum and instruction.
Comparison of Definitions

Researchers in the domains of mathematics and science education give various, yet related, definitions of RF (e.g., Bieda & Nathan, 2009; Heid & Blume, 2008; Huntley & Davis, 2008; Nathan, et al., 2010; Nathan & Kim, 2007; Sandoval et al., 2000; Suh & Moyer, 2007; Zbiek, et al., 2007). Some researchers use the words RF without providing a clear definition (e.g., Spitzer, 2008), others use the words representational flexibility and adaptability (e.g., Aceuedo Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009; Heinze, Star, & Verschaffel, 2009; Yerushalmy, 2006), and representational versatility (Hong and Thomas, 2002). Others yet discuss ideas that are related to the construct of RF without actually giving it that name (e.g., Adu-Gyamfi, 2007; Knuth, 2000).

An advantage to using the word fluency (over others such as versatility) is that it conveys both aspects of skill and understanding. Without regard to a specific discipline, one’s fluency denotes a quality or condition of being fluent—an ability to communicate with ease and accuracy. Admittedly, this word is appropriate to the context of MR in mathematics, for it necessitates accuracy in expression and fluidity in movement between and across representations. Consider Table 2.1, which includes a sample of definitions of RF. Listed in chronological order by publication date, these definitions were chosen for inclusion based on the criteria that they are commonly used and referenced in the literature, and/or the definition itself builds on related literature and constructs. The definition for RF adopted for the present study is listed last (row 6 of Table 2.1).
Table 2.1
Definitions of Representational Fluency (RF)

<table>
<thead>
<tr>
<th>Definitions</th>
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<tbody>
<tr>
<td>RF is defined as the ability to “interpret and construct various disciplinary representations, and to be able to move between representations appropriately. This includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims. This also includes an ability to link [MR] in meaningful ways.” (Sandoval et al., 2000, p. 6).</td>
</tr>
<tr>
<td>Suh and Moyer (2007) draw on research by Cramer (2003) who describes RF in terms of a model types of representation systems (real life experience, manipulative models, pictures or diagrams, spoken symbols, and written symbols) presented by Lesh, Landau, and Hamilton (1983): “The model suggests that the development of deep understanding of mathematical ideas requires experience in different modes, and experience making connections between and within these modes of representation. A translation requires a reinterpretation of an idea from one mode of representation to another” (p. 1).</td>
</tr>
<tr>
<td>Zbiek, Heid, Blume and Dick (2007) posit that, “[RF] includes the ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations” (p. 1192).</td>
</tr>
<tr>
<td>Heid and Blume (2008) define RF as “a reference to a user’s representation-related abilities, including the ability to construct various external representations and the ability to interpret the features of one representation in the context of another representation of the same mathematical entity or in the context of the real-world situation being represented. [RF] also refers to a user’s knowledge of particular representations that can be employed for selected illustrations or explanations, to a user’s facility in using the representations to justify claims, and to a user’s ability to link [MR] meaningfully. Finally, [RF] includes not only the ability to translate across representations, but also the ability to extract meaning about a mathematical entity from the coordination of and generalization of results from a variety of representations” (p. 68).</td>
</tr>
<tr>
<td>&quot;[RF], the ability to work within and translate among representations, is central to the enterprise of mathematical activity and knowledge construction&quot; (Bieda &amp; Nathan, 2009, p. 637)</td>
</tr>
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</table>

In light of the definitions shown in Table 2.1, the polysemic nature of RF could be viewed as a limitation of this construct. For instance, when multiple definitions of a construct are assumed, it makes it more difficult to generalize findings across various studies. However, despite the various nuances in how RF is defined and studied, a cross cutting characterization of all definitions is that students have certain abilities (e.g., skills, understandings) both within one type of representation and between MR. For the purpose of this study, RF will be measured by students’ abilities to create, work within, move
between, interpret, and connect tool-based representations, and to use representations to justify other claims and make generalizations.

The constructs of translations, connections, and coordinations are foundational to articulating the meaning of these abilities. The extended discussion of these constructs is seen as a purposeful way to convey the choice of definition for the present study, but also to inform the reader’s lens on interpreting empirical results that are posited to be related to the construct of RF. In other words, despite differences in definitions and meaning of related constructs, empirical findings presented in subsequent sections are not judged to be incommensurable.

Translations and Transpositions

The constructs of transposition and translation are owed to Janvier (1987a). He does not explicitly address transpositions in his proposed theoretical model, yet implicitly defines transpositions as processes to and from the same mode of representation (e.g., a graph). For this study, a transposition is defined as the creation and interpretation of MR within one representation type. On the other hand, Janvier (1987b) defines a translation process as a psychological conversion made from one type of representation to another. For this study, a translation is defined as the creation of and interpretation of the meaning of a target representation with respect to a source representation of a different type. Others have referred to translation processes between representational registers (or systems) as a process of conversion (Duval, 2006 as cited in Duncan, 2010). Key to both of these constructs is the change in form of the representation and the psychological processes of changing perspective.
In the literature, authors assume different degrees of transpositions and translations, especially with regards to the inclusion of both creation and interpretation. In particular, Sandoval et al. (2000) define RF in a way such that a student with RF engages in the activity of creating representations, moving between representations, and interpreting representations, and is able to explain these processes. On the other hand, like Bieda and Nathan (2009), Nathan et al. (2010) implicitly use the construct of transposition in the sense of a students’ ability to problem solve using a given representation type, without reference to the students’ ability to create that or another representation of that type. Moreover, Nathan and colleagues (2010) studied students’ ability to translate across representations through tasks that had them generate a new specified representation from a given representation. While this does involve a constructing activity, it does not involve an interpretation or explanation of that process, aspects of how Janvier defines a translation (and how Sandoval et al. [2000]) define RF).

Both the creative and interpretive activity seem to be important aspects of RF that should not be lost in the definition. For instance, when one creates tool-based representations with a representational toolkit such as a CAS, the creation of a representation may stop at the click of a few buttons. It is the interpretation of that activity that sheds light on the meaning of the mathematical object in question. On the other hand, absent of the creation of a tool-based representation there is potential for the user to be less active in doing mathematics. From the aforementioned distinctions between definitions, we learn that the constructs of transpositions and translations are common across definitions of RF, but are used in different ways. The definitions chosen for the present study were designed specifically to address the active role of the user. We
turn next to other important aspects of RF, such as those of making connections between MR, and coordinating information across representations.

**Connections and Coordinations**

The definitions of RF proposed by Heid and Blume (2008), Sandoval et al. (2000), and Zbiek et al. (2007) go beyond the constructs of transpositions and translations to include connections and coordinations. For example, Heid and Blume (2008) are explicit in the role of coordinating information: “[RF] includes not only the ability to translate across representations, but also the ability to extract meaning about a mathematical entity from the coordination of and generalization of results from a variety of representations” (p. 68). The language of “linking” MR is evident in the definition proposed by Sandoval et al. (2000), while Suh and Moyer (2007) use the language of “connecting” multiple representation types (or registers).

This study follows Adu-Gyamfi’s (2007) definition of making a connection between representations, which subsumes the construct of coordinations. Adu-Gyamfi defined that for a student to make a connection they must verbalize or write that they are *coordinating* information (i.e., invariant features of the object in question as evident in mathematically equivalent representations) in their interpretation of working within or among MR. In short, a *connection* between multiple tool-based representations is defined as giving a correct interpretation of an invariant feature of a mathematical object across MR or types. In this context, it should be noted that although CAS environments can act as representational toolkits, “students do not necessarily connect representations when operating in a multiple representation environment” (Heid & Blume, 2008, p. 98). Thus
availability of MR does not imply that students seek out and understand connections between representations.

The focus in the chosen definition for RF on connections over other practices such as generalizing and justifying was not meant to be exclusive of these important mathematical practices. Instead, justifying and generalizing are conceptualized as goals of mathematical activity that may guide one’s interpretation of their use of representations, or their articulation of particular invariant features across representation types. Although tacit in the statement of the definition, these practices are evident in the analysis of students’ change in RF as discussed in Chapter 4.

In sum, the key components of RF can be described by the constructs of transpositions, translations, and connections. In light of this discussion, the definition adopted for the present was chosen to convey the common theme that RF involves both the creation of and interpretation of multiple tool-based representations. These aspects are also evident in the framework for RF that was used to characterize students’ change in RF, and the instructional theory that was used as a lens to study the conditions of the CAS and P&P learning environment. Both of these are introduced in Chapter 3 as they were used as the primary lenses for data analysis.

Utility as a Construct

Zbiek et al. (2007) discuss the construct of RF as a tool for capturing change in curriculum, as a lens for examining mathematical activity in classrooms, and as a means to understanding students’ mathematical thinking. In Heid and Blume’s (2008) characterization of RF, they take the position that it is a construct that helps to
characterize the student-tool relationship, and in relation to students’ problem solving activity in technology-intensive algebra in particular, they take the position that RF impacts students’ strategies. Based on a review of relevant literature I have found that the notion of connections is key to the construct of RF (Sandoval et al., 2000; Suh & Moyer, 2007), especially in regards to using CAS as a representational toolkit.

The ensuing review of literature will be organized based on the multi-faceted nature of the construct of RF, including attention to the notion of connections. The major sections that organize this review include: RF and students’ mathematical activity and cognition, and RF and mathematics curriculum and instruction in classrooms. The user-tool relationship was not assumed as a major component to the way in which RF was characterized in the present study, yet literature on that topic is subsumed in the discussion of students’ mathematical activity because of the nature of the study.

RF and Students’ Mathematical Activity and Cognition

Students’ use of external representations can be used to characterize their mathematical activity, and also taken as indicators of their cognition. Results from a comparative study of college calculus students suggest that students prefer symbolic representations of functions when tasks are posed in a formal, non-contextualized manner and on the other hand, students prefer numeric representation types when tasks are posed with more informal and intuitive language (Keller & Hirsch, 1998). In light of these findings, Keller and Hirsch caution that care should be taken in the task design of problems so that the amount and nature of information that students need to wade through or interpret is carefully considered.
The importance of task design and the definition of RF are used to organize the following review in which researchers employed the use of specifically designed tasks to study students’ RF and mathematical activity. A section on solving equations posed in the symbolic representation type is followed by a consideration of studies focused on students’ use of verbal, graphic, and numeric representation types to solve problems. Students’ understandings of connections between symbolic and graphic representation types are considered third. At the close of this section, remarks are made to summarize and extend the literature with some connections to curriculum and instruction.

**Solving Equations Posed in the Symbolic Representation Type**

Herman (2007) studied the effect of a technology-enriched multi-representational instructional approach on college algebra students’ initial solution strategy choices and ability to employ multiple problem solving strategies to successfully solve algebra problems involving exponential, logarithmic, and polynomial functions. Equipped with paper, pencil, graph paper, and TI-83 calculators, 38 students completed pretest and posttest tasks that were presented in written form with a symbolic function rule. Herman predicted that given the symbolic function rule or equation such as

\[ z^4 + 2z^3 - 28z - 31 = 2z^3 - 44, \]

students might input the expressions related by the equal sign into the graphing calculator then solve the problem using graphs or tables (p. 50).

From the posttest data, complemented by verbal interpretations of students’ responses during semi-structured interviews with seven individuals, the predominant initial solution strategy (or choice of representation) was symbolic. Overall, the 38 students who completed a total of 6 tasks (with a total of 223 responses and 5 non-
attempts) used a symbolic representation 68.2% of the time, a graphic representation 31.8% of the time, and a numeric (table) representation 0% of the time. Use of the graphic representation type was reportedly done so to check solutions found using the symbolic representation type.

Herman (2007) attributed the reliance on symbolic approaches to students’ classroom experiences, instructor preferences, and the fact that students found algebraic approaches (symbolic representations) “more mathematical” (p. 36). The fact that no students used a tabular solution approach (numeric representation) “could have been a direct result of little use of tables demonstrated by instructors” (p. 38). Hence, the choice of initial representations in solving problems was seemingly an artifact of students’ knowledge of how to use technology, and instructor preferences for and exclusion of particular representational approaches.

Huntley and colleagues (Huntley & Davis, 2007; Huntley, Marcus, Kahan, & Miller, 2007) endeavored on a research project that aimed to move beyond previous studies (e.g., Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000) and gather more details on students’ problem solving abilities, use of MR, and the role of graphing calculator technology in these activities. A total of 44 pairs of higher achieving third-year high school students across several school sites, using either an integrated textbook such as IMP or CPMP (n = 30) or a traditional course sequence textbook such as UCSMP Algebra 2 (n = 14) participated in this study.

Huntley and Davis (2008) found that pairs of high achieving third-year high school students who were posed two TIMSS tasks in symbolic form (one a linear equality and another an equation involving a square root) used symbolic strategies as their first
approach in solving, yet many resorted to graphic or numeric table representations as resources to overcome barriers in their symbolic manipulations. Huntley and Davis speculated that students who are facile at translating between MR may be better at self-checking their strategies, needing less teacher support for "detecting and correcting their errors" (p. 387).

On the role of technology, Huntley and Davis (2008) reported that relatively few student pairs used graphing calculators in solving non-routine tasks posed in symbolic forms. Like Herman (2007), based on the nature of the tasks and students’ curriculum and instructional experiences the authors had expected more students to use graphing calculators in their problem solving activity. While it is indeed somewhat surprising that more students didn't use graphing calculators, Huntley and Davis gave insufficient details on the nature of the classroom practices that these students experienced in their third-year mathematics course. In order to better understand students' decisions about the use of MR and the choice to use P&P or graphing calculator tools, we need to know more about students' instructional experiences. Huntley and Davis' (2008) claim that students in this study learned from either a reform-oriented textbook or a less reform-oriented textbook (such as those referenced above) that was "being implemented according to the authors' intentions" (p. 382) was not well supported.

Huntley, Markus, Kahan, and Miller (2007) reported on these same students’ performance on three equation-solving tasks presented in symbolic form, all linear equations of the form $ax + b = cx + d$. One of the major findings of the research by Huntley and colleagues is that despite the fact that the upper level high school students had experienced enacted curriculum that emphasized technological approaches to solving
problems via MR, these students struggled in solving non-routine linear equations successfully. More specifically, linear equations with one solution posed the least amount of difficulties for student pairs, but students were less successful when solving linear equations with either no solutions or infinitely many solutions. Overall, students showed preference for using by hand symbolic techniques in each of the task situations, and often times students did not use MR such as a graphic representation type until the researcher motivated this activity with carefully crafted interview probes. Prompts such as “Could you solve these in another way?” and “What would be the relationship between the graphs of each side of the equation?” (Huntley et al., 2007, p. 120) were used.

Huntley and colleagues point out that another significant part of these students’ activity was with respect to the fact that they did not use graphing calculators in their initial approach to the problem, seldom using them as an alternative approach, and most often than not required probing from the interviewer to use the calculator at all. With the ease in which MR are available through the use of graphing calculator technology, these students were not found to capitalize on the strength of this tool to aid in their problem solving.

For each of Huntley and Davis (2008), Huntley et al. (2007), and Herman (2007), students were presented equation-solving tasks stated with a symbolic representation type, and students had graphing calculators available to aid in their problem solving. Students were persistent in their choice to first perform symbolic transpositions to attempt to solve equations, with only some students translating to the use of the graphic representation type. Also, the use of the graphing calculator to create representations (e.g., to move from a symbolic equation to a graph or table) was not prominent in
students’ mathematical activity, even though it was speculated as a means to support students’ mathematical activity. These findings suggest advanced algebra and college algebra students demonstrate limited RF in solving non-routine equations posed in the symbolic representation type, despite access to a graphing calculator and instructional experiences that support multi-representational, technology-based approaches.

*Strategies Involving Verbal, Graphic, and Numeric Representation Types*

In solving contextual word problems in paired interview situations Yersushalmy (2006) reported that lower ability beginning algebra students (eighth- and ninth-grade) who were learning algebra from multi-representational perspectives and had experience using computer-based graphing technology were found to use graphical and numerical approaches, often without the digital graphing tool, over symbolic approaches. Yerushalmy also reported that these students found the use of graphing technology to be a barrier to their problem solving efforts because common to many handheld and computer tools for mathematics, a correct explicit function rule was required in order to view its graph and table. The particular tool used in Yerushalmy’s research did not assist students in manipulating symbolic expressions or equations (such as the functionality available on a CAS).

At the time of the interviews the eighth- and ninth-graders participating in a three-year study that began in their seventh-grade were reportedly experienced in using the graphing technology, had experience solving contextualized problems, and had learned to perform simple symbolic manipulations on expressions. Despite these experiences, the lower ability algebra students (a subset of the class population) were found to struggle in
using symbolic approaches. Yerushalmy (2006) explains however that “Because of their flexibility in alternating among various representations and views of the problem, and because they understood the compatibility of these representations, they found a way out of these errors” (p. 383). In sum, Yerushalmy’s (2006) participants demonstrated RF through their ability to successfully select, move between, and communicate about the meanings of MR, with an emphasis on translations from verbal, contextual-based problems to graphic and numeric representations. It is important to note that student dialogue about these representations and meanings was sometimes a result of specific prompting by the interviewer, and may not have been evident without such specific prompting.

One result of Yerushalmy’s (2006) research is that lower ability algebra students can learn to be persistent in solving problems set in context and may tend to use numeric or graphic representations before resorting to symbolic approaches, if they can successfully tap a symbolic resource at all. This result is somewhat counter to other results that show college algebra and higher achieving high school algebra students tend to rely on symbolic representations (Herman, 2007; Huntley & Davis, 2008). However, besides the difference in ability level and age of these students, the task design used to assess students’ fluency with representations is paramount; Herman (2007) and Huntley and Davis (2008) presented students with tasks posed in symbolic representations, whereas Yerushalmy (2006) used contextually-based (verbal) representations.

Nathan and Kim (2007) examined middle grades students’ performance on pattern generalization tasks that combined graphic and verbal representations in both discrete and continuous forms. Cross sectional ($N = 372$) assessment data from sixth- through eighth-
grade students that captured aspects of students’ algebraic reasoning were analyzed with respect to students’ pattern generalizing ability in each of the representations individually, and taken together.

One of the findings that Nathan and Kim (2007) report is that for tasks stated verbally, students’ combined use of graphic and verbal representations was found to yield the highest overall performance, suggesting a “synergy” of complementary representations (p. 210). These researchers also report that verbal representations for continuous patterns of linear data are most accessible for young students (as compared to graphic representation types of discrete patterns, for example). In regards to RF, these middle school students demonstrated abilities to work within and interpret verbal and graphic representation types, with the most success in solving pattern generalization tasks when verbal and graphic representations were given together. Students’ abilities to create representations were not emphasized.

*Connecting Symbolic and Graphic Representation Types*

The studies reviewed in this section were selected based on the inclusionary criteria that the research attended to the nature of students’ connections (i.e., coordination and interpretation of information) between and across representations, an indicator of students’ RF. The Cartesian Connection, adaptability, and versatility are the key constructs that frame the following review.
Cartesian Connection

The Cartesian Connection was introduced by Moschkovich, Schoenfeld, and Arcavi (1993) in their theoretical discussion on how student come to understand the concept of function from both process and object perspectives in algebraic (symbolic), tabular (numeric), and graphic representations types. Moschkovich et al. (1993) define Cartesian Connection as the relationship between graphic and symbolic representations of a line: “A point is on the graph of the line L if and only if its coordinates satisfy the equation of L” (p. 73). In other words, there is an intimate bi-directional connection between graphic and symbolic representations of lines, or functions.

Moschkovich, Schoenfeld, and Arcavi (1993) advocate for an approach to curriculum development and assessment in which we are “seeing understanding as making connections” (p. 97). Moschkovich and colleagues summarize that one of the key lessons of this curricular program was that “Certain tables, verbal statements of relations between x and y values, verbal and algebraic expressions of the form $y = (something)x + (something)$, and linear graphs in the plane, are different ways of representing the same things” (p. 89). Others, including Kieran and Sfard (1999) had adopted a similar curricular focus, which seemed to promote more consistent discourse around the meanings of and connections between representations.

The primary data sources reported by Moschkovich and colleagues are two “data stories,” one from each of two sequences of tutoring sessions with eighth-grade students. These students’ mathematical activities and conceptions were captured as they worked

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2 It is noted that this is neither the earliest nor the only articulation of such a view on algebra curricula.
through curriculum units with access to the computer microworld GRAPHER (Schoenfeld, 1990) in several one-on-one tutoring sessions with a researcher. In the context of using digital technology and technology-based representations, Moschkovich and colleagues emphasize the importance of engaging students in tasks that have them predict, act, and reflect on their technology-based activity. As researchers (and teachers), we should not assume that students see the same things as experienced learners see when using technology. For example, Moschkovich et al. describe an instance in which a student focused on the jaggedness of lines drawn in the coordinate plane instead of more salient features of the graphic representations such as y-intercepts. Moschkovich et al. (1993) recount that both students encountered the greatest difficulty when the successful completion of tasks required them to demonstrate flexibility in switching between the process and object perspective of functions and/or different representations of these functions. In other words, secondary students seem to have demonstrated a lower level of RF in that they exhibited weaknesses in translating between representation types to successfully solve equation-solving tasks.

When asked to explain possible solution approaches to a task presented with symbolic equations and Cartesian graphs, Knuth (2000) found that based on responses from 178 high school students enrolled in college preparatory courses (e.g., algebra, advanced algebra, precalculus, and AP calculus) students overwhelmingly favored symbolic representations solution approaches, rarely recognizing the possibility of using graphic representations as a primary or alternative solution approach. Knuth (2000) posited that, “if students understood the Cartesian Connection, they should tend to select the more efficient (and often easier) solution strategy, which for this series of problems
was the graphical representation” (p. 503). Knuth thus came to the conclusion that secondary students have superficial connections between points on the graph of a line and ordered pairs satisfying the equation of lines.

The specific aspect of the Cartesian Connection that Knuth (2000) documented over three fourths of the students not demonstrating is the graphic to equation connection. Explicitly, “The majority of students did not suggest that the selection of any point on the graph of a line would be a solution to the equation of the line” (p. 506). Without data on the classroom curriculum and instruction, Knuth (2000) concludes that in light of the difficulties students demonstrated in connecting graphic to symbolic representations, the instructional conditions do not well support students’ ability to make graphic to equation connections, as more often than not students are taught symbolic to graphic connections (e.g., Leinhardt, Zaslavsky, & Stein, 1990), or simply emphasize algebraic representations of functions. Based on the limited amount of data collection, this preliminary study is quite restricted in what can be inferred about students’ abilities to successfully make connections between representations, and specific ties to curriculum and instruction are not well-supported with evidence. However, in light of the curriculum recommendations that students should be more adept at demonstrating connections between representations (e.g., NCTM, 1989), Knuth’s research does suggest a troubling situation in that many students are quite limited in these abilities.

Overall, Knuth characterized students as demonstrating superficial connections between representations especially when solving non-routine tasks. This finding suggests that an overreliance on symbolic solution approaches may be an artifact of curriculum
and instruction or students’ perceptions of the possible inaccuracies of using graphic approaches, such as an approximate solution.

Through task-based interviews with 38 middle school students from schools that use the curriculum *Connected Mathematics Project 2* (Lappan, Fitzgerald, Friel, & Phillips, 2006), Bieda and Nathan (2009) analyzed data from students’ speech, gestures, and writing during individual interviews to characterize students’ strategies in solving pattern generalization tasks in which a Cartesian graph was given. The tasks were designed to involve far prediction (FP) components in which the solution to the problems was not discernable from the static, graphic representation, as given. Thus, like Moschkovich et al. (1993), Bieda and Nathan specifically designed tasks so students would encounter a situation in which they would need to alter or otherwise change the representation and/or representation type to be successful in solving the task.

One outcome of Bieda and Nathan’s (2009) analysis was a definition for events of disfluencies: instances in which the student got stuck working in a given representation (i.e., they did not demonstrate an ability to use representations as resources to overcome barriers). Another outcome of this study was a tri-level framework for various types of (dis)fluencies. Specifically, according to Bieda and Nathan, students were disfluent if they did not shift from a given graphic representation to another representation type or merely modified the graphic representation; *physically grounded* or *spatially grounded*, respectively. On the other hand, students’ responses to FP tasks that involved some type of translation or shift from graphic to numeric or symbolic representations were categorized to be indicative of RF, or *interpretatively grounded*. The third category of
this framework is most significant for understanding students’ connections between representations, and hence students’ RF.

Students who are interpretively grounded “saw the graph as a portrayal of some pattern that could be reinterpreted through translation to generalized numeric structures such as arithmetic or algebraic formalisms, but that still maintained its connection back to the original graphical pattern” (Bieda & Nathan, 2009, p. 641). It seems fitting to say that students who are interpretively grounded have an understanding of the Cartesian Connection (cf. Moschkovich et al., 1993) because the mathematical strategies these students were found to demonstrate included abstracting a generalized algebraic rule and abstracting the rate of change between data points. These more sophisticated strategies were found to lead to the highest success rates out of all student strategies, 50% and 35% of the time correct strategies, respectively. In other words, Bieda and Nathan’s research suggests that strategies that are connected to students demonstrating a greater degree of abstraction and more sophisticated RF were found to be the most successful but their use does not guarantee that students will come to the correct response.

One contribution of Bieda and Nathan (2009) is a framework for characterizing students’ representational (dis)fluencies (i.e., physically grounded, spatially grounded, and interpretively grounded). They also gave a compelling example of the use of FP tasks that seemed to target students’ understanding of the connections between representations, and their development of RF. Beginning algebra students can demonstrate some level of RF and be somewhat successful in their translation strategies, but when they are bounded to a given (graphic) representation, students are less often successful in solving FP tasks (Bieda & Nathan, 2009). Finally, it is interesting to note that even though these middle
school students had access to calculators during the individual interviews, Bieda and Nathan (2009) focused on students’ gestures and speech, ignoring aspects of technology use in their discussions.

A commonality to the studies reviewed above on the Cartesian Connection is that working with the given representation, such as a graph, is not always the best approach to solving a problem, and students need to learn when and how to translate from a given representation to another representation to overcome barriers they encounter in problem solving (Bieda & Nathan, 2009; Knuth, 2000; Moschkovich et al., 1993). One contribution of the above studies is the identification and use of tools for categorizing specific types of connections between representations including bi-directional connections between graphic and symbolic representations of functions (Moschkovich et al., 1993; Knuth, 2000), and abstractions from patterns represented graphically to more generalized representations such as a symbolic rule (Bieda & Nathan, 2009). Some major limitations of the above studies include limited information on students’ instructional experiences that might have contributed to their fluencies or disfluencies in demonstrating competence in particular connections.

The mathematical domain of all of the studies reviewed on the Cartesian connection emphasized linear functions or patterns. Beyond limited attention to students’ mathematical experiences, both Bieda and Nathan (2009) and Knuth (2000) gave little to no attention to mathematics technology in their research. Moschkovich et al. (1993) provided the most details on students’ technology-based activities, as they were integral to the curricular approach advocated in their study. Overall, the Cartesian Connection is a difficult connection for students to make, especially from graphic to symbolic
representations. There is more to be explored in this domain, especially in regards to the role of technology in supporting students’ connections and development of RF.

Adaptability and Versatility

Upper level high school students with access to graphing calculator technology and who have experienced instruction that supports the use of multiple technology-based representations demonstrate difficulties in using MR to successfully overcome barriers they encounter in a given symbolic representation during task-based interviews (Huntley et al., 2007; Spitzer, 2008). Some details of each of these studies are given next.

Students’ versatility or ability to work within and move between representations was introduced previously with respect to students’ preferences for and choice of representations in solving linear equations tasks posed in symbolic forms. Huntley et al. (2007) found that for linear equations with a unique solution, the general trend in the data was that student pairs were successful in using P&P symbolic strategies. But when confronted with an identity or contradiction in their symbolic approaches (i.e., in the cases that the linear equations had infinitely or no solutions respectively), students had a difficult time interpreting the results of their own P&P symbolic manipulations and seldom resorted to using other representations in overcoming this barrier. The fact that most students required specific prompting to reason graphically in order to interpret their symbolic approaches led Huntley and colleagues to classify these students’ as having limited adaptability.

On understanding the connections between the graphic and symbolic representations, Huntley et al. summarized that “many students had difficulty
coordinating their solutions to the equations with their graphs (indicating limited versatility), and did not select or use tools well matched to the job at hand (indicating limited adaptability)” (p. 137). Specifically, Huntley et al. identified that more than half of the student pairs exhibited difficulties connecting symbolic and graphic representations of equations with no or infinitely many solutions. Students’ struggles with these translations were classified as either (a) from symbolic results to a graphic interpretation, (b) from using graphs to make sense of their symbolic results, or (c) both. This difficulty was not present for the linear equations with a unique solution, in which most students were successful in “predicting graphical representations based on results from symbolic manipulations” (p. 132), a successful translation. The results of Huntley et al. (2007) confirm Knuth’s (2000) findings that for non-routine tasks, students demonstrate superficial connections between graphic and symbolic representations of linear functions.³

Another important point to make in the context of students’ adaptability and versatility is that students’ abilities to work within and move between MR does not necessarily mean that students will be able to interpret the meaning of the mathematical object they are dealing with (i.e., make a connection). For instance, Spitzer’s (2008) dissertation study confirmed her hypothesis that high school algebra students who had access to a graphing calculator during task-based interviews used a more diverse range of strategies (and representations) than participants who did not have access to the technology, but this did not increase their level of performance. In other words, Spitzer’s

³ Thanks to a committee member’s knowledge of this research base, note that this finding is a subset of a larger set of studies on the NAEP results that students, in general, struggle in solving multistep or more sophisticated problems.
finding suggests that some high school algebra students might be using MR in their strategies to solve polynomial equations but do not necessarily understand how to synthesize or coordinate that information to come to a better understanding of the overall problem.

Spitzer (2008) claimed that the results of her study refute the idea that the use of graphing technology to access and use MR helps students to make connections across representations and to come to a better understanding of mathematics. However, it seems as though Spitzer overreached in drawing conclusions from her study results. Merely switching between technology-based representations (by literally pushing buttons on the graphing calculator) is not sufficient for bolstering students’ understanding of how to coordinate information and connect representations. It is the interpretation and reflection of the meaning of representations that is key to making connections, and to RF. Such interpretive activity was not evident in Spitzer’s interview data thus her finding that students demonstrated weak abilities in making connections across symbolic and graphic representation types is not necessarily attributable to presence (or absence) of the technology alone.

In sum, based on task-based interview data with upper secondary students (Algebra 2), both Spitzer (2008) and Huntley and colleagues (2007) present a mediocre image at best of students’ abilities to coordinate information across representations when completing equation solving tasks despite students’ abilities to work within and move between representations, with or without researcher prompting. Heid and Blume (2008) corroborate that the availability of MR does not imply that a connection between representations is understood.
The connection between symbolic and graphic representation types is illuminated by the Cartesian Connection as defined by Moschkovich et al (1993), and subsequently researched by Knuth (2000). This connection is foundational to students’ conceptual understanding of important algebraic concepts such as functions.

**Summary and Extensions**

When given an equation-solving task posed in more formal language (often void of context), upper secondary and beginning college students tend to prefer or choose to start solving within the symbolic representation type (Herman, 2007; Huntley & Davis, 2008; Huntley et al., 2007). This finding is consistent with Keller and Hirsch’s (1998) results on college calculus students’ preferences for representations. College algebra students interviewed in Herman’s (2007) study explained their reliance on symbolic representation type to be based on the notion that the symbolic representation type was considered “more mathematical” than other representation types.

The students studied in Yerushalmy’s (2006) research seem to be at the most comparable level with the population of students in the present study—ninth-grade algebra students. Yerushalmy reported results on students’ tendency to solve contextual word problems using numeric and graphic representation types instead of symbolic representation types. Again, Keller and Hirsh’s (1998) contention that contextually based problems seem to warrant a preference for more intuitive representation types such as tables and graphs rings true here. Finally, middle school students’ difficulties with far-prediction tasks and work within graphic representation types pinpoint possible aspects of disfluency that could be targeted in work with lower secondary students (Bieda &
Moreover, results of Nathan and Kim’s (2007) research suggests that the availability of both graphic and verbal representation types in pattern generalization tasks support students’ task completion more than tasks presented with those representation types in isolation.

Ample research supports the claim that secondary students, regardless of their access to graphing technology, have difficulties in articulating the connections between MR of patterns or functions (Bieda & Nathan, 2009; Huntley et al., 2007; Knuth, 2000; Moschkovich et al., 1993; Spitzer, 2008). These studies highlight that there is a great need for research aimed at supporting students’ understanding of the connections between representations. Moreover, the results presented above on students’ RF and their mathematical activity and cognition can be used to inform curriculum and instructional design. For example, findings by Moschcovich et al. (1993) and Knuth (2000) that secondary students seem to exhibit weak understandings of the bi-directional connections between symbolic equations and their graphic counterparts (and vice-versa) is a specific aspect of RF to target.

Many of the studies on students’ mathematical activity and cognition included attention to curriculum, instruction, and the role of technology. On task design (or more broadly, the role of curriculum), the contextual or non-contextual nature of the task and the absence or inclusion of a symbolic representation may influence students to choose to use the representation that seems most natural to the nature of the problem they are given (Keller & Hirsch, 1998). Beyond ideas for implications for future designs of curriculum and instruction, a summary of factors that seemed to influence students’ mathematical thinking in the studies reviewed above is given.
Factors that Influence Students’ Choice of Representation Type

Researchers have documented students’ preferences for representations (Keller & Hirsch, 1998) and in some cases, research has attempted to tie students’ classroom experiences to their representation-specific abilities and preferences (e.g., Herman, 2007; Huntley et al., 2007). Overall, the curriculum and instructional experiences were somewhat accounted for in the research reviewed above. For instance, Huntley and colleagues had data from schools on the high school curriculum adopted in the participants’ classrooms (Huntley & Davis, 2008; Huntley et al., 2007), Keller & Hirsch (1998) differentiated between calculus classes that used or had an absence of technology, and Herman (2007) relied on self-report data from students and teachers on classroom practices. Some details of how classroom instructional experiences relate to student performance will be expanded upon next, including the role of the teacher in privileging representations (e.g., Kendal & Stacey, 2001).

Curriculum and instruction. Recall that Herman (2007) found that none of the college algebra students used a tabular (numeric) approach in solving tasks presented in symbolic form, and hypothesized that these students may not have been exposed to such representations as a viable problem solving technique. In other words, Herman surmised that students’ reliance on symbolic representations (and sometimes graphic representations to check) in completing equation-solving tasks presented in contextualized situations with a symbolic equation may have been an artifact of the curriculum and instruction emphasizing the use of symbolic representations as the most appropriate and accurate method. Supported by data from interviews with college algebra
students and teacher-self report data, symbolic and graphic representations were the only ones students were expected to learn. Students reported that the symbolic representation seemed to be more “mathematical” thus was often the representation they chose to use.

While Herman was focused on instructional techniques, Huntley et al. (2007) drew attention to curriculum. Huntley and colleagues reported that, “more students who used an integrated textbook reasoned graphically prior to receiving prompts (which is an indicator of adaptability) than students using a non-integrated textbook.” (p. 136). The analysis did not support other curriculum-effect relationships, either in strategy choice or use of technology.

Both of the findings presented by Herman (2007) and Huntley et al. (2007) seem to suggest a correlation between instructional practice, curriculum, and students’ use (or lack thereof) of MR in solving tasks. However, a limitation of these studies was that insufficient attention was paid to the nature of instructional practice, and specifically, the experienced curriculum—correlation does not imply causation. Teacher or school self-report data were relied on as the primary means to judge the nature of curriculum implementation and role of representations in students’ learning experiences. One drawback of such methodological decisions is that detailed attention to patterns in use of tool-based representations and types of representations across various content foci may be difficult to capture without specifically designed instruments or other means of data collection (such as video) to capture such activity.

The role of technology. Based on the studies reviewed above, research has accumulated on the degree to which students use graphing calculators or other digital graphing devices in solving algebra tasks. For instance, Huntley et al. (2007) report that
the use of technology was not evident in any of upper secondary students’ first attempts to solve linear equations of the form $ax + b = cx + d$ (but was evident in later attempts, sometimes only after researcher-probing). In some cases, researchers expected students to use their graphing calculators to solve tasks posed in symbolic forms from a graphic or numeric representation, yet students’ persisted using non-calculator symbolic approaches as their first or dominant strategy, with technology-based graphical approaches a common second or alternative strategy (e.g., Herman, 2007; Huntley & Davis, 2008). In other cases, graphing technology did not support students’ approaches very well, and lower secondary students were found to use non-technology based numerical and graphical strategies instead (Yerushalmy, 2006).

In the case of Yerushalmy (2006), the affordances and constraints of the tools that are available in problem solving also seem to play a role in lower ability students’ choice or avoidance of using certain representations. For example, mathematics technology that is driven by symbolic representations (i.e., that requires an explicit rule to graph a continuous curve or view a table of values) and that does not support students’ abilities to create and manipulate equations and expressions is a type of constraint on students’ problem solving abilities. On the other hand, P&P can be used to create sketches of graphs and tables of values without requiring a symbolic representation. Various aspects of students’ representation-specific activities have been considered, including preferences for and choices of representations and factors that seem to influence those choices.
In Zbiek, Heid, Blume and Dick’s (2007) synthesis of research on technology in mathematics education, the construct of RF was discussed as a tool and lens for researchers in examining mathematical activity in classrooms and students’ mathematical thinking. The discussion of RF was also situated with respect to the tool-task relationship, yet the written, intended, and enacted curricula did not seem to be a major focus of their review. This interpretation seems to suggest a possible gap in the literature on RF, that the relationship between tool, curriculum, and student learning is not widely known.

For instance, in their review of algebra and function development in school mathematics, Heid and Blume’s (2008) synthesis of the research on students’ development of RF in the context of technology-based learning environments was centered on the student-tool relationship and students’ use of alternative strategies and/or representations in solving problems; little attention was paid to details of the learning environments in which students purportedly developed RF. Indeed, Heid and Blume call for greater attention to the context of the learning situation in research on technology in algebra; more explicitly, the learning environment, curricular tasks, and role of technology need to be carefully documented to better account for factors influencing students’ learning and development (of RF for example).

Task Design for CAS and P&P Environments

In this section, both CAS and P&P tools will be considered together, and an example of research that assumes the instrumental approach will be given, along with two frameworks that were built out of research aimed at bridging conceptual and technical aspects of tool use. The choice to review particular studies in this section as based on the
extent to which they contributed to our understanding of solving equations and of equivalence of expressions. Both empirical results and task design principles are discussed.

**RF and the User-Tool Relationship**

CAS and P&P tools can be used to facilitate the creation or selection of MR. It thus makes sense that some research on RF is focused on the specific relationship between a user and the tool. For instance, three constructs that have been researched in the context of user-CAS relationship—instrumental genesis (Guin & Trouche, 1999), orchestration metaphor (Drijvers & Trouche, 2008), and algebraic insight (Pierce & Stacey, 2002)—contribute to our understanding of the user-tool relationship with respect to RF. Specifically, instrumental genesis and algebraic insight put the mathematical thinking of students and the intricacies of the tool into focus. Instrumental genesis is concerned with how the user shapes the tool and how the tool shapes the user, and algebraic insight is concerned with the algebra needed by students to use CAS, including components of both algebraic expectation and the ability to link representations. The orchestration metaphor is useful for examining the role of the teacher in classroom situations in which students are using CAS.

These perspectives are not overtly employed in this study because the focus is more on the ways in which representations are positioned when learning mathematics; the tools play a tertiary role. However, some of the greatest value of the Pierce and Stacey’s (2002) algebraic insight framework is that it presents a perspective that emphasizes the importance of cultivating a disposition to predict or expect CAS generated results,
whether it be transposition or translation activity. Anticipating the invariance of properties of mathematical objects within and across representations (e.g., coordinating information), interpreting representations, and linking representations are all components of developing a more meaningful use of tool-based representations, and of developing RF. Thus this construct tacitly served as a component of the ways in which tasks were designed for the teaching experiment.

Several researchers have adopted an instrumental approach to understanding the user-tool relationship and have contributed what we know about students’ interactions with CAS (e.g., Artigue, 2002; Drijvers & Trouche, 2008; Guin & Trouche, 1999, 2000; Kieran & Drijvers, 2006). From an instrumental perspective, Drijvers and Trouche (2008) describe that instrumental genesis, the process of a tool becoming an instrument, involves co-implicative processes in which the tool shapes the user and the user shapes the tool. While the theoretical lens of instrumental genesis is not used to understanding students’ RF in this study, some research studies that employ an instrumental perspective have contributed to the research base on research-based techniques employed in combined CAS and P&P environments (e.g., Kieran & Drijvers, 2006; Kieran & Saldanha, 2008).

From an instrumental perspective, both the technical ability to use tools (e.g., to transpose within and translate between representations) and the conceptual activity involved in interpreting and making sense of the use of tools (e.g., the ability to communicate about and use representations to justify) involve negotiation between the user and the tool. The conceptual and technical aspects involved in using CAS highlights the main way in which a close user-tool relationship is significant to the development of RF.
In recent research, Carolyn Kieran and colleagues have assumed an instrumental approach in researching CAS and P&P environments, with attention to both technical and conceptual aspects of students’ work in learning and doing mathematics (e.g., Kieran & Drijvers, 2006; Kieran & Saldanha, 2008). This work has culminated in the presentation and use of significant lenses on combined CAS and P&P environments including the task-technique-theory framework (Kieran & Drijvers, 2006), and the notion of reconciling CAS and P&P work (Kieran & Saldanha, 2008). Following a discussion of these ideas, the user-tool relationships will be elaborated on according to the black-box/white-box didactic principle proposed by Buchberger (1990) and some contemporary variations that suggest a gray-box (Cedillo & Kieran, 2003) and step-by-step CAS techniques (Edwards, 2003) with lower secondary algebra students.

*Task-Technique-Theory Framework*

Kieran and Drijvers (2006) adopted an instrumental approach to tool use and the Task-Technique-Theory (TTT) framework to guide the design, implementation, and analysis of specifically designed tasks and lessons that were carried out in two teaching experiments with tenth-grade secondary students who were skilled in algebraic manipulation. The aim of this research was to explore the co-emergence of technique and theory in a combined CAS and P&P environment, and to analyze how it contributed to students’ development of algebraic thinking.

The TTT framework derives from an anthropological view put forth by Chevallard (1999), which has subsequently been built on by several French researchers who have investigated the relationship between technical and conceptual/theoretical
aspects of CAS use (e.g., Artigue, 2002; Lagrange, 2002, 2003). From the TTT view, the task is the mathematics that students are asked to do or complete, the technique is the means by which the task is completed (i.e., with P&P and/or CAS tools), and the theory includes the mathematical principles that undergird appropriate technique. Some significant ways in which the TTT was utilized by Kieran and Drijvers (2006) included the prominent role of tasks of a reflective nature, and the emphasis on making the mathematics associated with particular tool techniques salient in classroom discussions.

**Symbolic Transpositions with CAS and P&P**

Kieran and Drijvers (2006) found that the coordination of CAS technique and theoretical expectation prompted cognitive conflict in some students, and further reflection on the mathematics of the task. For instance, in one segment of the teaching experiment focused specifically on the theme of equivalence, equality, and equation, students had learned that the CAS technique of testing equivalence of expressions by relating them by an equal sign will result in “true” if the expressions are indeed equivalent (see line 1 of Figure 2.1). The rewriting of an equation that related non-equivalent expressions was an unexpected result that prompted further reflection on the theory underlying the CAS technique (shown in line 2 of Figure 2.1).

Besides conflicts between students’ expectations and results of CAS output, students encountered other difficulties in CAS and P&P environments aimed at the particular theme of equivalence, equality, and equation. The biggest challenge reported by Kieran and Drijvers (2006) was that of learning to use appropriate language to explain the meanings of equivalence, equality, solution, and solving. These discourse issues were
also reportedly tied to students’ experiences in dis-connecting their numerical reasoning skills from their symbolic or algebraic reasoning skills. For example, students justified equivalence of expressions based on equality in numeric form over common symbolic form.

![Figure 2.1](image)

*Figure 2.1.* Testing the equivalence of expression by relating them by an equal sign.

Kieran and Drijvers’ (2006) elaboration of specific tasks, techniques, and theoretical components of studying equivalence and equality are particularly beneficial for advancing research in this area because it allows their theory to be tested and refined in new research settings. Their articulation of a TTT framework will serve to be beneficial to the present study. Additionally, in accordance with the instrumental approach, Kieran and Drijvers contended that specifically designed tasks that encourage writing and reflection on CAS and P&P fostered the co-development of technique and theory, a goal of their research. Finally, in line with a major component of algebraic insight, these researchers also found that instances in which CAS results ran counter to students’ expectation served to be particularly productive learning experiences, as demonstrated in Figure 2.1.
Kieran and Drijvers (2006) recommend future research be carried out in combined CAS and P&P environments, especially with lower-ability algebra students. Furthermore, these researchers suggest that more careful attention to language and discourse in such learning environments must be accounted for to more fully investigate students’ development of algebraic thinking, especially in mathematical contexts such as equivalence and equality in which students may struggle with using appropriate language. The learner-tool relationship including both CAS and P&P is elaborated on next with respect to the construct of reconciling.

Reconciling

Reporting on the same project as discussed in Kieran and Drijvers (2006) with tenth-grade students, this time with a singular focus on examples from CAS task situations for factoring, Kieran & Saldanha (2008) present a task design framework for the use of CAS and P&P that purports to strike a balance between technical skill and conceptual understanding. The main features of this task design framework include: reconciling CAS and P&P work, reflecting on CAS results and the object that had been reconciled, and proving generalizations. Some of the foundational aspects of this framework are discussed next.

Following an instrumental approach, Kieran and Saldanha (2008) cite Rabardel (1995) in describing that when tools are appropriated by learners from a physical artifact to an instrument or psychological construction, “They are actually developing conceptually while they are perfecting their techniques with the tool” (p. 394). In their investigation of appropriate CAS tasks and techniques, Kieran and Saldanha (2008) draw
on existing literature in suggesting that CAS tasks should include components of anticipation (Guin & Trouche, 1999), as well as reflection on and connections to other tasks and previous knowledge (Zehavi & Mann, 2003).

Kieran and Saldanha’s (2008) construct of reconciling (also mentioned in Kieran & Drijvers’ [2006] discussion) builds on Thompson’s (2002) notion of reflective mathematical discourse that is aimed at productive reflection on specific mathematical “things.” Fittingly, Kieran and Saldanha’s (2008) research supports the notion that the responsibility to reconcile CAS and P&P does not stop at the statement of a curriculum task or task-situation in which students are merely checking their work. Instead, reconciling is specifically directed at making unexpected CAS results compatible with P&P forms; it is accomplished through tasks that required students to write about how they interpreted their work and class discussions specifically focused on the mathematical objects being studied.

The select research pieces reviewed above by Kieran and colleagues (Kieran & Djievers, 2006; Kieran & Saldanha, 2008) make headway in addressing Zbiek’s (2003) concern that the “absence of detail in research reports causes difficulty in using the reports to develop deeper insights into the kinds of CAS-related mathematical experiences that best support student learning” (p. 212). In sum, anticipation of results, reflecting on those results, interpretation of results of technology are important aspects of students’ effective use of CAS (Kieran & Saldanha, 2008; Kieran & Drijvers, 2006; Thomas, Monaghan, & Pierce, 2004). The processes of anticipation, reflection, and interpretation are connected to the constructs of algebraic insight and RF.
While the majority of the discussion above has focused mainly on symbolic representations, the general features of task design (with attention to elements of technique and theory as carried out in classroom practices) is applicable to situations in which CAS is treated as a representational toolkit (Dick & Edwards, 2008).

White-Box, Black-Box, Gray-Box

In Buchberger’s (1990) seminal article on *Should Students Learn Integration Rules*, the black-box/white-box didactic principle was introduced as a lens for addressing pedagogical questions about determining an appropriate balance between computing technology and by hand skills with P&P. Buchberger (1990) suggested that a particular area of mathematics should not be offloaded to CAS technology until students have learned and trivialized this area using P&P techniques (a white-box/black-box approach). This conservative design principle is the predominant approach to CAS use in contemporary high school textbooks (Davis & Fonger, 2010) and might be explained by the fact that the “appropriate” balance between CAS and P&P is not well understood (Heid, 2003; Kieran, 1992).

Two adaptations of the white-box/black-box principle are offered next. Both Cedillo and Kieran (2003) and Edwards (2003) report on research conducted with beginning algebra in which CAS was utilized during the learning process, instead of after mastery of P&P techniques.

beyond the white-box/black-box principle recommended by Buchberger (1990), suggesting that instead of considering white-box activity before black-box activity, the CAS can be used to learn algebra, what they call a “gray-box” approach. Specifically, Cedillo and Kieran’s envisioned role for CAS is one in which it “is not treated exclusively as a problem solving tool but is also considered as a mediator of algebra learning—a tool that helps create simultaneous meaning for the objects and transformations of algebra” (p. 221). The goal of the instructional sessions with these eighth-grade students was to build more meaningful connections between their experiences in arithmetic and generalizations with symbols. In other words, the activities that were produced and used as part of this research project aimed to intertwine the particular with the general by focusing on describing numeric patterns with symbolic representations.

The sequencing of the chunks of activities for this study followed a progression of topics: (a) describe number patterns, (b) produce number patterns, (c) produce equivalent algebraic expressions, (d) describe part-whole relationships algebraically, (e) explore inverse linear functions, and (f) confront problem situations that can be solved by creating an algebraic model. Through specifically designed task situations around these blocks, Cedillo and Kieran (2003) claim that students’ algebraic activity spanned all three aspects of algebra, as described by Kieran (1996): generational, transformational, and global meta-level mathematical activities.

All of the tasks presented by Cedillo and Kieran (2003) had students write a calculator program (or expression) that could be used to reproduce a numeric pattern (represented in a table or in geometric figures). For example, from a table representation,
students analyze data pertaining to the amount of wire sold (1.7, 2.4, 3.1, 4.06, 5.2) and the amount of wire remaining (8.3, 7.6, 6.9, 5.94, 4.8, respectively). They are then asked to “Make a program” and “Check it with your calculator” for these values (Cedillo & Kieran, 2003, p. 227). Figure 2.2 shows a CAS command that would address this task.

An important outcome of their work with beginning algebra students, Cedillo and Kieran (2003) report that “[CAS] permitted students to be introduced to algebra in such a way that the symbol-manipulating aspects were tightly tied to the students’ prior numerical experiences in arithmetic, thus making algebra a more meaningful activity for students” (p. 237). More specifically, Cedillo and Kieran highlight that students developed a meaningful conception of a variable.

Evidence that researchers are working closely with schools to develop and enact curricular tasks that move beyond the white-box/black-box principle, like that of Cedillo and Kieran (2003) is encouraging because it means that students have opportunities to use CAS as a powerful learning tool. Despite this, the research report were lacking details on the timeline of the activities and on the actual activities that were enacted with students (only a few examples that repeated the same CAS technique were given). For
instance, the fact that participating teachers of this study were asked to use the TI-92 CAS in at least two of the five 50-minute sessions of mathematics throughout the school week suggests that both CAS and P&P were used in these classrooms. However, the article did not contain comments on the P&P and CAS work of these students, which would have given a more complete picture of the nature of CAS activity in a combined CAS and P&P environment.

*White-box symbolic techniques with CAS.* Drawing on his dissertation research that involved a year-long comparison study of advanced algebra students in high school with and without access to CAS (Edwards, 2001), and excerpts from his personal teaching journal recorded while teaching beginning algebra students, Edwards (2003) discusses some specific CAS techniques and theoretical components of expression simplification and equation solving with the TI-92 CAS. Details of the tested learning progressions and examples of activities are given next.

On learning to simplify expressions and understand the properties of equivalent expressions, Edwards (2003) suggests activities and questioning strategies based on his professional insight from teaching with CAS. His recommendations include having students make predictions about algebraic forms, test conjectures, and identify patterns in their symbolic work. Consider the CAS results shown in Figure 2.3.
Figure 2.3. CAS can be used to look for patterns and test conjectures about symbolic expression simplification (Edwards, 2003).

Edwards suggests a developmental progression of activities that starts with looking for patterns and testing conjectures when combining positive terms in a single variable (Figure 2.3, lines 1-2), and moves to combining negative terms in a single variable (Figure 2.3, line 3-4), to examining the equivalence of more complicated expressions in which students should reason about the expected output before pressing Enter (Figure 2.3, line 5). Edwards (2003) posits that use of CAS is particularly helpful in learning the differences between the minus sign and negative sign (Figure 2.3, line 6).

In developing skill in equation solving techniques, Edwards (2003) distinguishes between a white-box approach in which the algebraic manipulation steps are visible to the user and a black-box approach in which the algebra manipulation steps are hidden from the user (cf. Buchberger, 1990). From Edwards’ experience teaching secondary algebra with CAS, novice algebra students need to see the intermediate steps of solving linear equations in order to understand the equation solving process. Consistent with Heid and Edwards (2001), Edwards (2003) posits that by using CAS to perform equation solving transformations one step at a time students are able to focus on the choice of appropriate symbolic transformations free from computations errors that may distract
students from understanding correct equation solving techniques by hand. Two examples of white-box solving techniques are shown in Figure 2.4.

![Figure 2.4. White-box solving technique applied to linear equations of the form $ax + b = cx + d$.](image)

While Edwards (2003) focused predominantly on students’ development of fluency and mastery with symbolic transformations in the context of simplifying expressions and solving equations, he made connections to MR to check the equation solving processes. For instance, the “with” operator on the CAS is used to check solutions found symbolically, and tables and graphs complement this approach as other appropriate ways to check students’ symbolic work (Figure 2.5). Overall, Edwards (2003) presents a seemingly productive progression of tasks from equivalence of expressions to transformations on equivalent equations that were tested in teaching experiences with algebra students. However, no evidence is presented to show student learning gains or how these approaches impact students’ understanding of symbolic transpositions (and multi-representational translations).
According to Cedillo and Kieran (2003) and Edwards (2003), the symbolic capabilities of CAS can be used to support students’ development of within-representation abilities (i.e., transpositions of symbolic representations). Some important details were provided with respect to the CAS techniques that were employed in these studies, and varying degrees of detail were offered on the theory that motivated the suggested tasks. For instance, Edwards (2003) was more thorough in illustrating how the mathematics of learning equivalence of algebraic expressions was developed with CAS, whereas Cedillo and Kieran (2003) merely referenced sketches of some related activities. Both studies extended the white-box/black-box principle in an important way: P&P symbolic manipulation skills were not assumed prior to using CAS, instead, CAS was predominantly used as a tool for learning transpositions within symbolic representations (Cedillo & Kieran, 2003, Edwards, 2003).

**Summary and Extensions**

Most of the CAS research reviewed on the user-tool relationship focused on the symbolic capabilities of CAS and generational and transformational aspects of symbolic
algebra (Cedillo & Kieran, 2003; Edwards, 2003) and the relationship between CAS and P&P tool use in task design (Kieran & Drijvers, 2006; Kieran & Saldanha, 2008).

One way that future research might help to inform curriculum design might be to follow the example set by Kieran and colleagues (Cedillo & Kieran, 2003; Kieran & Drijvers, 2006; Kieran & Saldanha, 2008) and Edwards (2003); they gave explicit attention to design principles for coordinating CAS and P&P activity in order to bolster the technical and conceptual aspects of CAS use. The principles tested to be productive in learning symbolic transpositions—anticipating, interpreting, reflecting, and reconciling CAS and P&P activity (Kieran & Drijvers, 2006; Kieran & Saldanha, 2008)—are tested in this research study. Design principles for symbolic transpositions can also be applied to work within other representation types. Research has accrued that supports the notion that work within technology-based graphic, numeric, and verbal representations can be used to support meaningful use of symbolic representations (Chazan & Yerushalmy, 2003; Heid, 1988; Kieran, 2007; Kieran & Sfard, 1999; Kieran & Yerushalmy, 2004).

Thus, an important extension of the reported research, especially in the context of equivalence and equations in which students have language difficulties (e.g., Kieran & Drijvers, 2006), is that a more prominent emphasis on graphic, numeric, and verbal representations should be given. As alluded to above, a multi-representational approach to the topics of equivalence and equations may support students in coming to better understand the mathematical objects they are working with; with graphs, tables, and words, students are afforded the opportunity to capitalize on their strengths in visual reasoning, quantitative reasoning, and language. Note that the specific CAS techniques, theory, and tasks that have been suggested on the theme of equivalence, expressions,
equations, and solving are also elaborated on in Chapter 3, as they formed the basis of the instructional theory and the design of tasks for the teaching experiment. Teaching experiment methodology is one important avenue to bringing closer ties between theory and practice and to uncover some of the intricacies of mathematical activity in the classroom that are tied to the progression of student learning.

**Sequencing Learning Progressions**

Researchers have devoted significant effort toward the goal of using mathematics technology as tools to amplify and reorganize curricula using technology-based MR and the core concept of functions (Fey & Good, 1985; Heid, 1988; Heid & Edwards, 2001; Kieran & Sfard, 1999; Sheets, 1993). Buchberger’s (1990) white box/black box didactic principle and the metaphors of amplifying and resequencing (Pea, 1985) are sometimes discussed with respect to the role of CAS as a tool that can transform mathematics curricula (Heid, 2003).

Early proponents for reform of mathematics education set ambitious agendas for research and curriculum development and supported revised priorities for a transformed school algebra curriculum that relied on the use of symbolic manipulation programs or CAS (Corbitt, 1985; Fey & Good, 1985; Fey & Heid, 1984; Ralston, 1985; Usiskin, 1985). Research studies that involved the use of symbolic manipulation software began to appear in the early 1980s, and were originally in the domain of the college calculus curriculum (Hart, 1991; Heid, 1983, 1988).

The above overview alluded to the role of CAS technology as a catalyst for transforming the nature of mathematics curriculum. In a related vein, another way to
make advances in the field of mathematics education is to challenge assumptions about the sequence of learning progressions that are appropriate for school mathematics. These progressions of learning become tangible in the form of curricular recommendations, and printed (or now digital) textbooks for school mathematics. Two learning progression sequences that are significant for this study will be elaborated on next.

*Functions-Based Approach*

Some experimental work in curriculum design was prominent in the 1980s and 1990s. Examples of empirical findings from studies in calculus and algebra are reviewed next. The role of technology, with capabilities to perform symbolic algebra, was a major part of the re-organization of the curriculum to follow a “concepts first” or “functions-based” approach.

*Concepts first approach.* In a comparative study between two experimental sections and a traditionally taught section, Heid (1988) employed the power of symbol manipulation computer technology as a catalyst to re-sequence the calculus curriculum to emphasize the concepts of calculus prior to formal symbolic manipulation. With respect to RF, a significant aspect of the experimental course was that a range of representations were used to “explore and explain the meaning of concepts” (p. 9) including derivatives. Heid was careful to detail that the experimental calculus class focused on graphic representations of functions in particular and were encouraged to reason from these representations (which was a departure from the traditionally taught students who focused mainly on symbolic representations). Classroom sessions and pedagogical decisions regarding the intended role of the computer technology in students’ learning
experiences, quiz and exam results, and interviews with students were also documented. This data set allowed the researcher to draw conclusions about students’ conceptual and procedural understandings with respect to the computer-based curriculum and instruction the students’ experienced.

Heid (1988) reported that students who used computer technology and learned from MR before emphasizing symbolic techniques using P&P outperformed control students on conceptual understanding of the mathematics and performed almost as well as their traditionally taught peers on procedural skills. Heid’s (1988) research has served as an “existence proof” that making drastic changes to traditional curricular approaches (with technology serving as the catalyst) are not necessarily detrimental to students’ symbolic manipulation performance in calculus.

Fey (1984) proposed that algebra should follow an inverted sequence of topics, offering applications first followed by algebraic transformations. In the curriculum vision proposed by Fey and Good (1985), students would be exposed to applications of algebra from a functional approach that affords opportunities for particular computing uses including numerical and graphical approaches. Fey and Good argued that presenting the applications first is a way for the concepts of algebra to be motivated prior to a formal introduction to symbolic manipulation, making the content more accessible and enticing to more students.

Increasing use of reform-oriented technology-rich algebra curricula became prominent in the 1990s and is a trend that continues today (Chazan & Yerushalmy, 2003; Fey, et al., 1995; Haimes, 1996; Kieran, 1992, 2007; Sheets, 1993). Also, a functions approach to the learning and teaching of algebra with a specific focus on the role of CAS
in linking MR of functions has become part of what is valued for school mathematics (e.g., Fey, 1989; Heid & Blume, 2008; Kaput, 1989; Zbiek & Heid, 2008).

Soon after Heid’s (1983, 1988) experimental work with CAS in the calculus curriculum, a collaborative research effort headed by Fey and Heid led to the development and testing of an experimental curriculum, Concepts in Algebra (CIA) (Fey, Heid, Good, Sheets, Blume, & Zbiek, 1995). The effects of this curriculum have been investigated through comparative studies with traditional algebra students by several researchers concerned with students’ development of an understanding of functions (O’Callahan, 1998; Sheets, 1993) and variable (Boers-van Oosterum, 1990). Some details of studies are discussed next.

Functions-based approach to algebra. At the college algebra level, O’Callahan’s (1998) experimental research suggests that CIA students performed better than students of the same level in a traditional algebra course on measures of the ability to translate between representations of functions. O’Callahan did not find significant differences between these groups of students’ understanding of functions on measures of other phenomena, such as reification. When ninth-grade algebra students who learned using CIA were compared to students in a traditional algebra class, Boers-van Oosterum (1990) found CIA students to have more robust understandings of variable. Both CIA and traditional algebra groups were proficient at manipulative skills. The research by O’Callahan and van Oosterum relied on pretest and posttest data in addition to individual

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Jim Fey spearheaded the curriculum development; Kathy Heid led the research.
interviews to produce their findings; evidence on the enacted curriculum was not provided.

In a similar vein, Sheets (1993) found that ninth-grade CIA students who learned conceptual and procedural underpinnings of functions with the use of “CAS” technology outperformed high school seniors taking a course on elementary functions on their flexibility in using MR in problem solving and were determined to have superior understandings of the behavior of functions. Sheets (1993) reported findings that suggest beginning algebra students who studied from the Concepts in Algebra curriculum outperformed students who were taught mathematics in a traditional way on their flexibility of understanding function concepts and MR. The high school seniors who participated in this study were college-intending and were selected by their teachers as having a grade of B or better in the current mathematics course. CIA students were high school freshmen who had been taught from the CIA curriculum for 8th and 9th grades, with constant access to what would today be considered a CAS (with capabilities to graph, fit functions to data, use spreadsheets, perform symbolic manipulation). The comparison students were senior level (twelfth-grade) high school students completing a traditional course in elementary functions who “had no formal (or informal) experiences with computer tool software for mathematical inquiry and problem solving prior to the interviews” (p. 39).

The methodology of this study included several task-based interviews with individual students from the treatment \(n = 6\) and control \(n = 4\) groups. CIA students

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5 The CAS used in this study afforded capabilities to perform symbolic manipulation, but are different from the handheld representational toolkit used in this study—the TI-Nspire CAS CX.
were permitted to use calculator and computer technology in the interviews, and it is left tacit that the control students were also permitted to use this technology (even though they presumably did not know how). With a focus on comparing students’ mathematical abilities and understandings of the function concept, the methodology was limited to an analysis of interviews. Data were not collected on the enacted curriculum in either the control or experimental classrooms.

Sheets (1993), O’Callahan (1989), and Matras (1988) used students’ successful translations as indicators of their problem solving abilities and understanding of functions, and found that students who studied from an experimental curriculum that emphasized a prominent role for computer technology and MR of function outperformed their “traditionally taught peers.” I agree with Zbiek and colleagues’ (2007) characterization of these studies (and Sheets’ study in particular) that evidence has accrued to support the notion that curriculum can be designed to support students’ RF.

Other research has been conducted on students’ flexibility in representation-specific abilities with more mainstream curricula with a functions approach, like Contemporary Mathematics in Context (Coxford, Fey, Hirsch, Schoen, Burrrill, Hart, & Watkins, 1997). For example, research by Huntley and colleagues (2000) found that CPMP students outperformed control students when asked to solve algebra problems posed in context and had access to graphing calculators. As discussed by Kieran and Yerushalmy (2004), one major finding of Huntley et al. (2000) was that the CPMP students did not perform as well as the control group on measures of skill in symbolic manipulation, but were better versed in alternative strategies that involved the use of a graphing calculator.
Some research studies support the finding that school algebra students can use MR in solving tasks and are more successful in translating between MR when they have experienced coursework with technology than without technology (Huntley et al., 2000; Ruthven, 1990; Sheets, 1993; O’Callahan, 1989). For instance, Ruthven (1990) found that on symbolization tasks (requiring translation from graphic to symbolic representations) and interpretation tasks (requiring translation from graphic to verbal descriptions) upper secondary students in the experimental class with regular access to graphing calculators outperformed students in a control class who did not have access to graphing calculators.

In Ruthven’s (1990) research, students completed written instrumentation without any identification of whether the technology was indeed used to complete the tasks. Thus, even though the experimental group reportedly had access to graphing calculator technology, it was not clear what role this technology played in students’ successful experiences with translating between representations. The fact that Huntley and colleagues (2007, 2008) found that relatively few students actually used graphing calculator technology when they had access to it in solving tasks begs the question of what role it actually played for students in Ruthven’s research. It also raises a question of what role graphing calculator technology played in the experienced curriculum of Huntley et al.’s study; little evidence was given on the nature of students’ instructional experiences with the graphing calculator.

Summary. In sum, Heid (2010) recounts a history of CAS research that stems from curriculum development efforts in calculus and algebra to promote a conceptual understanding of the concept of functions through the use of multiple technology-based
representations. Through experimental studies with college calculus students (Heid, 1983, 1988; Hart, 1991), beginning high school algebra students (Boers-van Oosterum, 1990; Matras, 1988; Sheets, 1993) and college algebra students (O‘Callahan, 1998), research has accumulated that seems to support the notion that mathematics curriculum can be reorganized to incorporate CAS as a representational toolkit so that students are supported in developing a more conceptual understanding of functions when compared to students who study from a traditional curriculum program. From a representational lens, the reorganization of curricula as discussed above seems to support students’ RF as measured primarily by their abilities to translate between MR of functions.

Research that has documented students’ successes in translating between representations with access to graphing calculators could be attributed to many factors, including the technology, and the instructional experiences of these studies. Research not tied to a particular curricular program, but simply comparing calculator and non-calculator classes of “parallel” mathematics content leaves many unanswered questions about the actual role that technology played in students’ successful problem solving performance and the kinds of mathematical experiences students encountered (e.g., Ruthven, 1990). On the other hand, research that specifically targets students who studied from an experimental curriculum (Matras, 1988; O‘Callahan, 1998; Sheets, 1993) or mainstream curricula (Huntley & Davis, 2008; Huntley et al., 2000) gives more insight into the structure of the mathematical topics and trajectories of learning main concepts.
such as functions, but still leaves open questions about how that curriculum was implemented with respect to the role of technology.6

From this review of research it is clear that to meet the goals of the present study a more careful documentation of the experienced curriculum is necessary; the mere “presence of technology” and “use of curriculum” are not adequate. The methodology was purposefully crafted to address the need to understand the conditions of the learning environment that support students’ change in RF, a current gap in the literature. Moreover, to push the field forward, it seems evident that both psychological and social aspects of classroom-based research need to be documented in tandem.

*Equivalence of Expressions and Solving Equations*

The concepts of equivalence and equations, both mathematical relations, are intertwined in intricate ways. There is not one single best approach to developing an understanding of these concepts; the tools that are used, representations that are emphasized, and sequencing of topics can vary greatly. The two approaches discussed here are based on Kieran’s research, some of which has been introduced in earlier sections. In the following review of both a graphical, functions-based approach (Kieran & Sfard, 1999) and a symbolic approach with CAS and P&P tools (Kieran & Drijvers, 2006), details on learning trajectory, tasks, and instructional approach are given.

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6 Thanks to a committee member’s insight on this literature base: “In the early [nineteen nineties], research was often about the strategies students used when given tech[ology]. Research done in the context of experimental curricula was often about how students handled the technology and not about the effects of the technology in the sense of curriculum evaluation projects.”
Graphical, functions-based approach. Kieran and Sfard (1999) discuss select components of a 30-day teaching experiment conducted with middle grades students on algebra and various representations of functions. The thrust of the excerpt discussed in this article is about how the students involved in this teaching experiment came to understand how to write equivalent expressions from experiences in graphing the addition of two functions and the multiplication of functions by a constant. Kieran & Sfard (1999) describe how students’ ability to perform such symbolic manipulations on equivalent expressions was supported through a functions approach to learning that began with graphical situations, and expanded to numeric and symbolic representations in which the table was the connecting representation between graphic and symbolic representations. The teaching experiment concludes with solving equations and inequalities by means of comparing the values of two functions and the graphs of two functions.

The subtle but important language choice that seemed to be significant to the design of the instruction is that the function was "presented as the mathematical object which ties the expression and the corresponding graph and table together: from now on, one would talk about them as representations of the same function" (Kieran & Sfard, 1999, p. 6). The fact that the table, graph, expression, and story are all viewed as representations of the same function seems to be the glue that tied together the understanding of students. Without referencing RF, the authors stressed the importance of the "development of students' ability to link expressions, graphs, tables, and stories" (Kieran & Sfard, 1999, p. 6), and students are prompted with the task to match three given expressions to their corresponding graph, table, and story.
To exemplify the ways in which these ideas were developed, Kieran and Sfard (1999) articulate that students discovered rules for simplifying expressions through tasks that had them graph several examples of functions and their sums (e.g., compare $f(x)$ and $g(x)$ to $f(x) + g(x)$) and expressions that had been multiplied by a constant (e.g., compare $h(x)$ to $5h(x)$). On solving equations, this particular teaching experiment stopped at graphical methods of solving (not going on to symbolic methods of solving) and it was reported that “the graphical approach alone proved sufficient for dealing effectively with all kinds of linear equations and inequalities” (p. 10) in addition to non-linear equations such as quadratics.

The major drawback of Kieran and Sfard’s (1999) publication is its brevity. We learn that the instructional models introduced the notion of graphs as a meaningful representation for students to come to understand and develop meaning for the symbolic representations. Example tasks are given from many of the instructional units focused on equivalence of expressions, but the design and implementation of this teaching experiment is not replicable based on this single publication. Furthermore, the role of technology was only sometimes apparent in the instructional activities. This is not to suggest that technology was expected to be used at all times, but instead, it is difficult to design instruction that effectively incorporates the role of technology. Thus the field would benefit from research reports on studies such as Kieran and Sfard’s work that give more attention to the appropriate role of the technology in promoting a functions approach to beginning algebra, and the difficult, albeit important, transition from expressions to equations. We see next how Kieran and Drijvers’ (2006) used the TTT framework to give more detailed attention to the role of technology in a teaching
experiment on the mathematical topic of the transition from equivalence of expressions to solving equations.

**Symbolic approach with CAS and P&P tools.** The teaching experiment by Kieran and Drijvers (2006) conducted with upper high school students in a combined CAS and P&P environment has been discussed previously. The focus here is on specific aspects of the teaching experiment that supported students’ progression of coming to understand the ideas of equivalence of equations and solving equations. In a broad overview, Kieran and Drijvers (2006) take an approach that assumed tight connections between students’ experiences in arithmetic and algebraic formalisms, ignored graphic representations, and focused instead on the meaning of equivalence and equality from a symbolic representation type.

Kieran and Drijvers (2006) experimentally tested the following developmental progression with upper high school students in a learning environment that utilized both CAS and P&P in studying equivalence, equality, and equation: (a) (dis-)connecting the numeric and the algebraic, (b) the notion of equivalence, (c) the issue of restrictions, and (d) coordination of solving an equation and the notion of equivalence. The tasks designed for Kieran and Drijvers’ (2006) teaching experiment spanned across three main activities (labeled as Activities 1, 2, and 3 in Table 2.2). Even though the tasks are not included here, each Activity had several parts (labeled as I, II, III, and IV as a sub-set of each Activity in Table 2.2). These activity parts together with the techniques (labeled as 1, 2, 3, 4, 5, and 6 in Table 2.2) with CAS and/or P&P are incorporated in Table 2.2 help to convey the nature of the tasks or problems with which students engaged. Note that Table
2.2 is a compilation of Kieran and Drijvers’ Tables 1 and 5. Reference to the theory (parts a-d) is also given next to each Task and Technique.

Read across each row of the table to see how the TTT elements are aligned. For example, Part I of Activity 1 utilizes the CAS in comparing expressions by numeric evaluation. The corresponding CAS technique is substitution using the with-operator ‘|’ and corresponds to the first part of the theoretical progression, (dis-)connecting the numeric and algebraic.

From an examination of Table 2.2, one notices that the developmental progression (a-d) spans several dimensions of the activities and corresponding techniques on CAS or P&P. In general, the ordering of the progression is sequential, with some overlap for particular parts of tasks. With explicit attention to the ways in which activities, technique, and theory were incorporated into Kieran and Drijvers’ (2006) teaching experiment we are able to glean specific information about the ways in which students’ learning of the concepts of expressions and equations were supported in coordinated CAS and P&P environments.
Table 2.2
*Tasks, Techniques, and Theory in Kieran and Drijvers’ (2006) Teaching Unit*

<table>
<thead>
<tr>
<th>Task (Activities, parts, and tools)</th>
<th>Technique (with CAS or P&amp;P variant)</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Activity 1 Equivalence of Expressions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I. Comparing expressions by numeric evaluation (CAS)</td>
<td>Substituting numeric values (with-operator ‘</td>
<td>’)</td>
</tr>
<tr>
<td>II. Comparing expressions by algebraic manipulation (P&amp;P)</td>
<td>Common form – by automatic simplification (manipulation by hand only to a limited extent)</td>
<td>c</td>
</tr>
<tr>
<td>III. Testing for equivalence by re-expressing the form of an expression – using the Expand command (CAS)</td>
<td>Common form – by expanding (expand command)</td>
<td>c</td>
</tr>
<tr>
<td>IV. Testing for equivalence without re-expressing the form of an expression – using a test of equality (CAS)</td>
<td>Test of equality (type equation then Enter key)</td>
<td>a, b, c</td>
</tr>
<tr>
<td><strong>Activity 2 Continuation of Equivalence of Expressions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I. Exploring and interpreting the effects of the Enter key and the Expand and Factor commands (CAS)</td>
<td>Common form – by automatic simplification (after pressing Enter key)</td>
<td>b, c</td>
</tr>
<tr>
<td>II. Showing equivalence of expressions by using various CAS approaches (CAS)</td>
<td>Common form – by factoring (factor command)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Common form – by expanding (expand command)</td>
<td></td>
</tr>
<tr>
<td><strong>Activity 3 Transition from Expressions to Equations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I. Introduction to the use of the Solve command (CAS)</td>
<td>Solving equations (solve command)</td>
<td>a, c, d</td>
</tr>
<tr>
<td>II. Expressions revisited and their subsequent integration into equations (CAS)</td>
<td>Test of equality (type equation then Enter key; manipulation by hand only to a limited extent)</td>
<td></td>
</tr>
<tr>
<td>III. Constructing equations and identities (P&amp;P)</td>
<td>Substituting numeric values (with-operator ‘</td>
<td>’; substitution, followed by evaluation by hand)</td>
</tr>
<tr>
<td>IV. Synthesis of various equation types (CAS)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One way to extend Kieran and Drijver’s (2006) study would be to incorporate elements of Kieran and Sfard’s (1999) learning trajectory (i.e., the graphical approach to equivalence) to better support students’ understanding of symbolic expressions. The graphic and numeric table counterparts to the symbolic representation types explored by Kieran and Drijvers were not included because they claimed that multi-representational
approaches to algebra had already been sufficiently covered in the literature. However, the use of CAS as a representational toolkit to address the progression from equivalence of expressions to solving equations with lower-ability algebra students is absent in the literature.

The Design of Instruction

Compatible with a goal of the design of the teaching experiment conducted in this study—to cultivate a change in students’ RF in the context of a CAS and P&P learning environment—the design of instruction matters. The research reviewed here informed the design of instructional support material and was tacitly embedded in data collection and other ongoing research activities throughout the collaborative teaching experiment, such as debriefing sessions.

