Imbedding Problems in Graph Theory

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IN
GRAPH THEORY

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INTRODUCTION

For some years there has been interest among mathematicians in determining the different ways in which certain graphs can be imbedded in given surfaces. M.P. Vanstraten [6], in 1948, determined that it is possible to imbed the graph $K_{3,3}$ (which is the graph representing the famous three houses, three utilities problem) in the torus in only two ways. She then used this fact to show that the graph representing the configuration of Desargues (containing $K_{3,3}$ as a subgraph) has genus two.

In 1937, I. N. Kagno [4] studied and analyzed $K_{3,3}$ and $K_5$ in the torus. In an article entitled "Configurations and Maps" [1] published in 1949, H.S.M. Coxeter studied some similar types of problems such as the configuration of Pappus, which has $K_{3,3,3}$ as its associated graph.

One major source of motivation for the work on imbedding problems has been their relation to coloring problems. For example the genus formula for the complete graphs $K_n$ was used by Ringel and Youngs in solving the Heawood Map Coloring Conjecture. Genus formulae as well as other imbedding results have been obtained in hopes of shedding some light on the Four Color Conjecture. This conjecture is by far the most famous coloring problem and has led to many results in graph theory.

Fairly recently, J. Sámonds [2] has developed a technique (algorithm) for determining all possible 2-cell imbeddings for a connected graph. Another very powerful technique for determining genus formulae was introduced by W. Gustin, developed by J.W.T. Youngs, and recently unified by A. Jacques. This technique is the method of quotient constellations, and reduced constellations.

I have had the opportunity (both through a Waldo-Sangren Scholar Award and Independent Study from the Honors College) to study both of these techniques in the last two years with Dr. A. T. White of the Mathematics Department. Dr. White has been interested in imbedding problems for some time and has used these tech-
niques in much of his research.

A second part of the Waldo-Sangren Scholar Award entailed assisting the Mathematics Department with the Graph Theory Conference held during the Spring Term of 1972. This experience also added to my understanding of graph theory in general and the Edmonds' technique in particular. The results of my studies will be presented in this paper.

DEFINITIONS

The purpose of this section is to present some definitions which are necessary to the understanding of imbedding problems. Also notation which will be used throughout the rest of this paper will be presented and explained.

A graph $G$ is a non-empty set of vertices, $V(G)$, and a set, $E(G)$, of unordered pairs of vertices. Each element of $E(G)$ is called an edge. This paper will deal only with graphs for which both $V(G)$ and $E(G)$ are finite.

The degree of a vertex $v$ is the number of edges to which $v$ belongs and is denoted by $\text{deg}(v)$.

A compact, orientable 2-manifold is a topological surface which is topologically equivalent to a sphere or a sphere with handles. Such manifolds will often be referred to as surfaces.

From what has been stated previously, it is seen that graphs can be considered to be sets of vertices with relations between certain vertices. Imbedding problems occur when one attempts to give a geometric realization of a graph. It is not hard to see that any finite graph can be realized in Euclidian three-space. However once the restriction that the imbedding be in a compact, orientable 2-manifold is made, the situation becomes much more interesting.

The graph $G$ with vertex set $V(G) = \{v_1, \ldots, v_m\}$ and edge set $E(G) = \{e_1, \ldots, e_n\}$ is imbedded in a 2-manifold $M$, if there exists a subspace $G(M)$ such that 

$$G(M) = \bigcup_{i=1}^{m} v_i(M) \bigcup_{j=1}^{n} e_j(M),$$

where
(i) $v_1(M),\ldots,v_m(M)$ are $m$ distinct points of $M$,
(ii) $e_1(M),\ldots,e_n(M)$ are $n$ mutually disjoint open arcs in $M$
(iii) $e_j(M) \cap v_i(M) = \emptyset$, for $i=1,\ldots,m$ and $j=1,\ldots,n$, and
(iv) if $e_j = (v_{j1},v_{j2})$, then the open arc $e_j(M)$ has $v_{j1}(M)$ and $v_{j2}$ as end points for $j=1,\ldots,n$.

In this definition, an arc in $M$ is a homeomorphic image of the closed unit interval. An open arc is an arc without its end-points.

A surface which is topologically equivalent to a sphere with $n$ handles is said to have genus $n$. It is now possible to define the concept of genus of a graph. If a graph $G$ can be imbedded in a surface of genus $n$, but cannot be imbedded in a surface of lower genus, the genus of the graph is defined to be $n$; $\gamma(G) = n$. If $\gamma(G) = n$, an imbedding of $G$ in a surface of genus $n$ is said to be minimal.

Let $G$ be a graph imbedded in a surface $M$. The points of $M$ remaining after the points of $M$ representing $G$ have been removed compose the complement of $G$ in $M$. The components of the complement of $G$ in $M$ are called regions or faces of $G$. A region which is homeomorphic to an open disk is called a 2-cell. If every region of an imbedding of a graph is a 2-cell, the imbedding is called a 2-cell imbedding.

There is an important formula which applies to 2-cell imbeddings. For a given imbedding of a graph in a surface $M$, let $F$ be the number of faces, $V$ be the number of elements in $V(G)$, $E$ be the number of elements in $E(G)$, and $\gamma$ be the genus of $M$. Then

\[(*) \quad F + V = E + 2(1 - \gamma).\]

This formula is a generalization of the Euler polyhedral formula.

Another important result is one which can be found in J. W. T. Youngs [7]. He has shown that any minimal imbedding of a graph is a 2-cell imbedding (and thus the Euler formula applies).

Given a graph $G$, imbedded in a surface $M$, a face distribution of $G$ is the finite sequence $F_3,F_4,F_5,\ldots$, where $F_i$ is the
number of faces having *i* edges in their boundary.

It is possible to rewrite (*) and obtain the following:

(i) \( 2\chi - 2 + V - E = -F = -\sum_{i=3} F_i \).

