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A Pattern in the Lusternik-Schnirelmann Category of Rational Spaces

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by

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The Lusternik-Schnirelmann (or LS) category of a space is one less than the number of contractible open sets with which we can cover the space. If we look at the LS categories of the skeleta of a CW complex, we find a sequence of dimensions where the LS category changes. I discuss whether certain of these "category sequences" (defined in the paper, "Categorical Sequences", by Nendorf, Scoville, and Strom) could be realized as the categorical sequences of rational spaces. I first reduce from looking at all rational spaces to only Postnikov sections of finite wedges of spheres. Using the Leray-Serre Spectral Sequence, I show that certain sequences can be rationally realized with a Postnikov section of a wedge of enough spheres. I finally conjecture that these are the only sequences in this pattern to be rationally realizable.

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Chapter 1

Introduction

The Lusternik-Schnirelmann category of a space X is one less than the smallest number of pieces contractible in X into which the space can be divided. The LS-category of a manifold gives a lower bound for the number of critical points possible for a function on that manifold. This can be useful information to have, but it is not so easy to find.

One restriction found early in the study of LS category says that if X is an $(n-1)$-connected CW complex, then

$$\operatorname{cat}(X) \;\leq\; \frac{\dim(X)}{n}, \tag{1.1}$$

for $n \geq 1$. This may be shown using the Whitehead definition of category (see [1]).

In [9], Nendorf, Scoville, and Strom define a sequence for a space X, denoted σ_X, which indicates at which dimensions the changes in category occur. These authors called this the *categorical sequence* for X, but this term is in use already, so we will deem this sequence for X to be its *category sequence*. Specifically, $\sigma_X(k)$ is the smallest dimension n such that the relative category of the n-skeleton X^n in X is at least k. We will easily see that when we get $\sigma_X(k) = n$, we know

$$\begin{aligned} \operatorname{cat}_X(X^{n-1}) &= k-1 \qquad \text{and} \\ \operatorname{cat}_X(X^n) &= k. \end{aligned}$$

This sequence has the following property, known as "superadditivity":

$$\sigma_X(k+l) \;\; \geq \;\; \sigma_X(k) + \sigma_X(l)$$

for all $k, l \geq 0$, which we can see quickly gives us the same result found in the inequality in (1.1). Given the dimension of X is finite, let X be $(n-1)$-connected and let the LS category of X be $k \in \mathbb{Z}$. Then the category sequence for X will have k finite entries with the first one at least n. So from only the superadditivity we would have

$$\sigma_X(k) \;\; \geq \;\; k \cdot \sigma_X(1) \; \geq \; k \cdot n,$$

which directly implies the inequality in (1.1) since $\dim(X) \geq \sigma_X(k)$.

The authors in [9] also proved that we can use the cup product and the superadditivity of σ_X to ease the calculation of the LS category of a space. We use this paper [9] as our starting point, and find further restrictions on the category sequence of a rational space. We want to know what sequences $(a,\, b,\, c,\, ?,\, \ldots)$ can be rationally realized.

We restrict our attention to spaces X which satisfy

$$\sigma_X(3) \;\; = \;\; \sigma_X(1) + \sigma_X(2).$$

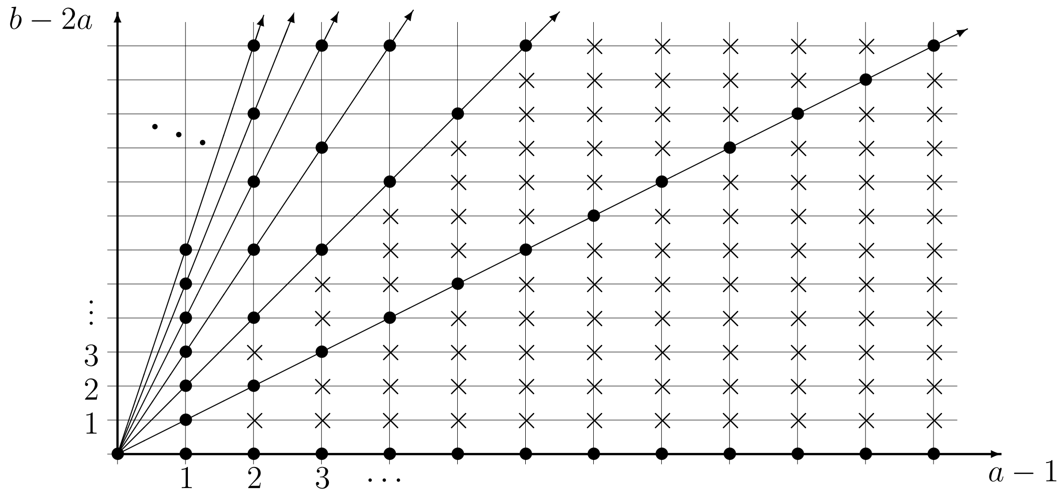
That is, we study category sequences of the form $(a,\, b,\, a+b,\, ?,\, \ldots)$. These tell us about the realizability of other sequences too; given sequence $(a,\, b,\, a+b,\, ?,\, \ldots)$ is realized by a rational space X, we can realize $(a,\, c,\, a+b,\, ?,\, \ldots)$ for all c such that $2a \leq c \leq b$ by taking $X \vee (S^a \times S^{c-a})$.

For a sequence of the form $(a,\, b,\, a+b,\, ?,\, \ldots)$ we refer to its "slope" as the quotient

$$m(a,b) \;\; = \;\; \frac{b-2a}{a-1}.$$

This $m(a,b)$ is the slope in Figure 1.1. In this chart, each point represents a category sequence. For example, if we take the point (3, 6), this is of slope $6/3 = 2$. This point corresponds to a sequence with $a-1=3$ and $b-2a=6$, which is the category sequence (4, 14, 18). Or, if we look at the point with non-integer slope, such as (2, 3), this corresponds to a sequence with $a-1=2$ and $b-2a=3$. That is, we have the sequence (3, 9, 12).

Figure 1.1: Graph of Sequences $(a, b, a+b)$



In this thesis, we look to answer the question of what sequences are rationally realizable. We have found that the slope determines realizability at least to a point. Let σ be the category sequence $(a,\ b,\ a+b,\ ?,\ \ldots)$.

Theorem 5.1.1. *If σ has slope $m(a,b) \in \mathbb{Z}$, then the sequence is rationally realizable.*

Theorem 6.2.1. *If σ has slope $m(a,b) \in (0,1)$, then the sequence is not rationally realizable.*

Theorem 6.3.1. *If σ has slope $m(a,b) \in (1,2)$, then the sequence is not rationally realizable.*

We convey the realizability of a slope in Figure 1.1 by labeling rationally unrealizable slopes by $\times$ symbols and rationally realizable slopes by $\bullet$ symbols.

These results about realizability by slope have led us to conjecture the following:

Conjecture 1.0.1. *The sequence* $(a,\, b,\, a+b,\, ?,\, \ldots)$ *is rationally realizable if and only if the slope* m *is a nonnegative integer.*

Note that the sequence $(3, 7, 10, \ldots)$ is realizable, but not rationally, as it is the sequence for Sp(2).

Before we prove any of these results, we simplify our work. In Theorem 4.3.1 we narrow our view of rational spaces by showing that any rational space X with $\sigma_X = (a,\, b,\, a+b,\, ?,\, \ldots)$ will have the same category sequence up to category three as a Postnikov section of some wedge of spheres taken at dimension $\sigma_X(2) - 1$.

We are working with rational spaces, which allows us to view the homotopy groups of a space directly through its models. In [2], Félix, Halperin, and Thomas present a theorem (which we state as Theorem 2.2.4) which gives us, for a space X, a bijection between a basis for $\pi_*(X) \otimes \mathbb{Q}$ and the set of generators V, where $(\Lambda V, d)$ is the Sullivan model for X. We use this bijection throughout this thesis to relate a nice basis for the homotopy groups of X (made up of "basic products") and the generators in our model.

Using the Witt formula and the Hilton-Milnor theorem, we are able to count basic products to show that the family of category sequences which have nonnegative integer slopes are rationally realizable with enough spheres.

We then look at sequences of slope strictly between 0 and 1 and see from their models that these sequences are not rationally realizable. Following that, we use our understanding of basic products and some results from [2], including Corollary 2.2.6 (which relates the differential to the Whitehead product), to get information about the differential in our model so we can show the family of category sequences with slopes strictly between 1 and 2 are not rationally realizable.

In order to show that a non-integer slope is not realizable, we have reduced the problem to a linear algebra check. We look at two-fold products in the model whose corresponding basic products can be formed from all distinct factors. We then form a matrix, M, which represents the restriction of the differential to $\Lambda^2 V$, as well as certain vectors, τ, which represent sums of three-fold products in the model which are also cycles. We find that the slope is realizable if and only if the matrix equation $M\overline{x} = \tau$ has no solutions. (See Section 6.1.)

Chapter 2

Preliminaries

In this chapter we give the basics of localization and rationalization of spaces, giving some examples. We define models and discuss their relationship to homotopy groups and cohomology. We define the Lusternik-Schnirelmann category and the LS category sequence and give a brief rundown of basic products and a Witt formula we use to prove Theorem 5.1.1.

2.1 Localization and Rationalization of Spaces

Since we would like to talk about rational spaces, we need to discuss rationalization and localization. Here we give the basic definitions of localization and rationalization, using [7] and [1] as primary sources.

We begin with a fixed set of primes $P \subseteq \mathbb{Z}$. Then the *localization of the integers at* P is given by

$$\mathbb{Z}_P \;=\; \left\{ \frac{a}{b} \mid (a,b) = 1 \text{ and } p \nmid b \;\; \forall p \in P \right\} \subseteq \mathbb{Q}.$$

For example, if we take $P = \{2\}$, then we get

$$\mathbb{Z}_P \;=\; \left\{ \frac{a}{b} \mid (a,b) = 1 \text{ and } b \text{ is odd } \right\}.$$

It is also easy to see that if P is the set of all primes, then $\mathbb{Z}_P$ is the integers: $\mathbb{Z}_{\{\text{all primes}\}} = \mathbb{Z}$. Also notice that $\mathbb{Z}_\emptyset = \mathbb{Q}$. So rationalization is localization with $P = \emptyset$.

We use this to build up to more general localization. Given an abelian group

G, we can localize this at P by tensoring with $\mathbb{Z}_P$. We write $G_P = G \otimes \mathbb{Z}_P$ for the *localization of* G *at* P.

A simply-connected space X is a P*-local space* (or *rational* when $P = \emptyset$) when its homotopy groups are $\mathbb{Z}_P$-modules. Further, a map $\rho : X \to Y$ into a P-local space Y is a P*-localization map* (or *rationalization map* for $P = \emptyset$) if it induces a bijection $\rho^* : [Y, W] \to [X, W]$ for all P-local spaces W.

We also have this important result from Sullivan ([10]):

Theorem 2.1.1. *1. Every simply-connected space X admits a P-localization map $X \to X_P$, where X_P is the P-localization of X at P.*

2. Further, this localization is functorial in that given a map $f : X \to Z$ of spaces, we get induced maps between their respective localizations such that the diagram in Figure 2.1 commutes up to homotopy.

Figure 2.1: Induced Map on Localizations

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_X"'] & Y \arrow[d, "\rho_Y"] \\ X_P \arrow[r, "f_P"'] & Y_P \end{tikzcd}$$

Note too that if ρ_X is a rationalization map then it induces algebraic rationalization maps $\pi_n(X) \to \pi_n(X_\mathbb{Q}) \cong \pi_n(X) \otimes \mathbb{Q}$ and $H_n(X) \to H_n(X_\mathbb{Q})$ ([7]).

2.2 DGAs and Sullivan Models

A *differential graded algebra*, or *DGA*, is a graded algebra A with a differential $d : A^n \to A^{n+1}$ for all n such that $d^2 = 0$ and

$$d(ab) = d(a)b + (-1)^{|a|} a d(b).$$

A DGA A is a *commutative DGA*, or *CDGA*, if it is graded-commutative. That is, for any $a, b \in A$, we get $ab = (-1)^{|a||b|}ba$.

