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On Eulerian Irregularity and Decompositions in Graphs

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ON EULERIAN IRREGULARITY AND DECOMPOSITIONS IN GRAPHS

by

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An Eulerian walk in a connected graph G is a closed walk that contains every edge of G at least once, while an irregular Eulerian walk in G is an Eulerian walk that encounters no two edges of G the same number of times. The minimum length of an irregular Eulerian walk in G is called the Eulerian irregularity of G and is denoted by $EI(G)$.

For a nontrivial connected graph G of size m, it is shown that $\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}$ and that $EI(G) = 2\binom{m+1}{2}$ if and only if G is a tree of size m. A necessary and sufficient condition has been established for all pairs k, m of positive integers for which there is a nontrivial connected graph G of size m with $EI(G) = k$. A formula for the Eulerian irregularity of a graph in terms of the size of certain even subgraph of the graph has been established. Furthermore, Eulerian irregularities are determined for all graphs of cycle rank 2 and all complete bipartite graphs as well as all prisms, grids and powers of cycles. Some general results on Eulerian irregularities of circulants are also presented.

For a set S of graphs and a graph G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called an S-maximal k-decomposition if $H_i \cong H$ for some $H \in S$ for each integer i with $1 \leq i \leq k$ and R contains no subgraph isomorphic to any subgraph in S. Let $\text{Min}(G, S)$ and $\text{Max}(G, S)$ be the minimum and maximum k, respectively, for which G has an S-maximal k-decomposition. A set S of graphs without isolated vertices is said to possess the intermediate decomposition property if for every connected graph G and each integer k with $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. All graphs of size 3 or less are determined that possess the intermediate decomposition property. Furthermore, the sets of graphs having size 3 that possess the intermediate decomposition property are determined as well as some sets of graphs having having size more than 3.

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Eric Andrews

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Chapter 1

Introduction

1.1 The Königsberg Bridge Problem

The city of Königsberg was the capital of German East Prussia in the 13th century and home of the Prussian Royal Castle. The River Pregel flowed through the city separating it into four land regions. Seven bridges were built over the river. Figure 1.1 displays a map of Königsberg, showing these four land regions (labeled A, B, C and D), the location of the river and the seven bridges (labeled a, b, c, d, e, f and g). The story goes that during the 1730s, some of its citizens enjoyed strolling about the city and wondered whether it was possible to go for a walk and pass over each bridge exactly once. This problem eventually became known as the *Königsberg bridge problem.*

Leonhard Euler, the great Swiss mathematician of the 18th century, became aware of this problem but initially did not think that this problem was particularly mathematical in nature. He discovered a method for solving not only the Königsberg bridge problem but a generalization of the problem. In a famous 1736

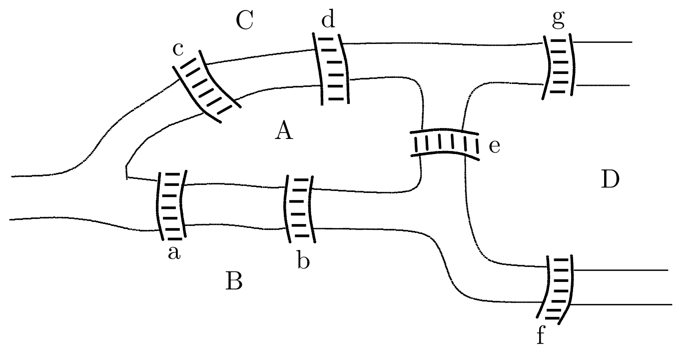


Figure 1.1: A map of Königsberg

paper by Euler [15], he described the problem, a generalization of the problem and the solutions of these problems. While Euler's reasoning in his paper was graph theoretic in nature, the term "graph" never appeared in the paper. Indeed, the term "graph," as used in this context, was not introduced until 1878, when the British mathematician James Joseph Sylvester first used this word.

In terms of graphs, Euler's paper contained the following result:

> *Suppose that there is a city consisting of land regions, some pairs of which are joined by one or more bridges. This town can then be represented by a graph or multigraph G whose vertices are the land regions where every two vertices of G are joined by a number of edges equal to the number of bridges joining the corresponding land regions. There is a round-trip in the town passing over each bridge exactly once if and only if G is connected and every vertex of G has even degree.*

Such a graph or multigraph is referred to as Eulerian. Since each vertex in the multigraph representing the map of Königsberg has odd degree (as shown in

Figure 1.2), it follows by Euler's result that it was not possible to walk about Königsberg and pass over each bridge exactly once. Euler's paper not only solved the Königsberg bridge problem, it marked the beginning of graph theory.

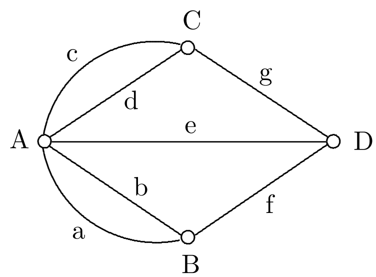


Figure 1.2: A multigraph representing Königsberg

1.2 The Chinese Postman Problem

If every edge of a nontrivial connected graph G of size m is replaced by two parallel edges, then the resulting multigraph is Eulerian, which implies that G contains a closed walk in which every edge of G appears exactly twice. Euler made this observation in his paper. Of course, G contains a closed walk in which every edge of G appears exactly once if and only if G itself is Eulerian. A weighted graph H can be obtained from G by assigning a positive integer weight to each edge of G. The degree of a vertex v in H is the sum of the weights of the edges incident with v. From the observation above, every vertex of H is even if every edge of G is assigned the weight 2. If every edge of G is assigned the weight 1, then every vertex of H is even if and only if G is Eulerian. Thus it is always possible to assign each edge of G a weight 1 or 2 in such a way that every vertex in the resulting weighted graph is even. A problem of interest is that of determining the minimum sum of all positive integer weights assigned to the edges of G so that every vertex in the resulting weighted graph is even. This is equivalent to determining the minimum length of a closed walk in G that contains every edge of G at least once. A solution to this problem also provides a solution to the so-called Chinese Postman Problem, named by Alan Goldman for the Chinese mathematician Meigu Guan (often known as Mei-Ko Kwan) who introduced this problem in 1962. Suppose that a postman starts from the post office and has mail to deliver to the houses along each street on his mail route. Once he has completed delivering the mail, he returns to the post office. The problem is to find the minimum length of a round trip that accomplishes this, as we state next.

The Chinese Postman Problem *Determine the minimum length of a round trip that traverses every road in a mail route at least once.*

The minimum length of a closed walk that contains every edge of a connected graph G of size m at least once is m if G is Eulerian. If G is not Eulerian, then G contains $2k$ odd vertices for some positive integer k. Suppose that the $2k$ odd vertices are divided into k pairs and the distance between the vertices in each pair is determined and these k numbers are summed. If d is the minimum value of all such sums over all partitions of these $2k$ odd vertices into k pairs, then the minimum length of such a closed walk is $m+d$. Suppose that $\{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}\}$ is a partition of the $2k$ odd vertices such that $\sum_{i=1}^{k} d(u_i, v_i) = d$ and P_i is a $u_i - v_i$ geodesic for $i = 1, 2, \ldots, k$. Then the paths P_i are pairwise edge-disjoint. If we replace each edge that belongs to one of these paths by two parallel edges, then we obtain an Eulerian multigraph M of size $m+d$. An Eulerian circuit in M gives rise then to a closed walk of minimum length in G that contains every edge of G at least once – a solution to the Chinese Postman Problem.

1.3 Decompositions in Graphs

One of the major topics in graph theory concerns graph decompositions. A problem of primary interest in this case has been to determine for graphs G and H whether it is possible to decompose G into subgraphs, each isomorphic to H, that is, whether G is H-decomposable. A classic historical problem in this context is the determination of those integers $n \geq 3$ for which the complete graph K_n is K_3-decomposable. This is equivalent to the problem of determining those integers $n \geq 3$ for which there is a Steiner triple system S_n, a problem initiated and solved in 1847 by the famous combinatorialist Thomas Kirkman [17], who showed that this occurred if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. Another familiar result of this type is that K_n can be decomposed (actually factored in this case) into Hamiltonian cycles if and only if n is odd, a result attributed to Walecki [3]. Another result of this type, dealing with paths P_3 of order 3, appeared in [12] and is stated below.

Theorem 1.3.1 *A nontrivial connected graph G is P_3-decomposable if and only if G has even size.*

Not all decomposition problems have dealt with decomposing a graph into subgraphs, each isomorphic to the same graph. The following theorem, due to Bryant, Horsley and Pettersson [7], verified a conjecture on cycle decompositions made by Alspach [2] in 1981.

Theorem 1.3.2 *Suppose that* $n \geq 3$ *is an odd integer and that* $m_1, m_2, \ldots, m_t$ *are integers such that* $3 \leq m_i \leq n$ *for each* i $(1 \leq i \leq t)$ *and* $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$*. Then* K_n *can be decomposed into the cycles* $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$*. Furthermore, for every even integer* $m \geq 4$ *and integers* $m_1, m_2, \ldots, m_t$ *such that* $3 \leq m_i \leq n$ *for each* i $(1 \leq i \leq t)$ *with* $m_1 + m_2 + \cdots + m_t = (n^2 - 2n)/2$*, there is a decomposition of* K_n *into a* 1*-factor and the cycles* $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$*.*

The famous topologist Oswald Veblen [21] proved that every Eulerian graph can be decomposed into cycles. A conjecture involving cycle decompositions of Eulerian graphs was introduced in [10].

The Eulerian Cycle Decomposition Conjecture (ECDC) *Let* G *be an Eulerian graph of size* m*, where* a *is the minimum number of odd cycles in a cycle decomposition of* G *and* b *is the maximum number of odd cycles in a cycle decomposition of* G*. For every integer* ℓ *such that* $a \leq \ell \leq b$ *and* ℓ *and* m *are of the same parity, there exists a cycle decomposition of* G *containing exactly* ℓ *odd cycles.*

It is therefore a consequence of the theorem by Bryant, Horsley and Pettersson that the ECDC is true for all complete graphs of odd order. This conjecture was verified for several classes of graphs in [10] but remains open in general.

Another decomposition problem, introduced in [1], involves subgraphs, no two of which are isomorphic.

The Ascending Subgraph Decomposition Conjecture *Let G be a graph of size m, where $\binom{k+1}{2} \leq m < \binom{k+2}{2}$ for some positive integer k. Then G can be decomposed into k subgraphs $G_1, G_2, \ldots, G_k$ where G_i has size m_i $(1 \leq i \leq k)$, $m_i < m_{i+1}$ for $i = 1, 2, \ldots, k-1$ and G_{i+1} contains a subgraph isomorphic to G_i.*

1.4 Basic Definitions and Well-Known Results

In this section, we formally present basic definitions and notation involved our research. We refer to [11] for graph theory notation and terminology not described in this paper. All graphs under consideration are nontrivial connected graphs.

For vertices u and v in a graph G, a $u - v$ *walk* W in G is a sequence

$$W = (u = v_0, v_1, v_2, \ldots, v_k = v) \tag{1.1}$$

of vertices in G such that $v_{i-1}v_i$ is an edge of G for each i $(1 \leq i \leq k)$. If $e_i = v_{i-1}v_i$, then the walk W in (1.1) can also be denoted by

$$W = (e_1, e_2, \ldots, e_k). \tag{1.2}$$

The length of the walk W is denoted by $L(W)$ and so $L(W) = k$ for the walk W in (1.1) and (1.2).

If G is a multigraph rather than a graph, then some pairs of vertices are joined by more than one edge. In this case, it is necessary to denote a walk as a sequence of edges as in (1.2) rather than a sequence of vertices as in (1.1) to avoid confusion. If $u = v$, then the $u - v$ walk is *closed*; while if $u \neq v$, then the $u - v$ walk is *open*. If there is no repetition of edges in a walk, then the walk is a *trail*. A closed nontrivial trail is a *circuit*. A $u-v$ walk W as in (1.1) is a $u-v$ *path* if the vertices $v_0, v_1, \ldots, v_k$ are distinct. If W is a circuit for which the vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct, then W is a *cycle*.

A circuit in a graph G that contains every edge of G is an *Eulerian circuit*, while an open trail containing every edge of G is an *Eulerian trail*. A graph containing an Eulerian circuit is an *Eulerian graph* and a graph containing an Eulerian trail is a *traversable graph*. In 1736, Leonhard Euler established the following characterization of Eulerian graphs [15].

Theorem 1.4.1 (**Euler's Theorem**) A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

In 1912, Oswald Veblen [21] also obtained an interesting characterization of Eulerian graphs in terms of graph decompositions. A *decomposition* $\mathcal{D}$ of a graph G is a collection $\{H_1, H_2, \ldots, H_t\}$ of nonempty subgraphs such that $H_i = G[E_i]$ for some (nonempty) subset E_i of $E(G)$, where $\{E_1, E_2, \ldots, E_t\}$ is a partition of $E(G)$. Thus no subgraph H_i in a decomposition of G contains isolated vertices. If $\mathcal{D}$ is a decomposition of G, then we say G is decomposed into the subgraphs $H_1, H_2, \ldots, H_t$. If $\mathcal{D} = \{H_1, H_2, \ldots, H_t\}$ is a decomposition of a graph G such that $H_i \cong H$ for some graph H for each i $(1 \leq i \leq t)$, then $\mathcal{D}$ is an *H-decomposition* of G or an *isomorphic decomposition* of G. If there exists an H-decomposition of a graph G, then G is said to be *H-decomposable*. If each H_i in a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_t\}$ is a cycle, then $\mathcal{D}$ is called a *cycle decomposition* or a *cyclic* decomposition. Similarly, if each H_i in $\mathcal{D}$ is a path, then $\mathcal{D}$ is called a *path decomposition*.

Theorem 1.4.2 (**Veblen's Theorem**) A nontrivial connected graph G is Eulerian if and only if G has a cycle decomposition.

An important corollary of Theorem 1.4.1 is the following characterization of traversable graphs.

Corollary 1.4.3 *A nontrivial connected graph G is traversable if and only if G contains exactly two vertices of odd degree. Any Eulerian trail in G then begins at one of these vertices and terminates at the other.*

If a graph G has four or more odd vertices, then G contains neither an Eulerian circuit nor an Eulerian trail, which again explains why there was no journey about Königsberg that crossed each bridge exactly once. Even though these graphs do not contain Eulerian circuits or Eulerian trails, there are some interesting properties that these graphs possess. We saw in the discussion related to finding a solution to the Chinese Postman Problem in Section 1.2 that if a connected graph G contains $2k$ odd vertices, then these $2k$ odd vertices can be partitioned into k pairs resulting in k pairwise edge-disjoint paths in G, each connecting pairs of odd vertices. In fact, it is well-known that G itself can be decomposed to k open trails connecting odd vertices and more can be said.

Theorem 1.4.4 *If G is a connected graph containing $2k \geq 4$ odd vertices, then G can be decomposed into k open trails connecting odd vertices but no fewer.*

Chapter 2

Irregular Eulerian Walks

While the Chinese Postman Problem asks for the minimum length of a closed walk in a connected graph G such that every edge of G appears on the walk once or twice, another problem of interest is that of determining the minimum length of a closed walk in G in which no two edges of G appear the same number of times. Such walks in a graph G distinguish the edges of G by their occurrences on the walk, which gives rise to the concept of irregular Eulerian walks in graphs.

2.1 Eulerian Walks in Graphs

Let G be a nontrivial connected graph of size m. By an *Eulerian walk* in G, we mean a closed walk that contains every edge of G. Thus the length of an Eulerian walk W in G is m if and only if W is an Eulerian circuit. In general then, the minimum length of an Eulerian walk in G is $m + d$ for some nonnegative integer d. We saw that if every edge of G is replaced by two parallel edges, then the resulting multigraph M is Eulerian and each Eulerian circuit in M gives rise to an

Eulerian walk in G that encounters every edge of G exactly twice. Hence if G is not Eulerian, then the minimum length of an Eulerian walk in G is more than m but not more than $2m$ and every edge appears once or twice in such an Eulerian walk in G.

Let H be a weighted graph obtained by assigning weights (positive integers) to the edges of a connected graph G. Then the *degree* $\deg_H v$ of a vertex v in H is the sum of the weights of the edges incident with v. Determining the minimum length of an Eulerian walk in G is then equivalent to determining an assignment of the weights 1 or 2 to the edges of G such that the sum of these weights is minimum and the degree of every vertex in H is even. The subgraph induced by the edges labeled 2 is the union of edge-disjoint paths in G. As we mentioned before, this problem is directly related to a well-known problem called the Chinese Postman Problem, which is the problem of determining the minimum length of a round trip that traverses every road in a mail route at least once.

2.2 Eulerian Irregularity

For every nontrivial connected graph G of size m, there is always an Eulerian walk in which each edge of G is encountered the same number of times. An *irregular Eulerian walk* in G is an Eulerian walk that encounters no two edges of G the same number of times. Thus the length of an irregular Eulerian walk in G is at least $1+2+\cdots+m = \binom{m+1}{2}$. If $E(G) = \{e_1, e_2, \ldots, e_m\}$ and each edge e_i $(1 \leq i \leq m)$ of G is replaced by $2i$ parallel edges, then the resulting multigraph M is Eulerian and each Eulerian circuit in M gives rise to an irregular Eulerian walk in which each edge e_i of G appears exactly $2i$ times in the walk. Thus G contains an irregular Eulerian walk of length $2+4+6+\cdots+2m = 2\binom{m+1}{2} = m^2+m$. The length of a walk W is denoted by $L(W)$. If W is an irregular Eulerian walk of minimum length in a connected graph G of size m, then $\binom{m+1}{2} \le L(W) \le 2\binom{m+1}{2}$. A problem of interest here is that of determining the minimum length of an irregular Eulerian walk in G, which we refer to as the *Eulerian irregularity* of G, which is denoted by $EI(G)$. Therefore, if G is a connected graph of size m, then

$$\binom{m+1}{2} \le EI(G) \le 2\binom{m+1}{2}. \tag{2.1}$$

Both bounds in (2.1) are sharp. First, we show that the lower bound in (2.1) is sharp. For an odd integer $n \ge 5$, let $G = C_n^2$ be the square of the n-cycle C_n. That is, if $C_n = (v_1, v_2, \ldots, v_n, v_1)$, then

$$\begin{aligned} E(G) &= \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \cup \\ &\quad \{v_1v_3, v_3v_5, \ldots, v_{n-2}v_n, v_nv_2, v_2v_4, \ldots, v_{n-3}v_{n-1}, v_{n-1}v_1\}. \end{aligned}$$

Thus, G is a 4-regular graph of size $m = 2n$ and

$$C = (v_1, v_2, v_3, \ldots, v_n, v_1, v_3, v_5, \ldots, v_n, v_2, v_4, \ldots, v_{n-3}, v_{n-1}, v_1)$$

is an Eulerian circuit of G. Suppose that C encounters the edges e_1, e_2, $\ldots$, e_m in this order and each edge e_i $(1 \le i \le m)$ is replaced by i parallel edges. Then the resulting multigraph M is Eulerian and so each Eulerian circuit in M gives rise to an irregular Eulerian walk in which each edge e_i of G appears exactly i times in the walk. Thus $EI(G) = \binom{m+1}{2}$ and so the lower bound in (2.1) is sharp. To see that the upper bound in (2.1) is sharp, we first state a theorem due to Mei-Ko Kwan [18].

Kwan's Theorem *Let G be a connected graph and let W be a closed walk of minimum length containing every edge of G at least once. Then W encounters no edge of G more than twice and no more than half of the edges in any cycle appear twice.*

Theorem 2.2.1 *For a connected graph G of size $m \geq 1$,*

$$EI(G) = 2\binom{m+1}{2} \text{ if and only if } G \text{ is a tree.}$$

Proof. Assume first that G contains a bridge uv. Let W be an Eulerian walk of G with initial vertex u. Then the first time that v is encountered on W, it is preceded by u and the next time that u is encountered on W, it is preceded by v. Therefore, uv occurs an even number of times on W. If G is a tree, then every edge of G is a bridge and so each edge of G is encountered an even number of times on W. Therefore, $EI(G) \geq 2\binom{m+1}{2}$. It then follows by (2.1) that $EI(G) = 2\binom{m+1}{2}$.

Suppose next that G is not a tree. Then G contains at least one cycle. By Kwan's theorem, there is an Eulerian walk W in which no edge of G occurs on W more than twice and some edges occur on W exactly once. Let $e_1, e_2, \ldots, e_k$ ($k \geq 1$) be those edges occurring exactly once on W and let $f_1, f_2, \ldots, f_\ell$ be those edges occurring exactly twice on W. By assigning each edge e_i ($1 \leq i \leq k$) the weight $2i - 1$ and each edge f_j $(1 \leq j \leq \ell)$ the weight $2j$ if $\ell \geq 1$, we obtain a weighted graph in which every vertex is even. Thus there is an Eulerian walk in G where e_i appears $2i - 1$ times and f_j appears $2j$ times. Since there is an irregular Eulerian walk of length less than $2\binom{m+1}{2}$, it follows that $EI(G) < 2\binom{m+1}{2}$. ■

If we were to consider all sets S of m positive integers and label the edges of a connected graph G with distinct elements of S so that every vertex is even in the resulting weighted graph, then the minimum of the sums of the elements of all such sets S is $EI(G)$.

2.3 Optimal Irregular Eulerian Walks

An irregular Eulerian walk W in a connected graph G of size m is said to be *optimal* if $L(W) = \binom{m+1}{2}$. In this case, the edges of G can be ordered as $e_1, e_2, \ldots, e_m$ such that e_i $(1 \leq i \leq m)$ is encountered exactly i times in W. As we have seen, there are graphs that possess an optimal irregular Eulerian walk. A graph G is *optimal* if it contains an optimal irregular Eulerian walk. First, we present a characterization of such connected graphs.

Theorem 2.3.1 *Let G be a connected graph of size m. Then G contains an optimal irregular Eulerian walk if and only if G contains a subgraph of size $\lceil m/2 \rceil$, every vertex of which is even.*

Proof. First, assume that G contains a subgraph F of size $\lceil m/2 \rceil$ such that every vertex of F is even. Then

$$E(G) = \{e_1, e_2, \ldots, e_{\lceil m/2 \rceil}\} \cup \{e'_1, e'_2, \ldots, e'_{\lfloor m/2 \rfloor}\},$$

where $E(F) = \{e_1, e_2, \ldots, e_{\lceil m/2 \rceil}\}$. We construct an Eulerian multigraph M by replacing each edge e_i where $1 \le i \le \lceil m/2 \rceil$ by $2i - 1$ parallel edges and replacing each edge e'_j, where $1 \le j \le \lfloor m/2 \rfloor$, by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W in G such that each edge e_i of G appears exactly $2i - 1$ times in W, where $1 \le i \le \lceil m/2 \rceil$, and each edge e'_j of G appears exactly $2j$ times in W where $1 \le j \le \lfloor m/2 \rfloor$. Then the length of W is $1 + 2 + 3 + \cdots + m = \binom{m+1}{2}$ and W is an optimal irregular Eulerian walk in G.

For the converse, suppose that G contains an optimal irregular Eulerian walk W. We may assume that $E(G) = \{f_1, f_2, \ldots, f_m\}$, where f_i appears exactly i times $(1 \le i \le m)$ on W. Let F be the subgraph of G of size of $\lceil m/2 \rceil$ induced by the set $\{f_1, f_3, \ldots, f_{2\lfloor (m-1)/2 \rfloor + 1}\}$ and let F' be the subgraph of G induced by the set $\{f_2, f_4, \ldots, f_{2\lceil (m-1)/2 \rceil}\}$. Thus $\{F, F'\}$ is a decomposition of G. We claim that every vertex of F is even. Let M be the weighted graph obtained by assigning the weight i $(1 \le i \le m)$ to each edge f_i of G. Let H be the weighted subgraph of M induced by the edges of F and let H' be the weighted subgraph of M induced by the edges of F'. Since G has an Eulerian walk in which each edge f_i appears exactly i times, every vertex of M has even degree. Since $\deg_M v = \deg_H v + \deg_{H'} v$ for every vertex v of G and $\deg_M v$ and $\deg_{H'}$ are both even, it follows that $\deg_H v$ is even. Suppose that $\deg_F v = k$. Then v is incident with k edges in H, each of odd weight. Since $\deg_H v$ is even, k is even and so v is an even vertex in F. ■

By Theorem 2.3.1, the graphs G_1 and G_3 of Figure 2.1 contain optimal irregular Eulerian walks while G_2 and G_4 do not. Since the Petersen graph P has size 15 and P contains an 8-cycle, it follows by Theorem 2.3.1 that P contains an optimal irregular Eulerian walk. On the other hand, by Theorem 2.3.1, no cycle contains an optimal irregular Eulerian walk. In fact, $EI(C_m) = 1+3+5+\cdots+(2m-1) = m^2$ for each $m \geq 3$. We have seen that if $n \geq 5$ is odd, then C_n^2 contains an optimal irregular Eulerian walk. Since the size of C_n^2 is $2n$ and C_n is a 2-regular graph of size n in C_n^2, it follows by Theorem 2.3.1 that C_n^2 contains an optimal irregular Eulerian walk for each integer $n \geq 4$.

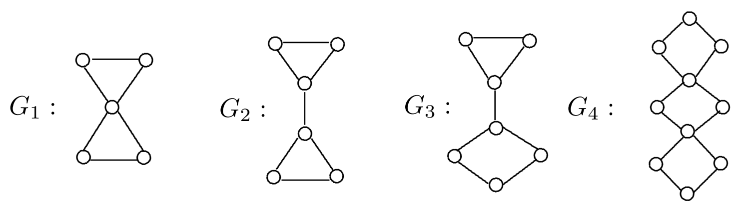


Figure 2.1: Illustrating Theorem 2.3.1

By Theorem 2.3.1, if G is a connected graph of size m, then $EI(G) = \binom{m+1}{2}$ if and only if G contains a subgraph of size $\lceil m/2 \rceil$, every vertex of which is even. The following is also a consequence of Theorem 2.3.1.

Corollary 2.3.2 *If G is a connected bipartite graph of size $m \geq 1$ such that $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$, then G does not contain an optimal irregular Eulerian walk.*

Proof. If $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$, then $\lceil m/2 \rceil$ is odd. If G contains an optimal irregular Eulerian walk, then by Theorem 2.3.1, G contains a subgraph

H of size $\lceil m/2 \rceil$, each of whose vertices is even. Therefore, H is a bipartite graph of odd size, each vertex of which is even. This is impossible. ■

Next, we determine all those complete graphs and complete bipartite graphs containing an optimal irregular Eulerian walk. In order to this, we first present two well-known results about complete graphs (see [11, p. 424-426]).

Theorem 2.3.3 *Let $n \geq 3$ be an integer.*

(1) *If n is odd, then K_n is Hamiltonian-factorable.*

(2) *If n is even, then K_n can be factored into $\frac{n}{2} - 1$ Hamiltonian cycles and a 1-factor.*

Theorem 2.3.4 *For each integer $n \geq 2$, the complete graph K_n contains an optimal irregular Eulerian walk if and only if $n \geq 4$.*

Proof. Since neither K_2 nor K_3 has an optimal irregular Eulerian walk, it remains to show that K_n contains an optimal irregular Eulerian walk for $n \geq 4$. By Theorem 2.3.1, it suffices to show that K_n contains a subgraph of size $\lceil m/2 \rceil$ where $m = \binom{n}{2}$ such that each vertex of this subgraph is even. We consider the cases when $n \equiv r \pmod 4$ for $r = 0, 1, 2, 3$.

Case 1. $n \equiv 0 \pmod 4$. Then $n = 4k$ for some positive integer k. The size m of K_{4k} is $\binom{4k}{2} = 2k(4k-1) = 8k^2 - 2k$ and so $m/2 = 4k^2 - k$. Let $\{H_1, H_2, \ldots, H_{2k-1}\}$ be a Hamiltonian-factorization of K_{4k-1}. Then the subgraph

H of K_{4k-1} with $E(H) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$ is a $2k$-regular subgraph of size $k(4k-1) = 4k^2 - k = m/2$ in K_{4k}.

Case 2. $n \equiv 1 \pmod 4$. Then $n = 4k+1$ for some positive integer k. The size m of K_{4k+1} is $\binom{4k+1}{2} = 2k(4k+1) = 8k^2 + 2k$ and so $m/2 = 4k^2 + k$. Let $\{H_1, H_2, \ldots, H_{2k}\}$ be a Hamiltonian-factorization of K_{4k+1}. Then the subgraph H of K_{4k+1} with $E(H) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$ is a $2k$-regular subgraph of size $k(4k+1) = m/2$ in K_{4k+1}.

Case 3. $n \equiv 2 \pmod 4$. Then $n = 4k+2$ for some positive integer k. The size m of K_{4k+2} is $\binom{4k+2}{2} = (2k+1)(4k+1) = 8k^2+6k+1$ and so $\lceil m/2 \rceil = 4k^2+3k+1$. For $k = 1$, the graph $K_{2,4}$ is a subgraph of size 8 in K_6, each vertex of which is even. Thus, we may assume that $k \geq 2$. Let $\{U, V\}$ be a partition of the vertex set of K_{4k+2}, where $U = \{u_1, u_2, \ldots, u_{2k+1}\}$ and $V = \{v_1, v_2, \ldots, v_{2k+1}\}$, and let H be the complete subgraph of order $2k+1$ with vertex set U and let F be the complete subgraph of order $2k+1$ with vertex set V. Suppose first that $k+1$ is even. Then $k+1 = 2p$ for some integer $p \geq 2$. Then the subgraph consisting of H, F and the $(k+1)$-cycle $(u_1, v_1, u_2, v_2, \ldots, u_p, v_p, u_1)$ has size $2k(2k+1)+(k+1) = 4k^2 + 3k + 1 = m/2$, each vertex of which is even. Suppose next that $k+1 \geq 3$ is odd. Then $k+4$ is even and so $k+4 = 2q$ for some integer $q \geq 3$. Let $H' = H - \{u_1u_2, u_2u_3, u_3u_1\}$. Then the subgraph consisting of H', F and the $(k+4)$-cycle $(u_1, v_1, u_2, v_2, \ldots, u_q, v_q, u_1)$ has size $2k(2k+1) - 3 + (k+4) = 4k^2+3k+1 = m/2$, each vertex of which is even.

Case 4. $n \equiv 3 \pmod 4$. Then $n = 4k+3$ for some positive integer k. The size m of K_{4k+3} is $\binom{4k+3}{2} = (2k+1)(4k+3) = 8k^2+10k+3$ and so $\lceil m/2 \rceil = 4k^2+5k+2$. For $k = 1$, the graph $K_{1,1,5}$ is a subgraph of size 11 in K_7, each vertex of which is even. Thus, we may assume that $k \geq 2$. Let $\{U, V, W\}$ be a partition of the vertex set of K_{4k+3}, where $U = \{u_1, u_2, \ldots, u_{2k+1}\}$, $V = \{v_1, v_2, \ldots, v_{2k+1}\}$ and $|W| = 1$. Let H be the complete subgraph of order $2k+1$ with vertex set U and let F be the complete subgraph of order $2k+1$ with vertex set V. If $3k+2$ is even, so $3k+2 = 2p$ for some integer $p \geq 3$, then the subgraph consisting of H, F and the $(3k+2)$-cycle $(u_1, v_1, u_2, v_2, \ldots, u_p, v_p, u_1)$ has size $2k(2k+1) + (3k+2) = 4k^2 + 5k + 2 = m/2$, each vertex of which is even. If $3k+2$ is odd, then $k \geq 3$ and $3k+5$ is even. Thus $3k+5 = 2q$ for some integer $q \geq 4$. Let $H' = H - \{u_1u_2, u_2u_3, u_3u_1\}$. Then the subgraph consisting of H', F and the $(3k+5)$-cycle $(u_1, v_1, u_2, v_2, \ldots, u_q, v_q, u_1)$ has size $2k(2k+1) - 3 + (3k+5) = 4k^2 + 5k + 2 = m/2$, each vertex of which is even. In each case, K_n contains a subgraph of size $\lceil m/2 \rceil$, each vertex of which is even. By Theorem 2.3.1, K_n contains an optimal irregular Eulerian walk. ■

Theorem 2.3.5 *For integers r and s with $2 \leq r \leq s$, the complete bipartite graph $K_{r,s}$ contains an optimal irregular Eulerian walk if and only if*

(*i*) *r and s are both even and $(r, s) \neq (2, 4k+2)$ for any nonnegative integer k or*

(*ii*) *at least one of r and s is odd and $rs \not\equiv 1, 2 \pmod 4$.*

Proof. Let $G = K_{r,s}$ whose partite sets are

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } W = \{w_1, w_2, \ldots, w_s\}.$$

The size of G is $m = rs$. First, assume that r and s are both even and $(r, s) = (2, 4k + 2)$ for some nonnegative integer k. We claim that $K_{2,4k+2}$ does not have an optimal irregular Eulerian walk; for otherwise, by Theorem 2.3.1, the graph $K_{2,4k+2}$ contains a subgraph H of size $4k + 2 = 2(2k + 1)$, each vertex of which is even. Suppose that the partite sets of H are U' and W' where then $U' \subseteq U$ and $W' \subseteq W$. Since $\deg_G w = 2$ for each $w \in W$ and each vertex of H is even, it follows that $\deg_H w = 2$ for each $w \in W'$ and so $U' = U$. Furthermore, the size of H is $2|W'| = 2(2k + 1)$ and so $|W'| = 2k + 1$. However then, $H = K_{2,2k+1}$ with partite sets U and W' and $\deg_H u = 2k + 1$ for each $u \in U$, which is a contradiction. Therefore, as claimed, $K_{2,4k+2}$ does not have an optimal irregular Eulerian walk. Next, assume that at least one of r and s is odd and $rs \equiv 1, 2 \pmod 4$. It then follows by Corollary 2.3.2 that G does not have an optimal irregular Eulerian walk.

For the converse, consider three cases, according to the parity of r and s.

Case 1. *r and s are both even and $(r, s) \neq (2, 4k + 2)$ for any nonnegative integer k.* Thus $4 \leq r \leq s$. First, suppose that at least one of r and s is congruent to 0 modulo 4. If $r \equiv 0 \pmod 4$, then let $H = K_{r/2,s}$; while if $s \equiv 0 \pmod 4$, than let $H = K_{r,s/2}$. In each case, H is a subgraph of size $\lceil m/2 \rceil = m/2 = rs/2$ in G, each vertex of which is even. Next, suppose that neither r nor s is congruent to 0 modulo 4. Thus each of r and s is congruent to 2 modulo 4 and so $6 \leq r \leq s$. Then $r = 4a + 2$ and $s = 4b + 2$ for some positive integers a and b where $a \leq b$. So $m/2 = 2(2a + 1)(2b + 1)$. Now let $F = (u_1, w_1, u_2, w_2, \ldots, u_{2a+1}, w_{2a+1}, u_1)$

be a cycle of order $2(2a+1)$ in G and let $F' = K_{4a+2,2b}$ be the subgraph in G induced by U and W' where $W' \subseteq W - \{w_1, w_2, \dots, w_{2a+1}\}$ with $|W'| = 2b$. Now consider the subgraph H consisting of F and F', where the partite sets of H are U and $W' \cup \{w_1, w_2, \dots, w_{2a+1}\}$ and $E(H) = E(F) \cup E(F')$. Then the size of H is $(4a+2)(2b) + 2(2a+1) = 2(2a+1)(2b+1)$ and each vertex of H is even.

Case 2. *r and s are both odd and $rs \not\equiv 1, 2 \pmod 4$.* If r and s are both congruent to 1 modulo 4 or r and s are both congruent to 3 modulo 4, then $m = rs \equiv 1 \pmod 4$. Thus exactly one of r and s is congruent to 1 modulo 4 and the other is congruent to 3 modulo 4. There are two subcases.

Subcase 2.1. *$r \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 4$.* Then $r = 4a+1$ and $s = 4b+3$ for some positive integers a and b and so $\lceil m/2 \rceil = 8ab + 6a + 2b + 2$. Let $b = a + k$, where $k \geq 0$. For each i with $0 \leq i \leq k$, define the $k+1$ graphs $G_0, G_1, G_2, \dots, G_k$ recursively such that $G_i = K_{4a+1,4(a+i)+3}$ for $0 \leq i \leq k$ and $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$. Let $G_0 = K_{4a+1,4a+3}$ with partite sets $U_0 = \{u_1, u_2, \dots, u_{4a+1}\}$ and $W_0 = \{w_1, w_2, \dots, w_{4a+3}\}$. For $i \geq 1$, let $G_i = K_{4a+1,4(a+i)+3}$ with partite sets $U_i = U_0$ and $W_i = W_{i-1} \cup \{a_i, b_i, c_i, d_i\}$. Denote the size of G_i by m_i for $0 \leq i \leq k$. Then

$$\lceil m_i/2 \rceil = 8a(a+i) + 6a + 2(a+i) + 2 = (8a^2 + 8a + 2) + 2(4a)i + 2i.$$

We now define the $k+1$ graphs $H_0, H_1, H_2, \dots, H_k$ recursively such that H_i ($0 \leq i \leq k$) is a subgraph of G_i, the size of H_i is $m_{H_i} = \lceil m_i/2 \rceil$, each vertex of H_i is even and $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_k$.

- Let H_0 be the graph obtained from $K_{4a,2a+2}$ with partite sets

$$U' = \{u_1, u_2, \ldots, u_{4a}\} \text{ and } W' = \{w_1, w_2, \ldots, w_{2a+2}\} \tag{2.2}$$

by (i) deleting the edge $u_{4a}w_{2a+2}$ and (ii) adding the three edges $u_{4a}w_{2a+3}$, $w_{2a+3}u_{4a+1}$, $u_{4a+1}w_{2a+2}$. The size of H_0 is $m_{H_0} = 8a^2+8a+2$ and each vertex of H_0 is even.

- We now construct $H_0 \subseteq H_1$ as follows. Let F_1 be the subgraph of G_1 constructed from the graph $K_{4a,2}$ with partite sets U' as described in (2.2) and $\{a_1, b_1\}$ by (i) deleting the edge $u_{4a}a_1$ and (ii) adding the three edges $u_{4a}c_1, c_1u_{4a+1}, u_{4a+1}a_1$. Then the graph H_1 consists of F_1 and H_0, that is, $V(H_1) = V(F_1) \cup V(H_0)$ and $E(H_1) = E(F_1) \cup E(H_0)$. The size of H_1 is $m_{H_1} = m_{H_0} + 2(4a) + 2 = (8a^2 + 8a + 2) + 2(4a) + 2$ and each vertex of H_1 is even.

- Suppose that H_i has been constructed for some integer i with $0 \leq i < k$ and H_i has the desired properties. We now construct $H_i \subseteq H_{i+1}$ as follows. Let F_{i+1} be the subgraph of G_{i+1} constructed from the graph $K_{4a,2}$ with partite sets U', as described in (2.2), and $\{a_{i+1}, b_{i+1}\}$ by (i) deleting the edge $u_{4a}a_{i+1}$ and (ii) adding the three edges $u_{4a}c_{i+1}, c_{i+1}u_{4a+1}, u_{4a+1}a_{i+1}$. Then the graph H_{i+1} consists of F_{i+1} and H_i, that is, that is, $V(H_{i+1}) = V(F_{i+1}) \cup V(H_i)$

and $E(H_{i+1}) = E(F_{i+1}) \cup E(H_i)$. The size of H_{i+1} is

$$\begin{aligned} m_{H_{i+1}} &= m_{H_i} + 2(4a) + 2 \\ &= [(8a^2 + 8a + 2) + (2i)(4a) + 2i] + 2(4a) + 2 \\ &= (8a^2 + 8a + 2) + 2(i+1)(4a) + 2(i+1). \end{aligned}$$

and each vertex of H_{i+1} is even.