(ii) \( 2E = \sum_{i=3} iF_i \) (in this summation, each edge is counted exactly twice). By adding three times equation (i) to equation (ii), the following is obtained:

(**) \( 6\chi - 6 + 3V - E = \sum_{i=4} (i-3)F_i \).

Equation (***) is very useful in calculating possible face distributions for a graph. Once the number of vertices and edges is known, the number of faces (corresponding to different values of \( \chi \)) can easily be found. A face distribution for a graph which satisfies both (*) and (***) is said to be compatible. A face distribution which corresponds to a 2-cell imbedding which can be exhibited in a surface is said to be realized. Thus if a face distribution is realized, it is compatible. However, the converse is not necessarily true, as will be seen later.

A pseudograph is a variation of a graph in which multiple edges (two or more edges consisting of the same unordered pair of vertices) and loops (an edge in which each element of the unordered pair of vertices is the same vertex) are allowed.

Given a plane graph \( G \), its geometric dual \( G^* \) is constructed in the following manner: place a vertex in each region of \( G \) and, if two regions have an edge \( e \) in common, join the corresponding vertices by an edge \( e^* \) crossing only \( e \). The geometric dual is always a pseudograph (although it may possibly be a graph).

The least integer greater than or equal to \( x \) is written as \( \lceil x \rceil \).

A group \( \Gamma \) is a set of elements together with an operation \( \circ \), satisfying the following conditions:

(i) \((a \circ b) \) is in \( \Gamma \), for all \( a, b \) in \( \Gamma \),

(ii) There exists an element \( e \) in \( \Gamma \) such that \((e \circ a) = a\).
(aoe) = a for all a in \( \Gamma \),

(iii) For every a in \( \Gamma \) there exists \( a^{-1} \) in \( \Gamma \) such that

\[ (a \circ a^{-1}) = e, \]

and

(iv) \( (a \circ b) \circ c = a \circ (b \circ c) \) for all a, b, c in \( \Gamma \).

A subgroup of a group is a set of elements of the group that forms a group itself. If \( N \) is a subgroup of the group \( \Gamma \), the right coset of \( N \) in \( \Gamma \) is the set \( \{g \mid \omega \in N\} \). This is denoted by \( \Gamma g \).

For further terms often encountered in graph theory, see Graph Theory by Harary 3.

EDMONDS' PERMUTATION TECHNIQUE

There is an algebraic method of representing 2-cell imbeddings. This method is known as Edmonds' permutation technique.

Suppose a graph \( G \) has n vertices; \( V(G) = \{1, 2, \ldots, n\} \). Let \( V(i) = \{k \mid [i, k] \in E(G)\} \). Let \( p_i : V(i) \rightarrow V(i) \) be a cyclic permutation of \( V(i) \), \( i = 1, 2, \ldots, n \) (of length \( n_i = V(i) \)). The advantages of this representation become apparent when one sees the following theorem.

Theorem (Edmonds): each choice \( (p_1, p_2, \ldots, p_n) \) determines a 2-cell imbedding \( G(M) \) of \( G \) in an orientable 2-manifold \( M \), such that there is an orientation on \( M \) which induces a cyclic ordering of the edges \( (i, k) \) at \( i \) in which the immediate successor to \( (i, k) \) is \( (i, p_i(k)) \), \( i = 1, 2, \ldots, n \). In fact, given \( (p_1, p_2, \ldots, p_n) \) there is an algorithm which produces the determined imbedding. Conversely, given a 2-cell imbedding \( G(M) \) of \( G \) in an orientable 2-manifold \( M \) with a given orientation, there is a corresponding \( (p_1, p_2, \ldots, p_n) \) determining that imbedding. Furthermore, the face boundaries of the imbedding can be computed as follows.

Let \( D = \{(a, b) \mid [a, b] \in E(G)\} \). Define \( \Phi : D \rightarrow D \) by:

\[ \Phi((a, b)) = (b, p_b(a)) \]

Then the orbits under the permutation \( \Phi \) cor-
respond to the face boundaries. (An orbit is a set of the form \( \{(a,b), P((a,b)), P^2((a,b)), \ldots, P^s((a,b))\} \), where \( s \) is the minimum positive integer such that \( P^{s+1}((a,b)) = (a,b) \).

The algorithm which was mentioned is very useful in imbedding problems. An example will help to clarify what the theorem states and will demonstrate the algorithm. Consider the graph \( K_4 \) (the graph with four vertices and all possible edges). Then \( V(K_4) = \{1,2,3,4\} \), \( V(1) = \{2,3,4\} \), \( V(2) = \{1,3,4\} \), \( V(3) = \{1,2,4\} \), \( V(4) = \{1,2,3\} \). A particular choice for \( (p_1, p_2, p_3, p_4) \) is the following:

- \( p_1 : (2,3,4) \)
- \( p_2 : (1,4,3) \)
- \( p_3 : (1,4,2) \)
- \( p_4 : (3,1,2) \).

The theorem states that this choice will determine a 2-cell imbedding of \( K_4 \) which satisfies the properties mentioned. The procedure for finding the imbedding is as follows:

**face #1:**

- \( P(((1,4)) = (4, p_4(1)) = (4,2) \)
- \( P(((4,2)) = (2, p_2(4)) = (2,3) \)
- \( P(((2,3)) = (3, p_3(2)) = (3,1) \)
- \( P(((3,1)) = (1, p_1(3)) = (1,4) \)

This orbit describes a 4-sided face (i.e. \( s = 3 \)).

**face #2:**

- \( (1,2), (2,4), (4,3), (3,2), (2,1), (1,3), (3,4), (4,1), (1,2) \)

This describes an 8-sided face.

