Rank Based Procedures for Ordered Alternative Models

Yuanyuan Shao, Joseph W. McKean

Statistics Department, Western Michigan University, Kalamazoo MI

Western Michigan University Department of Statistics

Western Michigan University Department of Statistics

Abstract

The ordered alternatives in a one-way layout with k ordered treatment levels are appropriate for many applications, especially in psychology and medicine. There is an extensive literature in this area, and many parametric and nonparametric approaches have been introduced. This study uses rank based estimators to robustify the Abelson-Tukey method. The approach extends to the two-way layout with b blocks and k treatments. One of the two statistics having maximum asymptotic local power and the greatest efficiency when the alternative hypothesis consists of a specific pattern, and extended research has been completed on detecting ordered alternatives with unknown peaks.

Introduction

The null hypothesis of interest is that there are no differences in locations. The alternatives considered that there is a trend (increasing, decreasing or umbrella) in the locations. For example, consider a randomized group experiment with five levels of the independent variables, where the treatments are doses of a drug (say, 10, 20, 30, 40 and 50 ml). With increasing doses level, the performance of the treatment tends to improve. This is the alternative of interest.

Suppose that Y_{ij} is distributed with continuous cdf $F(y-\theta_j)$ for $j=1,\ \ldots\ ,\ k$ and $i=1,\ \ldots\ ,\ n_j$ and the observations are mutually independent. The null and alternative hypotheses are

$$H_0:\ \theta_1=\cdots=\theta_k \text{ versus}$$
$$H_A:\ \theta_j=\theta_0+\theta c_j, (\theta>0;\ j=1,\ \ldots\ ,\ k),$$

where $c_1,\ \ldots\ ,\ c_k$ is a given set of constants that specifies the pattern of the alternative.

Initial empirical studies on existing methods show that the Abelson-Tukey (AT) method, though, is not robust and not more powerful than the Jonckheere-Terpstra (JT), the Spearman (SP) and the Hettmansperger-Norton (HN) nonparametric tests at normal errors for moderate sample sizes, and is even worse with heavy-tailed errors. But the AT test, is easily extended to general linear and mixed models.

