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Classifying Spaces of Symmetric Groups and Wreath Products

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CLASSIFYING SPACES OF SYMMETRIC GROUPS AND WREATH PRODUCTS

by

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This thesis was motivated by a desire to better understand the structure of classifying spaces of symmetric groups. The results contained in this thesis fall into two categories: general results about stable splittings or the groups we will work with, and specific results about the stable splittings for $B\mathfrak{S}_8$ and $B(Z_2 \wr Z_2 \wr Z_2)$, completed at the prime 2. Regarding the splittings, $Z_2 \wr Z_2 \wr Z_2$ is the largest 2-group outside of some specific families to have its classifying space completely split, and this document offers an example of the theory on splittings developed by Martino and Priddy applied to the symmetric group $\mathfrak{S}_8$, a group which is not a p-group.

Amongst the general results are group theoretic results about the n-fold iterated wreath product of Z_p with itself, denoted ${Z_p}^{\wr n}$. We explore the structure of its automorphism group and its maximal elementary abelian subgroups. Using this information, we determine the number of original summands of $B{Z_p}^{\wr n}$ and which of those summands appear in $B\mathfrak{S}_{p^n}$. We also relate their cohomological structure to an algebra of Dickson invariants.

We also present two examples involving linkage. It has been hypothesized that all linkage is strong linkage; we present a counterexample. It was also thought that isomorphic summands of a classifying space would either all be linked in that space or all be linked in the spaces for some collection of subgroups. Again, we identify a counterexample.

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Dedicated to my Mother and Father

for being proud of me

even when I was not

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Notation

The following notation will be used:

$X := Y$	A Definition: X is defined to be Y
Z_n	The cyclic group on n letters
$\mathfrak{S}_n$	The symmetric group on n letters
$\mathbb{Z}_p^{\hat{}}$	The p-adic integers
U	The group of upper triangular matrices in $GL_n(\mathbb{F}_2)$
$G^{\wr n}$	The n-fold iterated wreath product of G with itself
$\mathbb{F}_n$	The finite field of n elements
τ_n	The permutation $\prod_{i=1}^{p^{(n-1)}}(i\ p^{n-1}+i \ldots (p-1)p^{n-1}+i)$
Δ	The diagonal embedding of H into the base of $H \wr G$
ζ	The automorphism of $Z_2^{\wr n}$ defined on generators by $(\tau_1, \tau_2, \tau_3, \ldots, \tau_n) \mapsto (\tau_2, \tau_1, \tau_3, \ldots, \tau_n)$
c_x	The group homomorphism induced by conjugation by x
$A <_G B$	A is a subgroup of B up to conjugation in G
BG	The classifying space for the group G
$X_p^{\hat{}}$	The p-completion of the spectrum X
X_{abcd}	The original summand of $(Z_2)^n$ corresponding to the simple $GL_n -$ module with tableaux column lengths $abcd$
$L(n)$	The Steinberg summand of BZ_2^n
tr	The transfer map

Chapter 1

Introduction

For a group G, there are several constructions of the classifying space BG, a target object which classifies principal G-bundles over paracompact topological spaces. This construction is functorial, taking groups to spaces and group homomorphisms to continuous maps ([16]). They are often studied stably, and BG is used to denote both the suspension spectrum and the space. The cohomology theory which comes from mapping into the spectrum BG is the same as the group cohomology with coefficients from G, where applicable ([5]). Our area of interest will be the study of these spectra, specifically when G is the symmetric group $\mathfrak{S}_{p^n}$ or its Sylow p-subgroup.

Recall that given a group G and a permutation group $H \leq \mathfrak{S}_n$, the wreath product $G \wr H$ is the semidirect product

$$\prod_{i=1}^{n} G \rtimes H$$

where the conjugation action of H permutes the factors of $\prod G$ according to the given embedding $H \leq \mathfrak{S}_n$. The normal subgroup $\prod G$ is called the *base* of the wreath product. The Sylow subgroup of $\mathfrak{S}_{p^n}$ is the n-fold "iterated wreath product" $Z_p \wr Z_p \wr \ldots \wr Z_p$. This n-fold iterated wreath product will be denoted $Z_p^{\wr n}$.

A *stable splitting* of a spectrum X is a homotopy equivalence

$$X \simeq \bigvee_{i \in I} Y_i \tag{1.1}$$

between X and a wedge product of other spectra. If any stable splitting of X has only one nontrivial wedge summand, X is said to be *indecomposable*. A stable splitting is called *complete* if all of the Y_i are indecomposable. If a complete stable splitting exists, it is unique up to permutation of factors. In the case of classifying spaces of finite groups, it is known (see [4]) that there is a stable splitting

$$BG \simeq \bigvee_{p||G|} BG\hat{_p} \tag{1.2}$$

where $BG\hat{_p}$ denotes the p-completion (or in this case equivalently the p-localization) for primes p dividing the order of G. As the p-completions are more accessible, we shall focus our attention on those: for the rest of the paper we will deal exclusively with spectra rather than topological spaces. All spectra are assumed to be completed at a chosen prime p, and cohomology is taken with Z_p coefficients. By Equation 1.2, we can reconstruct a splitting for any BG if we can determine splittings for the p-completions, so it makes sense to work one prime at a time.

For any spectrum $\mathcal{A}$, the collection of stable maps $\{\mathcal{A}, \mathcal{A}\}$ has a ring structure, where the addition comes from $\mathcal{A}$ being a cogroup object and the multiplication from composition. There is an intimate connection between stable splittings and this ring:

given a finite stable splitting as in Equation 1.2, we can produce a decomposition of the identity into idempotents

$$id_{\mathcal{A}} = \sum_{i \in I} e_i$$

where e_i is given by projection onto and inclusion of the corresponding wedge summand. Similarly, given a decomposition of the identity id_X into idempotents e_i, we can recover Y_i as the colimit of

$$\mathcal{A} \stackrel{e_i}{\to} \mathcal{A} \stackrel{e_i}{\to} \mathcal{A} \stackrel{e_i}{\to} \dots$$

This gives a one to one correspondence between idempotents and summands, and it is immediate that the summand is indecomposable if and only if the idempotent is primitive. One of the important aspects of this correspondence is that the identity map of $\{Y_i, Y_i\}$ factors through e_i. In a more general situation, if the identity map of $\{Y_i, Y_i\}$ factors through a stable map $f : \mathcal{A} \to \mathcal{B}$, we say that f *carries* Y_i, and we must have that if there exist complete stable splittings for $\mathcal{A}$ and $\mathcal{B}$, then Y_i appears as one of the wedge factors of both.

For G a finite group, $\{BG, BG\}$ is an Artinian ring, so it is often of value to keep in mind the classical correspondence between primitive idempotents and simple modules. This perspective forms the underpinnings of some of the major theorems in the field ([24], [2]), and throughout this thesis we will change perspectives between indecomposible stable summands, primitive idempotents, and simple modules. Of particular note is the idempotent e_X corresponding to the trivial module: the summand X associated to this idempotent is called the *principal original summand*.

Carlsson's proof of the Segal conjecture In the mid 1980s ([7]) describes the set of stable maps $\{BG, BG'\}$ between the classifying spaces of two groups G and G'. In particular, if $G = P$, $G' = P'$ are p-groups then $\{BP, BP'\}$ has as a $\mathbb{Z}_p\hat{}$ basis maps of the form

$$BP \xrightarrow{tr} BQ \xrightarrow{Bf} BP'$$

where tr denotes the transfer map between spectra (which corresponds with the classical transfer from group cohomology after applying H^*), and $f : Q \to P'$ is a group homomorphism. Of particular interest, for G a group with Sylow p-subgroup P and inclusion $i : P \to G$, the composition

$$BP \xrightarrow{tr} BG \xrightarrow{Bi} BP$$

is an essential idempotent, so BG is a summand of BP, contrary to one's usual intuition. As a consequence, much of the emphasis on stable splittings for classifying spaces of finite groups begins with splittings for classifying spaces of p-groups.

Note that for any term of this Segal conjecture basis, there is a particular subgroup Q' of P' of minimum order that the map factors through. From this foundation, Nishida ([31]) established that for every stable summand X of a space BG, there is a p-subgroup P of G of minimum order which contains X, and that X cannot appear in BG' unless G' contains a subgroup isomorphic to P. He did this by defining, for every p-group P, an ideal $J(P)$ of $\{BP, BP\}$ generated by all the maps which factor through proper subgroups of Q. He then distinguished between those summands X of BP whose idempotents lay in $J(P)$ and those whose idempotents survived the quotient mapping

$$\{BP, BP\} \xrightarrow{\text{mod } J} Z_p\hat{}\text{Out }(P) \longrightarrow \mathbb{F}_p\text{Out }(P)$$

He called the summands whose idempotents mapped to $\mathbb{F}_p\mathrm{Out}\ (P)$ *dominant* in P. We shall say that they are *original* to P or BP, or that they *originate in* P or BP.

This connection between the automorphism group of p-groups and stable summands has resulted in some interest in automorphism groups ([9], [8]). There are two theorems of this thesis which bear directly on this topic. The first is a broad theorem relating the automorphism group of Q for certain p-groups to the automorphism group of the product $Q \times Q \times \ldots Q$:

Theorem (3.4.1)**.** *Let Q be a p-group satisfying the following conditions:*

- *The center of Q lies inside the Frattini subgroup (i.e. $Z(Q) \leq \Phi(Q)$)*
- *Q is indecomposable; e.g. $Q \ncong Q_1 \times Q_2$ for Q_i nontrivial*
- *Q has a finite generating set*

Then Aut $(\prod_{i=1}^{m} Q)$ is isomorphic to an extension of Aut $(Q) \wr \mathfrak{S}_m$ by a p-group.

This theorem provides a weak compliment to the papers [15] and [25], which cover cases where the factors are elementary abelian or distinct, respectively.

The second theorem deals specifically with the groups of interest, $BZ_p^{\wr n}$ and $B\mathfrak{S}_{p^n}$. Keeping the correspondence between original summands of BP and idempotents of $\mathbb{F}_p\mathrm{Out}\ (P)$ in mind, it would be nice to be able to describe at least the number of the original summands of $BZ_p^{\wr n}$, and better yet, to describe which of these appear in $B\mathfrak{S}_{p^n}$. A complete result in this direction is given:

Theorem (3.3.4)**.** *There are $(p-1)^n$ nonisomorphic original summands of $BZ_p^{\wr n}$, each with multiplicity one. Of these original summands, $B\mathfrak{S}_{p^n}$ contains only the principal original summand.*

Unfortunately, this theorem and its method of proof tell us very little about the the topological structure of these summands. One way to describe them would be to learn something about their cohomology. There has been significant interest in the cohomology of the symmetric groups ([29],[11]). One approach to finding the cohomology of symmetric groups is to use detection theorems: for G a symmetric group or iterated wreath product, for every class x of $H^*(BG)$ there is some maximal elementary abelian subgroup E with inclusion $j : E \to G$ such that $j^*(x) \neq 0$. One particular maximal elementary abelian subgroup of $\mathfrak{S}_{p^n}$ is of particular importance: the one corresponding to the Cayley theorem embedding of ${Z_p}^n$ into $\mathfrak{S}_{p^n}$, denoted V_n. The image of $j^* : H^*(\mathfrak{S}_{p^n}) \to H^*(V_n)$ contains the entire Dickson polynomial algebra of GL_n invariants (and, depending on p and n, portions of the exterior algebra). Using this information, we prove

Theorem (4.1.6)**.** *When $p = 2$, The induced mapping j^* factors through the cohomology of the original summand of $BZ_2^{\wr n}$:*

$$H^*(BZ_2^{\wr n}) \to H^*(Orig(Z_2^{\wr n})) \to H^*(BV_n).$$

There is an analogous result for $p \neq 2$ involving the principal original summand of $BZ_p^{\wr n}$.

Another method of approaching the structure of a classifying space is to be able to describe carefully certain lattices of subgroups, such as the lattice of elementary abelian subgroups or the lattice of p-subgroups. Treating such a lattice as a category with morphisms given by inclusion and conjugation, the classifying space can be realized as the nerve of the category [10]. From this perspective, the elementary abelian subgroups of iterated wreath products are of some interest. We explore these subgroups; $\Delta : G \to G \times G$ denotes the diagonal embedding $\Delta(g) = (g, g)$.

Theorem (3.2.2)**.** *A maximal elementary abelian subgroup E of the $Z_p^{\wr n}$ is built from maximal elementary abelian subgroups A_i of $Z_p^{\wr n-1}$ in one of three ways:*

1. *E is of the form $\prod_{i=1}^{p} A^{g_i}$. In other words, all the factors are the conjugate to the same subgroup of $Z_p^{\wr n-1}$, and live in the base of $Z_p^{\wr n-1} \wr Z_p$.*

2. *E is of the form $\prod_{i=1}^{p} A_i$, with differing factors, not all of which are conjugate, again living in the base unless $p = 2$.*

3. *E is conjugate to $\Delta(A) \times \langle \tau_n \rangle$.*

Theorem (3.2.4)**.** *Let $N_k(P)$ denote the normalizer of P in $Z_p^{\wr k}$. The maximal elementary abelian subgroups listed above have normalizers*

1. $\left(\prod_{i=1}^{p} N_{n-1}(A)^{\tau_n^i}\right) \rtimes \langle (g_1, g_1^{-1} g_2, \ldots, \left(\prod_{i=1}^{n-1} g_{n-i}^{-1}\right) g_n; \tau_n) \rangle \cong N_{n-1}(A) \wr Z_p$.

2. $\left(\prod_{i=1}^{p} N_{n-1}(A)^{\tau_n^i}\right)$.

3. *If $p = 2$, conjugate to $(\Delta(N_{n-1}(A)) \times \langle \tau_n \rangle)\, i(A)$, where i embeds A into the first coordinate of the base.*

While this does describe the maximal elementary abelian subgroups, it is insufficient to describe their intersections, especially when we are considering subgroups of $\mathfrak{S}_{p^n}$. Therefore these results were not applied directly to the problem of splitting $BZ_p^{\wr n}$ or $B\mathfrak{S}_{p^n}$.

Apart from the general results about splittings for these families of groups, one of the stated goals of the thesis was to produce complete stable splittings of $B\mathfrak{S}_8$ and its Sylow 2-subgroup, $BZ_2^{\wr 3}$. The splittings can be found at the end of

Chapter 5. The hope was that these splittings would provide enough examples to illuminate how the families would split. Unfortunately, this did not happen: the appearance of summands from subgroups both large and small did not fit into a recognizable pattern. However, in the process some examples of unexpected behavior were unearthed. We will need some additional definitions to describe the results.

Let X be an original summand of BP. There is a corresponding $\{BP, BP\}$ idempotent e_X, and its image $\hat{e}_X$ in $\mathbb{F}_p\text{Out}\ (P)$. There is also an inclusion ι : $\mathbb{F}_p\text{Out}\ (P) \to \{BP, BP\}$. The idempotent $\iota(\hat{e}_X)$ is no longer necessarily primitive. It contains e_X as a summand, but may have other summands e_Y where Y is original to some proper subgroup $Q \leq P$. These summands Y are said to be *associated* to X. As ι maps the identity to the identity, every summand is either original or associated to some original summand.

Given Y an indecomposable summand of BP, we say Y is *linked* in BP if, when we express the idempotent e_Y corresponding to Y in the Segal Conjecture basis, all the terms which carry Y do not contain a transfer to a proper subgroup of BP. If Y is not original to BP, then we say Y is *linked to* the original summand X that it is associated to. The reason for the terminology is that if the composition

$$BP \xrightarrow{Bi} BG \xrightarrow{tr} BP$$

carries X, then it carries any summand linked to X. If P is abelian, the converse is true: if the above composition carries Y, a summand linked in BP, then it also carries X, the original summand Y is linked to. It was hypothesized that this converse was true in general. In the process of identifying where summands of $BZ_2^{\wr 3}$ are linked, we found that an original summand of BD_8 is linked to the original summand of $BD_8 \times D_8$, and that the former appears in $BD_8 \wr Z_2$ but the latter does not; see Section 3.5 for details.

It was also thought that if Y was linked in BP, then all summands isomorphic to Y would be linked in BP. We also found that this is not the case; a particular counterexample is summands isomorphic to $L(2)$ (the Steinberg summand of $B{Z_2}^2$) appearing in $B(D_8 \times Z_2 \times Z_2)$.

Before passing to the body of the thesis, we would like to note its organization. Generally, results are grouped by their proof technique. The second chapter (Preliminaries) covers the details of other work that we will need, or straightforward calculations based on other work. The third chapter covers group theoretic results, the fourth cohomological ones, and the fifth chapter contains the splitting calculation for $BZ_2^{\wr 3}$ and $B\mathfrak{S}_8$. At several points during the development of this thesis, the computer algebra package `GAP` was relied upon; specialized code developed for our needs is given in the first appendix.

Chapter 2

Preliminaries

The aim of this chapter is a detailed introduction to the concepts, theorems and details which underlie the work done in this thesis. The knowledgable reader may be able to skim this Chapter with the exception of subsection 2.1.4.

2.1 Prerequisite Results: Stable Splittings

2.1.1 The Segal Conjecture

A complete decomposition of BG into indecomposable summands corresponds to a primitive decomposition of the identity in $\{BG, BG\}$. With Carlsson's solution of the Segal Conjecture [7], the structure of this ring became more accessible by the previous work of Lewis, May and McClure [18]. Lewis, May and McClure begin with a generalization of the Grothendieck group of finite left G-sets. For G' another finite group, define $A(G, G')$ to be the Grothendieck group of isomorphism classes of finite $G \times G'$ sets with free right G' action. Addition in $A(G, G')$ is given by disjoint union of sets. If $G = G'$ the Cartesian product gives a multiplication. Then

$A(G, G')$ is the free abelian group with basis the transitive $G \times G'$ sets. These sets have the form

$$G \times_f G' = (G \times G')/H_f$$

for a choice of subgroup $H < G$ and homomorphism $f : H \to G'$, where $H_f := \{(h, f(h))|h \in H\}$.

Letting BG_+ denote BG with a disjoint basepoint, we can use the transfer to define a homomorphism

$$\alpha : A(G, G') \to \{BG, BG'\}$$

by

$$\alpha(G \times_f G') := BG_+ \xrightarrow{tr_+} BH_+ \xrightarrow{Bf_+} BG'_+.$$

The behavior of the disjoint basepoint is not of interest, and therefore it is eliminated from the ring structure as follows. The usual Burnside ring $A(G)$ is $A(G, \{1\})$, which defines an augmentation homomorphism $\epsilon : A(G, G') \to A(G)$ by $\epsilon(G \times_f G') := G/H$. Define $\tilde{A}(G, G') := \ker \epsilon$. Then $\tilde{A}(G, G')$ is free abelian on classes $G \times_f G - (G/H \times G')$, with multiplication given by the following form of the Mackey Formula.

Proposition 2.1.1 (The Mackey Formula)**.** *Let $f : G \to H$ be a group homomorphism, and K a subgroup of H. Let t run through a set of double coset representatives for $f(G)$ and K in H and let that set of representatives be finite. Denote by c_t the group homomorphism induced by conjugation by t. Then the following diagram* *commutes:*

$$\begin{array}{ccccc}
BG & \xrightarrow{Bf} & BH & \xrightarrow{tr} & BK \\
{\scriptstyle pinch}\downarrow & & & & \\
\bigvee_t BG & & & & \uparrow {\scriptstyle \sum Bi} \\
{\scriptstyle \bigvee tr}\downarrow & & & & \\
\bigvee B\left(f^{-1}(f(G)\cap K^{t^{-1}})\right) & \xrightarrow{\bigvee Bf} & \bigvee B\left(f(G)\cap K^{t^{-1}}\right) & \xrightarrow{\bigvee Bc_t} & \bigvee B\left(f(G)^t\cap K\right)
\end{array}$$

Returning to $\tilde{A}(G,G')$, α then induces a homomorphism

$$\tilde{\alpha} : \tilde{A}(G,G') \to \{BG, BG'\}.$$

If $G = G'$, α induces a ring homomorphism. To get an isomorphism, a certain completion is required. For P a p-group, that completion is easy to work with:

Theorem 2.1.2 ((Segal Conjecture [7]), [18])**.** *If P is a p-group, $\tilde{\alpha}$ induces an isomorphism*

$$\tilde{\alpha}\hat{} : \tilde{A}(P,P') \otimes \mathbb{Z}\hat{}_p \to \{BP, BP'\}$$

In other words, for any p-group P, $\{BP, BG\}$ is freely generated as a $\mathbb{Z}\hat{}_p$-module by maps of the form

$$BP \xrightarrow{tr} BQ \xrightarrow{Bh} BG \tag{2.1}$$

subject to equivalence induced by conjugation by some $x \in P$, $y \in G$:

$$\begin{array}{ccccc}
BP & \xrightarrow{tr} & BQ & \xrightarrow{Bh} & BG \\
\downarrow {\scriptstyle c_x} & & \downarrow {\scriptstyle c_x} & & \downarrow {\scriptstyle c_y} \\
BP & \xrightarrow{tr} & BQ^x & \xrightarrow{Bh'} & BG
\end{array}$$

There is an important observation involving transfers and elementary abelian subgroups which we shall use often: namely that transfers to proper subgroups of an elementary abelian group are trivial on cohomology.

2.1.2 Original Summands

Goro Nishida used the Segal Conjecture to describe in great detail the types of summands which could possibly occur in a decomposition of BG for G a finite group [31]. Let P be a finite p-group, and define J_P to be the ideal generated by all elements of $\{BP, BP\}$ of the form

$$BP \to BQ \to BP$$

for some proper subgroup $Q < P$. Then using Theorem 2.1.2, Nishida shows that

Proposition 2.1.3 ([31])**.** *The composite*

$$\hat{\mathbb{Z}_p} Out\,(P) \xrightarrow{i} \{BP, BP\} \overset{mod\ J_P}{\longrightarrow} \{BP, BP\}/J_P$$

is an isomorphism of rings.

For a primitive idempotent e, eBP originates in BP if $e \notin J_P$. Nishida called these summands dominant in BP. By the Segal Conjecture and the fact that for an indecomposable summand Y, $\{Y, Y\}$ is a local ring, every primitive idempotent e_Y of $\{BP, BP\}$ factors through BQ_Y for some particular subgroup $Q_Y \leq P$, where Y is original to Q_Y. This argument gives the claimed description: every summand of BP for P a p-group is original to some subgroup of P. To extend this result to all finite groups, let P denote the Sylow p-subgroup of G with inclusion $i : P \to G$.

Now apply the Mackey formula 2.1.1 to

$$BP \xrightarrow{Bi} BG \xrightarrow{tr} BP.$$

The result mod $J_{(G)_p}$ is a sum of isomorphisms induced by conjugation by elements of G. The number of terms in this sum is not divisible by p, so the sum is an essential idempotent. Therefore BG is a summand of $B(G)_p$, contrary to one's intuition that the total group "contains" it's Sylow p-subgroup. Instead, the indecomposable summands of BG must be indecomposable summands of $B(G)_p$. Therefore, once again we have that all the stable summands of BG are original to some subgroup of G.

2.1.3 The $A(G, M)$ matrix

Using Nishida's work, it is possible to determine a list of potential isomorphism types of stable summands for any classifying space BG: the original summands of BQ, as Q runs through the p-subgroups of G. It remains to determine the multiplicity of these summands. That is the focus of a paper by Martino and Priddy [24]. Several of the ideas which went into the proof of the result will be useful elsewhere in this thesis, so we will explore several of the ingredients in detail.

One important technique is the examination of how an idempotent gets expressed in the Segal conjecture basis (Equation 2.1). Let Y be an indecomposable spectrum original to BQ and appearing in BP. We know that there is an idempotent

$$e_Y : BP \xrightarrow{f} BQ \to Y \to BQ \xrightarrow{g} BP.$$

Expressing $f = \sum f_i$ and $g = \sum g_j$ in the Segal conjecture basis, we get a collection of maps

$$e_{i,j} : BP \xrightarrow{f_i} BQ \to Y \to BQ \xrightarrow{g_j} BP.$$

As Y is an indecomposable spectrum, $\{Y, Y\}$ is a local ring [19], so either

$$Y \to BQ \xrightarrow{g_j} BP \xrightarrow{f_i} BQ \to Y$$

is a unit or it is nilpotent. While we are considering a particular summnd Y we are primarily interested in maps which are units in $\{Y, Y\}$, so we shall focus on those.

Let us first examine g_j. We find that we may presume g_j to be an identity transfer followed by an injection. If g_j began with a transfer to a proper subgroup, that transfer could not carry Y as Y is original to BQ, contradicting the assumption that the pair we are working with form a unit of $\{Y, Y\}$. A similar argument shows that the remaining group homomorphism should have a trivial kernel. Therefore we can express g as a sum of inclusion maps and maps which do not carry Y.

Let us now examine f in a similar fashion. Here we divide our attention into two cases. First let us assume that the transfer portion of f_i is the identity transfer to BP. Then f_i is induced by a surjective group homomorphism. As g_j is induced by an inclusion of groups, $f_i \circ g_j$ must be induced by an isomorphism of Q: it carries the summand Y original to BQ, so it cannot factor through the classifying space of a proper subgroup of Q. Therefore Q is a retract of P. For some summands, all the terms of the corresponding idempotent which carry the summand will begin with the identity transfer, followed by a split surjection. These are the summands which are linked in P.

In the second case, where f_i does contain a nontrivial transfer, say to BP_α, we still must have that Q is a retract of P_α if g_j factors through P_α. Otherwise, we know the composition will not carry Y by the following result:

Proposition 2.1.4. *Let P be a finite p-group, and Q_α, Q_β subgroups such that $Q_\alpha \not\leq Q_\beta^t$ for t a double coset representative for Q_α and Q_β in P. Then when we express the composition*

$$BQ_\alpha \stackrel{Bi}{\to} BP \stackrel{tr}{\to} BQ_\beta$$

in terms of the Segal conjecture basis, it is a sum of maps each with a nonidentity transfer. In particular, no unit map of a summand original to BQ_α or linked in BQ_α can factor through this composition. In particular if Q_α is elementary abelian, the composition is trivial in cohomology.

<u>Proof:</u> Applying the Mackey formula 2.1.1, the above map is equivalent to

$$\sum \left(BQ_\alpha \stackrel{tr}{\to} BQ_\alpha \cap Q_\beta^t \to BQ_\beta \right).$$

We have assumed that $Q_\alpha \cap Q_\beta^t$ is a proper subgroup of Q_α, so the composition has the desired form. Let X be an indecomposable summand of BQ_α. If X is original, then no unit map of $\{X, X\}$ can factor through any of the terms of the above sum because they all factor through proper subgroups. Therefore that unit of $\{X, X\}$ cannot factor through the sum itself as X is indecomposable. Similarly, if X is linked in BQ_α, then by the definition of linkage no unit map of X factors through a transfer to a subgroup.