These two faces account for all the possible directed edges in \( K_4 \) (each edge being counted once in each of the two possible directions). Thus there are only two faces in this particular imbedding. Using the Euler formula:

\[ 2 + 4 = 6 + 2(1 - \chi) \]

it is seen that \( \chi = 1 \). This means it should be possible to find a 2-cell imbedding of \( K_4 \) in the torus with an 8-sided face and a 4-sided face. Figure I confirms this.

To show that not all compatible face distributions are realized, consider \( K_5 \) (the graph with five vertices and all possible edges). The following face distribution is compatible: \( F_3 = 3, F_5 = F_6 = 1 \).
Thus \( F = 5, \ E = 10, \) and \( 5 + 5 = 10 + 2(1 - ) \). Therefore \( = 1 \).

To imbed this in the torus, it is necessary to have a 5-sided face. There are two possibilities for such a face:

The second possibility will result in a vertex of degree one but every vertex of \( K_5 \) has degree four. Thus it must be discarded as a possibility. Once the first situation is chosen, there are two ways to get a 3-sided face:

Again one possibility is immediately eliminated. The first one results in a vertex of degree two. Obviously it cannot be used. The situation shown in Figure IV has now been obtained.

The blanks in \( p_3 \) and \( p_4 \) can be immediately filled in since there is only one choice for them. The result is: \( p_3 : (1, 4, 2, 5) \) and \( p_4 : (5, 3, 1, 2) \). This leaves the blanks in \( p_2 \) and \( p_5 \) to be filled in. Notice \((1, 5), (5, 4), (4, 3), (3, 2), (2, 1), (1, 5)\) gives a 5-sided face and \((3, 4), (4, 1), (1, 3), (3, 4)\) gives a 3-sided face. Still required are two 3-sided faces and one 6-sided face.
There are two choices for $p_2$ and two choices for $p_5$. This gives a total of four possibilities.

(i) $p_2: (3,1,5,4)$ and $p_5: (1,4,3,2)$. Then $(3,1),(1,2),(2,5),(5,1),(1,4),(4,2),(2,3),(3,5),(5,2),(2,4),(4,5),(5,3),(3,1)$ gives a 12-sided face — obviously a different imbedding.

(ii) $p_2: (3,1,5,4)$ and $p_5: (1,4,2,3)$. Then $(3,1),(1,2),(2,5),(5,3),(3,1)$ describes a 4-sided face. Again a different imbedding has been obtained.

(iii) $p_2: (3,1,4,5)$ and $p_5: (1,4,2,3)$. Here a 12-sided face is obtained by $(3,1),(1,2),(2,4),(4,5),(5,3),(3,5),(5,1),(1,4),(4,2),(2,5),(5,3),(3,1)$.

(iv) $p_2: (3,1,4,5)$ and $p_5: (1,4,3,2)$. Then $(3,1),(1,2),(2,4),(4,5),(5,3),(3,1)$ describes a 5-sided face. This is not the desired imbedding either. Thus an imbedding of the type $F_3 = 3,F_5 = F_6 = 1$ for $K_5$ cannot be realized.

The last few examples have shown that Edmonds' algorithm can be used to determine which compatible imbeddings for a graph are realized and which ones are not. Once this is known, the genus of the graph can be easily found.

There is another reason why this algorithm is so useful. It is possible to program a computer to perform the necessary calculations for determining which face distributions are realized, and with what frequency. The use of the computer makes this technique much more practical than it would otherwise be. Quite often the number of possible choices for the permutation $P$ is too large to be done practically by hand. For example, for $K_5$ there are 7776 choices for $P$ (six choices for $p_i$ for each of the five vertices). Even when, due to the symmetries inherent in this problem, only 1296 possible choices need be considered, the number of calculations to be performed is very large. The computer cuts down the calculation time drastically. A.T. White has catalogued the 2-cell imbeddings of $K_5$. The results are in Appendix I.

I have used the computer to aid in investigating the graph $K_{3,4}$ (pictured in Figure V). For this graph, there are six choices.

Figure V: $K_{3,4}$
for $p_1$ for the vertices of degree four and two choices for $p_1$ for the vertices of degree three. This gives a total of 3,456 possible choices for $P$. However, there are symmetries inherent in this problem which allow fewer possibilities to be considered.

Consider any 2-cell imbedding of $K_{3,4}$. Since the labeling of the vertices of the graph is arbitrary, choose vertex 1 to be a vertex of degree three and then choose $p_1$: (2,4,6). This determines the labeling of three more vertices. So far vertices 1, 2, 4, and 6 have been labeled. Now choose $p_2$: (1,3,5,7). This determines the labeling of the last three vertices. This fixing of $p_1$ and $p_2$ did not depend on which face distribution the imbedding was, so only one choice for $p_1$ and $p_2$ need be considered. This reduces the number of possible choices for $P$ to be considered by a factor of twelve. Hence only 288 possible choices need to be considered. The runtime for the program that calculated each of these possibilities was 16.47 seconds (see Appendix II for the actual program).

For $K_{3,4}$ $V = 7$ and $E = 12$; so using equation (***) the following is obtained:

$$6\gamma+3 = \sum_{1\leq r \leq 4} (i-3)F_i.$$  

It is known that any graph that contains $K_{3,3}$ as a subgraph cannot be imbedded in the sphere. Since $K_{3,4}$ contains $K_{3,3}$ as a subgraph, it is only necessary to consider three cases.

(i) $\gamma = 1$. Then $F = 12-7+0 = 5$ and $6(1)+3 = 9 = \sum_{1\leq r \leq 4} (i-3)F_i$. Thus $9 = F_4+2F_5+3F_6+\ldots+6F_{11}+9F_{12}$.

(ii) $\gamma = 2$. Then $F = 3$ and $6(2)+3 = 15 = F_4+2F_5+\ldots+15F_{18}$.

(iii) $\gamma = 3$. Then $F = 1$ and $6(3)+3 = 21 = F_4+\ldots+21F_{24}$.