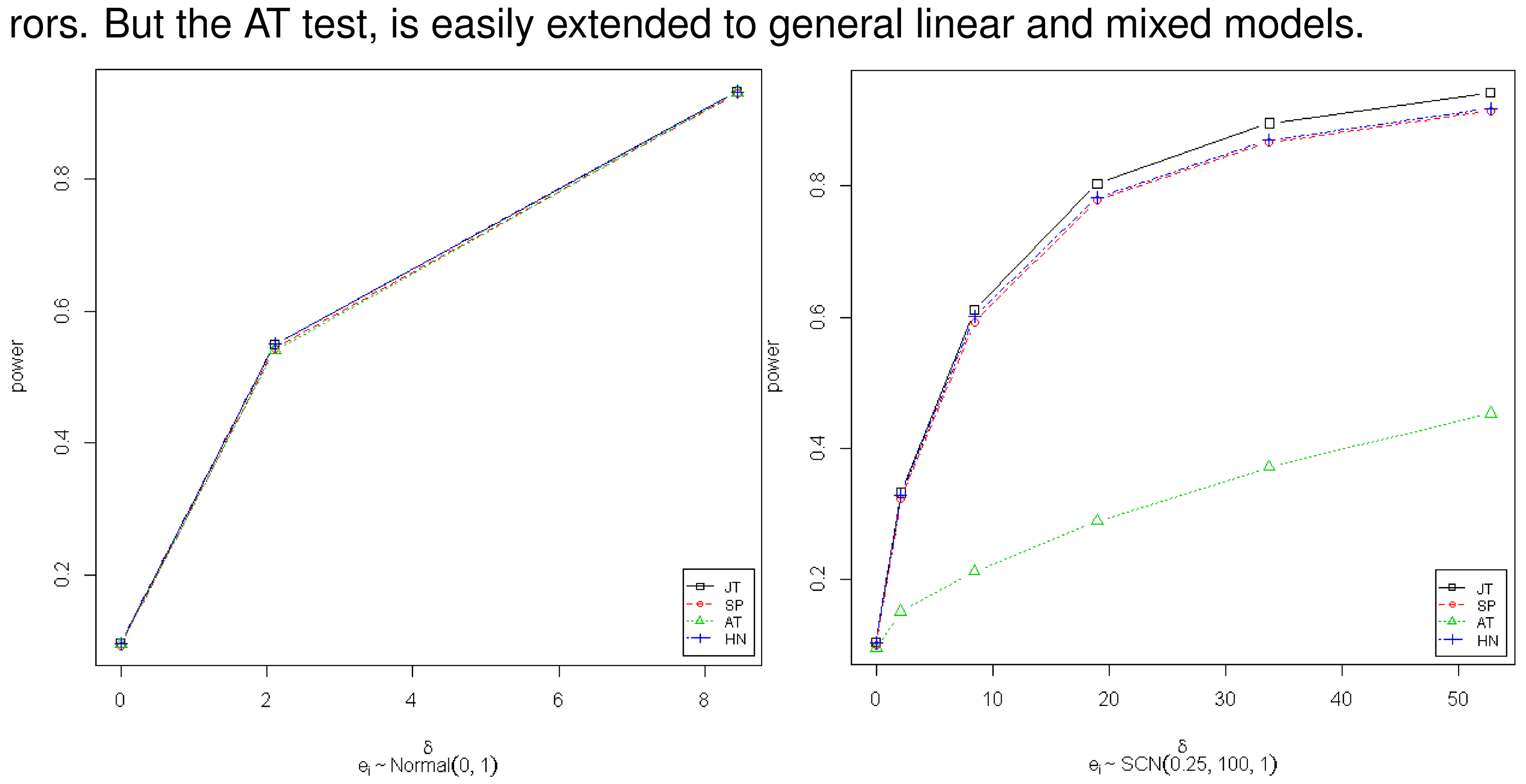

Figure 1: Empirical power curves for 10% tests

Both proposed methods, Robust Abelson-Tukey (RAT) test and Robust Shao-McKean (RSM) test, are based on the same test statistics form showed below

$$T=\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(a_j-a_i)\hat{\Delta}_{ij}$$

but with different estimates of the Δ_{ij}. a_js are $\lambda_j(c_j-\bar{c}_w)$, which maximize the Pitman efficacy of the test. $\bar{c}_w=\sum\lambda_j c_j$, $\lambda_j=\frac{n_j}{n}$, and $n=\sum_{j=1}^{k}n_j$.

Method I: RAT

A general linear model can be written as

$$\boldsymbol{Y}=\boldsymbol{\alpha}+\boldsymbol{X\beta}+\boldsymbol{e},$$

where $\boldsymbol{Y}$ is the $n\times 1$ vector of responses, $\boldsymbol{X}$ is the $n\times(k-1)$ design matrix, and $\boldsymbol{e}$ is the $n\times 1$ vector of error terms. Based on the dispersion function of Jaeckel (1972), a rank-based estimator of $\boldsymbol{\beta}$ is consistent and asymptotically normal (Hettmansperger and McKean, 2011), and can be summarized as

$$\hat{\boldsymbol{\beta}}_{\varphi}\dot{\sim}N(\boldsymbol{\beta},\ \tau_{\varphi}^2(\boldsymbol{X}'\boldsymbol{X})^{-1}), \qquad (1)$$

where τ_{φ} is the scale parameter which depends on f and the score function φ, and is given by

$$\tau_{\varphi}^{-1}=\int\varphi(u)\varphi_f(u)du, \text{ where } \varphi_f(u)=-\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}$$

Lemma 1. Given $\Delta_{ji}=\Delta_{j1}-\Delta_{i1}$, when $\sum_{j=1}^{k}a_j=0$ and $\Delta_{11}=0$, we have

$$\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(a_j-a_i)\Delta_{ji}=k\sum_{i=1}^{k}a_i\Delta_{i1}$$

The test statistic is

$$T=k\boldsymbol{a}'\hat{\boldsymbol{\beta}} \qquad (2)$$

where $\hat{\boldsymbol{\beta}}$ is a vector of the estimates of the shifts from group i to group 1, $\boldsymbol{a}=(a_1\ a_2\ \ldots\ a_k)'$. Based on (1), the asymptotic distribution of T is

$$T\dot{\sim}N(k\boldsymbol{a}'\boldsymbol{\beta},\ k^2\tau_{\varphi}^2\boldsymbol{a}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{a}), \qquad (3)$$

When $H_0:\theta_1=\cdots=\theta_k$ is true, $E(T)=0$. Under H_0

$$T^*=\frac{1}{\tau_{\varphi}}\boldsymbol{a}'\hat{\boldsymbol{\beta}}\left[\boldsymbol{a}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{a}\right]^{-\frac{1}{2}}\xrightarrow{D}Z\sim N(0,\ 1).$$

Hence, reject $H_0:\theta_1=\cdots=\theta_k$ when $T^*\geq Z_{\alpha}$.