If Q_α is an elementary abelian subgroup, every transfer is trivial in cohomology, so the above map will be trivial in cohomology. □

Therefore if a summand original to BQ appears in BP, there must be some split surjection $P_\alpha \to Q_\alpha$ for $Q_\alpha \leq P_\alpha \leq P$. Combining these observations, we see that summands Y original to $Q \cong Q_\alpha < P$ appear by including up from Q_α to P and then transferring down to some subgroup P_β and mapping via split surjection down to some $Q_\beta \cong Q$. This argument suggests that one might examine these split surjections carefully to derive data about the number of copies of a particular summand appearing in BP. That is precisely what Martino and Priddy did to achieve their splitting result.

To state the result, we need some terminology. Let Y be an original summand of BQ, with corresponding simple $R := \mathbb{F}_p\mathrm{Out}\,(Q)$-module M. Let

$$\mathrm{Split}(Q) = \{q_\alpha : P_\alpha \to Q_\alpha\}$$

denote the collection of conjugacy classes of split surjections q_α, with $Q_\alpha \cong Q$. Two such are said to be conjugate if for some $x \in P$, there is a commutative diagram

$$\begin{array}{ccc} P_\alpha & \xrightarrow{q_\alpha} & Q_\alpha \\ \downarrow{\scriptstyle c_x} & & \downarrow{\scriptstyle c_x} \\ P'_\alpha & \xrightarrow{q'_\alpha} & Q'_\alpha \end{array}$$

Let

$$\overline{W}_{\alpha\beta} := \sum_x q_\beta c_x \bmod J_Q$$

for $x \in \{x \in G | Q_\alpha^x < P_\beta\}/P_\beta$. By the isomorphisms $Q_\alpha \cong Q$, this sum defines an element of R. Letting k denote the field of endomorphisms of M, $n := |\mathrm{Split}(Q)|$ and $m := \dim_k(M)$, we can define $A(Q) := (\overline{W}_{\alpha\beta})$ to be an $n \times n$ matrix over R which catalogs all the different ways to include from $Q \cong Q_\alpha$ to P (indexed by α)

and the ways to get back down from P to P_β and onto $Q_\beta \cong Q$ by split surjection. This $A(Q)$ matrix accounts for all the ways for Y to appear in P. Some of them, however, may be redundant; while we have n different ways to include and n to map back down, that does not mean that we can factor a unit of n copies of Y through these maps. To identify the level of redundancy, we consider $A(Q)$ to be a matrix with entries in R, acting on n copies of the simple R-module M which corresponded to Y. This gives an $m \times n$ matrix over k, with rows given by the effect of the ith column of $A(Q)$ on the ith copy of M. Call this matrix $A(Q, M)$; its rank should capture how many independent ways there are to factor a unit of $\{Y, Y\}$ through BP. The result of Martino and Priddy states this formally.

Theorem 2.1.5 (Martino-Priddy [24])**.** *Let G be a finite group with Sylow p-subgroup P. Let Y be original to BQ, $Q \leq P$, with corresponding simple $R = \mathbb{F}_p Out\ (Q)$-module M. Then the multiplicity of Y in BG is*

$$m(Y, BG) = rank_k A(Q, M)$$

Theorem 2.1.5 restates the problem of finding indecomposable stable summands of BG in terms of more calculable objects, namely matrices and simple $\mathbb{F}_p\mathrm{Out}\ (Q)$-modules. It should be noted that in the same issue of *Topology*, Benson and Feshbach provide an alternative approach to calculating a complete stable splitting [2].

Theorem 2.1.5 result has several useful corollaries, noted in the original paper. The first deals with summands Y original to subgroups Q for which there is no split surjection $P_\alpha \to Q_\alpha$ for $P_\alpha \ncong Q$. The only way for such a summand to appear in P

is for there to be an idempotent

$$BP \xrightarrow{tr} BQ_\alpha \hookrightarrow BP.$$

These summands are somewhat easier to work with as the corresponding entries of the $A(P)$ matrix are independent of each other, so determining the multiplicity of Y is as simple as determining how many copies of Y come from each of the Q_α, and adding them up.

This concept of summands only being carried by transfers is formalized in the following hypothesis, which is used in several of the following propositions.

Hypothesis 2.1.1. *Let G be a finite group, P its Sylow p-subgroup, and let Q be a p-group not isomorphic to a subretraction of P. Furthermore, let X be an original summand of BQ, M_X the corresponding $\mathbb{F}_p Out\ (Q)$-module, and $k := End(M_X)$.*

Corollary 2.1.6. *Assume hypothesis 2.1.1. Let $Q_1, Q_2, \ldots, Q_n$ be a complete set of conjugacy classes of subgroups isomorphic to Q, and let*

$$\overline{W_i} = \sum_{x \in N_G(Q_i)/Q_i} c_x.$$

Then the multiplicity of X in BG is

$$\sum_{i=1}^{n} dim_k \overline{W_i} M$$

A useful application of the above is for summands X whose simple modules M_X are essentially the Steinberg module of GL_n. In this case, the possible actions of $\overline{W}_i$ are known, which leads to the following corollary we use repeatedly in our splitting calculations.

Corollary 2.1.7 ([24])**.** *Assume hypothesis 2.1.1, and further suppose there exists a normal subgroup N of Out (Q) such that Out $(Q)/N \cong GL_n(\mathbb{F}_p)$ for some n, with M_X the corresponding Steinberg module and X the corresponding original summand of BQ. Let $W_P(Q) := N_P(Q)/Q$. If Q is self-centralizing and $W_P(Q) \cap N = \{e\}$, then X is a summand of BP and the multiplicity of X from Q is $p^{\binom{n}{2}}/|W_P(Q)|$.*

2.1.4 Linkage

There is another phenomenon, first observed in [20], which allows us to reduce $A(G, M)$ to a sparser matrix. Let P be a p-group. Let X be an original summand of BQ with corresponding $\mathbb{F}_p\text{Out }(Q)$ simple module M_X and idempotent $e_X \in \mathbb{F}_p\text{Out }(Q)$. The summand $e_X BQ$ is not necessarily indecomposable as we chose an idempotent corresponding to X in $\mathbb{F}_p\text{Out }(Q)$ rather than in $\{BP, BP\}$. If Y is an indecomposable summand of $e_X BQ$ then Y is associated to X. Note that Y is not original unless $Y = X$; if Y were original then Y would have to correspond to a different indecomposable idempotent of $\mathbb{F}_p\text{Out }(Q)$, necessarily orthogonal to e_X.

Amongst associated summands, it is valuable to distinguish those whose idempotents can be written in Segal Conjecture basis using a transfer from those where doing so is not possible. This divides those summands which are linked from those which are not. An idempotent which does not involve a transfer must be of the form

$$BP \xrightarrow{Bq} BQ \xrightarrow{\alpha} BQ \xrightarrow{Br} BP$$

where q is a split surjection with splitting r and $\alpha \in \mathbb{F}_p\text{Out }(Q)$. Such idempotents (or their corresponding summands) are linked to their associated idempotent (resp.

summand). The reason for the terminology is the following proposition.

Proposition 2.1.8 ([24]). *Assume hypothesis 2.1.1. Let $m(X, BG)$ denote the multiplicity of X (resp. $m(Y, BG)$ the multiplicity of Y). Let X be original to Q, and let Y be linked to X. Then $m(X, BG) \leq m(Y, BG)$.*

The proof strategy is to take advantage of the known multiplicity of X. That multiplicity gives us a mapping

$$\bigvee_n X \to \underbrace{\bigvee_n BQ_\alpha \to BG \to \bigvee_n BQ_\alpha} \to \bigvee_n X.$$

The association of Y to X gives us a way to build a unit

$$\bigvee_n Y \to \bigvee_n BQ_\alpha \to \bigvee_n Y$$

which we can combine with the braced section above to give us a mapping guaranteeing that there are at least n copies of Y in BG. If Y is linked, the entries in $A(P, M_X)$ for Q_α and $A(P, M_Y)$ will be the same. As those entries are the only ones that contribute to the multiplicity of X by hypothesis 2.1.1 and that X is original to BQ_α, we get the claimed result.

Every summand is original to some isomorphism class of group, but each particular summand of BG is linked in one corresponding subgroup, as the following lemma shows.

Lemma 2.1.9 (Replacement Lemma, [23]). *Let X be a particular indecomposable summand of BG. Suppose there is a unit in $\{X, X\}$ of the form*

$$X \longrightarrow BG \xrightarrow{tr} BQ \longrightarrow X$$

where Q is a p-subgroup of G and X is not linked in BQ. Then there is a subgroup $R < Q$ and a unit of $\{X, X\}$ of the form

$$X \longrightarrow BG \xrightarrow{tr} BR \longrightarrow X$$

where X is linked in BR.

Original summands of BP correspond to simple $\mathbb{F}_p\text{Out}\ (P)$ modules and indecomposable summands of BP correspond to simple $\{BP, BP\}$ modules. These two correspondances were used by Martino-Priddy ([24]) and Benson-Feshbach ([2]) to achieve theorems which provided complete splittings. We would like to find a middle ground where summands linked in BP correspond to simple modules for some ring associated with P, with the hope that the correspondance would provide us with another splitting technique. We begin with a definition which parallels the definition of original summands. Define $J_\tau(P)$ to be the right ideal of $\{BP, BP\}$ generated by all maps beginning with a transfer to a proper subgroup. Note that $J_\tau(P)$ is also a left $\mathbb{F}_p\text{Out}\ (P)$ module: an application of the Mackey formula demonstrates that after rewriting in the Segal Conjecture basis, all terms will still begin with a nontrivial transfer.

The kernel of the ring homomorphism

$$\{BP, BP\} \to \text{Hom}\ (H_*(BP), H_*(BP))$$

lies inside the radical of $\{BP, BP\}$. Therefore we may factor the ring homomorphism of the Wedderburn theorem as

$$\{BP, BP\} \to \text{Hom }(H_*(BP), H_*(BP)) \to \{BP, BP\}/rad \cong \bigoplus \text{Mat}_{n_i,n_i}(k_i)$$

where i runs over the simple components of $\{BP, BP\}/rad$. Let $\overline{J}_{\tau,i}$ denote the image of J_τ in $\text{Mat}_{n_i,n_i}(k_i)$; this will remain a (Hom $(H_*(BP), H_*(BP))$, $\mathbb{F}_p$Out (P))-bimodule.

As quotients of semi-simple modules are semi-simple, we can express the quotient as a direct sum of simple left Hom $(H_*(BP), H_*(BP))$-modules:

$$\text{Mat}_{n_i,n_i}(k_i)/\overline{J}_{\tau,i} \cong \bigoplus_{j=1}^{m} \text{Mat}_{n_i,n_i}(k_i)e_j.$$

However, we can also use the bimodule structure to consider the quotient as a composition series of n_i isomorphic simple right $\mathbb{F}_p$Out (P)-modules $M_{P,i}$. This module is not necessarily semisimple over $\mathbb{F}_p$Out (P) as the radical is not necessarily contained in the radical of the larger ring $\{BP, BP\}$ (in fact, the counterexample to the Strong Linkage Hypothesis of subsection 2.1.5 is an occasion in which this happens; see Section 3.5). They are, however, all isomorphic modules as the idempotents for the various Y_i must all be associated to the same $\mathbb{F}_p$Out (P) idempotent. The k_i-dimension of this simple $\mathbb{F}_p$Out (P) module is m. It is this m which gives the number of copies of Y_i linked in BP.

It should be noted that our definition of linked summands using the Segal Conjecture Basis and this description of linked summands in terms of their corresponding bimodules does not match that given in [24] and [23]; it is due to Martino and Priddy and was communicated to me privately in advance of the publication of

([22]). The reason for the change is that the description given in the aforementioned papers does not capture well the behavior of linked summands in groups such as $Z_2 \times Z_2 \times D_8$. In this group there are 20 $L(2)$s, of which 2 are linked and the other 18 have idempotents which can be expressed using transfers to maximal elementary abelian subgroups (see Section 3.5). Using the previous definition, none of the 20 were said to be linked: there is a stable map which swaps any pair of isomorphic summands, so one could write the idempotent for any one of these $L(2)$s by using an idempotent for a different $L(2)$ involving a transfer. However, it is not possible to write an idempotent for all 20 which uses transfers in every term; one can only craft such an idempotent for 18 at once. Equivalently, J_τ does not cover the simple component corresponding to $L(2)$; its image decomposes into a direct sum of 18 copies of $M_{L(2)}$.

Consider the composition

$$\phi : BQ \xrightarrow{Bi} BG \xrightarrow{tr} BQ.$$

We would like to know how many summands isomorphic to Y_i that are linked in BQ are carried by ϕ. By the Mackey formula,

$$\phi = \sum_{x \in N_G(Q)} Bc_x + \sum Bf \circ tr \tag{2.2}$$

where all the latter terms begin with a nontrivial transfer, so they are all in J_τ. Therefore we have an action of ϕ on the *right* of $\mathrm{Mat}_{n_i,n_i}(k_i)/\overline{J}_{\tau,i}$: it acts as the $\mathbb{F}_p\mathrm{Out}\ (Q)$ element $\sum c_x$. The dimension of the result over k_i is the number of copies of Y_i which factor through ϕ.

In the $Z_2 \times Z_2 \times D_8 \leq Z_2^{\wr 3}$ example given above, we find that the normalizer in $Z_2^{\wr 3}$ is $D_8 \times D_8$, so the corresponding Weyl sum is $1 + c_x$, where c_x interchanges the two Z_2s. In this case the $M_{Z_2 \times Z_2 \times D_8, L(2)}$ module is the regular representation of $GL_2(\mathbb{F}_2)$, and $M_{Z_2 \times Z_2 \times D_8, L(2)}\overline{W}$ is one dimensional over $\mathbb{F}_2$, corresponding to the inclusion-transfer carrying one of the two linked $L(2)$s. This matches what we see in my calculations in Chapter 4.

We will now describe how linkage can be used to develop an alternative to the $A(P, M)$ matrix given above. Let X be a particular stable summand of BG, with corresponding idempotent e_X and field $k_X := \{X, X\}/rad$. The replacement lemma 2.1.9 tells us that there is a particular (conjugacy class of) subgroup corresponding to X where X is linked. We wish to compute the multiplicity of summands isomorphic to X by examining the subgroups in which they are linked.

For each of the conjugacy classes of subgroups of G, choose a representative Q_v. Consider all such classes such that BQ_v contains a summand isomorphic to X. Define a_{uv} to be the matrix with entries in k_X corresponding to the mappings

$$X \to BQ_v \xrightarrow{Bi} BG \xrightarrow{tr} BQ_u \to X$$

where the columns are indexed by inclusions $X \to BQ_v \to BG$ and the rows by the projections $BG \xrightarrow{tr} BQ_u \to X$.

We assemble these matrices into a larger matrix a_X to be the square matrix with (u, v) submatrix a_{uv}, with the subgroups listed by nondecreasing order. This matrix encapsulates all the different units $X \to BG \to X$; its rank will be the number of copies of X appearing in BG. Our strategy will be to demonstrate that the rank of this matrix is the same as the rank of a block-diagonal matrix $A_\tau(X)$, and that

that rank can be calculated by its action on a suitable collection of modules.

The first step in transforming the matrix a_X to $A_\tau(X)$ is to take the entries of the submatrix a_{uv} to be in $\{X, X\}/rad + J_\tau(Q_v)$. This affects the column rank; we are zeroing out entries of the form

$$X \to BQ_v \xrightarrow{tr} BQ' \xrightarrow{Bi} BG \to X.$$

Taking our entries modulo $J_\tau(Q_v)$ does not change the column rank, as for every X in BG, there is some isomorphism class of subgroup R such that X is original to BR, and the inclusion $X \to BG$ can be expressed by factoring through BR_X for some $R_X \cong R$. Certainly that inclusion cannot be expressed using a transfer to a proper subgroup of BR.

We do this because if Q_v is not conjugate to a subgroup of Q_u, the argument of proposition 2.1.4 tells us that the entries of a_{uv} modulo $J_\tau(Q_v)$ will be zero.

Next, we wish to zero out certain columns entirely. For the moment let us consider Q_u and Q_v as column positions, although we retain the notation as it is suggestive of the reason for the argument. If Q_v is conjugate to a proper subgroup of Q_u, then any column

$$X \longrightarrow BQ_v \xrightarrow{\qquad Bi \qquad} BG$$

can be factored as

$$X \longrightarrow BQ_v \xrightarrow{Bi_v^u} BQ_u \xrightarrow{Bi_u^G} BG.$$

This factorization shows that there is a linear dependence between the corresponding columns. We can eliminate columns of this sort in such a way so that the rank of the upper left block submatrices is unchanged, and the subdiagonal entries corresponding to

$$X \to BQ_v \xrightarrow{Bi} BG \xrightarrow{tr} BQ_u \to X$$

are all zero.

By zeroing out the blocks corresponding to $BQ_v \not\leq_G BQ_u$ and $BQ_v <_G BQ_u$, we have reduced our matrix to one with entries only in the block diagonal positions. Call this matrix $A_\tau(X)$. Its rank is the same as the rank of a_X: the multiplicity of X in BG. This argument is the basis of

Theorem 2.1.10 ([23])**.**

$$m(X, BG) = rank_k\ A_\tau(X).$$

By the transformations we have performed above, the rank of the block a_{uu} corresponds to only the summands X linked in BQ_u. To determine the rank of this block, we can have it act on the $M_{Q_u,X}$ module developed above. As observed above in equation 2.2, the mapping

$$\phi : BQ_u \xrightarrow{Bi} BG \xrightarrow{tr} BQ_u$$

acts as a Weyl sum modulo $J_\tau(Q_u)$. The mapping ϕ corresponds to the a_{uu} block, so the k-rank of $M_{Q_u,X}\phi$ will give the multiplicity of X linked in Q_u factoring through ϕ. Let us arrange the modules $M_{Q_u,X}$ as u varies in a row vector $v_\tau(X)$, in the same order as they appear in $A_\tau(X)$. Then

$$rank\ A_\tau(X) = dim_k\ v_\tau(X)A_\tau(X) = dim_k\ \bigoplus_u (M_{Q_u,X}\overline{W}_u)$$

2.1.5 The Linkage Hypothesis

The elementary abelian case is particularly noteworthy. In this case, for a given summand X original to $BQ := BZ_p^n$, there is exactly one associated summand Y original to $BR := BZ_p^m$ for some $m < n$, and Y is linked to X (see section 2.2.1). What is unusual is that in this case, the simple $\{BQ, BQ\}$-modules M_Y and M_X are isomorphic as $\mathbb{F}_p\text{Out}\ (Q)$-modules. Therefore the inequality in 2.1.8 becomes an equality, because the action of a Weyl sum on the $\mathbb{F}_p\text{Out}\ (Q)$-modules will be identical as the modules are isomorphic.

If Y is linked to X, the summand Y is said to be *strongly linked* when Y appears if and only if X does via transfer. It has been hypothesized that all linkage is strong linkage. One of the achievements of this thesis is to present a counterexample to that hypothesis (Section 3.5).

2.1.6 Additional Splitting Results

There is a general result about splittings that we will use due to Martino, Priddy and Douma [25]. That paper describes the summands of $BG \times H$ in terms of smash products of BG and BH. For any groups G and H, the spectrum $BG \times H$ splits as $BG \vee BH \vee (BG \wedge BH)$, but this splitting is not necessarily complete. They explored conditions under which if X was an indecomposable summand of BG and Y an indecomposable summand of BH then $X \wedge Y$ was indecomposable. One such condition which can be used involves the Frattini subgroups of the relevant groups. If ϕ_P denotes the quotient $P \to P/\Phi(P)$, then the group Out $(P \times Q)$ is said to be *parabolic* if its image under $\phi_{P\times Q}$ in Aut $(P/\Phi(P) \times Q/\Phi(Q)) \cong GL(Z_p)$ consists of block upper triangular matrices. The theorem we need is

Theorem 2.1.11 ([25]). *If Out $(M \times N)$ is parabolic and $\mathbb{F}_p$ is a splitting field for $\mathbb{F}_p Out\ (M)$ or $\mathbb{F}_p Out\ (N)$, then the simple modules of $\mathbb{F}_p Out\ (M \times N)$ are the tensor products of the simple modules of $\mathbb{F}_p Out\ M$ with the simple modules of $\mathbb{F}_p Out\ N$. Therefore the corresponding original summands of $M \times N$ are in one to one correspondence with the smash products of the original summands of M and the original summands of N.*

2.2 Prerequisite Results: Known Splittings

There are several specific group classifying spaces whose stable splittings are known and utilized in the process of splitting $BZ_2^{\wr 3}$. Sometimes only the splitting is needed, but in several cases the cohomological structure of the specific summands is necessary as well.

2.2.1 Elementary Abelian p-groups

The abelian p-group was split in [15], using algebraic techniques. One point of Theorem A of that paper states that for an elementary abelian p-group $P \cong {Z_p}^n$, the summands of BP correspond to the simple modules of the ring of $n \times n$ matrices $M_n(\mathbb{F}_p)$. An ingredient of their Theorem A is that the simple modules of $\mathbb{F}_p M_n(\mathbb{F}_p)$ are induced from simple $\mathbb{F}_p GL_k(\mathbb{F}_p)$-modules for $k \leq n$ using a quotient map

$$ {Z_p}^n \xrightarrow{q} {Z_p}^k. $$

When $k = n$, the corresponding summand is original.

Briefly, isomorphism classes of simple $\mathbb{F}_p GL_n(\mathbb{F}_p)$ modules are in one to one correspondence with p-regular Young Tableaux with n columns. For details on the construction of projective covers of these modules, please see [17]. Let us now take $p = 2$. Let M_λ denote the simple $GL_n(\mathbb{F}_2)$-module corresponding to the transpose λ of a 2-regular Young Tableaux. We use the transpose of the traditional orientation for the tableaux because it makes the notation for how these modules are linked much simpler. To identify the linked summand, we can simply delete the first column of the conjugate tableaux. To give a specific example, let $p = 2$ and $n = 4$. One possible simple $GL_4(\mathbb{F}_2)$ module corresponds to the 2-regular tableaux with column lengths $(3, 2, 1, 1)$. The conjugate of this tableaux will have column lengths $(4, 2, 1)$, and the corresponding summand would be X_{421}. The linked summand will correspond to a tableaux with (conjugate) column lengths $(2, 1)$. This tableaux corresponds to a simple $\mathbb{F}_2 GL_2(\mathbb{F}_2)$-module because its conjugate has two columns. Therefore the corresponding summand X_{21} is original to the subgroup ${Z_2}^2$. In general, the summand $X_{(\lambda_2,\lambda_3,\ldots,\lambda_m)}$ is linked to the summand $X_{(\lambda_1,\lambda_2,\ldots,\lambda_m)}$.

There is another important observation about linked summands in elementary abelian subgroups when $p = 2$. Let det denote the one-dimensional determinant module for $\mathbb{F}_2 M_n(\mathbb{F}_2)$. For any simple $\mathbb{F}_2 M_n(\mathbb{F}_2)$ module M, there is an isomorphism of $\mathbb{F}_2 GL_n(\mathbb{F}_2)$-modules

$$(M \otimes det) \downarrow^{M_n}_{GL_n} \cong M \downarrow^{M_n}_{GL_n} . \tag{2.3}$$

Therefore if M is simple, $M \otimes det$ is simple. Furthermore, M is induced from a $\mathbb{F}_p GL_n(\mathbb{F}_p)$-module if and only if singular matrices act as zero on M. Such modules correspond to the original summands. For these modules, $M \otimes det \cong M$ as $\mathbb{F}_p M_n(\mathbb{F}_p)$-modules. Otherwise, equation 2.3 tells us that the summand corresponding to M is associated to the summand corresponding to $M \otimes det$. As transfers to subgroups of elementary abelian subgroups are trivial in cohomology, associated summands are linked. As the two modules are actually isomorphic as $\mathbb{F}_2GL_n(\mathbb{F}_2)$-modules, the linkage here is strong; if

$$f : BZ_2^n \to BG \to BZ_2^n$$

is an element of $\{BZ_2^n, BZ_2^n\}$ for some group $G > Z_2^n$, the dimension of fM and $fM \otimes det$ are the same, so by theorem 2.1.5 the one summand appears if and only if the other does.

Of particular note is the Steinberg summand, corresponding to the Steinberg module $M_{n,n-1,\ldots,1}$. The summand is commonly referred to as $L(n)$ when the particular prime p is understood, and has been shown by Mitchell and Priddy to be isomorphic to $\sum^{-n} Sp^{p^n} S^0 / Sp^{p^{n-1}} S^0$, where $Sp^{p^n} S^0$ denotes the p^n fold symmetric product spectrum over the sphere spectrum S^0 [28].

A small example may be illustrative here. Let $p = 2$ and $n = 2$, so $P = {Z_2}^2$. There are two admissible tableaux, with column lengths 21 and 2, so there are three $\mathbb{F}_2M_2$-modules, $M_2 \cong M_2 \otimes det$, $M_{21} \otimes det$, and M_{21}. The two modules M_2 and $M_2 \otimes det$ are isomorphic because in this setting both are the one dimensional trivial module; this always happens for the module M_n when $p = 2$. The other modules are two dimensional over $\mathbb{F}_2$, corresponding to the usual two dimensional representations of $M_2(\mathbb{F}_2)$ and $GL_2(\mathbb{F}_2)$ on $\mathbb{F}_2^2$. Nonzero singular matrices act as zero on $M_{21} \otimes det$, whereas they act "as themselves" on M_{21}. The M_{21} modules correspond the two projection idempotents $e_1 : Z_2^2 \to Z_2$. Therefore the corresponding summand is original to BZ_2, so it is BZ_2. The other idempotents are original; M_2 is trivial and $M_{21} \otimes det$ is the Steinberg Module. As the multiplicity of a module is equal to its dimension, this gives a complete splitting

$$BZ_2^2 \cong X_2 \vee 2L(2) \vee 2L(1).$$

We could equivalently write

$$BZ_2^2 \cong X_2 \vee 2(X_{21} \vee X_1) \tag{2.4}$$

to emphasize the linkage and the module structure.

The primary difficulty in pursuing this approach is determining the structure of $\mathbb{F}_p GL_n$-modules; although there is a construction of their projective covers involving Weyl modules [17], the structure of the simple modules is an important open problem. We will be using a calculation based on some work by Franjou and Schwartz [13], see section 2.4.2. For low dimensional cases, direct computation can yield some details we will be using about specific splittings of elementary abelian groups for $p = 2$:

- BZ_2 is one summand as there is only one module for $M_1(\mathbb{F}_2)$. It is therefore isomorphic to both X_1 and $L(1)$ by the definitions of those summands.
- BZ_2^2 splits as given above. By examination of particular groups G which have BZ_2^2 as a proper subgroup, we find that $X_2 \cong BA_4$, the classifying space for the alternating group on four letters. Also, in general $B(G \times H) \cong BG \vee BH \vee BG \wedge BH$; taking $G = H = Z_2$ and comparing with equation 2.4 yields the complete splitting

$$BZ_2 \wedge BZ_2 \cong 2L(2) \vee X_2.$$

- BZ_2^3 splits as

$$BZ_2^3 \cong X_3 \vee 3(X_{31} \vee X_1) \vee 3(X_{32} \vee X_2) \vee 8(X_{321} \vee X_{21})$$

with $X_3 \cong BJ_1$ (the first Janko group)[21]. The summand X_{321} is $L(3)$ and X_{21} is $L(2)$, but the other original summands are not known to be associated with any other structures. In particular, note that we have $6X_{21}$ appearing in the three distinct pairings of $X_1 \wedge X_1$. The two remaining copies are summands of $X_1 \wedge X_2$, which also contains X_3.