2.2.1 Models for Spaces

We will be looking at models of spaces to help understand the spaces themselves; in particular, we will use Sullivan models. We refer to "Rational Homotopy Theory" [2] for the basic properties we use about Sullivan models. Let ΛV be the exterior algebra of a set V.

Definition 2.2.1. A simply-connected *minimal Sullivan algebra* over $\mathbb{Q}$ is a commutative differential graded algebra $\mathcal{A} = \Lambda V$ over $\mathbb{Q}$ such that

- $\mathcal{A}^0 \cong \mathbb{Q}$ and $\mathcal{A}^1 = 0$, and we have
- the image of the differential is contained in the span of two-fold and higher products.

We will write *CDGA* for the category of commutative differential graded algebras.

Definition 2.2.2. For a path-connected space X, a *Sullivan model for X* is a quasi-isomorphism

$$m : (\Lambda V, d) \longrightarrow A_{PL}(X)$$

from Sullivan algebra $(\Lambda V, d)$, where A_{PL} is a functor

$$A_{PL} : \ Spaces \longrightarrow CDGA.$$

Note that we will call ΛV the Sullivan model for X. Also, we have

$$H(\Lambda V, d) \cong H^*(X, \mathbb{Q}),$$

as we do with other commutative models.

In [2, Proposition 12.1], we have our existence theorem:

Theorem 2.2.3. *Every path-connected commutative cochain algebra has a Sullivan model.*

2.2.2 Models and Homotopy Groups

There is a bracket

$$\langle\ ;\ \rangle : V \otimes \pi_*(X) \to \mathbb{Q} \tag{2.1}$$

such that we get the following theorem (see [2, p. 172]):

Theorem 2.2.4. *Given the minimal Sullivan model $\mathcal{M}(X)$ of a space X of finite type, if we write*

$$\mathcal{M}(X) \quad = \quad (\Lambda V,\, d),$$

then the bracket in (2.1) is nondegenerate, so we get the following isomorphism φ:

$$\varphi : (\pi_*(X))^* \xrightarrow{\quad\cong\quad} V.$$

So if $\mathcal{B}$ is a basis for $\pi_*(X)$ (with X again of finite type), then we write $\mathcal{B}^*$ for the dual basis for $\big(\pi_*(X)\big)^*$. Further, we have $\mathcal{C} = \varphi(\mathcal{B}^*)$ is a basis for V.

For $v \in \mathcal{C}$ and $\mu \in \mathcal{B}$, we have

$$\langle v; \mu\rangle \quad = \quad \begin{cases} 1, & \text{if } v = \varphi(\mu^*), \\ 0, & \text{else.} \end{cases} \tag{2.2}$$

2.2.3 Interpreting the Differential

Given two generators $\mu, \eta \in \pi_*(X)$, we can take a Whitehead product in two ways (see [8] for details on Whitehead products). The following is the relationship between these two ways:

$$[\mu, \eta] \quad = \quad (-1)^{|\mu||\eta|}[\eta, \mu]. \tag{2.3}$$

Let X be a space with minimal Sullivan model $(\Lambda V, d)$. Then from [2], we have a bracket

$$\langle\ ;\ ,\ \rangle : \Lambda^2 V \times \pi_*(X) \times \pi_*(X) \quad \longrightarrow \quad \mathbb{Q}$$

given by

$$\langle v \wedge w; \mu, \eta\rangle \quad = \quad \langle v; \eta\rangle\langle w; \mu\rangle + (-1)^{|w||\mu|}\langle v; \mu\rangle\langle w; \eta\rangle,$$

where the two-fold bracket is defined in (2.2).

We also get the following showing how the differential fits in:

Theorem 2.2.5. ***([2, Proposition 13.16])*** *Given $v \in V$ and $\mu, \eta \in \pi_*(X)$, we get*

$$\langle d_1 v; \mu, \eta\rangle \quad = \quad (-1)^{|\mu|+|\eta|-1}\langle v; [\mu, \eta]\rangle,$$

where $[\ ,\ ]$ indicates the Whitehead product, and d_1 is the quadratic part of the differential.

Corollary 2.2.6. *If X is a wedge of spheres, then given $v \in V$ and $\mu, \eta \in \pi_*(X)$, we get*

$$\langle dv; \mu, \eta\rangle \quad = \quad (-1)^{|\mu|+|\eta|-1}\langle v; [\mu, \eta]\rangle.$$

This is true because if X is a wedge of spheres, the differential is the same as its quadratic part.

Let us look at the model for a small wedge of spheres.

Example 2.2.7. Let $W = S^3 \vee S^3$. Then the model for W must have generators x_3, y_3, where subscripts denote dimension. Further, it will have generator u_5 and differential d such that $du_5 = x_3 y_3$.

In dimension 7 we have v_7, w_7, with $dv_7 = x_3 u_5$ and $dw_7 = y_3 u_5$. The model will continue to have generators in all odd dimensions, and the images of these generators will be two-fold products which are cycles. □

2.3 Lusternik-Schnirelmann Category

We first define the LS-category of a map by how few pieces we can break up the domain into so that the restrictions are null-homotopic. That is,

Definition 2.3.1. The *Lusternik-Schnirelmann category* of a map $f : X \to Y$, denoted $\mathrm{cat}(f)$, is the smallest n such that X has an open cover

$$X = X_0 \cup X_1 \cup \cdots \cup X_n,$$

where each restriction of f to X_i is null-homotopic.

In the same vein we define the LS-category of a space as follows:

Definition 2.3.2. The *Lusternik-Schnirelmann category* (or *LS category*) of a space X, denoted $\mathrm{cat}(X)$, is the smallest n such that X has an open cover

$$X = X_0 \cup X_1 \cup \cdots \cup X_n,$$

where each inclusion $X_i \hookrightarrow X$ is homotopic to the constant map.

Notice that the category of a space is the category of the identity map on that space. Also, if we have a subspace $A \subseteq X$, then we define

Definition 2.3.3. The *relative category of A in X* is the LS-category of the inclusion map $i : A \to X$ and is denoted $\mathrm{cat}_X(A)$.

We list here a couple of important and easy results on category which can be found in [1]:

Theorem 2.3.4. *1. Given a map $f : X \to Y$ of spaces, we have*

$$\mathrm{cat}(f) \leq \min\{\mathrm{cat}(X), \mathrm{cat}(Y)\}.$$

2. Given two maps, f and g, we have

$$\mathrm{cat}(f \circ g) \leq \min\{\mathrm{cat}(f), \mathrm{cat}(g)\}.$$

3. If Y is a retract of X, that is if there exist maps f and g such that

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[rr, bend left, "\simeq \mathrm{id}_Y"] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

then $\mathrm{cat}(Y) \leq \mathrm{cat}(X)$.

One of the major appeals of LS category is that it is a homotopy invariant, which is shown in [1] (it follows easily from Lemma 1.29).

Now we define the cup length, which will help bound the LS category.

Definition 2.3.5. Given a space X, the *cuplength of* X *with* $\mathbb{Q}$ *coefficients* is the least $k \in \mathbb{Z}$ (or infinity) such that every $(k+1)$-fold cup product is zero in $\tilde{H}^*(X;\mathbb{Q})$. Write $\mathrm{cup}(X)$ for the cup length of X with rational coefficients.

Then [1] gives us the following theorem:

Theorem 2.3.6. *For spaces X and Y, and any map $f : X \to Y$, we have*

1. $\mathrm{cat}(X) \geq \mathrm{cup}(X)$*, and*
2. $\mathrm{cat}(f) \geq \mathrm{cup}(\mathrm{Im}(H^*(f)))$.

Here $\mathrm{Im}(H^*(f))$ is the image of the induced map in cohomology, $H^*(f) : H^*(Y) \to H^*(X)$.

We also can bound the category of a mapping cone C_f by one more than the category of the target of f. That is, given $f : X \to Y$, we have

$$\mathrm{cat}(C_f) \quad \leq \quad \mathrm{cat}(Y) + 1.$$

2.3.1 Skeleta

We are not just looking at the LS category of entire spaces in this thesis, rather we are interested in finding which dimensions have a change in LS category by looking at the CW skeleta of the space. First we have the n-skeleton for a space X.

Definition 2.3.7. An *n-skeleton* for a space X is a map $i : X^n \to X$ from CW complex X^n such that i is an n-equivalence and X^n is n-dimensional up to homotopy.

We can build the $(n+1)$-skeleton from the n-skeleton. Given maps $\alpha_n : \bigvee S^n \to X^n$, we get

$$X^{n+1} = X^n \cup_{\alpha_n} \bigvee D^{n+1}. \tag{2.4}$$

We also get the following properties of a skeleton (see [9, Lemma 1.2]):

Theorem 2.3.8. *Given $i : A \to X$ a map of simply-connected spaces, we have i is an n-skeleton if and only if*

1. *$H^k(A) = 0$ for $k > n$ and any coefficients, and*
2. *the induced map $H^k(i) : H^k(X) \to H^k(A)$ is an isomorphism for $k < n$ and injective for $k = n$ in any coefficients.*

Since we are working only with rational spaces in this thesis, we will also want a notion of rational n-skeleton.

Definition 2.3.9. If X is a simply-connected rational space, then $i : X^n \to X$ is an *n-skeleton* for X if X^n is a simply-connected rational space such that

1. $H^k(X^n; \mathbb{Q}) = 0$ for $k > n$, and
2. the induced map $H^k(i) : H^k(X; \mathbb{Q}) \to H^k(X^n; \mathbb{Q})$ is an isomorphism for $k < n$ and injective for $k = n$.

2.3.2 LS Category Sequence

The category sequence of a space indicates the dimensions at which the LS category changes. Our definition for this sequence is given by

Definition 2.3.10. The *category sequence* of a space X is the function $\sigma_X : \mathbb{N} \to \mathbb{N} \cup \infty$ defined by

$$\sigma_X(k) = \begin{cases} \min\{n \mid \mathrm{cat}_X(X^n) \geq k\}, & \text{if it exists, } \textit{and} \\ \infty, & \text{if the minimum does not exist.} \end{cases}$$

We will generally write $\sigma_X = (\sigma_X(1), \sigma_X(2), \sigma_X(3), \ldots)$ for the sequence.

We know from (2.4) that we can nontrivially add at most one contractible piece to X^{n+1} from X^n. Thus

$$\mathrm{cat}_X(X^n) \le \mathrm{cat}_X(X^{n+1}) \le \mathrm{cat}_X(X^n) + 1.$$

So if we have $\sigma_X(k) = n < \infty$ for some $n \in \mathbb{N}$, then we know that $\mathrm{cat}_X(X^{n-1}) < k$, but $\mathrm{cat}_X(X^n) \ge k$. That is, there is a jump in category from X^{n-1} to X^n as such:

$$\begin{array}{rcll} \mathrm{cat}_X(X^{n-1}) & = & k-1 & \qquad \text{and} \\ \mathrm{cat}_X(X^n) & = & k. & \end{array}$$

The following theorem from [9] is quite useful in calculating category sequences:

Theorem 2.3.11. *For a space X,*

1. $\sigma_X(k+l) \ge \sigma_X(k) + \sigma_X(l)$ *for all* $k, l \ge 0$*,*
2. *if X is simply-connected and $\sigma_X(k) = n$, then there exists some coefficient group A such that*

$$H^n(X;A) \neq 0.$$

3. *if $\sigma_X(k+l) = \sigma_X(k) + \sigma_X(l)$ and X is simply-connected, then there exist coefficient groups A and B such that the cup product*

$$H^{\sigma_X(k)}(X;A) \otimes H^{\sigma_X(l)}(X;B) \longrightarrow H^{\sigma_X(k+l)}(X;A \otimes B)$$

is nontrivial.

For our purposes, we will be concerned only with rational spaces, so [9] says that the coefficient group A in Theorem 2.3.11(2) (and similarly those in part (3) as well) is given by

$$\pi_n\Big(X/X^{n-1}\Big) \cong \bigoplus \mathbb{Q}.$$

This allows us to apply Theorem 2.3.11 for cohomology with $\mathbb{Q}$ coefficients.

