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WEIGHTED SHIFTS OF FINITE MULTIPLICITY

by

Dan Sievewright

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WEIGHTED SHIFTS OF FINITE MULTIPLICITY

Dan Sievewright, Ph.D.

Western Michigan University, 2013

We will discuss the structure of weighted shift operators on a separable, infinite dimensional, complex Hilbert space. A weighted shift is said to have multiplicity n when all the weights are $n \times n$ matrices. To study these weighted shifts, we will investigate which operators can belong to the Deddens algebras and spectral radius algebras, which can be quite large. This will lead to the necessary and sufficient conditions for these algebras to have a nontrivial invariant subspace.

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Dan Sievewright

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Chapter 1

Introduction

1.1 Invariant Subspace Problem

One of the main open problems in Operator Theory is to understand the structure of a bounded linear operator on Hilbert space $\mathcal{H}$. This entails, in the very least, finding answers to the following two questions. For any operator T, does there always exist a subspace $\mathcal{M} \subset \mathcal{H}$ such that $\mathcal{M}$ is invariant under T, i.e. $T\mathcal{M} \subset \mathcal{M}$? If so, what form can the restriction $T|_{\mathcal{M}}$ take? Of course, the zero subspace and the entire space are invariant for every operator, so we are really interested in the existence of a *nontrivial invariant subspace* (n.i.s.), i.e., an invariant subspace which is neither $\{0\}$ nor $\mathcal{H}$. The problem of finding a n.i.s. is known as the *invariant subspace problem.*

For the case where $\mathcal{H}$ is finite dimensional, both questions have been well studied and fully answered. In particular, we can decompose T into a direct sum of its Jordan blocks. If $\mathcal{H}$ is infinite dimensional, much less is known and even the first question remains an open problem. However, we can give an answer for some classes of operators.

If N is a *normal* operator, meaning that N commutes with its adjoint N^*, then the spectral theorem (cf., [9, p. 289]) shows that N has a n.i.s. An operator K is called *compact* if the image of the unit ball under K is compact. In 1954, Aronszajn and Smith proved in [2] that every compact operator on a Banach space has a n.i.s. Prior to this, J. von Neumann claimed to have a proof of the existence of a n.i.s. for compact operators on Hilbert space, but never showed it to anyone.

Not much progress was made on this problem until 1966 when Bernstein and Robinson proved in [3] that if $p(T)$ is a nonzero compact operator for some polynomial p, then T has a n.i.s. It should be noted that they used Nonstandard Analysis as the basis for their paper. When Halmos read this work, he translated the paper into the language of Operator Theory in [12]. From this latter paper, another important class of operators which emerged were the *quasitriangular* operators.

The next major result came in 1973 in [17] and is due to V. Lomonosov. There are many ways of wording Lomonosov's Theorem. Most simply, it states that if T commutes with a nonscalar operator A, and A commutes with a nonzero compact operator, then T has a n.i.s. More commonly, it is said that if A is a nonscalar operator which commutes with a nonzero compact operator, then A has a *nontrivial hyperinvariant subspace*. That is, there exists a nontrivial subspace which is invariant for all operators which commute with A. We will be most interested in phrasing the result in terms of the *commutant* of A. The commutant is denoted by $\{A\}'$ and it represents the algebra of operators which commute with A. Then, Lomonosov's Theorem states that if A is a nonscalar operator and there exists $K \in \{A\}'$ which is a nonzero compact operator, then $\{A\}'$ has a n.i.s. (the subspace is invariant for every operator in $\{A\}'$).

This theorem is such a strong and general statement that, for a while, it looked like it might answer the invariant subspace problem. In other words, it was not known that if given an operator T, there existed a nonscalar operator $A \in \{T\}'$, such that A commutes with a nonzero compact operator. If so, then Lomonosov's Theorem implies that T must have a n.i.s. However, a counterexample was provided in 1980. It was proved in [11] that if A is a *quasi-analytic shift*, then there does not exist a nonzero compact operator which commutes with any operator in $\{A\}'$.

Lomonosov's Theorem and the theory of quasitriangular operators inspired a lot of the research on the invariant subspace problem throughout the 1970's. The next big theorem about the existence of a n.i.s. was the result of Scott Brown about subnormal operators in 1978. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that, relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus (K \ominus \mathcal{H})$, there exists an operator

$$N = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$$

acting on $\mathcal{K}$, which is normal. In other words, T is the restriction of a normal operator to an invariant subspace. It is easy to see that every normal operator is subnormal by taking $\mathcal{H} = \mathcal{K}$.

In [6], Brown proved that every subnormal operator has a n.i.s. This is a true extension of the spectral theorem which proved that all normal operators have a n.i.s. The methods introduced in this paper were very useful and it turned out that they could be applied to other classes of operators, not just subnormal operators. This activity was capped in 1988 by a theorem of Brown, Chevreau, and Pearcy in

[7]. It states that every *contraction* whose spectrum contains the unit circle in the complex plane has a n.i.s. We call an operator T a contraction if $\|T\| \leq 1$. Since a subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for T if and only if $\mathcal{M}$ is invariant for λT, where $\lambda \neq 0$, then, without loss of generality, one can always assume that T is a contraction.

The result of Brown, Chevreau, and Pearcy was then generalized in 2004 by Ambrozie and Müller in [1]. They proved that if T is a *polynomially bounded operator* on a Banach space and the spectrum of T contains the unit circle, then T has a n.i.s. An operator T is polynomially bounded if there exists an $M > 0$ such that $\|p(T)\| \leq M\|p\|_\infty$ for all polynomials p, where $\|p\|_\infty = \sup\{|p(z)| : |z| \leq 1\}$. The von Neumann's Inequality (cf., [8, p. 200]) states that if T is a contraction on Hilbert space, then T is polynomially bounded with $M = 1$. Furthermore, if T is similar to a contraction C, (i.e., there exists an invertible X such that $XTX^{-1} = C$) then T is polynomially bounded. The result of Ambrozie and Müller is then a real generalization of Brown, Chevreau, and Pearcy since not every polynomially bounded operator is similar to a contraction (see [22]).

We now return to Lomonosov's Theorem. Recall that it can be stated in the form: if A is a nonscalar operator and $\{A\}'$ contains a nonzero compact operator, then $\{A\}'$ has a n.i.s. One can now consider what happens if we replace $\{A\}'$ with a larger algebra, and ask whether this larger algebra has a n.i.s. Such algebras were introduced in [16].

Let $A \in \mathcal{L}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$, and let $r(A)$ denote the spectral radius of A. Define $d_m = m/(1 + mr(A))$ and

$$R_m = \left(\sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n \right)^{\frac{1}{2}}.$$

Later, we will show that the above series converges in norm and the operators R_m are well-defined and invertible. The *spectral radius algebra* associated to A is defined by

$$\mathcal{B}_A = \{T \in \mathcal{L}(\mathcal{H}) : \sup_{m \in \mathbb{N}} \|R_m T R_m^{-1}\| < \infty\}.$$

In the beginning of Chapter 3, we will establish that $\mathcal{B}_A$ is an algebra which contains the commutant of A. This means that if $\mathcal{B}_A$ has a n.i.s., then so does $\{A\}'$. Of course, for this to be of any interest, we would need the containment $\{A\}' \subset \mathcal{B}_A$ to be proper. In all known cases, if A is nonscalar then $\{A\}' \neq \mathcal{B}_A$, but this is still an open question in general.

When K is a nonzero compact operator, it was shown in [16] that $\{K\}'$ is a proper subalgebra of $\mathcal{B}_K$ and that $\mathcal{B}_K$ has a n.i.s. The natural question that follows is, does every operator $T \in \mathcal{L}(\mathcal{H})$ belong to $\mathcal{B}_K$ for some nonzero compact operator K? This is still an open question. Further research led to spectral radius algebras associated to normal operators [4], operators on finite dimensional spaces [5], C_0 contractions [18] and [19], shifts [20], and weighted shifts [21].

As this research evolved, another type of algebra became interesting to investigate. These were originally studied by J. Deddens in [10] so we refer to them as Deddens algebras. Let $A \in \mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{L}(\mathcal{H})$ belongs to the Deddens algebra $\mathcal{D}_A$ if there exists $M > 0$ such that

$$\|A^n T x\| \leq M \|A^n x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Later, we will prove that $\mathcal{D}_A$ is an algebra and that $\{A\}' \subset \mathcal{D}_A \subset \mathcal{B}_A$. The Deddens algebras were particularly interesting when associated with a unilateral shift or a weighted shift. Since our work centers on weighted shifts, we will study both the Deddens and spectral radius algebras associated to these operators. In the next section, we will carefully define weighted shifts and look at some of the interesting properties that they have.

1.2 Weighted Shifts

Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers, let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and let W be an operator on $\mathcal{H}$ defined by $We_n = a_n e_{n+1}$. Such an operator W is called a *weighted shift* with weight sequence $\{a_n\}_{n\in\mathbb{N}}$. Historically, the main shift of interest was the *unilateral forward shift* S defined by $Se_n = e_{n+1}$, for all $n \in \mathbb{N}$.

To study the structure of this shift, it is useful to exploit the natural isomorphism between ℓ^2 and H^2. The space ℓ^2 is the Hilbert space of complex sequences $\{c_n\}_{n\in\mathbb{N}}$, such that $\sum_{n=1}^{\infty} |c_n|^2$ converges and the inner product on ℓ^2 is defined by

$$\langle \{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}} \rangle = \sum_{n=1}^{\infty} b_n \overline{c_n}.$$

The Hardy space H^2 consists of the complex functions defined on the unit circle that are square integrable,

$$\int_0^{2\pi} |f(e^{it})|^2 dt < \infty,$$

and whose negative Fourier coefficients are zero,

$$\int_0^{2\pi} f(e^{it}) e^{-int} dt = 0$$

for $n = -1, -2, \ldots$. The space ℓ^2 has a natural basis consisting of sequences with precisely one nonzero entry whose value is 1. Meanwhile, H^2 has a natural basis of $\{e^{int}\}_{n\geq 0}$. The Fourier transform identifies these bases and, under this isomorphism, the unilateral forward shift can be viewed as the multiplication operator M_z, defined by $M_z f(z) = zf(z)$. This approach was very successful in the study of this shift.

In particular, it can be used to establish that $T \in \{S\}'$ if and only if T is the limit of a sequence of polynomials in S in the *strong operator toplogy*, see [13, Problem 148]. This means that if $T \in \{S\}'$, then there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p_n(S)x = Tx$, for all $x \in \mathcal{H}$, where the limit is taken in the norm topology on $\mathcal{H}$. In terms of matrices, we can represent S as

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

relative to the basis $\{e_n\}_{n\in\mathbb{N}}$. If T commutes with S, then T is an *analytic Toeplitz operators*, that is

$$T = \begin{bmatrix} c_0 & 0 & 0 & \cdots \\ c_1 & c_0 & 0 & \\ c_2 & c_1 & c_0 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

is lower triangular and is constant along the diagonals.

For weighted shifts with weight sequence $\{a_n\}_{n\in\mathbb{N}}$, a similar strategy can be used. For these operators, we assume that $a_n \geq 0$ because W is unitarily equivalent to a weighted shift with weights $\{|a_n|\}_{n\in\mathbb{N}}$. Furthermore, if $a_n = 0$, for some $n \in \mathbb{N}$, then we can write W as a direct sum of weighted shifts $W = W_1 \oplus W_2$. The operator W_1 has the weights $\{a_1, a_2, \ldots, a_{n-1}\}$ and is a finite dimensional shift so its study belongs to linear algebra. The operator W_2 will be a weighted shift with weights $\{a_{n+k}\}_{k\in\mathbb{N}}$. Thus, we can assume without loss of generality that $a_n > 0$, for all $n \in \mathbb{N}$.

Let $\{\beta_n\}_{n\geq 0}$ be a sequence of positive numbers and define $H^2(\beta)$ to be the set of formal power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

such that

$$\|f\|_\beta^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$

This space becomes a Hilbert space under the inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle := \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2.$$

Define the operator M_z by

$$M_z \left(\sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=0}^{\infty} a_n z^{n+1}.$$

This operator is continuous if and only if $\{\beta_n/\beta_{n-1}\}_{n\in\mathbb{N}}$ is a bounded sequence.

The space $H^2(\beta)$ can be used in the study of weighted shifts as follows. Let W be a weighted shift with weights $\{a_n\}_{n\in\mathbb{N}}$, let $\beta_n := a_1 a_2 \cdots a_n$ for $n \in \mathbb{N}$ and $\beta_0 := 1$. Define $U : \ell^2 \to H^2(\beta)$ by

$$U(x_1, x_2, x_3, \ldots) = \sum_{n=0}^{\infty} \frac{x_{n+1} z^n}{\beta_n}$$

Then U is a unitary operator and $UWU^{-1} = M_z$. Therefore, every weighted shift is unitarily equivalent to the multiplication operator M_z on $H^2(\beta)$. Moreover, M_z can be viewed as a weighted shift with weights $\{\beta_n/\beta_{n-1}\}_{n\in\mathbb{N}}$.

Once again, the study of M_z allows us to determine $\{W\}'$. An operator T commutes with W if and only if there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$, such that $p_n(W)$ converges to T in the strong operator topology. Now, T is no longer a Toeplitz operator. Instead, if T has a matrix $\{t_{ij}\}_{i,j\in\mathbb{N}}$, then T is lower triangular and we have that the kth diagonal, i.e., the sequence $\{t_{k+n,1+n}\}_{n\geq 0}$, must be a constant multiple of the sequence $\{\frac{\beta_{k+n-1}}{\beta_n}\}_{n\geq 0}$.

We now turn our attention to the Deddens and spectral radius algebras associated to weighted shifts. It is not too hard to verify that both $\mathcal{D}_S$ and $\mathcal{B}_S$ contain all the bounded linear operators on $\mathcal{H}$. For other weighted shifts, these algebras become much more interesting. In [21], Petrovic studied the Deddens and spectral radius algebras associated to weighted shifts. The main result of that paper was that $\mathcal{D}_W$ and $\mathcal{B}_W$ are *weakly dense* in either $\mathcal{L}(\mathcal{H})$ or (LT), the algebra of operators with lower triangular matrices in $\{e_n\}_{n\in\mathbb{N}}$. A sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}(\mathcal{H})$ is said to converge weakly to $A \in \mathcal{L}(\mathcal{H})$ if $\lim_{n\to\infty}\langle (A_n - A)x, y\rangle = 0$, for all $x, y \in \mathcal{H}$.

More precisely, it was shown that $\mathcal{D}_W$ is weakly dense in (LT) if W is not *bounded below*, and it is weakly dense in $\mathcal{L}(\mathcal{H})$ if W is bounded below. Recall that an operator A is said to be bounded below if there exists $C > 0$ such that $\|Ax\| \geq C\|x\|$, for all $x \in \mathcal{H}$. As for the spectral radius algebra, $\mathcal{B}_W$ is weakly dense in (LT) if $r(W) = 0$ and it is weakly dense in $\mathcal{L}(\mathcal{H})$ if $r(W) > 0$. It should be noted that an operator $A \in \mathcal{L}(\mathcal{H})$ is called *quasinilpotent* whenever $r(A) = 0$. Hence, $\mathcal{D}_W$ has a n.i.s. if and only if W is not bounded below and $\mathcal{B}_W$ has a n.i.s. if and only if W is quasinilpotent.

The purpose of this manuscript is to generalize these results to weighted shifts of higher multiplicity. Later, we will give a precise definition of these operators. In the meantime, we will just point out that the matrix of such an operator is given by

$$W = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ A_1 & 0 & 0 & \\ 0 & A_2 & 0 & \\ 0 & 0 & A_3 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

where each entry is an $n \times n$ matrix (for a *shift of multiplicity* n) or an infinite matrix (for a *shift of infinite multiplicity*).

Once again, such an operator can be identified with the operator of multiplication by z on some function space, at least when the weights are invertible. This result is due to Keough in [15], where it was shown that these function spaces are generalized Hardy spaces, consisting of formal power series $\sum_{n=0}^{\infty} x_n Z^n$ where $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ which satisfies $\sum_{n=0}^{\infty} \|S_n x_n\|^2 < \infty$. The operators S_n are defined by $S_0 = I$, and $S_n = A_n A_{n-1} \cdots A_1$. Then, a weighted shift with invertible weights $\{A_n\}_{n\in\mathbb{N}}$, is unitarily equivalent to the multiplication operator M_Z on this generalized Hardy space.

The commutants for these weighted shifts can now be found in the same manner as the the commutants for weighted shifts of multiplicity one. Hence, we now turn our attention to the Deddens and spectral radius algebras. For weighted shifts of multiplicity one, these algebras were studied in [21] by considering the rank one operators $e_i \otimes e_j$ defined by $e_i \otimes e_j(x) := \langle x, e_j \rangle e_i$. The weak closure of $\mathcal{D}_W$ and

$\mathcal{B}_W$ are then determined based on whether these rank one operators belong to the algebra or not.

In particular, it was shown that $e_i \otimes e_j \in \mathcal{D}_W \subset \mathcal{B}_W$, for all $i \geq j$. Furthermore, if $i < j$, then $e_i \otimes e_j \in \mathcal{D}_W$ if and only if W is bounded below, and $e_i \otimes e_j \in \mathcal{B}_W$ if and only if $r(W) > 0$. Unfortunately, this approach cannot be taken for weighted shifts of higher multiplicity. For example, we will show later that it is possible for $\mathcal{D}_W$ to contain no rank one operators. It turns out that the operators we will be interested in are the ones whose block matrix contains at most one nonzero entry. If such an operator T belongs to $\mathcal{D}_W$, respectively $\mathcal{B}_W$, then we look at its nonzero entry T_{ij}, and say that it belongs to $\mathcal{D}_W(i,j)$, respectively $\mathcal{B}_W(i,j)$. We will show that $\mathcal{D}_W(i,j)$ and $\mathcal{B}_W(i,j)$ are vector spaces and their dimension will play an important role in the study of $\mathcal{D}_W$ and $\mathcal{B}_W$. In the case of weighted shifts of multiplicity one, these spaces are either zero or one dimensional and the exact size of $\mathcal{D}_W(i,j)$, respectively $\mathcal{B}_W(i,j)$, is determined by whether or not the operator $e_i \otimes e_j$ belongs to $\mathcal{D}_W$, respectively $\mathcal{B}_W$. In this manuscript, we will extend these results to weighted shifts of higher multiplicity.

In Chapter 2, we will consider the Deddens algebras associated to weighted shifts of arbitrary multiplicity. The first section will be devoted to carefully defining $\mathcal{D}_W$ and $\mathcal{D}_W(i,j)$ and looking at the properties of these spaces. Starting in Section 2.2, the results that are established about the Deddens algebra were results obtained by the author and has been accepted for publication in the Houston Journal of Mathematics. We will establish our first result about membership in $\mathcal{D}_W(i,j)$. From this, we obtain that $\{W\}'$ is a proper subalgebra of $\mathcal{D}_W$. Starting with Section 2.3, we will focus on weighted shifts of finite multiplicity. As a consequence, the dimension of $\mathcal{D}_W(i,j)$ will be finite and we can study the relations between various

$\mathcal{D}_W(i,j)$ where $i, j \in \mathbb{N}$. In particular, we will show that, when W is injective, there exist injective homomorphisms from $\mathcal{D}_W(i,j)$ to $\mathcal{D}_W(i+1,j)$ and $\mathcal{D}_W(i,j-1)$. Furthermore, there is an isomorphism between $\mathcal{D}_W(i,j)$ and $\mathcal{D}_W(i+1,j+1)$ whenever W is injective. In fact, when W is bounded below, these homomorphisms all become isomorphisms implying that the dimension of $\mathcal{D}_W(i,j)$ remains constant as i and j vary. Using these isomorphisms, we will show that if W is a weighted shift of multiplicity N which is bounded below, then $\mathcal{D}_W$ has a n.i.s. if and only if the common dimension of $\mathcal{D}_W(i,j)$ is less than N^2.

When W is injective but not bounded below, these homomorphisms are no longer necessarily invertible, but they are still injective. Hence, the dimension of $\mathcal{D}_W(i,j)$ cannot decrease as i increases or j decreases. Even though we lose invertibility of our homomorphisms, we will be able to show that there will always exist a n.i.s. for $\mathcal{D}_W$. The case when W is not injective is much less clear though. The homomorphisms between various $\mathcal{D}_W(i,j)$ are no longer injective many of the results established for injective weighted shifts no longer hold. The good news is that, when W is not injective, $\mathcal{D}_W$ has a n.i.s.

In the final section of Chapter 2, we will consider only weighted shifts of multiplicity two. We start with weighted shifts which are bounded below and give a complete list of the possible structures for $\mathcal{D}_W$. As mentioned before, the dimension of each $\mathcal{D}_W(i,j)$ will be the same. It will never be the zero space, but we will give examples to show that $\dim(\mathcal{D}_W(i,j))$ can take on any value between 1 and 4. We will then move onto injective weighted shifts which are not bounded below and show that the results established in Section 2.3 cannot be strengthened.

In Chapter 3, we begin our study of the spectral radius algebra. Section 3.1 will be devoted to introduction of these algebras and the properties that they have. The remainder of Chapter 3 consists of original research done by the author. The outline of Section 3.2 follows that of Section 2.2. However, we will now be interested in the additional question: when is $\mathcal{D}_W$ a proper subalgebra of $\mathcal{B}_W$? For injective weighted shifts of multiplicity one, it was shown in [21] that the weak closure of $\mathcal{D}_W$ differs from the weak closure of $\mathcal{B}_W$ if and only if W is not bounded below and $r(W) > 0$. For shifts of higher multiplicity, only one direction remains. Namely, we will show that if W is not bounded below and $r(W) > 0$, then $\overline{\mathcal{D}_W} \neq \overline{\mathcal{B}_W}$. However, this condition is no longer necessary. At the end of Section 3.2, we will give an example of a weighted shift W for which $r(W) > 0$ and W is bounded below, but the weak closure of $\mathcal{D}_W$ is properly contained in $\mathcal{B}_W$. In Section 3.4, we will go even further and prove that nearly every noninjective weighted shift of finite multiplicity with positive spectral radius also has the property that $\overline{\mathcal{D}_W} \neq \overline{\mathcal{B}_W}$.

Just like in Chapter 2, we will be mainly interested in the vector spaces $\mathcal{B}_W(i, j)$, which are the natural analogues of $\mathcal{D}_W(i, j)$. In Section 3.3, we restrict our attention to the case where W is an injective weighted shift of finite multiplicity. Here, the similarities to the Deddens algebra will be very apparent even though the algebras can be very different. Once again, we will utilize the homomorphisms between $\mathcal{B}_W(i, j)$ and $\mathcal{B}_W(k, l)$. The natural question to ask is "when are these homomorphisms invertible?" This happens precisely when $r(W) > 0$ and the study of these homomorphisms will be useful in the characterizing the existence of a n.i.s. for $\mathcal{B}_W$. Specifically, if W is an injective weighted shift of multiplicity N, then $\mathcal{B}_W$ has a n.i.s. unless $r(W) > 0$ and the common dimension of $\mathcal{B}_W(i, j)$ is N^2.

In Section 3.4, we will consider noninjective weighted shifts. Now the homomorphisms among various $\mathcal{B}_W(i, j)$ are no longer necessarily injective. However, we will still be able to give the necessary and sufficient conditions for $\mathcal{B}_W$ to have a n.i.s.

when W is any weighted shift of finite multiplicity.

In the final section of Chapter 3, we will study quasinilpotent weighted shifts of multiplicity two and provide insight on the structure of operators in $\mathcal{B}_W$. We will show that there are some algebraic structures which existed for $\mathcal{D}_W$, but cannot possibly exist for $\mathcal{B}_W$! However, we will not be able to give a specific listing of the possible structures of the spectral radius algebra, even though we could for the Deddens algebra. In the final chapter, we will look at some open problems and possible areas of future research in this topic.

Chapter 2

Deddens Algebras

2.1 Definitions and Properties

We start by carefully defining the Deddens algebra. Let $\mathcal{H}$ be a separable, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. In [10], J. Deddens considered the set

$$\{T \in \mathcal{L}(\mathcal{H}) : \sup_{n \in \mathbb{N}} \|A^n T A^{-n}\| < \infty\}$$

associated to an invertible operator $A \in \mathcal{L}(\mathcal{H})$. In that paper, Deddens showed that this set is an algebra, so we will refer to it as the *Deddens algebra associated to* A and denote it by $\mathcal{D}_A$. This definition requires $\mathcal{D}_A$ to be invertible, but we can extend it to noninvertible operators as well. To do this, we require the following proposition.

Proposition 2.1 *Let $A \in \mathcal{L}(\mathcal{H})$ be invertible, let $M > 0$, let $n \in \mathbb{N}$, and let $T \in \mathcal{L}(\mathcal{H})$. Then $\|A^n T A^{-n}\| \leq M$ if and only if $\|A^n T x\| \leq M\|A^n x\|$, for all $x \in \mathcal{H}$.*

PROOF: First, let $\|A^n T A^{-n}\| \leq M$. Then $\|A^n T A^{-n} y\| \leq M\|y\|$, for all $y \in \mathcal{L}(\mathcal{H})$. Let $x \in \mathcal{H}$. Substituting $y = A^n x$ into the previous inequality yields $\|A^n T x\| \leq M\|A^n x\|$, for all $x \in \mathcal{H}$.

Now, assume that $\|A^n T x\| \leq M\|A^n x\|$, for all $x \in \mathcal{H}$. Let $y \in \mathcal{H}$ and substitute $x = A^{-n} y$. Then we see that $\|A^n T A^{-n} y\| \leq M\|y\|$, for all $y \in \mathcal{H}$. Therefore, $\|A^n T A^{-n}\| \leq M$ which completes the proof. □

We can now state the "extended" definition of Deddens algebra.

Definition 2.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{L}(\mathcal{H})$ belongs to the Deddens algebra associated to A if there exists $M > 0$ such that $\|A^n T x\| \leq M\|A^n x\|$, for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$.*

Proposition 2.1 showed that this definition is equivalent to Deddens's definition when A is an invertible operator. For our purposes, we will solely be interested in Definition 2.2 since weighted shifts are not invertible. Next, we show that $\mathcal{D}_A$ is indeed an algebra, that is, it is a complex vector space which is closed under multiplication. In fact, we will also prove that the commutant is contained in the Deddens algebra. Recall that the commutant of A is denoted $\{A\}'$ and consists of the operators which commute with A.

Proposition 2.3 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $\mathcal{D}_A$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ containing the identity and $\{A\}' \subset \mathcal{D}_A$.*

PROOF: First, we will show that $\{A\}' \subset \mathcal{D}_A$. If $T \in \{A\}'$, then

$$\|A^n T x\| = \|T A^n x\| \leq \|T\|\|A^n x\|$$

so $T \in \mathcal{D}_A$, with $M = \|T\|$.

