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# On Eulerian Irregularity and Decompositions in Graphs

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ON EULERIAN IRREGULARITY AND  
DECOMPOSITIONS IN GRAPHS

by

Eric Andrews

A dissertation submitted to the the Graduate College  
in partial fulfillment for the requirement  
for the Degree of Doctor of Philosophy  
Mathematics  
Western Michigan University  
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# ON EULERIAN IRREGULARITY AND DECOMPOSITIONS IN GRAPHS

Eric Andrews, Ph.D.

Western Michigan University, 2014

An Eulerian walk in a connected graph  $G$  is a closed walk that contains every edge of  $G$  at least once, while an irregular Eulerian walk in  $G$  is an Eulerian walk that encounters no two edges of  $G$  the same number of times. The minimum length of an irregular Eulerian walk in  $G$  is called the Eulerian irregularity of  $G$  and is denoted by  $EI(G)$ .

For a nontrivial connected graph  $G$  of size  $m$ , it is shown that  $\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}$  and that  $EI(G) = 2\binom{m+1}{2}$  if and only if  $G$  is a tree of size  $m$ . A necessary and sufficient condition has been established for all pairs  $k, m$  of positive integers for which there is a nontrivial connected graph  $G$  of size  $m$  with  $EI(G) = k$ . A formula for the Eulerian irregularity of a graph in terms of the size of certain even subgraph of the graph has been established. Furthermore, Eulerian irregularities are determined for all graphs of cycle rank 2 and all complete bipartite graphs as well as all prisms, grids and powers of cycles. Some general results on Eulerian irregularities of circulants are also presented.

For a set  $S$  of graphs and a graph  $G$ , a decomposition  $\mathcal{D} = \{H_1, H_2, \dots, H_k, R\}$  of  $G$  is called an  $S$ -maximal  $k$ -decomposition if  $H_i \cong H$  for some  $H \in S$  for each integer  $i$  with  $1 \leq i \leq k$  and  $R$  contains no subgraph isomorphic to any subgraph in  $S$ . Let  $\text{Min}(G, S)$  and  $\text{Max}(G, S)$  be the minimum and maximum  $k$ , respectively, for which  $G$  has an  $S$ -maximal  $k$ -decomposition. A set  $S$  of graphs without isolated vertices is said to possess the intermediate decomposition property if for every connected graph  $G$  and each integer  $k$  with  $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$ , there exists an  $S$ -maximal  $k$ -decomposition of  $G$ . All graphs of size 3 or less are determined that possess the intermediate decomposition property. Furthermore, the sets of graphs having size 3 that possess the intermediate decomposition property are determined as well as some sets of graphs having having size more than 3.

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# Chapter 1

## Introduction

### 1.1 The Königsberg Bridge Problem

The city of Königsberg was the capital of German East Prussia in the 13th century and home of the Prussian Royal Castle. The River Pregel flowed through the city separating it into four land regions. Seven bridges were built over the river. Figure 1.1 displays a map of Königsberg, showing these four land regions (labeled A, B, C and D), the location of the river and the seven bridges (labeled a, b, c, d, e, f and g). The story goes that during the 1730s, some of its citizens enjoyed strolling about the city and wondered whether it was possible to go for a walk and pass over each bridge exactly once. This problem eventually became known as the *Königsberg bridge problem*.

Leonhard Euler, the great Swiss mathematician of the 18th century, became aware of this problem but initially did not think that this problem was particularly mathematical in nature. He discovered a method for solving not only the Königsberg bridge problem but a generalization of the problem. In a famous 1736

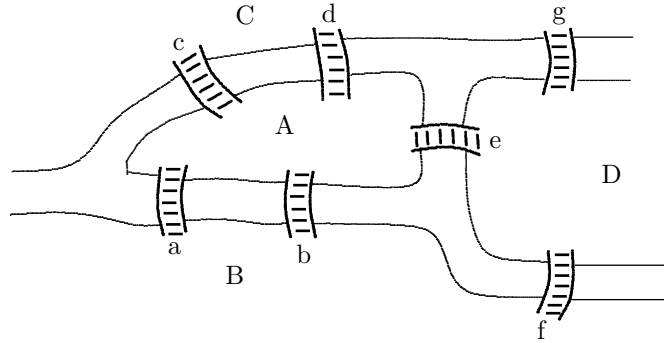


Figure 1.1: A map of Königsberg

paper by Euler [15], he described the problem, a generalization of the problem and the solutions of these problems. While Euler’s reasoning in his paper was graph theoretic in nature, the term “graph” never appeared in the paper. Indeed, the term “graph,” as used in this context, was not introduced until 1878, when the British mathematician James Joseph Sylvester first used this word.