Analogous to our category sequence of a space, we define the following sequence of an algebra:

Definition 2.3.12. For a commutative graded algebra A, the *product length sequence* of A is defined for $k \in \mathbb{Z}^{\geq 0}$ by

$$\sigma_A(k) \quad = \quad \inf\{n \mid \text{ there exist nontrivial } k\text{-fold products in } A^n\}.$$

Let us define another sequence related to the LS-category of X. We write

$$\epsilon_X(k) = \inf\{n \mid \exists \text{ class} \in H^n(X;\mathbb{Q}) \text{ rep'd in the model by sums of } k\text{-fold prods}\}.$$

for the *category weight sequence.*

Given any two sequences σ_1 and σ_2, we will write $\sigma_1 \leq \sigma_2$ if and only if $\sigma_1(k) \leq \sigma_2(k)$ for all $k \geq 0$. Then we have the following results from [9] which help to find a category sequence:

Theorem 2.3.13. *For a space X, we have the following:*

1. *If X is $(a-1)$-connected but not a-connected then $\sigma_X(0) = 0$ and $\sigma_X(1) = a$.*
2. *The finite values of σ_X are strictly increasing.*
3. *For any ring R, we have $\sigma_X \leq \sigma_{H^*(X;R)}$.*
4. *If X is a simply-connected rational space, and A is any model for X, then $\sigma_X \geq \sigma_A$.*

From this we get the following inequalities for simply-connected rational X:

$$\sigma_A \leq \sigma_X \leq \epsilon_X \leq \sigma_{H^*(X;R)},$$

where R is any ring and A is any rational model for the space X.

What we are interested in for this thesis is when sequences line up with spaces. We use the following definition:

Definition 2.3.14. A given sequence σ is called *(rationally) realizable* if there exists a (rational) space X such that $\sigma_X = \sigma$.

Now we focus our interest on spaces which have category sequences beginning $(a, b, a+b)$. A space X with $\sigma_X = (a, b, a+b, ?, ?, \ldots)$ will have

$$\begin{aligned}
0 &= \mathrm{cat}_X(X_0) = \mathrm{cat}_X(X_1) = \cdots = \mathrm{cat}_X(X_{a-1}), \\
1 &= \mathrm{cat}_X(X_a) = \mathrm{cat}_X(X_{a+1}) = \cdots = \mathrm{cat}_X(X_{b-1}), \\
2 &= \mathrm{cat}_X(X_b) = \mathrm{cat}_X(X_{b+1}) = \cdots = \mathrm{cat}_X(X_{a+b-1}), \text{ and} \\
3 &= \mathrm{cat}_X(X_{a+b}).
\end{aligned}$$

Further, we will look at these category sequences through the lens of the "slope" given by

$$m(a,b) = \frac{b-2a}{a-1}. \tag{2.5}$$

2.4 Ganea Construction

In the Ganea construction, we begin with a space X and the path fibration on X, and then we build new "Ganea" spaces and fibrations. The reason we are interested is that there exists a section of the kth Ganea fibration if and only if $\mathrm{cat}(X) \leq k$ (see [1, Proposition 1.57] for a proof).

We lay out the detailed construction below:

1. Let
$$F_0(X) \xrightarrow{i_0} G_0(X) \xrightarrow{p_0} X$$
be the path fibration on X, $\Omega X \to \mathcal{P}X \to X$.

2. Suppose we have fibration sequences
$$F_k(X) \xrightarrow{i_k} G_k(X) \xrightarrow{p_k} X$$
constructed for $k \leq n$. Let $C(i_n) = G_n(X) \cup_{i_n} C(F_n(X))$ be the mapping cone of i_n. Now define $q_n : C(i_n) \to X$ by
$$q_n(x) = \begin{cases} p_n(x), & \text{if } x \in G_n(X), \\ *, & \text{if } x = [y,t] \in C(F_n(X)) \end{cases}$$
(Here $[y,t]$ is the equivalence class in the cone.)

3. Next we convert q_n to a fibration $p_{n+1} : G_{n+1}(X) \to X$ such that the following diagram commutes:

$$\begin{tikzcd} G_n(X) \arrow[r] \arrow[dr, bend right, "p_n"'] & C(i_n) \arrow[r, "\simeq"] \arrow[d, "q_n"] & G_{n+1}(X) \arrow[dl, bend left, "p_{n+1}"] \\ & X & \end{tikzcd}$$

If we repeat this process, we get the following commutative diagram:

$$\begin{tikzcd} F_0(X) \arrow[r] \arrow[d, "i_0"] & F_1(X) \arrow[r] \arrow[d, "i_1"] & \cdots \arrow[r] & F_n(X) \arrow[r] \arrow[d, "i_n"] & \cdots \\ G_0(X) \arrow[r] \arrow[d, "p_0"] & G_1(X) \arrow[r] \arrow[d, "p_1"] & \cdots \arrow[r] & G_n(X) \arrow[r] \arrow[d, "p_n"] & \cdots \\ X \arrow[r, Leftrightarrow, "\mathrm{id}_X"] & X \arrow[r, Leftrightarrow] & \cdots \arrow[r, Leftrightarrow] & X \arrow[r, Leftrightarrow, "\mathrm{id}_X"] & \cdots \end{tikzcd}$$

The following are two results we will use about this construction. First we see what the fibers of each of these fibrations looks like. Ganea proves this in [3].

Theorem 2.4.1. *The Ganea fibers $F_k(X)$ are given by*

$$F_k(X) \ \simeq \ (\Omega X)^{*(k+1)},$$

where $(\Omega X)^{*(k+1)}$ is the join of $k+1$ copies of ΩX.

The next theorem is used in the proof of Lemma 4.2.2.

Theorem 2.4.2. *For a map $f : X \to Y$, we have that $\mathrm{cat}(f) \leq n$ if and only if there exists a lift $\lambda : X \to G_n(Y)$ such that*

$$\begin{tikzcd} & G_n(Y) \arrow[d, "p_n"] \\ X \arrow[ur, bend left, "\lambda"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

2.5 Basic Products and a Witt Formula

By the Hilton-Milnor theorem (see Theorem A in [6]), we can write the nth homotopy group of a wedge of spheres W as a direct sum

$$\pi_n(W) \;=\; \bigoplus_{i=1}^{\infty} \pi_n(S^{k_i}),$$

for specific $k_i \geq 2$. The dimensions of these spheres come from what are called "basic products". These basic products are an ordered subset of the Whitehead products. Each basic product has a *weight* which indicates how "long" the product is. We start with some number of weight one basic products; for a wedge of spheres these products of weight one are in one-to-one correspondence with the sphere summands of the wedge. These are the factors we could count to determine the weight of other basic products.

This is an ordered set of products, and the ordering may be arbitrarily chosen among basic products of equal weight, but all basic products of higher weight must be ordered larger than those of lower weight. Now, say δ is a basic product of weight bigger than one. Then we must have $\delta = [\mu, \eta]$, where both μ and η are themselves basic products and $\mu > \eta$. Furthermore, in order to be basic, if μ is of weight bigger than one, then $\mu = [\mu_1, \mu_2]$ and $\eta \geq \mu_2$.

For example, suppose we have three weight one basic products and call them α, β, γ with ordering

$$\alpha < \beta < \gamma.$$

Then the weight two basic products are

$$[\beta, \alpha],\ [\gamma, \alpha],\ \text{and}\ [\gamma, \beta].$$

The weight three basic products are then products of weight two basic products with weight one basic products:

$$[[\beta, \alpha], \alpha],\ [[\beta, \alpha], \beta],\ [[\beta, \alpha], \gamma],\ [[\gamma, \alpha], \alpha],\ [[\gamma, \alpha], \beta],\ [[\gamma, \alpha], \gamma],$$
$$[[\gamma, \beta], \beta],\ \text{and}\ [[\gamma, \beta], \gamma].$$

Some of the weight four basic products are products of weight three basic products with weight one basic products, these are listed here:

$$\begin{gathered}[[[\beta,\alpha],\alpha],\alpha],\ [[[\beta,\alpha],\alpha],\beta],\ [[[\beta,\alpha],\alpha],\gamma],\ [[[\beta,\alpha],\beta],\beta],\ [[[\beta,\alpha],\beta],\gamma],\\ [[[\beta,\alpha],\gamma],\gamma],\ [[[\gamma,\alpha],\alpha],\alpha],\ [[[\gamma,\alpha],\alpha],\beta],\ [[[\gamma,\alpha],\alpha],\gamma],\ [[[\gamma,\alpha],\beta],\beta],\\ [[[\gamma,\alpha],\beta],\gamma],\ [[[\gamma,\alpha],\gamma],\gamma],\ [[[\gamma,\beta],\beta],\beta],\ [[[\gamma,\beta],\beta],\gamma],\text{ and }[[[\gamma,\beta],\gamma],\gamma].\end{gathered}$$

The rest of the weight four basic products are of two weight two basic products. Let us assume we have been listing our products in increasing order.

$$[[\gamma,\alpha],[\beta,\alpha]],\ [[\gamma,\beta],[\beta,\alpha]],\text{ and }[[\gamma,\beta],[\gamma,\alpha]].$$

If $\delta = [\mu,\eta]$ where μ and η are basic with $\mu > \eta$, but no other conditions, then we call δ a *simple product*. We will later want to rewrite some non-basic but simple products in terms of basic ones. We use the Jacobi identity (given in (2.6) to do this rewriting.

$$[[\mu,\eta],\nu] \;=\; -(-1)^{|\mu||\nu|}\Big((-1)^{|\eta||\mu|}[[\eta,\nu],\mu] + (-1)^{|\nu||\eta|}[[\nu,\mu],\eta]\Big) \tag{2.6}$$

The Hilton-Milnor theorem breaks down the homotopy groups as sums generated by basic products. The Witt formula ([4, Theorem 11.2.2]) lets us count those basic products. Together we are able to count generators of the homotopy groups of our wedge of spheres, which we use to prove Theorem 5.1.1:

Theorem 2.5.1. *The number of n-fold basic products on r generators is given by*

$$p_n(r) \;:=\; \frac{1}{n}\sum_{d|n}\mu(d)r^{n/d},$$

where μ is is the Möbius function defined by

- $\mu(1) = 1$,
- *μ of any square is zero, and*
- *μ of a product of k distinct primes is $(-1)^k$.*

For example, if we take r to be three and we look for four-fold basic products, we find that we have

$$\begin{aligned} p_4(3) &= \frac{1}{4}\sum_{d|4}\mu(d)3^{4/d} \;=\; \frac{1}{4}\Big(3^4 - 3^2 + 0\Big) \\ &= \frac{1}{4}(81-9) \;=\; 72/4 \;=\; 18 \end{aligned}$$

of them, which is the number we listed out above (fifteen from weight three products with weight one products, and three more from two weight two products.)

Chapter 3

Postnikov Fibration and the Leray-Serre Spectral Sequence

We will assume throughout this thesis that our spaces are rational spaces.

Given a wedge

$$W = \bigvee_{\alpha} S^{n_\alpha},$$

and $a = \min\{n_\alpha\}$, we define a Postnikov section, $P = P_{b-1}(W)$, where $b \geq 2a$ and $b - a \geq \max\{n_\alpha\}$. Then let F be the fiber of the Postnikov map $W \to P$.

In Section 3.2, we will apply the Leray-Serre spectral sequence to this fibration sequence.

3.1 Cohomology Lines Up With Homotopy

Lemma 3.1.1. *The cohomology of the fiber matches the homotopy groups of the wedge in the range of $b-1$ through $2b-4$; that is*

$$H^k(F) \cong \big(\pi_k(W)\big)^*$$

for all k such that $b - 1 \leq k \leq 2b - 4$.

Proof. To see this lemma is true, we start by noting that because of our fiber sequence, we have $\pi_k(P) \cong \pi_k(W)$ for $k < b - 1$, and we also get $\pi_k(W) \cong \pi_k(F)$

for all $k \geq b - 1$.