Now, we show that $\mathcal{D}_A$ is an algebra. Let $T_1, T_2 \in \mathcal{D}_A$. Then there exists $M_1 > 0$ and $M_2 > 0$ such that $\|A^n T_k x\| \leq M_k \|A^n x\|$, for all $n \in \mathbb{N}$, $x \in \mathcal{H}$, and $k = 1, 2$. Thus,

$$\|A^n(T_1 + T_2)x\| \leq \|A^n T_1 x\| + \|A^n T_2 x\|$$

$$\leq M_1 \|A^n x\| + M_2 \|A^n x\| = (M_1 + M_2)\|A^n x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Hence $\mathcal{D}_A$ is closed under addition. Also,

$$\|A^n T_1(T_2 x)\| \leq M_1 \|A^n T_2 x\| \leq M_1 M_2 \|A^n x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Thus, $\mathcal{D}_A$ is closed under multiplication. Furthermore, for any $\lambda \in \mathbb{C}$, we have that $\lambda I \in \{A\}' \subset \mathcal{D}_A$ and it follows from this that $(\lambda I)T = \lambda T \in \mathcal{D}_A$, for all $T \in \mathcal{D}_A$. Therefore, $\mathcal{D}_A$ is a subspace of $\mathcal{L}(\mathcal{H})$ which is closed under multiplication and contains the identity. $\square$

Now that we have established that $\{A\}' \subset \mathcal{D}_A$, it is essential to ensure that this containment is proper. Otherwise, our results about $\mathcal{D}_W$ would be, in fact, about the commutant of a weighted shift, and these are already well understood. To accomplish this task, we will introduce the spaces $\mathcal{D}_W(i, j)$ which were mentioned in the introduction.

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots$ be complex Hilbert spaces and consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ consisting of sequences $(x_1, x_2, \ldots)$ such that $x_k \in \mathcal{H}_k$ and $\sum_{k=1}^{\infty} \|x_k\|^2$ is finite. The inner product of $x = (x_1, x_2, \ldots) \in \mathcal{H}$ and $y = (y_1, y_2, \ldots) \in \mathcal{H}$ is defined by

$$\langle x, y\rangle_{\mathcal{H}} = \sum_{k=1}^{\infty} \langle x_k, y_k\rangle_{\mathcal{H}_k}.$$

To each operator $T \in \mathcal{L}(\mathcal{H})$, we can associate a matrix $\{T_{ij}\}_{i,j\in\mathbb{N}}$ where $T_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. Meanwhile, for an operator $A \in \mathcal{L}(\mathcal{H}_k, \mathcal{H}_l)$, we can define its extension $\overline{A} \in \mathcal{L}(\mathcal{H})$ to be the operator whose matrix $\{\overline{A}_{ij}\}_{i,j\in\mathbb{N}}$ is given by

$$\overline{A}_{ij} = \begin{cases} 0 & \text{if } (i,j) \neq (l,k) \\ A & \text{if } (i,j) = (l,k) \end{cases} \tag{2.1}$$

Thus, $\|\overline{A}\| = \|A\|$ and, if $B \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_k)$, then $\overline{AB} = \overline{A}\,\overline{B}$. To determine the structure of the Deddens algebra associated to a weighted shift, we will focus on the spaces

$$\mathcal{D}_W(i,j) := \{T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) : \overline{T} \in \mathcal{D}_W\}.$$

They can be viewed as the restriction of $\mathcal{D}_W$ to operators whose block matrix is zero outside of the (i,j) block. These spaces will play a crucial role in the proof that $\mathcal{D}_W$ and $\{W\}'$ are different, as well as in the study of the structure of $\mathcal{D}_W$. The results contained in this chapter are original research done by the author. They have also been accepted for publication in the Houston Journal of Mathematics.

2.2 General Weighted Shifts

In this section we begin an investigation of Deddens algebras associated to weighted shifts. Our first step is to define these operators. Let $\mathcal{H}_1, \mathcal{H}_2, \dots$ be separable, complex Hilbert spaces such that $\dim(\mathcal{H}_k) = \dim(\mathcal{H}_1)$, for all $k \in \mathbb{N}$, and, just as in Section 2.1, let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of operators such that $A_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_{n+1})$, for all $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \|A_n\| < \infty$. Then we can define an operator $W \in \mathcal{L}(\mathcal{H})$ by

$$W(x_1, x_2, x_3, \dots) = (0, A_1x_1, A_2x_2, \dots).$$

We call W a *weighted shift with weight sequence* $\{A_n\}_{n\in\mathbb{N}}$. To emphasize the fact that W has a weight sequence $\{A_n\}_{n\in\mathbb{N}}$, we will write $W \sim (A_n)$.

Relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$, the matrix of W is given by

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ A_1 & 0 & 0 & \\ 0 & A_2 & 0 & \\ 0 & 0 & A_3 & \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

where each block is either an $n \times n$ matrix, when $\dim(\mathcal{H}) = n$, or an infinite matrix when $\dim(\mathcal{H}) = \aleph_0$. We will then say that W *has multiplicity* n, respectively *infinite multiplicity*, when $\dim(\mathcal{H}) = n$, respectively $\dim(\mathcal{H}) = \aleph_0$. Without loss of generality, we can assume that $A_n \neq 0$, for all $n \in \mathbb{N}$, because if $A_n = 0$, for some $n \in \mathbb{N}$, then W can be written as the direct sum $W_1 \oplus W_2$, where W_1 has a finite weight sequence $\{A_1, A_2, \dots, A_{n-1}\}$, and W_2 has a weight sequence $\{A_{n+k}\}_{k\in\mathbb{N}}$.

Furthermore, the matrix of W^n is given by

$$W^n = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ A_nA_{n-1}\cdots A_1 & 0 & 0 & \\ 0 & A_{n+1}A_n\cdots A_2 & 0 & \\ 0 & 0 & A_{n+2}A_{n+1}\cdots A_3 & \\ \vdots & & & \ddots \end{bmatrix}.$$

where the first nonzero entry of this matrix is in the $(n+1, 1)$ block.

For this section, we will make no assumptions about the multiplicity of W. nor the injectivity of its weights (with the exception of Theorem 2.8). We will establish that each weight $A_k \in \mathcal{D}_W(k+1, k)$ and we will state the necessary and sufficient conditions for A_k^{-1} to belong to $\mathcal{D}_W(k+1, k)$. In the process we will establish that the inclusion $\{W\}' \subset \mathcal{D}_W$ is always proper.

Proposition 2.4 *Let $W \sim (A_n)$ be a weighted shift and let $T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. Then $T \in \mathcal{D}_W(i, j)$ if and only if there exists $M > 0$ such that*

$$\|A_{i+n-1}A_{i+n-2}\cdots A_iTx_j\| \leq M\|A_{j+n-1}A_{j+n-2}\cdots A_jx_j\|, \tag{2.2}$$

for all $x_j \in \mathcal{H}_j$ and all $n \in \mathbb{N}$.

Proof: Let $x = (x_1, x_2, x_3, ...) \in \mathcal{H}$ such that $x_k = 0$, for all $k \neq j$. For $T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$ and $n \in \mathbb{N}$, we have $\|W^n\overline{T}x\| = \|A_{i+n-1}A_{i+n-2}\cdots A_iTx_j\|$ and $\|W^nx\| = \|A_{j+n-1}A_{j+n-2}\cdots A_jx_j\|$. Thus, if $T \in \mathcal{D}_W(i, j)$, then there exists $M > 0$ such that $\|A_{i+n-1}A_{i+n-2}\cdots A_iTx_j\| \leq M\|A_{j+n-1}A_{j+n-2}\cdots A_jx_j\|$, for all $n \in \mathbb{N}$ and

$x_j \in \mathcal{H}_j$.

In the other direction, let $x = (x_1, x_2, \dots) \in \mathcal{H}$ where $x_k \in \mathcal{H}_k$, for all $k \in \mathbb{N}$. Then $\|W^n x\|^2 = \sum_{k=1}^{\infty} \|A_{n+k-1}A_{n+k-2}\cdots A_k x_k\|^2$, because $A_{n+k-1}A_{n+k-2}\cdots A_k x_k$ is orthogonal to $A_{n_j-1}A_{n+j-2}\cdots A_j x_j$, for all $k, j \in \mathbb{N}$. If there exists $M > 0$ such that (2.2) holds, for all $x_j \in \mathcal{H}_j$ and $n \in \mathbb{N}$, then

$$\|W^n \overline{T} x\|^2 = \|A_{i+n-1}A_{i+n-2}\cdots A_i T x_j\|^2 \leq M^2 \|A_{j+n-1}A_{j+n-2}\cdots A_j x_j\|^2$$

$$\leq M^2 \sum_{k=1}^{\infty} \|A_{n+k-1}A_{n+k-2}\cdots A_k x_k\|^2 = M^2 \|W^n x\|^2.$$

Therefore, $T \in \mathcal{D}_W(i,j)$, which completes the proof. □

Using this result, we can quickly find many operators in $\mathcal{D}_W$.

Corollary 2.5 *Let $W \sim (A_n)$ be a weighted shift and let $k \in \mathbb{N}$, then the identity operator $I_k : \mathcal{H}_k \to \mathcal{H}_k$ belongs to $\mathcal{D}_W(k,k)$ and $A_k \in \mathcal{D}_W(k+1,k)$.*

Proof: Both results follow from (2.2). For I_k, we can take $M = 1$, because $\|A_{k+n-1}\cdots A_{k+1}A_k I_k x\| = \|A_{k+n-1}\cdots A_{k+1}A_k x\|$, for all $x \in \mathcal{H}_k$ and $n \in \mathbb{N}$. Also, if we let $M = \sup_{n>k} \|A_n\| \leq \|W\|$, then

$$\|A_{k+n}A_{k+n-1}\cdots A_{k+1}A_k x\| \leq M \|A_{k+n-1}A_{k+n-2}\cdots A_k x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_k$. Hence, $A_k \in \mathcal{D}_W(k+1,k)$. □

An important consequence of this is that we will always have $\mathcal{D}_W \neq \{W\}'$.

Corollary 2.6 *Let $W \sim (A_n)$ be a weighted shift. Then $\overline{I}_k$ belongs to $\mathcal{D}_W$ but not to $\{W\}'$, for all $k \in \mathbb{N}$ Thus, $\mathcal{D}_W \neq \{W\}'$.*

Proof: It is a simple calculation to check that $W\overline{I_k} = \overline{A_k}$ while $\overline{I_k}W = \overline{A_{k-1}}$. Hence, $\overline{I_k} \notin \{W\}'$ because the $(k, k-1)$ blocks of $\overline{A_{k-1}}$ and $\overline{A}_k$ are A_{k-1} and 0, respectively. Since we can assume that all the weights are nonzero, it follows that $W\overline{I_k} = \overline{A_k} \neq \overline{A_{k-1}} = \overline{I_k}W$. □

This means that the Deddens algebra is always strictly larger than the commutant for weighted shifts. Hence, any results about the Deddens algebra, particularly about the existence of a n.i.s., is a stronger result than the corresponding statement for the commutant.

Let $T \in \mathcal{L}(\mathcal{H})$, and let $i, j \in \mathbb{N}$. Then $\overline{T_{ij}} = P_i T P_j$ where P_i and P_j are the orthogonal projections on $\mathcal{H}_i$ and $\mathcal{H}_j$ respectively. Since $P_k = \overline{I_k} \in \mathcal{D}_W$, we have the following result.

Corollary 2.7 *Let W be a weighted shift and let $T \in \mathcal{D}_W$. Then $T_{ij} \in \mathcal{D}_W(i, j)$.*

Due to Corollary 2.7, an operator $T \in D_W$ can be written as a weak limit of operators of the form

$$T_n = \sum_{i,j=1}^{n} \overline{T_{ij}}, \quad \text{where } T_{ij} \in \mathcal{D}_W(i, j).$$

In other words, the weak closure of $\mathcal{D}_W$ is completely determined by the spaces $\mathcal{D}_W(i, j)$. In Section 2.3, we study the relationships between various $\mathcal{D}_W(i, j)$ and this will lead to a description of the weak closure of $\mathcal{D}_W$. Our method will be based on the fact that the multiplication on the right by A_j provides a linear transformation from $\mathcal{D}_W(i, j+1)$ to $\mathcal{D}_W(i,j)$ and multiplication on the left by A_i is a linear transformation from $\mathcal{D}_W(i,j)$ to $\mathcal{D}_W(i+1,j)$. Naturally, we would like to know when these homomorphisms are isomorphisms. This will happen precisely when $\overline{A_k^{-1}}$ belongs to $\mathcal{D}_W$.

Theorem 2.8 *Let $W \sim (A_n)$ be a weighted shift such that A_n^{-1} exists, for all $n \in \mathbb{N}$. The following are equivalent:*

(a) $A_n^{-1} \in \mathcal{D}_W(n, n+1)$, for all $n \in \mathbb{N}$.

(b) There exists $k \in \mathbb{N}$ such that $A_k^{-1} \in \mathcal{D}_W(k, k+1)$.

(c) W is bounded below.

Proof: We start by establishing the fact that the inequality

$$\|A_{k+n-1}A_{k+n-2}\cdots A_{k+1}x\| \leq M\|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\| \tag{2.3}$$

holds, for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_{k+1}$, if and only if $\sup_{n>k}\|A_n^{-1}\|$ is finite and $M \geq \sup_{n>k}\|A_n^{-1}\|$. Let $n \in \mathbb{N}$ and $x \in \mathcal{H}_{k+1}$. Then

$$A_{k+n-1}A_{k+n-2}\cdots A_{k+1}x = A_{k+n}^{-1}A_{k+n}A_{k+n-1}\cdots A_{k+1}x.$$

Substituting $y = A_{k+n}A_{k+n-1}\cdots A_{k+1}x$, (2.3) becomes $\|A_{k+n}^{-1}y\| \leq M\|y\|$. Since $x \in \mathcal{H}_{k+1}$ is arbitrary and the operator $A_{k+n}A_{k+n-1}\cdots A_{k+1}$ is invertible, then y can be any vector in $\mathcal{H}_{k+n+1}$. Therefore, (2.3) holds if and only if $\sup_{n\in\mathbb{N}}\|A_{k+n-1}^{-1}\|$ is finite and is no larger than M.

Now we will prove the announced equivalences. The claim that (a) implies (b) is obvious. We will now show that (b) implies (c). Notice that, by Proposition 2.4,

$A_k^{-1} \in \mathcal{D}_W(k, k+1)$ if and only if (2.3) holds, for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_{k+1}$. By the statement we established in the first paragraph, this is equivalent to $M = \sup_{n>k} \|A_n^{-1}\| < \infty$. Since $\|A_n^{-1}\| = 1/(\inf_{\|x\|=1} \|A_n x\|)$, $\|A_n x\| \geq \frac{1}{M}\|x\|$, for all $x \in \mathcal{H}_n$ and $n > k$. Also, there exists m_n such that $\|A_n x\| \geq m_n \|x\|$, for all $x \in \mathcal{H}_n$, because A_n^{-1} exists, for all $n \in \mathbb{N}$. Let $m = \min\{m_1, m_2, \ldots, m_k, \frac{1}{M}\}$. Then $\|A_n x\| \geq m\|x\|$, for all $x \in \mathcal{H}_n$ and $n \in \mathbb{N}$. It follows that $\|Wx\| \geq m\|x\|$, for all $x \in \mathcal{H}$.

Finally, we assume that W is bounded below and we will show that $A_n^{-1} \in \mathcal{D}_W(n, n+1)$, for all $n \in \mathbb{N}$. Since W is bounded below, then $\{A_n\}_{n\in\mathbb{N}}$ is uniformly bounded below, or equivalently, $\{\|A_n^{-1}\|\}_{n\in\mathbb{N}}$ is a bounded sequence. Therefore, $\sup_{n>k} \|A_n\|^{-1} < \infty$ for each $k \in \mathbb{N}$. Hence (2.3) holds with $M = \sup_{n>k} \|A_n\|^{-1}$ and $A_k^{-1} \in \mathcal{D}_W(k, k+1)$, for all $k \in \mathbb{N}$, by Proposition 2.4. □

2.3 Weighted Shifts of Finite Multiplicity

For the rest of this chapter, we will assume that W has finite multiplicity. This will allow us to determine explicitly the structure of $\mathcal{D}_W$. For injective weighted shifts, we will describe a relationship between $\mathcal{D}_W(i, j)$ and $\mathcal{D}_W(k, l)$. However, this relationship disappears when W is not injective and we will discuss what can be said about such shifts.

2.3.1 Injective weighted shifts

If $W \sim (A_n)$ is an injective weighted shift of finite multiplicity, then A_k is an invertible operator for each $k \in \mathbb{N}$. Indeed, if there exists an $n \in \mathbb{N}$ such that A_n were not invertible, then there exists $x_n \in \mathcal{H}_n$ such that $A_n x_n = 0$ but $x_n \neq 0$. Thus, for $x = (0, \dots, 0, x_n, 0, \dots)$, we would have that $x \neq 0$ and $Wx = (0, A_1 0, \dots, A_{n-1}0, A_n x_n, A_{n+1}0, \dots) = 0$. Thus, W would have a nontrivial kernel and could not be injective. The next two theorems take advantage of this fact and they demonstrate that some of the structure of $\mathcal{D}_W(i, j)$ is inherited by $\mathcal{D}_W(i + 1, j)$, $\mathcal{D}_W(i, j - 1)$, and $\mathcal{D}_W(i + 1, j + 1)$.

Theorem 2.9 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity and let $i, j \in \mathbb{N}$. Then multiplication on the left by A_i is an injective linear transformation from $\mathcal{D}_W(i, j)$ to $\mathcal{D}_W(i + 1, j)$. Also, multiplication on the right by A_j is an injective linear transformation from $\mathcal{D}_W(i, j)$ to $\mathcal{D}_W(i, j - 1)$.*

Proof: Let $T \in \mathcal{D}_W(i, j)$. Then $\overline{A_i T} \in \mathcal{D}_W$ by Corollary 2.5 and $A_i T : \mathcal{H}_j \to \mathcal{H}_{i+1}$. Hence, multiplication by A_i on the left is a well-defined linear transformation from $\mathcal{D}_W(i, j)$ to $\mathcal{D}_W(i + 1, j)$. Furthermore, this function is injective because A_i is an invertible operator. Similarly, multiplication by A_j on the right is an injective linear transformation $\mathcal{D}_W(i, j) \to \mathcal{D}_W(i, j - 1)$. □

Theorem 2.10 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity and let $i, j \in \mathbb{N}$. Then the mapping $\varphi : \mathcal{D}_W(i, j) \to \mathcal{D}_W(i + 1, j + 1)$ defined by $\varphi(T) = A_i T A_j^{-1}$ is an isomorphism of vector spaces.*

Proof: Let $T \in \mathcal{D}_W(i, j)$. Since W is an injective weighted shift of finite multiplicity, then A_n^{-1} exists, for all $n \in \mathbb{N}$, and it is not hard to see that the mapping $T \mapsto A_i T A_j^{-1}$ is an injective linear transformation $\mathcal{D}_W(i, j) \to \mathcal{L}(\mathcal{H}_{j+1}, \mathcal{H}_{i+1})$. We will now show that $A_i T A_j^{-1} \in \mathcal{D}_W(i + 1, j + 1)$. Since $T \in \mathcal{D}_W(i, j)$, there exists

$M > 0$ such that

$$\|A_{i+n-1}A_{i+n-2}\cdots A_iTx\| \le M\|A_{j+n-1}A_{j+n-2}\cdots A_jx\|,$$

for all $x \in \mathcal{H}_j$ and for all $n \in \mathbb{N}$. Let $y \in \mathcal{H}_{j+1}$. Then

$$\|A_{i+n-1}A_{i+n-2}\cdots A_{i+1}(A_iTA_j^{-1})y\| = \|A_{i+n-1}A_{i+n-2}\cdots A_iT(A_j^{-1}y)\|$$

$$\le M\|A_{j+n-1}A_{j+n-2}\cdots A_j(A_j^{-1}y)\| = M\|A_{j+n-1}A_{j+n-2}\cdots A_{j+1}y\|,$$

for all $y \in \mathcal{H}_{j+1}$ and $n \in \mathbb{N}$. By Proposition 2.4, $\overline{A_iTA_j^{-1}} \in \mathcal{D}_W$, which shows that φ is well-defined. Also, it is not hard to see that φ is injective, since A_n is invertible, for all $n \in \mathbb{N}$. This leaves us with verifying that φ is surjective.

Let $X \in \mathcal{D}_W(i+1, j+1)$. Since $\varphi(A_i^{-1}XA_j) = X$, it suffices to prove that $A_i^{-1}XA_j \in \mathcal{D}_W(i,j)$. By Proposition 2.4, there exists $M > 0$ such that

$$\|A_{i+n}A_{i+n-1}\cdots A_{i+1}Xx\| \le M\|A_{j+n}A_{j+n-1}\cdots A_{j+1}x\|,$$

for all $x \in \mathcal{H}_{j+1}$ and for all $n \in \mathbb{N}$. Let $y \in \mathcal{H}_j$ and let $M' = \max\{M, \|X\|\}$. Then

$$\|A_{i+n}A_{i+n-1}\cdots A_i(A_i^{-1}XA_j)y\| = \|A_{i+n}A_{i+n-1}\cdots A_{i+1}X(A_jy)\|$$

$$\le M'\|A_{j+n}A_{j+n-1}\cdots A_{j+1}A_jy\|,$$

for all $n \ge 0$, so the result now follows from Proposition 2.4. $\square$

The following corollary is a reformulation of Theorems 2.9 and 2.10 in terms of a relationship between dimensions of various $\mathcal{D}_W(i,j)$.

Corollary 2.11 *Let W be an injective weighted shift of finite multiplicity. Then*

1. $\dim(\mathcal{D}_W(i,j)) \leq \dim(\mathcal{D}_W(l,k))$ *for* $k \leq j$ *and* $i \leq l$*;*

2. $\dim(\mathcal{D}_W(1,j)) = \dim(\mathcal{D}_W(1+n, j+n))$*, for all* $j, n \in \mathbb{N}$*;*

3. $\dim(\mathcal{D}_W(i,1)) = \dim(\mathcal{D}_W(i+n, 1+n))$*, for all* $i, n \in \mathbb{N}$*.*

These results show that if we can find $\mathcal{D}_W(1,j)$ and $\mathcal{D}_W(i,1)$, then we can determine the weak closure of $\mathcal{D}_W$. The next figure helps give a picture of what is happening.

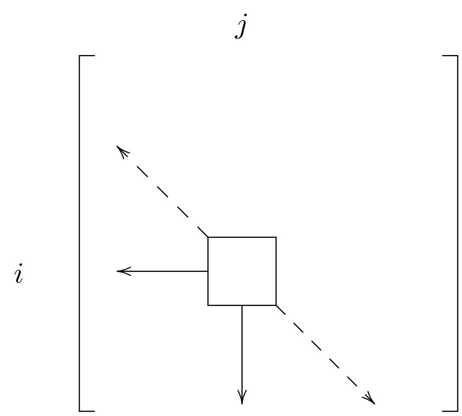

Figure 2.1

As we move from the (i,j) block in the direction of the solid arrows, the dimension of $\mathcal{D}_W(i,j)$ cannot decrease. While moving in the direction of dashed arrows, the dimension of $\mathcal{D}_W(i,j)$ is constant. The situation improves when the weighted shift is bounded below.

Theorem 2.12 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity. Then $\dim(\mathcal{D}_W(i,j)) = \dim(\mathcal{D}_W(l,k))$, for all $i,j,k,l \in \mathbb{N}$, if and only if W is bounded below.*

PROOF: Assume that $\dim(\mathcal{D}_W(i,j)) = \dim(\mathcal{D}_W(l,k))$, for all $i,j,k,l \in \mathbb{N}$. In particular, $\dim(\mathcal{D}_W(1,1)) = \dim(\mathcal{D}_W(1,2))$. By Theorem 2.9, multiplication by A_1 on the right provides an injective linear transformation $\mathcal{D}_W(1,2) \to \mathcal{D}_W(1,1)$. Since the dimensions are the same and finite, this multiplication is an isomorphism of vector spaces. From Corollary 2.5, we have that $I_1 \in \mathcal{D}_W(1,1)$ implying that there exists $X \in \mathcal{D}_W(1,2)$ such that $XA_1 = I_1$. Therefore $X = A_1^{-1} \in \mathcal{D}_W(1,2)$ and W is bounded below by Theorem 2.8.

Now assume that W is bounded below. Then Theorem 2.8 implies that $\overline{A_k^{-1}} \in \mathcal{D}_W$, for all $k \in \mathbb{N}$. If $T \in \mathcal{D}_W(i,j)$, then $A_{i-1}^{-1}T \in \mathcal{D}_W(i-1,j)$ so the multiplication by A_{i-1}^{-1} from the left is an imbedding of $\mathcal{D}_W(i,j)$ into $\mathcal{D}_W(i-1,j)$ and $\dim(\mathcal{D}_W(i-1,j)) \geq \dim(\mathcal{D}_W(i,j))$. Similarly, if $T \in \mathcal{D}_W(i,j)$, then $TA_j^{-1} \in \mathcal{D}_W(i,j+1)$ whence multiplication by A_j^{-1} on the right is an injective homomorphism from $\mathcal{D}_W(i,j)$ to $\mathcal{D}_W(i,j+1)$. Therefore, $\dim(\mathcal{D}_W(i,j+1)) \geq \dim(\mathcal{D}_W(i,j))$. Since the opposite inequalities are given in Part 1 of Corollary 2.11, we can now conclude that $\dim(\mathcal{D}_W(i,j)) = \dim(\mathcal{D}_W(l,k))$, for all $i,j,l,k \in \mathbb{N}$. □

When W is not bounded below, the dimensions of $D_W(i,j)$ cannot all be equal. The following result shows at least one place where the inequality is strict.

Corollary 2.13 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity which is not bounded below. Then $\dim(\mathcal{D}_W(1,2)) < \dim(\mathcal{D}_W(1,1))$. Also, $\mathcal{D}_W(1,k)$ contains no invertible operators for any $k > 1$.*

Proof: By Corollary 2.11, $\dim(\mathcal{D}_W(1,2)) \leq \dim(\mathcal{D}_W(1,1))$. Suppose, to the contrary, that $\dim(\mathcal{D}_W(1,2)) = \dim(\mathcal{D}_W(1,1))$. It would then follow that multiplication by A_1 on the right is a vector space isomorphism from $\mathcal{D}_W(1,2)$ to $\mathcal{D}_W(1,1)$. In particular, there must exist an operator $X \in \mathcal{D}_W(1,2)$ such that $XA_1 = I_1$. Thus, $A_1^{-1} \in \mathcal{D}_W(1,2)$ and, by Theorem 2.8, W cannot be bounded below. Therefore, $\dim(\mathcal{D}_W(1,2)) < \dim(\mathcal{D}_W(1,1))$.

Let $k > 1$ and assume, to the contrary, that there exists an invertible operator $X \in \mathcal{D}_W(1,k)$. Then multiplication by X on the right yields an injective linear transformation $\mathcal{D}_W(1,1) \to \mathcal{D}_W(1,k)$. However, this implies that $\dim(\mathcal{D}_W(1,1)) \leq \dim(\mathcal{D}_W(1,k)) \leq \dim(\mathcal{D}_W(1,2)) < \dim(\mathcal{D}_W(1,1))$. This contradiction shows that there cannot be any invertible operators in $\mathcal{D}_W(1,k)$, for any $k > 1$. $\square$

The relationships that we have established among the various $\mathcal{D}_W(i,j)$ can now be used to prove the main result of this chapter.