Orchestration Metaphor

Drijvers and Trouche (2008) discuss instrumental orchestration as an extension of the instrumental approach that is specifically concerned with the role of the teacher, artifacts, and other resources available in classroom situations, and the practices that develop therein. The importance of the orchestration metaphor is that the teacher’s role is that of fine-tuning a set of instruments, allowing flexibility for students’ development yet giving structure to the mathematical learning situation. This metaphor provided professional insight into the design of CAS and P&P learning environments, but like the construct of instrumental genesis, was not used explicitly as a means to analyze the learning environment.
These researchers discuss various classroom configurations that revolve around the involvement of a “sherpa-student” (p. 378). In brief, the sherpa-student has his or her technology screen displayed so that it is socially prominent in the classroom (e.g., projected on a screen in the front of the classroom). Drijvers and Trouche (2008) outline several possible variants of this configuration that may support more or less productive roles for both teachers and students: (a) student work and mathematical representations can be the subject of classroom discussion or debate, (b) the teacher can guide students’ use of technology, (c) the teacher can draw connections between technology and P&P work, and (d) more students can be involved in classroom learning situations that may promote means of formatively assessing students’ progress and understanding (p. 378-379).

It is feasible that such a construct would serve as a useful lens for analyzing interactions in a classroom situation. More specifically, the orchestration metaphor highlights the role of the teacher as a “conductor” of classroom practices with a central focus on the ways in which technology can be situated to support students’ mathematical learning and instrumental genesis.

*Privileging Representation Types in Instruction*

Kendal and Stacey (2001) reported on the differences between two teacher’s privileging of representations and use of CAS in the teaching and learning topics in differentiation. They found that despite the fact that these teachers worked from a common curriculum plan, over a two-year program, the teachers tended to prefer different aspects of the mathematics, emphasize different representations, and use CAS
functionality in different ways. The students’ performance data generally reflected the privileging that was dominant in each teacher’s instruction.

For instance, one teacher (Teacher B) emphasized the conceptual underpinnings and meaning of differentiation and tended to use CAS less, whereas the other teacher (Teacher A) tended to emphasize the rules of differentiation and used CAS more, such as for computing the derivatives to non-routine problems. Kendal and Stacey (2001) videotaped the instruction of these teachers for two consecutive iterations of a twenty-lesson introduction to calculus sequence. Interviews with each teacher were also conducted at the beginning and end of the lesson sequence, and student questionnaires and assessment data were also collected. Each teacher’s privileging of representations, use of CAS, and connections to student learning outcomes will be discussed in turn.

Kendal and Stacey (2001) reported that Teacher B privileged both graphic and symbolic representations, but believed that symbolic representations were the most important representation. Students were expected to develop proficiency in by hand computations and the symbolic features of CAS were used mainly for discovery (e.g., pattern identification). This teacher also used CAS for pedagogical reasons, especially in linking symbolic and graphic representations (185 minutes of instruction) and linking graphic and numeric representations (5 minutes of instruction), and allowed students to use CAS for graphing.

Kendal and Stacey (2001) described that Teacher A had a strong preference for symbolic representations with some graphic and numeric representations being used for

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7 Kendal and Stacey (2001) discuss that although these characterizations are general, both teachers were found to make some changes their privileging of representations and CAS use from the first to the second implementation of the lesson sequence (e.g., see p. 159).
approximating derivatives on the CAS. In particular, the CAS was mainly used by the teacher and students for symbolic computations of derivatives, but was also for pedagogical purposes in linking graphic and numeric representations (85 minutes of instruction) and in linking graphic and symbolic representations (60 minutes of instruction).

On student outcomes, students of Teacher B, who Kendal and Stacey (2001) characterized as being content-focused with emphasis on conceptual understanding, showed superior performance on tasks requiring interpretation, and showed preference for symbolic representations and higher performance on such tasks. In comparison, the students of Teacher A, who were characterized as content-focused with an emphasis on performance, demonstrated superior performance at translating between representations (i.e., moving between representations for the purpose of formulating or reformulating problems).

In conclusion, Kendal and Stacey (2001) confirmed that the nature of instruction, use of CAS technology, and privileging of representations were all significant factors in measures of the student outcomes in this study. This finding puts into perspective the importance of understanding teacher’s instructional practices, preferences for representations, and the role of technology, in interpreting data on student learning outcomes.

*Lesson Design Principles*

In a teaching experiment conducted with nine year-ten teachers across two different schools, Pierce, Stacey, Wander, and Ball (2011) investigated effective lesson
design elements that utilized dynamically linked representations on TI-Nspire CAS. In Cycle 1 of the design experiment, one two-hour lesson was taught by each of six teachers, a focus group debriefing session between teachers and researchers informed revisions to the lesson, and three new teachers taught the lesson in Cycle 2 of the experiment. The lesson involved dynamic modeling and came at the end of a unit on quadratic functions, with the mathematical aim to “reinforce” students’ understanding of what they had already studied.

As a result of this study, Pierce et al. (2011) articulate four main design principles for lessons that capitalize on multiple, dynamically-linked representations: variable naming, reducing clutter and cognitive load, maintaining motivation, and keeping a clear mathematical focus. In brief, Pierce and colleagues warn that variable naming or the use of certain strings of letters for quantities that vary can interfere with the processes of linking representations, and suggest that care should be taken when choosing to use (and show or hide) the names of objects such as parameters or functions.8 In a related vein, it may be appropriate to use pre-designed sketch environments with students, recognizing the need to provide students ample time to “appreciate” the mathematical context in an environment they themselves did not create. Motivation and focus are discussed in more depth next.

Representational toolkits afford a multitude of possible approaches, yet time constraints are of real concern for classroom teachers. One lesson that was learned during

8While this point is well taken, the naming of objects in TI-Nspire technology can also be seen as an affordance of the technology. The same name can be referenced in multiple different mathematical environments (e.g., geometry, statistics, calculation), providing assistance in linking representations.
a debriefing focus group session between the two rounds of instructional experimentation with teachers in the design experiment was that “The most valued activities need to go near the beginning of the activity” (p. 108). In other words, the purposeful selection and sequencing of representations was an area on which the research team carefully reflected; as a consequence select representations and sequences were revised between cycles of the teaching experiment. As a design principle, the mathematical goals of a lesson can help to focus the lesson by informing decisions about the representations that most important.

Another issue that the research team encountered during Cycle 1 of the teaching experiment was motivating the need for alternative solution approaches. For instance, after a solution had been found, students were reportedly “reluctant” to solve the problem using other representations, indicating a lack of motivation for MR. Pierce and colleagues (2011) were careful to account for this issue in the Cycle 2 instructional design by making certain that “each new representation would be used to solve a new part of the problem and that the differences would be explicitly identified” (p. 110). For example, the manipulation of a dynamic figure was for the purpose of exploring a general conjecture about the problem. This open-ended conjecture based on empirical investigation was found to motivate the need to verify the conjecture with a symbolic algebra approach.

Pierce and colleagues (2011) give helpful details on the design principles that can be employed in lesson development that utilizes multiple dynamically linked representations with mathematics analysis software (e.g., CAS). Reflected in these principles, and evidenced by focus group debriefing sessions with researchers and teachers, some of the issues of implementation include: time constraints, teacher
preferences for representations, and student motivation. However, little is learned about
the learning outcome of students because of incomplete data from student surveys and,
possibly, the short duration of the study.

**Summary and Discussion of Future Directions**

The previous sections have illustrated how research that employs teaching experiment methodology can afford powerful connections between theory (e.g., on student learning) and practice (e.g., classroom activities). Some of the major contributions of such developmental research are the articulation of learning trajectories with specifically designed activities supported by instructional practices. Some of the most useful studies offer sufficient detail in their methods that allow for replication. With respect to CAS technology in particular, Pierce et al. (2011) articulated several lesson design principles that could be used to guide the use of MR in classroom learning situations and Kieran and Drijvers (2006) articulated aspects of tasks, technique, and theory that were tested with high school students learning concepts of equivalence and equations in combined P&P and CAS environments.

Overall, the complexity of capturing classroom-based mathematical activity and instruction is a daunting task, and all studies have a certain level of detail that could be better explained. In particular, some studies do not give sufficient details about the tasks in which students engaged (Arzarello & Robutti, 2010; Kieran & Sfard, 1999). As a whole, other aspects of the enacted curriculum such as classroom discussions are not always well-captured depending on the means of data collection and reporting of
classroom events. The research questions and goals of a study are recognized to guide the methodology and the decisions made to report on such details.

To support students’ meaning making and connections between representations, researchers advocate for the importance of classroom discussions with a specific focus on the role of representations. For instance, according to Suh, Johnston, Jamieson, and Mills (2008), “Representations can support connections, reasoning, communication, and problem solving. However, without promoting these mathematical ideas and verbalizing them in class discussion, the rich potential of learning can be lost” (p. 46). Representations are not imbued with meaning; teachers, students, and researchers all bring their own experiences and perspectives to the table when interpreting representations. To support students’ mathematical activity in classrooms teachers must recognize those differences and focus on making the mathematics salient (e.g., Goldin & Shteingold, 2001).

In a similar vein, NCTM (2000) advocates that “It is important for teachers to highlight ways in which different representations of the same objects can convey different information and to emphasize the importance of selecting representations suited to the particular mathematical task at hand (Yerushalmy & Schwartz, 1993; Moschkovich, Schoenfeld, & Arcavi, 1993)” (p. 363). Having discussions with students about why particular representations are chosen for a particular problem solving purpose may help them to understand that selection of representations is not an arbitrary decision, but can be done intentionally.

Teachers should strive to emphasize the reasoning behind why decisions are made to use a single representation or to call on MR either sequentially or simultaneously, and
to highlight the connections between representations. For studies concerned with RF, researchers should strive to be more explicit about the classroom conditions in which purposefully designed tasks are enacted, especially the nature of classroom discussions and other pedagogical techniques that may or may not be supportive of students’ development of RF. For example, in a synthesis of CAS research, Zbiek (2003) suggests that “additional studies are needed to define the role and balance of graphic, numeric, and symbolic representations in learning environments where CAS are used” (p. 211). It is conjectured that the analytic framework assumed for the present study will help in attending to some of these deficiencies, especially in the analysis of classroom lessons.

As reviewed in the previous sections, researchers have made significant contributions to positing specific design principles for instruction and curricular tasks that aim to support students’ development of RF. Using this as a foundation, there is a need for establishing hypothetical learning trajectories for lower ability algebra students to help them develop RF using both CAS and P&P tools. The research design and methodology of the proposed study is detailed in Chapter 3.
CHAPTER III

RESEARCH DESIGN AND METHODOLOGY

The following research questions were investigated from an approach that closely followed design research (Gravemeijer & Cobb, 2006):

1. How does students’ RF in solving problems involving linear equations change as a result of learning how to solve linear equation problems within a CAS and P&P environment?

2. Under what conditions does a group of ninth-grade algebra students change their RF in solving problems involving linear equations within a CAS and P&P environment?

The purpose of this research was: (a) to document students’ change in RF from the beginning to the end of a teaching experiment, (b) to come to understand characteristics of the learning ecology in which this change is situated, and (c) to develop an empirically grounded instructional theory of students’ change in RF in a CAS and P&P environment.

The specific methodologies employed to address these aims included both a teaching experiment conducted in collaboration with a classroom teacher (Cobb, 2000) and case studies of individual students (Stake, 1995).¹

Consistent with an underlying motivation for design research, the aim was to contribute to both research (e.g., domain-specific instructional theory) and practice (e.g.,

¹ The research methodology discussed in this paper was carried out in the manner approved by HSIRB (Appendix A).
heuristics for practical wisdom) in the field of mathematics education (cf. Gravemeijer & Cobb, 2006). To frame this chapter the motivation and theoretical orientation are elaborated first. This is followed by a discussion of the methodology involved in the teaching experiment and case studies.

Motivation and Theoretical Orientation

Leaders in the field of mathematics education posit that forging tighter links between research and practice is a priority (Arbaugh et al., 2010). Using the metaphor “border crossing” and exchange of “currency,” Silver (2003) highlighted what might seem like disparate goals of research and practice: researchers are concerned with theory development and practitioners are concerned with applications to instructional practice. For the present study, the theoretical intent was to understand the learning ecology of a combined CAS and P&P environment in which students’ change in RF might be well supported. This particular goal resonated as an issue significant to the advancement of research and the improvement of practice. Moreover, inspired by the contemporary issue of linking research and practice in mathematics education, and in light of the theoretical intent of this research, it was fitting to structure the study as design research.

Drawing on research by Cobb, Confrey, diSessa, Lehrer, and Schauble (2003), The Design-Based Research Collective (2003), and van den Akker, Gravemeijer, McKenney, and Nieveen (2006), Markworth (2010) summarized that “design research is the intensive and systematic study of an intervention in context. This process relies on cycles of design, implementation, and revision. Rather than ignoring or controlling issues of context, it incorporates these in the process. It is both practice- and theory-oriented, as
it attends to practical issues of implementation at the same time contributing to theories of learning” (p. 59). Design research—sometimes referred to as developmental research (Gravemeijer, 1994)—involves three main phases including instructional design, classroom-based teaching experiment, and retrospective analyses.

I assume an emergent perspective (Cobb & Yackel, 1996) in which both psychological and social aspects of learning in classroom situations are of importance. This theoretical orientation is particularly well-suited for classroom-based developmental research as it allows for a dual perspective on both individual students’ activity and cognition, and the related practices of the classroom community—it is assumed that the individual and community co-evolve. Guided by the research questions and goals of this study, both classroom practices and individual students’ activity and cognition were foci (see Row 3 of Figure 3.1).

<table>
<thead>
<tr>
<th>Social Perspective</th>
<th>Psychological Perspective</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom Social Norms</td>
<td>Beliefs about our own role, others’ role, and the general nature of mathematical activity</td>
</tr>
<tr>
<td>Socio-Mathematical Norms</td>
<td>Specifically mathematical beliefs and values</td>
</tr>
<tr>
<td>Classroom Mathematical Practices</td>
<td>Mathematical conceptions and activity</td>
</tr>
</tbody>
</table>

*Figure 3.1.* An interpretive framework (Gravemeijer & Cobb, 2006) that captures the reflexive relation between social and psychological aspects of the emergent perspective (Cobb & Yackel, 1996).

This interpretive lens helped ground several methodological decisions including the overall design and conduct of the study, the identification of pertinent data sources and collection activities, and the ongoing and retrospective interpretation of results. A classroom-based teaching experiment conducted in collaboration with a classroom
teacher (Cobb, 2000) was the methodology pursued from a social perspective, and case studies (Stake, 1995) of individual students within this classroom was the methodology that focused on the psychological perspective. These methodological decisions are consistent with an important assumption about the emergent perspective in that social practices are impacted by individual (psychological) processes and practices, and vice versa. In other words, the teaching experiment provided the social context to study classroom practices and conditions of the environment and the case studies of individuals who were involved in this environment provided a complementary psychological perspective on the changes that occurred in their activity and conceptions. The teaching experiment methodology, discussed in a later section, was purposefully crafted to link theoretical considerations of the learning environment with instructional design in the classroom.

Overall, following the design research approach of Gravemeijer and Cobb (2006), the research was conducted in three main phases: (a) development and preparation of a conjectured instructional theory, (b) testing of an instructional theory during a teaching experiment, and (c) revision of an instructional theory based on retrospective analysis of classroom practices. The teaching experiment was conducted in collaboration with a classroom teacher (Cobb, 2000). The collaborative nature of this research spanned the first two phases of research, with more intensive involvement of the teacher during the second phase in which she actually taught the instructional unit. Including the teacher in the retrospective analysis phase was beyond the scope of this study. Each phase of the design experiment is elaborated on in turn.
Phase One: Preparation for the Experiment

In the first phase a conjectured instructional theory of students’ development of RF in solving linear equations was developed based on a synthesis of the literature review as detailed in Chapter 2. This proposed theory included a preliminary account of the beginning and end goals of students’ learning. Based on this foundation, the thrust of this theory involved both a conjectured learning process and learning progression, together with means of support for that learning process. The beginning and end learning goals are discussed first.

End Learning Goals and Starting Points

The endpoints or “core ideas in this domain” (Gravemeijer & Cobb, 2006, p. 19) included understanding and solving linear equations with RF. As will be elaborated in the next section, this core idea was investigated from an approach that a solid understanding of the equals sign as an equivalence relation was a necessary grounding to having a meaningful experience in solving equations (Knuth, Stephens, McNeil, & Alibali, 2006). Consistent with a representational lens on mathematical activity, it was assumed that students’ representation-specific abilities would be necessarily intertwined with their mathematical abilities and could be taken as indicators of their mathematical cognition (Hiebert & Carpenter, 1992). Following Pea (1993), it was also conjectured that the CAS and P&P tools utilized by the teacher and students’ might influence the nature of students’ use of and communication about external representations.

Thus while the targeted learning processes of this design experiment span three intertwined dimensions—understanding mathematics content, developing RF, and using
mathematics tools—the central learning processes that students were to develop were those related to the construct of RF. Recall that RF is defined as the ability to create, interpret, transpose within, translate between, and connect tool-based graphic, symbolic, numeric, and verbal representations in doing and communicating about mathematics.

The mathematical learning goals were negotiated based on conversations with the participating teacher, the schools’ current algebra program, an examination of related literature, standards documents including the Common Core State Standards for Mathematics (Common Core State Standards Initiative [CCSSI], 2010), a learning trajectory display for the Common Core Standards for high school mathematics (Confrey, Maloney, & Nguyen, 2011), and various textbook curricula including reform-oriented, integrated, and traditionally-sequenced. The learning goals are described in four main points:

I. Develop RF with linear expressions and equations.

II. Develop RF in solving problems involving linear equations in one and/or two variables.

III. Understand the meaning of the equals sign as a statement of equality between two expressions.

IV. Understand solving equations as a process of reasoning and explain that reasoning.  

Leaders at the participating school site had expressed interest in piloting the model pathways put forth by CCSSI (2010) in the 2011-2012 school year. At the start of this research, this plan was put on hold due to changes in the district. Knowing that common assessments will be based on the CCSSM standards starting in 2014, the collaborating teacher was still interested in aligning instructional goals to meet these standards.

Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original
The instructional starting points for these learning goals were assessed based on students’ performance on a pre-test created by the researcher and teacher, and interviews with select students at the beginning of the study. The pre-test was designed to measure the aforementioned learning goals, and aspects of the learning progression described in the next section. This “unit” pre-test was given only to the algebra sections taught by the participating teacher and was distinct from the “department-wide” pre-test given across all algebra sections in the school.

The goals put forth by CCSSI should be noted here because it is possible that incoming freshmen had achieved these goals (as there were no evident instructional starting points posited by the school district to reference instead). “By the end of eighth grade, students have learned to solve linear equations in one variable and have applied graphic and algebraic methods to analyze and solve systems of linear equations in two variables” (CCSSI, 2010, p. 15). In light of this prediction of the instructional starting points, it was important to discern the nature of students’ understanding of these ideas (e.g., procedurally oriented and/or conceptually oriented).

Working under the assumption that students would have at least been exposed to the ideas of solving equations, students may indeed be well ready to understand these ideas at a deeper level, gaining mastery in solving techniques, and using CAS technology to support a multi-representational, conceptual approach to learning. It should also be noted that the third and fourth learning goals of understanding the meaning of the equals sign and explaining the process of solving as reasoning are significant extensions of the work done in prior grades.

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equation has a solution. Construct a viable argument to justify a solution method” (CCSSI, 2010, p. 65).
The conjectured learning process, and means of support are introduced in the next section as the conjectured instructional theory. Foreshadowing the second phase of the research, this structured instructional design also served as an object of analysis during the classroom-based teaching experiment and was revised through cyclical processes of experimentation and analysis.

**Conjectured Instructional Theory**

A conjectured instructional theory is defined to include “conjectures about a possible learning process together with conjectures about possible means of supporting that learning process” (Gravemeijer & Cobb, 2006, p. 21). Building on the learning goals specified above, each aspect of this theory is detailed next.

**Learning Process**

In brief, the conjectured learning process through which students were to learn to solve linear equations with RF was: to study equivalence of expressions, to study equations as equivalence relations that are true or false, and to study equation solving through MR approaches throughout. The main rationale for assuming the aforementioned sequence of learning progressions is based on the fact that in order to be successful in solving equations, students need to have a solid understanding of the meaning of the equals sign as an equivalence relation (Knuth et al., 2006). In the conjectured progression, an understanding of the equals sign was developed through the notion of equivalence because equations are viewed as equivalence relations. Hence it was conjectured that students who gain meaningful experiences with the notion of
equivalence would be primed to solve tasks involving linear equations, and to better understand and explain the equation solving process.

The conjectured learning process is introduced next in three “chapters” of the intended curriculum that guided the enactment of the teaching experiment. These three chapters are considered to be the main thrust of the conjectured instructional theory as they provided the framework for the instructional design and give a solid outline of the more detailed conjectured learning progression. This discussion is followed by an elaboration of seven different aspects of the conjectured learning progression. As a whole, this conjectured learning process and learning progression were woven together with a complementary sequence of goal-directed activities that was conjectured as a means of support, discussed in a subsequent section.

*Equivalent expressions and linear equations of functions.* The first chapter of the teaching experiment was planned to lay the foundation for some important experiences in understanding the notion of equivalence from numeric, graphic, and symbolic representations, and also in describing contextually based situations. Algebraic properties such as the distributive and commutative properties, and procedures based on algebraic properties such as “combining like terms” were to be introduced in meaningful ways, closely connected to the context and meaning of particular variables and corresponding numeric and/or graphic representations. The goal was for students to sharpen their skills

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4 The instructional unit was divided into three pieces, called chapters. In this manuscript they are referred to as Chapter 1, Chapter 2, and Chapter 3.

5 During the enactment of this teaching unit, such procedures were called “Strategies” and are referred to as “techniques” in the Task-Technique-Theory framework. On this distinction in language, the former is an artifact of the vernacular (and teacher preference), while the latter is an artifact of the research-guiding framework.
in reasoning quantitatively wherein contextually based numeric and graphic representations of expression drove reasoning and sense making with symbols.

It was anticipated that CAS would play an important role as students learned to experiment with examining the equivalence of expressions from MR. Students were to learn to use both CAS and P&P tools to transpose within and translate between MR of expressions. Bi-directional translations would be emphasized so that students could build intuition about the symbolic forms as they used CAS to test their conjectures. After students had examined equivalence from graphical, numerical, and contextual situations, tasks would focus on transpositions within symbolic representations with both CAS and P&P. Students would be introduced to the importance of reconciling results between representation types (e.g., graphic and symbolic) and between tools (e.g., symbolic transformations with P&P and CAS).

Equations in one variable as equivalence relations. The second chapter of the teaching experiment marks an important transition from expressions to equations. As students continue to explain contextually-based situations it was anticipated that they would use both numeric and graphic approaches to scaffold their creation of symbolic equations. Then, through experiences with highlighting the similarities and differences between equivalent and non-equivalent expressions, and testing for equivalence, it was conjectured that students would have a more meaningful foundation for creating and interpreting equations. In other words, by building on students’ understanding of equivalence of expressions from MR, the goal was to coordinate that understanding and knowledge with the meaning of the equals sign and solving equations in one variable. It was conjectured that students would progress to understand equations in one variable as
equivalence relations. Careful attention was to be given to solidifying the ideas of and language used to describe equivalence, equations, expressions, and the equal sign.

The main techniques that were to be used in this grouping of tasks included describing patterns of numbers with symbolic rules and equations and utilizing the symbolic capabilities and language of CAS (e.g., evaluating equations to determine their “truth” using the with-operator). Connections among graphic, symbolic, and numeric representations were to be emphasized as students used both CAS and P&P in constructing and interpreting equations as equivalence relations.⁶

Creating and solving linear equations. The first goal-directed learning task during the third chapter of the teaching experiment was designed to continue students’ experiences in creating and solving equations from MR in which students will be allowed (but not required) to use numeric and/or graphic representations to support their translation from contextual to symbolic representations. It was conjectured that students would come to the pinnacle of their equation solving experience with linear equations when they engaged in finding and generalizing a pattern for general linear equations, and learned to explain the process of solving equations by relying on the notion of equivalent equations. It was theorized that by building on experiences with solving equations from MR that students would be better equipped at predicting solutions to equations, and understanding the solving process as a process of reasoning that is meaningful).

⁶ In retrospect, this portion of the learning process was not well-specified; more could have been done to target specific connections in advance of the teaching experiment. However, as will be introduced later, the Cartesian Connection eventually became one area of focus for the revised theory.
As with most mathematics technology, common experiences with graphing and viewing tables of linear equations are driven by a symbolic representation. It was anticipated that students might focus on creating equations from graphic sketches and tables of values written by hand, then use CAS to check or reconcile this activity once a symbolic rule or equation was conjectured. In other cases, students might start with a given symbolic representation and use their CAS to perform transformations on symbolic representations, such as when identifying patterns and a general solution to linear equations.

When students have meaningful experiences with individual representations, and use other representations to support that meaning making (through making connections) students are developing their RF. As the ultimate goal of this instructional unit was for students to gain fluency and mastery in solving linear equations with MR, the work developed with equivalent expressions, and solving equations from MR was conjectured to be productive in supporting students’ abilities to explain the equation solving process.

*Learning progression.* In addition to the three main facets of the learning process described above, a more detailed trajectory of learning or learning progression was posited based on the research. The first four aspects of the learning progression (A1, A2, B, C) were initially targeted in Chapter 1 of the teaching unit. It was planned to introduce the next two aspects (D1, D2) in Chapter 2. The final aspect (E) was conjectured to occur in Chapter 3.

A1. *Connecting and generalizing the quantitative, visual, and verbal with symbols.*

Rooted in the numeric world, symbolic expressions generalize numeric, graphic, and verbal patterns by allowing for compact, abstract notation.
A2. Different representations/representation types can signify the same object.

Different representations/representation types of the same linear expressions
and/or equations signify the same relationship, pattern, or function from different
yet complementary perspectives.\(^7\)

B. *Equivalence of expressions from MR*. Expressions are equivalent if they define the
same relationship, pattern, or function. Thus equivalent expressions (a) can be
transformed to have identical symbolic forms/rules, (b) can be graphed as
equations of functions (e.g., \(f(x) = y = \exp 1\), \(g(x) = y = \exp 2\)) that have identical
graphs, (c) yield the same numeric output for a given numeric input, and (d) can
model the same situation.

C. *Domain and range restrictions*. For linear expressions and equations the issue of
restrictions arises mainly in contextual situations in which the domain and range
are sometimes restricted to integers or non-negative values. Assumptions about
domain should be considered when determining equivalence.

D1. *Solutions to equations can be determined by equality of expressions*. Linear
equations are relations between linear expressions that are sometimes, always, or
never equal in value. Thus linear equations have one, infinitely many, or zero
solutions, respectively. Solutions can be represented verbally, symbolically,
graphically, or numerically.

\(^7\)This perspective is not to diminish the fact that different representations and
representation types of the same mathematical object may present a loss or gain in
information. Instead, the goal is for students to recognize different
representations/representation types of the same mathematical entity as providing
differing perspectives yet complementary information.
D2. Solving equations in one variable is conceptualized as a comparison of two functions. Linear equations in one variable such as \( ax + b = cx + d \) for real-valued parameters \( a, b, c, \) and \( d \), can be solved for the variable \( x \) by comparing the functions \( f(x) = ax + b \) and \( g(x) = cx + d \) for the value of \( x \) that makes the equation \( ax + b = cx + d \) true. Graphical, tabular, or symbolic methods can be used.

E. Equivalence of equations. Equations are equivalent if they have the same solution set. Represented graphically, solution sets of equivalent equations are \( x \)-coordinates of the intersection points in the coordinate plane. Represented in tables, solution sets of equivalent equations are the inputs for which the outputs are the same.

In considering the seven aspects of the learning progression as a whole it is important to note that Aspects A1, B, C and D1 were based on an adaptation of a developmental progression that was posited and tested by Kieran and Drijvers (2006) in classroom-based research with tenth-grade students in a learning environment that utilized both CAS and P&P in studying equivalence, equality, and equation within the symbolic representation type. The structure of these components flows from expressions and equivalence relations to solving equations. The main way that these aspects have been adapted is with respect to MR. Specifically, Kieran and Drijver’s statement of the progression in Parts A1, B, and D1 were restricted to symbolic interpretations, whereas the conjectured progression here involves MR.

Kieran and Drijvers (2006) purposefully excluded graphic approaches (citing Heid, 1996; Kieran & Yerushalmy, 2004) and focused on symbolic and numeric
approaches only in non-contextualized situations. Kieran and Drijver’s decision to focus on CAS and P&P technique with symbolic representation types alone was based on the contention that limited research on the transformational aspects of algebra has been conducted in the context of the co-emergence of technique with both media (p. 211). In the present study, it was decided to extend Kieran and Drijver’s research by incorporating attention to verbal, situational, numeric, and graphical and symbolic representations. This decision is based on the assumption that ninth-grade algebra students are a critical population of students for which the power of CAS technology and its many affordances, when taken as a representational toolkit, can be used to facilitate more meaningful experiences with symbolic approaches through the use of MR.

Support for starting this learning progression sequence with A1 on graphic, numeric, verbal, and symbolic interpretations of expressions is based on Yerushalmy (2006). In solving contextually-based problems, lower ability algebra students have been found to use more intuitive approaches that involve numeric and graphic representations before using more sophisticated approaches involving symbolic representations (Yerushalmy, 2006). This sentiment is echoed by Yerushalmy and Chazan (2002) who cite Herscovics and Linchevski’s (1994) research that beginning algebra students tend to avoid to use symbolic representation types as initial strategies. Thus the beginning of this learning progression was aimed at building on students’ experiences with arithmetic so that when students encounter the difficult task of translating from contextual and/or verbally stated problems students will be able to use their intuitive notions of numbers and even of graphs before the abstract symbolism is expected.
The topic of A2 followed Kutzler (2010) and Kaput (1989) in that different representations/representation types of the same mathematical object give different yet complementary information about the structure of that object. From a multi-representational lens, this part of the progression was posited to be evident in the examination of equivalence of expressions, solving equations, and equivalence of equations.

On B, Schwartz and Yerushalmy (1992) agree that technology can be used as a representational toolkit to promote meaning in the equivalence of expressions from MR. In using MR to determine the equivalence of expressions, it is important to consider any domain restrictions, topic C. This issue was made evident in Kieran and Drijver’s (2006) learning progression that focused mainly on non-linear expressions and equations. When focused on linear expressions and equations, the concern mainly arises in contextual situations (e.g., integer domain).

On D1 (and intimately connected to learning goal III), Knuth et al. (2006) researched middle grade students’ understanding of the meaning of the equals sign and ability to solve a linear equation. Drawing on this research, it is possible that students develop an understanding of the equal sign in elementary and middle school grades. However, the results of Knuth et al.’s research support the conclusion that middle grades students’ understanding is likely an operational understanding (the equal sign does something), not a relational understanding (the equal sign represents equality of the expressions it relates). And more importantly, Knuth and colleagues (2006) contend that, "Students must understand the equal sign as expressing a relation in order to make sense of the transformations performed on such an equation" (p. 299). It is conjectured that
with an understanding of equations as equivalence relations, that students will come to
develop a more meaningful understanding of the equation solving processes.

In D2, the characterization of algebraic techniques discussed by Yerushalmy and Chazan (2002) is followed where an equation in one variable is viewed as the relation
between two functions in one variable. Parts D1 and D2, when considered together, yield
a more robust approach to understanding equation solving and equivalence from MR.

One source of tension that may arise from this view is that when functions in one variable
are graphed on graphing calculators (e.g., \( f(x) = x + 2 \)) they can also be viewed as
equations in two variables (e.g., \( y = x + 2 \)) (Yerushalmy & Chazan, 2002). In the former
view the solution set is in terms of \( x \) and can be represented as points on a line, whereas
in the later view the solution set includes ordered pairs \((x, y)\) and can be represented as
points in the coordinate plane. Ultimately, this issue will be more prominent when
students are studying solutions to linear equations in two variables.

Finally, the equivalence of equations, E, is intimately tied to the fourth learning
goal on understanding solving equations as a process of reasoning. This final part of the
progression was included based on a consideration of the mathematics that is involved in
reasoning through equation solving from a multi-representational perspective (e.g.,
Davis, 2005). This “capstone” goal was conjectured to build on students’ prior
understandings as evidenced in earlier parts of the learning progression. It was
conjectured that graphical, tabular, and verbal approaches to examining equivalence and
solving equations would prime students for being better able to understand and explain
the equation solving process. The means of support for this conjectured learning process
are now introduced.
**Means of Support**

The conjectured means of support for the specified learning process were essentially pedagogical resources informed by research. The specific means of support that were conjectured to facilitate students’ change in RF in a CAS and P&P environment span four dimensions. These types of means of support were based on an adaptation of the structure employed in a collaborative design experiment on measurement concepts discussed by Cobb (2003): (a) the tools students use to represent mathematical ideas, (b) classroom culture and proactive role of the teacher, (c) classroom activity structure, and (d) a sequence of goal-directed activities and instructional tasks.

The tools students and teachers used in this instructional experiment have already been discussed. On the second aspect, classroom expectations are norms or standards for classroom practice and discourse that are cultivated and negotiated between the teacher and students in the context of instruction and assessment. The conjectured design principles related to classroom expectations span each of the intertwined components of mathematics, tools, and representations:

1. *Focused on mathematics*. Students should be scaffolded to maintain a clear mathematical focus (Pierce, Stacey, Wander, & Ball, 2011). This is particularly apt in situations that are focused on: (a) reflecting on results, (b) reconciling CAS and P&P and (c) articulating features of the mathematical object that are “the same” across representations and the features that might be “masked” in certain representations.
2. **Strategic user of tools.** Students should learn the appropriate and strategic use of CAS technology (CCSSI, 2010) (e.g., testing conjectures, searching for patterns, and examining links between representations) and the need to bridge P&P generated and CAS generated results (Kieran & Saldanha, 2008).

3. **Representationally flexible.** Students should come to expect the need to be fluent and flexible within and among graphic, numeric, symbolic, verbal and situational representations (Rider, 2007).

The classroom activity structure constituted the third means of support that defined the conjectured instructional theory. The engagement in classroom activities was conjectured to encompass a sequence of activities that were both cognitively and activity oriented:

1. **Anticipate.** Anticipate or predict the results of translating between and transposing within representations (Pierce & Stacey, 2002).


3. **Reflect.** Reflect on the actions performed on tool-based representations and reconcile CAS and P&P results (Kieran & Saldanha, 2008).

4. **Connect.** Coordinate information across representations and make connections (Adu-Gyamfi, 2007).

This sequence of anticipating, acting, reflecting, and connecting representations was conjectured as a way to support students’ development of RF.

The fourth conjectured means of support involved a sequence of goal-directed activities. This research specifically drew on the Task-Technique-Theory (TTT)
framework explained by Kieran and Drijvers (2006) in which the task is the mathematics that students were asked to do or complete, the technique is the means by which the task was to be completed (i.e., how P&P and/or CAS tools are used), and the theory included the mathematical principles that undergird appropriate technique. This framework has been demonstrated to be useful in situations in which both CAS and P&P are used as tools in the teaching and learning of equivalence, equations, and solving techniques (see Table 2.1 in Chapter 2).

Tables 3.1, 3.2, and 3.3 outline several aspects of the TTT framework which served to frame the sequence of instructional activities, together with a consideration of the learning goals (I-IV) and learning progression (A-E), in the left-hand and right-hand columns, respectively. The sequence of activities was conjectured at the onset of the teaching experiment and was organized around the three main clusters of activity (or chapters) that were introduced above: (1) equivalent expressions and linear equations of functions, (2) equations as equivalence relations, and (3) creating and solving linear equations.
### Table 3.1

**Chapter 1: Equivalent Expressions, Linear Equations of Functions**

<table>
<thead>
<tr>
<th>Days (1-4), Task, Technique/Tool, Theory</th>
<th>Mathematical Goal (I-V), Learning Progression (A-E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Compare expressions by numeric evaluation and graphs; Translate verbal/contextual situations into number patterns; Translate verbal/contextual situations into symbolic rules; and Test symbolic rules using CAS with-operator ‘(</td>
<td>)’, P&amp;P substitution.</td>
</tr>
<tr>
<td><strong>Vocab:</strong> Variable, Equivalent Expressions, Function, Linear Function</td>
<td></td>
</tr>
<tr>
<td>2. Compare expressions by algebraic manipulation (P&amp;P common form); and Test for equivalence by re-expressing the form of an expression (CAS Expand, Factor command; Reconciling symbolic representations with P&amp;P and CAS).</td>
<td>I. Develop RF in solving problems involving linear equations in one and/or two variables. A2. B.</td>
</tr>
<tr>
<td><strong>Vocab:</strong> Equivalent Expressions, Distributive Property, Commutative Property of Addition/Multiplication, Expand, Factor</td>
<td></td>
</tr>
<tr>
<td>3. Interpret and write expressions and equations for contextualized situations that are both mathematically-based and real-world situations (P&amp;P); and Justify symbolic representations with graphs and tables.</td>
<td>I. II. B. C. Domain and range restrictions.</td>
</tr>
<tr>
<td><strong>Vocab:</strong> Rule, Verbal Representation, Symbolic Representation</td>
<td></td>
</tr>
<tr>
<td>4. Interpret equivalence through MR to solve problems (CAS symbolic—the Enter key and the Expand and Factor Commands, CAS graph—graphs page, CAS numeric—tables of graphs).</td>
<td>I. II. A. A2. B. C.</td>
</tr>
</tbody>
</table>
Table 3.2  
*Chapter 2: Equations as Equivalence Relations*

<table>
<thead>
<tr>
<th>Days (5-8). Task, Technique/Tool, Theory</th>
<th>Mathematical Goal (I-V), Learning Progression (A-E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. Create expressions to describe contextually based situations; Verify equivalence of expressions by using various approaches and representations (CAS, P&amp;P); and Introduce examination of equations from equivalence perspective.</td>
<td>I. Develop RF with linear expressions and equations. II. Develop RF in solving problems involving linear equations in one and/or two variables. III. Understand the meaning of the equals sign as a statement of equality between two expressions. A1. Connecting and generalizing the quantitative, visual, and verbal with symbols. A2. Different representation/representation types can signify the same object. B. Equivalence of expressions from MR. C. Domain and range restrictions.</td>
</tr>
</tbody>
</table>

6. Test for equivalence without re-expressing the form of an expression (CAS test of equality—type an equation and press Enter, result will be “true” if expressions are equivalent, result will be a rewritten equation if expressions are non-equivalent); and Explore and interpret the “truth” of an equation / the equality of two expressions with graphic and numeric (table, CAS with-operator ‘l’) representations (P&P, CAS).  
*Vocab: Equation as Equivalence Relation; Solution to Equation*

7-8. Construct equations from equivalent, non-equivalent expressions; and Interpret solutions to equations from equivalence perspective.  
*Vocab: Solving Equations*  
I.  
II.  
III.  
B.  
D1.  
D2. *Solving equations in one variable is conceptualized as a comparison of two functions.*
### Table 3.3

**Chapter 3: Creating and Solving Linear Equations**

<table>
<thead>
<tr>
<th>Days (9-15), Task, Technique/Tool, Theory</th>
<th>Mathematical Goal (I-V), Learning Progression (A-E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. Create equations to model contextual situations; Determine solutions to equations from tables and graphs; and Solve problems involving linear equations from MR (P&amp;P; Reconciling P&amp;P and CAS; coordinating graphic, numeric, symbolic representations)</td>
<td>I. Develop RF with linear expressions and equations. II. Develop RF in solving problems involving linear equations in one and/or two variables. III. Understand the meaning of the equals sign as a statement of equality between two expressions. A1. Connecting and generalizing the quantitative, visual, and verbal with symbols. C. Domain and range restrictions. D2. Solving equations is one variable is conceptualized as a comparison of two functions.</td>
</tr>
<tr>
<td>12-13. Use CAS to identify patterns in solving linear equations; Conjecture and prove the general solution to a linear equation.</td>
<td>II. IV. Understand solving equations as a process of reasoning and explain that reasoning.</td>
</tr>
<tr>
<td>14-15. Solve linear equations and explain the reasoning process based on: Pattern generalizations for general linear equations; and Equivalence of equations</td>
<td>II. IV. E.</td>
</tr>
</tbody>
</table>
The formulation of this sequence was created in three main cycles of research and development. First Kieran and Drijvers’ (2006) framework was used as an overarching structure and foundation for the sequence. Secondly, the mathematical progression and learning goals were articulated to expand beyond those considered by Kieran and Drijvers to those more appropriate to the present study. Third, a review of relevant literature informed further refinement and articulation of the revised sequence of activities. In this third stage, most of the research and expository pieces that were consulted were selected based on the merit that they focused on tasks and design principles in the research domains of CAS and other digital technology, MR, and RF, including mathematical contexts such as equivalence, equations, equation solving, and functions (e.g., Cedillo & Kieran, 2003; Davis, 2005; Edwards, 2003; Heid & Edwards, 2001; Kieran & Drijvers, 2006; Yerushalmy & Chazan, 2002). Recall that for the following explanation of the three chapters of the teaching experiment, these ideas were posited before the teaching experiment and thus they were conjectures as to how the instructional sequence would unfold with students; they are not representative of what actually took place.

Prior to the conduct of the proposed study, the teacher and researcher engaged in several informal piloting situations together and often met to discuss and clarify the project goals, and to learn about the demands and needs of the school site and students in the context of introducing TI-Nspire CAS technology. Based on these collaborative experiences it was decided to adhere to the following plan to facilitate the collaborative development of lesson plans and activities prior to the teaching experiment: (a) discuss the initial sequence of learning goals and tasks, techniques, and theory; (b) articulate and
elaborate on how CAS technology would be utilized and how students’ RF would be targeted; (c) work independently on select components or aspects of lesson development, communicating via email for consultation and clarification as necessary; and (d) meet to discuss what had been developed individually.

This process was repeated until both the researcher and teacher were familiar enough with the proposed plan for the teaching experiment, as indicated by a verbal discussion that both parties were ready for the next steps. It was also recognized that leaving room for flexibility in these plans was acceptable. For planning purposes related to teacher compensation for participating in the study, it was anticipated that this would require approximately 15 hours on the part of the teacher, and approximately 30-45 hours on the part of the researcher. Once further details of lesson structures and plans were decided between the researcher and teacher, these plans were communicated with a dissertation committee member before each lesson was enacted. The purpose of this consultation was to seek input from other experts, and to verify that the plans met the research goals.

Summary

Following the perspectives of Gravemeijer and Cobb (2006), Cobb (2003), and Kieran and Drijvers (2006), several elements were key to the posited instructional theory in a combined CAS and P&P environment: (a) an elaboration of learning goals with attention to starting points, (b) theoretical aspects including processes by which student thinking and learning may unfold, (c) learning activities including tasks and techniques that were planned to be enacted in the classroom to support learning, and (d) other means
of support for students’ learning including classroom expectations and activity structure. All of these aspects of design have been accounted for in the above discussion.

It is important to recall that the conjectured instructional theory was a prediction about how students might change their RF in the context of solving linear equations with CAS and P&P. The proposed means of support, especially classroom expectations and activity structure were infused into planning sessions, lesson and activity design, and other communications between the researcher and teacher. Recall however, that throughout the cyclic process of development, revision, and testing of ideas, aspects of the context such as instructional strategies of the teacher were incorporated into design research rather than ignored or controlled for (cf. Markworth, 2010).

Phase Two: Experimentation

The initial phase of preparing for the experiment was followed by a secondary phase of ongoing experimentation in the classroom via teaching experiment methodology conducted in collaboration with a classroom teacher (Cobb, 2000). Both the researcher and teacher collaborated on the design and development of a sequence of instructional activities based on a conjectured learning progression and means of support. The classroom teacher served as the primary instructor for all class sessions. While the teaching experiment was in progress, conjectures about conditions under which students change their RF were tested and revised on a daily and weekly basis. These daily cycles of instructional experimentation and ongoing analysis were supported by regular debriefing sessions that focused on analysis of the hypothetical learning trajectory and changes to this trajectory in relation to the conjectured instructional theory.
The first subsection that follows outlines the timeline of all research activities that took place during the second phase of the research. The second and third subsections elaborate on the research context of the study and the data collection, respectively. The discussion of phase two closes with an elaboration of the ongoing analysis and instructional experimentation.

**Timeline of Research Activities**

Including assessments (prior to, during, and after the instructional unit), debriefing sessions, and interviews with students (initial and final), it was anticipated that the entire teaching experiment would span twenty days. The justification for a total of four weeks for this unit was based on the planning structure at the school site. Algebra 1 is taught in two trimesters; there are a total of twelve weeks of instructional time per trimester and this is one unit out of three that would be taught in this first trimester. In total, the study spanned twenty-seven days with five weeks of instructional activity, pre- and post-tests, and interviews with students. The researcher worked in close collaboration with the teacher in all aspects of planning and scheduling. A timeline of the main activities of research and practice is given in Table 3.4.
Table 3.4
Timeline of Research and Teaching Activities

<table>
<thead>
<tr>
<th>Phase of Research</th>
<th>Dates</th>
<th>Research and Teaching Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase One</td>
<td>August 18</td>
<td>Develop curriculum and plans with teacher</td>
</tr>
<tr>
<td></td>
<td>September 7</td>
<td>First full day of school, acquire student/parent consent</td>
</tr>
<tr>
<td>Phase Two</td>
<td>Days 0-26 (Sept. 9–Oct. 18)</td>
<td>Teaching experiment, student interviews, ongoing analysis</td>
</tr>
<tr>
<td></td>
<td>Day 0 (Sept. 9)</td>
<td>Administer pre-test to class</td>
</tr>
<tr>
<td></td>
<td>Day 1 (Sept. 12)</td>
<td>Begin instructional unit (Chapters 1-3)</td>
</tr>
<tr>
<td></td>
<td>Days 2–4 (Sept. 13-15)</td>
<td>Initial interviews with three select students</td>
</tr>
<tr>
<td></td>
<td>Day 23 (Oct. 13)</td>
<td>End of instructional unit</td>
</tr>
<tr>
<td></td>
<td>Day 24 (Oct. 14)</td>
<td>Administer post-test to class</td>
</tr>
<tr>
<td></td>
<td>Days 25–26 (Oct. 17-18)</td>
<td>Final interviews with three select students</td>
</tr>
<tr>
<td>Phase Three</td>
<td>October–Spring</td>
<td>Retrospective analyses of data</td>
</tr>
<tr>
<td></td>
<td>Stage 1</td>
<td>Individual’s change in RF</td>
</tr>
<tr>
<td></td>
<td>Stage 2</td>
<td>Conditions of environment and classroom practices</td>
</tr>
<tr>
<td></td>
<td>Stage 3</td>
<td>Reconstruct a revised instructional theory</td>
</tr>
</tbody>
</table>

As shown in the timeline of research activities and teaching activities, the data collection activities associated with the teaching experiment and case studies were conducted simultaneously. Details on this coordinated methodology will become evident in the ensuing discussion of the research context, data collection, and ongoing analysis and experimentation. A discussion of the retrospective analysis of the teaching experiment is given in the next section, phase three of the research. Details on the case studies appear thereafter.

Research Context

The school site where the teaching experiment took place, hereafter referred to by the pseudonym “South High,” is one of three public high schools in an urban district near
a large university in the Midwest of the United States. This site was selected based on successful recruiting of a teacher, hereafter referred to as “Ms. L,” who had expressed interest in the goals of the study⁸. One of Ms. L’s Algebra A classes, the first trimester of ninth-grade algebra, was purposefully selected based on Ms. L’s schedule, representing a typical class. The teacher’s and her students’ roles in this research context are described next. The role of the researcher is discussed in a later section focused on the ongoing analysis and experimentation.

Teacher Participant

The main selection criterion for recruiting a teacher included: (a) the teacher had been teaching for approximately four years so that the teacher had established routines but were willing to experiment with new ideas, (b) he/she had some experience teaching with handheld graphing calculator technology and had experience with or was interested in incorporating TI-Nspire CAS technology into their instruction, and (c) they shared the goal of developing students’ RF with technology-based and P&P representations. Ms. L fit all these criteria and was compensated for participating in this study. Specifically, the teacher earned university credit for this professional development experience.

The 2011-2012 school year was Ms. L’s fifth year of teaching. During her first four years of teaching she had taught all freshman mathematics courses of which the adopted textbooks included Algebra Connections (Dietiker, Kysh, Sallee, & Hoey, 2006a), and Geometry Connections (Dietiker, Kysh, Sallee, & Hoey, 2006b) from the

⁸ The teacher recruitment process was initially based on email communication with a teacher who was a former student of a dissertation committee member. The teacher and researcher subsequently developed a relationship through experiences in collaborating on lesson development and technology integration.
College Preparatory Mathematics curriculum series. South High had also recently adopted Holt McDougal Algebra 1 (Burger et al., 2011) and it was a department wide expectation that teachers would coordinate the use of both curriculum series in their instruction. Ms. L has a classroom set of TI-84+ Silver edition calculators and based on classroom observations and self-report data prior to the experiment she regularly used them with her students for activities including creating graphs and tables.

Student Participants

Approximately 68 percent of the student population at South High qualifies for free and reduced lunch.\(^9\) In general, the students in the incoming ninth-grade class attended one of two middle schools, both of which had adopted the *Connected Mathematics* curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) for use in their mathematics classes for several years prior to this study. The students taking Ms. L’s fifth-hour Algebra I course were all “pure freshmen,” with no upper class students.

The researcher recruited all students in this select classroom to use classroom video for research purposes and to obtain achievement data from their course assessments. The researcher also recruited three select individuals within this participant pool to be followed more closely during daily class sessions and in interviews. The selection of these three participants is discussed in the Case Studies section.

Both a mailing and in-person recruitment script were delivered to students as part of the recruitment process (see Appendix B for both scripts). Students who chose to

\(^9\) These are approximations based on teacher-report data (Ms. L, personal communication, June 28, 2011).
participate in the study were awarded extra credit points toward their course grade. Students turned in parent consent forms and student assent forms to the researcher during the first week of class, regardless of their intent to participate (see Appendix C for consent and assent documentation). For students who chose not to participate they were not interviewed, their work was not collected, and if captured on the classroom video their faces were edited out so as to conceal their identity.

**Data Collection**

The main data collection activities were organized around two main research methodologies: the teaching experiment and case studies. At a broad level, data for the teaching experiment were collected during daily instructional experimentation and daily and weekly thought experiments. For each of the case studies, data were collected at the beginning and end of the teaching experiment through semi-structured interviews based on students’ pre-test and post-test performance. Data for each case were also collected throughout the teaching experiment on a daily basis. The data sources and their main points of analysis are summarized in Figure 3.2.

Guided by the interpretive lens assumed in this study, the primary data sources that were collected with a stronger emphasis on the social aspects of the learning environment included (1-7): (1) observational field notes, (2) classroom artifacts (including annotated lesson plans, and instructional activities), (3) written and audio records of daily and weekly debriefing sessions between the researcher, (4) daily class

10 Note that all students in this class, regardless of their decision to participate, had similar opportunities to earn extra credit points toward their course grade.
summaries, (5) personal reflection journals, (6) weekly class summaries, and (7) whole-
class classroom video.

<table>
<thead>
<tr>
<th></th>
<th>Daily Ongoing Analysis</th>
<th>Weekly Ongoing Analysis</th>
<th>Retrospective Analysis of Teaching Experiment</th>
<th>Select Students for Case Studies</th>
<th>Retrospective Analysis of Case Studies</th>
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<td>Class Summary</td>
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<td>Classwork</td>
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</tr>
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<td>(12) Student</td>
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<tr>
<td>Interviews</td>
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</tbody>
</table>

*Figure 3.2. Data sources and their main points of analysis.*
The primary data sources that were collected with a greater emphasis on understanding the psychological aspects of the analysis included (8-12): (8) student artifacts (including students’ homework, activities, TI-Nspire files, and minor assessments), (9) video records of individual students’ classroom activity, (10) pre-test, (11) post-test, and (12) an initial and final video-recorded semi-structured interview with three select students.

Data sources 1-5, and 7-9 were collected on a daily basis throughout the teaching experiment whereas data source 6 was created on a weekly basis. Data sources 10-12 were collected at the beginning and end of the teaching experiment only. A more articulated discussion of how select data sources were used during the ongoing and retrospective analyses of the teaching experiments and case studies is evident in the appropriate sections that follow.

Experimentation and Ongoing Analysis

Throughout the conduct of the classroom-based teaching experiment, the researcher was engaged in ongoing analysis and experimentation in collaboration with the classroom teacher (Cobb, 2000). Each day the researcher spent approximately 3-5 ½ hours at the school site. On a daily basis, the teacher-researcher team worked together on "conjecturing, enacting, and revising hypothetical learning trajectories" (Gravemeijer & Cobb, 2006, p. 25). A hypothetical learning trajectory (HLT) includes the goals, sequence of goal directed activities, and conjectures about students’ learning (Simon, 1995). In this sense, it is a more specific instantiation of the conjectured instructional theory. Indeed, this process can be depicted as daily cycles of experimentation and ongoing analysis that
are directly related to and informed by the conjectured instructional theory (see Figure 3.3).

![Diagram of conjectured local instruction theory](image)

Figure 3.3. The reflexive relation between theory and practice is shown by Gravemeijer and Cobb’s (2006, p. 28) depiction of daily cycles of instructional experimentation and thought experiments.

As shown in Figure 3.3, both instructional experimentation and ongoing analysis (i.e., thought experiments) occurred throughout the teaching experiment. Cobb (2000) explained the importance of analyzing how the teaching experiment unfolded so that it could inform both the refinement to the learning trajectory and the articulation of students’ learning processes as situated in the social context of the classroom. For the present study, the ongoing analysis of classroom activities was documented on a daily and weekly basis.

Also note that prior to the conduct of the teaching experiment, the teacher and researcher met to engage in collaborative planning and development of conjectured HLTs. The intent of these meetings was to: (a) involve the teacher in clarifying the goals and trajectory of learning posited in the conjectured instructional theory, (b) support the teacher in coming to learn TI-Nspire CAS technology for teaching particular content, and
(c) ensure that district curriculum expectations including attention to the middle school to high school transition were carefully considered. However, most of the collaborative activity around the conjectured, enacted, and revised HLT, occurred on a daily basis after the day’s instruction. Before articulating each of these activities of the second phase of the research it is important to recall the interpretative framework that guided the researchers’ activities.

*Interpretive Framework*

During ongoing analysis and experimentation the researcher identified critical moments using the social aspect of mathematical practices from Cobb and Yackel’s (1996) interpretive framework to guide this analysis. According to Gravemeijer and Cobb (2006), "A mathematical practice can be described as the normative ways of acting, communicating and symbolizing mathematically at a given moment in time ... mathematical practices are specific to particular mathematical ideas or concepts. In addition, mathematical practices necessarily evolve in the course of an experiment" (p. 32). In light of this definition, one mechanism to keep track of initial conjectures about conditions of the learning environment (or mathematical practices) was to identify a critical moment (CM) during ongoing analysis and experimentation.

CMs were defined to involve interactions at the classroom level\(^\text{11}\) that appeared to support or contradict the conjectured instructional theory (i.e., learning goals, learning progression, and means of support). In other words, there were two types of CMs: (a)

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\(^{11}\) The lens on classroom-level interactions was limited to those that were addressed to the entire class. Thus student-to-student (and other small-group interactions) were excluded. The decision to limit the analysis in this way is justified by the interpretive lens adopted for this study.
those that are consistent with the instructional theory, and (b) those that contradict or provide a counterexample to the proposed instructional theory. CMs were first identified during the enactment of the teaching experiment; they also informed discussion during the debriefing sessions, helped identify segments of classroom video to review during ongoing analysis, and were recorded in daily and weekly class summaries. By design, these CMs informed the conjectured HLTs, were identified during enactment of HLTs, and were objects of analysis and reflection to revise HLTs.

*Conjectured HLTs*

Based on experience in conducting classroom-based teaching experiments Cobb (2000) stated that, “we have found it counterproductive to plan the details of specific instructional activities more than a day or two in advance” (p. 320). Thus as expected, the planning of details of classroom activities (e.g., specific lesson notes) occurred during the teaching experiment and throughout the ongoing analysis on a daily basis. As alluded to above, the development and refinement of classroom activities was largely a collaborative process between the researcher and teacher both before and during the teaching experiment. However, the researcher took leadership in creating instructional activities in that most of the writing of new tasks were done on her personal computer, unless the teacher offered to use her classroom computer to write them while we met together. It was often the case that the teacher did not have time to write curriculum materials on a daily basis yet she often provided feedback on activity design during meetings with the researcher or on the weekends. The activity design was always discussed between the teacher and researcher before enactment.
Specific conjectures about how students might think about these activities, ways to support that student thinking through instruction, and the basis of activities in research or instructional resources were specified for each activity. In particular, each instructional activity was specifically designed to fit a component of the Task-Technique-Theory framework. From this framework, tasks (e.g., mathematical activities), techniques with tools (e.g., what you do to accomplish the mathematical activities), and theoretical ideas (e.g., the research that undergirds the proposed activities and theory) were specified. The conjectured local instruction theory, the enacted HLT from previous days, issues of enactment, and the conjectured instructional theory as a whole were discussed and taken into consideration during the development process.

Enact HLTs

Before each class session, the daily lesson plans and activities were communicated via email and/or in person. Once at the school site, the researcher and teacher set up appropriate technology including any TI-Nspire documents, hard copies of activity sheets, and video recording equipment. The researcher and teacher sometimes informally discussed issues, ideas, or particulars about the day’s lesson in person. When appropriate, the researcher focused these daily discussions with the teacher (both before and after class, discussed later) toward students’ representation-specific activity and discourse related to students’ connections between representations. Due to the collaborative nature of this research, the teacher often had additional priority issues to discuss including design and enactment of activities or other student-related concerns.
Summaries of these discussions were recorded on the observation protocol and/or typed into the researcher’s journal at the end of each day.

The teacher was the primary instructor and led all whole-class instruction, discussions, and orchestration of technology. During teacher-led whole-class discussions and instructional time the researcher was situated in the back of the classroom near where the classroom video camera and individual video cameras were set up. This allowed for the researcher to operate the video camera, and to record observational field notes. Daily classroom field notes were recorded on a specifically designed protocol (Appendix D). This protocol was developed for three main purposes: (a) to document evidence for conditions under which students develop RF in classroom practices (i.e., activities, tools, discourse, activity structure); (b) to take note of student conceptions and activity that seemed to support change in students’ RF over time (using the definition of and measures for RF); and (c) to keep records of data collection and management including video records of individual and class activity and what was displayed and recorded on the board. Note that the first two purposes are directly connected to the research questions.

During individual or small group work time the researcher often circulated around the room to more closely observe how students were engaging with tasks and talking about their mathematical activity. As the researcher gained familiarity with the students, she would address students, answer questions related to the mathematical activities of the day, and refocus their attention to stay on task. The researcher also sometimes asked students questions to elicit their thinking, but did not re-teach or funnel students’ engagement with particular tasks. The researcher also sometimes assisted students and/or
the teacher with using CAS technology. Notes on these types of researcher participation were recorded on the observation protocol.

In light of the aforementioned role of the researcher in the classroom, there were two anticipated exceptions that provoked the researcher to play a more active role during instruction: (a) for CAS-related issues or technical difficulties that the teacher and/or students were not able to reconcile, or (b) for on-the-fly consultations between the teacher and researcher that may have impacted the teacher’s decisions. Both types of participation occurred sporadically throughout the teaching experiment. For example, the teacher sometimes requested that the researcher help to identify student volunteers to share their thinking with the class for warm-up activities. Sometimes the teacher also requested assistance from the researcher in demonstrating the use of TI-Nspire CAS teacher edition software. It was recognized that when the researcher interacted with the teacher and students in these ways that it may have altered how the lessons unfolded. The researcher documented such interactions in her field notes, and noted potential issues to be addressed for activity and instructional design in future lessons.

*Revise HLTs*

Debriefing sessions and thought experiments informed the revision of the daily HLTs. The main method of documenting these processes was through audio-recorded daily and weekly meetings with the teacher, written summaries of those sessions, and records of the researchers’ reflections throughout the teaching experiment. The thought experiments that occurred on a daily and weekly basis aimed to connect daily experimentation with theory (see Figure 3.3 from Gravemeijer & Cobb, 2006). Following
Cobb (2000), the primary purpose of having short daily debriefings immediately after
teaching and longer weekly meetings (discussed later) is:

...to develop, consensual or ‘taken-as-shared’ interpretations of what might be
going on in the classroom. These ongoing analyses of individual children’s
activity and of classroom social processes inform new thought experiments in the
course of which conjectures about possible learning trajectories are revised
frequently. As a consequence, there is often an almost daily modification of local
learning goals, instructional activities, and social goals for the classroom
participation structure (p. 320).

Indeed, the revision of HLTs occurred throughout the ongoing experimentation. In
particular, the researcher engaged in several thought experiments on the progress of the
classroom mathematical practices and individual’s mathematical activities and
conceptions. To varying degrees, the researcher, teacher, and one dissertation committee
member were involved in these conversations.

After each class session, the teacher-researcher team spent time debriefing the
current lesson to understand what was enacted, and also engaged in the creative process
of revising the HLT based on specific learning processes and means of support (including
more fine-grained daily learning goals and tasks). To inform these conversations,
oftentimes a variety of student work was examined, and differing perspectives on student
engagement and activity during the lesson were shared. Oftentimes these debriefing
sessions also involved the creation of new activities, a refinement of daily learning goals,
and the sharing of CAS techniques.

As soon as possible after each class session and debriefing meeting, the researcher
wrote a brief “Daily Class Summary” about the current and subsequent lessons to capture
these thought experiments. The two foci for these daily summaries were: (a) give a
summary of “Critical Moments” related to learning goals, learning progression, and
means of support, and (b) provide a summary of and rationale for changes to the learning trajectory. Several data sources informed this writing including the observational field notes, classroom artifacts (from both the teacher and select students), and classroom video. This reflection was shared with the teacher via email as soon as possible that evening with the intent to facilitate better communication between the teacher and researcher. This communication was improved when the teacher read these summaries on a daily or weekly basis (at the beginning of the experiment the teacher admitted to not having time to read them).

In addition to shorter daily debriefing sessions, the researcher and teacher met on a weekly basis for an extended debriefing session. The researcher recorded field notes during these audio-recorded debriefing meetings. The main purpose of these meetings was to document planned changes and reflections on the learning trajectory as a whole. Ideas for modifications to the next week’s planned lessons and activities were also discussed. In most cases the teacher recorded these plans discussed during our meeting in a weekly lesson plan template that was also shared with the school principal. The weekly lesson structure included attention to the Introduction, Lesson Title/Objective, Activity, Homework Assignment, and Closure.

Finally, at the end of each week and after the weekly debriefing between the teacher and researcher, the researcher also met with one dissertation committee member to discuss the progress of the teaching experiment. The primary goal of these conversations was to give an additional interpretation of how the teaching experiment was unfolding. These conversations often included reflection on particular challenges and
struggles faced throughout the week, and on how the instructional theory was changing.

Audio and written records of these meetings were recorded.

After both weekly debriefing sessions the researcher wrote a “Weekly Class Summary.” This summary was based on an analysis of the notes taken during the daily and weekly debriefing sessions with the teacher and dissertation committee member, the written “Daily Class Summaries,” classroom and individual artifacts, and classroom video. There were four specific foci for these weekly thought experiments: (a) document the progress of teaching experiment with respect to addressing learning goals and learning progression, (b) provide rationale for changes to learning trajectory or conjectured theory, (c) record thought experiments on how well instruction seems to be supporting students’ learning, and (d) give interpretations of “Critical Moments” based on debriefing sessions with the teacher and a dissertation committee member.

The aforementioned summarizing techniques were intended to aid in the process of analyzing the learning progression and served to frame and orient the retrospective analysis in Phase Three. Additionally, conjectures about students’ change in RF were often incorporated into the writing of daily and/or weekly class summaries. These conjectures were based on the researcher and teacher’s analysis of student work that was collected on a daily basis. Other purposes of these reflections were to be immersed in the data, to be well informed of the classroom events, and to further facilitate making strategic and informed decisions about the learning trajectory, as discussed with the teacher. The researcher also kept a daily personal journal during the teaching experiment that was not shared. As a reflective practitioner, this journal served as an important
avenue to tell the story of the teaching experiment without reservations or a structure to follow.

In summary, through iterative cycles of development, implementation, and analysis, various levels of conducting and analyzing the teaching experiment formed cohesive layers that spanned both aspects of research and practice. The cycle of planning, enactment, and revision discussed above soon became a routine in which thought experiments about student and class development were of focus. The third phase of the design experiment, retrospective analysis, is introduced next.

Phase Three: Retrospective Analysis

Once the teaching experiment concluded, the final phase of design research began—retrospective analysis of data. The aims of the retrospective analysis were to (a) develop an empirically grounded instructional theory of students’ change in RF in a CAS and P&P environment, and (b) come to better understand the characteristics of the learning ecology in which this change was situated. To systematically address these aims, all data sources were analyzed chronologically to confirm and refute conjectures about the conjectured instruction theory.

The retrospective analysis occurred in two phases: (1) initial conjectures about conditions under which students seemed to change their RF were confirmed and refuted, and (2) a sequence of instructional activities and other conditions of the learning environment were reconstructed to represent a revised instructional theory. Both phases, discussed in turn next, were geared toward answering the second research question.
Phase One: Constant Comparative Analysis of Chronological Episodes

To understand the conditions of the learning environment of the teaching experiment, the objective of the first phase of the retrospective analysis was to develop a sequence of conjectures and refutations that were tied to specific episodes, or in this case, days and activities of the teaching experiment (Gravemeijer & Cobb, 2006, p. 39). Initial conjectures were developed and documented during the teaching experiment itself as CMs. New conjectures were also formulated during the retrospective analysis. All conjectured conditions of the learning environment were subject to analysis to confirm or refute their merit based on the available data sources and emergent trends and patterns in the data.

Following what Gravemeijer and Cobb (2006) described as resembling a constant comparative method (Glaser & Strauss, 1967), initial conjectures about the conditions of the learning environment were confirmed or refuted based on evidence from the episode; these conditions were then tested against the subsequent episode. This analysis was geared toward "look[ing] for patterns that may explain the progress of the students" (Gravemeijer & Cobb, 2006, p. 44), that ultimately resulted in a refined instructional theory. The data sources examined for each episode included: (1)\textsuperscript{12} researcher observational field notes, (2) annotated lesson plans and instructional activities, (3) teacher-researcher debriefing session audio and/or notes, (4) researcher daily class summary, (5) researcher daily reflection journal, (6) researcher weekly class summary,

\textsuperscript{12} This numbering, like that referenced below, corresponds to the data sources listed in Figure 3.2.
(7) segments of whole-class classroom video, (8) select classwork from three students, and (9) segments of individual video from three students. Recall that during the ongoing analysis and experimentation CMs were specifically identified and discussed in each of the Daily and Weekly class summaries. The systematic analysis of conjectures about conditions of the learning environment involved (a) the compilation and analysis of initial CMs that were documented during the teaching experiment and (b) the identification of new CMs based on analysis of data sources and new perspectives and patterns developed during retrospective analysis. This analysis occurred in cycles that involved five main processes for each episode (described in Steps a, b, c, d, and e below):

(a) Analyze primary textual data sources including (1) classroom field notes, (2) classroom artifacts (annotated lesson plans, and instructional activities), and (3) debriefing field notes to identify specific time-stamped segments of video to view. Code segments of (7) daily classroom video in Studiocode as CMs and annotate with a brief note or partial transcript of key phrases and mathematical ideas. If classroom field notes were not robust, view classroom video as described in Step c before referring to secondary data sources.

(b) Analyze secondary textual data sources including (4) daily class summaries, (5) daily reflection journals, and (6) weekly class summaries for rationale for initial conjectures about CMs. Annotate existing CMs from Part a with additional justification, summaries, notes on the researcher’s role, and/or verbatim transcripts to support the identified CMs. Code and annotate any new CMs that are identified. Reference (8) student classwork, (9) student video, and/or (3)
debrieﬁng session audio as needed to provide additional evidence to help clarify
the nature of classroom-level interactions of identiﬁed CMs.13

(c) Review remaining segments of whole-class classroom video to identify additional
CMs.14 Code and annotate new CMs with brief notes or partial transcripts as in
Steps a-b. At this point in the analysis, the entire daily classroom video will have
been viewed and analyzed for initial and new CMs.

(d) Compile all annotations of identiﬁed CMs: export Studiocode Transcript as a text
ﬁle and import text ﬁle into HyperResearch as a source for the given episode;
create a new case. Analyze annotated CMs identiﬁed in Steps a-c and use
HyperResearch to code these CMs according to the instructional theory as
summarized in Table 3.6.

(e) Create a Code Report within HyperResearch to sort CMs by their code name;
consider all codes across episodes within a given group (i.e., section of
instructional theory) at a time. Use evidence from transcripts and summaries that
are tagged within the HyperResearch source ﬁle, the compilation of previously
coded and annotated CMs of the same code name (available within the code
report), and the code deﬁnition to conﬁrm or refute initial conjectures about CMs.
Add to each annotation the decision to conﬁrm or refute the conjecture. If a CM is
refuted, recode it as “retrospective refute” and add previous code name to
annotation.

13 This was especially helpful during segments of classroom video where there were no
classroom-level interactions such as during small group work time.
14 This recommendation was made during the dissertation proposal defense.
As outlined above, in this first sub-phase I became re-immersed in the data to inform the compilation of CMs and to confirm or refute initial conjectures about the conditions of the learning environment. The method for analyzing classroom video (Steps a-c) was largely informed by other data sources. The use of classroom artifacts (Step a) before summaries and reflection journals (Step b) was done to allow a fresh perspective on the classroom video before incorporating the interpretations recorded in the ongoing analysis and daily thought experiments. Data from individual students was used to support the identification of and justification of classroom practices. In other words, consistent with the definition of a CM, student-student interactions during small group work were not coded, but rather were taken as evidence of the ways in which individual students’ conceptions and activity may have informed the interactions at the classroom-level.

The use of HyperResearch to code CM according to the instructional theory strengthened the analysis process because the coding and sorting capabilities allowed for greater consistency in coding within and across episodes. All refinements to the instructional theory were kept track of in the HyperResearch code window. The five-step process described above was repeated for all episodes in chronological order. During the constant comparative process, Step e, the comparison across episodes allowed for patterns to become salient as the instructional theory was being refined. There were two types of nuances to this coding process that are worth noting here to clarify how difficulties in coding were handled.

First, it was sometimes difficult to determine if a particular CM was to be confirmed or refuted based on the available data. In these cases, two processes were
followed. First, additional evidence from data sources such as individual classroom video and/or student artifacts were referenced. If this did not clarify the issue, the CM remained in the summary file until additional evidence was gathered from subsequent episodes to which they could be compared.

Second, the analysis sometimes warranted the naming of a new type of CM that was not evident in the conjectured instructional theory or the clarification of a definition for an initially conjectured condition of the learning environment. If a code was re-named, given an updated description, or added as a new element of the instructional theory, all previous CMs that were coded within that code category and group were re-examined and confirmed/refuted.

In Phase d of the retrospective analysis the convention used to name episodes changed midway through the analysis. For Days 1-14, episodes were initially named by day (Day 1, Day 2, etc.); for Days 15-24, episodes were named by activity and day (Activity 8 Day 15 Day 16, Activity 9 Day 16 Day 17, etc). The decision to change the naming of episodes allowed for more efficient data management and reporting based on the Activities that were enacted because they almost always spanned more than one day of the teaching experiment. The episodes previously named Days 1-14 were later re-named by Activity to match the new convention.

As summarized in Table 3.5, the groups of codes were based on the structure of the instructional theory: Activity Sequence, Activity Structure, Learning Progression, Classroom Expectations, and other. The process of considering all code assignments within a code group at once, Step e, allowed for more internal consistency in the code assignments. The process of confirming or refuting CMs by group was then repeated until
no new codes were assigned or changed and no descriptions needed to be updated. The finalized and refined code name descriptions are given as a result in Chapter 5.

During the ongoing analysis, a dissertation committee member and fellow doctoral student coded an episode according to the conjectured instructional theory. Conversations with these coders and written feedback on the framework informed further clarification of the coding methodology and descriptions of code names. For example, some code descriptions had duplicate or overlapping information that made it difficult to code as one particular type. These issues were clarified in later refinements of the framework. A measure of inter-rater reliability was not computed for that comparative analysis because the primary focus of their reflection was to provide feedback on the framework and how it was being used, not to check the agreement across coders. During the retrospective coding process, the researcher regularly had conversations with graduate students and faculty members familiar with the study to help make sense of the data with respect to the evolving instructional theory and framework.

By design, all instructional theory components and descriptions were refined throughout the ongoing and retrospective analysis. The revised activity structure component of the instructional theory is expanded upon in Table 3.6. This table reflects revisions to both activity structure components and descriptions that were made during the analysis. Other components of the instructional theory are elaborated elsewhere: the learning progression was introduced above in the section of the same name, classroom expectations are detailed in Chapter 5 under a section of the same name, and the conjectured, enacted, and revised activities sequence is discussed in Chapter 6.
Table 3.5
Critical Moments Code Window for HyperResearch

<table>
<thead>
<tr>
<th>Group</th>
<th>Code Name</th>
</tr>
</thead>
</table>
| Activities Sequence | Daily Learning Goal*  
| | Creating and Moving Between MR*  
| | Creating and solving linear equations  
| | Equivalent expressions linear equations  
| | Linear equations as equivalence relations  
| Activity Structure | Act (memo with tool and Rule of Four)  
| | Anticipate  
| | CAS Check*  
| | Connect  
| | Generalize*  
| | Interpret  
| | Justify*  
| | Reconcile  
| | Reflect  
| | Translate  
| | Transpose  
| Classroom Expectation | Conventional Representation*  
| | Focused on mathematics  
| | Representationally flexible  
| | Strategic user of MR*  
| | Strategic user of tools  
| Instructional Theory | Reflection  
| | Revision  
| Learning Progression | A1 Connecting and generalizing the quantitative visual and verbal with symbols  
| | A2 Different representations representation types can signify the same object  
| | B Equivalent of expressions from MR  
| | C Domain and range restrictions  
| | C1 Role of Equals Sign Assigns Variables Rules or Names for Patterns that can be graphed and viewed as Tables for Contextual Situations  
| | C2 Role of Equals Sign Identity between equivalent expressions  
| | CC1 If a point P is on the line L P makes the equation of L true  
| | CC2 If a point P makes the equation of L true P is on the graph of L  
| | D1 Solutions to equations can be determined by equality of expressions that are sometimes always or never equal in value  
| | D2 Solving equations is one variable is conceptualized as a comparison of two functions Linear equations in one variable can be solved by comparing two functions for the value of x that makes the equation true  
| | Equivalence of equations  
| Other | New Classroom Practice*  
| | Researcher Role*  
| | Retrospective Analysis Refute*  

*Codes indicated with an asterisk were added during the retrospective analysis
### Table 3.6
*Activity Structure Elements and Revised Descriptions*

<table>
<thead>
<tr>
<th>Activity Structure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act</td>
<td><em>Create a representation and possibly explain the process of how one works within or moves between tool-based representations or types.</em></td>
</tr>
<tr>
<td>Anticipate</td>
<td>Predict the result of creating tool-based representations.</td>
</tr>
<tr>
<td>CAS Check**</td>
<td>Use the CAS to check or verify P&amp;P representations (often times within Symbolic representation type).</td>
</tr>
<tr>
<td>Connect</td>
<td>Give a correct interpretation of an invariant feature across multiple representations or types.</td>
</tr>
<tr>
<td>Generalize**</td>
<td>Make a generalization across several representations or representation types (e.g., abstract notation a(x+b)=ax+ab.</td>
</tr>
<tr>
<td>Interpret</td>
<td>Convey the meaning of the act/result of creating a tool-based representation; a basic, quick remark, thoughtful but not deep.</td>
</tr>
<tr>
<td>Justify**</td>
<td>Representations are used to confirm or ascertain a particular result or conclusion; “use representations as justifications for other claims” (Sandoval et al., 2000; reasoning must be present); formal/rigorous explanation, objective, based on set practices.</td>
</tr>
<tr>
<td>Reconcile</td>
<td>Negotiate differences between CAS and P&amp;P representations.</td>
</tr>
<tr>
<td>Reflect</td>
<td>React to or think deeply about representations/representation types with respect to equivalence and/or equations; heavy thought, detailed in response, subjective and developmentally oriented.</td>
</tr>
<tr>
<td>Translate**</td>
<td>Create and interpret the meaning of a target representation with respect to a source representation of a different type.</td>
</tr>
<tr>
<td>Transpose**</td>
<td>Create and interpret multiple representations within one representation type.</td>
</tr>
</tbody>
</table>

** Denotes component added during retrospective analysis

### Phase Two: Reconstruct Revised Instructional Theory

In the second phase of the retrospective analysis, a revised instructional theory was reconstructed. This revised sequence is given in Chapter 5. Again I followed Gravemeijer and Cobb (2006) to guide this analysis:

Because of the testing and revising of conjectures while the experiment is in progress, a revised, potentially optimal instructional sequence has to be discerned by conducting a retrospective analysis. It does not make sense, for example, to include instructional activities that did not live up to the expectations of the researcher, but the fact that these activities were enacted in the experiment will nonetheless have affected the students' learning ... Consequently, the instructional sequence will be put together by focusing on and reconstructing the instructional activities that proved to constitute the effective elements of a sequence. This
reconstruction of an optimal sequence will be based on the observations and inferences made during the design experiment complemented by the insights gained by conducting retrospective analyses. In this manner, it can be claimed that the results of a design experiment are empirically grounded (p. 42).