2.2.2 Nonabelian Group Splittings

There are some other splittings which we will use. Foremost among these is

$$BD_8 \cong BA_6 \vee 2(L(2) \vee L(1)).$$

Note the similarity with the splitting for BZ_2^2. In this case, the $L(2)$s are not original but appear by transferring down to two maximal Z_2^2 subgroups of D_8 and including back; they carry with them linked $L(1)$s. In particular, the retraction $BD_8 \to BZ_2 \cong L(1)$ can be written using a transfer.

We also need the splitting of $B(D_8 \circ D_8)$, the classifying space of the central product of two D_8s. The spectrum $BD_8 \circ D_8$ is split in [12] by mapping into spectra whose splittings were known and comparing Poincaré series. It splits as

$$B(D_8 \circ D_8) \cong B(SL_2(\mathbb{F}_3) \circ SL_2(\mathbb{F}_3)) \vee 4eT(\Delta_4) \vee 4X_{D_8 \circ D_8} \vee 4(2L(2) \vee L(1))$$

Of these, all except the $L(2)$ and $L(1)$ are original. The $eT(\Delta_4)$ is an indecomposable summand of $T(\Delta_4)$, the Thom Spectrum of a four dimensional representation for $\mathbb{F}_p\text{Out }(D_8 \circ D_8)$. We find that Out $(D_8 \circ D_8) \cong GL_2(\mathbb{F}_2) \wr Z_2$ using direct computation. The ring $\mathbb{F}_2 GL_2(\mathbb{F}_2) \wr Z_2$ has three isomorphism types of simple modules: the trivial module (corresponding to the summand $BSL_2(\mathbb{F}_3) \circ SL_2(\mathbb{F}_3)$) and two four dimensional modules, corresponding to the summands $eT(\Delta_4)$ and $X_{D_8 \circ D_8}$. Also of use to us is a Poincaré series given for $X_{D_8 \circ D_8}$, namely $(t^2+t^3)/(1-t)(1-t^3)(1-t^4)$. In particular, this summand has a cohomology class in degree 2. We also wish to note that the idempotents for $eT(\Delta_4)$ are products of the Steinberg idempotents of the two $GL_2(\mathbb{F}_2)$s, whereas the idempotents for $X_{D_8 \circ D_8}$ are products of a Steinberg idempotent and the trivial module idempotent. We note these facts to assist us in differentiating the modules later.

Using the fact that $GL_2(\mathbb{F}_2) \cong \mathfrak{S}_3$, we can express Out $(D_8 \circ D_8) \cong \mathfrak{S}_3 \wr \mathfrak{S}_2 \leq \mathfrak{S}_6$. This isomorphism allows us to intelligibly express a structure for the two simple 4-dimensional modules. Intuitively, they are built out of two copies the regular representation for $GL_2(\mathbb{F}_2)$ by either tensoring or direct sum, and then finding a way for the last wreath factor to act compatibly. One choice of images of generators in $GL_4(\mathbb{F}_2)$ is given below.

(Direct Sum Representation) (Tensor Product Representation)

$$(1,2) \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1,2) \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(1,2,3) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1,2,3) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$(1,4)(2,5)(3,6) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (1,4)(2,5)(3,6) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These are computationally confirmed to be simple and nonisomorphic using `GAP`. We can say that the tensor product representation corresponds to $eT(\Delta_4)$ and the direct sum corresponds to $X_{D_8 \circ D_8}$ because the matrix action corresponds to the construction of the idempotents.

We also use a splitting of BU_4, where U_4 denotes the group of upper triangular matrices in $GL_4(\mathbb{F}_2)$. This splitting was the primary content of [21].

$$\begin{aligned} BU \cong\ & Triv(BU) \vee 2St(BU) \vee 3St(B(Z_2^2 \wr Z_2)) \vee 2eT(\Delta_4) \\ & \vee 2X_{BD_8 \circ D_8} \vee 16L(4) \vee 21L(3) \vee 12L(2) \vee 3BZ_2 \vee 5X_{432} \\ & \vee 5X_{32} \vee BA_4 \vee 3X_{431} \vee 3X_{31} \vee 5X_{421} \vee X_{43} \vee BJ_1 \vee X_{41} \vee 3BA_6. \end{aligned}$$

There are several pieces of notation used in the splitting. $Triv(P)$ denotes the principal original summand of P corresponding to the trivial $\mathbb{F}_2\mathrm{Out}\ (P)$-module, and $St(P)$ denotes a summand corresponding to a Steinberg module (in both cases above, because there is a surjective mapping $\mathbb{F}_2\mathrm{Out}\ (P) \to \mathbb{F}_2 GL_2(\mathbb{F}_2)$ with a nilpotent kernel).

Lastly, we would like to note a specific example given in [25] of a case where the wedge product of indecomposable summands is not indecomposable. Inside $BZ_2 \times D_8$, $BZ_2 \wedge BA_6$ is not indecomposable. Much like what happens in $BZ_2 \times Z_2^2 \cong BZ_2^3$ where $BZ_2 \wedge BA_4 \cong BX_3 \vee 2X_{21} \vee \bigvee Y$ where the Y are summands of connectivity ≥ 4, $BZ_2 \wedge BA_6 \cong X_{Z_2 \times D_8} \vee 2L(2) \vee \bigvee Y'$ where the connectivity of Y' is ≥ 4. Unlike what occurs in BZ_2^3, the Y' are not original; $\mathrm{Out}\ (Z_2 \times D_8) \cong Z_2 \times D_8$, a 2-group, so there is only one original summand.

2.3 Prerequisite Results: Group Theory

2.3.1 Wreath Products

Introductory information about wreath products which follows can be found in [26]. Let X, Y be sets, let $G \leq \mathfrak{S}_X$ act on X and $H \leq \mathfrak{S}_Y$ act as permutation groups. Let H^X denote the set of all functions from X to H; this group is naturally isomorphic to $\prod_{i \in X} H$. It has a natural $\mathfrak{S}_X$ action: if $\sigma \in \mathfrak{S}_X$, we define $\sigma^{-1}(h_1, h_2, \ldots) :=$

$(h_{\sigma(1)}, h_{\sigma(2)}, \ldots)$. We define the *permutational wreath product* of H with G, written $(H,Y) \wr (G,X)$ (or $H \wr G$ when the group actions are understood) to be $H^X \rtimes G$, where the G action is as above. Using the isomorphism $H^X \cong \prod_{i \in X} H$, an element of the wreath product is often expressed as $(h_1, h_2, \ldots; g)$. The group $(H,Y) \wr (G,X)$ acts on the set $Y \times X$: if $f \in H^X$ and $g \in G$, then $fg(y,x) = (y^{f^g(x)}, x^g)$. This action is faithful if both G and H act faithfully. The subgroup H^X is called the *base* of the wreath product.

Theorem 2.3.1 ([26])**.** *The permutational wreath product is associative: for permutation groups (G,X), (H,Y), (K,Z) the there is an isomorphism of permutation groups $G \wr (H \wr K) \cong (G \wr H) \wr K$, each acting on $X \times Y \times Z$.*

The proof is a tedious but straightforward calculation.

From here forward, when we say wreath product we shall be referring to a permutational wreath product. We will refer to G as the first wreath factor, H as the second wreath factor, and so on.

There is an important embedding of $G \times H$ into $H \wr G$. Let $\Delta : H \to H \wr G$ send each element h to $(x_h; id)$, where $x_h(x) = h$ for all $x \in X$. We also can include G in the obvious fashion, $g \mapsto (x_{id}; g)$. Note that the images of these two maps centralize each other giving us the mentioned embedding.

These groups are of interest for our purposes because they form the Sylow p-subgroups of $B\mathfrak{S}_{p^n}$. We define the *iterated wreath product* to be

$$Z_p^{\wr n} := \underbrace{Z_p \wr Z_p \wr \ldots \wr Z_p}_{n \text{ factors}}$$

If we let Z_p act in the obvious fashion on the set $\{0, 1, \ldots, p-1\}$, then the iterated wreath product acts faithfully on $\{1, 2, \ldots, p^n\}$ via bijective correspondence with $\prod_{i=1}^{n}\{0, 1, \ldots p-1\}$, where $m \in \{1, 2, \ldots, p^n\}$ is sent to the ordered tuple of its digits in p-ary expansion. For example, $5 \in \{1, 2, \ldots, 2^4\}$ is sent to $(1, 0, 1, 0) \in \prod_{i=1}^{4}\{0, 1\}$. Therefore the iterated wreath product is a subgroup of $\mathfrak{S}_{p^n}$.

The order of the iterated wreath product can be shown to be $p^{(p^{n+1}-1)/(p-1)}$ using induction, which is precisely the power of p which divides $p^n! = |\mathfrak{S}_{p^n}|$. Therefore the iterated wreath product is the Sylow p-subgroup as it is of maximal p power order. Considering iterated wreath products as Sylow subgroups of symmetric groups suggests a generating set: let τ_i be the generator for the i-th factor of the wreath product,

$$\tau_i = \prod_{i=1}^{p^{n-1}} (i \quad i + p^{n-1} \quad i + 2p^{n-1} \quad \ldots \quad i + (p-1)p^{n-1}).$$

The generator τ_i is an element of order p comprised of p^{n-1} p-cycles. Inductively, $\langle \tau_1, \tau_2, \ldots, \tau_n \rangle$ is a generating set for $Z_p^{\wr n}$.

An example would be illuminative. Let us take $p = n = 3$. Then

$$\begin{aligned} \tau_1 &:= (1, 2, 3) \\ \tau_2 &:= (1, 4, 7)(2, 5, 8)(3, 6, 9) \\ \tau_3 &:= (1, 10, 19)(2, 11, 20) \ldots (9, 18, 27) \end{aligned}$$

Note that $\langle \tau_1 \rangle \cong Z_3 \leq \mathfrak{S}_{\{1,2,3\}}$, and conjugation by τ_2 cycles this to $\langle \tau_1 \rangle^{\tau_2} \cong Z_3 \leq \mathfrak{S}_{\{4,5,6\}}$ and $\langle \tau_1 \rangle^{\tau_2^2} \cong Z_3 \leq \mathfrak{S}_{7,8,9}$, giving $\langle \tau_1, \tau_2 \rangle \cong Z_3 \wr Z_3 \leq \mathfrak{S}_{\{1,2,\ldots,9\}}$, where τ_i generates the ith wreath factor. Similarly, conjugating $\langle \tau_1, \tau_2 \rangle$ by τ_3 yields conjugates acting on $\{10, \ldots 18\}$, and $\{19, \ldots, 27\}$, yielding $Z_3^{\wr 3} \leq \mathfrak{S}_{27}$.

In fact, subsets of this collection of generators always generate an iterated wreath product:

Proposition 2.3.2. *When considered as a subgroup of $\mathfrak{S}_\infty$, $\{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n}\}$ is a generating set for a subgroup isomorphic to $Z_p^{\wr n}$. If $i_k < i_{k+1}$ for all k, the generator of the kth Z_p in the wreath product is τ_{i_k}.*

This is proven in the section on Group structures as Proposition 3.1.1.

There are several less well-known results which are important for our purposes, concerning the automorphisms of a wreath product and the base.

Theorem 2.3.3 ([3])**.** *The base of a wreath product $G \wr H$ is characteristic unless the base is an elementary abelian 2-group and $H \cong Z_2$.*

Theorem 2.3.4 ([30])**.** *The collection of automorphisms of $G \wr Z_2$ which leaves the base invariant is either the entire automorphism group or a subgroup of index 2.*

Combining these, we get the following useful result.

Theorem 2.3.5. *The base of a wreath product is a characteristic subgroup unless the base is an elementary abelian 2-group. In that case, there is a subgroup of the automorphism group of index 2 which leaves the base invariant.*

2.3.2 Group Automorphisms

We also find ourselves regularly in need of some theory on automorphisms of p-groups. The material in this section can be found in Marshall Hall's book [14].

For a group G and element g of G, we call g a *nongenerator* if for any generating set $G = \langle g, g_1, g_2, \ldots \rangle$ we have $G = \langle g_1, g_2, \ldots \rangle$. We define the Frattini subgroup $\Phi(G)$ to be the intersection of all the maximal subgroups of G with G itself. The subgroup $\Phi(G)$ is quite useful.

Proposition 2.3.6. *$\Phi(G)$ has the following properties:*

- *$\Phi(G)$ is precisely the set of nongenerators of G if $G \neq \{e\}$.*
- *If G is a finite p-group, $\Phi(G)$ contains both $[G, G]$ and $\{x^p | x \in G\}$*
- *$\Phi(G)$ is a characteristic subgroup*

Perhaps most importantly, it stars in the Burnside Basis Theorem, which elucidates a way in which finite p-groups behave like finite vector spaces.

Theorem 2.3.7 (Burnside Basis Theorem)**.** *Let P be a p-group of order p^n. Then $P/\Phi(P)$ is an elementary abelian subgroup. If $P/\Phi(P)$ has rank r, then for every set of elements $\{p_1, \ldots, p_s\}$ that generates P, there is a subcollection $\{q_1, \ldots, q_r\}$ which also generates P. Under the mapping $P \to P/\Phi(P)$, $\{q_1, \ldots, q_r\}$ is mapped to a generating set for $P/\Phi(P)$. Moreover, if any other collection $\{q'_1, \ldots, q'_r\}$ maps onto a generating set for $P/\Phi(P)$, then that collection was a generating set for P.*

The proof technique can be examined more carefully to gain information on the automorphism group of P:

Theorem 2.3.8 (P. Hall)**.** *Let P be a p-group of order p^n, and $P/\Phi(P)$ have order p^r. Let $\theta(p^r) = |GL_r(\mathbb{F}_p)|$ denote the order of the automorphism group of $P/\Phi(P)$. Then $|Aut\ (P)|$ divides $p^{r(n-r)}\theta(p^r)$, and the order of $A_1(P)$, the subgroup of automorphisms fixing $P/\Phi(P)$ elementwise, is a divisor of $p^{r(n-r)}$.*

As $\Phi(P)$ is a characteristic subgroup, the quotient mapping $P \to P/\Phi(P)$ induces a mapping of automorphism groups

$$\phi : \text{Aut}\ (P) \to \text{Aut}\ (P/\Phi(P)) \cong GL_r(\mathbb{F}_p).$$

Also, as the commutator subgroup of P is contained in the Frattini subgroup, every inner automorphism of P is sent under ϕ to the identity in $GL_r(\mathbb{F}_p)$, so ϕ factors

$$\phi : \text{Aut }(P) \to \text{Out }(P) \to \text{Aut }\ (P/\Phi(P)) \cong GL_r(\mathbb{F}_p). \tag{2.5}$$

Hall's theorem tells us that the kernel of ϕ is a p-group.

2.4 Prerequisite Results: Cohomology

There are two results involving cohomology of groups which will be of use to us. In particular, we would like to know more about the cohomology of wreath products, and there is a result coming from the theory of unstable Steenrod modules which allows us to simultaneously identify simple $GL_4(\mathbb{F}_2)$-modules and calculate the Weyl action on them. We also would like to know about the cohomological structure of some commonly appearing summands in low degrees.

2.4.1 Cohomology of Wreath Products

In his paper on symmetric groups [29], Nakaoka uses algebraic techniques to develop the structure of the cohomology rings of wreath products. For (H, Y) and (G, X) permutation groups, he proves the following two rings are isomorphic:

$$H^*\left(B(H \wr G); Z_p\right) \cong H^*(BG, \bigotimes_{|X|} H^*(BH; Z_p)).$$

This isomorphism is valuable in the study of iterated wreath products in that it gives a framework for understanding the cohomology of $G^{\wr n}$ in terms of the cohomology of G and of $G^{\wr n-1}$. The resulting ring has algebraic generators of three types.

The first two types are coefficients in $H^*(BG; \bigotimes_{i=1}^{|X|} H^*(BH; Z_p))$. The structure of these can be better understood by looking at the inclusion map from the base $i : \prod H \to H \wr G$. It is relatively easy to work out i^* because it is induced by the mapping $\{e\} \to Z_p$ in the last wreath factor. Applying Nakaoka's theorem:

$$\begin{aligned} H^*i : H^*(BZ_p^{\wr n}; Z_p) \cong \quad & H^*(BZ_p, H^*(\prod BZ_p^{\wr n-1})) \\ \to \quad & H^*(B\{e\}, H^*(\prod BZ_p^{\wr n-1})) \cong \bigotimes H^*(BZ_p^{\wr n-1}; Z_p). \end{aligned}$$

There are two obvious types of invariants; those that map to elements which are symmetric products (e.g. $a \otimes a$ for $p = 2$) and those which map to symmetrized sums (e.g., $a \otimes b + b \otimes a$). As there is a bijection between $i^*(H^*(\prod BZ_p^{\wr n-1}))$ and the Z_p invariants in $H^*(\prod BZ_p^{\wr n-1})$, we will often drop the i^* and refer to the elements of $H^*(BZ_p^{\wr n}; Z_p)$ by their images under i^*.

The third type of generator comes from the cohomology of the last wreath factor. For our purposes that will almost always be $H^*(BZ_2; Z_2)$, so this third generator will be the generator of the entire algebra, e.g. $z \in H^1(BZ_2; Z_2)$.

The relations similarly divide into those that are visible in the base and those involving the last wreath factor. Those that are induced from relations in $H^*(BG)$, are detected exclusively in the base, and do not involve any classes from the last wreath factor. The other relations result from the coefficient action, and are products of symmetric sums with classes from $H^*(BH)$. For our purposes these relations will always take the form

$$z \cup (a \otimes b + b \otimes a) = 0$$

for some $a, b \in H^*(BZ_2^{\wr n-1}; Z_2)$.

The case we are primarily interested in is $p = 2$ and $n = 3$. We will work up to it inductively.

Let $p = 2$, $n = 1$.

$$H^*(BZ_2^{\wr 1}) = H^*(BZ_2) \cong \mathbb{F}_2[x_1]$$

Let $p = 2$, $n = 2$. In this case, we should have one symmetrized sum generator $(x \otimes 1 + 1 \otimes x)$, one symmetrized product generator $(x \otimes x)$, and one generator from the last wreath factor (y). They are of degrees one, two and one, respectively: examine the degree of the corresponding class in the base or the last wreath factor. It is customary to denote $x \otimes x$ by w, as it is the first Chern class. We will also use $\bar{x}$ or often just x to denote $x \otimes 1 + 1 \otimes x$. We also acquire one relation, $\bar{x}y = 0$. We have rederived the well known

$$H^*(BZ_2^{\wr 2}) \cong \mathbb{F}_2[x_1, y_1, w_2]/(x_1 y_1).$$

Let $p = 2$, $n = 3$. At this stage working out the details becomes more involved. For each pair chosen from $\{x, y, w\}$, we gain a new generator, either a symmetrized sum if the pair is distinct or a product if they are the same. All of these classes will be detected by the base.

The 'sum' invariant generators will be denoted

Element of $H^*(BZ_2^{\wr 3})$	i^* of that element
$\bar{x}$	$x \otimes 1 + 1 \otimes x$
$\bar{y}$	$y \otimes 1 + 1 \otimes y$
$\bar{w}$	$w \otimes 1 + 1 \otimes w$
α	$x \otimes w + w \otimes x$
α'	$y \otimes w + w \otimes y.$

The element mapping to $x \otimes y + y \otimes x$ is not listed, as it is not algebraically independent of the others. It is equal to $\bar{x} \cup \bar{y}$ because of the relation in $H^*\left(B(Z_2 \wr Z_2)\right)$.

The 'product' invariant generators will denoted

Element of $H^*(BZ_2^{\wr 3})$	i^* of that element
χ	$x \otimes x$
η	$y \otimes y$
ω	$w \otimes w.$

Relations among these generators are completely determined by their images in $H^*(BZ_2^{\wr 2} \times BZ_2^{\wr 2})$, which yields the following list of relations.

Relation	Relation Degree
$(\bar{x}\eta)$	3
$(\bar{y}\chi)$	3
$(\eta\chi)$	4
$(\bar{x}\alpha' + \bar{y}\alpha + \bar{x}\bar{y}\bar{w})$	4
$(\chi\alpha')$	5
$(\eta\alpha)$	5
$(\alpha\alpha' + \bar{x}\bar{y}\omega)$	6

There is also the degree one generator z from the cohomology of the last wreath factor, and the product of this class with any of the sum classes is zero. This yields

$$H^*(BZ_2^{\wr 3}) \cong \mathbb{F}_2[\bar{x}_1, \bar{y}_1, z_1, \chi_2, \eta_2, \bar{w}_2, \alpha_3, \alpha_3', \omega_4]/(rels)$$

where the relations include those given above.

Verifying that our list of relations is exhaustive is not trivial, and we will only need knowledge of the cohomology in low degrees. Peter Webb applied Nakaoka's result in [32] to the calculation of Poincaré series for wreath products. He computes the Poincaré series of a wreath product $F \wr G$ in terms of the Poincaré series for $H^*(F)$, the Poincaré series for subgroups of G, and a Möbius function. In the case of $G = Z_2$, the situation is particularly simple. We apply his result below to the wreath product $F \wr G = (D_8) \wr Z_2$: $f(t)$ denotes the Poincaré series for the first wreath factor (e.g. $\frac{1}{(1-t)^2}$), g_1 is the Poincaré series for cohomology of the subgroup $\{1\} \leq G$, g_G the Poincaré series for the total group of the last wreath factor (in this case $\frac{1}{1-t}$), f_1 is the Poincaré series for orbits of the trivial group on the base $D_8 \times D_8$, and f_G is the Poincaré series for the trivial orbits of $G = Z_2$ on the cohomology of the base. The $\frac{1}{2}$s are Möbius function values.

$$\begin{array}{cc} g_1 = 1 & f_1 = f(t)^2 = \left(\frac{1}{1-t}\right)^4 \\ g_G = \frac{1}{1-t} & f_G = f(t^2) = \left(\frac{1}{(1-t^2)^2}\right). \end{array}$$

So the Poincaré series is

$$\begin{aligned} \Phi(t) &= g_G f_G - \frac{1}{2} g_1 f_G + \frac{1}{2} g_G f_1 \\ &= \left(\frac{1}{1-t}\right)\left(\frac{1}{(1-t^2)^2}\right) - \left(\frac{1}{2(1-t^2)^2}\right) + \frac{1}{2}\left(\frac{1}{1-t}\right)\left(\frac{1}{(1-t)^4}\right) \\ &= \frac{2+2t}{2(1-t)^2(1-t^2)^2} = \frac{1}{(1-t)^3(1-t^2).} \end{aligned}$$

Intuitively, we can understand g_G as capturing cohomology classes z^k, f_G as capturing classes from the base of the form $a \otimes a$, and f_1 as capturing classes of the form $a \otimes b + b \otimes a$. The different terms are capturing the interactions between conjugation by elements of the last wreath factor and the cohomology of the base. In particular, the lack of a $g_G f_1$ term above coincides with the relation $z(a \otimes b + b \otimes a) = 0$ in $H^*(D_8 \wr Z_2)$. The negative $-\frac{1}{2} g_1 f_G$ term is an inclusion-exclusion term preventing double-counting of the $a \otimes a$ classes.

The coefficients of this Poincaré series in low dimensions are as follows:

$$\Phi(t) = 1 + 3t + 7t^2 + 13t^3 + 22t^4 + 34t^5 + 50t^6 + 69t^7 + \ldots$$

Given the above, we can simply compare the vector space generators for homogeneous elements that come from our algebraic generators and relations with the coefficients of the Poincaré series to guarantee that we have all the relations of that degree or lower identified. We do this for degrees one through four:

Degree 1 (3):

$$\bar{x}, \bar{y}, z$$

Degree 2 (7):

$$\bar{x}^2, \bar{y}^2, z^2, \bar{x}\bar{y}, \bar{w}, \chi, \eta$$

Degree 3 (13):

$$\bar{x}^3, \bar{y}^3, z^3, \alpha, \bar{x}\bar{w}, \alpha', \bar{y}\bar{w}, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, z\chi, z\eta, \bar{x}\chi, \bar{y}\eta$$

Degree 4 (22):

$$\bar{x}^4, \bar{x}(\alpha), \bar{x}^2\bar{w}, \bar{x}(\alpha'), \bar{x}^3\bar{y}, \bar{x}^2\bar{y}^2, \bar{x}\bar{y}^3, \bar{x}^2\chi$$

$$\bar{y}^4, \bar{y}(\alpha'), \bar{y}^2\bar{w}, \bar{y}(\alpha), \bar{y}^2\eta, z^2\chi, z^2\eta, z^4, \chi\bar{w}, \eta\bar{w}, \bar{w}^2, \chi^2, \eta^2, \omega$$

2.4.2 Cohomology and Simple $GL_n(\mathbb{F}_p)$-Modules

As discussed in section 2.2.1 on splitting elementary abelian groups, the indecomposable summands of such groups correspond to simple $\mathbb{F}_pGL_k(\mathbb{F}_p)$-modules for $k \leq n$, which are in turn categorized by p-regular tableaux with k columns. Unfortunately, while there is a construction for their projective cover, the structure of these modules is an important open problem. In order to apply 2.1.5 to these modules, we apply some theory coming from work on unstable Steenrod Modules due to Franjou and Schwartz [13]. This approach was used in [21], where the structure we will use here was identified with the exception of the Steinberg Module.

The theorems of Franjou and Schwartz that we use are about indecomposable Steenrod modules appearing in BZ_2^n. By other work in the same paper they show that these modules are in one to one correspondence with simple GL_n-modules, both indexed by column 2-regular partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i < \lambda_{i+1}$ and $\lambda_k \leq n$. They denote the conjugate partition by λ' and the simple Steenrod module in question by E_λ. The associated $GL_n(\mathbb{F}_2)$ representation they denote by F_λ; we have been denoting this module by $M_{\lambda'}$.

Theorem 2.4.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ denote the column 2-regular partition corresponding to the simple Steenrod Algebra module E_λ. Then the first degree in which E_λ is nonzero is $\lambda'_1 + 2\lambda'_2 + \ldots + 2^k\lambda'_j + \ldots$*

Theorem 2.4.2. *F_λ first appears in the composition series for the $GL_d(\mathbb{F}_2)$-module $H^*(BZ_2^d)$ in degree $\lambda'_2 + 2\lambda'_3 + \ldots + 2^{k'}\lambda_{j'} + \ldots$*

For our purposes $E_\lambda = H^*(X_{\gamma'})$, the cohomology of an indecomposable summand, and Theorem 2.4.1 tells us the lowest degree in which it has nonzero cohomology. Theorem 2.4.2 applies the first theorem to Harris and Kuhn's theorem on indecomposable summands of BZ_2^n. Let $\hat{\lambda}'$ denote the partition derived from λ' by deleting λ'_1. We know from the work of Harris and Kuhn that $X_{\hat{\lambda}'}$ is linked to $X_{\lambda'}$; in particular their simple modules are isomorphic as $\mathbb{F}_p GL_n(\mathbb{F}_2)$ modules. Theorem 2.4.2 can be interpreted to say that this module occurs amongst the bottom cells of the linked summand.