Method II: RSM

The Mann-Whitney-Wilcoxon estimates of shifts (Δ_{ij}) showed as follows are used in RSM.

$$\hat{\Delta}_{ij}=med\{Y_j-Y_i\}$$
$$=\tau_{\varphi}\left\{\frac{1}{n_j}\sum_{m=n_i+1}^{n_{ij}}\varphi[F(Z_m)]-\frac{1}{n_i}\sum_{m=1}^{n_i}\varphi[F(Z_m)]\right\}+o_p(\frac{1}{\sqrt{n}})$$

When $H_0:\theta_1=\cdots=\theta_k$ is true, $E(T)=0$, as $n\longrightarrow\infty$,

$$T^*=\frac{T}{\tau_{\varphi}\sqrt{k\sum\sum_{i<j}(a_j-a_i)(\frac{a_j}{n_j}-\frac{a_i}{n_i})}}\xrightarrow{D}Z\sim N(0,\ 1)$$

Hence, reject $H_0:\theta_1=\cdots=\theta_k$ when $T^*\geq Z_{\alpha}$.

Under $H_n:\theta_j=\theta_0+c_j\theta n^{-1/2}, (\theta>0, j=1,\ldots,k)$, and F has a density f with $\int f^2(x)dx<\infty$,

$$\Delta_{ij}=(c_j-c_i)\theta n^{-1/2},\ \Delta_n=\boldsymbol{c}\theta n^{-1/2}$$

where $\boldsymbol{c}$ is $p\times 1$ vector of $(c_j-c_i)'s$, $p=\binom{k}{2}$.

Suppose $\hat{\Delta}_0$ is the MWW estimate of Δ (a $p\times 1$ vector of Δ_{ij}) under H_0, and $\hat{\Delta}_n$ is the MWW estimate of Δ under H_n. Since $\hat{\Delta}$ is translation equivariant, thus

$$\hat{\Delta}_n=\hat{\Delta}_0+\Delta_n, \text{ and } Var(\hat{\Delta}_n)=Var(\hat{\Delta}_0)$$

The asymptotic power of the test based on RSM is given by

$$P_{\Delta_n}[T^*\geq Z_{\alpha}]=1-\Phi(Z_{\alpha}-\theta c_{\varphi})$$

where $c_{\varphi}=\frac{\sum_{i=1}^{k}a_ic_i}{[\sum_{i=1}^{k}\frac{a_i^2}{\lambda_i}]^{1/2}}(12)^{1/2}\int f^2(x)dx$ is the Pitman efficacy of the test, which is the same as the HN procedure.

Results

Setting $k=5$, $n_j=5$, $n=25$, $\alpha=0.05$ and $p_0=0.15$. 25 random errors were generated from right skewed continimated normal distribution $SCN(.25,\ 100,\ 1)$, and were randomly assigned into 5 groups. All groups shift by noncentrality parameter δ from previous group consistently, which is calculated based on $p_0=P\{F(\delta,\ k-1,\ n-k)\geq F(1-\alpha,\ k-1,\ n-k)\}$. 10,000 simulations were run.

As following figures showing that for a 10% test, when $e_i\sim SCN(.25,\ 100,\ 1)$ the RAT and the RSM methods with the bent1 scores work better than methods with the Wilcoxon scores, since the Wilcoxon scores are recommended when the errors come from a moderate tailed distribution and the bent1 scores are recommended whent the errors come from a highly right skewed distribution. With the bent1 scores RSM works as well as HN, and RAT works even better than HN.