By Theorem 2.3.1, G contains an optimal irregular Eulerian walk in this case.

Subcase 2.2. $r \equiv 3 \pmod 4$ *and* $s \equiv 1 \pmod 4$. Then $r = 4a + 3$ and $s = 4b + 1$ for some positive integers a and b and so $\lceil m/2 \rceil = 8ab + 2a + 6b + 2$. In this case, $a < b$. Let $b = a + k$, where $k \geq 1$. For each i with $0 \leq i \leq k$, define the k graphs $G_1, G_2, \ldots, G_k$ recursively such that $G_i = K_{4a+3,4(a+i)+1}$ for $1 \leq i \leq k$ and $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k$. Let $G_1 = K_{4a+3,4(a+1)+1}$ with partite sets $U_1 = \{u_1, u_2, \ldots, u_{4a+3}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{4(a+1)+1}\}$. For $i \geq 2$, let $G_i = K_{4a+3,4(a+i)+1}$ with partite sets $U_i = U_1$ and $W_i = W_{i-1} \cup \{a_i, b_i, c_i, d_i\}$. Denote the size of G_i by m_i for $1 \leq i \leq k$. Then

$$\begin{aligned} \lceil m_i/2 \rceil &= 8a(a+i) + 2a + 6(a+i) + 2 \\ &= (8a^2 + 16a + 8) + 2(4a+2)(i-1) + 2(i-1). \end{aligned}$$

We define the k graphs $H_1, H_2, \ldots, H_k$ recursively such that H_i $(1 \leq i \leq k)$ is a subgraph of G_i, the size of H_i is $\lceil m_i/2 \rceil$, each vertex of H_i is even and $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_k$.

- Let H_1 be the graph consisting of $F = K_{2a+4,4a}$ with partite sets

$$\{u_1, u_2, \ldots, u_{2a+4}\} \text{ and } \{w_1, w_2, \ldots, w_{4a}\}$$

and $F' = K_{2,4}$ with partite sets

$$\{u_{2a+5}, u_{2a+6}\} \text{ and } \{w_{4a+1},\ w_{4a+2},\ w_{4a+3},\ w_{4a+4}\};$$

that is, $V(H_1) = V(F) \cup V(F')$ and $E(H_1) = E(F) \cup E(F')$. Then the size of H_1 is $m_{H_1} = (2a+4)(4a) + 8 = 8a^2 + 16a + 8$ and each vertex of H_1 is even.

- We now construct H_2 such that $H_1 \subseteq H_2$ as follows. Let F_2 be the subgraph of G_2 constructed from the graph $K_{4a+2,2}$ with partite sets $\{u_1, u_2, \ldots, u_{4a+2}\}$ and $\{a_2, b_2\}$ by (i) deleting the edge $u_{4a+2}a_2$ and (ii) adding the three edges $u_{4a+2}c_2, c_2u_{4a+3}, u_{4a+3}a_2$. Then the graph H_2 consists of F_2 and H_1, that is, $V(H_2) = V(F_2) \cup V(H_1)$ and $E(H_2) = E(F_2) \cup E(H_1)$. The size of H_2 is $m_{H_2} = m_{H_1} + 2(4a+2) + 2 = (8a^2 + 16a + 8) + 2(4a+2) + 2$ and each vertex of H_2 is even.

- Suppose that H_i has been constructed as desired for some integer i with $1 \leq i < k$ and H_i has the desired properties. We now construct H_{i+1} such that $H_i \subseteq H_{i+1}$ as follows. Let F_{i+1} be the subgraph of G_{i+1} constructed from the graph $K_{4a+2,2}$ with partite sets $\{u_1, u_2, \ldots, u_{4a+2}\}$ and $\{a_{i+1}, b_{i+1}\}$ by (i) deleting the edge $u_{4a+2}a_{i+1}$ and (ii) adding the three edges $u_{4a+2}c_{i+1}, c_{i+1}u_{4a+3}, u_{4a+3}a_{i+1}$. Then the graph H_{i+1} consists of F_{i+1} and H_i.

The size of H_{i+1} is

$$\begin{aligned} m_{H_{i+1}} &= m_{H_i} + 2(4a+2) + 2 \\ &= [(8a^2 + 16a + 8) + 2(4a+2)(i-1) + \\ &\quad 2(i-1)] + 2(4a+2) + 2 \\ &= (8a^2 + 16a + 8) + 2(4a+2)i + 2i. \end{aligned}$$

and each vertex of H_{i+1} is even.

By Theorem 2.3.1, G contains an optimal irregular Eulerian walk in this case.

Case 3. *Exactly one of r and s is odd and $rs \not\equiv 1, 2 \pmod 4$.* There are two subcases.

Subcase 3.1. *r is odd and s is even.* If $s \equiv 2 \pmod 4$, then $m = rs \equiv 2 \pmod 4$. Thus we assume that $s \equiv 0 \pmod 4$. Then $r = 2a + 1$ and $s = 4b$ for some positive integers a and b and so $\lceil m/2 \rceil = 2b(2a+1)$. Let $U = \{u_1, u_2, \ldots, u_{2a+1}\}$ and $W = \{w_1, w_2, \ldots, w_{4b}\}$ be the partite sets of $K_{2a+1,4b}$ and let $F_1 = K_{2a,2b}$ be the subgraph of $K_{2a+1,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$. If b is even, then let $F_2 = K_{2,b}$ be the subgraph of $K_{2a+1,4b}$ with partite sets $\{u_1, u_2\}$ and $\{w_{2b+1}, w_{2b+2}, \ldots, w_{3b}\}$. If b is odd, then let F_2 be the subgraph of $K_{2a+1,4b}$ constructed from the graph $K_{2,b-1}$ with partite sets $\{u_1, u_2\}$ and $\{w_{2b+1}, w_{2b+2}, \ldots, w_{3b-1}\}$ by (i) deleting the edge u_2w_{3b-1} and (ii) adding the three edges $u_2w_{3b}, w_{3b}u_{2a+1}, u_{2a+1}w_{3b-1}$. In each case, the graph H consists of F_1 and F_2. Then H has size $2b(2a+1)$ and each vertex of H is even.

Subcase 3.2. *r is even and s is odd.* If $r \equiv 2 \pmod 4$, then $m = rs \equiv 2 \pmod 4$. Thus we assume that $r \equiv 0 \pmod 4$. Then $r = 4a$ and $s = 2b + 1$ for some positive integers a and b and so $\lceil m/2 \rceil = 2a(2b + 1)$. Let $U = \{u_1, u_2, \ldots, u_{4a}\}$ and $W = \{w_1, w_2, \ldots, w_{2b+1}\}$ be the partite sets of $K_{4a,2b+1}$ and let $F_1 = K_{2a,2b}$ be the subgraph of $K_{4a,2b+1}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$. If a is even, then let $F_2 = K_{a,2}$ be the subgraph of $K_{4a,2b+1}$ with partite sets $\{u_{2a+1}, u_{2a+2}, \ldots, u_{3a}\}$ and $\{w_1, w_2\}$. If a is odd, then let F_2 be the subgraph of $K_{4a,2b+1}$ constructed from the graph $K_{a-1,2}$ with partite sets $\{u_{2a+1}, u_{2a+2}, \ldots, u_{3a-1}\}$ and $\{w_1, w_2\}$ by (i) deleting the edge w_2u_{3a-1} and (ii) adding the three edges $w_2u_{3a}, u_{3a}w_{2b+1}, w_{2b+1}u_{3a-1}$. In each case, the graph H consists of F_1 and F_2. Then H has size $2a(2b + 1)$ and each vertex of H is even.

By Theorem 2.3.1, G contains an optimal irregular Eulerian walk in this case. ■

2.4 A Realization Result on the Eulerian Irregularity of a Graph

We have seen that if G is a nontrivial connected graph of size m, then $\binom{m+1}{2} \le EI(G) \le 2\binom{m+1}{2}$. This gives rise to the following question:

> *For given positive integers k and m with $\binom{m+1}{2} \le k \le 2\binom{m+1}{2}$, is there a connected graph G of size m such that $EI(G) = k$?*

In this section, we present a necessary and sufficient condition for a pair k, m of positive integers such that there is a nontrivial connected graph G of size m with $EI(G) = k$. In order to do this, we first present some preliminary results.

Recall that a *weighted graph* is a graph in which each edge e is assigned a positive integer called the *weight* of the edge and denoted by $w(e)$. The *degree of a vertex* v in a weighted graph H is the sum of the weights of the edges incident with v and is denoted by $\deg_H v$ (or $\deg v$ if the weighted graph H under consideration is clear). A weighted graph H is *Eulerian* if H is connected and every vertex has even degree. For an Eulerian walk W of a connected graph G, let G_W be the weighted graph obtained from G by assigning to each edge uv of G the number of times uv is encountered on W. In this case, G_W is said to be *induced* by W. Consequently, the vertex set of G_W is $V(G)$ and every vertex in G_W has even degree. Thus, the weighted graph G_W induced by an Eulerian walk W in G is Eulerian. Furthermore, for an Eulerian walk W of a connected graph G, let M be the multigraph obtained from G by replacing each edge uv of G by the number of parallel edges equal to the number of times uv is encountered on W. In this case, M is said to be *induced* by W. Consequently, M is an Eulerian multigraph whose vertex set is $V(G)$.

For a connected graph G of size m with edge set $\{e_1, e_2, \ldots, e_m\}$ and an Eulerian walk W, let a_i be the number of times that e_i is encountered in W for $1 \leq i \leq m$. If W is an Eulerian walk of minimum length, then $a_i \in \{1, 2\}$, while if W is an irregular Eulerian walk of minimum length, then $a_i \in \{1, 2, \ldots, 2m\}$ and $a_i \neq a_j$ for all i, j with $1 \leq i \neq j \leq m$. In general, a multiset $S = \{a_1, a_2, \ldots, a_m\}$ of positive integers is *Eulerian realizable* if there is a connected graph G of size m, an ordering $e_1, e_2, \ldots, e_m$ of the edges of G and an Eulerian walk W in G such that e_i is encountered exactly a_i times in W for $1 \leq i \leq m$. We now present a necessary and sufficient conditions for a multiset S of $m \geq 3$ positive integers to be Eulerian realizable.

Theorem 2.4.1 *For an integer $m \geq 3$, a multiset $S = \{a_1, a_2, \ldots, a_m\}$ of positive integers is Eulerian realizable if and only if either (i) no element in S is odd or (ii) at least three elements in S are odd.*

Proof. First, suppose that exactly one or exactly two elements of S are odd. For any connected graph F of size m and any ordering $f_1, f_2, \ldots, f_m$ of the edges of F, let H be the weighted graph obtained from F by assigning the weight a_i to f_i for $1 \leq i \leq m$. Since either exactly one edge of F is assigned an odd weight or exactly two edges of F are assigned odd weights, it follows that H must have at least two vertices of odd degree. Hence F cannot have a closed walk in which f_i is encountered a_i times for $i = 1, 2, \ldots, m$.

To verify the converse, first suppose that no element in S is odd. Let G be any connected graph of size m with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let H be the weighted graph obtained by assigning the weight a_i to e_i for $1 \leq i \leq m$. Since every element in S is even, each vertex of H has even degree and so G has a closed walk in which e_i is encountered a_i times for $i = 1, 2, \ldots, m$. Next, suppose that exactly $k \geq 3$ elements in S are odd. If $k = m$, then let $G = C_m$; while if $k < m$, then let G be the graph obtained from the k-cycle C_k of order k and the path P_{m-k} of order $m - k$ by joining an end-vertex of P_{m-k} to a vertex of C_k. Then the size of G is m. Let H be the weighted graph obtained by assigning the k odd weights to the k edges of C_k and the $m - k$ even weights to the remaining $m - k$ edges of G.

Then each vertex of H is even and so there is an ordering $e_1, e_2, \ldots, e_m$ of edges of G and a closed walk W in G such that e_i is encountered a_i times in W for $i = 1, 2, \ldots, m$. ■

In the problem of finding an Eulerian walk W of minimum length in G, we minimize the number of edges that are encountered exactly twice in W. In the problem of finding an irregular Eulerian walk W of minimum length in G, we have a different situation. For an Eulerian walk W in G, let $m_1 = m_1(W)$ be the number of edges that are encountered exactly once in W and $m_2 = m_2(W)$ the number of edges that are encountered exactly twice in W, where then $m = m_1 + m_2$. Let $e_1, e_2, \ldots, e_{m_1}$ be those edges occurring exactly once on W and let $f_1, f_2, \ldots, f_{m_2}$ be those edges occurring exactly twice on W. We construct an Eulerian multigraph M by replacing each edge e_i $(1 \leq i \leq m_1)$ by $2i-1$ parallel edges and replacing each edge f_j $(1 \leq j \leq m_2)$ by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W^* in G such that e_i $(1 \leq i \leq m_1)$ appears exactly $2i-1$ times in W^* and f_j $(1 \leq j \leq m_2)$ appears exactly $2j$ times in W^*. Thus, the length of W^* is

$$[1 + 3 + \cdots + (2m_1 - 1)] + [2 + 4 + \cdots + 2m_2] = m_1^2 + m_2(m_2 + 1)$$

where $m = m_1 + m_2$. Therefore, in the problem of finding an irregular Eulerian walk of minimum length in G, we investigate those connected graphs G that minimize $|m_1(W) - m_2(W)|$ over all Eulerian walks W in G. In the view of this observation, we present the following lemma.

Lemma 2.4.2 *Let G be a nontrivial connected graph of size m. If G contains an even subgraph F of size x, then there is an irregular Eulerian walk of length $x^2 + (m-x)(m-x+1)$ in G and so $EI(G) \leq x^2 + (m-x)(m-x+1)$.*

Proof. Let F be an even subgraph of size x in G and let

$$E(G) = \{e_1, e_2, \ldots, e_x\} \cup \{f_1, f_2, \ldots, f_{m-x}\},$$

where $E(F) = \{e_1, e_2, \ldots, e_x\}$. We construct an Eulerian multigraph M by replacing each edge e_i where $1 \leq i \leq x$ by $2i-1$ parallel edges and replacing each edge f_j where $1 \leq j \leq m-x$ by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W in G such that each edge e_i of G appears exactly $2i-1$ times in W where $1 \leq i \leq x$ and each edge f_j of G appears exactly $2j$ times in W where $1 \leq j \leq m-x$. Then the length of W is $x^2 + (m-x)(m-x+1)$ and so $EI(G) \leq L(W) = x^2 + (m-x)(m-x+1)$. ■

With the aid of Lemma 2.4.2, we determine the Eulerian irregularity of a special class of connected graphs. A graph G is *unicyclic* if G is connected and contains exactly one cycle. The next result provide the Eulerian irregularity of a unicyclic graph in terms of its size and the size of its unique cycle.

Proposition 2.4.3 *If G is a unicyclic graph of size $m \geq 3$ and the unique cycle in G is a k-cycle for some integer $k \geq 3$, then*

$$EI(G) = k^2 + (m-k)(m-k+1).$$

In particular, if $G = C_n$, then $EI(C_n) = n^2$.

Proof. Since G contains an even subgraph of size k, namely C_k, it follows by Lemma 2.4.2 that $EI(G) \leq k^2 + (m-k)(m-k+1)$. Now, let $C_k = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ be the unique cycle in G and let $E(G) - E(C_k) = \{f_1, f_2, \ldots, f_{m-k}\}$. Let W be an irregular Eulerian walk of minimum length in G. Since each edge $f_j \in E(G) - E(C_k)$ is a bridge in G for $1 \leq j \leq m-k$, it follows that f_j must be encountered an even number of times on W. Furthermore, either every edge on C_k is encountered an odd number of times on W or every edge on C_k is encountered an even number of times on W. Thus

$$\begin{aligned} EI(G) &= L(W) \geq [1+3+\cdots+(2k-1)] + [2+4+\cdots+2(m-k)] \\ &= k^2 + (m-k)(m-k+1), \end{aligned}$$

giving the desired result. In particular, if $G = C_n$, then an irregular Eulerian walk of minimum length encounters each edge of C_n an odd number of times and so $EI(C_n) = 1+3+\cdots+(2n-1) = n^2$. ■

If W is an irregular Eulerian walk of minimum length in a nontrivial connected graph G, then the set of occurrences of edges of G in W satisfies certain conditions, which are described in the next result.

Lemma 2.4.4 *Let G be a nontrivial connected graph G of size m and let W be an irregular Eulerian walk of minimum length in G. If there are x edges of G that are encountered an odd number of times in W and there are $m-x$ edges of G that are encountered an even number of times in W, then the numbers of times of the edges of G encountered in W are $1, 3, \ldots, 2x-1, 2, 4, \ldots, 2(m-x)$ and so $EI(G) = x^2 + (m-x)(m-x+1)$.*

Proof. Let W be an irregular Eulerian walk of minimum length in G, where then $L(W) = EI(G)$. For each edge e of G, let $w(e)$ be the number of times that e is encountered in W. Let $\{e_1, e_2, \ldots, e_x\}$ be the set of edges of G that are encountered an odd number of times in W and $\{f_1, f_2, \ldots, f_y\}$ the set of edges that are encountered an even number of times in W, where $y = m - x$. We may assume that $w(e_1) < w(e_2) < \cdots < w(e_x)$ and $w(f_1) < w(f_2) < \cdots < w(f_y)$. Thus $w(e_i) \geq 2i - 1$ for $1 \leq i \leq x$ and $w(f_j) \geq 2j$ for $1 \leq j \leq y$, which implies that $L(W) \geq x^2 + y(y+1)$. Now consider the Eulerian multigraph M obtained from G by replacing each edge e_i $(1 \leq i \leq x)$ by $2i - 1$ parallel edges and each edge f_j $(1 \leq j \leq y)$ by $2j$ parallel edges. An Eulerian circuit in M gives rise to an irregular Eulerian walk W^* in G such that e_i $(1 \leq i \leq x)$ appears exactly $2i - 1$ times in W^* and f_j $(1 \leq j \leq y)$ appears exactly $2j$ times in W^*. Thus, the length of $W^* = x^2 + y(y+1)$. Since W is an irregular Eulerian walk of minimum length, $L(W) \leq L(W^*) = x^2 + y(y+1)$. Therefore, $L(W) = x^2 + y(y+1)$ and so $w(e_i) = 2i - 1$ for $1 \leq i \leq x$ and $w(f_j) = 2j$ for $1 \leq j \leq y$. Therefore, the numbers of times of the edges of G encountered in W are $1, 3, \ldots, 2x-1, 2, 4, \ldots, 2(m-x)$ and so $EI(G) = x^2 + (m-x)(m-x+1)$. ■

We are now prepared to present the following realization result.

Theorem 2.4.5 *Let k and m be positive integers with $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$. Then there exists a nontrivial connected graph G of size m with $EI(G) = k$ if and only if there exists an integer x with $0 \leq x \leq m$ and $x \neq 1, 2$ such that $x^2 + (m-x)(m-x+1) = k$.*

Proof. First, suppose that G is a nontrivial connected graph of size m such that $EI(G) = k$. Let W be an irregular Eulerian walk of length $EI(G)$ in G. Suppose that there are $x \geq 0$ edges of G that are encountered an odd number of times in W and $m - x$ edges that are encountered an even number of times in W. It then follows by Lemma 2.4.4 that $L(W) = x^2 + (m - x)(m - x + 1)$. Furthermore, $x \neq 1, 2$ by Theorem 2.4.1.

For the converse, let k and m be positive integers with $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ and let x be an integer such that $0 \leq x \leq m$, $x \neq 1, 2$, and $x^2 + (m-x)(m-x+1) = k$. By Theorem 5.1.1, we may assume that $\binom{m+1}{2} < k < 2\binom{m+1}{2}$. Thus $x > 0$ and so $x \geq 3$. Let G be a unicyclic graph of size m that contains the cycle C_x of order x. It then follows by Proposition 2.4.3 that $EI(G) = x^2 + (m - x)(m - x + 1) = k$. ■

By Theorem 2.4.5, a pair k, m of positive integers with $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ can be realized as the Eulerian irregularity and the size of some nontrivial connected graph if and only if there exists an integer x with $0 \leq x \leq m$ and $x \neq 1, 2$ such that $x^2 + (m-x)(m-x+1) = k$. To determine the possible values of such integers x, we consider the real-valued function

$$L(x) = x^2 + (m - x)(m - x + 1) = 2x^2 - (2m + 1)x + m^2 + m. \tag{2.3}$$

Since $L(x)$ is a concave-up parabola which has the minimum value at $x_0 = \frac{2m+1}{4}$, it follows that the closer x is to x_0, the closer $L(x)$ is to $L(x_0)$. For a positive integer m, let $[0..m]$ be the set of all integers x with $0 \leq x \leq m$. We list the elements of $[0..m]$ as an ordered sequence s of length $m + 1$ where

$$s = (x_1, x_2, \ldots, x_{m+1}) \tag{2.4}$$

such that

$$L(x_1) \leq L(x_2) \leq \cdots \leq L(x_{m+1}), \tag{2.5}$$

where then

$$L(x_1) = \binom{m+1}{2},\ L(x_2) = \binom{m+1}{2} + 1,\ L(x_3) = \binom{m+1}{2} + 3,\ \ldots,\ L(x_{m+1}) = 2\binom{m+1}{2}.$$

The sequence s in (2.4) that satisfies (2.3) and (2.5) is referred to as the *Eulerian irregular sequence of* m. We now state a useful observation on Eulerian irregular sequences.

Observation 2.4.6 *Let m be a positive integer.*

- *If m is even, then the Eulerian irregular sequence of m is*

$$\left(\left\lceil\frac{m}{2}\right\rceil, \left\lceil\frac{m}{2}\right\rceil+1, \left\lceil\frac{m}{2}\right\rceil-1, \left\lceil\frac{m}{2}\right\rceil+2, \left\lceil\frac{m}{2}\right\rceil-2, \ldots, \left\lceil\frac{m}{2}\right\rceil+\left(\left\lfloor\frac{m}{2}\right\rfloor-1\right), \left\lceil\frac{m}{2}\right\rceil-\left(\left\lfloor\frac{m}{2}\right\rfloor-1\right), m, 0\right); \tag{2.6}$$

- *If m is odd, then the Eulerian irregular sequence of m is*

$$\left(\left\lceil\frac{m}{2}\right\rceil, \left\lceil\frac{m}{2}\right\rceil-1, \left\lceil\frac{m}{2}\right\rceil+1, \left\lceil\frac{m}{2}\right\rceil-2, \left\lceil\frac{m}{2}\right\rceil+2, \ldots, \left\lceil\frac{m}{2}\right\rceil-\left\lfloor\frac{m}{2}\right\rfloor, \left\lceil\frac{m}{2}\right\rceil+\left\lfloor\frac{m}{2}\right\rfloor = m, 0\right) \tag{2.7}$$

We next present a formula for the Eulerian irregularity $EI(G)$ of a graph G in terms of the size of G and the size of a certain even subgraph of G.

Theorem 2.4.7 *Let G be a nontrivial connected graph of size m and $(x_1, x_2, \ldots, x_{m+1})$ the Eulerian irregular sequence of m. If*

$$\alpha = \min\{i :\ G \text{ contains an even subgraph } F \text{ of size } x_i,\ 1 \leq i \leq m+1\},$$

then $EI(G) = x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$.

Proof. By Theorem 5.1.1, we may assume that G is not a tree. Since G contains an even subgraph of size x_α, it follows by Lemma 2.4.2 that

$$EI(G) \leq x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1).$$

Let W be an irregular Eulerian walk of length $EI(G)$ in G. Let E' be the set of edges of G that are encountered an odd number of times in W and let E'' be the set of edges of G that are encountered an even number of times in W. Since G is not a tree, it follows by Kwan's Theorem that $E' \neq \emptyset$. Let F' be the subgraph induced by E' and F'' the subgraph induced by E''. We claim that every vertex of F' is even. Let M be the weighted graph obtained by assigning the weight $w(e)$ to each edge e of G, where $w(e)$ is the number of times that e is encountered in W. Let H' be the weighted subgraph of M induced by the edges of F' and let H'' be the weighted subgraph of M induced by the edges of F''. Since G has an Eulerian walk in which each edge e appears exactly $w(e)$ times, every vertex of M has even degree. Since $\deg_M v = \deg_{H'} v + \deg_{H''} v$ for every vertex v of G and $\deg_M v$ and $\deg_{H''}$ are both even, it follows that $\deg_{H'} v$ is even. Suppose that $\deg_{F'} v = k$. Then v is incident with k edges in G, each of odd weight. Since $\deg_{H'} v$ is even, k is even and so v is an even vertex in F'. Therefore, F' is an even subgraph. Suppose that the size of F' is x, where then $1 \leq x \leq m$. It then follows by Lemma 2.4.4 that $EI(G) = L(W) = x^2 + (m - x)(m - x + 1)$. By the defining property of x_α and Observation 2.4.6, it follows that $x = x_\alpha$ and so $EI(G) = x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$. ∎

2.5 Eulerian Irregularities of Complete Bipartite Graphs

To illustrate the results obtained in Section 2.4, we determine Eulerian irregularities of two classes of graphs, namely the complete graphs and complete bipartite graphs. We have seen in Theorems 2.3.4 and 2.3.5 that (1) for each integer $n \geq 2$, the complete graph K_n is optimal if and only if $n \geq 4$ and (2) for integers r and s with $2 \leq r \leq s$, the complete bipartite graph $K_{r,s}$ is optimal if and only if (i) r and s are both even and $(r, s) \neq (2, 4k+2)$ for any nonnegative integer k or (ii) at least one of r and s is odd and $rs \not\equiv 1, 2 \pmod 4$.

Since K_2 is a tree and K_3 is a cycle, the Eulerian irregularities of complete graphs can be determined by Theorem 5.1.1, Proposition 2.4.3 and Theorem 2.3.4, which we state as follows.

Theorem 2.5.1 *For each integer $n \geq 2$,*

$$EI(K_n) = \begin{cases} 2 & \text{if } n = 2 \\ 9 & \text{if } n = 3 \\ \binom{\binom{n}{2}+1}{2} & \text{if } n \geq 4. \end{cases}$$

We now determine the Eulerian irregularity of a complete bipartite graph.

Theorem 2.5.2 *If the complete bipartite graph $K_{r,s}$ is not optimal where $2 \leq r \leq s$, then*

$$EI(K_{r,s}) = \begin{cases} \binom{rs+1}{2} + 6 & \text{if } r \text{ and } s \text{ are both even} \\ \binom{rs+1}{2} + 1 & \text{if at least one of } r \text{ and } s \text{ is odd.} \end{cases}$$

Proof. Suppose that $G = K_{r,s}$ is not optimal where $2 \leq r \leq s$. By Theorem 2.3.5, either

- r and s are both even and $(r, s) = (2, 4k + 2)$ for some $k \geq 0$ or
- at least one of r and s is odd and $rs = 1, 2 \pmod 4$.

Let $m = rs$ be the size of G.

First, suppose that r and s are both even and $(r, s) = (2, 4k + 2)$ for some $k \geq 0$. Then $m = rs = 8k + 4$ and so $\frac{m}{2} = 4k + 2$. Since G contains the even subgraph $H = K_{2,2k+2}$ of size $4k + 4 = \frac{m}{2} + 2$, it follows by Lemma 2.4.2 that

$$EI(G) \leq (4k + 4)^2 + (4k)(4k + 1) = \binom{rs + 1}{2} + 6.$$

Since G contains neither even subgraph of odd size $4k + 3$ nor even subgraph of odd size $4k + 1$, it follows by Theorem 2.4.5, Observation 2.4.6 and Theorem 2.4.7 that $EI(G) \geq \binom{rs+1}{2} + 6$ and so $EI(G) = \binom{rs+1}{2} + 6$.

Next, suppose that at least one of r and s is odd and $rs = 1, 2 \pmod 4$. Denote the partite sets of G by

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } W = \{w_1, w_2, \ldots, w_s\}.$$

We consider three cases, according to the parity of r and s.

Case 1. *r is odd and s is even.* Since $rs \equiv 2 \pmod 4$ and $r \leq s$, it follows that $r = 2a + 1$ and $s = 4b + 2$, where $a, b \geq 1$ and $a \leq 2b$. Since G is not optimal by Theorem 2.3.5 and $m = rs$ is even, it follows by Observation 2.4.6 that

$EI(G) \geq L(\frac{m}{2}+1) = \binom{m+1}{2}+1$ where $L(x)$ is defined in (2.3) for an integer x. That is,

$$EI(G) \geq \left(\frac{m}{2}+1\right)^2 + \left(\frac{m}{2}-1\right)\frac{m}{2}.$$

By Lemma 2.4.2, it remains to show that G contains an even subgraph of size $\frac{m}{2}+1$. Observe that

$$\frac{m}{2}+1 = (2a+1)(2b+1)+1 = 4ab+2a+2b+2.$$

We consider two subcases, according to whether $a+b$ is odd or $a+b$ is even.

Subcase 1.1. *$a+b$ is odd.* First, suppose that $a \leq b$. Then $3b+a+1 \leq 4b+2$ and $\frac{m}{2}+1 = 4ab+2(a+b+1)$. Let $F_1 = K_{2a,2b}$ be the subgraph of G induced by $\{u_1, u_2, \ldots, u_{2a}\} \cup \{w_1, w_2, \ldots, w_{2b}\}$ and let $F_2 = K_{2,a+b+1}$ the subgraph of G induced by $\{u_1, u_2\} \cup \{w_{2b+1},\ w_{2b+2},\ \ldots,\ w_{3b+a+1}\}$. Then let H be the even subgraph consisting of F_1 and F_2 whose vertex set is $V(F_1) \cup V(F_2)$ and whose edge set $E(F_1) \cup E(F_2)$. Then the size of H is $\frac{m}{2}+1$.

Next, suppose that $b < a \leq 2b$. If a is even and b is odd, then $\frac{m}{2}+1 = a(4b+2)+2(b+1)$. Let $F_1 = K_{a,4b+2}$ with partite sets $\{u_1, u_2, \ldots, u_a\}$ and W and let $F_2 = K_{2,b+1}$ with partite sets $\{u_{a+1},\ u_{a+2}\}$ and $\{w_1, w_2, \ldots, w_{b+1}\}$. Then let H be the even subgraph consisting of F_1 and F_2. If a is odd and b is even, then $\frac{m}{2}+1 = (a-1)(4b)+6b+2(a+1)$. Let $F_1' = K_{a-1,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{a-1}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2' = K_{2,3b}$ with partite sets $\{u_a, u_{a+1}\}$ and $\{w_1,\ w_2,\ \ldots,\ w_{3b}\}$ and let $F_3' = K_{a+1,2}$ with partite sets $\{u_1, u_2, \ldots, u_{a+1}\}$ and $\{w_{4b+1}, w_{4b+2}\}$. Then let H be the even subgraph consisting of F_1', F_2' and F_3' and the size of H is $\frac{m}{2}+1$.

Subcase 1.2. *$a+b$ is even.* Then a and b are of the same parity. First, suppose that a and b are both odd, say $a = 2p+1$ and $b = 2q+1$ for some integers $p, q \geq 0$. Then $\frac{m}{2} + 1 = (2a)(2b) + 2(2q) + 2(2p) + 6$. If $b = 1$, then $a = 1$ (since $r \leq s$) and so $G = K_{3,6}$. The even subgraph of G consisting of $C_4 = (u_1, w_1, u_2, w_2, u_1)$ and $C_6 = (u_1, w_3, u_2, w_4, u_3, w_5, u_1)$ has size 10. Thus, we may assume that $b \geq 2$ and so $3b + 4 \leq 4b + 2$. Let $F_1 = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$, let $F_2 = K_{2,2q}$ with partite sets $\{u_1, u_2\}$ and $\{w_{2b+1}, w_{2b+2}, \ldots, w_{3b-1}\}$, let $F_3 = K_{2p,2}$ with partite sets $\{u_1, u_2, \ldots, u_{2p}\}$ and $\{w_{3b}, w_{3b+1}\}$ and let

$$F_4 = C_6 = (u_1, w_{3b+2}, u_2, w_{3b+3}, u_3, w_{3b+4}, u_1).$$

Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m}{2} + 1$.

Next, suppose that a and b are both even, say $a = 2p$ and $b = 2q$ for some integers $p, q \geq 1$. Then $\frac{m}{2} + 1 = (2a)(2b) + 2[2(p-1) + 2q] + 6$. If $b = 2$, then $a = 2$ or $a = 4$. Thus $G = K_{5,10}$ or $G = K_{9,10}$. For $K_{5,10}$, it follows that $\frac{m}{2} + 1 = 26$ and let the even subgraph of G be consisted of $K_{2,10}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_{10}\}$ and $C_6 = (u_3, w_1, u_4, w_2, u_5, w_3, u_3)$. For $K_{9,10}$, it follows that $\frac{m}{2} + 1 = 46$ and let the even subgraph of G be consist of $K_{4,10}$ with partite sets $\{u_1, u_2, u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_{10}\}$ and $C_6 = (u_5, w_1, u_6, w_2, u_7, w_3, u_5)$. In each case, G has an even subgraph of size $\frac{m}{2} + 1$.

We now assume that $b \geq 4$ and so $3b + 5 \leq 4b + 2$. Let $F_1 = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$, let $F_2 = K_{2(p-1),2}$ with partite sets $\{u_1, u_2, \ldots, u_{2(p-1)}\}$ and $\{w_{2b+1}, w_{2b+2}\}$, let $F_3 = K_{2,2q}$ with partite sets $\{u_1, u_2\}$ and

$\{w_{2b+3},\ w_{2b+4},\ \ldots,\ w_{3b+2}\}$ and let $F_4 = C_6 = (u_1, w_{3b+3}, u_2, w_{3b+4}, u_3, w_{3b+5}, u_1)$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m}{2}+1$.

Case 2. *r is even and s is odd.* In this case, the size $m = rs$ is even. Furthermore, $r = 4a+2$ and $s = 2b+1$, where $a \geq 0$ and $b \geq 1$. Then

$$\frac{m}{2} + 1 = (2a)(2b) + 2(a+b+1).$$

Since $r \leq s$, it follows that $a < b$ and so $a+b+1 \leq 2b$. First, suppose that $a+b$ is odd. Let $F_1 = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$ and let $F_2 = K_{2,a+b+1}$ with partite sets $\{u_{2a+1}, u_{2a+2}\}$ and $\{w_1, w_2, \ldots, w_{a+b+1}\}$. Then let H be the even subgraph consisting of F_1 and F_2 and the size of H is $\frac{m}{2}+1$.

Next, suppose that $a+b$ is even. First, assume that a and b are both odd, say $a = 2p+1$ and $b = 2q+1$ for some integers $p, q \geq 0$. If $a = 1$, then $G = K_{6,2b+1}$, where $b \geq 3$, and $\frac{m}{2}+1 = 4b+2(b+2) = 4b+2(b-1)+6$. Let $F_1 = K_{4,b}$ with partite sets $\{u_1, u_2, u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_b\}$, let $F_2 = K_{2,b-1}$ with partite sets $\{u_5, u_6\}$ and $\{w_1, w_2, \ldots, w_{b-1}\}$ and let

$$F_3 = C_6 = (u_1, w_{b+1}, u_2, w_{b+2}, u_3, w_{b+3}, u_1).$$

Then let H be the even subgraph consisting of F_1, F_2 and F_3 and the size of H is $\frac{m}{2}+1$. Thus, we may assume that $a \geq 3$ and so $3a+4 \leq 4a+2$. Then $\frac{m}{2}+1 = (2a)(2b)+2(2p)+2(2q)+6$. Let $F_1' = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$, let $F_2' = K_{2p,2}$ with partite sets $\{u_{2a+1}, u_{2a+2}, u_{3a-1}\}$ and $\{w_1, w_2\}$, let $F_3' = K_{2,2q}$ whose partite sets $\{u_{3a}, u_{3a+1}\}$ and $\{w_1, w_2, \ldots, w_{2q}\}$ and let

$$F_4' = C_6 = (u_{3a+2}, w_1, u_{3a+3}, w_2, u_{3a+4}, w_3, u_{3a+2}).$$

Then let H be the even subgraph consisting of F_1', F_2', F_3' and F_4' and the size of H is $\frac{m}{2} + 1$.

Next, suppose that a and b are both even, say $a = 2p$ and $b = 2q$ for some integers $p, q \geq 1$. Then $\frac{m}{2} + 1 = (2a)(2b) + 2[2(p-1) + 2q] + 6$. Let $F_1 = K_{2a,2b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{2b}\}$, let $F_2 = K_{2(p-1),2}$ whose partite sets $\{u_{2a+1}, u_{2a+2}, \ldots, u_{3a-2}\}$ and $\{w_1, w_2\}$, let $F_3 = K_{2,2q}$ with partite sets $\{u_{3a-1}, u_{3a}\}$ and $\{w_1, w_2, \ldots, w_{2q}\}$ and let

$$F_4 = C_6 = (u_{3a+1}, w_1, u_{3a+2}, w_2, u_{3a+3}, w_3, u_{3a+1}).$$

Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m}{2} + 1$.

Case 3. *r and s are both odd.* Since $m = rs$ is odd, it follows by Observation 2.4.6 that $EI(G) \geq L(\lceil \frac{m}{2} \rceil - 1) = \binom{m+1}{2} + 1$. By Lemma 2.4.2, it remains to show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$. Since $rs \equiv 1 \pmod 4$, either $r, s \equiv 3 \pmod 4$ or $r, s \equiv 1 \pmod 4$. We consider these two subcases.

Subcase 3.1. *$r, s \equiv 3 \pmod 4$.* Then $r = 4a+3$ and $s = 4b+3$, where $0 \leq a \leq b$. Thus

$$\left\lceil \frac{m}{2} \right\rceil - 1 = \frac{m-1}{2} = 8ab + 6a + 6b + 4 = (2a)(4b) + 2(3a + 3b) + 4.$$

First, suppose that a and b are both even. If $a = b = 0$, then $G = K_{3,3}$ and $\frac{m-1}{2} = 4$. Let $H = C_4 = (u_1, w_1, u_2, w_2, u_1)$. Thus, we now assume that

$b \geq a \geq 2$. Let $F_1 = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2 = K_{3a,2}$ with partite sets $\{u_1, u_2, \ldots, u_{3a}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3 = K_{2,3b}$ with partite sets $\{u_{3a+1}, u_{3a+2}\}$ and $\{w_1, w_2, \ldots, w_{3b}\}$ and let

$$F_4 = C_4 = (u_{3a+3}, w_1, u_{3a+4}, w_2, u_{3a+3}).$$

Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m-1}{2}$.

Next, suppose that exactly one of a and b is odd. First, assume that a is even and b is odd, where then $a \geq 0$ and $b \geq 1$. If $a = 0$, then $G = K_{3,4b+3}$ and $\frac{m-1}{2} = 6b + 4 = 6(b-1) + 10$. Let $F_1 = K_{2,3(b-1)}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_{3b-3}\}$, let $F_2 = C_4 = (u_1, w_{3b-2}, u_2, w_{3b-1}, u_1)$ and $F_3 = C_6 = (u_1, w_{3b}, u_2, w_{3b+1}, u_3, w_{3b+2}, u_1)$. Then let H be the even subgraph consisting of F_1, F_2 and F_3 and the size of H is $\frac{m-1}{2}$. If $a = 2$, then $G = K_{11,4b+3}$ where $b \geq 3$ and $\frac{m-1}{2} = 4(4b) + 6(b+1) + 10$. Let $F_1' = K_{4,4b}$ with partite sets $\{u_1, u_2, u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2' = K_{2,3(b+1)}$ with partite sets $\{u_5, u_6\}$ and $\{w_1, w_2, \ldots, w_{3(b+1)}\}$,

$$F_3' = C_{10} = (u_7, w_1, u_8, w_2, u_9, w_3, u_{10}, w_4, u_{11}, w_5, u_7).$$

Then let H be the even subgraph consisting of F_1', F_2' and F_3' and the size of H is $\frac{m-1}{2}$. We now assume $a \geq 4$ and $3a + 7 \leq 4a + 3$. Then

$$\frac{m-1}{2} = (2a)(4b) + 2(3a) + 2[3(b-1)] + 10.$$

Let $F_1'' = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2'' = K_{3a,2}$ with partite sets $\{u_1, u_2, \ldots, u_{3a}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3'' = K_{2,3(b-1)}$ with partite sets $\{u_{3a+1}, u_{3a+2}\}$ and $\{w_1, w_2, \ldots, w_{3b-3}\}$ and let $F_4'' = C_{10}$ be a cycle of order 10 in the subgraph $K_{5,5}$ of G with partite sets $\{u_{3a+3}, u_{3a+4}, \ldots, u_{3a+7}\}$ and $\{w_1, w_2, \ldots, w_5\}$. Then let H be the even subgraph consisting of F_1'', F_2'', F_3'' and F_4'' and the size of H is $\frac{m-1}{2}$.