Theorem 2.14 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity. The following are equivalent:*

(a) W is bounded below and $\mathcal{D}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$.

(b) The weak closure of $\mathcal{D}_W$ is $\mathcal{L}(\mathcal{H})$.

(c) $\mathcal{D}_W$ has no n.i.s.

Proof: We start by showing that (a) implies (b). Let W be bounded below and $\mathcal{D}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$. By Theorem 2.12, we have $\mathcal{D}_W(i,j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $i, j \in \mathbb{N}$. If $T \in \mathcal{L}(\mathcal{H})$, then T has a matrix $\{T_{ij}\}_{i,j\in\mathbb{N}}$ with $T_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) = \mathcal{D}_W(i,j)$.

Therefore, T is the weak limit of $\sum_{i,j=1}^{n} \overline{T}_{ij}$ implying that T is in the weak closure of $\mathcal{D}_W$.

Next, we will prove that (b) implies (c). Let $\mathcal{D}_W$ be weakly dense in $\mathcal{L}(\mathcal{H})$ and let $\mathcal{M}$ be an invariant subspace for $\mathcal{D}_W$. The first step is to show that $\mathcal{M}$ is invariant for $\mathcal{L}(\mathcal{H})$ as well. Let $T \in \mathcal{L}(\mathcal{H})$. Then there exists a sequence $\{T_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}_W$ such that $\langle (T - T_n)x, y\rangle$ converges to zero, for all $x, y \in \mathcal{H}$. Let $x \in \mathcal{M}$ and $y \in \mathcal{M}^{\perp}$. Then $T_n x \in \mathcal{M}$, for all $n \in \mathbb{N}$, implying that $\langle T_n x, y\rangle = 0$ and $\langle (T - T_n)x, y\rangle = \langle Tx, y\rangle - \langle T_n x, y\rangle = \langle Tx, y\rangle$ converges to zero, for all $x \in \mathcal{M}$ and $y \in \mathcal{M}^{\perp}$. Thus, $Tx \in (\mathcal{M}^{\perp})^{\perp} = \mathcal{M}$, for all $x \in \mathcal{M}$. Since T was arbitrary in $\mathcal{L}(\mathcal{H})$, it follows that $\mathcal{M}$ is invariant for $\mathcal{L}(\mathcal{H})$. We can now show that $\mathcal{M}$ must be trivial. Assume that $\mathcal{M} \neq 0$ and let $x \in \mathcal{M}$ be nonzero. Then $y \otimes x \in \mathcal{L}(\mathcal{H})$ has $\mathcal{M}$ as an invariant subspace, for all $y \in \mathcal{H}$. Hence, $\langle x, x\rangle y = y \otimes x(x) \in \mathcal{M}$ and it follows that $y \in \mathcal{M}$, for all $y \in \mathcal{H}$ because $x \neq 0$. Therefore, $\mathcal{M} = \mathcal{H}$ implying that $\mathcal{M}$ cannot be a n.i.s.

We will now complete the proof by showing that (c) implies (a). We will actually prove the contrapositive: "if W is not bounded below or $\mathcal{D}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$ then there exists a n.i.s. for $\mathcal{D}_W$". Furthermore, we will take advantage of the tautology, "p $\vee$ q $\Leftrightarrow$ (p $\wedge$ $\neg$q) $\vee$ q". In other words, we will prove the existence of a n.i.s. under each of the assumptions: "W is not bounded below and $\mathcal{D}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$" and "$\mathcal{D}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$".

First, if $\mathcal{D}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$ and W is not bounded below, we will show that $\mathcal{D}_W(1,j) = 0$, for all $j > 1$. Assume, to the contrary, that $\mathcal{D}_W(1,j_0) \neq 0$, for some $1 < j_0 \in \mathbb{N}$. Since $\mathcal{D}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$, then $\mathcal{D}_W(i,j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $i \geq j$. Let $A \in \mathcal{L}(\mathcal{H}_1) = \mathcal{D}_W(1,1)$, let $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{j_0}) = \mathcal{D}_W(j_0, 2)$, and let $0 \neq X \in \mathcal{D}_W(1, j_0)$. Since $\mathcal{D}_W$ is closed under multiplication, it follows that $\overline{AXT} \in \mathcal{D}_W$ with AXT

taking $\mathcal{H}_2$ to $\mathcal{H}_1$. In other words, we have that $AXT \in \mathcal{D}_W(1,2)$. We will now show that this leads to $\mathcal{D}_W(1,2) = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$.

Let $y \in \mathcal{H}_1$, let $z \in \mathcal{H}_2$, and let $x \in \mathcal{H}_{j_0}$ such that $Xx \in \mathcal{H}_1$ is nonzero. Let $A = y \otimes (Xx) \in \mathcal{L}(\mathcal{H}_1)$ and $T = x \otimes z \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{j_0})$. Then it is not hard to verify that $AXT = \|Xx\|^2 (y \otimes z) \in \mathcal{D}_W(1,2)$. Since $Xx \neq 0$ and y and z were arbitrary, then $\mathcal{D}_W(1,2)$ contains all the rank one operators in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. Therefore, $\mathcal{D}_W(1,2)$ contains all the operators in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ because they are finite sums of rank one operators. In particular, $A_1^{-1} \in \mathcal{D}_W(1,2) = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and, by Theorem 2.8, W must be bounded below. However, W is not bounded below so we cannot have that $\mathcal{D}_W(1, j_0) \neq 0$, for some $j_0 > 1$. This implies that $\mathcal{D}_W$ consists of block lower triangular matrices, by Corollary 2.11, and thus has a n.i.s.

Next, we assume that $\mathcal{D}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$ and construct a n.i.s. for $\mathcal{D}_W$. Notice that $\mathcal{D}_W(1,1)$ is a finite dimensional algebra over the field of complex numbers. By Burnside's Theorem, there exists a nontrivial subspace $\mathcal{M}_1 \subset \mathcal{H}_1$, invariant for $\mathcal{D}_W(1,1)$. Define $\mathcal{M}_k$ to be the subspace of $\mathcal{H}_k$ generated by $\{Ax : x \in \mathcal{M}_1, A \in \mathcal{D}_W(k,1)\}$ and define $\mathcal{M}$ to be the closure of $\bigoplus_{k\in\mathbb{N}} \mathcal{M}_k$. We will show that $\mathcal{M}$ is a n.i.s. for $\mathcal{D}_W$. It suffices to show that $\mathcal{M}$ is invariant for $\overline{T}_{ij}$ whenever $T_{ij} \in \mathcal{D}_W(i,j)$, for all $i, j \in \mathbb{N}$. Indeed if $T \in \mathcal{D}_W$, then T is a weak limit of $\sum_{i,j=1}^{n} \overline{T_{ij}}$ and if $\mathcal{M}$ is invariant for each $\overline{T_{ij}}$, then $\mathcal{M}$ is invariant for the sum $\sum_{i,j=1}^{n} \overline{T_{ij}}$ and also for T. Also, $\mathcal{M}$ is nontrivial because we assumed that $0 \neq \mathcal{M}_1 \neq \mathcal{H}_1$. Thus, it remains to show that $\mathcal{M}$ is invariant for $\overline{T}_{ij}$. Let $T \in \mathcal{D}_W(i,j)$ and let $y \in \mathcal{M}_j$. Then there exists $m \in \mathbb{N}$, $B_k \in \mathcal{D}_W(j,1)$, and $x_k \in \mathcal{M}_1$, $1 \leq k \leq m$, such that $y = \sum_{k=1}^{m} B_k x_k$ and $Ty = \sum_{k=1}^{m} TB_k x_k$. Since $TB_k \in \mathcal{D}_W(i,1)$, the definition of $\mathcal{M}_i$ shows that $TB_k x_k \in \mathcal{M}_i$, for each $1 \leq k \leq m$. We conclude that $Ty \in \mathcal{M}_i$ and the proof is complete. □

2.3.2 A word on noninjective weighted shifts

When W is noninjective, Theorems 2.9 and 2.10 do not hold. This means that knowing the structure of $\mathcal{D}_W(i,j)$ yields no information about the spaces, $\mathcal{D}_W(i+1,j), \mathcal{D}_W(i,j-1)$, or $\mathcal{D}_W(i+1,j+1)$. We will construct an operator W that will illustrate that phenomenon. Before we do that, let us note that, when W is noninjective, the algebra $\mathcal{D}_W$ has a n.i.s. In fact, this result is true for any operator A, not only a shift.

Theorem 2.15 *Let $A \in \mathcal{L}(\mathcal{H})$ be a noninjective nonzero operator. Then* $\ker(A)$ *is invariant for $\mathcal{D}_A$, so $\mathcal{D}_A$ has a n.i.s.*

PROOF: Let $T \in \mathcal{D}_A$. Then there exists $M > 0$ such that $\|A^n T x\| \leq M \|A^n x\|$, for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. If $x \in \ker(A)$, then setting $n = 1$ yields $\|ATx\| \leq 0$. Hence $Tx \in \ker(A)$, for all $x \in \ker(A)$ and $T \in \mathcal{D}_A$. Therefore, $\ker(A)$ is invariant for $\mathcal{D}_A$ and is nontrivial precisely when A is a noninjective nonzero operator. □

Example

In this example we show that none of the statements in Corollary 2.11 hold if we do not assume that W is injective. Let $W \sim (A_n)$ be a weighted shift of multiplicity 2 such that

$$A_1 = A_5 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{otherwise.}$$

First, we will demonstrate that $\dim(\mathcal{D}_W(1,1)) \leq 3$ while $\dim(\mathcal{D}_W(6,6)) = 4$. This will contradict equality in dimension along diagonals obtained in Corollary 2.11. Recall from Proposition 2.4 that $T \in \mathcal{D}_W(1,1)$ if and only if there exists $M > 0$

such that

$$\|A_n A_{n-1} \cdots A_1 T x\| \le M \|A_n A_{n-1} \cdots A_1 x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_1$. Let $x = (0,1)^T$ and

$$T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}.$$

For $n = 1$, $\|A_1 T x\| = |t_2|$ and $\|A_1 x\| = 0$. Therefore, $T \in \mathcal{D}_W(1,1)$ implies that $t_2 = 0$ and $\mathcal{D}_W(1,1)$ is at most three dimensional. Meanwhile, $\|A_{n+5}A_{n+4} \cdots A_6 y\| = \|y\|$, for all $y \in \mathcal{H}_6$ and $n \in \mathbb{N}$. So for $T \in \mathcal{L}(\mathcal{H}_6)$, the inequality

$$\|A_{n+5}A_{n+4} \cdots A_6 T x\| = \|Tx\| \le \|T\|\|x\| = \|T\|\|A_{n+5}A_{n+4} \cdots A_6 x\|$$

holds, for all $x \in \mathcal{H}_6$ and $n \in \mathbb{N}$. By Proposition 2.4, $\mathcal{D}_W(6,6) = \mathcal{L}(\mathcal{H}_6)$.

Next, we will show that $\dim(\mathcal{D}_W(1,6)) = 4 > \dim(\mathcal{D}_W(1,1))$. This contradicts the statement that $\dim(\mathcal{D}_W(i,j))$ increases as j decreases. Let $T \in \mathcal{L}(\mathcal{H}_6, \mathcal{H}_1)$ and let $x \in \mathcal{H}_6$. Then

$$\|A_n A_{n-1} \cdots A_1 T x\| \le \|Tx\| \le \|T\|\|x\| = \|T\|\|A_{n+5}A_{n+4} \cdots A_6 x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_6$. By Proposition 2.4, $T \in \mathcal{D}_W(1,6)$ and $\mathcal{D}_W(1,6) = \mathcal{L}(\mathcal{H}_6, \mathcal{H}_1)$.

Finally, we will show that $\dim(\mathcal{D}_W(i,j))$ can decrease as i increases. In fact, we will show that $\mathcal{D}_W(3,1)$ consists of just the zero operator. This will imply that $\dim(\mathcal{D}_W(1,1)) > \dim(\mathcal{D}_W(3,1))$ because $\mathcal{D}_W(1,1)$ contains the identity operator by

Corollary 2.5. If $T \in \mathcal{D}_W(3,1)$ then there exists $M > 0$ such that

$$\|Tx\| = \|A_4 A_3 T x\| \leq M \|A_2 A_1 x\| = 0,$$

for all $x \in \mathcal{H}_1$. Hence T must be the zero operator.

2.4 Weighted Shifts of Multiplicity Two

This section will illustrate that, if W is a weighted shift of multiplicity 2, we cannot strengthen the results obtained in Section 2.2. For weighted shifts which are bounded below, Theorem 2.12 asserts that $\dim(\mathcal{D}_W(i,j))$ is the same, for all $i, j \in \mathbb{N}$. We will give examples where each number between 1 and 4 is realized as $\dim(\mathcal{D}_W(i,j))$. Clearly, this common dimension cannot be zero since $I_1 \in \mathcal{D}_W(1,1)$ is nonzero. For weighted shifts which are not bounded below, we have the inequalities stated in Corollary 2.11. Our examples will show that almost any choice of these dimensions that does not contradict these inequalities can be achieved. This shows that the structure of the Deddens algebra for a weighted shift of multiplicity 2 can vary a lot. This is in sharp contrast to weighted shifts of multiplicity 1 where $\mathcal{D}_W$ is weakly dense in either (LT) or $\mathcal{L}(\mathcal{H})$. Before we proceed with examples, we make one further simplification about the structure of the weights.

Let $W \sim (A_n)$ be a weighted shift and let U be the operator defined by

$$U(x_1, x_2, \dots) = (U_1 x_1, U_2 x_2, \dots).$$

Then UWU^* is a weighted shift with weight sequence $\{U_{n+1} A_n U_n^*\}_{n \in \mathbb{N}}$. By the QR-decomposition (assuming $\dim(\mathcal{H}) < \infty$, see [14]), there exists a unitary matrix

Q_k and an upper triangular matrix R_k such that $A_k U_k^* = Q_k R_k$. Moreover, we can assume that the diagonal entries of R_k are nonnegative real numbers. Let $U_1 := I$ and $U_{k+1} := Q_k^{-1}$. Then $U_{k+1} A_k U_k^* = R_k$ is an upper triangular matrix. Under these definitions, U is a unitary operator and this shows that W is unitarily equivalent to a weighted shift whose weights are upper triangular matrices with nonnegative diagonal entries. In this section, we will consider injective weighted shifts so we will assume that each weight is also invertible. Finally, we restrict our attention to weighted shifts of multiplicity two so now the weights will look like

$$A_k = \begin{bmatrix} a_k & b_k \\ 0 & c_k \end{bmatrix} \tag{2.4}$$

where $a_k, c_k > 0$. By induction, one could show that, for all $k, n \in \mathbb{N}$,

$$A_{k+n-1} A_{k+n-2} \cdots A_k = \begin{bmatrix} \prod_{t=0}^{n-1} a_{k+t} & \beta_{k,n} \\ 0 & \prod_{t=0}^{n-1} c_{k+t} \end{bmatrix} \tag{2.5}$$

where

$$\beta_{k,n} = \sum_{t=0}^{n-1} c_k c_{k+1} \cdots c_{k+t-1} b_{k+t} a_{k+t+1} \cdots a_{k+n-1}. \tag{2.6}$$

Note that $c_k c_{k+1} \cdots c_{k+t-1} = 1$, when $t = 0$, and $a_{k+t+1} a_{k+t+2} \cdots a_{k+n-1} = 1$, when $k = n - 1$.

Alternatively, we could consider the weighted shifts (of multiplicity 1) with weight sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$, and $\{c_n\}_{n \in \mathbb{N}}$. If we call these W_1, W_2, and W_3 respectively, then there exists a decomposition $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that, relative to this decomposition,

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix}.$$

Let $\{e_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ be the orthonormal bases (o.n.b.) for $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $W_1 e_n = a_n e_{n+1}$ and $W_3 f_n = c_n f_{n+1}$. Under the inclusions $\mathcal{M}_i \hookrightarrow \mathcal{H}$, the set $\{e_n, f_n\}_{n\in\mathbb{N}}$ forms an o.n.b. for $\mathcal{H}$.

2.4.1 Weighted shifts which are bounded below

Let W be a weighted shift of finite multiplicity which is bounded below. Theorem 2.12 established that $\dim(\mathcal{D}_W(i,j))$ is constant with respect to i and j. In this section, we will show that $\mathcal{D}_W(i,j)$ can be either $1, 2, 3$, or 4 dimensional. We start with an example where $\dim(\mathcal{D}_W(1,1)) = 4$.

Example

Let S be the unilateral shift of multiplicity 2 (each weight is the identity). Then S is an isometry and

$$\|S^n Tx\| = \|Tx\| \leq \|T\|\|x\| = \|T\|\|S^n x\|,$$

for all $n \in \mathbb{N}$ and $T \in \mathcal{L}(\mathcal{H})$. Therefore $\mathcal{D}_S = \mathcal{L}(\mathcal{H})$.

Example

Let W be a weighted shift with weight sequence $\{A_n\}_{n\in\mathbb{N}}$ defined by

$$A_k = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

for all $k \in \mathbb{N}$. For this operator, $\dim(\mathcal{D}_W(i,j)) = 3$, for all $i, j \in \mathbb{N}$.

PROOF: Since W is bounded below, it suffices to show that $\mathcal{D}_W(1,1)$ is three dimensional. Let $x = (x_1, x_2)^T$ and

$$T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1).$$

Then

$$A_n A_{n-1} \cdots A_1 T x = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2^n t_1 x_1 + 2^n t_2 x_2 \\ t_3 x_1 + t_4 x_2 \end{bmatrix},$$

while

$$A_n A_{n-1} \cdots A_1 x = \begin{bmatrix} 2^n x_1 \\ x_2 \end{bmatrix}.$$

By Proposition 2.4, $T \in \mathcal{D}_W(1,1)$ if and only if there exists $M > 0$ such that

$$\|A_n A_{n-1} \cdots A_1 T x\| \leq M \|A_n A_{n-1} \cdots A_1 x\|,$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}_1$. That is, $T \in \mathcal{D}_W(1,1)$ if and only if

$$(4^n |t_1 x_1 + t_2 x_2|^2 + |t_3 x_1 + t_4 x_2|^2)^{\frac{1}{2}} \leq M (4^n |x_1|^2 + |x_2|^2)^{\frac{1}{2}},$$

for all $n \in \mathbb{N}$ and $x_1, x_2 \in \mathbb{C}$. Letting $x_1 = 0$ and $x_2 = 1$, we see that $t_2 = 0$, so

$$(4^n|t_1x_1|^2 + |t_3x_1 + t_4x_2|^2)^{\frac{1}{2}} \leq M(4^n|x_1|^2 + |x_2|^2)^{\frac{1}{2}}.$$

If we let $t_1 = 1$ while $t_3 = t_4 = 0$, this inequality holds with $M = 1$. Also, this inequality holds if $t_3 = 1$ and $t_1 = t_4 = 0$, or if $t_4 = 1$ and $t_1 = t_3 = 0$. Therefore,

$$\mathcal{D}_W(1,1) = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$$

and $\dim(\mathcal{D}_W(1,1)) = 3$. □

Example

Our next example will show that $\mathcal{D}_W(1,1)$ can be two dimensional. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

and define W to be the weighted shift with weight sequence

$$(A, B, B, A, A, A, B, B, B, B, \dots).$$

Let

$$T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1).$$

We will show that $T \in \mathcal{D}_W(1,1)$ if and only if T is a diagonal matrix. Since W is bounded below, then this will prove that $\dim(\mathcal{D}_W(i,j)) = 2$, for all $i, j \in \mathbb{N}$.

Proof: Let $x = (x_1, x_2)^T$ and let

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

be operators in $\mathcal{L}(\mathcal{H}_1)$. Then

$$A_n A_{n-1} \cdots A_1 X x = A_n A_{n-1} \cdots A_1 \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

and we have $\|A_n A_{n-1} \cdots A_1 X x\| \leq \|A_n A_{n-1} \cdots A_1 x\|$, for all $x \in \mathcal{H}_1$, implying that $X \in \mathcal{D}_W(1,1)$. Similarly, $Y \in \mathcal{D}_W(1,1)$. Let

$$T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \in \mathcal{D}_W(1,1).$$

Then

$$T' = T - t_1 X - t_4 Y = \begin{bmatrix} 0 & t_2 \\ t_3 & 0 \end{bmatrix} \in \mathcal{D}_W(1,1).$$

It now suffices to show that $t_2 = t_3 = 0$. Let $n = 1 + 2 + 3 + \cdots + (2k+1) = (2k+1)(k+1)$. Then

$$A_n A_{n-1} \cdots A_1 = \begin{bmatrix} 2^{1+3+\cdots+2k+1} & 0 \\ 0 & 2^{2+4+\cdots 2k} \end{bmatrix} = \begin{bmatrix} 2^{(k+1)^2} & 0 \\ 0 & 2^{k(k+1)} \end{bmatrix}.$$

If $x = (0, x_2)^T$, then

$$A_n A_{n-1} \cdots A_1 T' x = \begin{bmatrix} 2^{(k+1)^2} & 0 \\ 0 & 2^{k(k+1)} \end{bmatrix} \begin{bmatrix} 0 & t_2 \\ t_3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2^{(k+1)^2} t_2 x_2 \\ 0 \end{bmatrix}$$

while

$$A_n A_{n-1} \cdots A_1 x = \begin{bmatrix} 2^{(k+1)^2} & 0 \\ 0 & 2^{k(k+1)} \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2^{k(k+1)} x_2 \end{bmatrix}.$$

Therefore

$$\|A_n A_{n-1} \cdots A_1 T' x\| = 2^{k+1} |t_2| \|A_n A_{n-1} \cdots A_1 x\|$$

implying that t_2 must be zero or T' does not belong to $\mathcal{D}_W$. Similarly, by considering $n = 1 + 2 + 3 + \cdots + 2k$ and $x = (x_1, 0)^T$, one can show that $t_3 = 0$. □

Finally, we will construct an operator such that $\dim(\mathcal{D}_W(i,j)) = 1$, for all $i, j \in \mathbb{N}$. The proof that W has this property will rely upon the following result.

Proposition 2.16 *Let W be an injective weighted shift of multiplicity 2 such that W is bounded below and* $\dim(\mathcal{D}_W(1,1)) > 1$. *Then there exists a rank one operator, in $\mathcal{D}_W$.*

PROOF: By Corollary 2.5, the identity $I : \mathcal{H}_1 \to \mathcal{H}_1$ is in $\mathcal{D}_W(1,1)$. If $\{I, A\}$ is a linearly independent set in $\mathcal{D}_W(1,1)$ and λ is an eigenvalue of A, then $A - \lambda I \in \mathcal{D}_W(1,1)$ with $\ker(A - \lambda I) \neq 0$. Since $A - \lambda I$ is an operator on $\mathcal{H}_1 \cong \mathbb{C}^2$, then $A - \lambda I$ is either a rank one operator or the zero operator. The assumption that $\{I, A\}$ is a linearly independent set implies that $A - \lambda I \neq 0$ so $\overline{A - \lambda I}$ is a rank one operator in $\mathcal{D}_W$.

□

Proposition 2.16 shows that if W is a weighted shift such that $\mathcal{D}_W$ contains no rank one operators, then $\dim(\mathcal{D}_W(1,1)) = 1$. The next proposition gives a sufficient condition for a weighted shift to have this property.

Proposition 2.17 *Let $W \sim (A_n)$ be a weighted shift of multiplicity 2 which is bounded below. If a_k, b_k, c_k and β_{kn} are as in (2.4)-(2.6) and if there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that*

$$\lim_{i\to\infty} \prod_{k=1}^{n_i} \frac{a_k}{c_k} = \infty \qquad \textit{and} \qquad \lim_{i\to\infty} \frac{\beta_{1,n_i}}{\prod_{k=1}^{n_i} a_k} = \infty, \tag{2.7}$$

then there are no rank one operators in $\mathcal{D}_W$.

Proof: If there exists a rank one operator $F \in \mathcal{D}_W$, then we can find positive integers, $i_0, j_0 \in \mathbb{N}$, such that $P_{i_0}FP_{j_0}$ is nonzero, where P_k denotes the projection on $\mathcal{H}_k$. Since $P_k = \overline{I}_k$ where I_k is the identity on $\mathcal{H}_k$, we have that $P_k \in \mathcal{D}_W$, for all $k \in \mathbb{N}$. Thus, $P_{i_0}FP_{j_0} \in \mathcal{D}_W$ and the matrix of $P_{i_0}FP_{j_0}$ consists of precisely one nonzero entry in the (i_0, j_0) block. In other words, there exists a nonzero X in $\mathcal{D}_W(i_0, j_0)$ such that $\overline{X} = P_{i_0}FP_{j_0}$. Since F is rank one, it follows that $X = y \otimes z$ is a rank one operator as well. By Theorem 2.8,

$$A_1^{-1}A_2^{-1}\cdots A_{i_0-1}^{-1}(y\otimes z)A_{j_0-1}A_{j_0-2}\cdots A_1$$

is a rank one operator in $\mathcal{D}_W(1,1)$. Therefore, it suffices to show that there are no rank one operators in $\mathcal{D}_W(1,1)$.

Next we will show that $y \otimes z \in \mathcal{D}_W(1,1)$, for any $y, z \in \mathcal{H}_1$, if and only if the sequence

$$\{\|A_n \cdots A_1 y\| \|A_n^{*-1} \cdots A_1^{*-1} z\|\}_{n\in\mathbb{N}}$$

is bounded. So let $y \otimes z \in \mathcal{L}(\mathcal{H}_1)$. Then $y \otimes z \in \mathcal{D}_W(1,1)$ if and only if there exists $M > 0$ such that

$$\|A_n A_{n-1} \cdots A_1 (y \otimes z) x\| \leq M \|A_n A_{n-1} \cdots A_1 x\|,$$

for all $x \in \mathcal{H}_1$ and $n \in \mathbb{N}$. This is equivalent to the existence of M such that

$$\|A_n A_{n-1} \cdots A_1 (y \otimes z) A_1^{-1} A_2^{-1} \cdots A_n^{-1}\| \leq M,$$

for all $n \in \mathbb{N}$. However,

$$\|A_n A_{n-1} \cdots A_1 (y \otimes z) A_1^{-1} A_2^{-1} \cdots A_n^{-1}\| = \|A_n \cdots A_1 y\| \|A_n^{*-1} \cdots A_1^{*-1} z\|,$$

so $y \otimes z \in \mathcal{D}_W(1,1)$ if and only if $\|A_n \cdots A_1 y\| \|A_n^{*-1} \cdots A_1^{*-1} z\|$ is a bounded sequence.