In terms of graphs, Euler’s paper contained the following result:

*Suppose that there is a city consisting of land regions, some pairs of which are joined by one or more bridges. This town can then be represented by a graph or multigraph  $G$  whose vertices are the land regions where every two vertices of  $G$  are joined by a number of edges equal to the number of bridges joining the corresponding land regions. There is a round-trip in the town passing over each bridge exactly once if and only if  $G$  is connected and every vertex of  $G$  has even degree.*

Such a graph or multigraph is referred to as Eulerian. Since each vertex in the multigraph representing the map of Königsberg has odd degree (as shown in

Figure 1.2), it follows by Euler's result that it was not possible to walk about Königsberg and pass over each bridge exactly once. Euler's paper not only solved the Königsberg bridge problem, it marked the beginning of graph theory.

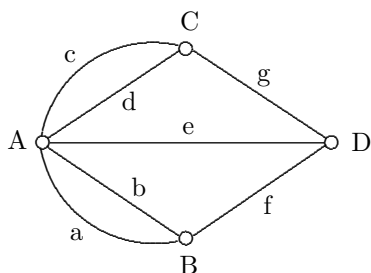


Figure 1.2: A multigraph representing Königsberg

## 1.2 The Chinese Postman Problem

If every edge of a nontrivial connected graph  $G$  of size  $m$  is replaced by two parallel edges, then the resulting multigraph is Eulerian, which implies that  $G$  contains a closed walk in which every edge of  $G$  appears exactly twice. Euler made this observation in his paper. Of course,  $G$  contains a closed walk in which every edge of  $G$  appears exactly once if and only if  $G$  itself is Eulerian. A weighted graph  $H$  can be obtained from  $G$  by assigning a positive integer weight to each edge of  $G$ . The degree of a vertex  $v$  in  $H$  is the sum of the weights of the edges incident with  $v$ . From the observation above, every vertex of  $H$  is even if every edge of  $G$  is assigned the weight 2. If every edge of  $G$  is assigned the weight 1, then every vertex of  $H$  is even if and only if  $G$  is Eulerian. Thus it is always possible to assign each edge of  $G$  a weight 1 or 2 in such a way that every vertex in the resulting weighted

graph is even. A problem of interest is that of determining the minimum sum of all positive integer weights assigned to the edges of  $G$  so that every vertex in the resulting weighted graph is even. This is equivalent to determining the minimum length of a closed walk in  $G$  that contains every edge of  $G$  at least once. A solution to this problem also provides a solution to the so-called Chinese Postman Problem, named by Alan Goldman for the Chinese mathematician Meigu Guan (often known as Mei-Ko Kwan) who introduced this problem in 1962. Suppose that a postman starts from the post office and has mail to deliver to the houses along each street on his mail route. Once he has completed delivering the mail, he returns to the post office. The problem is to find the minimum length of a round trip that accomplishes this, as we state next.

**The Chinese Postman Problem** *Determine the minimum length of a round trip that traverses every road in a mail route at least once.*

The minimum length of a closed walk that contains every edge of a connected graph  $G$  of size  $m$  at least once is  $m$  if  $G$  is Eulerian. If  $G$  is not Eulerian, then  $G$  contains  $2k$  odd vertices for some positive integer  $k$ . Suppose that the  $2k$  odd vertices are divided into  $k$  pairs and the distance between the vertices in each pair is determined and these  $k$  numbers are summed. If  $d$  is the minimum value of all such sums over all partitions of these  $2k$  odd vertices into  $k$  pairs, then the minimum length of such a closed walk is  $m+d$ . Suppose that  $\{\{u_1, v_1\}, \{u_2, v_2\}, \dots, \{u_k, v_k\}\}$  is a partition of the  $2k$  odd vertices such that  $\sum_{i=1}^k d(u_i, v_i) = d$  and  $P_i$  is a  $u_i - v_i$  geodesic for  $i = 1, 2, \dots, k$ . Then the paths  $P_i$  are pairwise edge-disjoint. If we

replace each edge that belongs to one of these paths by two parallel edges, then we obtain an Eulerian multigraph  $M$  of size  $m + d$ . An Eulerian circuit in  $M$  gives rise then to a closed walk of minimum length in  $G$  that contains every edge of  $G$  at least once – a solution to the Chinese Postman Problem.

### 1.3 Decompositions in Graphs

One of the major topics in graph theory concerns graph decompositions. A problem of primary interest in this case has been to determine for graphs  $G$  and  $H$  whether it is possible to decompose  $G$  into subgraphs, each isomorphic to  $H$ , that is, whether  $G$  is  $H$ -decomposable. A classic historical problem in this context is the determination of those integers  $n \geq 3$  for which the complete graph  $K_n$  is  $K_3$ -decomposable. This is equivalent to the problem of determining those integers  $n \geq 3$  for which there is a Steiner triple system  $S_n$ , a problem initiated and solved in 1847 by the famous combinatorialist Thomas Kirkman [17], who showed that this occurred if and only if  $n \equiv 1 \pmod{6}$  or  $n \equiv 3 \pmod{6}$ . Another familiar result of this type is that  $K_n$  can be decomposed (actually factored in this case) into Hamiltonian cycles if and only if  $n$  is odd, a result attributed to Walecki [3]. Another result of this type, dealing with paths  $P_3$  of order 3, appeared in [12] and is stated below.