Next, assume that a is odd and b is even, where then $1 \leq a < b$. If $a = 1$, then $G = K_{7,4b+3}$ and $\frac{m-1}{2} = 2(4b) + 2(3b+2) + 6$. Let $F_1 = K_{2,4b}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2 = K_{2,3b+2}$ with partite sets $\{u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_{3b+2}\}$, let $F_3 = C_6 = (u_5, w_1, u_6, w_2, u_7, w_3, u_5)$. Then let H be the even subgraph consisting of F_1, F_2 and F_3 and the size of H is $\frac{m-1}{2}$. Thus, we now assume that $a \geq 3$. Then $\frac{m-1}{2} = (2a)(4b) + 6(a-1) + 6b + 10$. Let $F_1' = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2' = K_{3(a-1),2}$ with partite sets $\{u_1, u_2, \ldots, u_{3a-3}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3' = K_{2,3b}$ with partite sets $\{u_{3a-2}, u_{3a-1}\}$ and $\{w_1, w_2, \ldots, w_{3b}\}$, let $F_4' = C_4 = (u_{3a}, w_1, u_{3a+1}, w_2, u_{3a})$ and let $F_5' = C_6 = (u_{4a+1}, w_1, u_{4a+2}, w_2, u_{4a+3}, w_3, u_{4a+1})$. Then let H be the even subgraph consisting of F_1', F_2', F_3', F_4' and F_5' and the size of H is $\frac{m-1}{2}$.

Final, suppose that a and b are both odd. Let $a = 2p+1$ and $b = 2q+1$ for some integers $p, q \geq 0$. Then $\frac{m-1}{2} = (2a)(4b) + 6(a-1) + 6(b-1) + 16$. Let $F_1 = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2 = K_{3(a-1),2}$ with partite sets $\{u_1, u_2, \ldots, u_{3a-3}\}$ and $\{w_{4b+1}, w_{4b+2}\}$, let $F_3 = K_{2,3(b-1)}$ with partite sets $\{u_{3a-2}, u_{3a-1}\}$ and $\{w_1, w_2, \ldots, w_{3b-3}\}$ and let $F_4 = K_{4,4}$ with $\{u_{3a}, u_{3a+1}, u_{3a+2}, u_{3a+3}\}$ and $\{w_1, w_2, w_3, w_4\}$. Then let H be the even subgraph consisting of F_1, F_2, F_3 and F_4 and the size of H is $\frac{m-1}{2}$.

Subcase 3.2. $r, s \equiv 1 \pmod 4$. Then $r = 4a+1$ and $b = 4b+1$ where $1 \le a \le b$. Thus

$$\left\lceil \frac{m}{2} \right\rceil - 1 = \frac{m-1}{2} = 8ab + 2a + 2b.$$

First, suppose that $a+b$ is even. Then $\frac{m-1}{2} = (2a)(4b) + 2(a+b)$. Let $F_1 = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$ and let $F_2 = K_{2,a+b}$ with partite sets $\{u_{2a+1}, u_{2a+2}\}$ and $\{w_1, w_2, \ldots, w_{a+b}\}$. Then let H be the even subgraph consisting of F_1 and F_2 and the size of H is $\frac{m-1}{2}$.

Next, suppose that $a+b$ is odd and so $a+b \ge 3$. If $a+b = 3$, then $G = K_{5,9}$ and $\frac{m-1}{2} = 22$. Let $F_1 = K_{2,8}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_8\}$ and $F_2 = C_6 = (u_3, w_1, u_4, w_2, u_5, w_3, u_3)$. Then let H be the even subgraph of size 22 consisting of F_1 and F_2 and the size of H is $\frac{m-1}{2}$. Thus we may assume that $a+b \ge 5$. If $a = 1$, then $b \ge 4$ and $G = K_{5,4b+1}$. Now $\frac{m-1}{2} = 2(4b) + 2(b-2) + 6$. Let $F_1' = K_{2,4b}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, $F_2' = K_{2,b-2}$ with partite sets $\{u_3, u_4\}$ and $\{w_1, w_2, \ldots, w_{b-2}\}$ and $F_3' = C_6 = (u_3, w_{b-1}, u_4, w_b, u_5, w_{b+1}, u_3)$. Then let H be the even subgraph consisting of F_1', F_2' and F_3' and the size of H is $\frac{m-1}{2}$. Now assume that $a \ge 2$. Then $\frac{m-1}{2} = (2a)(4b) + 2(a+b-3) + 6$. Let $F_1'' = K_{2a,4b}$ with partite sets $\{u_1, u_2, \ldots, u_{2a}\}$ and $\{w_1, w_2, \ldots, w_{4b}\}$, let $F_2'' = K_{2,a+b-3}$ with partite sets $\{u_{2a+1}, u_{2a+2}\}$ and $\{w_1, w_2, \ldots, w_{a+b-3}\}$ and let $F_3'' = C_6 = (u_{2a+3}, w_1, u_{2a+4}, w_2, u_{2a+5}, w_3, u_{2a+3})$. Then let H be the even subgraph consisting of F_1'', F_2'' and F_3'' and the size of H is $\frac{m-1}{2}$. ∎

Chapter 3

Eulerian Irregularity in Graphs

In this chapter, we study the Eulerian irregularities of some well-known classes of graphs, namely graphs of cycle rank 2, prisms, grids, powers of cycles and circulants, beginning with graphs of cycle rank 2.

3.1 Graphs of Cycle Rank 2

For a connected graph G of order n and size m, the number of edges that must be deleted from G to obtain a spanning tree of G is $m-n+1$. The number $m-n+1$ is called the *cycle rank* of G. Thus the cycle rank of a tree is 0 and the cycle rank of a *unicyclic* graph is 1. The cycle rank of a connected graph of order n and size $m = n + 1$ is therefore 2. In this section, we study the Eulerian irregularity of graphs of cycle rank 2.

Let G be a connected graph of order $n \geq 5$ and cycle rank 2. Then G contains one of the following three graphs of Figure 3.1 as a subgraph. If G contains two edge-disjoint cycles, then we say that G *is of type I*; otherwise, G is *is of type II.*

If G is of type I, then G contains a subgraph H_1 obtained from two edge-disjoint cycles C_{k_1} and C_{k_2} by either identifying a vertex of C_{k_1} with a vertex of C_{k_2} or by connecting a vertex of C_{k_1} and a vertex of C_{k_2} by a path as shown in Figure 3.1(a) and (b). In this case, H_1 is called a (k_1, k_2)-*subgraph* of G. If G is of type II, then G contains a subgraph H_2 obtained from three internally disjoint $u-v$ paths $P_{k_1+1}, P_{k_2+1}, P_{k_3+1}$ of lengths k_1, k_2, k_3, respectively, as shown in Figure 3.1(c). In this case, H_2 is called a (k_1, k_2, k_3)-*subgraph* of G.

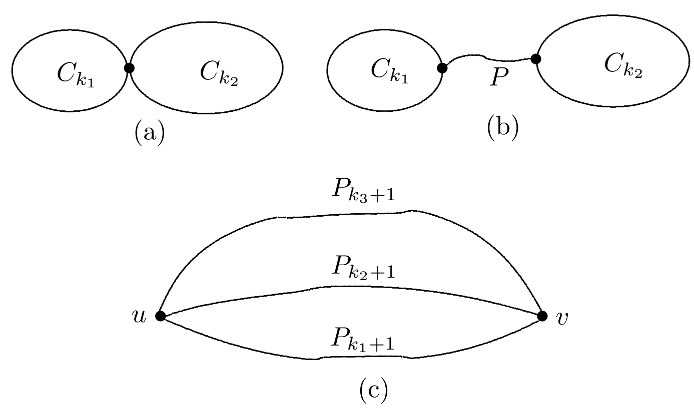


Figure 3.1: Subgraphs in a graph of cycle rank 2

Theorem 3.1.1 *Let G be a graph of order $n \geq 5$, size m and cycle rank 2. If G is of type I and contains a (k_1, k_2)-subgraph, where $3 \leq k_1 \leq k_2$, then*

$$EI(G) = \begin{cases} (k_1 + k_2)^2 + 2\binom{m-k_1-k_2+1}{2} & \text{if } m - (k_1 + k_2) \geq k_2 \\ k_2^2 + 2\binom{m-k_2+1}{2} & \text{if } m - (k_1 + k_2) < k_2. \end{cases}$$

Proof. Since G is of cycle rank 2, $m = n + 1$. First, we make an observation. Since each bridge of G is encountered an even number of times in an irregular Eulerian walk W, it follows that either all edges on C_{k_1} are encountered an odd number of times in W or all edges on C_{k_1} are encountered an even number of times in W. Similarly, this is the case for all edges on C_{k_2}. Divide the edge set $E(G)$ into three sets E_1, E_2 and E_3, where $E_i = E(C_{k_i})$ for $i = 1, 2$ and $E_3 = E(G) - (E_1 \cup E_2)$ is the set of all bridges of G. Thus $\{E_1, E_2, E_3\}$ is a partition of $E(G)$ if $E_3 \neq \emptyset$. Let W be an irregular Eulerian walk of minimum length in G. Now let $E'(W)$ be the set of edges of G that are encountered an odd number of times in W and $E''(W)$ the set of edges that are encountered an even number of times in W. As we indicated above, $E_3 \subseteq E''(W)$. Since W is an irregular Eulerian walk of minimum length in G, we may assume that $E'(W) = \{e_1, e_2, \ldots, e_a\}$ and $E''(W) = \{f_1, f_2, \ldots, f_b\}$ for some nonnegative integers a and b such that e_i $(1 \leq i \leq a)$ appears exactly $2i - 1$ times in W and f_j $(1 \leq j \leq b)$ appears exactly $2j$ times in W. Therefore,

$$\begin{aligned} L(W) &= [1 + 3 + \cdots + (2a - 1)] + (2 + 4 + \cdots + 2b) \\ &= a^2 + b(b + 1). \end{aligned}$$

Let $p = m - (k_1 + k_2)$. We consider two cases.

Case 1. $p \geq k_2$. There are four possibilities for W, according to the sets $E'(W)$ and $E''(W)$.

- If $E'(W) = E_1 \cup E_2$ and $E''(W) = E_3$, then $L(W) = (k_1 + k_2)^2 + p(p + 1)$.

- If $E'(W) = E_1$ and $E''(W) = E_2 \cup E_3$, then, since $p \geq k_2 \geq k_1$,

$$\begin{aligned} L(W) &= k_1^2 + (k_2 + p)(k_2 + p + 1) = k_1^2 + k_2^2 + 2pk_2 + p^2 + k_2 + p \\ &\geq k_1^2 + 2k_1k_2 + k_2^2 + p(p + 1) = (k_1 + k_2)^2 + p(p + 1). \end{aligned}$$

- If $E'(W) = E_2$ and $E''(W) = E_1 \cup E_3$, then, since $p \geq k_2$,

$$\begin{aligned} L(W) &= k_2^2 + (k_1 + p)(k_1 + p + 1) = k_1^2 + k_2^2 + 2pk_1 + p^2 + k_1 + p \\ &\geq k_1^2 + 2k_1k_2 + k_2^2 + p(p+1) = (k_1 + k_2)^2 + p(p+1). \end{aligned}$$

- If $E''(W) = E_1 \cup E_2 \cup E_3$, then

$$L(W) = (k_1 + k_2 + p)(k_1 + k_2 + p + 1) \geq (k_1 + k_2)^2 + p(p+1).$$

Thus $L(W) = (k_1+k_2)^2+p(p+1)$ is minimum when $E'(W) = E_1 \cup E_2$ and $E''(W) = E_3$, in which case, the difference between $|E'(W)|$ and $|E''(W)|$ in absolute value is the minimum. Therefore, $EI(G) = (k_1 + k_2)^2 + p(p+1)$.

Case 2. $p < k_2$. Again, there are four possibilities for W, according to the sets $E'(W)$ and $E''(W)$.

- If $E'(W) = E_2$ and $E''(W) = E_1 \cup E_3$, then $L(W) = k_2^2 + (k_1+p)(k_1+p+1)$.

- If $E'(W) = E_1$ and $E''(W) = E_2 \cup E_3$, then, since $k_2 \geq k_1$,

$$\begin{aligned} L(W) &= k_1^2 + (k_2 + p)(k_2 + p + 1) = k_1^2 + k_2^2 + 2pk_2 + p^2 + k_2 + p \\ &\geq k_2^2 + k_1^2 + 2pk_1 + p^2 + k_1 + p \\ &= k_2^2 + (k_1 + p)^2 + k_1 + p = k_2^2 + (k_1 + p)(k_1 + p + 1). \end{aligned}$$

- If $E'(W) = E_1 \cup E_2$ and $E''(W) = E_3$, then, since $k_1 \geq 3$ and so $2k_1 > k_1$,

$$\begin{aligned} L(W) &= (k_1+k_2)^2 + p(p+1) = k_1^2 + 2k_1k_2 + k_2^2 + p^2 + p \\ &\geq k_2^2 + k_1^2 + 2k_1(p+1) + p^2 + p \ \text{ since } p+1 \leq k_2 \\ &= k_2^2 + (k_1+p)^2 + 2k_1 + p \\ &> k_2^2 + (k_1+p)^2 + k_1 + p = k_2^2 + (k_1+p)(k_1+p+1). \end{aligned}$$

- If $E''(W) = E_1 \cup E_2 \cup E_3$, then

$$\begin{aligned} L(W) &= (k_1+k_2+p)(k_1+k_2+p+1) \\ &= (k_1+k_2+p)^2 + (k_1+k_2+p) \\ &= (k_1+k_2)^2 + 2(k_1+k_2)p + p^2 + k_1 + k_2 + p \\ &\geq (k_1+k_2)^2 + k_1 + p(p+1) > k_2^2 + (k_1+p)^2 + k_1 + p \\ &= k_2^2 + (k_1+p)(k_1+p+1) \end{aligned}$$

Thus $L(W) = k_2^2 + (k_1+p)(k_1+p+1)$ is minimum when $E'(W) = E_2$ and $E''(W) = E_1 \cup E_3$. Therefore, $EI(G) = k_2^2 + (k_1+p)(k_1+p+1)$. ■

Theorem 3.1.2 *Let G be a graph of order $n \geq 4$, size m and cycle rank 2. Suppose that G is of type II and contains a (k_1, k_2, k_3)-subgraph, where $1 \leq k_1 \leq k_2 \leq k_3$. Let*

$$M = \min\left\{\left|(k_i + k_j) - \left\lceil \frac{m}{2} \right\rceil\right| : i, j \in \{1,2,3\}, i \neq j\right\}. \tag{3.1}$$

(1) *If $M = 0$, then $EI(G) = \binom{m+1}{2}$;*

(2) *For* $M \geq 1$*,*

- *if there exist at least two distinct pairs* (i, j) *where* $i, j \in \{1, 2, 3\}$ *such that* $\left|(k_i + k_j) - \lceil \frac{m}{2} \rceil\right| = M$ *and* $(k_i + k_j) - \lceil \frac{m}{2} \rceil$ *are different in signs; that is, there are at lest two pairs* (r, s) *and* (ℓ, t)*, where* $r, s, \ell, t \in \{1, 2, 3\}$ *and* $(r, s) \neq (\ell, t)$*, such that*

$$(k_r + k_s) - \lceil \tfrac{m}{2} \rceil = M \text{ and } \lceil \tfrac{m}{2} \rceil - (k_\ell + k_t) = M,$$

then

$$EI(G) = \begin{cases} (k_r + k_s)^2 + (m - k_r - k_s)(m - k_r - k_s + 1) & \text{if } m \text{ is even} \\ (k_\ell + k_t)^2 + (m - k_\ell - k_t)(m - k_\ell - k_t + 1) & \text{if } m \text{ is odd.} \end{cases} \tag{3.2}$$

- *if for all pairs* (i, j) *where* $i, j \in \{1, 2, 3\}$ *such that* $\left|(k_i + k_j) - \lceil \frac{m}{2} \rceil\right| = M$ *either* $k_i + k_j > \lceil \frac{m}{2} \rceil$ *for all such pairs* (i, j) *or* $k_i + k_j < \lceil \frac{m}{2} \rceil$ *for all such pairs* (i, j) *and* (r, s) *is one of such pairs, then*

$$EI(G) = (k_r + k_s)^2 + (m - k_r - k_s)(m - k_r - k_s + 1). \tag{3.3}$$

Proof. If $M = 0$, then there is an even subgraph $C_{k_i+k_j}$ of order $k_i + k_j$ of G, where $i, j \in \{1, 2, 3\}$. Since the size of $C_{k_i+k_j}$ is $k_i + k_j = \lceil \frac{m}{2} \rceil$, it follows that $EI(G) = \binom{m+1}{2}$.

Now let $M \geq 1$. First assume that there exist at least two distinct pairs (r, s) and (ℓ, t) such that $(k_r + k_s) - \lceil \frac{m}{2} \rceil = M$ and $\lceil \frac{m}{2} \rceil - (k_\ell + k_t) = M$. Thus $k_r + k_s = \lceil \frac{m}{2} \rceil + M$ and $k_\ell + k_t = \lceil \frac{m}{2} \rceil - M$. Let $H_1 = C_{k_r+k_s}$ and $H_2 = C_{k_\ell+k_t}$. Then H_1 and H_2 are even subgraphs of size $k_r + k_s$ and $k_\ell + k_t$, respectively. By Observation 2.4.6, it follows that (3.2) holds.

Next, suppose that for all pairs (i, j) such that $\left|(k_i + k_j) - \lceil \frac{m}{2} \rceil\right| = M$ either $k_i + k_j > \lceil \frac{m}{2} \rceil$ for all such pairs (i, j) or $k_i + k_j < \lceil \frac{m}{2} \rceil$ for all such pairs (i, j). Let (r, s) be one of pairs. Then $H = C_{k_r + k_s}$ is an even subgraph of size $k_r + k_s$ in G. By Observation 2.4.6, it follows that (3.3) holds. ■

3.2 Prisms and Grids

The *Cartesian product* $G \,\square\, H$ of two graphs G and H has vertex set $V(G) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \,\square\, H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. The graph $C_n \,\square\, K_2$ where $n \geq 3$ is called a *prism* while $P_n \,\square\, P_q$ where $n \geq q \geq 2$ is called a *grid*. In this section, we determine the Eulerian irregularities of all prisms and grids. In order to do this, we first recall two useful lemmas in Chapter 2, the first of which is a consequence of Theorem 2.3.1 (also see Corollary 2.3.2), while the second one is a consequence of the proof of Theorem 2.4.5 (also see Lemma 2.4.2).

Lemma 3.2.1 *If G is a connected bipartite graph of size $m \geq 1$ such that $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$, then G is not optimal.*

Lemma 3.2.2 *Let G be a nontrivial connected graph of size m. If G contains an even subgraph of size x, then there is an irregular Eulerian walk of length $x^2 + (m - x)(m - x + 1)$ in G and so $EI(G) \leq x^2 + (m - x)(m - x + 1)$.*

Theorem 3.2.3 *For each integer $n \geq 3$, the prism $C_n \,\square\, K_2$ is optimal if and only if $n \not\equiv 2 \pmod 4$. Furthermore, if $C_n \,\square\, K_2$ is not optimal, then $EI(C_n \,\square\, K_2) = \binom{3n+1}{2} + 1$.*

Proof. For $G = C_n \,\square\, K_2$, let $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ be two disjoint copies of C_n in G such that $u_iv_i \in E(G)$ for $1 \leq i \leq n$. If $n \equiv 2 \pmod 4$, then G is a cubic bipartite graph of size $m = 3n$. Since m is congruent to 2 modular 4, it follows by Lemma 3.2.1 that G is not optimal.

For the converse, suppose that $n \not\equiv 2 \pmod 4$. Then $n \equiv r \pmod 4$, where $r = 0, 1, 3$. We consider these three cases. In each case, we show that G contains a 2-regular subgraph of size $\lceil m/2 \rceil$, where m is the size of G.

Case 1. $n \equiv 0 \pmod 4$. Then $n = 4k$ for some positive integer k and so the size of G is $m = 3n = 12k$. Thus $m/2 = 6k$. The $6k$-cycle

$$C_{6k} = (v_1, v_2, \ldots, v_{3k}, u_{3k}, u_{3k-1}, \ldots, u_1, v_1).$$

is a 2-regular subgraph of size $6k$ in G.

Case 2. $n \equiv 1 \pmod 4$. Then $n = 4k + 1$ for some positive integer k and so the size of G is $m = 3n = 12k + 3$. Thus $\lceil m/2 \rceil = 6k + 2$. The $(6k + 2)$-cycle

$$C_{6k+2} = (u_1, u_2, \ldots, u_{3k+1}, v_{3k+1}, v_{3k}, \ldots, v_1, u_1)$$

is a 2-regular subgraph of size $6k + 2$ in G.

Case 3. $n \equiv 3 \pmod 4$. Then $n = 4k + 3$ for some integer $k \geq 0$ and the size of G is $m = 3n = 12k + 9$ and so $\lceil m/2 \rceil = 6k + 5$. The $(6k + 5)$-cycle

$$\begin{aligned} C_{6k+5} = \ & (v_1, v_2, \ldots, v_{2k+1}, v_{2k+2}, u_{2k+2}, u_{2k+3}, v_{2k+3}, \\ & v_{2k+4}, u_{2k+4}, u_{2k+5}, v_{2k+5}, v_{2k+6}, u_{2k+6}, u_{2k+7}, \ldots, \\ & v_{4k+1}, v_{4k+2}, u_{4k+2}, u_{4k+3}, v_{4k+3}, v_1) \end{aligned}$$

is a 2-regular subgraph of size $6k+5$ in G. In each case, G contains a 2-regular subgraph of size $\lceil m/2 \rceil$, By Theorem 2.3.1, G is optimal.

We now assume that $n \equiv 2 \pmod 4$ and so $EI(G) \geq \binom{m+1}{2} + 1$. Since m is even, to show that $EI(G) \leq \binom{m+1}{2} + 1$, it suffices to show that G contains an even subgraph of size $\frac{m}{2} + 1$. Let $n = 4k + 2$ for some positive integer k and so $\frac{m}{2} + 1 = 6k + 4$. Note that G contains vertex-disjoint $H_1 = C_4$ and $H_2 = C_{6k}$ as subgraphs. Thus, G has an even subgraph of size $\frac{m}{2} + 1 = 6k + 4$ and so $EI(G) \leq \binom{m+1}{2} + 1$ by Lemma 3.2.2, giving the desired result. ■

We next determine the Eulerian irregularities of all grids $P_n \,\Box\, P_q$ where $n \geq q \geq 2$, beginning with the case when $q = 2$.

Theorem 3.2.4 *For each integer $n \geq 3$, the Cartesian product $P_n \,\Box\, K_2$ of P_n and K_2 is optimal if and only if $n \equiv 2, 3 \pmod 4$. Furthermore, if $P_n \,\Box\, K_2$ is not optimal, then $EI(P_n \,\Box\, K_2) = \binom{3n-1}{2} + 1$.*

Proof. If $n \equiv 0, 1 \pmod 4$, then the size $m = 3n - 2$ of $P_n \,\Box\, K_2$ is congruent to 2 or 1 modulo 4. It then follows by Lemma 3.2.1 that $P_n \,\Box\, K_2$ is not optimal. For the converse, suppose that $n \equiv 2, 3 \pmod 4$. Let $G = P_n \,\Box\, K_2$, where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are the two copies of P_n in G such that $u_i v_i \in E(G)$ for $1 \leq i \leq n$. The size m of G is $3n - 2$. For $n \equiv 2 \pmod 4$, let $n = 4k + 2$ for some positive integer k and $\lceil m/2 \rceil = 6k + 2$. The cycle

$$C_{6k+2} = (u_1, u_2, \ldots, u_{3k+1}, v_{3k+1}, v_{3k-1}, \ldots, v_2, v_1, u_1)$$

is a 2-regular subgraph of size $\lceil m/2 \rceil$ in G. For $n \equiv 3 \pmod 4$, let $n = 4k+3$ for some nonnegative integer k and $\lceil m/2 \rceil = 6k+4$. The cycle

$$C_{6k+4} = (u_1, u_2, \ldots, u_{3k+2}, v_{3k+2}, v_{3k+1}, \ldots, v_2, v_1, u_1)$$

is a 2-regular subgraph of size $\lceil m/2 \rceil$ in G. By Theorem 2.3.1, G is optimal if $n \equiv 2, 3 \pmod 4$.

We now assume that G is not optimal. Then either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ and $EI(G) \geq \binom{m+1}{2} + 1$. It remains to show that $EI(G) \leq \binom{m+1}{2} + 1$. For $n \equiv 0 \pmod 4$, let $n = 4k$ for some positive integer k. Since $m = 3n - 2$ is even, it suffices to show that G contains an even subgraph of size $\frac{m}{2} + 1 = 6k$. This is true as G contains C_{6k} as a subgraph and so $EI(G) = \binom{m+1}{2} + 1$ if $n \equiv 0 \pmod 4$. For $n \equiv 1 \pmod 4$, let $n = 4k+1$ for some positive integer k. Since $m = 3n - 2 = 12k + 1$ is odd, it suffices to show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2} = 6k$. This is true as G contains C_{6k} as a subgraph and so $EI(G) = \binom{m+1}{2} + 1$ if $n \equiv 1 \pmod 4$. ■

We now consider grids $P_n \,\square\, P_q$ for $n \geq q \geq 3$ in general.

Theorem 3.2.5 *For each pair (n, q) of integers with $n \geq q \geq 3$, the grid $P_n \,\square\, P_q$ is optimal if and only if (n, q) satisfies one of the following conditions:*

(i) If n and q are even, then either both n and q are congruent to 0 modulo 4 or both n and q are congruent to 2 modulo 4;

(ii) If n is even and q is odd, then $n \equiv 0 \pmod 4$ and $q \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$;

(*iii*) *If* n *is odd and* q *is even, then* $n \equiv 1 \pmod 4$ *and* $q \equiv 0 \pmod 4$ *or* $n \equiv 3 \pmod 4$ *and* $q \equiv 2 \pmod 4$.

Proof. Suppose that $G = P_n \,\square\, P_q$ consists of q paths of order n, which we denote by $P_{n,i} = (v_{1,i}, v_{2,i}, \ldots, v_{n,i})$ for $1 \leq i \leq q$ such that $v_{t,i}$ is adjacent to $v_{t,j}$ $(1 \leq t \leq n)$ when $|i - j| = 1$. The size of G is $m = n(q-1) + (n-1)q$ and G is a bipartite graph. Write $n = 4k + r_n$ and $q = 4\ell + r_q$, where $r_n, r_q \in \{0, 1, 2, 3\}$. Let $G' = P_{4k} \,\square\, P_{4\ell}$ be the induced subgraph of G with

$$V(G') = \{v_{a,b} : 1 \leq a \leq 4k, 1 \leq b \leq 4\ell\}. \tag{3.4}$$

That is, G' is the induced subgraph in G consisting of the 4ℓ paths $P_{4k,i}$ of order $4k$ where

$$P_{4k,i} = (v_{1,i}, v_{2,i}, \ldots, v_{4k,i}) \text{ for } 1 \leq i \leq 4\ell \tag{3.5}$$

such that $v_{t,i}$ is adjacent to $v_{t,j}$ $(1 \leq t \leq 4k)$ when $|i - j| = 1$. Then G' contains $k\ell$ vertex-disjoint copies (or blocks) of $P_4 \,\square\, P_4$, denoted by $B_1, B_2, \ldots, B_{k\ell}$ as shown in Figure 3.2.

B_1	B_{k+1}	$\cdots$	$B_{(\ell-1)k+1}$
B_2	B_{k+2}	$\cdots$	$B_{(\ell-1)k+2}$
B_3	B_{k+3}	$\cdots$	$B_{(\ell-1)k+3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
B_k	B_{2k}	$\cdots$	$B_{k\ell}$

Figure 3.2: The subgraph $G' = P_{4k} \,\square\, P_{4\ell}$ in G

In particular, for $1 \leq i \leq k$, the vertices of B_i appear in the way as shown in Figure 3.3. Note that $B_i = P_4 \,\square\, P_4$ contains each of the even cycles

$$C_4, C_6, C_8, C_{10}, C_{12}, C_{14}, C_{16}$$

as a subgraph.

$v_{4i-3,1}$	$v_{4i-3,2}$	$v_{4i-3,3}$	$v_{4i-3,4}$
$v_{4i-2,1}$	$v_{4i-2,2}$	$v_{4i-2,3}$	$v_{4i-2,4}$
$v_{4i-1,1}$	$v_{4i-1,2}$	$v_{4i-1,3}$	$v_{4i-1,4}$
$v_{4i,1}$	$v_{4i,2}$	$v_{4i,3}$	$v_{4i,4}$

Figure 3.3: The block $B_i = P_4 \square P_4$ in G' for $1 \le i \le k$

We consider three cases, according to the parities of n and q.

Case 1. *n and q are even.* If one of n and q is congruent to 0 modulo 4 and the other is congruent to 2 modulo 4, then $m \equiv 2 \pmod 4$ and so G is not optimal by Lemma 3.2.1. For the converse, suppose that both n and q are congruent to 0 modulo 4 or both n and q are congruent to 2 modulo 4. We consider two subcases. In each subcase, we construct an even subgraph H of size $\lceil m/2 \rceil$.

Subcase 1.1. *$n \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 4$.* Then $n = 4k$ and $q = 4\ell$ for some positive integers k and ℓ with $k \ge \ell$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \frac{m}{2} = 16k\ell - 2k - 2\ell.$$

The graph $G' = P_{4k} \square P_{4\ell}$ contains $k\ell$ vertex-disjoint copies $B_1, B_2, \ldots, B_{k\ell}$ of $P_4 \square P_4$ as shown in Figure 3.2.

- For $\ell = 1$, there are k vertex-disjoint blocks $B_1, B_2, \ldots, B_k$ of $P_4 \square P_4$ in G. Let $H_1 = C_{12}$ in B_1 and $H_i = C_{14}$ in B_i for $2 \le i \le k$. Now let H be the union of these vertex-disjoint subgraphs H_i $(1 \le i \le k)$ in G. Then H is a 2-regular graph of the size $m_H = 12 + 14(k-1) = 14k - 2$.

- For $\ell = 2$, there are $2k$ vertex-disjoint blocks $B_1, B_2, \dots, B_{2k}$ of $P_4 \,\square\, P_4$ in G. If $k = 2$, let $H_1 = C_8$ in B_1 and $H_i = C_{16}$ in B_i for $2 \le i \le 4$. Now let H be the union of these vertex-disjoint subgraphs H_i $(1 \le i \le 4)$ in G. Then H is a 2-regular graph of the size $m_H = 3 \cdot 16 + 8 = 56 = m/2$. If $k \ge 3$, let $H_i = C_{16}$ in B_i for $1 \le i \le k-2$ and let $H_i = C_{14}$ in B_i for $k-1 \le i \le 2k$. Now let H be the union of these vertex-disjoint subgraphs H_i $(1 \le i \le 2k)$ in G. Then H is a 2-regular graph of the size $m_H = 16(k-2) + 14(k+2) = 30k - 4$.

- For $\ell \ge 3$, let $H_i = C_{16}$ in B_i for $1 \le i \le k\ell - k - \ell$ and let $H_i = C_{14}$ in B_i for $k\ell - k - \ell + 1 \le i \le k\ell$. Now let H be the union of these vertex-disjoint subgraphs H_i $(1 \le i \le k\ell)$ in G. Then H is a 2-regular graph of the size $m_H = 16(k\ell - k - \ell) + 14(k + \ell) = 16k\ell - 2k - 2\ell$.

Subcase 1.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Then $n = 4k+2$ and $q = 4\ell+2$ for some integers k and ℓ with $k \ge \ell \ge 1$. In this case, the size m_H of a graph H with the desired properties is

$$\begin{aligned} m_H &= \frac{m}{2} = (4k+2)(4\ell+2) - (2k+1) - (2\ell+1) \\ &= (4k+2)(4\ell+2) - 2(k+\ell+1) = 16k\ell + 6k + 6\ell + 2. \end{aligned}$$

Let $G' = P_{4k} \,\square\, P_{4\ell}$ be the induced subgraph of G as defined in (3.4) or (3.5) that contains the $k\ell$ vertex-disjoint blocks $B_1, B_2, \dots, B_{k\ell}$ of $P_4 \,\square\, P_4$ as shown in

Figure 3.2. Let $H_i = C_{16}$ in B_i for $1 \leq i \leq k\ell$ and let $C = C_{6k}$ and $C' = C_{6\ell+2}$ be two vertex-disjoint cycles of orders $6k$ and $6\ell + 2$ respectively in $G - E(G')$, where

$$\begin{aligned} C &= (v_{1,4\ell+1}, v_{2,4\ell+1}, \ldots, v_{3k,4\ell+1}, v_{3k,4\ell+2}, v_{3k-1,4\ell+2}, \ldots, \\ &\quad v_{1,4\ell+2}, v_{1,4\ell+1}) \\ C' &= (v_{4k+1,1}, v_{4k+1,2}, \ldots, v_{4k+1,3\ell+1}, v_{4k+2,3\ell+1}, v_{4k+2,3\ell}, \ldots, \\ &\quad v_{4k+2,1}, v_{4k+1,1}). \end{aligned}$$

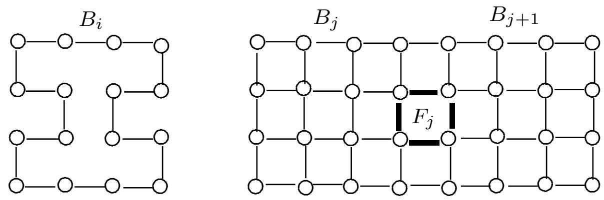
Now let H be the union of these vertex-disjoint subgraphs H_i ($1 \leq i \leq k\ell$), C and C' in G. Then H is a 2-regular graph of the size $m_H = 16k\ell + 6k + 6\ell + 2$.

Case 2. *n is even and q is odd.* If $n \equiv 0 \pmod 4$ and $q \equiv 3 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$, then $m \equiv 1 \pmod 4$ and so G is not optimal by Lemma 3.2.1. For the converse, suppose that either $n \equiv 0 \pmod 4$ and $q \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$. We consider two subcases. In each subcase, we construct an even subgraph H of size $\lceil m/2 \rceil$ in G.

Subcase 2.1. *$n \equiv 0 \pmod 4$ and $q \equiv 1 \pmod 4$.* Then $n = 4k$ and $q = 4\ell + 1$ for some positive integers k and ℓ with $k \geq \ell + 1$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = 4k(4\ell + 1) - 2k - 2\ell = 16k\ell + 2k - 2\ell.$$

For each i with $1 \leq i \leq k\ell$, let $H_i = C_{16}$ in B_i as shown in Figure 3.4(a), where the edges not belonging to C_{16} are not drawn; while for each j with $1 \leq j \leq k-1$, let $F_j = C_4$ lying between B_j and B_{j+1} as shown in Figure 3.4(b). Then H_i ($1 \leq i \leq k\ell$) and F_j ($1 \leq j \leq k - 1$) are edge-disjoint subgraphs of G.



(a) C_{16} in B_i (b) C_4 betweenB_j and B_{j+1}

Figure 3.4: The cycles C_{16} and C_4

- If $k-\ell$ is even, then $k-\ell = 2p$ for some integer $p \geq 1$. Let $H_i = C_{16}$ in B_i for $1 \leq i \leq k\ell$, where H_i is shown as in Figure 3.4(a). For each j with $1 \leq j \leq p \leq k-1$, let $F_j = C_4$ as defined in Figure 3.4(b) lying between B_j and B_{j+1}. Now let H consist of these edge-disjoint subgraphs H_i $(1 \leq i \leq k\ell)$ and F_j $(1 \leq j \leq p)$. That is,

$$\begin{aligned} V(H) &= \left(\cup_{i=1}^{k\ell} V(H_i)\right) \cup \left(\cup_{j=1}^{p} V(F_j)\right) \\ E(H) &= \left(\cup_{i=1}^{k\ell} E(H_i)\right) \cup \left(\cup_{j=1}^{p} E(F_j)\right) \end{aligned}$$

  Then H is a graph of the size $m_H = 16k\ell + 4p = 16k\ell + 2(k-\ell)$ and each vertex of H has degree 2 or 4.

- If $k-\ell$ is odd, then $k-\ell = 2p+1$ for some integer $p \geq 0$. Then $p+1 \leq k-1$. Let $H_i = C_{16}$ in B_i as shown in Figure 3.4(a) for $1 \leq i \leq k\ell-1$ and $H_{k\ell} = C_{14}$ in $B_{k\ell}$. For each j with $1 \leq j \leq p+1 \leq k-1$, let $F_j = C_4$ that lies between B_j and B_{j+1} as shown in Figure 3.4(b). Now let H consist of these edge-disjoint subgraphs H_i $(1 \leq i \leq k\ell)$ and F_j $(1 \leq j \leq p+1)$. Then H is a graph of the size $m_H = 16(k\ell-1)+14+4(p+1) = 16k\ell+2(k-\ell)$ and each vertex of H has degree 2 or 4.

Subcase 2.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 3 \pmod 4$. Then $n = 4k+2$ and $q = 4\ell+3$ for some positive integers k and ℓ with $k \geq \ell + 1$. In this case, the size m_H of a graph H with the desired properties is

$$\begin{aligned} m_H &= \left\lceil \frac{m}{2} \right\rceil = (4k+2)(4\ell+3) - (2k+1) - (2\ell+1) \\ &= 16k\ell + 10k + 6\ell + 4. \end{aligned}$$

For each i with $1 \leq i \leq k\ell$, let $H_i = C_{16}$ in B_i where for $1 \leq i \leq k$, the graphs B_i and H_i are defined as shown in Figure 3.4(a). For each j with $1 \leq j \leq k-1$, let $F_j = C_4$ between B_j and B_{j+1} are defined in Figure 3.4(b). Furthermore, let $C = C_{6\ell}$ and $C' = C_{6k+8}$ where

$$\begin{aligned} C &= (v_{4k+1,1}, v_{4k+1,2}, \ldots, v_{4k+1,3\ell}, v_{4k+2,3\ell}, v_{4k+2,3\ell-1}, \ldots, \\ &\quad v_{4k+2,1}, v_{4k+1,1}), \\ C' &= (v_{1,4\ell+2}, v_{2,4\ell+2}, \ldots, v_{3k+4,4\ell+2}, v_{3k+4,4\ell+3}, v_{3k+3,4\ell+3}, \ldots, \\ &\quad v_{1,4\ell+3}, v_{1,4\ell+2}). \end{aligned}$$

Since $k \geq \ell + 1$, it follows that $3k + 4 \leq 4k + 2$ and so such a cycle C' of order $6k + 8$ exists. Now let H consist of these edge-disjoint subgraphs H_i $(1 \leq i \leq k\ell)$, F_j $(1 \leq j \leq k-1)$, C and C'. Then H is a graph of the size

$$m_H = 16k\ell + 4(k-1) + 6\ell + 6k + 8 = 16k\ell + 10k + 6\ell + 4$$

and each vertex of H has degree 2 or 4.