Both theorems 2.4.1 and 2.4.2 follow from an insightful filtering of monomials of $H^*(BZ_2^d)$. For an integer i, let $\alpha(i)$ denote the number of 1's in the dyadic expansion of i; for example $\alpha(4) = 1$, $\alpha(5) = 2$. For a sequence or partition $I = (i_1, \ldots i_n)$, denote $\alpha(I) = \sum_j \alpha(i_j)$. Let us use I to list the sequence of exponents of a monomial; e.g. $x^I = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ in $H^*(BZ_2^n)$. For any element σ of the Steenrod algebra, if

$$\sigma(x^I) = \sum_{i=1}^{m} x^{J_i},$$

then $\alpha(I) \geq \alpha(J_i)$ for all i. Therefore monomials of weight less than or equal to any particular integer form a Steenrod submodule of $H^*(BZ_2^d)$.

Using an algebraic approach, Carlisle and Kuhn recreated the results of Franjou and Schwartz in [6].

2.4.3 Cohomology of Certain Indecomposable Summands

In [21], Martino took the results of the previous subsection and examined the Steenrod structure of the cohomology of $H^*(BZ_2^4) \cong \mathbb{F}_2[w, x, y, z]$ to calculate exactly where the simple GL_4 modules first appear. The relations are largely induced by elements which are not Steenrod primitive; these imprimitive elements cannot correspond to bottom cells of an indecomposable summand.

M_4	is the trivial module
M_{41}	is 4 dimensional with basis
	w, x, y, z
M_{42}	is 6 dimensional with basis
	$wx, wy, \ldots, yz$
M_{43}	is 4 dimensional with basis
	wxy, wyx, wyz, xyz
M_{421}	is 20 dimensional with basis
	$w^3x, \ldots, yz^3, w^2xy, \ldots xyz^2$
	with relations $Sq^1(wxy), \ldots Sq^1(xyz)$
M_{431}	is 14 dimensional with basis
	$w^3xy, \ldots, xyz^3, w^2xyz + wxy^2z, w^2xyz + wx^2yz, w^2xyz + w^2xyz^2$
	with relations $Sq^1(wxyz)$
M_{421}	is 20 dimensional with basis
	$w^3x^2yz, \ldots, wxy^2z^3, w^3x^3y, \ldots xy^3w^3$
	with relations $Sq^1(w^3xyz), \ldots Sq^1(wxyz^3)$

The situation for the Steinberg module M_{4321} is similar, but more complicated. Using Theorem 2.4.1, we find that the first appearance of this module (and the lowest classes of the corresponding summand) should appear in degree 11 with weight 6. I will list representatives from the $\mathfrak{S}_4$ orbits, along with the size of the orbit. We also separate monomials using three factors from those using four, as they can then be used for identifying $L(3)$s in BZ_2^3.

Representative	Size of Orbit
w^7x^3y	24
$w^5x^3y^3$	12
w^7x^2yz	12
w^6x^3yz	12
$w^5x^3y^2z$	24
$w^4x^3y^3z$	12

This basis generates a 96 - dimensional subspace, of which the Steinberg module is a subquotient. Classes which are not Steenrod primitive with a source of weight less than or equal to 6 cannot belong to these summands, so we examine the images of Sq^1 and Sq^2 projected onto the subspace spanned by the above. We use π to denote the projection. We again use representatives of S_4 orbits.

Representative	Size of Orbit
$\pi \circ Sq^2(w^3x^3y^3) = w^5x^3y^3 + w^3x^5y^3 + w^3x^3y^5$	4
$\pi \circ Sq^1(w^7xyz) = w^7x^2yz + w^7xy^2z + w^7xyz^2$	4
$\pi \circ Sq^1(w^5x^3yz) = w^6x^3yz + w^5x^3y^2z + w^5x^3yz^2$	12
$\pi \circ Sq^2(w^3x^3y^2z) = w^5x^3y^2z + w^3x^5y^2z + w^3x^3y^4z$	12

These can be taken as relations to give us a 64-dimensional quotient. As the module is 64-dimensional (by Corollary 2.1.7), the quotient must be the module itself.

Therefore we can simplify the above to the following generators

Representative	Size of Orbit
w^7x^3y	24
$w^5x^3y^3$	12
w^7x^2yz	12
$w^5x^3y^2z$	24

and relations

Representative	Size of Orbit
$w^5x^3y^3 + w^3x^5y^3 + w^3x^3y^5$	4
$w^7x^2yz + w^7xy^2z + w^7xyz^2$	4

Martino did not derive the result for M_{4321} because he was using Corollary 2.1.7 to calculate the multiplicity of the corresponding summand. We need that data because some of the multiplicities we are calculating are for groups which are not p groups.

An additional note is that throughout our calculations, none of the relations we have listed survive the Weyl group action, so the calculation of their images are omitted.

We will also need corresponding results for $H^*(BZ_2^3) \cong \mathbb{F}_2[x, y, z]$. The simple modules are M_3, M_{31}, M_{32}, and M_{321}, first appearing in degrees $0, 1, 2$, and 4. The weights of the associated monomials are $0, 1, 2, 3$. Therefore

$$\begin{aligned} M_3 &\quad \text{is the trivial module} \\ M_{31} &\quad \text{is a quotient of } x, y, z \\ M_{32} &\quad \text{is a quotient of } xy, xz, yz \\ M_{321} &\quad \text{is a quotient of } x^3y, \dots yz^3, x^2yz, xy^2z, xyz^2 \end{aligned}$$

Examining the span of each of these collections of monomials, we find that they are all Steenrod primitive with the exception of $Sq^1(xyz)$. Therefore the dimensions of the corresponding modules are $1, 3, 3$, and 8. One can also discern that the classes correspond to the lowest cells of the linked summands (modulo classes from summands of lower connectivity); for example X_{31} should be linked to $X_1 = BZ_2$, and we see M_{31} appearing with basis corresponding to the classes corresponding to the bottom cells of the three BZ_2s appearing in BZ_2^3.

If we identify cohomology classes associated with summands of low connectivity, we can then use that information to tell us something about the cohomological structure of summands of higher connectivity by eliminating classes which we can associate with the summands of low connectivity. In particular, we will use the following data, derived from the above:

$$\begin{aligned} H^*(X_1) &= \mathbb{F}_2[z_1] \\ H^2(X_2) \leq H^2(BZ_2^2) &= \langle xy \rangle \\ H^3(X_2) \leq H^2(BZ_2^2) &= \langle x^2y, xy^2 \rangle \\ H^4(X_2) \leq H^2(BZ_2^2) &= \langle x^2y^2 \rangle \\ H^4(2L(2)) \leq H^2(BZ_2^2) &= \langle x^3y, xy^3 \rangle \end{aligned}$$

These basis elements are again given modulo classes from summands of lower connectivity; for example the class generating $H^2(X_2)$ should be a $GL_2(\mathbb{F}_2)$ invariant, as X_2 corresponds to the trivial module for this group. Therefore it can be identified specifically as x^2+xy+y^2, the Dickson invariant. However, x^2 and y^2 belong to the $2L(1)$, appearing in $BZ_2 \times Z_2$, so modulo the classes from these other summands, these two generators are the same. In particular, if one considers the composition $H^*(BZ_2^2) \stackrel{i^*}{\to} X_2 \stackrel{\pi^*}{\to} H^*(BZ_2^2)$, xy will not be in the kernel. When we associate a cohomology class to a summand in this fashion, we say that the class belongs to the summand *modulo summands of lower connectivity*, or sometimes that the class belongs to the summand *modulo other summands*.

We also use the fact that $H^*(X_3) \cong H^*(BJ_1) \cong \mathbb{F}_2[x_3, y_4, z_7]/(\text{rels})$ [21], and all other summands X from elementary abelian groups have $H^*(X) = 0$ for $* \in \{1, 2, 3, 4\}$ (which can be derived from the Theorem 2.4.1). For other summands, we have $H^*(BA_6) \cong H^*(BA_4)$ as graded vector spaces by examining Poincaré series; to preserve the parallel if $H^*(BD_8) = \mathbb{F}_2[x_1, y_1, w_2]$ we choose our basis so that the bottom class of $H^*(BA_6)$ is $x^2 + y^2 + w$.

The last summand we wish to learn about is the original summand of $B(Z_2 \times D_8)$. There is only one original summand as the outer automorphism group of D_8 is a 2-group. Thinking about this as a product of classifying spaces $BZ_2 \times BD_8$ and recalling that we are working stably, we can decompose this as follows:

$$\begin{aligned} BZ_2 \times BD_8 &= (BZ_2 \wedge BD_8) \vee (BZ_2) \vee (BD_8) \\ &= (L1 \wedge (BA_6 \vee 2L1 \vee 2L2)) \vee (L1) \vee (BA_6 \vee 2L1 \vee 2L2) \\ &= ((L1 \wedge BA_6) \vee 2(L1 \wedge L1) \vee 2(L1 \wedge L2)) \\ &\qquad \vee (L1) \vee (BA_6 \vee 2L1 \vee 2L2)\,. \end{aligned}$$

We have denoted $L(1)$ and $L(2)$ by $L1$ and $L2$ for ease of reading.

Examining Poincaré series for all the resultant summands, we find they account for all the dimensions of $H^*(BZ_2 \times D_8)$ for $* \leq 4$ except for one dimension in H^3 and one dimension in H^4. As we know the exact cohomology for all the summands with lowest cohomology class in degree one or two, we can see that the class

$$z(x^2 + w + y^2) \in H^3(BZ_2 \times D_8) \cong \mathbb{F}_2[z_1] \otimes \mathbb{F}_2[x_1, y_1, w_2]/(xy)$$

does not lie in the span of classes lying in other summands, so it has a nonzero image under the composition of including to $H^*(X_{Z_2 \times D_8})$ and then projecting back to $H^*(BZ_2 \times D_8)$.

2.4.4 The Gysin Sequence

It will often be of use to us to understand the transfer under H^*. While this is a very natural setting for it, computations can still be challenging. When $p = 2$, there is a consequence of the Gysin sequence which simplifies matters considerably.

Theorem 2.4.3 ([27]). *Let P denote a finite 2-group, Q a subgroup of index 2,* *and $i : Q \to P$ the inclusion. Let $z \in H^1(BP, Z_2) \cong Hom(P, Z_2)$ correspond to the group homomorphism with kernel Q. Then the following is a long exact sequence of vector spaces.*

$$\cdots \xrightarrow{tr} H^*(BP) \xrightarrow{\cup z} H^*(BP) \xrightarrow{i^*} H^*(BQ) \xrightarrow{tr} H^*(BP) \xrightarrow{\cup z} \cdots$$

In particular, if one is looking for the cohomology classes of summands which are linked in BP, it is useful to examine classes which have no relations involving $H^1(BP)$; such classes cannot be in the image of the transfer.

2.4.5 The Inclusion $BG \times H \hookrightarrow BG \wr H$

We will have need of a good understanding of the inclusion map of the diagonal subgroup

$$\begin{aligned} i : G \times Z_2 \quad &\to \quad G \wr Z_2 \\ &\text{given by} \\ (g, z) \quad &\mapsto \quad (g, g; z) \end{aligned} \tag{2.6}$$

This map i develops more importance when examined from the perspective of the Borel construction: recall that given a space X with a G-action, we can form a product over G

$$X \times_G EG = (X \times EG)/G.$$

For $X = *$, this construction yields BG. Importantly for our purposes, for $X = BH^n$ and $G \leq \mathfrak{S}_n$, the construction yields $BH \wr G$.

Adem and Milgram [1] construct an operation Γ which takes an element of $H^t(Y)$ and produces an element in $H^{nt}((Y^n) \times_G EG)$. In terms of Nakaoka's basis for $G \wr Z_p$, $\Gamma(a)$ is $a \otimes a \otimes \ldots \otimes a$. In particular, they derive

$$i^*(\Gamma(a)) = \lambda_t \sum_{t=0}^{t} \mathrm{P}^j(a) \otimes e^{t-j} \tag{2.7}$$

where t denotes the degree of a, P denotes the jth summand of the total powering operation, e^{t-j} is the vector space generator for $H^{t-j}(BZ_p)$, and λ_t is a nonzero constant from $\mathbb{F}_p$. When $p = 2$, P^j is simply Sq^j and e^{t-j} is the $(t-j)$th power of the algebraic generator for $H^*(BZ_2)$.

Equation 2.7 along with the awareness that $i^*(a \otimes b + b \otimes a) = 0$ and the knowledge that the cohomology of the last wreath factor is carried unchanged by i^* allow us to completely determine i^*.

Recall that the cohomology of a group G is said to be *detected* by a collection of subgroups $\{H_1, \ldots, H_n\}$ with inclusions $\{i_1, \ldots, i_n\}$ if the direct sum mapping of cohomology

$$H^*(BG) \xrightarrow{\oplus i_j} \bigoplus_{j=1}^{n} H^*(H_j)$$

is injective. We extend this usage of the term *detected* to particular cohomology classes as well. Adem and Milgram show that $H^*(BG \wr Z_p)$ is detected by the cohomology of the base $H^*(B \prod_p G)$ and the diagonal detecting subgroup $H^*(B(G \times Z_p))$ ([1]). Applying this result inductively, we find that $H^*(BZ_p^{\wr n})$ is detected by the base and the 'iterated diagonal' detecting subgroup, denoted here by $V(n)$. If we consider all our groups as sitting inside $\mathfrak{S}_{p^n}$, $V(n)$ corresponds to the image of Z_p^n in $\mathfrak{S}_{p^n}$ given by Cauchy's theorem. All generators have the same cycle structure of p^{n-1} p-cycles.

The subgroup $V(n)$ is particularly noteworthy because the image of the induced map of the inclusion to the symmetric group contains the polynomial Dickson algebra of GL_n invariants by an application of the Cardenas-Kuhn theorem (see [1]). This result will allow us to derive some information about the cohomology of the principal original summand of $BZ_p^{\wr n}$.

Chapter 3

Group Structures and Their Consequences

In this section we shall develop several results about the structure of the groups we are dealing with, and the immediate consequences for stable splittings. We derive results about the relationship between iterated wreath products and symmetric groups, the structure of maximal elementary abelian subgroups and their normalizers, and the automorphism group of iterated wreath products. Using this information, we derive the incidence of original summands in iterated wreath products and symmetric groups. We also present a theorem describing the automorphism group of $P \times P \times \cdots \times P$ in terms of the automorphism group of P. As a consequence of this thoerem, we derive a counterexample to the strong linkage hypothesis, and also produce an example of two isomorphic summands in one subgroup, one of which is linked and the other is not.

3.1 Iterated Wreath Products

We derive a couple of structural results about iterated wreath products that were not found in the literature. These involve generating sets, split surjections, and the center. The first result describes the isomorphism type of subgroups generated by subsets of our preferred generating set.

Proposition 3.1.1. *When considered as a subgroup of* $\mathfrak{S}_\infty$, $\{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n}\}$ *is a generating set for a subgroup isomorphic to* $Z_p^{\wr n}$. *If* $i_k < i_{k+1}$ *for all* k*, the generator of the* k*th* Z_p *in the wreath product is* τ_{i_k}.

Proof: We argue by induction. When $n = 1$, the proposition is apparent as all of the τ generate cyclic subgroups. Assume it is true for all lists of length n. Considering an arbitrary list $\{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_{n+1}}\}$, observe that the group generated by $\{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n}\}$ and the group generated by all of those generators conjugated by $\tau_{i_{n+1}}$ are disjoint as they act on disjoint subsets of $\mathbb{N}$. Similarly, we have p $\tau_{i_{n+1}}$ conjugates of the group generated by $\{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n}\}$, all disjoint, each is assumed isomorphic to $Z_p^{\wr n}$, and that conjugating by $\tau_{i_{n+1}}$ cycles between them. □

As split surjections play an important role in our theory, it would be nice to know a little more about what they look like. It turns out that describing the center is an important step towards that goal.

Proposition 3.1.2. *The center of* $Z_p^{\wr n}$ *is isomorphic to* Z_p *and is generated by* $[\ldots[[\tau_1, \tau_2], \tau_3], \ldots, \tau_n]$.

Proof: Recall that an element is central if and only if it commutes with every element of the generating set. Consider the group element $\prod_{i=0}^{p^{n-1}-1}(pi + 1, pi + 2, \ldots, pi + p)$. This product commutes with τ_1 because the two elements overlap only on the set $\{1, 2, \ldots, p\}$, where they have the same cyclic action. It commutes with the other τs because conjugation by a higher τ will simply permute the different cycles in the product, resulting in the same group element. Therefore this element and its powers are central.

To show no other element is central, consider the set

$$X := \{\tau_i^{\prod_{j=1}^{k} \tau_{i_j}} | 0 \leq k < n, 0 < i < i_1 < \ldots < i_k \leq n\}$$

This set contains each of the τ generators, along with their translations by appropriate powers of p. If $x \in Z_p^{\wr n}$ is central, x either contains some power of the cycle $(kp+1, kp+2, \ldots, kp+p)$ or it leaves all these elements fixed, as this cycle is an element of X and x must commute with all such elements. Next, observe that the subgroup $\langle \tau_2, \tau_3, \ldots, \tau_n \rangle$ acts transitively on the cycles $(kp+1, kp+2, \ldots, kp+p)$, so if a power of one of these is in x, then all of these to that power have to show up in the cycle decomposition of x in order for x to commute with every element of that subgroup. Therefore central elements are the identity or powers of $\prod_{i=0}^{p^{n-1}-1}(pi+1, pi+2, \ldots, pi+p) = [\ldots[[\tau_1, \tau_2], \tau_3], \ldots, \tau_n]$.□

Note in particular that the center has order p, which leads us to the following observation:

Corollary 3.1.3. *Let $q : Z_p^{\wr n} \to Q$ be a split surjection with nontrivial kernel K. Then the center $Z(Z_p^{\wr n})$ is a subgroup of the kernel K.*

Proof: This is a consequence of the order of $Z(Z_p^{\wr n})$: by applying the Orbit Stabilizer Theorem to the conjugation action of $Z_p^{\wr n}$ on K, we see that $Z(Z_p^{\wr n}) \cap K$ is nontrivial, therefore it must be all of $Z(Z_p^{\wr n})$.□

3.2 Elementary Abelian Subgroups

We would like to know more about elementary abelian subgroups. The structure of the lattice of elementary abelian subgroups is closely tied to the structure for the classifying space of the corresponding group (see [10]). In addition to its general usefulness, we will use some of the results of this section to identify summands appearing in $BZ_2^{\wr 3}$ via transfer.

We begin with a lemma which leads us quickly to the structure of the maximal elementary abelian subgroups.

Lemma 3.2.1. *An element of order p in $Z_p^{\wr n-1} \wr Z_p$ is either in the base or it is conjugate to a power of τ_n.*

Proof: Let g be an element that is not in the aforementioned subgroup, e.g. $g = (g_1, g_2, \ldots, g_p; \tau_n^a)$. Consider g^p. If $g^p = id$, then the mth coordinate function $\prod_{n=0}^{p-1} g_{m-an}$ must be the identity for each m. Thus $g_m = \left(\prod_{n=1}^{p-1} g_{m-an}\right)^{-1}$, and $g = g_1 \tau_n^a g_1^{-1}$. □

This lemma is also found in a slightly different format in [1].

Proposition 3.2.2. *The maximal elementary abelian groups of $Z_p^{\wr n} = Z_p^{\wr n-1} \wr Z_p$ are of one of two forms: either they are maximal elementary abelian subgroups of the base, or they are conjugate to a maximal elementary abelian subgroup of the diagonal $Z_p^{\wr n-1} \times \langle \tau_n \rangle$.*

Proof: Let E be a maximal elementary abelian subgroup. For $g \in E$, either g is in the base or by Lemma 3.2.1 g is conjugate to τ_n^a. If g is conjugate to τ_n^a, then the centralizer $C_{Z_2^{\wr n}}(g)$ is a subgroup of the diagonal subgroup, so E is conjugate to a subgroup of $\Delta(Z_p^{\wr n-1} \times \langle \tau_n \rangle)$. Otherwise E is contained in the base.□

This proposition gives an alternate proof that the base of $Z_p \wr Z_p^{\wr n-1}$ is characteristic for $p \neq 2$:

Corollary 3.2.3. *An elementary abelian subgroup of $Z_p^{\wr n}$ of maximum order is unique (and therefore characteristic) for $p \neq 2$, and has order $p^{p^{n-1}}$.*

<u>Proof:</u> We argue by induction. The result is obvious for $n = 1$. Assuming it for n, by the previous result maximal elementary abelian subgroups of $Z_p^{\wr n+1}$ are either of the form a maximal elementary abelian subgroup of $Z_p^{\wr n}$ times Z_p (order less than or equal to $p^{p^{n-1}+1}$) or are a subgroup of the base, $\prod_{a=0}^{p-1}(Z_p^{\wr n})^{\tau_{n+1}^a}$. Considering maximal elementary abelian subgroups lying in the base, maximal elementary abelian subgroups of a product are products of maximal elementary abelian subgroups of the factors. In this case, these have order less than or equal to $\prod_{a=0}^{p-1} p^{p^{n-1}} = (p^{p^{n-1}})^p = p^{p^n}$, with equality in the case where we are looking at a product of the unique previous maximum (the base of $Z_p \wr Z_p^{\wr n-1}$) with itself, therefore this new maximum is unique amongst subgroups of the base. This product is the base of $Z_p \wr Z_p^{\wr n}$. As $p^{p^n} > p^{p^{n-1}+1}$ unless $p = n = 2$, the new maximum is unique. □

We need $p \neq 2$ because if $p = 2$, then the induction fails at $n = 2$, as the two maximal elementary abelian subgroups have orders $2^{2^0} \cdot 2 = 4$ and $2^{2^1} = 4$, so the order argument limiting them being characteristic fails. For $p = 2$ a similar argument does yield that maximum subgroups have order $2^{2^{n-1}}$.

Recall that maximal elementary abelian subgroups of a product are products of maximal elementary abelian subgroups of the factors. Therefore maximal elementary abelian subgroup E of the nth wreath product is built up from maximal elementary abelian subgroups A_i of the $n-1$st wreath product in one of three ways:

1. They are products of the form $\prod_{i=1}^{p} A_i$, where $A_i = A^{g_i}$. In other words, all the factors are the conjugate to the same subgroup of $Z_p^{\wr n-1}$, and live in the base of $Z_p^{\wr n-1} \wr Z_p$.

2. They are products of the form $\prod_{i=1}^{p} A_i$, with differing factors, not all of which are conjugate, again living in the base unless $p = 2$.

3. They are conjugate to $\Delta(A) \times \langle \tau_n \rangle$.

These three types are distinguished by their normalizers.

Theorem 3.2.4. *Let $N_k(P)$ denote the normalizer of P in $Z_p^{\wr k}$. The maximal elementary abelian subgroups listed above have normalizers*

1. $\left(\prod_{i=1}^{p} N_{n-1}(A)^{\tau_n^i}\right) \rtimes \langle (g_1, g_1^{-1} g_2, \ldots, \left(\prod_{i=1}^{n-1} g_{n-i}^{-1}\right) g_n; \tau_n) \rangle \cong N_{n-1}(A) \wr Z_p$.

2. $\left(\prod_{i=1}^{p} N_{n-1}(A)^{\tau_n^i}\right)$.

3. *If $p = 2$, conjugate to $(\Delta(N_{n-1}(A)) \times \langle \tau_n \rangle)\, i(A)$, where i embeds A into the first coordinate of the base.*

It becomes necessary to discuss centralizers and normalizers in the various wreath products; $C_n(E)$ will denote the centralizer of E in $Z_p^{\wr n}$, similarly $N_n(E)$ would denote the normalizer of E in $Z_p^{\wr n}$. To approach this theorem, first a Lemma is needed:

Lemma 3.2.5. *All of the maximal elementary abelian subgroups of $P := Z_p^{\wr n-1} \wr Z_p$ are maximal abelian subgroups; equivalently they are equal to their own centralizers.*

Proof: An important observation is that if $g \in C_n(E)$, then as E is a maximal elementary abelian subgroup, we must have that g^{p^k} is a nontrivial element of E for some integer k. Starting from this observation, we will prove our lemma using an inductive argument. The case $n = 1$ is trivially true. If E is of the first or second form (e.g. a product of subgroups of the base) then the centralizer of the product is the product of the centralizers of the projections. By the inductive hypothesis this centralizer is therefore equal to the product of the projections, so the lemma is proven in these cases. We turn our attention to the last case, $E = (\Delta(A) \times \langle \tau_n \rangle)^g$. Without loss of generality, let $g = id$. Let $h = f\sigma \in C_n(E)$, an arbitrary element of the centralizer expressed as an element of the wreath product. As $\tau_n \in C_n(E)$, we may assume without loss of generality that $\sigma = id$. We have

$$(\Delta(a); \tau^\epsilon) = h^{-1}(\Delta(a); \tau^\epsilon)h = ((f^{-1}\Delta(a)f)^\sigma; \tau^\epsilon)$$

which implies that all the coordinate functions f_i have images in $C_{n-1}(A)$, so by the inductive hypothesis they have images in A. As h must commute with τ_n, all these coordinate functions must be identically the same, so $f \in \Delta(A)$. $\square$

With knowledge of the structure of the centralizers in hand, we can proceed to the proof of Theorem 3.2.4.

Proof of Theorem 3.2.4: In the first or second case, let h normalize E. If h is in the base of $Z_p^{\wr n-1} \wr Z_p$, we are working inside of a direct product, in which case the normalizer decomposes into a product of normalizers of the factor groups, so h is of the given form. If h is not in the base, then without loss of generality assume that $h = (h_1, h_2, \ldots, h_{p^n}; \tau_n)$; we may do this as there is an automorphism of $Z_p^{\wr n}$ carrying any element not in the base to an element of this form. Conjugation by this element carries $A_i^{\tau_n^i}$ into $A_i^{\tau_n^i h_i \tau_n}$. This implies that $A_i^{h_i} \subseteq A_{i+1}$. Therefore all the A_i are conjugate, which implies the form of the normalizer in the first case. In the second case, we have a contradiction, so we must have that h must lie in the base, implying the given form of the normalizer.