From these equations and the fact that a bipartite graph has no faces with an odd number of edges, it is simple to determine that there are ten compatible face distributions. (For $\gamma \geq 4$, (***) determines that there must be a negative number of faces. This is obviously not possible.) The following are the ten compatible face distributions:

\[
\begin{align*}
\gamma = 1 & \quad F_8 = 1, \quad F_4 = 4 \\
& \quad F_6 = 2, \quad F_4 = 3 \\
\gamma = 2 & \quad F_{16} = 1, \quad F_4 = 2
\end{align*}
\]
It was determined (by means of the computer) that four of the ten compatible face distributions were not realized. The following table summarizes the results of my investigation of $K_{3,4}$.

<table>
<thead>
<tr>
<th>Genus of Surface</th>
<th>Face Distribution</th>
<th>No. of Occurences</th>
<th>% of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$F_8 = 1, F_4 = 4$</td>
<td>108</td>
<td>3.12</td>
</tr>
<tr>
<td>1</td>
<td>$F_6 = 2, F_4 = 3$</td>
<td>48</td>
<td>1.39</td>
</tr>
<tr>
<td>2</td>
<td>$F_6 = 1, F_4 = 2$</td>
<td>864</td>
<td>25.00</td>
</tr>
<tr>
<td>2</td>
<td>$F_6 = 1, F_6 = 1, F_4 = 1$</td>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>$F_8 = 1, F_6 = 2$</td>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>$F_8 = 1, F_6 = 1, F_6 = 1$</td>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>$F_8 = 3$</td>
<td>12</td>
<td>0.35</td>
</tr>
<tr>
<td>3</td>
<td>$F_{24} = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>2394</td>
<td>68.40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3456</td>
<td>100.00</td>
</tr>
</tbody>
</table>

The proportion of non-realized face distributions was much higher (40%) than had been expected. As of the moment, there is no efficient way of knowing which face distributions (if any) are not realized, until calculations such as the above are performed.

From the information obtained, it is noted that the genus of $K_{3,4}$ is one, since that is the smallest genus of any surface that $K_{3,4}$ can be 2-cell imbedded in. This agrees with the formula for the genus of a complete bipartite graph:

$$\gamma(K_{m,n}) = \left\lfloor \frac{(m-2)(n-2)}{4} \right\rfloor$$

giving

$$\gamma(K_{3,4}) = \left\lfloor \frac{(3-2)(4-2)}{4} \right\rfloor = 1.$$ Also, the maximum genus (the largest genus of any surface that a graph can be 2-cell imbedded in) is three.

It is interesting to note the relative number of occurences of each face distribution. If a permutation $P$ were chosen at random,
there would be a good (68\%) chance of obtaining a face distribution consisting of one 24-sided face. However, there would only be a 0.35\% chance of obtaining a $P_g = 3$ distribution. Also a minimal imbedding results only 4.51\% of the time.

**CONSTELLATIONS, QUOTIENT CONSTELLATIONS, AND REDUCED CONSTELLATIONS**

Although the Edmonds' method does apply to every graph, quite often the necessary calculations simply become so numerous that the method becomes impractical to use. There is another very useful technique, known as the method of quotient graphs and quotient manifolds. This technique was introduced by Gustin in 1963. Since then it has been developed by Youngs and used in the solution of the Heawood Map Coloring Conjecture. In his Ph.D. thesis (1969) A. Jacques \[4\] unified this method, using the terminology of constellations, quotient constellations, and reduced constellations.

A very important concept necessary to the understanding of this method is that of a Cayley color-graph. For a finite group $\Gamma$ and a generating set $\Delta$, for $\Gamma$, there is associated a graph known as the **Cayley color-graph**, denoted by $C_\Delta(\Gamma)$. The vertex set of $C_\Delta(\Gamma)$ is $\{\varepsilon \mid \varepsilon \in \Gamma\}$. $(g, g')$ is a directed edge of $C_\Delta(\Gamma)$ (labeled with generator $\delta_i$) if and only if $g' = g\delta_i$. It is assumed that unless an element $\delta_i$ in $\Gamma$ has order two, $\delta_i^{-1}$ is not in $\Delta$. If $\delta_i$ does have order two, then the two directed edges, $(g, g\delta_i)$ and $(g\delta_i, g)$ are represented as a single undirected edge $[g, g\delta_i]$, labeled $\delta_i$. By deleting all labels and directions from the edges of $C_\Delta(\Gamma)$, a graph is obtained which is called the **Cayley graph**, $G_\Delta(\Gamma)$. This graph has all edges of the form $[g, g\delta_i]$, for $g$ in $\Gamma$ and $\delta_i$ in $\Delta$. As an example of a Cayley color graph, consider the graph $K_4$ with the group $\mathbb{Z}_4$. ($\mathbb{Z}_n$ denotes the group of positive integers modulo $n$.) Its representation as a Cayley color graph is pictured in Figure VI.

- **Figure VI:** The Cayley color-graph $K_4$
Suppose there is a Cayley graph $G_\Delta(\Gamma)$ imbedded in a surface $M$. It is possible to study the imbedding of the Cayley color-graph $C_\Delta(\Gamma)$, which determined $G_\Delta(\Gamma)$. Since a Cayley color-graph is being considered, there is a set of generators, $\Delta$, associated with the graph. Let $\Delta^{-1} = \{ \delta^{-1} | \delta \in \Delta \}$. A new set $\Delta^*$ is now formed, $\Delta^* = \Delta^{-1} \cup \Delta$. Each element of $\Delta^*$ is called a current.

At every vertex $g$, let $\sigma_g = \rho_g$, the local vertex permutation discussed in Edmonds' technique. Since every vertex adjacent to $g$ is an element of the set $g\Delta^* = \{ g\delta | \delta \in \Delta^* \}$, $\sigma_g$ is a permutation which maps the set $g\Delta^*$ onto itself. As was noted before, this permutation was determined by the orientation.