Case 3. *n is odd and q is even.* If $n \equiv 1 \pmod 4$ and $q \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 0 \pmod 4$, then $m \equiv 1 \pmod 4$ and so G is not optimal by

Lemma 3.2.1. For the converse, suppose that either $n \equiv 1 \pmod 4$ and $q \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 2 \pmod 4$. We consider these two subcases. In each subcase, we construct an even subgraph H of size $\lceil m/2 \rceil$ in G. Let $G' = P_{4k} \,\square\, P_{4\ell}$ be the induced subgraph in G consisting of the 4ℓ paths of order $4k$ as defined in (3.5) and let $B_1, B_2, \ldots, B_{k\ell}$ are the $k\ell$ vertex-disjoint blocks of $P_4 \,\square\, P_4$ in G' as shown in Figure 3.2.

Subcase 3.1. $n \equiv 1 \pmod 4$ *and* $q \equiv 0 \pmod 4$. Then $n = 4k+1$ and $q = 4\ell$ for some positive integers k and ℓ with $k \geq \ell$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = (4k+1)4\ell - 2k - 2\ell = 16k\ell - 2k + 2\ell.$$

Let $H_i = C_{14}$ in B_i if $1 \leq i \leq k - \ell$ and let $H_i = C_{16}$ in B_i if $k - \ell + 1 \leq i \leq k\ell$. Let H be the union of these vertex-disjoint subgraphs H_i for $1 \leq i \leq k\ell$. Then H is a 2-regular subgraph of G and the size of H is

$$14(k-\ell) + 16[k\ell - (k-\ell)] = 16k\ell - 2k + 2\ell.$$

Subcase 3.2. $n \equiv 3 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Then $n = 4k+3$ and $q = 4\ell+2$ for some positive integers k and ℓ with $k \geq \ell$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = (4k+3)(4\ell+2) - (2k+1) - (2\ell+1) = 16k\ell + 6k + 10\ell + 4.$$

Let $H_i = C_{16}$ in B_i if $1 \leq i \leq k\ell$ which are defined as shown in Figure 3.4(a) and for each j with $1 \leq j \leq \ell - 1 \leq k - 1$, let $F_j = C_4$ between B_j and B_{j+1} as defined in Figure 3.4(b). Furthermore, let $C = C_{6\ell}$ and $C' = C_{6k+8}$ where

$$\begin{aligned} C &= (v_{4k+2,1}, v_{4k+2,2}, \ldots, v_{4k+2,3\ell}, v_{4k+3,3\ell}, v_{4k+3,3\ell-1}, \ldots, \\ &\quad v_{4k+3,1}, v_{4k+2,1}), \\ C' &= (v_{1,4\ell+1}, v_{2,4\ell+1}, \ldots, v_{3k+4,4\ell+1}, v_{3k+4,4\ell+2}, v_{3k+3,4\ell+2}, \ldots, \\ &\quad v_{1,4\ell+2}, v_{1,4\ell+1}). \end{aligned}$$

Since $3k+4 \leq 4k+3$, such a cycle C' of order $6k+8$ exists. Now let H consist of these edge-disjoint subgraphs H_i $(1 \leq i \leq k\ell)$, F_j $(1 \leq j \leq \ell - 1)$, C and C'. Then H is a graph of the size

$$m_H = 16k\ell + 4(\ell - 1) + 6\ell + 6k + 8 = 16k\ell + 10\ell + 6k + 4$$

and each vertex of H has degree 2 or 4. ■

Theorem 3.2.6 *For integers n, p with $n \geq p \geq 3$, if $P_n \,\square\, P_q$ is not optimal, then*

$$EI(P_n \,\square\, P_q) = \binom{n(q-1) + (n-1)q + 1}{2} + 1.$$

Proof. By Theorem 3.2.5, if $P_n \,\square\, P_q$ is not optimal, then n and q satisfy one of the following:

(i) If n and q are even, then either $n \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$;

(ii) If n is even and q is odd, then either $n \equiv 0 \pmod 4$ and $q \equiv 3 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$;

(*iii*) If n is odd and q is even, then $n \equiv 1 \pmod 4$ and $q \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 0 \pmod 4$.

Suppose that $G = P_n \,\square\, P_q$ consists of q paths of order n, which we denote by

$$P_{n,i} = (v_{1,i}, v_{2,i}, \ldots, v_{n,i}) \text{ for } 1 \leq i \leq q \tag{3.6}$$

such that $v_{t,i}$ is adjacent to $v_{t,j}$ $(1 \leq t \leq n)$ when $|i - j| = 1$. The size of G is $m = n(q-1)+(n-1)q$. Then $n = 4k+r_n$ and $q = 4\ell+r_q$, where $r_n, r_q \in \{0, 1, 2, 3\}$. Let $G' = P_{4k} \,\square\, P_{4\ell}$ be the subgraph of G with

$$V(G') = \{v_{a,b} : 1 \leq a \leq 4k, 1 \leq b \leq 4\ell\}. \tag{3.7}$$

The graph G' is else defined in (3.4) and (3.5). Then G' contains $k\ell$ vertex-disjoint copies (or blocks) of $P_4 \,\square\, P_4$, denoted by $B_{i,j}$ where $1 \leq i \leq k$ and $1 \leq j \leq \ell$. These blocks $B_{i,j}$ appear in G' in the way as shown in Figure 3.5.

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$\cdots$	$B_{1,\ell}$
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$\cdots$	$B_{2,\ell}$
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$\cdots$	$B_{3,\ell}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{k,1}$	$B_{k,2}$	$B_{k,3}$	$\cdots$	$B_{k,\ell}$

Figure 3.5: The subgraph $G' = P_{4k} \,\square\, P_{4\ell}$ in G

In each $B_{i,j} = P_4 \,\square\, P_4$, the vertices of $B_{i,j} = P_4 \,\square\, P_4$ appear in the way as shown in Figure 3.6.

$v_{4i-3,4j-3}$	$v_{4i-3,4j-2}$	$v_{4i-3,4j-1}$	$v_{4i-3,4j}$
$v_{4i-2,4j-3}$	$v_{4i-2,4j-2}$	$v_{4i-2,4j-1}$	$v_{4i-2,4j}$
$v_{4i-1,4j-3}$	$v_{4i-1,4j-2}$	$v_{4i-1,4j-1}$	$v_{4i-1,4j}$
$v_{4i,4j-3}$	$v_{4i,4j-2}$	$v_{4i,4j-1}$	$v_{4i,4j}$

Figure 3.6: The block $B_{i,j} = P_4 \,\square\, P_4$ in G'

Note that $B_{i,j}$ contains five edge-disjoint copies of C_4, namely

$$\begin{aligned}
Q_1 &= (v_{4i-3,4j-3}, v_{4i-3,4j-2}, v_{4i-2,4j-2}, v_{4i-2,4j-3}, v_{4i-3,4j-3}) \\
Q_2 &= (v_{4i-3,4j-1}, v_{4i-3,4j}, v_{4i-2,4j}, v_{4i-2,4j-1}, v_{4i-3,4j-1}) \\
Q_3 &= (v_{4i-1,4j-1}, v_{4i-1,4j}, v_{4i,4j}, v_{4i,4j-1}, v_{4i-1,4j-1}) \\
Q_4 &= (v_{4i-1,4j-3}, v_{4i-1,4j-2}, v_{4i,4j-2}, v_{4i,4j-3}, v_{4i-1,4j-3}) \\
Q_5 &= (v_{4i-2,4j-2}, v_{4i-2,4j-1}, v_{4i-1,4j-1}, v_{4i-1,4j-2}, v_{4i-2,4j-2})
\end{aligned}$$

where Q_5 is at the center of $B_{i,j}$ and surrounded clockwise by Q_1, Q_2, Q_3, Q_4. For each pair i, j with $1 \le i \le k$ and $1 \le j \le \ell$, let $F_{i,j}$ be the even subgraph of $B_{i,j}$ consisting of the five edge-disjoint subgraphs Q_1, Q_2, Q_3, Q_4, Q_5, each of which is a copy of C_4 and let $F'_{i,j}$ be the even subgraph of $B_{i,j}$ consisting of the four edge-disjoint subgraphs Q_1, Q_2, Q_3, Q_4. Thus, the size of $F_{i,j}$ is 20 and the size of $F'_{i,j}$ is 16 for all i, j with $1 \le i \le k$ and $1 \le j \le \ell$. We consider three cases.

Case 1. *n and q are even.* Since m is even, it suffices to show that G has an even subgraph of size $\frac{m}{2} + 1$. There are two subcases.

Subcase 1.1. $n \equiv 0 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Let $n = 4k$ and $q = 4\ell + 2$, where then $k > \ell \ge 1$. Note that $\frac{m}{2} + 1 = 16k\ell + 4k + 2(k - \ell)$. Let $G' = P_{4k} \,\square\, P_{4\ell}$

be the subgraph of G as described in (3.7) and $G^* = P_n \,\square\, P_2$ be the subgraph of G which is the Cartesian product of $P_{n,4\ell+1}$ and $P_{n,4\ell+2}$ as described in (3.6).

- If $k = \ell + 1$, then for $1 \leq i \leq k-1$ and $j = 1$, let $H_{i,1} = F_{i,1}$ and $H_{k,1} = F'_{k,1}$, while for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$.

- If $k \geq \ell + 2$, then for $1 \leq i \leq k$ and $j = 1$, let $H_{i,1} = F_{i,1}$, for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ and let $H_{k,\ell+1} = C_{2(k-\ell)}$ be a cycle of order $2(k-\ell)$ in G^* (which is possible since $2 \leq k - \ell \leq 4\ell + 1$).

In each case, let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of m_H is $\frac{m}{2} + 1$.

Subcase 1.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 0 \pmod 4$. Let $n = 4k+2$ and $q = 4\ell$, where then $k \geq \ell \geq 1$. Note that $\frac{m}{2} + 1 = 16k\ell - 2k + 6\ell$. In this case, we consider the subgraph $G'' = P_{4k+1} \,\square\, P_{4\ell}$ of G with vertex set

$$V(G'') = \{v_{a,b} : 1 \leq a \leq 4k+1, 1 \leq b \leq 4\ell\}. \tag{3.8}$$

Then G'' contains $(k-1)\ell$ vertex-disjoint copies (or blocks) $P_4 \,\square\, P_4$, which are denoted by $B_{i,j}$ where $1 \leq i \leq k-1$ and $1 \leq j \leq \ell$ and ℓ vertex-disjoint copies of $P_5 \,\square\, P_4$, which are denoted by B'_j where $1 \leq j \leq \ell$. These blocks $B_{i,j}$ and B'_j appear in G'' in the way as shown in Figure 3.7.

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$\cdots$	$B_{1,\ell}$
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$\cdots$	$B_{2,\ell}$
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$\cdots$	$B_{3,\ell}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{(k-1),1}$	$B_{(k-1),2}$	$B_{(k-1),3}$	$\cdots$	$B_{(k-1),\ell}$
B'_1	B'_2	B'_3	$\cdots$	B'_ℓ

Figure 3.7: The subgraph $G'' = P_{4k+1} \square P_{4\ell}$ in G

For each j with $1 \leq j \leq \ell$, the vertices of $B'_j = P_5 \square P_4$ appear in the way as shown in Figure 3.8.

$v_{4k-3,4j-3}$	$v_{4k-3,4j-2}$	$v_{4k-3,4j-1}$	$v_{4k-3,4j}$
$v_{4k-2,4j-3}$	$v_{4k-2,4j-2}$	$v_{4k-2,4j-1}$	$v_{4k-2,4j}$
$v_{4k-1,4j-3}$	$v_{4k-1,4j-2}$	$v_{4k-1,4j-1}$	$v_{4k-1,4j}$
$v_{4k,4j-3}$	$v_{4k,4j-2}$	$v_{4k,4j-1}$	$v_{4k,4j}$
$v_{4k+1,4j-3}$	$v_{4k+1,4j-2}$	$v_{4k+1,4j-1}$	$v_{4k+1,4j}$

Figure 3.8: The block $B'_j = P_5 \square P_4$ in G''

Note that each B'_j $(1 \leq j \leq \ell)$ contains each of C_{14} and C_{18} as a subgraph. For $i = 1$ and $1 \leq j \leq \ell$, let $H_{1,j} = F_{1,j}$ in $B_{1,j}$, for each pair i, j with $2 \leq i \leq k-1$ and $1 \leq j \leq \ell - 1$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$, for $2 \leq i \leq k$ and $j = \ell$, let $H_{i,\ell} = C_{14}$ (in $B_{i,\ell}$ if $1 \leq i \leq k-1$ and in B'_k if $i = k$) and for $i = k$ and $1 \leq j \leq \ell - 1$, let $H_{k,j} = C_{18}$ in B'_j. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of m_H is $\frac{m}{2} + 1$.

Case 2. *n is even and q is odd.* Since m is odd, it suffices to show that G has an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$. Let $G' = P_{4k} \square P_{4\ell}$ be the subgraph of G as described in (3.7). There are two cases.

Subcase 2.1. $n \equiv 0 \pmod 4$ *and* $q \equiv 3 \pmod 4$. Let $n = 4k$ and $q = 4\ell + 3$, where then $k > \ell \geq 1$. Note that $\frac{m-1}{2} = 16k\ell + 4k + 2(3k - \ell - 1)$ and $2 \leq 3k - \ell - 1 \leq 4k$. Let $G^* = P_n \,\square\, P_2$ be the subgraph of G which is the Cartesian product of $P_{n,4\ell+1}$ and $P_{n,4\ell+2}$ as described in Subcase 1.1. For $1 \leq i \leq k$ and $j = 1$, let $H_{i,1} = F_{i,1}$ in $B_{i,1}$, for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$ and let $H_{k,\ell+1} = C_{2(3k-\ell-1)}$ be a subgraph in G^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of H_m is $\frac{m-1}{2}$.

Subcase 2.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 1 \pmod 4$. Let $n = 4k+2$ and $q = 4\ell+1$, where then $k \geq \ell \geq 1$. Note that $\frac{m-1}{2} = 16k\ell+2(k+3\ell)$ and $2 \leq k+3\ell \leq 4\ell+1 = q$. Let $F^* = P_2 \,\square\, P_q$ be the subgraph of G which is the Cartesian product of the two paths

$$(v_{n-1,1}, v_{n-1,2}, \ldots, v_{n-1,q}) \text{ and } (v_{n,1}, v_{n,2}, \ldots, v_{n,q}). \tag{3.9}$$

For each pair i, j with $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$ and let $H_{k,\ell+1} = C_{2(k+3\ell))}$ be a subgraph in F^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of H_m is $\frac{m-1}{2}$.

Case 3. *n is odd and q is even.* Since m is odd, we are seeking for an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$ in G. Let $G' = P_{4k} \,\square\, P_{4\ell}$ and $G'' = P_{4k+1} \,\square\, P_{4\ell}$ be the subgraphs of G as described in (3.7) and (3.8), respectively. There are two cases.

Subcase 3.1. $n \equiv 1 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Let $n = 4k+1$ and $q = 4\ell+2$, where then $k > \ell \geq 1$. Note that $\frac{m-1}{2} = 16k\ell + 4k + 2(k + \ell)$ and $2 \leq k + \ell \leq 4k$. Let $G^* = P_n \,\square\, P_2$ be the subgraph of G as described in Subcase 1.1. For $1 \leq i \leq k$

and $j = 1$, let $H_{i,1} = F_{i,1}$ in $B_{i,1}$, for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$ and let $H_{k,\ell+1} = C_{2(k+\ell)}$ be a subgraph in G^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of H_m is $\frac{m-1}{2}$.

Subcase 3.2. *$n \equiv 3 \pmod 4$ and $q \equiv 0 \pmod 4$.* Let $n = 4k+3$ and $q = 4\ell$, where then $k \geq \ell \geq 1$. Let $F^* = P_2 \,\square\, P_q$ be the subgraph of G which is the Cartesian product of the two paths described in (3.9).

- If $\ell = 1$, then $G = P_{4k+3} \,\square\, P_4$ and $\frac{m-1}{2} = 14k + 8$. For $1 \leq i \leq k$, let $H_i = C_{14}$ in $B_{i,1}$ and let H_{k+1} be a cycle C_8 of order 8 where

$$\begin{aligned} H_{k+1} \quad = \quad & (v_{4k+1,1}, v_{4k+1,2}, v_{4k+1,3}, v_{4k+1,4}, v_{4k+2,4}, v_{4k+2,3}, \\ & v_{4k+2,2}, v_{4k+2,1}, v_{4k+1,1}). \end{aligned}$$

  Let H be the even subgraph of G consisting of edge-disjoint subgraphs H_i and then the size of H_m is $\frac{m-1}{2}$.

- If $\ell \geq 2$, then $\frac{m-1}{2} = 16k\ell - 2k + 6\ell + 2(2\ell - 1)$. Let H_1 be the even subgraph of size $16k\ell - 2k + 6\ell$ in $P_{4k+1} \,\square\, P_{4\ell}$ (which is described in Subcase 1.2) and let $H_2 = C_{2(2\ell-1)}$ be a subgraph of F^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs H_1 and H_2 and then the size of H_m is $\frac{m-1}{2}$. ■

3.3 Optimal Powers of Cycles

For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u, v) \leq k$.

The graph G^2 is called the *square* of G and G^3 is the *cube* of G. If $k \geq \text{diam}(G)$, then G^k is a complete graph. We have seen in Theorem 2.3.4 that all complete graphs of order at least 4 are optimal.

By Theorem 2.3.1, the n-cycle C_n is not optimal; while by Theorem 2.3.4, the complete graph K_n is. Thus if $k = 1$, then $C_n^1 = C_n$ is not optimal; while if $k \geq \lfloor n/2 \rfloor$, then C_n^k is. We show, in fact, that C_n^k is optimal for each integer $k \geq 2$. In order to do this, we introduce an additional definition. For a positive integer t, the *t-step* $G^{[t]}$ of a connected graph G is that graph whose vertex set is $V(G)$ such that uv is an edge of $G^{[t]}$ if $d_G(u, v) = t$. In particular, $G^{[1]} = G$. Furthermore, if $t \leq k$, then $G^{[t]}$ is a subgraph of G^k and

$$E(G^k) = E(G^{[1]}) \cup E(G^{[2]}) \cup \cdots \cup E(G^{[k]}).$$

For the n-cycle $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $n \geq 3$ and each integer i with $1 \leq i \leq n$, the vertex v_i is adjacent to v_{i+t} and v_{i-t} in $G^{[t]}$, where the subscripts are expressed as integers modulo n. Thus $C_n^{[t]}$ is a 2-regular graph of order n if $t \neq n/2$ and $C_n^{[t]} = \frac{n}{2} K_2$ if $t = n/2$ where then n is even. The kth power C_n^k of C_n is then a $2k$-regular graph of order n and size kn if $k < n/2$.

Theorem 3.3.1 *For each pair k, n of integers, where $2 \leq k \leq \lfloor n/2 \rfloor$ and $n \geq 4$, the kth power C_n^k of the n-cycle is optimal.*

Proof. If $k = \lfloor n/2 \rfloor$, then $C_n^k = K_n$, which is optimal by Theorem 2.3.4. Thus, we now assume that $k < \lfloor n/2 \rfloor$. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $n \geq 4$. The size of C_n^k is $m = kn$. If k is even, say $k = 2p$ for some positive integer p, then the subgraph C_n^p is a $(2p)$-regular graph of size $pn = \lceil m/2 \rceil$. It then follows by Theorem 2.3.1 that C_n^k is optimal if k is even. Thus, it remains to consider the case when $k \geq 3$ is odd. Since $k < \lfloor n/2 \rfloor$, it follows that $n \geq 8$. We show that C_n^k contains a subgraph H_k of size $\lceil m/2 \rceil$, each of whose vertex is even. We begin with the cube C_n^3 of C_n. There are two cases, according to whether n is even or n is odd.

Case 1. *n is even.* Let $C^* = (v_1, v_3, v_5, \ldots, v_{n-1}, v_1)$ be the cycle of order $n/2$ in C_n^3 and let H_3 be the spanning subgraph of G with $E(H_3) = E(C_n) \cup E(C^*)$. Then the size of H_3 is $3n/2$ and each vertex of H_3 has degree 2 or 4. By Theorem 2.3.1, C_n^3 is optimal if n is even.

Case 2. *n is odd.* Let $n = 2\ell + 1$ for some integer $\ell \geq 4$. Then $\lceil m/2 \rceil = \lceil 3n/2 \rceil = 3\ell + 2$. First, suppose that ℓ is even. Let C' be the cycle of order $n - 4$ in G defined by

$$C' = (v_2, v_3, \ldots, v_{\ell-1}, v_{\ell+2}, v_{\ell+1}, v_{\ell+4}, v_{\ell+5}, \ldots, v_{n-1}, v_2)$$

and let C'' be the circuit in G defined by

$$C'' = (v_1, v_3, \ldots, v_{\ell-1}, v_\ell, v_{\ell+2}, v_{\ell+3}, v_{\ell+1}, v_\ell, v_{\ell+3}, v_{\ell+4}, v_{\ell+6}, \ldots, v_{n-1}, v_1).$$

Figure 3.9(a) shows C' and C'' for $n = 9$ and $n = 13$, where the edges of C' are drawn in solid lines and the edges of C'' are drawn in dashed lines. Let H_3 be the subgraph of G induced by $E(C') \cup E(C'')$. Then the size of H_3 is $|E(C')| + |E(C'')| = (n-4) + 7 + (n-5)/2 = 3\ell + 2 = \lceil 3n/2 \rceil$ and each vertex of H_3 has degree 2 or 4.

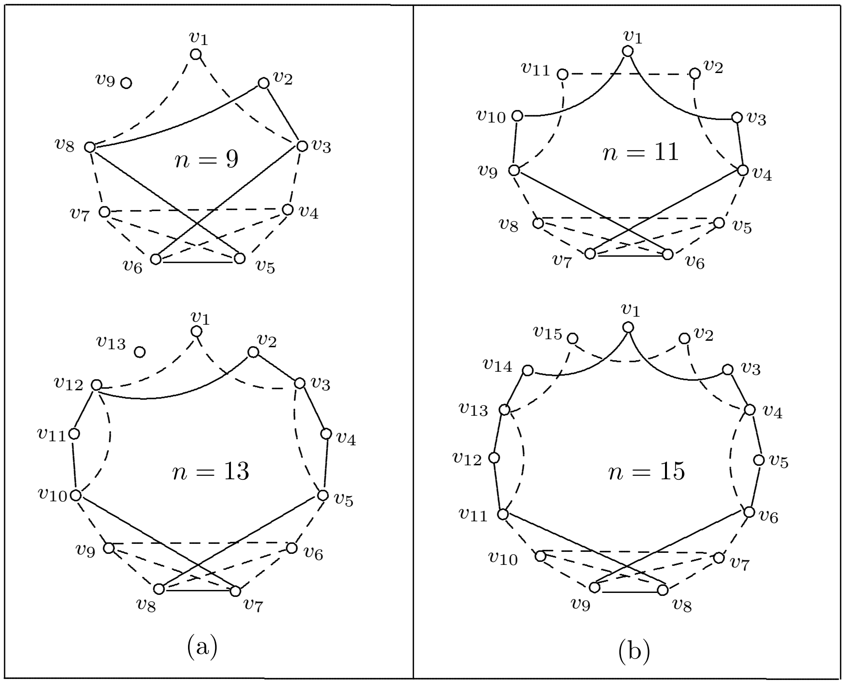


Figure 3.9: Subgraphs C' and C'' in C_n^3 for $n = 9, 11, 13, 15$

Next suppose that ℓ is odd. Let C' be the cycle of order $n-4$ in G defined by

$$C' = (v_1, v_3, v_4, \ldots, v_{\ell-1}, v_{\ell+2}, v_{\ell+1}, v_{\ell+4}, v_{\ell+5}, \ldots, v_{n-1}, v_1)$$

and let C'' be the circuit in G defined by

$$C'' = (v_2, v_4, \ldots, v_{\ell-1}, v_\ell, v_{\ell+2}, v_{\ell+3}, v_{\ell+1}, v_\ell, v_{\ell+3}, v_{\ell+4}, v_{\ell+6}, \ldots, v_n, v_2).$$

Figure 3.9(b) shows C' and C'' for $n = 11$ and $n = 15$, where the edges of C' are drawn in solid lines and the edges of C'' are drawn in dashed lines. Let H_3 be the subgraph of G induced by $E(C') \cup E(C'')$. Then the size of H_3 is $|E(C')| + |E(C'')| = (n-4) + 7 + (n-5)/2 = 3\ell + 2 = \lceil 3n/2 \rceil$ and each vertex of H_3 has degree 2 or 4.

In general, if $k \geq 5$ is odd and $n = 2\ell + 1$, then $\lceil m/2 \rceil = \lceil kn/2 \rceil = k\ell + \lceil k/2 \rceil$. For $k = 5$, let H_5 consists of H_3 and $C_n^{[4]}$. Since each vertex in H_3 and $C_n^{[4]}$ is even, every vertex of H_5 is even and the size of H_5 is $|E(H_3)|+n = (3\ell+2)+(2\ell+1) = 5\ell+3 = 5\ell + \lceil 5/2 \rceil$. More generally then, for an odd integer $k \geq 7$ with $k \leq \lfloor n/2 \rfloor - 3$, the subgraph H_{k+2} consists of H_k and $C_n^{[k+1]}$, where every vertex of H_k and $C_n^{[k+1]}$ is even and the size of H_k is $k\ell + \lceil k/2 \rceil$. Hence, every vertex of H_{k+2} is even and the size of H_{k+2} is $|E(H_k)| + n = (k\ell + \lceil k/2 \rceil) + (2\ell + 1) = (k+2)\ell + \lceil (k+2)/2 \rceil$.

Therefore, C_n^k is optimal for each integer odd integer k with $3 \leq k \leq \lfloor n/2 \rfloor$. ■

3.4 Circulants

We saw that the n-cycle C_n is not optimal for each $n \geq 3$. On the other hand, the kth power C_n^k of the n-cycles is optimal for each $k \geq 2$. The kth power of C_n is a special case of a more general class of graphs. For each integer $n \geq 3$ and $k \geq 1$ distinct integers $n_1, n_2, \ldots, n_k$ where $1 \leq n_1 < n_2 < \ldots < n_k \leq \lfloor n/2 \rfloor$, the *k-circulant* $C_n(n_1, n_2, \ldots, n_k)$ is that graph with n vertices $v_1, v_2, \ldots, v_n$ such that v_i $(1 \leq i \leq k)$ is adjacent to $v_{i \pm n_j (mod\ n)}$ for each j with $1 \leq j \leq k$. The integers n_i $(1 \leq i \leq k)$ are called the *jump sizes* of the circulant. The circulants $C_{10}(1,3)$, $C_{10}(1,5)$ and $C_{12}(1,2,5)$ are shown in Figure 3.10. In particular, $C_n(1) = C_n$ and $C_n(1,2,\ldots,k)$ is the *kth power* C_n^k of C_n. Furthermore, $C_n^k = K_n$ for all $k \geq \left\lfloor \frac{n}{2} \right\rfloor$.

For a circulant $C_n(n_1, n_2, \ldots, n_k)$, if $n_k < n/2$, then $C_n(n_1, n_2, \ldots, n_k)$ is $2k$-regular and so Eulerian, while if $n_k = n/2$, then n is even and $C_n(n_1, n_2, \ldots, n_k)$ is $(2k-1)$-regular. Thus circulants $C_n(n_1, n_2, \ldots, n_k)$ are symmetric classes of regular

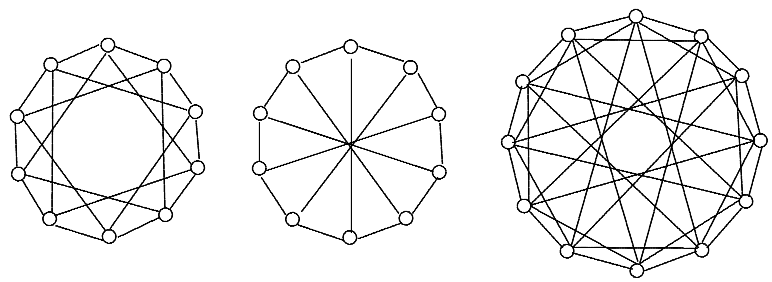


Figure 3.10: The circulants $C_{10}(1,3)$, $C_{10}(1,5)$ and $C_{12}(1,2,5)$

graphs. In this section, we investigate the problem concerning which k-circulants are optimal.

3.4.1 k-Circulants

In this section, we establish some general results on k-circulants.

Theorem 3.4.1 *For integers n, k, t where $n \geq 4$, $k \geq 2$ and $t \geq 0$, let n_1, n_2, $\ldots$, n_{k+2t} be $k+2t$ distinct integers such that $1 \leq n_i \leq n/2$ for $1 \leq i \leq k$ and $1 \leq n_i < n/2$ for $k+1 \leq i \leq k+2t$. If the k-circulant $C_n(n_1, n_2, \ldots, n_k)$ is optimal, then the $(k+2t)$-circulant $C_n(n_1, n_2, \ldots, n_{k+2t})$ is optimal.*

Proof. Let $G_0 = C_n(n_1, n_2, \ldots, n_k)$ and $G = C_n(n_1, n_2, \ldots, n_{k+2t})$. Then G_0 is a subgraph of G. If $t = 0$, then $G_0 = G$ is optimal. Thus, we may assume that $t \geq 1$. Suppose that the size of G_0 is m_0 and the size of G is m. Since G_0 is optimal, it follows by Theorem 2.3.1 that G_0 has an even subgraph H_0 of size $\lceil \frac{m_0}{2} \rceil$. Since $1 \leq n_i < n/2$ for $k+1 \leq i \leq k+2t$, it follows that $m = m_0 + 2tn$. Let H be the even subgraph consisting of the edge-disjoint subgraphs H_0 and

$H_1 = C_n(n_{k+1}, n_{k+2}, \ldots, n_{k+t})$ of G, that is, $E(H) = E(H_0) \cup E(H_1)$. Then the size of $H = \lceil \frac{m_0}{2} \rceil + nt = \lceil \frac{m_0+2nt}{2} \rceil = \lceil \frac{m}{2} \rceil$. Thus G is optimal by Theorem 2.3.1. ■

3.4.2 Eulerian Circulants

We first consider Eulerian k-circulants $C_n(n_1, n_2, \ldots, n_k)$, where then $n_i < n/2$ for each $i = 1, 2, \ldots, k$. With the aid of Theorem 3.4.1, we are able to extend the result on kth powers of cycles in Theorem 3.3.1 to certain Eulerian circulants.

Proposition 3.4.2 *Let $G = C_n(n_1, n_2, \ldots, n_k)$ where $k \geq 2$ and $1 \leq n_1 < n_2 < \ldots < n_k < n/2$. If k is even or k is odd and there exist three distinct elements $n_r, n_s, n_t \in \{n_1, n_2, \ldots, n_k\}$ such that $C_n(n_r, n_s, n_t)$ is optimal, then G is optimal.*

Proof. The graph G is a $2k$-regular graph of order n and size $m = nk$. First, suppose that $k \geq 2$ is even and consider $G_0 = C_n(n_1, n_2)$. Since $1 \leq n_1 < n_2 < n/2$, the subgraph $H = C_n(n_1)$ is an even subgraph of size $n = \lceil \frac{m}{2} \rceil$ in G_0 and so G_0 is optimal by Theorem 2.3.1. Thus G is optimal by Theorem 3.4.1. Next, suppose that $k \geq 3$ is odd and there exist three distinct elements $n_r, n_s, n_t \in \{n_1, n_2, \ldots, n_k\}$ such that $C_n(n_r, n_s, n_t)$ is optimal. Since $1 \leq n_1 < n_2 < \ldots < n_k < n/2$, it follows by Theorem 3.4.1 that G is optimal. ■

In order to determine other Eulerian optimal circulants, we present a lemma.

Lemma 3.4.3 *For each pair a, n of integers, where $2 \leq a < n/2$, if $d = \gcd((a, n)$ and $p = n/d$, then $C_n(a)$ can be decomposed into d p-cycles.*

Proof. Let $C_n = (v_0, v_1, \ldots, v_{n-1}, v_n = v_0)$ where $n \geq 3$. Since $d = \gcd(a, n)$ and $d \mid a$, it follows that a/d is an integer and so $n \mid pa$. Hence $i \equiv i+pa \pmod{n}$ for all integers i. Furthermore, since $p = n/d$ and $d = \gcd(a, n)$, if $n \mid xa$ for any positive integer x, then $p \mid x$. Thus p is the smallest positive integer x such that $i \equiv i + xa \pmod{n}$. Moreover, $p = n/d \geq 3$ as $d \leq a < n/2$. Now for each integer i with $0 \leq i \leq d-1$, define a p-cycle Q_i in $C_n(a)$ as $Q_i = (v_i, v_{i+a}, v_{i+2a}, \ldots, v_{i+pa}, v_i)$. We claim that $V(Q_i) \cap V(Q_j) = \emptyset$ for all pairs i, j with $0 \leq i \neq j \leq d-1$. Assume, to the contrary, that $V(Q_i) \cap V(Q_j) \neq \emptyset$ for some pair i, j, say $i > j$. Then there exists $u \in V(Q_i) \cap V(Q_j)$ such that $u = v_{i+\ell a} = v_{j+ta}$. Hence $i + \ell a \equiv j + ta \pmod{n}$ and so $(i-j) + (\ell - t)a = nq$ for some integer q. Since $d \mid a$ and $d \mid n$, it follows that $d \mid (i-j)$, which is impossible as $1 \leq i - j < d-1$. Therefore, $Q_0, Q_2, \ldots, Q_{d-1}$ are vertex-disjoint (and edge-disjoint) p-cycles in $C_n(a)$. Since $n = dp$ is the size of $C_n(a)$, it follows that $\{Q_0, Q_2, \ldots, Q_{d-1}\}$ is a decomposition of $C_n(a)$ into p-cycles. Figure 3.11 shows such a cycle decomposition of $C_{15}(6)$ into three 5-cycles. ■

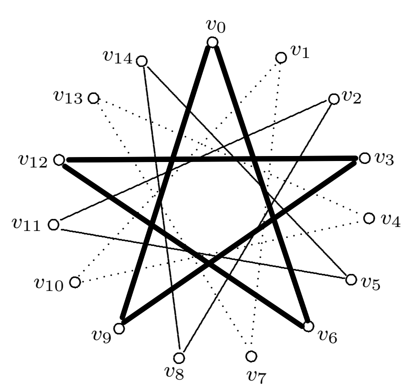


Figure 3.11: A cycle decomposition of $C_{15}(6)$ into three 5-cycles

Proposition 3.4.4 *Let $G = C_n(n_1, n_2, \ldots, n_k)$ where n is even, $k \geq 2$ and $1 \leq n_1 < n_2 < \ldots < n_k < n/2$. If there is $i \in \{1, 2, \ldots, k\}$ such that n_i is even, then G is optimal.*

Proof. By Proposition 3.4.2, we may assume that $k \geq 3$ is odd. Furthermore, by Theorem 3.4.1, it suffices to consider $k = 3$ and $G_0 = C_n(n_1, n_2, n_3)$. Assume, without loss of generality, that n_1 is even and $d = \gcd(n, n_1) = 2t$ for some positive integer t. By Lemma 3.4.3, $C_n(n_1)$ can be decomposed into $2t$ cycles Q_1, Q_2, ..., Q_{2t} of length $n/2t$. Thus G_0 can be decomposed into Q_1, Q_2, ..., Q_{2t}, $C_n(n_2)$, $C_n(n_3)$. The even subgraph H consisting of $C_n(n_2)$ and Q_i $(1 \leq i \leq t)$ has size $n + \frac{n}{2} = \lceil \frac{m}{2} \rceil$. It then follows by Theorem 2.3.1 that G_0 is optimal and so is G by Theorem 3.4.1. ∎

3.4.3 Non-Eulerian Circulants

By Proposition 3.4.2, for all integers n, n_1, n_2 where $n \geq 4$ and $1 \leq n_1 < n_2 < n/2$, the circulant $C_n(n_1, n_2)$ is optimal. This, however, is not the case when $n_2 = n/2$. In general, if $1 \leq n_1 < n_2 < \ldots < n_k = n/2$, where $k \geq 2$, then n is even and $C_n(n_1, n_2, \ldots, n_k)$ is a $(2k-1)$-regular graph of order n and size $m = n(2k-1)/2$. Thus these circulants are not Eulerian. In particular, $C_n(1, n/2)$ is a 3-regular graph of order n and size $m = 3n/2$. If $n = 4$, then $C_4(1, 2) = K_4$; while if $n = 6$, then $C_6(1, 3) = K_{3,3}$. Recall that all optimal complete graphs and complete bipartite graphs have determined in Theorems 2.3.4 and 2.3.5. which we state next.

- The complete graph K_n of order n is optimal if and only if $n \geq 4$.

- For integers r and s with $2 \leq r \leq s$, the complete bipartite graph $K_{r,s}$ is optimal if and only if (i) r and s are both even and $(r, s) \neq (2, 4k+2)$ for any nonnegative integer k or (ii) at least one of r and s is odd and $rs \not\equiv 1, 2 \pmod 4$.

By Theorems 2.3.4 and 2.3.5, the graph $C_4(1, 2)$ is optimal while $C_6(1, 3)$ is not. In fact, more can be said. Also, recall that if G is a connected bipartite graph of size $m \geq 1$ such that $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$, then G is not optimal.

Proposition 3.4.5 *For each even integer $n \geq 4$, the graph $C_n(1, n/2)$ is optimal if and only if $n \not\equiv 6 \pmod 8$. Furthermore, if $n \equiv 6 \pmod 8$, then $EI(G) = \binom{m+1}{2} + 1$ where $m = 3n/2$ is the size of $C_n(1, n/2)$.*

Proof. Since the statement is true when $n = 4$ or $n = 6$, we assume that $n \geq 8$ is even. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Then $C_n(1, n/2)$ is a 3-regular graph of order n and size $m = 3n/2$. First, assume that $n \equiv 6 \pmod 8$ and so $n = 8p + 6$ for some positive integer p. For each i with $1 \leq i \leq n$, the vertex v_i is adjacent to v_{i-1}, v_{i+1} and $v_{i+n/2}$ in $C_n(1, n/2)$, where the subscript of each vertex is expressed as an integer modular n. Since $n/2 = 4p + 3$ is odd, the integers $i - 1, i + 1, i + n/2$ are of the same parity and i and j where $j \in \{i - 1, i + 1, i + n/2\}$ are of the opposite parity. Thus $C_n(1, n/2)$ is a bipartite graph whose partite sets are the same as those of C_n. Since $m = 12p + 9 = 4(3p + 2) + 1$, it follows by Lemma 3.2.1 that $C_n(1, n/2)$ is not optimal.