Now given any rational space X that is $(n-1)$-connected, the dual of the homotopy groups of X are zero up through dimension $n-1$, so the first place any products could occur would be in dimension $2n$. Thus the differential must be zero up through dimension $2n-2$, and we get

$$H^k(X) \cong \big(\pi_k(X)\big)^*,$$

for $k \leq 2n - 2$.

This means since F is $(b-2)$-connected, we know

$$H^k(F) \cong \big(\pi_k(F)\big)^*$$

for $k \leq 2b - 4$. So then we get our result. □

3.2 Differentials Determine Realizability

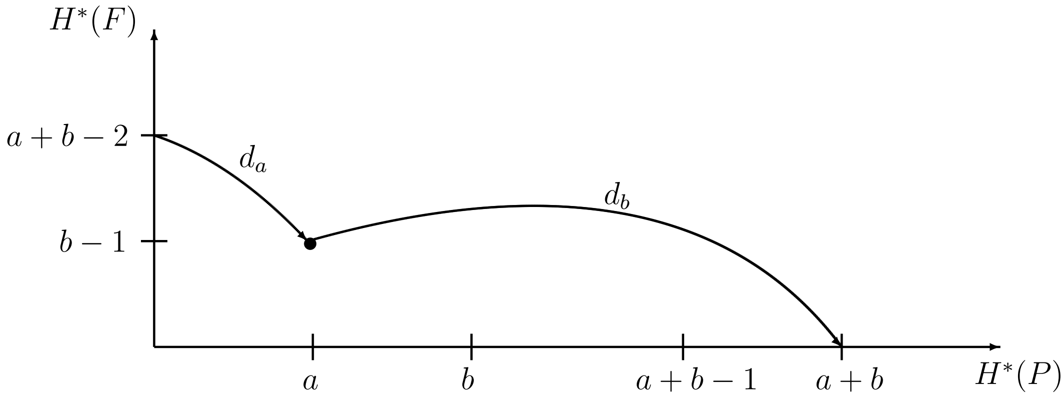
We apply the Leray-Serre spectral sequence to the following fiber sequence:

$$F \longrightarrow W \longrightarrow P.$$

This means we get the page shown in Figure 3.1 in the corresponding Leray-Serre spectral sequence converging to $H^*(W)$.

From this we determine some helpful equivalent conditions for the Postnikov section P realizing the category sequence $(a,\, b,\, a+b,\, ?,\, \ldots)$.

Figure 3.1: Differentials in the Spectral Sequence



Theorem 3.2.1. *The following are equivalent:*

1. *The space P has sequence $\sigma_P = (a, b, a+b, ?, ?, \ldots)$,*
2. *There exists a nonzero product*
$$H^a(P) \otimes H^b(P) \to H^{a+b}(P),$$
3. *The differential*
$$d_b : H^{b-1}(F) \otimes H^a(P) \to H^{a+b}(P)$$
is nonzero,
4. *The differential*
$$d_a : H^{a+b-2}(F) \to H^a(P) \otimes H^{b-1}(F)$$
is not onto.

Proof. First we show #2 implies #3 and then #3 implies #2. Since $H^k(F) = 0$ for $k < b-1$ and $H^{b-1}(F) \neq 0$, we know d_b is the only differential which has $H^{b-1}(F)$ as its domain (or target). Since $H^b(W) = 0$, it must be that none of $H^{b-1}(F)$ nor any of $H^b(P)$ survives to the infinity page, thus $d_b : H^{b-1}(F) \to H^b(P)$ is an isomorphism.

Now if the product in #2 is nonzero, then there is some $u \in H^a(P)$ and some $v \in H^b(P)$ such that $0 \neq u \cdot v \in H^{a+b}(P)$. If we let $\overline{v} \in H^{b-1}(F)$ such that $d_b(\overline{v}) = v$, then

$$u \otimes \overline{v} \in H^a(P) \otimes H^{b-1}(F),$$

and $d_b(u \otimes \overline{v}) = \pm u \cdot d_b(\overline{v}) = \pm uv \neq 0$. Since d_b takes this element to something nonzero, the differential d_b is nonzero.

On the other hand, if the differential d_b is nonzero, then there is some element

$$u \otimes \overline{v} \in H^a(P) \otimes H^{b-1}(F)$$

such that $d_b(u \otimes \overline{v}) \neq 0$. That is,

$$d_b(u \otimes \overline{v}) \;=\; d_b(u)\overline{v} - u d_b(\overline{v}) \;=\; 0 - uv,$$

and so uv is a nonzero product in $H^{a+b}(P)$ with $u \in H^a(P)$ and $v \in H^b(P)$.

Next we see #1 if and only if #3. Assume P realizes the sequence. We know that $H^{b-1}(F) \cong H^b(P)$ by spectral sequence argument, but we also know we have this isomorphism because of the one-to-one correspondence between generators in $\pi_{b-1}(W)$ and the two-fold products in dimension b of the minimal model for W which are cycles. Since $H^{b-1}(F) \cong \pi_{b-1}(F) \cong \pi_{b-1}(W)$, and $H^b(P)$ is precisely the two-fold products in dimension b of the model for W (as they are not boundaries as they were in W), we get $H^{b-1}(F) \cong H^b(P)$.

Then we know that σ_P satisfies $\sigma_P(3) = \sigma_P(2) + \sigma_P(1)$, so we can use Theorem 2.3.11(3) to get our nonzero differential.

On the other hand, if the differential is nonzero, then we have some $d_b(uv) \in H^{a+b}(P)$ where $u \in H^a(P)$ and $v \in H^b(P)$. Then v is represented in the model by a sum of twofold or higher products since P has no generators above dimension $b-1$. Thus uv must be represented by a sum of threefold or higher products, which means $\sigma_P \leq \epsilon_P \leq (a, b, a+b, ?, ?, \ldots)$. Since we have superadditivity, we know $\sigma_P(3) \geq a+b$, so it must be $\sigma_P = (a, b, a+b, ?, ?, \ldots)$.

Next, to show #3 if and only if #4, we first suppose that $d_b \neq 0$. We know that everything in $T = H^a(P) \otimes H^{b-1}(F)$ must either kill something else off or it must be killed by the infinity page. Note that d_a and d_b are the only differentials with T as domain or target. So since $b > a$, we know that d_b being nonzero means that some generators in T were not killed by d_a. Thus d_a is not onto.

Conversely, if we suppose d_a is not onto, then there is some $u \in H^a(P) \otimes H^{b-1}(F)$ which is not killed off by d_a. Since it must kill or be killed by the infinity page, and $H^k(F) = 0$ for $k < b$, the only differential which is able to do either is d_b. That is, we have d_b must be nonzero on u.

$\square$

Chapter 4

Ease up with Postnikov

In this chapter we begin by describing the model for a generic Postnikov section of a wedge of spheres. Then we go on to show that if we have a rational space with sequence $(a, b, a+b)$, (which is the only type of sequence we are investigating in this thesis,) then we can find a Postnikov section of a wedge of spheres such that the sequence of the Postnikov section is also $(a, b, a+b)$ in the first three entries. Finally we show that if we have only one dimension where we have spheres, then we must get an integer slope from the category sequence.

4.1 Postnikov Model Versus Wedge Model

Let W be a wedge of spheres with the lowest dimensional sphere of dimension a. Suppose W has two-fold products in dimension b and let $P = P_{b-1}(W)$ be the Postnikov section of W taken at dimension $b-1$.

Start with the model, $(\Lambda V, d)$ for W. This model has generators from spheres and generators whose images are products which are in the kernel of the differential. When we take the Postnikov section P, we are removing all homotopy groups in dimension $b-1$ and above, which corresponds to removing all generators from the model in those dimensions. Thus we have that our model for P is $(\Lambda(V^{\leq b-2}), d')$, where d' is the restriction of d to $\Lambda(V^{\leq b-2})$.

This means P has elements which are in the kernel of the differential d and not in the image of d'. Thus we get cohomology classes in $H^*(P)$ which were not in $H^*(W)$.

In Theorem 3.2.1, we found that we know $d_b \neq 0$ if and only if the sequence is realized by this P. Thus P realizes the sequence if and only if three is some $\delta \in \pi_{b-1}(W)$ and some $\alpha \in \pi_a(W)$ such that $d\big(\varphi(\delta^*)\big) \cdot (\varphi(\alpha^*)) \neq 0$ and that image is not a boundary. Hence we look only at products of this form. That is, we see that P realizes the category sequence if and only if some $d(\varphi(\delta^*)) \cdot (\varphi\alpha^*)$ is not a boundary.

4.2 A Couple of Lemmas

In this section we show two results which we will use later in this chapter.

4.2.1 Induced Equivalence

For the result we will use in Theorem 4.3.1 in Section 4.3, we need the following result about the equivalence we get from repeated joins.

Lemma 4.2.1. *Given X is $(c-1)$-connected and $f : Z \to X$ is an n-equivalence ($n \geq c$), the induced map*

$$(\Omega f)^{*(k+1)} : F_k(Z) \to F_k(X)$$

is an $(n + kc - 1)$-equivalence for $k \geq 0$ (where F_k gives the kth fiber in the Ganea construction).

Proof. First, Ωf is easily seen to be an $(n-1)$-equivalence since from this fiber sequence

$$\Omega X \longrightarrow \mathcal{P}X \longrightarrow X$$

we get a long-exact sequence in homotopy groups, and we know that since the path space is contractible that its homotopy groups are trivial, so we have $\pi_{n-1}(\Omega X) \cong \pi_n(X)$. Thus an n-equivalence $f : X \to Y$ induces an $(n-1)$-equivalence on the loop spaces.

Now we need to find out what happens when we join a map f with itself. We use that the map $f * f$ is homotopy equivalent to the suspension of the smash product, $f * f \simeq \Sigma(f \wedge f)$, and we work by induction on k to show that $f^{*(k+1)}$ is an $(n + k(c+1))$-equivalence for $k \geq 0$. (We get to Ωf later.)

For $k = 0$ we have just f, which is an n-equivalence and so gives us the base case. Thus we suppose that we have f^{*k} is an $(n + (k-1)(c+1))$-equivalence.

Now we know that since all relevant spaces are simply-connected, f is an n-equivalence if and only if its cofiber is n-connected. So in Figure 4.1 we start with the top left square and then take cofibers horizontally and vertically. We get the space in the bottom right corner by taking the cofiber of the bottom row.

Figure 4.1: Cofibers of Smashes

$$\begin{tikzcd}
Z \wedge Z^{*k} \arrow[r, "{[\mathrm{id},\, *^k f]}"] \arrow[d, equal] & Z \wedge X^{*k} \arrow[r] \arrow[d, "{[f,\, \mathrm{id}]}"] & Z \wedge C_{*^k f} \arrow[d] \\
Z \wedge Z^{*k} \arrow[r, "{[f,\, *^k f]}"] \arrow[d] & X \wedge X^{*k} \arrow[r] \arrow[d] & C_{f \wedge (*^k f)} \arrow[d, dotted] \\
* \arrow[r] & C_f \wedge X^{*k} \arrow[r] & C_f \wedge X^{*k}
\end{tikzcd}$$

Notice that we have only written "$C_{f\wedge(*^k f)}$" for the cofiber of the middle row. We want to know the connectivity of this cofiber to determine the level of equivalence of the corresponding map. Since we must get the same cofiber from both the far right column and the bottom row, we know that the connectivity of $C_{f\wedge(*^k f)}$ is at least the minimum of the connectivities of $Z \wedge C_{*^k f}$ and $C_f \wedge X^{*k}$.

To find the connectivity of $Z \wedge C_{*^k f}$, we know that Z is at least $(c-1)$-connected and $C_{*^k f}$ is at least $(n + (k-1)(c+1))$-connected by inductive hypothesis, so we have that the smash product of these spaces has connectivity at least

$$(c-1) + (n + (k-1)(c+1)) + 1 \quad = \quad n + k(c+1) - 1.$$

On the other hand, the connectivity of $C_f \wedge X^{*k}$ is at least

$$n + (kc + k - 2) + 1 \quad = \quad n + k(c+1) - 1,$$

since the connectivity of X^{*k} is at least $kc + k - 2$. Thus we have that the connectivity of the cofiber $C_{f\wedge(*^k f)}$ is at least $n+k(c+1)-1$, and so by suspending we get that $*^{k+1}f$ is at least an $(n + k(c + 1))$-equivalence.