Case Studies

The cases of ninth-grade algebra students identified for this study were three students from Ms. L’s Algebra I class. The criterion used to select these student participants is detailed first. Second, the semi-structured interview methodology is explained. Finally, the coding process followed in analyzing these cases is detailed.

Student Participants

The selection of three students for individual case studies from the entire class population was based on an analysis of students’ pre-test results and teacher discretion. All students in this class took two pre-tests as part of their regular classroom instruction—one department wide trimester assessment, and one unit assessment designed by the research team. These exams were taken on the second and third full days of the 2011-2012 school year, respectively. Students were given approximately 40-50 minutes to complete each exam.

Pre- and Post-Test

All students in this class took a pre-test and post-test designed by the researcher and teacher as part of their regular classroom instruction (see Appendix E and F). The primary purpose of the pre-test was to select three students to study as case studies. Another purpose of the pre-test was to assess the instructional starting points of the
learning trajectory. However, the time constraints at the beginning of the teaching experiment limited the use of the exam for this purpose and the analysis of class performance as a whole was more informal and not documented as a specific means of informing the learning trajectory. The pre- and post-tests were also used to inform the selection of tasks for the initial and final interviews with students, respectively.

Note that the pre-test and post-test, each with 16 tasks, followed the same general structure with slightly different tasks. The decision to update the post-test was based largely on the intent to clarify the exam questions. While the structure of the exam was the same, the numbers (in equations, symbolic expressions) in the problems were changed with the intent of not changing the cognitive demand of the tasks (e.g., careful attention to negatives/positives, decimals and fraction versus whole number solutions). The initial representation type that students were given was not changed (including careful consideration of using the same 'real-world' contexts). More details on a few specific problems are summarized next.

Question 9 was changed to ask for, but not necessarily "require," students to determine a symbolic equation given a table or graph (this was beyond the focus of the content covered during the teaching experiment). Question 10 was changed to specifically refer to the third row in the table that gives students an equation and has them create a graph and a table. Students were then to use each of these representations to indicate the solution to an equation. Both Questions 11 and 12 were written to be in the variable x. Question 12 was changed from an equation with infinite solutions to an equation with no solutions (on the pre-test, the "no solutions" case was absent—an oversight). Finally, Question 13 was changed so that the "error" in the students' reasoning
is on the step in which 4 is being added to both sides (on the pre-test, it was a combining like terms error). Finally, students were allowed to choose which tools they used during the exam, with TI-Nspire CX CAS, TI-84+ graphing calculators, and P&P available. To keep track of students' use of CAS, a graphic was added next to each item on the post-test and students were prompted to circle if they used the calculator page, graphs page, and/or tables page.

Selection Criteria

A total of eighteen students volunteered to be captured on classroom video and to have their pre- and post-tests collected and analyzed. From this population, six students also agreed to have their daily classwork recorded and to participate in two 45-minute interviews with the researcher. The criteria used to select three participants from the six volunteers was threefold: (a) capture a range of mathematical abilities (high, middle, and low), (b) be representative of both genders, and (c) include participants that demonstrate “good” communication skills (e.g., clarity in writing). The teacher’s input in helping to decide which students to select was based on the desire to have the sample be representative of the racial diversity in the classroom. The researcher shared this value and used this as an additional consideration in selecting the three students. Of the 31 students enrolled in the class, there were 23 male and 8 female students, and 15 white and 16 students of color. The aforementioned criteria guided the methodology of choosing three of the six students for individual case studies.

First, the department wide and unit assessments were analyzed and scored for the six student volunteer participants, Student 3, 9, 12, 19, 20 and 26. The department wide
assessment was scored based on an item-by-item basis, with a total of 56 items possible. The unit assessment was scored based on a pre-determined scoring scale that the teacher decided prior to administering the exam, with a total of 70 points possible across 26 items (with a 5 point extra credit opportunity that involved two items). Across both exams, items were determined by the exam numeration system (e.g., Problem 1a and 1b counted as two items). Students received two percentage scores for each exam. The first was calculated based on the number of correct items/points out of the total items/points possible, and the second was calculated based on the number of correct items/points out of the items/points attempted. These data are summarized in Table 3.7.

Table 3.7
Unit and Trimester Pre-Test Performance for Case Study Candidates

<table>
<thead>
<tr>
<th>Student</th>
<th>Unit Pre-Test</th>
<th>Trimester Pre-Test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Overall</td>
<td>Attempted</td>
</tr>
<tr>
<td>Low</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>7/70 (10%)</td>
<td>7/14 (50%)</td>
</tr>
<tr>
<td>3</td>
<td>9/70 (13%)</td>
<td>9/14 (64%)</td>
</tr>
<tr>
<td>26</td>
<td>15/70 (21%)</td>
<td>15/42 (36%)</td>
</tr>
<tr>
<td>Middle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>26/70 (37%)</td>
<td>26/44 (59%)</td>
</tr>
<tr>
<td>High</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>31/70 (44%)</td>
<td>31/53 (58%)</td>
</tr>
<tr>
<td>12</td>
<td>31/70 (44%)</td>
<td>31/33 (94%)</td>
</tr>
</tbody>
</table>

From the data in Table 3.7, it is possible that the low performance levels across all students are due to many factors. First, no mathematics instruction took place during the week that these exams were administered. The most recent mathematics instruction of these students was either during summer school or eighth grade in the 2010-2011 school year. Second, as mandated by district policy, the teacher reported that students had been taking several assessments in all of their core subjects throughout the week, not just in
mathematics. Third, during each test-taking situation, the overall tone of the students in the classroom was somewhat negative with many students expressing mannerisms and behavior that was characteristic of feeling despondent and disconsolate, despite teacher intervention. During the Unit pre-test in particular, several (but not all) students outwardly expressed a fair amount of frustration, lack of engagement, or distraction, as evidenced by verbal and/or non-verbal cues (e.g., not attempting many assessment items and/or finishing the exam after a short amount of time—15 minutes). This third component may possibly be a consequence of the first two contextual factors surrounding the pre-test administration.

In light of the aforementioned context, it seemed fitting to add exclusionary criteria to further refine the sample based on student disposition and perseverance in regards to attempting difficult mathematical material. Students who were found to disengage after a short period of time and/or not attempt at least half of either assessment were excluded from the sample. Thus Student 3 and Student 9 were excluded. From the four remaining candidates, the goal was to select a stratified sample of students at high, middle, and low levels of ability that encompassed both male and female students.

Student 19 was selected based on the criterion that she demonstrated a middle level ability as compared to the other students, and was the only female volunteer of the six. Student 26 was selected based on the criteria that he demonstrated a low ability level in comparison to the other students. Finally, both Student 12 and Student 20 demonstrated similar communication skills, and performed comparably overall on both the unit pre-test (44% and 44%, respectively) and end of trimester pre-test (41% and 41%, respectively). Student 12 scored higher than Student 20 on the attempted points on
the unit test (94% versus 58%) and attempted items on the trimester test (66% versus 49%). Both Student 12 and Student 20 were considered high ability; the original decision to select Student 12 was based on a consideration of the overall and relative percentages (in which he demonstrated higher percentages\textsuperscript{15}). However, after the first day of the teaching experiment, Student 12 decided not to continue with his participation. Student 20 was selected instead starting on the second day of the teaching experiment.

Student 20 (high-ability) and Student 26 (low-ability) are both male; Student 20 is white and Student 26 is a student of color. Student 19 (middle-ability) is a white female. This selection fits the criteria of having a range of mathematical abilities, gender, and also aligns with the value shared between the teacher and researcher to have a sample of three students that reflected the racial diversity of the classroom. The pseudonyms of Annie, Bryon, and Carlos will be used instead of Student 19, 20, and 26, respectively.

\textit{Semi-Structured Task-Based Interviews}

Following Goldin (2000), a strength of structured task-based interviews is the ability to gain insight into students’ conceptual understanding, helping to “infer and describe deeper understandings in students” (p. 524). Thus the intent of the semi-structured interviews was to uncover students’ thinking about how they solved tasks (or analogous tasks) from each of the pre- and post-tests. Two separate semi-structured interviews were conducted with each of the three select individual students. Each interview occurred as soon as possible after each of the pre- and post-tests.

\textsuperscript{15}The relative percentage in this case is simply being used as a tool to further compare these students. It is recognized Student 20 attempted more problems than Student 12 on each exam, but was not successful on them.
Think Aloud Protocol

An analysis of students’ performance on the pre-test and post-test tasks served to inform the conduct of the task-based interviews. Depending on the nature of students’ responses on the assessment items, the researcher may have asked the student to articulate why they gave a particular response to a question to better uncover the students’ thinking and rationale.

The researcher followed a “think-aloud” protocol that aimed to elicit verbal dialogue from students as they worked through a task situation. Specific probes were intended to scaffold the semi-structured interviews to uncover alternative solution approaches or ways of thinking that the student may not have otherwise verbalized, but that shed light on students’ RF. See Appendix G for the interview protocol. Note that this protocol was inspired by the interview protocol discussed in Huntley, Marcus, Kahan, and Miller (2007) and was tested and subsequently refined based on the interview protocol that was tested in a pilot study (Fonger, 2011).

Task Selection

Prior to administering the assessments, the researcher had selected several tasks from the unit pre-assessment that seemed fitting to assess students’ RF during the interviews. In particular, tasks that had students work within, translate between, and make connections across MR in solving linear equations were selected as potentially good tasks to focus on for the semi-structured interviews. To further refine the selection of interview
tasks the researcher more carefully analyzed Annie’s, Bryon’s, and Carlos’ unit pre-test responses with respect to the analytic framework for RF and definition of RF.

In a 45-minute interview conducted at the end of the school day, approximately 3-6 tasks were prioritized to discuss with the participants. During this interview, the researcher asked the student to follow a think-aloud protocol as they either explained why they gave the response they did to a particular problem on the pre-test, or to attempt an analogous task.

By design, students considered equation-solving tasks of a similar structure during the post-test and final interview. Despite this consistency in design, there are two main differences to note. The task to solve a linear equation with all real solutions [Task 12] was presented in a different format with a symbolic solution approach shown, requiring students to analyze that approach and explain what the solution was. Secondly, a new task was added to the final interview that was not considered in the initial interview. This task asked students to solve a linear equation with no real solutions [Task 13]. Table 3.8 summarizes the comparison of tasks that were used to measure students’ initial and final RF in solving problems involving linear equations.
Table 3.8
Task Type Compared to Measure Student’s Initial and Final RF

<table>
<thead>
<tr>
<th>Interview</th>
<th>Task 10</th>
<th>Task 12</th>
<th>Task 13</th>
<th>Task 14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>Solve an equation</td>
<td>Solve a linear equation with all real solutions</td>
<td>n/a</td>
<td>Solve an equation</td>
</tr>
<tr>
<td></td>
<td>( y = ax + b ) at ( y = c ) for ( x ), or solve ( c = ax + b )</td>
<td></td>
<td></td>
<td>( ax + b = cx + d ) for ( x ) with one real solution</td>
</tr>
<tr>
<td>Final</td>
<td>Solve an equation</td>
<td>Analyze and interpret the solution to an equation with all real solutions</td>
<td>Solve a linear equation with no real solutions</td>
<td>Solve an equation</td>
</tr>
<tr>
<td></td>
<td>( y = ax + b ) at ( y = c ) for ( x )</td>
<td></td>
<td></td>
<td>( ax + b = cx + d ) for ( x ) with one real solution</td>
</tr>
</tbody>
</table>

Characterizing RF

One goal of this study was to characterize how students’ changed their RF, if at all. The SOLO taxonomy (Biggs & Collis, 1982) structured the analytic framework for RF. With the use of this framework, students’ RF could be characterized according to various levels of sophistication. These levels of sophistication in representation-specific abilities (specific to various task situations) were then compared at the beginning and end of the study; this was the basis for describing students’ change (or lack thereof) in RF.

Analytic Framework for RF

Based on pilot study research, a framework for RF was proposed a priori to coding (Fonger, 2011). However, while coding the pre-tests and initial interviews of the three cases, the framework proved to be difficult to use and created inconsistencies when coding within a case and across cases. The levels of the framework were subsequently revised using Biggs and Collis (1982) as a primary source in that update, but also paying
close attention to a more coherent way to incorporate all aspects of the definition of RF. In particular, the correct and complete processes of creating and interpreting representations were used to signify each level of the framework (as indicated in the top row of Table 3.9 and as parenthetical remarks within each row). In the following, I give some background information that helps to elaborate why the SOLO taxonomy was targeted as an appropriate tool to use, and detail how it was used to refine the framework. The coding process is then elaborated.

Biggs and Collis (1982) developed the SOLO taxonomy to characterize the hierarchical nature of students’ development of higher order skills and thinking. The SOLO taxonomy can be used to categorize a students’ attainment or level of development of particular abilities at a particular time (e.g., for a particular task in a specified domain). The levels of this framework are prestructural, unistructural, multistructural, relational, and abstract. These levels, described in more detail next, each have a basic structure that involves a focus on a particular number of assessed items, and the relationship between those items. This structure of observed learning outcomes was interpreted to be logically consistent with the basic aspects of RF, namely, creating and interpreting different representations and representation types. The fusion of SOLO levels with the aspects of the definition of RF is evident across the rows of Table 3.9 and is elaborated on below.¹⁶

¹⁶ These characterizations were also informed by the data, related research, and other adaptations of the SOLO taxonomy for use in characterizing students’ translations and coordinations (Adu-Gyamfi, 2007) and translations and connections (Rider, 2004).
### Table 3.9
#### Analytic Framework for Representational Fluency

<table>
<thead>
<tr>
<th>SOLO Level</th>
<th>Indicator of RF (create, interpret)</th>
<th>Description and Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prestructural</strong> lacks knowledge of the meaning of a representation or type, no/little understanding is conveyed</td>
<td>P-0. Work Within (<em>both incorrect</em>)&lt;br&gt;P-1. Representationally grounded or bounded = Work Within (<em>one incorrect</em>)</td>
<td>Some procedural work within a given representation type may be demonstrated but no understanding is conveyed, not able to create and/or interpret a representation type with success to complete a task (Bieda &amp; Nathan, 2009).</td>
</tr>
<tr>
<td><strong>Unistructural</strong> focuses on a single representation or type, no/limited understanding of connections</td>
<td>U-0. Transposition = Work Within (<em>both correct</em>)</td>
<td>Creates and/or interprets within a single representation or type to make progress toward or successfully complete a task but demonstrates no/little understanding of meaning, nor connections to other representation types.</td>
</tr>
<tr>
<td><strong>Multistructural</strong> focuses on several representation types with little to no understanding of how they are related or what they mean</td>
<td>M-1. Move Between (<em>both incorrect</em>)&lt;br&gt;M-2. Move Between (<em>one incorrect</em>)</td>
<td>Moves between more than one representation type to make progress toward or successfully complete a task but demonstrates no or incomplete understanding of how these representation types are related or what they mean to solve the problem.</td>
</tr>
<tr>
<td><strong>Relational</strong> relates at least two different representation types together with a more sophisticated understanding of how they are related, what they mean, and/or of their significance to the whole.</td>
<td>R-1. Translation (<em>both correct</em>)</td>
<td>Create and interpret the meaning of a target representation with respect to a source representation of a different type.</td>
</tr>
<tr>
<td></td>
<td>R-2. Uni-Directional Connection (<em>both correct + invariant feature</em>)</td>
<td>Create and give a correct interpretation of an invariant feature across MR or types.</td>
</tr>
<tr>
<td></td>
<td>R-3. Bi-Directional Translation (<em>both correct</em>)</td>
<td>Perform translation and complementary translation processes.</td>
</tr>
<tr>
<td></td>
<td>R-4. Bi-Directional Connection (<em>both correct + invariant feature</em>)</td>
<td>Create and interpret a source representation with respect to a target representation and vise-versa, and recognize invariant features across the two types.</td>
</tr>
<tr>
<td><strong>Connection</strong> Give a correct interpretation of an invariant feature across MR or types</td>
<td>R-5. Multi-Directional Translation (<em>both correct</em>)</td>
<td>More than two representation types are related by translation processes.</td>
</tr>
<tr>
<td></td>
<td>R-6. Multi-directional Connection (<em>both correct + invariant feature</em>)</td>
<td>More than two representation types are related by translation processes and involve a correct interpretation of an invariant feature.</td>
</tr>
<tr>
<td><strong>Extended Abstract</strong> seeing the concept from an overall viewpoint</td>
<td>A. Abstract Connection</td>
<td>A generalization is made across different representations or types (i.e., MR are used in reasoning about characteristics of a mathematical entity).</td>
</tr>
</tbody>
</table>
At the prestructural level, students do not demonstrate knowledge of the assessed component. In the case of RF, students do not demonstrate an ability to create or interpret representations—the two foundational practices for translations, transpositions, and connections. Drawing on Bieda and Nathan’s (2009) research, students at this level were characterized to demonstrate some procedural work in certain representations, but were unsuccessful in completing particular tasks.

Students classified at the unistructural level were focused on a single assessed component or with no or little understanding of its relationship to other assessed components. In the case of RF, this is logically equivalent to a transposition in which students are successful at creating and interpreting MR within one representation type. At this level, multiple representation types and the connections between representation types are not considered.

The multistructural level is characterized by a focus on several assessed components with little to no understanding conveyed of the relationship between them. Correspondingly, students’ RF at this level is characterized by students’ movement between representations in either the creation or interpretation (but not both) of representations. At this level, progress is made toward successfully completing the task at hand, but the relationship or connection between these representations is not well conveyed.

At the relational level, two or more assessed components are related and a more sophisticated understanding of their meaning is conveyed. This level of the framework

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17 This is the lowest level of the SOLO taxonomy, yet in the framework for RF an additional code was added for situations in which a student did not attempt a task. It is tacit that the prestructural level assumes an attempt on a task.
for RF is typified by translations and connections between representations. Recall that a translation is defined as the creation and interpretation of meaning of a target representation with respect to a source representation of a different type. Additionally, a connection is defined as a correct interpretation of an invariant feature across MR or types. Using the Rule of Four framework (cf. Huntley et al., 2007), the direction of translations between representation types can be in one direction, the reverse direction, or multiple directions. These directions of the translation between representation types signify the sub-levels of uni-directional, bi-directional, and multi-directional, respectively. Also, within each of these directional translations, the interpretation of an invariant feature across representation types signifies the connection of sub-levels.

To clarify the levels of the SOLO taxonomy, consider the illustrations of the framework for RF through use of the Rule of Four model (Figure 3.4). Such abbreviations and images are be used throughout Chapter 4.

*Figure 3.4. Illustrations of the Analytic Framework for RF according to the Rule of Four.*

The abbreviations of P-0, P-1, U-0, M-1, M-2, R-1, R-2, etc. are used to indicate the specific level of the Framework at which particular episodes are coded. The capital letters of S, V, G, and N indicate the representation type (symbolic, verbal, graphic, and numeric). A solid line with an arrowhead (→) indicates a complete translation or
transposition from one representation type to another. A dashed line with an arrowhead (- ->) indicates that either the creation of or interpretation of one representation type to another was incorrect or incomplete in some way. Finally, a dashed line with an “X” and an arrowhead (-X->) indicates that both the creation and interpretation of the representation types were incorrect or incomplete. Notice that the use of a solid, dashed, or broken line with an “X” correspond to the Rule of Four web graphics in each case. A double-headed arrow on solid lines is used to indicate connections at the relational level of RF (e.g., R-2).

In addition to the aforementioned abbreviations, for the episodes discussed in Chapter 4, the tool(s) used to create the representation(s) are also indicated as “GC,” “CAS,” or “P&P” for a TI-84 Plus graphing calculator, a TI-Nspire CX CAS, or P&P, respectively.

Coding Process

Once a revised version of the framework was decided, all coding instances for each of the three initial interviews and written work were re-coded. When problematic instances in coding arose, the researcher first compared the coding decision across the cases and the rationale provided for each coding decision. If this did not allow the researcher to come to a resolved conclusion, other coders were involved. Specifically, one mathematics education faculty and one mathematics education doctoral student familiar with the study were trained to use the framework and were asked to independently code these particular problematic segments according to the prestructural, unistructural, multistructural, and relational levels.
All code disagreements were discussed until the discrepancy was resolved in either a re-assignment of a code or an adaptation to the framework. Before discussion there was 58% (moderate) agreement in coding as computed using the online Kappa calculator http://justusrandolph.net/kappa/; the interpretation of the Fleiss value was based on http://en.wikipedia.org/wiki/Fleiss%27_kappa. After discussion of the inconsistencies in coding there was 81% (almost perfect) agreement. This discussion also led to modification of the framework for clarity in coding. The adapted framework was then used to recode all data a final time; no problematic instances arose in this coding.

As soon as possible after they were conducted, the interviews with select individuals were viewed, coded and memoed according to the Analytic Framework for RF (Table 4.8) using Studiocode. A separate “Interview Case Summary” was created per interview to capture key aspects of the students’ RF. To create these case summaries, a cyclical process of coding proceeded in the following manner: each task was coded across the three cases, the coding of each task across Annie, Bryon, and Carlos was reanalyzed for the initial interview, then across Annie, Bryon, and Carlos for the final interview. Finally, the change in RF across Annie, Bryon, and Carlos were compared across the cases. This process increased the consistency in coding using the framework across the cases and throughout the entire data set as a whole.

Initial and Final Levels of RF

In coding and analyzing student work during both initial and final interviews, the researcher used a method of comparison that involved taking one task at a time across all three students per interview. Once the initial interview was coded in this manner, the
analysis of the second interview followed a similar approach. The researcher subsequently cycled back through a task across all students’ initial and final interviews before finalizing code decisions. This method of comparing across cases increased the consistency in coding across cases.

To capture a measurement of a student’s RF at the end of the study, both the post-test and final interview were coded simultaneously. Moreover, for consistency in coding across cases, each student’s post-test and the corresponding task(s) in each student’s interviews were coded one problem at a time. This afforded a much clearer cross-case analysis, and within case analyses with respect to students’ initial measures of RF.

*Change in RF*

To capture change in students’ RF after the initial and final interviews were analyzed, a shorthand summary of each student’s problem solving performance per task was created. These summaries were then compared across cases. All code assignments were revisited to check for consistencies in coding across the cases. In other words, if there seemed to be inconsistencies across cases in how the individual’s responses were coded, those code assignments and justifications were compared against each other to verify the original coding decisions. Once all code decisions were checked and clarified across the cases, elaborated descriptions of each student’s change in RF were written.
CHAPTER IV

FINDINGS: CASE STUDIES

Students’ Change in Representational Fluency

Recall representational fluency (RF) is defined as the ability to create, interpret, transpose within, translate between, and connect representations. RF is measured by one’s ability to create, work within, move between, interpret, justify, connect, and generalize. Each student’s RF was measured according to the five distinct levels posited in the Analytic Framework for RF, introduced in Chapter 3, Table 3.9.

To inform the characterization of change in students’ RF in solving problems involving linear equations as a result of learning how to solve linear equations within a CAS and P&P environment, I detail each students’ initial and final RF as measured at the onset and close of the study, respectively. For the initial characterization of each student’s RF, results are based on analyses of both the pre-test and initial interview where data collected during the initial interview are primary, with the pre-test as secondary. Similarly, the final characterization is based on the post-test and final interview. Results of coding using the analytic framework for RF (Table 3.9) are then summarized with a description of the observed change in each student’s RF.

Findings for each of three case studies are presented in the next sections. For each case, the same organizational structure is followed: an overall description of the students’ initial RF, final RF, and change in RF in solving linear equations. All descriptions are
supported with data from interviews and unit exams that were coded according to the Analytic Framework for RF. In the discussion of results, students’ abilities to create and interpret representations are discussed in relation to their equation solving abilities (i.e., one may be able to create a representation but not be able to use it to solve an equation). In other words, students’ RF is seen as specific to solving problems involving linear equations.

With access to students’ external inscriptions and verbalizations about that work, the researcher made interpretations about what the students seemed to understand. Throughout the characterization of each student’s RF, attempts are made to clarify what seem like ambiguous uses of representations. For instance, it may not always be clear what the student sees the external representation as signifying. While meaning resides in the eye of the beholder, a researcher can only surmise what the true meaning of the representation are from the perspective of the student.

There are four main equation solving tasks that are discussed for each case, for which \( x \) is the variable to solve and \( a, b, c, \) and \( d \) represent real-valued numbers:

(a) solving an equation of the form \( y = ax + b \) at \( y = c \) [Task 10], (b) solving a linear equation with all real solutions [Task 12], (c) solving an equation of the form \( ax + b = cx + d \) with one solution [Task 14], and (d) solving an equation with no solutions [Task 13].

By design, students considered equation-solving tasks of a similar structure during the post-test and final interview. Despite this consistency in design, there are two main differences to note. The task to solve a linear equation with all real solutions [Task 12] was presented in a different format with a symbolic solution approach shown,
requiring students to analyze that approach and explain what the solution was. Secondly, a new task was added to the final interview that was not considered in the initial interview. This task asked students to solve a linear equation with no real solutions [Task 13]. Table 4.1 summarizes the comparison of tasks that were used to measure students’ initial and final RF in solving problems involving linear equations.

<table>
<thead>
<tr>
<th>Table 4.1</th>
<th>Task Type Compared to Measure Student’s Initial and Final RF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interview</td>
<td>Task 10</td>
</tr>
<tr>
<td>Initial</td>
<td>Solve an equation</td>
</tr>
<tr>
<td></td>
<td>( y = ax + b ) at ( y = c ) for ( x ), or solve ( c = ax + b )</td>
</tr>
<tr>
<td>Final</td>
<td>Solve an equation</td>
</tr>
<tr>
<td></td>
<td>( y = ax + b ) at ( y = c ) for ( x )</td>
</tr>
</tbody>
</table>

For the three case studies, the pseudonyms of Annie (abbreviated SA in references to data transcriptions), Bryon (SB), and Carlos (SC) are used throughout this manuscript. In alphabetical order, the case of Annie is presented first.

**Annie’s Initial RF**

Annie was an interesting ninth-grader to interview; she conveyed a sense of thoughtfulness and frankness in her tone and mannerisms. Provided with a GC, P&P, and the prerogative to complete three distinct equation-solving tasks (each with symbolic equations included), Annie was initially grounded to the symbolic representation type.
With specific prompting from the researcher, she had made at least one valiant attempt at using her GC to graph and consider tables of values to solve an equation posed in the symbolic representation type, but did not convey a very meaningful understanding of how these representations were specifically related to her P&P equation-solving techniques within the symbolic representation type (Task 12). Moreover, the use of this tool to move between representations was not determined to be a consistent part of Annie’s fluency, for on a subsequent task such a multi-representational approach was seemingly “impossible.” Overall, Annie’s initial RF was typified by failed attempts to create symbolic equations and incomplete interpretations of the meaning of her work within this representation type. Annie demonstrated limited fluency in her ability to create and interpret graphic and numeric table representations.

Solving an Equation \( c = ax + b \) With One Solution: Task 10

On the pre-test, Annie did not attempt the task that asked her to solve an equation \( y = ax + b \) at \( y = c \) for \( x \) using the symbolic, graphic, and numeric representations. During the initial interview, the researcher scaffolded the task by first giving Annie the correct equation \( y = 8 - 3x \) and second by helping Annie interpret that to solve this equation at \( y = 5 \), she needed to consider the equation \( 5 = 8 - 3x \). This scaffolding was deemed appropriate based on the goal of Task 10 to elicit the students’ ability to solve an equation using MR, not necessarily the ability to determine the equation that is to be solved (see Figure 4.1).
9.) Given one of the representations below, create the other two.

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

10.) Refer back at the first row in Question 9. In this row, clearly indicate the solution to the equation when \( y = 5 \) for each of the table, graph, and symbolic equation.

Figure 4.1. Annie’s attempt to solve the equation \( 5 = 8 - 3x \) in a symbolic equation and the identification of this solution in the table.

The decision to scaffold the set-up of this task was also based on the fact that at an earlier point in the interview, when prompted, Annie was not able to translate from a table and/or graph to create a symbolic equation on her own.

I normally won't be able to create an equation but I can normally work off from an equation to create a table but I can normally use graphs sometimes. So I don't think I could work backwards, I normally work forwards on making things.

(SA_day2_InitialInterview_8:13:21)

When given the equation of \( y = 8 - 3x \), Annie explained that usually she can solve these equations when there is a value for \( y \). After the researcher repeated the problem statement of identifying the solution when \( y = 5 \), Annie did not make progress and was bounded to this symbolic equation without using the problem statement and additional symbolic equation of \( y = 5 \) as a resource to overcome this barrier. The researcher-directed scaffolding in setting up this equation was not coded as an aspect of Annie’s RF, had she not been given this set-up she likely would not have attempted this task.
With the correct symbolic equation set up, Annie proceeded to demonstrate mixed abilities using P&P in her transposition within the symbolic representation type as shown in the top-right corner of Figure 4.1. While correct operations were performed on the equation, (e.g., subtract 8 from both sides of the equation), Annie introduced an incorrect intermediate step by writing down the equation \(-3 = 3x\) instead of \(-3 = -3x\); this is an incorrect creation within the symbolic representation type. Annie had also incorrectly interpreted \(x = -1\) to be the solution to the equation. Annie was not successful at solving the equation within the symbolic representation type with correct creations or interpretations, thus Annie’s RF in solving this equation within the symbolic representation type was at the prestructural level (P-0, S-X->S). Reminding Annie of the goal of this task, the researcher prompted Annie to consider the solution to the equation in the table or the graph.

1 R: Can you also see your answer in the table or the graph?
2 SA: Yeah, right there (points to a \(y\)-value of \(-1\) in the table [Figure 4.1]). Yay!
3 R: So is that the value of \(x\) when \(y\) equals five?
4 SA: Mm Hmm. I am positive.
5 R: Okay.
6 SA: Can you circle it?
7 SA: Wait, on the graph (points to the table) or, over here (points to the equation \(x = 1\))?
8 R: Either. Both.
9 SA: (circles a \(y\)-value of \(-1\) in the table and the value of \(-1\) in the symbolic equation box [Figure 4.1])

(SA_day2_InitialInterview_16:08:13)

Despite Annie’s previous contention that she should be able to work from symbolic representation to tables and graphs, Annie did not demonstrate this ability here in solving the equation (lines 2, 4 and 9). Annie merely circled the value of “\(-1\)” in the table (supposedly corresponding to the solution of \(x = -1\)), but did not allude to the columns of the table. This exchange is not an indicator of Annie’s RF because she did not
actually use or move to the numeric (table) representation type. A successful movement to the numeric table representation would entail an indication of the identified value of \( x \) with a specific reference to the \( x \)-column of the table. In other words, Annie would have needed to allude to the fact that she was indicating an \( x \)-value of \(-1\); above she was merely indicating a value of \(-1\) as if she had picked it off of the page without any reference to the structure of the table representation type.

Overall, Annie’s RF in solving an equation of this type was classified at a prestructural level within the symbolic representation type. Annie’s representation-specific activity in this task was characterized by an incorrect creation of and interpretation of symbolic representations, and incorrect attempts at translating to the numeric (table) representation type.

_Solving a Linear Equation With All Real Solutions: Task 12_

The statement of Task 12 was purposefully designed to give students options in the way in which they solved the equation and showed their work (Figure 4.2).

For 11-12, solve for the variable and check your solution using a table, graph, or equation (your choice). In each question, sketch the section of the graph (with the scale marked) or the table and indicate where you found the answer. If you solve the problem using equations, show all of your work.

12. \[ t - 2 + 3t = -6 + 4t + 4 \]

_Figure 4.2._ Task 12 from the pre-test and initial interview.

The characterization of Annie’s RF in solving the equation \( t - 2 + 3t = -6 + 4t + 4 \) for \( t \) with all real solutions is divided into two sections. First I discuss Annie’s work within the symbolic representation type. This is followed by a discussion of a multi-representational approach to solving this equation that was prompted by the researcher.
Work within the symbolic representation type. During the initial interview, Annie was presented with Task 12 as a copy of her pre-test work with the following sequence of equations:

\[
\begin{align*}
    t - 2 + 3t &= -6 + 4t + 4 \\
    3t + -2 &= -2 + 4t \\
    1t + -2 &= -2 \\
    + 2 &+ 2 \\
    1/1 &= 0/1
\end{align*}
\] (1)

The researcher had hoped that Annie would re-consider this work during the initial interview, and possibly change her solution approach or demonstrate a higher level of RF. In Annie’s initial attempt to solve the equation \( t - 2 + 3t = -6 + 4t + 4 \) for the variable \( t \), she worked within the symbolic representation and created incorrect representations. Specifically, she started with equation 2, \( 3t + -2 = -2 + 4t \) (Figure 4.3, Line 1), an incorrect equation from her pre-test work, in which it appeared as though Annie had dropped the “\( t \)” on the left hand side of the equation and combined the \(-6\) and \(+4\) on the right hand side of the equation. Instead of re-considering this first step or other subsequent steps, Annie worked from equation 2, combined the \( 3t \) and the \( 4t \) to get \( 7t \), and likewise combined the \(-2\) and the \(-2\) to get \(-4\) (Figure 4.3, Line 2). Annie did not know what to do from here, explaining, “I normally do get stuck” (SA_day2_initialinterview_19:38). The incorrect creation of an expression and inability to interpret this expression as meaningful to solving the problem is evidence of a prestructural level of RF within the symbolic representation type (P-0, S-X->S, P&P).
Later near the close of working on this task, Annie had considered a symbolic solution approach again (see Figure 4.4). This time, Annie re-recorded her incorrect first step from her pre-test work (equation 2, Figure 4.3 Line 1) then continued to work within the symbolic representation type to create incorrect equations based partly on a mistake in adding like terms (e.g., $-2 + 2 = -4$, see Figure 4.4). This work led Annie to conclude that $t = -4$ was the solution to the equation. The fact that $t = -4$ is a solution to the equation (which has infinite solutions) was not recognized. In this additional solution approach, Annie had not made progress and the classification of her RF within the symbolic representation type is not changed from the prestructural level (P-0, S-X→S, P&P).

Figure 4.3. Annie’s initial approach to solving the equation $t - 2 + 3t = -6 + 4t + 4$ in the symbolic representation type.

Figure 4.4. Annie’s second take at solving the equation $t - 2 + 3t = -6 + 4t + 4$ within the symbolic representation type.
In retrospect, the set-up of Task 12 during the interview may have caused Annie unnecessary difficulty in working within this representation type, especially because the incorrect first step of “\(3t + -2 = -2 + 4t\)” (equation 2) was used by Annie in both the first and second solution approaches within the symbolic representation type (Figures 4.3 and 4.4, respectively). Another possible factor contributing to Annie’s prestructural level of RF was her inconsistent and incorrect procedure for combining like terms in which the common terms on either side of the equal sign were added together to yield an expression.

Multi-representational approach. As discussed above, Annie seemed to demonstrate some understanding of her work within the symbolic representation type. For instance, she correctly performed some procedures such as combining like terms, and her problem-solving goal seemed to be to isolate the variable in the equation. Despite this understanding, Annie was reportedly “stuck” in working within this representation type. Indeed, when the researcher prompted, “Is there a different way to solve [the equation]?”, Annie replied, matter-of-factly, “I have no idea” (SA_day2_initialinterview_19:58). At this point in the interview, the researcher decided to probe Annie’s understanding of the use of a graphic representation type, and scaffolded Annie’s use of this representation. Specifically, the researcher asked “Could you solve this equation if viewed as a graph?” to which Annie did not reply (there was at least a 5 second pause). The researcher then gave more scaffolding and asked Annie to consider the following set-up: “If you viewed this side of the equation (pointing to \(t - 2 + 3t\)) as one graph and this side of the equation (pointing to \(-6 + 4t + 4\)) as another graph, could you solve it that way?”
The following exchange then ensued after this prompt by researcher.

SA: Possibly.
R: Could you show me what that would look like?
SA: Oh, dang, I'm gonna’ be needin’ a table first.
R: You need a table, OK.
SA: Yeah, but I don't even know what the table is going to look like.

Based on the fact that the researcher told Annie how to view the symbolic equation from a graphical perspective, Annie’s movement to the numeric (table) representation and later to a graphic representation are excluded from being considered at the relational level of RF. Annie would have needed to demonstrate an understanding of how to move from the symbolic representation type to the other types on her own, for which she did not have an opportunity once the researcher told her how (i.e., view the equation as $f_1(x) = f_2(x)$ and graph each of $f_1(x)$ and $f_2(x)$). Nonetheless, Annie was facile at the use of her GC to move to each of a numeric table (Figure 4.5a) and graphic representation (Figure 4.6a).

Figure 4.5. Annie’s use of the numeric (table) representation for the equation $y = -6 + 4x + 4$. 
Figure 4.6. Annie’s creation of graphic representations of the equation 
\(-6 + 4t + 4 = t - 2 + 3t\) based on researcher probing.

Despite the assistance provided by the researcher in creating these representations, Annie’s interpretations of her movement from the symbolic representation type to the numeric table and graph do give some indicator of her initial RF.

1 SA: (Types \(x - 2 + 3x\) into \(y1\); Presses Control + Table [Figure 4.5a]. Records the table on her paper [Figure 4.5b]; Types \(-6 + 4x + 4\) to replace the previous expression in \(y1\); Presses Control + Table)
2 SA: Why is it the same thing? Except for \(x\) equals 0. ( Cursors through table so that \(x = 0\) is shown at the top of the screen.) Oh, no, never mind.
3 SA: (Presses Graph [Figure 4.6a]) Well, there's your graph. (Sketches a graph on her paper [left hand side of Figure 4.6b]).
4 SA: There’s that graph, but they're normally the same so it’s pretty much going to be the same graph. (Sketches a second graph on her paper [right hand side of Figure 4.6b])
5 SA: And it might be bigger than the other, but you know what I'm trying to get to.
6 SA: How they equal the same thing I have no idea.

(SA_day2_initialinterview_Task12_21:36)

Annie was successful in creating the numeric (table) representation and correctly recognized that the table for each of the expressions was “the same thing,” but was not successful at interpreting what this meant in regards to solving the equation, nor was she able to explain why this was the case (lines 1–2 above). Recognizing that the researcher scaffolded Annie in interpreting the symbolic equation in a way so that she could use her
GC to view additional representations, Annie had correctly used her GC to create a numeric table, yet did not correctly interpret the meaning of this table in a manner that helped her to solve the equation. Thus Annie’s movement from the symbolic to the numeric (table) representation type was classified as multistructural (M-2, S- ->N, GC). Appeasing the researcher’s request, Annie was also successful in using her GC to create the graph for $y = -6 + 4x + 4$ (line 3, Figure 4.6a) yet was unable to articulate why the graphs “equal the same thing” (line 6). Like her movement to the numeric table, this movement to the graphic representation is evidence of a multistructural level of RF (M-2, S- ->G, GC).

In the above exchange, Annie had also correctly anticipated that the graph of $y = x - 2 + 3x$ would be identical to the graph of $y = -6 + 4x + 4$ (line 4). From this explanation it is clear that Annie used the numeric table representations of $y = x - 2 + 3x$ and $y = -6+ 4x + 4$ to inform her creation of the graphic representation of $y = x - 2 + 3x$ without the use of her GC (i.e., two lines with the same table will have the same graph\(^1\)). The fact that Annie correctly created a “copy cat” graph (SA_day2_initialinterview_23:07) yet did not interpret the meaning of this representation with respect to the meaning of the solution to the equation is again evidence of a multistructural level of RF (M-2, N- ->G, GC, P&P, anticipate).

As was alluded to above, Annie did not interpret the meaning of the tables and graphs to signify “infinite solutions.” The researcher reminded Annie of the task of solving this equation and attempted to refocus her explanations on the solution to the equation.

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\(^1\) It is noted however that the numeric table is discrete whereas the graph is continuous.
R: So what does that [the fact that you got the same table] tell you about the solution to this equation (gestures back and forth across the equation $t - 2 + 3t = -6 + 4t + 4$), this original equation?

SA: They should probably equal the same thing. Or like, both sides are even (uses pen to gesture across the equation $t - 2 + 3t = -6 + 4t + 4$), and they like, equal the same thing in table (uses pen to point to table on paper [Figure 4.5b]) and in graph (uses pen to point to graphs drawn on paper [Figure 4.6b]).

R: So if you were to write down what's the solution to that equation, what would you write down?

SA: I...the solution would either be... uh.... uhm...

R: And it's OK if you just say it in words.

SA: I have no idea what the solution would be. The solution's like, it's the same thing, it's the same thing, it should probably equal the same thing in graph and in table, but right here (points to the symbolic work that shows $7t + -4$ [Figure 4.3]), it doesn't actually equal to the same thing, it's going like, what did I do? What did I get into? Why is it doing that? Is it adding wrong? Am I sure that's the answer? Is it?

Illustrated in lines 2 and 6 above, Annie did not make further progress in interpreting the meaning of the graphic nor the numeric (table) representation types in regards to solving the equation (supporting the multistructural classification of RF). Immediately following this exchange, the researcher offered Annie a new sheet of paper with the original problem printed on it, and Annie proceeded to work within the symbolic representation type, albeit unsuccessfully (Figure 4.4, discussed previously). Annie was focused on several representation types, expressing some understanding of how they are connected—the expressions $t - 2 + 3t$ and $-6 + 4t + 4$ are the same in a graphic representation, and the same in a numeric (table) representation (line 2)—but no understanding of what this means for solving the equation (line 6).

It is significant that Annie self-prompted a reflection on the symbolic equation in an apparent attempt to reconcile the “sameness” that she observed within the numeric (table) and graphic representations. This is not coded as an additional level using the framework for RF because, despite a focus on several representation types, Annie did not
make additional progress in solving the task. Her problem solving goals of reflecting and reconciling are still noted.

Summary of Task 12. During the initial interview, Annie’s progress in solving an equation that has infinite solutions was dominated by the symbolic representation type. Within this type, Annie’s RF was characterized at the prestructural level. It is possible that had Annie been given a problem distinct from the pre-test, or a copy of the same problem without her incorrect attempt during the pre-test, that she may have been more successful in working within this representation type. However, as evidenced in other tasks during the initial interview, Annie’s work within the symbolic representation type was often misguided by incorrect strategies (e.g., combining like terms across the equal sign in Task 10).

The researcher’s prompting to consider a graphical approach to solving the equation led Annie to use her GC to first move from the symbolic equation to two numeric tables. Although able to recognize that the tables were the same, Annie did not convey an understanding of why, nor how this related to solving the equation, a multistructural level of RF from the symbolic to the numeric (table) representation type. Annie also used her GC to create one graph then correctly anticipated that both graphs would be the same. The movement from symbolic to graphic and numeric to graphic representation types was at the multistructural level because again Annie did not correctly interpret these graphs to represent infinite solutions.

Finally Annie considered all three representation types in her reflection on the perceived “sameness” across the graphic and numeric table representations in an attempt to reconcile the revisited symbolic representation type. However, the promise of these
problem-solving activities did not help Annie make additional progress in solving the equation. Her conclusion that \( t = -4 \) was the solution was based on incorrect work within the symbolic representation type (a prestructural level of RF), and not well connected to the numeric table nor graphic representation types. Overall with respect to this particular task Annie’s RF in solving the equation was low despite her small successes in recognizing the representations of the equations \( y = t - 2 + 3t \) and \( y = -6 + 4t + 4 \) as being the same in the table and in the graph.

*Solving an Equation \( ax + b = cx + d \) with One Solution: Task 14*

The final means by which Annie’s initial level of RF will be characterized is based on her performance on a task that involved solving an equation of the form \( ax + b = cx + d \) for \( x \) which had only one real solution. Annie’s pre-test work on this task (Lines 1–3, and 5) and her addition to this work during the interview (Line 4) are shown in Figure 4.7.

Despite evidence in other equation solving tasks that with specific prompting Annie could be successful in translating from symbolic to graphic and numeric representations, Annie did not demonstrate these abilities in this particular equation-solving task. Prompts to “circle all values in the table that represent solutions to the equation 100 + 2x = 12x − 10” and to “explain how graphs can be used to solve” the aforementioned equation were intended to elicit higher levels of RF, yet Annie persisted at the prestructural level, bounded to use of the symbolic representation, and did not attempt to use tables or graphs to solve the equation.
Figure 4.7. Annie’s work within the symbolic representation type to solve the equation 100 + 2x = 12x – 10 for x.

For example, prompted to use the table in Part a, Annie expressed “I didn't know what to do in Part a, I'm like OK, how? And when? And what happened in the equation?” (SA_InitialInterview_day2_Task14_35:40). For the graphs, Annie posited, “Graphs. I really didn't think a graph could be used to solve that equation. I was like, how am I going to do that. It's a teeny bit impossible” (SA_initialinterview_day2_Task14_38:13).

As evidenced by her pre-test work, Annie was limited in her success in working within the symbolic representation type to solve the equation. During the interview, the procedural nature of Annie’s understanding of the meaning of the symbolic equation and the solution to that equation were evident. For example, when Annie was explaining how to check her solution she said,

Like if it's my last part, and I'm like, ‘Am I sure that's my answer?’ I would just re-type it in and try and see if it's right. [Brief pause] Ten divided by (types 10/ into calculator that is not turned on), hold on (writes on paper and divides both sides of the equation 10x = –110 by 10; then types on calculator as she explains
[Figure 4.7]) negative one-ten (−110) divided by (/) ten (10) equals (Enter) negative eleven. So I would try and re-do that to see if I was correct.

(SA_InitialInterview_Day2_36:41:84)

After this explanation, Annie verbally confirmed that $x = -11$ is the solution to this equation. To Annie, the use of the graphing calculator for arithmetic computations on numbers was sufficient for checking and verifying the solution to the equation. The use of the GC to perform numeric computations was not considered to be a movement to the numeric representation type, and instead, was a sub-procedure that was part of her symbolic solution attempt. Based on her incorrect manipulation techniques and interpretation of $x = -11$ to be the solution to the equation, Annie demonstrated a prestructural level of RF in regards to solving this equation within the symbolic representation type (P-0, S-X>S, P&P). Annie did not think it was possible to use tables or graphs, despite her previous work in Task 12 with these representation types.

**Summary of Annie’s Initial RF**

Across all of the equation solving tasks at the onset of the study, Annie’s RF was the weakest in tasks that she worked within the symbolic representation type (the prestructural level). All of Annie’s equation solving was grounded in symbolic approaches that yielded incorrect solutions. When prompted to consider graphical and tabular approaches to equation solving, Annie showed mixed abilities. She recognized that when two equations yield the same table and thus the same graph, that both sides of the equal sign are the same (i.e., equivalence of expressions). However, she did not use correct language in interpreting the graphic and numeric equivalence in terms of the solution to the symbolic representation (i.e., all values of $x$ make the equation true).
In a later problem, when prompted to use a table to solve an equation, Annie resorted to the symbolic approach. In the same problem, when prompted to think of solving the equation graphically, she did not think it was possible to do so. It seemed as though Annie considered the symbolic representation type to be the only legitimate way to represent the solution to an equation, or maybe the only (albeit weak) tool that she had available for solving problems involving linear equations.

*Annie’s Final RF*

Annie’s performance during the final interview was rich in the creation and interpretation of MR. Equipped with a CAS, P&P, and a set of tasks comparable to those administered during the initial interview (see Table 4.1), Annie demonstrated skill in creating graphs, numeric tables, and verbal representations from symbolic equations. Having several representation types at her disposal when solving linear equations, Annie seemed to prefer and put more emphasis on the creation of and meaning of graphs and tables over the creation and meaning of the symbolic equations.

Although Annie did not generalize across representations, she did demonstrate an ability to connect representations. Building off her understanding of the Cartesian Connection, Annie did verbalize the coordination of solutions to equations from symbolic to verbal, symbolic to numeric table, symbolic to graphic, and from graph to numeric table. Again, her success in translating between representations outweighed her ability to transpose within representations, especially within the symbolic representation type. Finally, as will be evident in the following detailed characterizations of Annie’s final RF based on her completion of a variety of tasks, Annie’s justifications for solutions to
equations seemed to be based on her understanding of the different cases of the number of solutions an equation has—one, infinite, and none.

Solving an Equation \( c = ax + b \) With One Solution: Task 10

During the final interview, Annie was successful in using MR to solve an equation of the form \( y = ax + b \) at \( y = c \) (Task 10 is shown in Figure 4.8). Her work on this task first involved the correct creation of a graph and table representation using her CAS, and recorded on her paper.

Annie first used the symbolic equation to determine that \( x = 2 \) is the solution when \( y = 4 \) (see Part b in Figure 4.8). As Annie worked within the symbolic representation type, she explained,

\[ \text{Um, I'm thinking that if I plus two to what negative two, which is actually minus two, I'd cross that out, add two to the other side, and get six and I'd divide it by three to get my answer, which is two.} \]

(\text{SA\_day26\_finalinterview\_10:03})

Annie’s work and interpretation of this work within the symbolic representation type is evidence of a unistructural level of RF (U-0, S→S, P&P).

When prompted to show the solution in the table (Task b), she was quick to recognize and circle the row “2, 4” in the table (Figure 4.8). Prompted to explain, “Why did you circle that?” Annie referenced how her symbolic work informed her identification of the solution in the tabular representation: “If \( y \) equals four (points to original problem statement) then \( x \) would equal two (points to \( x = 2 \) in symbolic work, then the circled row in the table [Figure 4.8])” (\text{SA\_day26\_finalinterview\_10:28–10:48}).

To Annie, the solution found within the symbolic representation was used to justify or
explain her use of the numeric tabular representation to identify the solution to the
equation, a uni-directional connection at the relational level of RF (R-2, S→N, P&P).

10.) An equation is given in Row 3, below.

a.) Create a graph and a table for the symbolic equation \( y = 3x - 2 \).

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td>( y = 3x - 2 )</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

b.) Find the solution to the equation \( y = 3x - 2 \) when \( y = 4 \).

Show the solution to the equation using the table

Show the solution to the equation using the graph

Show the solution to the equation using the symbolic equation.

\[
\frac{4 + 2}{3} \cdot \frac{1}{2} \cdot x = \frac{6}{3} \cdot x = 2
\]

Figure 4.8. Annie’s work during the final interview on solving an equation of the form
\( y = ax + b \) at \( y = c \) for \( x \) using the table and the symbolic equation.

In a later exchange shown below, Annie’s understanding of the connection
between the solution to the equation as shown in the symbolic equation and as identified
in the numeric table is elaborated. This gives additional evidence for this classification of
RF.
R: What about, does it make sense what you circled on the table and what you found in your equation?
SA: (with confidence) Yes.
R: Tell me, can you tell me why?
SA: Because (inaudible) Because it makes sense because if my \( x \) (points to solution of \( x = 2 \) in symbolic work [Figure 4.8b]) is two, there's my \( x \) (points to value of 2 in \( x \)-column of table [Figure 4.8a]) and \( y \) (points to value of 4 in \( y \)-column of table [Figure 4.8a]) is (points to the four in the equation \( 4 = 3x - 2 \) [Figure 4.8a]) four. That's how it easily makes sense to me. (SA then points to circled values in table, then back and forth between the "4" and the "\( x = 2 \)" in the symbolic representation Figure 4.8a and b).
SA: I find my answers from here (points to symbolic equation work), and then I find them there (points to table), circle, and then, yeah.
R: Mm hm.
R: Could you have found the answers in a different order? Like could you have found the answer in the table first? Like I know that you solved it in the symbols first (points to students' symbolic work on paper).
SA: Possibly. Yeah.
R: Why do you think so?
SA: Because if I'm looking, if my \( y \) is already there (points to the \( y = 4 \) in the below to Problem statement "Find the solution to the equation \( y = 3x - 2 \) when \( y = 4 \)) then all I have to do is look for my \( x \), which is right next to it (moves pen to the left in blank space on paper).
R: Ok. Ok.
SA: I just chose to take the equations, because it's, pretty much a habit of just going with the equations first and then just doing the rest.
R: I see. Something that you've learned to start with the equation?
SA: Mm. Hm.

In line 4, Annie’s identification of solution \( x = 2 \) in the symbolic work was identified as an invariant feature of the numeric table representation, and in this tabular representation, the corresponding \( y \)-value was four, which was the given value in the statement of Task 10. The directional nature of this translation from the symbolic to numeric (table) representation is evidenced in lines 5 and 12 when Annie explicitly referenced the order in which she approached this task.

Although Annie recognized that the solution could have been shown in the numeric (tabular) representation before performing symbolic transpositions (line 8), there
was not sufficient evidence to support a bi-directional connection here. Line 10 is more of a hypothetical movement from a problem statement within the symbolic representation to be able to solve within the numeric representation type before solving within the symbolic representation type. Overall, Annie’s “habit” of starting to solve an equation within the symbolic representation before translating to other representation types was clear (line 12).

Despite the ease with which Annie worked within the symbolic representation to solve, and subsequently used this solution as a means to justify her translation to the numeric (table) representation, Annie struggled a bit more with the graphic representation. Annie had been successful using her CAS to create a graph of the given equation $y = 3x - 2$ and sketching it on her paper (Figure 4.9). However, when working within this representation type, she was able to point to $(2,4)$ on her CAS screen, but not on her paper. Her inability to reconcile the graphic representation type between the P&P and CAS inscriptions caused some discomfort for Annie that she was not able to resolve.

Figure 4.9. Annie’s use of the CAS to graph the equation $y = 3x - 2$ and the corresponding P&P graph.

The following exchange is used to classify Annie’s RF on this part of the equation solving task.
R: And can you show that [solution] on the graph?

SA: I didn't number it. Wait, wait, wait [...(irrelevant exchange related to saving the students' work on the CAS document)…]

SA: (Types f3(x) = 3x – 2 into the Graphs line of a CAS graph page, Enter, moves cursor along the line, moves horizontally to the y-axis [Figure 4.9a]) four (uses pen and hovers over quadrant one of graph drawn on her paper) is right there (points to the placement of her cursor on the graph).

R: Alright.

SA: If only I could find that on my graph [in reference to P&P inscription, Figure 4.9b]. (Sighs).

Annie was able to move from the symbolic representation type to create a correct graphic representation type on her CAS (line 3, above), but was not able to correctly use this representation to identify the solution of x = 2—the point (2,4) was identified on the CAS inscription but the value of x = 2 was not identified as the solution to the equation. Annie had correctly moved from the symbolic to the graphic representation of y = 3x – 2 using her CAS (creation) but did not give a completely correct interpretation of the solution to the equation, a multistructural level of RF in solving the equation from the symbolic to graphic representation types (M-2, S- ->G, CAS). Finally, based on Annie’s inability to work within the graphic representation type to create a complete P&P representation of the graph from the CAS inscription (with appropriate labels, for example) (line 5), her RF within the graphic representation type was characterized as prestructural. The CAS representation Annie created was not well connected to the P&P graph, Annie was not able to correctly work within this representation type to identify the point (2, 4) nor articulate the solution of x = 2 in the P&P graph (P-0, G-X->G, CAS, P&P).

On Task 10 Annie demonstrated a unistructural level in working within the symbolic representation type, and a relational level of RF in articulating a connection
from the symbolic solution type to the numeric representation type. To complete this task, Annie’s RF in solving the equation was also characterized as multistructural from the symbolic to the graphic representation, and prestructural within the graphic representation type.

*Solving a Linear Equation With All Real Solutions: Task 12*

Task 12 was designed to have students consider and analyze the correctness of a symbolic solution approach to solving an equation with infinite solutions, and to state the solution to the equation (see Figure 4.10). This task was not included as part of the post-test examination, only the final interview. Annie’s work on this task during the final interview is divided into two sections: work within the symbolic representation type and a resourceful multi-representational approach.

*Work within the symbolic representation type.* In approaching this task, Annie initially struggled to interpret Andy’s work of combining like terms (Step 2). When she tried to check his work, Annie questioned the “Algebra” in Step 2, seemingly referring to the right hand side of the equation despite the error in her interpretations, “How did he get the two $x$ plus two?” (SA_day26_finalinterview_41:03). She continued to mumble to herself something about the $x$ and three $x$ as she verified what she referred to as “two $x$” on the right hand side of the equation (which is actually a negative two $x$). Shown in Line 3 of Figure 4.11, Annie also used her CAS to verify that $8 + -6 = 2$, giving her the “$+ 2$” on the right hand side of the equation in Step 2.
178

Figure 4.10. Annie’s work on Task 12 during the final interview involved verbal interpretations and CAS-based inscriptions (no use of P&P).

<table>
<thead>
<tr>
<th>Steps</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.) Original problem</td>
<td>(2 - x - x = x + 8 - 3x - 6)</td>
</tr>
<tr>
<td>2.) Combine like terms ′</td>
<td>(2 - 2x = -2x + 2)</td>
</tr>
<tr>
<td>3.) Add 2x to both sides</td>
<td>(2 = 2)</td>
</tr>
<tr>
<td>4.) Subtract 2 from both sides</td>
<td>(0 = 0)</td>
</tr>
<tr>
<td>5.) Solution</td>
<td>(x=0) and (x=2)</td>
</tr>
</tbody>
</table>

Andy worked out the problem above. When he substituted his solutions into the original equation, he discovered that both \(x = 0\) and \(x=2\) make the equation \(2 - x - x = x + 8 - 3x - 6\) true.

a) Was Andy’s work correct? Explain why or why not.

b) What is the solution to the equation \(2 - x - x = x + 8 - 3x - 6\)?

Figure 4.10. Annie’s work on Task 12 during the final interview involved verbal interpretations and CAS-based inscriptions (no use of P&P).

Figure 4.11. Annie’s work within a CAS Calculator page to make sense of transpositions on the equation \(2 - x - x = x + 8 - 3x - 6\).

As shown in Line 2 of Figure 4.11, Annie also used her CAS to combine the like terms of \(-2x\) and \(2x\), then immediately grabbed this line and changed it on Line 1 to \(-2x + -2x\). Based on this work, Annie explained,

If he would have added those two together (pointing to the \(-2x\) and \(-2x\) on either side of the equal sign) he would have gotten negative four \(x\) (based on CAS work [Figure 4.11, Line 1]). How he got two I really have no idea.

(SA_day26_finalinterivew_Task12_42:00–43:01)
To Annie, the step of adding $2x$ to both sides of the equation seemed to be synonymous with combining the like terms on either side of the equal sign (Figure 4.11, Line 1).

Annie seemed convinced that Andy should have gotten $-4x$, not 2 in the equation $2 = 2$.

The fact that Annie was not able to reflect on the created representation and interpreted it to be incorrect is evidence of a prestructural level of RF within the symbolic representation type (P-0, S-X->S, P&P, CAS).

It is interesting to note that later in the interview, after considering alternative solution approaches (including a graphical approach, numerical [table] approach, and symbolic/verbal approach within the Graphs, Table, and Calculator pages, respectively) Annie articulated a more meaningful understanding of the symbolic representation type.

Oh I get what they're doing (points to the third step in the table on her paper), they're adding two $x$ to both sides. They're pretty much the same […] Oh I get it, because it's infinite solution. They're both the same, except for with minuses. Emm... Yeah, they're infinite solutions it's hard to tell what $x$ equals.

(SA_day26_finalinterview_45:09, 46:35).

Somewhat tacit in this exchange is Annie’s recognition of the correct result of Step 3 and that the expressions related by the equal sign were the same in Step 2, namely $2 - 2x$ and $-2x + 2$. Annie also reasoned that the equation had infinite solutions and the symbolic representations (the expressions relating the equal sign) were the same. She did however struggle with explaining what this meant for the value(s) of $x$ to signify the solution.

With the available data, one can only conjecture that some aspect of Annie’s multi-representational approach with her CAS informed a reflection on the previous interpretation of the equation (e.g., the graphic representation was the first form in which she identified that the equation had “infinite solutions”). Despite this ambiguity, it is clear that Annie had a reflective stance; she recognized the created symbolic transposition of
“add 2x to both sides” and correctly justified that because the equation had infinite solutions it made sense to her that the expressions related by an equal sign were “the same.” Thus after a multi-representational approach Annie’s RF improved to the unistructural level within the symbolic representation type; she correctly interpreted the given series of symbolic equations that she had previously thought to be incorrect. Namely,

\[
\begin{align*}
2 - x - x &= x + 8 - 3x - 6 \\
2 - 2x &= -2x + 2 \\
2 &= 2
\end{align*}
\]

This interpretation allowed Annie to make progress toward successfully completing the task because an “identity” equation in which the expressions related by the equal sign are “the same” is one way to signify infinite solutions, which Annie had verbalized (U-0, S\(\rightarrow\)S, reflection).

Annie’s resourceful multi-representational approach. After coming to a sticking point within the symbolic representation type early in the interview, namely, “How he got two I really have no idea” (SA_day26_finalinterivew_42:00–43:01), Annie was resourceful in creating a graphic then numeric table representation to overcome the barrier experienced within the symbolic representation type. Annie’s creation of these representations was completed using her CAS, as shown in Figure 4.12a and b.
Figure 4.12. Annie used her CAS to create graphic and numeric (tabular) representations of the equation \(2 - 2x = -2x + 2\).

It is important to note that the researcher did not prompt Annie to consider this multi-representational approach, exemplified below.

1 SA: But if you take (trails off…) Cause I know, hold on, I need a graph for this. (Presses Home, Add Graph page)
2 R: Why do you need a graph? Can you tell me about that?
3 SA: Because if I take both of them (points to \(2 - 2x\) and \(-2x + 2\)) which is probably going to be equal in the graph, it may help.
4 SA: (Types \(2 - 2x\) into \(f_6(x)\), Enter, Tab, types \(-2x + 2\) into \(f_7(x)\), Enter, mumbles to self as she types).
5 SA: [Looking at Figure 4.12a] (Deep inhalation of breath) That's my problem!
6 R: What do you mean?
7 SA: Well, they have (presses control T) infinite solutions (cursors in table from \(f_6(0) = 2\) to \(f_7(0) = 2\) [Figure 4.12b]).

(SA_day26_finalinterview_43:01–43:46)

In the above exchange, and evidenced in her CAS work, Annie demonstrated a relational level of RF from the symbolic to the graphic representation. First, Annie anticipated that the graph of the equation \(2 - 2x = -2x + 2\) would yield two lines that are “equal in the graph” (line 3). Annie then used her CAS to correctly create a graph of the equation \(2 - 2x = -2x + 2\) as two lines, \(f_6(x) = 2 - 2x\) and \(f_7(x) = -2x + 2\) (line 4, Figure 4.12a). Immediately, Annie recognized her “problem” from the symbolic representation type, and used the graph to explain that the equation \(2 - 2x = -2x + 2\) has infinite
solutions (line 7, Figure 4.12a and b). The fact that Annie translated from a symbolic equation to a graph to overcome a barrier in the symbolic representation type and interpreted there to be infinite solutions is evidence of a uni-directional connection, which allowed her to make progress in successfully solving the task (R-2, S\rightarrow G, CAS, representationally resourceful). Note that although Annie had created the numeric tabular representation as she was interpreting the graphic representation (line 9), the use of this representation is not considered to be a factor at this point in Annie’s understanding of “infinite solutions.” The speed with which Annie created the table (i.e., split screen graph/table view) seemed to be more of a technical strategy or preference that she learned in using the CAS than something that informed her interpretation of the equation in the above exchange.

For the remainder of the interview, Annie was focused on the relationships between the symbolic representation (as shown in the series of equations in Steps 1–5 in Figure 4.10), the numeric table representation that was created with her CAS (see Figures 4.12b and 4.13), and the verbal representation created in a CAS Calculator page (Figure 4.14).

\footnote{This correct movement to the numeric table representation is included in later codes that also involve specific interpretations of solutions to the equation from the symbolic to the numeric table representation}
Figure 4.13. Annie worked within the numeric (table) representation type in an attempt to explain the solutions to the equation $2 - 2x = -2x + 2$.

In the following dialogue, Annie moved back and forth between the symbolic and numeric table representation types in what seemed to be an attempt to justify Andy’s conclusion that $x = 0$ is a solution.

1 SA: But how he got zero, well, zero is correct (points to $x = 0$ in Step 5 on paper), two isn't (points to $2 = 2$ in Step 3 on paper).
2 SA: Well it is correct (pointing to $2 = 2$) but it's not really the answer, it's zero (pause) (cursors to an $x$-value of 0 in the table [Figure 4.13a])
3 R: Can you tell me more about that?
4 SA: Because two (points to $2 = 2$ on paper) is "y," but it's like infinite solutions (points to CAS handheld).

(SA_day26_finalinterview_Task12_43:57)

In lines 1–4 above, and as shown in Figures 4.12a, 4.13a and 4.12b, Annie had correctly created the numeric table representation and correctly used this representation to justify the conclusion that $x = 0$ is a solution. Her specific reference to the $x$- and $y$-values in the P&P symbolic representation of $x = 0$ (line 1) and in the numeric table (lines 2 and 4) help to clarify the fact that this is a uni-directional connection because she
focused on the invariant feature of $x = 0$ (a symbolic equation) and a corresponding $x$-value of 0 in the table with $y$-values of 2 (i.e., $f6(0) = f7(0) = 2$) (R-2, S→N, CAS, P&P).  

Annie then continued to reference the symbolic equations and tabular representations in her attempt to explain Andy’s solution of $x = 2$.

So, how did he get two? Because (cursors in table) two only shows up once in this, but what really gets me is the zero (points to $x = 0$ in Figure 4.13a) because zero is what equals it (then points to $0 = 0$ in Step 4 of Figure 4.10), but the two equals the negative two (cursors down to $f7(2)$ and points to it in the table [Figure 4.13b]). This equation is getting me messed up. (Sigh).  

(SA_day26_finalinterview_43:57)

Annie had correctly recognized that at an $x$ value of 2, the corresponding $y$-values (or function values) are $-2$ (line 5, Figure 4.13b), yet she was unable to interpret this as a way to justify that $x = 2$ is a solution to the equation. Likewise, she struggled to see the equation $0 = 0$ as corresponding to an $x$-value (or solution) of 1. Here, her creation and use of the numeric table representation did not support a correct interpretation to justify or explain the meaning of this symbolic representation. Thus for the solution of $x = 2$ Annie’s RF was at the multistructural level (M-2, S-→N, CAS, P&P).

When prompted to explain what the solution to the equation is (Task 12 b), Annie said “I’m believing it’s zero” (SA_day26_finalintervie_Task12_45:45). Annie then decided to insert a CAS calculator page and used the “with” (l) operator to check both of Andy’s purported solutions of $x = 0$ and $x = 2$ (Figure 4.14a).

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3 This is one of two uni-directional connections that comprise a multi-directional connection for the solution of $x = 0$. 


Figure 4.14. Annie used a CAS Calculator page to check the correctness of the solutions $x = 0$ and $x = 2$ in the equation $2 - x - x = x + 9 - 3x - 6$ then referenced the table to hint at the fact that there can be more solutions than just $x = 0$ and $x = 2$.

After checking $x = 2$, she exclaimed, “That’s always true!” and seemed more confident about why this was the case. Consider the following dialogue on what this CAS work meant to Annie.

1 R: And you found what you just did here--
2 SA: They're both true.
3 R: So what does that mean if it tells you true?
4 SA: That each one is correct.
5 R: Ok. So if you had to explain to someone what the solution was what would you say?
6 SA: It could be either zero, or two.
7 R: Could it be anything else?
8 SA: Yes, including that it's a very infinite solution I believe. (Switches back to split graph and table view on CAS and scrolls down to f7(3) [Figure 4.14b])

(SA_day26_finalinterview_Task12_46:35–47:22 + )

From Annie’s dialogue, the CAS calculator page was used as a means to justify that both $x = 0$ and $x = 2$ are “correct” solutions to the equation (lines 2–6, above). The fact that Annie was focused on the invariant feature of the solutions $x = 0$ and $x = 2$ makes this a connection at the relational level of RF. In other words, Annie used a CAS Calculator page to translate from the symbolic representation of the original equation to a verbal result of “true” in two separate creations. The verbal confirmation of “true” was interpreted as a means to justify the correctness of both solutions $x = 0$ (R-2, S→V,
and $x = 2$ (R-2, $S \rightarrow V$, CAS). For the solution of $x = 0$ in particular, this uni-directional code is subsumed in a higher code category because Annie had also interpreted this invariant feature from the symbolic to the numeric (table) representation (R-6, $S \rightarrow N$, $S \rightarrow V$, CAS, P&P).

It is also significant here that Annie recognized that the solution could be more than just $x = 0$ and $x = 2$ (lines 7–8). She used her CAS numeric table representation to confirm that at $x = 3$ (Figure 4.14b) “they have the same answers” (47:51). However, in the absence of additional data, this utterance is not coded according to the analytic framework; Annie would have needed to explicitly reference other values of $x$ as being solutions to the equation. It was not clear whether or not Annie’s understanding of “infinite solutions” meant that all values of $x$ are solutions to the equation, instead, it seemed as though infinite solutions meant that the graphic and table representations showed two identical linear relationships. Indeed, despite her progress in using the verbal representation to justify the fact that $x = 2$ is a solution to the equation, Annie expressed some apprehension about how both $x = 0$ and $x = 2$ could be solutions (a subset of the entire solution set).

I feel Ok about what work I’ve shown so far and I have a little bit of questions but that’s just, well, that’s just (short pause) I’m still boggling about the zero and the two […] I’m debating on how it could equal two, and I know that it’s an infinite but, why not choose one, then the other.

(SA_day26_finalinterview_48:23; 49:11)

One possible interpretation of this statement is that Annie is confident that there are infinite solutions, but is not confident about what that means in relation to needing to choose particular values of $x$ or multiple values of $x$ as solutions. In particular, based on

4 This is the second of two uni-directional connections for the solution of $x = 0$. 
Annie’s CAS work (e.g., Figure 4.14a) she seemed to believe that both $x = 0$ and $x = 2$ were solutions to the equation, yet her movement from the symbolic to the numeric table for the solution of $x = 2$ was less confident. Another possible interpretation of Annie’s expressed confusion in the above dialogue is the fact that the solution of $x = 0$ corresponds to the equation $2 = 2$ yet the solution of $x = 2$ corresponds to an equation of $-2 = -2$, which is not explicitly included in the statement of Task 12. Moreover, the equation $0 = 0$ corresponds to a solution of $x = 1$, also not reflected in Andy’s work in Task 12.

It seemed as though Annie’s use of the CAS calculator page with the symbolic and verbal representations (Figure 4.14a) was only helpful in a limited way. It is possible that had Annie checked these solutions by hand (and substituted values into the equation) she may have come to better understand the meaning of the equations and the meaning of the result of true by comparing the numeric equations and numeric table. Annie had struggled in explaining the solution of $x = 2$, which also attests to the fact that connections across representations are not bi-directional or multi-directional on their own. Understanding one connection (in this case, from the symbolic to the verbal) was not sufficient to give Annie confidence in the overall conclusion of the solution to the equation.

*Summary of Task 12.* Annie had originally struggled to make sense of the symbolic representation type at the prestructural level, later coming to a more complete understanding and recognition of the equivalent expressions related to the equal sign (a uni-structural level). It is speculated that this change occurred as a result of Annie’s multi-representational approach, but a specific relationship was not determined. Also,
Annie was not completely convinced of nor able to explain that all values of $x$ make the equation true, which would have been a more sophisticated way to articulate the meaning of the infinite solution case within the symbolic representation type.

It seems as though Annie’s self-prompted use of the CAS was to overcome barriers in the symbolic representation type. In this particular task, Annie used her CAS as a representational toolkit to recognize the case of infinite solutions from the graph, and to support a multi-directional connection at the relational level of RF from the symbolic to the numeric table representation and from the symbolic to the verbal representation (in justifying the solution of $x = 0$). Although Annie made a uni-directional connection from the symbolic to the verbal representation for the solution of $x = 2$, she was less confident in justifying this solution and was not successful at using the numeric table to explain this symbolic result, a multistructural level of RF. Despite these successes and higher levels of RF, Annie had some disconnected understandings, with no bi-directional connections across representation types.

*Solving an Equation $ax + b = cx + d$ With One Solution: Task 14*

When solving the equation $9 + x = 6 + 4x$ for $x$ during the final interview, Annie demonstrated a prestructural level of RF in the numeric tabular representation type.

While she correctly used the table to circle the row “1, 10, 10” (see Figure 4.15) she gave a hesitant and incomplete interpretation of what the solution was.
14.) a. Circle all values in the table that represent solutions to $9 + x = 6 + 4x$.

<table>
<thead>
<tr>
<th></th>
<th>$y = 9 + x$</th>
<th>$y = 6 + 4x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>6</td>
<td>-6</td>
</tr>
<tr>
<td>-2</td>
<td>7</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
<td>6</td>
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<td>1</td>
<td>10</td>
<td>10</td>
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<tr>
<td>2</td>
<td>11</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>18</td>
</tr>
</tbody>
</table>

*Figure 4.15. Annie’s work within the numeric representation type to identify the solution to the equation $9 + x = 6 + 4x$.*

Consider the following transcription from the final interview of Task 14 Part a.

1  R: Can you do part a?
2  SA: Yeah (as she immediately circles the row in the table "1, 10, 10" [Figure 4.15]).
3  R: How did you know to circle that?
4  SA: Because if your $y$ equals the same thing, they can intersect at least once.
5  R: And what about in part b--before doing part b, what do you think the solution to the equation is?
6  SA: Hmm. Either it's one, or huh, I have no idea until I do it.
7  R: Why did you say it might be one?
8  SA: Because the $x$ is one, and that's where both of them, kind of like, intersect (gestures with hands to show a cross or intersection like an "X" in the air).
9  R: Mm hm.
10 SA: Eh, this is that sometimes thing.
11 R: How can you tell that?
12 SA: Well the other numbers don't hit the same number, it only hits there once, which is right at ten. So you've only got like one answer from it. (SA_day26_finalinterview_30:11–31:40)

In line 6, Annie hesitated as she stated that the solution is “one” yet conjectured that the solution might be something other than one. She continued in line 8 to explain that the “$x$ is one” because that is where the graphs of both $f(x) = 9 + x$ and $g(x) = 6 + 4x$ intersect. Annie was also successful in identifying that there is only one solution (lines 10 and 12). Overall, despite Annie’s correct use of the numeric table (lines 2–4), Annie was not convincing in her interpretation of $x = 1$ as the one solution to the equation $9 + x = 6 + 4x$, a prestructural level of RF within the numeric table.
representation (P-1, N- >N, P&P). To Annie, the notion of a solution represented numerically seemed to be tied to the graphic representation type and the notion of an “intersection” (lines 4 and 12), of which we’ll see more evidence of next.

In her subsequent work, despite the prompt in Part b to “Solve the equation $9 + x = 6 + 4x$ symbolically” she did not work within the symbolic representation type to solve the equation, and instead graphed the equations $f_4(x) = 9 + x$ and $f_5(x) = 6 + 4x$ using her CAS in a graph/table split screen view (see Figure 4.16a). Based on this initial view of the CAS-based representations that she created, Annie moved from the symbolic equation to a graph and a table in her explanation that, “It [the CAS graph] tells me that it [the graph of the equation] will intersect soon (motions along positive y-axis of the CAS graph screen) and with my table it [the CAS function table] tells me right there (points to $f_5(1) = 10$ in table)” (SA_day26_finalinterview_33:37). Annie demonstrated a multistructural level of RF as she correctly created a graphic and numeric table representation of the symbolic equation $9 + x = 6 + 4x$, and identified the invariant feature of the “intersection point” in each representation type, but did not isolate the value of $x$ to give a completely correct interpretation of the solution to this equation in either the graph or the table (M-2, S- >G- >N, CAS).

Despite Annie’s movement between these representation types, Annie expressed confusion about what the solution was.

R: So, can you check the solution to the equation? What did you find the solution was?
SA: My solution is either ten or one. I'd have to find out, I really don't know right now.