Each permutation $\sigma_g$ induces another permutation $\sigma_g^*$ which maps the set $\Delta^*$ onto itself. The permutation $\sigma_g^*$ is induced by the action of $\sigma_g$ on the set $g\Delta$ (see Figure VII).

\begin{align*}
\sigma_0: & (1,2,3,4) \\
\sigma_1: & (0,3,4,2) \\
\sigma_0^*: & (1,2,3,4) \\
\sigma_1^*: & (4,2,3,1)
\end{align*}

Figure VII : Obtaining $\sigma_g^*$ from $\sigma_g$.

Note that the elements of $\sigma_0$ are elements of the set $0\Delta^* = \{1,2,3,4\}$ and the elements of $\sigma_1$ are the elements of $1\Delta^* = \{2,3,4,0\}$. The elements of both $\sigma_1^*$ and $\sigma_0^*$ are elements of $\Delta^*$. Let $\Lambda$ be a subgroup of $\Gamma$ such that if $\Lambda g = \Lambda g'$, then $\sigma_g^* = \sigma_{g'}^*$. There is always at least one such subgroup. Let $\Lambda = \{ \text{identity} \} \cap \Gamma \cap \Delta^*$, then $\Lambda$ is a subgroup of $\Gamma$ and $\Lambda g = \{ g \}$. So if $\Lambda g = \Lambda g'$, then $\{ g \} = \{ g' \}$. This implies that $g = g'$, so obviously $\sigma_g^* = \sigma_{g'}^*$. However as will be seen later, choosing $\Lambda$ as large as possible is a distinct advantage.

Given a Cayley color-graph, $C_\Delta(\Gamma)$, and a subgroup $\Lambda$ of $\Gamma$, the Schreier Coset graph, $C_{\Delta}(\Gamma/\Lambda)$, is given by: the vertex set is the set of all right cosets of $\Lambda$ in $\Gamma$, and $(\Lambda g, \Lambda g')$ is an edge labeled $\delta \in \Delta$ if and only if $(\Lambda^{-1}) \delta = \Lambda g'$. The imbedding determined by the collection $\{ \sigma_g^* \}$, for any collection $\{ g \}$ of right coset representatives, of the Schreier Coset graph is called
(using the terminology of Jacques) a quotient constellation, C', for the constellation \( C_\Delta(\Gamma) \) in \( \mathbb{M} \).

Note that in the quotient constellation, loops and multiple edges are allowed. Figure VIII shows a quotient constellation and the constellation it was obtained from.

\[
\begin{align*}
\text{Figure VIII} & : \\
\Gamma = 2^6 & \quad \Delta = \{2, 3\} \\
\Delta^* = \{2, 3, 4\} & \\
C_\Delta(\Gamma) : & \\
\sigma_0^* : (2, 4, 3) & \quad \sigma_2^* : (2, 4, 3) & \quad \sigma_4^* : (2, 4, 3) \\
\sigma_1^* : (2, 3, 4) & \quad \sigma_3^* : (2, 3, 4) & \quad \sigma_5^* : (2, 3, 4)
\end{align*}
\]

In this example the choice for \( \Lambda \) is \( \{0, 2, 4\} \). Note that this is a subgroup of \( \Gamma \). Since \( \Lambda = \{0, 2, 4\} \), \( \Lambda 0 = \Lambda 2 = \Lambda 4 \). Also \( \sigma_0^* = \sigma_2^* = \sigma_4^* = (2, 4, 3) \). Thus \( \Lambda \) is a subgroup of \( \Gamma \) with the desired properties.

In this example there are two loops in the quotient constellation. There are only two vertices because there are only two distinct right cosetes of \( \Lambda \) in \( \Gamma \). The direction of the loops is determined by the permutations \( \sigma_g^* \). At vertex \( \Lambda \), the direction was necessarily clockwise since this resulted in the local vertex permutation \( (2, 4, 3) \). This must be the permutation at this vertex since \( \sigma_g^* = (2, 4, 3) \) for each \( g \) in \( \Lambda \). Similarly, the direction of the loop at vertex \( \Lambda 1 \) was necessarily counterclockwise.

Consider now the dual of the quotient constellation, denoted by \( (C')^* \). This pseudograph is referred to as the reduced constellation or quotient graph in its quotient manifold. This is a
2-cell imbedding of a pseudograph which has each edge labeled with the current of its dual edge in a surface. The surface in which it is imbedded is called the quotient manifold.

Continuing the previous example, the reduced constellation associated with C' (Figure VIII) is shown below.

The direction of the edges was chosen so that the edge boundaries of $\mathcal{N}$ and $\mathcal{N}_1$ in $(C')^*$ were the same as the local vertex permutations of $\mathcal{N}$ and $\mathcal{N}_1$ (respectively) in C'.

A brin is an ordered pair $(g, g\delta^*)$ where $g$ is in $\Gamma$ and $\delta^*$ is in $\Delta^*$.

The reduced constellation satisfies the following five properties:

1) each brin carries a current from $\Delta^*$,
2) two opposing brins, $x = (g, g\delta^*)$ and $x^{-1} = (g\delta^*, g)$ carry inverse currents (if $x = x^{-1}$, the current must be of order two),

![Figure X: Two opposing brins.](image)

3) the regions are in one-to-one correspondence with the right cosets of $\mathcal{N}$ in $\Gamma$,
4) the currents appearing in a region boundary are in one-to-one correspondence with the elements of $\Delta^*$, and
5) if a brin $x$ appears in the boundary of a region associated with $\mathcal{N}_g$ and its opposite brin, $x^{-1}$, in the boundary of a region associated with $\mathcal{N}_g'$, then the current carried by $x$ is in the set $g^{-1}\mathcal{N}_g'$.