Finally we put these pieces together with the knowledge that connectivity of Ωf is one less than the connectivity of f and see that if f were instead Ωf, that we get $(\Omega f)^{*(k+1)}$ is at least an $(n - 1 + k(c - 1 + 1))$-equivalence, which is what we wanted to prove. □

4.2.2 Category of a Map Matches its Domain

This lemma is the statement of a remark in [9] following their Proposition 2.2.

Lemma 4.2.2. *If X is $(c-1)$-connected and $f : Z \to X$ is an n-equivalence $(n \geq c)$ with $\mathrm{cat}(f) = k$ and $\dim(Z) \leq n + kc - 1$, then $\mathrm{cat}(Z) = \mathrm{cat}(f)$.*

Proof. First we see that $\mathrm{cat}(f) \leq \mathrm{cat}(Z)$ from Theorem 2.3.4(1). Then let $\mathrm{cat}(f) = k$ and we try to show that $\mathrm{cat}(Z) \leq k$. We know $n \geq c$, so

$$\pi_c(Z) \xrightarrow{\quad f_* \quad} \pi_c(X)$$

is an isomorphism. Since it is nontrivial, f is not of category 0, so $k \geq 1$.

By Theorem 2.4.2, there is a lift $\lambda : Z \to G_k(X)$ of f, where $G_k(X)$ is the kth Ganea space of X. See the Figure 4.2, where P is the pullback to make the bottom right square a pullback square. Note that $F_k(X)$ is the fiber over $P \to Z$ since we are taking a pullback.

Also, since P is the pullback, we get the unique section $\tau : Z \to P$ (see Figure 4.3 for definition).

Now we see what kind of equivalences we have in the fiber spaces of the Ganea construction. We know from Theorem 2.4.1, we have

$$F_k(X) \quad \cong \quad (\Omega X)^{*(k+1)};$$

Figure 4.2: Getting a Section $Z \to G_k(Z)$

$$\begin{tikzcd}
F_k(Z) \arrow[r, "(\Omega f)^{*(k+1)}"] \arrow[d] & F_k(X) \arrow[r, equal] \arrow[d] & F_k(X) \arrow[d] \\
G_k(Z) \arrow[r, "j"] \arrow[d] & P \arrow[r] \arrow[d] & G_k(X) \arrow[d] \\
Z \arrow[r, equal] & Z \arrow[r, "f"'] \arrow[ul, dotted, "\sigma"] \arrow[u, dashed, bend left, "\tau"] \arrow[ur, "\lambda"'] & X
\end{tikzcd}$$

Figure 4.3: First Get a Section $Z \to P$

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! \tau"] \arrow[drr, bend left, "\lambda"] \arrow[ddr, bend right, "id_Z"'] & & \\
& P \arrow[r] \arrow[d] & G_k(X) \arrow[d] \\
& Z \arrow[r, "f"'] & X
\end{tikzcd}$$

and we want to find the connectivity of $(\Omega f)^{*(k+1)}$, given that f is an n-equivalence.

It follows from Lemma 4.2.1 that $F_k(Z) \to F_k(X)$ is an $(n+kc-1)$-equivalence, and then by looking at the long exact sequences in homotopy groups, we get that j is an $(n+kc-1)$-equivalence as well.

So the dimension of Z being less or equal to $n+kc-1$ implies that the map

$$[Z,\, G_k(Z)] \xrightarrow{j_*} [Z,\, P],$$

is surjective, and thus $\mathrm{cat}(Z) \leq k$. $\square$

Corollary 4.2.3. *If X is $(c-1)$-connected then for $n \geq c$ we get*

$$\mathrm{cat}_X(X^n) \;=\; \mathrm{cat}(X^n).$$

4.3 Simplify Our Spaces

In this section we show that for our pursuit of rational spaces which realize the sequence $(a,\, b,\, a{+}b,\, ?,\, \ldots)$, we can focus on Postnikov sections of wedges of spheres rather than all rational spaces. The following theorem makes this precise:

Theorem 4.3.1. *If X is a rational space such that $\sigma_X = (a, b, a+b)$ for $a \geq 2$, then there exists a finite set $\{n_i\}$ (where for each i, $a \leq n_i \leq b-a\}$) such that for*

$$P \;:=\; P_{b-1}\big(\bigvee S^{n_i}\big),$$

we have $\sigma_P = \sigma_X$ in the first three entries.

Proof. First, using Corollary 4.2.3 we know that

$$\mathrm{cat}_X(X^n) \;=\; \mathrm{cat}(X^n) \;=\; 1$$

for all n such that $a \leq n < b$. So, in particular, since X is a rational space, X^{b-1} being of category one makes it homotopy equivalent to a wedge of rational spheres ([5]). Let W be this wedge of spheres. Then we take $P = P_{b-1}(W)$.

Thus it suffices to show that $\sigma_P(3) = a+b$ since, in this case, superadditivity implies $\sigma_P(2) \leq b$, and, further, that we have equality since

$$\mathrm{cat}(P^{b-1}) = \mathrm{cat}(W^{b-1}) = \mathrm{cat}(X^{b-1}) = 1.$$

We know that $\sigma_P(3) \geq a+b$ by superadditivity, so we want to show that we do get category 3 in dimension $a+b$.

Suppose we do not; that is, suppose $\mathrm{cat}(P^{a+b}) < 3$. Then we have two cases to consider: first if $\mathrm{cat}(f) = 1$, then if $\mathrm{cat}(f) = 2$. So we assume $\mathrm{cat}(f) = 1$.

Let $\tilde{P} = P_{b-1}(X)$ and $\tilde{f} : X \to \tilde{P}$ be the Postnikov map which gives us an isomorphism in homotopy groups up through dimension $b-2$ and a surjection in dimension $b-1$. Denote by f the restriction of $\tilde{f}$ to X^{a+b}:

$$f : X^{a+b} \longrightarrow (\tilde{P})^{a+b}$$

Also, since W is a wedge of spheres which come from the cells of X, if we look at the minimal models of these spaces, we have that the generators for the model of W must be in bijective correspondence with the generators for the model of X up through dimension $b-1$. So we have W is a retract of X, and so $(\tilde{P})^{a+b}$ is a retract of P^{a+b}. Then we use Theorem 2.3.4 parts (1) and (3) to see

$$\operatorname{cat}(f) \leq \operatorname{cat}\left((\tilde{P})^{a+b}\right) \leq \operatorname{cat}(P^{a+b}) < 3.$$

Now we know from Theorem 2.3.6 that the category of a map f is at least as large as the length of the longest nonzero product in the image of $H^*(f)$. Since we know that $\sigma_X(2) = b$, then $H^b(X) \neq 0$, and so there is a nonzero product in the image of $H^*(f)$.

This implies $\operatorname{cat}(f) \geq 2$. So we explore the case when $\operatorname{cat}(f) = 2$.

Since $P_{b-1}(X)$ is $(a-1)$-connected and the dimension of X^{a+b} is no more than $(b-1) + ka - 1$ for $k = \operatorname{cat}(f) = 2$, we have from Lemma 4.2.2 that $\operatorname{cat}(X^{a+b}) = \operatorname{cat}(f) \leq 2$. This is a contradiction, and thus $\operatorname{cat}(P^{a+b}) \geq 3$.

This means we have $\sigma_P = \sigma_X$ in the first three entries.

What remains to be argued is that the spheres in the wedge W can be in dimensions no higher than $b-a$. This is simply because any generators in $\pi_*(W)$ above dimension $b-a$ cannot be factors in basic products below dimension b. □

Lemma 4.3.2. *If we can realize a sequence with a noninteger slope $m \in (n, n+1)$, then we can realize this sequence using a wedge of $n+3$ spheres.*

Proof. If we have a slope m with $n < m < n+1$, then we get

$$(n+2)a - n < b < (n+3)a - (n+1).$$

Thus in the model there is not enough room to get any products or sums of products in dimension b which corresponds under the isomorphism in Theorem 2.2.4 to basic product which are more than $(n+2)$-fold basic products. That is, if we have a three-fold product in dimension $a+b$ of the model given by $d(\delta^*)\alpha^* \neq 0$, then the δ must be no more than a $(n+2)$-fold basic product. Hence there is not enough space to involve more than $n+3$ spheres up through dimension $a+b$. $\square$

4.3.1 Sufficient to Use Products With Distinct Factors

In this section we would like to restrict the number of basic products which we need to consider when testing realizability. Throughout this section we use $*$ to indicate the dual.

Theorem 4.3.3. *Let $W = \bigvee_i^r S^{n_i}$, and suppose the category sequence $(a, b, a+b, ?, \ldots)$ is realized by Postnikov section $P = P_{b-1}(W)$. Suppose further that $\delta \in \pi_{b-1}(W)$ is a k-fold basic product involving each S^{n_i} sphere r_i times, and that $r_j \geq 2$ for some j such that $1 \leq j \leq r$.*
Define a new wedge

$$W' = \left(\bigvee_{i=1}^{r} S^{n_i}\right) \vee S^{n_j}.$$

Then $P' = P_{b-1}(W')$ will have the same category sequence up to category three as P.

Proof. We look at the models of the different wedges; let $\mathcal{M} = (\Lambda V, d)$ be a model for P and let $\mathcal{M}' = (\Lambda V', d')$ be a model for $P' = P_{b-1}(W')$. Then we get

$$V = \{v_1, v_2, \ldots, v_r\},$$

for v_i in dimension n_i for all i, and

$$V' = V \cup \{v_j'\}.$$

So we define a map

$$W \longrightarrow W'$$

to be the identity on the spheres S^{n_i} for $i \neq j$ and to be the pinch map on the sphere S^{n_j}. This induces the map

$$\begin{array}{rcll} f:\pi_*(W) & \longrightarrow & \pi_*(W') & \qquad \text{defined by} \\ \nu_i & \mapsto & \nu_i & \qquad \text{for } i \neq j, \text{ and} \\ \nu_j & \mapsto & \nu_j + \nu'_j. & \end{array}$$

We then dualize this map and get

$$f^* : \big(\pi_*(W')\big)^* \longrightarrow \big(\pi_*(W)\big)^*,$$

where $f^*(\nu'^*)(\nu_l) = \nu'^*\big(f(\nu_l)\big)$ for all $\nu' \in \pi_*(W')$ and all l, $1 \le l \le r$. So let $\nu' \in \pi_*(W')$, let $j \neq l$, and we get

$$f^*(\nu'^*)(\nu_l) = \begin{cases} 1, & \text{if } \nu' = \nu_l, \text{ and} \\ 0, & \text{else.} \end{cases}$$

Also, for $j = l$, we have

$$f^*(\nu'^*)(\nu_j) = \nu'^*(\nu_j + \nu'_j) = \begin{cases} 1, & \text{if } \nu' = \nu_j \text{ or } \nu' = \nu'_j, \text{ and} \\ 0, & \text{else.} \end{cases}$$

That is, we get

$$f^*(\nu'^*) = \begin{cases} \nu'^*, & \text{if } \nu' \notin \{\nu_j, \nu'_j\} \\ \nu_j^*, & \text{else.} \end{cases}$$

Notice that we have the following commutative diagram:

$$\begin{tikzcd} \big(\pi_*(W')\big)^* \arrow[r, "\varphi'"] \arrow[d, "f^*"'] & V' \arrow[r, "d"] \arrow[d, "f^*"'] & \Lambda^2 V' \arrow[d, "f^*"'] \\ \big(\pi_*(W)\big)^* \arrow[r, "\varphi"'] & V \arrow[r, "d"'] & \Lambda^2 V \end{tikzcd}$$

Now we see that $f(\delta)$ splits into a sum of four Whitehead products: two with one factor each of ν_j and ν'_j, one with two factors of ν_j, and one with two factors of ν'_j. The latter two will always be basic since we are assuming δ is basic. Sometimes one of the former two is not basic and can be rewritten as a sum of two basic products. In the case where all four are basic, we take δ' to be any one of them and we will have

$$f^*(\delta'^*) \;=\; \delta^*.$$

In the case where one of the four is not basic, we rewrite that nonbasic product as a sum of basic products. If all of these basic products canceled then $f(\delta)$ would be zero, but since f is injective and $\delta \neq 0$, this could not be true. So let δ' be one of the basic products which does not cancel. Then, again, we have $f^*(\delta'^*) = \delta^*$.