Let $y = (y_1, y_2)^T \in \mathcal{H}_1$ and $z = (z_1, z_2)^T \in \mathcal{H}_1$ and assume that $y \otimes z \in \mathcal{D}_W$. Then

$$\|A_n A_{n-1} \cdots A_1 y\|^2 = \left| \prod_{k=1}^{n} a_k y_1 + \beta_{1,n} y_2 \right|^2 + |y_2|^2 \prod_{k=1}^{n} c_k^2$$

and

$$\|A_n^{*-1} A_{n-1}^{*-1} \cdots A_1^{*-1} z\|^2 = \frac{|z_1|^2}{\prod_{k=1}^{n} a_k^2} + \left| \frac{-\beta_{1,n} z_1}{\prod_{k=1}^{n} a_k c_k} + \frac{z_2}{\prod_{k=1}^{n} c_k} \right|^2.$$

Therefore

$$\|A_n A_{n-1} \cdots A_1 y\| \|A_n^{*-1} A_{n-1}^{*-1} \cdots A_1^{*-1} z\|$$

$$\geq \left| \prod_{k=1}^{n} a_k y_1 + \beta_{1,n} y_2 \right| \cdot \left| \frac{-\beta_{1,n} z_1}{\prod_{k=1}^{n} a_k c_k} + \frac{z_2}{\prod_{k=1}^{n} c_k} \right|$$

$$= \prod_{k=1}^{n} \frac{a_k}{c_k} \left| y_1 + \frac{\beta_{1,n}}{\prod_{k=1}^{n} a_k} y_2 \right| \left| \frac{-\beta_{1,n}}{\prod_{k=1}^{n} a_k} z_1 + z_2 \right|.$$

The first condition in (2.7) implies that

$$\left| y_1 + \frac{\beta_{1,n_i}}{\prod_{k=1}^{n_i} a_k} y_2 \right| \to 0 \quad \text{or} \quad \left| \frac{-\beta_{1,n_i}}{\prod_{k=1}^{n_i} a_k} z_1 + z_2 \right| \to 0. \tag{2.8}$$

However, the second condition in (2.7) shows that $y = 0$ or $z = 0$, which implies that $y \otimes z = 0$. □

We now give an example of a weighted shift W such that $\mathcal{D}_W(i,j)$ is one dimensional, for all $i, j \in \mathbb{N}$.

Example

Let $\{a_k\}_{k \in \mathbb{N}}$ be the sequence

$$(a_1, a_2, a_3, \dots) = (2, 1, 1, 2, 2, 2, 1, 1, 1, 1, \dots),$$

let $\{c_k\}_{k \in \mathbb{N}}$ be the sequence

$$(c_1, c_2, c_3, \dots) = (1, 2, 2, 1, 1, 1, 2, 2, 2, 2, \dots),$$

and let $b_k = 1$, for all $k \in \mathbb{N}$. The Deddens algebra for the weighted shift $W \sim (A_n)$ contains no rank one operators.

Proof: Let $n_i = 1 + 2 + \cdots + (2i+1) = (2i+1)(i+1)$. We will investigate the product $A_{n_i} \cdots A_2 A_1$ In this case,

$$\prod_{k=1}^{n_i} a_k = 2^{1+3+5+\cdots+(2i+1)} = 2^{(i+1)^2} \quad \text{and} \quad \prod_{k=1}^{n_i} c_k = 2^{2+4+6+\cdots+2i} = 2^{i(i+1)}.$$

Therefore,

$$\prod_{k=1}^{n_i} \frac{a_k}{c_k} = 2^{i+1} \to \infty$$

satisfying the first condition in (2.7). Using induction on i, we will show that

$$\beta_{1,n_i} = 2^{(i+1)^2} + 2^{i(i+1)}[2^{2i+3} - 2^{i+3} - 3(2^i - 1)], \tag{2.9}$$

for all $i \geq 0$. Indeed, it is not hard to see that $\beta_{1,n_0} = 1$. So now assume that (2.9) holds for $n_i = n_{j-1}$. Then

$$\begin{bmatrix} 2^{(j+1)^2} & \beta_{1,n_j} \\ 0 & 2^{j(j+1)} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{2j+1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{2j} \begin{bmatrix} 2^{j^2} & \beta_{1,n_{j-1}} \\ 0 & 2^{j(j-1)} \end{bmatrix}$$

$$= \begin{bmatrix} 2^{2j+1} & 2^{2j+1} - 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2^{2j} - 1 \\ 0 & 2^{2j} \end{bmatrix} \begin{bmatrix} 2^{j^2} & \beta_{1,n_{j-1}} \\ 0 & 2^{j(j-1)} \end{bmatrix}$$

$$= \begin{bmatrix} 2^{2j+1} & 2^{4j+2} - 2^{2j+1} - 2^{2j} \\ 0 & 2^{2j} \end{bmatrix} \begin{bmatrix} 2^{j^2} & \beta_{1,n_{j-1}} \\ 0 & 2^{j(j-1)} \end{bmatrix}.$$

Therefore,

$$\beta_{1,n_j} = 2^{2j+1}\beta_{1,n_{j-1}} + 2^{j(j-1)}(2^{4j+2} - 2^{2j+1} - 2^{2j})$$

$$= 2^{2j+1}\beta_{1,n_{j-1}} + 2^{j(j-1)}2^{2j}(2^{2j+2} - 2 - 1)$$

$$= 2^{2j+1}[2^{j^2} + 2^{j(j-1)}(2^{2j+1} - 2^{j+2} - 3(2^{j-1} - 1))] + 2^{j(j+1)}(2^{2j+2} - 3)$$

$$= 2^{(j+1)^2} + 2^{2j+1}2^{j(j-1)}(2^{2j+1} - 2^{j+2} - 3(2^{j-1} - 1)) + 2^{j(j+1)}(2^{2j+2} - 3)$$

$$= 2^{(j+1)^2} + 2^{j(j+1)}(2^{2j+2} - 2^{j+3} - 3(2^j - 2)) + 2^{j(j+1)}(2^{2j+2} - 3)$$

$$= 2^{(j+1)^2} + 2^{j(j+1)}(2^{2j+2} + 2^{2j+2} - 2^{j+3} - 3(2^j - 2) - 3)$$

$$= 2^{(j+1)^2} + 2^{j(j+1)}(2^{2j+3} - 2^{j+3} - 3(2^j - 1)).$$

We have now completed our induction to show that (2.9) holds for any n_i, where $i \geq 0$. A final computation shows that

$$\frac{\beta_{1,n_i}}{\prod_{k=1}^{n_i} a_k} = \frac{2^{(i+1)^2} + 2^{i(i+1)}(2^{2i+3} - 2^{i+3} - 3(2^i - 1))}{2^{(i+1)^2}}$$

$$= 1 + 2^{-i-1}(2^{2i+3} - 2^{i+3} - 3(2^i - 1))$$

$$= 1 + 2^{i+2} - 2^2 - 2^{-i-1} \cdot 3(2^i - 1) \to \infty$$

as $i \to \infty$. By Proposition 2.17, it follows that $\mathcal{D}_W$ contains no rank one operators. Hence, $\mathcal{D}_W(1,1)$ is one dimensional by Proposition 2.16 and it follows that $\dim(\mathcal{D}_W(i,j)) = 1$, for all $i, j \in \mathbb{N}$, since W is bounded below. □

2.4.2 Weighted shifts which are not bounded below

In this section, we assume that $W \sim (A_n)$ is an injective weighted shift of multiplicity 2 which is not bounded below. We have shown that the dimension of $\mathcal{D}_W(i,j)$ is constant along diagonals so it suffices to investigate $\mathcal{D}_W(1,j)$ and $\mathcal{D}_W(i,1)$ for $i, j \in \mathbb{N}$. It might seem that if i increases or j decreases, then $\dim(\mathcal{D}_W(i,1))$ and $\dim(\mathcal{D}_W(1,j))$ may increase in an arbitrary way. The following two theorems show that this is not quite the case.

Theorem 2.18 *Let $W \sim (A_n)$ be an injective weighted shift of multiplicity two which is not bounded below. Then* $\dim(\mathcal{D}_W(1,2)) \leq 2$.

Proof: Assume, to the contrary, that $\dim(\mathcal{D}_W(1,2)) > 2$. Let $\{B_1, B_2, B_3\}$ be a linearly independent set in $\mathcal{D}_W(1,2)$. If any of these are invertible, then Corollary 2.13 implies that W is bounded below. Therefore, we can assume that $B_k = x_k \otimes y_k$, $k = 1, 2, 3$, is a rank one operator.

Notice that the nonzero vectors, x_1, x_2, x_3, cannot be multiples of each other. Otherwise, we would have that $\operatorname{span}\{B_1, B_2, B_3\} \subset \{x_1 \otimes y : y \in \mathcal{H}_2\}$ is at most two dimensional, contradicting the assumption that $\{B_1, B_2, B_3\}$ forms a linearly independent set. Similarly, we can show that y_1, y_2, y_3 are not all multiples of each other. It is now an exercise in logic to see that there exists k, j such that x_k is not a multiple of x_j and y_k is not a multiple of y_j.

Next, we will show that, for such k, j, $x_k \otimes y_k + x_j \otimes y_j$ is an invertible operator in $\mathcal{D}_W(1,2)$. Suppose, to the contrary, that there exists $z \in \mathcal{H}_2$ such that $(x_k \otimes y_k + x_j \otimes y_j)z = 0$. Then $\langle z, y_k\rangle x_k = -\langle z, y_j\rangle x_j$. Hence $\langle z, y_k\rangle = 0 = \langle z, y_j\rangle$ because x_k is not a multiple of x_j. By linearity of the inner product, it follows that $\langle z, y\rangle = 0$, for all $y \in \mathcal{H}_2$. Therefore, $z = 0$ and $x_k \otimes y_k + x_j \otimes y_j$ is invertible. Now, Corollary 2.13 implies that W is bounded below and this contradiction completes the proof. $\square$

Now that we have determined that $\dim(\mathcal{D}_W(1,2)) \leq 2$, we start with the possibility that $\dim(\mathcal{D}_W(1,2)) = 2$. The following theorem shows that, in this case, $\dim(\mathcal{D}_W(i,j))$ cannot vary too much.

Theorem 2.19 *Let $W \sim (A_n)$ be an injective weighted shift of multiplicity two which is not bounded below and suppose that* $\dim(\mathcal{D}_W(1,2)) = 2$. *Then* $\dim(\mathcal{D}_W(1,k)) = 2$, *for all* $k > 1$ *and* $\dim(\mathcal{D}_W(k,1)) = 3$, *for all* $k \in \mathbb{N}$.

PROOF: Assume, to the contrary, that $\dim(\mathcal{D}_W(k,1)) = 4$, for some $k \in \mathbb{N}$. We will show that $\dim(\mathcal{D}_W(k-1,1)) = 4$. Let $\mathcal{B} = \{B_1, B_2\}$ be a basis for $\mathcal{D}_W(1,2)$. By Corollary 2.13, we know that B_1 and B_2 are not invertible so they must be rank one operators. Let $B_j = x_j \otimes y_j$, for $1 \le j \le 2$. We now consider two cases.

First, suppose that x_1 and x_2 are scalar multiples of each other. Then, without loss of generality, we can assume that $x_1 = x_2$. Furthermore, $\operatorname{span}\{y_1, y_2\} = \mathcal{H}_2$. Let $z_1 \in \mathcal{H}_k$ and $z_2 \in \mathcal{H}_2$. Then there exists $c_1, c_2 \in \mathbb{C}$ such that $z_2 = c_1y_1 + c_2y_2$ and there exists $A \in \mathcal{D}_W(k,1) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_k)$ such that $Ax_1 = z_1$. Therefore, $z_1 \otimes z_2 = A(x_1 \otimes (c_1y_1 + c_2y_2)) \in \mathcal{D}_W(k,2)$. Hence $\mathcal{D}_W(k,2)$ contains all the rank one operators and is four dimensional. By Corollary 2.11, $\dim(\mathcal{D}_W(k-1,1)) = 4$.

Now, suppose that x_1 and x_2 are not scalar multiples of each other. Applying Theorem 2.10 $k-2$ times, there is an isomorphism $\varphi : \mathcal{D}_W(1,2) \to \mathcal{D}_W(k-1,k)$ defined by $\varphi(T) = A_{k-2}A_{k-3}\cdots A_1 T A_2^{-1}A_3^{-1}\cdots A_{k-1}^{-1}$. Let $w_j = A_{k-1}^{-1*}A_{k-2}^{-1*}\cdots A_2^{-1*}y_j$ and $u_j = A_{k-2}A_{k-3}\cdots A_1x_j$, for $1 \le j \le 2$. Then $\{u_1 \otimes w_1, u_2 \otimes w_2\}$ forms a basis for $\mathcal{D}_W(k-1,k)$. Furthermore, u_1 and u_2 are not scalar multiples of each other. Let $z_1 \in \mathcal{H}_{k-1}$ and $z_2 \in \mathcal{H}_1$. Then there exists $B_1, B_2 \in \mathcal{D}_W(1,k) = \mathcal{L}(\mathcal{H}_k, \mathcal{H}_1)$ such that $B_1^*w_1 = z_2$ and $B_2^*w_2 = z_2$. Also, there exists $c_1, c_2 \in \mathbb{C}$ such that $z_1 = c_1u_1 + c_2u_2$. Therefore,

$$z_1 \otimes z_2 = c_1(u_1 \otimes z_2) + c_2(u_2 \otimes z_2)$$

$$= c_1(u_1 \otimes w_1)B_1 + c_2(u_2 \otimes w_2)B_2 \in \mathcal{D}_W(k-1,1).$$

In both cases, we have now shown that the assumption of $\dim(\mathcal{D}_W(k,1)) = 4$ implies that $\dim(\mathcal{D}_W(k-1,1)) = 4$, whence $\dim(\mathcal{D}_W(1,1)) = 4 = \dim(\mathcal{D}_W(2,2))$. Let $x \otimes y \in \mathcal{D}_W(1,2)$. If $z_1 \otimes z_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, then there exists $B \in \mathcal{D}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$, and $C \in \mathcal{D}_W(2,2) = \mathcal{L}(\mathcal{H}_2)$ such that $z_1 \otimes z_2 = B(x \otimes y)C \in \mathcal{D}_W(1,2)$. Hence, $\mathcal{D}_W(1,2)$ contains all rank one operators and is four dimensional. This contradiction shows that $\dim(\mathcal{D}_W(k,1)) \leq 3$, for all $k \in \mathbb{N}$. Since W is not bounded below, Corollaries 2.11 and 2.13 imply that $\dim(\mathcal{D}_W(k,1)) \geq \dim(\mathcal{D}_W(1,1)) > \dim(\mathcal{D}_W(1,2)) = 2$. Thus, $\dim(\mathcal{D}_W(k,1)) = 3$, for all $k \in \mathbb{N}$.

We now show by induction that $\dim(\mathcal{D}_W(1,n)) = 2$, for all $n > 1$. It is true for $n = 2$ by assumption, so we assume that it is true for $n = k$. Let $\dim(\mathcal{D}_W(1,k)) = 2$. By Corollary 2.13, operators in $\mathcal{D}_W(1,k)$ are not invertible so we can find a basis, $\{u_k \otimes v_k, x_k \otimes y_k\}$, for $\mathcal{D}_W(1,k)$. If $u_k \notin \operatorname{span}\{x_k\}$ and $v_k \notin \operatorname{span}\{y_k\}$, then the same argument as in the proof of Theorem 2.18 shows that $u_k \otimes v_k + x_k \otimes y_k$ is invertible. This leaves two cases, $\mathcal{D}_W(1,k) = \{x \otimes y_k : x \in \mathcal{H}_1\}$, where $y_k \in \mathcal{H}_k$ is a fixed vector, or $\mathcal{D}_W(1,k) = \{x_1 \otimes y : y \in \mathcal{H}_k\}$, where $x_1 \in \mathcal{H}_1$ is fixed. We will work under the assumption that $\mathcal{D}_W(1,k) = \{x \otimes y_k : x \in \mathcal{H}_1\}$ and leave the second case to the reader. Let $z_k = A_k^{*-1} y_k$. By Theorem 2.10, $A_1 x \otimes z_k \in \mathcal{D}_W(2, k+1)$. Since A_1 is invertible and $x \in \mathcal{H}_1$ is arbitrary, $u \otimes z_k \in \mathcal{D}_W(2, k+1)$, for all $u \in \mathcal{H}_2$. In particular, if $y_2 = A_2^* A_3^* \cdots A_{k-1}^* y_k$, then $y_2 \in \mathcal{H}_2$ and $y_2 \otimes z_k \in \mathcal{D}_W(2, k+1)$. Furthermore, $x \otimes y_2 \in \mathcal{D}_W(1,2)$, for all $x \in \mathcal{H}_1$, so $(x \otimes y_2)(y_2 \otimes z_k) = \|y_2\|^2 x \otimes z_k \in \mathcal{D}_W(1, k+1)$, for all $x \in \mathcal{H}_1$. Hence $\dim(\mathcal{D}_W(1, k+1)) \geq 2$ and $2 = \dim(\mathcal{D}_W(1,2)) \geq \dim(\mathcal{D}_W(1, k+1))$ implies that $\dim(\mathcal{D}_W(1, k+1)) = 2$. By induction on k we have proved that $\mathcal{D}_W(1,k)$ is two dimensional, for all $k > 1$. $\square$

Example

Define $W \sim (A_n)$ by

$$A_n = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{bmatrix}.$$

We will prove that W satisfies the assumptions of Theorem 2.19 and thus, $\mathcal{D}_W(i,j)$ is two dimensional for $i < j$ and $\dim(\mathcal{D}_W(i,j)) = 3$ for $i \geq j$.

Proof: First, W is an injective weighted shift which is not bounded below. It remains to show that $\dim(\mathcal{D}_W(1,2)) = 2$. If $T \in \mathcal{D}_W(1,2)$, then by Corollary 2.13, T is not invertible so $\operatorname{rank}(T) < 2$. Thus, $T = x \otimes y$, for some $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. An argument, analogous to the one in the proof of Proposition 2.17, shows that $x \otimes y \in \mathcal{D}_W(1,2)$ if and only if there exists $M > 0$ such that

$$\|A_n A_{n-1} \cdots A_1 x\|^2 \|A_{n+1}^{*-1} A_n^{*-1} \cdots A_2^{*-1} y\|^2 \leq M,$$

for all $n \in \mathbb{N}$. Letting $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$, then this is equivalent to

$$\left(\left| \frac{x_1}{n!} \right|^2 + |x_2|^2 \right) (|(n+1)! y_1|^2 + |y_2|^2) \leq M,$$

for all $n \in \mathbb{N}$. It is easy to see this is true if and only if $y_1 = 0$. Thus, $\mathcal{D}_W(1,2) = \{x \otimes f_2 : x \in \mathcal{H}_1\}$ is two dimensional. □

Now, we look at examples where $\dim(\mathcal{D}_W(1,2)) = 1$. Let $p, q, r \in \mathbb{N}$ be fixed such that $p > 1$, $q > 1$, and $r \geq p + q - 1$. The main result of this section is the following construction.

Theorem 2.20 *There exists a weighted shift of multiplicity two, W, such that*

1. $\mathcal{D}_W(1+n, k+n)$ *is zero dimensional for* $k > p$, *and* $n \geq 0$,

2. $\mathcal{D}_W(1+n, k+n)$ *is one dimensional if* $1 < k \leq p$ *and* $n \geq 0$,

3. $\mathcal{D}_W(k+n, 1+n)$ *is two dimensional if* $1 \leq k < q$ *and* $n \geq 0$,

4. $\mathcal{D}_W(k+n, 1+n)$ *is three dimensional if* $q \leq k < r$ *and* $n \geq 0$, *and*

5. $\mathcal{D}_W(k+n, 1+n)$ *is four dimensional if* $r \leq k$ *and* $n \geq 0$.

The following matrix gives a clearer picture of what is happening with $\dim(\mathcal{D}_W(i,j))$.

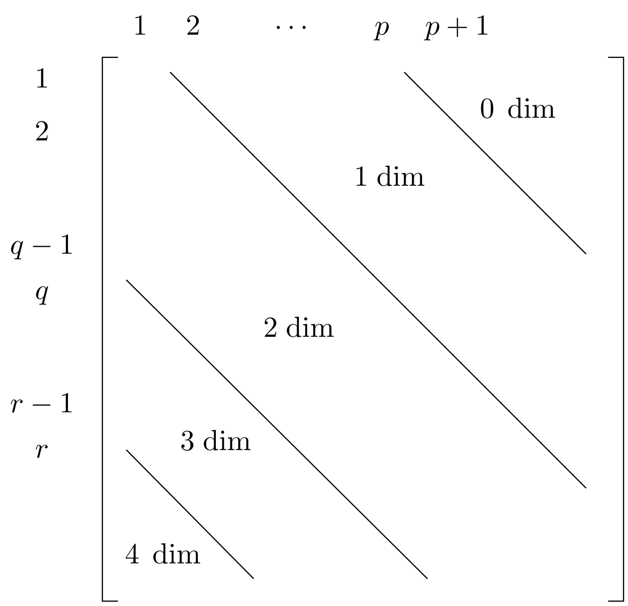

Figure 2.2

First, we will explain why the assumption $r \geq p+q-1$ is necessary. If $x \otimes y \in \mathcal{D}_W(1,p)$ and $A \in \mathcal{D}_W(r,1) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_r)$, then $Ax \otimes y \in \mathcal{D}_W(r,p)$. Since A can be chosen arbitrarily, this shows $\dim(\mathcal{D}_W(r,p)) \geq 2$ whence $r \geq p$. Also, we are assuming that W is injective, so $A_{r-1}A_{r-2}\cdots A_p : \mathcal{H}_p \to \mathcal{H}_r$ is an invertible operator in $\mathcal{D}_W(r,p)$. Considering this invertible operator along with the rank one operators of the form $Ax \otimes y$, we conclude that $\dim(\mathcal{D}_W(r,p)) \geq 3$ and, by Corollary 2.11, $\dim(\mathcal{D}_W(r-p+1,1)) \geq 3$. It follows that $r-p+1 \geq q$ so $r \geq p+q-1$.

Before defining a weighted shift whose Deddens algebra has the above properties, we will prove several results of a technical nature.

Lemma 2.21 *Let $W \sim (A_n)$ be a weighted shift with weights A_n given in (2.4) and suppose that the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ are not bounded below. Let $k \geq 2$ and let the sequence*

$$\left\{\frac{c_1c_2\cdots c_n}{a_ka_{k+1}\cdots a_{k+n-1}}\right\}_{n\in\mathbb{N}}$$

be unbounded. Then $\mathcal{D}_W(1,k) \subset \operatorname{span}(e_1 \otimes f_k)$ and the equality holds if and only if the sequence

$$\left\{\frac{a_1a_2\cdots a_n}{c_kc_{k+1}\cdots c_{k+n-1}}\right\}_{n\in\mathbb{N}} \tag{2.10}$$

is bounded.

Proof: Since the sequence $\{a_n\}_{n\in\mathbb{N}}$ is not bounded below, W is not bounded below as well. By Corollary 2.13, $\mathcal{D}_W(1,k)$ contains no invertible operators, for any $k \in \mathbb{N}$. Let $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_k$. Then $x \otimes y \in \mathcal{D}_W(1,k)$ if and only if there exists $M > 0$ such that

$$\|A_nA_{n-1}\cdots A_1x\|\|A^{*-1}_{k+n-1}A^{*-1}_{k+n-2}\cdots A^{*-1}_k y\| \leq M, \tag{2.11}$$

for all $n \in \mathbb{N}$. Let $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$. Then (2.11) is equivalent to the boundedness of the sequence

$$\left(\left|\prod_{i=1}^{n} a_i x_1 + \beta_{1n} x_2\right|^2 + \left|\prod_{i=1}^{n} c_i x_2\right|^2\right)\left(\left|\frac{y_1}{\prod\limits_{i=k}^{k+n-1} a_i}\right|^2 + \left|\frac{-\beta_{kn} y_1}{\prod\limits_{i=k}^{k+n-1} a_i c_i} + \frac{y_2}{\prod\limits_{i=k}^{k+n-1} c_i}\right|^2\right).$$

Since

$$\|A_n A_{n-1} \cdots A_1 x\|^2 \|A_{k+n-1}^{*-1} A_{k+n-2}^{*-1} \cdots A_k^{*-1} y\|^2 \geq \left|\prod_{i=1}^{n} c_i x_2\right|^2 \left|\frac{y_1}{\prod\limits_{i=k}^{k+n-1} a_i}\right|^2,$$

and $\{(c_1 c_2 \cdots c_n)/(a_k a_{k+1} \cdots a_{k+n-1})\}_{n \in \mathbb{N}}$ is unbounded, the assumption that $x \otimes y \in \mathcal{D}_W(1, k)$ implies that $x_2 = 0$ or $y_1 = 0$. Suppose first that $x_2 = 0$. Then

$$\|A_n A_{n-1} \cdots A_1 x\|^2 \|A_{k+n-1}^{*-1} A_{k+n-2}^{*-1} \cdots A_k^{*-1} y\|^2 \geq \left|\prod_{i=1}^{n} a_i x_1\right|^2 \left|\frac{y_1}{\prod\limits_{i=k}^{k+n-1} a_i}\right|^2.$$

Note that the sequence

$$\frac{\prod_{i=1}^{n} a_i}{\prod\limits_{i=k}^{k+n-1} a_i} = \frac{a_1 a_2 \cdots a_{k-1}}{a_{n+1} a_{n+2} \cdots a_{k+n-1}}$$

is unbounded because $\{a_n\}_{n \in \mathbb{N}}$ is not bounded below. It follows that $y_1 = 0$ or $x_1 = 0$. Of course, x is a nonzero vector, so the assumption that $x_2 = 0$ implies $y_1 = 0$. An analogous argument shows that $y_1 = 0$ implies $x_2 = 0$. Consequently,

$y_1 = x_2 = 0$ and

$$\mathcal{D}_W(1,k) \subset \{\alpha e_1 \otimes f_k : \alpha \in \mathbb{C}\}. \tag{2.12}$$

Finally, equality will hold in (2.12) if and only if $e_1 \otimes f_k \in \mathcal{D}_W(1,k)$. Since

$$\|A_n A_{n-1} \cdots A_1 e_1\|^2 \|A_{k+n-1}^{*-1} A_{k+n-2}^{*-1} \cdots A_k^{*-1} f_k\|^2 = \frac{a_1 a_2 \cdots a_n}{c_k c_{k+1} \cdots c_{k+n-1}},$$

then it follows that $e_1 \otimes f_k \in \mathcal{D}_W(1,k)$ if and only if $\frac{a_1 a_2 \cdots a_n}{c_k c_{k+1} \cdots c_{k+n-1}}$ is bounded. □

Lemma 2.22 *Let $W \sim (A_n)$ be a weighted shift with weights A_n given in (2.4). Let $\mathcal{D}_W(1,2) = \mathrm{span}(e_1 \otimes f_2)$. Then $\mathcal{D}_W(k,1)$ is at least two dimensional, for all $k \in \mathbb{N}$, and exactly two dimensional if and only if*

$$\left\{ \frac{\beta_{kn}}{c_1 c_2 \cdots c_n} \right\}_{n \in \mathbb{N}} \tag{2.13}$$

is an unbounded sequence.