**Theorem:** Let $\mathcal{M}(\Gamma/\mathcal{N})$ be a pseudograph 2-cell imbedded in a sur-
face, with edges directed and labeled by elements from $\Delta$ (a generating set for $\Gamma$) and the regions labeled by the right cosets of $\Lambda$ in $\Gamma$ satisfying the five properties listed above; then there exists a 2-cell imbedding $C$ of $C_\Delta(\Gamma)$ such that $(C')^* = M(\Gamma/\Lambda)$.

Let $a$ be a vertex in a reduced constellation. Let $\tau$ be a product of the currents directed away from the vertex, in the order of orientation. The valence, $\mathcal{V}$, of $a$ is the order of $\tau$ in $\Gamma$. Even though the product $\tau$ is not necessarily unique, the order of every such product will be the same (if $\tau$ and $\tau'$ are two such products, the order of $\tau$ is the same as the order of $\tau'$). This now leads to the main theorem of this section.

**Theorem (Jacques):** A vertex of degree $k$ and valence $\mathcal{V}$ in a reduced constellation; $(C')^*$, determines $\frac{|\mathcal{V}|}{\mathcal{V}}$ faces of length $k\mathcal{V}$ in the imbedding of the Cayley color-graph $C_\Delta(\Gamma)$ in a surface.

These last two theorems are what is truly important for imbedding problems. They allow imbeddings to be studied in terms of simpler structures—reduced constellations (Jacques) or, equivalently, quotient graphs and quotient manifolds (Youngs). Some examples will demonstrate the strength of the theory.

Suppose it is desired to use this method to determine if the graph $K_{2,2,2,2}$ (see Figure XI) can be imbedded with every face a triangle. To apply this theory, an appropriate group must first be chosen. Since $K_{2,2,2,2}$ has eight vertices, the group must have eight elements. One of the most natural groups to consider is $\mathbb{Z}_8$.

![Figure XI: $K_{2,2,2,2}$](image-url)
Let \( \Gamma = \mathbb{Z}_8 \) and try to apply the theory with \( \mathcal{L} = \mathbb{Z}_8 \). Since each vertex is adjacent to six others, the set \( \Delta^* \) must contain six elements. Let \( \Delta = \{1, 2, 3\} \). Then \( \Delta^* = \{1, 2, 3, 5, 6, 7\} \). Note that 4 (an element of order 2) is the only non-identity element missing. Hence \( G_{\Delta}(\Gamma) \) will consist of every possible edge except four, no two of which have a common end point (i.e. \( G_{\Delta}(\Gamma) = K_{2,2,2,2} \)). Because each face (in \( K_{2,2,2,2} \)) must be triangular, \( k \mathcal{V} \) must equal three. Thus there are two cases to consider:

(i) \( k = 1, \mathcal{V} = 3 \). This case can be dismissed because there are no elements of order three in \( \mathbb{Z}_8 \).

(ii) \( k = 3, \mathcal{V} = 1 \). This says that every vertex in \((\mathcal{C}')^*\) has degree three. Because \( \mathcal{L} = \mathbb{Z}_8 \), there is only one face in \((\mathcal{C}')^*\). This face has length six.

By using the equation \( 2E = \sum_{i \in \mathcal{F}} iF_i \), it is seen that \( E = 3 \). Also (since \((\mathcal{C}')^*\) is cubic) \( 2E = 3V \). Thus \( V = 2 \). Now using the Euler formula, \( 1+2 = 3+2(1-\mathcal{O}) \). Hence \( \chi = 1 \).

What is known now is that a pseudograph with two vertices and three edges must be found and properly imbedded in the torus. If this can be done (with the proper labeling), it will have been shown that \( K_{2,2,2,2} \) does have an imbedding with every region a triangle. Figure XII shows such a pseudograph. This determines \( 1(\mathcal{O}) + 1(\mathcal{E}) = 16 \) faces of length three.

![Figure XII: \((\mathcal{C}')^*\) for \( K_{2,2,2,2} \)](image_url)

In this case, the method of quotient graphs and quotient manifolds was found to be applicable. It is obviously much easier to discuss and study a pseudograph with two vertices and three edges rather than a graph with eight vertices and twenty-four edges. This method is capable of making such simplifications reasonably often.
In my studies with Dr. White, I have applied this theory to several graphs.

The graph $K_{6,6}$ has twelve vertices and thirtysix edges. According to the Euler formula, it is possible to imbed it in a sphere with four handles, $S_4$, with every region being a 4-sided one. Let $\Gamma = \mathbb{Z}_{12}$, $\Delta = \{1, 3, 5\}$, and $\Lambda = \{0, 4, 8\}$. Since $\Delta = \{1, 3, 5\}$, $\Delta^* = \{1, 5, 7, 9, 11\}$. Note that $\Delta^*$ connects two vertices with an edge if and only if one vertex is an odd integer and the other an even integer. This results in two sets of six vertices each, with a vertex in one set being adjacent to all the vertices in the other set, but adjacent to none in its own set (i.e. $G_\Delta(\Gamma) = K_{6,6}$). Figure XIII shows that this imbedding does exist.

The following example illustrates an interesting fact about this method. Although in some cases this technique does not work, in others it not only works, but works in more than one way. The following three figures show three different reduced constellations, each of which determines an imbedding of $K_{3,3,3}$ in the torus with $F = F_3 = 18$. $K_{3,3,3}$ is shown in Figure XIV.

Let $\Gamma = \mathbb{Z}_9$, $\Lambda = \{0, 3, 6\}$ and $\Delta = \{1, 2, 4\}$. Note that 3 and 6 are the only non-identity elements not in $\Delta^*$. This means every edge is going to be present except the edges forming the 3-sided faces 0, 3, 6; 1, 4, 7; 2, 5, 8. Hence $G_\Delta(\Gamma) = K_{3,3,3}$. Figure XIV shows $K_{3,3,3}$ and the first reduced constellation that determines the desired
imbedding of $K_{3,3,3}$.