For the converse, assume that $n \equiv 0, 2, 4 \pmod 8$. For $n \equiv 0 \pmod 8$, let $n = 8p$ for some positive integer p. Then $m = 12p$ and $m/2 = 6p = (6p-2)+2 = 2(3p-1)+2$. The cycle $(v_1, v_2, \ldots, v_{3p}, v_{7p}, v_{7p-1}, \ldots, v_{4p+1}, v_1)$ is a cycle of size $6p$ in $C_n(1, n/2)$. For $n \equiv 2 \pmod 8$, let $n = 8p+2$ for some positive integer p. Then $m = 12p+3$ and $\lceil m/2 \rceil = 6p+2$. The cycle $(v_1, v_2, \ldots, v_{3p+1}, v_{7p+2}, v_{7p+1}, \ldots, v_{4p+2}, v_1)$ is a cycle of size $6p+2$ in $C_n(1, n/2)$. For $n \equiv 4 \pmod 8$, let $n = 8p+4$ for some positive integer p. Then $m = 12p+6$ and $\lceil m/2 \rceil = m/2 = 6p+3 = 3(2p+1)$. Let $d = n/2 = 4p+2$ be the diameter of C_n. The cycle $(v_1, v_2, \ldots, v_{2p+2}, v_{2p+2+d}, v_{2p+2+d+1}, v_{2p+2+2d+1}, v_{2p+2+2d+2}, \ldots, v_{n-2}, v_{n-1}, v_{n-1+d}, v_{n-1+d+1}, v_{n-1+d+2}, v_1)$ is a cycle of size $6p+3$ in $C_n(1, n/2)$, where the subscript of each vertex is expressed as an integer modular n. Figure 3.12 shows such cycles in the circulants $C_n(1, n/2)$ for $n = 12, 20, 28$. Therefore, $C_n(1, n/2)$ is optimal by Theorem 2.3.1 if $n \equiv 0, 2, 4 \pmod 8$.

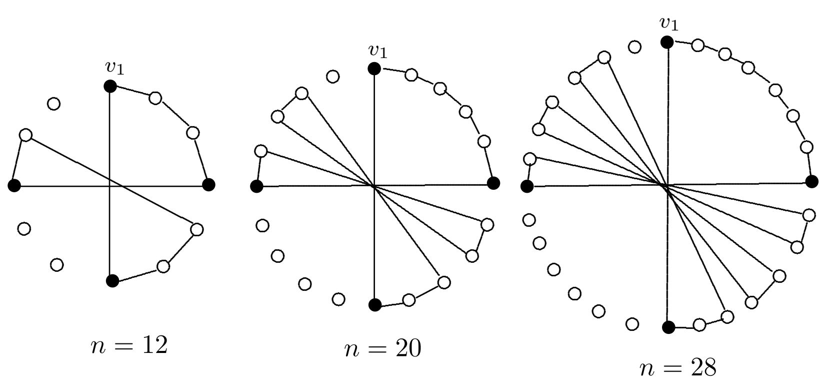


Figure 3.12: Circulants $C_n(1, n/2)$ are optimal $n = 12, 20, 28$

Now, suppose that $n \equiv 6 \pmod 8$. Since $m = 3n/2$ is odd, it suffices to show that G contains an even subgraph of size $\frac{m-1}{2}$. Let $n = 8k+6$ for some integer

$k \geq 0$. Then $\frac{m-1}{2} = 6k + 4$. Note that G contains C_{6k+4} as a subgraph. For example, $(v_1,\ v_2,\ \ldots,\ v_{3k+2},\ v_{7k+5},\ v_{7k+4},\ v_{7k+3},\ \ldots,\ v_{4k+4},\ v_1)$ is such a cycle of order $6k + 4$, where each subscript is expressed as an integer module n. Thus $EI(G) = \binom{m+1}{2} + 1$. ■

By Proposition 3.4.2, if $k \geq 2$ is even, then all Eulerian k-circulants $C_n(n_1, n_2, \ldots, n_k)$ are optimal. For non-Eulerian k-circulants when k is even, we have the following, which is a consequence of Theorem 3.4.1 and Proposition 3.4.5.

Corollary 3.4.6 *If n and k are even integers with $n, k \geq 2$ and $n \not\equiv 6 \pmod 8$, then the graph $C_n(1, n_2, n_3, \ldots, n_k)$ is optimal for all integers $n_2, n_3, \ldots, n_k$ with $2 \leq n_2 < n_3 < \ldots < n_k = n/2$.*

3.4.4 Circulants $C_n(n_1, n_2, n_3)$

We saw that 3-circulants play an important role in determining which k-circulants are optimal when $k \geq 3$ is odd. In this section we investigate 3-circulants $C_n(n_1, n_2, n_3)$ for small values of n_1, n_2, n_3, namely at least two of n_1, n_2, n_3 are $1, 2$ or 3. First, we establish some definitions and notation. Let $G = C_n(n_1, n_2, n_3)$ where $C_n = (v_1, v_2, \ldots, v_n = v_0, v_{n+1} = v_1)$. In what follows, all subscripts are expressed as integers modulo n. For a positive integer b, a *block B of order b* in C_n is an ordered set of b consecutive vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+b}$ of C_n for some i with $1 \leq i \leq n$ and is denoted by $B = [v_{i+1}, v_{i+b}]$ or $B = (v_{i+1}, v_{i+2}, \ldots, v_{i+b})$. Two blocks B and B' of order b and b', respectively, are *consecutive* in C_n if $B = [v_{i+1}, v_{i+b}]$ and $B' = [v_{j+1}, v_{j+b'}]$ for some integers i and j where $j \in \{i + b - 1, i + b\}$ and

$1 \leq i, j \leq n$. Thus two consecutive blocks of C_n have at most one vertex in common. If $B_1, B_2, \ldots, B_s$ ($s \geq 2$) are pairwise disjoint consecutive blocks of C_n such that $B_1 \cup B_2 \cup \cdots \cup B_s = V(C_n)$, then $\{B_1, B_2, \ldots, B_s\}$ is referred to as a *partition* of C_n. For a block B of C_n, let $G[B]$ denote the subgraph of G induced by the vertices in B.

Circulants $C_n(1, 2, n_3)$

We first show that all 3-circulants $C_n(1, 2, n_3)$ are optimal for each integer $n_3 \geq 3$.

Theorem 3.4.7 *For each integer $n \geq 8$, the graph $C_n(1, 2, n_3)$ is optimal for all integers n_3 with $3 \leq n_3 \leq n/2$.*

Proof. Let $G = C_n(1, 2, n_3)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. By Proposition 3.4.4, if $n_3 < n/2$, then G is optimal. Thus, we may assume that $n_3 = n/2$ and so $n \geq 8$ is even. Let $n = 2k$ for some integer $k \geq 4$. The size of G is $m = 2n + \frac{n}{2} = 5k$. The subgraph $C_n(1, 2)$ of G contains k edge-disjoint triangles $T_1, T_2, \ldots, T_k$ defined by

$$T_i = (v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i-1}) \text{ for } 1 \leq i \leq k, \tag{3.10}$$

where each subscript is expressed as an integer modulo n. We consider three cases.

Case 1. $\lceil \frac{m}{2} \rceil \equiv 0 \pmod 3$. Then $\lceil \frac{m}{2} \rceil = 3q$ for some integer $q \geq 4$. If k is even, then $\lceil \frac{m}{2} \rceil = \frac{5k}{2} = 3q$ and so $q = \frac{5k}{6} < k$. If k is odd, then $\lceil \frac{m}{2} \rceil = \frac{5k+1}{2} = 3q$ and so $q = \frac{5k+1}{6} < k$. The even subgraph consisting of q edge-disjoint triangles $T_1, T_2, \ldots, T_q$ as defined in (3.10) has size $\lceil \frac{m}{2} \rceil = 3q$. Thus G is optimal.

Case 2. $\lceil \frac{m}{2} \rceil \equiv 1 \pmod 3$. Then $\lceil \frac{m}{2} \rceil = 3q+1 = 3(q-1)+4$ for some integer $q \geq 4$. If k is even, then $\lceil \frac{m}{2} \rceil = \frac{5k}{2} = 3q+1$ and so $q = \frac{5k-2}{6} < k$. If k is odd, then $\lceil \frac{m}{2} \rceil = \frac{5k+1}{2} = 3q+1$ and so $q = \frac{5k-1}{6} < k$. Since $q \leq k-1$, it follows that $2q-1 \leq 2k-3$ and so $2q+2 \leq 2k$. Thus the 4-cycle $Q = (v_{2q-1}, v_{2q+1}, v_{2q+2}, v_{2q}, v_{2q-1})$ and the first $q-1$ triangles $T_1, T_2, \ldots, T_{q-1} = (v_{2q-3}, v_{2q-2}, v_{2q-1}, v_{2q-3})$ as defined in (3.10) are edge-disjoint. The even subgraph consisting of $T_1, T_2, \ldots, T_{q-1}$ and the 4-cycle Q has size $\lceil \frac{m}{2} \rceil = 3q+1$ and so G is optimal.

Case 3. $\lceil \frac{m}{2} \rceil \equiv 2 \pmod 3$. Then $\lceil \frac{m}{2} \rceil = 3q+2$ for some integer $q \geq 3$. If $k = 4$, then $G = C_8(1,2,4)$ and $\lceil \frac{m}{2} \rceil = 10$. The even subgraph of G consisting of T_1, T_2 and the 4-cycle $(v_5, v_7, v_8, v_6, v_5)$ has size 10. If $k = 5$, then $G = C_{10}(1,2,5)$ and $\lceil \frac{m}{2} \rceil = 13$. The even subgraph of G consisting of T_1, T_2, T_3 and the 4-cycle $(v_7, v_9, v_{10}, v_8, v_7)$ has size 13. We now assume that $k \geq 6$ and so $\lceil \frac{m}{2} \rceil = 3(q-2)+8$. If k is even, then $\lceil \frac{m}{2} \rceil = \frac{5k}{2} = 3q+2$ and so $q = \frac{5k-4}{6}$. If k is odd, then $\lceil \frac{m}{2} \rceil = \frac{5k+1}{2} = 3q+2$ and so $q = \frac{5k-3}{6}$. In either case, $2q+3 \leq 2k$. Then the two edge-disjoint 4-cycles $Q_1 = (v_{2q-3}, v_{2q-1}, v_{2q}, v_{2q-2}, v_{2q-3})$ and $Q_2 = (v_{2q}, v_{2q+2}, v_{2q+3}, v_{2q+1}, v_{2q})$ and the first $q-2$ triangles $T_1, T_2, \ldots, T_{q-2} = (v_{2q-5}, v_{2q-4}, v_{2q-3}, v_{2q-5})$ as defined in (3.10) are edge-disjoint. The even subgraph consisting of $T_1, T_2, \ldots, T_{q-2}$ and the 4-cycles Q_1 and Q_2 has size $\lceil \frac{m}{2} \rceil = 3q+2$. Therefore, G is optimal. ■

The following is a consequence of Theorem 3.4.1, Proposition 3.4.2 and Theorem 3.4.7.

Corollary 3.4.8 *For integers n and k where $n \geq 8$ and $k \geq 2$, $C_n(1,2,n_3,n_4,\ldots,n_k)$ is optimal for all integers $n_3, n_4, \ldots, n_k$ with $3 \leq n_3 < n_4 < \ldots < n_k \leq n/2$.*

Circulants $C_n(1, 3, n_3)$

We first consider Eulerian circulants $C_n(1, 3, n_3)$ where then $4 \leq n_3 < n/2$.

Theorem 3.4.9 *For each integer $n \geq 8$, the graph $C_n(1, 3, n_3)$ where $4 \leq n_3 < n/2$ is optimal if and only if either $n \not\equiv 2 \pmod 4$ or n_3 is even. Furthermore, if $C_n(1, 3, n_3)$ is not optimal, then*

$$EI(C_n(1, 3, n_3)) = \binom{3n+1}{2} + 1.$$

Proof. Let $G = C_n(1, 3, n_3)$ where $4 \leq n_3 < n/2$ and $C_n = (v_1, v_2, \ldots, v_n = v_0, v_{n+1} = v_1)$. The size of G is $m = 3n$. First, suppose that $n \equiv 2 \pmod 4$ and n_3 is odd. Then G is a bipartite graph of size $m \equiv 2 \pmod 4$. It then follows by Lemma 3.2.1 that G is not optimal and so $EI(G) \geq \binom{3n+1}{2} + 1$. To show that $EI(G) \leq \binom{3n+1}{2} + 1$, it suffices to show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil + 1$ (since m is even). Let $n = 4k + 2$ for some integer $k \geq 2$ and so $\lceil \frac{m}{2} \rceil + 1 = 6k + 4 = (4k + 2) + (2k + 2)$.

- First, suppose that k is even, say $k = 2s$ for some positive integer s Then $2k + 2 = 4s + 2 = 4(s - 1) + 6$ and $n = 8s + 2$. Partition the cycle C_n into s consecutive blocks $B_1, B_2, \ldots, B_s$, where each B_i has order 8 $(1 \leq i \leq s - 1)$ and B_s has size 10. For each i with $1 \leq i \leq s - 1$, let Q_i be a 4-cycle in $G[B_i]$ and let Q_s be a 6-cycle in $G[B_s]$, each of which uses only edges in the subgraph $C_n(1, 3)$ of G. Hence all subgraphs Q_i $(1 \leq i \leq s)$ and $C_n(n_3)$ are pairwise edge-disjoint. Thus the subgraph consisting of Q_i $(1 \leq i \leq s)$ and $C_n(n_3)$ is even and has size $(4k + 2) + (2k + 2) = \lceil \frac{m}{2} \rceil + 1$ in G.

- Next, suppose that k is odd, say $k = 2s + 1$ for some positive integer s. Then $2k + 2 = 4s + 4 = 4(s + 1)$ and $n = 8s + 6$. Partition the cycle C_n into $s + 1$ consecutive blocks $B_1, B_2, \ldots, B_{s+1}$ where each B_i has order 8 $(1 \leq i \leq s)$ and B_{s+1} has size 6. For each i with $1 \leq i \leq s + 1$, let Q_i be a 4-cycle in $G[B_i]$ using only edges in the subgraph $C_n(1, 3)$ of G. Then all subgraphs Q_i $(1 \leq i \leq s + 1)$ and $C_n(n_3)$ are pairwise edge-disjoint and the subgraph consisting of Q_i $(1 \leq i \leq s + 1)$ and $C_n(n_3)$ is even and has size $(4k + 2) + (2k + 2) = \lceil \frac{m}{2} \rceil + 1$ in G.

For the converse, assume that if $n \not\equiv 2 \pmod 4$ or n_3 is even. We show that G is optimal. By Theorem 2.3.1, it suffices to show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil$. Suppose that $n \equiv i \pmod 4$ for $i = 0, 1, 2, 3$. We consider these four cases.

Case 1. $n \equiv 0 \pmod 4$. Let $n = 4k$ for some integer $k \geq 2$. Then $m = 12k$ and $\lceil \frac{m}{2} \rceil = 6k$. Consider the $(2k)$-cycle C defined by

$$C = (v_0, v_1, v_4, v_5, v_8, \ldots, v_{4i}, v_{4i+1}, v_{4i+4}, \ldots, \\ v_{4(k-1)}, v_{4(k-1)+1}, v_{4(k-1)+4} = v_0, v_1),$$

where the distances between two consecutive vertices of C alternate 1 and 3 in G. Note that $C_n(n_3)$ and C are edge-disjoint. Hence the subgraph consisting of $C_n(n_3)$ and C is even and size $n + 2k = 6k = \lceil \frac{m}{2} \rceil$.

Case 2. $n \equiv 1 \pmod 4$. Let $n = 4k + 1$ for some integer $k \geq 2$. Then $m = 12k + 3$ and $\lceil \frac{m}{2} \rceil = 6k + 2 = (4k + 1) + (2k + 1)$. Consider the $(2k + 1)$-cycle defined by

$$C = (v_0, v_1, v_2, \ldots, v_{k+1}, v_{(k+1)+3}, v_{(k+1)+6}, \ldots,$$
$$v_{(k+1)+3i}, v_{(k+1)+3(i+1)}, \ldots, v_{(k+1)+3k} = v_0).$$

Then $C_n(n_3)$ and C are edge-disjoint and the subgraph consisting of $C_n(n_3)$ and C is even and has size $n + (2k+1) = \lceil \frac{m}{2} \rceil$.

Case 3. $n \equiv 2 \pmod 4$. Then n_3 is even and so G is optimal by Proposition 3.4.4.

Case 4. $n \equiv 3 \pmod 4$. Let $n = 4k+3$ for some integer $k \geq 2$. Then $m = 12k+9$ and $\lceil \frac{m}{2} \rceil = 6k+5 = (4k+3) + (2k+2)$.

- First, assume that k is odd and so $k = 2t+1$ for some integer $t \geq 1$. For each i with $1 \leq i \leq t+1$, let Q_i be a 4-cycle in G defined by

  $$Q_i = (v_{3(i-1)+1}, v_{3(i-1)+2}, v_{3(i-1)+3}, v_{3(i-1)+4}, v_{3(i-1)+1}). \tag{3.11}$$

  In particular, $Q_1 = (v_1, v_2, v_3, v_4, v_1)$ and

  $$Q_{t+1} = (v_{3t+1}, v_{3t+2}, v_{3t+3}, v_{3t+4}, v_{3t+1}).$$

  Note that all 4-cycles Q_i $(1 \leq i \leq t+1)$ and $C_n(n_3)$ are pairwise edge-disjoint. Hence the subgraph consisting of $C_n(n_3)$ and Q_i $(1 \leq i \leq t+1)$ is even and has size $n + 4(t+1) = n + 2k + 2 = \lceil \frac{m}{2} \rceil$.

- Next, assume that k is even and so $k = 2t$ for some integer $t \geq 1$. For each i with $1 \leq i \leq t$, let Q_i be a 4-cycle in G described in (3.11) and let Q be a 6-cycle defined by

$$Q = (v_{3t+1}, v_{3t+2}, v_{3t+3}, v_{3t+6}, v_{3t+5}, v_{3t+4}, v_{3t+1}).$$

Then all 4-cycles Q_i $(1 \leq i \leq t)$, Q and $C_n(n_3)$ are pairwise edge-disjoint. Hence the subgraph consisting of $C_n(n_3)$, Q_i $(1 \leq i \leq t)$ and Q is even and size $n + 4(t-1) + 6 = n + 2k + 2 = \lceil \frac{m}{2} \rceil$. ■

We now consider non-Eulerian circulants $C_n(1, 3, n/2)$ where then $n \geq 8$ is even.

Theorem 3.4.10 *For each even integer $n \geq 8$, the graph $C_n(1, 3, n/2)$ is optimal if and only if $n \not\equiv 2 \pmod 8$. Furthermore, if $n \equiv 2 \pmod 8$, then*

$$EI(C_n(1, 3, n/2)) = \binom{m+1}{2} + 1$$

where $m = \frac{5n}{2}$ is the size of $C_n(1, 3, n/2)$.

Proof. Let $n = 2k$ for some integer $k \geq 4$ and let $G = C_{2k}(1, 3, k)$, where

$$C_{2k} = (v_1, v_2, \ldots, v_{2k}, v_{2k+1} = v_1).$$

We consider two cases, according to whether k is odd or k is even.

Case 1. *k is odd.* We claim that G is a bipartite graph with partite sets

$$\begin{aligned} U &= \{v_i : i \text{ is odd and } 1 \leq i \leq 2k-1\} \\ W &= \{v_i : i \text{ is even and } 2 \leq i \leq 2k\}. \end{aligned}$$

To see this, let $e \in E(G)$ be any edge of G and we show that e joins a vertex in U and a vertex in W. Assume, without loss of generality, that $e = v_1 v_t$ where then

$v_1 \in U$ and $t = 2, 3, \ldots, 2k$. It then follows by the defining property of G that either $t \in \{2, 2k\}$, or $t \equiv 1 \pm 3 \pmod{2k}$ or $t = 1 + k$. In each case, t is even and so $v_t \in W$. Therefore, G is a bipartite graph with partite sets U and W, as claimed.

First, we define two subgraphs $C_6 \star C_4$ and C_4 in G in a block of size 8 and size 4, respectively.

(i) For the block $[v_{i+1}, v_{i+8}] = (v_{i+1}, v_{i+2}, \cdots, v_{i+8})$ of order 8 on C_{2k}, let $C_6 \star C_4$ denote the subgraph of $G[v_{i+1}, v_{i+8}]$ that consists of the 6-cycle $(v_{i+1},\ v_{i+2},\ v_{i+5},\ v_{i+8},\ v_{i+7},\ v_{i+4},\ v_{i+1})$ and the 4-cycle $C_4 = (v_{i+3},\ v_{i+4},\ v_{i+5},\ v_{i+6},\ v_{i+3})$. The graph $C_6 \star C_4$ is shown in Figure 3.13(a) where the edges of C_4 are drawn in dashed lines. Hence $C_6 \star C_4$ consists of two edge-disjoint cycles C_6 and C_4 and the size of $C_6 \star C_4$ is 10.

(ii) For the block $[v_{i+1}, v_{i+4}] = (v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$ of order 4 on C_{2k}, let $C_4 = (v_{i+1},\ v_{i+2},\ v_{i+3},\ v_{i+4},\ v_{i+1})$ be a 4-cycle in $G[v_{i+1}, v_{i+4}]$. Note that if B is a block of order $4 + 3q$ for some nonnegative integer q, then the subgraph $G[B]$ induced by the vertices of B contains $q + 1$ edge-disjoint copies of the 4-cycle C_4.

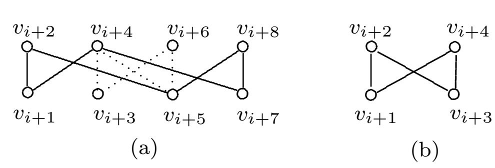


Figure 3.13: The subgraphs $C_6 \star C_4$ and C_4

Since k is odd, either $k \equiv 1 \pmod 4$ or $k \equiv 3 \pmod 4$. We consider these two subcases.

Subcase 1.1. $k \equiv 1 \pmod 4$. Since $m = 5k \equiv 1 \pmod 4$, it follows by Lemma 3.2.1 that G is not optimal. Thus $EI(G) \geq \binom{m+1}{2} + 1 = \binom{5k+1}{2} + 1$. Next, we show that $EI(G) \leq \binom{m+1}{2} + 1$. Since m is odd, it suffices to show G contains an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$. Let $k = 4s + 1$ for some positive integer s and so $n = 2k = 8(s-1) + 10$. Then $\frac{m-1}{2} = 10s + 2$. Consider the s consecutive blocks $B_1, B_2, \ldots, B_s$ of C_n defined by $B_i = [v_{7(i-1)+1}, v_{7i+1}]$ for $1 \leq i \leq s-1$ and $s \geq 2$ and $B_s = [v_{7s-6}, v_n]$ of C_{2k}. Then the order of B_i is 8 for $1 \leq i \leq s-1$ and the order of B_s is at least 10. For each i with $1 \leq i \leq s-1$, let $H_i = C_6 \star C_4$ be the subgraph of $G[B_i]$ as defined in (i). Since the order of B_s is greater than 10, there are three edge-disjoint copies of C_4 in $G[B_s]$, which we denoted by Q_1, Q_2 and Q_3 by (ii). Then the even subgraph H consisting of H_i $(1 \leq i \leq s-1)$, Q_1, Q_2 and Q_3 has size $10(s-1) + 12 = 10s + 2$.

Subcase 1.2. $k \equiv 3 \pmod 4$. Let $k = 4s + 3$ for some positive integer s and so $n = 2k = 8s + 6$ and $m = 5k = 20s + 15$. We show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil = 10s + 8$. Consider the $s + 1$ consecutive blocks $B_1, B_2, \ldots, B_{s+1}$ where $B_i = [v_{7(i-1)+1}, v_{7i+1}]$ for $1 \leq i \leq s$ and $B_{s+1} = [v_{7s+1}, v_n]$ of C_{2k} Then the order of B_i is 8 for $1 \leq i \leq s$ and the order of B_{s+1} is at least 7. For each i with $1 \leq i \leq s$, let $H_i = C_6 \star C_4$ be the subgraph of $G[B_i]$ as defined in (i). Since the order of B_{s+1} is at least 7, there are two edge-disjoint copies of C_4 in $G[B_{s+1}]$, which we denoted by Q_1 and Q_2 by (ii). Then the even subgraph H consisting of H_i $(1 \leq i \leq s)$,Q_1 and Q_2 has size $10s + 8$.

Case 2. *k is even.* First, we define eight subgraphs A^*, B^*, C^*, D^*, E^*, F^*, G^*, H^* in G. To simplify notation, let $B = (u_1, u_2, \ldots, u_b) = (v_{i+1}, v_{i+2}, \cdots, v_{i+b})$ be a block of order $b \geq 4$ in C_n as shown in Figure 3.14.

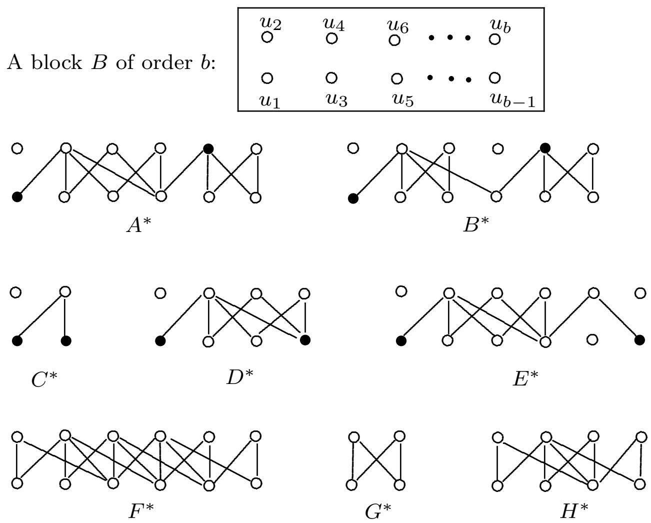


Figure 3.14: The eight subgraphs

- In a block B of order 12, let A^* be the subgraph of $G[B]$ consisting of

$$C_6 = (u_3,\ u_4,\ u_5,\ u_8,\ u_7,\ u_6,\ u_3),$$

$$C_4 = (u_9, u_{10}, u_{11}, u_{12}, u_9)$$

and

$$P_4 = (u_1, u_4, u_7, u_{10}).$$

That is, $E(A^*) = E(C_6) \cup E(C_4) \cup E(P_4)$. There are two odd vertices in A^*, namely u_1 and u_{10}. The vertex u_1 is called the initial vertex of A^* and the vertex u_{10} is called the terminal vertex of A^*. Then the size of A^* is 13.

- In a block B of order 12, let B^* be the subgraph of $G[B]$ consisting of

$$C_4{=}(u_3,\ u_4,\ u_5,\ u_6,\ u_3),$$

$$C_4 = (u_9, u_{10}, u_{11}, u_{12}, u_9)$$

  and $P_4 = (u_1, u_4, u_7, u_{10})$. The two odd vertices in B^* are u_1 and u_{10}. The vertex u_1 is called the initial vertex of B^* and the vertex u_{10} is called the terminal vertex of B^*. Then the size of B^* is 11.

- In a block B of order 4, let $C^* = P_3 = (u_1, u_4, u_3)$ The vertex u_1 is called the initial vertex of C^* and the vertex u_3 is called the terminal vertex of C^*. Then the size of C^* is 2.

- In a block B of order 8, let D^* be the subgraph of $G[B]$ consisting of

$$C_6{=}(u_3,\ u_4,\ u_5,\ u_8,\ u_7,\ u_6,\ u_3) \text{ and } P_3 = (u_1, u_4, u_7).$$

  The two odd vertices in D^* are u_1 and u_7. The vertex u_1 is called the initial vertex of D^* and the vertex u_7 is called the terminal vertex of D^*. Then the size of D^* is 8.

- In a block B of order 12, let E^* be the subgraph of $G[B]$ consisting of

$$C_6{=}(u_3,\ u_4,\ u_5,\ u_8,\ u_7,\ u_6,\ u_3) \text{ and } P_5 = (u_1, u_4, u_7, u_{10}, u_{11}).$$

  The two odd vertices in E^* are u_1 and u_{11}. The vertex u_1 is called the initial vertex of E^* and the vertex u_{11} is called the terminal vertex of E^*. Then the size of E^* is 10.

- In a block B of order 12, let F^* be the subgraph of $G[B]$ consisting of

$$C_4 = (u_3,\ u_4,\ u_5,\ u_6,\ u_3),$$

$$C_4 = (u_7, u_8, u_9, u_{10}, u_7) \text{ and}$$

$$C_{10} = (u_1, u_2, u_5, u_8, u_{11}, u_{12}, u_9, u_6, u_7, u_4, u_1).$$

  Then F^* is an even subgraph of size 18.

- In a block B of order 4, let $G^* = C_4 = (u_1, u_2, u_3, u_4, u_1)$.

- In a block B of order 8, let H^* be the subgraph of $G[B]$ consisting of

$$C_6 = (u_1,\ u_2,\ u_5,\ u_8,\ u_7,\ u_4,\ u_1) \text{ and } C_4 = (u_3, u_4, u_5, u_6, u_3).$$

  Then H^* is an even subgraph of size 10.

These eight subgraphs are shown in Figure 3.14 where the odd vertices in each subgraph are drawn in solid vertices. Note that the subgraphs G^* and H^* are also shown in Figure 3.13 and we include these two subgraphs in Figure 3.14 for completion.

Since either $k \equiv 0 \pmod 4$ or $k \equiv 2 \pmod 4$, we consider these two subcases.

Subcase 2.1. $k \equiv 0 \pmod 4$. Let $k = 4s$ for some positive integer s and so $n = 2k = 8s$ and $m = 5k = 20s$. We show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil = 10s$. Partition C_n into s blocks $B_1, B_2, \ldots, B_s$, each of which has order 8. For each i with $1 \leq i \leq s$, let $H_i = H^*$ be the subgraph of $G[B_i]$, each of which has size 10. Then the even subgraph consisting of H_i ($1 \le i \le s$) has size $10s$.

Subcase 2.2. $k \equiv 2 \pmod 4$. Let $k = 4s + 2$ for some positive integer s and so $n = 2k = 8s + 4$ and $m = 5k = 20s + 10$. Suppose that $s \equiv i \pmod 3$ for $i = 0, 1, 2$, where then $s = 3t + i$ for $i = 0, 1, 2$ for some nonnegative integer t. First, suppose that $t = 0$. Then $i = 1, 2$.

- If $i = 1$, then $s = 1$ and so $n = 12$ and $m = 30$. Hence $\lceil \frac{m}{2} \rceil = 15$. Consider the block $B = [v_7, v_2]$ and let H^* be the subgraph of $G[B]$ of size 10 in B, where H^* is shown in Figure 3.14. Let H be the subgraph consisting of $C_5 = (v_1, v_4, v_5, v_6, v_7, v_1)$ and H^*. Then H is an even subgraph of G and has size 15, as shown in Figure 3.15(a).

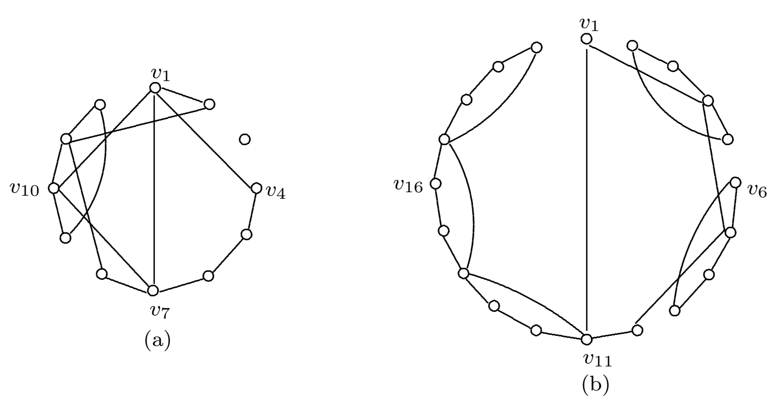


Figure 3.15: Two subgraphs in the proof of Subcase 2.2

- If $i = 2$, then $s = 2$ and so $n = 20$ and $m = 50$. Hence $\lceil \frac{m}{2} \rceil = 25$. Let $B_1 = [v_2, v_5]$, $B_2 = [v_6, v_9]$, $B_3 = [v_{11}, v_{14}]$, $B_4 = [v_{14}, v_{17}]$ and $B_5 = [v_{17}, v_{20}]$. For each i with $1 \le i \le 5$, let $H_i = C_4$ in $G[B_i]$. The even subgraph H of

G consists of H_i $(1 \le i \le 5)$ and $C_5 = (v_1, v_4, v_7, v_{10}, v_{11}, v_1)$ has size 25, as shown Figure 3.15(b).

Thus, we may assume that $t \ge 1$. We consider three subcases.

Subcase 2.2.1. $s = 3t$ *for some integer* $t \ge 1$. In terms of t then, $k = 12t + 2$, $n = 2k = 24t + 4$ and $\lceil \frac{m}{2} \rceil = 30t + 5$. Partition C_n into $2t + 1$ consecutive blocks $B_1, B_2, \ldots, B_{2t+1}$, where $|B_i| = 12$ for $1 \le i \le 2t + 1$ and $i \ne t + 1$ and $|B_{t+1}| = 4$. For each i with $1 \le i \le 2t + 1$, let H_i be the subgraph of $G[B_i]$ defined by

$$H_i \cong \begin{cases} A^* & \text{if } i = 1 \\ B^* & \text{if } 2 \le i \le t \\ C^* & \text{if } i = t + 1 \\ F^* & \text{if } t + 2 \le i \le 2t + 1. \end{cases}$$

Let H be the subgraph consisting of $H_1, H_2, \ldots, H_{2t+1}$ by joining (1) the terminal vertex of H_i to the initial vertex of H_{i+1} for $1 \le i \le t$ and (2) the terminal vertex of H_{t+1} to the initial vertex of H_1. Then H is an even subgraph of G and has size $30t + 5$.

Subcase 2.2.2. $s = 3t+1$ *for some integer* $t \ge 1$. In terms of t then, $k = 12t+6$, $n = 2k = 24t+12$ and $\lceil \frac{m}{2} \rceil = 30t+15$. Partition C_n into $2t+2$ consecutive blocks $B_1, B_2, \ldots, B_{2t+2}$, where $|B_i| = 12$ for $1 \le i \le t$, $|B_{t+1}| = 8$, $|B_i| = 12$ for $t + 2 \le i \le 2t + 1$ and $|B_{2t+2}| = 4$. For each i with $1 \le i \le 2t + 2$, let H_i be the subgraph of $G[B_i]$ defined by

$$H_i \cong \begin{cases} A^* & \text{if } i = 1 \\ B^* & \text{if } 2 \le i \le t \\ D^* & \text{if } i = t+1 \\ F^* & \text{if } t+2 \le i \le 2t+1 \\ G^* & \text{if } t = 2t+2. \end{cases}$$

Let H be the subgraph consisting of $H_1, H_2, \ldots, H_{2t+2}$ by joining (1) the terminal vertex of H_i to the initial vertex of H_{i+1} for $1 \le i \le t$ and (2) the terminal vertex of H_{t+1} to the initial vertex of H_1. Then H is an even subgraph of G and has size $30t + 15$.

Subcase 2.2.3. $s = 3t+2$ *for some integer* $t \ge 1$. In terms of t then, $k = 12t+10$, $n = 2k = 24t+20$ and $\lceil \frac{m}{2} \rceil = 30t+25$. Partition C_n into $2t+1$ consecutive blocks $B_1, B_2, \ldots, B_{2t+1}$, where $|B_i| = 12$ for $1 \le i \le 2t+1$ and $|B_{2t+2}| = 8$. For each i with $1 \le i \le 2t+2$, let H_i be the subgraph of $G[B_i]$ defined by

$$H_i \cong \begin{cases} A^* & \text{if } i = 1, 2 \\ B^* & \text{if } 3 \le i \le t \\ E^* & \text{if } i = t+1 \\ F^* & \text{if } t+2 \le i \le 2t+1 \\ H^* & \text{if } t = 2t+2 \end{cases}$$

Let H be the subgraph consisting of $H_1, H_2, \ldots, H_{2t+2}$ by joining (1) the terminal vertex of H_i to the initial vertex of H_{i+1} for $1 \le i \le t$ and (2) the terminal vertex of H_{t+1} to the initial vertex of H_1. Then H is an even subgraph of G and has size $30t + 25$. ■

The following is a consequence of Theorem 3.4.1, Proposition 3.4.2 and Theorems 3.4.9 and 3.4.10.

Corollary 3.4.11 *Let $G = C_n(1, 3, n_3, n_4, \ldots, n_k)$ where*

$$4 \leq n_3 < n_4 < \cdots < n_k \leq n/2 \text{ and } k \geq 3.$$

- *For $n_k < n/2$, if $n \not\equiv 2 \pmod 4$ or n_i is even for some i with $3 \leq i \leq k$, then G is optimal.*
- *For $n_k = n/2$, if $n \not\equiv 2 \pmod 8$, then G is optimal.*

Circulants $C_n(2, 3, n_3)$

As with 3-circulants $C_n(1, 2, n_3)$, all 3-circulants $C_n(2, 3, n_3)$ are optimal for each $n_3 \geq 4$.

Theorem 3.4.12 *For each integer $n \geq 8$, the graph $C_n(2, 3, n_3)$ is optimal for all integers n_3 with $4 \leq n_3 \leq n/2$.*

Proof. Let $G = C_n(2, 3, n_3)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. We consider two cases, according to whether $n_3 < n/2$ or $n_3 = n/2$.

Case 1. $4 \leq n_3 < n/2$. If n is even, then G is optimal by Proposition 3.4.4. Thus, we may assume that n is odd. Let $n = 2t + 1$ for some integer $t \geq 4$. Then the size of G is $m = 3n = 6t + 3$ and so $\lceil \frac{m}{2} \rceil = 3t + 2 = (2t + 1) + (t + 1)$. First, we make an observation. Let $B = (u_1, u_2, \ldots, u_b)$ be a block of order b (that is, $u_1, u_2, \ldots, u_b$ are b consecutive vertices in C_n), which is shown in Figure 3.14.

(i) If $b \geq 8$, then $G[B]$ contains a contains a k-cycle for $k = 4, 5$ using only edges in $C_n(2,3)$. For example,

$$C_4 = (u_1, u_3, u_6, u_4, u_1) \text{ and } C_5 = (u_1, u_3, u_5, u_7, u_4, u_1).$$

(ii) if $b \geq 10$, then $G[B]$ contains a k-cycle for $k = 6, 7$ or $2C_4$ using only edges in $C_n(2,3)$. For example,

$$C_6 = (u_1, u_3, u_6, u_9, u_7, u_4, u_1),\ C_7 = (u_1, u_3, u_5, u_7, u_9, u_6, u_4, u_1)$$

and $2C_4$ consists of

$$(u_1, u_3, u_6, u_4, u_1) \text{ and } (u_5, u_7, u_{10}, u_8, u_5).$$

Let $t = 4s + r$ for some positive integer s where $r = 0, 1, 2, 3$. Then $n = 8s + 2r + 1$ and $t + 1 = 4s + r + 1$. Partition C_n into s consecutive blocks $B_0, B_1, \ldots, B_{s-1}$ defined by $B_i = [v_{8i+1}, v_{8(i+1)}]$ for $0 \leq i \leq s-2$ and $s \geq 2$ and $B_{s-1} = [v_{8(s-1)+1}, v_n]$. By (i), the subgraph $G[B_i]$ induced by B_i contains C_4 as a subgraph using only edges in $C_n(2,3)$ and if $r = 0$, then $|B_{s-1}| = 9$ and so $G[B_{s-1}]$ contains C_5 as a subgraph using only edges in $C_n(2,3)$. By (ii), if $r \neq 0$, then $G[B_{s-1}]$ contains C_6, C_7 or $2C_4$ as a subgraph using only edges in $C_n(2,3)$. We now define a subgraph H_i of $G[B_i]$ using only edges in $C_n(2,3)$ define as follows: For each i with $0 \leq i \leq s-2$, let $H_i \cong C_4$ (when $s \geq 2$) and

$$H_{s-1} \cong \begin{cases} C_5 & \text{if } r = 0 \\ C_6 & \text{if } r = 1 \\ C_7 & \text{if } r = 2 \\ 2C_4 & \text{if } r = 3. \end{cases}$$

Hence all subgraphs H_i $(0 \leq i \leq s-1)$ and $C_n(n_3)$ are pairwise edge-disjoint. Therefore, the subgraph consisting of H_i $(0 \leq i \leq s-1)$ and $C_n(n_3)$ is an even subgraph of size $(2t+1)+(t+1) = \lceil \frac{m}{2} \rceil$.