PROOF: First, we have that $\mathcal{D}_W(1,1)$ contains the identity operator I_1 and the rank one operator $(e_1 \otimes f_2)A_1$, hence $\mathcal{D}_W(1,1)$ is at least two dimensional. By Corollary 2.11, $\dim(\mathcal{D}_W(k,1)) \geq \dim(\mathcal{D}_W(1,1)) \geq 2$.

Next, we will prove that $\dim(\mathcal{D}_W(k,1)) = 2$ if and only if the sequence (2.13) is unbounded. First, we notice that

$$\|A_{k+n-1} A_{k+n-2} \cdots A_k f_k\|^2 \|A_n^{*-1} A_{n-1}^{*-1} \cdots A_1^{*-1} f_1\|^2$$

$$= \left(|\beta_{kn}|^2 + \left| \prod_{i=k}^{k+n-1} c_i \right|^2 \right) \left(\frac{1}{c_1 c_2 \cdots c_n} \right)^2, \tag{2.14}$$

and the second product

$$\frac{c_k c_{k+1} \cdots c_{k+n-1}}{c_1 c_2 \cdots c_n} = \frac{c_{n+1} c_{n+2} \cdots c_{n+k-1}}{c_1 c_2 \cdots c_{k-1}} \leq \frac{\|W\|^{k-1}}{c_1 c_2 \cdots c_{k-1}}$$

is bounded with respect to n. Thus, sequence (2.13) is bounded if and only if $f_k \otimes f_1 \in \mathcal{D}_W(k,1)$. Therefore, the boundedness of (2.13) implies that $\dim(\mathcal{D}_W(k,1)) \geq 3$.

Suppose now that the sequence (2.13) is unbounded. It is not hard to see that

$$A_{k-1} A_{k-2} \cdots A_1 (e_1 \otimes f_2) A_1 = a_1 a_2 \cdots a_{k-1} e_k \otimes c_1 f_1.$$

Since all a_i and c_i are nonzero, and $e_1 \otimes f_2 \in \mathcal{D}_W(1,2)$, it follows that $e_k \otimes f_1 \in \mathcal{D}_W(k,1)$. Also, $A_{k-1} A_{k-2} \cdots A_1 \in \mathcal{D}_W(k,1)$ which means that $\mathcal{D}_W(k,1)$ contains operators of the form

$$T = \begin{bmatrix} a & b \\ 0 & \gamma_k a \end{bmatrix} \tag{2.15}$$

where $a, b \in \mathbb{C}$, $\gamma_1 = 1$, and

$$\gamma_k = \frac{c_1 c_2 \cdots c_{k-1}}{a_1 a_2 \cdots a_{k-1}}$$

for $k \geq 2$. On the other hand, $f_k \otimes f_1 \notin \mathcal{D}_W(k,1)$. Further, as we noticed above, $e_1 \otimes f_1 \in \mathcal{D}_W(1,1)$. Since $f_k \otimes f_1 = (f_k \otimes e_1)(e_1 \otimes f_1)$, it follows that $f_k \otimes e_1 \notin \mathcal{D}_W(k,1)$. As a consequence, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{D}_W(k,1),$$

then the same is true of

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ 0 & \gamma_k a \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = cf_k \otimes f_1$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ 0 & \gamma_k a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d - a\gamma_k \end{bmatrix} = cf_k \otimes e_1 + (d - a\gamma_k)f_k \otimes f_1.$$

We see that $c = 0$, $d = a\gamma_k$, and

$$\mathcal{D}_W(k,1) = \left\{\begin{bmatrix} a & b \\ 0 & a\gamma_k \end{bmatrix} : a, b \in \mathbb{C}\right\}$$

is two dimensional. □

Remark: This proof shows that

$$\left\{\begin{bmatrix} a & b \\ 0 & a\gamma_k \end{bmatrix} : a, b \in \mathbb{C}\right\} \subset \mathcal{D}_W(k,1),$$

for all $k \in \mathbb{N}$, whenever $\mathcal{D}_W(1,2)$ contains the operator $e_1 \otimes f_2$.

Lemma 2.23 *Let $W \sim (A_n)$ be a weighted shift with weights A_n given in (2.4). Let $\mathcal{D}_W(1,2) = \text{span}(e_1 \otimes f_2)$. Then $\mathcal{D}_W(k,1)$ is 3 dimensional if and only if the sequence*

$$\left\{\frac{\beta_{kn}}{c_1 c_2 \cdots c_n}\right\}_{n \in \mathbb{N}}$$

is bounded and $f_k \otimes e_1 \notin \mathcal{D}_W(k,1)$.

PROOF: First, we assume that $\{\beta_{kn}/(c_1c_2\cdots c_n)\}_{n\in\mathbb{N}}$ is bounded and $f_k \otimes e_1 \notin \mathcal{D}_W(k,1)$. By Lemma 2.22, $\mathcal{D}_W(k,1)$ is at least three dimensional. Hence, $\mathcal{D}_W(k,1)$ is three dimensional because $f_k \otimes e_1 \notin \mathcal{D}_W(k,1)$.

Now we assume that $\dim(\mathcal{D}_W(k,1)) = 3$. By Lemma 2.22, $\{\beta_{kn}/(c_1c_2\cdots c_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. Further, the proof of Lemma 2.22 shows that $\mathcal{D}_W(k,1)$ contains $f_k \otimes f_1$ and all operators of the form (2.15), whence all the upper triangular matrices. Since we assumed that $\mathcal{D}_W(k,1)$ is three dimensional, we have that $f_k \otimes e_1 \notin \mathcal{D}_W(k,1)$. $\square$

Lemma 2.24 *Let* $W \sim (A_n)$ *be a weighted shift with weights* A_n *given in (2.4). Let* $\mathcal{D}_W(1,2) = \mathrm{span}(e_1 \otimes f_2)$. *Then* $\mathcal{D}_W(k,1)$ *is four dimensional if and only if* $f_k \otimes e_1 \in \mathcal{D}_W(k,1)$.

PROOF: If $\mathcal{D}_W(k,1)$ is four dimensional, then it clearly contains $f_k \otimes e_1$. In the other direction, it was established in the remark after Lemma 2.22 that $\mathcal{D}_W(k,1)$ contains operators of the form (2.15). Assuming that $f_k \otimes e_1 \in \mathcal{D}_W(k,1)$ gives us a third dimension and $f_k \otimes f_1 = (f_k \otimes e_1)(e_1 \otimes f_2)A_1 \in \mathcal{D}_W(k,1)$ is the fourth. $\square$

These four lemmas will allow us to show that there is a weighted shift W with $\mathcal{D}_W$ as given in Theorem 2.20.

Example

Let p, q, and r be fixed integers such that $p \geq 2, q \geq 2$, and $r \geq p+q-1$. Let $m_k = k(p+r-2)$ and $n_k = p-1+m_k$. We define a weighted shift $W \sim (A_n)$ such that the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ are defined by

$$a_{m_k} = a_{n_k+q-1} = \frac{1}{k}, \; c_{n_k} = c_{n_k+r-1} = \frac{1}{k}, \text{ for all } k \in \mathbb{N}, \text{ and } a_n = c_n = 1 \text{ otherwise},$$

and the sequence $\{b_n\}_{n\in\mathbb{N}}$ is defined by

$$b_{n_1+q-1} = 1, \quad b_{n_k+r-1} = \frac{1}{k(k+1)}, \; k \geq 1, \text{ and } \quad b_n = 0 \text{ otherwise.}$$

Notice that, for any $k \in \mathbb{N}$, $m_k < n_k + q - 1 < m_{k+1}$ and $n_k < n_{k+r-1} < n_{k+1}$. Therefore,

$$a_1 a_2 \cdots a_{m_k+t} = \begin{cases} \frac{1}{(k-1)!k!} & \text{if } 0 \leq t < p+q-2, \\ \frac{1}{k!^2} & \text{if } p+q-2 \leq t < r+p-2, \end{cases} \tag{2.16}$$

and

$$c_1 c_2 \cdots c_{n_k+t} = \begin{cases} \frac{1}{(k-1)!k!} & \text{if } 0 \leq t < r-1, \\ \frac{1}{k!^2} & \text{if } r-1 \leq t < r+p-2. \end{cases} \tag{2.17}$$

The first thing we will show is that $\mathcal{D}_W(1,p)$ is one dimensional while $\mathcal{D}_W(1,p+1)$ is zero. We start by noting that neither of the sequences, $\{a_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$, are bounded below. Also, the sequence

$$\left\{ \frac{c_1 c_2 \cdots c_n}{a_1 a_2 \cdots a_n} \right\}_{n\in\mathbb{N}}$$

is unbounded (take $n = m_k$ and notice that $m_k < n_k$). Hence

$$\frac{c_1 c_2 \cdots c_n}{a_k a_{k+1} \cdots a_{k+n-1}} = \frac{c_1 c_2 \cdots c_n}{a_1 a_2 \cdots a_n} \cdot \frac{a_1 a_2 \cdots a_{k-1}}{a_{n+1} \cdots a_{k+n-1}}$$

is a product of unbounded sequences. This shows that

$$\left\{\frac{c_1c_2\cdots c_n}{a_ka_{k+1}\cdots a_{k+n-1}}\right\}_{n\in\mathbb{N}}$$

is unbounded for any choice of k, so all hypotheses of Lemma 2.21 are satisfied. Let $0 \le t < p+r-2$ and $n = m_k + t$. Then $p+n-1 = p+m_k+t-1 = n_k+t$. Also, $c_1 = c_2 = \cdots = c_{p-1} = 1$, so

$$\frac{a_1a_2\cdots a_n}{c_pc_{p+1}\cdots c_{p+n-1}} = \frac{a_1a_2\cdots a_{m_k+t}}{c_1c_2\cdots c_{n_k+t}} = \begin{cases} 1 & \text{if } 0 \le t < p+q-2, \\ \frac{1}{k} & \text{if } p+q-2 \le t < r-1, \\ 1 & \text{if } r-1 \le t < p+r-2. \end{cases}$$

Thus, the sequence, $\{(a_1a_2\cdots a_n)/(c_pc_{p+1}\cdots c_n)\}_{n\in\mathbb{N}}$, is bounded and Lemma 2.21 implies that $\mathcal{D}_W(1,p)$ is one dimensional. On the other hand,

$$\frac{a_1a_2\cdots a_{m_k-1}}{c_{p+1}c_{p+2}\cdots c_{p+m_k-1}} = \frac{a_1a_2\cdots a_{m_k-1}}{c_{p+1}c_{p+2}\cdots c_{n_k}} = \frac{1/((k-1)!)^2}{1/(k!(k-1)!)} = k$$

so the sequence $\{(a_1a_2\cdots a_n)/(c_{p+1}c_{p+2}\cdots c_{p+n})\}_{n\in\mathbb{N}}$ is unbounded and $\mathcal{D}_W(1,p+1)$ is zero.

By Theorems 2.18 and 2.19 we know that $\dim(\mathcal{D}_W(1,2)) < 2$. Since $\mathcal{D}_W(1,2)$ contains $(e_1\otimes f_p)A_{p-1}A_{p-2}\cdots A_2 = e_1\otimes f_2$, then $\mathcal{D}_W(1,2) = \text{span}\{e_1\otimes f_2\}$. Thus, all hypotheses of Lemmas 2.22−2.24 are satisfied.

Next, we turn our attention to β_{qn}. The first step is to show that the sequence, $\{\beta_{q-1,n}/(c_1c_2\cdots c_n)\}_{n\in\mathbb{N}}$, is unbounded while $\{\beta_{qn}/(c_1c_2\cdots c_n)\}_{n\in\mathbb{N}}$ is bounded. It will follow from Lemmas 2.22 and 2.23 that $\mathcal{D}_W(q-1,1))$ is two dimensional while

$\dim(\mathcal{D}_W(q,1)) \geq 3$. We use the relation

$$\begin{bmatrix} a_k a_{k+1} \cdots a_{k+n-1} & \beta_{k,n} \\ 0 & c_k c_{k+1} \cdots c_{k+n-1} \end{bmatrix} = A_{k+n-1} A_{k+n-2} \cdots A_k$$

$$= \begin{bmatrix} a_{k+n-1} & b_{k+n-1} \\ 0 & c_{k+n-1} \end{bmatrix} \begin{bmatrix} a_k a_{k+1} \cdots a_{k+n-2} & \beta_{k,n-1} \\ 0 & c_k c_{k+1} \cdots c_{k+n-2} \end{bmatrix}$$

to obtain

$$\beta_{k,n} = \beta_{k,n-1} a_{k+n-1} + c_k c_{k+1} \cdots c_{k+n-2} b_{k+n-1}. \tag{2.18}$$

Replacing k with q, and n with $n_k + t$, we find that

$$\beta_{q,n_k+t} = \beta_{q,n_k+t-1} a_{q+n_k+t-1} + c_q \cdots c_{q+n_k+t-2} b_{q+n_k+t-1}. \tag{2.19}$$

For $1 \leq t < r-q$, we have that $b_{q+n_k+t-1} = 0$ and $a_{q+n_k+t-1} = 1$, whence $\beta_{q,n_k+t} = \beta_{q,n_k+t-1}$. Similarly, for $r-q < t < r+p-2$, $b_{q+n_k+t-1} = 0$ and $a_{q+n_k+t-1} = 1$ so $\beta_{q,n_k+t} = \beta_{q,n_k+t-1}$. Therefore,

$$\beta_{q,n_k} = \beta_{q,n_k+1} = \cdots = \beta_{q,n_k+r-q-1} \quad \text{and} \quad \beta_{q,n_k+r-q} = \beta_{q,n_k+r-q+1} = \cdots = \beta_{q,n_{k+1}-1}. \tag{2.20}$$

Now, we will use induction on k to prove that

$$\beta_{q,n_k+t} = \begin{cases} \frac{1}{k!(k-1)!} & \text{if } 0 \leq t < r-q, \\ \frac{1}{(k!)^2} & \text{if } r-q \leq t < r+p-2. \end{cases} \tag{2.21}$$

The first time b_i is nonzero is when $i = n_1 + q - 1$, so all weights A_i are diagonal, for $1 \leq i < n_1 + q - 1$, and $\beta_{q,n_1-1} = 0$. Consequently, (2.18) implies that $\beta_{q,n_1} =$

$c_q c_{q+1} \cdots c_{q+n_1-2} b_{q+n_1-1}$. Since $c_n = 1$, for all $n < n_2$, and $b_{q+n_1-1} = 1$,we have that $\beta_{q,n_1} = 1$ whence (2.20) implies that $\beta_{q,n_1+r-q-1} = 1$. Also, using (2.19),

$$\beta_{q,n_1+r-q} = \beta_{q,n_1+r-q-1} a_{n_1+r-1} + c_q c_{q+1} \cdots c_{n+1+r-2} b_{n_1+r-1} = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1.$$

Next, assume that $\beta_{q,n_k} = \frac{1}{k!(k-1)!}$. We will complete the proof of (2.21) by showing that $\beta_{q,n_k+r-q} = \frac{1}{k!^2}$ and $\beta_{q,n_{k+1}} = \frac{1}{k!(k+1)!}$. Note that

$$a_{n_k+r-1} = a_{m_{k+1}} = \frac{1}{k+1}, \ c_q c_{q+1} \cdots c_{n_k+r-2} = \frac{1}{k!(k-1)!}, \text{ and } b_{n_k+r-1} = \frac{1}{k(k+1)}.$$

Thus, using (2.19) and (2.20),

$$\beta_{q,n_k+r-q} = \beta_{q,n_k+r-q-1} a_{n_k+r-1} + c_q c_{q+1} \cdots c_{n_k+r-2} b_{n_k+r-1} \tag{2.22}$$

$$= \frac{1}{k!(k-1)!} \cdot \frac{1}{k+1} + \frac{1}{k!(k-1)!} \cdot \frac{1}{k(k+1)} = \frac{1}{(k!)^2}.$$

Finally, combining (2.19), (2.20), (2.22), and the definition of b_n, we get that

$$\beta_{q,n_{k+1}} = \beta_{q,n_{k+1}-1} a_{n_{k+1}+q-1} + c_q c_{q+1} \cdots c_{q+n_{k+1}-2} b_{q+n_{k+1}-1}$$

$$= \frac{1}{(k!)^2} \cdot \frac{1}{k+1} + c_q c_{q+1} \cdots c_{q+n_{k+1}-2} \cdot 0 = \frac{1}{(k+1)!k!},$$

and we have established (2.21).

Combining (2.17) and (2.21), the sequence $\{\beta_{qn}/(c_1 c_2 \cdots c_n)\}_{n \in \mathbb{N}}$ is bounded and $\mathcal{D}_W(q,1)$ is at most three dimensional by Lemma 2.23. To show that $\mathcal{D}_W(q-1,1)$

is two dimensional, we use

$$\begin{bmatrix} a_{q-1}a_q\cdots a_{q+n-2} & \beta_{q-1,n} \\ 0 & c_{q-1}c_q\cdots c_{q+n-2} \end{bmatrix} = A_{q+n-2}A_{q+n-3}\cdots A_qA_{q-1}$$

$$= \begin{bmatrix} a_q\cdots a_{q+n-2} & \beta_{q,n-1} \\ 0 & c_q\cdots c_{q+n-2} \end{bmatrix} \cdot \begin{bmatrix} a_{q-1} & b_{q-1} \\ 0 & c_{q-1} \end{bmatrix},$$

which allows us to deduce that

$$\beta_{q-1,n} = c_{q-1}\beta_{q,n-1} + b_{q-1}a_qa_{q+1}\cdots a_{q+n-2}.$$

Since $b_{q-1} = 0$ and $c_{q-1} = 1$, then $\beta_{q-1,n_k} = \beta_{q,n_k-1} = 1/(k-1)!^2$. It follows that

$$\frac{\beta_{q-1,n_k}}{c_1c_2\cdots c_{n_k}} = \frac{1/(k-1)!^2}{1/(k!(k-1)!)} = k$$

and $\{\beta_{q-1,n}/(c_1c_2\cdots c_n)\}_{n\in\mathbb{N}}$ is unbounded. By Lemma 2.22, $\mathcal{D}_W(q-1,1)$ is two dimensional.

Next we prove that $f_r \otimes e_1 \in \mathcal{D}_W(r,1)$ but $f_{r-1} \otimes e_1 \notin \mathcal{D}_W(r-1,1)$. This will imply that $\dim(\mathcal{D}_W(i,1)) = 4$, for $i \geq r$, while $\dim(\mathcal{D}_W(i,1)) \leq 3$, for $i < r$, and the proof will be complete. The sequence

$$\|A_{r+n-1}A_{r+n-2}\cdots A_rf_r\|^2\|A_n^{*-1}A_{n-1}^{*-1}\cdots A_1^{*-1}e_1\|^2$$

$$= \left(|\beta_{rn}|^2 + |c_rc_{r+1}\cdots c_{r+n-1}|^2\right)\left(\left|\frac{1}{a_1a_2\cdots a_n}\right|^2 + \left|\frac{-\beta_{1n}}{a_1a_2\cdots a_nc_1c_2\cdots c_n}\right|^2\right) \quad (2.23)$$

is bounded in n if and only if $f_r \otimes e_1 \in \mathcal{D}_W(r,1)$. Equations (2.16) and (2.17) imply that the sequence $\{(c_r c_{r+1} \cdots c_{r+n-1})/(a_1 a_2 \cdots a_n)\}_{n \in \mathbb{N}}$ is bounded. For β_{rn}, we use the product

$$(A_{r+n-1} A_{r+n-2} \cdots A_r)(A_{r-1} A_{r-2} \cdots A_q)$$

to obtain the relation

$$\beta_{q,n+r-q} = a_r a_{r+1} \cdots a_{r+n-1} \beta_{q,r-q} + \beta_{rn} c_q c_{q+1} \cdots c_{r-1}.$$

Since $r - q < n_1$, then $\beta_{q,r-q} = 0$ and $c_q c_{q+1} \cdots c_{r-1} = 1$, implying that

$$\beta_{r,n_k+t} = \beta_{q,n_k+r-q+t} = \begin{cases} \frac{1}{k!^2} & \text{if } 0 \leq t < p+q-2, \\ \frac{1}{(k+1)!k!} & \text{if } p+q-2 \leq t < r+p-2. \end{cases} \tag{2.24}$$

A similar computation based on the decomposition

$$A_{n_k+t} A_{n_k+t-1} \cdots A_1 = (A_{n_k+t} A_{n_k+t-1} \cdots A_q)(A_{q-1} A_{q-2} \cdots A_1),$$

shows that

$$\beta_{1,n_k+t} = \beta_{q,n_k-q+t+1} = \begin{cases} \frac{1}{(k-1)!^2} & \text{if } 0 \leq t < q-1, \\ \frac{1}{k!(k-1)!} & \text{if } q-1 \leq t < r-1, \\ \frac{1}{k!^2} & \text{if } r-1 \leq t < r+p-2. \end{cases} \tag{2.25}$$

From (2.16) and (2.24), we conclude that the sequence $\{\beta_{rn}/(a_1a_2\cdots a_n)\}_{n\in\mathbb{N}}$ is bounded. Meanwhile, Equations (2.16), (2.17), and (2.25) imply that

$$\frac{\beta_{1,n_k+t}}{a_1a_2\cdots a_{n_k+t}c_1c_2\cdots c_{n_k+t}} = \begin{cases} k!^2 & \text{if } 0 \le t < r-1, \\ (k+1)!k! & \text{if } r-1 \le t < r+p-2. \end{cases} \tag{2.26}$$

Therefore, the sequence

$$\left\{\frac{c_rc_{r+1}\cdots c_{r+n-1}\beta_{1n}}{a_1a_2\cdots a_nc_1c_2\cdots c_n}\right\}_{n\in\mathbb{N}}$$

is bounded by applying Equations (2.17) and (2.26) and noting that $r > p$. Finally, Equations (2.24), (2.26), and the fact that $r \ge p+q-1$ imply that the sequence

$$\left\{\frac{\beta_{rn}\beta_{1n}}{a_1a_2\cdots a_nc_1c_2\cdots c_n}\right\}_{n\in\mathbb{N}}$$

is bounded. Therefore, the expression in (2.23) is bounded and $\mathcal{D}_W(r,1)$ is four dimensional.

The only thing left to show is that $f_{r-1}\otimes e_1 \notin \mathcal{D}_W(r-1,1)$, or equivalently, that the sequence

$$\begin{aligned} &\|A_{r+n-2}A_{r+n-3}\cdots A_{r-1}f_{r-1}\|^2\|A_n^{*-1}A_{n-1}^{*-1}\cdots A_1^{*-1}e_1\|^2 \\ &= \left(|\beta_{r-1,n}|^2 + |c_{r-1}c_r\cdots c_{r+n-2}|^2\right)\left(\left|\frac{1}{a_1a_2\cdots a_n}\right|^2 + \left|\frac{-\beta_{1n}}{a_1a_2\cdots a_nc_1c_2\cdots c_n}\right|^2\right) \end{aligned} \tag{2.27}$$

is unbounded. It will suffice to show that

$$\left\{ \frac{\beta_{r-1,n}\beta_{1n}}{a_1 a_2 \cdots a_n c_1 c_2 \cdots c_n} \right\}_{n \in \mathbb{N}} \tag{2.28}$$

is unbounded. Similar to how we found the relation between $\beta_{q-1,n}$ and $\beta_{q,n-1}$, we have

$$\beta_{r-1,n} = b_{r-1} a_r a_{r+1} \cdots a_{r+n-1} + c_{r-1}\beta_{r,n-1}.$$

Since $b_{r-1} = 0$ and $c_{r-1} = 1$, this simplifies to $\beta_{r-1,n} = \beta_{r,n-1}$ Letting $n = n_k$, and using (2.24) and (2.26),

$$\frac{\beta_{r-1,n_k}\beta_{1,n_k}}{a_1 a_2 \cdots a_{n_k} c_1 c_2 \cdots c_{n_k}} = \frac{1/[k!(k-1)!] \cdot 1/[(k-1)!^2]}{1/[k!(k-1)!] \cdot 1/[k!(k-1)!]} = k,$$

so $f_{r-1} \otimes e_1 \notin \mathcal{D}_W(r-1, 1)$, and the proof is complete.

Remarks: In this example we can take $q = 1$ and $\dim(\mathcal{D}_W(i, j))$ would change from one to three at the main diagonal. Or we could take $p = 1$ but we would need to adjust the sequence $\{c_n\}_{n \in \mathbb{N}}$ as it would not be well defined ($n_k + r - 1 = n_{k+1}$). Defining $c_{n_k} = \frac{1}{k(k-1)}$ and $c_n = 1$ otherwise would fix the problem and the proof would continue as normal. If $p = q = 1$, we adjust $\{a_n\}_{n \in \mathbb{N}}$ by taking $a_{m_k} = a_{n_k+q-1} = \frac{1}{k^2}$. If we wanted $r = p = q = 1$, we could take $W \sim (A_n)$ where

$$A_n = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{bmatrix}.$$

We can further generalize this example by having $\dim(\mathcal{D}_W(1, 2)) = 0$ and letting changes in dimension happen arbitrarily below the main diagonal. Let $1 < p \le q \le r$

and $n_k = m_k - p + 1$ where the definition m_k along with the definition of A_n is the same as in the example. We could then prove $\mathcal{D}_W \subset (LT)$ and $\dim(\mathcal{D}_W(i, 1))$ changes when $i = p, q$, and r. The full details are left to the reader but the proof follows the same reasoning as in the previous example.

Chapter 3

Spectral Radius Algebras

3.1 Definitions and Preliminaries

Recall from the first chapter that for an operator $A \in \mathcal{L}(\mathcal{H})$, we denote its spectral radius by $r(A)$ and, for each $m \in \mathbb{N}$, we define $d_m = \frac{m}{1+mr(A)}$ and

$$R_m = \left(\sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n \right)^{\frac{1}{2}}.$$

Before going any further, we will prove that the operators R_m are well defined and that they are invertible.

Proposition 3.1 *The series in the definition of R_m converges in the norm topology. Also, R_m is invertible.*

Proof: It is well known that $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = r(A)$. Also, note that $\|A^{*n}A^n\| = \|A^n\|^2$, for all $n \in \mathbb{N}$. Hence, there exists N such that, for all $n \geq N$, we have that

$\|A^{*n}A^n\|^{\frac{1}{2n}} = \|A^n\|^{\frac{1}{n}} \leq r(A) + \frac{1}{2}$. It now follows that

$$d_m^{2n}\|A^{*n}A^n\| = \left(\frac{m\|A^{*n}A^n\|^{\frac{1}{2n}}}{1+mr(A)}\right)^{2n} \leq \left(\frac{m(r(A)+\frac{1}{2})}{1+mr(A)}\right)^{2n}.$$

Therefore, the series $\sum_{n=N}^{\infty} d_m^{2n}\|A^{*n}A^n\|$ is bounded above by a convergent geometric series, implying that $\sum_{n=0}^{\infty} d_m^{2n}A^{*n}A^n$ converges in the norm topology of $\mathcal{L}(\mathcal{H})$.