Thus, in either case, assuming $\alpha \in \pi_a(W)$ and $\alpha' \in \pi_*(W')$ such that $f^*(\alpha'^*) = \alpha^*$, we get that if $d\big(\varphi(\delta^*)\big)(\varphi(\alpha^*)) \neq 0$, then

$$\begin{aligned}
0 \;\neq\; d\Big(\varphi(\delta^*)\Big)\big(\varphi(\alpha^*)\big) \;&=\; d\Big(\varphi\big(f^*(\delta'^*)\big)\Big)(\varphi(\alpha^*)) \\
&=\; f^*\Big(d\big(\varphi'(\delta'^*)\big)\Big)(\varphi(\alpha^*)) \\
&=\; f^*\Big(d\big(\varphi'(\delta'^*)\big)\Big)\big(\varphi(f^*(\alpha'^*))\big) \\
&=\; f^*\Big(d\big(\varphi'(\delta'^*)\big)\Big)\big(f^*(\varphi'(\alpha'^*))\big) \\
&=\; f^*\Big(\big(d(\varphi'\delta'^*)\big)\big(\varphi'\alpha'^*\big)\Big).
\end{aligned}$$

Thus we have $d(\varphi'\delta'^*)\big)\big(\varphi'\alpha'^*\big) \neq 0$ since f^* is a homomorphism. This gives us that P' has a nonzero three-fold product in dimension $a+b$, and thus P' also realizes the sequence $(a,\, b,\, a+b,\, ?,\, \ldots)$. $\square$

Corollary 4.3.4. *If we take W' to be a wedge where we add in extra spheres for all duplicate factors in δ, then $P' = P_{b-1}(W')$ will have the same category sequence up to category three as P.*

This is an induction argument from Theorem 4.3.3.

Now that we have this result, we can return to our discussion of rational realizability from Section 4.1 and get the following:

Theorem 4.3.5. *Let* $W = \bigvee_i^r S^{n_i}$. *Then the Postnikov section* $P = P_{b-1}(W)$ *realizes the sequence* $(a,\, b,\, a+b,\, ?,\, \ldots)$ *if and only if there is some* $\delta \in \pi_{b-1}(W)$ *with all distinct factors and some* $\alpha \in \pi_a(W)$ *which is not a factor in* δ *such that the three-fold product*

$$d\big(\varphi(\delta^*)\big) \cdot (\varphi(\alpha^*))$$

is not a boundary in P. *(The map* φ *is that of Theorem 2.2.4 and* δ^* *represents the image of* δ *in* $\big(\pi_*(X)\big)^*$.*)*

4.4 Integral Slopes Emerge

We start with a Postnikov section of a wedge of a-dimensional spheres and we show that any possible b value we choose will give an integer slope. Note that the sequence in the lemma below does not indicate where category three first shows up.

Lemma 4.4.1. *Every Postnikov section of a wedge of at least two* a*-dimensional spheres will have category sequence* $(a,\, b,\, ?,\, ?,\, \ldots)$ *where* $m(a, b) \in \mathbb{Z}$.

Proof. First, if we have at least two a-spheres, then we know that in the model of the wedge, we get products in dimensions $2a$, $3a-1$, ... , $ka-(k-2)$. This means we only have homotopy groups in dimensions one below those, so for wedge W we see

$$\begin{array}{rcl}
P_a(W) & = & P_{a+1}(W) = P_{a+2}(W) = \cdots = P_{2a-1}(W) \\
P_{2a}(W) & = & P_{2a+1}(W) = \cdots = P_{3a-2}(W) \\
& \vdots & \\
P_{(k-1)a-(k-3)}(W) & = & P_{(k-1)a-(k-4)}(W) = \cdots = P_{ka-(k-3)}(W) \\
P_{ka-(k-2)}(W) & = & P_{ka-(k-3)}(W) = \cdots = P_{(k+1)a-(k-1)}(W) \\
& \vdots &
\end{array}$$

Thus we know that b must be b_k given by

$$b_k \;=\; ka - (k-2),$$

for some $k \in \mathbb{Z}$. So we get a slope of

$$\begin{aligned}
m &= \frac{b_k - 2a}{a-1} \\
&= \frac{ka - (k-2) - 2a}{a-1} \\
&= \frac{k(a-1) - 2(a-1)}{a-1} = k - 2.
\end{aligned}$$

That is, we get an integer slope. $\square$

Chapter 5

Integer Slopes

5.1 Integer Slopes Are Eventually Realizable

Using some results on differentials alongside Witt's formulae for the number of basic products, we can prove sequences with integer slopes are rationally realizable:

Theorem 5.1.1. *When $a \geq 2$ and the slope of a sequence $(a,\, b,\, a+b,\, ?,\, \ldots)$ is a nonnegative integer, there is a Postnikov section of a wedge of a-spheres which will realize the sequence.*

Proof. First, since we want $m(a,b) \geq 0$, this means $b \geq 2a$. Now let W be a wedge of r a-spheres and let $P = P_{b-1}(W)$. Recall that from Theorem 3.2.1 that a sequence is realized if and only if the differential d_a in the spectral sequence of Figure 3.1 is not onto.

If we count the dimension of $H^a(P) \otimes H^{b-1}(F)$ and find that it is larger than the dimension of $H^{a+b-2}(F)$, then we know that d_a could not be onto. That is, if

$$\dim(H^a(P)) \cdot \dim(H^{b-1}(F)) - \dim(H^{a+b-2}(F))$$

is positive, then the given sequence can be realized.

Using our knowledge of how to count the dimension of homotopy groups of the wedge W and our Lemma 3.1.1, we can specifically say that the sequence is realized if the following expression is positive:

$$r \cdot p_{m+2}(r) - p_{m+3}(r),$$

where r is the number of a-spheres, and $m = m(a, b)$.

Now if we think about this expression as $r \to \infty$, we know we can just focus on the highest-degree terms, which gives

$$\frac{1}{m+2}(\mu(1)r \cdot r^{m+2}) - \frac{1}{m+3}(\mu(1)r^{m+3}),$$

that is, we have

$$\left(\frac{1}{m+2} - \frac{1}{m+3}\right)r^{m+3},$$

which is clearly positive. So this proves that with enough spheres, we can realize any nonnegative integer slopes with a Postnikov section of a wedge of sufficiently many a-spheres. $\square$

Chapter 6

Non-Integer Slopes

Now that we have seen we can realize the integer slopes, we want to know how close to the Conjecture we can get. In Section 6.1 we lay out the basic algorithm for determining realizability by brute force. In Sections 6.2 and 6.3 we see that the non-integer slopes between 0 and 2 fail to be realiable.

6.1 Realizability by Force

Here we describe a conceptually simple, but increasingly-computationally-difficult algorithm for determining realizability.

As usual, we let P be the Postnikov section of a wedge of spheres taken at dimension $b-1$, where the lowest dimensional sphere has dimension a.

Let $\{w_i\}$ be the set of all two-fold products (representing basic products with no repeated factors and no factors in common) which exist in the model of P in dimension $a+b-1$. Then let $\{v_i\}$ be all possible three-fold products (representing basic products with no repeated factors and no factors in common) in dimension $a+b$ of the model. Form a matrix, M, with the two-fold products along the columns and the three-fold products down the rows, such that if $dw_k = \sum_i^n c_i \cdot v_i$

for $c_i \in \mathbb{Z}$, then we get that the kth column of M is

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Next we find all sums of three-fold products in the model for P which are cycles. Let $\{\overline{u_i}\}$ be vectors representing these cycles.

Then the sequence $(a,\, b,\, a+b,\, ?,\, \ldots)$ is realized by P if and only if there exists an i such that the matrix equation

$$M\overline{x} \;=\; \overline{u_i}$$

has no solutions.

Note that this process amounts to finding the differentials on generators and using the product rule. Then by Corollary 2.2.6 this comes down to rewriting these simple products as sums of basic products.

6.2 Slopes Between 0 and 1 Are Not Realizable

In this section we find that the smallest positive noninteger slopes are not rationally realizable.

Theorem 6.2.1. *There is no rational space whose category sequence has slope* $0 < m < 1$.

Proof. We use Theoreom 4.3.5 and since we are looking at slopes between 0 and 1, Lemma 4.3.2 says that if we can realize a sequence of this slope that we should be able to realize it with a Postnikov section of a wedge of no more than three spheres.

So suppose we have a nonzero $\tau = d(\delta^*)\alpha^*$ in the model for our Postnikov section, where $\delta \in \pi_{b-1}(W)$ has no factors of α and no repeated factors, and

$\alpha \in \pi_a(W)$. The only possible values for δ are two-fold basic products, say $[\gamma, \beta]$. Recall that since we have a non-integer slope, it must be that $|\gamma| > a$. This means that we know $[\beta, \alpha]^*$ is of small enough dimension to be in the Postnikov section. Furthermore, the only three-fold products in the model can be $\pm\gamma^*\beta^*\alpha^*$.

If $|\beta| > a$, then we also have $[\gamma, \alpha]^*$ being small enough to be in the Postnikov section, but $[\beta, \alpha]^*$ will always be there and as we can see in the following calculation, the latter is sufficient to kill off our only three-fold product.

We do this simple check:

$$\begin{aligned} d\Big(([\beta, \alpha])^*\gamma^*\Big) &= (-1)^{(|\alpha|+1)(|\beta|+1)}\beta^*\alpha^*\gamma^* \\ &= (-1)^{(|\alpha|+1)(|\beta|+1)+|\gamma|(|\alpha|+|\beta|)}\gamma^*\beta^*\alpha^*, \end{aligned}$$

and so this sequence is not realized. □

Notice that the above proof shows that for any two-fold basic product δ, that products $d(\delta^*)\alpha^*$ in the model for P will be boundaries in all higher non-integer slopes as well.

6.3 Slopes Between 1 and 2 Are Not Realizable

We move up now to the noninteger slopes which are larger than one and smaller than two. From Lemma 4.3.2, we know that if we realize a sequence of this slope then it must be done with no more than four spheres. We sort through the details of the differential calculations in Appendix A.

Theorem 6.3.1. *There is no rational space whose category sequence has slope* $1 < m < 2$.

Proof. Suppose the sequence is realizable. Then there is a nonzero class in $H^{a+b}(P)$ which is represented by a sum of three-fold products in the model and which is of the form $\tau = d\delta^*\alpha^*$, where $\delta \in \pi_{b-1}(W)$ is basic without repeated factors. By Lemma 4.3.2 and Corollary 4.3.4, we need only look at three-fold products which are of this form. That is, we need only use linear combinations of those products which have a single factor of α^*. From this we see that τ must be one of the following:

$$\tau \in \left(\begin{bmatrix} (-1)^{(z+1)(w_2+1)} \\ (-1)^{(z+1)(w_1+1)+w_2w_1+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (-1)^{(z+1)(w_2+1)} \\ 0 \\ 0 \\ (-1)^{(w_2+1)(w_1+1)+z(w_2+w_1)+1} \\ 0 \\ 0 \end{bmatrix} \right). \tag{6.1}$$

We have the matrix M which we find in Figure A.1 of Section A.1.2. We want to know whether we can solve the equation $M\overline{x} = \tau$ for each of these vectors. To this end, we augment M by inserting these two τ vectors as the last columns of M to give us the matrix in Figure B.2. Then we can determine whether we can solve $M\overline{x} = \tau$ by row-reducing this matrix M'. (For details on the row-reduction, see Appendix B.)