Case 2. $n_3 = n/2$. Then n is even and so $n = 2t$ for some integer $t \geq 4$. Let $t = 4s + r$ for some positive integer s where $r = 0, 1, 2, 3$. Then $n = 8s + 2r$ and $m = 20s + 5r$. Hence

$$\left\lceil \frac{m}{2} \right\rceil = \begin{cases} 10s & \text{if } r = 0 \\ 10s+3 & \text{if } r = 1 \\ 10s+5 & \text{if } r = 2 \\ 10s+8 & \text{if } r = 3. \end{cases} \tag{3.12}$$

First, we make an observation. Let $B = (u_1, u_2, \ldots, u_b)$ be a block of order b as shown in Figure 3.14.

(i) If $b \geq 8$, then $G[B]$ contains an even subgraph F of size 10 using only edges in $C_n(2, 3)$. For example, let F consist of two edge-disjoint 5-cycles

$$(u_1,\ u_3,\ u_5,\ u_7,\ u_4,\ u_1) \text{ and } (u_2, u_4, u_6, u_8, u_5, u_2).$$

(ii) If $b \geq 10$, then $G[B]$ contains an even subgraph X of size 13 using only edges in $C_n(2, 3)$. For example, let X consist of the 7-cycle

$$(u_1, u_3, u_5, u_7, u_9, u_6, u_4, u_1)$$

and the 6-cycle

$$(u_2, u_4, u_7, u_{10}, u_8, u_5, u_2).$$

(iii) If $b \geq 12$, then $G[B]$ contains an even subgraph Y of size 15 using only edges in $C_n(2,3)$. For example, let Y consist of the 9-cycle

$$(u_1, u_3, u_5, u_7, u_9, u_{11}, u_8, u_6, u_4, u_1)$$

and the 6-cycle

$$(u_2,\ u_4,\ u_7,\ u_{10},\ u_8,\ u_5,\ u_2)$$

as defined in (ii).

(iv) If $b \geq 14$, then $G[B]$ contains an even subgraph Z of size 18 using only edges in $C_n(2,3)$. For example, let Z consist of the even subgraph F defined in (i) and the two edge-disjoint 4-cycles

$$(u_7, u_9, u_{12}, u_{10}, u_7) \text{ and } (u_8, u_{11}, u_{13}, u_{10}, u_8).$$

These four subgraphs F, X, Y, Z are shown in Figure 3.16.

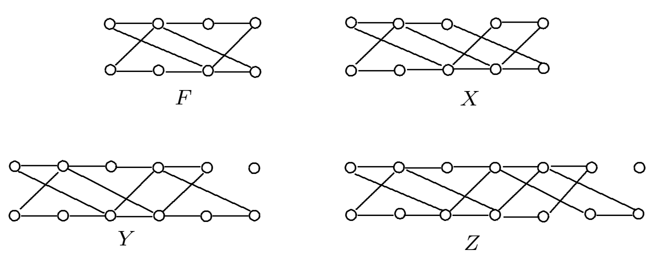


Figure 3.16: The four subgraphs F, X, Y, Z

Partition C_n into s consecutive blocks $B_0, B_1, \ldots, B_{s-1}$ defined by $B_i = [v_{8i+1}, v_{8(i+1)}]$ for $0 \leq i \leq s-2$ and $s \geq 2$ and $B_{s-1} = [v_{8(s-1)+1}, v_n]$. By (i), the subgraph $G[B_i]$

induced by B_i contains an even subgraph F of size 10 using only edges in $C_n(2,3)$. For each i with $0 \leq i \leq s-2$ and $s \geq 2$, let $H_i \cong F$ as defined in (i). For $r=0$, let $H_{s-1} \cong F$ as defined in (i); for $r=1$, let $H_{s-1} \cong X$ as defined in (ii); for $r=2$, let $H_{s-1} \cong Y$ as defined in (iii) and for $r=3$, let let $H_{s-1} \cong Z$ as defined in (iv). Therefore, the subgraph consisting of H_i $(0 \leq i \leq s-1)$ is an even subgraph of size $\lceil \frac{m}{2} \rceil$ as described in (3.12). ■

The following are consequences of Theorem 3.4.1, Proposition 3.4.2 and Theorem 3.4.12.

Corollary 3.4.13 *For each integer $n \geq 8$, the graph $C_n(2,3,n_3,n_4,\ldots,n_k)$ is optimal for all $4 \leq n_3 < n_4 < \cdots < n_k \leq n/2$.*

3.4.5 Summary

We now summarize what we have obtained. Let $G = C_n(n_1,n_2,\ldots,n_k)$ where $n_1,n_2,\ldots,n_k$ are k distinct integers with $1 \leq n_i \leq n/2$ for $1 \leq i \leq k$.

I. $k \geq 2$ is even and $n \geq 6$

- The Eulerian circulant G is optimal if $1 \leq n_i < n/2$ for all i with $1 \leq i \leq k$.
- The non-Eulerian circulant $C_n(1,n/2)$ is optimal if and only if $n \not\equiv 6 \pmod 8$.
- If $n \not\equiv 6 \pmod 8$, then the non-Eulerian circulant $C_n(1,n_2,n_3,\ldots,n_k)$ is optimal for all $k \geq 2$ and $2 \leq n_2 < n_3 < \ldots < n_{k-1} < n_k = n/2$. Note that the converse of this statement is not true. For example, the non-Eulerian circulant $C_{22}(1,3,11)$ is optimal by Theorem 3.4.10, while $n = 22 \equiv 6 \pmod 8$.

II. $k \geq 3$ **is odd and** $n \geq 8$

- If there exist $n_r, n_s, n_t \in \{n_1, n_2, \ldots, n_k\}$ such that $C_n(n_r, n_s, n_t)$ is optimal and $n_i < n/2$ for each $i \in \{1, 2, \ldots, k\} - \{r, s, t\}$, then G is optimal.

- The graph $C_n(1, 2, n_3, n_4, \ldots, n_k)$ is optimal for all $3 \leq n_3 < n_4 < \ldots < n_k \leq n/2$.

- The graph $C_n(2, 3, n_3, n_4, \ldots, n_k)$ is optimal for all $4 \leq n_3 < n_4 < \ldots < n_k \leq n/2$.

- The graph $C_n(1, 3, n_3)$ $(n_3 < n/2)$ is optimal if and only if $n \not\equiv 2 \pmod 4$ or n_3 is even; while $C_n(1, 3, n/2)$ is optimal if and only if $n \not\equiv 2 \pmod 8$.

Chapter 4

Decompositions in Graphs

4.1 Trail, Circuit and Cycle Decompositions

Recall that in his famous 1736 paper [15], Leonhard Euler not only solved the Königsberg Bridge Problem, he described a generalization of this problem and solved this problem as well (although the proof was only completed in 1873 in a paper by Hierholzer [16]). This led to the class of graphs called *Eulerian graphs*, namely those graphs containing an Eulerian circuit. In terms of graphs, what Euler proved was that a connected graph G is Eulerian if and only if every vertex of G is even. Euler also showed that a connected graph G has an Eulerian trail if and only if G contains exactly two odd vertices, in which case any Eulerian trail begins at one of the odd vertices and ends at the other. Therefore, if a connected graph G has more than two odd vertices, then G has neither an Eulerian circuit nor an Eulerian trail. On the other hand, by Theorem 1.4.4, if G is a connected graph containing $2k$ odd vertices for some positive integer k, then G can be decomposed into k open trails but no fewer. In 1973, Chartrand, Polimeni and Stewart [12] proved the following.

Theorem 4.1.1 *If G is a connected graph containing $2k$ odd vertices for some positive integer k, then G can be decomposed into k open trails, at most one of which has odd length.*

We now present a generalization of Theorem 4.1.1. In order to do this, we present an additional definition. The *distance between two subgraphs F and H* in a connected graph G is

$$d(F, H) = \min\{d(u, v) : u \in V(F), v \in V(H)\}.$$

Theorem 4.1.2 *Let G be a connected graph of size m containing $2k$ odd vertices $(k \geq 1)$. Among all decompositions of G into k open trails, let s be the maximum number of such trails of odd length.*

(a) *If m is even, then s is even and for every even integer a such that $0 \leq a \leq s$, there exists a decomposition of G into k open trails, exactly a of which have odd length.*

(b) *If m is odd, then s is odd and for every odd integer b such that $1 \leq b \leq s$, there exists a decomposition of G into k open trails, exactly b of which have odd length.*

Proof. We only verify (a) as the proof of (b) is similar. Since the size of G is even, s is even and $0 \leq s \leq k$. If $s = 0$, then the result is true trivially. Thus we may assume that $s \geq 2$. It suffices to show that there exists a decomposition of G into

k open trails, exactly $s-2$ of which have odd length. Among all decompositions of G into k open trails, consider those decompositions containing exactly s trails of odd length; and, among those, consider one, say $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$, where the distance between some pair T_i, T_j of trails of odd length is minimum. We may assume that T_r is a $u_r - v_r$ trail for $1 \leq r \leq k$. We claim that this minimum distance is 0. Assume that this is not the case. Suppose that P is a path of minimum length connecting a vertex w_i in T_i and a vertex w_j in T_j, and let $w_i x$ be the edge of P incident with w_i (where it is possible that $x = w_j$). Then $w_i x$ belongs to a trail T_p among $T_1, T_2, \ldots, T_k$. Necessarily, T_p has even length, for otherwise, the distance between T_i and T_p is 0, producing a contradiction. Since T_i and T_p have the vertex w_i in common, T_i and T_p may be replaced by trails T_i^* and T_p^* connecting odd vertices such that T_i^* has even length, T_p^* has odd length,

$$E(T_i) \cup E(T_p) = E(T_i^*) \cup E(T_p^*)$$

and $w_i x$ belongs to T_p^*. Hence the distance between T_j and T_p^* is smaller than the distance between T_i and T_j, a contradiction. Thus, as claimed, the distance between T_i and T_j is 0 and so these two trails have a vertex w in common. Either the $u_i - w$ subtrail T_i' or the $w - v_i$ subtrail T_i'' has odd length, say the former. We may also assume that the $u_j - w$ subtrail T_j' of T_j has odd length and the $w - v_j$ subtrail T_j'' has even length. Then the $u_i - u_j$ trail T_{ij}' formed from T_i' and T_j' and the $v_i - v_j$ trail T_{ij}'' formed from T_i'' and T_j'' both have even length. Then $(\mathcal{T} - \{T_i, T_j\}) \cup \{T_{ij}', T_{ij}''\}$ is a decomposition of G into k open trails, exactly $s-2$ of which have odd length. ∎

Theorem 4.1.1 is then a consequence of Theorem 4.1.2. In [10] a circuit decomposition theorem (similar to Theorem 4.1.2) on an Eulerian graph was established, as we state next. For completion, we also include a proof of this theorem presented in [10].

Theorem 4.1.3 *For an Eulerian graph G of size m, let s be the maximum number of circuits of odd length in a circuit decomposition of G.*

(*a*) *If m is even, then s is even and for every even integer a such that $0 \leq a \leq s$, there exists a circuit decomposition of G, exactly a of which have odd length.*

(*b*) *If m is odd, then s is odd and for every odd integer b such that $1 \leq b \leq s$, there exists a circuit decomposition of G, exactly b of which have odd length.*

Proof. We only verify (a) as the proof of (b) is similar. Since the size of G is even, s is even. If $s = 0$, then the result is true trivially. Thus we may assume that $s \geq 2$. It suffices to show that there exists a circuit decomposition of G, exactly $s - 2$ of which have odd length. Among all circuit decompositions of G, consider those circuit decompositions containing exactly s circuits of odd length; and, among those, consider one, say $\mathcal{D} = \{C_1, C_2, \ldots, C_k\}$ for some positive integer k, where the distance between some pair C_i, C_j of circuits of odd length is minimum. We claim that this minimum distance is 0. Assume that this is not the case. Suppose that P is a path of minimum length connecting a vertex w_i in C_i and a vertex w_j in C_j, and let $w_i x$ be the edge of P incident with w_i (where it is possible that $x = w_j$). Then $w_i x$ belongs to a circuit C_p among $C_1, C_2, \ldots, C_k$. Necessarily, C_p has even length, for otherwise, the distance between C_i and C_p is 0, producing a contradiction. Since C_i and C_p have the vertex w_i in common, C_i and C_p may be replaced by the circuit C' consisting of C_i and C_p (that is, $E(C') = E(C_i) \cup E(C_p)$) and C' has odd length. However then, the circuit decomposition $\mathcal{D}' = (\{C_1, C_2, \ldots, C_k\} - \{C_i, C_p\}) \cup \{C'\}$ has exactly s circuits of odd length and the distance between C_j and C' in $\mathcal{D}'$ is smaller than the distance between C_i and C_j in $\mathcal{D}$, which contradicts the defining property of $\mathcal{D}$. Thus, as claimed, the distance between C_i and C_j is 0 and so C_i and C_j have a vertex in common. Hence the circuit C^* consisting of C_i and C_j has even length. Then $(\{C_1, C_2, \ldots, C_k\} - \{C_i, C_j\}) \cup \{C^*\}$ is a circuit decomposition of G, exactly $s - 2$ of which have odd length. ■

In 1912 Oswald Veblen [21], one of the important early figures in topology, presented another characterization of Eulerian graphs (Theorem 1.4.2) which we restate as follows.

Veblen's Theorem *A connected graph G is Eulerian if and only if G has a cycle decomposition.*

Over the years there has been a host of theorems and conjectures dealing with the characteristics of cycles in a cycle decomposition of Eulerian graphs. Certainly one of the best known is due to Thomas Kirkman [17] and concerns decompositions of Eulerian complete graphs into 3-cycles (Steiner triple systems).

Kirkman's Theorem *The complete graph K_n with $n \geq 3$ has a C_3-decomposition if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.*

At the other extreme is the following theorem by Walecki (see [3]).

Walecki's Theorem *The complete graph K_n with $n \geq 3$ is C_n-decomposable (Hamiltonian factorable) if and only if n is odd.*

A well-known conjecture in this area was made by Brian Alspach [2] in 1981.

Alspach's Conjecture *Suppose that $n \geq 3$ is an odd integer and that $m_1, m_2, \ldots, m_t$ are integers such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$. Then K_n can be decomposed into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$. Furthermore, for every even integer $m \geq 4$ and integers $m_1, m_2, \ldots, m_t$ such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ with $m_1 + m_2 + \cdots + m_t = (n^2 - 2n)/2$, there is a decomposition of K_n into a 1-factor and the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$.*

Alspach's Conjecture was verified in its entirety by Bryant, Horsley and Pettersson [7] in 2012. The results mentioned above led to another conjecture involving cycle decompositions of Eulerian graphs, introduced in [10].

The Eulerian Cycle Decomposition Conjecture (ECDC) *Let G be an Eulerian graph of size m, where a is the minimum number of odd cycles in a cycle decomposition of G and b is the maximum number of odd cycles in a cycle decomposition of G. For every integer ℓ such that $a \leq \ell \leq b$ and ℓ and m are of the same parity, there exists a cycle decomposition of G containing exactly ℓ odd cycles.*

It is therefore a consequence of the theorem by Bryant, Horsley and Pettersson that the ECDC is true for all Eulerian complete graphs. This observation was made in [10]. For positive integers $n \geq 2$ and r, let $K_{n(r)}$ be the regular complete

n-partite graph each of whose partite sets consists of r vertices. Since removing the edges of a 1-factor from K_{2n}, where $n \geq 2$, produces the graph $K_{n(2)}$, it follows that the ECDC holds for $K_{n(2)}$ for every integer $n \geq 2$. This conjecture was verified in [10] for Eulerian kth powers of cycles for $k = 2, 3, 4$ and all Eulerian complete 3-partite graphs.

In this work we study, for graphs G and H, decompositions of G into $k+1 \geq 1$ subgraphs, k of which are isomorphic to H and where the remaining subgraph contains no subgraph isomorphic to H.

4.2 On H-Maximal Decompositions

A graph H is said to *divide* a graph G, often expressed by writing $H \mid G$, if G is H-decomposable, that is, if G has a decomposition $\{H_1, H_2, \ldots, H_k\}$, where $H_i \cong H$ for $i = 1, 2, \ldots, k$. If G has size m, H has size m' and $H \mid G$, then certainly $m' \mid m$. On the other hand, if $H \nmid G$, then either G does not contain a subgraph isomorphic to H or G contains a decomposition of $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ where $H_i \cong H$ for each i ($1 \leq i \leq k$) and R is a nonempty subgraph of G containing no subgraph isomorphic to H. The subgraph R may be referred to as the *remainder subgraph* for this decomposition. This observation may be considered as a graph theory analogue of the famous Division Algorithm for integers, where if the positive integer b is divided by the positive integer a, then there exist integers q and r with $0 \leq r < a$ such that $b = aq + r$. Unlike the Division Algorithm for integers where q and r are unique, in this so-called Division Algorithm for graphs G and

H, resulting in a decomposition $\mathcal{D}$ (above) of G in terms of H, the integer k and remainder graph R need not be unique. This observation leads us to the subject of this chapter, namely that of determining all graphs H such that for every graph G the integers k in such decompositions consist of a set of consecutive integers.

For two graphs H and G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called *H-maximal* or an *H-maximal k-decomposition* if $H_i \cong H$ for $1 \leq i \leq k$ and R contains no subgraph isomorphic to H. If G contains no subgraph isomorphic to H, then $k = 0$ and $R = G$. For graphs H and G, let

$$\begin{aligned} \mathrm{Min}(G, H) &= \min\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\} \\ \mathrm{Max}(G, H) &= \max\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}. \end{aligned}$$

Obviously, $\mathrm{Min}(G, H) \leq \mathrm{Max}(G, H)$. Throughout this section, we assume that H is a graph without isolated vertices. A graph H is said to possess the *intermediate decomposition property* (IDP) and H is called an *ID-graph* if for each graph G and each integer k with $\mathrm{Min}(G, H) \leq k \leq \mathrm{Max}(G, H)$, there exists an H-maximal k-decomposition of G. Trivially, the graph K_2 is an ID-graph. On the other hand, neither the claw $K_{1,3}$ nor the triangle K_3 is an ID-graph. For example, the graph G of Figure 4.1 has a $K_{1,3}$-maximal 1-decomposition and a $K_{1,3}$-maximal 3-decomposition but has no $K_{1,3}$-maximal 2-decomposition. Similarly, the graph F of Figure 4.1 has a K_3-maximal 1-decomposition and a K_3-maximal 3-decomposition but has no K_3-maximal 2-decomposition.

These observations lead to the following problem.

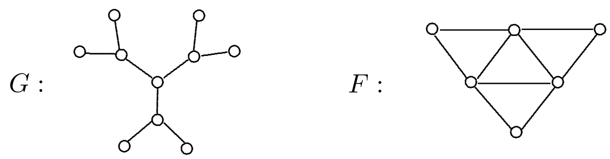


Figure 4.1: Illustrating that $K_{1,3}$ and K_3 are not ID-graphs

The Intermediate Value Problem for H-Maximal Decompositions

Which graphs (without isolated vertices) are ID-graphs?

We first show that each of the two graphs of size 2 (shown in Figure 4.2), namely the path P_3 of order 3 and $2K_2$ consisting of two components of order 2, is an ID-graph.

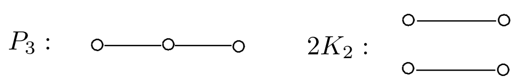


Figure 4.2: The two graphs of size 2

Recall the following result of Chartrand, Polimeni and Stewart [12] dealing with paths P_3 of order 3 will be useful to us (also see Theorem 1.3.1).

Theorem 4.2.1 *A nontrivial connected graph G is P_3-decomposable if and only if G has even size.*

For two edges $e = u_1u_2$ and $f = v_1v_2$ in a nontrivial connected graph G, an $e - f$ path has e as its initial edge and f as its terminal edge. The *distance* $d(e, f)$ between e and f is defined as

$$\min\{d(u, v) : u \in \{u_1, u_2\} \text{ and } v \in \{v_1, v_2\}\}.$$

Proposition 4.2.2 *The graph P_3 is an ID-graph.*

Proof. Let $a = \text{Min}(G, P_3)$ and suppose that $\mathcal{D}_a = \{H_1, H_2, \ldots, H_a, R\}$ is a P_3-maximal a-decomposition of G where $H_i = P_3$ for $1 \leq i \leq a$ and $R = rK_2$ for some integer $r \geq 0$. If $r = 0$ or $r = 1$, then $\text{Max}(G, P_3) = a$ and the result follows. Thus, we may assume that $r \geq 2$ and show that G has a P_3-maximal $(a+1)$-decomposition. Let e_1 and e_2 be two distinct edges in R such that

$$d(e_1, e_2) = \min\{d(e, f) : e, f \in R\}. \tag{4.1}$$

Necessarily, $d(e_1, e_2) \geq 1$. Let P be an $e_1 - e_2$ path of shortest length in G. It then follows by (4.1) that each $e \in E(P) - \{e_1, e_2\}$ belongs to H_i for some i with $1 \leq i \leq a$. Let

$$W = \{i : H_i \cap E(P) \neq \emptyset \text{ and } 1 \leq i \leq a\} \tag{4.2}$$

and let F be the subgraph induced by the set $\{e_1, e_2\} \cup \left(\bigcup_{i \in W} E(H_i)\right)$ of edges. We may assume, without loss of generality, that $W = \{1, 2, \ldots, t\}$ for some positive integer t. Since F is a connected graph of size $2(t+1)$, it follows by Theorem 4.2.1 that F is P_3-decomposable. Suppose that $\{H'_1, H'_2 \ldots, H'_{t+1}\}$ is a P_3-decomposition of F and $R' = R - \{e_1, e_2\}$. Then $\mathcal{D}_{a+1} = \{H'_1, H'_2 \ldots, H'_{t+1}, H_{t+1}, \ldots, H_a, R'\}$ is a P_3-maximal $(a+1)$-decomposition of G. Continuing in this manner, we see that for each integer k with $a \leq k \leq \text{Max}(G, P_3)$, there exists a P_3-maximal k-decomposition of G. ∎

Proposition 4.2.3 *The graph $2K_2$ is an ID-graph.*

Proof. Assume, to the contrary, that $2K_2$ is not an ID-graph. Then there exists an integer a with $\mathrm{Min}(G, 2K_2) \leq a < a+1 < b = \mathrm{Max}(G, 2K_2)$ such that G has a $2K_2$-maximal a-decomposition $\mathcal{D}_a = \{H_1, H_2, \ldots, H_a, R\}$ but no $2K_2$-maximal $(a+1)$-decomposition. Since G has a $2K_2$-maximal b-decomposition and R contains no subgraph isomorphic to $2K_2$, it follows that $R = K_{1,t}$ for some integer $t \geq 4$. Hence the size of G is $m = 2a + t$. First, we verify the following:

> For each integer i with $1 \leq i \leq a$, exactly one edge in H_i is adjacent to every edge in R; that is, for each integer i with $1 \leq i \leq a$, there is exactly one edge in H_i such that this edge and the edges of R form a star $K_{1,t+1}$.

Suppose that this statement is false for some subgraph H_i in $\mathcal{D}_a$, say $H_i = H_1$ where $E(H_1) = \{e_1, e_2\}$. First, since $H_1 = 2K_2$, it is impossible that each of the two edges in H_1 is adjacent to every edge of R. Thus neither of the two edges e_1 and e_2 of H_i is adjacent to all edges of R. Then e_1 and e_2 can be adjacent to at most two edges of R. Let $E(R) = \{f_1, f_2, \ldots, f_t\}$. Since $t \geq 4$, there are two edges of R, say f_1 and f_2, such that neither e_1 nor e_2 is adjacent to f_j for $j = 1, 2$. Let H'_j consist of e_j and f_j for $j = 1, 2$ and let $R' = R - \{f_1, f_2\}$. Then $\mathcal{D} = \{H'_1, H'_2, H_2, \ldots, H_a, R'\}$ is a $2K_2$-maximal $(a+1)$-decomposition of G, which is impossible. Thus, as claimed, for each integer i with $1 \leq i \leq a$, exactly one edge in H_i is adjacent to every edge in R.

Next, for each i with $1 \leq i \leq a$, let $f'_i \in E(H_i)$ such that f'_i and $E(R)$ form a $K_{1,t+1}$ in G. Thus G contains $K_{1,t+a}$ as a subgraph. Since the size of G is $2a + t$, there are at most a sets of two nonadjacent edges of G. Therefore, G cannot have a $2K_2$-maximal b-decomposition for $b \geq a+2$, which is a contradiction. ■

While the graph $2K_2$ is an ID-graph, the graph $3K_2$ is not. Let $G = K_3 \,\square\, K_2$ be the Cartesian product of K_3 and K_2 whose edges are labeled as shown in Figure 4.3. The graph G has a $3K_2$-maximal 1-decomposition and a $3K_2$-maximal 3-decomposition. For example, let $D_1 = \{H_1, R_1\}$ where $H_1 = G[\{e_1, e_2, e_2\}]$ and $D_3 = \{L_1, L_2, L_3, R_3\}$ where $L_i = G[\{e_i, f_i, g_i\}]$ and R_3 is an empty graph. Since the size of G is 9, it follows that $\text{Min}(G, 3K_2) = 1$ and $\text{Max}(G, 3K_2) = 3$.

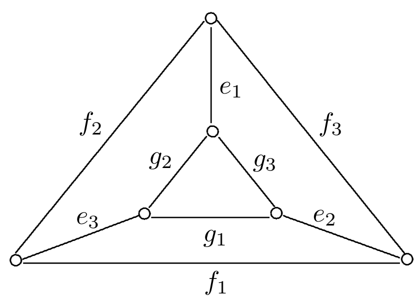


Figure 4.3: Showing that $3K_2$ is not an ID-graph

It remains to show that G does not have a $3K_2$-maximal 2-decomposition. Assume, to the contrary, that G has a $3K_2$-maximal 2-decomposition $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ where $F_1 \cong F_2 \cong 3K_2$ and R_2 contains no subgraph isomorphic to $3K_2$. First, we observe that if $F \in \{F_1, F_2\}$ such that F contains an edge in $\{f_i, g_i\}$ where $i = 1, 2, 3$, then $E(F) = \{e_i, f_i, g_i\}$. At least one of F_1 and F_2 contains an edge in $\{f_i, g_i : 1 \leq i \leq 3\}$, say $f_1 \in F_1$ and so $E(F_1) = \{e_1, f_1, g_1\}$. Since F_1 and F_2 are edge-disjoint and $F_2 = 3K_2$, it follows that F_2 must contain an edge in $\{f_i, g_i : i = 2, 3\}$. By symmetry, we may assume that $f_2 \in E(F_2)$. However then,

$E(F_2) = \{e_2, f_2, g_2\}$ and so $R_2 = 3K_2$ with edge set $E(R_2) = \{e_3, f_3, g_3\}$, which is a contradiction.

4.3 The ID-Graphs of Size 3

We have seen that none of the three graphs $K_{1,3}, K_3, 3K_2$ of size 3 is an ID-graph. In this section, we show that the two remaining graphs of size 3 (without isolated vertices) are both ID-graphs. To do this, we introduce a decomposition concept involving sets of graphs without isolated vertices. For a set S of graphs and a graph G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called *S-maximal* or an *S-maximal k-decomposition* if $H_i \cong H$ for some $H \in S$ for each integer i with $1 \leq i \leq k$ and R contains no subgraph isomorphic to any subgraph in S. For a set S of graphs without isolated vertices and a graph G, let

$$\begin{aligned} \text{Min}(G, S) &= \min\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\} \\ \text{Max}(G, S) &= \max\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\}. \end{aligned}$$

A set S of graphs without isolated vertices is said to possess the *intermediate decomposition property* (IDP) or S is called an *ID-set* if for every graph G and each integer k with $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. By Propositions 4.2.2 and 4.2.3, if $S = \{P_3\}$ or $S = \{2K_2\}$, then S is an ID-set. On the other hand, the set $S = \{K_{1,3}, K_3\}$ is not an ID-set. For example, the graph G of Figure 4.1 has an S-maximal 1-decomposition and an S-maximal 3-decomposition but has no S-maximal 2-decomposition. (On the other hand, the graph F of Figure 4.1 has an S-maximal k-decomposition for

$k = 1, 2, 3.$)

Observe that if S is the set of all graphs (connected or disconnected) of the same size m, then S is an ID-set. To see this, let G be a graph, $a = \text{Min}(G, S)$ and let $\mathcal{D} = \{H_1, H_2, \ldots, H_a, R\}$ be any S-maximal a-decomposition of G. Since R contains no subgraph that is isomorphic to any graph in S, it follows that $0 \le |E(R)| \le m-1$. Thus $\text{Min}(G, S) = \text{Max}(G, S) = a$. We state this observation below.

Observation 4.3.1 *For each positive integer m, the set S_m of all graphs (connected or disconnected) of size m is an ID-set.*

By Observation 4.3.1, the sets $S_2 = \{P_3, 2K_2\}$ of all graphs of size 2 and $S_3 = \{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ of all graphs of size 3 are ID-sets, where $P_3 + K_2$ is the union of P_3 and K_2. It is the following problem, however, that is of more general interest to us.

The Intermediate Value Problem for S-Maximal Decompositions

Which sets of graphs (without isolated vertices) are ID-sets?

For a set S of graphs, a graph G is said to have the *intermediate decomposition property with respect to S* (IDP-S) if for each integer k with $\text{Min}(G, S) \le k \le \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. In this case, the graph G is referred to as an *IDP-S* graph; otherwise, G is a *non-IDP-S* graph. Of course, if every graph is an IDP-S graph, then S is an ID-set.

The following theorem concerning sets of graphs that are not ID-sets will be useful in showing that both P_4 and $P_3 + K_2$ are ID-graphs.

Theorem 4.3.2 *Let S be a set of graphs without isolated vertices that is not an ID-set and let $\mathcal{F}_S$ be the set of all non-IDP-S graphs, where G is a graph of minimum size in $\mathcal{F}_S$. Moreover, let a and b be the smallest integers with $1 \leq a < b-1$ such that (i) G has an S-maximal a-decomposition $\mathcal{D}_a = \{H_1, H_2, \ldots, H_a, R_a\}$ and an S-maximal b-decomposition $\mathcal{D}_b$ =$\{L_1,\ L_2,\ \ldots,\ L_b,\ R_b\}$ but (ii) G has no S-maximal k-decomposition for every integer k with $a < k < b$.*

- *(I) If $\mathcal{D}_c$ is an S-maximal c-decomposition of G where $c \geq b$, then $H_i \notin \mathcal{D}_c$ for all i with $1 \leq i \leq a$.*
- *(II) For all pairs i, j where $i \in \{1, 2, \ldots, a\}$ and $j \in \{1, 2, \ldots, b\}$, it follows that $E(H_i) \cap E(L_j) \neq \emptyset$.*
- *(III) The number b satisfies $b \leq \min\{|E(H_i)| : 1 \leq i \leq a\}$.*

Proof. To verify (I), we first assume that $a = 1$. Then $\mathcal{D}_a = \{H_1, R_1\}$. Since R_1 contains no subgraph isomorphic to any graph in S, the statement is true for $a = 1$. Next, suppose that $a \geq 2$. Assume, to the contrary, that G has an S-maximal c-decomposition $\mathcal{D}_c$ for some integer $c \geq b \geq a + 2$ such that $H_i \in \mathcal{D}_c$ for some i $(1 \leq i \leq a)$, say $H_1 \in \mathcal{D}_c$. Let $G' = G - E(H_1)$. Then $\mathcal{D}_a - \{H_1\}$ is an S-maximal $(a-1)$-decomposition and $\mathcal{D}_c - \{H_1\}$ is an S-maximal $(c-1)$-decomposition of G'. Since $1 \leq a-1 < b-2 < b-1 \leq c-1$ and G' does not have an S-maximal k'-decomposition for each k' with $a-1 < k' < b-1$, it follows that this decomposition together with H_1 forms an S-maximal $(k'+1)$-decomposition of G where $a < k'+1 < b$. This, however, is impossible by property (ii) possessed by a and b. Thus $G' \in \mathcal{F}_S$. Since the size of G' is smaller than that of G, this contradicts the defining property of G and so (I) holds.

Suppose that statement (II) is false. Then there exist i, j where $i \in \{1, 2, \ldots, a\}$ and $j \in \{1, 2, \ldots, b\}$ such that $E(H_i) \cap E(L_j) = \emptyset$, say $E(H_1) \cap E(L_1) = \emptyset$. Let $G^* = G - E(L_1)$. Thus, G^* contains H_1 as a subgraph. Hence G^* has an S-maximal k-decomposition $\mathcal{D}^*$ for some positive integer k such that $H_1 \in \mathcal{D}^*$. We claim that $k \leq a-1$. Suppose that this is not the case. Then $k \geq a$ and so $k+1 > a$. Observe that (1) $\mathcal{D}^* \cup \{L_1\}$ is an S-maximal $(k+1)$-decomposition of G, where, necessarily, $k+1 \geq b$, (2) $\mathcal{D}^* \cup \{L_1\}$ contains H_1 and (3) $H_1 \in \mathcal{D}_a$. This contradicts (I). Therefore, $k \leq a-1$ as claimed.

Since $1 \leq k \leq a-1$, it follows that $a \geq 2$. Observe that $\mathcal{D}_b - \{L_1\}$ is an S-maximal $(b-1)$-decomposition of G^*. We now claim that G^* has no S-maximal k^*-decomposition for each integer k^* with $a-1 < k^* < b-1$. If this is not the case, then there exists an S-maximal k^*-decomposition. However, this decomposition together with L_1 forms an S-maximal (k^*+1)-decomposition of G. Since $a < k^*+1 < b$, this is impossible by (ii). Thus $G^* \in \mathcal{F}_S$. However, the size of G^* is smaller than the size of G, which contradicts the defining property of G and so (II) holds.

It remains to verify (III). Observe that if $i \in \{1, 2, \ldots, a\}$. It then follows by (II) that $E(H_i) \cap E(L_j) \neq \emptyset$ for every $j \in \{1, 2, \ldots, b\}$. Since $L_1, L_1, \ldots, L_b$ are pairwise edge-disjoint, $|E(H_i)| \geq b$ for $1 \leq i \leq a$ and (III) holds. ■

By Theorem 4.3.2, every graph of size 2 is an ID-graph. Therefore, Propositions 4.2.2 and 4.2.3 are consequences of Theorem 4.3.2.

For a set S of graphs without isolated vertices that is not an ID-set, a graph G of minimum size that is not an IDP-S graph (as described in Theorem 4.3.2) is referred to as a *minimum non-IDP-S graph*. If $S = \{H\}$ consists of a single graph H, then a minimum non-IDP-S graph is also referred to as a *minimum non-IDP-H graph*.

Theorem 4.3.3 *The graph P_4 is an ID-graph.*

Proof. Assume, to the contrary, that P_4 is not an ID-graph. Then there exists a graph G of minimum size that is not an IDP-P_4 graph. Then there are smallest integers a and b where $1 \leq a \leq b-2$ such that G has a P_4-maximal a-decomposition $\mathcal{D}_a$ and a P_4-maximal b-decomposition $\mathcal{D}_b$ but G has no P_4-maximal k-decomposition for every integer k with $a < k < b$. By Theorem 4.3.2(III), $b \leq 3$, which implies that $a = 1$ and $b = 3$. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where $H_1 \cong P_4$, $L_i \cong P_4$ ($i = 1, 2, 3$) and neither R_1 nor R_3 contains a subgraph isomorphic to P_4. By assumption, G has no P_4-maximal 2-decomposition.

Let $H_1 = (v_1, v_2, v_3, v_4)$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, 3$. Since, by Theorem 4.3.2(II), each subgraph L_i must contain an edge of H_1, we may assume that $e_i \in E(L_i)$ for $i = 1, 2, 3$. In particular, $e_2 = v_2 v_3 \in E(L_2)$ and so at least one of v_2 and v_3 is an interior vertex of L_2, say v_2 is an interior vertex of L_2. Then

(u, v_2, v_3) is a subpath of L_2 for some vertex u of G. Thus, either

$$L_2 = (u, v_2, v_3, w) \quad \text{or} \quad L_2 = (w, u, v_2, v_3) \tag{4.3}$$

for some vertex w of G. Next, we claim that either v_1 is an interior vertex of L_1 or v_4 is an interior vertex of L_3. If this were not the case, then

$$L_1 = (v_1, v_2, x, y) \quad \text{and} \quad L_3 = (v_4, v_3, x', y')$$

for some vertices x, y, x', y' of G. We consider two cases, according to the two possible choices of L_2 described in (4.3).

Case 1. $L_2 = (u, v_2, v_3, w)$. Since L_1 and L_2 are edge-disjoint, $x \neq u$. We now show that $y = u$. Suppose that $y \neq u$. Then $(u, v_2, x, y) = P_4$ is a subgraph of R_1, which is impossible. Consequently, $y = u$. Since L_1 and L_3 are edge-disjoint, $x' \neq w$. Next, we show that $y' = w$. Suppose that $y' \neq w$. Then (w, v_3, x', y') is a subgraph of R_1, again impossible. Therefore, $y' = w$. Hence G contains the subgraph shown in Figure 4.4, where the bold edges are the edges of H_1. Let $F_1 = (v_1, v_2, x, y)$ and $F_2 = (u, v_2, v_3, v_4)$. Then $F_1 \cong F_2 \cong P_4$ and F_1 and F_2 are edge-disjoint. Since $E(H_1) \subseteq E(F_1) \cup E(F_2)$, it follows that $R_2 = G - [E(F_1) \cup E(F_2)]$ is a subgraph of R_1 and so R_2 contains no subgraph isomorphic to P_4. Therefore, $\{F_1, F_2, R_2\}$ is a P_4-maximal 2-decomposition of G, which produces a contradiction.

Case 2. $L_2 = (w, u, v_2, v_3)$. Since L_1 and L_2 are edge-disjoint, $x \neq u$. We show that $x = w$. Suppose that $x \neq w$. Then $(x, v_2, u, w) = P_4$ is a subgraph of R_1, which is impossible. Consequently, $x = w$. Next, we show that $y = u$. If $y \neq u$, then $(y, x, v_2, u) = P_4$ is a subgraph of R_1, which is impossible. Hence $y = u$. Since $x = w$ and $y = u$, it follows that $wu = xy \in E(L_1) \cap E(L_2)$, which is impossible.

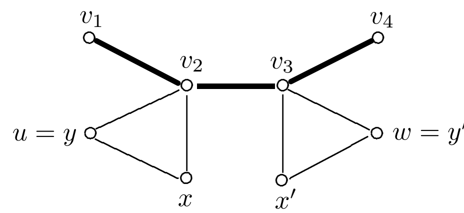


Figure 4.4: A step in the proof of Theorem 4.3.3

Therefore, as claimed, either v_1 is an interior vertex of L_1 or v_4 is an interior vertex of L_3, say the former. Then (z, v_1, v_2) is a subpath of L_1 for some vertex z of G. Define a decomposition $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ of G where $F_1 = (z, v_1, v_2, v_3)$, $F_2 = L_3$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Then $F_1 \cong F_2 \cong P_4$ and $E(F_1) \cap E(F_2) = \emptyset$. Since $E(H_1) \subseteq E(F_1) \cup E(F_2)$, it follows that R_2 is a subgraph of R and so R_2 contains no subgraph isomorphic to P_4. Hence $\mathcal{D}_2$ is a P_4-maximal 2-decomposition, which is a contradiction. ∎

We next show that $P_3 + K_2$, the remaining graph of size 3, is also an ID-graph.