Furthermore, $A^{*n}A^n$ is a *positive operator* for each $n \geq 0$. Recall that $T \in \mathcal{L}(\mathcal{H})$ is a positive operator if $T^* = T$ and $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$. This implies that the sum $\sum_{n=0}^{\infty} d_m^{2n}A^{*n}A^n$ is a positive operator on $\mathcal{H}$. Since we have that $d_m^0 A^{*0}A^0 = I$, the identity on $\mathcal{H}$, then $\langle\sum_{n=0}^{\infty} d_m^{2n}A^{*n}A^n x, x\rangle \geq \langle x, x\rangle$, for all $x \in \mathcal{H}$. By the spectral theorem (cf,. [9, p.289]), we can conclude that $\sum_{n=0}^{\infty} d_m^{2n}A^{*n}A^n$ is an invertible operator with a unique positive square root which is also invertible. □

We now know that R_m is well-defined and invertible, so we can concretely define the spectral radius algebra associated to $A \in \mathcal{L}(\mathcal{H})$ by

$$\mathcal{B}_A = \{T \in \mathcal{L}(\mathcal{H}) : \sup_{m\in\mathbb{N}} \|R_m T R_m^{-1}\| < \infty\}.$$

Our first task is to verify that $\mathcal{B}_A$ is indeed an algebra.

Proposition 3.2 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $\mathcal{B}_A$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ which contains the identity.*

PROOF: First, every scalar operator $\lambda I \in \mathcal{L}(\mathcal{H})$ commutes with R_m, for all $m \in \mathbb{N}$. Thus, $\|R_m(\lambda I)R_m^{-1}\| = |\lambda|$, for all $m \in \mathbb{N}$, and it follows that $\lambda I \in \mathcal{B}_A$. It now suffices to show that $\mathcal{B}_A$ is closed under addition and multiplication. Let $T_1, T_2 \in \mathcal{B}_A$. Then there exists $M_1, M_2 > 0$ such that $\|R_m T_k R_m^{-1}\| \leq M_k$, for all $m \in \mathbb{N}$ and $k = 1, 2$.

Thus,

$$\|R_m(T_1+T_2)R_m^{-1}\| = \|R_mT_1R_m^{-1} + R_mT_2R_m^{-1}\| \leq M_1 + M_2,$$

for all $m \in \mathbb{N}$, implying that $T_1 + T_2 \in \mathcal{B}_A$. Furthermore,

$$\|R_m(T_1T_2)R_m^{-1}\| = \|R_mT_1(R_m^{-1}R_m)T_2R_m^{-1}\| \leq \|R_mT_1R_m^{-1}\|\|R_mT_2R_m^{-1}\| \leq M_1M_2,$$

for all $m \in \mathbb{N}$. Therefore, $\mathcal{B}_A$ is closed under multiplication and scalar multiplication since $\lambda I \in \mathcal{B}_A$. □

Although our definition allowed for a straightforward proof that $\mathcal{B}_A$ is an algebra, calculating R_m and R_m^{-1} can be very difficult. Hence, we will use the inequality introduced in the following proposition to make our work easier.

Proposition 3.3 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{B}_A$ if and only if there exists $M > 0$ such that*

$$\sum_{n=0}^{\infty} d_m^{2n}\|A^nTx\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n}\|A^nx\|^2, \tag{3.1}$$

for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$.

Proof: First, we note that there exists $M > 0$ such that $\|R_mTR_m^{-1}\| \leq M$ if and only if $\|R_mTx\| \leq M\|R_mx\|$, for all $x \in \mathcal{H}$. Equivalently, $T \in \mathcal{B}_A$ if and only if there exists $M > 0$ such that $\|R_mTx\|^2 \leq M\|R_mx\|^2$, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$. It now suffices to show that $\|R_mx\|^2 = \sum_{n=0}^{\infty} d_m^{2n}\|A^nx\|^2$. Since R_m is positive, $R_m^* = R_m$ and

$$\|R_mx\|^2 = \langle R_mx, R_mx\rangle = \langle R_m^2x, x\rangle.$$

Hence,

$$\|R_m x\|^2 = \left\langle \sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n x, x \right\rangle = \sum_{n=0}^{\infty} d_m^{2n} \langle A^{*n} A^n x, x\rangle$$

$$= \sum_{n=0}^{\infty} d_m^{2n} \langle A^n x, A^n x\rangle = \sum_{n=0}^{\infty} d_m^{2n} \|A^n x\|^2$$

and the proof is complete. □

Proposition 3.3 allows us to state an equivalent definition of $\mathcal{B}_A$. Throughout this chapter, we will find it more useful.

Definition 3.4 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{L}(\mathcal{H})$ belongs to the spectral radius algebra associated to A if there exists $M > 0$ such that (3.1) holds, for all $m \in \mathbb{N}$ and $x \in \mathcal{H}$.*

Our next result will demonstrate the effectiveness of this new definition.

Proposition 3.5 *Let $A \in \mathcal{L}(\mathcal{H})$. Then $\{A\}' \subset \mathcal{D}_A \subset \mathcal{B}_A$.*

Proof: The inclusion $\{A\}' \subset \mathcal{D}_A$ was proved in Proposition 2.3, so we just need to verify that $\mathcal{D}_A \subset \mathcal{B}_A$. Let $T \in \mathcal{D}_A$ and let $M > 0$ such that $\|A^n T x\| \leq M\|A^n x\|$, for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. Then it quickly follows that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A^n T x\|^2 \leq \max\{M^2, \|T\|^2\} \sum_{n=0}^{\infty} d_m^{2n} \|A^n x\|^2$$

where the factor $\|T\|^2$ appears due to the $n = 0$ term. This holds, for all $m \in \mathbb{N}$ and $x \in \mathcal{H}$, whence $T \in \mathcal{B}_A$ and $\mathcal{D}_A \subset \mathcal{B}_A$. □

To study the structure of $\mathcal{B}_W$, we introduce the vector spaces

$$\mathcal{B}_W(i,j) := \{T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) : \overline{T} \in \mathcal{B}_W\},$$

where $i, j \in \mathbb{N}$, and $\overline{T}$ has the same meaning as in (2.1). These are the natural analogues of the space $\mathcal{D}_W(i,j)$. Let $T \in \mathcal{B}_W(i,j)$ and $x = (x_1, x_2, \dots) \in \mathcal{H}$. If $\overline{T}$ is as in (2.1), then

$$\|W^n\overline{T}x\| = \|A_{i+n-1}\cdots A_i T x_j\|$$

for $n \geq 1$. Since we are interested in verifying (3.1), we will often come across the sum

$$\sum_{n=0}^{\infty} d_m^{2n}\|W^n\overline{T}x\|^2 = \|Tx_j\|^2 + \sum_{n=1}^{\infty} d_m^{2n}\|A_{i+n-1}\cdots A_i T x_j\|^2.$$

Instead of writing $\|Tx_j\|^2$ separately, we will say that

$$\sum_{n=0}^{\infty} d_m^{2n}\|W^n\overline{T}x\|^2 = \sum_{n=0}^{\infty} d_m^{2n}\|A_{i+n-1}\cdots A_i T x_j\|^2$$

with the understanding that the operator $A_{i+n-1}\cdots A_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{H}_{i+n-1})$ is the identity on $\mathcal{H}_i$, when $n = 0$. It is noted here that the remainder of this chapter is original research done by the author.

3.2 General Weighted Shifts

In this section, we will derive basic results about the vector spaces $\mathcal{B}_W(i,j)$, $i, j \in \mathbb{N}$. We will then use them to present a class of operators for which $\mathcal{D}_W \neq \mathcal{B}_W$. For such operators, the existence of a n.i.s. for $\mathcal{B}_W$ is a strictly stronger result than the analogous statement for $\mathcal{D}_W$. We start with a simple characterization of the membership in $\mathcal{B}_W(i,j)$.

Proposition 3.6 *Let $W \sim (A_n)$ be a weighted shift and let $T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. Then $T \in \mathcal{B}_W(i,j)$ if and only if there exists $M > 0$ such that*

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i T x\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j x\|^2, \tag{3.2}$$

for all $x \in \mathcal{H}_j$ and $m \in \mathbb{N}$.

Proof: Let $x = (x_1, x_2, \dots)$, with $x_k = 0$, for all $k \neq j$, and let $T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. Notice that

$$\|W^n \overline{T} x\| = \|A_{i+n-1}A_{i+n-2}\cdots A_i T x_j\|$$

and

$$\|W^n x\| = \|A_{j+n-1}A_{j+n-2}\cdots A_j x_j\|.$$

From this, it is easy to see that if $T \in \mathcal{B}_W(i,j)$ then (3.2) holds.

In the other direction, let $x = (x_1, x_2, \dots) \in \mathcal{H}$ where $x_k \in \mathcal{H}_k$, for all $k \in \mathbb{N}$. Since $\{A_{k+n-1}A_{k+n-2}\cdots A_k x_k\}_{k\in\mathbb{N}}$ forms an orthogonal set, we can write $\|W^n x\|^2 = \sum_{k=1}^{\infty} \|A_{k+n-1}A_{k+n-2}\cdots A_k x_k\|^2$. If there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i T x_j\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j x_j\|^2,$$

for all $x_j \in \mathcal{H}_j$, then

$$\sum_{n=0}^{\infty} d_m^{2n} \|W^n \overline{T} x\|^2 = \sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i T x_j\|^2$$

$$\leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1} A_{j+n-2} \cdots A_j x_j\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|W^n x\|^2.$$

Therefore $T \in \mathcal{B}_W(i,j)$, which completes the proof. $\square$

Since $\mathcal{D}_W \subset \mathcal{B}_W$, the following corollary is a consequence of Proposition 2.5.

Corollary 3.7 *Let $W \sim (A_n)$ be a weighted shift and let I_k be the identity on $\mathcal{H}_k$. Then $I_k \in \mathcal{B}_W(k,k)$ and $A_k \in \mathcal{B}_W(k+1,k)$.*

Let P_k be the orthogonal projection on $\mathcal{H}_k$. Then $P_k = \overline{I}_k \in \mathcal{B}_W$. For $T \in \mathcal{B}_W$, it follows that $\overline{T_{ij}} = P_i T P_j \in \mathcal{B}_W$, and we obtain the following result.

Corollary 3.8 *Let W be a weighted shift and let $T \in \mathcal{B}_W$. Then $T_{ij} \in \mathcal{B}_W(i,j)$.*

Due to Corollary 3.8, an operator $T \in B_W$ can be written as a weak limit of operators of the form

$$T_n = \sum_{i,j=1}^{n} \overline{T_{ij}}, \quad \text{where } T_{ij} \in \mathcal{B}_W(i,j).$$

In Section 3, we study the relationships among the subspaces $\mathcal{B}_W(i,j)$ and this will lead to a description of the weak closure of $\mathcal{B}_W$.

We have seen in Chapter 2 that knowing when $\overline{A_k^{-1}} \in \mathcal{D}_W$ played an important role in the structure of $\mathcal{D}_W$. For similar reasons, the membership of $\overline{A_k^{-1}}$ in $\mathcal{B}_W$ will be crucial in the study of spectral radius algebras. Not only that, but it will give us a class of operators for which the weak closures of $\mathcal{D}_W$ and $\mathcal{B}_W$ can have wildly different structures. Later on, we will show that if W is an injective weighted shift of finite multiplicity and $A_k^{-1} \in \mathcal{B}_W(k,k+1)$, then $\overline{\mathcal{B}_W}$ is completely determined by $\mathcal{B}_W(1,1)$.

Theorem 3.9 *Let $W \sim (A_n)$ be a weighted shift such that A_k^{-1} exists, for some $k \in \mathbb{N}$. Then $A_k^{-1} \in \mathcal{B}_W(k, k+1)$ if and only if $r(W) > 0$.*

PROOF: Let $k \in \mathbb{N}$ such that $A_k^{-1} \in \mathcal{B}_W(k, k+1)$. By Proposition 3.6, there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n-1}A_{k+n-2}\cdots A_k A_k^{-1}x\|^2 \le M \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2,$$

for all $x \in \mathcal{H}_{k+1}$ and $m \in \mathbb{N}$. The left hand side of this inequality can be rewritten as

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n-1}A_{k+n-2}\cdots A_k A_k^{-1}x\|^2$$

$$= \|A_k^{-1}x\|^2 + d_m^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2. \tag{3.3}$$

This implies that

$$d_m^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2 \le M \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2, \tag{3.4}$$

whence $\{d_m\}_{m\in\mathbb{N}}$ must be a bounded sequence. This happens if and only if $r(W) > 0$.

To complete the proof, we will assume that $r(W) > 0$ and that A_k^{-1} exists, for some $k \in \mathbb{N}$, and we will show that $A_k^{-1} \in \mathcal{B}_W(k, k+1)$. Since $r(W) > 0$, the sequence $\{d_m\}_{m\in\mathbb{N}}$ is bounded by $\frac{1}{r(W)}$ and (3.3) implies that, for all $x \in \mathcal{H}_{k+1}$,

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n-1}A_{k+n-2}\cdots A_k A_k^{-1}x\|^2$$

$$\leq \|A_k^{-1}x\|^2 + \frac{1}{r(W)^2}\sum_{n=0}^{\infty} d_m^{2n}\|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2$$

$$\leq \|A_k^{-1}\|^2\|x\|^2 + \frac{1}{r(W)^2}\sum_{n=0}^{\infty} d_m^{2n}\|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2$$

$$\leq \|A_k^{-1}\|^2\sum_{n=0}^{\infty} d_m^{2n}\|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2 + \frac{1}{r(W)^2}\sum_{n=0}^{\infty} d_m^{2n}\|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2$$

$$= \left(\|A_k^{-1}\|^2 + \frac{1}{r(W)^2}\right)\sum_{n=0}^{\infty} d_m^{2n}\|A_{k+n}A_{k+n-1}\cdots A_{k+1}x\|^2.$$

Therefore, Proposition 3.6 implies that $A_k^{-1} \in \mathcal{B}_W(k, k+1)$ by taking $M = \|A_k^{-1}\|^2 + \frac{1}{r(W)^2}$. □

Now, we can finally present a class of operators for which the weak closure of $\mathcal{D}_W$ is strictly smaller than that of $\mathcal{B}_W$.

Corollary 3.10 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity such that $r(W) > 0$ and W is not bounded below. Then $\overline{\mathcal{D}_W} \neq \overline{\mathcal{B}_W}$.*

Proof: It was shown in Theorem 2.8 that if W is injective, then $A_n^{-1} \in \mathcal{D}_W(n, n+1)$, for all $n \in \mathbb{N}$, if and only if W is bounded below. If W has finite multiplicity and $T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, then $\dim(\mathcal{D}_W(i,j)) < \infty$ and $\overline{T}$ belongs to $\overline{\mathcal{D}_W}$ if and only if $T \in \mathcal{D}_W(i,j)$. Hence, $\overline{A_n^{-1}}$ does not belong to $\overline{\mathcal{D}_W}$, for any $n \in \mathbb{N}$. On the other hand, $r(W) > 0$ so Theorem 3.9 shows that $A_k^{-1} \in \mathcal{B}_W(k, k+1)$ whenever A_k is invertible. □

There are many examples of such operators. Let B_n be a sequence of operators which are not uniformly bounded below. Define a weight sequence A_n by $A_{2^n} = B_n$

and A_n is the identity otherwise. Then $W \sim (A_n)$ is not bounded below. However, due to arbitrarily long sequences of the identity operator, we have that $\|W^n\| \geq 1$, for all $n \in \mathbb{N}$. Hence, the spectral radius formula implies that $r(W) \geq 1$.

The converse of Corollary 3.10 is true if W is a shift of multiplicity one (see [21]). However, in the next example, we will demonstrate that there exists an injective weighted shift of finite multiplicity which is bounded below and $\overline{\mathcal{D}_W} \neq \overline{\mathcal{B}_W}$. In Sections 3.4 and 3.5, we will further generalize this result.

Example

Let $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right], B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, and define W to be the weighted shift with weight sequence

$$(A, B, B, A, A, A, B, B, B, B, \dots).$$

It is clear from this definition that W is an injective weighted shift which is bounded below. Also, we can compute that $\|W^n\| = 2^n$, for all $n \in \mathbb{N}$, whence $r(W) = 2$.

It was shown in the third example of Section 2.4.1 that

$$\mathcal{D}_W(1,1) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1) : a, b \in \mathbb{C} \right\} \subset \mathcal{B}_W(1,1).$$

We will show that

$$T := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}_W(1,1).$$

Since $\mathcal{D}_W(1,1) \subset \mathcal{B}_W(1,1)$, $\mathcal{B}_W(1,1)$ will contain all the upper triangular matrices and $\overline{\mathcal{D}_W}$ is a proper subalgebra of $\overline{\mathcal{B}_W}$.

Let $x = (0,1)^T \in \mathcal{H}_1$. We will prove that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 x\|^2 = 1 + \sum_{k=0}^{\infty} 2^{2k(k+1)} d_m^{2k(2k+1)+2} \left(\frac{1 - d_m^{2(2k+1)}}{1 - d_m^2} \right)$$

$$+ \sum_{k=0}^{\infty} 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1 - (2d_m)^{2(2k+2)}}{1 - (2d_m)^2} \right). \tag{3.5}$$

The first term on the right side comes from the fact that, for $n = 0$, we have that $d_m^{2n} \|A_n \cdots A_1 x\|^2 = \|x\|^2 = 1$. We now explain how to obtain the first series on the right side of (3.5).

Let $k \geq 0$ be fixed and $1 + 2 + \cdots + 2k < n \leq 1 + 2 + \cdots + 2k + (2k+1)$. Then $A_n = A$ implying that $A_n A_{n-1} \cdots A_1 x = A_{n-1} \cdots A_1 x$ and $d_m^{2n} \|A_n A_{n-1} \cdots A_1 x\|^2 = d_m^2 (d_m^{2(n-1)} \|A_{n-1} A_{n-2} \cdots A_1 x\|^2)$. Note that $1 + 2 + \cdots + 2k = k(2k+1)$ and $1 + 2 + \cdots + 2k + (2k+1) = (k+1)(2k+1)$. Hence,

$$\sum_{n=k(2k+1)+1}^{(k+1)(2k+1)} d_m^{2n} \|A_n A_{n-1} \cdots A_1 x\|^2$$

$$= d_m^{2(k(2k+1)+1)} \|A_{k(2k+1)+1} A_{k(2k+1)} \cdots A_1 x\|^2 \sum_{n=0}^{2k} d_m^{2n}. \tag{3.6}$$

Substituting the formulas,

$$A_{k(2k+1)+1} A_{k(2k+1)+1} \cdots A_1 = \begin{bmatrix} 2^{1+3+\cdots+(2k-1)} & 0 \\ 0 & 2^{2+4+\cdots+2k} \end{bmatrix},$$

$$\sum_{n=0}^{2k} d_m^{2n} = \left(\frac{1 - d_m^{2(2k+1)}}{1 - d_m^2} \right),$$

and $2+4+\cdots+2k = k(k+1)$ into (3.6), we obtain

$$\sum_{n=k(2k+1)+1}^{(k+1)(2k+1)} d_m^{2n} \|A_n A_{n-1} \cdots A_1 x\|^2 = 2^{2k(k+1)} d_m^{2k(2k+1)+2} \left(\frac{1-d_m^{2(2k+1)}}{1-d_m^2} \right)$$

which are the terms in the first series on the right hand side of (3.5). Applying this same procedure to each $k \in \mathbb{N}$, we obtain the entire first series of (3.5).

Similarly, we can fix $k \geq 0$ and consider $1+2+\cdots+(2k+1) < n \leq 1+2+\cdots+(2k+2)$. Summing up over such n, we can show in the same manner as above that

$$\sum_{n=(k+1)(2k+1)+1}^{(k+1)(2(k+1)+1)} d_m^{2n} \|A_n \cdots A_1 x\|^2 = 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1-(2d_m)^{2(2k+2)}}{1-(2d_m)^2} \right).$$

Therefore, (3.5) does indeed hold.

Next we consider $y = (1,0)^T$. We can repeat the same process, used to establish (3.5), to show that $\sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 y\|^2$ can be written as

$$1 + \sum_{k=0}^{\infty} 2^{2(k+1)^2} d_m^{(2k+1)(2k+2)+2} \left(\frac{1-d_m^{2(2k+2)}}{1-d_m^2} \right)$$

$$+ \sum_{k=0}^{\infty} 2^{2(k^2+1)} d_m^{2k(2k+1)+2} \left(\frac{1-(2d_m)^{2(2k+1)}}{1-(2d_m)^2} \right). \tag{3.7}$$

We can now prove that

$$T := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}_W(1,1).$$

Let $z \in \mathcal{H}_1$ and write $z = z_1 y + z_2 x$, where $y = (1,0)^T$ and $x = (0,1)^T$ as above. Then

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 T z\|^2 = |z_2|^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 y\|^2$$

and

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 z\|^2$$

$$= |z_1|^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 y\|^2 + |z_2|^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 x\|^2.$$

This last equation holds because A_k is a diagonal matrix, for all $k \in \mathbb{N}$, whence $A_n A_{n-1} \cdots A_1 x$ is orthogonal to $A_n A_{n-1} \cdots A_1 y$. We will now prove the inequality

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 y\|^2 \leq 2 \sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 x\|^2. \tag{3.8}$$

This will imply that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 T z\|^2 \leq 2 \sum_{n=0}^{\infty} d_m^{2n} \|A_n A_{n-1} \cdots A_1 z\|^2,$$

for all $z \in \mathcal{H}_1$, which shows that $T \in \mathcal{B}_W(1,1)$. To prove (3.8) we start by showing that

$$2^{2(k+1)^2} d_m^{(2k+1)(2k+2)+2} \left(\frac{1 - d_m^{2(2k+2)}}{1 - d_m^2} \right) \leq 2 \left(2^{2k(k+1)} d_m^{2k(2k+1)+2} \left(\frac{1 - d_m^{2(2k+1)}}{1 - d_m^2} \right) \right).$$

Dividing through by the factor in parentheses on the right hand side and canceling terms, this is equivalent to showing that

$$\frac{2^{2k+2} d_m^{4k+2} (1 - d_m^{2(2k+2)})}{1 - d_m^{2(2k+1)}} \leq 2.$$

Since $\frac{1}{3} = d_1 < d_m < \frac{1}{r(W)} = \frac{1}{2}$,

$$\frac{2^{2k+2} d_m^{4k+2}(1 - d_m^{2(2k+2)})}{1 - d_m^{2(2k+1)}} < \frac{1 - d_m^{2(2k+2)}}{1 - d_m^{2(2k+1)}} \leq \frac{1}{1 - d_m^{2(2k+1)}}$$

$$\leq \frac{1}{1 - d_m^2} < \frac{1}{1 - (1/2)^2} = \frac{4}{3}.$$

In a similar manner, we now prove that

$$\sum_{k=0}^{\infty} 2^{2(k^2+1)} d_m^{2k(2k+1)+2} \left(\frac{1 - (2d_m)^{2(2k+1)}}{1 - (2d_m)^2} \right)$$

$$\leq 2 \sum_{k=0}^{\infty} 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1 - (2d_m)^{2(2k+2)}}{1 - (2d_m)^2} \right). \tag{3.9}$$

For the first term on the left hand side, we have that

$$2^2 d_m^2 \left(\frac{1 - (2d_m)^2}{1 - (2d_m)^2} \right) \leq \sum_{k=0}^{\infty} 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1 - (2d_m)^{2(2k+2)}}{1 - (2d_m)^2} \right),$$

so (3.9) will be proved if we show that

$$\sum_{k=1}^{\infty} 2^{2(k^2+1)} d_m^{2k(2k+1)+2} \left(\frac{1 - (2d_m)^{2(2k+1)}}{1 - (2d_m)^2} \right)$$

$$\leq \sum_{k=0}^{\infty} 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1 - (2d_m)^{2(2k+2)}}{1 - (2d_m)^2} \right).$$

Converting the left hand side to a sum starting at zero, we have to show that

$$\sum_{k=0}^{\infty} 2^{2((k+1)^2+1)} d_m^{(2k+2)(2k+3)+2} \left(\frac{1 - (2d_m)^{2(2k+3)}}{1 - (2d_m)^2} \right)$$

$$\leq \sum_{k=0}^{\infty} 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1-(2d_m)^{2(2k+2)}}{1-(2d_m)^2} \right).$$

It is left to the reader to check that we indeed have

$$2^{2((k+1)^2+1)} d_m^{(2k+2)(2k+3)+2} \left(\frac{1-(2d_m)^{2(2k+3)}}{1-(2d_m)^2} \right)$$

$$\leq 2^{2k(k+1)+2} d_m^{(2k+2)(2k+1)+2} \left(\frac{1-(2d_m)^{2(2k+2)}}{1-(2d_m)^2} \right)$$

and thus (3.9) holds.

Therefore (3.8) holds and $T \in \mathcal{B}_W(1,1)$. This proves that $\mathcal{B}_W(1,1)$ contains all upper triangular matrices. It can also be shown in a similar manner that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathcal{B}_W(1,1).$$

Therefore, $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$. In the next section, we will show that this equality, along with the fact that $r(W) > 0$, implies that $\mathcal{B}_W$ is weakly dense in $\mathcal{H}$.

3.3 Injective Weighted Shifts of Finite Multiplicity

From now on, we will make a simplifying assumption that W has a finite multiplicity. This will allow us to explicitly determine the structure of $\mathcal{B}_W$. For injective weighted shifts, we will describe a relationship between $\mathcal{B}_W(i,j)$ and $\mathcal{B}_W(k,l)$. However, this relation disappears when W is not injective and we will discuss what can be said about such shifts in Section 3.4. The main result of this section is Theorem 3.16 that gives the necessary and sufficient conditions for $\mathcal{B}_W$ to have a n.i.s.

When W is injective, A_k is an invertible operator, for each $k \in \mathbb{N}$. The next two theorems take advantage of this fact and they demonstrate that some of the structure of $\mathcal{B}_W(i,j)$ is inherited by $\mathcal{B}_W(i+1,j)$, $\mathcal{B}_W(i,j-1)$, and $\mathcal{B}_W(i+1,j+1)$.

Theorem 3.11 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity and let $i,j \in \mathbb{N}$. Then multiplication on the left by A_i is an injective linear transformation from $\mathcal{B}_W(i,j)$ to $\mathcal{B}_W(i+1,j)$. Also, multiplication on the right by A_j is an injective linear transformation from $\mathcal{B}_W(i,j)$ to $\mathcal{B}_W(i,j-1)$.*

PROOF: Let $T \in \mathcal{B}_W(i,j)$. By Corollary 3.7, $\overline{A_iT} \in \mathcal{B}_W$ and $A_iT : \mathcal{H}_j \to \mathcal{H}_{i+1}$. Hence, multiplication by A_i on the left is a well-defined linear transformation from $\mathcal{B}_W(i,j)$ to $\mathcal{B}_W(i+1,j)$. Furthermore, this function is injective because A_i is an invertible operator. Similarly, multiplication by A_j on the right is an injective linear transformation from $\mathcal{B}_W(i,j)$ to $\mathcal{B}_W(i,j-1)$. $\square$

This previous theorem is very similar to Theorem 2.9. The next one follows in the footsteps of Theorem 2.10.