(SA_day26_finalinterview_34:15).
Immediately following this exchange, and possibly a consequence of the researcher’s prompt to “check” her solution, Annie decided to insert a CAS calculator page (Figure 4.16b) as a resource to overcome the barrier she encountered in the split screen Graph/Table CAS page (Figure 4.16a).

Consider the following dialogue and interpretation of Annie’s work.

1 SA: (From Graph page [Figure 4.16a]) OK get back to that. (Home, Insert Calculator page, types as she says) nine plus x equals six plus four x (inserts "|" operator) x equals 1 (input: "9 + x = 6 + 4x| x = 1," Enter, output: "true" [Figure 4.16b]) True!
2 R: What did you try?
3 SA: I put nine plus x equals six plus four x, and that slash thingy, then x equals one, and it came out true.
4 R: Were you expecting it to give you true?
5 SA: Eh, eh,... (tilts hand back and forth)
6 R: Maybe, maybe not?
7 SA: It's like you just hope it's true and it's like, yeah.

(SA_day26_finalinterview_Task14_32:29)

Annie seemed satisfied with the verbal confirmation of “true” given as a result of executing $9 + x = 6 + 4x| x = 1$ (Figure 4.16b, line 1), a movement from the symbolic to the verbal representation type. When prompted to reflect on this result in relation to Annie’s previous work (line 3) Annie expressed uncertainty (lines 5–6). A better probing
question would have been to ask Annie what the result of “true” meant to her in this situation. With the uncertainty in Annie’s voice and her “hopeful” result (lines 5–7), it is not clear whether Annie would have said that the value of \( x = 1 \) is the solution or not. Annie did not specifically interpret the meaning of this symbolic representation type to mean that the solution was \( x = 1 \), thus Annie’s RF in this exchange is coded as a multistructural level of RF (M-2, S- ->V, CAS).

Finally, when prompted to explain any relationships to her previous work, Annie first went back to work within the graphic portion of the split screen Graph/Table view on her CAS (Figure 4.16a) as the following exchange ensued.

1 R: So in Part b, where it asked you to solve, you did a graph and a table and did that graph and table help you solve?
2 SA: Yeah, it actually did.
3 R: So can you see the solution on your graph?
[irrelevant exchange related to getting the cursor to move from the table to the graph]
4 SA: (drags CAS graph screen to show the area of the graph in which the lines \( f_4(x) = 9 + x \) and \( f_5(x) = 6 + 4x \) intersect, moves cursor over intersection point, [Figure 4.17]) Right here.
5 R: How do you know?
6 SA: Hold on, my axes, my axes should tell me (moves cursor near the point (0, 10) then near the point (1, 0) on CAS graph).
[irrelevant exchange related to "blinking" lines on graph...]
7 SA: (counts as she moves her finger up along y-axis of CAS graph) four, five, six, seven, eight, nine, ten. There I counted it. And it hits right at ten.
8 R: So from your graph you think the solution is ten?
9 SA: Eh, wait no, wait, what? Oh; (talking to self, repeating the researcher) the solution is ten. Sorry. Um...
10 It ends up right at one (points to (1,0) on CAS graph) and then goes to ten (points to (1,10) on CAS graph). Yeah.
11 R: So from the graph what can you tell?
12 SA: Where one is (points to (1,0) on CAS graph), and where the tens (points to y-values of 10 in circled row on P&P table), and where it intersects right there (points to (1,10) on CAS graph).

(SA_day26_finalinterview_32:29; 36:00–37:06)
Prompted by the researcher to “see the solution in the graph,” Annie’s change of the window dimension (line 4) and reading of points on the axes of the CAS graph (e.g., lines 6–7) is evidence of her work within the graphic representation type. Annie’s RF in using this graphic representation to solve the equation was classified at the prestructural level because although she correctly identified the point (1, 10) as the intersection of the two lines (lines 7, 10, and 12) she did not correctly interpret the x-value of this ordered pair to be the solution to the equation \(9 + x = 6 + 4x\) (P-1, G- ->G, CAS). Instead, Annie was still focused on both the x- and y-values of the intersection point. The researcher probed Annie’s understanding of the solution within the graphic representation type by asking if she thought the solution was ten (line 8), to which Annie clarified that the point of intersection is (1,10), on both the graph and table (lines 9–10, and 12). This is further evidence of Annie’s groundedness to the graphic representation type by not understanding the meaning of the x-value of the point as the solution to the equation.

Note that the researcher had prompted Annie to go back to the graphic representation type (that she had already created) and Annie’s movement from the symbolic to the graphic representation type was already coded above as a multistructural level. Also, Annie’s brief reference to the values of 10 in the table on her paper were not significant to warrant a movement between the graph and numeric representation types either (line 12).
Figure 4.17. Annie understood the invariant feature of (1,10) as evident in the screen CAS representation of a table and graph.

Overall, Annie demonstrated a prestructural level of RF in working within the numeric (tabular) representation and when working within the graphic representation type. The task was directed at using MR to solve the equation \(9 + x = 6 + 4x\) yet Annie’s movement between representations was specifically focused on the ordered pair of the intersection point (i.e., the point \((1, 10)\)) and not the value of \(x = 1\), a multistructural level of RF. Despite these lower levels of fluency within and between representation types, Annie did demonstrate some understanding that the point \((1, 10)\) was an invariant feature of the graphic and numeric (table) representation types, evidence of her understanding of the Cartesian Connection.

At the close of this task, Annie’s expressed confusion as to whether the solution was 1 or 10 was somewhat reconciled, but not completely.

1 R: And what about what you showed on your calculator page, is that related to what you’re showing in your table and graph?
2 SA: Oui (French for yes).
3 R: How so?
4 SA: (struggles to verbalize coherently, points to “1” in x-column of table on paper, motions to values of “10” and back to “1” as she says) My one is x (presses Control Left to move back to split screen Graph/Table on CAS) I don’t like going back and forth, oh hey it [the graph] stopped blinking.
5 SA: And my one's right there (taps pen on point (1, 0) on CAS graph), for x (uses pen to motion up and down along the line \(x = 1\) from \(y = 0\) to \(y = 10\) on CAS graph page) to get ten (Control Right to move back to CAS Calculator page),
SA: And then, all my Xs (points to column of x values on P&P table, stopping at $x = 1$) point toward one (taps Calculator page $x = 1$).

R: Does this problem make sense to you?

SA: Yeah, but it's starting to make me confused.

R: Do you feel pretty confident about the solution that you found?

SA: Yeah, as long as it has true (points to CAS calculator page).

(SA_finalinterview_Task14_38:37)

What is interesting to notice in this exchange is that Annie seemed to switch her focus to be more on the value of $x = 1$ than on the value of $y = 10$ (lines 4 and 6). However, the “extraneous” information of the value of $y$ at the solution of $x = 1$ was not stripped from Annie’s explanation (line 5). More specific probing would have been needed to clarify what Annie really thought the solution to be, or to characterize her RF at levels beyond the prestructural and multistructural.

_Solving a Linear Equation With No Real Solutions: Task 13_

In solving the equation $x + 2 + 2x = 5 + 3x - 1$ for $x$, which has no solutions, Annie’s performance on the post-test was mirrored in her initial solution approaches during the final interview; she was grounded to the symbolic representation type and not successful at working within this representation type to solve the equation, a prestructural level of RF. During the final interview, Annie used the CAS Calculator page to perform arithmetic computations, work within the symbolic representation type, and used the “with” operator (l) to check her answers that were found by using P&P. From this work, Annie persisted in the symbolic representation type, moving back and forth between P&P and CAS representations, and despite the verbal feedback of “false” for several tested values of $x$, she was not successful at interpreting the fact that this equation had no
solutions. A discussion of Annie’s written work (Figure 4.18) and CAS work (Figures 4.19 and 4.20) during the final interview is given next.

Solve for the variable and check your solution.

If you use a graph, sketch the section of the graph.
If you use a table, include the table and indicate where you found the answer.
If you use equations, show all of your work.

Figure 4.18. Annie’s attempt to solve \( x + 2 + 2x = 5 + 3x - 1 \) and her incorrect work within the symbolic representation type during the final interview.

Annie made several attempts to work within the symbolic representation type using P&P (Figure 4.18) and also used her CAS to work within the symbolic representation type and to move from the symbolic to the verbal representation type within a CAS Calculator page (Figure 4.19).
After these several incorrect attempts, the researcher asked for Annie’s interpretation of her progress on this equation-solving task.

1 R: So let's think about this for a minute. What do you think is happening in this equation—this original equation (points to \( x + 2 + 2x = 5 + 3x - 1 \)) in which you were to solve for the variable (points to the problem statement Solve for the variable and check your solution)?

2 SA: I really don't know. I've tried so many ways and it doesn't work.

3 S: Could you solve it in a different way?

4 SA: I've been trying, and unsucceeding [sic]. ( Cursors up through previous commands and outputs of CAS Calculator screen).

5 SA: I mean, I got most of my answer which is (highlights the result of line 6, \( 3x + 2 = 3x + 4 \) [Figure 4.19a], which had previously been computed on line 11 as the result of \( x + 2 + 2x = 5 + 3x - 1 \) [Figure 4.19b], and presses Enter to put the result as the input on line 0 [Figure 4.19c]) right there which is three \( x \) plus two equals three \( x \) plus four (looking at CAS screen).

6 R: Mm. Hmm.

7 SA: From there, I don’t know.

(SA_day26_finalinterview_Task13_27:32–28:10)

Evidenced in Figures 4.18 and 4.19, Annie demonstrated mixed abilities in working within the symbolic representation type; her P&P written work was wrought with inconsistencies and buggy algorithms, and her CAS work did not help her to be successful in using this representation type. Annie was stuck working within the symbolic representation (lines 2, 4, 6) and not able to successfully solve the equation

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5 The work shown in Figure 4.19a-c was originally done in the CAS Calculator portion of the Scratchpad and later saved to document Page 2.1 by the researcher.
Moreover, her use of the CAS to correctly create verbal representations from a symbolic equation (Lines 7–8, Line 12, Line 1–2 in Figure 4.19a, b, and c), in each case, resulting in “false,” did not help her come to better understand the solution to this equation (M-2, S- ->V, CAS). It was not until Annie was prompted, that she considered other representations, including a CAS-generated graph and split screen view with the table (Figure 4.20) and eventually came to a resolution on this task.

Figure 4.20. Prompted to use a graph to solve the equation $3x + 2 = 3x + 4$ for $x$, Annie used her CAS to create a graph and split screen table.

Annie’s creation of and interpretation of the graphic and numeric table representation for the equation $3x + 2 = 3x + 4$ is discussed next.

1. R: Could you use a graph?
2. SA: (paused for two seconds, and enthusiastically responded) Yes (as she raised her hand with her index finger signaling upwards, as if she had an idea.)
3. SA: (Within the Scratchpad she moved from the Calculator page to the Graphs page and typed $3x + 2$ Enter into the line for f1(x))
4. R: So tell me what you're thinking now.
5. SA: (as she is typing $3x + 4$ into the line for f2(x)) I'm thinking if I hit Enter the, both of the equations and then get my table (both lines are now plotted in one graphs page).
6. SA: (Briefly pauses as she looks at the graph screen with f1(x) and f2(x) plotted [Figure 4.20a]) Oh that's why.
8. SA: ( Cursors from f1(0) to f2(0) so that both columns are visible [Figure 4.20b]) They don't even touch, not once. (Continues to scroll down through f2(x) column
of table on CAS) Yeah, that's why; they don't even touch.

9 R: So what does that mean?

10 SA: It doesn't have not one, uh, solution. ...(pauses)... I'm taking this back to the uh, like the solution, it has some, none, or infinite; this has none.

11 R: And how do you know for sure?

12 SA: The lines are parallel each other (sic) and they won't intersect.

13 SA: I'm going to just circle all of these (circles icons for calculator, graph, and table pages on paper) because I actually used all of them.

14 R: So are you satisfied with that problem?

15 SA: Yes finally.

16 R: Ok.

17 SA: My answer was nothing.

(SA_day26_finalinterview_Task13_28:10–29:53)

In characterizing Annie’s RF in this part of her solution approach it is first recognized that the researcher provoked Annie’s resourceful use of the graphic representation type (line 1), to which Annie also created a numeric table representation (lines 7–8) on her own accord. Annie was not strategic in her use of CAS as a representational toolkit to overcome the barrier she faced when working in the symbolic representation type and using the verbal representation type to check her purported solutions on her own. It is conjectured that without this scaffolding Annie likely would not have considered this approach. Next consider the nature of Annie’s movement from the symbolic equation to the graph and the numeric table representation and her interpretations of that work more carefully.

Annie’s first reaction to the graph was one of clarity, as if she now understood something that she hadn’t before (line 6). She later gave evidence of using the graph in her interpretation that the lines are parallel and won’t intersect (line 12), and to Annie, this was a justification for why the equation had no solutions (lines 10–11). Based on Annie’s creation and movement to the numeric table, and scrolling activity to see several values within both columns of the table, it is interpreted that Annie was using the table
representation as she explained that “they don’t even touch not once” and “they don’t even touch” (line 8). Annie alluded to the fact that there are no $x$-values for which the $y$-values match (i.e., “not once”); this is interpreted to mean that Annie was using the table representation to explain why the equation had no solution (line 10). From this analysis based on Annie’s activity and interpretations, it seemed as though the graph and table were intricately connected in her justification that the equation had no solutions. Hence, Annie’s RF in this episode is coded as a connection at the relational level whereby she used the graph and numeric table to justify that the equation has no solutions, an invariant feature across these representations. More specifically, Annie alluded to the fact that no values of $x$ make the equation true or are solutions to the equation (lines 10, 17). To Annie, this was evident in the graph since no values of $x$ have matching $y$-values or a point of intersection (i.e., “won’t intersect,” line 12), and in the table because no values of $x$ have matching $y$-values (i.e., “don’t even touch not once,” line 8) (R-6, S→G→N, CAS).

In this task, Annie was successful in moving from the symbolic equation to using the graph and table representation to justify that the equation had no solutions. However, this connection was not bi-directional. Annie did not revisit the symbolic equation and interpret it to determine that the equation had no solutions (e.g., same slope lines). However, Annie’s utterance at the very end that her “answer was nothing” (line 17) could be interpreted to mean that Annie reflected on her earlier work within the symbolic representation type, recognizing that she shouldn’t have gotten a solution (maybe explaining why the verbal representation was always “false”). However, Annie did not reconcile her solution from the graph and numeric representation to the symbolic nor
verbal representation types; she seemed to put confidence in the graphic representation type to be satisfied with this problem despite her prestructural level of working within the symbolic representation type, and multistructural movement to the verbal representation type.

**Summary of Annie’s Final RF**

Considering the collection of final interview tasks as a whole, Annie demonstrated both prestructural and unistructural levels of RF within the symbolic representation type, and multistructural and relational levels of RF in moving between representations. Annie’s success with the symbolic representation type was limited to her transpositions when solving an equation of the form \( y = ax + b \) at \( y = c \), or solving \( ax + b = c \) for \( x \). In the tasks that specifically asked Annie to use MR to solve equations (Task 10, Task 14), Annie was more successful in identifying the \( x \)-value as the solution in Task 10 than in Task 14 in which she was focused on the \((x, y)\) ordered pair. Although both equations had one solution, the difference in the equation types and presentation of the equations might have made the difference in her varied levels of success.

For the tasks that did not “require” the use of MR to solve, but were an option (Task 12, Task 13) Annie used graphic and numeric (table) representations on her own, and when prompted. Annie’s use of the graphic representation was mostly used to identify the solution case (as either infinite solutions or no solutions, respectively). Annie’s movement from the symbolic to the numeric representation type in Task 12 was to identify the solutions \( x = 0 \) and \( x = 2 \) (a subset of the solution set), and to recognize that the expressions related by an equal sign “don’t even touch not once” in Task 13.
Annie ranged in her ability to interpret and explain the meaning of MR in identifying the solutions (if applicable) as $x$ values that made the equation true. Often, the movement from symbolic to graphic and/or numeric table representations was not complemented by a return to the symbolic representation type to deepen the connection across representations.

**Annie’s Change in RF**

One of the most striking changes in Annie’s RF in solving equations from the initial to the final interview was in her increased use of MR to (attempt to) solve linear equations. Annie’s initial RF was characterized by incorrect transpositions within the symbolic representation type, while movement from the symbolic representation to numeric tables, graphs, or verbal representations was more typical of Annie’s final RF. While Annie demonstrated much difficulty in creating representations within the symbolic representation type at the onset of the study, she had made some progress in creating and interpreting successful transpositions within this type by the end of the study. In regards to Annie’s movement between MR, Annie’s RF in solving linear equations had increased to higher levels of the framework in all tasks attempted.

Each pair of tasks that Annie completed at the beginning and end of the teaching experiment are considered in turn. Annie’s change in RF is summarized according to her performance on these tasks in the aforementioned analysis.
Solving an Equation $c = ax + b$ With One Solution: Task 10

Figure 4.21 illustrates Annie’s change in RF as characterized based on her activity during the initial and final interviews while solving an equation $y = ax + b$ at $y = c$ for $x$. The comparison of her initial and final RF is shown in the Rule of Four webs in Figure 4.21. The corresponding coding categories from the Analytic Framework for RF are listed below each image in the order in which they occurred during the interview.

Annie’s RF during the initial interview was limited to incorrect work within the symbolic representation type. The researcher had scaffolded Annie in setting up the equation, and when prompted to identify the solution in the table and graph, Annie did not use these representations. By the final interview, Annie’s RF in working within the symbolic representation type had improved to the unistructural level (with successful transpositions on the symbolic equations). Moreover, Annie correctly connected this symbolic solution to identifying it in the numeric table representation and had also
created a graph of the given equation. Annie’s movement to and work within the graphic representation had not been evident in the initial interview, and Annie demonstrated little progress in this area. She seemed to struggle most in identifying the $x$-value in the graph, and in comparing her CAS and P&P work.

Solving a Linear Equation With All Real Solutions: Task 12

On the pre-test and initial interview, Annie worked on solving the equation $t - 2 + 3t = -6 + 4t + 4$ for $t$. The prompt was for her to show the equations, the graphs, or the tables that she used in solving this equation and to check the solution. On the post-test, such a task was not included. However, during the final interview, Annie considered a task that involved analyzing the steps of a symbolic solution approach to the equation $2 - x - x = x + 8 - 3x - 6$ for $x$. Five steps of “Andy’s” work were shown and Annie was prompted to consider (a) was the work correct and (b) what is the solution to the equation. In both cases, the solution to the equation was “all values of $x$;” which represented the case of “infinite solutions” in the sense that the expressions related by the equal sign were “the same” (i.e., mathematically equivalent). Figure 4.22 summarizes the characterization of Annie’s initial and final RF based on her work on these tasks.

Annie had made some progress from the initial to the final interviews in her RF in solving a linear equation with infinite solutions. Most notably, by the final interview, Annie seemed to use MR as resources to overcome barriers in the symbolic representation type. In both interviews, Annie initiated and concluded her work on Task 12 by working within a symbolic representation type. Annie demonstrated progress by the final interview in that she was able to see how the case of infinite solutions could be
signified by a symbolic identity equation in which the expressions related by the equal sign were “the same.”

Figure 4.22. Annie’s initial and final levels of RF while solving an equation with infinite solutions (Task 12).

Different from the first interview, Annie self-prompted the use of a graphic representation to overcome barriers she had encountered in her first approach within the symbolic representation type. Annie’s resourcefulness in using the CAS as a representational toolkit is considered to be a change in her RF, from the multistructural level to a relational connection. There was also evidence of a change in her RF from the initial to the final interview in that she was successful in making connections from symbolic equations to a numeric table representation and to verbal representations during the final interview, and not in the initial interview. Annie’s change in making a connection from the symbolic equation to the verbal representation was specific to her
CAS activity and interpretation of that activity. Thus the fact that Annie did not have access to such a tool during the initial interview means that such a representation was not available for her to make that connection.

Overall for the case of an equation with infinite solutions, Annie showed the greatest change in being able to identify specific values as part of the solution set of the equation, yet she did not demonstrate a well-connected understanding of how these values were related to the meaning of “infinite solutions.” In other words, Annie did change in being able to interpret the “sameness” of graphs, numeric tables, and symbolic expressions to signify that there were infinite solutions, yet Annie was not successful at identifying that all values of $x$ make the equation true. A highlight from Annie’s work during the final interview was her choice to use the graphic representation to overcome the barrier she originally encountered within the symbolic representation, then her later reflection on the symbolic representation. In most episodes coded for Task 12, Annie had improved her RF from the prestructural and multistructural levels to the unistructural and relational levels of RF.

_Solving an Equation $ax + b = cx + d$ With One Solution: Task 14_

The characterizations of Annie’s initial and final levels of RF in solving an equation of the form $ax + b = cx + d$ for $x$ (Task 14) are illustrated in Figure 4.23.
Figure 4.23. Annie’s initial and final levels of RF while solving an equation of the form $ax + b = cx + d$ for $x$ with one solution (Task 14).

From the initial to the final interview, Annie was more facile at creating and interpreting graphic and numeric table representation types to explore the solution set to an equation of the form $ax + b = cx + d$ with one solution. In the initial interview Annie had only considered a symbolic representation type; Annie demonstrated a change in her RF because by the final interview she used graphs and numeric tables to identify the $x$- and $y$-values that corresponded to an intersection point. However, Annie emphasized both the $x$- and $y$-values of the intersection point when only the $x$-value is considered the solution to the equation. Annie’s emphasis on both the $x$- and $y$-values of the intersection point made it difficult for her to be successful in correctly articulating the solution to the equation in the numeric table and graph.

With respect to Annie’s use of the symbolic representation type, Annie’s initial RF was prestructural, and at the end of the teaching experiment Annie included the
symbolic representation in her movement between representation types. Annie did not however attempt to solve the equation via transpositions on the symbolic representation type. Thus no change in RF with respect to using the symbolic representation type to solve the equation can be reported here. It is possible that the absence of an attempt to work within the symbolic representation type during the final interview made it more difficult for Annie to see the value of $x$ as the solution to the equation (and not the value of $y$).

_Solving a Linear Equation With No Real Solutions: Task 13_

The pre-test and initial interview were not designed to consider the case of “no solutions” in a linear equation. Consequently, it is only possible to discuss a “final” characterization of Annie’s RF in this particular equation solving context (see Figure 4.18).

![Diagram](image)

*Figure 4.24.* Annie’s final level of RF while solving an equation of the form $ax + b = cx + d$ for $x$ with no solutions (Task 13).
Annie’s use of the symbolic representation type was prestructural, yet with scaffolding during the interview to consider “Could you use a graph?” Annie was successful in creating a split screen graph/table view on her CAS and interpreted the graphic representation type to mean that the equation had no solutions and that they “won’t intersect,” and the numeric table representation type to mean that they “don’t touch not once.” Annie had justified the invariant feature of “no solutions” from the symbolic to the graph and numeric table representations, a relational level of RF and connection. Annie seemed satisfied with these justifications and did not go back to reconcile these findings with her prestructural work within the symbolic representation type, and multistructural movement to the verbal representation type.

Summary of Annie’s Change in RF

Two highlights of Annie’s change in RF include resourceful use of the graphic representation when stuck in working within the symbolic representation type, and her connection between the numeric and symbolic representation type in interpreting $x = 0$ and $x = 2$ to be solutions to an equation with infinite solutions (Task 12). Annie did not make much progress in correctly using the symbolic representation type to solve equations, still demonstrating inconsistencies in combining like terms (Task 12, Task 13). The symbolic manipulation capabilities and the “with” operator played a prominent role during the final interview, but didn’t always assist Annie in coming to clear and correct conclusions about solutions to a given equation (Task 13, Task 14).

Overall, Annie seemed more confident in using MR to attempt to solve problems involving linear equations, yet was not always clear in her interpretations of what the
solution was or wasn’t. The identification of the three possible cases of solutions to linear equations—sometimes true (one solution, Task 14), always true (infinite solutions, Task 12), and never true (no solutions, Task 13)—seemed to resonate with Annie as she used MR to identify these cases during the final interview, often a result of reflecting on CAS-based inscriptions.

_Bryon’s Initial RF_

During the initial interview, Bryon was very symbolically oriented. He demonstrated great persistence in using this representation type to solve linear equations, despite the fact that he wasn’t always successful in the attempts. Bryon did not seem to recognize graphic nor numeric table representations as legitimate ways to identify or solve linear equations. The only time Bryon demonstrated some movement between representations was when it was specifically prompted by the researcher with additional scaffolding on how to use a GC. Overall Bryon seemed satisfied with using P&P to attempt to solve all tasks within the symbolic representation type, despite some difficulties in working within this representation type.

More specific evidence of Bryon’s initial RF is detailed below with respect to three distinct equation-solving tasks. A task with an equation of the form $c = ax + b$ with one solution, a task with an equation with infinite solutions, and a task with an equation of the form $ax + b = cx + d$ with one solution are considered in turn.
Solving an Equation $c = ax + b$ With One Solution: Task 10

In a task that involved solving an equation $y = ax + b$ at $y = c$ for $x$, Bryon demonstrated a low level of RF. During the initial interview Bryon had created a graph from a given table, and had also attempted to create an equation that matched (see Task 9 in Figure 4.25). Presented with this work during the initial interview, Bryon was prompted to consider Task 10 in which he was to use the three representations of Task 9 to solve the equation at the value of $y = 5$ (Task 10 in Figure 4.25).

9.) Given one of the representations below, create the other two.

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

10.) Refer back at the first row in Question 9. In this row, clearly indicate the solution to the equation when $y = 5$ for each of the table, graph, and symbolic equation.

Figure 4.25. During the initial interview Bryon was presented with his pre-unit work and prompted to indicate the solution when $y = 5$ (Task 10).

Bryon admitted that he unintentionally overlooked Task 10 when taking the pre-unit assessment. So upon his first approach during the initial interview he was initially successful in substituting the value of $y = 5$ into the equation he had created, $y = (1/4)x - 4$, to yield $5 = (1/4)x - 4$ (SB_day4_initialinterview_08:04). This seemed to indicate that Bryon understood that the task entailed solving for a particular $x$ value (given a particular $y$-value). Bryon’s P&P work within the symbolic equation during the
initial interview was messy and involved several incorrect computations (see Figure 4.26).

![Figure 4.26](image)

*Figure 4.26. Bryon correctly substituted a value of y = 5 into an incorrect equation and continued with incorrect transformations on this equation.*

Bryon explained his equation solving technique as he wrote down the corresponding computations.

So five (writes 5 below the y in the original equation) I'd do the same, and instead I'd add four (writes + 4 below –4), which cancels that out (crosses through –4 and + 4), which equals zero (writes 0), plus four which equals nine (writes + 4 below the 5 and records 9 as the result), divide by four (draws horizontal line and four below the 9 with an equals sign), and I'm pretty sure that equals two point two five, but let me check. (Types 9/4 Enter on his calculator.) Yep, two point two five (writes 2.25). (He then moved his pen to the symbolic equation in Row 2, signifying that he was finished with Task 10).

(SB_day4_initialinterview_08:14).

One incorrect step within this approach was dividing the left hand side of the equation by four instead of multiplying by four. Bryon was also inconsistent in his use of the equal sign because the original equation was transformed with a sequence of operations without keeping the equal sign, and later replaced by the equation \( \frac{9}{4} = 2.25 \). The fact that Bryon moved on to Row 2 on his own accord after writing the response of 2.25 indicates that he saw this as the solution to the equation. From Bryon’s incorrect work within the symbolic
representation type and incorrect conclusion of the solution to the equation, this is a
prestructural level of RF (P-0, S-X->S, P&P). Bryon’s difficulty in solving this equation
within the symbolic representation type is attributed to the fact that first he was working
from an incorrect equation, and second this equation involved fractions. Had Bryon been
given a correct symbolic equation (which did not involve fractions), he may have been
more successful at using the symbolic representation type to come to a solution. In
retrospect, this would have been an appropriate intervention on the part of the researcher.

After it seemed as though Bryon was satisfied with his symbolic solution
approach, the researcher prompted Bryon to reconsider the other aspects of Task 10.

1 R: So before you do row 2 [shown partially in Figure 4.26], can you find the
solution when y equals five for the table and the graph? Could you find that
solution?
2 SB: Um (...5 second pause...), well (...6 second pause...) (Presses clear on GC,
types 2.25 × .25, which is .5625)
3 R: Can you tell me what you're thinking?
4 SB: Um what I was thinking is that since it's one fourth (points to 1/4x in the
equation y = 1/4x – 4 [Figure 4.26]), and one fourth is point two five (i.e., .25), I'd
just times that by the two point two five (records × 0.25 on paper [Figure 4.26])
which together equals five point six two five (records .5625 on paper [Figure
4.26]), but since I learned (draws an arrow underneath .5626 [Figure 4.26]) is that
you can move it over a digit to make it an actual five number (writes 5.625
[Figure 4.26]) which would equal five point six two five.

(BS_day4_initialinterview_08:46)

Bryon had apparently interpreted the prompt in line 1 to show the solution to the
equation in the graph and table that he was to substitute the value he found into the
original equation (line 4). The fact that Bryon merely multiplied his purported solution of
2.25 by .25, and did not subtract four from this result is an incorrect movement to the
numeric representation type. Beyond the recording of the incorrect value 5.625, Bryon
did not convey a meaningful sense of how these numeric results were related to solving
the equation. This attempt to use MR was at the multistructural level (M-1, S-X->N,
P&P, GC). Prompted again to consider the specific task of identifying solutions to the equation using graphs and tables, Bryon did not think it was possible to work using the graphic and numeric table representations to identify solutions to the equation.

R: So what question 10 is asking, is can you find the value of x, or the solution when y = 5 in the table and the graph.
SB: No.

(SB_day4_initialinterview_10:09)

In solving an equation \( y = ax + b \) at \( y = c \) for \( x \), or an equation of the form \( c = ax + b \), Bryon incorrectly worked within the symbolic representation type at the prestructural level, and his attempt to use MR involved a movement to and use of numeric computations. Bryon did not attempt to work within the given table or graph to identify the solution.

Solving a Linear Equation With All Real Solutions: Task 12

In both the pre-unit assessment and initial interview, Bryon was prompted to solve a linear equation with all real solutions. During the initial interview Bryon was asked to reconsider the same task and the (incorrect) work he had completed within the symbolic representation type during pre-test (see Figure 4.27).

For 11-12, solve for the variable and check your solution using a table, graph, or equation (your choice). In each question, sketch the section of the graph (with the scale marked) or the table and indicate where you found the answer. If you solve the problem using equations, show all of your work.

12.) \[ x - 2 + 3t = -6 + 4t + 4 \]

\[ \frac{-12 + 8t}{4} = \text{solution} \]

Figure 4.27. Bryon’s attempt to solve an equation with infinite solutions during the pre-test.
In his first attempt during the initial interview, Bryon was not able to move forward with nor correctly interpret this incorrect work from the pre-test so he attempted to re-write the original equation and work within the symbolic representation type as his first solution approach. As shown in Figure 4.28, this approach actually involved correct transformations on the symbolic representation, stopping with the equation $3T = 3T$.

![Figure 4.28](image).

*Figure 4.28. Bryon’s first attempt during the initial interview to solve an equation with infinite solutions within the symbolic representation type.*

As explained by Bryon, the correct symbolic work was interpreted to be incorrect.

(Writes $3T = 3T$ on paper, closes pen cap, mumbles to self, then takes pen cap off and motions over symbolic steps on paper [Figure 4.28]) Oh, I think I messed up somewhere in there. [...] If I mess up I just get rid of it all and just do it again (SB_InitialInterview_day4_11:07:47).

Despite the progress Bryon had made in creating a series of correct equivalent equations (Figure 4.28), Bryon interpreted this work to be incorrect, a prestructural level of RF within the symbolic representation type (P-1, S- ->S, P&P). Bryon persisted in the symbolic representation type for two more solution attempts (see Figure 4.29).
In his second attempt, Bryon had made an error that violated the equality of the original equation, yielding an equation of $T + 4 = T$ (Figure 4.29a). Bryon recognized that this was an incorrect attempt and did not continue this solution approach. The fact that Bryon saw this as a distinct solution approach, created incorrect representations, yet correctly interpreted it to be incorrect is again evidence of a prestructural level of RF (P-1, S-\rightarrow-S, P&P). On his own accord, Bryon continued with a third attempt at solving the equation $t - 2 + 3t = -6 + 4t + 4$ for $t$ within the symbolic representation type (Figure 4.29b). In this third attempt, Bryon added two to both sides of the equation, subtracted two $T$ from both sides of the equation, and combined the $-6$ and $+6$ on the right hand side. The equivalence of equations was violated by the fact that $2T$ was subtracted twice from the left hand side and only once from the right hand side of the equation; this yielded an equation of $0 = 2T$. After recording this equation (Figure 4.29b) Bryon stated matter-of-factly, “Two $t$ divided by zero is zero” (SB_day4_initialinterview_18:30) and closed his pen cap, signifying the end of his work on this task. Prompted later to explain his thinking, Bryon pointed to this work on his paper and explained, “I figured that all the $t$’s are zero” (19:26). In this third distinct solution approach, Bryon’s incorrect creation
within the symbolic representation type and incorrect interpretation that $T = 0$ was the solution did not move him beyond the prestructural level of RF (P-0, S-$X \rightarrow S$, P&P). The next aspects of Bryon’s equation solving involve the numeric equation representation shown in Figure 4.30.

![Figure 4.30](image.png)

*Figure 4.30.* Bryon correctly moved from a symbolic to a numeric representation type.

To better understand Bryon’s thinking, the researcher prompted Bryon to articulate what the solution to the equation was, which led Bryon to move to the numeric representation type.

1 R: So what's your solution after those three attempts?
2 SB: Well, zero minus two plus zero equals negative six plus zero plus four (records $0 - 2 + 2 = -6 + 0 + 4$ on paper [Figure 4.30]). Alright. So in other words to simplify that all I got is a negative two and negative two (writes $-2 = -2$ below first equation [Figure 4.30])
3 SB: and you put those together and you've got left is, I think it's a positive one, I'm not sure if you put em--(Types $-2/-2$, Enter, yields 1 on GC) Because usually when you put them together you want to divide them by each other which equals the positive one (records 1 below the equation $-2 = -2$ [Figure 4.30]).

(SB_day4_initialinterview_18:33)

Prompted to identify the solution to the equation (line 1) Bryon correctly substituted a value of 0 in for the variable T and moved from the symbolic equation to a numeric equation (Figure 4.30, line 2). In this work, he had correctly created the numeric equations $0 - 2 + 0 = -6 + 0 + 4$ and $2 = 2$ but incorrectly came to the conclusion of “positive one;” this is a multistructural level of RF (M-2, S- $\rightarrow$ N, P&P). Prompted again to explain his thinking on this task, Bryon then reconsidered his numeric equations, and chose to work within the numeric representation type.
R: So what did you just show right here in these steps?
SB: Well, since I re-did this and I figured that all the t's are zero, I figured on that side I'd only have negative two left, and this is a positive and this is a negative […] break in transcript; he then changed his mind about his work within the numeric representation type, crossed out 1, and changed his response to be .2 [Figure 4.30] after computing –2/–10 Enter .2 on his GC…

R: Why did you think to put that value of zero back into your original equation?
SB: To see what the actual final prod--final answer in the whole thing.
R: So T equals zero is not your final answer?
SB: No. The zero point two or the negative zero point two would be my final answer.

(SB_day4_initialinterview_19:26, 21:35)

It is unclear why Bryon thought that his movement to the numeric equation was incorrect, and what specifically prompted him to work within the numeric representation type to change his solution to .2 (line 2). The interesting thing about this exchange is that this work within the numeric representation type involved both an incorrect creation of numeric equations to yield .2, but also an incorrect interpretation that this was his final answer (line 6). This prestructural level of RF within the numeric representation type (P-0, N-X->N, P&P) was convincing to Bryon and had replaced his earlier conclusion that the solution was \( t = 0 \). To Bryon it seemed as though the process of substituting a value into the original equation was how to get the “final” answer (lines 4–6).

Finally, the researcher prompted Bryon to consider solving this equation with a graphic representation. After some guidance on how to access the \( y = \) menu and to use the variable button (instead of “Alpha T”), Bryon correctly reasoned that he should enter \( y_1 = X - 2 + 3X \) and \( y_2 = -6 + 4X + 4 \). The researcher also answered his question as to how to show the graphs (by pressing the Graph button) (Figure 4.31a). Bryon then drew the graph on his paper, motioning to the left-hand side of the equation as what was shown on the graph (Figure 4.31b). He went back to the \( y = \) menu to cursor over \( y_2 \), then pressed the Graph button again (seemingly thinking that he needed to do this to show the second
graph). After a pause, he realized that it was the same thing (i.e., the same graph) but didn’t know why (Figure 4.31c). Prompted by the researcher, “Does that make sense?” Bryon articulated, “Not to me. I don’t really like calculators” (SB_InitialInterview_Day4_24:19:26).

Figure 4.31. Bryon used his GC to move from the symbolic equation \( t - 2 + 3t = -6 + 4t + 4 \) to a graphic representation.

With assistance from the researcher, Bryon had correctly moved from the symbolic to the graphic representation type, but did not give a correct interpretation of what this movement meant in terms of solving the equation (M-2, S- \( \rightarrow \) G, GC). It was evident that Bryon did not consider graphs as a tool to solve the equation.

In solving the equation \( t - 2 + 3t = -6 + 4t + 4 \) for \( t \), Bryon did not come to a resolve on the solution to this equation to recognize that the equation was true for all values of \( t \). Despite several attempts (some of which involved correct transformations on the equation), Bryon was unsuccessful interpreting this work and thus within the symbolic representation was classified at the prestructural level of RF. Bryon’s
movement to and work within numeric equations was classified at the multistructural and prestructural levels of RF respectively, and did not seem to support his understanding of the solution to this equation; Bryon had interpreted the use of this representation to signify the “final solution,” and did not view it as a way to check his solution nor did he recognize that the equation had infinite solutions. When prompted to consider a graphical approach, the researcher supported Bryon with technical assistance and he was then able to use the graphing calculator to graph the equation. While Bryon correctly recognized how each symbolic expression was related to a corresponding Cartesian graph, he did not make sense of why these graphs were the same; a multistructural level from symbolic to graphic.

*Solving an Equation* \( ax + b = cx + d \) *with One Solution: Task 14*

Bryon had not considered Task 14 on the pre-test, so the Initial Interview was his first attempt at solving a given equation in the form \( ax + b = cx + d \) for \( x \) with one solution. When prompted to solve the equation using the table (Task 14a), Bryon first performed correct symbolic transpositions on the equation until he came to a solution of “11,” he then circled “11” in the table that was given (Figure 4.32).
Consider the following transcription of Bryon’s work on Task 14a.

1. R: Can you do part a?
2. SB: Yeah. OK, circle all the values in the table that represent solution, represent solutions to.
3. SB: OK, minus two x, minus two x (writes \(-2x\) on either side of the equation), what's this, which is one hundred, which is \(10x\) (writes \(100 = 10x-10\), plus ten--see, what I figured out is that if you, what I think is that you can add ten, which makes this a zero (records \(+10\) below \(-10\), a horizontal line, and 0), and you add ten to this, which is one one zero (writes \(+10\) below 100 and a horizontal line, and 110). And if you divide it by the ten that is right here (points to the coefficient 10 on the term \(10x\) from first line),
4. SB: all you get is eleven (writes \(110/10 = 11\)). Which is right here (circles value of 11 in table).
5. R: OK. So when you look at that table, so you circled the value of \(x = 11\) in the table, why is that a solution?
6. SB: Um, because, see what I always do is, let's say there's a lower amount of \(x\) on this side than there is on this side I'd always take the lower \(x\) from this one which makes it a ten, same as this, but it'd be like the only answer that I could possibly think of.

(SB_day4_initialinterview_27:55)

Evident in Figure 4.32 and lines 3–4 above, Bryon’s work within the symbolic representation type involved correct transpositions and thus is classified at the unistructural level (U-0, S→S, P&P). Bryon merely circled the value of 11 in the first column of the table without noting the corresponding \(y\)-values, and without noting the
significance of the \( x \)-column of the table (line 4). Moreover, when explaining the correctness of his work he was limited to the symbolic representation type and did not reference the numeric representation (lines 5–6). From this episode, Bryon’s RF in solving an equation of the form \( ax + b = cx + d \) for \( x \) with one solution is classified at the unistructural level for the symbolic representation type only. Bryon’s identification of the value of 11 is not coded as using the numeric table representation, even at the prestructural level. Beyond the identification of the number 11, the tabular representation did not appear to be meaningfully connected to the symbolic work that he had done.

When prompted to consider Part c of this problem, “Explain how a graph can be used to solve the equation” Bryon was unable to do so. It is interesting to note just prior to working on Task 12, Bryon worked on Task 14 in which he was able to use the graph to at least consider each side of the equation as a function and look at their graphs. He did not transfer this skill into looking at this problem, and simply said it wasn’t possible. Overall, Bryon seemed satisfied with the original symbolic work and justified the correctness of this method based on the steps that he performed.

*Summary of Bryon’s Initial RF*

Considering Bryon’s progress on the various equation solving tasks explained above, Bryon demonstrated split success in working within the symbolic representation types, evidenced by both prestructural and unistructural levels of RF in creating and interpreting the symbolic representation type to solve problems involving linear equations. In each of the three tasks of solving an equation of the form \( y = ax + b \) at \( y = c \) for \( x \), and \( ax + b = cx + d \) for \( x \) with infinite and one solution, Bryon persisted in using
the symbolic representation type. His ability to correctly work within the symbolic representation type was not consistently complemented by a correct interpretation of that work (e.g., when he got to $3T = 3T$ while solving an equation with infinite solutions).

Bryon also demonstrated limited ability to move between representations to solve equations, and was the most successful in explaining the connection between the graphic and numeric tabular representations for the goal of creating representations (not solving). Bryon’s success in moving from a symbolic representation type to a graphic representation type in Task 12 was not meaningful to him, nor did he transfer this skill to working on Task 14 to use a graph to solve. It is important to note that the aforementioned instances of moving between representations were specifically prompted by the task statement or by the researcher’s additional probing. Bryon had also demonstrated a multistructural level of RF from a symbolic equation to a numeric equation by a process of substitution, but again, this work was not meaningfully connected to solving the equation and did not help Bryon be successful on the equation-solving task. Overall, Bryon’s initial RF was dominated by his work within the symbolic representation type.

*Bryon’s Final RF*

The characterization of Bryon’s final RF is based on his completion of a comparable set of tasks during the final interview at the end of the instructional unit. Possibly due to fact that the task design always included a symbolic equation, Bryon had worked within the symbolic representation type for each of the interview tasks. Moreover, he was mostly successful in solving the linear equations, demonstrating a
unistructural level of RF for three of the four tasks. Beyond the work within the symbolic representation type, Bryon’s equation solving strategies were quite diverse in terms of his use of MR. His approaches on each task involved at least two movements from a symbolic to a numeric, graphic, or verbal representation type. Bryon often used his CAS to help create correct representations from a given symbolic equation, but overall he persisted at the multistructural level of RF in moving between representations with incorrect or incomplete interpretations of this work with respect to the solution to the equation. Given next is a detailed classification of Bryon’s final RF according to the Analytic Framework for RF and is organized by his work on four distinct tasks that involved solving linear equations.

Solving an Equation \( c = ax + b \) With One Solution: Task 10

During the final interview, Bryon was successful in using his CAS to create graphic and numeric table representations of a given equation (Figure 4.33). These creations are taken as movement from the symbolic to the graphic and symbolic to the numeric table representation, respectively, and we see later how Bryon interpreted this work with respect to solving the equation \( y = 3x - 2 \) for \( y = 4 \).
10.) An equation is given in Row 3, below.

a.) Create a graph and a table for the symbolic equation \( y = 3x - 2 \).

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td>( y = 3x - 2 )</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

b.) Find the solution to the equation \( y = 3x - 2 \) when \( y = 4 \).

Show the solution to the equation using the **table**
Show the solution to the equation using the **graph**
Show the solution to the equation using the **symbolic equation**

*Figure 4.33.* In task 10, Bryon used his CAS to create a table and graph given the equation \( y = 3x - 2 \), then checked his table using the symbolic equation.

In Bryon’s initial attempt at solving Task 10b (shown in Figure 4.33), he first solved the equation \( y = 3x - 2 \) for \( x = 4 \), instead of the desired \( y = 4 \) (not shown here). He later realized his mistake and worked to solve the equation \( y = 3x - 2 \) for \( y = 4 \) (Figure 4.34). After his first incorrect attempt, when prompted to reply to the researcher as to what equation he had just solved, Bryon re-read Part b and articulated, “For b, oh, when \( y \) equals four? Oh, ok. Let me redo all of that […] I thought it was asking for \( x \).”

(SB_day25_finalinterview_17:55, 18:21).
Figure 4.34. Bryon solved the equation $4 = 3x - 2$ for $x$ using the symbolic equation, a graph, and a table.

Beginning anew in the symbolic representation type, Bryon correctly performed transpositions on the symbolic equation and stated the solution to be $x = 2$ (Figure 4.34a).

While using P&P to work within the symbolic representation type, Bryon articulated:

Ok, well if I wanted to find that, I'd add two there, and then 6 equals three $x$, three, three, two for my $x$.

(SB_day25_finalinterview_8:48)

Based on Bryon’s focus on and correct use of the symbolic representation type to solve the equation his RF within the symbolic representation type is at the unistructural level (U-0, S $\rightarrow$ S, P&P).

After a correct symbolic solution was found, Bryon attempted to “Show the solution using the graph” (Task 10b). Considering the fact that Bryon had originally been successful at creating a graph from the given equation (Figure 4.33), it was surprising that Bryon did not refer to this created representation again. To Bryon, the activity of creating the graph of $y = 3x - 2$ was seemingly not connected to using this representation to identify a solution at $y = 4$. Bryon did not interpret this representation and instead created
Based on his creation of a graphic representation and interpretation of that work, it seemed as though Bryon was attempting to follow similar “equation solving steps” like those he demonstrated within the symbolic representation type (lines 1–2 above, Figure 4.33a). In particular, Bryon decided that the point (6, 6) was meaningful and proceeded to subtract two from six to get four and divide six by three to get two (line 2). The operations of subtracting two and dividing by three were seemingly inspired by the operations performed on the equation (Figure 4.34a). Although a tacit connection, Bryon had also correctly drawn line segments representing the lines $y = 4$ and $x = 2$, which were determined in the symbolic representation type just before moving to a graphic representation type. Both the creation of and interpretation of representations that Bryon used in the above exchange support the fact that Bryon was focused on both the graphic representation type and the symbolic representation type without a clear understanding of
the connection between these representations in relation to solving the equation $4 = 3x - 2$ for $x$, a multistructural level of RF (M-1, S-X->G, P&P).

Although Bryon’s reported computations in lines 1–2 above “work” in the sense that one can work within the graphic representation type to move between ordered pairs that are related by arithmetic computations, it appears as though Bryon was focused on the computations and numeric results and was “finding” these relationships within the graph he was creating. After working within this graph, Bryon had expressed confusion about what the solution was or how this use of the graphic representation helped him to show the solution to the equation (lines 4–6). Thus Bryon’s RF is classified as using the graphic representation type without complete success. The point (2,4) was correctly identified on the graph but the $x$-value of 2 was not isolated as representing the solution to the equation, as it was in the symbolic representation type (P-1, G- ->G, P&P).

At the start of his work on Part b, Bryon had expressed curiosity and maybe bewilderment at the question of showing the solution in the table, because

To use the table, all I'd have to do is use the equation. Which is what is confusing, because it tells me to use the table but in order to get the table you have to use the equation to make sure the table is right.

(SB_day25_finalinterview_13:56)

After showing the solution using the equation and graph, Bryon continued,

And then you might as well have, four, two four, I guess (writes “2,4” and boxes it, then records “x, y” below it).

(SB_day25_finalinterview_Task10_21:47)

To Bryon, it doesn’t seem significant that he was given the value of $y = 4$ and was then supposed to determine the value of $x = 2$. The ordered pair of (2, 4) was intimately connected to the context of solving this equation.
In this case, the value of \( y = 4 \) is not necessarily part of the solution, but rather a “given” constraint on solving the equation. Thus, despite Bryon’s identification of the correct \( x \) value, he did not clarify that the \( x \)-value was the real “solution” to this equation.

It is surmised that Bryon’s original movement from the symbolic to the numeric table (Figure 10.B.1) informed his use of the table to solve the equation (Figure 10.B.2.c).

Thus, despite a correct movement from symbolic to numeric representation in the representation that was created, Bryon did not give a complete interpretation of the solution \( (x = 2) \), a multistructural level of RF (M-2, S-\( \rightarrow \)N, CAS, P&P).

Overall, Bryon showed the highest level of fluency in working within the symbolic representation type. He had clearly indicated the solution of \( x = 2 \), a unistructural level of RF. When he moved to the graphic and numeric representation types, Bryon worked much more hesitantly, and was not clear in expressing the solution of \( x = 2 \), but rather seemed focused on the ordered pair \((2, 4)\). The multistructural level of RF from the symbolic to the graphic and symbolic to the numeric table is further supported in Bryon’s explanation below.

1 R: Are there any relationships between the solution to the equation as shown symbolically, in the graph, and in the table?
2 SB: They've all got twos and fours in them.  
(SB_day25_finalinterview_Task10_21:33)

Bryon did not clearly express how the solution to this equation was the \( x \)-value of two, while the corresponding \( y \)-value of four was needed to inform his identification of this solution. The superficial level of identifying the “twos and fours” (line 2) across his work in the symbolic, graphic, and numeric table representations is not sufficient to classify Bryon’s level of RF at the relational level; it was not clear that this relationship was meaningful to Bryon in terms of solving the equation.
Solving a Linear Equation With All Real Solutions: Task 12

The intent of Task 12 was for students to consider and make sense of another students’ solution approach to solving an equation with infinitely many solutions, and to explain the solution to the equation (Figure 4.35). Even though this was the first time Bryon had seen this particular task, he started working on this problem before the researcher had a chance to prompt him to do anything. In the symbolic work he wrote to the side of Andy’s solution approach, Bryon had made an error in not correctly combining like terms on the right hand side of the equation (Figure 4.35). The fact that Bryon made a mistake in the creation of the equations and stopped after coming to an equation of $4 = 0x$ is evidence of a prestructural level of RF (P-0, S-X->S, P&P).

<table>
<thead>
<tr>
<th>Steps</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.) Original problem</td>
<td>$2 - x - x = x + 8 - 3x - 6$</td>
</tr>
<tr>
<td>2.) Combine like terms</td>
<td>$2 - 2x = -2x + 2$</td>
</tr>
<tr>
<td>3.) Add 2x to both sides</td>
<td>$2 = 2$</td>
</tr>
<tr>
<td>4.) Subtract 2 from both sides</td>
<td>$0 = 0$</td>
</tr>
<tr>
<td>5.) Solution</td>
<td>$x = 0$ and $x = 2$</td>
</tr>
</tbody>
</table>

Andy worked out the problem above. When he substituted his solutions into the original equation, he discovered that both $x = 0$ and $x = 2$ make the equation $2 - x - x = x + 8 - 3x - 6$ true.

a) Was Andy’s work correct? Explain why or why not.

Figure 4.35. Bryon’s first attempt at solving the equation $2 - x - x = 8 - 2x - 6$ for $x$.

To refocus Bryon’s work on the task at hand, the researcher prompted him to consider whether or not Andy’s solution approach was correct. Bryon explained, “I’m just trying to see how [Andy] got two,” he paused and then exclaimed “Oh” before starting a new approach to solving the equation within the symbolic representation type
Although tacit in Bryon’s dialogue, it is possible that he used Andy’s symbolic work as a resource in his subsequent symbolic solution approach.

In this second approach (Lines 1–4 of the symbolic work shown in the lower right hand corner of Figure 4.36 and Line I of the response in Part a) he correctly created a series of equivalent equations within the symbolic representation type, yet incorrectly interpreted the meaning of this work. In particular, he did recognize that the expressions in Line 2 of Figure 4.36 were equivalent, or in Bryon’s words, “These are the exact same problems” (SB_day25_finalinterview_Task12_45:00), and continued to show how adding 2x to both sides and subtracting 2 from both sides of the equation would yield 0 = 0 (Lines 3–4 of Figure 4.36). However, with respect to Bryon’s RF in solving equations, he did not use this work within the symbolic representation type to conclude that the equation had infinitely many solutions. Instead, when prompted to state what the solution to the equation was (Task 12 part b), Bryon responded matter-of-factly, “Zero equals zero” (SB_day25_finalinterview_45:46). Despite the correct creation of equivalent equations, Bryon was grounded to the symbolic representation type and not able to correctly identify the solution to the equation (P-1, S- ->S, P&P). This is not a unistructural level of RF because Bryon did not allude to the fact that this equation has infinitely many solutions.
a) Was Andy’s work correct? Explain why or why not.

In order to get 0 on the right hand side of the equation, he put 0 in all the x spots.

b) What is the solution to the equation \(2 - x - x = x + 8 - 3x - 6\)?

\[
\begin{align*}
\text{Line 1} & : & \text{He did get the 0 right but not the 2 = 2,} \\
\text{Line 2} & : & \text{In order to get 0 he put 0 in all the x spots.} \\
\text{Line 3} & : & \text{2 = 2} \\
\text{Line 4} & : & \text{O = 0}
\end{align*}
\]

*Figure 4.36.* Bryon worked within the symbolic representation type in an attempt to solve the equation \(2 - x - x = x + 8 - 3x - 6\).

Immediately following Bryon’s purported solution (zero equals zero), Bryon used his CAS to move to a graphic representation of the equation \(2 - x - x = x + 8 - 3x - 6\) (Figure 4.37). This movement from a symbolic to a graphic representation type was unprompted by the researcher.

*Figure 4.37.* Bryon used his CAS to move from the symbolic to the graphic representation of the equation \(2 - x - x = x + 8 - 3x - 6\).
It seemed as though Bryon used this representation to “check” or help justify the asserted solution, but it is not clear from the exchange given below.

1  SB: Cause… OK, control I (types ctrl I on CAS), sweet, graph (selects 2: Add Graphs), two minus x minus x (types into f3(x) = 2 − x − x, Enter), OK. And then you've got x plus eight minus three x minus six (types into f4(x) = x + 8 − 3x − 6 [Figure 4.37]). And they overlap the whole time.

2  R: Mm hm.

3  SB: What is the solution (in reference to Task 12 Part b)? (Draws an arrow from original equation in Part b to equation work written to side [Figure 4.36]) I don't feel like writing it again.

4  R: That's Ok. So...

5  R: They overlap the whole time (points to CAS graph screen [Figure 4.37]), that means that the solution is... (leading question)

6  SB: (immediate response) Zero equals zero. At least I think.  

(SB_day25_finalinterview_Task12_45:50–46:39)

Bryon’s conclusion that “they overlap the whole time” was seemingly referring to the functions f3(x) and f4(x) yet was not well explained (line 1). Moreover, he seemed to interpret the meaning of the graphic representation as signifying the solution zero equals zero (lines 5–6). Bryon’s movement from the symbolic to the graphic representation was classified at the multistructural level because he correctly created a graphical representation from the symbolic equation yet did not convey a meaningful understanding of how these representation types were connected with respect to the solution to the equation (M-2, S- ->G, CAS). Indeed, Bryon remained focused on the symbolic representation in answering Task 12 Part b, not referencing his graphical work at all (line 3) in explaining the solution to the equation.

Another way that Bryon used his CAS was to check Andy’s purported solutions of x = 0 and x = 2 by using the “with” operator in a Calculator page (Figure 4.38, Lines 1–5).
After an incorrect creation of a verbal representation from a symbolic representation that was “an accident” (Figure 4.38, Line 3), Bryon eventually created the correct representation (Figure 4.38, Line 2) to verify that $x = 0$ is “true.” Without a verbalized coordination of the meaning of this representation, Bryon’s RF is classified at the multistructural level with a correct creation yet incomplete interpretation of the meaning of these representations (M-2, S- ->V, CAS). Consider the following transcription that is a continuation of Bryon’s work discussed above.

1 SB: Ok. That's true (looking at result of Figure 4.38, Line 2). (Grabs Line 2 to execute Line 1 with $x = 2$, presses Enter) Ok I guess it is true. I accidentally changed it, the wrong way.
2 SB: Yeah I don't know how he got that.
3 R: So do you think that $x = 2$ is a solution?
4 SB: It is a solution, but I don't know…(Moves back to graphs page [Figure 4.37]) how. So that, yeah I don't know how.

(SB_day25_finalinterview_48:06–48:46)

For the solution of $x = 2$, Bryon did create a corrected CAS inscription (Figure 4.38, Line 2 then Line 1) that signified a movement from the symbolic to the verbal representation type. With a clarification question prompted by the researcher, he also verified verbally this time that $x = 2$ is a solution to the equation (line 4 of above transcript). The correct creation and interpretation of these representations for the
invariant feature of \( x = 2 \) is coded as a uni-directional connection at the relational level of RF (R-2, S→V, CAS). The somewhat uncertain tone that Bryon expressed (lines 2 and 4) does not discount the symbolic to verbal connection. Instead, it is interpreted to mean that Bryon was unable to make a bi-directional connection to justify or explain why these values are solutions to the symbolic equation. It is also interesting to note that Bryon switched the CAS view back to the graph (line 4) yet did not reference this representation to explain why \( x = 0 \) and \( x = 2 \) were two of the infinitely many solutions to the equation.

The final way in which Bryon’s RF is characterized in this task is with respect to his unprompted movement from the symbolic equation to various numeric expressions and equations (shown in Figure 4.39). As the following analysis will make clear, Bryon was focused on both the symbolic and the numeric representation types, yet did not convey a clear understanding of how these representations are related or what they mean with respect to solving the equation.

Illustrated in Part a of Figure 4.39 and explained in Line II of Figure 4.36, Bryon substituted a value of \( x = 0 \) into the equation \( 2 - x = x - 8 - x - 6 \) and incorrectly interpreted this to signify how Andy had obtained the equation \( 2 = 2 \) in his equation solving process (instead of as a way to verify that \( x = 0 \) is a solution to the equation). This was an incorrect interpretation based on a correct movement from a symbolic to a numeric representation type (M-2, S- ->N, P&P). Secondly, Bryon worked to “see if \( x \) could equal two” (SB_day26_finalinterview_51:46) as shown in his work in Figure 4.39. First Bryon’s substitution of \( x = 2 \) into the expression \( 2 - 2x \) was correct (Part b of Figure 4.39), but then Bryon explained that similar to the right hand side of the equation in part a, “instead of zeros I put in two” (53:10), yielding an incorrect conclusion of 2 instead of
Based on this work, Bryon came to the conclusion that \( x = 2 \) was wrong, or not a solution to the equation\(^6\).

\[
\begin{align*}
2 + 8 - 6 \cdot 6 &= 2 - 2(2) \\
10 - 6 \cdot 2 &= 2 - 4 \\
4 - 6 &= -2 \\
-2 &= 2
\end{align*}
\]

Figure 4.39. Bryon moved from the symbolic to the numeric representation type.

Bryon subsequently recognized that he had done something wrong (in Part c) then after correcting it and writing the equation \(-2 = -2\) (the conclusion of Parts d and b)

Bryon concluded, “Ok, maybe. But that's opposite what his final answer was. Because he got negative twos” (SB_day26_finalinterview_52:41). Based on this apparent confusion, Bryon did not express a meaningful understanding of \( x = 2 \) as a solution to the equation.

Bryon had created correct numeric representations from the symbolic equation, but demonstrated little understanding of what they meant in regards to solving the equation (M-2, S- ->N, P&P).

Across the several solution attempts Bryon made in completing this task, he demonstrated prestructural, multistructural, and relational levels of RF. Bryon’s work within the symbolic representation alone was prestructural because despite correct transformations on the equation, Bryon gave an incorrect interpretation that the equation

---

\(^6\) Because this was incorrect work with an expression (and not an equation) it is not coded separately as part of the framework, and instead used as context to explain how Bryon persisted to move from the symbolic to the numeric representation type until the creation was correct.
0 = 0 was the solution. Bryon used his CAS to facilitate the creation of correct representations as he moved from the symbolic to the graphic and the symbolic to the verbal representation types, yet did not give complete and correct interpretations of these representation types to come to a conclusion that the equation had infinite solutions (despite his conclusion that \( x = 2 \) is a solution). Bryon’s final attempt to move to the numeric representation type from the symbolic equation was also unsuccessful in that he didn’t seem to gain any understanding of what the full solution to the equation was.

Bryon’s last words on this problem give further evidence that he persisted in taking his symbolic equation as the final answer despite work in other representations that might have suggested otherwise.

R: So you first said the solution was zero equals zero (points to SB's recorded work on paper of 0 = 0), have you changed your mind, or do you still think that's the solution?
SB: Oh, I still think that's the same solution.
R: Final answer?
SB: Zero equals zero.

(SB_day26_finalinterview_Task12_53:00–53:14)

The fact that Bryon had demonstrated higher levels of RF across the symbolic, verbal, graphic, and numeric representation types for the (partial) solutions of \( x = 0 \) and \( x = 2 \) was overall not meaningful to him. Again, it seemed as though Bryon’s P&P symbolic solution approach took precedence over the multi-representational approaches with CAS and P&P, despite the fact that Bryon had created correct representations from the given symbolic equation.
Solving an Equation $ax + b = cx + d$ With One Solution: Task 14

Bryon demonstrated comparable performance during the post-test and final interview in solving an equation of the form $ax + b = cx + d$ for $x$ with one solution in the numeric and symbolic representation types, with expanded abilities using the graphic representation type during the final interview. During the final interview in particular, when prompted to “circle all values in the table that represent solutions to the equation $9 + x = 6 + 4x$” Bryon first started by working within the symbolic representation type, correctly solving the equation to yield “1,” a unistructural level of RF (U-0, S→S, P&P) (Figure 4.40a).

Figure 4.40. Bryon’s work within the symbolic and numeric representation types to solve the equation $9 + x = 6 + 4x$ in Task 14.
As soon as Bryon finished writing the “1” to the left of the equal sign in Task 14a, Bryon moved to the numeric table representation and the following dialogue took place in regards to his written work (see Figure 4.40a).

1 SB: Um, OK, The numbers are the same, that'd be one (circles row on paper with 1, 10, 10 [Figure 4.40a])
2 R: Can you say more how you knew to circle there?
3 SB: Um, both of these (motions down each of the columns \( y = 9 + x \) and \( y = 6 + 4x \) [Figure 4.40a]) had ten in 'em.
4 SB: And when I did that (motions to symbolic transpositions to the left of table [Figure 4.40a]), my \( x \) equaled one (points to value of 1 to the left of the equals sign on paper [Figure 4.40a]), so it basically just told me.

(SB_day25_finalinterview_31:45–31:56)

The original task (Part a) had asked to use the table that was given to solve the equation, but what transpired instead was a movement from the symbolic to the numeric table representation. Bryon seemed to use two key pieces of information to inform his use of the numeric table representation. First, Bryon stated that “the numbers are the same” (line 1) and clarified this to mean that the columns for \( y = 9 + x \) and \( y = 6 + 4x \) both “had ten” in them (line 3). Second, Bryon’s solution of “1” from the symbolic representation type “told” him that this was the correct thing to circle in the table (line 4). The fact that Bryon circled the entire row in the table instead of the solution of \( x = 1 \) in the table is evidence of a multistructural level of RF from the symbolic to the numeric representation type (M-2, S- ->N, P&P), the only value in the table that represents a solution to the equation \( 9 + x = 6 = 4x \) is 1.

On his own accord, Bryon continued with Task b to check his work in which he was focused on both the symbolic and numeric representation types (see Figure 4.40b).

1 SB: Nine plus one, basically, six plus (writes \( 9 + 1 = 6 + \))—cause that's nine plus \( x \) equals six plus four \( x \) (writes \( 9 + x = 6 + 4x \) above)—and then, four times \( x \) so that basically equals four too (writes 4) so that's ten equals ten (writes 10 = 10) and then put them together and you get one (writes 1) [Figure 4.40b].
From Bryon’s written work and verbalizations he used both the symbolic equation
\[ 9 + x = 6 + 4x \] (lines 1 and 3) and the numeric table (line 3) and the numeric equations (line 1) to check his solution (with the caveat that the researcher prompted Bryon to consider how his numeric equations specifically related to the table, line 2). The fact that Bryon substituted a value of “1” for “x” in the equation \( 9 + x = 6 + 4x \) is evidence of a movement from the symbolic to the numeric representation types (line 1). This exchange is coded at the multistructural level of RF because of Bryon’s focus on both the value of “1” (the solution) and the invariant feature of “10” or “\(10 = 10\)” (M-2, S-\(\rightarrow\)N, P&P). It was not clear from Bryon’s exchange, especially in how he put the \(10 = 10\) “together to get 1” that he understood that the sequence of numeric equations that he created signified that the solution of \(x = 1\) was correct. The focus on both the \(x\) and \(y\) values is evidence of an understanding of the Cartesian Connection, that the value of \(x = 1\) satisfies the equation \(9 + x = 6 + 4x\) with a corresponding \(y\)-value of 10 for both \(9 + x = 6 + 4x\) (in the numeric equation, and the numeric table). To be coded at the relational level of RF Bryon would have needed to focus on the \(x\)-value only, being clear about this as the solution.

Prompted in Task c to “explain how a graph can be used to solve the equation \(9 + x = 6 + 4x\)” Bryon correctly used his CAS to move from the symbolic equation to
graphic representation of \( f_1(x) = 9 + x \) and \( f_2(x) = 6 + 4x \). Bryon’s initial reaction to this graph was “Whoa. Did I do that right? (pauses) That’s weird” (SB_day25_finalinterview_35:12). Prompted to explain himself, Bryon continued to talk to himself, “Oh, that’s what happened” and proceeded to ask for technical assistance in changing the window of the graph. Despite his successful creation of the graphic representation from the symbolic equation, the fact that Bryon seemed surprised by the graph of the equation \( 9 + x = 6 + 4x \) and gave an incomplete interpretation of what the graph meant at this point is evidence of a multistructural level of RF (M-2, S- ->G, CAS).

Bryon did not convey a relational level of RF of the relationship between these representations despite his having already solved this equation in the symbolic equation and identified the invariant feature of \((1, 10)\) in both a numeric table and numeric equations. Bryon’s subsequent work with the graph is considered to be working within this representation because he was focused singly on this representation type, and not dually on both the symbolic and graphic representation types.

At Bryon’s request for technical assistance to “get this thing to go up a little higher” (SB_day25_finalinterview_35:26) the researcher suggested that he use the drag feature on the window, from which he dragged the screen until he saw fit, releasing the drag feature when the point of intersection was shown in the window. The initial and secondary graph views are shown in Figure 4.41a and b, respectively.
Figure 4.41. Bryon’s use of his CAS to solve the equation $9 + x = 6 + 4x$ using a graph.

Bryon’s interpretation of this graphic representation involved a focus on both the $x$- and $y$-axes.

They only intersect at one spot right here (moves cursor over point 1,10 on graph [Figure 4.41]) because let’s see ...(mumbles to self, then counts, one, two three four..., eight nine ten, as he traces his pen along $y$-axis of the CAS graph [Figure 4.41]). I figured if I just counted up I'd get ten, which I did. And then I noticed that it’s right about the one $x$ (tracing vertically along the line $x = 1$ [Figure 4.41]). So it gave me, one $x$, one, one $x$ ten...(pauses)... one ten I guess.  
(SB_day25_finalinterview_37:28:41)

Similar to use of the numeric representation type, Bryon’s use of the graph was focused on both the $x$- and $y$-values of the intersection point. Bryon’s conclusion that “it gave me...one ten” is an indicator that he was not able to separate the solution of $x = 1$ from the $y$-value of 10. Thus the use of the graphic representation type here did not help him to come to the correct conclusion that the value of $x = 1$ is the solution; he was grounded to the graphic representation type and did not use other representations (e.g., the symbolic) to inform his identification of the solution to the equation to be an $x$ value of one (P-1, G- ->G, CAS).

Finally, Part d of Task 14 was designed to elicit students’ understandings of the connections between the table and graph (see Figure 4.42).
d. Reflect on your work in Parts a-c. How does the graphical approach (Part c) relate to the values circled on the table (Part a)?

well I looked at the graph and then the table and the cross at 1, 10. Then I looked up at the table in A and saw that it had the same y-axis numbers (10) and the x-axis is 1.

Figure 4.42. Bryon explained the relationship between the graph and table for the equation $9 + x = 6 + 4x$.

To build on Bryon’s use of the graphic and table representations, he was prompted by the researcher, “So did you know before graphing that they were going to intersect?” to which he explained, “Yeah, just by looking at the table and the work I did above it told me it was going to intersect” (SB_day25_finalinterview_38:56). Together with his written work (Figure 4.42), Bryon’s RF in solving the equation $9 + x = 6 + 4x$ with the graphic and numeric representation types was classified at the multistructural level; he correctly identified the point $(1,10)$ to be an invariant feature of the equation $9 + x = 6 + 4x$ in both the graph and the table, yet did not focus on the fact that the $x$-value was the solution to the equation (M-2, G- ->N, CAS, P&P); Bryon’s written work further supports this conclusion. Bryon’s identification of the intersection point on the graph (i.e., “the[y] cross at 1, 10”) was followed by an identification of this point in the table in which he specifically focused on the “y-axis numbers (10) and the x-axis is 1” (Figure 4.42).
To summarize Bryon’s solution approaches in solving the equation $9 + x = 6 + 4x$ for $x$, Bryon was clear in identifying the solution of $x = 1$ in the symbolic representation type (unistructural), yet his movement to numeric (equation and table) and graphic representation types did not exclude the extraneous information of $y = 10$, conflating any conclusions about the actual solution to the equation (multistructural). In particular, it is not clear if Bryon really understood that his movement between representation types was geared toward an identification of the solution, or an identification of the invariant feature of the point $(1,10)$. For instance, both the $x$- and $y$-coordinates of the intersection point of the graphs $f_1(x) = 9 + x$ and $f_2(x) = 6 + 4x$ were identified, without a direct indication that the $x$-value is the solution to the equation $9 + x = 6 + 4x$. Moreover, the entire row was circled in the table (including both $x$- and $y$-values) when only the $x$-value of 1 should have been circled for a true representation of the solution to the equation. It also seemed clear that Bryon understood that the point $(1,10)$ was represented in what he identified as the intersection point of the graph and what he circled in the numeric table (multistructural). Thus Bryon’s RF in solving the equation was strongest in the symbolic representation type, and less strong across numeric and graphic representation types.

*Solving a Linear Equation With No Real Solutions: Task 13*

During the final interview only, students considered solving a linear equation with no real solutions. As shown in Figure 4.43, Bryon initially attempted to solve the equation $x + 2 + 2x = 5 + 3x – 1$ for the variable $x$ by working within the symbolic representation type.
Solve for the variable and check your solution.

If you use a graph, sketch the section of the graph.
If you use a table, include the table and indicate where you found the answer.
If you use equations, show all of your work.

\[
x + 2 + 2x = 5 + 3x - 1
\]

\[
3x + 2 = 4 + 3x
\]

\[
-2 \quad -2
\]

\[
3x = 0 + 3 > 2 \quad -3 >
\]

\[
-3x
\]

\[
0 = 2
\]

Figure 4.43. Bryon worked within the symbolic representation type to conclude that \(0 = 2\) then substituted \(x = 0\) into the equation to yield \(2 = 4\) or 2.

Bryon performed correct transformations on the symbolic equation, working down to an equation of \(0 = 2\). The researcher then prompted Bryon to interpret this work within the symbolic representation type.

1 R: So, what's the solution to this equation?
2 SB: Um, it's got none.
3 SB: Well there's zero Xs and it equals two, so it'd be, um, [...] segment of transcript omitted [...] if there's no Xs there's no solution.
4 R: Uh huh.
5 SB: Unless it's a zero equals zero.
6 R: Uh huh.
7 SB: But there isn't a zero equals zero, it's a zero equals two. So that's a no solution.

(SB_day26_finalinterview_25:23-25:58)

Bryon’s correct creation (Figure 4.43) and interpretation of the solution to the equation (line 3) is evidence of a unistructural level of RF within the symbolic representation type.
(U-0, S→S, P&P). Somewhat tacit in Bryon’s response is that he interprets this equation as falling into a class of equations with no solutions (line 7). Additional evidence for this is discussed later.

Next, prompted to check his solution (consistent with the Task statement), Bryon first expressed uncertainty in how he might go about doing that. He then moved from the symbolic representation to the numeric representation type (evidenced in the work shown to the right in Figure 4.43 and at the top just above the equation in the same figure above). Despite this movement, Bryon carried out an unnecessary step of dividing both sides of the equation $2 = 4$ by 2, and also expressed confusion about what this meant with respect to solving the equation. He then seemed to use the CAS as a resource to help explain the meaning of this numeric work, moving from a symbolic equation to a verbal statement of the truth of the equation at $x = 0$ (Figure 4.44).

![Figure 4.44](image)

*Figure 4.44.* Bryon used his CAS Calculator page to check the veracity of the equation $x + 2 + 2x = 5 + 3x - 1$ evaluated at $x = 0$.

Bryon’s interpretation of the movement from the symbolic to the numeric representation type is given next, followed by his interpretation of the movement from the symbolic to the verbal representations.
SB: And since two equals four (uses pen cap to circle 2 = 4 on paper), I figured, well that's not right, so I divided both sides by the smaller number to get the same number I just divided it by, to get two (points to 2 on paper). Basically.

R: So you went from the final equation of 2 = 4 just to an answer of 2?

SB: Mm hmm. (pause)

SB: Because (presses CAS Scratchpad Calculator page) If I go like this [...]segment of transcript omitted related to Line 2 of CAS screen [Figure 4.44] [...] (Grabs line 2 and types | x = 0 into CAS Line 1 [Figure 4.44]) False. (Pause)

R: So what does that tell you? (In reference to Line 1 on CAS calculator page [Figure 4.44].)

SB: That there's no solution to it. Because there's zero Xs in it all together, I guess, (mumbles to self "I don't know")

R: So are you convinced?--

SB: No--

R: --that there are no solutions?

SB: --I was completely confused on this whole little thing.

R: Is there a different way that you could be convinced that there are no solutions?

SB: Let's see… (mumbles to self, then silent for about 8 seconds)… Nope. Because no matter which I put it there's always zero x.

(SB_day25_finalinterview_26:48–28:48)

In line 1 of the above transcript, although Bryon had correctly substituted a value of $x = 0$ into the equation $x + 2 + 2x = 5 + 3x – 1$ to yield $2 = 4$, he did not correctly interpret that this meant there are no solutions to the equation. Instead, he incorrectly divided the false equation $2 = 4$ by two, despite the fact that he recognized that the equation was “not right.” Overall, Bryon’s movement from the symbolic to the numeric equation is coded at the multistructural level based on his (partially) incorrect creation and incorrect interpretation (M-1, S-X->N, P&P). In an apparent attempt to explain this work, Bryon used his CAS to move from the symbolic representation to the verbal representation (Figure 4.44, Line 1, transcript line 4) yet incorrectly concluded based on this one point that the original equation has no solutions (line 6) (M-2, S- ->V, CAS); all one can determine from this result is that $x = 0$ is not a solution. At this point in the interview, he was not very confident in his solution (lines 7–10), and did not recognize an
alternative to the symbolic representation to determine that there are no solutions (lines 11–12).

Bryon was apparently stuck in being able to completely explain this task, thus the researcher prompted Bryon to consider if he could use a graph to solve. Bryon used his CAS to graph the equation $x + 2 + 2x = 5 + 3x – 1$ (see Figure 4.44b).

![Figure 4.45](image)

*Figure 4.45.* Bryon used his CAS to graph the equation $x + 2 + 2x = 5 + 3x – 1$.

In the following exchange, Bryon explained what the graphic representation meant to him, and also alluded to the general case of no solutions in his response.

1 SB: Yeah they never mat--they never line up with each other so it's no solution, they're [sic.] never be tou—they'll never touch.
2 R: So are you more convinced?
3 SB: Yeah.
4 R: You're more convinced by this (points to CAS graph [Figure 4.45]) than by this (points to P&P work)? Can you tell me about that?
5 SB: Well since I got that (referring to P&P work), and now I just put, since you had me put that in there (referring to CAS graph), it convinces me even more because those lines don't touch at all, there's no intersections. I've just proved my point that if there's zero Xs […] segment of transcript omitted […] if it's zero and a different number besides zero, it will never interact.