In the third case, without loss of generality assume that E is in fact $\Delta(A) \times \langle \tau_n \rangle$. It is clear that elements of $\Delta(N_{n-1}(A))$ normalize, and τ_n has to normalize because it is in the subgroup. That elements of $i(A)$ also normalize is less obvious: this requires $p = 2$. An element of E is of the form $(a, a; \tau)$ or $(a, a; id)$. Let $(b, id; id)$ be an element of $i(A)$. Recalling that A is elementary abelian and $p = 2$, we have $ba = ab$ and $b = b^{-1}$, so

$$(b, id; id)(a, a; id)(b, id; id) = (bab, a; id) = (a, a; id) \in E$$

$$(b, id; id)(a, a; \tau)(b, id; id) = (ba, ab; \tau) = (ab, ab; \tau) \in E$$

To show that elements of these forms comprise the entire normalizer, we consider the mapping of the normalizer into a subgroup of $GL_n(\mathbb{F}_2)$, where each element of the normalizer is sent to the matrix expression of the automorphism coming from conjugation by that element.

The kernel of this homomorphism is the centralizer of E in Z_2^{lk}, which by the above lemma is E itself.

The base of a wreath product is always normal, so no element of the normalizer can send $\Delta(N_{n-1}(A))$ to an element with a nonidentity τ_n coordinate. This implies that the bottom row of the matrix expression of an element has 0s except in the last spot. The n, n minor of such a matrix represents an element of $\Delta(N_{n-1}(A))$. Lastly, conjugation by an element $a \in i(A)$ is represented by a matrix which is the identity except the last column has been replaced by a with a 1 in the τ_n position.

To summarize, a normalizer element looks like:

$$\begin{pmatrix} \ddots & & & \vdots \\ \Delta(N_{n-1}(A)) & & & i(A) \\ & & \ddots & \vdots \\ \cdots & 0 & \cdots & 1 \end{pmatrix}$$

As the bottom row is compelled to be zero, and $N_{n-1}(A)$ contained every element normalizing A, every element normalizing E must be generatable by elements of the given forms. □

There is an interesting interaction between split surjections and a particular one of the elementary abelian subgroups. Let $V(n)$ denote the subgroup built inductively using the third way to construct elementary abelian subgroups of $Z_p^{\wr n}$ from those of $Z_p^{\wr n-1}$; $V(n) := \Delta(V(n-1)) \times Z_p)$. This coincides with the image of Z_p^n in $\mathfrak{S}_{p^n}$ under the inclusion given by Cayley's theorem.

Proposition 3.2.6. *Let $q : Z_p^{\wr n} \to Q$ be a split surjection with nontrivial kernel K. Then if $\tau_n \in K$, $V(n) \in K$.*

Proof: We argue inductively on n. The proposition is vacuously true when $n = 1$. Assuming it for $n-1$, recall that the surjection π with kernel the base $Z_p^{p^{n-1}}$ is also split, so that we have

$$\begin{array}{ccc} Z_p^{\wr n} & \xrightarrow{q} & Q \\ \downarrow & & \downarrow \\ Z_p^{\wr n-1} & \xrightarrow{q} & \pi(Q). \end{array}$$

$V(n) \cap Z_p^{\wr n-1}$ is of rank $n-1$, and one choice for the missing generator is an element of the center. However, the center lies in K by proposition 3.1.3, so $V(n) \leq K$.□

3.3 Automorphism Groups of Wreath Products

By Nishida's work, the original summands of $BZ_p^{\wr n}$ are in one to one correspondance with the simple modules for $\mathbb{F}_p\text{Out }(Z_p^{\wr n})$, justifying some interest in the latter. In this section we derive what we need to know to determine the number of original summands of $BZ_p^{\wr n}$, and how many of those appear in $B\mathfrak{S}_{p^n}$.

When $p \neq 2$, it is clear that the outer automorphism group of $Z_p^{\wr n}$ is not a p-group as $\text{Out }(Z_p) \cong Z_{p-1}$, so there will be multiple original summands as $\mathbb{F}_p\text{Out }(Z_p^{\wr n})$ will have nontrivial simple modules. However, when $p = 2$ things are easier to work out:

Proposition 3.3.1. *Out* $(Z_2^{\wr n})$ *is a* 2 *group.*

Proof: By 2.3.5, there is a subgroup of the Automorphism group of index 2 which leaves the base of $Z_2 \wr Z_2^{\wr n-1}$ invariant. Let $\mathcal{A}$ denote this automorphism group. By Hall's theorem 2.3.8, there is a group homomorphism

$$\phi : \text{Aut }((Z_2)^{\wr n}) \to \text{Aut }((Z_2^{\wr n})/\Phi(Z_2^{\wr n})) \cong GL_n(\mathbb{F}_2)$$

with kernel a 2 group. Let us consider the image of $\mathcal{A}$ under ϕ. As the base of each wreath product $Z_2^{\wr k} \wr Z_2^{\wr n-k}$ is characteristic for $k \geq 1$, the image of $\mathcal{A}$ must consist entirely of upper triangular matrices. However, this subgroup of $GL_n(\mathbb{F}_2)$ is a 2 group, so $\mathcal{A}$ is an extension of a 2 group by a 2 group, and is therefore itself a 2 group. It has index 2 in Out $(Z_2^{\wr n})$, which is therefore a 2 group. $\square$

This result is implied by the work in [3].

Corollary 3.3.2. $Z_2^{\wr n}$ *has one original summand.*

The automorphism group for $p \neq 2$ can be handled in a related fashion.

Proposition 3.3.3. *Out* $(Z_p)^{\wr n}$ *is an extension of* $\prod_{i=1}^{n}$ *Aut* (Z_p) *by a* p *group.*

<u>Proof:</u> $p = 2$ is handled above. By 2.3.5, the base of $Z_p \wr Z_p^{\wr n-1}$ is characteristic. By a consequence of Hilbert's basis theorem, there is a group homomorphism

$$\phi : \text{Aut } ((Z_p)^{\wr n}) \to \text{Aut } ((Z_p)^{\wr n}/\Phi(Z_p)^{\wr n}) \cong GL_n(\mathbb{F}_p)$$

with kernel a p group. Let us consider $\phi(\text{Aut } (Z_p^{\wr n}))$. As above, the image of $\mathcal{A}$ must consist entirely of upper triangular matrices. The diagonal matrices are the image of $\prod_{i=1}^{n} \text{Aut } (Z_p)$, where i indexes the factor in the iterated wreath product. The group of upper triangular matrices are an extension of the diagonal matrices by the strictly upper triangular matrices, and this last has order $p^{n(n-1)/2}$. Therefore as above, our original Aut $(Z_p^{\wr n})$ must be an extension of Aut $(Z_p^{\wr n})$ by a p-group. □

It is important to note that the diagonal matrices referred to above are in the image of the subgroup $N_{\mathfrak{S}_{p^n}}(Z_p^{\wr n}) \leq \text{Aut } (Z_p^{\wr n})$, as the automorphisms $\tau_i \mapsto \tau_i^k$ can be realized by conjugation in the symmetric group. Therefore these must be the only outer automorphisms induced by conjugation in the symmetric group: the Sylow Subgroup is already using up a p component. This means that $N_{\mathfrak{S}_{p^n}}(Z_p^{\wr n})/Z_p^{\wr n} \cong \prod_{i=1}^{n} \text{Aut } (Z_p)$.

Furthermore, the collection of strictly upper triangular matrices is normal in the collection of all upper triangular matrices, and has the collection of diagonal matrices as a complement. Therefore the simple $\mathbb{F}_p$Out $(Z_p^{\wr n})$-modules are n-fold tensors of the simple $\mathbb{F}_p$Out (Z_p)-modules. The Weyl sum for a Sylow Subgroup in the Symmetric group adds over representatives of the p' component of $\mathbb{F}_p$Out $(Z_p^{\wr n})$, which will annihilate each of these simple modules other than the trivial one. This proves

Theorem 3.3.4. *There are $(p-1)^n$ nonisomorphic original summands of $BZ_p^{\wr n}$, each with multiplicity one. Of these original summands, $B\mathfrak{S}_{p^n}$ contains only the principal original summand.*

3.4 Automorphisms of Products

In this section we present a result about the automorphism groups of products which is a key ingredient in one of the counterexamples of the following section.

Theorem 3.4.1. *Let Q be a p group satisfying the following conditions:*

- *The center of Q lies inside the Frattini subgroup (i.e. $Z(Q) \leq \Phi(Q)$)*
- *Q is indecomposible; e.g. $Q \ncong Q_1 \times Q_2$ for Q_i nontrivial*
- *Q has a finite generating set*

Then Aut $(\prod_{i=1}^m Q)$ is isomorphic to an extension of Aut $(Q) \wr \mathfrak{S}_m$ by a p-group.

Remarks: The first condition is satisfied by any nilpotent nonabelian p-group such as finite nonabelian p-groups. The nilpotency class of such a group is at least 2, and the first term of the lower central series $Z(Q)$ will be contained in the first term of the upper central series $[Q, Q]$, which is in turn contained in the Frattini subgroup of Q. The smallest group satisfying the conditions is $Z_2 \wr Z_2$.

For the case $m = 2$ the result for finite Q can be found in Jason Douma's thesis [9], where it is proved using similar techniques.

Proof: Let us begin by choosing a minimal generating set $\{q_1, \ldots, q_n\}$ for Q. We utilize this to produce a minimal generating set of size mn for $\prod Q$: let $q_{i,j}$ denote the generator which is q_i in the jth coordinate and the identity elsewhere. We will also work regularly with the product $\prod_{j=1}^{m} Q$; it will be convenient to denote the inclusion of the jth factor by ι_j and the projection onto the jth factor by π_j.

I claim that the conditions ensure that every element $(a_1, a_2, \ldots, a_p)$ in the orbit of $q_{i,j}$ under the action of the automorphism group has all but one of the a_j in $Z(Q)$. Consider the centralizers of these two elements in $\prod Q$. If they are in the same automorphism orbit, the automorphism of $\prod Q$ will restrict to an isomorphism of the centralizers.

$$C_{\prod Q}(q_{i,1}) \cong C_Q(q_i) \times \prod_{i=2}^{p} Q$$

whereas

$$C_{\prod Q}(a_1, a_2, \ldots, a_m) = \prod C_Q(a_i).$$

These two groups can be isomorphic only if exactly one of the a_i is not in $Z(Q)$ because Q cannot be written as a nontrivial direct product and q_i is not in $Z(Q)$.

We can say even more about the images of $q_{i,j}$. Let us temporarily fix a particular j, say $j = 1$. Then the collection $\{q_{i,1}\}_{1 \leq i \leq n}$ generate a subgroup isomorphic to Q (in fact, the first factor of the direct product). Observe that for any given automorphism f, we have that if we define $a_{i,k}$ by $f : q_{i,1} \to (a_{i,1}, a_{i,2}, \ldots, a_{i,m})$, there can be at most one coordinate k such that $a_{i,k}$ is noncentral in Q for all i. If not, we can partition the generators into collections depending on which coordinate is noncentral. The generators from each collection generate a subgroup of $\prod Q$, call them $A_1, \ldots A_l$. All the $f(q_{i,1})$ together generate a subgroup of $\prod Q$ isomorphic to Q. All the generators in the each group commute with those in all of the others, yielding a nontrivial factoring $Q \cong \prod A_k$. This factoring contradicts the assumption that Q is not a direct product. Furthermore, if for our given f, $f(q_{i,j})$ has a noncentral kth component for each i, then that coordinate is in fact a generator for the kth factor of Q. If it were not, then all the coordinates of $f(q_{i,j})$ would be nongenerators for Q, so $f(q_{i,j}) \in \Phi(\prod Q)$, a contradiction. For similar reasons, these kth components must all be independant as i varies, so in fact the composition

$$Q \xrightarrow{f} \prod Q \xrightarrow{\pi_k} Q$$

is in fact an automorphism of Q.

We are nearly ready to describe Aut $(\prod Q)$ as a semidirect product. Let $\mathcal{A}$ denote the subgroup of automorphisms α of $\prod Q$ which for each generator $q_{i,j}$, the $\alpha(q_{i,j})$ has a noncentral jth coordinate. In other words, if we consider the composite

$$Q \xrightarrow{\iota_j} \prod Q \xrightarrow{\pi_j} Q.$$

we see that it sends q_i to a generator for Q, but under any other composition with $j \neq j'$

$$Q \xrightarrow{\iota_j} \prod Q \xrightarrow{\pi_{j'}} Q$$

it will send q_i to a central element (which is a nongenerator by the second assumption). Therefore for any automorphism $f : \prod Q \to \prod Q$, we can define a permutation $\sigma \in \mathfrak{S}_m$ by letting $\sigma(j)$ be the coordinate of $f(q_{i,j})$ with a noncentral entry. In other words, we can factor $f = \sigma\alpha$, where $\alpha \in \mathcal{A}$ and $\sigma \in \mathfrak{S}_m$ regarded as acting on the m factors of $\prod Q$. Furthermore, $\mathfrak{S}_m \cap \mathcal{A} = id$ as each element of $\mathcal{A}$ sends generators of the first factor to generators of the first factor, but every nonidentity element of $\mathfrak{S}_p$ does not have this property. Lastly, $\mathfrak{A}$ can be shown to be normal in $Aut(\prod Q)$

by the following calculation. If $a \in \mathcal{A}$ and $f = \sigma\alpha \in \text{Aut }(\prod Q)$,

$$faf^{-1} = \sigma\alpha a\alpha^{-1}\sigma^{-1} = \sigma a'\sigma^{-1}$$

for $a' \in \mathcal{A}$, and

$$\sigma a'\sigma^{-1}(q_{i,j}) = \sigma a'(q_{i,\sigma^{-1}(j)}) = \sigma(x_{\sigma^{-1}j}) = x_j$$

where $x_j, x_{\sigma^{-1}j}$ are generators with noncentral j, $\sigma^{-1}j$ coordinates, respectively. Therefore $faf^{-1}(q_{i,j})$ has a noncentral j coordinate for any $q_{i,j}$, so this element is in $\mathcal{A}$, and thus $\mathcal{A} \trianglelefteq \text{Aut }(\prod Q)$. We conclude that $\text{Aut }(\prod Q) \cong \mathcal{A} \rtimes \mathfrak{S}_m$.

Next, observe that each element a of $\mathcal{A}$ induces an element of $\prod_{j=1}^{m} \text{Aut }(Q)$ via

$$Q \xrightarrow{\iota_j} \prod Q \xrightarrow{a} \prod Q \xrightarrow{\pi_j} Q.$$

on each coordinate of the direct product, giving a group homomorphism

$$q : \mathcal{A} \to \prod_{j=1}^{p} \text{Aut }(Q).$$

The homomorphism q is surjective as the homomorphism splits: the splitting r is the coordinatewise embedding of $\prod_{j=1}^{m} \text{Aut }(Q)$ into $\text{Aut }(\prod_{j=1}^{m} Q)$. Note that the image of r lies in $\mathcal{A}$. The proposition will be proven if we can show that the kernel of q is a p-group. We achieve this goal by returning to the machinery of the Hilbert basis theorem. If we restrict the domain of ϕ_* to $\mathcal{A}$, we see that for an automorphism $f \in \mathcal{A}$, f and $q(f)$ induce the same automorphism of the Frattini quotient $\prod Q/\Phi(\prod Q)$ as they differ only by elements in the center of $\prod Q$. Therefore the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{A} & & \xrightarrow{\phi_*} & & \text{Aut } (\prod Q/\Phi(\prod Q)) \\ \big\downarrow q & & & & \big\uparrow id \\ \prod \text{Aut } (Q) & \xrightarrow[r]{} & \text{Aut } (\prod_{j=1}^{p}(Q)) & \xrightarrow{\phi_*} & \text{Aut } (\prod Q/\Phi(\prod Q)) \end{array}$$

This shows that ϕ_* factors through q. Therefore the kernel of q is contained in the kernel of $\phi|_{\mathcal{A}} : \mathcal{A} \to \text{Aut } (\prod Q/\Phi(\prod Q))$. Recall that

$$\phi_* : \text{Aut } (\prod Q) \to \text{Aut } (\prod Q/\Phi(\prod Q))$$

has a p-group for a kernel (Hall's Theorem, 2.3.8). This means that the kernel of q is a p-group. We conclude that $\mathcal{A}$ is an extension of $\prod_{j=1}^{p} \text{Aut } (Q)$ by a p-group, so Aut $(\prod Q)$ is an extension of $(\prod \text{Aut } Q) \rtimes \mathfrak{S}_m$ by a p-group. □

For the purposes of stable splittings at the prime 2, the above theorem has an important corollary:

Corollary 3.4.2. *Let Q be a 2-group satisfying the conditions for the previous proposition, and furthermore let Aut (Q) be a 2-group. Then Aut $(Q \times Q)$ is a 2-group.*

Proof: Observe that $Aut(Q)$ is a 2-group and $\mathfrak{S}_2$ is a 2-group, so using the previous proposition we can write down a composition series for Aut $(Q \times Q)$ where all the factors are 2-groups.□

3.5 Linkage Counterexamples

3.5.1 Strong Linkage Hypothesis

So for Q a finite, nonabelian 2-group with an automorphism group which is a 2-group (so BQ has only one original summand), $B(Q \times Q)$ also has only one original summand. I hypothesize that under these conditions, the original summand for BQ is linked to that for $B(Q \times Q)$. Some evidence for this is the following:

Proposition 3.5.1. *The principal original summand BA_6 of BD_8 is linked to the principal original summand of $B(D_8 \times D_8)$.*

<u>Proof:</u> We will show that there is a unit

$$BA_6 \to B(D_8 \times D_8) \to BA_6$$

but that no such unit can be factored

$$BA_6 \overset{Bf}{\to} B(D_8 \times D_8) \overset{tr}{\to} BQ \to BA_6$$

for Q any proper subgroup of $D_8 \times D_8$ and $f \in \mathbb{F}_p\text{Rep}\ (A_6, D_8 \times D_8)$, the collection of group homomorphisms up to conjugation. It suffices to demonstrate this for Q of index 2, as a transfer to any other Q' would factor through a transfer to a subgroup of index 2. We will combine detailed knowledge of the cohomology of $B(D_8 \times D_8)$ and a couple of other relevant groups with a version of the Gysin exact sequence to demonstrate that any composition $BA_6 \to B(D_8 \times D_8) \to BQ$ induces the trivial mapping in cohomology. If this is the case, no unit of $\{BA_6, BA_6\}$ could factor through this map, demonstrating that BA_6 is linked in $B(D_8 \times D_8)$ as claimed.

We have that

$$H^*(BD_8) \cong \mathbb{F}_2[x_1, y_1, w_2]/(xy),$$

and that $BD_8 \cong BA_6 \vee 2L(2) \vee 2L(1)$, with the bottom cells of the summands being $w + (x+y)^2$ for BA_6, x^2w or y^2w for $L(2)$ and x or y for the $L(1)$. Also relevant is the splitting $B(Z_2 \times Z_2) \cong BA_4 \vee 2L(2)$, where

$$H^*\left(B(Z_2 \times Z_2)\right) \cong \mathbb{F}_2[x_1, y_1]$$

and the lowest cells for $L(1)$ are again x and y, for $L(2)$ are x^3y and xy^3, and the lowest cell for BA_4 is $xy + x^2 + y^2$. Lastly,

$$H^*(B(D_8 \times D_8)) \cong H^*(BD_8) \otimes H^*(BD_8) \cong \mathbb{F}_2[x_1, y_1, x'_1, y'_1, w_2, w'_2]/(xy, x'y')$$

where subscript indicates degree.

Let us specifically examine $H^2(B(D_8 \times D_8))$. It is 10-dimensional, with the following as one choice of basis:

$$x^2, y^2, x'^2, y'^2, xx', xy', x'y, yy', w, w'$$

Observe that the first eight elements of this basis are in

$$H^2((L(1) \vee L(1)) \times (L(1) \vee L(1))) \leq \mathbb{F}_2[x, y, x', y']/(xy, x'y'),$$

which contains $4L(1) \vee 4BA_4$. These summands account for all the mentioned classes. We know $B(D_8 \times D_8)$ contains $2BA_6$ because there are two independant ways to retract off D_8, and BA_6 is the original summand of BD_8. Therefore the first eight basis elements are in the kernel of the projection

$$H^*(i) : H^*(B(D_8 \times D_8)) \to H^*(BA_6 \vee BA_6)$$

and the remaining two, w and w', map injectively onto a basis for $H^2(BA_6 \vee BA_6)$. If we have a map $BA_6 \to B(D_8 \times D_8)$, we know it can be made to factor through the aforementioned inclusion as those must be the only two copies of that summand.

The next observation is that for any subgroup Q of index 2 in $B(D_8 \times D_8)$, the composition $BA_6 \vee BA_6 \xrightarrow{i} BD_8 \times BD_8 \xrightarrow{tr} BQ$ induces the trivial map in H^2. A variant of the Gysin sequence [27] gives that the following sequence of graded vector spaces is exact for P a 2-group, Q a subgroup of index 2, and $\alpha \in H^1(BP) \cong Hom(P, Z_2)$ corresponding to the group homomorphism with kernel Q:

$$\cdots \xrightarrow{\cup\alpha} H^*(BP) \xrightarrow{in^*} H^*(BQ) \xrightarrow{tr} BP \xrightarrow{\cup\alpha} H^*(BP) \xrightarrow{in^*} \ldots$$

In our case, taking $P = D_8 \times D_8$, observe that α will be some sum of x, y, x', y', and cupping with any of these elements will not annihilate w or w', so the image of the transfer lies in the span of $x^2, y^2, x'^2, y'^2, xx', xy', x'y, yy'$, which we have shown to be in the kernel of i^*. Therefore any composition

$$BA_6 \to BA_6 \vee BA_6 \xrightarrow{i} B(D_8 \times D_8) \xrightarrow{tr} BQ \to BA_6$$

is trivial under H^2. In particular, such a composition cannot be a unit. Therefore BA_6 is linked in $B(D_8 \times D_8)$. By the previous proposition, there is only one original summand of $B(D_8 \times D_8)$, so BA_6 must be linked to that summand. □

These two propositions together allow us to construct a counterexample to the strong linkage hypothesis. Observe that the composition

$$B(P \times P) \hookrightarrow B(P \wr Z_2) \stackrel{tr}{\to} B(P \times P)$$

is equivalent to the Weyl sum $1 + \tau$, where τ is the automorphism of $P \times P$ that interchanges the two factors. Therefore the composition

$$BP \stackrel{in_L}{\to} B(P \times P) \hookrightarrow B(P \wr Z_2) \stackrel{tr}{\to} B(P \times P) \stackrel{\pi_L}{\to} BP$$

is the identity in cohomology, demonstrating that summands of BP appear in $B(P \wr Z_2)$. In particular, with $P = D_8 \cong Z_2 \wr Z_2$, we have that the original summand BA_6 appears in $BZ_2^{\wr 3}$, but that the summand that it is linked to in $B(D_8 \times D_8)$ does not appear because the unit of a principal original summand will be annihilated by an inclusion-transfer composition, and $D_8 \times D_8$ is not a quotient of $Z_2^{\wr 3}$ because they are both rank 4.

3.5.2 Linkage of Multiple Summands

It was thought that for summands of BP isomorphic to a particular summand X, either they would all be linked in BP or they would all be linked in some subgroup. In this subsection we demonstrate that it is possible for some but not all summands of a particular isomorphism class to be linked. The example we develop is the number of $L(2) = X_{21}$ summands linked in $BP := B(Z_2 \times Z_2 \times D_8)$. Our approach is to calculate the number of summands appearing by decomposing this as a product, in a manner similar to 2.1.11. From here it is easy to see which cohomology classes correspond to which summands, and to observe that of the 20 dimensions corresponding to $L(2)$s, 2 are independant of the image of the transfer, but that the remaining 18 correspond to summands linked in proper subgroups. Therefore for any idempotent of $\{BP, BP\}$, if every term of that idempotent contains a transfer, then that idempotent carries no more than 18 $L(2)$.

First off, we decompose this classifying space as a product $B(Z_2 \times Z_2) \times BD_8$. Decomposing each of the factors, we have that BP is equivalent to

$$(BA_4 \vee 2L(2) \vee 2L(1)) \times (BA_6 \vee 2L(2) \vee 2L(1))$$

As we are working stably, a product $A \times B$ decomposes as $A \vee B \vee (A \wedge B)$. Therefore we can rewrite the above as

$$\begin{gathered}(BA_4 \vee 2L(2) \vee 2L(1)) \wedge (BA_6 \vee 2L(2) \vee 2L(1)) \\ \vee(BA_4 \vee 2L(2) \vee 2L(1)) \vee (BA_6 \vee 2L(2) \vee 2L(1))\end{gathered}$$

This distributes to

$$\begin{array}{rcccc}(BA_4 \wedge BA_6) & \vee & 2(BA_4 \wedge L(2)) & \vee & 2(BA_4 \wedge L(1)) \\ \vee 2(L(2) \wedge BA_6) & \vee & 4(L(2) \wedge L(2)) & \vee & 4(L(2) \wedge L(1)) \\ \vee 2(L(1) \wedge BA_6) & \vee & 4(L(1) \wedge L(2)) & \vee & 4(L(1) \wedge L(1)) \\ \multicolumn{5}{c}{\vee(BA_4 \vee 2L(2) \vee 2L(1)) \quad \vee \quad (BA_6 \vee 2L(2) \vee 2L(1))}\end{array}$$

Let us examine the multiplicity of $L(2)$s in each of the listed summands above. Recall that all summands are connected, so summands of the form $L(2) \wedge X$ contain no $L(2)s$. The summand $BA_4 \wedge BA_6$ is indecomposible by Theorem 2.1.11, as worked out in [25]. Therefore it contains no $L(2)$s. $BA_4 \wedge L(1)$ and $BA_6 \wedge L(1)$ contain

$2L(2)$ each, as argued in subsection 2.2.2. These facts allow us to conclude that we get the following after projecting onto $L(2)$ summands.

$$\begin{array}{ccccc} (*) & \vee & (*) & \vee & 2(2L(2)) \\ \vee(*) & \vee & (*) & \vee & (*) \\ \vee 2(2L(2)) & \vee & (*) & \vee & 4(2L(2)) \\ & \vee(2L(2)) & \vee & (2L(2)) & \end{array}$$

This gives a multiplicity of 20 $L(2)$s in BP.

One of the advantages of this decomposition is that it makes it easy to see the cohomological structure of the summands. Letting

$$H^*(BP) \cong \mathbb{F}_2[a_1, b_1, x_1, y_1, w_2]/(xy),$$

we can label the above $L(2)$s with their bottom cohomology class modulo other summands.

$$\begin{array}{ccccc} (*) & \vee & (*) & \vee & (a^2bx, ab^2x, a^2by, ab^2y) \\ \vee(*) & \vee & (*) & \vee & (*) \\ \vee(axw, ayw, bxw, byw) & \vee & (*) & \vee & \left(\begin{array}{c} a^3x, a^3y, ax^3, ay^3, \\ b^3x, b^3y, bx^3, by^3 \end{array} \right) \\ & \vee(a^3b, ab^3) & \vee & (x^2w, y^2w) & \end{array}$$

Note that differentiating the $L(2)$s appearing in $BA_4 \wedge L(1)$ and $BA_6 \wedge L(1)$ from the original summands of BZ_2^3 and $B(Z_2 \times D_8)$ using cohomology can be tricky. One could also list a^2w, b^2w, x^2ab, and y^2ab as $L(2)$s, but we need to leave room for $2X_3$, with degree 4 classes $(a+b+x)abx$ or $(a+b+y)(aby)$, and also the two original summands of $Z_2 \times D_8$, with degree 4 classes $(a+x+y)(aw)$ and $(b+x+y)(bw)$. The above is a complete list of classes modulo other summands.