Theorem 3.12 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity and let $i,j \in \mathbb{N}$. Then the mapping $\varphi : \mathcal{B}_W(i,j) \to \mathcal{B}_W(i+1,j+1)$ defined by $\varphi(T) = A_iTA_j^{-1}$ is an isomorphism of vector spaces.*

PROOF: Let $T \in \mathcal{B}_W(i,j)$. Since W is an injective weighted shift of finite multiplicity, A_n^{-1} exists, for all $n \in \mathbb{N}$, and it is not hard to see that the mapping $T \mapsto A_iTA_j^{-1}$ is an injective linear transformation $\mathcal{B}_W(i,j) \to \mathcal{L}(\mathcal{H}_{j+1}, \mathcal{H}_{i+1})$. We will now show that $A_iTA_j^{-1} \in \mathcal{B}_W(i+1,j+1)$. Since $T \in \mathcal{B}_W(i,j)$, by Proposition 3.6 there exists

$M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i T x\|^2 \le M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j x\|^2,$$

for all $x \in \mathcal{H}_j$ and for all $n \in \mathbb{N}$. Let $y \in \mathcal{H}_{j+1}$. Then

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_{i+1}(A_i T A_j^{-1})y\|^2$$

$$= \sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i T A_j^{-1} y\|^2$$

$$\le M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j A_j^{-1} y\|^2$$

$$= M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_{j+1} y\|^2.$$

Therefore, $\overline{A_i T A_j^{-1}} \in \mathcal{B}_W$ by Proposition 3.6, so the range of φ is a subspace of $\mathcal{B}_W(i+1, j+1)$. It remains to show that φ is surjective.

Let $X \in \mathcal{B}_W(i+1, j+1)$. Since $\varphi(A_i^{-1} X A_j) = X$, it suffices to prove that $A_i^{-1} X A_j \in \mathcal{B}_W(i, j)$. By Proposition 3.6, there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n}A_{i+n-1}\cdots A_{i+1} X x\|^2 \le M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n}A_{j+n-1}\cdots A_{j+1} x\|^2,$$

for all $x \in \mathcal{H}_{j+1}$ and for all $n \in \mathbb{N}$. Let $y \in \mathcal{H}_j$ and let $M' = M d_1^{-2} + \|A_i^{-1} X\|$. Note that $d_m \ge d_1$, whence $d_m^{-2} \le d_1^{-2}$, for all $m \in \mathbb{N}$. Thus,

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i (A_i^{-1} X A_j) y\|^2$$

$$= \|A_i^{-1} X A_j y\|^2 + \sum_{n=1}^{\infty} d_m^{2n} \|A_{i+n-1} A_{i+n-2} \cdots A_{i+1} X A_j y\|^2$$

$$= \|A_i^{-1} X A_j y\|^2 + d_m^{-2} \sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n} A_{i+n-1} \cdots A_{i+1} X A_j y\|^2$$

$$\leq \|A_i^{-1} X\|^2 \|A_j y\|^2 + M d_1^{-2} \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n} A_{j+n-1} \cdots A_{j+1} A_j y\|^2$$

$$\leq M' \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n} A_{j+n-1} \cdots A_{j+1} A_j y\|^2,$$

for all $m \in \mathbb{N}$ and $y \in \mathcal{H}_j$. The result follows from Proposition 3.6. □

We will now use Theorems 3.11 and 3.12 to describe a relationship between the dimensions of various $\mathcal{B}_W(i, j)$.

Corollary 3.13 *Let W be an injective weighted shift of finite multiplicity. Then*

1. $\dim(\mathcal{B}_W(i, j)) \leq \dim(\mathcal{B}_W(l, k))$ *for* $k \leq j$ *and* $i \leq l$,
2. $\dim(\mathcal{B}_W(1, j)) = \dim(\mathcal{B}_W(1 + n, j + n))$, *for all* $j, n \in \mathbb{N}$, *and*
3. $\dim(\mathcal{B}_W(i, 1)) = \dim(\mathcal{B}_W(i + n, 1 + n))$, *for all* $i, n \in \mathbb{N}$.

According to Corollary 3.13, if we know the structure of $\mathcal{B}_W(1, j)$ and $\mathcal{B}_W(i, 1)$, then we can determine the weak closure of $\mathcal{B}_W$. This is just as it was in the case with the Deddens algebra. We also have the same picture as before to explain how the dimensions change as we move away from the (i, j) block.

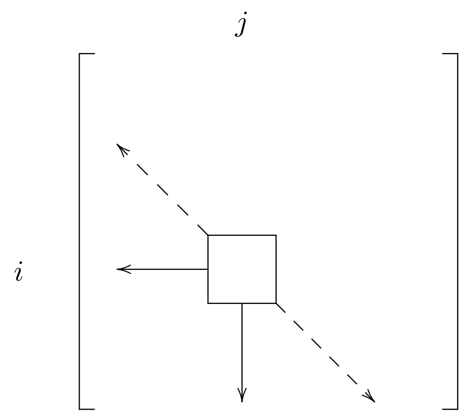

Figure 3.1

In the direction of the solid arrows, the dimension of $\mathcal{B}_W(i,j)$ cannot decrease. While moving in the direction of dashed arrows, the dimension of $\mathcal{B}_W(i,j)$ is constant. The homomorphisms in Theorem 3.11 are invertible precisely when $A_k^{-1} \in \mathcal{B}_W(k,k+1)$, for all $k \in \mathbb{N}$. In this case, we have equality among dimensions and the weak closure of $\mathcal{B}_W$ is determined by $\mathcal{B}_W(1,1)$.

Theorem 3.14 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity. Then* $\dim(\mathcal{B}_W(i,j)) = \dim(\mathcal{B}_W(1,1))$ *holds, for all $i,j \in \mathbb{N}$, if and only if $r(W) > 0$.*

Proof: Assume that $\dim(\mathcal{B}_W(i,j)) = \dim(\mathcal{B}_W(1,1))$, for all $i,j \in \mathbb{N}$. In particular, $\dim(\mathcal{B}_W(1,1)) = \dim(\mathcal{B}_W(1,2))$. By Theorem 3.11, multiplication by A_1 on the right provides an injective linear transformation $\mathcal{B}_W(1,2) \to \mathcal{B}_W(1,1)$. Since $\dim(\mathcal{B}_W(1,2)) = \dim(\mathcal{B}_W(1,1)) < \infty$, this multiplication is an isomorphism of vector spaces. From Proposition 3.7, we have that $I_1 \in \mathcal{B}_W(1,1)$ implying that there exists $X \in \mathcal{B}_W(1,2)$ such that $XA_1 = I_1$. Therefore, $X = A_1^{-1} \in \mathcal{B}_W(1,2)$ and $r(W) > 0$ by Theorem 3.9.

If $r(W) > 0$, then Theorem 3.9 implies that $\overline{A_k^{-1}} \in \mathcal{B}_W$, for all $k \in \mathbb{N}$. If $T \in \mathcal{D}_W(i,j)$, then $A_{i-1}^{-1}T \in \mathcal{B}_W(i-1,j)$ and $TA_j^{-1} \in \mathcal{B}_W(i,j+1)$. Thus, multiplication by A_{i-1}^{-1} from the left is an injective linear transformation $\mathcal{B}_W(i,j) \to \mathcal{B}_W(i-1,j)$ so $\dim(\mathcal{B}_W(i-1,j)) \geq \dim(\mathcal{B}_W(i,j))$. Applying the same argument to the multiplication on the right by A_j^{-1}, we see that $\dim(\mathcal{B}_W(i,j+1)) \geq \dim(\mathcal{B}_W(i,j))$. The opposite inequalities are given in Part 1 of Corollary 3.13. Therefore, $\dim(\mathcal{B}_W(i,j)) = \dim(\mathcal{B}_W(1,1))$, for all $i,j \in \mathbb{N}$. □

When $r(W) = 0$, the dimensions of $B_W(i,j)$ cannot all be equal. The main diagonal will once again provide us with at least one place where the inequality is strict.

Corollary 3.15 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity with $r(W) = 0$. Then $\dim(\mathcal{B}_W(1,2)) < \dim(\mathcal{B}_W(1,1))$. Also, $\mathcal{B}_W(1,k)$ contains no invertible operators for any $k \geq 2$.*

Proof: By Corollary 3.13, $\dim(\mathcal{B}_W(1,2)) \leq \dim(\mathcal{B}_W(1,1))$. If $\dim(\mathcal{B}_W(1,2)) = \dim(\mathcal{B}_W(1,1))$, then multiplication by A_1 on the right is a vector space isomorphism. Thus, $A_1^{-1} \in \mathcal{B}_W(1,2)$ and, by Theorem 2.8, $r(W) \neq 0$. Therefore, $\dim(\mathcal{B}_W(1,2)) < \dim(\mathcal{B}_W(1,1))$.

Let $k \geq 2$ and assume, to the contrary, that there exists $X \in \mathcal{B}_W(1,k)$ such that X is an invertible operator. Then multiplication by X on the right yields an injective linear transformation $\mathcal{B}_W(1,1) \to \mathcal{B}_W(1,k)$. However, this would imply that $\dim(\mathcal{B}_W(1,1)) \leq \dim(\mathcal{B}_W(1,k)) \leq \dim(\mathcal{B}_W(1,2))$. □

To finish up this section, we will describe weighted shifts such that $\mathcal{B}_W$ is weakly dense in $\mathcal{L}(\mathcal{H})$. Then, for operators *without* this property, we will find a n.i.s. for $\mathcal{B}_W$.

Theorem 3.16 *Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity. The following are equivalent:*

(a) $r(W) > 0$ *and* $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$.

(b) The weak closure of $\mathcal{B}_W$ *is* $\mathcal{L}(\mathcal{H})$.

(c) $\mathcal{B}_W$ *has no n.i.s.*

Proof: We start by showing that (a) implies the (b). Let $r(W) > 0$ and $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$. By Theorem 3.14, $\mathcal{B}_W(i,j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $i, j \in \mathbb{N}$. If $T \in \mathcal{L}(\mathcal{H})$, then T has a matrix $[T_{ij}]_{i,j\in\mathbb{N}}$ with $T_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) = \mathcal{B}_W(i,j)$. Therefore, T is the weak limit of $\sum_{i,j=1}^{n} \overline{T}_{ij}$ implying that T is in the weak closure of $\mathcal{B}_W$.

Next, we will show that (b) implies (c). If $\mathcal{M}$ is invariant for $\mathcal{B}_W$, then it is also invariant for $\overline{\mathcal{B}}_W = \mathcal{L}(\mathcal{H})$. Thus, $\mathcal{M}$ must be trivial. It remains to show that (c) implies (a). We will actually prove the contrapositive, "if $r(W) = 0$ or $\mathcal{B}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$, then there exists a n.i.s. for $\mathcal{B}_W$".

First, if $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$ and $r(W) = 0$, we will show that $\mathcal{B}_W(1,j) = 0$ for $j > 1$. This will imply that $\mathcal{B}_W$ is block lower triangular, by Corollary 3.13, and has many n.i.s. Since $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$, we know that $\mathcal{B}_W(i,j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $i \geq j$. Assume, to the contrary, that $\mathcal{B}_W(1, j_0) \neq 0$, for some $1 < j_0 \in \mathbb{N}$. Let $A \in \mathcal{L}(\mathcal{H}_1)$, let $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{j_0})$, and let $0 \neq X \in \mathcal{B}_W(1, j_0)$. Then $\overline{AXT} \in \mathcal{B}_W$ with $AXT : \mathcal{H}_2 \to \mathcal{H}_1$. In other words, we have that $AXT \in \mathcal{B}_W(1,2)$. Since A and T were arbitrary, $\mathcal{B}_W(1,2)$ contains every rank one operator in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, whence $\mathcal{B}_W(1,2) = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. In particular, $A_1^{-1} \in \mathcal{B}_W(1,2) = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and, by Theorem 2.8, $r(W) > 0$.

Next, we assume that $\mathcal{B}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$ and construct a n.i.s. for $\mathcal{B}_W$. Since $\mathcal{B}_W(1,1)$ is a finite dimensional algebra over the complex numbers, Burnside's Theorem guarantees that there exists a n.i.s. $\mathcal{M}_1 \subset \mathcal{H}_1$ for $\mathcal{B}_W(1,1)$. Define $\mathcal{M}_k$ to be the subspace of $\mathcal{H}_k$ generated by $\{Ax : x \in \mathcal{M}_1, A \in \mathcal{B}_W(k,1)\}$ and define $\mathcal{M}$ to be the weak closure of $\bigoplus_{k\in\mathbb{N}} \mathcal{M}_k$. We will show that $\mathcal{M}$ is a n.i.s. for $\mathcal{B}_W$. It suffices to show that, for any $i, j \in \mathbb{N}$, $\mathcal{M}$ is invariant for $\overline{T}_{ij}$ whenever $T_{ij} \in \mathcal{B}_W(i,j)$. Indeed if $T \in \mathcal{B}_W$, then T is a weak limit of $\sum_{i,j=1}^{n} \overline{T_{ij}}$ and, if $\mathcal{M}$ is invariant for each $\overline{T_{ij}}$, then $\mathcal{M}$ is invariant for the sum $\sum_{i,j=1}^{n} \overline{T_{ij}}$ and hence for T. Also, $\mathcal{M}$ is nontrivial because we assumed that $0 \neq \mathcal{M}_1 \neq \mathcal{H}_1$.

Thus, it remains to show that $\mathcal{M}$ is invariant for $\overline{T}_{ij}$. Let $T \in \mathcal{B}_W(i,j)$ and let $y \in \mathcal{M}_j$. Then there exists $m \in \mathbb{N}$, $B_k \in \mathcal{B}_W(j,1)$, and $x_k \in \mathcal{M}_1$, $1 \leq k \leq m$, such that $y = \sum_{k=1}^{m} B_k x_k$. Thus, $Ty = \sum_{k=1}^{m} TB_k x_k$. Since $TB_k \in \mathcal{B}_W(i,1)$, the definition of $\mathcal{M}_i$ shows that $TB_k x_k \in \mathcal{M}_i$. We conclude that $Ty \in \mathcal{M}_i$ and the proof is complete. □

3.4 Noninjective Weighted Shifts

When W is noninjective, Theorems 3.11 and 3.12 do not hold and knowing the structure of $\mathcal{B}_W(i,j)$ yields no information about $\mathcal{B}_W(i+1,j), \mathcal{B}_W(i,j-1)$, or $\mathcal{B}_W(i+1,j+1)$. In order to understand the structure of spectral radius algebras associated to a noninjective shift, we will consider consider separately the case where W has a positive spectral radius and the case where W is quasinilpotent.

3.4.1 Noninjective weighted shifts with positive spectral radius

In this section, we will show that Theorem 3.16 does not extend to noninjective shifts with positive spectral radius. Before we construct an example of such a shift, we will establish some results.

Theorem 3.17 *Let $W \sim (A_n)$ be a noninjective weighted shift with $r(W) > 0$. Let $i, k \in \mathbb{N}$, and let $y \in \mathcal{H}_i$ be such that $A_{i+k-1}A_{i+k-2}\cdots A_i y = 0$. Then $y \otimes x \in \mathcal{B}_W(i,j)$, for all $j \in \mathbb{N}$ and all $x \in \mathcal{H}_j$.*

Proof: Let $y \in \mathcal{H}_i$ satsify $A_{i+k-1}A_{i+k-2}\cdots A_i y = 0$ and let x be an arbitrary vector in $\mathcal{H}_j$. Define M to be

$$M = \|x\|^2 \sum_{n=0}^{k-1} \frac{1}{r(W)^{2n}} \|A_{i+n-1}A_{i+n-2}\cdots A_i y\|^2.$$

Let $z \in \mathcal{H}_j$. Note that $d_m \le \frac{1}{r(W)}$ when $r(W) > 0$, whence

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i (y\otimes x) z\|^2 = \sum_{n=0}^{k-1} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i y\|^2 |\langle z, x\rangle|^2$$

$$\le \sum_{n=0}^{k-1} \frac{1}{r(W)^{2n}} \|A_{i+n-1}A_{i+n-2}\cdots A_i y\|^2 |\langle z, x\rangle|^2 \le M\|z\|^2$$

$$\le M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j z\|^2.$$

By Proposition 3.6, $x \otimes y \in \mathcal{B}_W(i,j)$. □

This theorem allows us to extend Corollary 3.10 to apply to nearly all noninjective weighted shifts.

Corollary 3.18 *Let W be a noninjective weighted shift of finite multiplicity. If $r(W) > 0$ and there exists $n \in \mathbb{N}$ such that $\ker(W^n) \neq \ker(W)$, then $\overline{\mathcal{D}_W} \neq \overline{\mathcal{B}_W}$.*

PROOF: Since W is not injective, there exists $j \in \mathbb{N}$ and a nonzero vector $x \in \mathcal{H}_j$ such that $A_j x = 0$. Also, there exists $y \in \mathcal{H}$ such that $y \in \ker(W^n)$ but $y \notin \ker(W)$. We can write $y = (y_1, y_2, \dots)$, with $y_k \in \mathcal{H}_k$, for all $k \in \mathbb{N}$. Furthermore, $y \notin \ker(W)$, so there exists $i \in \mathbb{N}$ such that $y_i \notin \ker(A_i)$. However, $W^n y = 0$ implies that $A_{i+n-1}A_{i+n-2}\cdots A_i y_i = 0$. By Theorem 3.17, $y_i \otimes x \in \mathcal{B}_W(i,j)$. On the other hand, there does not exist $M > 0$ such that $0 \neq \|A_i(y_i \otimes x)x\| \leq M\|A_j x\| = 0$. Thus, Proposition 2.4 implies that $y_i \otimes x \notin \mathcal{D}_W(j,k)$. □

If W is nilpotent when restricted to the vectors whose only nonzero component lies in $\mathcal{H}_i$, then Theorem 3.17 shows that $\mathcal{B}_W(i,j)$ contains all rank one operators. Since $\mathcal{B}_W(i,j)$ is a vector space, then we actually have $\mathcal{B}_W(i,j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$.

Corollary 3.19 *Let $W \sim (A_n)$ be a weighted shift of finite multiplicity with $r(W) > 0$. Let $i, n \in \mathbb{N}$ be such that $A_{i+n-1}A_{i+n-2}\cdots A_i = 0$. Then $\mathcal{B}_W(i,j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $j \in \mathbb{N}$.*

We now give an example of a weighted shift W of multiplicity 2 such that $r(W) > 0$, $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$, and $\mathcal{B}_W$ has a n.i.s.

Example

Define $W \sim (A_n)$ by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and } A_n = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ \text{for } n \geq 3.$$

From this, we see $r(W) = 2$ and $A_2A_1 = 0$. Hence, Corollary 3.19 implies that $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$. For injective shifts, these two conditions guarantee that $\mathcal{B}_W$ is weakly dense in $\mathcal{L}(\mathcal{H})$. However, as we will show, when W is defined in this manner, the algebra $\mathcal{B}_W$ has a n.i.s.

First, we will show that

$$\mathcal{B}_W(3,3) \subset \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a,b,c \in \mathbb{C} \right\}. \tag{3.10}$$

Let

$$T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \in \mathcal{B}_W(3,3).$$

Then, by Proposition 3.6, there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{n+2}A_{n+1}\cdots A_3Tx\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{n+2}A_{n+1}\cdots A_3x\|^2, \tag{3.11}$$

for all $x \in \mathcal{H}_3$. We will take $x = (0,1)^T$. When evaluating the left hand side of the equation, we obtain

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{n+2}A_{n+1}\cdots A_3Tx\|^2 = \sum_{n=0}^{\infty} \left(\frac{m}{1+2m}\right)^{2n} (|2^n t_2|^2 + |t_4|^2)$$

$$= \frac{|t_2|^2}{1-(\frac{2m}{1+2m})^2} + \frac{|t_4|^2}{1-(\frac{m}{1+2m})^2}.$$

Meanwhile, the right hand side simplifies to

$$M \sum_{n=0}^{\infty} d_m^{2n} \|A_{n+2}A_{n+1}\cdots A_3x\|^2 = M \sum_{n=0}^{\infty} \left(\frac{m}{1+2m}\right)^{2n} = \frac{1}{1-(\frac{m}{1+2m})^2}$$

Therefore, (3.11) becomes

$$\frac{|t_2|^2}{1-(\frac{2m}{1+2m})^2}+\frac{|t_4|^2}{1-(\frac{m}{1+2m})^2}\leq\frac{M}{1-(\frac{m}{1+2m})^2}.$$

However,

$$\lim_{m\to\infty}\frac{1}{1-(\frac{2m}{1+2m})^2}=\infty,\quad\text{and}\quad\lim_{m\to\infty}\frac{1}{1-(\frac{m}{1+2m})^2}=2.$$

Thus, $|t_2|$ must be zero for (3.11) to hold. Now that we have established that $\mathcal{B}_W(3,3)\neq\mathcal{L}(\mathcal{H}_3)$, it must have a n.i.s. $\mathcal{N}_3$. From here we proceed just like in the proof of Theorem 3.16. Namely, define $\mathcal{N}_k=\operatorname{span}\{Ax : A\in\mathcal{B}_W(k,3), x\in\mathcal{N}_3\}$, for $k\neq 3$, and $\mathcal{N}$ to be the closure of $\oplus_{k\in\mathbb{N}}\mathcal{N}_k$. Repeating the proof in Theorem 3.16, we obtain that $\mathcal{N}$ is a n.i.s. for $\mathcal{B}_W$.

Our next example features a shift that, at first glance, should not behave too differently from the previous one. Namely, they are both noninjective weighted shifts of multiplicity two with positive spectral radii. The difference is that this one does not have a n.i.s.

Example

Let

$$X=\begin{bmatrix}1&0\\0&0\end{bmatrix}\text{ and }Y=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

Define the weight sequence for $W \sim (A_n)$ by

$$A_n = \begin{cases} X & \text{for } n = 2^k, k \geq 1, \\ Y & \text{for } n = 2^k + 1, k \geq 1, \\ I & \text{otherwise,} \end{cases}$$

where I is the 2×2 identity matrix. This sequence contains arbitrarily long finite subsequences of the form $(I, I, I, \ldots, I)$ which implies that $r(W) = 1$. Since $XY = YX = 0$, then for every $k \in \mathbb{N}$, there exists n_k such that $A_{k+n_k} A_{k+n_k-1} \cdots A_k = 0$. Thus, we can apply Corollary 3.19 to conclude that $\mathcal{B}_W(k, j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_k)$, for all $j, k \in \mathbb{N}$. Therefore, the algebra $\mathcal{B}_W$ is weakly dense in $\mathcal{L}(\mathcal{H})$ and does not have a n.i.s.

The last two examples show that, when W is a noninjective weighted shift and $r(W) > 0$, the equality $\mathcal{B}_W(1, 1)$ is not the proper condition to guarantee that $\mathcal{B}_W$ is weakly dense in $\mathcal{L}(\mathcal{H})$. We will return to this issue after we conduct a study of quasinilpotent weighted shifts

3.4.2 Noninjective quasinilpotent weighted shifts

In this section, we turn our attention to the study of a weighted shift W which is noninjective and quasinilpotent. Let $x \in \mathcal{H}_j$, for some $j \in \mathbb{N}$. We will use n_x to denote the smallest positive integer such that $A_{j+n-1} A_{j+n-2} \cdots A_j x = 0$ and we will say that $n_x := \infty$ if $A_{j+n-1} A_{j+n-2} \cdots A_j x \neq 0$, for all $n \in \mathbb{N}$. We start off with a general theorem which applies to all noninjective quasinilpotent weighted shifts, even those of infinite multiplicity.

Theorem 3.20 *Let W be a noninjective quasinilpotent weighted shift and let $T \in \mathcal{B}_W(i,j)$. Then $n_{Tx} \leq n_x$, for all $x \in \mathcal{H}_j$.*

PROOF: If $n_x = \infty$, then this inequality trivially holds. Let $n_x \in \mathbb{N}$. Since $T \in \mathcal{B}_W(i,j)$, by Proposition 3.6 there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i Tx\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j x\|^2, \quad (3.12)$$

for all $m \in \mathbb{N}$. Further, W is quasinilpotent, i.e., $r(W) = 0$, so $d_m = m$. Therefore, (3.12) becomes

$$\sum_{n=0}^{n_{Tx}} m^{2n} \|A_{i+n-1}A_{i+n-2}\cdots A_i Tx\|^2 \leq M \sum_{n=0}^{n_x} m^{2n} \|A_{j+n-1}A_{j+n-2}\cdots A_j x\|^2. \quad (3.13)$$

Now both sides of (3.13) are polynomials in m. So n_x cannot be less than n_{T_x}, i.e., $n_x \geq n_{Tx}$. □

From here, we get an important result about the existence of a n.i.s. for $\mathcal{B}_W$.

Corollary 3.21 *Let W be a noninjective quasinilpotent weighted shift. Then* $\ker(W)$ *is a n.i.s. for $\mathcal{B}_W$.*

PROOF: Let $T \in \mathcal{B}_W$ and let $\{T_{ij}\}_{i,j\in\mathbb{N}}$ be its matrix relative to the decomposition $\mathcal{H} = \oplus_{n=1}^{\infty}\mathcal{H}_n$. By Corollary 3.8, $T_{ij} \in \mathcal{B}_W(i,j)$, and it suffices to show that $\ker(W)$ is invariant under $\overline{T_{ij}}$, for all $i, j \in \mathbb{N}$. Let $x = (x_1, x_2, \dots) \in \ker(W)$, with $x_n \in \mathcal{H}_n$, for $n \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$, $n_{x_k} = 1$ if $x_k \neq 0$ and $n_{x_k} = 0$ if $x_k = 0$. Since $\overline{T_{ij}}x = T_{ij}x_j$, by Theorem 3.20, $n_{T_{ij}x_j} \leq n_{x_j} \leq 1$. Therefore, either $T_{ij}x_j = 0$ or $n_{T_{ij}x_j} = 1$. In either case, $W\overline{T_{ij}}x = A_iT_{ij}x_j = 0$. It follows that $\ker(W)$ is invariant for $T \in \mathcal{B}_W$, and it is nontrivial because W is a nonzero noninjective operator. $\square$

Corollary 3.21 established that $\ker(W)$ is a n.i.s. for $\mathcal{B}_W$ if W is a quasinilpotent, noninjective weighted shift. Earlier, Theorem 3.16 offered an analogous result for quasinilpotent, injective weighted shifts of finite multiplicity. Combining these two, we obtain the following theorem.

Theorem 3.22 *Let W be a quasinilpotent weighted shift of finite multiplicity. Then $\mathcal{B}_W$ has a n.i.s.*

Finally, we are in position to establish our main result, a characterization of weighted shifts of finite multiplicity such that the associated spectral radius algebra possesses a n.i.s.