(SB_day25_finalinterview_29:53–30:52)

Despite the fact that the researcher prompted Bryon to “use graphs,” Bryon was able to create a graphic representation on his CAS (Figure 4.45) and interpret this to mean that there are no solutions (line 1, above). Bryon expressed confidence in his
interpretation of the meaning of the symbolic equation from a graphic representation type, apparently using the fact that for the graph “they’ll never touch” (line 1) and “never intersect” (line 5) to explain the meaning of the equation “0[x] = 2” or “zero Xs” (line 5). The invariant feature of “no solutions” was coordinated from the symbolic to the graphic representation, a relational level of RF (R-2, S→G, P&P, CAS).

In solving an equation with no solutions, Bryon correctly worked within the symbolic representation type, a unistructural level of RF. Prompted to check his solution, Bryon moved from the symbolic to the numeric, and from the symbolic to the verbal representation types, yet was not confident nor completely correct in what these representations helped him to explain with respect to the solution to the equation, a multistructural level of RF in both cases. Finally, prompted by the researcher to consider a graphical approach to solving, Bryon demonstrated a relational level of RF from the symbolic to the graphic representation type based on his interpretation that the graphs don’t intersect or interact in the case of no solutions. Bryon seemed to rely most on the fact that he had “zero Xs” in the symbolic representation type and that the graphs don’t intersect to justify the fact that the equation had no solution. It is important to note that Bryon was not resourceful in using the graph to help explain the no solution case—the researcher prompted this translation.

Summary of Bryon’s Final RF

Considering all four equation solving tasks that Bryon attempted during the final interview, Bryon’s RF was typically at the unistructural levels of RF within the symbolic representation types (for all but one equation), and at the multistructural levels of RF
from the symbolic to the graphic, numeric, and verbal representations (for all but two episodes). Bryon’s strength in creating and interpreting symbolic equations was prominent in all interview tasks, and often outweighed his attempts to use other representation types to solve the equations, especially in Tasks 10 and 14 in which there was one solution. It is possible that when Bryon was using graphs, tables, and numeric equations in these two tasks and he did not really see this activity as solving or identifying the solution to the equation. Instead, Bryon seemed to demonstrate an understanding of the Cartesian Connection in the sense that the ordered pair of the solution was interpreted to be an invariant feature across MR. Despite this connection, Bryon was not successful at isolating the $x$-value as the solution in graphic and numeric table representations, evidence of his multistructural level of RF in solving these equations.

In the tasks that had infinite and no solutions, Tasks 12 and 13, respectively, Bryon demonstrated differing degrees of RF with respect to his interpretations of representations. In both tasks he was successful in creating verbal, graphic, and numeric representations. However, in Task 12 Bryon failed to recognize that the equation had infinite solutions. The overlapping graphs, the result of “true” in the verbal representation type, and the identification of “the same” expressions related by an equal sign did not provoke Bryon to understand that this equation had infinite solutions. On the other hand, in solving Task 13, it was much more immediate to Bryon that the equation of $0 = 2$, the graphs that never intersect, and the result of “false” in the verbal representation signified the case of “no solutions” for the particular equation he was asked to solve. The
differences in Bryon’s RF across these tasks are further compared in the following discussion of Bryon’s change in RF.

**Bryon’s Change in RF**

Evident across all equation solving tasks during the initial and final interviews was Bryon’s persistence in working within the symbolic representation type. Within this type, by the final interview Bryon had changed his RF from a mainly prestructural level to a mainly unistructural level, with more consistency in his creation of correct representations and interpretations of the solutions to equations. Bryon had also changed in his ability to move between MR when solving equations posed in symbolic form. During the initial interview Bryon had demonstrated little evidence of this ability, while by the final interview, Bryon had improved in his ability to create these representations, with little improvement in his ability to correctly interpret these representations with respect to correctly identifying the solution to equations.

Further evidence of this purported change in Bryon’s RF is discussed next with respect to the differences he exhibited from the initial to the final interview. His overall change is analyzed with respect to his RF on a task-by-task basis.

**Solving an Equation** \( c = ax + b \) **With One Solution: Task 10**

Bryon’s initial and final levels of RF when solving a task of the form \( c = ax + b \) for \( x \) are illustrated in Figure 4.46. Recall that the illustration using the Rule of Four Web corresponds to the specific levels of the Analytic Framework for RF that are listed below each figure in the order in which they occurred during each interview. The intent of this
figure is to summarize the findings discussed in the aforementioned sections for both the initial and final characterizations of Bryon’s RF.

![Diagram of initial and final levels of RF while solving an equation]

**Figure 4.46.** Bryon’s initial and final levels of RF while solving an equation of the form $y = ax + b$ at $y = c$ for $x$ with one solution using MR (Task 10).

During the initial interview, Bryon struggled to work within the symbolic representation type to solve the equation $5 = (1/4)x – 4$ for $x$, demonstrating a prestructural level of RF. Bryon also did not think it was possible to use a table nor a graph to solve the equation, despite his ability to correctly create a graph from a given table, and articulate the connection between these representations as having the same points. It is possible that two specific nuances of this problem set up may have confounded Bryon’s engagement with this task. First, the equation Bryon created did not correspond to the table and graph next to it, and second, the equation itself involved a fraction. Had Bryon had the correct original equation ($5 = 8 – 3x$) it may have supported his ability to correctly work within the symbolic representation type, and possibly also his ability to use the graph and table to identify the solution.
Based on his performance on the final interview, Bryon made progress in his ability to work within the symbolic representation type to solve an equation; he was successful in solving the equation \( 4 = 3x - 2 \) for \( x \). Prompted to show the solution to the equation using the graph and table, Bryon demonstrated multistructural levels of RF in his movement from the symbolic representation to these two other types. Bryon seemed to view the ordered pair \((2, 4)\) as a similarity that was evident across all three representations, but did not specifically focus on the fact that the solution to this equation as \( x = 2 \)—the \( x \)-coordinate of the point on the graph, and in the table. Bryon had also not interpreted his originally created graph as being a meaningful representation from which to illustrate the solution to the equation. He chose to create a new graphic representation to work within in order to demonstrate where the \((2, 4)\) came from. It is possible again that the wording of the task may have confounded the situation. Had Bryon been asked to use the graph to “identify” the solution (as opposed to “show” the solution) he may have used the original graph instead of creating a new graph that didn’t seem to have a meaningful connection to the original.

*Solving a Linear Equation With All Real Solutions: Task 12*

In both the initial interview and final interview, Bryon made progress in creating representations that moved him closer to the goal of solving an equation with all real solutions—the case of infinite solutions—yet in neither interview did Bryon correctly identify that the equation indeed had infinite solutions. Bryon’s initial and final levels of RF are summarized in Figure 4.47.
Figure 4.47. Bryon’s initial and final levels of RF while solving an equation of the form $ax + b = cx + d$ for $x$ with infinite solutions (Task 12).

During the initial interview, Bryon was not able to use the symbolic representation type to correctly solve the equation, nor was he able to interpret the meaning of the graphic or numeric equation representations to mean that the equation had infinite solutions. The researcher had prompted Bryon to consider solving the equation using graphs, to which Bryon needed technical assistance in using his GC. Although he correctly identified the graph for each expression related by an equal sign, Bryon’s movement from the symbolic to graphic representation was coded as multistructural because he recognized that the graphs were the same, yet this movement between representations did not seem to be very meaningful in that he did not demonstrate a greater understanding of the solution to the equation.

On a related task during the final interview, Bryon still operated at the prestructural level of RF within the symbolic representation type. Despite his success in transforming the equation to yield $0 = 0$, Bryon interpreted this to be the solution, instead of an indicator that the equation had infinite solutions. This surface-level understanding
of the meaning of the symbolic representation type did not improve despite subsequent approaches in moving between MR, at both multistructural and relational levels of RF. For instance, Bryon used his CAS to move from the symbolic to the graphic representation type, with little explanation of the meaning of the fact that the graphs “overlap” with respect to the solution. Likewise, in verifying the solutions of $x = 0$ and $x = 2$, Bryon used the CAS calculator page to evaluate the equation using the “with” operator, yielding a verbal confirmation of “true,” demonstrating multistructural and relational levels of RF respectively based on his verbalization that $x = 2$ is a solution to the equation. Bryon’s RF was limited however because he did not demonstrate a bi-directional connection to be able to explain why $x = 0$ and $x = 2$ make the equation true. From this work, the greatest change in Bryon’s RF was with respect to using the CAS to verbally confirm that $x = 2$ is a solution.

During both the initial and final interviews, Bryon moved from the symbolic to the numeric representation type by substituting values into the original equation. In both cases, Bryon’s RF was coded at the multistructural level, with little change demonstrated by the final interview. In both cases the numeric equations were not well connected to the meaning of the symbolic equation nor to the solution to the equation in any of the other representation types that Bryon had considered. For instance, during the initial interview Bryon had correctly substituted a value of $t = 0$ into the equation, yet later came to the conclusion (based on incorrect work in the numeric representation) that 0.2 was the final solution. During the final interview by substituting a value of $x = 2$ into the equation Bryon again was successful at creating a numeric equation from the symbolic equation yet did not make sense of the meaning of the resultant equation $-2 = -2$. 
At no point during Bryon’s attempts to solve an equation with infinite solutions did he show evidence that he understood all values of \( x \) to make the equation true. Based on Bryon’s work during the initial interview he did not seem to recognize the possibility of an “identity” equation (such as \( 3T = 3T \)). By the final interview, Bryon seemed satisfied with the equation “\( 0 = 0 \)” as a solution in and of itself, yet did not convey additional meaning of this representation type on its own, nor after recognizing that the graphs “overlap.”

**Solving an Equation** \( ax + b = cx + d \) **With One Solution: Task 14**

Bryon’s initial and final levels of RF when solving an equation of the form \( ax + b = cx + d \) for \( x \) that has one solution are illustrated using the Rule of Four in Figure 4.48.

**Figure 4.48.** Bryon’s initial and final levels of RF while solving an equation of the form \( ax + b = cx + d \) for \( x \) with one solution (Task 14).
Bryon’s initial attempt to solve the equation $100 + 2x = 12x - 10$ using the table was dominated by the symbolic representation type. After correctly transposing within the symbolic representation type (unistructural level), Bryon did not use the table representation when prompted. The fact that Bryon circled the value of “11” within a format that to an experienced other would be interpreted to be a table was not sufficient evidence that he was actually using this numeric table representation. He did not reference the fact that he was circling an $x$-value of 11, nor did he mention the corresponding $y$-values. Also, when prompted, Bryon was also not able to use the graphic representation to solve this equation. In this task he seemed satisfied with his symbolic work and made no real attempt to use MR to solve this equation.

Bryon’s RF in working within the symbolic representation type to solve a comparable task during the final interview did not change, but Bryon did show growth in his ability to move between representations to make progress toward solving an equation with one solution. Prompted first to use a table to solve the equation $9 + x = 6 + 4x$ for $x$, Bryon instead first correctly transposed within the symbolic representation type to solve the equation, a unistructural level of RF (no change from his initial interview). Bryon’s RF in moving between representations to solve the equation was classified at the multistructural level in all cases—from the symbolic representation to the given numeric table representation, and from the symbolic to a numeric equation representation created using P&P, and from a symbolic to a graphic representation created using CAS. Bryon struggled most in giving correct interpretations of the solutions in each of these representations.
The theme in his work was a focus on both the $x$-values of 1 and the $y$-value of 10, evidenced by his circling of the entire row, 1, 10, 10 in the numeric table, his satisfaction with $10 = 10$ put together to get 1 in the numeric equations, and the $x$-axis value of 1 and $y$-axis value of 10 as the intersection point on the graph. The identification of this point (1, 10) as an invariant feature across these representation types was more supportive of a classification of Bryon’s understanding of the Cartesian Connection, than of his RF in solving the equation. Bryon did not give evidence that he clearly understood the value of $x = 1$ to be the solution across the numeric and graphic representation types, despite his success in identifying this value in the symbolic representation type.

In solving a linear equation with one solution, Bryon did show growth in his ability to move between representations of the linear equation. Specifically, his RF changed in his ability to successfully create and use a graph and a table to identify the $x$-value of 1, and also the $y$-value of 10. It seems appropriate that a stronger understanding of the Cartesian Connection across these representation types is an intermediate “step” in building one’s RF in solving equations. Without an understanding of the invariant feature of (1, 10) across the symbolic, numeric, and graphic representation types, it would be difficult to clearly justify the solution. It is conjectured that for Bryon, the extraneous information of $y = 10$ seemed to muddle the solution to the equation rather than clarify it’s meaning. To recognize a greater change in RF in his ability to solve the equation, there would have needed to be more concrete evidence that Bryon did not see the value of $y = 10$ as the solution or a part of the solution to the equation.
Solving a Linear Equation With No Real Solutions: Task 13

Recall that the task of solving an equation with no solutions was presented during the final interview only. A summary of Bryon’s RF in solving the equation $x + 2 + 2x = 5 + 3x - 1$ for $x$ is given in Figure 4.49.

![Diagram](image)

*Figure 4.49. Bryon’s final level of RF while solving an equation with no solutions (Task 13).*

In solving an equation with no solutions, Bryon’s first approach within the symbolic representation type was successful, a unistructural level of RF. When prompted to check his solution, Bryon moved to the numeric and verbal representations using P&P and CAS, respectively, yet was not able to correctly justify why “no solutions” made sense in these representations, evidence of multistructural levels of RF. However, when prompted to consider using a graphic representation, Bryon was successful in using his CAS to create a graph representing the equation, and correctly interpreted that, because the graphs do not intersect, the equation had no solutions—a relational level of RF.
Bryon’s success in using the symbolic representation to solve the equation rested most prominently on his symbolic and graphic representations. It seemed that to Bryon, he had used the graphic representation to help justify or explain his conclusion within the symbolic representation type that an equation of $0 = 2$ meant that there are zero Xs equal to two, which when shown in the graph, means that the lines never intersect.

*Summary of Bryon’s Change in RF*

Measured at both the onset and close of the instructional unit, Bryon’s RF changed the most in his successful creation and interpretation of solutions within the symbolic representation type, and also in his successful creation of graphic, numeric, and verbal representations from a given symbolic equation. There were few instances in which Bryon seemed to use MR to justify or explain his solutions, and more often, Bryon’s multi-representational activity in solving equations (or in identifying solutions in multiple representation types) was somewhat confounded by the Cartesian Connection.

In particular, Bryon had difficulties interpreting the $x$-value as the salient feature of the graph and numeric table representations that corresponded to a symbolic equation, even when he had been successful in solving within the symbolic representation type first. With strengths in creating representations, the difficulty of correctly interpreting solutions within and across MR limited Bryon in not demonstrating well-understood connections across representations. There was also limited evidence that Bryon had an ability to generalize across representations, with the exception of some hints at the general form of equations or types of solution cases (i.e., no solutions).
Carlos’ Initial RF

Carlos was a quiet and timid student in one-on-one interview situations, and more vibrant in personal (non-math related) interactions. He expressed at least twice during the interview situations that he was not good at math; he possibly felt intimidated when the tasks he was to work were identical to or very similar to what he had worked on during in-class assessments (i.e., the tasks felt like tests). It was oftentimes difficult to engage Carlos in working on tasks, but nonetheless, the researcher did her best to probe his understandings to get a fuller picture of his RF.

At the onset of the study Carlos’ RF was very weak in that he demonstrated at best a prestructural level of RF in the symbolic and numeric representation types. Often prompted to consider the use of MR, Carlos did not engage in linear equation solving activity. Carlos’ work on three tasks during the initial interview involved both incorrect creations of and incorrect interpretations of the representations he used to attempt to solve equations.

_Solving an Equation c = ax + b With One Solution: Task 10_

On the pre-test, and during the initial interview, Carlos did not attempt the problem that involved solving an equation of the form $y = ax + b$ at $y = c$. Based on Carlos’ pre-test work as shown in Figure 4.50, when prompted if he could use the table, graph, and symbolic equation to identify the solution to the equation when $y = 5$, Carlos responded “Uh, uh. No” (SC_day3_initialinterview_10:28). Despite prompting to work with these representations, he did not make an effort toward completing this task.
As Carlos did not attempt solving an equation of the form $y = ax + b$ at $y = c$, consider Carlos’ work on an equation of the form $ax + b = c$ when asked to solve for the variable $x$. After careful prompting and encouragement, Carlos attempted Task 11, as shown in Figure 4.51.

Based on an incorrect application of the distributive property (as indicated by the arrows that he drew from 3 to $-6$ and from $x$ to 12 in Figure 4.51), Carlos transformed the equation $3(x - 6) = 12$ into the expression $12x - 18$. He then used his GC to type $18 - 12x$.
which yielded –10.085 (because a value was stored for the variable $x$). Carlos explained this work in a procedural manner, and did not make progress toward determining the solution to this equation. Based on an incorrect rule that he remembered, he said “Well, you can't minus twelve from eighteen so I flip-flopped them, and that's how I got this answer (writes –10.085 on paper [Figure 4.51]), I'll just stop there” (SC_day3_initialinterview_11:22). Carlos’ incorrect work within the symbolic representation type to complete and solve the equation did not involve a complete interpretation of the meaning of the solution either. This is a prestructural level of RF within the symbolic representation type (P-0, S-X→S, P&P). The researcher’s prompting to have Carlos consider the use of graphs and/or tables were unsuccessful.

_Solving a Linear Equation With All Real Solutions: Task 12_

In Task 12, Carlos did not attempt solving an equation of the form $ax + b = cx + d$ with all real solutions in either the pre-test nor after prompting during the initial interview (he simply refused to attempt to solve this equation). However, in the context of discussing the meaning of the equal sign during the initial interview, Carlos was directed to reconsider the equation in Task 12 (Figure 4.52).

For 11-12, solve for the variable and check your solution using a table, graph, or equation (your choice). In each question, sketch the section of the graph (with the scale marked) or the table and indicate where you found the answer. If you solve the problem using equations, show all of your work.

12.) $t – 2 + 3t = -6 + 4t + 4$

*Figure 4.52. Carlos did not solve the equation $t – 2 + 3t = -6 + 4t + 4$ for $t$ but did give some indication of the meaning of the equal sign.*
In describing the meaning of the equal sign, Carlos explained that in equations such as \( t - 2 + 3t = -6 + 4t + 2 \) that it can be either true or false. This language was taken to be meaningful with respect to Carlos’ RF in solving equations because by definition, a solution to an equation makes the equation “true.”

1  SC: Well, they're just trying to like express, uh, something to see if it is either correct, or, like, add it up to see if it's correct, true-false. Or you just break it down into (taps pen) what the real answer is (pointing to previous work on Task 11 [Figure 4.52]).
2  R: I see. So when you say true-false (points to Task 12), if we think about problem 12, um, how would we know if it is true or false?
3  SC: Well, usually, I would type it in the calculator and then see if the answer is right or not.
4  R: Can we try it?
5  SC: Well yeah… (grabs calculator, and is re-directed to leave the calculator on the table within view; types \(-2 + 3\), points to \( t - 2 + 3t = -6 + 4t + 2 \) on paper) you don't need to put the variables in there (presses Enter, yields 1)
6  R: Why don't you need to put the variables in?
7  SC: Well that's what my old, um, eighth-grade teacher told me you don't need to put the variables in. So you don't have to put a bunch of variables and letters in there. You just add the numbers.
8  R: So you were adding the numbers, so I interrupted you, you were adding the numbers negative two and three? Is that it? Or do you have to do more than that?
9  SC: I think that's it.
10 R: So do you know if this equation is true or false?
11 SC: I believe that it's false maybe, because, well, we all know one equation can't equal another.
12 R: Why's that? What do you mean by that?
13 SC: Well if one equation equaled another equation, it just sort of doesn't make sense to me.
14 R: And do you feel like that's what we have here (points to the equation in Task 12)?
15 SC: Yeah.

(SC_day3_initialinterview_18:51)

Although Carlos did not create a symbolic representation in this exchange, he did give an interpretation of the meaning of this equation with respect to whether it was true or false. In line 1 he gave an indication that he was going to decide whether the given equation is true or false. Despite some incomplete numeric computations using his GC
Carlos concluded that the equation was false because one equation can’t equal another (line 11). One way to interpret this dialogue is that perhaps Carlos did not recognize \( t - 2 + 3t = -6 + 4t + 2 \) to be a legitimate equation, and thus thought it was a false equation where indeed, this equation is always true. This incorrect interpretation of the symbolic representation is evidence of a prestructural level of RF within the symbolic representation type because no understanding of the meaning of the solution to this equation was conveyed, and Carlos’ interpretation of the equation (with respect to its solution) was incorrect (P-0, S-X->S). Carlos’ contention that an equation can be “true” or “false” was promising at the onset (line 1), but misguided by incorrect procedures that he had learned in an earlier grade (lines 6–8). This ultimately led him to reflect on the form of the equation as a whole, concluding that it must be false.

_Solving an Equation \( ax + b = cx + d \) With One Solution: Task 14_

When prompted to use the table to circle all values that represent solutions to a given equation, Carlos incorrectly identified four values (as shown in Figure 4.53).

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7 Note that Carlos’ use of his GC to compute the sum of the constant –2 and the coefficient 3, this is not coded as a movement from the symbolic to the numeric representation type because he was focused on the expression on the left-hand side of the equation only and did not make progress toward successfully completing the task.
Carlos first circled the values, 110, 120, 128 then 146. Prompted to explain his thinking, Carlos said:

“Well they are all even numbers, and if you add ten to one hundred you get one hundred and twenty. And if you added ten like two more times (uses pen to swish across the equation $100 + 2x = 12x - 10$), or three, you'd get one hundred twenty eight. And the same up here (points to the column heading in the table with values for $y = 12x - 10$).”

(SC_InitialInterview_Day3_21:47:53)

Despite Carlos’ unsuccessful attempt to identify the solution to the equation, his gestures and explanations give evidence that he was using this table. First, Carlos was focused on the relationship between the values 110 and 120, in the same row in the table. Second, Carlos motioned to the original equation when explaining how he moved from 110 to the next value of 128. This is taken as evidence that he saw the table as representing the original equation and attempted to use it as a way to explain the relationship between the
numbers in the table (i.e., the values in the table were generated by the original equation). Third, Carlos pointed to the third column header to explain why he circled the last value of 146. It seemed as though Carlos was attempting to find a pattern or relationship between the values in the table but was inconsistent in his explanation of what that meant to him, demonstrating a prestructural level of RF in the numeric representation type (P-0, N-X→N, P&P).

Carlos did not attempt to complete Task 14b, which asked for a symbolic solution approach and check. However, the table, graph and start of an equation inscription shown next to Part b were created when Carlos was asked to explain how a graphic representation could be used to solve the equation. He explained that the axes and graph that he drew (shown in Figure 4.53) “was just an example” and that “no” he could not create a graph that matched the equations $y = 100 + 2x$ and $y = 12x - 10$. Carlos recognized that it was feasible to create such representations that were seemingly related. For example, he articulated, “you can make a graph, and plot the points from [the table], then make an equation to see what the answer comes out to be” (SC_day3_initialinterview_24:04). However, because he did not actually complete any of these representations specific to the given equation, this activity did not warrant a level of RF according to the analytic framework for RF. Carlos’ RF in solving an equation with one solution was limited to his incorrect use and interpretation of the numeric table representation—a prestructural level of RF.
Summary of Carlos’ Initial RF

Carlos did not attempt three out of the five problems that were posed as equation-solving tasks, two of which explicitly involved tables, graphs, and symbolic equations (the other three of which were posed in a symbolic representation type only). For the two tasks that Carlos did attempt during the Initial Interview, he demonstrated a prestructural level of RF in working within the symbolic and numeric representation types. In each task, Carlos was prompted to move between representations to create and interpret the solution. However, Carlos did not demonstrate an ability to create or interpret graphical, numerical, or verbal representations with success during the initial interview.

Carlos’ initial RF in solving problems involving linear equations was peppered with few successful events of creating and interpreting work within symbolic and numeric representation types. Attempts to use one representation type to justify the creation of another were largely unsuccessful, and Carlos demonstrated no successful transpositions within a representation type.

Carlos’ Final RF

During the final interview, Carlos was equipped with a CAS and P&P versions of several equation solving tasks. These tasks were specifically designed to target his RF in working within and moving between MR of linear equations. During the final interview, Carlos demonstrated limited abilities in creating and interpreting representations. All tasks were designed to involve the statement of a symbolic equation, and Carlos’ RF was the strongest in his ability to work within this representation type (at the prestructural
level) with comparable skill in working within the numeric table representation type.

Carlos did not successfully demonstrate any abilities to move between representation types during the final interview. A more detailed characterization of Carlos’ RF is given next based on his attempts to complete four equation-solving tasks.

**Solving an Equation c = ax + b With One Solution: Task 10**

During the final interview, Carlos refused to attempt to solve the equation 
\[ y = 3x - 2 \] at \( y = 4 \). He was also not successful at creating graphic and numeric representations from the given symbolic equation (see Figure 4.54).

10.) An equation is given in Row 3, below.

a.) Create a graph and a table for the symbolic equation \( y = 3x - 2 \).

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
</table>
| \[ \begin{array}{c|c}
| x & y \\
| 1 & 3 \\
| 3 & -2 \\
| 3 & 3 \\
| 4 & 3 \\
| 5 & -6 \\
| \end{array} \] | ![Graph](image) | \( y = 3x - 2 \) |

b.) Find the solution to the equation \( y = 3x - 2 \) when \( y = 4 \).

*Figure 4.54.* Carlos used P&P to create an incorrect table and graph for the equation \( y = 3x - 2 \).

Despite the incorrect representations created in part a, the researcher prompted Carlos to work on Part b.

1 R: Can you find the solution to the equation \( y = 3x-4 \) when \( y = 4 \)?
2 SC: Uhhhh… No.
The fact that Carlos did not attempt to solve the equation (line 2, above) might be explained by some “anxiety” that Carlos appeared to have with the words “solve” and “solution,” and maybe the fact that it was posed in a symbolic representation type. Perhaps had the researcher given more specific verbal prompting to show the solution using the table and graph, Carlos may have attempted to do so. However, it didn’t seem like Carlos was willing to think more about this task after creating the table and graph in part a.

_Solving a Linear Equation With All Real Solutions: Task 12_

It was difficult to engage Carlos in working on Task 12 during the final interview (Figure 4.55); also recall that this task was not included on the post-test. Overall, there was little evidence to suggest that Carlos was working from the given symbolic representations to make interpretations about this equation. He had a very despondent tone and was hesitant to respond to questions about this task situation.
Figure 4.55. Carlos considered Task 12 during the final interview but did not make progress in determining the correct solution to the equation $2 - x - x = x + 8 - 3x - 6$.

<table>
<thead>
<tr>
<th>Steps</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.) Original problem</td>
<td>$2 - x - x = x + 8 - 3x - 6$</td>
</tr>
<tr>
<td>2.) Combine like terms</td>
<td>$2 - 2x = -2x + 2$</td>
</tr>
<tr>
<td>3.) Add $2x$ to both sides</td>
<td>$2 = 2$</td>
</tr>
<tr>
<td>4.) Subtract 2 from both sides</td>
<td>$0 = 0$</td>
</tr>
<tr>
<td>5.) Solution</td>
<td>$x = 0$ and $x = 2$</td>
</tr>
</tbody>
</table>

Andy worked out the problem above. When he substituted his solutions into the original equation, he discovered that both $x = 0$ and $x = 2$ make the equation $2 - x - x = x + 8 - 3x - 6$ true.

a) Was Andy’s work correct? Explain why or why not.

b) What is the solution to the equation $2 - x - x = x + 8 - 3x - 6$?

Carlos first thought that Andy’s work was correct, explaining, “honestly I’m just following what he did” (SC_day26_finalinterview_32:03). In his responses, Carlos did not allude to the “Steps” that Andy had performed, and instead was focused on the “Algebra” in the third and fourth lines and the purported solution in the fifth line (Figure 4.55). When prompted to “predict” how many solutions the equation had, Carlos promptly said “None” (SC_day26_finalinterview_32:34). Prompted again to explain his thinking so that it might convince someone else, Carlos said:

Since the $x$ equals zero and $x$ equals two, because if $x$ equals zero how can $x$ equal two? Because (pointing to Andy's work in Steps 3 and 4 [Figure 4.55]) zero would equal zero and two would equal two.

(SC_day26_finalinterview_33:08)

Carlos’ incorrect interpretation that this equation had no solutions was based on a reflection on Andy’s final solution stated in Step 5, and also on a reflection of Steps 3–4. The fact that Carlos was incorrect in interpreting two solutions to mean that the equation
had no solutions is evidence of a prestructural level of RF within the symbolic representation type (P-0, S-X→S). Hinted at above, and in the remainder of the interview, Carlos seemed to be focused on the numeric equations $0 = 0$ and $2 = 2$ to explain why the original equation had no solutions, or rather, why the case of ‘two solutions’ ($x = 0$ and $x = 2$) was seemingly incorrect. After some prompting, Carlos attempted to use his CAS (Line 1–2 from bottom in Figure 4.56).

![Image](image_url)

*Figure 4.56.* Carlos used his CAS to type 0, Enter, and 2, Enter, seemingly related to Andy’s work of $0 = 0$ and $2 = 2$ in steps 4 and 3, respectively.

As is evident below, despite attempts to probe Carlos to explain more, he gave little in return for responses.

1 R: So do you think in this problem that you could use your CAS to convince someone?
2 SC: Ok. (Types 0, Enter, Types 2, Enter [Lines 1–2 from bottom, Figure 4.56]) [...break in transcript...]
3 R: (Points to CAS screen) Does Andy show that in his work?
4 SC: Hmm. Wait, yeah. See (points to the algebra associated with Steps 3 and 4 [Figure 4.55]). [...break in transcript...]
5 R: So you think that those are solutions or that they aren’t solutions? (Referring to 0 and 2)
6 SC: Aren’t. (pause)
7 R: Because there are two of them? Is that what you were saying earlier?
8 SC: Yes.

(SC_day26_finalinterview_34:02–35:18)
Albeit simple, Carlos did use his CAS to create numeric representations within a Calculator page (Lines 1–2, Figure 4.56). He incorrectly interpreted this CAS work to be consistent with Andy’s created representations of $2 = 2$ and $0 = 0$ (line 4 of transcript). Moreover, as evident in the above exchange, Carlos persisted to give an incorrect interpretation that neither “0” nor “2” were solutions to the equation (lines 5–8). Based on the severely limited scope of Carlos’ creation and interpretation of the numeric representation type, his RF was classified at the prestructural level (P-0, N-X>N, CAS).

Carlos had not used his CAS as a tool to work within or move between representations, despite additional prompts to do so. Overall, Carlos did not make further progress in coming to a correct solution to the equation, nor did he demonstrate any higher levels of RF. Carlos was ultimately not motivated to engage with this task. The coding of Carlos’ RF in solving an equation with infinite solutions was primarily based on his interpretations of the given symbolic and numeric representation types.

*Solving an Equation $ax + b = cx + d$ With One Solution: Task 14*

Carlos demonstrated limited abilities and low levels of RF in solving an equation of the form $ax + b = cx + d$ for $x$ with one solution during both the post-test and final interview. In both cases, given the prompt to “Circle all values in the table that represent solutions to the equation” Carlos had worked within the numeric table representation in the same way, circling the $y$-values that matched (see Figure 4.57).
The *x*-value (solution) was not identified in either of these recorded inscriptions (Figure 4.57), an indication of an incorrect use of the numeric table representation to identify a solution to an equation. However, during the final interview, Carlos verbalized that the solution is “some” and the answer is “one,” explained in his own words below.

<table>
<thead>
<tr>
<th><em>x</em></th>
<th><em>y</em> = 11 − 2<em>x</em></th>
<th><em>y</em> = 6 − 7<em>x</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>17</td>
<td>27</td>
</tr>
<tr>
<td>-2</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>-1</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>0</td>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>-8</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>-15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><em>x</em></th>
<th><em>y</em> = 9 + <em>x</em></th>
<th><em>y</em> = 6 + 4<em>x</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>6</td>
<td>-6</td>
</tr>
<tr>
<td>-2</td>
<td>7</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

*Figure 4.57.* Carlos’ use of the numeric representation type to identify solutions to an equation on the post-test and during the final interview, respectively.

Carlos’ RF in using the numeric table to solve the equation $9 + x = 6 + 4x$ is classified at the prestructural level (P-1, N- ->N, P&P). Carlos had incorrectly circled the
y-values instead of the $x$-values, yet had correctly (albeit hesitantly) interpreted the answer to be one (line 8). Carlos had correctly identified that the solution was signified by the point with the same $x$ and $y$ values (line 4) yet did not correctly identify a value of $x$ equals one in his inscription or in his verbalizations (he referenced the “category” in line 12 which could be interpreted to mean “column”).

The second part of the task involved using a symbolic representation type to solve the equation. With some encouragement from the researcher, Carlos reluctantly attempted some symbolic manipulations on the equation $9 + x = 6 + 4x$ (Figure 4.58). When he got to the point of $x = 3x$, he stopped, and asserted that “No” he could not check his solution. Without stating what his solution would be, Carlos’ incorrect creation of equations within the symbolic representation type is representative of a prestructural level of RF (P-0, S-X->S, P&P).

Carlos made no further progress on this task or related sub-tasks that requested a graphical solution approach. Overall, Carlos’ RF within the symbolic representation type was classified at the prestructural level and Carlos’ RF within the numeric table representation type was classified at the prestructural level as well. Despite his success at
identifying the solution within the numeric table, Carlos did not demonstrate any understanding of this work as being connected to his attempt to solve the equation within the symbolic representation type.

Solving a Linear Equation With No Real Solutions: Task 13

On a task that required solving an equation with no solutions (Figure 4.59), Carlos demonstrated a prestructural level of RF within the symbolic representation type. Specifically, Carlos was stuck in the symbolic representation type, not able to perform correct transpositions to solve the equation with success, nor was he able to interpret this work to establish a solution to the equation. Consider the symbolic work Carlos started with during the final interview shown in Figure 4.59.

Solve for the variable and check your solution.

If you use a graph, sketch the section of the graph.
If you use a table, include the table and indicate where you found the answer.
If you use equations, show all of your work.

\[
\begin{align*}
13) \quad x + 2 + 2x &= 5 + 3x - 1 \\
\quad -2x &\quad -2x \\
\quad x + 5 + 1x - 1 \\
\end{align*}
\]

*Figure 4.59.* Carlos performed incorrect transpositions on the symbolic equation \(x + 2 + 2x = 5 + 3x - 1\) and was not successful at solving.

After subtracting \(2x\) from each side of the equation, Carlos recorded the expression \(x + 2 + 5 + 1x - 1\), reasoning because he “combined like terms” (SC_day26_finalinterview_14:34). He then stopped working on the problem and
outwardly expressed that he was stuck and not able to solve this equation using a different approach.

1 R: Can you tell me what you're thinking now?
2 SC: (puts down pen) No, I'm stuck.
3 R: You're stuck?
4 SC: Umm hmm. […]break in transcript…
5 R: Do you think that, um, you could try solving the equation in a different way?
6 SC: Um (15 second pause) hmmm, I don't think so, that I know of.

(SC_day26_finalinterview_14:30/14:53)

Carlos incorrectly wrote an expression instead of an equivalent equation, and was not able to make further progress or try a different approach (lines 2 and 6, above). Thus Carlos’ RF was classified at the prestructural level within the symbolic representation type (P-0, S-X->S, P&P).

The researcher then prompted Carlos to consider using his CAS (without specifying the type of representation to use). Carlos then inserted a Calculator page and performed some symbolic transpositions (Figure 4.60).

![Figure 4.60](image)

Figure 4.60. Carlos used a CAS Calculator page to perform symbolic transpositions but was unsuccessful in interpreting this work.

Despite making some progress in working in the Calculator page, Carlos persisted to be stuck in solving this equation. The researcher prompted Carlos to explain his thinking after executing Lines 7 and 6 (Figure 4.60).
Carlos was able to use his CAS to make some progress toward solving the equation (line 1), and showed some persistence in applying more transformations on the equation (line 8). In both cases however, Carlos felt “stuck” and not able to solve the equation with success (lines 5 and 8, above). Based on the correct creation of equivalent equations using the CAS and his inability to correctly interpret this work, Carlos’ RF was classified at the prestructural level within the symbolic representation type (P-1, S- -> S, CAS). The use of the CAS in this task did not help Carlos solve the equation successfully.

To see if Carlos could use other representations to help overcome this barrier, the researcher explicitly prompted him to use graphs, and to use a table. Carlos’ responses indicated that he could (and would) not attempt such approaches.

Carlos’ purported inability to use either graphs (lines 2–4, 6) or tables (line 6) was not coded at a particular level of RF. Finally, the researcher attempted to assess Carlos
understanding of how many solutions this equation had by using the language of
“prediction.”

1 R: If you had to make a prediction, how many solutions do you think this equation
has?
2 SC: Um, I'd probably say some. Very few.
3 R: Mm. hmm.
4 SC: But other than that I don't think so.
5 R: Why do you think it has some?
6 SC: Because if it was, like exact same, infinite, then it'd be like the same numbers
on the side but if it was never then they'd be just completely random numbers on
the sides of the equals sign.

(SC_day26_finalinterview_19:29–20:04)

Carlos gave some evidence that he was comparing the expressions on either side
of the equal sign to make a conclusion about the number of solutions that the equation
had (line 6, above). However, Carlos did not recognize from his CAS work (Lines 7 or 6,
Figure 4.60) that the coefficients on the variables were the same, and only the constant
terms differed. Again, this exchange did not result in additional evidence of a higher level
of RF, but it does give some indication of Carlos’ understanding of the equation-solving
task, and the fact that there are different possibilities for the number of solutions. Carlos
had drawn on the symbolic representation to correctly interpret that the expressions on
either side of the equal sign were not equivalent, yet came to an incorrect conclusion
about the number of solutions.

Summary of Carlos’ Final RF

Based on Carlos’ attempts during the final interview to solve equations, his final
RF was classified at best at the prestructural level within the symbolic and numeric
representation types. Carlos was very hesitant in working on the tasks during the final
interview yet demonstrated some strengths in attempting to create symbolic
representations, and interpret numeric representations. For example, within the symbolic representation type Carlos was unable to identify a legitimate “first step” toward obtaining a simpler equivalent equation, but would drop the equal sign and write an expression instead of an equivalent equation (e.g., Task 14). Also, Carlos seemed to express an understanding of the different cases of solving equations, but did not correctly identify these with particular equations that he attempted to solve.

Carlos’ Change in RF

In comparing Carlos’ RF from the initial to final interviews, there was generally no change in his RF. In both the initial and final interviews, Carlos’ confidence was quite low, and he made very few attempts to solve the given tasks. In both cases, Carlos’ RF within the symbolic and numeric representation types was classified at the prestructural level. A more detailed examination of Carlos’ (lack of) change in RF in solving problems involving linear equations is determined by a comparative analysis of his RF in solving equations during the initial interview and the final interview. A task-by-task analysis is given next.

Solving an Equation \( c = ax + b \) With One Solution: Task 10

Carlos’ RF in solving an equation of the form \( c = ax + b \) for \( x \) is compared across the initial and final interview and illustrated in Figure 4.61. Similar to Annie and Bryon’s cases, the Rule of Four diagrams are representations of the coding according to the Analytic Framework for RF, listed in the order in which they occurred during the initial and final interviews.
Carlos’ initial and final levels of RF while solving an equation of the form $y = ax + b$ at $y = c$ for $x$ with one solution using MR (Task 10).

Carlos struggled during the initial interview when prompted to use his created representations to solve the equation at $y = 5$. He refused to work on this task. In a later interview task, Carlos attempted to solve the equation $3(x - 6) = 12$ by first incorrectly applying the distributive property to yield an expression, then concluded his work on this problem with an incorrect solution, a prestructural level of RF. During the final interview, Carlos first created an incorrect graph and table for the equation $y = 3x - 2$. When prompted if he could solve the equation $y = 3x - 2$ at $y = 4$, Carlos replied “No” and did not attempt to solve this equation. A similar equation solving task of the form $ax + b = c$ was not presented to Carlos during the final interview, thus no further comparison is made here.

Overall, Carlos didn’t show much progress in demonstrating higher levels of RF in solving an equation of the form $y = ax + b$ at $y = c$ or $ax + b = c$ from the initial to the final interview. In general, during both interviews, Carlos was quite hesitant in solving equations, and despite the relatively “simple” nature of this type of equation, Carlos refused to attempt it.
Solving a Linear Equation With All Real Solutions: Task 12

The characterization of Carlos’ RF is summarized in Figure 4.62 in solving an equation with all real solutions is primarily based on his interpretations of given representations of numeric and symbolic equations, and less on his creation of representations within these types.

![Initial and Final RF Levels](image)

*Figure 4.62. Carlos’ initial and final levels of RF while solving an equation of the form $ax + b = cx + d$ for $x$ with infinite solutions (Task 12).*

During the initial interview, Carlos did not attempt to solve an equation that was posed in symbolic form with infinite solutions. After some additional prompting, Carlos did allude to the meaning of the equal sign in this equation to signify that the equation could be either true or false. However, possibly based on his interpretation of the form of the equation, Carlos came to an incorrect conclusion that the equation was false. Based on the definition of a solution to an equation, the use of the language “true” and “false” was taken to indicate a prestructural interpretation of the meaning of the symbolic
equation. In a similar manner, Carlos did not engage much during the final interview when prompted to interpret a students’ symbolic solution approach to a task that had infinite solutions. Prompted to predict the number of solutions, Carlos said “None” based on his interpretation of the solution which was stated as \( x = 0 \) and \( x = 2 \). The fact that this equation had two purported solutions did not make sense to him, a prestructural level within the symbolic representation. Related to the fact that both “0” and “2” were identified to be solutions and that the numeric equations “0 = 0” and “2 = 2” were evident in the given representations, Carlos used his CAS to create numeric representations, but again was not successful at completing the task, and demonstrated little to no understanding of the meaning of these numeric representation types—a prestructural level of RF.

Based on Carlos’ earlier work in the final interview, and performance during class, it was surprising that he did not use his CAS Calculator page with the actual equations instead of just when dealing with expressions on this task. The similarity between the initial and final interviews was that Carlos had incorrectly interpreted the equations both times to be false or have no solutions. This interpretation seemed to be based initially on his view of the form of the equation and finally on the statement that there were two solutions. Overall, there is no evidence to support that Carlos demonstrated a change in his RF from the initial to the final interview on solving an equation with infinite solutions.
**Solving an Equation** \( ax + b = cx + d \) **With One Solution: Task 14**

In the case of Carlos, his initial and final levels of RF when solving an equation of the form \( ax + b = cx + d \) for \( x \) that has one solution are illustrated using the Rule of Four in Figure 4.63.

![Figure 4.63](image)

*Figure 4.63. Carlos’ initial and final levels of RF while solving an equation of the form \( ax + b = cx + d \) for \( x \) with one solution (Task 14).*

During the initial interview, Carlos attempted to use the numeric table representation type to identify solutions to an equation with one solution, but was largely unsuccessful. It seemed as though Carlos made reference to the pattern observed in the symbolic equation itself to identify a series of numbers circled in the table, but none of these values represented the solution to the equation. Prompted to use other representations to solve the equation, Carlos gave a general statement that it is possible (to use a graph, for example) but did not make progress in using other representation
types. By the final interview, Carlos had given a correct interpretation of the solution to the equation $9 + x = 6 + 4x$ to be “one” but had incorrectly circled the $y$-values in the given table instead of the $x$-value.

Within the numeric table representation, Carlos demonstrated a change in his RF in that he was able to correctly reason that for the table row that had the same $x$ and $y$ values that row contains the solution. However, it was not made clear by Carlos’ circling of the $y$-values (and not the $x$-values) that he really understood the use of this representation to solve an equation with one solution. His understanding seemed to be procedurally based since he was not very confident despite his reasoned conclusion.

In neither the initial nor final interview did Carlos consider using a graphic representation to solve an equation of the form $ax + b = cx + d$ with one solution. However, during the final interview, Carlos did attempt to use the symbolic representation type (albeit unsuccessfully) to solve the given equation. This attempt to solve within the symbolic representation type is a change in Carlos’ RF, for he did not even attempt such an approach during the initial interview.

*Solving a Linear Equation With No Real Solutions: Task 13*

Based solely on his performance on a task during the final interview, Carlos attempted to solve a linear equation with no real solutions. Consider the summary of Carlos’ RF as illustrated in Figure 4.64.
Carlos’ work on solving the equation $x + 2 + 2x = 5 + 3x - 1$ was bounded to the symbolic representation type only. His first approach to this task involved incorrect movement within the symbolic representation type, and he was not able to interpret the meaning of this work with success. After prompting by the researcher to consider using his CAS, Carlos created some correct representations using his CAS, yet again did not interpret this work to be meaningful nor did it help him to solve the equation. In an attempt to move Carlos beyond this prestructural level within the symbolic representation type, the researcher had prompted Carlos to use a graph and/or numeric table representation to solve the equation. Carlos expressed doubt in his ability to do so and ultimately did not attempt using MR.

In summary, Carlos “predicted” that this equation had some solutions, based on the fact that the expressions on either side of the equal sign were not the exact same. While impossible to assess the “change” in RF that Carlos demonstrated in solving an equation with no solutions, Carlos did attempt to work within the symbolic representation type, and made some progress in using his CAS to perform symbolic transpositions.
Overall, the meaning of the symbolic representation type was not meaningful to Carlos nor was it connected to the use of other representations.

**Summary of Carlos’ Change in RF**

Despite the fact that Carlos persisted at the prestructural level of RF across all equation solving tasks, the above analysis highlighted a few episodes that indicated some level of change, albeit small, in Carlos’ RF. For example, during the initial interview Carlos’ identification of values within the table representation type was not significant to the meaning of the solution. By the final interview, Carlos had recognized the significance of the $y$-values being the same for a corresponding $x$-value, despite the incorrect circling within this representation type. Another example of a change in Carlos’ RF was in regards to his attention to the different cases for possible solutions (infinite, some, or none). Carlos was not successful at identifying the equation in Task 12 as having infinite solutions, nor was he successful in identifying the equation in Task 13 as having no solutions. Carlos did convey that Task 14 had “some” solutions, instead of actually only having “one” solution, at a value of $x = 1$.

Finally, Carlos’ did not use his CAS to graph or create tables for the given symbolic equations, and instead persisted in the CAS calculator page for work within symbolic and numeric equations. In neither the initial nor final interviews did Carlos use the digital tool available to really aid in his ability to solve the equations.
CHAPTER V

FINDINGS: TEACHING EXPERIMENT

This chapter is devoted to answering the research question:

*Under what conditions does a group of ninth-grade algebra students change their RF in solving problems involving linear equations within a CAS and P&P environment?*

There are three components involved in answering this question:

a) What was developed as the conjectured instructional theory?

b) What instructional theory was tested during the teaching experiment?

c) What is a revised instructional theory based on empirical evidence from the experimentation?

The first component was detailed in Chapter 3 in the section titled *Conjectured Instructional Theory*. The second component will be addressed below as the results of the teaching experiment. The final component will be addressed in the discussion in Chapter 6, but is also reflected in the refinements that were made in the components of the instructional theory that are presented below. That is, revisions to the instructional theory were made during the retrospective analysis process (i.e., code refinements) and are also a result of that process.
Conditions of Learning Environment

Conditions of the learning environment in which a group of ninth-grade algebra students changed their RF in solving problems involving linear equations with both CAS and P&P are summarized according to the instructional theory. Two main components of this theory were the activity structure and learning progressions. A summary of these components across the entire set of 14 Activities and the 24 Days of the teaching experiment are included in Appendix H and Appendix I, respectively. Note that learning progression component “E. Equivalence of equations”:

Equations are equivalent if they have the same solution set. Represented graphically, solution sets of equivalent equations are x-coordinates of the intersection points in the coordinate plane. Represented in tables, solution sets of equivalent equations are the inputs for which the outputs are the same. was excluded from the analysis because the teaching experiment did not cover this content. More specific results from each of the Activities are outlined in subsequent sections to give a great sense of the expansive variety of and relationships between tasks, techniques, and theoretical components of the learning trajectory that were tested.

Note that in the presentation of results sometimes verbatim student dialogue was not able to be included (it was not always audible from the video camera, nor from an additional audio-recording device placed at the front of the room). However, the researcher’s observational field notes and the teacher’s practice of a “re-voicing” strategy were drawn on to give the most accurate depiction of student thinking possible. On this point, as will become evident in the presentation of classroom interactions, teacher-dominated mathematical activity was the norm. In some cases, the teacher interpreted the meaning of representations without student input. In other cases, the teacher may have re-
voiced student ideas in a way that extended or moved beyond their thinking without additional student input. This point is noted here as a pre-cursor to interpreting the results of the teaching experiment; It will also be discussed in Chapter 6 in relation to how the teacher may have impacted the instructional theory. Finally, pseudonyms for student names are used, and in some cases, dialogue is paraphrased instead of directly quoted.¹

Tested Instructional Theory

The presentation of the tested instructional theory will be organized according to a Task-Technique-Theory framework that articulates critical moments evident in the daily instructional experimentation. The structure shown in Table 5.1 will be used for each activity discussed below.

Table 5.1
Template for Task-Technique-Theory (TTT) Framework

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective of classroom mathematical practices</td>
<td>Means by which students accomplished the task</td>
<td>Mathematical ideals underpinning tasks</td>
</tr>
</tbody>
</table>

Data: Daily learning goal, lesson plan, classroom field notes, enacted learning progression

Data: Enacted Learning Progression Summary, Activity Structure Rule of Four, student activity work, transcription from class discussion

Data: Activity Sequence and Learning Progression with evidence from transcript and field notes, Connections to Literature

Task Summary

Mathematical activity that was enacted as a whole class (discussed in classroom interactions); what was enacted as a whole class

Data: Summary of activity design including HW, Excerpts from activity/student work, board work, screen shot, CAS pages

¹ Not all students in the class gave assent nor had parental consent to be captured on the classroom video. In instances in which those students played a significant role in classroom interchanges, the interaction was paraphrased instead of directly quoted.
Each of fourteen activities will be discussed in turn with attention to the learning goal(s), CAS and P&P techniques, theoretical underpinnings of design, and tasks that were discussed in classroom interactions. Consistent with the designed instructional theory, the activities are grouped into three chapters, each with 7, 4, and 4 activities each, across a total of 12 ½, 2 ½, and 7 days, respectively. See Table 5.2 for a summary of the sequence of activities by chapter and day.

Table 5.2
Outline of Teaching Unit by Chapter, Activities, and Day(s)

<table>
<thead>
<tr>
<th>Chapter 1</th>
<th>Create and Interpret the Meaning of Equivalent Expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity 1</td>
<td>Multiple Representations of Tile Patterns (Days 1–4)</td>
</tr>
<tr>
<td>Activity 2</td>
<td>Equivalent Expressions with Symbolic Representations (Days 4–7)</td>
</tr>
<tr>
<td>Activity 2.5</td>
<td>Graphs, Tables, and Symbols for Equivalence of Expressions (Days 4, 6–8)</td>
</tr>
<tr>
<td>Activity 3</td>
<td>Translations Among Words, Numbers, &amp; Symbols (Days 7–8)</td>
</tr>
<tr>
<td>Activity 4</td>
<td>Evaluating Expressions with CAS and Paper-and-Pencil (Days 8–10)</td>
</tr>
<tr>
<td>Activities 5–6</td>
<td>The Distributive Property With P&amp;P and CAS (Days 11–14)</td>
</tr>
<tr>
<td>Chapter 2</td>
<td>Equations are Equivalence Relations that are Sometimes, Always, Never True</td>
</tr>
<tr>
<td>Activity 7</td>
<td>Equations – Sometimes, Always or Never Part I (Day 13)</td>
</tr>
<tr>
<td>Activity 8</td>
<td>Equations – Sometimes, Always or Never True (Days 15–16)</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Solving Linear Equations with Graphs and Tables, and Algebraic Symbols</td>
</tr>
<tr>
<td>Activity 8.5</td>
<td>The “Cartesian Connection” (Days 16–17)</td>
</tr>
<tr>
<td>Activity 9</td>
<td>Linear Equations: Solving with Tables &amp; Graphs (Days 16–17, 20)</td>
</tr>
<tr>
<td>Activity 10</td>
<td>Solving Problems Involving Linear Equations (Day 18–19)</td>
</tr>
<tr>
<td>Activities 11–12</td>
<td>Solving Equations with CAS as a Representational Toolkit (Day 19–21)</td>
</tr>
<tr>
<td>Activities 12–14</td>
<td>Strategies for Solving Linear Equations with One, None, and Infinite Solutions (Days 22–23)</td>
</tr>
</tbody>
</table>

On a daily basis, the structure of each lesson tended to follow the same general flow: (1) warm-up, (2) review, (3) lesson, (4) closure, and (5) homework. This structure was reflected in the teacher’s daily lesson plan. A sample of which is provided in Figure 5.1 for Day 4.

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2 For clarity in the presentation of results, some activities were combined (e.g., Activities 5–6) and some new Activities were “created” (e.g., Activity 2.5). The original activity names are 1–14, and a total of 13 clusters of activities are discussed as separate sections.
Lesson Plan - Algebra A

Lesson Title: Equivalent Expressions with Symbolic Representations

Lesson Goals: Students compare expressions by algebraic manipulation and test for equivalence.

<table>
<thead>
<tr>
<th>Time</th>
<th>Activity/Procedures</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 mins.</td>
<td>1. Warm-Up: Take out HW – Big C Pattern – Write the equation for the pattern (if you have not done so) and explain how you came up with it.</td>
</tr>
<tr>
<td>20 mins.</td>
<td>2. Review: HW (Big C Pattern) Return T-pattern</td>
</tr>
<tr>
<td>5 mins.</td>
<td>3. Lesson: • Vocab: variable: a quantity that can change expression: a mathematical phrase that contains variables and/or numbers equation: a mathematical sentence that relates two expressions by an equals sign equivalent expressions: define the same pattern and can be shown by the same rule, graph and table • Activity 2: Equiv. Expressions w/Symbolic Representations ▪ #1 and #2 ▪ Debrief: CAS screen shows expressions are equivalent</td>
</tr>
<tr>
<td>5 mins.</td>
<td></td>
</tr>
<tr>
<td>15 mins.</td>
<td></td>
</tr>
<tr>
<td>10 mins.</td>
<td>4. Closure: Preview HW Turn in CAS Exit Ticket: Simplify expressions</td>
</tr>
<tr>
<td></td>
<td>5. HW: Handout: Equivalence of Algebraic Expressions</td>
</tr>
</tbody>
</table>

Assessment Strategy: Circulate to answer questions – visual cues

Special Needs Accommodation: Pencil/paper handout clearly written

Materials: CW and HW handouts, vocab sheets CAS

Figure 5.1. Sample of the teacher’s daily lesson plan with typical daily structure.
After each daily lesson, the teacher and researcher met for a daily debriefing session. The structure of this session was captured in the following excerpt from the researcher’s journal on Day 16:

I first ask Ms. L how she thinks the lesson went, and why. Then I share some comments building on hers. Then we usually talk about some of the understandings of the students, where they are at and what we can build on for the next day’s lesson. During this part of the conversation I try to infuse as much “theory” as possible with respect to the learning trajectory, learning progression, particular means of support, etc. For instance, when Ms. L mentioned that we were using the CAS strategically, I asked her what that meant to her. She explained that she thought the CAS was affording access to some information that was not possible with other tools (such as TI-83 or paper and pencil).

Recall that the goal of design research is to “give situated accounts of learning that relate learning to the means by which it can be supported and organized” (Cobb et al., 2003, p. 13). Thus the instructional theory that was tested during the teaching experiment is presented next with evidence from the interactions evident in the learning environment.

A table that outlines the tasks, techniques, and theory is given to set the stage for the results of each activity; means of support such as classroom expectations are discussed across the set of activities as a whole.

Create and Interpret the Meaning of Equivalent Expressions

The first thirteen days of the teaching experiment were concerned with MR of equivalent expressions. This work involved two major subtopics including: (a) the creation of and movement between words, symbols, graphs, and tables of equations and expressions, and (b) the verification of equivalence of expressions using graphs, tables, numeric and symbolic representations. Seven activities were designed and implemented during the teaching experiment.
Activity 1: Multiple Representations of Tile Patterns

The overall design of Activity 1, summarized in Table 5.3, was intended to provide a meaningful context for students to create and interpret tables, graphs, and symbolic representations of equivalent expressions with both CAS and P&P. The use of tile patterns as a motivation for understanding equivalence of expressions was inspired by CMP materials that students might have encountered in middle school\(^3\) and it was conjectured that the introduction of equivalent expressions using graphs and tables would support students in coming to have a meaningful understanding of equivalence.

The first activity spanned three days in which the focus of classroom activities was more on creating correct representations and making connections between tables, graphs, and symbolic equations that represent tile patterns than on the equivalence of expressions. To complement the first problem of the activity (Figure 5.2), warm-ups, exit tickets, and homework were specifically directed at translation activities with P&P. CAS use was limited to creating graphical representations (Figure 5.2, Part c) and to a reconciling activity between CAS and P&P graphs (Figure 5.6, Part h).

\(^3\) Recall that from teacher report data, most student participants took an eighth-grade math class in which CMP was the adopted curriculum series.
Table 5.3

TTT for Activity 1: Multiple Representations of Tile Patterns

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Students use P&amp;P and a CAS to make connections between linear tile patterns, tables, graphs and equations. (Day 1, Day 2, Day 3)</td>
<td>• Translate contextual situations (i.e., linear tile patterns) into patterns represented graphically and numerically with P&amp;P and CAS; • Make connections between graphic and numeric representations with P&amp;P; and • Translate from numeric and graphic representations of linear tile patterns to symbolic rules with P&amp;P.</td>
<td>A1. Connecting and generalizing the quantitative, visual, and verbal with symbolic. A2. Different representations/representation types can signify the same object. C. Domain and range restrictions. C1. Equal Sign: Assigns Variables Rules/Names for Patterns</td>
</tr>
</tbody>
</table>

Task

Day 1 – Extend a tile pattern and create a graph and table to represent the number of tiles around the perimeter of the “garden” \( T = 4S + 8 \).

Day 2 – Create a table from a graph, translate to a symbolic rule that represents both \( y = 3x + 3 \); solve problems involving the garden tile pattern \( T = 4S + 8 \).

Day 3 – Translate from a table \( \{(-2,9), (-1,7), (0,5), (1,3), (2,1)\} \) to a graph; Create a tile pattern and move to numeric and symbolic representations \( T = 3S + 5 \); use CAS to graph \( f1(x) = 4x + 8 \) and reconcile differences with P&P graph.

Day 4 – Given Figures 1, 2, and 3 of a “Big C” tile pattern, create Figures 0 and 5. Complete the table and graph of Figure Number and Number of Tiles for Figures 0–5. Write an equation that represents the pattern at the bottom of the page [“Big C” pattern]. Tell how you figured it out \( T = 6S + 3 \).

Translations with P&P and CAS

The translations in Activity 1 were primarily accomplished using P&P, with some introductory activity using CAS to graph symbolic equations. During a discussion of the warm-up activity on Day 2, the table representation was created and interpreted, highlighting a translation from the graph to the table representation (shown in Figure 5.3). The teacher also elicited student responses to translate from graphic and numeric table representations to a symbolic equation.
Activity Day 1: Multiple Representations of Tile Patterns

1. Suppose you are planning to create a square vegetable garden and want to use square tiles around the border. The number of tiles you use depends on the size of the garden. The first three stages of this pattern are shown below.

![Stage 0 to Stage 3](image)

a. Draw the next picture that fits this pattern.

b. Complete the table for the Stage (S) of the tile pattern and the Number of Tiles (T) needed around the border.

<table>
<thead>
<tr>
<th>Stage of Pattern (S)</th>
<th>Number of Tiles (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

c. Next to the table, create a graph that matches the table. Be sure to include appropriate labels on each axis.

d. Is your graph connected (your pencil would not need to lift off the page to trace it) or disconnected (a collection of points that are not touching each other)? Explain why.

Figure 5.2. The first tasks of Activity 1 involved the creation of tile patterns, numeric tables, and Cartesian graphs.

1  Ms. L: We're talking about how would you write an equation for this [...]4 What about the table? David?
2  S: It's adding by three.
3  Ms. L: So you're saying this looks linear like a straight line [sketches line on disconnected graph, Figure 5.3], this is going up by three [writes + 3 next to y-values in table]. So what kind of equation would describe this? Kenneth?
4  Kenneth: $y = 3x + 3$ Ms. L: Wow. […]
5  Ms. L: What is the y-intercept? [points to the point (0, 3) on the graph in Figure 5.3]
6  Kenneth: At three.
7  Ms. L: How can you see that in the table?
8  Kenneth: Where the 0 equals 3.
9  Ms. L: You got it. Wherever x equals zero.

(Day 2, 00:14:14:24)

4 Indicates a break in the transcript.
Ms. L scaffolded students’ experiences in identifying the slope or growth factor (lines 1–3) and y-intercept or constant term (lines 6–9); both the graph and the table were used to move to the symbolic representation and in the interpretation of the meaning of the symbolic representation with respect to its slope or constant growth rate and y-intercept (line 3, lines 7–9). This is a translation from the graph and table to a symbolic equation.

Figure 5.3. The creation of a table from a graph, and the use of both representations to identify slope and y-intercept in the translation to a symbolic equation.

For another P&P example, in the following exchange, a student had translated from a geometric tile pattern to the numeric values representing the number of tiles. This representation was then used to translate to the symbolic representation.

The stage was going up by three, S is for stage, T is for tiles […] the stage is going up by three each time, the number of tiles we started with was 5.  
(Day 3, 00:54:40:38)

Ms. L later follows up with a comment that “It's really helpful to put those numbers (of tiles) next to the pictures” (Day 3, 00:57:32:30). It seems as though the students’ translation to a symbolic representation was based on the translation from a geometric tile
pattern to a numeric representation. The creation of these representations is illustrated in one student’s homework exercises (Figure 5.4). Note the numbers written below the tile patterns in the image.

![Image of a geometric tile pattern](sb_day3_activity.pdf)

**Figure 5.4.** The creation of a numeric pattern from a geometric tile pattern was used to translate to a symbolic equation.

Another translation activity occurred during the classroom discussion of the warm-up for Day 3 which was focused on the features that were the same across a table and graph, namely that the points matched. Ms. L explained:

But we do have, the points do match up [points to each table value and each corresponding point on the graph] he's got seven and negative one (–1,7), he's got x equals 0, y equals 5.

(Day 3, 00:17:33:66)

Recognizing the invariant feature of having the same points in the graph and the table is evidence of a connection across the numeric table and the Cartesian graph representations. An example of a student-generated graph that corresponds to the warm-up task is shown Figure 5.5b.
Figure 5.5. The warm-up activity from Day 3 (a) made the numeric to graphic translation explicit and (b) correct student work shows a graphic representation that was created from the numeric table.

Overall, the warm-up design was intended to showcase connections between the graphs and tables with opposite translations (first graph to table, second table to graph).

The use of CAS was evident during the activity on Day 2 in which the teacher followed the task design in helping students learn the technique of using CAS to graph an equation (Figure 5.6).
2. You have discovered a relationship between two variables, $S$ and $T$, in which the number of tiles depends on the stage. An equation relating variables that depend on each other is called a function. Let's graph this function.
   a. Turn on your CAS and press Home $\Rightarrow$.
   b. Press $\mathbf{1}$: New Document.
   c. Do you want to save changes? Press No.
   d. Add a $\mathbf{2}$: Graphs Page.
   e. This is where you type in expressions using the variable $x$.
   \[
   \begin{array}{c}
   n(x) = \quad 5 \\
   \end{array}
   \]
   How would you enter the equation you wrote in part (g) in this box? $-x + \frac{5}{3}$ Try it!
   f. Press Enter to graph.
   g. Hide or show this equation line by pressing $\Rightarrow$ G for graph.
   h. Split the screen with a table by pressing $\Rightarrow$ T. Compare your graph and table with the ones you created in Problem 1. What do you notice?

Figure 5.6. Instructions students followed on how to use CAS to graph an equation.

The activity structure surrounding the CAS work to move from a symbolic representation to a graphic representation was focused on reconciling the differences between the P&P and CAS based representations. In particular, the graph created in Problem 2f (Figure 5.6) was compared against the graph created by hand using P&P in Problem 1b (Figure 5.2). The goal was to get the CAS graph to “look like” the graph they had drawn on their paper with a focus on the window dimensions (not on the connectedness of the graph).

Ms. L: What's the difference between what you see on your calculator and what you see on the front page? Is there any difference or is it exactly the same? Angela said that it's exactly the same. Anyone have a different opinion?
S: It's not exactly the same.
Ms. L: Why?
S: Because.
Ms. L: How is that different than what the drawing was?
S: They're not the same.
Ms. L: Why not. What doesn't look the same?
S: The number line.
Ms. L: The scale is a little different. This one goes from negative 6.67 to positive 6.67. Does the table match up? Yes. Anything else different about the graph, besides the scale?
S: Quadrants.
Ms. L: Ok, this one [points to CAS graph] doesn't have all four quadrants, this one [points to P&P graph] is just in the first quadrant. The other one is just the first quadrant. [interruption] Can you make this graph [CAS graph] look like your other one [P&P graph]? It's kind of tricky actually. [Ms. L shows how to drag axes using teacher software] See if you can move your graph around. See if you can get your graph to show the first quadrant. [...] We wanted to make a connection between what we did by hand and what we can do with the calculator. And we wanted to see that it was really the same thing but it's giving us just another tool to use it.

(Day 2, 00:44:24:51 & 01:06:48:93)

Seeking consistency between P&P and CAS graphs seemed important. However, the quadrants of the Cartesian graph were emphasized over the notion of connectedness, possibly a more salient point of the graphs that was not discussed in relation to the context of the tile pattern. In other words, the teacher saw the P&P and CAS graphs to be the same despite the fact that the CAS showed a connected graph with no domain or range restrictions while the P&P graph showed a disconnected graph with domain and range restrictions to whole numbers only.

*Theoretical Underpinnings*

Overall, the work on Activity 1 was centered on the activity sequence of creating and moving between graphic, geometric, tabular, numeric, and contextual (verbal) representations of tile patterns. The translation activity from numeric and graphic representations to symbolic representations (see Figure 5.3) was also grounds for the learning progression aspect of “A1. Connecting and generalizing the quantitative, visual, and verbal with symbols.” In one notable exchange, a student had realized that it would be simpler to use the symbolic representation as opposed to continuing with repeated iterations of using the numeric pattern. Ms. L commended the students for such insight.
So smart. [...] It would take a long time to add that all up in the table. So I can use the equation to figure it out. The equation is a shortcut that represents the same values in the table and the pattern you're seeing in the tiles.

(Day 3, 00:58:10:04)

In the above excerpt, the efficiency of using the symbolic representation to generalize the numeric pattern was recognized.

During the warm-up on Day 2 the task had prompted students to explicitly identify things that were the same and different across the graphic and numeric table representations they had created for the tile pattern (Figure 5.2). In the whole-class discussion of these prompts, the learning progression component of “A2. Different representation types signify the same object” was evident for the linear equation of focus.

1 Ms. L: So we have two different representations that give us the same information [...] Anything else that’s similar?
2 S: [inaudible]
3 Ms. L: He’s saying that you can write an equation from the table, and you can write an equation from the graph.
4 S: I said that they both have an x and a y.
5 Ms. L: Yes in the graph in axes form and in the table an x-column and a y-column.

(Day 2, 00:10:16:75)

Ms. L had articulated the “sameness” of the graph and table representation (line 1), and with additional prompting a student recognized that both of these representations could be used to write an equation (lines 2–3). The recognition of the x- and y-axes or column labels is part of the convention of creating representations that was made explicit by Ms. L (lines 4–5).

The tile pattern context served as an impetus behind addressing an aspect of the learning progression that dealt with “C. Domain and range restrictions.” The teacher explicitly noted that the graph of the stage number and number of tiles should be disconnected to more accurately portray the situation (there can be no ½ stages) (Day 1,
In a later exchange, the teacher asked students why they connected the points on their graph, to which a student remarked that they are linear. Ms. L then summarized “It looks like a straight-line pattern [...] I didn't tell you the situation so you wouldn't know if they should be connected or not connected” (Day 3, 00:19:01:39). By the nature of the activity design, the domain and range restrictions were related to the context of the linear tile patterns in which it didn’t make sense to have non-whole number values.

The domain and range restriction aspect of the learning progression that was also addressed continued in the discussion of the “C” problem homework task in which students created a table and graph and symbolic rule. The creation of the graphic representation was discussed during the warm-up of Day 4. One student discussed their point plotting strategy, then after a quick poll of the class the teacher explained why the graphs should be disconnected.

Yeah, ok kind of the impulse is to connect them. But let's look at the context. Let's look at the situation. Do we have figure 1.5 on this, in this example? Do we have a figure 1 and 1/2? No, we don't have a figure 2 1/2 either. So when we've got tile patterns lots of times we leave it disconnected, so if you have points on your graph it's probably more accurate than connecting it. Just like our garden tiles, if we just have this size and this size and no sizes in between then we're just going to have discrete or individual points.

(Day 4, 00:10:44:15)

The teacher emphasized thinking about the discrete nature of the tile pattern context in creating a correct graphic representation.

Finally, the equal sign was used to describe relationships between the number of tiles and the stage number. Students wrote equations such as \( T = 3S + 5 \) to signify this relationship (Part b in Figure 5.4). These rules were also used to create a numeric table of values and a graph (see Figure 5.5). This is connected to the learning progression of “C1.
Role of Equal Sign: Assigns Variables Rules/Names for Patterns that can be Graphed, and viewed as Tables for Contextual Situations.”

Activity 2: Equivalent Expressions with Symbolic Representations

The instructional design tested on Days 4–8 spanned several tasks, techniques, and theoretical underpinnings regarding equivalent expressions (see Table 5.4). Two clusters of activities arose from the retrospective analysis: (a) equivalence of expressions within the symbolic representation type, and (b) equivalence of expressions in graphs, tables, and symbols. These topics were deliberately separated in the presentation of results into Activity 2 and Activity 2.5, respectively. This decision was made because equivalence of expressions from graphic and numeric representation types was intended to foreshadow equivalence of expressions from the symbolic representation type, but was tested in the reverse order.

There were three tasks within Activity 2 in which students used both CAS and P&P to perform the same technique of combining like terms, two of which were directed at the goal of simplifying expressions to test for equivalence (by common symbolic form). The first was enacted as the P&P and CAS work on Task 1 (Figure 5.7), the second was enacted as part of a warm-up activity on Day 6 (Figure 5.8). The third spanned Day 4 and Day 6 as Tasks 3 and 4 (Figure 5.10) and will be discussed as part of Activity 2.5 as the goal in that activity was to use graphs, tables, and symbols to verify equivalence.
Table 5.4
*TTT for Activity 2: Equivalent Expressions with Symbolic Representations*

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Students compare expressions by algebraic manipulation and test for</td>
<td>• Compare expressions by algebraic manipulation (P&amp;P common form,</td>
<td>B. Equivalence of expressions from multiple representations.</td>
</tr>
<tr>
<td>equivalence. (Day 4)</td>
<td>combining like terms, simplifying expressions);</td>
<td>C2. Role of Equal Sign: Identity between equivalent expressions.</td>
</tr>
<tr>
<td>• Students compare expressions by algebraic manipulation, graphs, and</td>
<td>and</td>
<td></td>
</tr>
<tr>
<td>tables and test for equivalence. Emphasis on Predict/Act/Reflect/Connect.</td>
<td>• Test for equivalence by re-expressing the form of an expression (CAS</td>
<td></td>
</tr>
<tr>
<td>(Day 6)</td>
<td>auto-simplify to identify pattern).</td>
<td></td>
</tr>
</tbody>
</table>

Task Summary

Day 4 – Use P&P to combine like terms \((x, x + x, x + x + x, x + x + x + x; x + y, x + y + x + y, x + y + x + y + x + y),\) use CAS to help identify a pattern in the simplified forms; and define vocabulary terms “variable,” “expression,” “equation,” and “equivalent expressions.”

Day 5 – Determine if \(2x + 7\) and \(x + x + 3 + 4\) are equivalent expressions or not, explain.

Day 6 – Use P&P to write an equivalent and simplified form, and use CAS to check \((x + 3x + 2x, x + x + x + y + x + x, 4x – y – 2x + y, –2x + 2x, 0*y).\)

Day 7 – Activity 2 HW Simplify the expressions; identify two pairs of expression that are equivalent; and explain why they are equivalent: a. \(2x + 3x + 4x + 5x,\) b. \(x + x + x + x + x + x + x + x + x + x + x + x + x + x + x + x,\) c. \(-7x + 5 – 3x – 8,\) and d. \(-7x – 3x + 5 – 8.\)

The homework tasks for Activity 2 were discussed on Day 7 from a symbolic perspective only with an emphasis on combining like terms using P&P, but no CAS.

Because of the emphasis on symbolic transpositions, these are listed as Tasks in Table 5.4. However, the discussion of what was enacted on Day 7 is further elaborated in Activity 2.5 with respect to the definition of equivalent expressions. The focus of the next section is on the combined use of CAS and P&P to perform symbolic transpositions on linear expressions.
**Activity Structure: Transpose CAS and P&P**

First, for Task 1, the activity design called for P&P work before CAS work. Ms. L was specific in the direction to “not use your calculator right away. Fill out column one and column two and then we're going to do problem two and use our CAS to see the pattern” (Day 4, 00:44:25:74). After individual activity time, the teacher led a discussion of students’ P&P work in Column 2 (see Figure 5.8) before assisting them with CAS techniques. This interchange focuses on Katrina and Ms. L’s interpretations.

1. Katrina: That'd be $x + y$, and two $x$ plus $y$ [2x + y] and three $x$ plus [3x + ] oh I mean two $x$ plus two $y$ [2x + 2y] and three $x$ plus three $y$ [3x + 3y] and four $x$ plus four $y$ [4x + 4y].
2. Ms. L: Ok [writes 2x + 2y, 3x + 3y, and 4x + 4y on overhead] [...] So Katrina how did you figure that out?
3. Katrina: Because two $x$s is two $x$ and like three $x$s [x + x + x] is three $x$ [3x].
4. Ms. L: So part of it is just like the previous one [Column 1, Figure 5.7]. So what's different between this one and this one [Column 1 and Column 2, Figure 5.7]?
5. Katrina: That one has another variable [Column 2, Figure 5.7].
6. Ms. L: Ok, so it has another variable [Column 2, Figure 5.7], it has a $y$. So we were adding apples in Column 1 and adding apples and bananas in Column 2. (Day 4, 00:51:17:14)

The above interchange in an example of transposition within the symbolic representation type because the focus is on how Katrina created these representations (or simplified the expressions, lines 1–2) and what it means (combining like terms, lines 3 and 6). In the subsequent interchange, the teacher focused on the creation of symbolic representations without interpreting their meaning, an action within the symbolic representation and using CAS to check:

I want you to put in everything you did in column 2 and see if it matches up with what we wrote. So I'm going to put in $x + y$ and hit Enter, and what does the CAS tell you? [...] Explains that CAS work should be written in CAS screen on paper] Now we want to put in 2x + 2y, well, actually, $x + y + x + y$. [...] teacher demonstrates CAS technique of using previous result to add more terms before
combining like terms again for $x + y + x + y$ and $x + y + x + y + x + y$] OK so for the next part, oh the question says, describe the pattern you see. So on the lines next to the pattern, what's changing. Put it in words.

The teacher’s call to “see if it matches up with what we wrote” was using CAS to check P&P results, and the dialogue was directed at the technique of how to enter expressions into CAS using previously computed results.

It is important to note that despite the correct interpretation from Katrina and the correct demonstration of CAS technique by Ms. L, this did not seem sufficient for all students to obtain the correct simplified expressions for this task. For instance, each of Annie, Bryon, and Carlos had incorrect results recorded in the P&P portion of this worksheet (Column 2) despite having correct CAS results (e.g., Figure 5.7). In other words, the differences obtained in the CAS work did not prompt students to go back and fix their P&P transpositions. The subsequent discussion of this work also did not get into this matter.
Activity 2: Equivalent Expressions with Symbolic Representations

1. Use paper and pencil to simplify the expressions below (combine like terms) Share the work with a partner.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Column 1</th>
<th>Expression</th>
<th>Column 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>x</td>
<td>xy</td>
<td>x+y</td>
</tr>
<tr>
<td>x+x</td>
<td>x²</td>
<td>x+y+xy</td>
<td>x+y³</td>
</tr>
<tr>
<td>x+x+x</td>
<td>x³</td>
<td>x+y+xy+xy</td>
<td>x³+y³</td>
</tr>
<tr>
<td>x+x+x+x</td>
<td>x⁴</td>
<td>x+y+xy+xy+xy</td>
<td>x⁴+y⁴</td>
</tr>
</tbody>
</table>

2. Let's use our CAS to help see the pattern in the Simplified Expression above in Column 2. Type in each expression in Column 2 and describe the pattern you see.