Now observe that of these 24 classes, only 20 lie in the span of any transfers: apply the Gysin sequence (theorem 2.4.3) and observe that the classes a^3b, ab^3, a^2w, b^2w are not involved in any relations, and so they do not lie in the span of images of transfer maps. Therefore they must correspond to linked summands. Of these, a^3b and ab^3 correspond to the $L(2)$s which factor through the projection onto the $Z_2 \times Z_2$ factor of P. The other two correspond to two copies of the original summand of $B(Z_2 \times D_8)$, each of which comes up via retraction but cannot come up by transfer because the associated transfer is trivial in cohomology.

Now let E_1, E_2 denote the two subgroups of P isomorphic to Z_2^4, with inclusions f_1, f_2, dual to y and x, respectively. We know that

$$BE_i \xrightarrow{Bf_i} BP \xrightarrow{tr} BE_i \qquad i \in \{1,2\}$$

acts as $\overline{W}$, both on summands and in cohomology as the E_i are normal. A direct examination of the classes listed above yields that $9L(2)$s and $1X_3$ factor through each composition, so there are 18 $L(2)$s linked in maximal elementary abelian subgroups.

Chapter 4

Cohomology of Summands

Cohomology is an important tool in analyzing any topological space, especially when working stably. In our case, results about the cohomology of $B(Z_2)^{\wr 3}$ and $BZ_p^{\wr n}$ will be used to place bounds on the number of summands and to describe the structure of some of those summands.

4.1 The Principal Original Summand of $BZ_p^{\wr n}$

The goal of this section is to provide some information on $H^*(X)$, where X is the principal original summand of $BZ_p^{\wr n}$. To do this, we shall develop results on the detection maps $i_j : E_j \to BZ_p^{\wr n}$ for E_j a maximal elementary abelian subgroup, and on Steenrod's Γ construction. These results will give us better information about summands appearing via transfer. We will then apply several results appearing in Adem and Milgram's book about the maximal elementary abelian subgroup $V(n)$ to give us information about summands appearing via retraction. Lastly, we will derive some information about $H^*(X)$.

We will deal transfers first. In order to do this, we will pay special attention to the effect of the Γ construction on detection.

Proposition 4.1.1. *For $a \in H^*(BZ_p^{\wr n-1})$, a is detected by all maximal elementary abelian subgroups of $Z_p^{\wr n-1}$ if and only if $\Gamma(a)$ is detected by all maximal elementary abelian subgroups of $Z_p^{\wr n}$.*

Proof: First, let a be detected by all maximal elementary abelian subgroups of $Z_p^{\wr n-1}$. By the structure theorem for maximal elementary abelian subgroups 3.2.2, every elementary abelian subgroup lies in the base or is conjugate to $\Delta(E) \times Z_p$. Under the induced map for inclusion into the base, $\Gamma(a)$ maps to $a \otimes a \otimes \ldots \otimes a$, which is detected by every product $\prod E_j$, as a is detected by each of the factors. Turning to the inclusion map from $Z_p^{\wr n-1} \times Z_p$, by 2.7, if we let z_n^k denote a class of $H^k(Z_p)$ (Note $z_n^k \neq (z_n^1)^k$ for $p \neq 2$), λ an appropriate coefficient and P^i and appropriate element of the Steenrod Algebra, then

$$i^*(\Gamma(a)) = \lambda a z_n^{(p-1)\deg\ a} + \sum_{i>0} \lambda_i' P^i(a) z_n^{(p-1)\deg\ a-i}.$$

In particular, the term $\lambda a z_n^{(p-1)\deg\ a}$ will be detected by every maximal elementary abelian subgroup of $Z_p^{\wr n-1} \times Z_p$, and its image in that subgroup will be different than those of the other terms of the sum because the degrees of the factors from $Z_p^{\wr n-1}$ differ.

Arguing the other direction, if $\Gamma(a)$ is detected by every elementary abelian subgroup of $Z_p^{\wr n-1} \wr Z_p$, then we simply observe that in the base $\Gamma(a)$ is sent to $a \otimes a \otimes \ldots a$, which can be detected by every $\prod E_i$ if and only if a is detected by every E_j.□

Corollary 4.1.2. *Using Nakaoka's basis, the only basis elements of $H^*(Z_p^{\wr n})$ which are detected by every elementary abelian subgroup are those of the form $\Gamma^n(z_1^k)$, where $z_1^k \in H^k(Z_p)$, the cohomology of the first wreath factor.*

Proof: Symmetric sum classes are not detected by subgroups of $Z_p^{\wr n-1} \times Z_p$ and classes of the form $\Gamma(a) \cup z_n^m$ are not detected by the base, leaving only those of the form $\Gamma(a)$. Inductively applying the previous lemma 4.1.1, the result follows.$\square$

Next, we show that the classes $\Gamma^n(z_1^k)$ are unique in the sense that no element in the span of the other basis elements can have the same image in *any* elementary abelian subgroup.

Proposition 4.1.3. *Let B denote Nakaoka's basis for $H^{kp^{n-1}}(Z_p^{\wr n})$, and $S :=$ span $(B - \{\Gamma^n(z_1^k)\})$. Then for any $a \in S$ and any elementary abelian subgroup E_j, $i_j^*(\Gamma^n(z_1^k) + a) \neq 0$.*

Proof: The argument is inductive on n. When $n = 1$, B is empty, so the result is vacuously true. Assuming the result for $n - 1$, we again look at the inclusion maps into the base and $Z_p^{\wr n-1} \times Z_p$. First considering the induced map into the cohomology of the base, the kernel is elements with a factor of z in them, all of which lie in S, and then we can apply the inductive hypothesis on each factor of $\Gamma^{n-1}(z_1^k) \otimes \Gamma^{n-1}(z_1^k) \otimes \ldots \otimes \Gamma^{n-1}(z_1^k)$ to see that $i_j^*(\Gamma^n(z_1^k) + a) \neq 0$ for any E_j in the base.

For E_j lying in $Z_p^{\wr n-1} \times Z_p$, we first observe that the kernel of the mapping is all the symmetric sum classes, which again lie in S. We may then use an argument similar to that in the second paragraph of 4.1.1, observing that the image of $\Gamma^n(z_1^k)$ consists of a bunch of distinct terms, each with differing degree in the $Z_p^{\wr n-1}$ factor, and that we may apply the inductive hypothesis to the first term to conclude that

$i_j^*(\Gamma^n(z_1^k)+a) \neq 0$. □

Corollary 4.1.4. *The image of every transfer* $H^*(BQ) \xrightarrow{tr} B^*(BZ_p^{\wr n})$ *lies in* S*, so the span of such transfers is a subspace of* S*.*

Proof: As $Z_p^{\wr n}$ has a basis consisting of elements of order p, for any proper subgroup Q there must be some maximal elementary abelian subgroup E_Q such that $E_Q \not\subseteq Q$. By 2.1.4, the composition

$$H^*(BQ) \xrightarrow{tr} H^*(BZ_p^{\wr n}) \xrightarrow{i_j^*} H^*(BE_j)$$

is trivial. Therefore the image of any particular transfer will have to be undetected by some maximal elementary abelian subgroup. As every element of $S+\Gamma^n(z_1^k)$ is detected by all maximal elementary abelian subgroups, the image of any particular transfer must lie inside S. As S is a subspace, the span of all these images still lies inside S. □

Next we turn our attention to split retractions. We are going to focus on a particular maximal elementary abelian subgroup $V(n)$, the one built up by choosing the diagonal subgroup each time in the structure theorem for maximal elementary abelian subgroups 3.2.2. It has order p^n, and when $p \neq 2$ it is the unique subgroup of this order. Let $i_{V(n)} : V(n) \to Z_p^{\wr n}$ denote the inclusion. We shall see that this map will allow us to perceive the existence of a class from an original summand.

Adem and Milgram observe that the smallest Dickson invariant that is part of the polynomial algebra (in degree $\alpha := 2^{n-1}$ for $p=2$ and $\alpha := 2p^{n-1}$ for $p>2$) is in the image of the inclusion map, and $\Gamma^n(z_1^1)+(z_n^\alpha)$ (resp $\Gamma^n(z_1^2)+(z_n^\alpha)$ for $p \neq 2$) maps to it. Letting $i_{(p)} : Z_p^{\wr n} \to \mathfrak{S}_{p^n}$ denote the inclusion of the Sylow subgroup, they also show $\ker(i_{(p)} \circ i_{V(n)})^* : H^\alpha(B\mathfrak{S}_{p^n}) \to H^\alpha(BV(n))$ has codimension 1, because this is the only $GL_n(\mathbb{F}_p)$ invariant in degree α. Also note that because of the previous proposition 4.1.3, the kernel of $i^*_{V(n)}$ in degree α is contained in S.

Lemma 4.1.5. *If $Z_p^{\wr n} \stackrel{q}{\to} Q$ is a split retraction, then the composition*

$$BV(n) \xrightarrow{i_{V(n)}} BZ_p^{\wr n} \xrightarrow{i_{(p)}} B\mathfrak{S}_{p^n} \xrightarrow{tr} Z_p^{\wr n} \stackrel{q}{\to} Q$$

is trivial after applying $H^\alpha(\_)$.

First, consider the commutative diagram

$$\begin{array}{ccc} Z_p^{\wr n} & \xrightarrow{q} & Q \\ \uparrow & & \uparrow \\ V(n) & \xrightarrow{q|_{V(n)}} & q(V(n)). \end{array}$$

As q is a split retraction, $Z(Z_p^{\wr n})$ is in $\ker q$ 3.2.6, so $q(V(n))$ has a lower rank than $V(n)$. Therefore after applying H^*, for any class $x \in H^*(BQ)$, $i_V(n)^* \circ q^* x \neq d_{n,n-1}$, as the map factors through $Bq(V(n))$.

In a similar fashion, when we consider

$$\begin{array}{ccccc} \mathfrak{S}_{p^n} & \xrightarrow{i_{(p)}} & Z_p^{\wr n} & \xrightarrow{q} & Q \\ & & \uparrow & & \uparrow \\ & & V(n) & \xrightarrow{q|_{V(n)}} & q(V(n)) \end{array}$$

we wish to show that for $x \in H^*(BQ)$, and the corresponding class $x' := tr \circ q^*(x) \in H^*(B\mathfrak{S}_{p^n})$, that $i^*_{V(n)} \circ i^*_{(p)}(x') \neq d_{n,n-1}$. In this case, it's Dickson invariant or bust: the image is only the one dimensional subspace.

We cannot immediately use the approach we used in $Z_p^{\wr n}$ because there is a transfer in the way:

$$i_V(n)^* \circ i^*_{(p)}(x') = \underbrace{i_V(n)^* \circ i^*_{(p)} \circ tr}\circ q^*(x).$$

Therefore we are led to apply the Mackey formula to the braced section above. Any term with a nontrivial transfer will be the trivial map, as it involves a transfer to a proper subgroup of an elementary abelian subgroup. The remaining terms have the form

$$BV(n) \xrightarrow{c_t} BV(n)^t \to BZ_p^{\wr n} \xrightarrow{Bq} BQ.$$

However, maps of this form were studied in the previous paragraph; when cohomology is applied the Dickson invariant is not in the image. In particular, the subspace of $V(n)$ corresponding to $Z(Z_p^{\wr n})$ is always missing no matter what t we conjugate by, so no matter what terms are in this sum the Dickson invariant is still not in the image of the induced map on cohomology. Again, this map factors through $H^*(B\mathfrak{S}_{p^n})$, so either the Dickson invariant is in the image or the map is trivial. □

Theorem 4.1.6. *The principal original summand X of $BZ_p^{\wr n}$ satisfies that $H^*(X)$ contains a Steenrod submodule isomorphic to the polynomial subalgebra of GL_n invariants of $H^*(BV(n))$. In particular, $H^\alpha(X) \neq 0$, and there is some $x \in H^\alpha(X)$ such that $i_(p)^* \circ i^*_{V(n)}(x)$ is the Dickson invariant in degree α of $H^*(BV(n))$.*

Let $B\mathfrak{S}_{p^n} = X \vee \bigvee Y_i$, where Y_i are indecomposible summands original to proper subgroups $Q_i < Z_p^{\wr n}$ and X is the principal original summand of $BZ_p^{\wr n}$ (recall this is the only original summand appearing in $B\mathfrak{S}_{p^n}$ by theorem 3.3.4). We define the idempotent e to be the projection

$$e : BZ_p^{\wr n} \xrightarrow{i_{(p)}} B\mathfrak{S}_{p^n} \to \bigvee Y_i \to B\mathfrak{S}_{p^n} \xrightarrow{tr} BZ_p^{\wr n}.$$

Observe that $e \circ tr \circ i_{(p)}$ is equivalent to e up to a unit of $B\mathfrak{S}_{p^n}$. We shall work with this composition, and examine $(e \circ tr \circ i_{(p)} \circ i_{V(n)})$. It shall turn out that the image of the induced map in $H^*(V(n)$ will be trivial in the degree of the smallest Dickson invariant in the polynomial algebra (previously called α), but that this invariant is in the image of $i^*_{V(n)}$, giving us our desired result.

We shall rewrite e in two stages. First, we shall write it out summand by summand:

$$e = BZ_p^{\wr n} \xrightarrow{\sum f_i} Q_i \to Y_i \to Q_i \xrightarrow{\sum g_i} BZ_p^{\wr n}.$$

Then we shall rewrite each f_i in terms of the Segal Conjecture basis:

$$f_i = BZ_p^{\wr n} \xrightarrow{\sum f_k} Q_i$$

where each f_k is either a group homomorphism or a transfer followed by a group homomorphism. We will examine the resulting f_k composed with the appropriate g_i, and evaluate whether a unit of Y_i factors through the composition. If not, we discard the term from consideration as Y_i is not appearing in $BZ_p^{\wr n}$ or $B\mathfrak{S}_{p^n}$ in that fashion. Let $f := f_k$ and $g := g_i$.

First, let f contain a transfer to a proper subgroup Q'. Then considering the corresponding term of $e \circ tr \circ i_{(p)}$ we have

$$V(n) \xrightarrow{i_{V(n)}} \underbrace{BZ_p^{\wr n} \xrightarrow{i_{(p)}} B\mathfrak{S}_{p^n} \xrightarrow{tr} BQ'} \to BQ \xrightarrow{f} BZ_p^{\wr n}$$

for Q' a proper subgroup of $Z_p^{\wr n}$. However, applying the Mackey formula to the braced section, we see that it consists of a bunch of transfers to proper subgroups of $Z_p^{\wr n}$, followed by group homomorphisms In particular, the image in $H^\alpha(Z_p^{\wr n})$ will lie inside S by corollary 4.1.4. The whole sum will be zero after applying H^α because it has to land in the image of $i^*_{V(n)} \circ i^*_{(p)}$ (the subspace generated by the Dickson invariant) but the preimage of that subspace under $i_{V(n)^*}$ intersects trivially with S.

Next, let f be a group homomorphism. As a unit of Y_i factors through $f \circ g$ and Y_i is original to Q_i, we have that f is a split surjection with splitting g. But we have shown in lemma 4.1.5 that this composition is also trivial after applying H^α.

As each term is trivial, the entire sum is trivial. Therefore as claimed the image of $\bigvee Y_i$ is trivial, but the image of $B\mathfrak{S}_{p^n}$ is not, forcing X to have a nontrivial class in degree α mapping onto the Dickson invariant.□

Corollary 4.1.7. *The cohomology of the original summand X of $Z_p^{\wr n}$ contains a Steenrod submodule which maps surjectively onto the polynomial subalgebra of the Dickson invariants under $i^*_{V(n)} \circ i^*_{(p)}$*

Proof: By an application of the Cardenas-Kuhn theorem, these invariants are all in the image of the induced map of the inclusion from $V(n)$ to $\mathfrak{S}_{p^n}$, and as the invariant of lowest degree is in the image of X, they all must be in the image of classes from X as they are all connected by Steenrod operations.□

As there is only one original summand when $p = 2$, this yields immediately

Corollary 4.1.8. *The original summand X of $BZ_2^{\wr n}$ satisfies $H^{2^{n-1}}(X) \neq 0$, and $H^*(X)$ is detected by the cohomology of every elementary abelian subgroup.*

I further hypothesize that the class described above corresponds to the lowest class of the principal original summand, but the proof given does not allow us to rule out the possibility of classes of lower degree belonging to the original summand.

4.2 Principal Summands of $B\mathfrak{S}_m$

We can also describe the principal summand of Sylow 2-subgroups of arbitrary symmetric groups. The Sylow subgroup of $\mathfrak{S}_m$ is a product of these iterated wreath products; consider the binary expression of m, $m = a_0 2^0 + a_1 2^1 + \ldots + a_n 2^n$, where each a_i is 0 or 1. By the same construction and counting argument used for the Sylow subgroups of $\mathfrak{S}_{p^n}$, the Sylow subgroup of $\mathfrak{S}_m$ has the form $\prod_{i=1}^{n} \left(Z_2^{\wr n} \right)^{a_i}$. It turns out that there is only one such summand, by theorem 2.1.11. Recall that if ϕ_P denotes the quotient $P \to P/\Phi(P)$, then the group Out $(P \times Q)$ is said to be *parabolic* if its image under $\phi_{P \times Q}$ in Aut $(P/\Phi(P) \times Q/\Phi(Q)) \cong GL(Z_p)$ consists of block upper triangular matrices.

Proposition 4.2.1. *Out* $(Z_p^{\wr n} \times Z_p^{\wr m})$ *is parabolic for* $n > m$.

<u>Proof:</u> We will argue by induction on m, leaving n free. When $m = 1$, no matter how the generators are chosen, one generator of $Z_p^{\wr n} \times Z_p^{\wr 1}$ lies in a characteristic subgroup (the center). This generator is therefore essentially unique, and it generates the $Z_p^{\wr 1}$, so the outer automorphism group is parabolic.

Assuming the hypothesis holds for $m-1$, we consider $Z_p^{\wr n} \times Z_p^{\wr m}$. The base B of the second factor $Z_p \wr Z_p^{\wr m-1}$ contains one generator and is nearly characteristic: see Theorem 2.3.5. Either way B is normal, so there is a group homomorphism of onto $Z_p^{\wr n} \times Z_p^{\wr m-1}$. Furthermore under this group homomorphism exactly one generator is in the kernel.

If $p \neq 2$, then the generator in the base is in a characteristic subgroup, so its column is upper triangular, and all the others map onto the quotient group in a way which induces a map on the Frattini quotients. The inductive hypothesis can be applied in $Z_p^{\wr n} \times Z_p^{\wr m-1}$, so the columns for all the other generators are block upper triangular. Therefore the whole outer automorphism group is parabolic.

If $p = 2$, then let τ_1 denote the generator in the base and τ_2 denote an independent generator in the base of $Z_2^{\wr 2} \wr Z_2^{\wr m-2}$. B is still normal, so we can still find a mapping to $Z_p^{\wr n} \times Z_p^{\wr m-1}$. We still have a map of Frattini quotients, but the induced map of the quotients of the Automorphism groups can only be defined on the subgroup of index 2. However, this is sufficient for our needs because we know the structure of the unaccounted automorphisms; they swap τ_1 and τ_2. By applying the inductive hypothesis to the quotient, the column corresponding to τ_2 is block upper triangular (with potentially an entry in the τ_1 row) for any automorphism. The two generators are in the same orbit under the action of the automorphism group, so the column corresponding to τ_1 must have the same property. Therefore the outer automorphism group is parabolic. □

Corollary 4.2.2. *Out* $(\prod Z_2^{\wr a_i})$ *is parabolic.*

Proof: Use induction on i. The base case of one factor is immediate. For the inductive step, observe that the portion of the matrix corresponding to

$$\text{Aut}\left(\prod Z_2^{\wr a_i}/\Phi(\prod Z_2^{\wr a_i})\right)$$

involving the first factor and any other factor is block upper triangular by the previous proposition, and the rest of the matrix is block upper triangular by the inductive hypothesis. □.

It bears noting that assuming $p = 2$ in the previous proposition eliminates the possibility that two of the a_i are the same. The claim does not hold when $p \neq 2$. Instead, there are blocks corresponding to factors isomorphic to $\prod_k Z_p^{\wr a_i}$, and the matrix is parabolic with respect to these blocks. By 3.4.1, the blocks are isomorphic to an extension of $\prod_k \left(\prod_i \text{Aut}\ (Z_p)\right) \wr \mathfrak{S}_p$ by a p-group.

Theorem 4.2.3. *There is one principal summand of* $(B\mathfrak{S}_m)\hat{}_2$

Proof: We know that the Sylow 2 subgroup of $\mathfrak{S}_m$ is isomorphic to $\prod Z_2^{\wr a_i}$ By theorem 2.1.11, there will be only one principal summand if the simple modules of $\mathbb{F}_p \prod \left(Z_p^{\wr i}\right)^{a_i}$ are tensor products of the simple modules of the factors. But that is immediate as the simple modules in question consist only of the one dimensional trivial module: the tensor product of two one dimensional modules is one dimensional, and the trivial module is the only one dimensional module for a group ring when $p = 2$. □

Chapter 5

A Splitting Calculation for $BZ_2^{\wr 3}$ and $B\mathfrak{S}_{2^3}$

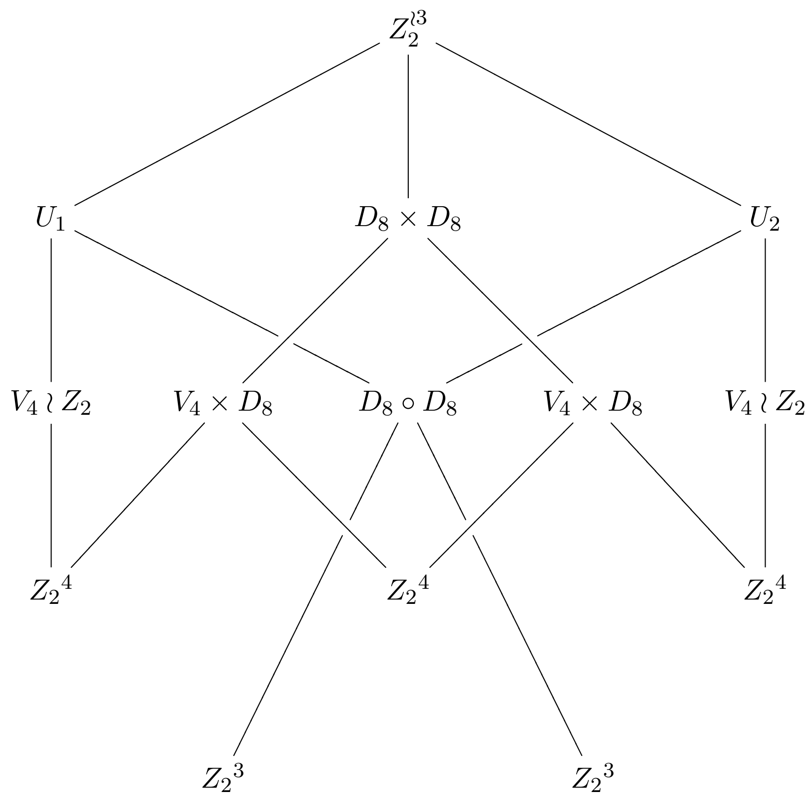
5.1 Overview

The goal of this chapter is to calculate a complete splitting of $BZ_2^{\wr 3}$ and $B\mathfrak{S}_{2^3}$, using the approach of Martino and Priddy ([24], [23]). We shall divide our summands into two types: summands linked in the subgroup which they originate or in elementary abelian subgroups (well-behaved summands), and summands potentially linked in other subgroups (ill-behaved summands). We organize both types by isomorphism class of subgroup rather than by summand.

For the first category, a calculation of the $A_\tau(X)$ matrix in theorem 2.1.10 is accessible, with the help of theorems 2.4.1 and 2.4.2. As linkage in elementary abelian subgroups is so well behaved, it is straightforward to calculate the multiplicities for these two types of summands simultaneously. The primary difficulty lies in the number of different conjugacy classes of subgroups of $Z_2^{\wr 3}$ to be checked. We narrow that list considerably using the `GAP` scripts `IsSelfCentralizing` and `IsOutGroup2Group` found in Appendix A. To contribute original summands via transfer, a group must contain its centralizer, or else the transfer is trivial in cohomology. It must also have original summands which do not correspond to the trivial module; a Weyl sum with p^k terms will annihilate the trivial module for $k > 1$. We subdivide this category into summands linked in abelian subgroups and summands from nonabelian subgroups linked in their own group because the calculations for the two subcategories are so different in character. For some of the nonabelian subgroups, they are described in terms of generators and their Small Groups Library number.

For summands linked in other subgroups, it can be difficult to identify where the summands are linked and to establish the structure of the corresponding modules there. Instead we return to the $A(Z_2^{\wr 3}, M)$ matrix of 2.1.5. We first limit the number of summands under consideration by using `GAP` to list all the isomorphism types of subretractions. Given that list, our approach for these summands is usually to establish a number of linearly independant columns, and then use knowledge of the cohomology of the summands and $BZ_2^{\wr 3}$ to demonstrate that the submatrix generated by these columns is of maximal rank. An exception to this plan is for summands original to BZ_2^3, where we instead demonstrate that all summands original to it are linked in elementary abelian subgroups, and therefore are otherwise accounted for.