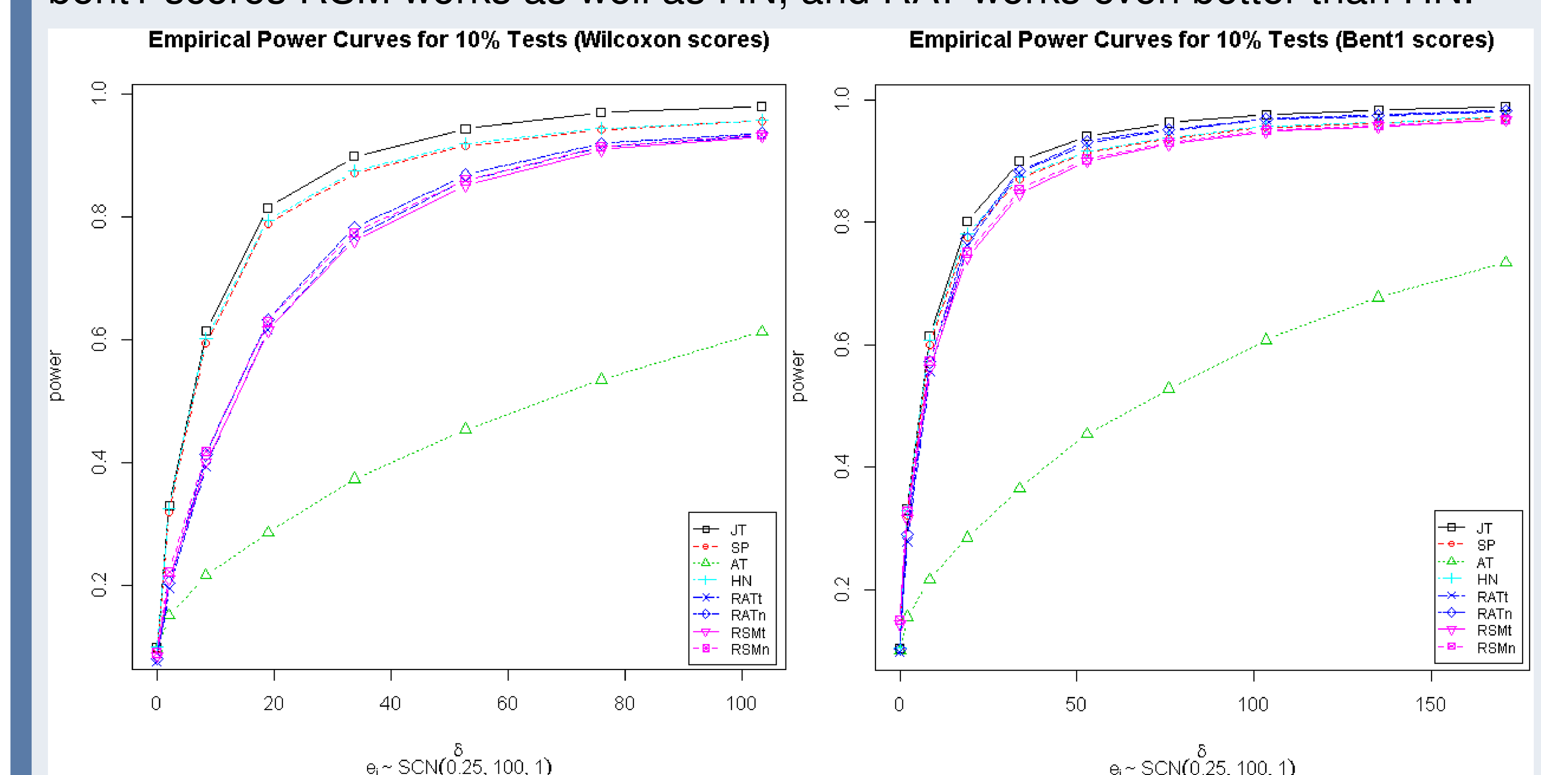

Figure 2: Simulation Results

To detect the ordered alternatives with unknown peaks, data sets with 10 groups and 16 observations in each group were generated. All groups data were generated from normal distribution with $\sigma=1$ and different μs. The values of μ were chosen from a set of numbers $\{1,\ 2,\ 3,\ \ldots,\ 10\}$, and the values of the μs depend on where the peak is located.

Table 1: Estimates of the peak by three methods

Method	peak
True peak	1 2 3 4 5 6 7 8 9 10
HN	1 2 3 4 5 6 7 8 9 10
RAT	1 2 3 4 5 6 7 8 9 10
RSM	1 2 3 4 5 6 7 8 9 10

References

[1] Abelson, R.P., Tukey, J.W., *Efficient Utilization of Non-Numerical Information in Quantitative Analysis: General Theory and The Case of Simple Order*, The Annals of Mathematical Statistics, volume 34, number 4, 1963, 1347-1369

[2] Hettmansperger, T.P. and Norton, R.M. , *Tests for Patterned Alternatives in k-Sample Problems*, Journal of the American Statistical Association, volume 82, number 397, 1987, 292-299

[3] Kloke, J.D. and McKean, J.W., *Rfit: Rank-based Estimation for Linear Models*, The R Journal, volume 4, number 2, 2012, 57-64