For the warm-up activity on Day 6 (see Figure 5.8), the design of the activity was enacted so that students used P&P to act or work within the symbolic representation then used the CAS to check the correctness of the simplified expression. Ms. L directed:

Adam says 3x + 2y. Now go to your CAS. I asked you to 'check it out' on your CAS so go to your calculator and see if you get 3x + 2y [interruption]. Is that the same thing that we wrote? Let's write it, 3*x + 2*y. Yeah, your CAS calculator automatically puts in a multiplication sign without even asking. I didn't type in the multiplication. So those mean the same thing.

Figure 5.7. P&P and CAS transpositions of symbolic representations were not reconciled by all students.

(Day 6, 00:20:23:63)
The CAS was explicitly treated as a tool to check P&P symbolic manipulations. Then Ms. L’s interpretation that the expressions with and without the multiplication mean the same thing (multiplication is implicit when written by hand, and made explicit when computed using CAS) supports the symbolic transposition code.

\[ x+y+x+y+x = 3x+2y = 3 \cdot x + 2 \cdot y \]  

(day6_T_activity_wu.pdf)

Figure 5.8. CAS task to check P&P technique of combining like terms.

Theory: Role of Equal Sign

Within the discussion of vocabulary, the teacher chose to illustrate the vocabulary term “Equivalent Expressions” with an equation that represented an identity between equivalent expressions (see Figure 5.9). This was the first introduction students had to the meaning of an equal sign as an equivalence relation or identity between two equivalent expressions. Note, however, that this particular role of the equal sign was not discussed in this nature thus its meaning as an equivalence relation remained tacit at this point (Day 4, 00:33:42:86).
However, in a later exchange related to the warm-up task on Day 5, a student justified equivalence based on an equation that represented an identity. Ms. L re-stated this reasoning as follows:

She says those are equivalent because it's exactly the same thing on each side [On board: $2x + 7 = x + x + 3 + 4$ then $2x + 7 = 2x + 7$]. They're exactly the same on each side of the equation so they're equivalent [interruption]. If I simplify it I get the same thing on both sides of the equals sign.

(Day 5, 00:07:31:88)

The learning progression aspect of the role of the equal sign as representing an identity between equivalent expressions was added during the teaching experiment because it was seen to be connected to the learning goal of understanding the equal sign as representing a relation between expressions. The seed idea of this learning goal is also evident in the way the term “Equation” was defined (see Figure 5.9).

Activity 2.5: Graphs, Tables, and Symbols for Equivalence of Expressions

Equivalence of linear expressions can be determined by reasoning about graphs, numeric tables, and symbolic representation types. The tasks discussed as part of Activity
2.5 specifically involved the use of more than one representation type to explain the equivalence of linear expressions. Table 5.5 summarizes this activity work.

Table 5.5  

<table>
<thead>
<tr>
<th>Learning Goals</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Define equivalent expressions; and • Gain experience in applying the definition of equivalent expressions.</td>
<td>• Verify equivalence using Graphs, Numbers (Tables or Numeric Evaluation), and Symbols (Rules); and • Verify non-equivalence of Symbolic Expressions or using CAS graphs and tables.</td>
<td>B. Equivalence of expressions from multiple representations.</td>
</tr>
</tbody>
</table>

Task Summary

Day 4 – Definition of equivalent expressions and example with T tile pattern, $3x + 5$ and $1x + 2x + 5$, P&P symbols, table, and graph. Verify the equivalence of $x + x + x + x + x$ and $2x + 3x$ with P&P Symbols.

Day 6 – Verify the equivalence of $x + x + x + x + x$ and $2x + 3x$ with CAS symbols, graph and table.

Day 7 – WU with $4x – (x + 2)$ and $3x – 2$ with CAS graph and tables, and P&P symbols.

Day 8 – WU with $5x – (–x + 2)$ and $2x + 2$, CAS graph and tables, equal in table, connection to graph.

Contrary to the conjectured instructional design, the verification of equivalence from graphic and numeric representations occurred after students had completed tasks that focused on equivalence from a symbolic representation type. The “Equivalence of expressions from MR” part of the learning progression was introduced on Day 4 in a discussion of a way to illustrate the vocabulary term “equivalent expression,” and also on Days 4 and 6 with a focus on the equivalence of “$x + x + x + x + x$” and “$2x + 3x$” using P&P symbols, and CAS graphs, tables, and symbols, respectively. Finally, the warm-up tasks on Day 7 and Day 8 were designed to emphasize the use of graphs and tables to justify equivalence of expressions.
The definition of equivalent expressions was discussed on Day 4 in the context of two expressions representing a “T” tile pattern. As the illustration in Figure 5.9 shows, Ms. L discussed this example with students:

1. Ms. L: I want to give you an example of what that means. […] I had students say that 3x + 5 is the same as 1x + 2x + 5.
2. Katrina: It is.
3. Ms. L: Why?
4. Katrina: Because 1x + 2x is 3x.
5. Ms. L: And a lot of students said that. And when we look at our table and plug that in, look what happened to our table. This is 3x + 5 and this is 1x + 2x + 5. We got exactly the same answers [pointing to two identical tables written in P&P on overhead].
6. Katrina: It's the same equation.
7. Ms. L: It's the same expression, you're right.
8. Ms. L: And what happened to our graph? It looked the same.
9. Ms. L: The reason I want to bring that up is because that goes with our definition of equivalent expressions. They show, you're right Katrina, symbolically they show the same thing, graphically they show the same thing, the table is going to show the same thing. […] So if you have equivalent expressions you're not only going to see it in symbols but you're going to see it in everything else too. So let's put an example down here. 3x + 5 and 1x + 2x + 5 [wrote 3x + 5 = 1x + 2x + 5] so those are equivalent expressions by our definition.

(Day 4, 00:33:42:86)

The fact that the “same answers” (line 5) were obtained in the table representation, the symbolic expressions were the same (line 7), and the graphs “looked the same” (line 8) were all combined in this interchange to be taken as an example of fitting the definition of equivalence of expressions (line 9). Katrina’s interpretation of the meaning of the expressions in line 6 was not justified with an explanation of why (e.g., combine like terms to recognize the same coefficient). However, Ms. L’s summary in line 9 counts as a justification for the equivalence of expressions because the reasoning is based on the use of the definition in which the symbols, graphs, tables, (and pattern, although not included here), are all “the same” for equivalent expressions.
The use of the language of “the same” in the above exchange was not taken as evidence of the component of the learning progression “A1. Different representations/representation types can signify the same object” because the focus seemed to be more on the fact that the graph of \( f_1 = \text{Exp1} \) and \( f_2 = \text{Exp2} \) are the same, the table of \( f_1 = \text{Exp1} \) and \( f_2 = \text{Exp2} \) are the same, and both \( \text{Exp1} \) and \( \text{Exp2} \) can be written in the same symbolic form. This is in contrast to an emphasis on the fact that the graphs, tables, and symbols all represent “the same” mathematical objects called expressions, and the expressions in question also happen to be “the same” or equivalent.

Tasks 3 and 4 from Activity 2 were designed to give students more experience in using the definition of equivalence of equations to verify the equivalence of two expressions. At the very end of the class on Day 4, Task 3 from Activity 2 was introduced in which the focus was on an explanation of why \( x + x + x + x + x \) and \( 2x + 3x \) are equivalent (see Figure 5.10 for a sample of student work).

Figure 5.10. Carlos’ explanation of why \( x + x + x + x + x \) and \( 2x + 3x \) are equivalent.

The intent of this task was to offer flexibility in student responses to draw on the graphical and/or numerical table and/or symbolic representation types in explaining or justifying the equivalence of the expressions \( x + x + x + x + x \) and \( 2x + 3x \). However, the focus of the discussion of Task 3, like the sample of student work shown at the top of Figure 5.10, was guided toward the symbolic representation only in which a combining like terms strategy was used to justify the equivalence (Day 4, 01:00:13:20 &
01:02:01:04; Figure 5.10). It was not until Day 6 that this work was reflected upon to include more representation types. Indeed, Task 4 was used as an impetus to include the use of CAS graphs, tables, and symbols to verify the equivalence of these expressions. Figure 5.11 shows an example of student work.

4. Use your CAS to verify whether $x+x+x+x+x$ and $2x+3x$ are equivalent. Use algebraic expressions, graphs, and tables.

(See next page for instructions and to record work!)

To verify if algebraic expressions are equivalent:
- Use Symbols (press Control "I" and 1: Add Calculator)
- Sketch their Graphs (press Control "I" and 2: Add Graphs; type $f1(x)=x+x+x+x+x$ Enter, Tab, $f2(x)=2x+3x$ Enter).
- From the graph, create a Table (press Control T).

![Graphs and Tables Example]

Do the expressions have the same Graphs? Same Tables? Can they be re-written to have the same algebraic expression (Symbols)?

(sc_day6_activity.pdf)

Figure 5.11. CAS prompt for verifying that the expressions $x + x + x + x + x$ and $2x + 3x$ are equivalent using symbols, graphs, and tables.

The enactment of Task 4 was completed on Day 6 in which the teacher walked students through the activity while demonstrating using TI-Nspire CAS Teacher Software projecting at the front of the classroom. The teacher had students use a CAS Calculator page first, followed by a CAS Graphs page, then a split screen Table page.
The following exchange highlights the teacher-student discourse surrounding the equivalence of these expressions using graphs:

1 Ms. L: Is that what you would have expected to have happen [once both \( f_1(x) = x + x + x + x + x \) and \( f_2(x) = 2x + 3x \) were graphed]? Why?
2 S: They're the same.
3 S: They're equal. [...] 
4 Ms. L: So I've got a line that looks like this and the other line is right on top of it. [pointing to graph of \( f_1(x) \) and \( f_2(x) \)]
5 S: They overlap.
6 Ms. L: You guys keep saying, they're the same, they're the same. But can you see from the graph and the symbolic that they're the same?
7 S: It’s the same no matter what you do, they are the same. 

(Day 6, 00:56:21:15)

Analogous to the discussion of the equivalence of \( 3x + 5 \) and \( 1x + 2x + 5 \), the language of “the same” is used by both students and the teacher as an informal way to discuss the equivalence of the expressions \( x + x + x + x + x \) and \( 2x + 3x \) (line 2, 6, and 7). Moreover, this example showcases how the graphic representation is used as a tool to explain why the expressions are equivalent (lines 4–5). Despite this explanation being developmentally oriented, it is not considered a true justification because mathematical principles are not used to ascertain the basis of the explanation, it is based on the looks of the graph that “overlap” for what is shown on the screen. Both of these examples were tested using symbolic, graphic, and numeric representation types to verify the equivalence of expressions, thus they are taken as evidence of learning progression part “B. Equivalence of Expressions from MR.” The warm-up activities on Day 7 and Day 8 were designed to reinforce the notion of equivalence from graphic and numeric tabular representation types. The activity structure components of reflect and justify were evident during these classroom interchanges.
In the Day 7 warm-up task (Figure 5.12), students translated from symbolic expressions to a graph and numeric table representation type, and interpreted the meaning of that work (two translations):

Ms. L: Think about what we did yesterday, if one graph is right on top of the other, what does that tell me about those equations?
S1: They're similar?
Ms. L: OK, sort of.
S2: They're equivalent factors?
Ms. L: Well I would call them equivalent equations.

(Day 7, 00:16:53:14)

Note the inconsistent use of “equations,” “equivalent factors,” and “equivalent equations” in this exchange. The technique required to complete the task of verifying the equivalence of the expressions from a graphic representation type involves the creation of function rules or equations (as defined in the Cartesian graph system). The equations

\[ f1(x) = 4x - (x + 2) \]

and

\[ f2(x) = 3x - 2 \]

are defined by the equivalent expressions

\[ 4x - (x + 2) \] and \[ 3x - 2 \]

yet equivalent equations is defined as equations with the same solution set. Here the mathematical context or perspective is on equivalence of expressions, not on the solution set of equations.

**Algebra WU**

Consider the expressions 4x – (x+2) and 3x-2.
Graph these expressions in your CAS. From Home:

- Press “Home” ( ), choose 1, “New Document”
- Choose, “No” to saving
- Choose “2”, “Add Graphs”
- Enter 4x-(x+2) in f1, then hit “tab” and enter 3x-2 in f2

1. **What does the graph tell you?** Write a sentence or two.
   - Choose “CRTL T” to view the tables.

2. **What do the tables tell you?** Write another sentence or two.

(Day 7 WU, day7_ActivityNotes.pdf)

*Figure 5.12. Warm-up task to determine equivalence of expressions using graphs and tables.*
In a continuation of the Day 7 Warm-Up discussion, both graphic and numeric tabular representations are considered together as a reflection to verify the equivalence of the expressions in question.

1  "Ms. L: You have exactly the same table and exactly the same graph, what does that tell you about the equations?
2  S: Are equivalent?
3  Ms. L: Yeah. Why--does that make sense to you? If you've got exactly the same table and exactly the same graph—[interruption]. If you have exactly the same table and exactly the same graph that tells us that these equations \( f_1(x) = 4x - (x + 2) \) and \( f_2(x) = 3x - 2 \) are equivalent.

(Day 7, 00:21:12:51)

Again, the language of equivalent equations was used by Ms. L in Lines 1 and 3. While she is correct that \( f_1(x) = 4x - (x + 2) \) and \( f_2(x) = 3x - 2 \) are equivalent equations, the mathematical purpose at hand is to confirm the equivalence of the expressions \( 4x - (x + 2) \) and \( 3x - 2 \) based on the fact that they can be viewed from both graphic and numeric representation types and are identical from those perspectives. After recognizing the equivalence of the expressions \( 4x - (x + 2) \) and \( 3x - 2 \) based on the graph and table, Ms. L reflected further on the original symbolic expressions. The “sameness” of representations within each of graphic, numeric, and symbolic representation types was coordinated to conclude that the original expressions in question were equivalent, as seen in the following excerpt:

If I just looked at these symbols that I gave you in the warm up, I gave you these two things as symbols [circles \( 4x - (x + 2) \) and \( 3x - 2 \) as shown on overhead] and what I'm trying to get you guys to see is, you could simplify this [first expression] in symbols and get that [second expression], or you could look at a table of the same values and know they are equivalent, or you could look at a graph that is exactly the same and know they are equivalent. And then I would know that these two expressions are equivalent even if I didn't simplify them first.

(Day 7, 00:21:38:22)
This summary was a reflection on the various representation types that could be used to verify the equivalence of the symbolic expressions. It is possible that the simplified equivalent forms were not immediately evident in the given expressions and thus motivated the use of graphic and numeric representations to justify their equivalence. This is a difference in the activity design from Activity 2 in which the expressions were possible too easy to warrant a need for using graphs or tables to verify the equivalence. In other words, there was a need for students to be fluent in their ability to translate to a graph and table and reflect on the meaning of those representations with respect to the equivalence of the expressions in question.

It is interesting to note that after this extended discussion of the warm-up task situation (approximately 15 minutes), the Activity 2 homework discussion of equivalence was focused on the symbolic representation only, without reference to analogous justification in graphic and/or numeric representation types. This seems to suggest some discontinuity between the warm-up task and the design and testing of the HW activity. These homework tasks were introduced in Activity 2 and listed in Table 5.4 because they specifically involve symbolic transpositions. The purpose of including the dialogue here is to juxtapose the flow of the activity design as it was tested in the classroom:

Ms. L: What can you say about a and b? [Task: Rewrite each expression in a simplified and equivalent form.]

a. 2x + 3x + 4x + 5x  b. x + x + x + x + x + x + x + x + x + x + x + x + x + x

Ms. L: They both simplify to 14x, we had that theme going on. David?

S: They would be equivalent expressions.

Ms. L: They would also be equivalent expressions because they simplify to the same thing.

(Day 7, 00:38:40:93)

Not surprisingly, the examples (Day 7 Warm Up and Activity 2 HW) above seem to suggest that the task design is a major factor in determining the representation types that
are used to justify equivalence. In other words, the warm-up task explicitly called for an interpretation of graphs and tables, whereas the homework tasks did not. So it seems as though the shift from equivalence among the graphic, numeric tabular, and symbolic representation types to equivalence within the symbolic representation type only was consistent with the task design. From another perspective, using graphs and numeric tables to justify equivalence may not have become a classroom mathematical practice yet in that the use of these other representation types may have required more justification than what the teacher wanted to allocate time to. So the use of symbolic representation types was more efficient than creating graphs and/or tables to discuss the solution.

The case of non-equivalence was considered in the warm-up on Day 8. In particular, students were instructed to engage with the following task:

Consider the expressions $5x - (-x + 2)$ and $2x + 2$. Graph these expressions in your CAS. What does the graph tell you. View the tables. What do the tables tell you?

This task was presented in an analogous way to the warm-up on Day 7—the CAS technique was made explicit on the warm-up sheet (see Figure 5.12). After using CAS to graph $f_1(x) = 5x - (-x + 2)$ and $f_2(x) = 2x + 2$, Ms. L initiated a discussion regarding the notion of equivalence:

1. Ms. L: What do the graphs and the tables tell us?
2. Stan: (inaudible)
3. Ms. L: So, the $2x$ times 2, you say it was going up by two, and this one is going up by six? OK so he's noticing from the table that they're increasing at different rates. So what does that tell us about equivalence? If it's changing at a different rate do you think those two expressions are equivalent?
5. Ms. L: Why?
6. Katrina: They're not equal. They're not the same at all.

(Day 8, 00:15:33:37)
After correctly creating the graph and table using their CAS, Ms. L directed students’ attention to articulate their interpretations of these representations. Stan’s apparent recognition of different rates of change in the table (line 2) was re-voiced by Ms. L in line 3. The use of the specific property of rate of change from the table representation constitutes the use of this representation to justify the non-equivalence of the expressions. Then, even though Katrina’s claim that the expressions in question are “not the same at all” (line 6) did not build on the idea of rate of change, Ms. L subsequently redirected the conversation to focus on the meaning of the graph and table representations; she reiterated that the expressions $5x - (−x + 2)$ and $2x + 2$ are not equivalent because “We're not seeing exactly the same graph, the lines aren't right on top of each other. We're not seeing the same values in the table, exactly” (Day 8, 00:19:29:45). Here, Ms. L reflected on the use of graphs and tables to confirm the non-equivalence, and used the information that these representations provided to determine that, according to the definition of equivalence of expressions, the expressions $5x - (−x + 2)$ and $2x + 2$ are not equivalent. This is also evidence of the learning progression component “B. Equivalence of Expressions from MR.”

Ms. L also took care to summarize the difference between equal and equivalence in order to clarify the different meanings of these terms.

We do have one spot where we have a point that's equal. But that doesn't mean they're equivalent. Equivalent means that they have to be equal for every point. That's why when we saw that graph before we saw exactly the same values in the table and the graph.

(Day 8, 00:22:06:56)
The clarification of language seemed important here because of the closeness of the words “equal” and “equivalent” in their sound and meaning (especially in the vernacular).

To extend this discussion beyond the focus on equivalence, Ms. L prompted a student to specify if there are any places where the table is the same. A student identified a point of intersection or overlap from which the teacher points to \( x = 1 \), \( y = 4 \) in the table. After a prompt from the teacher to explain “Where do you see that on the graph,” a student verbalized a connection between the graph and table as the point where they overlap (Day 8, 00:21:14:93). Here, the students were prompted to think deeply about the representations, a reflection on the graph and table, and the articulation of the point being the same across the table and graph is evidence of a connection across these representations.

Activity 3: Translations Among Words, Numbers, and Symbols

Activity 3 was directed at creating and moving between words, numbers, and symbols. Table 5.6 summarizes the learning goals and TTT framework for this activity that spanned Day 7 and Day 8.

Fitting with the theme of Activity 3 the activity structure will be discussed with respect to the activities of anticipating, translating, and using CAS to check. Tasks that focused on bi-directional translations between verbal and symbolic representations are also presented.
Table 5.6
	TTT for Activity 3: Translations Among Words, Numbers, and Symbols

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Given a verbal situation, students represent the relationship numerically, and then describe it with a symbolic rule. Students translate from words to symbolic expressions and back to words (Day 7)</td>
<td>• Given a Verbal Situation, represent the relationship Numerically then describe it by a Symbolic Rule; and • Interpret the meaning of a given Symbolic Expression or Rule and write a Symbolic Expression or Rule that fits a given Situation.</td>
<td>A1. Connecting and generalizing the quantitative, visual, and verbal with symbolic. C1. Role of Equal Sign: Assigns Variables Rules/Names for Patterns</td>
</tr>
</tbody>
</table>

**Task Summary**

Day 7 – Translate between verbal and symbolic representations of binary operations (multiplication, subtraction, addition, division); translate from verbal representations to a spreadsheet, and use that to write a rule.

Day 8 – Move back and forth between verbal and symbolic representations to solve problems.

*Activity Structure: Anticipate, Translate, CAS Check*

The first task in Activity 3 was directed at translating from a verbal situation to a numerical table or spreadsheet representation of that verbalized relationship. The conjecture that students would make more meaningful and correct translations to a symbolic situation if they first encountered a numerical representation was specifically tested here. Indeed, the enactment of Activity 3 involved a teacher-led discussion with students that, from a representational lens, moved from the numeric representation shown in the upper corner of Figure 5.13, to a symbolic representation of this relationship as an equation, the start of which is shown in Task 4 in Figure 4.3.1.
Activity 3: Translations Among Words, Numbers, & Symbols

Press Home, 2: My Documents, select folder “Algebra 5th hour” and file “Translate This.”

1. This table shows the amount of money that John has in his pocket on a given day (Column A).

2. If Mary has twice as much money in her pocket as John

Complete the table showing how much money Mary has on a given day. Type the values into Column B below the horizontal bar. Press Enter after each value.

3. Think about the pattern that relates John’s money and Mary’s money. Create an equation relating the variables “mary” and “john” that will determine how much money Mary has. mary = __________

4. Use your CAS to test your idea:
   • Move your cursor to the Gray “Diamond” cell in Column C. Press Enter.
   • Use the variable “john” and type an expression into this box that will determine the amount of money “mary” has. Press Enter.

5. Compare the values in Column C with the values that you typed in Column B. What did you find? ________________________________

Figure 5.13. Task design in Activity 3 emphasizing translations from verbal and numeric situations to a symbolic equation.

Both P&P and CAS were used in the following exchange in which the teacher orchestrated a discussion that included the anticipation of CAS output, a translation from verbal to numeric to symbolic representations, and the use of CAS to check P&P work.

Just prior to this conversation, students had filled out the spreadsheet table that explained the relationship between John and Mary’s money (see Figure 5.14, column B “marys_money” for an example of Annie’s CAS activity on this task).

1 Ms. L: Who has an equation?
2 Bryon: x times two or two x.
3 Ms. L: but we’re supposed to use the variables mary and john, how would I write it?
4 Bryon: mary = john*2.
5 Ms. L: So I’m going to write it just like that. […]
6 Ms. L: So, I want to put an equation into my CAS. So go back to your calculator
[discusses where to type john*2] So I’m going to type j-o-h-n, mary = john*2. And I’m going to hit Enter — what do you think is going to happen when we hit Enter? Before you hit it?
7 Quincy: It says error or something.
8 Ms. L: It might say error. What else might it say?
9 S: Mine says john*2 = 64. Ms. L: Ah (shrugs) so it’s working.
10 Angela: How is it doing that?
11 Ms. L: So what it’s doing in this column [pointing to “mary” column C] — Angela’s got a really good question, what’s, how’s it times-ing it by two? This program is taking all the numbers, the values that are in this column [“john” column A], and you’re telling it to take whatever is in John and multiply it by two. So it takes this value [points to cell A1 with a value of 32], multiplies it by two and gets 64. Takes this value [cell A2: “25”] and multiplies it by two and gets 50, this value [cell A3: “14”] gets 28.
12 Ms. L: So what do you notice about what we did by hand and what we did with the equation?
13 S: It’s all the same.
14 Ms. L: So is our equation correct?
15 S: Yeah.
16 Ms. L: OK so that was a great way to verify that our equation was correct.

(Day 7, 00:55:15:45)

In the first part of the exchange (lines 1–5), a student offered an equation to describe the verbal situation and table of values and the teacher prompted that student to use the appropriate variables for the situation at hand. The creation of the symbolic representation and change of variables is evidence of working within the symbolic representation. Second, in using the CAS to enter this equation the teacher specifically prompted students to anticipate or predict what they might see (line 6).
Figure 5.14. Annie’s CAS work for Activity 3 Problems 2 and 4.

Once executed, one student, Angela, asked about how the CAS created that particular list of numbers (line 10). The teacher reflected on the created representation and helped her to interpret the meaning of the equation “john = mary*2” with respect to the numbers in the spreadsheet (line 11). Finally, the CAS results were checked against the P&P results that were previously created. Both students and the teacher verbally verified that the results were the same with both tools, thus CAS was used to check (lines 12–16).

**Bi-Directional Translations Between Verbal and Symbolic Representation Types**

The theme of bi-directional translations was first evident in a teacher-led discussion of vocabulary terms. As shown in Figure 5.15, students were given a word, symbol, or meaning, and asked to complete the other components of the table in a back-and-forth teacher-student interaction pattern.
Figure 5.15. Word problem vocabulary worksheet required translations between verbal and symbolic representations of operations.

For example, in the first row, students translated the words “sum or more than” to the symbol “+” and meaning “adding” (Day 7, 00:42:33:53). This is an example of a translation from a verbal representation to a symbolic representation. The second row is an example of the reverse translation from the symbol “−” to the words “difference or less than” and meaning “subtract, take away” (Day 7, 00:43:34:42). The column expressing the meaning of the symbol and word further emphasizes the importance of verbalizing in one’s own words how they interpret what the symbolic representation means.

A final example from Activity 3 in which students practiced the back and forth translation from verbal to symbolic representations is illustrated by a sample of student work shown in Figure 5.16. All three parts were discussed as a class on both Day 7 and Day 8. The first line of each part is a verbal to symbolic translation, and the subsequent lines are the reverse translation back to a verbal meaning. As evident in the researcher’s
reflections on Day 7, “The bi-directional movement between verbal and symbolic representations was emphasized and seemed to encourage students to use alternative language in describing the meaning of expressions” (Day 7, Daily Class Summary).

8. Reflect on your work in the previous problems. Focus on translating from words to expressions and expressions back to words.

<table>
<thead>
<tr>
<th>a.</th>
<th>b.</th>
<th>c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>If John has “x” dollars, and Mary has twice as much money in her pocket as John, write an expression representing how much money Mary has:</td>
<td>If John has “x” dollars, and Kim has one and a half times as much money as John, write an expression representing how much money Kim has:</td>
<td>If John has “x” dollars, and Jud has 8 more than three times as much money as John, write an expression representing how much money Jud has:</td>
</tr>
</tbody>
</table>
| $2x$ or $x \cdot 2$ | $1.5x$ or $1.5 \cdot x$ | $x \cdot 3 + 8$
Write what this expression means in your own words: | Write what this expression means in your own words: |
| $x = \text{John}$ | $x = \text{John}$ |
| $x \cdot 2 = \text{Mary’s money}$ | $1.5 \cdot x = \text{Kim’s money}$ |
| multiply by two | multiply one and half |

Figure 5.16. Task reflection sheet designed to involve bi-directional translation between words and symbols.

Note that the tasks explained in Figure 5.16 call for “expressions” but should actually call for “equations” because we were looking to specify a relationship between two people and the amount of money they have. This is an error in the task design.

Activity Sequence Creating and Moving Between MR

The focus of Activity 3 was on creating and moving between verbal, numeric, and symbolic representations of linear expressions and equations. This general structure is evident in the enactment of Tasks 1–5 where students’ creation of the symbolic equation $(\text{mary} = \text{john} \times 2)$ was designed to generalize the verbal relationship between Mary and
John (Mary has twice as much money as John), and the numeric table representing this (shown in columns A and B or Columns A and C of Figure 5.14). Thus the learning progression component “A1. Connecting and generalizing the quantitative, visual, and verbal with symbolic” was evident when the teacher discussed both the verbal and numeric representations before discussing the equation that summarized this relationship.

Another aspect of the learning progression that occurred during the enactment of Activity 3 was “C1. Role of Equal Sign: Assigns Variables Rules/Names for Patterns.” An example of this role of the equal sign was evident in the class discussion of Problem 8 Part a (see Figure 5.16 for an example of Bryon’s work). The task itself called for an “expression” representing how much money Mary has. Bryon’s work clearly shows two possible symbolic expressions: “2x or x*2.” However, in an interaction between one individual student and the teacher that was addressed to the entire class, the solution to this task was said to be “x*2 = m” (Day 8, 00:25:39:54). Here, the variable “m” (for Mary) was used to name the rule of “twice as much” or “2x.” The appropriate role of equations instead of expressions for this type of problem is discussed in Chapter 5 as a potential revision to the learning trajectory.

**Activity 4: Evaluate Symbolic Expressions Using CAS and P&P**

On Day 8, Day 9, and Day 10, the technique of evaluating expressions was explored with both P&P and CAS representations. With respect to these mathematical tools, the teacher also articulated that, “We're going to practice moving between the two” (Day 8, Learning Goal, 00:24:11:42). Evaluation was formally defined on Day 14. See Table 5.7 for a summary of Activity 4.
Table 5.7
TTT for Activity 4: Evaluate Symbolic Expressions using CAS and P&P

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Students evaluate expressions using CAS with TI number line and test equivalence of symbolic rules using CAS with operator and P&amp;P substitution. (Day 8)</td>
<td>• Evaluate expressions at numeric values using CAS number line tool; and • Evaluate symbolic rules using P&amp;P substitution; and • Evaluate symbolic rules using the CAS with-operator ‘</td>
<td>’.</td>
</tr>
<tr>
<td>• Understand the meaning of evaluating an expression. (Day 10)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Task Summary

Day 8 - Move from symbolic expression \((2a + -7)\) to a number by prediction, action, reflection on a CAS number line representation.

Day 9 - Use the expression \(2x + -7\) to solve problems; Evaluate \(y = 3 - x\) with symbolic equation and Cartesian graph.

Day 10 - Evaluate the expression \(3C + 2\) at 14; Evaluate \(2l + 2w\) for the length and width of a rectangle; Use with operator ‘|’ to evaluate the expression \(a + b\).

Day 14 - Define “Evaluate” as “to substitute a value or values in for one or more variables.”

The predict, act, reflect components of the activity structure are discussed first. This is followed by a presentation of results that gives evidence of the learning progressions.

Activity Structure: Predict, Act, Reflect

Consider a static image of a CAS number line representation that was explored in Activity 4 (Figure 5.17).
Figure 5.17. TI number line sketch used to illustrate the process of evaluation of an expression.

The teacher had incorrectly interpreted the CAS screen to mean that we were solving an equation, when really it was illustrating the process of evaluation as connected to a number line representation. Ms. L prompted students to think about:

How are the points on the line and the expression related? You've got a = 0 and you've got this equation down here that is being solved. What do you think? How are these two things connected?

(Day 8, 00:41:49:32)

Despite the error in mathematical language that was used, this interchange was important because it was a prediction of how the CAS representation would behave before interacting with it. After exploring the use of the CAS number line representation (by dragging the value of a on the dynamic sketch), a student correctly interpreted that “Wherever ‘a’ is on the number line ‘a’ is in the equation" (Day 8, 00:45:44:16). This student may have interpreted the symbols below the number line to represent an equation because of the presence of the equal sign. Moreover, Ms. L’s utterance above about an “equation down here that is being solved” may have also influenced this students’ interpretation.
After predicting the result of using this representation, the activity of using this representation was open-ended in allowing students to give examples of what they noticed was the same and different when the representation was dragged. For example, Carlos’ responses are given in Figure 5.18. Some students in the class shared analogous ideas with the class.

2. **Act!** Now *drag “point a” one value at a time*. Be sure to hold down the “Click” key until the hand closes around the point.
   a. What numbers/expressions change when you drag point a? Be specific!
      i. One thing that changes is the bottom numbers on the problem
      ii. And the answer changes
   b. List two things that *stay the same* when you drag point a. Be specific!
      i. The negative 7 stays in the problem
      ii. And it has the same format

*Figure 5.18.* Carlos’ interpretation of the things that were the same and different when the number line/expression representation was dragged.

After students explored this CAS-based representation, the teacher directed a next task in which P&P was used to evaluate the expression, despite the intent of the task to offer students a choice of which tool to use (either CAS or P&P). The table representation shown in Figure 5.19 was most like what the teacher wrote on the overhead projector with the value of ‘a’ in parentheses below the table.
3. Evaluate the expression $2a+7$ at the values $a=-2, -1, 0, 1, 2$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2a+7$</td>
<td>-11</td>
<td>-9</td>
<td>-7</td>
<td>-5</td>
<td>-3</td>
</tr>
</tbody>
</table>

4. Reflect and Connect! What pattern(s) do you notice based on your work in Problem 3?

| It's mostly odd numbers and/or goes up by 2 |

Figure 5.19. The evaluation of $2a + 7$ task completed by a student using P&P.

The reflection on this numeric work was directed at identifying a pattern, to which many students were able to correctly interpret that it “goes up by two” (Day 8, 01:04:38:88).

Following the anticipate, act, reflect activity structure, the movement from symbolic to numeric representations was used to solve the problem of when the value of $2a + 7$ was zero, negative, and positive. One student presented her work to the class:

1. Katrina: I used the equation. I just put in the numbers for $a$. There's not much to it.
2. Ms. L: So did you do like a guess and check?
3. Katrina: I just like kept going, so I did 3 which was negative one, then 3.5 which was 0, and 4 which was positive one.
4. Ms. L: [Records work to extend number line on Katrina’s paper for the points (3, –1), (3.5, 0), and (4, 1)] What Katrina did is she turned $2a + 7$ into an equation [records $y = 2a + 7$] and she evaluated that equation until she got –1, 0, and 1 [records below equation: $1 = 2(4) + 7$, then $–1 = 2(3) + 7$] How did you know 3.5?
5. Katrina: I knew 4 would give me 1, and 3 would give me –1, so I guessed the number in the middle would give me zero.
6. Ms. L: Ok [records $0 = 2(3.5) + 7$].

(Day 9, 00:16:04:22)

In the above exchange, Katrina’s explanation (lines 3 and 5) and Ms. L’s interpretation of that explanation (line 4) constitute a translation from the symbolic to the numeric representation. The process of evaluating $2a + 7$ at numeric values was interpreted to signify the solution to the problem, or when the expression was positive, negative, and zero. Immediately following this exchange, the teacher attempted to make a connection between Katrina’s solution and the number line representation.
When I move the ‘a’ to 3, it’s evaluating this at $a = 3$. And this is what Katrina was doing with solving the equation.

(Day 9, 00:19:23:08)

It is significant that Ms. L interpreted the number line representation so that students might come to see how the process of evaluation was connected to that representation. However, there was also some confusion here in that the process of evaluation and solving an equation seemed to be conflated.

In addition to the P&P techniques that students learned with evaluation, students were introduced to an evaluation technique using the with ‘|’ operator on a CAS Calculator page. The teacher asked a student volunteer to show the class how to use this command at the front of the classroom. The CAS technique was also shown on students’ papers (Figure 5.20).

8. A CAS can evaluate expressions and equations using the “with” operator or vertical bar “[“.
   a. Press `ctrl i`, Add Calculator.
   b. Test your work in problem 7 in a CAS calculator page:
      
      \[
      \begin{align*}
      \frac{c+b}{a} &= 9 \text{ and } b = -2 \\
      2 \cdot a + 3 \cdot b &= a = 9 \text{ and } b = -2 \\
      (a - b) \cdot 5 &= a = 9 \text{ and } b = -2
      \end{align*}
      \]

      (day8_T_activity.docx)

*Figure 5.20*. The CAS “with” operator ‘|’ task used to introduce as a technique to evaluate symbolic expressions at numeric values.

Due to several classroom interruptions, including the introduction of a new student to class, the use of this CAS technique was not discussed thoroughly by the teacher, nor was it clearly introduced by the student. The researcher attempted to intervene by directing students to follow along with what the student demonstrator was
showing, but the class was unruly at this point. Feeling pressed for time to move on to a next activity, the teacher concluded by saying "the CAS ‘with’ operator is an evaluation tool" (Day 10, 00:29:33:72), then stopped the activity because we "needed to do some other things" (ibid.).

Learning Progression: Evaluation and the Cartesian Connection

A homework task that was discussed in class was designed to draw out the Cartesian Connection (see Figure 5.21): A point P satisfies the equation of a line L if and only if P is on the graph of line L.

3. Consider the equation $y=3-x$.
   
a. Evaluate the equation $y=3 - x$ at $x=-1$. ________________

   Fill in the missing coordinate of Point A on the graph below.

b. Evaluate the equation $y=3 - x$ at $x=5$. ________________

   Fill in the missing coordinate of Point C on the graph below.

c. Evaluate the equation $y=3 - x$ at $y=2$. ________________

   Fill in the missing coordinate of Point B on the graph below.

Figure 5.21. Task designed to connect symbolic equations with a Cartesian graph.
Ms. L and Bryon gave different explanations for how to complete this task and what it means:

1 Ms. L: We said we're going to evaluate this equation at \( x = -1 \). So that means I take \( y = 3 - x \) and in place of the \( x \) I put in negative one. So \( 3 - x \) is \( 3 - (-1) \), which is \( 3 + 1 \) which is 4. Then it says "Fill in the missing coordinate of Point A on the graph below." So on the graph, I make this point (1,4). What I want you to do is to make the connection, the equation of this line is \( y = 3 - x \) and if \( x = -1 \), \( y = 4 \), and on my graph this is my point (−1,4).

2 Bryon: If you use the suggestion to look at the graph you wouldn't even have to use the equation.

3 Ms. L: Really good point. He said if you're looking at the graph you don't need to use the equation.

(Day 9, 00:34:32:80)

In line 1 and Part a of Task 3 (Figure 5.21) the teacher articulated a nice connection between the symbolic and numeric representations (for evaluation) to include the graphic representation as a means to evaluate an equation at a given point. In the second part of the interaction (line 2) it was Bryon’s idea to just use the graphic representation (and not the symbolic equation) to do the evaluation. Moreover, it is tacit in the second part of the exchange, that the student recognized the graph and the equation as logically equivalent representations of the same equation, or that the process of reading points off the graph is the same as evaluating the equation at particular values of \( x \). Ms. L subsequently used this method to demonstrate a translation from a graphic representation to a numeric representation:

I'm going to do Bryon’s method. If you look at the graph, \( x \) equals 1, 2, 3, 4, 5, what's my \( y \)? What's my two? That's negative two. So point C. which goes with this problem [circles evaluate the equation \( y = 3 - x \) at \( x = 5 \] And I've got \( x = 5 \) and \( y = -2 \). So If I do \( y = 3 - 5 = -2 \), does that match? Yeah, [draws arrow from result of evaluating equation \( y = 3 - x \) at \( x = 5 \) to Point (5, −2) on graph].

(Day 9, 00:37:40:38)

In this exchange the point (5, −2) on the graph is identified then confirmed to satisfy the equation \( y = 3 - x \), which is evidence of the learning progression aspect “CC1. If a point
P is on line L, P makes the equation of L true.” This point or invariant aspect is coordinated across the graphic and symbolic representations, a connection across these representations. The teacher’s demonstration of this translation is not sufficient to determine that students in the class also made the same translation.

In the last part of the discussion of this Task, Ms. L directed the class to move from the symbolic equation to the numerical value for the evaluation process. This is then connected or “matched” to the graphic representation.

1  Ms. L: Evaluate the equation $y = 3 - x$ at $y = 2$. What's the value there?
2  Ms. L: Angela?
3  Angela: Three minus x two.
4  Ms. L: Three minus x two... Well let's do it, um, let's do it with the equation and see if it matches our graph. I've got [writes on board (Figure 5.22a)] "$y = 3 - x$" and I want to evaluate that, I'm on letter C here, at [writes on board (Figure 5.22a)] "$y = 2$". What do I plug in for $y$ here? I plug a 2 in for?
5  S: y.
6  Ms. L: y. So I take $2 = 3 - x$. Well what do you take away from 3 that gives you 2?
7  S: 1. [Ms. L writes $x = 1$ on board (Figure 5.22a)].
8  (2) Ms. L: 1. So you can kind of look at that and see that it gives you 1. Is that the point on our graph? [Points to B (Figure 5.22b)]
9  S: Yeah.
10 Ms. L: Sure is. So we're trying to see can I connect this algebra [points to equations on board (Figure 5.22a)] with the graph? [points to points on graph (Figure 5.22b)]
11 […]
12 Ms. L: That's how you make points and those points are on a line.

(Day 9, 00:38:27:43)

In the first part of the above exchange the point is evaluated in the symbolic equation (lines 1–7) and in the second part of the interaction it is confirmed to match the point on the graph (lines 8–10). This is evidence of the Cartesian Connection in that the point that satisfies the equation of a line lies on the graph of that line (Learning Progression component CC2). In general this is evidence of a connection because $y = 2$ and $x = 1$ are recognized in both the symbolic and graphic representation types as being
the same. Thus the point (1,2) is an invariant feature of the line $y = 3 - x$ that is recognized across symbolic and graphic representations.

Figure 5.22. The teacher showed (a) how to evaluate the equation $y = 3 - x$ at $y = 2$ and that it is connected to (b) reading the coordinates (1, 2) off the graph of this line.

Finally, somewhat tacit in this exchange that the graph and the equation both represent the same linear relationship of $y = 3 - x$. It seems clearest in the exchange when the student utters, "that how you make points" and the teacher confirms "That's how you make points and those points are on a line" (lines 11–12). This interaction is taken as evidence of the student’s generalization of the process of evaluation.
Various aspects of the activity structure will be highlighted with examples from Days 10–14 including predict, act, reconcile, interpret, and generalize. Examples will be given to showcase the learning progression elements of A2, B, and C2 that are also summarized in Table 5.8.

Table 5.8

<table>
<thead>
<tr>
<th>TTT for Activities 5–6: The Distributive Property with P&amp;P and CAS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Learning Goal</strong></td>
</tr>
<tr>
<td>Use geometric representations to explore the distributive property. Pictures ↔ Symbolic Expressions. (Day 11)</td>
</tr>
</tbody>
</table>

Task Summary

Day 10 – Write two equivalent expressions for the pool with dimensions $x + 4$ and 8.
Day 11 – Find efficient way to calculate area of field with length $10 + 100 + 10$ and width 30; write expressions to represent area of pools $8(x + 4) = 8^*x + 8^*4$; use CAS to expand $-3(5 + 2), 4(x - 2), 2(x - 1/2), -1(x + 1), 3(x + 2)$ and $0.5(x - 4)$.
Day 12 – use P&P and CAS to apply the distributive property and reconcile differences for $-3(2x + 2), 3/2(x - 2), 2(3x - 1), 0.1(5x + 100), 0(6x + 1002)$, and $a(x + b)$.
Day 13 – Interpret the CAS expand command technique for expressions $2(x + 1/2), -3(9x - 5)$, and $a(x + b)$.
Day 14 – Define the distributive property, coefficient, and constant; discuss HW tasks to Apply the Distributive property and combine like terms to write an equivalent expression that is simpler: (2) $4(3x - 2) - x + 5$, (3) $4(-5x + 4)$, (4) $-1(x + 1) + 2x + 1$, (5) $a(bx + c)$. Draw your own diagram to show that $4(x + 3) = 4x + 12$. 
Activity Sequence

*Predict and act.* Students were encouraged to think about and predict the result of using CAS before executing commands to produce simplified expressions. The teacher emphasized this kind of predictive activity while students worked by saying “Make a prediction about the next one and see if you get what you had here” (Day 11, 01:01:07:11) and “Did you predict what would happen before you did it? OK. In this part, what is happening? Write down what is happening here” (Day 11, 01:02:27:56).

Pre-designed worksheets delivered on CAS were used to guide students’ activity with a focused attention on the role of the constant term in the distributive property.

![Figure 5.23. Screen shot of completed CAS tasks to expand expressions.](sa_day11_Expand This! A.tns)

The tasks shown in Figure 5.23 were designed to have students reflect on their CAS work and interpret the meaning of the symbolic transposition with a prompt: “How is the original expression related to the expanded expression.” On an individual basis there were mixed interpretations from “It could just give the answer” (Annie, SA_day11_activity2.pdf) to “It gives the shortened or simplified expression” (Bryon, SB_day11_activity2.pdf). The meaning of the CAS expand command was not discussed
as a whole class. Thus the enactment of this activity was considered acting within the symbolic representation; had there been an associated interpretation it would have been considered transposing.

**Predict, act, and reconcile CAS and P&P.** The activity structure of making predictions recorded in P&P, acting on CAS-based representations, and reconciling the differences between these tool-based representations was tested with CAS-based tasks shown on Figure 5.24.

![Image](day11_T_activity_ExpandThisA.tns)

**Figure 5.24.** CAS-based tasks designed to engender the activity sequence of predict, act, reconcile.

On Day 12, students recorded their work in a table-like format that the teacher and researcher had demonstrated how to fill out at the beginning of the activity (see Figure 5.25).
Figure 5.25. A role-play designed to demonstrate the predict (with P&P), act (with CAS), and reconcile (“Fix”) activity structure.

The task designed to reconcile symbolic transpositions between CAS and P&P was discussed during a whole class interchange.

This teacher-student interchange highlighted the activity structure process of using P&P to predict what the CAS would produce (lines 3–4), then acting using CAS (lines 5–6). The reflection on the surprising CAS result (line 1) involved a negotiation of the differences between the CAS and P&P representations so that that P&P work was corrected (lines 7–11). The student seemed to understand the error and the articulation of that process was integrally connected to the distributive property (line 10). Overall, the CAS result seemed to support the recognition of the correct use of the distributive property as it provided the impetus to fix the P&P symbolic transposition. The focused
classroom interchange on the predict, act, reflect, and reconcile activity structure seemed productive based on the example given. However, it should also be noted that it did not seem clear to all students what they should write to “fix” the differences they noted between CAS and P&P. For instance, see Figure 5.26 for Carlos’ work on this task. His CAS and P&P differed, but notes on how to fix the differences were absent. It is possible that Carlos did not understand what to write to negotiate the differences in these representations.

<table>
<thead>
<tr>
<th>Original</th>
<th>Paper &amp; Pencil</th>
<th>CAS</th>
<th>Fix</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.) $3 \cdot (2 \cdot x + 2) = \frac{6x - 6}{2}$</td>
<td>$\frac{6x - 6}{2}$</td>
<td>$\frac{6x - 6}{2}$</td>
<td></td>
</tr>
<tr>
<td>b.) $\frac{3}{2} \cdot (x - 2) = \frac{3x - 6}{2}$</td>
<td>$\frac{3x - 6}{2}$</td>
<td>$\frac{3x - 6}{2}$</td>
<td></td>
</tr>
<tr>
<td>c.) $2 \cdot (3 \cdot x - 1) = 6x + 2$</td>
<td>$6x + 2$</td>
<td>$6x + 2$</td>
<td></td>
</tr>
<tr>
<td>d.) $0.1 \cdot (5 \cdot x + 100) = \frac{5x + 10}{10}$</td>
<td>$\frac{5x + 10}{10}$</td>
<td>$\frac{5x + 10}{10}$</td>
<td></td>
</tr>
<tr>
<td>e.) $0 \cdot (6 \cdot x + 1000) = 0$</td>
<td>$0$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>f.) $a \cdot (x + b) = ax + ab$</td>
<td>$ax + ab$</td>
<td>$ax + ab$</td>
<td></td>
</tr>
</tbody>
</table>

*Talk to your group. Look back - Be sure that your paper and pencil matches the CAS output!*

*(sc_day12_activityA.pdf)*

Figure 5.26. Carlos’ record of different P&P and CAS results.

Thus purposeful reflection and reconciling may require more articulated task design, or elaborated classroom discussions directed at students’ work. Based on their research with algebra students in a classroom teaching experiment, Kieran and Saldanha (2008) purport that activities that combine paper-and-pencil and CAS should include (1) reconciling, (2) generalizing, and (3) proving. The latter two components are alluded to in the following.

*Interpret and generalize.* The use of the distributive property was not clear in all cases. However, there was some evidence that students did seem to recognize the
distributive property at work. For example, consider the equation

“3(4 + 11) = 3*4 + 3*11” to which the teacher asked students to interpret “Do you see the distributive property here?” One student remarked that it just “looked like a distributive property problem” (Day 12, 00:18:19:43). The recognition of this equation as representing the distributive property is taken as evidence that the role of the equal sign here signifies identity between equivalent expressions—Learning Progression C2.

The distributive property was also recognized in the generalized form. Consider the following exchange related to part f as shown in Figure 5.26.

So you've got \(a \times x + a \times b\). Ok. And is that what the CAS tells you, too? [Writes this next to part f: \(a \times (x + b) = \ldots\)] Ok, so. [interruption] This is kind of the culminating, [interruption] this is the very last bit. [interruption] Ok. "a" and "b" here are variables that represent numbers. Ok. So when we say [interruption] this is like a definition of the distributive property because "a" and "b" can be any number. You saw a bunch of examples with numbers [gestures to parts a – e], and now you know that this property works with any number—it works with fractions, it works with positive numbers, it works with decimal numbers, it works with negative numbers; "a" and "b" can be any number and you're still going to get this same expression expanded [points to \(a \times x + a \times b\)], equal to the expression you start with [points to \(a \times (x + b)\) in equation \(a \times (x + b) = a \times x + a \times b\)]. So you've got some equivalent expressions.

(Day 12, 01:01:03:34)

The generalization in this exchange is evident in the fact that the teacher recognizes the generalized form with parameters \(a\) and \(b\) and variable \(x\). The role of the equal sign as representing an identity between the expressions \(a \times (x + b)\) and \(a \times x + a \times b\) is due to the distributive property. In a later exchange related to expanding the expression \(a \times (b \times x + c)\), the teacher posited “If you understand letter e [expand(\(a \times (b \times x + c)\))] you can do any of these [distributive property problems]” (Day 13, 00:28:53:67). Students were not afforded an explicit opportunity to reflect on this generalization, thus it is difficult to determine if they recognized the same generalization that the teacher had. Moreover,
many of the examples that have been cited above have involved the teacher doing a lot of the mathematics. This is reflective of the interactions at the classroom level in which there was little student involvement.

**Learning Progression Elements**

*B Equivalence of expressions from multiple representations.* Both linear expressions and area diagrams (rectangles) representing a swimming pool were used to represent equivalent expression. An example of the area diagram is shown in Figure 5.27.

![Pool 1](day10_lessonplan_notes.pdf)

*Figure 5.27.* Geometric area diagram prompting students to write two different but equivalent expressions for the total area of each pool.

Ms. L summarized the intent of these area diagrams with respect to the notion of equivalent expressions.

If this expression describes the sum of the areas and we know that the area of the sum is the same as the total we know that the expressions are equivalent. So we're using the areas of rectangles to say that these expressions are equivalent. And what's happening, is we're using the distributive property today and we'll see what is going on here.

(Day 11, 00:39:28:80)

Ms. L described one example of these representations being used to represent equivalent expressions:
8 times four plus 8 times \(x\), which is this area plus this area (points to two sections of diagram), equals the total outside which is \(8(4 + x)\). Again, you've got equivalent expressions.

(Day 11, 00:53:16:20)

Despite Ms. L’s involvement in the classroom discussions to give focused attention on making sense of the use of geometric area diagrams, students seemed to have some difficulty with using the geometric diagrams as tools to help understand the notion of equivalence of expressions. Incorrect expressions from Annie, Bryon and Carlos are shown in Figure 5.28 to illustrate the difficulty of using this representation. Misconceptions about area and length seemed to stymie these students.

One purpose of introducing the pool diagrams with equivalent expressions was to provide access to a visual proof of the distributive property. In the excerpt below, Ms. L discussed this purpose with students. One student’s work follows in Figure 5.29.

Now it says can you draw a pool that represents this as a visual proof. […] \(a^n(bx + c)\) [labels on length and width of pool] this gives us the total area. OK now over here we have the length of \(a\) again but the distance is \(b^kx\) and \(c\) so when
I find the area of this rectangle [points to part of diagram] it is $a \times b \times x$ and the smaller rectangle is $a \times c$. So he knows that if you add $a \times b \times x$ to $a \times c$ this is exactly what he had over there. So this is really kind of a neat way to look at the distributive property and to connect that to equivalence. Equivalent expressions.

(Day 13, 00:30:33:52)

Students’ difficulties with non-formulaic versions of the pool problems were likely compounded with the use of parameters $a$, $b$, and $c$ and variable $x$ in this abstract diagram. As shown in Figure 5.29, Bryon did have insight into the generalized nature of this diagram despite the difficulty in articulating the meaning.

![Diagram]

**Figure 5.29.** Bryon’s interpretation of an abstract area diagram linked to his symbolic representation.

The reconciling, generalizing, and proving activity structure was tested in activities that combined the use of CAS and P&P with a focus on symbolic transpositions. As alluded to above, students seemed to have difficulties in using geometric diagrams, and it took a considerable amount of time to teach and learn the use of this alternative representation.

For the teacher and researcher, this time might have been better spent in examining the distributive property in symbolic transpositions and making connections to a Cartesian
A2. Different representations signify the same object, and C2. Role of equal sign

identity between equivalent expressions. In the discussion of a homework task on Day 14, Ms. L elicited responses from students to help them see how different representations of an expression meant the same thing. In this particular example, students offered two different equivalent expressions to \(-1(x + 1): -1x + 1\) and \(-x + 1\). Ms. L emphasized that anything times one is just itself again so these two expressions are the same because the multiplication of 1 need not be written (Day 14, 00:31:03:26).

Another example of “A2. Different representations signify the same object” was also evidence of the learning progression component “C2. Role of the equal sign as indicating an identity between equivalent expressions.” Analogous to the above example, students were to apply the distributive property and combine like terms to write a simplified version of a given expressions. One student offered that \(4(3x - 2) - x + 5\) can be simplified to \(12x - 8 - x + 5\) and another student suggested that it can be simplified to \(11x - 3\) (Day 14, 00:32:01:33). Ms. L had recorded this on the board as a string of equations, \(4(3x - 2) - x + 5 = 12x - 8 - x + 5 = 11x - 3\). This symbolic representation indicates identity between the two expressions on either side of the equal sign, and by transitivity of equality, equivalence of the original and most simplified form. In determining equivalent expressions by symbolic transpositions it is natural for the symbolic language to be recorded in equations that relate equivalent expressions.
Equations Are Equivalence Relations that Are Sometimes, Always, or Never True

Chapter 1 of the teaching unit set the foundation for students’ abilities to create and interpret the equivalence of expressions from symbolic, graphic, numeric and verbal representation types. Through specifically designed tasks, techniques, and theory, several aspects of the instructional theory were tested and evident as conditions of the classroom learning environment, including translations and transpositions, and several aspects of the learning progression. While much of the focus in Chapter 1 was on equivalence of expressions, several roles of the equal sign were introduced.

One role of the equal sign was that it assigns variables rules or names for patterns such as a function. Another role of the equal sign was to represent an identity between equivalent expressions. There were also cases in which two non-equivalent expressions were considered. For example, “2x + 6 = 10” was given as an illustration for the definition of an equation, defined as “a mathematical sentence that relates two expressions by an equals sign” (day4_activity_vocab.pdf).

The definition of an equation as signifying a relationship between two expressions was seen as critical to the activities in Chapter 2 of the teaching experiment. This chapter was tested as a means to provide the groundwork for understanding what solving linear equations in one variable represented in symbolic, graphic, numeric, and verbal representation types means. In particular, the relationship between expressions related by an equal sign was explored as being sometimes, always, or never true in Activities 7 and 8, discussed next.
Activity 7: Equations – Sometimes, Always, Never Part I

It was stated at the top of the Activity sheet for Activity 7 that, “The equals [sic.] sign is a powerful symbol that is used in different ways, depending on the situation. Pay attention to when and how you use the equal sign in these tasks” (day13_T_activity_2.doc). At the beginning of Activity 7 the teacher articulated that the goal of Task 1.3 was to help students understand “how expressions and equations are related” (Day 13, 00:36:30:17). Students considered the evaluation of two separate expressions, and the evaluation of the equation at two separate values. The tasks, techniques, and theory associated with this activity are shown in Table 5.9.

Table 5.9

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Students interpret the truth of an equation. (Day 13)</td>
<td>• Evaluate linear expression and equations using the CAS with operator;</td>
<td>III. Understand the meaning of the equal sign as a statement of equality between two expressions.</td>
</tr>
<tr>
<td>• Explore equivalent and non-equivalent expressions and interpret the “truth” of an equation. (Activity 7)</td>
<td>• Interpret the meaning of expressions related by an equal sign;</td>
<td>C2. Role of Equal Sign: Identity between equivalent expressions</td>
</tr>
<tr>
<td></td>
<td>• Translate from symbolic equation to verbal statement; and</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Translate from symbolic expression to numeric table.</td>
<td></td>
</tr>
</tbody>
</table>

Task Summary

Day 13 – Evaluate the expression $2x + 3 - x + 7$ and $2(x + 6) - x - 2$; evaluate the equation $2x + 4 - x + 7 = 2(x + 6) - x - 2$; and interpret the meaning of an equation that is always true.
An Equation that is Always True

Since Day 4 students had been creating and using equations that are always true when the first equation that related equivalent expressions, \(3x + 5 = 1x + 2x + 5\), was introduced to illustrate the vocabulary term “equivalent expression” (00:33:42:86). In Task 1.3 of Activity 7, students were specifically confronted with making sense of the verbal output of the CAS after an equation was entered (Figure 5.30a) and later in Task 1.5 the numeric and verbal output of the CAS after expressions and an equation were evaluated at numeric values (Figure 5.30b).

![Figure 5.30. Verbal and numeric output of CAS for an equation that is always true.](image)

In making sense of the verbal output of “true” in Task 1.3, Annie was focused on the verbal representation, not expecting or able to make sense of the output in her written statement: “No cause its [sic] a word” (sa_day13_activityB.pdf). It seemed as though Byron made sense of the verbal output of “true” by relying on the symbolic representation in his written statement that: “[Y]es because if you put the numbers and variable together it makes sense” (sb_day13_activityB.pdf). Byron’s response alluded to the fact that the expressions \(2x + 3 - x + 7\) and \(2(x + 6) - x - 2\) could be simplified to
have a common symbolic form. Carlos relied on the authority of the CAS in the justification of the truth of the equation in his written statement: “I think it does because the CAS says the two equations are equil [sic]” (sc_day13_activityB.pdf).

In Task 1.5, students were to translate from a symbolic representation of an expression to a numeric expression and verbal output. An example of the table students completed is shown in Figure 5.31. The classroom dialogue that follows is focused on interpreting the meaning of this table representation when evaluated at $x = 3$.

![Table of expressions](sc_day13_activityB.pdf)

*Figure 5.31. Table of expressions $2x + 3 - x + 7$ and $2(x + 6) - x - 2$ and the equation that relates them where a student evaluated them at several values of $x$."

1  Ms. L: If this expression $[2x + 3 - x + 7]$ evaluates to 13 and this expression $[2(x + 6) - x - 2]$ evaluates to 13, what do you know about those two expressions? Ethan?
2  Ethan: That you're just adding 10.
3  Davon: Equivalent.
4  Ethan: Or that. Equivalent.
5  Davon: Equivalent.
6  Ms. L: Wait, let Ethan finish.
7  Ethan: Well, you can add ten, you can add ten to 3, and they're also equivalent.
8  Ms. L: Oh I see what you’re saying. He’s saying 10 plus three is thirteen, that’s what you’re saying that that pattern develops. But the other thing that Davon was focusing on too was that you can say that the expressions are equivalent.
9  Ms. L: So now when we evaluate the whole equation, the definition of an equation is you've got two expressions that are equivalent you're going to get this result, that the equation is true.

(Day 13, 01:01:24:58)
In the above exchange the table was conveyed to mean that the numeric pattern goes up by 10 (line 2). Also, the students (and the teacher confirmed) that the fact that one value matched for both expressions meant the expressions were equivalent (lines 3–8). This interpretation of the numeric representation to signify equivalence is incorrect and not a sufficient use of the numeric representation to justify the equivalence of the two expressions, hence this does not constitute a correct translation from symbolic to verbal representation. As recorded in the Daily Class Summary, the teacher and researcher reflected on this incorrect use of the numeric representation to justify equivalence in the daily debriefing session (Day13_DailyClassSummary.docx). The idea of equivalence of linear expressions from a numeric representation had been discussed in previous class periods but not discussed in a way that gave sufficient justification.

The teacher’s statement in line 9 introduced a definition for equations that are always true. Namely, equations that relate equivalent expressions are true equations. Ms. L later re-iterated the logic of the argument as expressed by her and the students in the class:

Ok, so we said any value of \( x \) is going to make the equation true, cause we know there's this pattern and we said that because we know these expressions are equivalent if we put the same thing in for \( x \) we're going to have a true statement no matter what.

(Day 13, 01:05:27:73)

Somewhat of a back and forth argument ensued here as the teacher went between equality in numeric evaluation, equivalence of symbolic form, and truth of an equation, using one idea to justify the other. Further attempts to justify why the equation \( 2x + 3 - x + 7 = 2(x + 6) - x - 2 \) is true for all values of \( x \) (not just a finite amount) were not well supported in classroom discourse. The researcher had observed that a particular student
wrote about how to use graphs, tables, and a symbolic approach to justifying the equivalence of these expressions but the use of MR to justify equivalence was not shared in the whole class discussion in a rigorous way (Day13_DailyClassSummary.doc). Instead this student’s work was used as impetus to simply state that one could use tables or a calculator to show that all values of $x$ make this equation true (Day 13, 01:07:00:16).

In summary, the special type of symbolic equation that relates two equivalent expressions was interpreted by the teacher to be “true” for all values of $x$. The numeric representation was used as a primary means to interpret the equivalence of these expressions. The meaning of the equal sign as relating two expressions seemed to become evident as part of the teacher’s definition of a “true” equation that relates two equivalent expressions. Thus the theoretical underpinning of understanding an equation as a relationship between two expressions was explored through the movement from symbolic expressions to numeric and verbal representation types.

**Activity 8: Equations – Sometimes, Always, or Never True**

The intent of Activity 8 was to serve as a bridge and necessary foundation to students’ activities with solving linear equations, before the vocabulary of “solution” to an equation was used. The inspiration for the task design was largely based on Kieran and Drijver’s design experiment in which “the relation between two expressions being equivalent or not, and the corresponding equation having many, some, or no solutions was explored in both CAS and paper-and-pencil tasks” (Kieran & Drijvers, 2006, p. 216). As outlined in Table 5.10, the enactment of Activity 8 spanned Day 15 and Day 16.
Table 5.10

TTT for Activity 8: Equations – Sometimes, Always, or Never True

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Explore and interpret the truth of an equation from graphic and numeric representations. (Day 15)</td>
<td>• Translate from symbolic equations to tables and graphs using P&amp;P or CAS; and • Use graphs and tables to determine and justify the truth of an equation relating two (non-)equivalent expressions as sometimes, always, or never true.</td>
<td>Understand the meaning of the equal sign as a statement of equality between two expressions. D1. Solutions to equations can be determined by equality of expressions.</td>
</tr>
</tbody>
</table>

Task Summary

Day 15 – Use a table to justify equivalence of $2x + 3 - x + 7$ and $2(x + 6) - x - 2$; complete a table and graph for $f_1(x) = x^*x$, $f_2(x) = 2^*x$, $f_3(x) = -x - 7$, $f_4(x) = -x - 3$; use graph, table, and/or symbols to explain why $x^*x = 2x$ and $-x - 7 = -x - 3$ are sometimes, always, or never true equations.

Day 16 – For each graph, determine if the equation is sometimes true, always true, or never true:

A. $5x + 7 + x = 6x$ with graphs of $f_5(x) = 5x + 7 + x$ and $f_6(x) = 6x$.

B. $2(x + 5) = 3x + 5$ with graphs of $f_2(x) = 2(x + 5)$ and $f_1(x) = 3x + 5$.

C. $-1(x + 4) = -x - 4$ with graphs of $f_4(x) = -1(x + 4)$ and $f_3(x) = -x - 4$.

For each table, determine if the equation is sometimes true, always true, or never true:

D. $2(x + 1) = 7 - 3x$ with a table for $y_5(x) = 2^*(x + 1)$ and $y_6(x) = 7 - 3x$.

E. $2x - 3 = -2(-x + 1.5)$ with a table for $y_1(x) = 2^*x - 3$ and $y_2(x) = -2(-x + 1.5)$.

F. $-x - 1 = -(x - 1)$ with a table for $y_3(x) = -x - 1$ and $y_4(x) = -(x - 1)$.

Creating Versus Interpreting (Reflection on Instructional Theory)

Students’ small group work time on Day 15 was observed to be focused on the creation of graphs and table representations for the equations $x^*x = 2x$ and $-x - 7 = -x - 3$, see Figure 5.32 and 5.33, respectively. Complementary to this activity, the whole-class discussion was directed at interpreting the meaning of these representations.
1. The equation \( x^x = 2x \) relates the expressions \( x^x \) and \( 2x \).

   a. Complete the table of values for \( y = x^x \) and \( y = 2x \) and sketch a graph.

   Use paper and pencil, or your CAS.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^x )</th>
<th>( y = 2x )</th>
<th>( x^x = 2x ) (True/False)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>6.7</td>
<td>-6</td>
<td>False</td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
<td>-4</td>
<td>False</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>False</td>
</tr>
<tr>
<td>0</td>
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<td>0</td>
<td>True</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>False</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>True</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>True</td>
</tr>
</tbody>
</table>

(Figure 5.32. Bryon’s creation of a table and graph for \( y = x^x \) and \( y = 2x \) with values that make the equation \( x^x = 2x \) true highlighted.)

In a summary of the activities on Day 15, the researcher recounted that

Bryon was observed to plot the graph point by point from the table [see Figures 5.32 and 5.33], other students followed along when Ms. L demonstrated how to use the scratchpad to graph. Students seemed confused about whether they needed to also record this graph on their paper (some left it on the CAS only) (approx. 37 min).

(Daily Class Summary, Day 15)

3. The equation \( -x - 7 = -x - 3 \) relates the expressions \( -x - 7 \) and \( -x - 3 \).

   a. Complete the table of values for \( y = -x - 7 \) and \( y = -x - 3 \) and sketch a graph. Use paper and pencil, or your CAS.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = -x - 7 )</th>
<th>( y = -x - 3 )</th>
<th>( -x - 7 = -x - 3 ) (True/False)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-10</td>
<td>-6</td>
<td>False</td>
</tr>
<tr>
<td>-2</td>
<td>-9</td>
<td>-5</td>
<td>False</td>
</tr>
<tr>
<td>-1</td>
<td>-9</td>
<td>-9</td>
<td>False</td>
</tr>
<tr>
<td>0</td>
<td>-7</td>
<td>-7</td>
<td>False</td>
</tr>
<tr>
<td>1</td>
<td>-6</td>
<td>-6</td>
<td>False</td>
</tr>
<tr>
<td>2</td>
<td>-5</td>
<td>-5</td>
<td>False</td>
</tr>
<tr>
<td>3</td>
<td>-4</td>
<td>-4</td>
<td>False</td>
</tr>
</tbody>
</table>

(Figure 5.33. Bryon’s creation of a table and graph for \( y = -x - 7 \) and \( y = -x - 3 \).
During the debriefing session after Day 15, the researcher and teacher reflected that the creation of representations takes a significant amount of time. We recognized that the use of CAS to create these representations can serve as a way to make that activity more efficient, but only if students are facile at that technique and choose to use that tool. For instance, students may decide, like Bryon did, that plotting points by hand is a preferred use of tools over the CAS; this is strategic thinking. It is also interesting to note here that in Bryon’s graphical representations (as shown in the right-hand sides of Figures 5.32 and 5.33) there seemed to be a literal relationship assumed between the points shown in the table and the points that were plotted in the graph. Bryon did not extend the graphs to capture a more true representation of the equations in question. Maybe Bryon saw the graph as a way to represent the table of values, but not the equations.

The point of this activity was to have students use these representations to interpret the meaning of the equation as representing a case of sometimes or never true equations (with two or no solutions, respectively). However, had the activity been designed to have the representations be given, it may have better supported students change in RF because both the interpretive activity could have been emphasized. Thus one aspect of task design that challenged us in this Activity was the appropriate balance and coordination of creating representations and interpreting their meaning, in essence, a translation.

To account for the relatively little attention devoted to interpreting these representations, the task on Day 16 was explicitly focused on reflecting on and writing about the meaning of these representations. Explicitly, Figure 5.34 corresponds to the
The “Type 2 writing” task in Figure 5.34 is part of the John Collins writing program that is required at South High. In brief, Type 2 calls for a correct response (as opposed to an opinion) and is intended as a way to formatively assess students on a topic or idea that they know about or have thought about.

Nicole said:

I. The expressions \(-x - 7\) and \(-x - 3\) are not equivalent so the equation \(-x - 7 = -x - 3\) can’t be always true.

II. The equations \(y = -x - 7\) and \(y = -x - 3\) will never give the same values in the table because they both go down by 1 each time and they have different start values (y-intercepts).

III. I can tell the equation \(-x - 7 = -x - 3\) is NEVER true because they are parallel lines that will never intersect.

Which argument makes the most sense to you? Explain why you chose that over the others.

Type 2 writing. 5 minutes. Skip lines when you write.

Figure 5.35. A second writing task to give a focused interpretation of the equation \(-x - 7 = -x - 3\).
Having students write about their interpretations independently held them accountable for conveying their interpretations, reflections, and/or justifications of the meaning of these representations. These aspects of the instructional theory are elaborated next.

**Interpret, Reflect, and Justify the Truth of an Equation**

After the warm-up on Day 16 the teacher orchestrated a fifteen-minute discussion around a homework page in which students were assigned to consider three graphs, three tables, and three symbolic equations. The task in each case was to decide if the linear equation was sometimes, always, or never true, and to circle all values of \( x \) that make the equation true. In other words, the primary activity that students were to engage with was to decide the meaning of a created representation of an equation.

In the instructional theory, each of interpret, reflect, and justify involve the conveyance of meaning of a representation for a mathematical purpose. The analysis of this particular homework discussion was a catalyst for refining the instructional theory to have clearer distinctions between these code categories. In brief, to interpret is to provide a basic quick remark, to reflect is more detailed in response and developmentally oriented, and to justify is to provide a formal explanation based on set practices. The revised definitions of these categories were given in Table 3.6 of Chapter 3.

**Interpret.** Of the three graphs and three tables discussed, two of the graphs were discussed at the level of an interpretation. For example, in considering the graph of the equation \( 5x + 7 + x = 6x \) with \( f_5(x) = 5x + 7 + x \) and \( f_6(x) = 6x \) the teacher remarks:

Ms. L: It's a never why [sic]? Are they ever going to intersect?  
Students: No.
Ms. L: So for letter A we should have never true. [interruption] Never true for number one, the lines don't intersect.

(Day 16, 00:23:45:42)

In this example the meaning of the graph as representing the fact that the equation \(5x - 7 + x = 6x\) is never true is stated without an elaboration on the reasoning. The fact that the lines don’t intersect was not verified by discussing the fact that the lines are parallel, nor that both lines have a linear coefficient (or slope) of 6. This quick remark is based on the teacher’s authority and visual reasoning of the picture more than a mathematical reason why the lines don’t intersect.

Reflect. One graph task and two of the table tasks involved a reflection on the given representations. The specific case in question here involved the technique of interpreting the table of \(y_3(x) = -x - 1\) and \(y_4(x) = -(x - 1)\) to decide if the equation \(-x - 1 = -(x - 1)\) was never true. The following is a classroom interchange around this idea.

1 Thomas: Never.
3 Abila: Because the numbers don't like, they're not going to come to be the same ever.
4 Ms. L: Ok I like what she said. She said it doesn't look like the numbers are ever going to be the same. What's this one changing by? [points to y-values of \(y_3(x) = -x - 1\)]
5 Students: One.
6 Ms. L: So it's going down by one. This one's \([y_3(x) = -x - 1]\) going down by one, but they're, what Abila’s saying is that they're both \([y_3(x) = -x - 1\ and \ y_4(x) = -(x - 1)]]\) going down by one but they're never going to match up because they're both going down by the same amount. That's really cool. So this one is going to be a never.

(Day 16, 00:33:32:24)

This interchange was coded as a reflection because it is an elaborate explanation that is developmentally oriented toward the idea of the equations \(y_3\) and \(y_4\) having the same rate
of change (lines 4–6). However the meaning of table as representing an equation that is never true is missing a piece of justification. Namely, the fact that the equations have different starting values despite having the same rate of change is not articulated. It is not clear that the student understands why the numbers are “not going to be the same ever” (line 3).

_Justify._ For two of the table tasks but none of the graph tasks, the classroom interchange involved a justification. Consider for example the task in which a table for $y_1(x) = 2x - 3$ and $y_2(x) = -2(-x + 1.5)$ was interpreted to signify that the equation $2x - 3 = -2(-x + 1.5)$ which is always true, as illustrated in the following exchange.

```
1 Ms. L: I'm going to highlight the values of $x$; what's happening to the values of $y$?
2 David: They're all going up by two, they're closer to the groups so it's always.
3 Ms. L: They're all going up by two, and what else? Are they all starting at the same point right?
4 David: They all match.
5 Ethan: Oh my god they are!
6 Ms. L: So all the numbers are exactly the same for every value of $x$. So that's going to be an always.
7 Student: That makes sense.
8 Ms. L: So that is always true because for every value of $x$ we have on the table we have the same Ys.
```

(Day 16, 00:32:48:52)

In line 1, Ms. L directed students’ attention to the rate of change of the functions $y_1(x) = 2x - 3$ and $y_2(x) = -2(-x + 1.5)$, to which a student correctly recognized the same rate of change (line 2). The teacher also pointed out to students that they have the same starting value (line 3). For the linear functions in question, this is sufficient information to conclude that the equation $2x - 3 = -2(-x + 1.5)$ is always true, summarized by the teacher in line 8.
The above discussion of critical moments related to the “truth” of an equation was designed and enacted to be foundational to solving linear equations. In particular, the view of solving equations posited in “D1. Solutions to equations can be determined by equality of expressions” was evident in these cases even though the language of “solution” was not used. The view that linear equations are relations that are sometimes, always, or never true was an informal way to introduce students to thinking about solutions from graphic, tabular, and symbolic representations before solving them. From a multi-representational lens students’ understanding of the Cartesian Connection was also supported by the teaching experiment through specifically designed tasks.

Solving Linear Equations with Graphs and Tables, and Algebraic Symbols

The third chapter of the teaching experiment included 6 activities—Activities 9, 10, 11, 12, 13, and 14—and spanned Day 16 through Day 23. An additional “Activity” named “Activity 8.5: The Cartesian Connection” was included retrospectively as a compilation of warm-up exercises that were discussed as pre-cursors to more elaborate discussions of solving linear equations with graphs, tables, and symbols between Activities 8 and 9.

The definition of a solution to an equation was introduced at the end of Day 16, after students had started Activity 9. Definitions for infinite solutions, no solution, and one solution were also introduced on this vocabulary sheet as a means to connect Chapter 2 and Chapter 3. Carlos’ vocabulary sheet is shown in Figure 5.36.
Figure 5.36. Carlos’ vocabulary sheet for solution to an equation, infinite solutions, no solution, and one solution.

Note that the symbolic illustrations in Figure 5.36 were given by the teacher on Day 17, and the graphic illustrations were added independently by each student as a way to signify the word with additional representation types. Students were not instructed to show “matching” illustrations (so that the symbolic equation and Cartesian graphs don’t necessarily represent the same linear functions), but rather to use their homework sheet from Activity 8 to give examples that made sense to them.

Activity 8.5: The “Cartesian Connection”

The Cartesian Connection was elaborated on in the warm-ups for both Day 16 and Day 17 (see Table 5.11). Captured as a thought experiment during a daily debriefing session, “[The Cartesian Connection] is an important part of the learning trajectory that
was added based on (a) students’ perceived difficulties in interpreting representations, and (b) the mathematics that is needed to understand what a graphical solution is and a tabular solution is” (Daily Class Summary, Day 17). On Day 16 students used the line \( y = 4 - 0.5x \) to determine ordered pairs from the symbolic and graphic representations.

On Day 17 students were given the line \( A = 5 + 0.5d \) with the task to explain “How are the points (3,6.5) and (30,20) related to the equation?” and “Find another point on the line that makes the equation true” (Day17_lessonplan_notes.pdf).