The Lattice of Subgroups of $Z_2^{\wr 3}$ which Contribute Summands via Transfer



5.2 Well-behaved Summands

5.2.1 GAP Output

We begin by eliminating most of the isomorphism types from consideration. Asking GAP for conjugacy classes of subgroups which are self-centralizing and have more than one original summand (see Appendix A for code) produces the following collection of representatives:

Subgroup Conjugacy Class Representative	Isomorphic To
$\langle(1,2)(5,6),(3,4)(7,8),(1,5)(2,6)(3,7)(4,8)\rangle$	$(Z_2)^3$
$\langle(1,2)(3,4)(5,6)(7,8),(1,5)(2,6)(3,7)(4,8),$ $(1,3)(2,4)(5,7)(6,8)\rangle$	$(Z_2)^3$
$\langle(7,8),(3,4),(5,6),(1,2)\rangle$	$(Z_2)^4$
$\langle(1,2)(3,4),(5,6)(7,8),(5,7,6,8),(1,3,2,4)\rangle$	$(Z_4)^2$
$\langle(5,6),(1,2)(3,4),(7,8),(1,3)(2,4)\rangle$	$(Z_2)^4$
$\langle(5,6),(1,2)(3,4),(7,8),(1,3,2,4)\rangle$	$Z_2\times Z_2\times Z_4$
$\langle(1,2)(3,4),(5,6)(7,8),(5,7)(6,8),(1,3)(2,4)\rangle$	$(Z_2)^4$
$\langle(1,2)(3,4),(5,6)(7,8),(5,7)(6,8),(1,3,2,4)\rangle$	$Z_2\times Z_2\times Z_4$
$\langle(1,2)(3,4),(5,6)(7,8),(1,5)(2,6)(3,7)(4,8),$ $(1,3,2,4)(5,7,6,8)\rangle$	$(Z_4\times Z_2)\rtimes Z_2$
$\langle(3,4)(5,6),(1,2)(7,8),(1,5)(2,6)(3,7)(4,8),$ $(1,3,2,4)(5,7,6,8)\rangle$	$(Z_4\times Z_2)\rtimes Z_2$
$\langle(1,2)(3,4),(5,6)(7,8),(5,7)(6,8),$ $(1,5)(2,6)(3,7)(4,8)\rangle$	$(Z_2\times Z_2)\wr Z_2$

Subgroup Conjugacy Class Representative	Isomorphic To
$\langle(1,2),(3,4),(5,6),(7,8),(1,3)(2,4)\rangle$	$D_8 \times Z_2 \times Z_2$
$\langle(5,6),(1,2)(3,4),(7,8),(5,7)(6,8),(1,3)(2,4)\rangle$	$D_8 \times Z_2 \times Z_2$
$\langle(5,6)(7,8),(3,4)(7,8),(1,2)(7,8),$ $(1,5)(2,6)(3,7)(4,8),(1,3)(2,4)(5,7)(6,8)\rangle$	Central Product $D_8 \circ D_8$
$\langle(5,6)(7,8),(3,4)(7,8),(1,2)(7,8),(5,7)(6,8),(1,3)(2,4)\rangle$	$(Z_2 \times Z_2) \wr Z_2$
$\langle(5,6),(1,2)(3,4),(7,8),(1,2)(5,7)(6,8),(1,3)(2,4)\rangle$	$(Z_2 \times Z_2) \wr Z_2$
$\langle(1,2),(3,4),(5,6),(7,8),(1,5)(2,6)(3,7)(4,8)\rangle$	$(Z_2 \times Z_2) \wr Z_2$
$\langle(1,2),(3,4),(5,6),(7,8),(1,3)(2,4)(5,7)(6,8)\rangle$	$(Z_2 \times Z_2) \wr Z_2$
$\langle(5,6)(7,8),(3,4)(7,8),(1,2)(7,8),$ $(1,2)(5,7)(6,8),(1,3,2,4)\rangle$	$(Z_4 \times Z_4) \rtimes Z_2$
$\langle(1,2),(3,4),(5,6),(7,8),$ $(1,5)(2,6)(3,7)(4,8),(1,3)(2,4)(5,7)(6,8)\rangle$	U_1
$\langle(5,6)(7,8),(3,4)(7,8),(1,2)(7,8),$ $(1,5)(2,6)(3,7)(4,8),(5,7)(6,8)\rangle$	U_2
$\langle(1,2),(3,4),(5,6),(7,8),(5,7)(6,8),$ $(1,3)(2,4),(1,5)(2,6)(3,7)(4,8)\rangle$	$Z_2^{\wr 3}$

The U_is denote subgroups isomorphic to the group of upper triangular matrices in $GL_4(\mathbb{F}_2)$. They are of particular note as most of the subgroups in the previous list are subgroups of one of the Us. If a subgroup $Q \leq U_i$ and Q is not a subretraction of $BZ_2^{\wr 3}$ then original summands of BQ can appear in $BZ_2^{\wr 3}$ only if they appear in BU, as the idempotent

$$BZ_2^{\wr 3} \xrightarrow{tr} BQ \hookrightarrow BZ_2^{\wr 3}$$

factors

$$BZ_2^{\wr 3} \xrightarrow{tr} BU \xrightarrow{tr} BQ \hookrightarrow BU \hookrightarrow BZ_2^{\wr 3}$$

and if

$$BU \xrightarrow{tr} BQ \hookrightarrow BU$$

is trivial, then the original idempotent must have been trivial as well. BU was split in [21] as follows:

$$\begin{aligned} BU \cong \quad & Triv(BU) \vee 2St(BU) \vee 3St(BZ_2^2 \wr Z_2) \vee 2eT(\Delta_4) \\ & \vee 2X(BD_8 \circ D_8) \vee 16L(4) \vee 21L(3) \vee 12L(2) \vee 3BZ_2 \vee 5X_{432} \\ & \vee 5X_{32} \vee BA_4 \vee 3X_{431} \vee 3X_{31} \vee 5X_{421} \vee X_{43} \vee BJ_1 \vee X_{41} \vee 3BA_6. \end{aligned}$$

Excepting summands appearing in elementary abelian subgroups, only summands original to the following isomorphism types appear:

$$D_8,\ (Z_2 \times Z_2) \wr Z_2,\ D_8 \circ D_8, \text{ and } U.$$

Therefore if a subgroup Q is contained in one of the two groups U_1, U_2, if original summands of Q do not appear in this list, we do not need to examine Q further for summands. This eliminates the groups isomorphic to $(Z_2 \times Z_4) \rtimes Z_2$ from consideration.

5.2.2 Abelian Subgroups

For these subgroups, calculation of the structure of the corresponding simple modules is given in section 2.4.3, and based on work of Franjou and Schwartz. I give the monomials of $H^*(Z_2^n)$ of appropriate weight in terms of a representative of a $\mathfrak{S}_n$

orbit $\mathcal{O}$, and then the rank of $\overline{W}\mathcal{O}$ modulo any relations. The sum of these ranks is the rank of $\overline{W}M$.

$Q = \langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,7)(6,8), (1,5)(2,6)(3,7)(4,8) \rangle$

This is the diagonal detecting subgroup, a maximal elementary abelian group of rank 3. By 3.2.4, $N_{Z_2^{\wr 3}}(Q)$ acts as the group of upper triangular matrices.

For the classifying space of this group, modules for original summands can be found with a basis given by the following cohomology classes (modulo other summands):

Module	$\mathfrak{S}_4$ Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_3	1	0
M_{31}	x	0
M_{32}	xy	0
M_{321}	x^3y	See Below
	x^2yz	See Below

We can determine the multiplicity of the Steinberg summand using corollary 2.1.7:

$$\text{Multiplicity of Steinberg Summand} = 2^{\binom{n}{2}} / \|W\| = 2^3/2^3 = 1$$

To summarize, this group contributes

$$L(3) \vee L(2)$$

to $BZ_2^{\wr 3}$ via transfer, and each is linked in this group or original.

When we examine which of these summands survive to $\mathfrak{S}_8$, we observe that all elements of this group have the same cycle structure, so $N_{\mathfrak{S}_8}(Q) = GL_3(\mathbb{F}_2)$. The Weyl action will therefore annihilate all $\mathbb{F}_2\mathrm{Out}\ (Q) = \mathbb{F}_2 GL_3(\mathbb{F}_2)$ modules.

$Q = \langle (1,2)(5,6), (3,4)(7,8), (1,5)(2,6)(3,7)(4,8) \rangle$

As this subgroup is the image of the above under the twist homomorphism, it contributes the same summands to $BZ_2^{\wr 3}$, so this subgroup contributes

$$L(3) \vee L(2)$$

via transfer, and each is linked in this group or original.

In $\mathfrak{S}_8$, this group is not self-centralizing; it is a subgroup of the elementary abelian subgroup $\langle (1,2)(5,6), (3,4)(7,8), (1,5)(2,6), (3,7)(4,8) \rangle$, which is not contained in the Sylow subgroup under examination. Therefore any summands from this subgroup are conjugate to those examined in the elementary abelian subgroup $\langle (1,3)(2,4), (1,2)(3,4), (5,7)(6,8), (5,6)(7,8) \rangle$.

$Q = \langle (1,3)(2,4), (1,2)(3,4), (5,7)(6,8), (5,6)(7,8) \rangle$

This maximal elementary abelian subgroup of rank 4 is the image of the base of $Z_2 \wr D_8$ under ζ. It is generated by $\tau_2, \tau_2^{\tau_1}, \tau_2^{\tau_3}, \tau_2^{\tau_1 * \tau_3}$. It does not contain τ_1, τ_3 or $\tau_3^{\tau_2}$. The Weyl Group in $Z_2^{\wr 3}$ is generated by the images of conjugation by τ_1 and τ_3, and is therefore isomorphic to D_8 as it has order 8. It acts as $Z_2 \wr Z_2 \leq \mathfrak{S}_4$ on the four generators of the group.

For the classifying space of this group, the modules for original summands can be found with a basis given by the following cohomology classes (modulo other summands):

Module	$\mathfrak{S}_4$Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_4	1	0
M_{41}	w	0
M_{42}	wx	0
M_{43}	wxy	0
M_{421}	w^3x	1
	w^2xy	1
M_{431}	w^3xy	1
	$w^2xyz + wx^2yz$	0
M_{432}	w^3x^2yz	1
	w^3x^3y	1
M_{4321}	*various*	See Below

We can determine the multiplicity of the Steinberg Module using corollary 2.1.7:

$$\text{Multiplicity of Steinberg Summand} = 2^{\binom{n}{2}}/\|W\| = 2^6/2^3 = 8$$

To summarize, this group contributes

$$8X_{4321} \vee 8X_{321} \vee 2X_{432} \vee 2X_{32} \vee X_{431} \vee X_{31} \vee 2X_{421} \vee 2X_{21}$$

to $BZ_2^{\wr 3}$ via transfer, and each is linked in this group or original.

The Weyl group in $\mathfrak{S}_8$ is $GL_2(\mathbb{F}_2) \wr \mathfrak{S}_2$, an extension of the previous $Z_2 \wr Z_2$ by the $2'$ group $Z_3 \times Z_3$. We recalculate using the larger Weyl sum:

Module	$\mathfrak{S}_4$Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_4	1	0
M_{41}	w	0
M_{42}	wx	0
M_{43}	wxy	0
M_{421}	w^3x	0
	w^2xy	0
M_{431}	w^3xy	0
	$w^2xyz + wx^2yz$	0
M_{432}	w^3x^2yz	1
	w^3x^3y	0
M_{4321}	w^7x^3y	0
	$w^5x^3y^3$	0
	w^7x^2yz	1
	$w^5x^3y^2z$	3

So this group contributes

$$4X_{4321} \vee 4X_{321} \vee X_{432} \vee X_{32}$$

to $B\mathfrak{S}_8$ via transfer, and each is linked in this group or original.

$Q = \langle (1,2),(3,4),(5,6),(7,8) \rangle$

This subgroup is the base of $Z_2 \wr (Z_2^{\wr 2})$, the image of the above under ζ_8, and so contributes

$$8X_{4321} \vee 8X_{321} \vee 2X_{432} \vee 2X_{32} \vee X_{431} \vee X_{31} \vee 2X_{421} \vee 2X_{21}$$

to $BZ_2^{\wr 3}$ via transfer, and each is linked in this group or original.

In $B\mathfrak{S}_8$, the Weyl group is $\mathfrak{S}_4$ acting on the set of four generators given. We recalculate using the larger Weyl sum:

Module	$\mathfrak{S}_4$Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_4	1	0
M_{41}	w	0
M_{42}	wx	0
M_{43}	wxy	0
M_{421}	w^3x	0
	w^2xy	0
M_{431}	w^3xy	0
	$w^2xyz + wx^2yz$	0
M_{432}	w^3x^2yz	0
	w^3x^3y	0
M_{4321}	w^7x^3y	1
	$w^5x^3y^3$	0
	w^7x^2yz	0
	$w^5x^3y^2z$	1

Therefore this group contributes

$$2X_{4321} \vee 2X_{321}$$

to $B\mathfrak{S}_8$ via transfer, and each is linked in this group or original.

$Q = \langle (1,2), (3,4), (5,7)(6,8), (5,6)(7,8) \rangle$

This group is isomorphic to Z_2^4, and has normalizer $D_8 \times D_8$, the base of $D_8 \wr Z_2$. The Weyl Group is therefore $Z_2 \times Z_2$, with the Weyl generators interchanging $(1,2)$ and $(3,4)$, and $(5,7)(6,8)$ and $(5,6)(7,8)$.

Again, examining $H^*(BQ) \cong \mathbb{F}_2[w,x,y,z]$, we see the simple $GL(4,2)$ modules by looking at the following elements:

Module	$\mathfrak{S}_4$Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_4	1	0
M_{41}	w	0
M_{42}	wx	1
M_{43}	wxy	0
M_{421}	w^3x	2
	w^2xy	2
M_{431}	w^3xy	2
	$w^2xyz + wx^2yz$	0
M_{432}	w^3x^2yz	2
	w^3x^3y	2
M_{4321}	*various*	See Below

We can determine the multiplicity of the Steinberg Module using corollary 2.1.7:

$$\text{Multiplicity of Steinberg Summand} = 2^{\binom{n}{2}}/\|W\| = 2^6/2^2 = 16$$

To summarize, this group contributes

$$16X_{4321} \vee 16X_{321} \vee 4X_{432} \vee 4X_{32} \vee 2X_{431} \vee 2X_{31} \vee 4X_{421} \vee 4X_{21} \vee X_{42} \vee X_2$$

to $BZ_2^{\wr 3}$ via transfer, and each is linked in this group or original.

The Weyl group for this subgroup in $\mathfrak{S}_8$ is $Z_2 \times GL_2(\mathbb{F}_2)$. We recalculate using the larger Weyl sum:

Module	$\mathfrak{S}_4$Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_4	1	0
M_{41}	w	0
M_{42}	wx	0
M_{43}	wxy	0
M_{421}	w^3x	0
	w^2xy	1
M_{431}	w^3xy	1
	$w^2xyz + wx^2yz$	0
M_{432}	w^3x^2yz	2
	w^3x^3y	1

Module	$\mathfrak{S}_4$Orbit Representative(s)	rank $\overline{W}\mathcal{O}$
M_{4321}	w^7x^3y	3
	$w^5x^3y^3$	1
	w^7x^2yz	2
	$w^5x^3y^2z$	6

Therefore this group contributes

$$12X_{4321} \vee 12X_{321} \vee 3X_{432} \vee 3X_{32} \vee X_{431} \vee X_{31} \vee X_{421} \vee X_{21}$$

to $B\mathfrak{S}_8$ via transfer, and each is linked in this group or original.

$Q = \langle(1,3,2,4),(5,7,6,8)\rangle$

This group is isomorphic to $Z_4 \times Z_4$. From [15], we know this classifying space splits in a way which is cohomologically isomorphic to how $B(Z_2 \times Z_2)$ splits, and that the linkage between summands is exactly the same. However, neither of the summands isomorphic to $L(1)$ can survive, as dim $H_1(S_3) = 3$, and we already have three $L(1)$ summands showing up via retraction. As the summands isomorphic to $L(1)$ are linked to the ones isomorphic to $L(2)$, then neither of the summands isomorphic to $L(2)$ can survive either. Lastly, the principal dominant summand cannot show up via transfer in any containing group. Therefore, no summands from this group survive to $BZ_2^{\wr 3}$ or $B\mathfrak{S}_8$.

$Q = \langle (1,3,2,4),(5,6),(7,8)\rangle$

This subgroup, isomorphic to $Z_4 \times Z_2 \times Z_2$, is normal in the diagonal $Z_2^{\wr 2} \times Z_2^{\wr 2}$, with Weyl group $Z_2 \times Z_2$ generated by sending the four cycle to its inverse (conjugating by τ_1) and swapping the two two cycles (conjugating by $(5,7)(6,8) = \tau_2^{\tau_3}$).

Looking at cohomology, one sees that no cohomology classes survive under $\overline{W}$ because one of the elements of $\overline{W}$ (namely the one which sends the four-cycle to its inverse) acts as the identity in cohomology, and thus every element in the Weyl sum will have an even number of elements which act in the same way, so no summands are passed from this subgroup via transfer.

$Q = \langle (1,3,2,4),(5,6)(7,8),(5,7)(6,8)\rangle$

This is the image of the above group under ζ_8, so it also fails to contribute any summands via transfer.

5.2.3 Nonabelian Subgroups

We shall group these by isomorphism type, as there are fewer summands per isomorphism type to consider. Occasionally we must subdivide to look at different conjugacy classes of subgroups in $BZ_2^{\wr 3}$ isomorphic to these isomorphism types.

$((Z_4 \times Z_2) \rtimes Z_2)$

There are two subgroups of this isomorphism type (up to conjugacy) and one is contained in U_1 and the other in U_2. For original summands from these subgroups to appear via $tr \circ incl$ in $BZ_2^{\wr 3}$, those summands would have to appear in BU. BU was split in [21], and it does not contain any summands from groups of this isomorphism type. Therefore no summands from this group appear in $BZ_2^{\wr 3}$ or $B\mathfrak{S}_8$.

$(Z_2 \times Z_2) \wr Z_2$

There are five conjugacy classes of subgroups of $Z_2^{\wr 3}$ isomorphic to this group. Two are normal, the other three are normal in subgroups of order 64. The outer automorphism group of this group is $\mathfrak{S}_4 \times Z_2$, which is an extension of S_3 by Z_2^3, so there are only two original summand types: the original and a Steinberg summand with a two dimensional module. The original summand does not appear as this group is not a subretraction of $BZ_2^{\wr 3}$ (see section 5.3), so only the multiplicity of the Steinberg summand needs to be considered. For the two subgroups of this isomorphism type that are normal, the order of the Weyl group in $Z_2^{\wr 3}$ is the dimension of the module times a power of 2, so $\overline{W}$ trivializes the module. The others require individual attention:

$Q = \langle (1,2), (3,4), (5,6), (7,8), (1,5)(2,6)(3,7)(4,8) \rangle$

This group is normal in U_1, with Weyl group generated by conjugation by $(1,3)(2,4)(5,7)(6,8)$, which has the effect of swapping the two Z_2s that are wreathed. Viewing the outer automorphism group as a subgroup of $\mathfrak{S}_6$ with generating set $\{(1,2), (1,2,3,4), (5,6)\}$, the nontrivial Weyl conjugation corresponds to $(1,3)(5,6)$, which is nontrivial under the quotient mapping $\mathfrak{S}_4 \times Z_2 \to S_3$, so by corollary 2.1.7,

$$\text{Multiplicity of Steinberg Summand} = 2^{\binom{n}{2}} / \|W\| = 2^3/2^3 = 1$$

So this group contributes one original summand

$$St((Z_2 \times Z_2) \wr Z_2)$$

and other (linearly dependent) summands from subgroups to $BZ_2^{\wr 3}$ via transfer. Also of note is that there is a degree one class which survives the transfer-inclusion process from this subgroup to $Z_2^{\wr 3}$ (aka the bottom cell of an $L(1)$ corresponding to τ_1), but that this class does not survive the transfer-inclusion from either maximal elementary abelian subgroup of this group. Therefore that $L(1)$ is likely linked in this subgroup.

Turning to $B\mathfrak{S}_8$, the Weyl group grows to order 128 (with additional generator represented by $(3,7)(4,8)$), so the Weyl sum trivializes the modules as above. Effectively, in $\mathfrak{S}_8$, this group can be regarded as a normal subgroup in a different Sylow subgroup for the purposes of counting contributed summands.

$Q = \langle (1,2)(3,4), (1,3)(2,4), (1,5)(2,6)(3,7)(4,8) \rangle$

This group is the image of the previous one under the ζ_8; as such it is normal in U_2 and it contributes one original summand

$$St((Z_2 \times Z_2) \wr Z_2)$$

and other (linearly dependent) summands from subgroups to $BZ_2^{\wr 3}$ via transfer.

Regarding this subgroup in $B\mathfrak{S}_8$, the Weyl group grows to $GL_2(\mathbb{F}_2)$. This Weyl sum annihilates the two dimensional Steinberg module, so no original summands are contributed via transfer.

$Q = \langle (5,6), (1,3)(2,4), (1,2)(5,7)(6,8) \rangle$

This group is not a subgroup of U_1 or U_2. Its Weyl generator is conjugation by $(5,7)(6,8)$. Viewing the outer automorphism group as a subgroup of $\mathfrak{S}_6$ with generators $(1,2), (1,2,3,4), (5,6)$, the nontrivial Weyl conjugation corresponds to $(1,2)(3,4)(5,6)$, which is in the kernel of the extension mapping

$$Z_2 \times Z_2 \times Z_2 \to \mathfrak{S}_4 \times Z_2 \to \mathfrak{S}_3,$$

so no copies of the Steinberg summand survive the inclusion-transfer as the Weyl sum will act on any simple module as a sum of two copies of the identity.

$D_8 \circ D_8 \cong U_1 \cap U_2$

There is only one subgroup of this isomorphism type, and it is the intersection of the two upper triangular groups. This characteristic subgroup is isomorphic to the central product of two copies of D_8. It contains neither τ_1 or τ_2, and in fact is the kernel for a retraction of $Z_2 \times Z_2$ with these generators, which is therefore the Weyl group. Its outer automorphism group is isomorphic to $\mathfrak{S}_3 \wr \mathfrak{S}_2$. Viewed as a subgroup of $\mathfrak{S}_6$, conjugation by τ_1 corresponds to $(1,2)$ and conjugation by τ_2 corresponds to $(4,5)$. Recall that in subsection 2.2.2, we have established that $\mathbb{F}_2\mathfrak{S}_3 \wr \mathfrak{S}_2$ has two nontrivial four dimensional simple modules, which can be represented in the following fashion:

(Direct Sum Representation) (Tensor Product Representation)

$$(1,2) \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1,2) \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(1,2,3) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1,2,3) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$(1,4)(2,5)(3,6) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (1,4)(2,5)(3,6) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Upon direct inspection of the actions of the Weyl sum, we see that a direct sum module is trivialized and the tensor representation is reduced to one dimension. Therefore we have one copy of the corresponding summand, $eT(\Delta_4)$, appearing in $BZ_2^{\wr 3}$.

In $\mathfrak{S}_8$, the Weyl Group extends to D_{12}, with an additional conjugation action corresponding to $(1,2,3) \in \mathfrak{S}_3 \wr \mathfrak{S}_2$. This means that there is a subgroup of the Weyl group which acts as $GL_2(\mathbb{F}_2)$ simultaneously on the first two dimensions and the last two dimensions of the tensor module, with basis according to the columns of the matrices given above. However, a sum from this subgroup will trivialize each of the unit vectors, and therefore will trivialize the entire module. Therefore the entire sum, which consists of this action and a coset representative times this action, will trivialize the entire module, and no copies of this summand appear in $B\mathfrak{S}_8$.

$B(Z_4 \times Z_4) \rtimes Z_2$

There is only one group of this isomorphism type, $\langle(1,3,2,4),(5,7,6,8),(1,2)(7,8)\rangle$ This group has Id#34. This subgroup is normal, with Weyl Group elements being conjugation by id, τ_1, τ_3, and $\tau_1 * \tau_3$. This subgroup is invariant under ζ_8.

Observe that conjugation by τ_1 acts trivially on the group's cohomology because it inverts one of the four-cycles and does nothing to the other generators. Therefore the Weyl action trivializes all the group's cohomology, and no summands show up from this group via transfer.

$B(D_8 \times Z_2 \times Z_2)$

There are two conjugacy classes of subgroups of this isomorphism type and ζ interchanges them. Choosing conjugacy class representatives carefully, one contains τ_1 but not τ_2, the other contains τ_2 but not τ_1. The outer automorphism group is an extension of $GL_2(\mathbb{F}_2)$ by a two group; this corresponds to the automorphism group of $Z_2 \times Z_2$. Furthermore, in [25] this group was shown to be smash decomposible, so there are two types of original summands; a principal summand and one corresponding to a Steinberg module.

For the representative containing τ_1, the Weyl action in $Z_2^{\wr 3}$ interchanges the two Z_2s. Therefore the Weyl action annihilates the principal original summand and reduces the dimension of the Steinberg module to 1. In $\mathfrak{S}_8$, the Weyl group is unchanged, so the subgroup contributes the exact same original summands.

For the representative containing τ_2, the Weyl action in $Z_2^{\wr 3}$ is the same as the group is an automorphic image of the previous one, so we again see one Steinberg summand. However, in $\mathfrak{S}_8$ the Weyl group grows to $GL_2(\mathbb{F}_2)$, which annihilates both of the modules under consideration.

U

There are two subgroups of index 2 are isomorphic to the group of strictly upper triangular 4×4 matrices. They are kernels for retractions of $\langle \tau_1 \rangle$ and $\langle \tau_2 \rangle$, respectively. Their outer automorphism group is isomorphic to D_{12}. Therefore this group has only two types of original summand, a Steinberg and a trivial. We need only worry about the Steinberg summand. The Steinberg summands correspond to a two dimensional module, and by corollary 2.1.7,

$$\text{Multiplicity of Steinberg Summand} = 2^{\binom{n}{2}}/\|W\| = 2^1/2^1 = 1$$

In $B\mathfrak{S}_8$, the Weyl group for U_1 is unchanged, but the Weyl group for U_2 expands to $GL_2(\mathbb{F}_2)$, whose action will trivialize the Steinberg module. Therefore there is only one summand of this type appearing in $B\mathfrak{S}_8$.

$BZ_2^{\wr 3}$

By theorem 3.3.2, the group we are splitting has only one original summand. As argued in section 5.3.8, its lowest cell is in degree 4, and corresponds (modulo other summands) to ω. This summand will also appear in $B\mathfrak{S}_8$ as the Weyl Group is trivial.

5.3 Ill-behaved Summands

As described at the beginning of the chapter, the multiplicity of summands which are linked in subgroups in which they do not originate can be more challenging to establish. On a group of this size, `GAP` can churn out a complete list of isomorphism types of subretractions. For the code used, see `ListSubretractionTypesID` in Appendix A. That list is:

$$\{e\},\ Z_2,\ Z_2 \times Z_2,\ Z_4,\ Z_2 \times Z_2 \times Z_2,\ Z_4 \times Z_2,\ Z_2 \wr Z_2,\ Z_2 \wr Z_2 \times Z_2$$

We will examine summands from the classifying spaces from each of these group by group.

5.3.1 BZ_2

The classifying space of this group has one original summand, BZ_2. $H^1(BZ_2^{\wr 3})$ is three dimensional, so rank $A(BZ_2^{\wr 3}, M_{BZ_2}) \leq 3$. Three distinct retractions of groups isomorphic to Z_2 off of $BZ_2^{\wr 3}$ can be described: one choice of kernels is U_1, U_2, and the base $Z_2 \wr Z_2 \times Z_2 \wr Z_2$, and generators for the complements are τ_1, τ_2 and τ_3, respectively. These are not conjugate, so rank $A(BZ_2^{\wr 3}, M_{BZ_2}) \geq 3$. Therefore there are exactly three copies of BZ_2 in $BZ_2^{\wr 3}$.

Turning to $B\mathfrak{S}_8$, only one of the three cohomology classes is in the image of the inclusion map, namely the class $\overline{x}$ corresponding to the generator τ_1. In $\mathfrak{S}_8$, the other generators τ_2, τ_3 are equal to products of an even number of conjugates of this generator (simply examine their cycle structure) so they cannot correspond to elements of $H^1(B\mathfrak{S}_8)$. Therefore there is only one copy of BZ_2 in $B\mathfrak{S}_8$.