Theorem 3.23 *Let $W \sim (A_n)$ be a weighted shift of finite multiplicity. Then $\mathcal{B}_W$ has a n.i.s. if and only if $r(W) = 0$ or $\mathcal{B}_W(n,n) \neq \mathcal{L}(\mathcal{H}_n)$, for some $n \in \mathbb{N}$.*

PROOF: We start off by showing that if $r(W) > 0$ and $B_W(n,n) = \mathcal{L}(\mathcal{H}_n)$, for all $n \in \mathbb{N}$, then $\mathcal{B}_W$ does not have a n.i.s. More precisely, we will show that if $\mathcal{M} \neq \{0\}$ is an invariant subspace for $\mathcal{B}_W$, then $\mathcal{M} = \mathcal{H}$. Since the projections on $\mathcal{H}_k$ belong to $\mathcal{B}_W$, for all $k \in \mathbb{N}$, $\mathcal{M}$ is necessarily of the form $\mathcal{M} = \bigoplus_{k\in\mathbb{N}} \mathcal{M}_k$ where $\mathcal{M}_k \subset \mathcal{H}_k$ is invariant for $\mathcal{B}_W(k,k) = \mathcal{L}(\mathcal{H}_k)$. If we assume that $\mathcal{M}$ is nonzero, then there exists $n \in \mathbb{N}$ such that $\mathcal{M}_n \neq 0$, thus $\mathcal{M}_n = \mathcal{H}_n$. Since $A_n \in \mathcal{B}_W(n+1,n)$, $A_n x \in \mathcal{M}_{n+1}$, for all $x \in \mathcal{M}_n = \mathcal{H}_n$. Therefore, $\mathcal{M}_{n+1} \neq 0$ and $\mathcal{M}_{n+1} = \mathcal{H}_{n+1}$. This shows that $\mathcal{M}_k = \mathcal{H}_k$, for all $k \geq n$, and it remains to establish that $\mathcal{M}_k = \mathcal{H}_k$, for $k < n$.

If A_{n-1} is injective, then A_{n-1}^{-1} exists because $\dim(\mathcal{H}_n) = \dim(\mathcal{H}_{n-1}) < \infty$. Since $r(W) > 0$, we can apply Theorem 3.9 to conclude that $A_{n-1}^{-1} \in \mathcal{B}_W(n-1,n)$

and $A_{n-1}^{-1}x \in \mathcal{M}_{n-1}$, for all $x \in \mathcal{M}_n = \mathcal{H}_n$. Therefore, $\mathcal{M}_{n-1} = \mathcal{H}_{n-1}$. A similar argument shows that, as long as A_k is injective, for all $k < n$, we obtain that $\mathcal{M}_k = \mathcal{H}_k$. On the other hand, if there exists $k < n$ such that A_{k+1} is injective, but A_k is not, let y be a nonzero vector in A_k. By Theorem 3.17, $y \otimes x \in \mathcal{B}_W(k,n)$ for all $x \in \mathcal{H}_n$. Since $x \in \mathcal{M}_n = \mathcal{H}_n$, then $(y \otimes x)x = \|x\|^2 y \in \mathcal{M}_k$. This shows that $\mathcal{M}_k$ is nonzero whence $\mathcal{M}_k = \mathcal{H}_k$ (because $\mathcal{M}_k$ is invariant for $\mathcal{B}_W(k,k) = \mathcal{H}_k$. It follows that $\mathcal{M}_k = \mathcal{H}_k$, for all $k \in \mathbb{N}$, so all invariant subspaces for $\mathcal{B}_W$ must be trivial.

In the other direction, we assume that $r(W) = 0$ or $\mathcal{B}_W(n,n) \neq \mathcal{L}(\mathcal{H}_n)$ for some $n \in \mathbb{N}$ and we will prove that $\mathcal{B}_W$ has a n.i.s. If $r(W) = 0$, then this immediately follows from Theorem 3.22. If $\mathcal{B}_W(n,n) \neq \mathcal{L}(\mathcal{H}_n)$, for some $n \in \mathbb{N}$, then there exists a n.i.s. $\mathcal{M}_n$ for $\mathcal{B}_W(n,n)$. We define

$$\mathcal{M}_k = \{Ax : A \in \mathcal{B}_W(k,n), x \in \mathcal{M}_n\} \subset \mathcal{H}_k,$$

for $k \in \mathbb{N}$, and $\mathcal{M} = \oplus \mathcal{M}_k$. We will show that $\mathcal{M}$ is a n.i.s. for $\mathcal{B}_W$. It suffices to show that $\mathcal{M}$ is invariant for $\overline{T}_{ij}$ whenever $T_{ij} \in \mathcal{B}_W(i,j)$, for all $i,j \in \mathbb{N}$. Indeed if $T \in \mathcal{B}_W$, then T is a weak limit of $\sum_{i,j=1}^n \overline{T_{ij}}$, and if $\mathcal{M}$ is invariant for each $\overline{T_{ij}}$, then $\mathcal{M}$ is invariant for the sum $\sum_{i,j=1}^n \overline{T_{ij}}$ and also for T. Also, $\mathcal{M}$ is nontrivial because we assumed that $\mathcal{M}_n$ is nontrivial. Thus, it remains to show that $\mathcal{M}$ is invariant for $\overline{T}_{ij}$. Let $T \in \mathcal{B}_W(i,j)$ and let $y \in \mathcal{M}_j$. Then there exists $m \in \mathbb{N}$, $B_k \in \mathcal{B}_W(j,n)$, and $x_k \in \mathcal{M}_n$ such that $y = \sum_{k=1}^m B_k x_k$ and $Ty = \sum_{k=1}^m TB_k x_k$. Since $TB_k \in \mathcal{B}_W(i,n)$, then the definition of $\mathcal{M}_i$ tells us that $TB_k x_k \in \mathcal{M}_i$ for each $1 \leq k \leq m$. We conclude that $Ty \in \mathcal{M}_i$ and the proof is complete. □

3.5 Weighted Shifts of Multiplicity Two

In this final section on spectral radius algebras, we look specifically at injective quasinilpotent weighted shifts of multiplicity two. As before, we will consider those shifts that have their weights of the form

$$A_k = \begin{bmatrix} a_k & b_k \\ 0 & c_k \end{bmatrix} \tag{3.14}$$

where $a_k, c_k > 0$, for all $k \in \mathbb{N}$. For such matrices A_k,

$$A_{k+n-1}A_{k+n-2}\cdots A_k = \begin{bmatrix} \prod_{t=0}^{n-1} a_{k+t} & \beta_{k,n} \\ 0 & \prod_{t=0}^{n-1} c_{k+t} \end{bmatrix}, \tag{3.15}$$

where

$$\beta_{k,n} = \sum_{t=0}^{n-1} c_k c_{k+1} \cdots c_{k+t-1} b_{k+t} a_{k+t+1} \cdots a_{k+n-1}. \tag{3.16}$$

Note that $c_k c_{k+1} \cdots c_{k+t-1} = 1$, when $t = 0$, and $a_{k+t+1}a_{k+t+2} \cdots a_{k+n-1} = 1$, when $k = n-1$. Our first theorem will give a complete description of the space $\mathcal{B}_W(1,2)$, at least when W is injective and quasinilpotent.

Theorem 3.24 *Let W be an injective quasinilpotent weighted shift of multiplicity two and let T be a nonzero operator $\mathcal{B}_W(1,2)$. Then there exists $x \in \mathcal{H}_2$ and $y \in \mathcal{H}_1$ such that $T = y \otimes x$ and $\langle A_1 y, x\rangle = 0$. Furthermore, $\mathcal{B}_W(1,2)$ is at most one dimensional and one of the following must hold:*

(a) $T = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, *for some* $a \in \mathbb{C}$.

(b) $T = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$, *for some* $a \in \mathbb{C}$.

(c) $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *satisfy* $x_1, y_2 \neq 0$ *and*

$$\liminf_{n\to\infty} \left| \frac{y_1}{y_2} + \frac{\beta_{1n}}{a_1 a_2 \cdots a_n} \right| = 0.$$

Proof: Let $T \in \mathcal{B}_W(1,2)$ be a nonzero operator. By Corollary 3.15, T is not invertible, so we must have that $T = y \otimes x \in \mathcal{B}_W(1,2)$, for some $x \in \mathcal{H}_2$ and $y \in \mathcal{H}_1$. Let $y = (y_1, y_2)$ and $x = (x_1, x_2)$. Then

$$T = y \otimes x = \begin{bmatrix} x_1 y_1 & x_2 y_1 \\ x_1 y_2 & x_2 y_2 \end{bmatrix}.$$

The condition that $T \in \mathcal{B}_W(1,2)$ implies that there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} m^{2n} \|A_n \cdots A_1 T z\|^2 \leq M \sum_{n=0}^{\infty} m^{2n} \|A_{n+1} \cdots A_2 z\|^2, \tag{3.17}$$

for all $m \in \mathbb{N}$ and all $z \in \mathcal{H}_2$. A calculation shows that the left hand side of (3.17) equals

$$L(z) := |\langle z, x\rangle|^2 \sum_{n=0}^{\infty} m^{2n} (|a_1 a_2 \cdots a_n y_1 + \beta_{1n} y_2|^2 + |c_1 c_2 \cdots c_n y_2|^2), \tag{3.18}$$

while the right hand side is

$$R(z) := \sum_{n=0}^{\infty} m^{2n} (|a_2 a_3 \cdots a_{n+1} z_1 + \beta_{2n} z_2|^2 + |c_2 c_3 \cdots c_{n+1} z_2|^2). \tag{3.19}$$

From the product $(A_{n+1}A_n \cdots A_2)A_1$, we can see that

$$\beta_{1,n+1} = b_1 a_2 \cdots a_{n+1} + c_1 \beta_{2n}. \tag{3.20}$$

Let $z = A_1 y = (a_1 y_1 + b_1 y_2, c_1 y_2)$. Then

$$R(A_1 y) = \sum_{n=0}^{\infty} m^{2n}(|a_1 a_2 \cdots a_{n+1} y_1 + \beta_{1,n+1} y_2|^2 + |c_1 c_2 \cdots c_{n+1} y_2|^2)$$

$$= |a_1 y_1 + b_1 y_2|^2 + |c_1 y_2|^2 + \frac{1}{m^2} \sum_{n=2}^{\infty} m^{2n}(|a_1 a_2 \cdots a_n y_1 + \beta_{1n} y_2|^2 + |c_1 c_2 \cdots c_n y_2|^2).$$

Now, the inequality $L(A_1 y) \leq R(A_1 y)$ implies that $\langle A_1 y, x \rangle = 0$

From this, we can now show that $\mathcal{B}_W(1,2)$ is at most one dimensional. Let $B_k = x_k \otimes y_k \in \mathcal{B}_W(1,2)$ for $k = 1, 2$. If x_1 and x_2 are not collinear and y_1 and y_2 are not collinear, then it not hard to verify that $B_1 + B_2$ is invertible. If x_1 and x_2 are collinear, then $\langle A_1 y_1, x_1 \rangle = 0$ implies that $\langle A_1 y_1, x_2 \rangle = 0$. Thus, $0 = \langle A_1 y_k, x_2 \rangle = \langle y_k, A_1^* x_2 \rangle$ for $k = 1, 2$. Hence, y_1 and y_2 are collinear. Similarly, if y_1 and y_2 are collinear, then so are x_1 and x_2. Therefore, B_1 and B_2 must be collinear and $\dim(\mathcal{B}_W(1,2)) \leq 1$.

Let $z = (1,0)$. Since $L(1,0) \leq R(1,0)$, using (3.18)–(3.19), we find that

$$|x_1|^2 \sum_{n=0}^{\infty} m^{2n}(|a_1 a_2 \cdots a_n y_1 + \beta_{1n} y_2|^2 + |c_1 c_2 \cdots c_n y_2|^2) \leq M \sum_{n=0}^{\infty} m^{2n} |a_2 a_3 \cdots a_{n+1}|^2.$$

It follows that

$$|x_1|^2 \sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2 \left| y_1 + \frac{\beta_{1n} y_2}{a_1 a_2 \cdots a_n} \right|^2 \leq M \sum_{n=0}^{\infty} m^{2n} |a_2 a_3 \cdots a_{n+1}|^2. \tag{3.21}$$

If $x_1 = 0$, then $0 = \langle A_1 y, x\rangle = c_1 y_2 x_2 = 0$, which implies that $y_2 = 0$ since $x \neq 0$. Therefore $T = x \otimes y$ is of the form as given in (a). Also, if $y_2 = 0$, then $0 = \langle A_1 y, x\rangle = a_1 y_1 x_1 = 0$, so $x_1 = 0$ and T is of the form as given in (b). We must now consider what happens when both $x_1 \neq 0$ and $y_2 \neq 0$.

Let x_1 and y_2 be nonzero. The right hand side of (3.21) can be written as

$$\frac{M}{m^2 a_1^2} \sum_{n=1}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2.$$

Suppose now that

$$\liminf_{n\to\infty} \left| \frac{y_1}{y_2} + \frac{\beta_{1n}}{a_1 a_2 \cdots a_n} \right| \neq 0.$$

Then there exists $N \in \mathbb{N}$ and $\delta > 0$ such that

$$\left| y_1 + \frac{\beta_{1n} y_2}{a_1 a_2 \cdots a_n} \right|^2 \geq \delta, \tag{3.22}$$

for $n \geq N$. It follows that

$$\sum_{n=N}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2 \leq \frac{1}{\delta} \sum_{n=N}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2 \left| y_1 + \frac{\beta_{1n} y_2}{a_1 a_2 \cdots a_n} \right|^2$$

$$\leq \frac{M}{\delta m^2 a_1^2} \sum_{n=1}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2,$$

which is impossible. Therefore,

$$\liminf_{n\to\infty} \left| \frac{y_1}{y_2} + \frac{\beta_{1n}}{a_1 a_2 \cdots a_n} \right| = 0.$$

In other words, $y \otimes x$ is as in (c). □

By Theorem 3.24, $\dim(\mathcal{B}_W(1,2)) \leq 1$ and, if this dimension is 1, then $\mathcal{B}_W(1,2)$ consists of operators that are described by $(a) - (c)$ in the same theorem. It is natural to ask what form $\mathcal{B}_W(1,1)$ must take in each of these cases. We will provide the answer for case (a), but we will make an additional assumption about the weight sequence $\{A_n\}_{n\in\mathbb{N}}$.

Theorem 3.25 *Let W be an injective quasinilpotent weighted shift of multiplicity two, and suppose that the sequence $\{\beta_{1n}/(a_1 a_2 \cdots a_n)\}_{n\in\mathbb{N}}$ is convergent. If*

$$\mathcal{B}_W(1,2) = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in \mathbb{C} \right\},$$

then

$$\mathcal{B}_W(1,1) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Proof: Let

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}_W(1,2).$$

Then

$$TA_1 = \begin{bmatrix} 0 & c_1 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}_W(1,1),$$

and $I \in \mathcal{B}_W(1,1)$. Hence, it suffices to show that

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}_W(1,1).$$

More precisely, we will show that there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} m^{2n} |a_1 \cdots a_n z_1|^2 \leq M \sum_{n=0}^{\infty} m^{2n} (|a_1 a_2 \cdots a_n z_1 + \beta_{1n} z_2|^2 + |c_1 c_2 \cdots c_n z_2|^2), \quad (3.23)$$

for all $z_1, z_2 \in \mathbb{C}$ and all $m \in \mathbb{N}$.

Let $z = (z_1, z_2) \in \mathcal{H}_2$. Then

$$\sum_{n=0}^{\infty} m^{2n} \|A_n A_{n-1} \cdots A_1 T z\|^2 = |z_2|^2 \sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2, \quad \text{and}$$

$$\sum_{n=0}^{\infty} m^{2n} \|A_{n+1} A_n \cdots A_2 z\|^2 = \sum_{n=0}^{\infty} m^{2n} (|a_2 a_3 \cdots a_{n+1} z_1 + \beta_{2n} z_2|^2 + |c_2 c_3 \cdots c_{n+1} z_2|^2).$$

Therefore, if we take $z_2 = 1$, then there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2$$

$$\leq M \sum_{n=0}^{\infty} m^{2n} (|a_2 a_3 \cdots a_{n+1} z_1 + \beta_{2n}|^2 + |c_2 c_3 \cdots c_{n+1}|^2), \quad (3.24)$$

for all $z_1 \in \mathbb{C}$ and all $m \in \mathbb{N}$.

If $z_1 = 0$ in (3.24), then we obtain

$$\sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2 \leq M \sum_{n=0}^{\infty} m^{2n} (|\beta_{2n}|^2 + |c_2 c_3 \cdots c_{n+1}|^2) \quad (3.25)$$

By the definition of β_{kn}, $\beta_{1,n+1} = b_1 a_2 a_2 \cdots a_{n+1} + c_1 \beta_{2n}$, so

$$\frac{\beta_{2n}}{a_2 a_3 \cdots a_{n+1}} = \frac{a_1}{c_1} \cdot \frac{\beta_{1,n+1}}{a_1 a_2 \cdots a_{n+1}} - \frac{b_1}{c_1}$$

It follows that the sequence $\{\beta_{2n}/(a_2a_3\cdots a_{n+1})\}_{n\in\mathbb{N}}$ is bounded and there exists $L>0$ such that

$$|\beta_{2n}| \le La_2a_3\cdots a_{n+1},$$

for all $n\in\mathbb{N}$. Thus, (3.25) implies that

$$\sum_{n=0}^{\infty} m^{2n}|a_1a_2\cdots a_n|^2$$

$$\le M\sum_{n=0}^{\infty} m^{2n}|c_2c_3\cdots c_{n+1}|^2 + L\sum_{n=0}^{\infty} m^{2n}|a_2a_3\cdots a_{n+1}|^2. \tag{3.26}$$

The last sum in (3.26) can be rewritten as

$$\frac{1}{a_1^2}\sum_{n=0}^{\infty} m^{2n}|a_1a_2\cdots a_{n+1}|^2 \le \frac{1}{a_1^2m^2}\sum_{n=0}^{\infty} m^{2n}|a_1a_2\cdots a_n|^2.$$

Therefore, there cannot exist $K>0$ such that

$$\sum_{n=0}^{\infty} m^{2n}|a_1a_2\cdots a_n|^2 \le K\sum_{n=0}^{\infty} m^{2n}|a_2a_3\cdots a_{n+1}|^2.$$

In view of (3.26), it follows that there exists $M'>0$ such that

$$\sum_{n=0}^{\infty} m^{2n}|a_1a_2\cdots a_n|^2 \le M'\sum_{n=0}^{\infty} m^{2n}|c_2c_3\cdots c_{n+1}|^2,$$

for all $m\in\mathbb{N}$.

In order to prove (3.23), we will assume that $z_1=1$, without loss of generality. Hence, we will prove that

$$\sum_{n=0}^{\infty} m^{2n}|a_1a_2\cdots a_n|^2$$

$$\leq M \sum_{n=0}^{\infty} m^{2n} \left(|a_1 a_2 \cdots a_n|^2 \left| 1 + \frac{\beta_{1n} z_2}{a_1 a_2 \cdots a_n} \right|^2 + |c_1 c_2 \cdots c_n z_2|^2 \right), \qquad (3.27)$$

for all $m \in \mathbb{N}$ and $z_2 \in \mathbb{C}$.

By assumption, the sequence $\{\beta_{1n}/(a_1 a_2 \cdots a_n\}_{n \in \mathbb{N}}$ is convergent, whence bounded so there exists $L' > 0$ such that $|\frac{\beta_{1n}}{a_1 a_2 \cdots a_n}| < L'$, for $n \in \mathbb{N}$. First consider $|z_2| \geq \frac{1}{2L'}$. Then

$$\sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2 \leq M' \sum_{n=0}^{\infty} m^{2n} |c_2 c_3 \cdots c_{n+1}|^2 \leq \frac{2L'M'\|W\|}{c_1} \sum_{n=0}^{\infty} m^{2n} |c_1 c_2 \cdots c_n z_2|^2,$$

so (3.27) holds. Suppose now that $|z_2| < \frac{1}{2L'}$. Fix z_2 and note that

$$\left| \lim_{n \to \infty} \frac{\beta_{1n} z_2}{a_1 a_2 \cdots a_n} \right| \leq \frac{1}{2},$$

for all $n \geq 0$. Then it quickly follows that

$$\left| 1 + \frac{\beta_{1n} z_2}{a_1 a_2 \cdots a_n} \right|^2 \geq \frac{1}{4},$$

for all $z_2 < \frac{1}{2L'}$ and all $n \geq 0$. Thus,

$$\sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2 \left| 1 + \frac{\beta_{1n} z_2}{a_1 a_2 \cdots a_n} \right|^2 \geq \frac{1}{4} \sum_{n=0}^{\infty} m^{2n} |a_1 a_2 \cdots a_n|^2.$$

Therefore (3.27) holds with

$$M = \max \left\{ 4, \frac{2L'M'\|W\|}{c_1} \right\},$$

and $\mathcal{B}_W(1,1)$ is as claimed in the statement of this theorem. $\square$

Chapter 4

Open Problems

In this final chapter, we will discuss possible areas of future study related to Deddens algebras, spectral radius algebras, and weighted shifts. We start off with an interesting question which can be asked about any operator, not just weighted shifts.

Problem 4.1 *Let* $\lambda \neq 0$*. Does* $\mathcal{B}_W = \mathcal{B}_{\lambda W}$*?*

For the Deddens algebra and the commutant this question is easily answered Indeed, for every operator $A \in \mathcal{L}(\mathcal{H})$, it can be shown that $\mathcal{D}_A = \mathcal{D}_{\lambda A}$ and $\{A\}' = \{\lambda A\}'$ whenever $\lambda \neq 0$. However, for spectral radius algebras, this is still an open question. In fact, even when $\lambda \in \mathbb{N}$, the answer to this question remains elusive. The only thing we can say for sure is the following proposition.

Proposition 4.2 *If* $k \in \mathbb{N}$ *and* $A \in \mathcal{L}(\mathcal{H})$*, then* $\mathcal{B}_A \subset \mathcal{B}_{kA}$*.*

PROOF: Let $T \in \mathcal{B}_A$. Then there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^{2n} \|A^n T x\|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A^n x\|^2, \tag{4.1}$$

for all $m \in \mathbb{N}$ and $x \in \mathcal{H}$. Let $\widehat{d}_m = \frac{m}{1+mr(kA)} = \frac{m}{1+kmr(A)}$. Some quick algebra shows that

$$\widehat{d}_m^{\,2n} \|(kA)^n y\|^2 = d_{km}^{2n} \|A^n y\|^2,$$

for all $y \in \mathcal{H}$. So, for any $k, m \in \mathbb{N}$, (4.1) implies that

$$\sum_{n=0}^{\infty} \widehat{d}_m^{\,2n} \|(kA)^n T x\|^2 = \sum_{n=0}^{\infty} d_{km}^{2n} \|A^n T x\|^2$$

$$\leq M \sum_{n=0}^{\infty} d_{km}^{2n} \|A^n x\|^2 = M \sum_{n=0}^{\infty} \widehat{d}_m^{\,2n} \|(kA)^n x\|^2$$

holds, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$. Therefore $T \in \mathcal{B}_{kA}$. □

Problem 4.3 *What do the Deddens and spectral radius algebras look like for weighted shifts of infinite multiplicity?*

In this work, we focused on weighted shifts of finite multiplicity, but did not really discuss this infinite multiplicity case. Much of the work that we did with the homomorphisms $\mathcal{B}_W(i,j) \to \mathcal{B}_W(k,l)$ can be generalized if we assume that all the weights are invertible. However, it no longer makes sense to look at dimension arguments. Furthermore, it would be nice to still consider weighted shifts whose weights are injective but might not be invertible.

Another topic of interest is weighted shifts with constant weight sequences. Let W be a weighted shift with weight sequence $A_n = A$, for all $n \in \mathbb{N}$. Upon identifying $\mathcal{H}_j$ with $\mathcal{H}$, for all $j \in \mathbb{N}$, it can be shown that $T \in \mathcal{B}_W(i,j)$ if and only if $T \in \mathcal{B}_A$. A similar statement can be made for the Deddens algebra. This means that $\mathcal{B}_W$, respectively $\mathcal{D}_W$, has a n.i.s. if and only if $\mathcal{B}_A$, respectively $\mathcal{D}_A$, does.

Problem 4.4 *For injective weighted shifts of multiplicity two, what are the possible structures for the weak closure of $\mathcal{B}_W$?*

First, for weighted shifts with a positive spectral radius, the answer to this question proved to be quite elusive. What is known is that for the unilateral shift of multiplicity two, we have that $\mathcal{B}_W(1,1) = \mathcal{L}(\mathcal{H}_1)$ is four dimensional. Meanwhile, for the weighted shift W with weights $\{A_n\}_{n\in\mathbb{N}}$ defined by

$$A_n = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{bmatrix},$$

one can quickly show that $\mathcal{B}_W(1,1)$ is three dimensional.

In Section 2.3.1, we gave examples to show that it was possible for $\mathcal{D}_W(1,1)$ to be either two dimensional or one dimensional. For both of these examples, it can be shown that $\mathcal{B}_W(1,1)$ is four dimensional. Indeed, at the end of Section 3.2, we proved that $\mathcal{B}_W(1,1)$ is four dimensional for the example given to demonstrate that $\mathcal{D}_W(1,1)$ can be two dimensional, This proof can be adapted to show $\mathcal{B}_W(1,1)$ is four dimensional for the other example as well. However, we were not able to prove that there did or did not exist injective weighted shifts with $r(W) > 0$ such that $\mathcal{B}_W(1,1)$ was either two or three dimensional.

For spectral radius algebras associated to quasinilpotent weighted shifts, we partially answered this question in Section 3.5. In particular, Theorem 3.24 tells us that there are precisely three possibilities for $\mathcal{B}_W(1,2)$ when it is nonzero. One might wonder if there exist weighted shifts which have $\mathcal{B}_W(1,2)$ consisting of operators satisfying condition 3 of Theorem 3.24. We were not able to answer this question here, so we state it as another open question.

Problem 4.5 *Does there exist a weighted shift W of multiplicity two such that $T \in \mathcal{B}_W(1,2)$ is nonzero but is not of the form given in parts (a) or (b) of Theorem 3.24?*

We do know of operators which satisfy part (a) of this theorem and we investigated these operators in Theorem 3.25. However, in that theorem, we made the additional assumption that $\{\beta_{1n}/(a_1a_2\cdots a_n)\}_{n\in\mathbb{N}}$ is a converging sequence. The last example of Chapter 2 demonstrated that this sequence may converge to ∞.

Problem 4.6 *Can Theorem 3.25 be extended to the case when the sequence*

$$\left\{\frac{\beta_{1n}}{a_1a_2\cdots a_n}\right\}_{n\in\mathbb{N}}$$

converges to ∞?

If we can, then we have found a large class of operators for which the weak closures of $\mathcal{D}_W$ and $\mathcal{B}_W$ differ. Also, it would be interesting to know if $\{\beta_{1n}/(a_1a_2\cdots a_n)\}_{n\in\mathbb{N}}$ does or does not have to converge in general.

Problem 4.7 *Does there exist a weighted shift W of multiplicity two such that W satisfies part (a) of Theorem 3.24, but $\{\beta_{1n}/(a_1a_2\cdots a_n)\}_{n\in\mathbb{N}}$ does not converge?*

If such a weighted shift exists, then one might hope that the example can be modified so that the analogue of Theorem 2.20 remains true for spectral radius algebras. Meanwhile, if we can show that such an example does not exist, then the analogue of Theorem 2.20 might not be true and this would be a very big step in that direction.

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