Once we reduce, we find that the row-reduced echelon form for M' is given in Figure 6.1, where N is a 5×4 matrix in $\mathbb{Z}$ and c_1 and c_2 are 5×1 column vectors in $\mathbb{Z}$.

Figure 6.1: Row-Reduced Echelon Form of M'

$$\left(\begin{array}{ccc|ccc|c|c} & I_5 & & & N & & c_1 & c_2 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{array} \right)$$

This tells us that the equations $M\overline{x} = \tau$ both have solutions and thus the sequence is not rationally realizable. $\square$

Appendix A

Differential Calculations for Four Spheres

In this Appendix we make calculations which we use in Chapter 6 to show that we cannot realize non-integer slopes using a wedge of four spheres.

A.1 Find the Differentials

Here we let $a, w_1, w_2, z \in \mathbb{Z}$ such that $2 \leq a \leq w_1 \leq w_2 \leq z$ and $a < z$. (From Lemma 4.4.1, we know that if all generators were in dimension a then we would get an integer slope.) Then take

$$W = S^a \vee S^{w_1} \vee S^{w_2} \vee S^z,$$

and α to be the generator for S^a, β_1 to be the generator for S^{w_1}, β_2 that for S^{w_2}, and γ for S^z. These are one-fold basic products ordered by $\alpha < \beta_1 < \beta_2 < \gamma$.

We have our relation given by Theorem 2.2.4 which gives a bijection $\varphi : \big(\pi_*(W)\big)^* \to V$. For this chapter we will not write the φ given in Theorem 2.2.4. That is, if given $\mu \in \pi_*(W)$, we will write $\varphi(\mu^*) = \mu^*$.

A.1.1 Preliminary Calculations

First we let μ and η be generators for spheres, and we determine $d(\mu\eta)^*$:

Theorem A.1.1. *For $\mu, \eta \in \{\alpha,\, \beta_1,\, \beta_2,\, \gamma\}$ assuming $\mu > \eta$ (ordering as basic products),*

$$d\big((\mu\eta)^*\big) \;=\; (-1)^{(|\mu|+1)(|\eta|+1)} \cdot \mu^*\eta^*.$$

Proof. We know that $d(\mu\eta)^* = c \cdot \mu^*\eta^*$ for some constant $c \in \mathbb{Q}$, and we determine c by first seeing

$$\begin{aligned}\langle d(\mu\eta)^*\,;\, \mu,\, \eta\rangle &= \langle c \cdot \mu^*\eta^*\,;\, \mu,\, \eta\rangle\\ &= c \cdot \big(\langle \mu^*,\, \eta\rangle\langle \eta^*,\, \mu\rangle + (-1)^{|\eta^*||\mu|}\langle \mu^*,\, \mu\rangle\langle \eta^*,\, \eta\rangle\big)\\ &= c \cdot (-1)^{|\mu||\eta|},\end{aligned}$$

but also

$$\langle d(\mu\eta)^*\,;\, \mu,\, \eta\rangle \;=\; (-1)^{|\mu|+|\eta|-1}\langle(\mu\eta)^*;\, \mu\eta\rangle \;=\; (-1)^{|\mu|+|\eta|-1}.$$

We find $c = (-1)^{(|\mu|+1)(|\eta|+1)}$, and so

$$d\big((\mu\eta)^*\big) \;=\; (-1)^{(|\mu|+1)(|\eta|+1)} \cdot \mu^*\eta^*.$$

□

Next we find $d((\mu\nu)\eta)^*$ and $d((\eta\nu)\mu)^*$.

Theorem A.1.2. *Given $\mu, \eta, \nu \in \{\alpha,\, \beta_1,\, \beta_2,\, \gamma\}$ with $\mu > \eta > \nu$,*

$$d\big(((\mu\nu)\eta)^*\big) \;=\; (-1)^{(|\eta|+1)(|\mu|+|\nu|)}(\mu\nu)^*\eta^* + (-1)^{|\mu||\nu|+|\mu|+|\eta|+1}(\mu\eta)^*\nu^*,$$

and

$$d\big(((\eta\nu)\mu)^*\big) \;=\; (-1)^{(|\mu|+1)(|\eta|+|\nu|)}(\eta\nu)^*\mu^* + (-1)^{(|\mu|+1)(|\eta|+|\nu|+1)}(\mu\eta)^*\nu^*$$

Proof. To determine $d((\mu\nu)\eta)^*$ and $d((\eta\nu)\mu)^*$, we first need to write the simple product $(\mu\eta)\nu$ as a sum of basic products:

$$\begin{aligned}(\mu\eta)\nu &= -(-1)^{|\mu||\nu|}\big[(-1)^{|\eta||\mu|}(\eta\nu)\mu + (-1)^{|\nu||\eta|}(\nu\mu)\eta\big]\\ &= (-1)^{|\mu|(|\eta|+|\nu|)+1}(\eta\nu)\mu + (-1)^{|\eta||\nu|+1}(\mu\nu)\eta\end{aligned} \tag{A.1}$$

and use this expansion in our calculations to determine the relevant coefficients. That is, we now can find $d((\mu\nu)\eta)^*$ and $d((\eta\nu)\mu)^*$. Start with $d((\mu\nu)\eta)^* = c_1 \cdot (\mu\nu)^*\eta^* + c_2 \cdot (\mu\eta)^*\nu^*$:

$$\langle d((\mu\nu)\eta)^* \,;\, \mu\nu,\, \eta\rangle \;=\; \langle c_1 \cdot (\mu\nu)^*\eta^* + c_2 \cdot (\mu\eta)^*\nu^* \,;\, \mu\nu,\, \eta\rangle \;=\; c_1 \cdot (-1)^{|\mu\nu||\eta|}$$

On the other hand, we get

$$\langle d((\mu\nu)\eta)^* \,;\, \mu\nu,\, \eta\rangle \;=\; (-1)^{|\mu\nu|+|\eta|-1} \;=\; (-1)^{|\mu|+|\eta|+|\nu|},$$

so we have

$$c_1 \;=\; (-1)^{|\mu|+|\eta|+|\nu|} \cdot (-1)^{(|\mu|+|\nu|-1)|\eta|} \;=\; (-1)^{(|\eta|+1)(|\mu|+|\nu|)}.$$

Now we find c_2. We first have

$$\langle d((\mu\nu)\eta)^* \,;\, \mu\eta,\, \nu\rangle \;=\; \langle c_1 \cdot (\mu\nu)^*\eta^* + c_2 \cdot (\mu\eta)^*\nu^* \,;\, \mu\eta,\, \nu\rangle \;=\; c_2 \cdot (-1)^{|\mu\eta||\nu|},$$

and we also know that

$$\begin{aligned}\langle d((\mu\nu)\eta)^* \;;\; \mu\eta,\, \nu\rangle \;&=\; (-1)^{|\mu\nu|+|\eta|-1}\langle((\mu\nu)\eta)^* \,;\, (\mu\eta)\nu\rangle\\ &=\; (-1)^{|\mu|+|\eta|+|\nu|}\langle((\mu\nu)\eta)^* \,;\, (-1)^{|\mu|(|\eta|+|\nu|)+1}(\eta\nu)\mu + (-1)^{|\eta||\nu|+1}(\mu\nu)\eta\rangle\\ &=\; (-1)^{|\mu|+|\eta|+|\nu|} \cdot (-1)^{|\eta||\nu|+1} \;=\; (-1)^{|\eta||\nu|+|\mu|+|\eta|+|\nu|+1}\end{aligned}$$

(Note that we used equation (A.1) in this step.)

Now we get

$$\begin{aligned}c_2 \;&=\; (-1)^{|\eta||\nu|+|\mu|+|\eta|+|\nu|+1} \cdot (-1)^{(|\mu|+|\eta|-1)|\nu|}\\ &=\; (-1)^{|\mu||\nu|+|\mu|+|\eta|+1}\end{aligned}$$

So we have found both coefficients and we have

$$d\big(((\mu\nu)\eta)^*\big) \;=\; (-1)^{(|\eta|+1)(|\mu|+|\nu|)}(\mu\nu)^*\eta^* + (-1)^{|\mu||\nu|+|\mu|+|\eta|+1}(\mu\eta)^*\nu^* \tag{A.2}$$

The only difference in determining the coefficients for $d((\eta\nu)\mu)^* = c_3 \cdot (\eta\nu)^*\mu^* + c_4 \cdot (\mu\eta)^*\nu^*$ is that when we use equation (A.1) to make a substitution, we pick up the other term's coefficient. So we will get c_3 looks just like c_1, but with $|\mu|$

swapped with $|\eta|$, and we have

$$c_4 \;=\; (-1)^{|\mu|+|\eta|+|\nu|} \cdot (-1)^{|\mu|(|\eta|+|\nu|)+1} \;=\; (-1)^{(|\mu|+1)(|\eta|+|\nu|+1)}.$$

So we have

$$d\big(((\eta\nu)\mu)^*\big) \;=\; (-1)^{(|\mu|+1)(|\eta|+|\nu|)}(\eta\nu)^*\mu^* + (-1)^{(|\mu|+1)(|\eta|+|\nu|+1)}(\mu\eta)^*\nu^* \tag{A.3}$$

□

A.1.2 Specific Calculations

Now we take our generators as in Section A.1. The calculations we made in Section A.1.1 allow us to easily calculate our differential for each of the two-fold products we have in dimension $a+b-1$ of the model for the Postnikov section.

Of those listed in Theorem A.1.3, we know the first five two-fold products must always be in the Postnikov section, since $|\gamma| > a$. The next two are only in there when $|\beta_2| > a$, and the last two are in there when $|\beta_1| > a$.

Theorem A.1.3. *The differential is defined as follows:*

$$d\big((\gamma\beta_2)^*(\beta_1\alpha)^*\big) = (-1)^{(z+1)(w_2+1)+(z+w_2)(w_1+a-1)}(\beta_1\alpha)^*\gamma^*\beta_2^* \\ + (-1)^{w_1a+z+w_2+w_1+a}(\gamma\beta_2)^*\beta_1^*\alpha^*,$$

$$d\big((\gamma\beta_1)^*(\beta_2\alpha)^*\big) = (-1)^{(z+1)(w_1+1)+(z+w_1)(w_2+a-1)}(\beta_2\alpha)^*\gamma^*\beta_1^* \\ + (-1)^{w_2a+z+w_2+w_1+a}(\gamma\beta_1)^*\beta_2^*\alpha^*,$$

$$d\big((\gamma\alpha)^*(\beta_2\beta_1)^*\big) = (-1)^{(z+1)(a+1)+(z+a)(w_2+w_1-1)}(\beta_2\beta_1)^*\gamma^*\alpha^* \\ + (-1)^{w_2w_1+z+w_2+w_1+a}(\gamma\alpha)^*\beta_2^*\beta_1^*,$$

$$d\big(((\beta_2\alpha)\beta_1)^*\gamma^*\big) = (-1)^{(w_1+1)(w_2+a)+zw_1}(\beta_2\alpha)^*\gamma^*\beta_1^* \\ + (-1)^{za+w_2a+w_2+w_1+1}(\beta_2\beta_1)^*\gamma^*\alpha^*,$$

$$d\big(((\gamma\alpha)\beta_1)^*\beta_2^*\big) = (-1)^{(w_1+1)(z+a)+w_2w_1}(\gamma\alpha)^*\beta_2^*\beta_1^* \\ + (-1)^{za+w_2a+z+w_1+1}(\gamma\beta_1)^*\beta_2^*\alpha^*,$$

$$d\big(((\gamma\alpha)\beta_2)^*\beta_1^*\big) = (-1)^{(w_2+1)(z+a)}(\gamma\alpha)^*\beta_2^*\beta_1^* \\ + (-1)^{za+w_1a+z+w_2+1}(\gamma\beta_2)^*\beta_1^*\alpha^*,$$

$$d\big(((\beta_1\alpha)\beta_2)^*\gamma^*\big) = (-1)^{(w_2+1)(w_1+a)+zw_2}(\beta_1\alpha)^*\gamma^*\beta_2^* \\ + (-1)^{za+w_2w_1+w_1a+w_2+w_1+1}(\beta_2\beta_1)^*\gamma^*\alpha^*,$$

$$d\big(((\beta_1\alpha)\gamma)^*\beta_2^*\big) = (-1)^{(z+1)(w_1+a)}(\beta_1\alpha)^*\gamma^*\beta_2^* \\ + (-1)^{zw_1+w_2a+w_1a+z+w_1+1}(\gamma\beta_1)^*\beta_2^*\alpha^*,$$

and

$$d\big(((\beta_2\alpha)\gamma)^*\beta_1^*\big) = (-1)^{(z+1)(w_2+a)}(\beta_2\alpha)^*\gamma^*\beta_1^* \\ + (-1)^{zw_2+w_2a+w_1a+z+w_2+1}(\gamma\beta_2)^*\beta_1^*\alpha^*.$$

Proof. We will need equation (2.3) for all calculations. Then, for the first three differentials, we also use Theorem (A.1.1). For the next three, we use equation (A.2) in Theorem (A.1.2). And for the final three differentials, we use equation (A.3) in Theorem (A.1.2).