It is important to note that the above argument can be generalized: There are (at least) two different ways to realize $BZ_p^{\wr n}$ as a wedge summand of $BZ_p^{\wr n+1}$. The first is a retraction of $Z_p^{\wr n}$ off of $Z_p^{\wr n+1}$, with kernel the base $Z_p^{p^n}$. The cohomology of the summand contains classes which map to $H^*(B\langle\tau_i\rangle)$ under the inclusions, for $1 < i \leq n+1$. There is also the composition

$$BZ_p^{\wr n} \to BZ_p^{\wr n+1} \xrightarrow{tr} BZ_p^{\wr n} \times BZ_p^{\wr n} \to BZ_p^{\wr n}$$

which induces the identity in cohomology, as described in 2.1.4. This copy of $BZ_p^{\wr n}$ contains classes which map to $H^*(B\langle\tau_i\rangle)$ for $1 \leq i < n+1$ under the inclusions maps. Inductively, this shows that there are at least n independent BZ_ps in $BZ_p^{\wr n}$, and as dim $H^1(BZ_p^{\wr n}) = n$ as there are n independent generators, we have that these account for all summands of this type and also of type BZ_{p^m} for $m \geq 1$. In $B\mathfrak{S}_{p^{n+1}}$, the same thing happens as above, where only the class dual to τ_1 survives the inclusion into the symmetric group, so there is only one BZ_p in $B\mathfrak{S}_{p^{n+1}}$.

5.3.2 BZ_4

This subgroup also has only one original summand, which is the entire classifying space. However, the lowest cell is also in degree 1, and we have exhausted $H^1(BZ_2^{\wr 3})$, so there are no copies of this summand in $BZ_2^{\wr 3}$ or $B\mathfrak{S}_8$; in fact, by the previous remark, there are no copies of BZ_{p^m} in $BZ_p^{\wr n}$ or $B\mathfrak{S}_{p^n}$ for $m > 1$.

5.3.3 $B(Z_2 \times Z_2)$

This subgroup has two original summands, BA_4 and $L(2)$. We will see that there is one $BA_4 \cong X_2$ linked to an X_{42} in a maximal elementary abelian subgroup. This is the only copy by exhaustion; $H^2(BZ_2^{\wr 3})$ is seven dimensional, 3 dimensions are accounted for by BZ_2s and three are accounted for by BA_6s, so there is only the one copy of this summand in $BZ_2^{\wr 3}$. It is worth noting that there is a retraction of $Z_2 \times Z_2$ off of $Z_2^{\wr 3}$ with kernel $U_1 \cap U_2$, which makes it easy to identify that the unit correspding to this summand sends the class in $H^2(BA_4)$ through $\bar{x}\bar{y} \in H^2(BZ_2^{\wr 3})$, modulo other summands.

The situation with the other summand, $L(2)$, is more complicated. We will see that there are 12 $L(2)$ in $BZ_2^{\wr 3}$. 10 appear via transfer from maximal elementary abelian subgroups (and are therefore linked there), and are identified in in the subsection on abelian subgroups. For the remaining two, recall that in 3.5 we identified that the identity mapping for a complete copy of $H^*(BD_8)$ sitting inside $H^*(BZ_2^{\wr 3})$, with the inclusion map induced by the inclusion $\langle \tau_1, \tau_2 \rangle \hookrightarrow \langle \tau_1, \tau_2, \tau_3 \rangle$. As

$$BD_8 \cong BA_6 \vee 2L(2) \vee 2L(1)$$

This gives two nonzero entries in $A(Z_2^{\wr 3}, M_{L(2)})$. I wish to show that the columns containing these entries are independant of the columns corresponding to transfering down to elementary abelian groups and quotienting from there. First, I wish to express an idempotent which carries exactly one of these $L(2)$s. Let me label some particular subgroups.

$$\begin{aligned}
R &:= \langle (1,2),(3,4) \rangle \cong Z_2 \times Z_2, \\
Q &:= \langle (1,2),(3,4),(5,6),(5,7)(6,8) \rangle \cong Z_2 \times Z_2 \times D_8, \\
E_1 &:= \langle (1,2),(3,4),(5,6),(7,8) \rangle \cong Z_2^4, \\
\text{and } E_2 &:= \langle (1,2),(3,4),(5,8)(6,7),(5,7)(6,8) \rangle \cong Z_2^4.
\end{aligned}$$

The composition

$$L(2) \to BR \to BZ_2^{\wr 3} \xrightarrow{tr} BD_8 \times D_8 \xrightarrow{q} BD_8 \xrightarrow{tr} BR \to L(2)$$

which is a unit of $\{L(2), L(2)\}$, is equivalent to

$$L(2) \to BR \to BZ_2^{\wr 3} \xrightarrow{tr} BQ \xrightarrow{q} BR \to L(2)$$

by applying the Mackey formula. I will now argue that the column containing this unit is independant of any column corresponding to transferring to an elementary abelian subgroup and quotienting from there.

Observe that E_1 and E_2 are the only maximal elementary abelian subgroups of either Q or $Z_2^{\wr 3}$ which contain R. Also recall that the Weyl group for E_1 in $Z_2^{\wr 3}$ is $Z_2 \wr Z_2 \leq \mathfrak{S}_4$, and for E_2 is $Z_2 \times Z_2 \leq \mathfrak{S}_4$, and in both cases the action trivializes the specific modules corresponding to the idempotents that the $L(2)$s from BR are linked to in the BEs. More directly, one can observe that the corresponding cohomology classes in $H^4(BE)$ are mapped to zero by the transfer to $H^*(BZ_2^{\wr 3})$. For any other maximal elementary abelian subgroup, 2.1.4 forces the entries corresponding to including from R and transferring down to that subgroup to be zero. Therefore the entries in $A(Z_2^{\wr 3}, M_{L(2)})$ corresponding to inclusion from this subgroup and transfer to any maximal elementary abelian subgroup are zero.

The above argument also applies to all of the subgroups after applying the automorphism ζ, yielding the other $L(2)$ from the embedded BD_8. Therefore we can conclude that the rank of $A(Z_2^{\wr 3}, M_L(2))$ is at least $10 + 2 = 12$, counting the 10 $L(2)$ from elementary abelian subgroups. As there are only 12 available dimensions in $H^4(BZ_2^{\wr 3})$ after other summands are accounted for, this proves that $12L(2)$ are wedge summands of $BZ_2^{\wr 3}$.

Turning to $B\mathfrak{S}_8$, we we will see that none of the BA_4s and only one of the 10 $L(2)$s from the maximal elementary abelian subgroups survive. Looking at R above, we see that the Weyl action is the same, so the $L(2)$ examined there survives as well. However, with R^ζ, the Weyl group grows to $GL_2(\mathbb{F}_2)$, which annihilates the appropriate module. Therefore we can conclude that there are $2L(2)$ in $B\mathfrak{S}_8$.

5.3.4 $B(Z_2 \times Z_2 \times Z_2)$

Subgroups of this isomorphism type fall into two categories; those which are self-centralizing, in which case they contribute summands via $tr \circ incl$, and those which are not, and so only contribute summands linked in some other subgroup. As this section is concerned with subgroups of the latter type, those are what we shall address here; those of the first type are addressed subgroup by subgroup in the Abelian Subgroups section. The two types will be independent of each other by proposition 2.1.4.

To contribute summands, subgroups of this isomorphism type which are not self-centralizing must retract off of some other subgroup. A run of the script `ListSubretractionZ23ID` demonstrates that subgroups of this isomorphism type only retract off of subgroups isomorphic to Z_2^4 or $Z_2 \times Z_2 \times D_8$. However, we know that the retractions of Z_2 off of D_8 can be written using transfers down to subgroups isomorphic to $Z_2 \times Z_2$, so as transfers commute with pullbacks, we know the retractions of $Z_2 \times Z_2 \times Z_2$ off of $Z_2 \times Z_2 \times D_8$ can be written using transfers to subgroups isomorphic to $Z_2 \times Z_2 \times Z_2 \times Z_2$.

As there is strong linkage in elementary abelian groups[15], one can establish the number of summands coming from Z_2^3 through Z_2^4 by examining which of the original summands from Z_2^4 appear in $BZ_2^{\wr 3}$ and $B\mathfrak{S}_8$ via transfer. These summands must be independent of each other by proposition 2.1.4. Therefore the remaining summands from Z_2^3 can also be found in the Abelian Subgroups section, but under the subsections for groups isomorphic to Z_2^4.

5.3.5 $(BZ_4 \times Z_2)$

The classifying space of this group has one original summand, $BZ_4 \wedge Z_2$, which is 2-connected. As $H^2(BZ_2^{\wr 3})$ is exhausted, no copies of this summand appear in $BZ_2^{\wr 3}$ or $B\mathfrak{S}_8$.

5.3.6 $BZ_2^{\wr 2}$

The classifying space of this group has one original summand, BA_6. Considering maps $BZ_2^{\wr 3} \to BZ_2^{\wr 2}$, there are two retractions and one transfer-retraction. The columns containing each are independent of the others because (when combined with appropriate inclusions) they factor through different classes of $H^*(BZ_2^{\wr 3})$. The retractions have kernels the base Z_2^4 and its automorphic image $(Z_2^4)^\zeta$. The inclusion-quotients factor through and the subalgebras $\mathbb{F}_2[\overline{y}, z, \eta]/(\overline{y}z)$ and $\mathbb{F}_2[\overline{x}, z, \chi]/(\overline{x}z)$, respectively. As groups, the quotients are generated by the image of $\langle \tau_2, \tau_3 \rangle$ and $\langle \tau_1, \tau_3 \rangle$, respectively. If we let q' and q denote the two retractions, then we find $q' = q \circ \zeta$.

I'd like to examine one of these split retractions in more detail. Let $R = \langle \tau_1, \tau_3 \rangle$. Observe that $R \leq U_1$. Moreover, observe that if Q is another subgroup if index 2 in $BZ_2^{\wr 3}$, $R \not\leq Q$, so the identity will be the only double coset representative for R and

Q in $Z_2^{\wr 3}$. Therefore, applying the Mackey formula to

$$BR \hookrightarrow BZ_2^{\wr 3} \xrightarrow{tr} BQ \xrightarrow{Bq|_Q} BR$$

we find that this composition lies in J, and applying the Mackey formula to

$$BR \hookrightarrow BZ_2^{\wr 3} \xrightarrow{tr} BU_1 \xrightarrow{Bq|_Q} BR$$

yields the sum of two identity maps, which is trivial. Therefore the row corresponding to this inclusion has zero entries in any column corresponding to a transfer. Combining that with the observation from section 3.5 that the original summand of BD_8 corresponding to

$$BD_8 \to BZ_2^{\wr 3} \xrightarrow{tr} B(D_8 \times D_8) \to BD_8$$

is linked in $B(D_8 \times D_8)$ tells us that we do not have to transfer far to lose all three of these summands.

The bottom cells of the BA_6s, modulo classes from other summands, are equivalent to χ and η. The transfer-retraction is discussed in section 3.5, and produces a complete copy of $H^*(BD_8)$ contained in $\mathbb{F}_2[\overline{x}, \overline{y}, \overline{w}]$. Note that the transfer does not preserve the ring structure, which is why the product $\overline{xy}$ can be nonzero in $H^*(BZ_2^{\wr 3})$, whereas it would be zero in $H^*(BD_8)$. This gives the bottom cell (modulo other classes) equivalent to $\overline{w}$. This gives three independent copies of BA_6, which together with three dimensions from $L(1)$s and one BA_4 exhaust $H^2(BZ_2^{\wr 3})$, meaning that these account for all the copies of BA_6 which appear in $BZ_2^{\wr 3}$.

Turning to $B\mathfrak{S}_8$, we observe that dim $H^2(B\mathfrak{S}_8) = 2$, so there is room for precisely one summand with lowest degree 2, and as there are only two types of summands satisfying this condition and BA_4 does not appear, we must have exactly one BA_6 in $B\mathfrak{S}_8$.

5.3.7 $B(D_8 \times Z_2)$

Recall that this classifying space has only one type of original summand, which has bottom cell in degree 3. Examining $H^3(BZ_2^{\wr 3})$, we have 13 dimensions available. Three are accounted for by $L(1)$s, corresponding (modulo other summands) to $\overline{x}^3$, $\overline{y}^3$, and z^3. Six are accounted for by $3BA_6$ (corresponding to $\eta\overline{y}, \eta z, \chi\overline{x}, \chi z, \overline{wx}, \overline{wy}$) and two by the BA_4 (corresponding to $\overline{x}^2\overline{y}, \overline{xy}^2$). This leaves two remaining dimensions to fill (α and α'). I claim these are occupied by two original summands from $B(Z_2^{\wr 2} \times Z_2)$s, that the units for each correspond to a composition

$$u : X \to B(Z_2 \wr Z_2 \times Z_2) \longrightarrow BZ_2^{\wr 3} \xrightarrow{tr} B(Z_2 \wr Z_2 \times Z_2 \times Z_2) \longrightarrow B(Z_2 \wr Z_2 \times Z_2) \to X$$

and that these summands are in fact linked in distinct $B(Z_2 \wr Z_2 \times Z_2 \times Z_2)$, meaning that they generate independent columns.

Let $Q = \langle (1,2), (3,4), (5,6), (7,8), (1,3)(2,4) \rangle \cong Z_2^{\wr 2} \times Z_2 \times Z_2$. The Weyl Group consists of the identity and conjugation by $\tau_2^{\tau_3}$, which interchanges the two copies of Z_2. The outer automorphism group is an extension of $\mathfrak{S}_3$ by a two group.

Taking u as given, consider what survives the $tr \circ incl$ through $Z_2^{\wr 3}$. As this group is smash decomposable [25], we can evaluate the action of the $tr \circ incl$ by looking at its action on tensor products of the simple modules of the two factors. We are interested in $M_{L(1)} \otimes M_{BA_6}$, which is a two dimensional module because there are two $L(1)$ in $B\langle (5,6), (7,8) \rangle$. The Weyl sum acts on the cohomology of this group by mapping $a, b \mapsto (a+b)$, where a and b are classes corresponding to the generators (5,6) and (7,8), so $\overline{W} M_{L(1)} \otimes M_{BA_6}$ is one dimensional. Therefore one copy of the original summand of $BZ_2 \times D_8$ factors through the aforementioned map. As the entire argument applies to a different collection of subgroups after applying the automorphism ζ of $Z_2^{\wr 3}$, we have that $BZ_2^{\wr 3}$ contains at least two copies of this summand. As there are only two dimensions of $H^3(BZ_2^{\wr 3})$ available, there are exactly two.

These two copies are linked in $B(Z_2 \times Z_2 \times D_8)$ because $B(Z_2 \times D_8)$ has only one original summand which must pass through a retraction before it can pass through a transfer, and as mentioned above the only retractions are from subgroups isomorphic to $B(Z_2 \times Z_2 \times D_8)$.

Turning to $B\mathfrak{S}_8$, we have that on the copy discussed above, the Weyl group is unchanged, but on the other copy the Weyl group grows to $GL_2(\mathbb{F}_2)$, which annihilates the appropriate module. Therefore there is one copy of the original summand of $B(Z_2 \times D_8)$ in $B\mathfrak{S}_8$.

Returning our attention to $BZ_2^{\wr 3}$, we saw above that we get one $Orig(B(Z_2 \times D_8))$ from Q. There are two conjugacy classes of subgroups of this isomorphism type, with representatives Q and $Q^\zeta = \langle (1,2), (1,3)(2,4), (5,6)(7,8), (5,7)(6,8) \rangle$, each of which must have the same behavior as they are automorphic images of each other. Therefore there are at least two copies of this summand in $BZ_2^{\wr 3}$, and as observed previously there are only two free dimensions in $H^3(BZ_2^{\wr 3})$, so this accounts for all the summands of this type, and the remaining basis elements must correspond (modulo other summands) to summands of this type.

5.3.8 Cohomology and Summands of Low Connectivity

It may be beneficial to the reader to describe which classes correspond (modulo other summands) to which of the summands mentioned above. Recall that

$$H^*(BZ_2^{\wr 3}) \cong \mathbb{F}_2[\overline{x}_1, \overline{y}_1, z_1, \overline{w}_2, \chi_2, \eta_2, \alpha_3, \alpha_3', \omega_4]/(rels)$$

where subscript indicates degree.

Degree 1

We have three dimensions to fill. There are three copies of $L(1)$ which appear; this exhausts the available dimensions.

$\bar{x}$	$L(1)$
$\bar{y}$	$L(1)$
z	$L(1)$

Degree 2

We have seven dimensions to fill. 3 come from $L(1)$s. We demonstrated above that there is one BA_4 and $3BA_6$; this exhausts the available dimensions.

$\bar{x}^2, \bar{y}^2, z^2$	$3L(1)$
η	BA_6
χ	BA_6
$\bar{w}$	BA_6
$\bar{x}\bar{y}$	BA_4

Degree 3

We have thirteen dimensions to fill. 3 come from $L(1)$s, 8 from the BA_4 and three BA_6s, leaving 2 remaining. As argued above, these must correspond to original summands of $BZ_2 \wr Z_2 \times Z_2$.

$\bar{x}^3, \bar{y}^3, z^3$	$3L(1)$
$\eta\bar{y}, \eta z$	BA_6
$\chi\bar{x}, \chi z$	BA_6
$\bar{w}\bar{x}, \bar{w}\bar{y}$	BA_6
$\bar{x}^2\bar{y}, \bar{y}^2\bar{x}$	BA_4
α	$X_{D_8 \times Z_2}$
α'	$X_{D_8 \times Z_2}$

Degree 4

We have 22 dimensions to fill. There are three dimensions filled by $L(1)$s, 4 by BA_4s and BA_6s, and 6 by $BA_6 \wedge BZ_2$, leaving 11 unaccounted for. 8 come from $L(2)$s appearing as described above in the sections on $BZ_2 \times Z_2$ and $BZ_2 \wr Z_2$. Generators independent of the above list are $\chi\overline{w}$, $\eta\overline{w}$, and ω. ω can be demonstrated to not be in the image of any transfer or any quotient map, so it corresponds (modulo other summands) to the bottom class of the original summand of $BZ_2^{\wr 3}$.

$\overline{x}^4, \overline{y}^4, z^4$	$3L(1)$
$\overline{x}^3\overline{y}, \overline{y}^3\overline{x}, \overline{x}^2\chi, \overline{y}^2\eta, \overline{x}^2\overline{w}, \overline{y}^2\overline{w}, z^2\chi, z^2\eta$	$8L(2)$
$\overline{w}^2, \chi^2, \eta^2$	$3BA_6$
$\overline{x}^2\overline{y}^2$	BA_4
$\overline{x}(\alpha), \overline{y}(\alpha), \chi\overline{w}$	$2L(2) \vee X_{D_8 \times Z_2}$
$\overline{y}(\alpha'), \overline{x}(\alpha'), \eta\overline{w}$	$2L(2) \vee X_{D_8 \times Z_2}$
ω	$Orig(BZ_2^{\wr 3})$

5.4 The Final Splitting

All the summands which were identified were demonstrated to be independent, so to come up with our final splitting we can simply count them:

$$\begin{aligned} B(Z_2^{\wr 3}) = \quad & Orig(Z_2^{\wr 3}) \vee 32X_{4321} \vee 34X_{321} \vee 8X_{432} \vee 8X_{32} \vee 8X_{431} \vee 8X_{31} \\ & \vee 8X_{421} \vee 12X_{21} \vee X_{42} \vee X_2 \vee 3X_1 \vee 2St(BU) \vee 2St(B(Z_2 \times Z_2) \wr Z_2) \\ & \vee eT(\Delta_4) \vee 2St(BD_8 \times Z_2 \times Z_2) \vee 2Orig(BZ_2 \times D_8) \vee 3BA_6 \end{aligned}$$

We can do the same for the symmetric group:

$$\begin{aligned} B\mathfrak{S}_8 = \quad & Orig(Z_2^{\wr 3}) \vee 18X_{4321} \vee 18X_{321} \vee 4X_{432} \vee 4X_{32} \\ & \vee X_{431} \vee X_{31} \vee X_{421} \vee 2X_{21} \vee X_1 \\ & \vee St(BU) \vee Orig(BZ_2 \times D_8) \vee St(BD_8 \times Z_2 \times Z_2) \vee BA_6 \end{aligned}$$

Appendix A

GAP Code

To explore iterated wreath products and to do some of the calculations I created some specialized `GAP` code. The code is included for completeness.

```
################################################################################
# Coded by David Arnold, 6/5/2011
# This file contains the functions:
# IsSelfCentralizing(g,h)
# IsOutGroup2Group(g)
# ListRetractionsID(Group)
# ListSubretractionTypesID(Group)
# ListSubretractionZ23ID(Group)
# SubgroupsThatTransferSummands(TotalGroup)
# WeylRepresentatives(TotalGroup,Subgroup)


################################################################################
# function IsSelfCentralizing(TotalGroup,SubGroup)
#
```

```
# Code for a function which determines whether a subgroup of a particular
# group contains its centralizer.  This is relevant if we are attempting
# to determine the Weyl group action on the subgroup; if the group does not
# contain its centralizer, the Weyl action with trivialize the subgroup in
# the group ring.  Also, the Transfer will be trivial in cohomology.

IsSelfCentralizing:=function(g,h)
  if not IsSubgroup(g,h) then
    return fail;
  fi;

  return IsSubset(h,Centralizer(g,h));
end;


################################################################################
# function IsOutGroup2Group(AnyGroup)
#
# Code for a function which checks to see if the OuterAutomorphismGroup is a two
# group, in which case transfers down to this group carry no summands to any
# higher group because the Weyl action will be trivial.

IsOutGroup2Group:=function(g)
  local out, temp;
  if not IsGroup(g) then
    return fail;
  fi;

  out:=Size(AutomorphismGroup(g))*Size(Center(g))/Size(g);
```

```
  if not IsEvenInt(out) then
    return false;
  else
    return IsPrimePowerInt(out);
  fi;
end;


################################################################################
# function ListRetractionsID(Group)
#
# This should list all retractions of a group, in an efficient fashion.  It
# returns a pair of lists, the first is a list of conjugacy classes of subgroups
# which retract off, and the second is the conjugacy classes of the kernels. The
# approach is to get a list of normal subgroups, a corresponding list of factor
# groups, and then check which subgroups of the appropriate size are isomorphic
# to the factor and intersect trivially with the normal: this will give a
# retraction.  Output is given as a list of the retracted subgroups, and a list
# of the corresponding normal subgroups.  It depends heavily on the SmallGroups
# library, and so is limited by it, especially the computation time on
# IdSmallGroup, which is used to identify isomorphisms.
#
# Note that while many calculations are done with representatives of conjugacy
# classes, if two representatives are compared one is always normal, so there
# is no harm in only using the representatives.

ListRetractionsID:=function(g)
local ccllist, replist, normlist, repIDlist, factorlist, factIDlist, temp,
retlist, retnlist,i,j;
```

```
# First, set up the ConjugacyClass data
ccllist:=ConjugacyClassesSubgroups(g);
replist:=List(ccllist,Representative);
repIDlist:=List(replist,IdSmallGroup);
normlist:=[];
factorlist:=[];
factIDlist:=[];
retlist:=[];
retnlist:=[];

# next set up the factor data
for i in replist do
  if Size(ccllist[Position(replist,i)])=1 then  #i is normal
    Add(normlist,i);
    temp:=FactorGroupNC(g,i);
    Add(factorlist,temp);
    Add(factIDlist,IdSmallGroup(temp));
  fi;
od;

# Computations!

for j in replist do
  if Size(j)=Size(g) then
    continue;
  fi;

  for i in factorlist do

    if factIDlist[Position(factorlist,i)]=repIDlist[Position(replist,j)] then
```

```
          # They're isomorphic.  Check intersection properties.

      temp:=Intersection(normlist[Position(factorlist,i)],j);
      if Size(temp)=1 then

                       # It's a retraction!

        Add(retlist,ccllist[Position(replist,j)]);
        Add(retnlist,normlist[Position(factorlist,i)]);
        break;               # Only list subgroups once
      fi;
    fi;
  od;
od;


return [retlist,retnlist];
end;


################################################################################
# function ListSubretractionTypesID(Group)
#
# This function lists all isomorphism types of subretractions of a group by
# iterating ListRetractionsID.  It's not terribly efficient, but it gets the job
# done for smaller groups.

ListSubretractionTypesID:=function(g)
local ccllist, replist, idlist, temp, i;
```

```
ccllist:=ConjugacyClassesSubgroups(g);
replist:=List(Unique(List(ccllist,g->IdSmallGroup(Representative(g)))),SmallGroup);
idlist:=[];

for i in replist do
  temp:=List(ListRetractionsID(i)[1],g->IdSmallGroup(Representative(g)));
  Append(idlist,temp);
od;

return(Unique(idlist));
end;


################################################################################
# function ListSubretractionZ23ID(Group)
#
# This function lists all subretractions where the group retracted off is
# isomorphic to Z_2^3.

ListSubretractionZ23ID:=function(g)
local ccllist, retlist, homelist, temp, temp2, i, j;

ccllist:=ConjugacyClassesSubgroups(g);
retlist:=[]; homelist:=[];

for i in ccllist do
  temp:=[];
  temp2:=[];
```

```
  if Size(Representative(i))<32 then      #I'm only interested in retractions
    continue;                             #off groups other than Z_2^4
  fi;

  temp:=Filtered(ListRetractionsID(Representative(i))[1],
    h->[8,5]=IdSmallGroup(Representative(h)));

  if not temp=[] then
    for j in [1..Length(temp)] do
      temp2[j]:=i;
    od;
  fi;

  Append(retlist,temp);
  Append(homelist,temp2);

od;

return [retlist,homelist];
end;


################################################################################
# function SubgroupsThatTransferSummands(TotalGroup)
#
# Code for a function which determines which conjugacy classes of subgroups of
# TotalGroup could possibly contribute summands only via transfer; this does not
# garuntee that they DO contribute summands, only that all others do not.

SubgroupsThatTransferSummands:=function(g)
```

```
  local temp, temp2, CandidateList;
  CandidateList:=[];

  for temp in ConjugacyClassesSubgroups(g) do
    temp2:=Representative(temp);
    if (not IsOutGroup2Group(temp2)) and IsSelfCentralizing(g,temp2) then
      Add(CandidateList,temp);
    fi;
  od;

  return CandidateList;
end;


################################################################################
# function WeylRepresentatives(TotalGroup,Subgroup)
#
# Produces a list of elements of TotalGroup (actually the normalizer of Subgroup
# in TotalGroup) which generate the Weyl Group.

WeylRepresentatives:=function(grp,sub)
  local temp, WeylList;
  WeylList:=[];
  for temp in RightTransversal(Normalizer(grp,sub),sub) do
    Add(WeylList,ConjugatorIsomorphism(sub,temp));
  od;

  return WeylList;
end;
```

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