### Table 5.11

**TTT for Activity 8.5: The “Cartesian Connection”**

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Articulate the Cartesian Connection between graphs and symbols and make connections to tables.</td>
<td>• Interpret graphs and tables; and Evaluate equations at values that correspond to points on a graph/in a table (CAS &amp; P&amp;P).</td>
<td>CC1. If a point ( P ) is on line ( L ), ( P ) makes the equation of ( L ) true.</td>
</tr>
<tr>
<td>• Articulate the Cartesian Connection between graphs and symbols and make connections to tables.</td>
<td>• Interpret graphs and tables; and Evaluate equations at values that correspond to points on a graph/in a table (CAS &amp; P&amp;P).</td>
<td>A2. Different representation types can signify the same object</td>
</tr>
</tbody>
</table>

**Task Summary**

Day 16 – Use the symbolic equation and graph of \( y = 4 - 0.5x \) to determine ordered pairs.
Day 17 – Explain how the points (3,6.5) and (30,20) are related to the equation \( A = 5 + 0.5d \) and find another point on the line that makes the equation true.

The classroom interactions surrounding these activities seemed to support the notion that by reading the coordinates of an ordered pair from a graph and/or table, and evaluating that value in the equation of the line/table, that value made the equation true. This is captured by the learning progression element “CC1. If a point \( P \) is on the line of \( L \), \( P \) makes the equation of \( L \) true.” The tabular representation was not explicitly captured in the statement of the Cartesian Connection (cf. Moschkovich, Schoenfeld, & Arcavi, 1993) yet it is included here because students were learning to articulate that points in the table are the same as those represented on the graph and vice versa, or that “A2. Different
representation types can signify the same object” (in this case, a linear equation). In the following interchange, a numeric representation is created and interpreted from a graph, and is subsequently identified in the numeric function table; then the point is verified in the equation.

Ms. L: Ok so he's looking at this point here where y is 3.5 and he said if he looks the x value right there is equal to one [draws line on graph from point (1,3.5) down to (0,3.5), then highlights the ordered pair (1,3.5) in orange] Is there another place where I see that on the table? David?
David: The one, three point five.
Ms. L: Yeah, and the x is right there. So you could use the table or the graph to find the x and y in both of those.

(Day 16, 00:14:05:99)

This translation from a graphic to a numeric representation type also signifies a connection between representation types because the point (1, 3.5) was specifically identified as an invariant feature across both representations. The recognition that both the table and graph can be used to determine the same information is an indicator of Learning Progression element A2. Subsequently thereafter, at the close of an interchange between Ms. L. and a different student in which they had evaluated the equation

\[ y = 4 - 0.5x \] at (1, 3.5) for the class, the teacher concluded:

Sure is [a true equation]. So there's a couple different ways to do it. The first one I plugged in, we evaluated this for x equals negative two. The second one we actually put both the x and the y in and we evaluated it and we got a true statement at the bottom. So those are both ways to verify the equations.

(Day 16, 00:17:14:91)

Overall, the ordered pair \( x = 1, y = 3.5 \) was verified to be on the graph of the line

\[ y = 4 - 0.5x \] and in the table of values, then it was verified to make the equation of the line true. The classroom interaction above is an example of the graph and table conveying the same information in two different representation types, which was used to convey part of the Cartesian Connection.
The task to describe “How are the points (3, 6.5) and (30, 20) related to the equation” was intended to elicit the reverse direction of the Cartesian Connection. However, instead of verifying points in the symbolic equation first and recognizing them on the graph, the teacher enacted the task in the reverse direction by first directing students’ attention to see that the points are on the graph and table, and second, evaluated the equation at these values to recognize a true statement (Day 17, 00:11:26:97_2). Similarly, Ms. L prompted students with the task “Is there another point on the graph or table that might make that \[ A = 5 + 0.5d \] equation true?” to which a student, David, identified (20, 15) from the graph and Ms. L recognized “Ok so he looked at the graph and saw that point on the graph. You could also see that it's right here on the table [points to ordered pair (20, 15) in table]” (Day 17, 00:14:51:81). The ordered pair was not evaluated in the symbolic equation to verify that it made a true equation before it was concluded to be on the line. Instead, Ms. L made a generalization about the relationship between points on the graph, points in the table, and the symbolic equation that generated them:

So any of those points that are in the table are points that are on the line. Ok, any points that are in the table are the same points that are on the line. And when we use it in the equation we get a true statement.

(Day 17, 00:14:51:81)

This reflection signifies one direction of the Cartesian Connection (CC2) and the element of the learning progression that different representation types—in this case, the table and graph—signify the same object (A2).

In closing, this multi-representational view of points (ordered pairs) that make an equation true was enacted to provide grounds for students to understand and determine solutions to linear equations from graphs and tables. The teaching experiment was
designed so that students would learn to see the $x$-value of the intersection point of two graphs, or point in common in a table, as the solution that made the symbolic equation true.

Activity 9: Linear Equations: Solving with Tables and Graphs

Activity 9 was designed so that students would use the language of solution to an equation in the context of using graphs and tables before they performed symbolic transpositions to solve linear equations. Table 5.12 gives a summary of this activity.

Some of the rationale for why students were encouraged to use graphs and tables to solve equations included wanting students to have opportunities to learn how to use MR to solve equations so that, if solving an equation using symbolic representations became a barrier, they could overcome that barrier by using a different representation. In other words, the tasks in Activity 9 were designed and sequenced to support students’ RF in solving linear equations.
Table 5.12

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Recognize solutions to linear equations in graphical, numeric, and symbolic representations. (Day 16)</td>
<td>• Link expressions and functions for equations of the form (c = ax + b); • Create and interpret graphs and tables of symbolic functions (f(x) = c) and (g(x) = ax + b) to identify the solution to (c = ax + b); and • Verify the solution to (c = ax + b) using CAS (test the truth of an equation) or transpose within symbolic representation type (P&amp;P).</td>
<td>D1. Solutions to equations can be determined by equality of expressions.</td>
</tr>
<tr>
<td>• Solve linear equations using graphical, numeric, and symbolic representations. (Day 17)</td>
<td></td>
<td>D2. Solving equations in one variable is conceptualized as a comparison of two functions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A2. Different representation types can signify the same object.</td>
</tr>
</tbody>
</table>

**Task Summary**

Day 16 – Use a graph and table of \(22 = 100 - 3x\) to determine the solution (x-value that makes the equation true); verify the x-value with the symbolic equation; introduce vocabulary for “solution to an equation,” “infinite solutions,” “no solution,” and “one solution.”

Day 17 – Re-interpret and use a graph and table of the symbolic equation \(22 = 100 - 3x\) to determine the solution \(x\); for an equation \(f1(x) = f2(x)\) create a graph and table of \(f1(x)\) and \(f2(x)\) to determine the solution and verify using symbols (repeat method for \(4x - 5 = -1\) and \(8 = 2x - 6\)).

Day 20 (homework) – Determine if the equation \(1.5 = 0.5x\) is sometimes always or never true and explain why. Use the distributive property and combine like terms to write an equivalent expression for \(x(-2 + 3)\).

**Linking Expressions and Functions**

In order to use graphs and tables to solve an equation of the form \(ax + b = c\) for \(x\) with real-valued parameters \(a, b,\) and \(c\), students need to view the expressions \(ax + b\) and \(c\) as functions of \(x\). This technique of viewing an equation as a relationship between two expressions is the essence of “D1. Solutions to equations can be determined by equality of expressions.” The terminology of “functions” was not used during the teaching experiment, and the teacher discussed graphing equations instead. The following exchange highlights an example of the teacher helping students to correctly translate from
the equation $4x - 5 = -1$ to graphs (and tables) of $f_1(x) = 4x - 5$ and $f_2(x) = -1$; this technique was part of “Markus’ method” in Activity 9.

1. Ms. L: Now if you're going to use Markus' method, that means you have to graph both of those expressions.
2. Quincy: I don't know how to do that.
3. Ms. L: I'm going to show you right now. So go back to the graph and we're going to graph $4x - 5$ and we're going to graph $y = -1$. So we're going to graph both of those. So in $f_1(x)$ write $4x - 5$ and you're going to get a line. And then I want you to hit tab and the other one under $f_2$ you're just going to type $-1$ because we're graphing both sides of this equation [writes on board as she says] $4x - 5 = -1$ so this is what's in your $f_1$ [writes $f(1)$ above $4x - 5$ and draws arrow down to it] and this is what's in your $f_2$ [writes $f(2)$ above $-1$ and draws arrow down to it] so you're going to have two lines, just like with Markus' method, you had two lines [see Figure 5.37].
4. Quincy: So do I do that over again?
5. Ms. L: This goes in $f_1$ [points to $4x - 5$, Figure 5.37], $-1$ goes in $f_2$ [points to $-1$]. So in $f_1$ you've got $4x - 5$ and in $f_2$ you're just going to put in $-1$.
6. Quincy: (in reference to graph of $f_2(x) = -1$) It just goes across the bottom.
7. Ms. L: Ah. You've got your graphs; you know how to get to the table.

(Day 17, 00:35:49:47)

The symbolic label given to “$f_1(x)$” and “$f_2(x)$” was incorrectly transposed by Ms. L to be “$f(1)$” and “$f(2)$”, as shown in Figure 5.37, despite the use of correct language to reference the functions (lines 3 and 5). This is a minor point, for the real focus was on the interpretation of the equation $4x - 5 = -1$ to be two separate expressions that can be graphed (as introduced in line 1).
Figure 5.37. Ms. L pointed out to students how to view the equation $4x - 5 = -1$ as the graphs of two equations “f1” and “f2.”

Connections Between Tables and Graphs

Another part of the instructional theory that was emphasized during Activity 9 was the activity structure of connecting different representation types, and the learning progression of “A2. Different representations/representation types can signify the same object.” During the debriefing session on Day 16 we discussed how we thought students were having difficulties interpreting the graph and table representations for $100 - 3x$. This resulted in part in Ms. L reconciling the CAS table with a conventional P&P table. The CAS function table shows one column for $x$ and one column for each function that is selected. A typical P&P table has pairs of columns for each function with both independent and dependent variables shown (as in Figure 5.38b and Figure 5.38c).
To emphasize the relationship between the graph and table inscriptions that were shown at the front of the room, Ms. L was deliberate about pointing out the location of the ordered pair (26, 22) in each of the three representations shown in Figure 5.38 (see Figure 5.39).

These representations were important to the following interchange in which Ms. L discussed the tasks “Use the graph to circle the value of $x$ that makes the equation true” and “Use the table to circle the value of $x$ that makes the equation true.”
Ms. L: ... so this point right here is $x = 26$, and this is [motions along y-axis] 10, 20--it's, it's 22, but how did I know that from the table?

Student: Because both the numbers were the same?

Ms. L: Right here, right [uses finger to circle across bottom rows in Figure 5.38b and Figure 5.38c]? $x$ is 26, $y$ is 22. And now that I split it up I see $x$ is 26, $y$ is 22 [Figure 5.38b]. So this point on the pink line [Figure 5.39a, Figure 5.39b] and this point on the orange line [Figure 5.39c] are right here [Figure 5.39d]. So the pink line and orange line have that point in common. That's their point of intersection.

In line 1, the Ms. L used the scale on the graph to identify the point (26, 22). The student’s question in line 2 that the point that makes the equation true is recognizable because both “numbers,” or values of $y$, are the same is a correct interpretation of the table representation. Finally, Ms. L makes an explicit connection across the numeric table and graphic representations (line 3). Note that the task still used the informal language of “true” without explicit reference to $x$-value of 26 as the solution to the equation $22 = 100 - 3x$.

A student, David, also articulated the connection between the graphs and the tables of $f_4(x) = 100 - 3x$ and $f_5(x) = 22$. He responded to Ms. L’s statement of the task “How are the graph, table, and equation related?” by explaining:

For the graph they were the same because the $y$-intersection was the same, [interruption] so for the graph, how they would be the same is because they intercepted at that one point right there, that's how they were the same. And then for the table they both lined up at like when $x$ was 26 across they were both 22 and 22, and in the equation [inaudible] is equals 100 – 3—can I just show how I did it?

It is interesting to note that the task design here was intended to elicit a connection across representation types and the student responded with a connection within each of the graph and table representation types.
Solutions to Linear Equations in Tables, Graphs, and Symbols

Ms. L had specifically referenced and pointed to an image of the Rule of Four web on Day 17 as a tool to articulate the importance of understanding the connections between MR. Much of the discussion conducted as a whole class was focused around solving the equation $22 = 100 - 3x$ by transposing within and translating between a table, graph, and symbols. Markus’ method was first to link expressions and function (with $f_4(x) = 100 - 3x$ and $f_5(x) = 22$), second to use a graph and a table to identify the value of $x$ that made the equation true (as partially shown in Figure 5.38), and third to verify that value of $x$ using a symbolic representation. All but the last component has been discussed. The two techniques shown in Figure 5.40 are (a) use of a CAS scratchpad to test the “truth” of the value of $x = 26$, and (b) transpositions within the symbolic representation.

Figure 5.40. A student used the symbolic representation to verify the solution of $x = 26$ in the equation $22 = 100 - 3x$ is true with (a) CAS and with (b) P&P at the white board.
After identifying the $x$-value of 26 from the graph and table, Ms. L showed students how to use the CAS scratchpad to use the symbolic representation type to verify this value. The technique involved evaluating the symbolic equation $22 = 100 - 3x$ at a numeric value of $x = 26$ which yielded a verbal representation of “true” on the CAS (Figure 5.40a, Day 17, 00:25:33:42). In a separate interchange, a student David offered to share how he used the symbolic representation to verify the solution of $x = 26$. Ms. L invited David to the board where he wrote and explained the series of equations shown in Figure 5.40b (Day 17, 00:30:09:83).

The numeric-graphic connection, and work within the table representation to reconcile CAS and P&P conventions were already discussed. The final piece of the activity that was enacted on Day 17 involved the teacher demonstrating how the CAS graph could be dragged to show an extension of the lines $f_4(x) = 100 - 3x$ and $f_5(x) = 22$ to support the claim that there is only one solution to the equation $22 = 100 - 3x$ (Day 17, 00:25:41:82).

Later when Activity 9 homework was discussed during class on Day 20, one student’s work was used as a discussion piece for the task of determining if the equation $1.5 = 0.5x$ is sometimes always or never true and why (see Figure 5.41). The reason why this equation is sometimes true relied on the fact that the expression $0.5x$ “goes up by .5 each time” (Day 20, 00:21:10:32). Ms. L. recognized that the result was correct but the explanation was incomplete so another student, Ethan, was prompted to explain why the equation $1.5 = 0.5x$ is sometimes true.

Ms. L: … how could you see sometimes true from the graph and the table besides that it goes up by .5? What are you looking for in the graph and the table to know that it is sometimes true? Ethan?
Ethan: You're looking for the intersection.
Ms. L: Yeah, yeah, on the graph you’re looking for the intersection point, and they circled it there so they knew that. And what else did they do on the table that shows me they knew?
Student: Look where the Ys are the same number.
Ms. L: Exactly, exactly. So they showed me on their graph and their table, it’s not exactly what they wrote, but they were correct on that.

(Day 20, 00:21:10:32)

1. Here is a graph and a table of the equation 1.5 = 0.5x.

a. Is the equation 1.5 = 0.5x sometimes, always, or never true?
   
   Student: I think it is sometimes because if you plug in a .5 each time the number increases.  

   (day20_lessonplan_notes.pdf)

Figure 5.41. Discussion piece using a graph and table to identify the x-value that makes the equation 1.5 = 0.5x true.

Other related homework tasks asked students to specifically identify the x-value of the intersection point as the solution to the equation. However this classroom discussion did not give a focused treatment of finding the solution, the language of “sometimes true” dominated the conversation.

Activity 10: Solving Problems Involving Linear Equations

The activity on Day 18 was intended to give students more experience in using graphs, tables, and symbols to solve linear equations (see Table 5.13). The warm-up task and Activity 10 did not well-support this goal; students were overwhelmingly in favor of
using symbolic representations only, yet Ms. L persisted in showing alternative methods of solving these equations.

Table 5.13

TTT for Activity 10: Solving Problems Involving Linear Equations

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Summarize our understanding of linear equations and show how to use multiple representations to solve them. (Day 18)</td>
<td>• Evaluate an equation at a numeric value and use CAS to evaluate the truth of an equation; • Solve an equation ( a = cx + d ) by inspection or symbolic transpositions; and • Create a graph and table to represent an equation ( y = ax + b ) and interpret ordered pairs as solutions.</td>
<td>D2. Solving equations in one variable is conceptualized as a comparison of two functions.</td>
</tr>
</tbody>
</table>

Task Summary

Day 18 – If Alana raises $10 from a sponsor, we want to know how many kilometers Alana walks. In the equation \( A = 5 + 0.5d \) this means that \( A = 10 \). The equation is now \( 10 = 5 + 0.5d \) (1) Which value of \( d \) will make this a true statement? (2) Could you solve this in more than one way? If so, how? Suppose Alana walks 23 kilometers (\( d = 23 \)). Show how you can find the amount of money (\( A \)) that Alana gets from each sponsor. Suppose Alana receives $60 from a sponsor (\( A = 60 \)). Show how you can find the number of kilometers (\( d \)) she walks.

Day 19 (homework) – Which equation has a graph that contains the point (7, –35)?

\[
\begin{align*}
y & = 2x, \\
y & = -5x, \\
y & = 2x - 6, \\
y & = -2x + 1, \\
y & = 7.
\end{align*}
\]

Decide whether each statement is always true or not. a. \( 15 - 3x = 15 + -3x \). Use the distributive property and combine like terms to write an equation that is always true. a. \( 6(x - 4) + 9 \).

The tasks in Activity 10 were to determine the value of \( A \) (Alana’s money) that would be earned if she walked 23 kilometers (i.e., evaluate \( A = 5 + 0.5(23) \)) and to determine the distance Alana would need to walk to earn 60 dollars (i.e., solve the equation \( 60 = 5 + 0.5x \)). The task was a contextual problem that students were familiar with because it was the focus of the previous day’s warm up, and the current day’s warm up. Students had examined this situation in tables, graphs, and symbols. Ms. L called on
students to share their work with the class. Each of the two students who shared showed a symbolic method that was procedurally oriented.

For example, in a demonstration at the front of the class one student correctly evaluated the equation $A = 5 + 0.5d$ at $d = 23$, then explained that $5 \times 23$ is $16.5$. Next, to explain how a solution of $110$ was found, the student explained that $60 = 5 + 16.5$ and instead of adding the student “did the equation” by taking $5$ away from both sides to get $60 = 16.5$. Then to check the solution the student multiplied $110$ by $0.5$ to get $55$. After this explanation, Ms. L prompted the students for alternative methods of solving, and proceeded to demonstrate how the CAS could be used as a representational toolkit with an emphasis on the table.

1 Ms. L: Did anyone use the tables and the graphs? You could have; that equation was already in your CAS. And you could have looked at the table and the graph. Just like we did in the warm-up you could have looked at the table and the graph and figured out OK, where is the value of $d = 23$ from your table or you could have looked at your graph. And you could have figured out where is the value of $A$ that equals $60$. [gets technology set up; types in $5 + 0.5x$ into CAS to look at graph, control T to get table] And we did this part in the warm-up. So we could have used the table and the graph to come up with the answer for number 1 and number 2. So the answer for number 1, it asks $d$, remember $d$ is the same as $x$, I could have scrolled down through here till I got to 23, and how much money did she make when I get to 23 kilometers? 16.5.

2 Student: Dollars.

3 Ms. L: And if I what to know how many kilometers to walk to get 60 bucks I've got to scroll all the way down through these numbers, a long, long way; but with your CAS it's pretty straight forward, and I've got to get to $110$ because you guys know the answer [interruption] there's $60$ and there's $110$ [points to CAS Table with $x$-value of 100, $f1(x)$ – value of 60].

4 Ethan: That's just too much work.

5 Ms. L: [Interruption] So really what we want you to be able to do, is do the symbols like we saw students did, we also want you to be able to look at a graph, but we'd have to make the scale a little bit bigger here on this graph, or look at a table. So all those representations are important for understanding how to solve equations.

(Day 18, 00:33:42:43)
In line 4 when Ethan recognizes the “work” involved in using a table, the efficiency of the solution method was in question. The fact that students were successful in working with the symbolic representation type, and substituted numeric values into the given equation seemed sufficient. Alternative solution methods such as using table that required a lot of “work” (line 4), were not well motivated in this task situation. Moreover, it is not clear whether Ms. L’s explanation of the use of the table (lines 1, 3) was satisfactory for students to interpret the values of 16.5 and 110 to be meaningful.

The graph was created (line 1) but was not used to solve the equation because the scale would have needed to be adjusted (line 5). Thus it was tacit in the teacher’s explanation that the graph would require more work than the table. The symbolic equation, the function table, and Cartesian graph were all mentioned (and demonstrated to some degree) as solution methods. This hints at the theoretical learning progression component of “D2. Solving equations in one variable is conceptualized as a comparison of two functions,” but this is not well-supported by the data.

In the above example the symbolic method was presented first, followed by the use of a table to verify or check the solution. In the warm-up task, 10 was determined to be the distance \( (d = 10) \) Alana would need to walk to earn 10 dollars \( (A = 10) \). This value of 10 had been found by students first working with the symbolic representation type, then by Ms. L’s demonstration of how to use the CAS to view a graph and a table of the situation. The table was used over the graph as a more efficient way to determine the value than re-scaling the graph. Finally, Ms. L argued that the CAS could be used to verify the truth of the equation:

Oh. So if we had chosen to use our CAS we would have found that \( f1(x) \) or \( y \) or \( A \), whatever variable we use, the value for \( x \) or \( d \) is 10. So that means that if I replace
this x with 10 I should have a true statement. Right? So if I do this, 
\[ 10 = 5 + 0.5 \times 10, \] that should be a true statement. Well can you do that in your 
CAS? Can you go to a calculator page and check it? [pause, no student response] 
Sure you can. So take your calculator and go to a calculator page, and I’m going 
to clear mine out, so I’m going to just clear out that page, and I’m going to enter 
\[ 10 = 5 + 0.5 \times 10, \text{ I'm going to check} \] and see if \( d = 10 \) do I get a true statement? [interruption] So you guys are 
expecting to see true, and let’s see [presses Enter], it's true. So if \( d = 10 \) then 
\( A = 10. \) 

(Day 18, 00:14:57:74) 

So the symbolic equation \( 10 = 5 + 0.5d \) was evaluated at \( d = 10 \) to determine that the 
equation \( 10 = 5 + 0.5(10) \) is a “true” statement. Note that the expectation to see “true” 
was the teacher’s prediction, and not necessarily consistent with student thinking. The 
efficiency of using the symbolic equation (over tables and graphs) may have been a 
motivator for students to prefer the use of this representation type. For many students 
they still seemed to attack a problem in the representation it was presented in, in this case, 
symbolic. It was then the teacher’s role to emphasize the importance of using graphs and 
tables to make sense of why the values that make the equation true should make sense to 
students. 

The primary focus of the mathematical activity for the final week of the teaching 
experiment was on using symbolic representations to transform equations with one 
variable for the purpose of solving them. Students used CAS and P&P to solve equations 
in a step-by-step manner, focusing on four main approaches: (1) prediction using graphs 
and tables, (2) combining like terms, (3) getting variables and numbers on either side of 
the equal sign, and (4) using the distributive property. These methods were investigated 
and summarized in Activities 11, 12, 13, and 14 on Days 19, 20, 21, 22, and 23. 
Separated into two sections, Activities 11–12 focus on the combined use of P&P and 
CAS as a representational toolkit to solve equations. Activities 13–14 focus on the
strategies for solving equations with an emphasis on symbolic transpositions. The theme of identifying the nature of solutions to a linear equation—one, none, or infinite—weaves throughout these activities thus connections to “D1. Solutions to equations can be determined by equality of expressions” and “D2. Solving equations is conceptualized as a comparison of two functions” in the learning progression are evident.

*Activities 11–12: Solving Equations with CAS as a Representational Toolkit*

Results from the enactment Activity 11: Solving Equations with CAS (Days 19–20) and Activity 12: Reasoning About Equations (Days 20–21) are combined because of the similarity in task, technique, and theory that underscores the lesson design for each Activity. See Table 5.14 for a summary of these activities.
Table 5.14
TTT for Activities 11–12: Solving Equations with CAS as a Representational Toolkit

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Solve equations by creating equivalent equations. (Day 19)</td>
<td>• Perform symbolic transpositions using CAS</td>
<td>The CAS can be used to execute white-box solving techniques (Edwards, 2003).</td>
</tr>
<tr>
<td>• Explain the reasoning process of solving linear equations of the form $ax + b = cx + d$. (Day 20)</td>
<td>and P&amp;P to create equivalent equations, solve linear equations including combine like terms and distributive property;</td>
<td>D1. Solutions to equations can be determined by equality of expressions.</td>
</tr>
<tr>
<td>• Explain the reasoning process in solving linear equations that involve the Distributive Property. (Day 21)</td>
<td>• Evaluate the truth of an equation at the value of its solution to check it using CAS with operator and CAS and P&amp;P substitution; and</td>
<td>D2. Solving equations in one variable is conceptualized as a comparison of two functions.</td>
</tr>
<tr>
<td>• Perform symbolic transpositions using CAS and P&amp;P to create equivalent equations, solve linear equations including combine like terms and distributive property;</td>
<td>• Create and interpret solutions to linear equations of the form $ax + b = cx + d$ from graphs, tables, and symbolic representation types.</td>
<td></td>
</tr>
</tbody>
</table>

Task Summary

Day 19 – Use the table and the graph to solve the equation $y = -3 + 3x$ when $y = 3$. Circle the value in the table, and the point on the graph. Check your solution using the equation. Solve the equation $8x - 12 = 4$ using CAS as a white box and simultaneously compare with P&P technique to make sense of solution process and to determine an efficient first step. Use CAS to check the solution $x = 2$.

Day 20 – Solve $5s + 2 = 21$ for $s$ using P&P or CAS or both and check the solution by evaluating the value using CAS.

Day 21 – Predict the number of solutions the equation $4x - 9 = -7x + 13$ has by translating to a graph and a table of the functions $f_1(x) = 4x - 9$ and $f_2(x) = -7x + 13$ and determining the point they have in common. Perform symbolic transpositions to solve $4x - 9 = -7x + 13$ for $x$.

CAS as a White-Box and Symbolic Transpositions with P&P

Symbolic transpositions were performed using both CAS and P&P tools. As part of the step-by-step or CAS white-box equation solving technique, Ms. L encouraged students to verbalize their steps and make a prediction about what would occur as a result.

1 Ms. L: What would be a smarter thing to do? A quicker thing to do? Instead of subtracting 8, what would help us get that $x$ all alone? David, what did you do?
David: Add 12.

Ms. L: [deletes calculator line] So I’m going to add 12 here [to the left side of the equal sign], add 12 here [to the right side of the equal sign], now let’s do that on the CAS [types \((8x - 12 = 4) + 12\)] and what do you think I’m going to get?

Student: 16.

Ms. L: [Pressed Enter on CAS] \(8x = 16\). So that’s what we predicted.

The dialogue in line 1 was in reference to a first attempt at solving in which 8 was subtracted from both sides of the equation and determined to be an inefficient step. David’s idea to add 12 was recorded on the board in what Ms. L had referred to as the “old fashioned way” (Day 19, 00:40:14:61), and then typed into the CAS. Before pressing Enter Ms. L prompted students to predict what would result (line 2). The result of “16” or the value on the right side of the equation was stated (line 4) to which Ms. L reiterated with the entire equation of \(8x = 16\), confirming that it met David’s prediction (line 5). Figure 5.42 shows both the board work and CAS work for this task.

Figure 5.42. “Old fashioned” board work and CAS work examples of solving the equation \(8x - 12 = 4\) via symbolic transpositions.

There were not situations in which students needed to reconcile their work, but the predict, act, reflect cycle was evident in classroom interactions, especially surrounding
the white-box solving techniques. Another way in which step-by-step symbolic
transpositions were explored on CAS was with the technique of automatic simplification.

At the beginning of Day 21, Ms. L guided students’ attention to act and reflect on
the first step in two tasks in Activity 12; the first step of the CAS work is reproduced in
Figure 5.43.

![Figure 5.43](image)

*Figure 5.43.* The CAS technique of automatic simplification provided impetus for the
equation-solving step of combining like terms.

Students were directed to do the first step in solving each of the equations shown in
Figure 5.43 and to reflect on “What did the CAS do in Step 1? How did that help to solve
the equation?” (Activity 12). The following classroom interchange illustrates the steps:

1. Ms. L: What did your CAS do on Step 1 or what did you do on the first line on
   number 3?
2. Davon: It simplified it.
3. Ms. L: OK so what does it look like?
5. Ms. L: So the first line on this one the CAS gave you $-5v = 5v + 5$ [writes $-5v = 5v + 5$] so it took the $6v$ plus a negative $v$, combined those like terms and
gave you $5v$ plus $5$. So this is a strategy that we use when we solve equations. If
we have lots of terms on one side of the equals sign, one strategy is that we
combine those terms that are alike. And your CAS automatically does that for
you. If you did it by hand maybe you did that, too.

(Day 21, 00:40:32:99)
Ms. L first reiterated that students had a choice of using CAS or P&P, but it was expected that the same technique would be performed. In other words, the CAS technique of automatic simplification is mathematically equivalent to the P&P technique of combining like terms (line 5). In this interchange, Davon had used his CAS to combine like terms. In the subsequent exchange regarding the equation $3x + 7 - 2x = 4x + 10$, Ethan and David had used P&P to perform the same technique of combining like terms:

6 Ms. L: … For number four, if this was our input [points to $3x + 7 - 2x = 4x + 10$] what was our output?
7 Ethan: I didn’t use the CAS.
8 Ms. L: Ok so how’d you do it by hand?
9 David: [inaudible]… I simplified; how we got it is I just added like the $3x$ and the negative $2x$ and got $1x$ and left the seven there and the rest of it stayed the same.
10 Ms. L: Very good. What David said is it [the CAS] simplified $3x$ plus negative $2x$ and got $1x + 7 = 4x + 10$. So first line of [number] 3 and first line of [number] 4 it combined terms that were the same. So that's combining like terms; that's a strategy.

(Day 21, 00:40:32:99 continued)

Ms. L summarized that the first step of simplifying the equation is called combining like terms and is a strategy that is used for solving equations (line 10). This strategy of performing symbolic transpositions and employing the technique of combining like terms was formalized on Day 22 with an example from Activity 12 homework, and without the use of CAS.

*Solutions to Equations in Graphs, Tables, and Symbols*

On Day 21, a multi-representational view of equation solving was prompted by a task in which students were to first “predict” the number of solutions to a given linear equation and then “identify the solution.” Graphs, a table, and symbolic equation were each used to solve the equation with connections across the three.
Ms. L: So, if I, I'm not sure by looking at these equations [in reference to \(4x - 9 = -7x + 13\)], I can graph these in my CAS. I can put \(4x - 9\) in my f1. Do that right now. Put \(-7x + 13\) in my f2 [interruption].

Ms. L: Open CAS, go to new document, add a graph. f1 we're going to put in \(4x - 9\), hit tab [interruption] Everyone needs to be doing this right now. It's going to graph it, ooh there's our line. Hit tab again and it's going to graph \(f2 = -7x + 13\). What happened to those lines?

Katrina: They overlap.

Ms. L: They cross don't they? They sure are intersecting. Alright. So these guys are crossing. They have a point of intersection. They have one solution. When you have linear equations if they intersect they only intersect at one point. Or they don't intersect at all. This is so important guys. [interruption] So number four, I should expect one solution, \(x\) is going to be, at only one point are they going to cross. And if I look at my graph can I figure out what value of \(x\) they're going to cross at?

Student: Yeah.

Ms. L: About where?

Abila: Negative one and two.

Ms. L: So \(x = 2, y = -1\). What if I look at the table?

Davon: You can check.

Ms. L: That's the way we do it too. So if I go to my table, Control T, go to your table, and look where my \(y\)-values are the same, OK, that's the only place where the \(y\)-values are the same. Right here [gestures to table] \(x = 2, y = -1\). That's the only place. So one solution.

(Day 21, 00:47:54:05)

The focus of lines 1–5 is on the use of a graphic representation type to predict the fact that the equation \(4x - 9 = -7x + 13\) has one solution. In line 5, the attention then shifts to identifying the value of that solution. Ms. L reiterated Abila’s remark stating the solution as \(x = 2, y = -1\) (lines 8–9), even though the solution is simply \(x = 2\). The ordered pair \((2, -1)\) was also identified in the table representation (line 11).

Ms. L explained that students would need to solve equations like this as homework and Katrina complained, “I don’t even have a calculator” (Day 21, 00:51:39:04). This seemingly prompted Ms. L to explain an equation solving technique that students could complete without using technology—paper-and-pencil symbolic transpositions. At the end of teacher-student dialogue about the correct techniques for
solving the equation—add nine to each side to get $4x$ by itself, add $7x$ to both sides to get $11x = 22$, then divide by 11 because it is the opposite of multiplication—Ethan concluded that the answer is “dos.” Figure 5.44 shows a reproduction of the CAS and P&P inscriptions that Ms. L created and projected at the front of the classroom.

![CAS and P&P inscriptions](day21_classroomfieldnotes.pdf)

**Figure 5.44.** Reproduction of (a) a CAS graph and table solution to $4x - 9 = -7x + 13$ and (b) P&P symbolic transpositions on a blank CAS screen outline.

The conclusion that $x = 2$ found in the symbolic representation type was immediately followed by a reflection on the graphs, tables, and symbols that were used to solve the equation $4x - 9 = -7x + 22$.

Ms. L: Now let’s look at our other representations [slides CAS screen over next to symbolic work] At $x = 2$, [points to $x$-value of 2 in table] that was my solution here [points to values of $f1(2)$ and $f2(2)$ in table], where these lines crossed [points to $(2, -1)$], and we did it symbolically. We did it all three ways. So we got the graph, we got the table and we can see from the rule, that’s the same thing as our equation [points to Rule of Four, Figure 5.45]. So that’s how you can approach the homework problems now the only thing you need to do is check your solution.

(Day 21, 00:54:36:22)

Ms. L articulated the solution as an invariant feature across the graph, table, and symbolic equation or rule, a connection across MR. The reference at the end of the statement to “check your solution” is seemingly in reference to the process of substituting the value of
$x = 2$ into the original equation $4x - 9 = -7x + 13$. One student had recognized above that creating the table is a way to check the solution found in the graph (lines 9–10), yet the process of evaluating the equation to check the solution was valued by Ms. L.

Figure 5.45. Rule of Four as displayed in the classroom.

**Linking Equations, Expression and Functions**

The linking of expressions and functions was evident especially when equations were viewed as graphs and tables (D2). In the case of solving the equation $4x - 9 = -7x + 13$ for $x$, linking expressions to functions introduced extraneous information that was identified as being part of the solution. Both $x$- and $y$-values were identified across the graph and table, but not the symbolic rule.

Finally, the shape and nature of linear functions was used as a means to explain that there are only three viable options for the number of solutions to a linear equation: one, none, or infinite. Ms. L used gestures of an “$X$” and “$||$” to signify the one and none cases, and described the lines as right on top of each other for the infinite case (Day 21, 00:44:56:62). The equations explored in Activities 12–13 all had one solution, so the
“sometimes, always, never” language was not prominent. However, reference to the nature of linear equations is evidence of D1.

The theoretical underpinnings of this sequence of lessons stemmed from research showing that high school students often get stuck in solving linear equations posed in symbolic form that have no or infinite solutions (Huntley et al., 2007). It was conjectured that if students come to see tables and graphs as viable representations for identifying solutions to linear equations then if/when they get stuck in a symbolic representation, they might use graphs and or tables to overcome that barrier. This idea was discussed during the debriefing session on Day 21.

It is anticipated that after students are scaffolded to use graphs and/or tables to determine if there are no, infinite, or one solution, that when confronted with a more difficult problem on their own, they might resort to these other representations to help interpret the symbolic work.

(Day 21, Daily Class Summary)

These ideas forms the groundwork for the final set of activities in which strategies for solving linear equations with one, none, and infinite solutions were central to the activity work in the final days of the teaching experiment.

Activities 12–14: Solving Linear Equations with One, None, and Infinite Solutions

Activity 12 was discussed in class with full access to CAS technology. The homework for this activity was done outside of class and thus students did not have access to CAS technology. Two homework tasks in particular were discussed in class on Day 22 as a means to summarize strategies for solving linear equations. These are discussed in this section because they were discussed in a manner that emphasized symbolic transpositions with P&P over using CAS as a Representational Toolkit.
Moreover, Activity 13: Solving Linear Equations and the Distributive Property was assigned as in-class work on Day 21 but was not discussed. One of the tasks on this activity was also discussed on Day 22 as a strategy for solving linear equations. Finally the discussion of Activity 14 and Activity 14 homework spanned Days 22 and 23 and emphasized linear equations with no and infinite solutions. Table 5.15 provides a summary.

Table 5.15  
TTT for Activities 12–14: Solving Linear Equations with One, None, and Infinite Solutions

<table>
<thead>
<tr>
<th>Learning Goal</th>
<th>Technique</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Solve and reason about linear equations that have no solutions or infinite solutions. (Day 22)</td>
<td>• Link expressions and equations by translating from symbolic to graphic representations of an equation of the form ax + b = cx + d;</td>
<td>D1. Solutions to equations can be determined by equality of expressions.</td>
</tr>
<tr>
<td>• Explain the reasoning process in solving linear equations that involve the Distributive Property. (Day 21)</td>
<td>• Interpret symbolic equations and graphs of functions to represent infinite, one, and no solutions; and</td>
<td>D2. Solving equations in one variable is conceptualized as a comparison of two functions.</td>
</tr>
<tr>
<td>• Summarize what we know about linear equations and equivalent expressions (Day 23)</td>
<td>• Transpose within symbolic representations on P&amp;P and CAS to solve equations with one, no, and infinite solutions.</td>
<td></td>
</tr>
</tbody>
</table>

Task Summary

Day 22 – The equation \(4x + 1 = 2(2x + 0.5) / x + 2 = 2x - x + 4 / x + 3 = -x + 3 + x/ \) has _____ solution(s). I know this because... [the graphs of each of the equations were shown as a warm-up task]; Solve each equation for the variable. Show your work. Check your solution. \(5x + 1 = 20x + 10 \) [use the Predict and variable = number strategy]; Solve each equation for the variable. Show your work. Check your solution. \(4r + 9 = 7r + 1 + r \) [use the predict, combine like terms, and variable = number strategy]; Solve \(5(x + 3) = 10x + 5 \) for \(x \) using the first step of expanding the equation (i.e., apply the distributive property). 1–2. Before solving the equation, make a prediction. a. use a graph or table to predict if the equation \(6x = 5x + 7 + x/-s - 4 = -1(s + 4) \) has zero, no, or infinite solutions. Provide a sketch to show why. b. Solve the equation \(6x = 5x + 7 + x/-s - 4 = -1(s - 4) \) for the variable \(x/s \). Show your work. c. Check your solution. Does it make sense? d. Was your prediction correct? Explain why or why not.

Day 23 – Solve each equation for the variable. Show your work. Check your solution. 1. \(2(r + 1) = 7 - 3t \). 2. \(-r - 5 = -(r + 5) \).
Predicting One, No, or Infinite Solutions

The ability to predict that a symbolic equation of the form $ax + b = cx + d$ had one solution (sometimes true), no solutions (never true), or infinite solutions (always true) was emphasized as a strategy that students should recognize when solving linear equations. Ms. L introduced the idea as follows:

So our first strategy is we want to predict. We want to predict how many solutions we might get. And if we're not sure, we can always do a table and graph to try to figure that out. And that kind of goes with our web [points to Rule of Four image on board]. Now if we're not sure how many we're going to get, we might want to go ahead and solve it. So the prediction is just something in your head. You might want to kind of imagine, and you'll get better at this.

(Day 22, 00:14:21:53)

Graphic, then symbolic, and tabular representations were the predominant representation types that were used in classroom interactions to interpret the nature of the solutions of an equation. The use of graphic representation types was often associated with an informal interpretation of the nature of solutions to linear equations.

1 Ms. L: How many solutions? Just looking at the graph, Evan, how many intersection points do you have in graph three?
2 Evan: One.
3 Ms. L: One. So how many solutions will there be?
4 Evan: One.
5 Ms. L: One. One intersection point, one solution. Very good. So maybe some people [interruption] so maybe some people said one solution because the intersection is at one point; maybe some people looked at the table and found there's one spot where the y's would be the same. Ok.

(Day 22, 00:12:32:34)

The difference between the interchange above and the classroom interaction regarding a similar problem recorded on Day 21 was that the solution was not determined.
The use of a symbolic representation seemed to afford a more sophisticated explanation of why the equation had one, no, or infinite solutions despite the presence of a graphic representation.

Ms. L: [...] how many solutions for number two?
Bryon: No solutions.
Ms. L: No solutions. Why?
Bryon: Because um, they both have one x, but they have a difference in the other number; one's 2 and one's 4, so they won't line up at all.
Ms. L: So they both have one x when you simplify it, and one ends up as x + 2 one ends up as x + 4. So that's how you know. So either by looking at the symbols---Ethan.
Ethan: I say because the lines are parallel.
Ms. L: And what does that mean? If the lines are parallel?
Students: They never touch. They never intersect.
Ms. L: They never intersect and so there's no place where there is a solution and the intersection is where we usually identify the solution. Very good. If you were going to look at a table of values for this, what would you see in the table? What would you see in a table of values for that [pointing at graph of $x + 2 = 2x - x + 4$]? What kind of numbers you gonna’ see? Any Ys that are the same?
Students: No.
Ms. L: No because your Ys are always different, these are going up by the same rate, and your Ys are always going to be different.
These explanations are taken as evidence to support RF because of the interpretation of graphic and symbolic representations, and the use of them to justify mathematical assertions. However, the identification of one, none, or infinite solution case was not always connected to solving the equation, if appropriate. For example, the solution to the equation $x + 3 = -x + 3 + x$ was not stated; it was only interpreted to have one solution because it has one point of intersection (Day 22, 00:12:32:34). The infinite solution case was also not connected to the fact that every value of $x$ makes the equation true or is a solution. Likewise, no values of $x$ are solutions, is the case of no solutions. Opportunities to link the case of one, none, or infinite solutions to a statement of the solution set (even if it was an empty set) may have better supported students’ RF in solving linear equations, as in D2.

In some cases students guessed that a linear equation had two solutions, or even three solutions (e.g., Day 22, 00:10:36:54). To this Ms. L would reiterate that the only possible cases for linear equations were one, none, or infinite solutions. This proclamation is again evidence of D1.

Symbolic Transpositions

Beyond the strategy of predicting solutions using graphs (or tables), three other techniques were summarized as strategies for solving linear equations through symbolic transpositions. Recall that as part of Activities 11–12, students had used CAS to motivate the strategy of combining like terms (CAS technique of automatic simplification, Day 21, 40:32:99). The strategy of using the distributive property was also the focus of Activity 13 which was worked on by students but not discussed as a whole class with respect to
students’ CAS work, only their P&P work. See Figure 5.47 for the task from Activity 13 that was the impetus behind the distributive property strategy. Note that this task was only discussed with respect to students’ P&P symbolic transpositions, and was not compared and contrasted with the alternative method of dividing the equation by 5 as a viable first step (as opposed to applying the distributive property) using CAS.

![Figure 5.47. Use of the distributive property contrasted with technique of dividing equation by a constant.](Activity 13, day22_lessonplan_notes.pdf)

The “variable = number” classroom strategy was achieved by performing operations to both sides of an equation. This involved CAS and P&P work on the part of the teacher and students. An example of the P&P technique employed to achieve this strategy is visible on the left-hand side of Figure 5.48. Note that the strategies summarized here consist of symbolic transpositions only, with graphs and table solution strategies being subsumed within the classroom strategy of “prediction.”
Infinite Solutions and No Solutions

During a homework discussion on Day 23, Ms. L used Ethan’s symbolic transpositions as a foundation for a discussion about what “the solution” to the equation 

\[-r - 5 = -(r + 5)\]

is:

1. Ms. L: Looks like he used the distributive property, [shows how Ethan used this technique to complete solving an equation, also added the same thing to both sides] and he ended up with zero equals zero \([0 = 0]\). What does zero equals zero mean?
2. Students: Zero. (laughter)
3. Ms. L: No solution?
5. [student utterance]
6. Ms. L: Infinite solutions. Why? Why is it infinite solutions? Cause it means the same thing on either side of the equal sign. Look back at this step right here. What do you see about the expression on the left and the expression on the right?
7. Ethan: They're the same.
8. Ms. L: They're exactly the same. So you have equivalent expressions. If your expressions on either side of the equals sign are exactly the same that means you have the same line, you have exactly the same line, so infinite solutions on
In line 5, the equivalence of the expressions that are related by the equal sign, namely, \(-r - 5\) and \(-(r + 5)\) was interpreted to mean that the lines are the same, and thus the equation \(-r - 5 = -1(r + 5)\) has infinite solutions. The identification of an equation relating equivalent expressions is evidence of learning progression C2. Moreover, the idea that these equivalent expressions could be graphed and would represent the same lines was taken as justification for a result of 0 = 0 means infinite solutions. It was not recognized that 0 and 0 are equivalent expressions, and the addition of \(r\) and 5 to both sides of the equation was completed in one step (see Figure 5.49) precluding any further analysis of equivalence of expressions during the equation solving process.

Figure 5.49. Student work solving an equation with infinite solutions via symbolic transpositions.

In solving the equation \(6x = 6x + 7\) for \(x\), the task design directed students to make a prediction about how many solutions the equation had and to sketch a graph to show why. This prediction was then related by Ms. L to a student’s strategy using the symbolic representation type.
He combined his like terms and when he subtracted 6x from both sides he got 0 = 7. And we said, what does that mean? OK. That means there's no solution. And what he did is he guessed from up here, he graphed the lines and he saw that they were going to be parallel, no solution. So if you get an answer that makes no sense, you have no solution. Answer with no sense, no solution.

(Day 22, 01:01:12.25)

Both the equation that makes “no sense” (i.e., 0 = 7) and the graphic representation of two parallel lines were used as resources to make sense of an equation that has no solutions. However, the symbolic equation 0 = 7 and the graphic representation of 6x = 6x + 7 were not related to one another. In a reflection on student understanding during the teaching experiment, the researcher recorded:

There was work within symbolic representations, and movement between symbolic and graphic representations, but I’m not sure how well students understood the connection between the equation 0 = 7 and the parallel lines. Getting down to an equation of 0x = 7 or 0 = 7 didn’t seem to be very meaningful for students, and only some seemed to recognize that this equation is not true (i.e., it is a false equation/never true). Some tried to force a solution out of something that doesn’t make sense (e.g., they change the 0x to a 1x so that x = 7).

(Day 22, Daily Class Summary)

There may have been an artificial level of student understanding here in that the idea that parallel lines means never true and no solutions, but a deep level of connection did not seem to be achieved.

Assessments and Review

During the first chapter of the teaching experiment students took three minor assessments, one each at the end of the first three weeks (Day 5, Day 9, Day 14). The fourth quiz of the unit was designed to tie the content of Chapter 2 to Chapter 3, administered at the end of the fourth week (Day 18). Also, students worked in groups on
an end-of-unit review that mimicked the end-of-unit test that would be given the next day (Day 23). The lesson goals on these days reflect the focus of each assessment (or review):

- Students demonstrate understandings of creating and using multiple representations of a linear pattern. (Day 5)
- Assess students' understanding of multiple representations and equivalent expressions from HW assignments. (Day 9)
- Students demonstrate understanding of and apply the Distributive Property by connecting geometric representations with expressions. (Day 14)
- Summarize our understanding of linear equations and show how to use multiple representations to solve them. (Day 18)
- Summarize what we know about linear equations and equivalent expressions. (Day 23)

In addition to the end-of-unit review activity on Day 23, some of the equation solving techniques that involved the use of graphs, tables, and symbolic representation types for the case of one and infinite solutions were reviewed with students just prior to taking the final assessment (Day 24, 00:13:06:41; Day 24, 00:16:11:60). Despite the fact that this formal review was not planned as part of the teaching experiment, Ms. L viewed this activity as an appropriate way to “jog students’ memory” about these tasks (Day 24, Daily Class Summary).

The teaching experiment was designed to begin and end with a unit assessment. As discussed in Chapter 3 the pre-test was used to select three participants from the eligible population to study as cases. Both the pre-test and post-test were used as a means to facilitate the semi-structured interviews that occurred at the beginning and end of the
teaching experiment. Most of the classroom time on Day 24 of the teaching experiment was directed toward reviewing for the post-test assessment. Students worked in groups to complete an activity handout that contained some of the same problems that were on their pre-test assessment. The post-test was crafted to be similar to the pre-test and at the same level of cognitive demand but not identical.

Summary of Activity Structure and Learning Progression

The prominent components of the instructional theory will be summarized here for clarity in the tested instructional theory. First, a summary of the number of critical moments (CMs) coded according to the Activity Sequence is given as a frequency chart in Table 5.16. Also, the data shown in Table 5.17 are a compilation of all CMs across the set of activities. These results help to reveal patterns in the enactment of the learning trajectory so as to support the fact that the activity structure was a major component of the conditions of the learning environment that appeared to support a group of ninth-grade algebra students’ change in their RF.

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<thead>
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<th>Code</th>
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</tr>
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<tr>
<td>Anticipate</td>
<td>18</td>
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<tr>
<td>CAS Check</td>
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<tr>
<td>Connect</td>
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<tr>
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<td>Justification</td>
<td>12</td>
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Table 5.17
Number of CMs Coded Per Activity Structure Across the Set of Activities

<table>
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<tr>
<th>Component / Activity</th>
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<th>9</th>
<th>10</th>
<th>11–12</th>
<th>12–14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>12</td>
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</tr>
<tr>
<td>Connect</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Generalize</td>
<td></td>
<td></td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interpret</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Justify</td>
<td>1</td>
<td>5</td>
<td></td>
<td></td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reconcile</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reflect</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Translate</td>
<td>5</td>
<td>5</td>
<td>12</td>
<td>8</td>
<td>5</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Transpose</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In addition to the Activity Structure, over the course of the teaching experiment, each element of the Learning Progression was evident (sans component E). The distribution of elements of the learning progression across the activities is shown in Table 5.19.

Table 5.18
Code Frequency for Learning Progression Elements

<table>
<thead>
<tr>
<th>Code</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1 Connecting and generalizing the quantitative, visual, and verbal with symbols</td>
<td>3</td>
</tr>
<tr>
<td>A2 Different representations/representation types can signify the same object</td>
<td>12</td>
</tr>
<tr>
<td>B Equivalence of expressions from multiple representations</td>
<td>17</td>
</tr>
<tr>
<td>C Domain and range restrictions</td>
<td>3</td>
</tr>
<tr>
<td>C1 Role equal sign: Assign variables rules/names for patterns</td>
<td>3</td>
</tr>
<tr>
<td>C2 Role of equal sign: Identity between equivalent expressions</td>
<td>20</td>
</tr>
<tr>
<td>CC1 If a point P is on the line L, P makes the equation of line L true</td>
<td>4</td>
</tr>
<tr>
<td>CC2 If a point P makes the equation of L true, P is on the graph of L</td>
<td>2</td>
</tr>
<tr>
<td>D1 Solutions to equations can be determined by equality of expressions</td>
<td>18</td>
</tr>
<tr>
<td>D2 Solving equations in one variable is conceptualized as a comparison of two functions</td>
<td>8</td>
</tr>
</tbody>
</table>

Combined with the breakdown of activities in Table 5.19 we see that for the most part, the learning progression occurred sequentially with some overlap throughout the teaching experiment.
Table 5.19
*Summary of Enacted Learning Progression Components and Activities*

<table>
<thead>
<tr>
<th>Theory</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1, A2, C, C1</td>
<td>1</td>
</tr>
<tr>
<td>B, C2</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2.5</td>
</tr>
<tr>
<td>A1, C1</td>
<td>3</td>
</tr>
<tr>
<td>A2, CC1, CC2</td>
<td>4</td>
</tr>
<tr>
<td>A2, B, C2; Activities that combine paper-and-pencil and CAS should include (1) reconciling, (2) generalizing, and (3) proving (Kieran &amp; Saldanha, 2008)</td>
<td>5, 6</td>
</tr>
<tr>
<td>II, C2</td>
<td>7</td>
</tr>
<tr>
<td>D1, Understand the meaning of the equal sign as a statement of equality between two expressions</td>
<td>8</td>
</tr>
<tr>
<td>CC1, A2</td>
<td>8.5</td>
</tr>
<tr>
<td>D1, D2, A2</td>
<td>9</td>
</tr>
<tr>
<td>D2</td>
<td>10</td>
</tr>
<tr>
<td>D1, D2, The CAS can be used to execute white-box solving techniques (Edwards, 2003)</td>
<td>11, 12</td>
</tr>
<tr>
<td>D1, D2</td>
<td>12, 13, 14</td>
</tr>
</tbody>
</table>

Another condition of the learning environment that was significant for the design of activities and tasks, was the CAS and P&P techniques that were employed. The following table (Table 5.20) summarizes the techniques that were enacted, and variants for both CAS and P&P technologies. This table helps to clarify the broad range of techniques that were employed with both P&P and CAS tools. Note that the structure of Table 5.20 was inspired by Kieran and Drijvers (2006). The final section of this chapter reports conditions of the learning environment that were classified as classroom expectations.
Table 5.20
Techniques for Comparing Expressions for Equivalence, or Solving Equations

<table>
<thead>
<tr>
<th>Technique</th>
<th>CAS variant</th>
<th>P&amp;P variant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substituting numeric values (to check solution to an equation)</td>
<td>With-operator ‘</td>
<td>’ or substitution followed by automatic evaluation, number line TNS file</td>
</tr>
<tr>
<td>Symbolic Transposition / Common form – by automatic simplification</td>
<td>Automatic simplification after Entering</td>
<td>Manipulation by hand to combine like terms</td>
</tr>
<tr>
<td>Symbolic Transposition / Common form – by expanding</td>
<td>Expand command</td>
<td>Apply distributive property of multiplication over addition; use geometric models</td>
</tr>
<tr>
<td>Test of equality</td>
<td>Type equation, press Enter</td>
<td>Perform symbolic transformations on an equation to maintain equality</td>
</tr>
<tr>
<td>Solving equations via symbolic transpositions</td>
<td>White-box technique, apply operation to entire equation (type equation, press Enter; type operation to perform to equation, press Enter, repeat until of the form variable = number)</td>
<td>Manipulation by hand, apply operation to each side of equation; by inspection</td>
</tr>
<tr>
<td>Make a connection between values that make an equation true and points on a line and/or in a table</td>
<td>Interpret a graph and/or table, evaluate an equation by substitution or with operator ‘</td>
<td>’</td>
</tr>
<tr>
<td>Translate from an equation to a graph and a table; link expressions and functions for equations of the form $ax + b = cx + d$ (or $c = ax + b$)</td>
<td>Type function rules $ax + b$ and $cx + d$ into $f1(x)$ and $f2(x)$ of graph line, choose Control T to view table; interpret the meaning with respect to the constant and coefficient, the relationship between the expressions of the original equation $ax + b = cx + d$ (i.e., equivalence), and/or the solution to the equation (or truth at particular value(s))</td>
<td>Sketch Cartesian axes and draw line with specified slope and y-intercept, write table of values at specified x-values with corresponding y-values; interpret the meaning with respect to the constant and coefficient, the relationship between the expressions of the original equation $ax + b = cx + d$ (i.e., equivalence), and/or the solution to the equation (or truth at particular value(s))</td>
</tr>
<tr>
<td>Translate to and from symbols and words (verbal representations)</td>
<td>Test of equality for equivalent expressions related by equal sign (true)</td>
<td>Write symbols to represent words, interpret the meaning of symbols for a situation or pattern</td>
</tr>
<tr>
<td>Translate from numeric and graphic representations to a symbolic rule</td>
<td>n/a in this teaching experiment</td>
<td>Identify the slope and y-intercept from the table of values and/or Cartesian graph</td>
</tr>
</tbody>
</table>
Classroom Expectations

Classroom practices develop over time. These CMs were identified based on the criteria that they were \textit{communicated as a normative aspect of the learning environment}. In other words, they occurred several times across days and activities throughout the duration of the teaching experiment. The conjectured classroom expectations are summarized in Table 5.21 with descriptions and examples. In presenting results of the classroom expectations, the question of \textit{“What would a learning environment look like to support this kind of thinking?”} Additional examples are given to help articulate the conditions of the learning environment as evidenced by these CMs.
Table 5.21

*Description and Examples of Classroom Expectations*

<table>
<thead>
<tr>
<th>Classroom Expectation</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional Representation**</td>
<td>Mathematical and classroom conventions determine the appropriate way of creating and interpreting tool-based representations</td>
<td>What do you call when two lines cross? Ethan: Collision. Abila: Intersection. Ms. L: Intersection, thank you. Ethan: Y-intercept? Ms. L: Nope. Actually what Abila said is correct, it's called an intersection. (Activity 8, Day 15, 00:57:50:38)</td>
</tr>
<tr>
<td>Representationally Flexible</td>
<td>Students should come to expect the need to be fluent and flexible within and among MR</td>
<td>Ms. L: This is what we call the web. As we're solving equations. This was our situation, was Alana walking, we had a graph we could have gotten information from, we had the rule the equation 5 + .5x, and we had the table. So we want you to put all those pieces together. Sometimes you're going to have a preferred method that's going to make more sense to you, and that's cool. We just want you to know that the other methods are out there. Ok, to use. (Activity 10, Day 18, 00:36:21:53)</td>
</tr>
<tr>
<td>Strategic user of MR**</td>
<td>The choice of using a particular representation type is done strategically (e.g., for efficiency, to overcome a barrier)</td>
<td>Ms. L: Now what I'm looking for is when y = 10 but I don't see that on this graph because my graph only goes up to 7.39. What could I look at quickly that I also have on my CAS if I don't see it on my graph very fast? Ethan: Your table. Ms. L: I could go to the table. So if you hit &quot;control T&quot; and go to your table. Ok, now, this is what I'm looking for. If this value is 10, what is the x-value? Student: Ten. Ms. L: Oh. So if we had chosen to use our CAS we would have found that f1(x) or y or A, whatever variable we use, the value for x or d is 10. So that means that if I replace this x with 10 I should have a true statement. Right? So if I do this, 10 = 5 + 0.5*10, that should be a true statement. Well can you do that in your CAS? Can you go to a calculator page and check it? [pause, no student response] Sure you can. (Activity 10, Day 18, 00:14:57:74)</td>
</tr>
</tbody>
</table>

** Denotes category added during retrospective analysis
The code categories and descriptions that were not evident in the CMs across the teaching experiment are summarized in Table 5.22.

Table 5.22

Classroom Expectations that Were Not Supported

<table>
<thead>
<tr>
<th>Classroom Expectation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Focused on Mathematics</td>
<td>Articulate features that are the same across representations, or masked in certain representations</td>
</tr>
<tr>
<td>Strategic User of Tools</td>
<td>Understand/learn when tool use is appropriate and for what purposes</td>
</tr>
</tbody>
</table>

The “Focused on Mathematics” code was determined to be logically equivalent with the “A2. Different representations/representation types signify the same object” so it was collapsed into that code only. This code was motivated from the recent research by Pierce and colleagues (2011) and their findings of lesson design principles. Possibly due to the ambiguity of the code description or code name, this aspect was not evident in the data as a possible classroom expectation or condition of the learning environment.

Moreover, there was insufficient evidence of the “Strategic User of Tools” code in the data and thus it was eliminated from the instructional theory framework. The genesis of this code was the CCSSM Mathematical Practice of Using Appropriate Tools Strategically (2010). In conversations with the teacher, and in reflections on daily classroom activity, the researcher identified that this notion is not well defined (Day 8, Daily Class Summary; Day 10, Week 2 Debriefing). Therefore, it is not surprising to see little evidence of this code when the focus was really on students’ RF.
Conventional Representations

The Conventional Representation code was added during retrospective analysis to capture the ways in which the normative aspects of creating and interpreting tool-based representations were established in classroom interactions. It became evident in the first days of retrospective analysis that some parts of Ms. L’s practice were geared toward the correct interpretation, or at least toward a view that seemed to become “socially accepted” in the classroom.

Authority of Mathematical Community

In a warm-up exercise on Day 3, Ms. L recognized a student’s graph as signifying the correct points, but not in the correct convention of a Cartesian graph. “In our normal convention, I want you to think about it the next time you do a graph...so we've got to think about maybe drawing a four-quadrant graph” (Activity 1, Day 3, 00:17:33:66_1). This utterance is evidence that the teacher, who is an important member of the classroom community, values the creation of a correct P&P graph. It is beyond the scope of the analysis to conclude that students also valued this as a classroom practice.

Correct Terminology

The following example focuses on the conventional or correct use of language to describe or interpret symbolic representations.

1 Ms. L: If we have two expressions that are exactly the same we say they are--
2 Ethan: Equivalent.
3 Ms. L: --Always true--yes, they're equivalent, that's a better word, and the equation then is always true. Well, let's look at this first one. I've got 15 – 3x and 15 + −3x. Is that the exact same expression on the right and the left?
Student: No, Yes.
Ms. L. Who said yes? Yes is correct. $15 - 3x$ is the same as $15 + -3x$. Plus a negative is the same as subtracting. So for that one I'm going to circle always true.

(Activity 10, Day 19, 00:25:02:88)

Evident in the above exchange, the choice of language to describe mathematical representations, in this case, symbolic expressions, was primary. For example, Ms. L recognized Ethan’s language of “equivalent” to be “more correct” than “always true” (lines 1–3). Similarly, a common algebraic notation is that subtracting is the same as adding a negative (line 4). The teacher’s declaration of the solution in line 5 is also an example that illustrates her role in the classroom and in classroom interactions. In particular, Ms. L sometimes made strong or direct statements, communicating an authority about the accepted or correct mathematical solutions.

Reconciling CAS and P&P Representations

One example of conventional representations was sparked by a reflection on the teaching experiment during a debriefing session between the teacher and researcher. The issue being addressed in the following was the way function tables are typically recorded using P&P (as in Figure 5.50a with separate $x$ and $y$ columns for each equation) differs from how it is conventionally shown on the CAS (as in Figure 5.50b with one column for $x$ and a column for each of the functions that are selected).
Figure 5.50. Mock-up of how two “function tables” are conventionally shown in (a) P&P and (b) CAS tools.

In the debriefing after Day 16, we had sensed that students didn’t really have adequate instruction on how to interpret [screen shots of CAS Function table] representations, so Ms. L devoted time to helping students with this interpretation. (Day 17, Daily Class Summary)

The following statement illustrates how a classroom interaction revolved around the explication of negotiating differences between tool-based representations.

Ms. L: Now what you didn't have yesterday is you didn't have this column here. I cut it apart so that you could see. [...] So no matter what value I pick for $x$ on this orange line I get 22 for $y$. Ok, does that make sense? Does this help you make more sense of the table from yesterday? Ok.
(Activity 9, Day 17, 00:19:46:64_3)

The Conventional Representation aspect of the instructional theory was considered significant to the construct of RF because the creation and interpretation of representations is necessarily judged based on what is acceptable in the mathematical community that one is working within. In this particular ninth-grade algebra classroom, the practices of creating and interpreting MR and representation types often were based on shared understandings of the “mathematics community” at large (often conveyed as Ms. L’s authority), or students’ experience and knowledge recall in creating “correct”
P&P representations and/or terminology. Other instances were prompted based on reconciling CAS and P&P representations.

It is also important to note that the “Conventional Representation” aspect of the instructional theory also spurred an additional code category during retrospective analysis—“Language Issue”—that is a topic of discussion in the next chapter. This code mostly captured the instances in classroom interactions in which there were inconsistencies and incorrect use of mathematical language or terminology (e.g., equivalent equation instead of equivalent expression).

**Representationally Flexible**

The representational flexibility aspect of the instructional theory was specifically designed for during the enactment of the teaching experiment and was an object of discussion during debriefing sessions (e.g., Week 2 Debriefing, Day 10). The following examples from classroom interactions are organized into categories—importance of MR, the use of MR in different solution approaches, and the role of the tasks and the researcher—and give evidence to support this classroom practice.

**MR are important.** The teacher often emphasized the importance of using more than one representation type during classroom discussions. For example, Ms. L acknowledged a student’s method of solving with symbols by showing student work; then she showed an additional method of solving using a table to encourage flexibility in use of representations to solve equations. She then summarized:

So really what we want you to be able to do, is do the symbols like we saw students did, we also want you to be able to look at a graph, but we'd have to make the scale a little bit bigger here on this graph, or look at a table. So all those representations are important for understanding how to solve equations.
Consider the following CM in which multiple representation types were used by various members of the class.

1. Ms. L: How many solutions for number two?
2. Bryon: No solutions.
3. Ms. L: No solutions. Why?
4. Bryon: Because um, they both have one \( x \), but they have a difference in the other number; one's 2 and one's 4, so they won't line up at all.
5. Ms. L: So they both have one \( x \) when you simplify it, and one ends up as \( x + 2 \) one ends up as \( x + 4 \). So that's how you know. So either by looking at the symbols---Ethan.
6. Ethan: I say because the lines are parallel.
7. Ms. L: And what does that mean? If the lines are parallel?
8. Students: They never touch. They never intersect.
9. Ms. L: They never intersect and so there's no place where there is a solution and the intersection is where we usually identify the solution. Very good.
10. Ms. L: If you were going to look at a table of values for this, what would you see in the table? What would you see in a table of values for that [pointing at graph of \( x + 2 = 2x - x + 4 \)]? What kind of numbers you gonna see? Any Ys that are the same?
11. Students: No.
12. Ms. L: No because your Ys are always different, these are going up by the same rate, and your Ys are always going to be different.

In lines 1–4, Bryon reasons from the symbolic representation type to explain why the equation \( x + 2 = 2x - x + 4 \) has no solutions. Then Ethan offers his perspective that the reason why this equation has no solutions is because the lines are parallel (line 6), or that they have no intersection points (lines 7–8). The interchange between the teacher and two students seems to signify a “taken as shared” value for using multiple representation types in classroom interactions.

Another reason MR are important is to check work. The teacher often encouraged students when they used representations in a way that seemed to promote an aspect of flexibility or fluency in the use of MR. For example, the use of multiple representation
types might be appropriate to check work in another representation type. Ms. L explained:

So yeah, you can solve it symbolically and get a solution, that's fine. The reason we're pushing you guys to use tables and graphs is that sometimes you might get a symbolic answer and wonder if you're right or not. And we saw some really good work on your partner quizzes from yesterday in that some people solved it symbolically and then when they looked at their table and graph it didn't agree and they wondered well why don't those numbers match? Because if you get a solution using symbols it should match what you see on your graph and what you see in the table.

(Activity 11, Day 19, 00:14:45:59)

Thus, one viable reason to use multiple representation types is to check work to be sure the results match.

While the teacher led many of the classroom interactions, there were instances in which the teacher showed that students’ use of different representation types were valued. For example, Ms. L began by evaluating an equation using the symbolic representation type, then Bryon interjected that the use of a graph might be a better route to go.

1 Ms. L: What I want you to do is to make the connection, the equation of this line is $y = 3 - x$ and if $x = -1$, $y = 4$, and on my graph this is my point ($-1, 4$).
2 Bryon: If you use the suggestion to look at the graph you wouldn't even have to use the equation.
3 Ms. L: Really good point. He said if you're looking at the graph you don't need to use the equation.

(Activity 4, Day 9, 00:34:32:80)

In line 3, Ms. L recognized Bryon’s proposed method as a viable way to evaluate the equation, even though it was different from her approach to solving this task (line 1).

**Rule of Four**

The Rule of Four web was often referenced when more than one representation type was used to solve a problem involving linear equations.
Ms. L: This is what we call the web [references a picture of the Rule of Four]. As we're solving equations—this was our situation, was Alana walking, we had a graph we could have gotten information from, we had the rule—the equation $5 + .5x$, and we had the table. So we want you to put all those pieces together. Sometimes you're going to have a preferred method that's going to make more sense to you, and that's cool. We just want you to know that the other methods are out there.

(Activity 10, Day 18, 00:36:21:53)

Ms. L emphasized the importance of seeing connections among multiple representation types, yet acknowledged students’ preferences in choosing representation types to solve problems. Ms. L seemed to use the web as a tool to showcase the variety of different types of representations that could be used to solve the problem, but was not telling the students that they had to use all of these methods all the time.

Tasks and Researcher’s Role

In closing, most of the examples shown above were drawn from specific classroom interactions between the teacher and the group of ninth-grade algebra students. It should be recognized that the task design and researcher also played a role in the conditions of the learning environment. Related to the Activity Structure, the task design often called for students to make explicit connections between representations (e.g., Activity 8, Day 15, 00:50:51:84), or to use each of graphs, symbols, and numeric representation types (e.g., Activity 9, Day 17, 00:37:45:72). The researcher’s role as a participant observer influenced the conditions of the learning environment because there were instances in which the researcher encouraged students to follow task design that called for more than one representation type. For example, in Activity 9, Markus’ method involved the use of graphs, numeric tables, and symbolic equations. The researcher observed a pair of students to only have the solution recorded in symbolic equations. She
then prompted the students, “Can you find the solution using the table and the graph?” (Day 17, 00:44:08:09). Those students then continued to work to show the solution in these other representation types.

**Strategic User of Multiple Representations**

There was some evidence to suggest that representation types were chosen or selected for efficiency in solving problems involving linear equations. Two brief examples of this code category are included below to highlight students’ views on the efficiency of using particular representation types.

Ms. L: Ethan how’d you do it?
Ethan: Like that (in reference to using CAS to create table then graph)
Ms. L: But Ethan asked me a question he said ‘couldn't I just solve it?’
Student: Nope.
Ms. L: Yeah, if you want to find the solution--
Abila: That takes longer [inaudible]
Ethan: No it doesn't.

(Activity 11, Day 19, 00:13:48:09)

Ms. L: And if I what to know many kilometers to walk to get 60 bucks I've got to scroll all the way down through these numbers, a long long way, but with your CAS it's pretty straight forward, and I've got to get to 110 because you guys know the answers [interruption] there's 60 and there's 110.
Ethan: That's just too much work.

(Activity 10, Day 18, 00:33:42:43)

These examples highlight some possible aspects or conditions of the learning environment that might help to acculturate a strategic disposition toward the use of multiple representation types. The time it takes to follow a particular approach or to use a particular representation type is one aspect that can be explicitly discussed. Other facets of strategic use of MR could be related to the choice of tools, although tacit in the classroom interactions cited here. Recall that the strategic choice of representations was
added as a code during the retrospective analysis, but was not an explicit design component that was tested during the design and conduct of the teaching experiment. It was still deemed to be an important aspect of the learning environment because the ability to use representations to overcome barriers encountered in other representation is an indicator of students’ RF.

The frequency of Classroom Expectations across the teaching experiment is summarized in Table 5.23. Note that, of the facts of the conditions of the learning environment, the classroom expectations had the least amount of codes overall.

Table 5.23
Frequency of Classroom Expectations Across All Days and Activities

<table>
<thead>
<tr>
<th>Code</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional Representation</td>
<td>11</td>
</tr>
<tr>
<td>Representationally Flexible</td>
<td>14</td>
</tr>
<tr>
<td>Strategic User of MR</td>
<td>7</td>
</tr>
</tbody>
</table>

Despite the relatively low frequency of CMs coded as Classroom Expectations, these aspects of the instructional theory are considered to be a significant part of the conditions of the learning environment because they give insight into the classroom interaction patterns that other aspects of the instructional theory fail to capture.
CHAPTER VI

DISCUSSION

Contemporary goals for school mathematics undergird the importance of the use of multiple representations (MR) and of representational toolkits such as CAS for learning and doing mathematics (CCSSI, 2010; NCTM, 2000; NRC, 2001). In the field of mathematics education, the objective to link research and practice around central issues that are important to both researchers and practitioners alike is a priority (Arbaugh et al., 2010). One such issue that this research undertook was the use of representational toolkits to support meaningful connections between representations of mathematical objects such as linear equations. Decades of research has shown that students demonstrate shortcomings in all or some aspects of representational fluency (RF) as defined in this study in solving problems involving linear equations (Bieda & Nathan, 2009; Heid & Blume, 2008; Huntley & Davis, 2007; Kieran, 1992, 2007; Knuth, 2000; Moschkovich, Schoenfeld, & Arcavi, 1993; Nathan et al., 2010; Nathan & Kim, 2007; Sfard & Linchevski, 1994; Spitzer, 2008; Suh & Moyer, 2007; Yerushalmy, 2006; Zbiek, Heid, Blume, & Dick, 2007). This study sought to clarify how students’ RF changed as a result of learning to solve linear equations within a CAS and P&P environment, and to elucidate conditions of the learning environment that may have helped to support such change.
Results and Contributions

The results of the case study analysis indicate that two of the three ninth-grade algebra students, Annie and Bryon, improved to unistructural and relational levels of RF as a result of learning how to solve linear equation problems within a CAS and P&P environment. These students also demonstrated an increase in the number of multistructural levels of RF and performed some unprompted movements between MR when solving linear equations posed in a symbolic representation type. The was no measured change in RF for the third student, Carlos, who persisted at the prestructural level of RF from the initial to the final interview. Results from these three cases suggest that RF is specific to the context of the task at hand including the number of solutions to the equation, the representation type(s) given and the tools that are available. A student’s RF may also be related affect and disposition toward mathematics.

The characteristics of the learning ecology in which ninth-grade algebra students changed their RF were guided by the enactment of a sequence of activities that were specifically designed according to a research-driven instructional theory posited and tested throughout a collaborative teaching experiment. The mathematical goal of learning to solve linear equations as a process of reasoning was approached through a learning progression and sequence of activities that included: (a) defining equivalent expressions from graphic, numeric, symbolic, and verbal representation types, (b) examining equations as equivalence relations that relate (non-)equivalent expressions, and (c) using each of the four main representation types to solve linear equations with both CAS and P&P tools.
Teacher and student interactions at the classroom level supported an activity structure of techniques including a predict-act-reflect sequence, and reconciling differences between CAS and P&P. Classroom activity was largely driven by teacher-led mathematics yet the expectations to create conventional representations, to be flexible in the use of MR, and to be a strategic user of MR was valued by the teacher and researcher and evident in classroom practice.

Some conditions of the learning environment might be explained by the fact that the teacher played a prominent role in interpreting the meaning of representations. Her authority in the classroom sometimes limited the number of opportunities students had to share and validate the completeness and correctness of their interpretations of representations. The dominance of teacher proclamations over publicly shared student thinking may have had an impact on students’ resulting change in RF. For instance, the fact that Annie, Bryon, and Carlos had difficulties in correctly interpreting the solution to linear equations in MR may be connected to their experiences in the classroom in which the teacher directed the interpretation of representations in public discourse.

It is recognized that the meaning of a representation is in the eye of the beholder. Many of the aforementioned results are based on students’ and Ms. L’s interpretations of representations. Beyond the analyses of data at the classroom and individual level, the researcher used a variety of interpretive lenses to draw conclusions about activity and cognition. It is possible however that these individuals held different conceptions in their minds about what the signified was, given certain signifiers (cf. Presmeg, 2006). Said otherwise, an individual’s interpretation of a representation need not necessarily convey the meaning of the object that this representation stands for.
Boundaries of Research

The boundaries of this research are cast with respect to both the design and conduct of the research. Both delimitations and limitations are considered.

Delimitations

Role of Researcher

Consistent with the planned methodology, the researcher’s role in the teaching experiment evolved from a non-interactive observer to a participant in the classroom. As the teacher enacted pedagogical strategies that were more student-centered (e.g., student group work), the researcher correspondingly was more interactive with the students during this work time. The researcher also became more involved with the teacher’s instructional decisions during the lessons. The researcher would sometimes cue the teacher with regard to timing of lesson elements (e.g., time to move on to next part of lesson plan), and with regard to selecting student work to share with the whole class. This involvement was negotiated between the teacher and the researcher, but sometimes occurred spontaneously. For example, during Activity 4, the researcher’s attempt to motivate students to stay on task involved an address to the entire class to follow along with a student presenter in order to learn a new CAS technique (Day 10, 00:29:33:72).

The researcher played a crucial role during debriefing sessions that also included attention to planning for future lessons. For instance, there were some mathematical ideas that were introduced with CAS technology that the teacher was not familiar with, such as using CAS as a white-box to solve linear equations. The debriefing sessions sometimes
turned into a one-on-one professional development (PD) session in which the researcher would lead discussions about particular CAS techniques and how the mathematical ideas might be taught using this new tool.