Theorem 4.3.4 *The graph $P_3 + K_2$ is an ID-graph.*

Proof. Assume, to the contrary, that $P_3 + K_2$ is not an ID-graph. Let G be a graph of minimum size that is not an IDP-P_4 graph. By Theorem 4.3.2 then, G has a (P_3+K_2)-maximal 1-decomposition $\mathcal{D}_1$ and a (P_3+K_2)-maximal 3-decomposition $\mathcal{D}_3$ but G has no $(P_3 + K_2)$-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where $H_1 \cong P_3 + K_2$, $L_i \cong P_3 + K_2$ $(i = 1, 2, 3)$ and neither R_1 nor R_3 contains a subgraph isomorphic to $P_3 + K_2$. Let $E(H_1) = \{e_1, e_2, e_3\}$ where e_1 and e_2 are adjacent edges in H_1 (see Figure 4.5). We may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| = t \geq 6$.

H_1 :

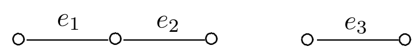


Figure 4.5: The graph H_1 in the proof of Theorem 4.3.4

We claim that $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ (with possibly additional isolated vertices, in which case we consider the subgraph of R_1 that consists of all nontrivial components of R_1). First, suppose that R_1 contains two or more nontrivial components. Since R_1 does not contain $P_3 + K_2$ as a subgraph, each nontrivial component is K_2 and so $R_1 = tK_2$. Next, suppose that R_1 is connected. Again, because R_1 does not contain $P_3 + K_2$ as a subgraph and the size of R_1 is at least 6, it follows that R_1 cannot contain vertex-disjoint copies of P_3 and K_2 and so either $R_1 = K_{1,t}$ or $R_1 = K_4$. Consequently, $R_1 = tK_2$, $R_1 = K_{1,t}$, or $R_1 = K_4$. We consider these three cases.

Case 1. $R_1 = tK_2$. Since $t \geq 6$ and $H_1 \cong P_3 + K_2$, there is at least one edge (say f_1) in R_1 that is not adjacent to any edge in H_1 and at least one edge (say f_2) in $E(R_1) - \{f_1\}$ that is not adjacent to e_3. We claim, in fact, that

$$\text{no edge in } R_1 \text{ is adjacent to } e_3. \tag{4.4}$$

Suppose that there is an edge f_3 in R_1 that is adjacent to e_3. Let $F_1 = G[\{e_1, e_2, f_1\}]$ and $F_2 = G[\{e_3, f_2, f_3\}]$. Then $F_1 \cong F_2 \cong P_3 + K_2$ and $E(F_1) \cap E(F_2) = \emptyset$. Since $E(H_1) \subseteq E(F_1) \cup E(F_2)$, it follows that $R_2 = G - [E(F_1) \cup E(F_2)]$ is a subgraph of R_1. Hence $\{F_1, F_2, R_2\}$ is a $(P_3 + K_2)$-maximal 2-decomposition of G, which is impossible. Thus (4.4) holds.

Next we claim that

$$\text{there are two edges in } R_1\text{, each of which is adjacent to } e_1 \text{ or } e_2. \tag{4.5}$$

Suppose this is not the case. Then there is at most one edge in R_1 that is adjacent to e_1 or e_2. It then follows by (4.4) that G is one of the graphs $P_3 + (t + 1)K_2, P_4 + tK_2, K_{1,3} + tK_2$. However then, G cannot have a $(P_3 + K_2)$-maximal 3-decomposition $\mathcal{D}_3$, which is a contradiction. Thus (4.5) holds. Let g_1 and g_2 be two distinct edges of R_1 that are adjacent to e_1 or e_2. Since g_1 and g_2 are nonadjacent, the subgraph $G[\{e_1, e_2, g_1, g_2\}]$ induced by $\{e_1, e_2, g_1, g_2\}$ is one of the two graphs shown in Figure 4.6, each of which can be decomposed into two copies T_1 and T_2 of P_3. Let $F_1 = G[E(T_1) \cup \{f_1\}]$ (where $f_1 \in E(R_1)$ is not adjacent to any edge in $E(H_1) \cup E(R_1)$) and $F_2 = G[E(T_2) \cup \{e_3\}]$ (where e_3 is not adjacent to any edge in $E(H_1) \cup E(R_1)$ by (4.4)). Then $F_1 \cong F_2 \cong P_3 + K_2$ and F_1 and F_2 are edge-disjoint. Since $E(H_1) \subseteq E(T_1) \cup E(T_2)$, it follows that $R_2 = G - (E(T_1) \cup E(T_2))$ is a subgraph of R_1 and so contains no subgraph isomorphic to $P_3 + K_2$. Therefore, $\{F_1, F_2, R_2\}$ is a $(P_3 + K_2)$-maximal 2-decomposition of G, which is a contradiction.

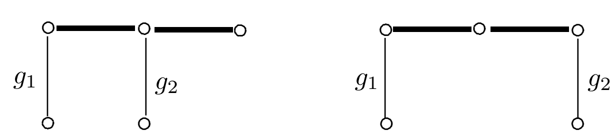


Figure 4.6: The subgraph $G[\{e_1, e_2, g_1, g_2\}]$ in Case 1

Case 2. $R_1 = K_{1,t}$. Note that if an edge of H_1 is adjacent with *all* edges in R_1, then this edge must be incident with the central vertex v of R_1 (see Figure 4.7). Since e_3 is not adjacent to e_1 or e_2, it is impossible that both e_1 and e_3 (or both e_2 and e_3) are adjacent to all edges of R_1. Thus at most two of the three edges e_1, e_2 and e_3 can be adjacent to all edges of R_1. Hence there are three possibilities, namely (i) both e_1 and e_2 are adjacent to all edges of R_1 (and e_3 is not), (ii) exactly one of e_1, e_2 and e_3 is adjacent to all edges of R_1 and (iii) none of e_1, e_2 and e_3 are adjacent to all edges of R_1. We consider these three situations.

$R_1 = K_{1,t}$:
$t \geq 6$

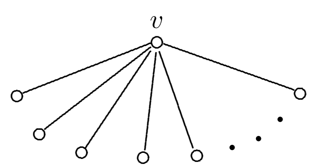


Figure 4.7: The graph R_1 in Case 2

Subcase 2.1. *Both e_1 and e_2 are adjacent to all edges of R_1.* Then $G - e_3 = K_{1,t+2}$. However then, G cannot have a $(P_3 + K_2)$-maximal 3-decomposition $\mathcal{D}_3$, which is impossible.

Subcase 2.2. *Exactly one of e_1, e_2 and e_3 is adjacent to all edges of R_1.* Let e and f are the edges of H_1 that are not adjacent to all edges of R_1. Then $G - e - f = K_{1,t+1}$. However then, G cannot have a $(P_3 + K_2)$-maximal 3-decomposition $\mathcal{D}_3$, which is impossible.

Subcase 2.3. *None of e_1, e_2 and e_3 are adjacent to all edges of R_1.* Thus none of e_1, e_2 and e_3 is incident with the central vertex v of R_1. Since the order of H_1 is 5 and the size t of R_1 is at least 6, there is an edge (say f_1) in R_1 that is not adjacent to any edge in H_1. Furthermore, there are at least two edges (say

f_2 and f_3) that are not adjacent to e_3. Let $F_1 = G[\{e_1, e_2, f_1\}] \cong P_3 + K_2$ and $F_2 = G[\{e_3, f_2, f_3\}] \cong P_3 + K_2$. Then $R_2 = G - (E(F_1) \cup E(F_2))$ is a subgraph of R_1. Therefore, $\{F_1, F_2, R_2\}$ is a $(P_3 + K_2)$-maximal 2-decomposition of G, which is a contradiction.

Case 3. $R_1 = K_4$. Observe that there exists $f \in E(R_1)$ such that f is adjacent to neither e_1 nor e_2. Let $F_1 = G[\{e_1, e_2, f\}] \cong P_3 + K_2$. The subgraph $G[(E(R_1) - \{f\}) \cup \{e_3\}]$ is one of the three graphs in Figure 4.8. In each case, there is a subgraph P_3 in $R_1 - f$ that is vertex-disjoint from e_3 (whose edges are drawn in bold in Figure 4.8). Let $F_2 = G[E(P_3) \cup \{e_3\}] \cong P_3 + K_2$. Then $R_2 = G - (E(F_1) \cup E(F_2))$ is a subgraph of R_1. Therefore, $\{F_1, F_2, R_2\}$ is a $(P_3 + K_2)$-maximal 2-decomposition of G, which is a contradiction. ■

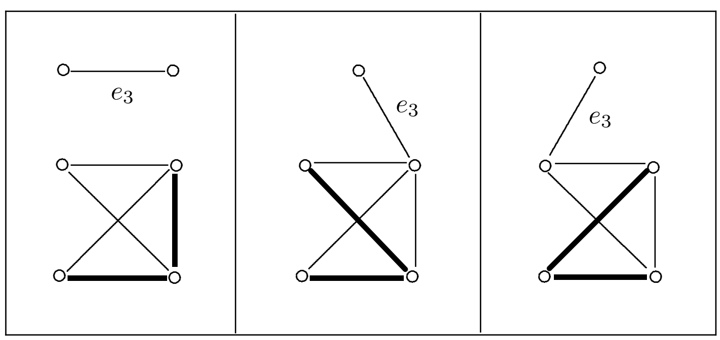


Figure 4.8: The subgraph $G[(E(R_1) - \{f\}) \cup \{e_3\}]$ in Case 3

The following result summarizes what we have discovered for all graphs of size 2 or 3.

Theorem 4.3.5 *A graph H of size 2 or 3 is an ID-graph unless $H \in \{K_3, K_{1,3}, 3K_2\}$.*

4.4 The ID-Sets of Graphs of Size 3

In this section, we turn to the problem of investigating whether certain sets of graphs of size 3 are ID-sets. Let $S_3 = \{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ be the set of graphs of size 3 without isolated vertices. By Observation 4.3.1, S_3 is an ID-set. Furthermore, by Theorem 4.3.5, we know all ID-sets consisting of exactly one graph of size 3. Therefore, we consider subsets S of S_3 with $2 \leq |S| \leq 4$. There are 25 such sets. We begin with whose sets consisting of two graphs of size 3 are ID-sets.

4.4.1 The 2-Element ID-Sets

While we already seen that $\{K_{1,3}, K_3\}$ is not an ID-set, it turns out that there two other sets of two graphs of size 3 that are non ID-sets, namely $\{3K_2, K_3\}$ and $\{3K_2, K_{1,3}\}$. For example, if $S = \{3K_2, K_3\}$, then the graph $K_3 + 2K_{1,3}$ is a non-IDP-S graph while if $S = \{3K_2, K_{1,3}\}$, then the graph $2K_3 + K_{1,3}$ is a non-IDP-S graph. Figure 4.9 provides a complete list of ID-sets and non-ID-sets consisting of one or two graphs of size 3.

	P_4	$P_3 + K_2$	K_3	$K_{1,3}$	$3K_2$
P_4	ID	ID	ID	ID	ID
$P_3 + K_2$	ID	ID	ID	ID	ID
K_3	ID	ID	NO	NO	NO
$K_{1,3}$	ID	ID	NO	NO	NO
$3K_2$	ID	ID	NO	NO	NO

Figure 4.9: ID-sets and non-ID sets of two graphs of size 3

We begin by verifying those 2-element ID-sets containing the graph $P_3 + K_2$.

The following observation will be useful, which is a consequence of Theorem 4.3.4.

Observation 4.4.1 *Suppose that R is a graph without isolated vertices having size $t \geq 6$. If R does not contain $P_3 + K_2$ as a subgraph, then $R = tK_2$, $R = K_{1,t}$ or $R = K_4$.*

Theorem 4.4.2 *The set $\{P_4, P_3 + K_2\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{P_4, P_3 + K_2\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where H_1, $L_i \in S$ ($i = 1, 2, 3$) and neither R_1 nor R_3 contains an subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$. We may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| = t \geq 6$. Since R_1 does not contain P_4 or $P_3 + K_2$ as a subgraph, $R_1 = K_{1,t}$ or $R_1 = tK_2$. We may assume $H_1 - e_1 = P_3 = (u, v, w)$ where $e_2 = uv$ and $e_3 = vw$.

$H_1 - e_1$:

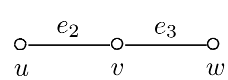


Figure 4.10: The graph $H_1 - e_1$ in the proof of Theorem 4.4.2

Assume first that there is an edge $e \in E(R_1) - E(L_1)$ such that e is adjacent to neither e_2 nor e_3. Let $F_1 = L_1$, $F_2 = G[\{e_2, e_3, e\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal

2-decomposition, which is impossible. Hence we may assume that every edge in $E(R_1) - E(L_1)$ is adjacent to e_2 or e_3. Since $|E(R_1) - E(L_1)| \geq 4$, it is impossible if $R_1 = tK_2$. Hence $R_1 = K_{1,t}$. Let $V(R_1) = \{x, x_1, x_2, \ldots, x_t\}$ and x is the central vertex of R_1 (see Figure 4.11). First, suppose that $x \in \{u, v, w\}$. If $x = u$ or $x = v$, then $G - \{e_1, e_3\} = K_{1,t+1}$; while if $x = w$, then $G - \{e_1, e_2\} = K_{1,t+1}$. In either case, G cannot have an S-maximal 3-decomposition $\mathcal{D}_3$, a contradiction. Next, suppose that $x \notin \{u, v, w\}$. We may assume, without loss of generality, that $xx_i \notin E(L_1)$ for $1 \leq i \leq 4$. Since each edge xx_i ($1 \leq i \leq 4$) is incident with exactly one vertex in $\{u, v, w\}$, it follows that $x_i \in \{u, v, w\}$ for $1 \leq i \leq 4$, this is impossible. ■

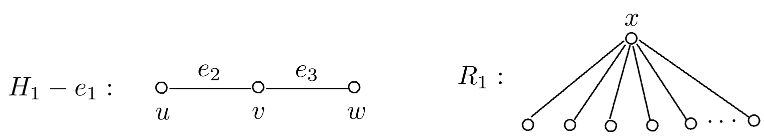


Figure 4.11: A step in the proof of Theorem 4.4.2

Theorem 4.4.3 *The set $\{K_3, P_3 + K_2\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{K_3, P_3 + K_2\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where H_1, $L_i \in S$ ($i = 1, 2, 3$) and neither R_1 nor R_3 contains an subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$ and we may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| = t \geq 6$. Since R_1 does not contain $P_3 + K_2$ and K_3 as a subgraph, $R_1 = tK_2$ or $R_1 = K_{1,t}$. We may assume that $H_1 - e_1 = P_3 = (u, v, w)$ where $e_2 = uv$ and $e_3 = vw$.

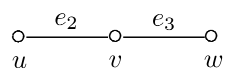


Figure 4.12: The graph $H_1 - e_1$ in the proof of Theorem 4.4.3

First suppose that there is an edge $e \in E(R_1) - E(L_1)$ such that e is adjacent to neither e_2 nor e_3. Let $F_1 = L_1$ and $F_2 = G[\{e_2, e_3, e\}] \in \{P_3 + K_2, K_3\}$. Then F_1 and F_2 are edge-disjoint and $E(H_1) \subseteq E(F_1) \cup E(F_2)$. Since $R_2 = G - (E(F_1) \cup E(F_2))$ is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible.

Next, suppose that each edge in $E(R_1) - E(L_1)$ is incident with exactly one vertex in $\{u, v, w\}$. Since $|E(R_1) - E(L_1)| = t - 2 \geq 4$, this is impossible when $R_1 = tK_2$. Hence $R_1 = K_{1,t}$. Let $V(R_1) = \{x, x_1, x_2, \ldots, x_t\}$ and x is the central vertex of R_1 (see Figure 4.11). First, suppose that $x \in \{u, v, w\}$, say $x = u$ or $x = v$. Then $G - \{e_1, e_3\} = K_{1,t+1}$. (In fact, G is the graph obtained from K_3 by adding t pendant edges at one vertex of K_3.) However then, G cannot have an S-maximal 3-decomposition $\mathcal{D}_3$, a contradiction. Next, suppose that $x \notin \{u, v, w\}$. We may assume, without loss of generality, that $xx_i \notin E(L_1)$ for $1 \leq i \leq 4$. Since each edge xx_i ($1 \leq i \leq 4$) is incident with exactly one vertex in $\{u, v, w\}$, it follows that $x_i \in \{u, v, w\}$ for $1 \leq i \leq 4$, this is impossible. ■

Theorem 4.4.4 *The set $\{K_{1,3}, P_3 + K_2\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{K_{1,3}, P_3 + K_2\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where H_1, $L_i \in S$ $(i = 1, 2, 3)$ and neither R_1 nor R_3 contains a subgraph isomorphic to any graph in S.

Let $E(H_1) = \{e_1, e_2, e_3\}$. We may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| = t \geq 6$. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$. Furthermore, R_1 does not contain $K_{1,3}$ as a subgraph and so $R_1 = tK_2$. Since each $L_i - e_i$ is a subgraph of R_1 for $i = 1, 2, 3$ and $R_1 = tK_2$, it follows that $L_i \cong P_3 + K_2$ for $i = 1, 2, 3$. Thus, $\mathcal{D}_3$ is in fact a $(P_3 + K_2)$-maximal 3-decomposition.

First, suppose that $H_1 \cong P_3 + K_2$. Thus $\mathcal{D}_1$ is a $(P_3 + K_2)$-maximal 1-decomposition. Since $\mathcal{D}_3$ is a $(P_3 + K_2)$-maximal 3-decomposition and $P_3 + K_2$ is an ID-graph, it follows that G has a $(P_3 + K_2)$-maximal 2-decomposition (and so an S-maximal 2-decomposition) which is impossible.

Next, suppose that $H_1 \cong K_{1,3}$ (see Figure 4.13). We show that there is an edge in R_1 that is adjacent to an edge in H_1 and there are two edges in R_1 that are not adjacent to any edge in H_1. If there were no edge in R_1 that is adjacent to an edge in H_1, then $G = K_{1,3} + tK_2$. However then, G has no S-maximal 3-decomposition, which is impossible. Furthermore, since $R_1 = tK_2$ (where $t \geq 6$) and H_1 has exactly four vertices, at most four edges in R_1 can be adjacent to an edge in H_1. Hence at least two edges in R_1 are not adjacent to any edges in H_1.

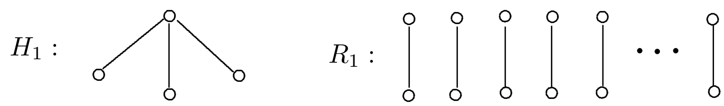


Figure 4.13: A step in the proof of Theorem 4.4.4

Let $f_1, f_2, f_3 \in R_1$ such that f_1 is adjacent an edge (say e_1) in H_1 and neither f_2 nor f_3 is adjacent to any edges in H_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$ and $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$. Since $E(H_1) \subseteq E(F_1) \cup E(F_2)$, it follows that $R_2 = G - E(F_1) \cup E(F_2)$ is a subgraph of R_1. Therefore, $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. ∎

Theorem 4.4.5 *The set $\{3K_2, P_3 + K_2\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{3K_2, P_3 + K_2\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where H_1, $L_i \in S$ ($i = 1, 2, 3$) and neither R_1 nor R_3 contains a subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$. We may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so

$|E(R_1)| = t \geq 6$. Since R_1 does not contain $P_3 + K_2$ or $3K_2$ as a subgraph, $R_1 = K_{1,t}$ or $R_1 = K_4$ (see Figure 4.14). We consider two cases, according to whether $H_1 \cong 3K_2$ or $H_1 \cong P_3 + K_2$.

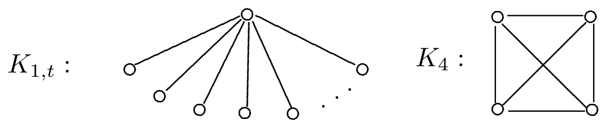


Figure 4.14: Two possible graphs for R_1 in the proof of Theorem 4.4.5

Case 1. $H_1 \cong 3K_2$. First, suppose that there is an edge $e \in E(R_1) - E(L_1)$ that is adjacent to neither e_2 nor e_3. Then let $F_1 = L_1$, $F_2 = G[\{e_2, e_3, e\}] \cong 3K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Next, suppose that every edge in $E(R_1) - E(L_1)$ is adjacent to e_2 or e_3. Since $|E(R_1) - E(L_1)| \geq 4$ and $R_1 = K_{1,t}$ or $R_1 = K_4$, there is an edge f such that f is adjacent to exactly one of e_2 and e_3. Let $F_1 = L_1$, $F_2 = G[\{e_2, e_3, f\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible.

Case 2. $H_1 \cong P_3 + K_2$. Let $H_1 - e_1 \cong P_3 = (u, v, w)$ where $e_2 = uv$ and $e_3 = vw$. First, suppose that there is an edge $e \in E(R_1) - E(L_1)$ such that e is adjacent to neither e_2 nor e_3. Let $F_1 = L_1$, $F_2 = G[\{e_2, e_3, e\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Hence we may assume that every edge in $E(R_1) - E(L_1)$ is adjacent to e_2 or e_3. Then R_1 cannot be K_4 and so $R_1 = K_{1,t}$ where $V(R_1) = \{x, x_1, x_2, \ldots, x_t\}$ and x is the central vertex of R_1 (see Figure 4.15). First, suppose that $x \in \{u, v, w\}$, say $x = u$ or $x = v$. Then $G - \{e_1, e_3\} \cong K_{1,t+1}$. However then, G cannot have an S-maximal 3-decomposition $\mathcal{D}_3$, a contradiction. Next, suppose that $x \notin \{u, v, w\}$. We may assume, without loss of generality, that $xx_i \notin E(L_1)$ for $1 \leq i \leq 4$. Since each edge xx_i ($1 \leq i \leq 4$) is incident with exactly one vertex in $\{u, v, w\}$, it follows that $x_i \in \{u, v, w\}$ for $1 \leq i \leq 4$, this is impossible. ■

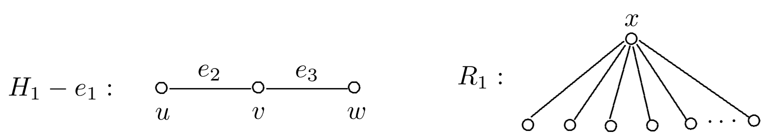


Figure 4.15: A step in the proof of Theorem 4.4.5

Theorem 4.4.6 *Each of the sets* $\{K_{1,3}, P_4\}$ *and* $\{K_3, P_4\}$ *is an ID-set.*

Proof. Let $S = \{K_{1,3}, P_4\}$ or $S = \{K_3, P_4\}$. Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2(III), G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but G has no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where each of H_1, L_1, L_2, L_3 is isomorphic to some graph in S and R_1 and R_3 contain no subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$ and we may assume that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| \geq 6$.

We now construct an S-maximal 2-decomposition $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ of G as follows. Let e_1 be an edge of H_1 such that $H_1 - e_1$ is connected. Then $H_1 - e_1 = P_3$

(see Figure 4.16) is a component of size 2 in R_1 at least one of whose two edges is adjacent to e_1. Let $F_1 = L_1$. To construct F_2, we consider two cases, according to whether $S = \{K_{1,3}, P_4\}$ or $S = \{K_3, P_4\}$.

$H_1 - e_1$:

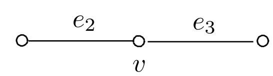


Figure 4.16: The graph $H_1 - e_1$ in the proof of Theorem 4.4.6

Case 1. $S = \{K_{1,3}, P_4\}$. For $H_1 - e_1 = P_3$, there is an edge in $L_2 - e_2$ (say f_2) that is adjacent to e_2 in $H_1 - e_1$ and an edge in $L_3 - e_3$ (say f_3) that is adjacent to e_3 in $H_1 - e_1$. If one of f_2 and f_3 is incident with the vertex v of degree 2 in $H_1 - e_1$, say f_2 is incident with v, then let $F_2 = G[\{e_2, e_3, f_2\}] \cong K_{1,3} \in S$. Thus, we may assume each of f_2 and f_3 is incident with the two end-vertices of $H_1 - e_1$. Then at least one of f_2 and f_3, say f_2, such that $G[\{e_2, e_3, f_2\}] \neq K_3$. Hence $G[\{e_2, e_3, f_2\}] \cong P_4$ and let $F_2 = G[\{e_2, e_3, f_2\}]$.

Case 2. $S = \{K_3, P_4\}$. We show that there is an edge in $E(R_1) - E(L_1)$ that is incident with an end-vertex of of $H_1 - e_1$. If this is not the case, then neither L_2 nor L_3 is K_3 and so $L_2 \cong L_3 \cong P_4$. Let $E(L_2) - e_2 = \{e_4, e_5\}$ and $E(L_3) - e_3 = \{e_6, e_7\}$, where then no edge in $\{e_4, e_5, e_6, e_7\}$ is incident with any end-vertex of $H_1 - e_1$. We may assume, without loss of generality, that L_2 and L_3 are the graphs shown in Figure 4.17. Since $G[\{e_4, e_5, e_6, e_7\}] \cong P_5$ is a subgraph of R_1, it follows that R_1 contains P_4 as a subgraph, which is a contradiction. Therefore, there is an edge $e \in E(R_1) - E(L_1)$ that is incident with an end-vertex of $H_1 - e_1$. Let $F_2 = G[e_2, e_3, e\}]$, which is either P_4 or K_3.

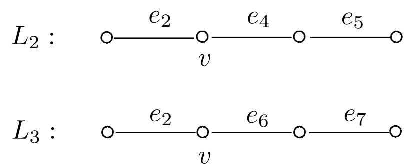


Figure 4.17: Graphs L_2 and L_3 in Case 2

In each case, $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$. Hence R_2 is a subgraph of R_1 and so R_2 contains no subgraph isomorphic to any graph in S. Therefore, $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. ■

Theorem 4.4.7 *The set* $\{3K_2, P_4\}$ *is an ID-set.*

Proof. Assume, to the contrary, that $S = \{3K_2, P_4\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where H_1, $L_i \in S$ $(i = 1, 2, 3)$ and neither R_1 nor R_3 contains an subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$ and we may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| = t \geq 6$. We claim the following:

$$H_1 = 3K_2 \text{ and } L_i = P_4 \text{ for } i = 1, 2, 3. \tag{4.6}$$

We first show that $H_1 = 3K_2$. Assume, to the contrary, that $H_1 = P_4 = (v_1, v_2, v_3, v_4)$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, 3$. We now show that $L_i \cong P_4$ for

$i = 1, 2, 3$. If this is not the case, then we may assume, without loss of generality, that $L_1 \cong 3K_2$ or $L_2 \cong 3K_2$. Consider these two cases.

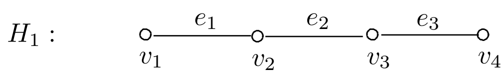


Figure 4.18: The graph H_1 in the proof of Theorem 4.4.7

Case 1. $L_1 \cong 3K_2$. Let $E(L_1) = \{e_1, f_1, f_2\}$ where $e_1 = v_1v_2$. Thus each f_i ($i = 1, 2$) is incident with neither v_1 nor v_2. We show that f_i ($i = 1, 2$) is not incident with v_3. For otherwise, we may assume that f_1 is incident with v_3. Let $F_1 = G[\{e_1, e_2, f_1\}] \cong P_4$, $F_2 = L_3$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Since f_1 and f_2 are nonadjacent, at most one of f_1 and f_2 can be incident with v_4. We may assume that f_1 is not incident with v_4. Let $F_1 = G[\{e_1, e_3, f_1\}] \cong 3K_2$, $F_2 = L_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Thus, $L_1 \cong P_4$. Similarly, $L_3 \cong P_4$.

Case 2. $L_2 \cong 3K_2$. Let $E(L_2) = \{e_2, g_1, g_2\}$ where $e_2 = v_2v_3$. Thus each g_i ($i = 1, 2$) is incident with neither v_2 nor v_3. We show that each g_i ($i = 1, 2$) is incident with neither v_1 nor v_4. For otherwise, we may assume that g_1 is incident with v_4. Let $F_1 = L_1$, $F_2 = G[\{g_1, e_2, e_3\}] \cong P_4$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Hence neither g_1 nor g_2 in L_2 is adjacent to any edge in $\{e_1, e_2, e_3\}$. Since $L_1 \cong P_4$ (by Case 1), there is an edge $f \in L_1 - \{e_1\}$

that is adjacent to $e_1 = v_1v_2$ and so f is incident with exactly one of v_1 and v_2. Thus G contains a subgraph F isomorphic to one of the graphs in Figure 4.19(a)–(e).

- If F is the graph in Figure 4.19(a), let $F_1 = G[\{f, e_1, e_2\}] \cong P_4$ and $F_2 = L_3$.
- If F is the graph in Figure 4.19(b)–(d), let $F_1 = G[\{e_1, f, e_3\}] \cong P_4$ and $F_2 = L_2$.
- If F is the graph in Figure 4.19(e), let $F_1 = G[\{f, e_2, e_3\}] \cong P_4$ and $F_2 = \{e_1, g_1, g_2\} \cong 3K_2$.

In each case, let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Thus, $L_2 \cong P_4$.

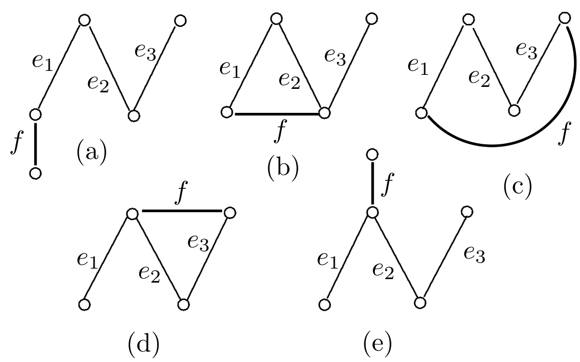


Figure 4.19: A step in the proof of Theorem 4.4.7

Therefore, if $H_1 \cong P_4$, then $L_i \cong P_4$ for $i = 1, 2, 3$. Hence $\mathcal{D}_1$ is a P_4-maximal 1-decomposition and $\mathcal{D}_3$ is a P_4-maximal 3-decomposition. However then, since P_4 is an ID-graph, G has a P_4-maximal 2-decomposition (and so an S-maximal 2-decomposition), which is impossible. Therefore, as claim, $H_1 = 3K_2$.

Next, we show that $L_i = P_4$ for $i = 1, 2, 3$. We consider two cases.

Case (i). At least two of L_1, L_2 and L_3 are isomorphic to $3K_2$, say $L_1 \cong L_2 \cong 3K_2$. Let $E(L_1) = \{e_1, f_1, f_2\}$ and $E(L_2) = \{e_2, g_1, g_2\}$. We show that each f_i ($i = 1, 2$) is adjacent to both e_2 and e_3 and each g_i ($i = 1, 2$) is adjacent to both e_1 and e_3. If this is not the case, we may assume that f_1 is not adjacent to e_2. Then let $F_1 = G[\{e_1, e_2, f_1\}] \cong 3K_2$, let $F_2 = L_3$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Therefore, we may assume that G contains a subgraph of Figure 4.20. Let $F_1 = G[\{g_1, e_1, g_2\}] \cong P_4$, $F_2 = G[\{e_2, f_1, e_3\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible.

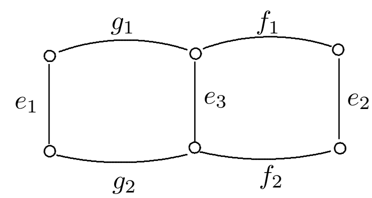


Figure 4.20: A step in Case (i) of the proof of Theorem 4.4.7

Case (ii). Exactly one of L_1, L_2 and L_3 is isomorphic to $3K_2$, say $L_1 \cong 3K_2$ and $L_i = P_4$ for $i = 2, 3$. Again, let $E(L_1) = \{e_1, f_1, f_2\}$. By the argument in Case (i), each of f_1 and f_2 is adjacent to e_2 and e_3. Thus, G contains the graph of Figure 4.21(a) as a subgraph. We first show that no edge in $L_2 - e_2$ can be adjacent to both e_1 and e_2 and no edge in $L_3 - e_3$ can be adjacent to both e_1 and e_3. If this is not case, we may assume that $g \in E(L_2 - e_2)$ that is adjacent to both e_1 and e_2. Let $F_1 = G[\{e_1, g, e_2\}] \cong P_4$, $F_2 = G[\{f_1, e_3, f_2\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Suppose that $E(L_2) = \{e_2, g_1, g_2\}$ and $E(L_3) = \{e_3, h_1, h_2\}$ where g_1 is adjacent to e_2 and h_1 is adjacent to e_3. Note that neither g_1 nor h_1 can be adjacent to both e_2 and e_3; for otherwise, we may assume that g_1 is adjacent to both e_2 and e_3. Then let $F_1 = L_1$, $F_2 = G[\{e_2, g_1, e_3\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Hence G contains one of the graphs of Figure 4.21(b)–(d) as a subgraph.

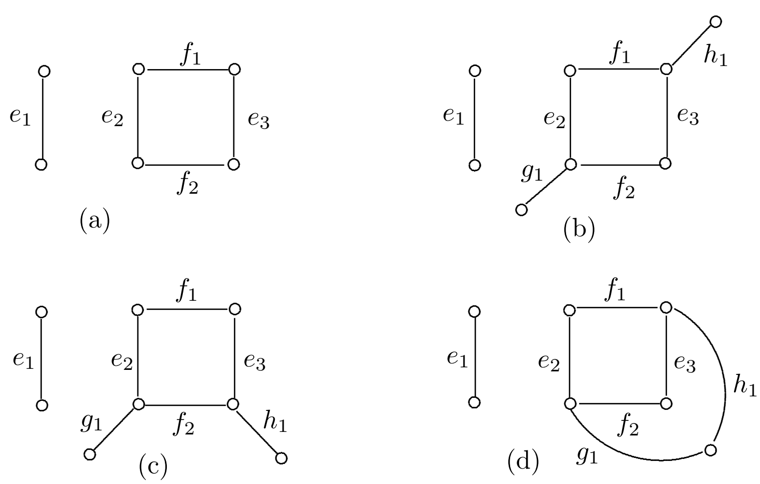


Figure 4.21: A step in Case (ii) of the proof of Theorem 4.4.7

First, suppose that G contains a subgraph isomorphic to the graph in Figure 4.21(b). Now let $F_1 = G[\{e_1, g_1, h_1\}] \cong 3K_2$, $F_2 = G[\{e_2, f_2, e_3\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Next, suppose that G contains a subgraph isomorphic to the graph in Figure 4.21(c) or in Figure 4.21(d). However then, R_1 contains the subgraph $G[\{g_1, f_2, h_1\}] \cong P_4$, which is a contradiction.

Therefore, $H_1 = 3K_2$ and $L_i = P_4$ for $i = 1, 2, 3$, as we claimed in (4.6). We now show that if an edge in $L_i - e_i$ that is adjacent to e_i, then this edge is not adjacent to any edges in $E(H_1) - \{e_i\}$ for $i = 1, 2, 3$. If this is not the case, we may assume that $f_1 \in E(L_1)$ is adjacent to e_1 and e_2. Let $F_1 = G[\{e_1, f_1, e_2\}] \cong P_4$, $F_2 = L_3$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Then $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Next, we show that each e_i is the interior edge of L_i for $i = 1, 2, 3$. If this is not the case, we may assume that $L_1 = (e_1, f_1, f_2)$. If f_2 is not adjacent to e_2, then let $F_1 = G[\{e_1, f_2, e_2\}] \cong 3K_2$, $F_2 = L_3$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Then $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Similarly, if f_2 is not adjacent to e_3, then there is an S-maximal 2-decomposition, which is impossible. Thus f_2 is adjacent to both e_2 and e_3. However then, since f_1 is adjacent to f_2, either f_1 and e_2 are adjacent or f_1 and e_3 are adjacent, which is impossible. Therefore, e_i is the interior edge of L_i for $i = 1, 2, 3$, as claimed.

Let $L_1 = (f_1, e_1, f_2)$, $L_2 = (g_1, e_2, g_2)$ and $L_3 = (h_1, e_3, h_2)$. It then follows from the argument above that L_1, L_2 and L_3 have the following properties:

(a) No edge in L_i $(i = 1, 2, 3)$ is adjacent to any edges in $\{e_1, e_2, e_3\} - \{e_i\}$.

(b) Since R_1 contains no subgraph isomorphic to P_4, it follows that $\{f_1, f_2\}$, $\{g_1, g_2\}$ and $\{h_1, h_2\}$ are sets of two independent edges.

Since R_1 contains no $3K_2$, there are adjacent edges in $\{f_1, f_2, g_1, g_2, h_1, h_2\}$. By (a) and (b), we may assume that f_1 and g_1 are adjacent. Let $F_1 = G[\{e_1, f_1, g_1\}] \cong P_3$, $F_2 = \{f_2, e_2, e_3\} \cong 3K_2$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Then $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. ∎

4.4.2 The 3-Element or 4-Element ID-Sets

By Observation 4.3.1, the set $\{P_4,\ K_3,\ K_{1,3},\ P_3 + K_2,\ 3K_2\}$ consisting of all graphs of size 3 is an ID-set. Also, by the table in Figure 4.9, we have the following:

(1) A graph H of size 2 or 3 is an ID-graph unless $H \in \{K_3, K_{1,3}, 3K_2\}$.

(2) Every 2-element subset S of $\{P_4,\ K_3,\ K_{1,3},\ P_3 + K_2,\ 3K_2\}$ is an ID-set unless S is a 2-element subset of $\{K_3, K_{1,3}, 3K_2\}$.

Therefore, it remains to determine the ID-sets and non-ID sets that are subsets S of $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ with $|S| = 3$ or $|S| = 4$.

We first show that neither of the sets $\{K_3, 3K_2, P_4\}$ and $\{K_{1,3}, 3K_2, P_4\}$ is an ID set. For $\{K_3, 3K_2, P_4\}$, let $G = K_3 + 2K_{1,3}$ be the union of K_3 and two copies of $K_{1,3}$. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$, where $H_1 \cong K_3$ and $R_1 \cong 2K_{1,3}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ where $L_1 \cong L_2 \cong L_3 \cong 3K_2$ and R_3 is an empty graph. However, G has no S-maximal 2-decomposition. For $\{K_{1,3}, 3K_2, P_4\}$, let $G = 2K_3 + K_{1,3}$ be the union of two copies of K_3 and $K_{1,3}$. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$, where $H_1 \cong K_{1,3}$ and $R_1 \cong 2K_3$ and an S-maximal 3-decomposition

$\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ where $L_1 \cong L_2 \cong L_3 \cong 3K_2$ and R_3 is an empty graph. However, G has no S-maximal 2-decomposition. Hence neither set is an ID-set.

Next, we show that if $S \subseteq \{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ with $|S| \in \{3, 4\}$ such that S is neither $\{K_3, 3K_2, P_4\}$ nor $\{K_{1,3}, 3K_2, P_4\}$, then S is an ID-set. In order to do this, we first present three useful observations (some of which we already saw earlier).

Observation 4.4.8 *Let S be a non-ID-set of graphs of size 3. If G is a minimum non-IDP-S graph having an S-maximal 1-decomposition $\{H_1, R_1\}$, then the size of R_1 is at least 6.*

Observation 4.4.9 *Suppose that R is a graph without isolated vertices having size $t \geq 6$.*

(a) If R does not contain $P_3 + K_2$ as a subgraph, then $R = tK_2$, $R = K_{1,t}$ or $R = K_4$.

(b) If R does not contain $3K_2$ as a subgraph, then R has at most two components and $R = K_{1,t}$, $R = K_{1,r} + K_{1,s}$ where $1 \leq r \leq s$ and $r + s = t$, $R = 2K_3$ or $R = K_3 + K_{1,t-3}$.

(c) If R does not contain P_4 as a subgraph, then each component of R is K_3 or stars.