Figure XIV: A reduced constellation for $K_{3,3,3}$ in the torus.

Let $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathcal{A} = \Gamma$, and $\Delta = \{(1,0), (1,1), (1,2)\}$. The reduced constellation must have one 6-sided face, because $\mathcal{A} = \Gamma$ and $\Delta^*$ has six elements. Hence there are three edges in the reduced constellation. This implies that $V = 4 - 2\chi$. Figure XV shows the required reduced constellation for $\chi = 0$. It determines $3(\frac{\chi}{2}) = 9$ faces of length $l(3) = 3$ and $l(\frac{\chi}{1}) = 9$ faces of length $3(1) = 3$, $(F = F_3 = 18)$.

Figure XV: Another reduced constellation for $K_{3,3,3}$ in the torus.

The reduced constellation when $\chi = 1$ is pictured below. It is easy to verify that $G_\Delta(\Gamma) = K_{3,3,3}$ for $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Delta$ as chosen. The chosen $\Delta$ induces a $\Delta^*$ that has all elements of $\Gamma$ except two elements of order three. An argument similar to the one used for $\Gamma = \mathbb{Z}_3$ and the $\Delta$ in that case gives this verification.

Figure XVI: A third reduced constellation for $K_{3,3,3}$ in the torus.

It is also instructive to see why and when this method does not work, as may occur. For example, for the graph $K_{6,6}$ the method...
does not work when $\Gamma = \mathbb{Z}_{12}$ and $\Delta = \{1, 3, 5\}$. In this situation, the required reduced constellation must have one 6-sided face. Then $
abla = 3$ and $\nabla = 4 - 2\chi$. Since the imbedding of $K_{6, 6}$ being sought has all 4-sided faces, there are only three possibilities for the degree and valence of the vertices in the reduced constellation:

(i) $k = 4$ and $\gamma = 1$
(ii) $k = 2$ and $\gamma = 2$
(iii) $k = 1$ and $\gamma = 4$.

When $\chi = 0$, $V = 4$. Also the sum of all the degrees must be six. The only possibility for the four vertices is two of degree two and two of degree one. However, this situation does not satisfy another restriction. The edge adjacent to each vertex of degree one must be labeled with an element of $\Delta$ of order four. Two such elements are required, but there is only one. This case then gives no further information.

When $\chi = 1$, $V = 2$. One vertex of degree four and one of degree two satisfy some of the restrictions. Now it is required to find a proper labeling for these edges. The valence of the vertex of degree four must be one while the valence of the other must be two. The next figure shows how these vertices must be placed in order to give rise to a 6-sided face.

$$\Delta = \{\delta_1, \delta_2, \delta_3\}$$

![Figure XVII](image)

The valence of the vertex of degree four is the order of $\Gamma = \delta_1 \delta_2 \delta_1^{-1} \delta_3^{-1}$. Since the group is $\mathbb{Z}_{12}$, $\Gamma = \delta_1 \delta_2 \delta_2 \delta_3^{-1} = \delta_2 \delta_3^{-1}$. Because the order of $\Gamma$ is one, $\Gamma$ must be the identity of $\mathbb{Z}_{12}$, and $\delta_2 = \delta_3$. But this is clearly impossible, since the three elements of the generating set must be unique.

Neither of the two cases where $\Gamma = \lambda = \mathbb{Z}_{12}$ provided the answers which were required. As was shown earlier, another case does prove to be satisfactory. If every possible case were to prove unsatisfactory, it would not be possible to decide whether
or not the desired imbedding was possible (on the basis of this method alone).

SUMMARY

Two of the main techniques for dealing with imbedding problems have been presented. There still remain many unanswered questions in this field. It is still not known how to effectively predict when a compatible face distribution will be realized. Concerning the Edmonds' technique, for example, could a practical computer program be written that would work for any graph in general (of order less than or equal to \( n \), say) instead of only one particular graph? This seems very possible.

Relating to the reduced constellation technique, for what graphs do all minimal imbeddings have = identity? This question is rather important because in this case no advantage is gained by using this method.

These are only a few of many questions that are being asked about imbedding problems. The answers will provide some interesting information and also no doubt raise more questions that will further develop graph theory and mathematics.


5. I.N. Kagno, "Graphs on Surfaces", *Journal of Mathematical Physics*, 16, 46-75 (1937)


APPENDIX I

The following table catalogues the 2-cell imbeddings of $K_5$.

<table>
<thead>
<tr>
<th>Genus of Surface</th>
<th>Face Distribution</th>
<th>No. of Occurences</th>
<th>% of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$F_3 = 4$, $F_8 = 1$</td>
<td>150</td>
<td>1.9</td>
</tr>
<tr>
<td>1</td>
<td>$F_3 = 3$, $F_4 = F_7 = 1$</td>
<td>120</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>$F_3 = F_4 = 2$, $F_6 = 1$</td>
<td>120</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>$F_3 = F_5 = 2$, $F_4 = 1$</td>
<td>60</td>
<td>0.8</td>
</tr>
<tr>
<td>1</td>
<td>$F_4 = 5$</td>
<td>12</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>$F_3 = 3$, $F_5 = F_6 = 1$</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>$F_3 = F_5 = 1$, $F_4 = 3$</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 = 2$, $F_{14} = 1$</td>
<td>960</td>
<td>12.3</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 = F_4 = F_{13} = 1$</td>
<td>960</td>
<td>12.3</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 = F_8 = F_9 = 1$</td>
<td>720</td>
<td>9.3</td>
</tr>
<tr>
<td>2</td>
<td>$F_4 = F_6 = F_{10}$</td>
<td>420</td>
<td>5.4</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 = F_7 = F_{10} = 1$</td>
<td>360</td>
<td>4.6</td>
</tr>
<tr>
<td>2</td>
<td>$F_4 = F_7 = F_9 = 1$</td>
<td>360</td>
<td>4.6</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 = F_5 = F_{12} = 1$</td>
<td>240</td>
<td>3.1</td>
</tr>
<tr>
<td>2</td>
<td>$F_3 = F_6 = F_{11} = 1$</td>
<td>240</td>
<td>3.1</td>
</tr>
<tr>
<td>2</td>
<td>$F_4 = 2$, $F_{12} = 1$</td>
<td>240</td>
<td>3.1</td>
</tr>
<tr>
<td>2</td>
<td>$F_4 = F_5 = F_{11} = 1$</td>
<td>120</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>$F_5 = F_6 = F_9 = 1$</td>
<td>120</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>$F_5 = F_7 = F_8 = 1$</td>
<td>120</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>$F_4 = 1$, $F_8 = 2$</td>
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<td>0.8</td>
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<tr>
<td>2</td>
<td>$F_6 = 2$, $F_8 = 1$</td>
<td>30</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>$F_5 = 2$, $F_{10} = 1$</td>
<td>24</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>$F_6 = 1$, $F_7 = 2$</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>$F_{20} = 1$</td>
<td>2340</td>
<td>30.1</td>
</tr>
</tbody>
</table>