The researcher and teacher each assumed a diverse expanse of roles. For instance, the researcher took on roles from logistical (camera person, recorder of field notes, technical assistant, technology keeper) to professional (curriculum designer and developer, PD orchestrator, leader, bricoleur of resources). The teacher assumed roles including teacher, learner, orchestrator of CAS use, and disciplinarian. This is a delimitation of the research because the myriad of roles that the two primary research team members assumed sometimes took attention away from the broader design of the experiment. However, there were two main useful tools that were used to keep the teaching experiment on track with the planned instructional theory, the classroom field notes protocol (Appendix D) and the sequence of activities summary (Tables 3.1, 3.2, and 3.3).

Emergent Perspective

The retrospective analysis of classroom data was primarily targeted at the social perspective on classroom interactions and mathematical practices with some attention to individual activity work as needed. On the other hand, the retrospective analysis of interview data was primarily targeted at the psychological perspective on individual students’ activity and cognitions while solving problems involving linear equations. Cobb (2000) explained a strength of the emergent perspective in teaching experiments conducted in collaboration with a classroom teacher is that:
accounts of [students’] mathematical development might involve the coordination of psychological analyses of their individual activities with social analyses of the norms and practices established by the classroom community (p. 310).

While some insights into the both the social and psychological aspects of the teaching experiment were identified in Chapter 5, a more coordinated effort of intertwining the psychological and social could have occurred during the retrospective analysis of classroom data. Finally, the retrospective analysis of cases studies and the classroom learning environment would have been stronger had more individuals been involved to a larger extent in the coding process. Member-checking with the participating teacher is another viable option.

Definitions and Frameworks

The definition of RF assumed for this study informed the choice of frameworks to characterize students’ RF and classroom conditions that seemed to support RF. As introduced in Chapter 2, the polysemic nature of the construct of RF required that choices be made to specify the meaning of this term for this study. Other definitions may have given greater emphasis to meaning making, and/or the role of justification and generalization. This change could have also been reflected in a revised analytic framework for RF in which the role of justification was more explicitly incorporated at the relational level.

Had the definition of RF been expanded to include components such as drawing meaning about a mathematical entity, and using resourceful work methods, this might have afforded greater connections to be made across research studies. For instance, Huntley and colleagues (2007) demonstrated how the constructs of adaptability and
versatility (à la Sfard & Linchevski, 1994) could be used to productively examine students’ algebraic thinking and reasoning within and across MR. In a similar manner, others such as Kendal and Stacey (2001) refer to the notions of formulating and interpreting as a way to characterize students’ achievement on tasks and understanding of the mathematical situation. Relationships between constructs could support a more synergistic view and purpose of the research results related to RF and students’ mathematical activity and cognition.

The fact that the instructional theory served as both an object of analysis and lens for the retrospective analysis greatly shaped the analysis and results. While the methodology was crafted in such a way that new components to this theory could be added (as some indeed were added), the fact that much of the theoretical lens was determined \textit{a priori} to the conduct of the study limited the scope of the interpretations of the data. In other words, the design of activities, the ongoing analysis, and the retrospective analysis all occurred under the auspices of this lens.

To account for this seemingly singular focus, the identification of CMs that surprised or seemed to contradict the theoretical lens occurred during both ongoing and retrospective analysis. In the ongoing analysis, these “surprising” moments proved to be difficult to communicate with the teacher because they were sometimes construed as attacks on the enacted curriculum, or weaknesses in the teacher’s pedagogy. Once realized, these reflections were then recorded in a personal journal that was not shared with the teacher. This imposed restrictions on communication between research team members and possibly on the modification to lesson design components during ongoing analysis. In the retrospective analysis, CMs that seemed to be counter to the instructional
theory were compiled and sometimes used to define a new element to the instructional
theory (e.g., classroom expectation ‘Conventional Use of Representations’).

Finally, in conventional use of the Rule of Four framework with solving
equations, the numeric representation signifies the solution to an equation. In this study,
the numeric representation type included numeric equations and numeric function tables.
One issue at play is the complexity in representations and representation types that are
introduced when the relationship between variables is explored, such as with the
Cartesian Connection. The distinction between tabular and non-tabular numeric
representation types of representations could be clarified in future research and use of
such a framework.

Limitations

According to Cobb and colleagues (2003), “Design experiments are conducted to
develop theories, not merely to empirically tune ‘what works’” (p. 9). Thus the results of
this research are not intended to provide a solution to the issue of cultivating students’ RF
in a combined CAS and P&P environments. That is, this study does not quantify or
qualify the instructional theory in the sense of working or not.

Instead, the intent was to characterize students’ change in RF and to catalog the
ways in which this change may have been supported in one particular learning
environment with a group of ninth-grade algebra students. Building on this notion, the
results of this study are unable to show the appropriate role of CAS and P&P in
supporting RF.
Another limitation of the design and enactment of the teaching experiment was that students did not have access to their TI-Nspire CX CAS outside of school. This imposed certain restrictions on the design of homework tasks, but also possibly on the student-tool interactions that were possible both in and outside of the classroom.

Empirically-Based Instructional Theory

Based on the empirical evidence from the teaching experiment, and insights gained from the analysis of the three cases, a revised instructional theory is proposed in Table 6.1. The first column of this table summarizes the conjectured activity sequence that was posited before the teaching experiment was conducted. The subsequent columns show the enacted sequence that was tested during the teaching experiment, and a proposal for a revised sequence of activities. The three rows of this table correspond to three “Chapters” of the teaching experiment. This table will serve to organize the ensuing discussion of the empirically-based instructional theory with connections to the literature.

One of the challenging complexities of instructional design aimed to change students’ RF in solving problems involving linear equations was the simultaneous effort to meet mathematical goals, while exercising appropriate mathematical practices with tools. Following the CCSSM (CCSSI, 2010), a mathematical learning goal was to “Understand solving equations as a process of reasoning and explain the reasoning” and an aim for the role of technology was to “and “Use appropriate tools strategically.”

In light of these goals, there seem to be two themes that are interwoven in the revised activity sequence that are also posited to support students in changing their RF. First, Knuth and colleagues (2006) suggest that to successfully solve equations, equations
need to be understood as equivalence relations. Second, linking expressions and functions so as to recognize solutions in symbolic, graphic, numeric, and verbal representation types was considered foundational to supporting students’ RF.

Table 6.1
Conjectured, Enacted, and Revised Activity Sequence

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Conjectured</th>
<th>Enacted</th>
<th>Revised</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Equivalent Expressions, Linear Equations of Functions</td>
<td>Multiple Representations of Equivalent Expressions</td>
<td>The “Cartesian Connection” in Graphs, Symbols, Tables, and Words</td>
</tr>
<tr>
<td>2</td>
<td>Equations as Equivalence Relations</td>
<td>Equations are Equivalence Relations that are Sometimes, Always, or Never True</td>
<td>Equations are Equivalence Relations that are Sometimes, Always, or Never True</td>
</tr>
<tr>
<td>3</td>
<td>Creating and Solving Linear Equations</td>
<td>Solving Linear Equations with Multiple Representations</td>
<td>Identifying Solution Sets of Linear Equations in Graphs, Tables, Symbols, and Words</td>
</tr>
</tbody>
</table>

The flow from equivalence of expressions, to equations as equivalence relations, and then to solving linear equations is maintained in the revised instructional theory with additions to consider the Cartesian Connection in Chapter 1 and Equivalent Equations in Chapter 3.

Chapter 1

The Cartesian Connection as a Foundation

The Cartesian Connection, as defined by Moschkovich, Schoenfeld, and Arcavi (1993) and researched by Knuth (2000), has historically been a difficult concept for
secondary mathematics students to master. With a focus on RF in solving linear
equations, the present study extended the notion of the Cartesian Connection from
symbolic and graphic connections to a connection across symbolic, graphic, and numeric
tabular representation types. As evident in Annie and Bryon’s final interviews, the
conditions of the learning environment seemed to support students’ abilities to articulate
the connections of a common element (in many cases, a single point or ordered pair)
across graphic, numeric, and symbolic representation types.

In the revised instructional theory the Cartesian Connection is emphasized as an
additional component of Chapter 1 so that students might become facile at linking
expression and functions, and gain ability in recognizing and articulating invariant
aspects of linear equations across representation types. This background is seen as a
necessary component to recognizing equivalence of expressions in MR, but also for
solution sets to equations in graphs, tables, symbols, and words.

*Equivalence of Expressions as a Review Topic*

Based on informal analyses of the group of ninth-grade algebra students’ pre-
tests, initial interviews with Annie, Bryon, and Carlos’, and in-class discussion and quick
polls, it seems as though students’ experiences prior to a high school algebra included
attention to symbolic transpositions to write equivalent expression and to use of symbolic
representations or methods of inspection to solve simple linear equations such as those in
the form $ax + b = c$. In the revised activity sequence it is recommended that the activities
are streamlined to address the goal of changing students’ RF in solving problems
involving linear equations by making the introductory material on equivalence less prominent.

At some points during the teaching experiment the researcher fell prey to using other curricular resources that were already developed instead of focusing carefully on the mathematical goals and Rule of Four lens of the current experiment. Two significant examples of this are Activity 4 and the TI-Nspire number line activity (from an online resource) and Activities 5-6 and the geometric area diagrams (from CMP materials on equivalence of expressions). The geometric area diagrams seemed to be particularly problematic:

The pre-requisite knowledge (from elementary school and middle school) of understanding why the operation of addition corresponds to length and the operation of multiplication corresponds to area is weak for many students. (This is tied to students’ conceptions or meanings of multiplication.) When compounded with abstract symbolism (e.g., with side length parts of \(x\) and 4), this became unmanageable for many (as evidenced in these students’ work on Days 10 and 11 with similar problems).

(Day 10, Daily Class Summary)

It may have been much more beneficial to stick to the conventional representations of Cartesian graphs, numbers, symbols, and words instead of introducing additional representation types that were not a part of the Rule of Four framework.

Another reason for sticking to the use of conventional CAS representations in the Activities in Chapter 1 is that it would have given students more experience to reason about and make sense of the link between expressions and function, and the ideas of equivalence in graphs, tables, symbols, and words.

Translating from expressions to Cartesian graphs to make sense of equivalent expressions may have helped to make the symbolic transpositions to determine equivalent expressions more meaningful. Note however that as introduced in the
discussion of the teaching experiment results, one purpose of introducing the pool

diagrams with equivalent expressions was to provide access to a visual proof of the
distributive property; one of three components for reconciling, generalizing, and proving
(Kieran & Saldanha, 2008). It was conjectured that by using an alternative visual

representation (other than a Cartesian graph of lines) that students might come to make

sense of why these expressions are equivalent.

In retrospect, seeing equivalence and non-equivalence from a graphical
perspective (such as in Activity 2.5) may have better supported students’ reasoning about

the symbolic transpositions. An example of a task that would be revised is “Draw your

own diagram to show that $4(x + 3) = 4x + 12$” (Day 14, Activity 6B HW). This could read

instead, “Use graphs, tables, and/or symbols to show why $4(x + 3) = 4x + 12$. Write in

words what this equation means to you.” Note how the revision to the task focuses on the

meaning of the equation with options for flexible representations of this mathematical

object.

Reconciling CAS and P&P to Support Transpositions

Several of the CAS and P&P techniques and theoretical components of the
learning progression that were tested in Kieran and Drijver’s (2006) experiments with

grade 10 high school students were tested in the present study with grade 9 high school
students (albeit at a lower level of mathematical sophistication, e.g., rational expressions

and equations versus linear expressions and equations, respectively). One compelling
example of a consistent result across these studies is the power of a “surprise element”
when an unexpected CAS result prompts further reflection on the part of the student.
Kieran and Drijvers (2006) found that the coordination of CAS technique and theoretical expectation prompted cognitive conflict in some students, and promoted further reflection on the mathematics of the task. In the present study, the distributive property activity design and enactment (Activities 5-6, Days 10-12) was more successful than others such as the combining like terms activity (Activity 2, Days 4-6) at encouraging a reconciling activity that promoted successful symbolic transpositions and meaning making. The reconciling notion had limited extensions into graphical and tabular representation types, with the most notable being the case of a linear function table showing two functions (Activity 9, Day 17). Despite this connection to Kieran and Drijvers, the present study did not assume an instrumental perspective on individual students’ tool use. Moreover, the reconciling notion, as expanded upon by Kieran and Saldanha (2008), played only a minor role in the teaching experiment discussed here (spanning 8 CMs across a total of 5 different days).

More research might be directed toward how graphical, tabular, and verbal representation types might be successfully reconciled across tool-based representations to support students’ abilities in performing transpositions in representation types other than symbolic. Learning such skills may promote a more balanced approach to RF instead of a heavy emphasis on the symbolic representation type. The reconciling activity within multiple representation types is also conjectured to make the learning transfer from CAS to P&P more meaningful.
Knowledge of Equal Sign

Research by Knuth, Stephens, McNeil and Alibali (2006) on students’ understanding of the “=” sign and its connections to equation solving was the primary impetus behind the flow of the instructional sequence to include a focus on understanding the equal sign as an equivalence relation. In particular, the present study was based on the notion that students should have a relational understanding of the equal sign in order to be successful in solving equations. The relational understanding of the equal sign is built into the activity sequence of the instructional design, but students’ knowledge of this symbol was not formally tested throughout the teaching experiment. Following Kieran and Drijvers (2006) the attempt to incorporate the meaning of an equation as an equivalence relation into the teaching experiment is based on first studying equivalence of expressions (Chapter 1), then examining the meaning of equations of the form \( ax + b = cx + d \) with respect to deciding if this equation is sometimes, always or never true (Chapter 2).

It is important to note that one area of difficulty that arose during the teaching experiment was the use of accurate mathematical language to describe equations, expressions, and equivalence. The use of informal language to describe this relationship was also acknowledged by Kieran and Drijvers (2006) as an area that students struggled with in their teaching experiment on a related progression of learning and that warranted future attention in research. In results from the present study, the teacher also struggled with the correct use of language including expressions and equations, and the activity
design did not support the transition from equivalence to equation solving. Based on the fact that one’s RF depends on his/her ability to successfully interpret the meaning of various representations and representation types it seems as though these language difficulties were conditions of the learning environment that did not help to support students’ change in RF.

*Connecting Equivalence and Solving*

The notion of introducing equations that are sometimes, always, or never true in Activities 7 and 8 was decided as a way to transition students from considering the equivalence of expressions to solving equations. However, the classroom interactions did not seem to be well connected to students’ experiences with equivalence of expressions, or were to the terminology of “solution.” This in turn made it difficult for the work with sometimes, always, or never true equations to be meaningful. In their experiment, Kieran and Drijvers had specifically discussed the importance of connecting equivalence to solutions of equations through purposeful reflection questions.

The reflection question that was raised after that concerned the relation between the nature of an equation’s solution(s) and the equivalence or non-equivalence of the expressions that form the equation.

(Kieran & Drijvers, 2006, p. 216)

This piece was absent in the present study and may have helped to clarify the intent of this important transition point in the teaching experiment.

It seemed as though the emphasis on the truth of an equation overshadowed the solution set to that equation. In other words, the informal connection in language—sometimes means one solution, always means infinite solutions, and never means no solutions—does not seem sufficient for helping students understand what the solution set
actually is. In the exchange below, the nuance between an equation being always true and
the fact that every value of $x$ is a solution to the equation needs to be better articulated.

Ms. L: I have equivalent expressions [in reference to $-s - 4$ and $-1(s + 4)$] so I
know that I have infinite solutions [for the equation $-s - 4 = -1(s + 4)$]. [...] Alright, here's the quick and dirty, how many people got $s = s$, or $-4 = -4$, or you
might have gotten $0 = 0$, you might have gotten any one of those answers when
you got down to the end. Which one did you get, Derek?
Derek: Um, I got $s = s$.
Ms. L: Yeah, is that sometimes, always, or never true?
Derek: Always
Ms. L: Always true. Infinite solutions. Is negative four equal to negative four? All
the time. Is zero equal to zero? All the time. Infinite solutions. Ok. So if your
symbols are always true, you've got infinite solutions.

(Activity 14, Day 22, 01:06:54:46)

It is conjectured that understanding the solution set may also lead to a better
understanding of equivalence of equations as a part of the reasoning process. In other
words, students need to make sense of the fact that “infinite solutions” means every value
of $x$ makes the equation true and that no solutions means no values of $x$ make the
equation true.

Chapter 3

Solution Sets to Equations in Graphs, Symbols, Tables

Based on the results of the teaching experiment not enough attention was given to
the process of solving and the recognition of solutions to linear equation within each of
graphic and numeric representation types. Moreover, based on in-depth analyses of
students’ RF in solving problems involving linear equations, students seemed to be better
at identifying an ordered pair or point as an invariant feature across representation types
and struggled in translating among representation types of the solution to an equation
(i.e., the $x$-value only). This suggests a need to pay greater attention to the nature of solutions in multiple representation types, and the ancillary aspects that are inherent in linking expressions and functions to view graphs and tables of equations.

Huntley and colleagues (2007) reported students’ difficulties in solving linear equations with an empty solution set and that students would often get stuck in the symbolic representation type, unable to overcome this barrier. One aspect of RF that Annie and Bryon demonstrated was the ability to overcome barriers in the symbolic representation type to be able to successfully identify a point that made an equation true (not the $x$-value of the solution, but the ordered pair that satisfied the equation). This ability was an indicator of their RF, and seems to be related to students’ experiences in the classroom in which connections were made across more than one representation type to identify this common point. More work needs to be done to make sense of the difference between seeing a point as an invariant aspect across these representation types, and seeing the $x$-coordinate as part of the solution set, with the $y$-value as extraneous information used to help solve the equation.

On a final note, the definition of equivalent expressions was crafted in such a way that it involved reference to a variety of representation types (numeric, graphic, symbolic, and context). Later experiences with equivalent expressions should have followed suit and not strayed to focus on other representation types, in this case, geometric area diagrams. Moreover, the definition of a solution to an equation should have been given in graphs, tables, symbols, and words. Instead, the intent of the activities in Chapter 3 was for students to make this assertion themselves after looking at particular cases. But in retrospect, to expect students to be fluent in using these representations to solve
equations, they should have been defined as such to parallel the definition given for equivalent expressions.

*Equivalent Equations*

In the enacted instructional theory, tasks that focused on equivalence of equations were not tested. Based on the above discussion the revised instructional theory includes this as a final component. The approach to equivalent equations suggested by Davis (2005) and Zbiek and Heid (2011) could be used as a starting point to explore task design in this aspect of the activity sequence.

*Conclusions*

This study extended Kieran and Drijver’s (2006) approach to equivalence and equations by extending the set of techniques that were tested and the learning progression that was enacted by including graphs, tables, and words as viable representation types to determine equivalence of expressions and to solve equations. With a focus on supporting students’ RF in solving problems involving linear equations, the present study was more concerned with a balanced approach to transpositions and translations across the Rule of Four web.

As alluded to in the above discussion, the learning goal and mathematical practice posited by CCSSI (2010) were not met in the teaching experiment—students did not come to see equation solving as a process of reasoning based on equivalence of equations, and the classroom expectation of being a strategic user of tools was not evident in the retrospective analyses. However, it seems as though progress was made
toward testing a learning progression that was founded on a representational lens to studying linear equations with a balanced use of CAS and P&P tools. In particular, the revised activity structure (Table 6.1), activity sequence (Table 3.6) and learning progression (Table 3.5) are proposed as elements of an empirically-based instructional theory that may help to support students achieve the aforementioned learning goal and higher levels of RF.

*Teacher Support*

More attention should be paid to teacher support materials and intended solutions to the activities in written form. The swift pace of the teaching experiment did not allow much time for explication of solution approaches as much as the design of the activities and intended sequencing. It is possible that had more attention been paid to this aspect of activity design (teacher support) that the theoretical components of the learning progression may have been more salient, especially in regards to the mathematical storyline of equivalence of expressions, to equations as equivalence relations, to solving and creating equations as a process of reasoning.

Most of the teacher guide materials were written from the perspective of making connections to the learning trajectory, that is, ‘How does the design of this activity fit with the research?’ It is possible that the teacher saw this writing as more useful for the researcher than as a guide to instruction or enactment of the curriculum tasks.
**Task Design**

The tasks that students engaged with could have been at a higher cognitive demand. A good example of this is in Activity 2 in which students compared the expressions \(x + x + x + x + x\) and \(2x + 3x\). The impetus for this activity design was from Edwards (2003) and the examples that he used with beginning algebra students on exploring the CAS technique of automatic simplification. In retrospect, such an activity could have been enacted as a brief warm-up activity or introduction to such CAS technique; a more difficult example in which the equivalence of expressions was not immediately obvious would have been better suited for a task that would have motivated the use of CAS as a representational toolkit to examine the equivalence of expressions.

We decided that in future uses of graphical and tabular representations to verify equivalence of expressions that students need to have a “need” to use the other representations (i.e., the symbolic simplification was not obvious in one step). This is part of our Warm-Up for Day 7.

(Day 6 Daily Class Summary)

See Figure 6.1 for an example of what this might look like.

**Figure 6.1.** Example of using CAS as a representational toolkit to verify equivalence that is not obvious from inspection of symbolic form.
Design Experiment

As alluded to in Chapter 3, the cyclical aspect of the design experiment occurred on a daily basis. The planning for instructional experimentation was based on the overarching instructional theory, while the revision and planning for the experimentation that occurred on each subsequent day was largely informed by what was enacted during the experimentation. This included the use of student work and the interpretations of the teacher and researcher on how well the daily learning goal and components of the instructional theory were being met. On most days, the teacher assigned some type of classwork that was collected; a quiz, or exit ticket, were the most common form of formative assessment. These artifacts were informally used to inform our thinking about ways in which the teaching experiment might be modified to inform the hypotheses that were tested next.

A few recommendations are made on how to improve the enactment of this aspect of the design experiment. First, the warm-up and exit tickets be designed to be similar to one another and be specifically connected to the day’s learning goal. This would allow for students’ progress to be measured more easily in relation to the goals of the teaching experiment. One problematic aspect of this form of data collection is that students did not always complete these tasks. Second, the Texas Instruments Navigation software and hardware may have been used to capture students’ screen shot representations throughout and at the close of a lesson so as to understand the nature of the representations and representation types they utilized during class, or to conduct a quick poll on student thinking. This form of data collection would have been an efficient way to formatively assess students related to the daily hypotheses. Finally, more notes could have been
captured during the thought experiment sessions held between the teacher and researcher on a daily basis. An improved instrument protocol for collecting notes during these meetings may have helped to summarize the conversations in a way to improve the future experimentation. One possible difficulty of using such a written instrument is that it may distract from personal interactions directed toward sensitive topics such as teacher pedagogy and how well the enactment of the curriculum seemed to meet the learning goals.

Significance of This Research

The tested and revised instructional theory captures a possible sequence of activities that may be used as a foundation for curriculum and instruction that prepares students to use graphs, tables, symbols, and words to solve linear equations. It seems as though this work would also lay the foundation for studying the topic of equivalence of equations, critical to explaining the reasoning process behind solving linear equations (cf. CCSSI, 2010).

The nature of this research was based on the need to establish links between research and practice as a means to study individual students within a classroom learning environment. For instance, the collaborative nature of the teaching experiment served as an intensive PD experience for the participating teacher with a focus on task design for CAS and P&P that were specifically guided by an instructional theory with an articulated learning progression. One unique aspect of this collaborative teaching experiment was the opportunity for “teachers and students to have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and
communications” (NCTM, 2011). Consistent with NCTM’s position statement on technology, the power and potential of computer technologies for enhancing student learning and understanding of mathematics has long been recognized (cf. Conference Board of the Mathematical Sciences, 1983; Fey et al., 1984; and more recently Fey, Cuoco, Kieran, McMullin, & Zbiek, 2003; Masalski, 2005; and Zbiek Heid, Blume, & Dick, 2007). The role of CAS and P&P technologies were integral to the study of students’ change in RF and the conditions of the learning environment in the present study.

In closing, it should be noted that simply having access to mathematical software tools like TI-Nspire is not sufficient for supporting students’ change in RF. Curriculum materials that integrate that software, and teachers with both the disposition toward and knowledge of using technology strategically, play critical roles in mediating students’ use of technological tools and what they ultimately learn (cf. Zbiek & Heid, 2011). Thus the articulation of an empirically-based instructional theory—including the activity structure, learning progression, techniques, and classroom expectations—is a step forward in addressing the need to understand the classroom conditions in a combined CAS and P&P environment that can support students’ change in RF, an indicator of their mathematical sophistication.

Directions for Future Research

Directions for future research are mentioned here with respect to particular research constructs, theoretical lenses, and contemporary initiatives in the field of mathematics education. Several ideas for ways to improve a next iteration of the teaching
experiment have already been discussed in earlier sections (e.g., in task design and teacher support materials).

External representations are created with tools, and are a necessary component to accessing and understanding students’ RF. Future research on students’ RF might take an instrumental approach to more fully understand the role of the tool in students’ representation specific abilities. From an emergent perspective on learning in the classroom setting, the orchestration metaphor of Drijvers and Trouche (2008) might offer a productive lens for studying classroom activity and the interaction patterns between teachers and students. On the role of tools, the mathematical practice to “Use appropriate tools strategically” (CCSSI, 2010) needs to be researched in more depth in classroom learning environments with specific attention to the relationship between CAS and P&P tools. One way students might become more strategic in their use of tools is to engage in an activity structure that incorporates Kieran and Saldanha’s (2008) reconciling notion with specific attention to anticipation, reflection, justification, and generalization.

Finally, to address the concern of appropriate use of language in discerning between the closely related objects of expressions and equations and topics of equivalence and solving, future research should give a more central focus on the role of the equal sign. In particular, future experimentation might start from students’ conceptions of the equal sign and use that to build a trajectory of learning that supports a relational understanding. The contemporary research by Matthews and colleagues (2012) would likely be productive in informing such an investigation.
Closing Comments

Kieran and Drijvers (2006) argued that the heart of algebra consists of symbolic transformations and that other research had already addressed the role of multiple representation types in making algebraic objects more meaningful (cf. Heid, 1996; Kieran and Yerushalmy, 2004). Based on the results of the teaching experiment and case studies, it is argued that a focus on the construct of RF gives the role of MR a renewed purpose in school mathematics. Students who are representationally fluent not only have an ability to make connections across representation types, but are flexible in their perspective and are well positioned to be resourceful when unable to solve a problem from a given representation or type. Thus from a representational lens, the value of having the ability to create, interpret, translate between, and transpose within graphic, numeric, symbolic, and verbal representation types in doing and communicating about mathematics is akin to being a competent problem solver, ready to solve new problems and persevere in solving them.

While the student participants in this study did not demonstrate consistently high levels of RF, they did show gains in their skills to transpose within and translate between MR when solving problems involving linear equations. This research has contributed to the field of mathematics education by characterizing students’ change in RF when solving problems involving linear equations to include higher levels of RF and a greater use of MR in their approach to these tasks. These characterizations of RF are also tied to a classroom learning ecology that was guided by an instructional theory in which a sequence of tasks, techniques, and theoretical components were tested and later refined.
There seems to be great promise in experimental research that focuses on cross-cutting aspects of mathematics such as representations, and brings research results and technological innovations into the classroom through collaborations with in-service teachers. School mathematics is instantiated in classrooms like Ms. L’s, in the minds of students like Annie, Bryon, and Carlos, and in the form of internal and external tool-based representations. The goal of supporting students in their learning environments to change their RF remains an important quest for both researchers and practitioners. This goal may be best achieved through a collaborative endeavor that capitalizes on the strengths of an empirically-based learning trajectory guided by contemporary goals for school mathematics including the combined use of P&P and representational toolkits such as CAS.
REFERENCES


Appendix A

HSIRB Approval Letter
Date: August 18, 2011

To: Jon Davis, Principal Investigator
    Nicole Fonger, Student Investigator for dissertation

From: Amy Naugle, Ph.D., Chair

Re: HSIRB Project Number 11-07-16

This letter will serve as confirmation that your research project titled “Algebra Students' Development of Representational Fluency in a Computer Algebra System and Paper and Pencil Environment” has been approved under the expedited category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may only conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition, if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: August 18, 2012
Appendix B

Subject Recruitment
Greetings,

You are receiving this letter because your child is scheduled to be enrolled in Mrs. L’s fifth period Algebra A class in the first trimester of the 2011-2012 school year at South High School. In this class, Mrs. L will be participating in a research study together with Nicole Fonger, a graduate student in mathematics education at Western Michigan University. This research study is aimed at improving the learning and instruction of algebra.

Your child may volunteer to participate in this study, and you may elect to grant permission for his or her participation. There are two enclosed forms that describe the details of the study. We ask that you and your child please read these forms. To be eligible to volunteer to participate in this research study, both you and your child must check a box and sign the forms.

Please have your child return these forms on the first day of class, September 6th, to Mrs. L, regardless of the decision that is made about volunteering to participate.

Thank you for your time.


*Student Recruitment Script Delivered by the Student Investigator During Class Time*

You are invited to participate in a research study that may help other teachers make improvements to teaching algebra with technology. It may also help algebra students successfully learn algebra with technology. There are two different levels of this research that you may volunteer to participate in: (1) you may elect to have your pre-/post-test work collected and be captured on classroom video recording for research purposes, and (2) you may elect to have your pre-/post-test work collected, be captured on classroom video, and volunteer to participate in two interviews with the researcher and have your daily class work recorded with individual video cameras and photo copies. If you elect to participate in the first level (pre-test work collection, and classroom video), you will automatically be involved in this research. If you elect to participate in the second level (pre-/post-test work collection, classroom video, two interviews, and daily class work recorded) you may not be chosen to participate in interviews and individual video-recording, even though your pre-test work and classroom video will still be collected.

To volunteer to participate in this research study, both you and your parent or guardian must read, check a box, and sign a form that gives more details about this research project. Also, there will be a separate form that Mrs. L will talk to you about and give to you that your parents must sign to allow for video to be captured in the classroom.

For those students who agree to participate in either the two levels of this study, Mrs. L will assign extra credit points toward your homework grade. You will not be penalized if you choose not to participate. Also, for those that choose not to participate, there will be other opportunities of a similar nature throughout the trimester for earning extra credit points toward your homework grade.

Mrs. L and myself will notify you if you are selected to participate in interviews and have your daily classwork and deskwork collected and video-recorded. If you are not notified, you have not been selected to participate in interviews.

<The researcher will then hand out the forms and address any questions the students may have to better ensure an informed consent process.>
Appendix C

Consent and Assent Documents
Student Assent Form

Western Michigan University
Department of Mathematics
Principal Investigator: Dr. Jon Davis
Graduate Student Investigator: Nicole Fonger
Project Title: Ninth Grade Algebra Students’ Development of Representational Fluency

Background Information About the Research Project:
Nicole Fonger is doing a research study as part of her Ph.D. program in the Mathematics Department under the direction of Dr. Jon Davis at Western Michigan University. A research study is a special way to find out about something. We want to find out about how you use various representations with technology and paper and pencil when learning algebra.

Your Participation in the Research Project:
There are two levels of participation that you may volunteer to participate in for this research project that is being conducted in Mrs. L’s Algebra class. If you are in the study at either level of participation you may be in the classroom videotape.

At the first level of participation, we would: collect your pre-test and post-test and video-record the whole classroom. Your participation at this first level would not be any different than what you regularly do in class.

At the second level of participation we would: collect your pre-test and post-test and video-record the whole classroom, and also select some students to be interviewed and to have their classwork recorded each day. If you agree to be interviewed and have your classwork documented, you may not be selected, not because of anything you did but because there is only enough time to interview about 3 students. If you are selected to be interviewed and have your classwork recorded each day, you would be asked to schedule two forty-five minute after-school interview sessions with Ms. Fonger near the beginning and middle of the trimester, save and submit your homework and classwork to be copied on a daily basis, and allow for a video camera and tripod to be positioned on your desk.

The two interviews between you and Ms. Fonger will be video-recorded so that Ms. Fonger can watch them later and learn what you know about using multiple representations, any writing that you do will also be collected. During the interviews you will be asked to explain your solution approaches to several mathematical tasks that you completed on the pre-test and post-test or on related tasks. Your daily classwork will be captured using a video camera to document what you write and how you use technology. During class time you will be asked to write clearly, verbalize your thinking if appropriate, and save all technology work that you do. Copies of your math papers will also be made after we remove your name from them.

A potential risk for your participation in interviews is the loss of time when they are conducted after school. Also, you may be uncomfortable have your work being video-taped and having your voice recorded. However the video camera will not be positioned to see your face, only your deskwork and your voice will be recorded. If you decide to be in this study it may help you to become better at explaining your thinking while solving algebra problems and become a better user of technology. Mrs. L will give every student who agrees to participate in interviews and classroom video-taping extra credit points toward their homework grade, even if you are not selected to be interviewed or have your classwork recorded. Only the principal and student investigators will have access to the information collected during this study. All video and class
documents will be kept locked in the office of the principal investigator for a minimum of three years after the interviews are conducted, then destroyed. When we are done with the study, we will write a report about what we found out, and it may be published and presented to other math teachers. We won’t use your name in the report.

You can choose to stop participating in the study at anytime for any reason. You will not suffer any prejudice or penalty by your decision to stop your participation. You will experience NO consequences either academically or personally if you choose to withdraw from this study. Should you have any questions prior to or during the study, you can contact the principal investigator, Jon Davis at 269-387-4591 or jon.davis@wmich.edu, or the graduate student investigator, Ms. Fonger at 269-387-4589 or nicole.m.lanie@wmich.edu. You may also contact the Chair, Human Subjects Institutional Review Board at 269-387-8293 or the Vice President for Research at 269-387-8298 if questions arise during the course of the study.

This consent document has been approved for use for one year by the Human Subjects Institutional Review Board as indicated by the stamped date and signature of the board chair in the upper right corner. Do not participate in this study if the stamped date is older than one year.

Please check the level of participation then sign after reading this form:

Your check(s) and signature below indicates that you agree to participate in the research as described on this form and that information collected can be used for research activities.

☐ I give permission to have my pre-test and post-test be collected and to be video-taped as part of regular classroom activities. The information collected as part of this study may be used for research purposes.

☐ I give permission to have my pre-test and post-test be collected, to be video-taped as part of regular classroom activities, to be interviewed, and to have my classwork recorded on a daily basis. The information collected as part of this study may be used for research purposes.

Print name here: ____________________________

Sign name here: ____________________________    Today’s Date _______
Parent Consent Form

Western Michigan University
Department of Mathematics
Principal Investigator: Dr. Jon Davis
Graduate Student Investigator: Nicole Fonger
Project Title: Ninth Grade Algebra Students’ Development of Representational Fluency

Background Information About the Research Project
Your child has been invited to participate in a research project that will serve as Nicole Fonger’s dissertation research in partial fulfillment of the requirements of a Ph.D. in the Mathematics Education under the direction of Dr. Jon Davis at Western Michigan University. The purpose of this study is (1) to better understand how ninth grade algebra students use and communicate about the representations they create using paper and pencil and mathematics technology when solving algebra problems; and (2) to determine how to better support students in learning algebra with paper and pencil and technology, such as computer algebra systems.

Your Child’s Participation in the Research Project
The research is being conducted in Mrs. L’s Algebra class. There are two levels of participation that you may give permission for your child to participate in. If your child is in the study at either level of participation they may be in the classroom videotape.

At the first level of participation, we would: collect your child’s pre-test and post-test and video-record the whole classroom. Your child’s participation at this first level would not be any different than what they regularly do in class.

At the second level of participation we would: collect your child’s pre-test and post-test and video-record the whole classroom, and also select some students to be interviewed and to have their classwork recorded each day. If your child is selected to be interviewed and have their classwork recorded each day, they would be asked to schedule two forty-five minute after-school interview sessions with Ms. Fonger near the beginning and end of the trimester, save and submit their homework and class work to be copied on a daily basis, and allow for a video camera and tripod to be positioned on their desk.

The two interviews between your child and Ms. Fonger will be video-recorded and any writing that your child does will also be collected. During the interviews your child will be asked to explain their solution approaches to several mathematical tasks that they completed on the pre-test and post-test or on related tasks. Your child’s daily classwork will be captured using a video camera to document what they write and how they use technology. During class time your child will be asked to write clearly, verbalize their thinking if appropriate, and save all technology work that they do. Copies of their math papers will also be made after we remove their name from them. Note: you may grant permission for your child to be interviewed and have their classwork recorded but they may not be selected in the final sample simply due to having too many volunteers.

Risks and Benefits of Your Child’s Participation
Regardless of your child’s participation, they will need to complete all tests and written assignments as part of the normal classroom practice. If you choose not to have your child participate, their work will simply not be photocopied and they will not be included in any videotaping or interviewing. Also, because the use of technology exposes students to an alternative
method of instruction, this investigation may benefit your child with their improved understanding of algebraic concepts. Children who agree to be interviewed may benefit from sharing their thinking about mathematics and develop additional confidence in their mathematical thinking. The collection and analysis of assessment data may provide Mrs. L with information on the potential benefits of certain instructional techniques. Mrs. L will give every student who agrees to participate in this study extra credit points toward their homework grade, even if they are not selected to be interviewed or have their classwork recorded.

A potential risk for your child’s participation in interviews is the loss of time when they are conducted after school. Also, your child may be uncomfortable having their work video-taped and having their voice recorded. However the video camera will not be positioned to see your child’s face, only their deskwork and voice will be recorded.

**Confidentiality and Dissemination of Data**

Only the principal and student investigators will have access to the information collected during this study. All video and class documents will be kept locked in the office of the principal investigator for a minimum of three years after the interviews are conducted, then destroyed. The results of this study will be presented at a dissertation defense presentation, and may also be presented at a national conference and in a journal, both designed for mathematics educators. Pseudonyms will be used so that confidentiality of identity is maintained.

You can choose to withdraw your child from participating in the study at anytime for any reason. Your child will not suffer any prejudice or penalty by your decision to stop their participation. They will experience NO consequences either academically or personally if you choose to withdraw them from this study. Should you have any questions prior to or during the study, you can contact the principal investigator, Jon Davis at 269-387-4591 or jon.davis@wmich.edu, or the graduate student investigator, Nicole Fonger at 269-387-4589 or nicole.m.lanie@wmich.edu. You may also contact the Chair, Human Subjects Institutional Review Board at 269-387-8293 or the Vice President for Research at 269-387-8298 if questions arise during the course of the study.

This consent document has been approved for use for one year by the Human Subjects Institutional Review Board as indicated by the stamped date and signature of the board chair in the upper right corner. Do not participate in this study if the stamped date is older than one year.

Please check the level of participation then sign after reading this form:

Your check(s) and signature below indicates that you give your permission for ________________________________ (child’s name) to participate in the research as described on this form and that information collected can be used for research activities.

☐ I give permission to have my child’s pre-test and post-test be collected and to be video-taped as part of regular classroom activities. The information collected as part of this study may be used for research purposes.

☐ I give permission to have my child’s pre-test and post-test be collected, to be video-taped as part of regular classroom activities, to be interviewed, and to have their classwork recorded on a daily basis. The information collected as part of this study may be used for research purposes.

Print name here: ________________________________________

Sign name here: ________________________________________  Today’s Date ________
Teacher Observation, Videotape, and Audiotape Consent Form

Western Michigan University
Department of Mathematics
Principal Investigator: Dr. Jon Davis
Graduate Student Investigator: Nicole Fonger

Project Title: Ninth Grade Algebra Students’ Development of Representational Fluency

Purpose of Study
You have been invited to participate in a research project titled "Ninth Grade Algebra Students’ Development of Representational Fluency." This project will serve as Nicole Fonger’s dissertation research in partial fulfillment of the requirements of a Ph.D. in K-12 Mathematics Education. The purpose of this research study is to (1) better understand how ninth-grade algebra students use and communicate about the representations they create using paper and pencil and computer algebra systems (CAS) when solving algebra problems, and (2) determine how to better support students during instructional situations with an emphasis on students’ development of representational fluency with paper and pencil and CAS as they learn to master solving equations.

Participation
You are the participating teacher for this study. Based on your discretion, students in one of your “average” algebra classes in the Freshman Academy at South High School will be selected to participate in this study. All students in this class will be recruited have the entire classroom video-taped and have their pre-test and post-test work collected. Only three students will be recruited to participate in two interviews and to have their classwork photocopied and deskwork recorded with video cameras on a daily basis.

The research activities that you will participate in for this study will involve approximately 10-15 hours of planning time prior to September 2011. The most time intensive part of this project will be carrying out the instructional unit for approximately the first four weeks of the first trimester in the 2011-2012 school year.

Planning time prior to the instructional unit would allow for us to discuss and interpret the specific mathematical content and goals of the target instructional unit, based roughly on Unit 1 of the Common Core Standards for School Mathematics. Together we would negotiate and carry out a feasible plan for developing and refining instructional materials and annotated lesson plans that incorporate both CAS and paper and pencil technologies.

Data Collection Activities
You will be the primary instructor for the planned instructional unit. The student investigator will observe, collect field notes, and video-record the instructional sessions from the back of the classroom on a daily basis throughout the instructional unit. Lesson plans and daily activities will also be collected. The purpose of documenting classroom activity with field notes, lesson plans, and video is to be able inform decisions about changes that might be made to the instructional unit and to document classroom practices that seem to contribute to students’ development of representational fluency. As a regular part of classroom practice, either the teacher or the student investigator may make suggestions to change and improve subsequent lesson plans and activities based on what transpired in earlier lessons. The responsibility to update these plans in electronic documents will be shared between both collaborators, and also used as a source of data.

To encourage regular communication about the instructional unit as it unfolds, and to better understand the teaching and learning situation, the teacher and student investigator will
likely engage in regular conversations before or after class about the planned or enacted lesson. In addition, the teacher and student investigator will meet on a weekly basis after school hours for no more than one hour. These meetings will be audio recorded for the purpose of capturing the ideas that were discussed so that they can be revisited later.

**Implications of Research**

An enormous amount of data will be generated and collected as part of this study. In particular, you may be uncomfortable being video-taped or having our debriefing meetings audio recorded. To account for this, this information will be protected. Pseudonyms will be used for all participants so that confidentiality of the identity is maintained, and only the principal and student investigators will have access to the data collected during this study. The results of this study will be presented at a dissertation defense presentation, and may also be presented at a national conference and in a journal, both designed for mathematics educators.

It is anticipated that as an outcome of this study, you will gain valuable professional development experiences regarding curriculum, instruction, CAS technology. You may gain deeper insights into student thinking and particular classroom practices that support students’ development of representational fluency with paper and pencil and CAS-based representations. A small stipend will be offered to support your work in this professional development (which may be in the form of University Credit).

You can choose to stop participating in the study at anytime for any reason. You will not suffer any prejudice or penalty by your decision to stop your participation. You will experience NO consequences if you choose to withdraw from this study.

Should you have any questions prior to or during the study, you can contact the principal investigator, Jon Davis or the student investigator, Nicole Fonger. You may also contact the Chair, Human Subjects Institutional Review Board at 269-387-8293 or the Vice President for Research at 269-387-8298 if questions arise during the course of the study.

Sincerely,

**Jon Davis**  
Principal Investigator  
jon.davis@wmich.edu  
269-387-4591

**Nicole Fonger**  
Student Investigator  
nicole.m.lanie@wmich.edu  
269-387-4589

This consent document has been approved for use for one year by the Human Subjects Institutional Review Board as indicated by the stamped date and signature of the board chair in the upper right hand corner. You should not sign this document if the corner does not have a stamped date and signature.

I have read this informed consent document. The risks and benefits have been explained to me and I agree to participate in this research study.

Printed Name ___________________________________________ Date __________________

Signature ___________________________________________ Date __________________
Appendix D

Observational Field Notes
**Observation Protocol "0" – Daily Classroom Instruction**

**Class and Lesson Details**  
Date: _________________  
Day of Unit: __________

**Artifact Collection and Management**

<table>
<thead>
<tr>
<th>Event Description</th>
<th>Folder (Teaching Experiment) Name</th>
<th>File Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson Activity</td>
<td>COPY PDF DOC</td>
<td>dayX_T_activity.pdf/doc:</td>
</tr>
<tr>
<td>Lesson Plan</td>
<td>COPY PDF DOC</td>
<td>dayX_T_lessonplan.pdf/doc:</td>
</tr>
<tr>
<td>Class Obs. Nicole</td>
<td>COPY PDF</td>
<td>dayX_classroomFieldNotes.pdf:</td>
</tr>
<tr>
<td>Classroom Video</td>
<td>ORIG SAVED</td>
<td>dayX_T_classroom.dv/mov:</td>
</tr>
<tr>
<td>Student Video</td>
<td>ORIG SAVED</td>
<td>dayX_sY_classroom.mov:</td>
</tr>
<tr>
<td>Student Homework</td>
<td>COPY PDF</td>
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</tr>
<tr>
<td>Student Activity Work</td>
<td>COPY PDF</td>
<td>sY_dayX_activity.pdf:</td>
</tr>
<tr>
<td>Student TI-Nspire File</td>
<td>ORIG SAVED</td>
<td>sY_dayX_TIwork.tns:</td>
</tr>
</tbody>
</table>
Time instruction actually began: ______________ Explain if different from scheduled time.

Time instruction actually ended: ______________ Explain if different from scheduled time.

Number of: Students=_________ Teacher / Teacher Aid= ______________

List any unusual aspects of the setting for the observation (e.g., last day before break, day of the school pay, day of conferences, etc.):

**Before Class Session**
Thought experiments related to conjectured local instruction theory:

Teacher-researcher conversations that occurred:

**During Class Session**

*Classroom Mathematical Practices & Individual Mathematical Conceptions and Activity*

<table>
<thead>
<tr>
<th>Time</th>
<th>Tools</th>
<th>Class / Indiv. Practices &amp; Activity</th>
<th>Classroom Practices (T-teacher, S-student), Activity &amp; Conceptions (e.g., transpositions, translations, making connections)</th>
</tr>
</thead>
</table>


**Classroom Mathematical Practices & Individual Mathematical Conceptions and Activity**

<table>
<thead>
<tr>
<th>Time</th>
<th>Tools</th>
<th>Class / Indiv.</th>
<th>Classroom Practices (T-teacher, S-student), Activity &amp; Conceptions (e.g., transpositions, translations, making connections)</th>
</tr>
</thead>
</table>
### Classroom Mathematical Practices & Individual Mathematical Conceptions and Activity

<table>
<thead>
<tr>
<th>Time</th>
<th>Tools</th>
<th>Class / Indiv.</th>
<th>Classroom Practices (T-teacher, S-student), Activity &amp; Conceptions (e.g., transpositions, translations, making connections)</th>
</tr>
</thead>
</table>


### Classroom Mathematical Practices & Individual Mathematical Conceptions and Activity

<table>
<thead>
<tr>
<th>Time</th>
<th>Tools</th>
<th>Class / Indiv.</th>
<th>Classroom Practices (T-teacher, S-student), Activity &amp; Conceptions (e.g., transpositions, translations, making connections)</th>
</tr>
</thead>
</table>
Conditions Under which Students Develop RF in Solving Problems Involving Linear Equations

<table>
<thead>
<tr>
<th>Time</th>
<th>Comments</th>
<th>Means of Support</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1. Tools</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a. CAS</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Paper and Pencil</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2. Classroom Expectations (Norms or standards for practice and discourse)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a. Focused on mathematics.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Strategic user of tools.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>c. Representationally flexible.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3. Classroom Activity Structure (Sequence of activities to organize engagement)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a. Anticipate.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Act.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>c. Reflect.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>d. Connect.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. Activities (Sequence of activities and instructional tasks)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a. Equivalent Expressions, Linear Equations of Functions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Linear Equations as Equivalence Relations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>c. Creating and Solving Linear Equations</td>
</tr>
</tbody>
</table>

Student Conceptions and Activity that Evidence Change in Students’ RF Over Time

<table>
<thead>
<tr>
<th>Time</th>
<th>Comments</th>
<th>Measures of RF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>• Creation of tool-based representations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Use of single representation type (work within)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Movement between tool-based representation types</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Interpretations of and communication about what representations illustrate or explain</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Use of representations in justifying mathematical assertions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Connections among representations/representation types</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Generalizations across representations/representation types</td>
</tr>
</tbody>
</table>
**Role of the Researcher**

During the lesson, the researcher played the role of a technical assistant  YES  NO

Whole-class (describe):

Individual basis (describe):

Include notes here about other pertinent aspects of the researcher’s participation:

**After Class Session**

<table>
<thead>
<tr>
<th>Daily Debriefing</th>
<th>MP3 TIMELN COPY PDF</th>
<th>Folder (Thought Experiments) Name (dailyDebriefing_NF_ML_dayX.mp3/pdf/TLcode):</th>
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<tbody>
<tr>
<td>Audio</td>
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<td></td>
</tr>
<tr>
<td>Written</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Researcher Reflection</th>
<th>MP3 TIMELN COPY PDF</th>
<th>Folder (Thought Experiments) Name (dailyReflection_NF_dayX.mp3/TLcode/doc/PDF):</th>
</tr>
</thead>
<tbody>
<tr>
<td>Audio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Written/Type</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Other notes about ongoing analysis of *this* lesson:

Other notes about *future* instructional experimentation:
Appendix E

Pre-Test
1. Linus has a savings account, in which he has ‘x’ dollars. Some of his friends also have accounts. Express the balance (amount of money) in each of his friend’s accounts as an algebraic expression in terms of the balance of Linus’ account (x).

a. Lucy has twice as much as Linus.

b. Pigpen has 6 more than 3 times as much as Linus.

2. If Linus has a balance of $80, find the balance of each person. Show your thinking.

a. Lucy

b. Pigpen

3. The following questions are about this statement:

\[ 3 + 4 = 7 \]

a. The arrow above points to a symbol. What is the name of the symbol?

b. What does the symbol mean?

c. Can the symbol mean anything else? If yes, please explain.
True or False

4.) _______________ $7s-13 + 3s - 5$ is a linear equation.

5.) _______________ -8 is a solution to the equation $x + 7 = 7 + x$.

Multiple Choice
Identify the choice that best completes the statement or answers the question.

ANSWER
letter only please

6.) Situation ______ can be described by the symbolic rule $4x$.

   A. Planted as a sapling at 6” tall, the tree grew 4’ a year.
   B. Katrina is four years older than her sister.
   C. The area of a square with side length $x$.
   D. Joe earns $4 an hour babysitting.

7.) The expression $3x + 14$ is not equivalent to:

   A. $x + x + x + 7 + 7$               C. $5 \cdot 2 + 2(1.5x) + 4$
   B. $3(x + 7)$                       D. $7 - (-3x - 14) - 7$

Short Answer

8.) Use the Distributive Property and combine like terms to simplify:

   $3(x^2 + x - 4 + 3x + 6)$


9.) Given one of the representations below, create the other two.

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
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</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

10.) Refer back at the first row in Question 9. In this row, clearly indicate the solution to the equation when \( y=5 \) for each of the table, graph, and symbolic equation.
For 11-12, solve for the variable and check your solution using a table, graph, or equation (your choice). Use graphs, tables, or equations to solve each equation in 11-12 for the given variable and check your solution. In each question, sketch the section of the graph (with the scale marked) or the table and indicate where you found the answer. If you solve the problem using equations, show all of your work.

11.) \[ 3(x - 6) = 12 \]

12.) \[ t - 2 + 3t = -6 + 4t + 4 \]

### Table 13.

<table>
<thead>
<tr>
<th></th>
<th>Steps</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.)</td>
<td>Original problem</td>
<td>[ 3x - 5 + 3x = x + 8 - x - 1 ]</td>
</tr>
<tr>
<td>2.)</td>
<td>Combine like terms</td>
<td>[ 6x - 5 = 2x + 7 ]</td>
</tr>
<tr>
<td>3.)</td>
<td>Subtract 2x from both sides</td>
<td>[ 4x - 5 = 7 ]</td>
</tr>
<tr>
<td>4.)</td>
<td>Add 5 to both sides</td>
<td>[ 4x = 12 ]</td>
</tr>
<tr>
<td>5.)</td>
<td>Divide both sides by 4</td>
<td>[ x = \frac{12}{4} ]</td>
</tr>
</tbody>
</table>

Shayla worked out the problem above, but when she substituted her solution into the original equation, she discovered that \( x = 3 \) is not correct.

Find her error. Explain where she made her mistake and why using complete sentences.

________________________________________________________________________
________________________________________________________________________
________________________________________________________________________
________________________________________________________________________
14.)
a. Circle all values in the table that represent solutions to \( 100 + 2x = 12x - 10 \).

<table>
<thead>
<tr>
<th>x</th>
<th>( y = 100 + 2x )</th>
<th>( y = 12x - 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>116</td>
<td>86</td>
</tr>
<tr>
<td>9</td>
<td>118</td>
<td>98</td>
</tr>
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<td>10</td>
<td>120</td>
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<td>11</td>
<td>122</td>
<td>122</td>
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<td>12</td>
<td>124</td>
<td>134</td>
</tr>
<tr>
<td>13</td>
<td>126</td>
<td>146</td>
</tr>
<tr>
<td>14</td>
<td>128</td>
<td>158</td>
</tr>
</tbody>
</table>

b. Solve the equation \( 100 + 2x = 12x - 10 \) symbolically, and check your answer. Show your work.

c. Explain how a graph can be used to solve the equations \( 100 + 2x = 12x - 10 \).

d. Reflect on your work in Parts a-c. How does the graphical approach relate to the values circled in the table?
15.) We want to learn more about when and how you used your calculator.

a. Did you use your calculator on this exam?  YES   NO

b. I mostly used my calculator for (circle all of the following that apply):
   
   A. Graphing  
   B. Calculating Numbers  
   C. Viewing Tables  
   D. Other: _____________________

   c. Go back to each problem (#1-14), and put a “*” next to it if you used your graphing calculator
      (Y= menu, graph, table, calculator).

Extra Credit

16.) Amy conjectured that the solution to any equation of the form $ax + b = c$ where $a$, $b$, and $c$
       are nonzero numbers, is $x = \frac{c - b}{a}$.

For the following equations, identify the values of $a$, $b$, $c$. Use the equation above to solve for $x$.

a. $5x - 3 = 7$  
   $a =$  
   $b =$  
   $c =$  
   $x =$

b. $2x - \frac{1}{2} = \frac{7}{4} + x$  
   $a =$  
   $b =$  
   $c =$  
   $x =$
Appendix F

Post-Test
Integrated Algebra 1

Mathematician ____________________

Hour _______ Date: ________________

Unit 1 Post-Assessment

*Show all your work for full credit.*

1.) Calvin has a bank account with ‘x’ dollars. Some of his friends also have accounts. Use the variable x to express the balance (amount of money) in each of his friend’s accounts as an algebraic expression in terms of the balance of Linus’ account (x).

a. Hobbes has three times as much as Calvin.

b. Marcus has 5 more than 2 times as much as Calvin.

2.) If Calvin has a balance of $70, find the balance of each person. Show your thinking.

a. Hobbes

b. Marcus

3.) The following questions are about this statement:

\[ 7 + 2 = 9 \]

a. The arrow above points to a symbol. What is the name of the symbol?

b. What does the symbol mean?

c. Can the symbol mean anything else? If yes, please explain.
True or False

4.) __________ $4z - 5 + z - 3$ is a linear equation.

5.) __________ -9 is a solution to the equation $4 + x = x + 4$.

Multiple Choice

Identify the choice that best completes the statement or answers the question.

6.) Situation _____ can be described by the symbolic rule $5x$.

   A. Sydney is five years older than her cousin.
   B. Ben earns $5 an hour walking dogs.
   C. Planted at 2” tall, the flower grew 5’ a year.
   D. The area of a pentagon with side length $x$.

7.) The expression $5x + 3$ is not equivalent to:

   A. $x + x + x + x + x + 2 + 1$
   B. $-3 - (-5x - 3) + 3$
   C. $5(x + 3)$
   D. $0.5 \cdot 2 + 2(2.5x) + 2$

Short Answer

8.) Use the Distributive Property and combine like terms to simplify:

   $2(x^2 + x - 1 + 2x + 7)$
9.) Create the missing graph and table in Row 1 and Row 2. If possible, write the equation.

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td><img src="image1" alt="Table" /></td>
<td><img src="image2" alt="Graph" /></td>
<td>y = 2x – 3</td>
</tr>
</tbody>
</table>

10.) An equation is given in Row 3, below.

a.) Create a graph and a table for the equation $y = 2x – 3$.

b.) Find the solution to the equation $y = 2x – 3$ when $y=5$ by using each of the table, graph, and symbolic equation. Clearly show your work in each of the boxes below.

<table>
<thead>
<tr>
<th>Table</th>
<th>Graph</th>
<th>Symbolic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td><img src="image3" alt="Table" /></td>
<td><img src="image4" alt="Graph" /></td>
<td>$y = 2x – 3$</td>
</tr>
</tbody>
</table>
For 11-12, solve for the variable and check your solution using a table, graph, or equation (your choice). If you use a graph, sketch the section of the graph (with the scale marked). If you use a table, include the table and indicate where you found the answer. If you use equations, show all of your work.

11.) \(4(x - 5) = 16\)  
12.) \(x + 4 + 2x = -1 + 3x + 7\)

13.)

<table>
<thead>
<tr>
<th>Steps</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.) Original problem</td>
<td>(7 + 2x + x - 3 = -1 + x + 7)</td>
</tr>
<tr>
<td>2.) Combine like terms</td>
<td>(3x + 4 = x + 6)</td>
</tr>
<tr>
<td>3.) Subtract (x) from both sides</td>
<td>(2x + 4 = 6)</td>
</tr>
<tr>
<td>4.) Subtract 4 from both sides</td>
<td>(2x = 6)</td>
</tr>
<tr>
<td>5.) Divide both sides by 2</td>
<td>(x = 3)</td>
</tr>
</tbody>
</table>

Ahmed worked out the problem above. When he substituted his solution into the original equation, he discovered that \(x = 3\) is not correct.

a) Find Ahmed's error. Explain where he made his mistake and why using complete sentences.

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________
14.)
e. Circle all values in the table that represent solutions to $11 - 2x = 6 - 7x$.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>$y=11-2x$</td>
<td>$y=6-7x$</td>
</tr>
<tr>
<td>-3</td>
<td>17</td>
<td>27</td>
</tr>
<tr>
<td>-2</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>-1</td>
<td>13</td>
<td>13</td>
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<td>0</td>
<td>11</td>
<td>6</td>
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<td>1</td>
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<td>-1</td>
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<tr>
<td>2</td>
<td>7</td>
<td>-8</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>-15</td>
</tr>
</tbody>
</table>

f. Solve the equation $11 - 2x = 6 - 7x$ symbolically. Show your work. Check your answer.

g. Explain how a graph can be used to solve the equation $11 - 2x = 6 - 7x$.

h. Reflect on your work in Parts a-c. How does the graphical approach (Part c) relate to the values circled on the table (Part a)?
15.) We want to learn more about when and how you used your CAS calculator.

a. Did you use your CAS calculator on this exam?  YES  NO

b. I mainly used my CAS calculator for (circle all of the following that apply):

   A. Graphing
   B. Calculating Numbers
   C. Viewing Tables
   D. Symbolic work on Calculator page
   E. Other: _____________________________________________________________

c. If you haven’t done so already, go back to each problem (#1-14), and circle the type of CAS page that you used.

d. List any other comments that you have about using the CAS calculator for Algebra:

   __________________________________________________________
   __________________________________________________________
   __________________________________________________________
   __________________________________________________________
   __________________________________________________________

16.) Amy conjectured that the solution to any equation of the form $ax + b = c$ where $a$, $b$, and $c$ are nonzero numbers, is $x = \frac{c-b}{a}$. Test Amy's conjecture for the following equations. First identify the values of $a$, $b$, $c$.

Then determine the solution using the equation $x = \frac{c-b}{a}$.

a. $12x - 2 = 4$  
   $a =$  
   $b =$  
   $c =$  
   $x =$

b. $3x - 0.5 = 1.25 + 2x$  
   $a =$  
   $b =$  
   $c =$  
   $x =$
Appendix G

Interview Protocol


**Interview Protocol for Researcher**

**Priority of Tasks**

- Task 9 – 10 Translating between multiple representations and solving equations using multiple representations
- Task 11 OR Task 12 Solving Equations using Representation(s) of choice
- Task 14 Solving equations and connecting Numeric, Symbolic, Graphical methods
- Task 13 Analysis of equation solving and justification of error
- Task 3 Meaning of Equals Sign

**Task Delivery:**

We’ll start by revisiting some of the tasks that you completed on the [pre-test, post-test] assessment. I am interested in your thinking, so I will ask that you talk out loud as you step through the solution process to these tasks.

You may use a graphing calculator at any point in our conversation. Paper and pencil are also available for you to use.

The video camera is recording what you do on your screen, and what you say, but not your face.

Do you have any questions before we begin?

**Reinforce Norms to Think Aloud:**

Explain how you are thinking about this problem.

Can you put into words what you are thinking as you solve this task?

**Scaffolding Prompts:**

*Probe 1:* “Can you explain how you figured out the solution to this problem?”

*Probe 2:* “Can you tell me why you thought to do it that way?”

*Probe 3:* “Could you solve this problem in another way?”

[Wait for the students to respond. Record the response and the strategies used. If the students did not use an alternative representation or persist to be stuck, then continue.]

*Probe 4:* “How could you solve this problem if viewed as…”

[as two graphs / in a table / in symbols / as a situation that you could describe]?
Appendix H

Activity Structure Summary
Table 4.AS
Activity structure descriptions and examples

<table>
<thead>
<tr>
<th>Activity Structure</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act</td>
<td>Create a representation and possibly explain the process of how one works within or moves between tool-based representations or types</td>
<td>Thomas: I put the equation on the graph, and then yeah. Ms. L: You put the equation in your CAS? Thomas: Yeah. Ms. L: And then you got the table. (Activity 11, Day19, 00:13:27:71)</td>
</tr>
<tr>
<td>Anticipate</td>
<td>Predict the result of creating tool-based representations</td>
<td>Ms. L: If I look at this, $5x + 1 = 20x + 10$, I want to think, how many solutions do I think I'm going to get? [...] What are our options? One, zero, and infinite. Is it exactly the same expression on the right and the left? No. So it's not infinite solutions. So it's probably one or none. So we're just going to put that in our heads. Now, I'm going to leave that prediction aside, and I want to get to variable = number, we're going to do it the old fashioned way. (Activity 12, Day 22, 00:22:19:93)</td>
</tr>
<tr>
<td>CAS Check**</td>
<td>Use the CAS to check or verify P&amp;P representations (often times within Symbolic representation type)</td>
<td>Ms. L: So let's check it. Grab original equation, hit Enter, and do control equals, such that, do this, so I want you to check that $-3$ over $5$, is that the right solution? Such that $x$ equals, negative three fifths [CAS line: $5x + 1 = 20x + 10 \mid x = -3/5$], what would you expect to see? What do you expect to see when you plug it back in the first? It's a true. (Activity 14, Day 22, 00:30:32:78)</td>
</tr>
<tr>
<td>Connect</td>
<td>Give a correct interpretation of an invariant feature across multiple representations or types</td>
<td>Ms. L: [...] OK. So he looked at this and he looked at point $a$ and he saw that $x$ is negative two and he counted up five so he said he knew the $y$-value was [motions along $y$-axis on graph of $y = 4 - 0.5x$] one, two, three, four, five. So he knew that this point right here [points to $(-2.5)$ on graph] was $x$ equals negative two, $y$ equals five. And then what did you do to double check that? Thomas: I looked at the table. Ms. L: Oh, and then he looked over here at the table and he saw that when $x$ equals negative two on the table $y$ is equal to five. So that's the value of $y$ when $x$ equals negative two. [interruption] So those are the two places where he saw the first one [highlights $(-2.5)$ on graph and row with $x = -2$ and $f12(x): = 4 - 0.5 \ast x = 5$ on table] (Activity 8.5, Day 16, 00:12:25:56)</td>
</tr>
<tr>
<td>Activity Structure</td>
<td>Description</td>
<td>Example</td>
</tr>
<tr>
<td>--------------------</td>
<td>-------------</td>
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</tr>
<tr>
<td><strong>Generalize</strong></td>
<td>Make a generalization across several representations or representation types (e.g., abstract notation $a(x + b) = ax + ab$).</td>
<td>Ms. L: […] Ok. So when we say [interruption] this is like a definition of the distributive property because &quot;a&quot; and &quot;b&quot; can be any number. You saw a bunch of examples with numbers [gestures to parts a-e], and now you know that this property works with any number—it works with fractions, it works with positive numbers, it works with decimal numbers, it works with negative numbers; &quot;a&quot; and &quot;b&quot; can be any number and you're still going to get this same expression expanded [points to $a * x + a * b$], equal to the expression you start with [points to $a * (x+b)$ in equation $a * (x + b) = a * x + a * b$]. Activities 5-6, Day 12, 01:01:03:34)</td>
</tr>
<tr>
<td><strong>Interpret</strong></td>
<td>Convey the meaning of the act/result of creating a tool-based representation; a basic, quick remark, thoughtful but not deep</td>
<td>Ms. L: What about this one? [points to expand $(-3(9x - 5)) = -27x + 15$] Is that doing the same thing? What is happening with this negative three? S: Times-ing it by 9 (Activity 5-6, Day 13, 00:06:01:58)</td>
</tr>
<tr>
<td><strong>Justify</strong></td>
<td>Representations are used to confirm or ascertain a particular result or conclusion; “use representations as justifications for other claims” (Sandoval et al., 2000; reasoning must be present); formal/rigorous explanation, objective, based on set practices</td>
<td>Solve each equation for the variable. Show your work. Check your solution. 2. $-r - 5 = - (r + 5)$ Ms. L: Looks like he used the distributive property. […] And he ended up with $0 = 0$. What does zero equals zero mean? […]; student response Ms. L: Infinite solutions. Why? Why is it infinite solutions? Cause it means the same thing on either side of the equal sign. Look back at this step right here. What do you see about the expression on the left and the expression on the right? Ethan: They're the same. Ms. L: They're exactly the same. So you have equivalent expressions. If your expressions on either side of the equals sign are exactly the same that means you have the same line, you have exactly the same line, so infinite solutions on number 2. (Activity 14, Day 23, 00:20:51:66)</td>
</tr>
<tr>
<td>Activity Structure</td>
<td>Description</td>
<td>Example</td>
</tr>
<tr>
<td>--------------------</td>
<td>-------------</td>
<td>---------</td>
</tr>
<tr>
<td>Reconcile</td>
<td>Negotiate differences between CAS and P&amp;P representations</td>
<td>Ms. L: I want to know which one of these surprised you? S: Part a. [Ms. L: What did you write for the paper-and-pencil part - 3 * (2 * x + 2)) = ____]? S: 3x + 12. Ms. L: What did the CAS tell you? […] S: -6x - 6. Ms. L: […] So what did you have to understand to fix it? S: that I have to multiply three all the way through to get the right answer. (Activities 5-6, Day 12, 00:57:53:83)</td>
</tr>
<tr>
<td>Reflect</td>
<td>React to or think deeply about representations/representation types with respect to equivalence and/or equations; heavy thought, detailed in response, subjective and developmentally oriented</td>
<td>Ms. L: [interruption] So yeah, you can solve it symbolically and get a solution, that's fine. The reason we're pushing you guys to use tables and graphs is that sometimes you might get a symbolic answer and wonder if you're right or not. […] Because if you get a solution using symbols it should match what you see on your graph and what you see in the table. (Activity 11, Day 19, 00:14:45:59)</td>
</tr>
<tr>
<td>Translate**</td>
<td>Create and interpret the meaning of a target representation with respect to a source representation of a different type</td>
<td>Ms. L: So, if I, I'm not sure by looking at these equations [in reference to 4x - 9 = -7x + 13], I can graph these in my CAS. I can put 4x - 9 in my f1. Do that right now. Put -7x + 13 in my f2 [interruption] Open CAS, go to new document, add a graph. f1 we're going to put in 4x - 9, hit tab [interruption] Everyone needs to be doing this right now. It's going to graph it, ooh there's our line. Hit tab again and it's going to graph f2 = -7x + 13. What happened to those lines? Katrina: They overlap Ms. L: They cross don't they? They sure are intersecting. Alright. So these guys are crossing. They have a point of intersection. They have one solution (Activity 12, Day 21, 00:47:54:05)</td>
</tr>
<tr>
<td>Activity Structure</td>
<td>Description</td>
<td>Example</td>
</tr>
<tr>
<td>--------------------</td>
<td>-------------</td>
<td>---------</td>
</tr>
<tr>
<td>Transpose**</td>
<td>Create and interpret multiple representations within one representation type.</td>
<td>Ms. L: Now while we're doing this, I'm going to solve this doing the, kind of the &quot;old fashioned way&quot; so that we can see what the CAS is doing. So I have $8x - 12 = 4$. So if you were doing that with paper-and-pencil you might have done it that way already. [...] I want to get a simpler equation. For example, to subtract 8 from both sides type $-8$. [...] So if I type $-8$ (&quot;minus 8&quot;), oh, what does it say? Student: Answer Ms. L: [...] Now what do you think is going to happen if I subtract 8 from that equation? [...] the 12 is going to change into a 20. [...] And what happens to the $8x$? [...] Students: Minus one. Two. Ms. L: Well this is his prediction. Student: Is that right? Ms. L: Well let's find out, press Enter. What does my CAS say? Student: Dude, that is not right. Ms. L: It says $8x - 20 = -4$. So how'd you know it was going to be $-4$? Student: Because you subtracted 8 from 4 Ms. L: Right. Ok. So he said I'm doing 4 minus 8 I'm going to get $-4$. So what's cool about the CAS is it does it automatically from both sides. We only put the minus 8 here, and it subtracted 8 from 12 and from 4. (Activity 11, Day 19, 00:40:10:72)</td>
</tr>
</tbody>
</table>

** Denotes component added during retrospective analysis
Appendix I

Learning Progression Summary
<table>
<thead>
<tr>
<th>Learning Progression</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A1</strong></td>
<td><strong>Connecting and generalizing the quantitative, visual, and verbal with symbols.</strong> Symbolic expressions generalize numeric, graphic, and verbal patterns by allowing for compact, abstract notation.</td>
<td>S: I was going to go all the way up to 50 but I forgot that I had that equation so I used that Ms. L: So smart. ... It would take a long time to add that all up in the table. So I can use the equation to figure it out. The equation is a shortcut that represents the same values in the table and the pattern you're seeing in the tiles... (Activity 1, Day 3, 00:58:10:04)</td>
</tr>
<tr>
<td><strong>A2</strong></td>
<td><strong>Different representations/representation types can signify the same object.</strong> Different representations/representation types of the same linear expressions and/or equations signify the same relationship, pattern, or function from different yet complementary perspectives.</td>
<td>Ms. L: So what I wanted to be clear with you guys is that this line $100 - 3 \times x$ [points to graph] goes with this first table [points to corresponding x- and y-columns] Here's all your x-values, here's all your y-values &amp; these points are the points on this line. Activity 9, Day 17, 00:19:46:64</td>
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<td><strong>B</strong></td>
<td><strong>Equivalence of expressions from multiple representations.</strong> Expressions are equivalent if they define the same relationship, pattern, or function.</td>
<td>&quot;Ms. L: If I just looked at these symbols that I gave you in the warm up, I gave you these two things as symbols (circles $4x - (x + 2)$ and $3x - 2$ as shown on overhead) and what I'm trying to get you guys to see is, you could simplify this (first expression) in symbols and get that (second expression), OR you could look at a table of the same values and know they are equivalent, OR you could look at a graph that is exactly the same and know they are equivalent. And then I would know that these two expressions are equivalent even if I didn't simplify them first.&quot; (Activity 2.5, Day 7, 00:21:38:22)</td>
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<td><strong>C</strong></td>
<td><strong>Domain and range restrictions may arise in contextual situations and should be considered when determining equivalence.</strong></td>
<td>Ms. L: Yeah, ok kind of the impulse is to connect [the points on our line]. But let's look at the context. Let's look at the situation. Do we have figure 1.5 on this, in this example? Do we have a figure 1 and 1/2? No, we don't have a figure 2 1/2 either. So when we've got tile patterns lots of times we leave it disconnected, so if you have points on your graph it's probably more accurate than connecting it. (Activity 1, Day 4, 00:10:44:15)</td>
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<td>Learning Progression</td>
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<td>C1*</td>
<td>Role of Equal Sign: Assign variables rules/names for patterns</td>
<td>David: $J$ times 1.5 equals $k$. Ms. L: $J$ times 1.5 equals $k$ [writes $J \times 1.5 = k$] So the amount of money John has times 1.5 equals the amount of money Kim has. (Activity 3, Day 8, 00:26:57:14).</td>
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<td>C2*</td>
<td>Role of Equal Sign: Identity between equivalent expressions</td>
<td>Ms. L: [On board: $2x + 7 = x + x + 3 + 4$ then $2x + 7 = 2x + 7$] She says those are equivalent because it's exactly the same thing on each side [...] they're exactly the same on each side of the equation so they're equivalent (Activity 2, Day 5, 00:07:31:88).</td>
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<td>CC1*</td>
<td>If a point $P$ is on the line $L$, $P$ makes the equation of $L$ true</td>
<td>Ms. L: I'm going to do Bryon's method. If you look at the graph [of $y = 3 - x$], $x$ equals 1, 2, 3, 4, 5, what's my $y$? What's my two? That's negative two. So point C, which goes with this problem [circles evaluate the equation $y = 3 - x$ at $x = 5$] And I've got $x = 5$ and $y = -2$. So If I do $y = 3 - 5 = -2$, does that match? Yeah. [draws arrow from result of evaluating equation $y = 3 - x$ at $x = 5$ to Point (5, –2) on graph] (Activity 4, Day 9, 00:37:40:38).</td>
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<td>CC2*</td>
<td>If a point $P$ makes the equation of $L$ true, $P$ is on the graph of $L$</td>
<td>Ms. L: We said we're going to evaluate this equation at $x = -1$. So that means I take $y = 3 - x$ and in place of the $x$ I put in negative one. So $3 - x$ is $3 - (-1)$ which is $3 + 1$ which is 4. Then it says &quot;Fill in the missing coordinate of Point A on the graph below.&quot; So on the graph, I make this point (1,4). What I want you to do is to make the connection, the equation of this line is $y = 3 - x$ and if $x = -1, y = 4$, and on my graph this is my point (–1,4). (Activity 4, Day 9, 00:34:32:80).</td>
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Learning Progression | Description | Example
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D1 | *Solutions to equations can be determined by equality of expressions.* Linear equations are relations between linear expressions that are sometimes, always, or never equal in value. Thus linear equations have one, infinitely many, or zero solutions, respectively. | Ms. L: You know what, yesterday, when we looked at your exit tickets, a lot of people said there would be two solutions or three solutions. These, these are straight lines. You can't have two solutions if you've got straight lines. Think back to Activity 8 HW, we have three cases. When you have linear equations you only have three different choices. Either the lines are going to intersect. [gestures an "X" with hands] If two lines cross how many solutions do you have? Ethan: One Ms. L: […] You're going to have no solutions where they're parallel [gestures 11 with hands and points to Activity 8 HW graph of $5x + 7 + x = 6x$], or what was the third case? Ethan: Always. Ms. L: Always [interruption] infinite solutions [gestures along Activity 8 HW graph of $-1(x + 4) = -x - 4$]. (Activity 12, Day 21, 00:44:56:62)

D2 | *Solving equations in one variable is conceptualized as a comparison of two functions.* Linear equations in one variable such as $ax + b = cx + d$ for real valued parameters $a$, $b$, $c$, and $d$, can be solved for the variable $x$ by comparing the functions $f(x) = ax + b$ and $g(x) = cx + d$ for the value of $x$ that makes the equation $ax + b = cx + d$ true. Graphical, tabular, or symbolic methods can be used. | Ms. L: So, if I, I'm not sure by looking at these equations [in reference to $4x - 9 = -7x + 13$], I can graph these in my CAS. I can put $4x - 9$ in my f1. Do that right now. Put $-7x + 13$ in my f2 [interruption]. What happened to those lines? Katrina: They overlap Ms. L: They cross don't they? They sure are intersecting. Alright. So these guys are crossing. They have a point of intersection. They have one solution. […] And if I look at my graph can I figure out what value of $x$ they're going to cross at? Student: Yeah. Ms. L: About where? Abila: Negative one and two. Ms. L: So $x = 2$, $y = -1$. (Activity 12, Day 21, 00:47:54:05)

* Denotes component was added during teaching experiment