Observation 4.4.10 *Suppose that S is a non-ID-set of graphs and S contains an ID-subset S_0. If G is a non-IDP-S graph such that G has an S-maximal a-decomposition $\mathcal{D}_a$ =$\{H_1,\ H_2,\ \ldots,\ H_a,\ R_a\}$ and an S-maximal b-decomposition*

$\mathcal{D}_b = \{L_1, L_2, \ldots, L_b, R_b\}$ but no S-maximal k-decomposition for every integer k with $a < k < b$, then either $H_i \in S - S_0$ for some $i \in \{1, 2, \ldots, a\}$ or $L_j \in S - S_0$ for some $j \in \{1, 2, \ldots, b\}$.

Proposition 4.4.11 *The set $\{K_{1,3}, K_3, P_4\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{K_{1,3}, K_3, P_4\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. Since each graph in S has size 3, it follows by Theorem 4.3.2(III) that G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but G has no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where each of H_1, L_1, L_2, L_3 is isomorphic to some graph in S and R_1 and R_3 contain no subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$. We may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 4.3.2(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| \geq 6$. Furthermore, each component of R_1 has size at most 2.

We now construct an S-maximal 2-decomposition $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ of G as follows. Let $F_1 = L_1 \in S$. Now, let $e \in E(L_2) - \{e_2\}$ that is adjacent to e_2. Then the subgraph $F_2 = G[\{e_2, e_3, e\}]$ induced by $\{e_2, e_3, e\}$ is a connected subgraph of size 3 and so $F_2 \in S$. Furthermore, $E(F_1) \cap E(F_2) = \emptyset$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$, it follows that R_2 is a subgraph of R_1 and so R_2 contains no subgraph isomorphic to any graph in S. Therefore, $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. ■

Proposition 4.4.12 *Each of the following sets is an ID-set:*

$$\{K_{1,3}, K_3, 3K_2\},\ \{K_{1,3}, K_3, 3K_2, P_4\} \text{ and } \{K_{1,3}, K_3, 3K_2, P_3 + K_2\}. \tag{4.7}$$

Proof. Let S be one of the sets in (4.7). Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $|E(R_1)| = t \geq 6$ by Observation 4.4.8. Since R_1 does not contain $3K_2$ as a subgraph, $R_1 = K_{1,t}$, $R_1 = K_{1,r} + K_{1,s}$ where $1 \leq r \leq s$ and $r + s = t$, $R_1 = 2K_3$ or $R_1 = K_3 + K_{1,t-3}$ by Observation 4.4.9(b). Since R_1 contains neither $K_{1,3}$ nor K_3 and $|E(R_1)| = t \geq 6$, this is impossible. ■

Proposition 4.4.13 *Each of the following sets is an ID-set:*

$$\{K_{1,3}, 3K_2, P_3 + K_2\} \text{ and } \{K_{1,3}, 3K_2, P_3 + K_2, P_4\}. \tag{4.8}$$

Proof. Let S be one of the sets in (4.8). Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 4.3.2 then, G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $|E(R_1)| = t \geq 6$ by Observation 4.4.8. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 4.4.9(a). Since R_1 contains neither $K_{1,3}$ nor $3K_2$ as a subgraph and $t \geq 6$, this is impossible. ■

Proposition 4.4.14 *The set $\{K_{1,3}, K_3, P_3 + K_2\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{K_{1,3}, K_3, P_3 + K_2\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$

but no S-maximal 2-decomposition. Then $|E(R_1)| = t \geq 6$ by Observation 4.4.8. Since R_1 does not contain P_3+K_2 as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 4.4.9(a). Since R_1 contains neither $K_{1,3}$ nor K_3 as a subgraph and $t \geq 6$, it follows that $R_1 = tK_2$. Let $E(H_1) = \{e_1, e_2, e_3\}$, where say $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. Since $R_1 = tK_2$, it follows that $L_i - e_i = 2K_2$ and so $L_i = P_3 + K_2$ for $i = 1, 2, 3$.

- If $H_1 = K_{1,3}$, then let $S' = \{K_{1,3}, P_3 + K_2\}$.
- If $H_1 = K_3$, then let $S' = \{K_3, P_3 + K_2\}$.
- If $H_1 = P_3 + K_2$, then let $S' = \{P_3 + K_2\}$.

In each case, S' is an ID-set. Since $\mathcal{D}_1$ is an S'-maximal 1-decomposition and $\mathcal{D}_3$ is an S'-maximal 3-decomposition, it follows that G has an S'-maximal 2-decomposition, which is a contradiction. ■

Proposition 4.4.15 *Each of the following sets is an ID-set:*

$$\{K_{1,3}, P_4, P_3 + K_2\} \text{ and } \{K_{1,3}, K_3, P_4, P_3 + K_2\}. \tag{4.9}$$

Proof. Let S be one of the sets in (4.9). Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Then $|E(R_1)| = t \geq 6$ by Observation 4.4.8. Since R_1 does not contain P_3+K_2 as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 4.4.9(a). Since R_1 contains neither $K_{1,3}$ nor

P_4 as a subgraph and $t \geq 6$, it follows that $R_1 = tK_2$. Let $E(H_1) = \{e_1, e_2, e_3\}$, where say $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. We consider two cases.

Case 1. $S = \{K_{1,3}, P_4, P_3 + K_2\}$. Then $L_i \in \{P_4, P_3 + K_2\}$. Since $\{P_4, P_3 + K_2\}$ is an ID-set, it follows that $H_1 \cong K_{1,3}$. Because $\{K_{1,3}, P_4\}$ and $\{K_{1,3}, P_3 + K_2\}$ are both ID-sets, at least one of L_i $(1 \leq i \leq 3)$ is P_4 and at least one of L_i $(1 \leq i \leq 3)$ is $P_3 + K_2$. We may assume that $L_1 \cong P_4 = (f_1, e_1, f_2)$ where e_1 is the middle edge of L_1 and $L_2 \cong P_3 + K_2$ where e_2 is adjacent an edge in L_2, say e_2 is adjacent to g in L_2. This implies that G contains a subgraph isomorphic to one of the graphs in Figures 4.22(a) and 4.22(b), where the edges in L_1 are drawn in bold. In each case, let $F_1 = L_1$ and $F_2 = G[\{e_2, e_3, g\}]$. Thus $F_2 \cong K_{1,3}$ or $F_2 \cong P_4$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$, $E(F_1) \cap E(F_2) = \emptyset$, and R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

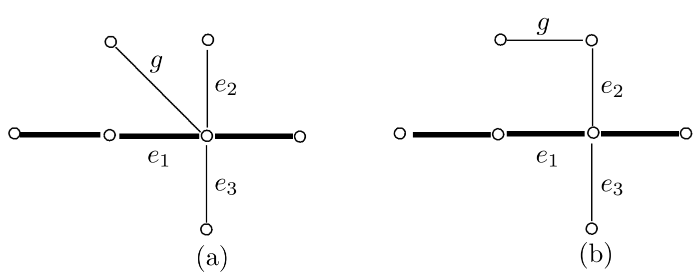


Figure 4.22: A step in the proof of Proposition 4.4.15

Case 2. $S = \{K_{1,3}, K_3, P_4, P_3 + K_2\}$. Then $L_i \in \{P_4, P_3 + K_2\}$, where $i = 1, 2, 3$. In particular, $L_i \not\cong K_3$ for $i = 1, 2, 3$. Furthermore, we claim that $H_1 \not\cong K_3$. If this were not the case, then observe that at least one edge of R_1 is adjacent to some edge of H_1; for otherwise, $G = K_3 + tK_2$ and G cannot have an S-maximal 3-decomposition. On the other hand, since $t \geq 6$, at least two edges of R_1 are not adjacent to any edge of H_1. Let $f_1, f_2, f_3 \in E(R_1)$ such that f_1 is adjacent to some edge of H_1, say f_1 is adjacent to e_1, while neither f_2 nor f_3 is adjacent to any edge of H_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$, $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$, $E(F_1) \cap E(F_2) = \emptyset$ and R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. Thus, $H_1 \ncong K_3$, as claimed. Hence $H_1, L_i \in S' = \{K_{1,3}, P_4, P_3 + K_2\}$ for $i = 1, 2, 3$. Since S' is an ID-set by Case 1, it follows that G has an S-maximal 2-decomposition, which, again, is impossible. ■

In order to show that the remaining sets of graphs of size 3 are ID-sets, we first present a lemma.

Lemma 4.4.16 *Let $S = \{K_3, P_4, P_3 + K_2\}$. If G is a minimum non-IDP-S graph and $\mathcal{D}_1 = \{H_1, R_1\}$ is an S-maximal 1-decomposition, then $R_1 \neq tK_2$ where $t \geq 6$.*

Proof. Assume, to the contrary, that $R_1 = tK_2$ where $t \geq 6$. Since G is a minimum non-IDP-S graph, G also has an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Let $E(H_1) = \{e_1, e_2, e_3\}$, where say $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. Since $R_1 = tK_2$, each e_i $(i = 1, 2, 3)$ must be adjacent to some edge of R_1; for otherwise, at least one of L_1, L_2, L_3 is $3K_2$, which is impossible.

First, suppose that $H_1 = K_3$ or $H_1 = P_4$. Then at least one edge of R_1 is adjacent to some edge of H_1. Since $R_1 = tK_2$ and $t \geq 6$, at least two edges of R_1 are not adjacent to any edge of H_1. Let $f_1, f_2, f_3 \in E(R_1)$ such that f_1 is adjacent to some edge of H_1, say f_1 is adjacent to e_1, while neither f_2 nor f_3 is adjacent to any edge of H_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$, $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$, it follows that R_2 is a subgraph of R_1 and $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

Next, suppose that $H_1 = P_3 + K_2$. We may assume that e_1 and e_2 are adjacent to edges in H_1. Since $t \geq 6$, there is an edge in R_1 that is not adjacent to any edge of H_1. Furthermore, since G is connected, at least one of e_1 and e_2 is adjacent to some edge of R_1 and e_3 is adjacent to some edge of R_1. Let $f_1, f_2, f_3 \in E(R_1)$ such that f_1 is not adjacent to any edge of H_1, f_2 is adjacent to e_2 and f_3 is adjacent to e_3. First, suppose that $f_2 \neq f_3$. Let $F_1 = G[\{e_2, f_1, f_2\}] \cong P_3 + K_2$, $F_2 = G[\{e_1, e_3, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Next, suppose that $f_2 = f_3$. Since $t \geq 6$, there is $f_4 \in E(R_1) - \{f_1\}$ such that f_4 is not adjacent to any edge of H_1. Let $F_1 = G[\{e_1, e_2, f_4\}] \cong P_3 + K_2$, $F_2 = G[\{e_3, f_1, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. In either case, $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$. Therefore, R_2 is a subgraph of R_1 and so $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. ■

Proposition 4.4.17 *Each of the following sets is an ID-set:*

$$\{K_3, 3K_2, P_3 + K_2\},\ \{K_3, P_4, P_3 + K_2\},$$

$$\{3K_2, P_4, P_3 + K_2\},\ \{K_3, 3K_2, P_4, P_3 + K_2\}.$$

Proof. Let S be one of the sets described above. Assume, to the contrary, that S is not an ID-set. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Then $|E(R_1)| = t \geq 6$ by Observation 4.4.8. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 4.4.9(a). For each set S under consideration, it follows that (i) either $3K_2 \in S$ or $S = \{K_3, P_4, P_3 + K_2\}$ and (ii) either $P_4 \in S$ or $K_3 \in S$. Hence $R_1 \neq tK_2$ (by Lemma 4.4.16) and $R_1 \neq K_4$. Therefore, $R_1 = K_{1,t}$. Let $E(R_1) = \{f_1, f_2, \ldots, f_t\}$ where $t \geq 6$ and let $E(H_1) = \{e_1, e_2, e_3\}$ where, say, $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. First, we make an observation. For $i = 1, 2, 3$, since (a) $L_i \in S$ and $K_{1,3} \notin S$ and (b) $L_i - e_i$ is a subgraph of R_1 and $R_1 = K_{1,t}$, it follows that e_i is not incident with the center vertex of R_1 (or e_i cannot be adjacent to all edges in R_1). Since $H_1 \in \{K_3, 3K_2, P_4, P_3 + K_2\}$, we consider four cases.

Case 1. $H_1 = K_3$. Then at least three edges in R_1 that are not adjacent to any edges in H_1, say f_1, f_2 and f_3 are three such edges in R_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2 \in S$, $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2 \in S$ and $R_2 = G - (E(F_1) \cup E(F_2))$.

Case 2. $H_1 = 3K_2$. Then $\{3K_2, P_3 + K_2\} \subseteq S$. We may assume that e_1 is adjacent to f_1 (and possibly to f_2), e_2 is adjacent to f_3 (and possibly to f_4) and e_3 is adjacent to f_5 (and possibly to f_6). More precisely, neither e_1 nor e_3 is adjacent to f_3 and e_2 is not adjacent to f_1 or f_2. Let $F_1 = G[\{e_1, e_3, f_3\}] \cong 3K_2 \in S$, $F_2 = G[\{e_2, f_1, f_2\}] \cong P_3 + K_2 \in S$ and $R_2 = G - (E(F_1) \cup E(F_2))$.

Case 3. $H_1 = P_4$. As we observed earlier, no edge in H_1 is incident with the center vertex of R_1. Thus at least two edges in R_1 are not adjacent to any edge in H_1, say f_1 and f_2 are two such edges in R_1. We may assume that $H_1 = (e_1, e_2, e_3)$. Then there is $f_3 \in E(R_1) - \{f_1, f_2\}$ such that f_3 is not adjacent to e_1. Let $F_1 = G[\{e_1, f_1, f_3\}] \cong P_3 + K_2 \in S$ where $E(P_3) = \{f_1, f_3\}$, $F_2 = G[\{e_2, e_3, f_2\}] \cong P_3 + K_2 \in S$, where $E(P_3) = \{e_2, e_3\}$, and $R_2 = G - (E(F_1) \cup E(F_2))$.

Case 4. $H_1 = P_3 + K_2$. Let e_1 and e_2 be the two adjacent edges in H_1. We may assume (i) f_1 is not adjacent to e_1 or e_2 and (ii) f_2 and f_3 are not adjacent to e_3. Let $F_1 = G[\{e_1, e_2, f_1\}] \cong P_3 + K_2 \in S$, $F_2 = G[\{e_3, f_2, f_3\}] \cong P_3 + K_2 \in S$ and $R_2 = G - (E(F_1) \cup E(F_2))$.

In each case, $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$. Hence R_2 is a subgraph of R_1 and so $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. ■

In summary, we have the following.

Theorem 4.4.18 *A subset S of the set $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ of all graphs of size 3 without isolated vertices is an ID-set if and only if S is not one of the following eight sets:*

$$\{3K_2\},\ \{K_3\},\ \{K_{1,3}\},\ \{3K_2, K_3\},\ \{3K_2, K_{1,3}\},$$
$$\{K_3, K_{1,3}\},\ \{3K_2, K_3, P_4\},\ \{3K_2, K_{1,3}, P_4\}.$$

Chapter 5

Topics for Further Study

5.1 Eulerian Irregularity Spectrum

Let G be a nontrivial connected graph of size m. Recall that an *Eulerian walk* in G is a closed walk that contains every edge of G. An *irregular Eulerian walk* in G is an Eulerian walk that encounters no two edges of G the same number of times. The minimum length of an irregular Eulerian walk in G, is referred to as the *Eulerian irregularity* of G, which is denoted by $EI(G)$. For a set $\mathcal{G}$ of connected graphs with a prescribed property, define the *Eulerian irregularity spectrum* $S(\mathcal{G})$ of $\mathcal{G}$ to be the set of all values of $EI(G)$ where $G \in \mathcal{G}$; that is, $S(\mathcal{G}) = \{EI(G) : G \in \mathcal{G}\}$. Recall the following result on the Eulerian irregularities of graphs in Chapter 2.

Theorem 5.1.1 *If G is a nontrivial connected graph G of size m, then*

$$\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}.$$

Furthermore,

(a) $EI(G) = 2\binom{m+1}{2}$ *if and only if G is a tree.*

(*b*) *$EI(G) = \binom{m+1}{2}$ if and only if G contains a subgraph of size $\lceil m/2 \rceil$, every vertex of which is even.*

For a positive integer m, let $\mathcal{G}_m$ be the set of all connected graphs of size m. It then follows by Theorem 5.1.1 that both $\binom{m+1}{2}$ and $2\binom{m+1}{2}$ are elements of $S(\mathcal{G}_m)$. Also, recall the following realization result on the Eulerian irregularities of graphs (Theorem 2.4.5 in Chapter 2).

Theorem 5.1.2 *Let k and m be positive integers with $\binom{m+1}{2} \le k \le 2\binom{m+1}{2}$. Then there exists a nontrivial connected graph G of size m with $EI(G) = k$ if and only if there exists an integer x with $0 \le x \le m$ and $x \ne 1, 2$ such that $x^2 + (m-x)(m-x+1) = k$.*

By Theorem 5.1.1, if $\mathcal{T}_n$ is the set of all trees of order n, then $S(\mathcal{T}_n) = \{n(n-1)\}$. For a nontrivial connected graph G that is not a tree, it follows by the proof of Theorem 5.1.1 that if e is a bridge of G and W is an Eulerian walk in G, then e must be encountered an even number of times on W. Thus every bridge of G must be encountered at least twice on any Eulerian walk in G. Also, recall the following theorem that appeared in Chapter 2.

Kwan's Theorem *Let G be a connected graph and let W be a closed walk of minimum length containing every edge of G at least once. Then W encounters no edge of G more than twice and no more than half of the edges in any cycle appear twice.*

We plan to study the Eulerian irregularity spectra for some classes of connected graphs that are not tree. In particular, we are interested in the Eulerian irregularity spectrum of the set of all 2-connected graphs having a fixed size.

5.2 Proper Eulerian Walks

An Eulerian walk W in a connected graphs G is *proper* if every two adjacent edges of G are encountered a different number of times in W. This is equivalent to assigning positive integer weights to the edges of G such that the resulting edge labeling is also a proper edge coloring of G and the degree of every vertex in the resulting weighted graph is even. This is shown in Figure 5.1. Since every connected graph has an irregular Eulerian walk and every irregular Eulerian walk is proper, it follows that every connected graph has a proper Eulerian walk.

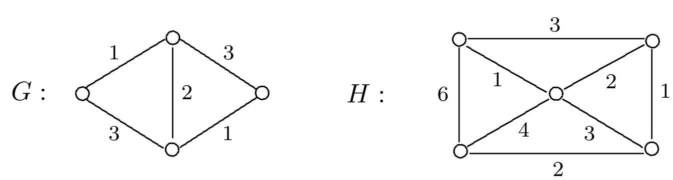


Figure 5.1: Proper Eulerian walks

The *Eulerian chromaticity* of G is the minimum length of a proper Eulerian walk W in G and is denoted by $EC(G)$. Since every irregular Eulerian walk is proper, $EC(G) \leq EI(G)$ for every connected graph G. This is equivalent to minimizing the *sum* of the weights that can be assigned to the edges of G such that the resulting edge labeling is also a proper edge coloring of G and the degree of every vertex in the resulting weighted graph is even.

Let G be a connected graph of size $m \geq 1$ with $E(G) = \{e_1, e_2, \ldots, e_m\}$. If W is an Eulerian walk in G such that e_i is encountered exactly a_i times in W for $i = 1, 2, \ldots, m$, then let $C_W = \{a_1, a_2, \ldots, a_m\}$. Then C_M is in general a multiset of positive integers. The cardinality $|C_W|$ of C_W is the number of distinct elements in C_W. If W is a proper Eulerian walk of G, then $\chi'(G) \leq |C_W| \leq m$ and $|C_W| = m$ if and only if W is irregular. This gives rise to two other concepts.

- The *chromatic Eulerian value* of G is the minimum positive integer p such that G has a proper Eulerian walk W in which no edge of G is encountered more than p times in W. If G is a connected graph of size $m \geq 1$ with $E(G) = \{e_1, e_2, \ldots, e_m\}$ and each edge e_i $(1 \leq i \leq m)$ of G is replaced by $2i$ parallel edges, then the resulting multigraph M is Eulerian and each Eulerian circuit in M gives rise to an irregular Eulerian walk in which each edge e_i of G appears exactly $2i$ times in the walk. This irregular Eulerian walk is proper in which every edge of G is encountered at most $2m$ times and so the chromatic Eulerian value of G is at most $2m$. This is equivalent to minimizing the *maximum value* of the weights that can be assigned to the edges of G such that the resulting edge labeling is also a proper edge coloring of G and the degree of every vertex in the resulting weighted graph is even.

- The *chromatic Eulerian index* of G is the minimum cardinality of C_W over all proper Eulerian walks W in G and is denoted by $\chi'_E(G)$. This is equivalent to minimizing the number of weights that can be assigned to the edges of G such that the resulting edge labeling is also a proper edge coloring of G and the degree of every vertex in the resulting weighted graph is even.

5.3 Consecutive Eulerian Walks

An Eulerian walk W in a connected graphs G is *consecutive* if W has the property that every two consecutive edges in W are encountered a different number of times. In this case, the graph G cannot contain an end-vertex. Thus, we only consider connected graphs with minimum degree at least 2. If W is a proper Eulerian walk in G, then W is consecutive but the converse is not true in general. A problem here concerns minimizing the *maximum value* of the weights or minimizing the *sum* of the weights that can be assigned to the edges of G to produce a consecutive Eulerian walk in G.

5.4 On Maximal Decompositions in Graphs

First, we review some definitions and notation described in Chapter 4. For two graphs H and G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called an *H-maximal k-decomposition* if $H_i \cong H$ for $1 \leq i \leq k$ and R contains no subgraph isomorphic to H. If G contains no subgraph isomorphic to H, then $k = 0$ and $R = G$. For graphs H and G, let

$$\begin{aligned}\text{Min}(G, H) &= \min\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}\\ \text{Max}(G, H) &= \max\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}.\end{aligned}$$

A graph H is said to possess the *intermediate decomposition property* (IDP) and H is called an *ID-graph* if for each graph G and each integer k with $\text{Min}(G, H) \leq$

$k \leq \text{Max}(G, H)$, there exists an H-maximal k-decomposition of G.

For a set S of graphs and a graph G, a decomposition $\mathcal{D}$ ={H_1, H_2, …, H_k, R} of G is called an *S-maximal k-decomposition* if $H_i \cong H$ for some $H \in S$ for each integer i with $1 \leq i \leq k$ and R contains no subgraph isomorphic to any subgraph in S. For a set S of graphs without isolated vertices and a graph G, let

$$\begin{aligned}\text{Min}(G, S) &= \min\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\}\\ \text{Max}(G, S) &= \max\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\}.\end{aligned}$$

A set S of graphs without isolated vertices is said to possess the *intermediate decomposition property* (IDP) and S is called an *ID-set* if for every graph G and each integer k with $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. As we have seen Theorem 4.4.18, a subset S of the set $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ of all graphs of size 3 without isolated vertices is an ID-set if and only if S is not one of the following eight sets:

$$\{3K_2\},\ \{K_3\},\ \{K_{1,3}\},\ \{3K_2, K_3\},\ \{3K_2, K_{1,3}\},$$
$$\{K_3, K_{1,3}\},\ \{3K_2, K_3, P_4\},\ \{3K_2, K_{1,3}, P_4\}.$$

For a set S of graphs, a graph G is said to have the *intermediate decomposition property with respect to S* (IDP-S) if for each integer k with $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. In this case, the graph G is referred to as an *IDP-S* graph; otherwise, G is a *non-IDP-S* graph. Recall that for a set S of graphs without isolated vertices that is not an ID-set, a graph G of minimum size that is not an IDP-S graph (as described in

Theorem 4.3.2) is referred to as a *minimum non-IDP-S graph.* If $S = \{H\}$ consists of a single graph H, then a minimum non-IDP-S graph is also referred to as a *minimum non-IDP-H graph.*

5.4.1 Path-Cycle Sets of Graphs Having Size at Most 4

We plan to investigate the problem of determining which subsets of the set of all graphs of size at most 4 without isolated vertices are ID-sets. Since this is a challenging problem in general, we begin with those 2-element subsets consisting of paths and cycles, which we referee to as *path-cycle sets.* The following is a consequence of Theorem 4.3.2(III) and Theorem 4.4.18.

Corollary 5.4.1 *Each of the sets* $\{P_3, C_3\}$, $\{P_4, C_3\}$ *and* $\{P_3, C_4\}$ *is an ID-set.*

Theorem 5.4.2 *The set* $\{P_4, C_4\}$ *is an ID-set.*

Proof. Assume, to the contrary, that $S = \{C_4, P_4\}$ is not an ID-set. Let G be a graph of minimum size that is not an IDP-S graph. Then there are smallest integers a and b where $1 \leq a \leq b - 2$ such that G has an S-maximal a-decomposition $\mathcal{D}_a$ and an S-maximal b-decomposition $\mathcal{D}_b$ but G has no S-maximal k-decomposition for every integer k with $a < k < b$. Since each graph in $\mathcal{D}_a$ has size at most 4, it follows by Theorem 4.3.2(III) that $b \leq 4$. Hence there are three possibilities for a and b, namely (i) $a = 1$ and $b = 3$, (ii) $a = 1$ and $b = 4$ or (iii) $a = 2$ and $b = 4$. First, we show that

$$P_4 \notin \mathcal{D}_a \text{ for any } S\text{-maximal } a\text{-decomposition } \mathcal{D}_a \text{ of } G. \tag{5.1}$$

Assume, to the contrary, that there is an S-maximal a-decomposition $\mathcal{D}_a$ of G such that $P_4 \in \mathcal{D}_a$. Since the size of P_4 is 3, it follows by Theorem 4.3.2(III) that $b = 3$ and so $a = 1$. Let $\mathcal{D}_1 = \{H_1, R_1\}$ where $H_1 = P_4$ and let $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ be an S-maximal 3-decomposition of G. Suppose that $E(H_1) = \{e_1, e_2, e_3\}$. By Theorem 4.3.2(II), we may assume that $e_i \in E(L_i)$ for $i = 1, 2, 3$ and then $L_i - e_i$ is a subgraph of R_1. If $L_i = C_4$ for some $i \in \{1, 2, 3\}$, say $L_1 = C_4$, then $L_1 - e_1 \cong P_4$ is a subgraph of R_1, which is impossible. Thus $L_i = P_4$ for $i = 1, 2, 3$. Thus $\mathcal{D}_1$ is a P_4-maximal 1-decomposition of G and $\mathcal{D}_3$ is a P_4-maximal 3-decomposition of G. Since P_4 is an ID-graph, G has a P_4-maximal 2-decomposition (and so an S-maximal 2-decomposition), which is impossible. Thus (5.1) holds. Next, we show that

$$C_4 \notin \mathcal{D}_b \text{ for any } S\text{-maximal } b\text{-decomposition } \mathcal{D}_b \text{ of } G. \tag{5.2}$$

Suppose that C_4 belongs to some S-maximal b-decomposition

$$\mathcal{D}_b = \{L_1, L_2, \ldots, L_b, R_b\}$$

of G where $b = 3, 4$. We may assume that $L_1 = C_4$. Let $\mathcal{D}_a = \{H_1, \ldots, R_1\}$ be an S-maximal a-decomposition of G where $a = 1$ or $a = 2$. Then $H_1 = C_4$ by (5.2). By Theorem 4.3.2(II), $|E(L_1) \cap E(H_1)| = 1, 2$. If L_1 contains exactly one edge e of H_1, then $L_1 - e = P_4 \in R_b$, which is a contradiction. Thus L_1 contains exactly two edges of H_1. This implies that the subgraph $F = G[E(H_1) \cup E(L_1)]$ induced by $E(H_1) \cup E(L_1)$ is one of the graphs shown in Figure 5.2, where $L_1 = (v_1, v_2, v_3, v_4, v_1)$ and the bold edges are edges of H_1. In each case, F can be decomposed into two edge-disjoint copies F_1 and F_2 of P_4. Since $E(H_1) \subseteq E(F_1) \cup E(F_2)$, it follows that $R_2 = G - E(F_1) \cup E(F_2)$ is a subgraph of R_1. Therefore, $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. Consequently, $L_i = P_4$ for each $i = 1, 2, \ldots, b$ and (5.2) holds.

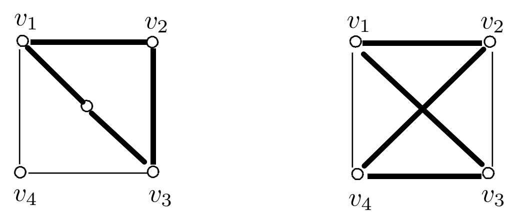


Figure 5.2: The subgraph $F = G[E(H_1) \cup E(L_1)]$

We consider two cases, according to whether $a = 1$ or $a = 2$.

Case 1. $a = 1$. By (5.1) and (5.2), let $\mathcal{D}_1 = \{H_1, R_1\}$ where

$$H_1 \cong C_4 = (v_1, v_2, v_3, v_4, v_1)$$

and $\mathcal{D}_b = \{L_1, L_2, \ldots, L_b, R_b\}$ where $L_i \cong P_4$ for $1 \leq i \leq b$ and $b = 3, 4$. By Theorem 4.3.2(II), it follows that $E(H_1) \cap E(L_i) \neq \emptyset$ for $1 \leq i \leq b$. Let $E(H_1) = \{e_1, e_2, e_3, e_4\}$. We may assume that $e_i \in E(L_i)$ for $i = 1, 2, 3$. Since $L_i \cong P_4$, there is an edge $f_i \in E(L_i)$ such that f_i is adjacent to e_i for $i = 1, 2, 3$. Hence G contains a connected subgraph F of size 6 that is isomorphic to one of the five graphs in Figure 5.3 and all of these five graphs contains C_4 as a subgraph. In each case, F can be decomposed into two copies F_1 and F_2 of P_4 (one of which is drawn in bold in Figure 5.3). Since $E(H_1) \subseteq E(F_1) \cup E(F_2)$, it follows that $R_2 = G - E(F_1) \cup E(F_2)$ is a subgraph of R_1. Therefore, $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

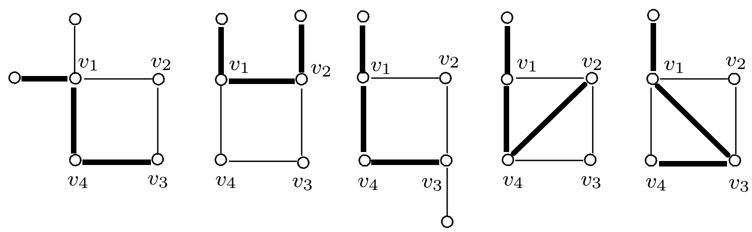


Figure 5.3: Subgraphs in Case 1

Case 2. $a = 2$. Let $\mathcal{D}_2 = \{H_1, H_2, R_2\}$ where $H_1 \cong H_2 \cong C_4$ and

$$\mathcal{D}_b = \{L_1,\ L_2,\ L_3, L_4,\ R_4\}$$

where $L_i \cong P_4$ for $1 \leq i \leq 4$. Suppose that $H_1 = (v_1, v_2, v_3, v_4, v_1)$. As in Case 1, $E(H_1) \cap E(L_i) \neq \emptyset$ for $1 \leq i \leq 4$. Let $E(H_1) = \{e_1, e_2, e_3, e_4\}$. We may assume that $e_i \in E(L_i)$. Since $L_i \cong P_4$, there is an edge $f_i \in E(L_i)$ such that f_i is adjacent to e_i for $i = 1, 2, 3, 4$. This implies that G contains a subgraph F that is isomorphic to one of the graphs in Figure 5.3, each of which contains C_4 as a subgraph. Let $F_1 \cong F_2 \cong P_4$ defined in Case 1 and let $F_3 = H_2$. Then F_1, F_2, F_3 are edge-disjoint and $E(H_1) \cup E(H_2) \subseteq E(F_1) \cup E(F_2) \cup E(F_3)$. Thus $R_3 = G - [E(F_1) \cup E(F_2) \cup E(F_3)]$ is a subgraph of R_2 and so R_3 contains no subgraph isomorphic to any graph in $\{P_4, C_4\}$. Therefore, $\{F_1, F_2, F_3, R_3\}$ is an S-maximal 3-decomposition of G, which is a contradiction. ∎

By Corollary 5.4.1 and Theorem 5.4.2, it remains to consider the following questions.

1. Is $\{P_5, C_3\}$ an ID-set?

2. Is $\{P_5, C_4\}$ an ID-set?

We also plan to investigate those *path-star sets* or *cycle-star sets*, which are defined as expected.

5.4.2 Graphs of Larger Sizes

We have seen that neither K_3 nor $K_{1,3}$ is an ID-graph. In fact, for each integer $n \geq 3$, none of $C_n, K_n, K_{1,n}$ are ID-graphs. To show this, we construct a non-ID-graph for each of $C_n, K_n, K_{1,n}$.

- For C_n, let $F_0, F_1, \ldots, F_n$ be the $n+1$ copies of C_n where

  $$F_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n}, v_{i,1})$$

  for $0 \leq i \leq n$. The graph G is then obtained from $F_0, F_1, \ldots, F_n$ by identifying the edge $v_{0,j}v_{0,j+1}$ in F_0 with the edge $v_{j,1}v_{j,2}$ for $1 \leq j \leq n$ where $v_{0,n}v_{0,n+1} = v_{0,n}v_{0,1}$. The graph G is shown in Figure 4.1 for $n = 3$ and in Figure 5.4(a) for $n = 4$. Then G has (1) a C_n-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $H_1 = F_0$ and R_1 is a cycle of order $n^2 - n > n$ and (2) a C_n-maximal n-decomposition $\mathcal{D}_n = \{L_1, L_2, \ldots L_n, R_n\}$ where $L_i = F_i$ for $1 \leq i \leq n$ and R_n is an empty graph. However, G does not have a C_n-maximal k-decomposition for each integer k with $2 \leq k \leq n-1$. Thus C_n is not an ID-graph.

- For K_n, let $F_0, F_1, \ldots, F_n$ be the $n+1$ copies of K_n where $(v_{i,1}, v_{i,2}, \ldots, v_{i,n}, v_{i,1})$ is a Hamiltonian cycle of F_i for $0 \leq i \leq n$. The graph G is then obtained from $F_0, F_1, \ldots, F_n$ by identifying the edge $v_{0,j}v_{0,j+1}$ in F_0 with the edge

$v_{j,1}v_{j,2}$ for $1 \le j \le n$ where where $v_{0,n}v_{0,n+1} = v_{0,n}v_{0,1}$. The graph G is shown in Figure 4.1 for $n = 3$ and in Figure 5.4(b) for $n = 4$. Then G has a K_n-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $H_1 = F_0$ and the clique number R_1 is $n - 1$ and so R_1 contains no K_n as a subgraph and a K_n-maximal n-decomposition $\mathcal{D}_n = \{L_1, L_2, \ldots L_n, R_n\}$ where $L_i = F_i$ for $1 \le i \le n$ and R_n is the graph $K_n - C_n$ However, G does not have a K_n-maximal k-decomposition for each integer k with $2 \le k \le n-1$. Thus K_n is not an ID-graph.

- For $K_{1,n}$, let $F_0, F_1, \ldots, F_n$ be $n+1$ copies of $K_{1,n}$ where

$$V(F_0) = \{u,\ u_1,\ u_2,\ \ldots,\ u_n\}$$

and u is the central vertex of F_0. For each i with $1 \le i \le n$, let v_i be the central vertex of F_i. The graph G is then obtained from $F_0, F_1, \ldots, F_n$ by identifying the end-vertex u_i in F_0 with the central vertex v_i in F_i for $1 \le i \le n$. The graph G is shown in Figure 4.1 for $n = 3$ and in Figure 5.4(c) for $n = 4$. Then G has a $K_{1,n}$-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $H_1 = F_0$ and $R_1 \cong nK_{1,n-1}$ and a $K_{1,n}$-maximal n-decomposition $\mathcal{D}_n = \{L_1, L_2, \ldots L_n, R_n\}$ where $L_i = F_i$ for $1 \le i \le n$ and R_n is an empty graph. However, G does not have a $K_{1,n}$-maximal k-decomposition for any integer k with $2 \le k \le n-1$. Thus $K_{1,n}$ is not an ID-graph.

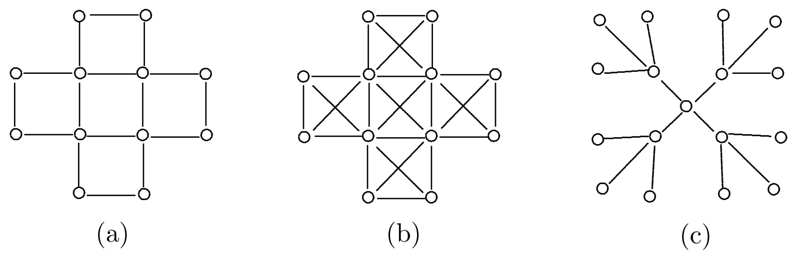


Figure 5.4: Illustrating that $C_n, K_n, K_{1,n}$ are not ID-graphs for $n = 4$

Problem 5.4.3 *For an integer $n \geq 5$, is P_n an ID-graph?*

Bibliography

[1] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdős and O. R. Oellermann, The ascending subgraph decomposition problem. *Congr. Numer.* **58** (1987) 7-14.

[2] B. Alspach, Research problems, Problem 3. *Discrete Math.* **36** (1981) 333.

[3] B. Alspach, The wonderful Walecki construction. *Bull. Inst. Combin. Appl.* **52** (2008) 7-20.

[4] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$. *J. Combin. Theory Ser B.* **81** (2001) 77-99.

[5] A. Benjamin, G. Chartrand and P. Zhang, *The Fascinating World of Graph Theory*, Manuscript.

[6] E. J. Billington, Decomposing complete tripartite graphs into cycles of lengths 3 and 4. 16th British Combinatorial Conference (London, 1997). *Discrete Math.* **197/198** (1999) 123-135.

[7] D. Bryant, D. Horsley and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths. *Proceedings of the London Mathematical Society.* To appear.

[8] D. Bryant and G. Martin, Some results on decompositions of low degree circulant graphs. *Australas. J. Combin.* **45** (2009) 251-261.

[9] N. J. Cavenagh, Decompositions of complete tripartite graphs into k-cycles. *Australas. J. Combin.* **18** (1998) 192-200.

[10] G. Chartrand, H. Jordon and P. Zhang, A cycle decomposition conjecture for Eulerian graphs. *Australas. J. Combin.* To appear.

[11] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs: 5th Edition*, Chapman & Hall/CRC, Boca Raton, FL (2010).

[12] G. Chartrand, A. D. Polimeni and M. J. Stewart, The existence of 1-factors in line graphs, squares, and total graphs. *Nedrl. Akad. Wetensch. Proc. Ser. A* 76 *Indag. Math.* **35** (1973) 228-232.

[13] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC, Boca Raton, FL (2008).

[14] C. J. Colbourn, D. G. Hoffman and R. Rees, A new class of group divisible designs with block size three. *J. Combin. Theory Ser. A* **59** (1992) 73–89.

[15] L. Euler, Solutio problematis ad geometriam situs pertinentis. *Comment. Academiae Sci. I. Petropolitanae* **8** (1736) 128-140.

[16] C. Hierholzer, Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechnung zu umfahren. *Math. Ann.* **6** (1873) 30-32.

[17] T. P. Kirkman, On a problem in combinatorics. *Cambridge and Dublin Math. J.* **2** (1847) 191-204.

[18] M. K. Kwan, Graphic programming using odd or even points, *Acta Math. Sinica* **10** (1960) 264-266 (Chinese); translated as *Chinese Math.* **1** (1960) 273-277.

[19] R. Laskar and W. Hare, Chromatic numbers for certain graphs. *J. London Math. Soc.* **4** (1971/72) 489-492.

[20] R. Rees, Two new direct product-type constructions for resolvable group-divisible designs. *J. Combin. Designs* **1** (1993) 15-26.

[21] O. Veblen, An application of modular equations in analysis situs. *Ann. of Math.* **14** (1912) 86-94.