Total 7776 100.0
The following is the computer program which was used to investigate the graph $K_{3,4}$.

```
DIMENSION IMAGE (7,6,7), IVTX (26), LTH(5)
DIMENSION IPFD (10,11), INF (24), NFD (10), FREQ (10)
WRITE (30,100)
100 FORMAT (1H, 'ENTER DATA')
    DO 11 I = 1,7
    DO 11 J = 1,6
11   READ (5,2), (IMAGE(I,J,K), K = 1,7)
2    FORMAT(7l1)
    DO 19 I = 1,10
19   READ(5,20), (IPFD(I,J), J = 4,24,2)
20   FORMAT(lll1)
    DO 25 I = 1,10
25   NFD(I) = 0
    IN = 1
    DO 14 J3 = 1,2
    DO 14 J4 = 1,6
    DO 14 J5 = 1,2
    DO 14 J6 = 1,6
    DO 14 J7 = 1,2
17   K = 1
    L = 1
    DO 51 I = 4,24,2
51   INF(I) = 0
    LTH = 0
    DO 18 I = 1,5
18   LTH(I) = 0
    DO 5 M = 1,6
      MM = M+1
      DO 5 N = MM,7,2
      IVTX(K) = M
      IVTX(K+1) = N
      IF(N.EQ.2) GO TO 10
      DO 3 J = 2,K-1
3    GO TO 11
```

IF(N-IVTX(J)) 3,11,3
11 IF(M.EQ.IVTX(J-1)) GO TO 5
3 CONTINUE
10 KK = K+2
   DO 141 I = KK,26
      I1 = IVTX(I-1)
      I2 = IVTX(I-2)
      IF(I1.EQ.1) J11 = 1
      IF(I1.EQ.2) J11 = 1
      IF(I1.EQ.3) J11 = J3
      IF(I1.EQ.4) J11 = J4
      IF(I1.EQ.5) J11 = J5
      IF(I1.EQ.6) J11 = J6
      IF(I1.EQ.7) J11 = J7
      IVTX(I) = IMAGE(I1,J11,I2)
      IP(IVTX(I)-N) 141,8,141
8 IF(IVTX(I-1).EQ.M) GO TO 9
141 CONTINUE
9 LTH(L) = I-K-1
   LNTH = LNTH+LTH(L)
   IS = LTH(L)
   INF(IS) = INF(IS) +1
   IF(LNTH-24) 15,6,16
15 L = L+1
   K = I-1
5 CONTINUE
6 WRITE(30,12) IN, (LTH(III),III= 1,5), LNTH
12 FORMAT(1H 'IMB. NO', 15,12X,5I3,8X,13)
   DO 21 I = 1,10
      DO 38 J = 4,24,2
         IF(INF(J)-IPFD(I,J)) 21,23,21
23 IF(J.EQ.24) GO TO 24
38 CONTINUE
21 CONTINUE
24 NFD(I) = NFD(I)+1
IN = IN+1
14 CONTINUE
   DO 70 I = 1,10
      FREQ(I) = NFD(I)/288.
      WRITE(30,26) I,NFD(I)
   26 FORMAT(1H, 'NFD(',I2,') = ',I4)
      WRITE(30,70) I,FREQ(I)
   70 FORMAT(1H, 'FREQ(',I2,') = ',F10.6)
      ISUM1 = 0
      DO 27 I = 1,2
   27 ISUM1 = ISUM1+NFD(I)
      ISUM2 = 0
      DO 28 I = 3,9
   28 ISUM2 = ISUM2+NFD(I)
      ISUM3 = NFD(10)
      ISUM = ISUM1+ISUM2+ISUM3
      WRITE(30,29)ISUM1, ISUM2
   29 FORMAT(1H, 'ISUM1 = '15,3X,'ISUM2 = '15)
      WRITE(30,500) ISUM3,ISUM
   500 FORMAT(1H,'ISUM3 = ' 15,3X 'ISUM = '15)
      FRQ1 = ISUM1/288.
      FRQ2 = ISUM2/288.
      FRQ3 = ISUM3/288.
      WRITE(30,30) FRQ1
   30 FORMAT(1H,'FRQ1 = ' F10.6)
      WRITE(30,400) FRQ2
   400 FORMAT(1H, 'FRQ2 = ' F10.6)
      WRITE(30,200) FRQ3
   200 FORMAT(1H,'FRQ3 = ' F10.6)
      IF(ISUM<288) 31,32,31
   31 WRITE(30,34)
            34 FORMAT(1H, 'GOSH DARN!!')
            GO TO 16
   32 WRITE (30,36)
   36 FORMAT(1H, 'RIGHT ON!!')
16 CALL EXIT
END