For example, in the first calculation, we have

$$\begin{aligned} d\big((\gamma\beta_2)^*(\beta_1\alpha)^*\big) &= (-1)^{(z+1)(w_2+1)}\gamma^*\beta_2^*(\beta_1\alpha)^* \\ &\qquad\qquad + (-1)^{z+w_2-1}\cdot(-1)^{(w_1+1)(a+1)}(\gamma\beta_2)^*\beta_1^*\alpha^* \\ &= (-1)^{(z+1)(w_2+1)+(z+w_2)(w_1+a-1)}(\beta_1\alpha)^*\gamma^*\beta_2^* \\ &\qquad\qquad + (-1)^{w_1a+z+w_2+w_1+a}(\gamma\beta_2)^*\beta_1^*\alpha^*, \end{aligned}$$

$\square$

Now that we have our differential figured out on the two-fold products involving no repeated factors, we can form our six-by-nine matrix of coefficients, M, but with the entries only labeled as the actual values will not fit.

Figure A.1: Differential Matrix

	$(\gamma\beta_2)^*(\beta_1\alpha)^*$	$(\gamma\beta_1)^*(\beta_2\alpha)^*$	$(\gamma\alpha)^*(\beta_2\beta_1)^*$	$((\beta_1\alpha)\beta_2)^*\gamma^*$	$((\beta_2\alpha)\beta_1)^*\gamma^*$	$((\gamma\alpha)\beta_1)^*\beta_2^*$	$((\beta_1\alpha)\gamma)^*\beta_2^*$	$((\gamma\alpha)\beta_2)^*\beta_1^*$	$((\beta_2\alpha)\gamma)^*\beta_1^*$
$(\gamma\beta_2)^*\beta_1^*\alpha^*$	m_{11}							m_{18}	m_{19}
$(\gamma\beta_1)^*\beta_2^*\alpha^*$		m_{22}				m_{26}	m_{27}		
$(\gamma\alpha)^*\beta_2^*\beta_1^*$			m_{33}			m_{36}		m_{38}	
$(\beta_2\beta_1)^*\gamma^*\alpha^*$			m_{43}	m_{44}	m_{45}				
$(\beta_2\alpha)^*\gamma^*\beta_1^*$		m_{52}			m_{55}				m_{59}
$(\beta_1\alpha)^*\gamma^*\beta_2^*$	m_{61}			m_{64}			m_{67}		

Figure A.1 has the following values for its entries:

$$m_{11} = (-1)^{w_1a+z+w_2+w_1+a}$$
$$m_{22} = (-1)^{w_2a+z+w_2+w_1+a}$$
$$m_{33} = (-1)^{w_2w_1+z+w_2+w_1+a}$$
$$m_{44} = (-1)^{za+w_2w_1+w_1a+w_2+w_1+1}$$
$$m_{45} = (-1)^{za+w_2a+w_2+w_1+1}$$
$$m_{26} = (-1)^{za+w_2a+z+w_1+1}$$
$$m_{27} = (-1)^{zw_1+w_2a+w_1a+z+w_1+1}$$
$$m_{18} = (-1)^{za+w_1a+z+w_2+1}$$
$$m_{19} = (-1)^{zw_2+w_2a+w_1a+z+w_2+1}$$

$$m_{61} = (-1)^{(w_2+1)(z+1)+(z+w_2)(w_1+a+1)}$$
$$m_{52} = (-1)^{(z+1)(w_1+1)+(z+w_1)(w_2+a+1)}$$
$$m_{43} = (-1)^{(z+1)(a+1)+(z+a)(w_2+w_1+1)}$$
$$m_{64} = (-1)^{(w_2+1)(w_1+a)+zw_2}$$
$$m_{55} = (-1)^{(w_1+1)(w_2+a)+zw_1}$$
$$m_{36} = (-1)^{(w_1+1)(z+a)+w_2w_1}$$
$$m_{67} = (-1)^{(z+1)(w_1+a)}$$
$$m_{38} = (-1)^{(w_2+1)(z+a)}$$
$$m_{59} = (-1)^{(z+1)(w_2+a)}$$

Appendix B

Matrix Reduction

We are going to start with the matrix in Figure A.1 and add two column vectors given by τ in (6.1) so that we can solve the equations $M\overline{x} = \tau$ for both of those τ vectors. We will row-reduce this to find that we will always have solutions.

B.1 Notation to Simplify

Before we get to the reduction, we will rewrite our matrix in terms of variables which are not exponents to make the matrices simpler to read. It may appear cumbersome, but it allows for easier row-reduction.

Define the following:

$$
\begin{aligned}
k &= (-1)^{a}, & l &= (-1)^{w_1}, \\
m &= (-1)^{w_2}, & n &= (-1)^{z}, \\
p &= klmn, & & \\
q &= (-1)^{w_1 a}, & r &= (-1)^{w_2 a}, \\
s &= (-1)^{za}, & t &= (-1)^{w_2 w_1}, \\
u &= (-1)^{z w_1}, & v &= (-1)^{z w_2}, \\
\pi &= qrstuv. & &
\end{aligned}
$$

Then our matrix can be written as that of Figure B.1.

Now if we insert the column vectors given by τ in (6.1), we get the matrix in Figure B.2.

Figure B.1: Simplified Matrix

	$(\gamma\beta_2)^*(\beta_1\alpha)^*$	$(\gamma\beta_1)^*(\beta_2\alpha)^*$	$(\gamma\alpha)^*(\beta_2\beta_1)^*$	$((\beta_1\alpha)\beta_2)^*\gamma^*$	$((\beta_2\alpha)\beta_1)^*\gamma^*$	$((\gamma\alpha)\beta_1)^*\beta_2^*$	$((\beta_1\alpha)\gamma)^*\beta_2^*$	$((\gamma\alpha)\beta_2)^*\beta_1^*$	$((\beta_2\alpha)\gamma)^*\beta_1^*$
$(\gamma\beta_2)^*\beta_1^*\alpha^*$	pq							$-mnqs$	$-mnqrv$
$(\gamma\beta_1)^*\beta_2^*\alpha^*$		pr				$-lnrs$	$-lnqru$		
$(\gamma\alpha)^*\beta_2^*\beta_1^*$			pt			$knqtu$		$knrv$	
$(\beta_2\beta_1)^*\gamma^*\alpha^*$			$-\pi t$	$-lmqst$	$-lmrs$				
$(\beta_2\alpha)^*\gamma^*\beta_1^*$		$-\pi r$			$kmqtu$				$kmsv$
$(\beta_1\alpha)^*\gamma^*\beta_2^*$	$-\pi q$			$klrtv$			$klsu$		

Figure B.2: Simplified Matrix

$$\left(\begin{array}{ccccccccc|cc}
pq & 0 & 0 & 0 & 0 & 0 & 0 & -mnqs & -mnqrv & -mnv & -mnv \\
0 & pr & 0 & 0 & 0 & -lnrs & -lnqru & 0 & 0 & lntu & 0 \\
0 & 0 & pt & 0 & 0 & knqtu & 0 & knrv & 0 & 0 & 0 \\
0 & 0 & -\pi t & -lmqst & -lmrs & 0 & 0 & 0 & 0 & 0 & lmtuv \\
0 & -\pi r & 0 & 0 & kmqtu & 0 & 0 & 0 & kmsv & 0 & 0 \\
-\pi q & 0 & 0 & klrtv & 0 & 0 & klsu & 0 & 0 & 0 & 0
\end{array}\right)$$

We will use this matrix in our proof of Theorem 6.3.1.

B.2 Time to Row-Reduce

The first step we will make will be to multiply each row by a value to get leading ones. That is, we perform these row operations:

$$\begin{aligned}(pq)\cdot R1 &\rightarrow R1\\(pr)\cdot R2 &\rightarrow R2\\(pt)\cdot R3 &\rightarrow R3\\(\pi t)\cdot R4 &\rightarrow R4\\(\pi r)\cdot R5 &\rightarrow R5\\(\pi q)\cdot R6 &\rightarrow R6\end{aligned}$$

This gives us

$$\left(\begin{array}{ccccccccc|cc}1 & 0 & 0 & 0 & 0 & 0 & 0 & -kls & -klrv & -klqv & -klqv\\0 & 1 & 0 & 0 & 0 & -kms & -kmqu & 0 & 0 & kmrtu & 0\\0 & 0 & 1 & 0 & 0 & lmqu & 0 & lmrtv & 0 & 0 & 0\\0 & 0 & \text{-1} & -lmrtuv & -lmquv & 0 & 0 & 0 & 0 & 0 & lm\pi uv\\0 & \text{-1} & 0 & 0 & kmsv & 0 & 0 & 0 & kmqtu & 0 & 0\\\text{-1} & 0 & 0 & klsu & 0 & 0 & klrtv & 0 & 0 & 0 & 0\end{array}\right).$$

Then we do

$$\begin{aligned}R4+R3 &\rightarrow R4\\R5+R2 &\rightarrow R5\\R6+R1 &\rightarrow R6\end{aligned}$$

followed by

$$\begin{aligned}(-lm\pi qs)\cdot R4 &\rightarrow R4\\(kmsv)\cdot R5 &\rightarrow R5\\(klsu)\cdot R6 &\rightarrow R6\end{aligned}$$

to get the matrix

$$\left(\begin{array}{ccccccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -kls & -klrv & -klqv & -klqv \\
0 & 1 & 0 & 0 & 0 & -kms & -kmqu & 0 & 0 & kmrtu & 0 \\
0 & 0 & 1 & 0 & 0 & lmqu & 0 & lmrtv & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & qrt & -\pi su & 0 & -u & 0 & 0 & -\pi rt \\
0 & 0 & 0 & 0 & 1 & -v & -\pi rt & 0 & \pi r & \pi q & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \pi q & -u & -\pi qt & -\pi rt & -\pi rt
\end{array}\right).$$

Next we do only $R4 - (qrt) \cdot R5 \to R4$ to give

$$\left(\begin{array}{ccccccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -kls & -klrv & -klqv & -klqv \\
0 & 1 & 0 & 0 & 0 & -kms & -kmqu & 0 & 0 & kmrtu & 0 \\
0 & 0 & 1 & 0 & 0 & lmqu & 0 & lmrtv & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \pi q & -u & -\pi qt & -\pi rt & -\pi rt \\
0 & 0 & 0 & 0 & 1 & -v & -\pi rt & 0 & \pi r & \pi q & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \pi q & -u & -\pi qt & -\pi rt & -\pi rt
\end{array}\right),$$

and finally, we subtract $R6 - R4 \to R6$ and have

$$\left(\begin{array}{ccccccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -kls & -klrv & -klqv & -klqv \\
0 & 1 & 0 & 0 & 0 & -kms & -kmqu & 0 & 0 & kmrtu & 0 \\
0 & 0 & 1 & 0 & 0 & lmqu & 0 & lmrtv & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \pi q & -u & -\pi qt & -\pi rt & -\pi rt \\
0 & 0 & 0 & 0 & 1 & -v & -\pi rt & 0 & \pi r & \pi q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

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