**Theorem 1.3.1** *A nontrivial connected graph  $G$  is  $P_3$ -decomposable if and only if  $G$  has even size.*

Not all decomposition problems have dealt with decomposing a graph into subgraphs, each isomorphic to the same graph. The following theorem, due to Bryant, Horsley and Pettersson [7], verified a conjecture on cycle decompositions made by Alspach [2] in 1981.

**Theorem 1.3.2** *Suppose that  $n \geq 3$  is an odd integer and that  $m_1, m_2, \dots, m_t$  are integers such that  $3 \leq m_i \leq n$  for each  $i$  ( $1 \leq i \leq t$ ) and  $m_1 + m_2 + \dots + m_t = \binom{n}{2}$ . Then  $K_n$  can be decomposed into the cycles  $C_{m_1}, C_{m_2}, \dots, C_{m_t}$ . Furthermore, for every even integer  $m \geq 4$  and integers  $m_1, m_2, \dots, m_t$  such that  $3 \leq m_i \leq n$  for each  $i$  ( $1 \leq i \leq t$ ) with  $m_1 + m_2 + \dots + m_t = (n^2 - 2n)/2$ , there is a decomposition of  $K_n$  into a 1-factor and the cycles  $C_{m_1}, C_{m_2}, \dots, C_{m_t}$ .*

The famous topologist Oswald Veblen [21] proved that every Eulerian graph can be decomposed into cycles. A conjecture involving cycle decompositions of Eulerian graphs was introduced in [10].

**The Eulerian Cycle Decomposition Conjecture (ECDC)** *Let  $G$  be an Eulerian graph of size  $m$ , where  $a$  is the minimum number of odd cycles in a cycle decomposition of  $G$  and  $b$  is the maximum number of odd cycles in a cycle decomposition of  $G$ . For every integer  $\ell$  such that  $a \leq \ell \leq b$  and  $\ell$  and  $m$  are of the same parity, there exists a cycle decomposition of  $G$  containing exactly  $\ell$  odd cycles.*

It is therefore a consequence of the theorem by Bryant, Horsley and Pettersson that the ECDC is true for all complete graphs of odd order. This conjecture was verified for several classes of graphs in [10] but remains open in general.



Another decomposition problem, introduced in [1], involves subgraphs, no two of which are isomorphic.

**The Ascending Subgraph Decomposition Conjecture** *Let  $G$  be a graph of size  $m$ , where  $\binom{k+1}{2} \leq m < \binom{k+2}{2}$  for some positive integer  $k$ . Then  $G$  can be decomposed into  $k$  subgraphs  $G_1, G_2, \dots, G_k$  where  $G_i$  has size  $m_i$  ( $1 \leq i \leq k$ ),  $m_i < m_{i+1}$  for  $i = 1, 2, \dots, k-1$  and  $G_{i+1}$  contains a subgraph isomorphic to  $G_i$ .*

## 1.4 Basic Definitions and Well-Known Results

In this section, we formally present basic definitions and notation involved our research. We refer to [11] for graph theory notation and terminology not described in this paper. All graphs under consideration are nontrivial connected graphs.

For vertices  $u$  and  $v$  in a graph  $G$ , a  $u - v$  walk  $W$  in  $G$  is a sequence

$$W = (u = v_0, v_1, v_2, \dots, v_k = v) \tag{1.1}$$

of vertices in  $G$  such that  $v_{i-1}v_i$  is an edge of  $G$  for each  $i$  ( $1 \leq i \leq k$ ). If  $e_i = v_{i-1}v_i$ , then the walk  $W$  in (1.1) can also be denoted by

$$W = (e_1, e_2, \dots, e_k). \tag{1.2}$$

The length of the walk  $W$  is denoted by  $L(W)$  and so  $L(W) = k$  for the walk  $W$  in (1.1) and (1.2).

If  $G$  is a multigraph rather than a graph, then some pairs of vertices are joined by more than one edge. In this case, it is necessary to denote a walk as a sequence

of edges as in (1.2) rather than a sequence of vertices as in (1.1) to avoid confusion. If  $u = v$ , then the  $u - v$  walk is *closed*; while if  $u \neq v$ , then the  $u - v$  walk is *open*. If there is no repetition of edges in a walk, then the walk is a *trail*. A closed nontrivial trail is a *circuit*. A  $u - v$  walk  $W$  as in (1.1) is a  $u - v$  *path* if the vertices  $v_0, v_1, \dots, v_k$  are distinct. If  $W$  is a circuit for which the vertices  $v_0, v_1, \dots, v_{k-1}$  are distinct, then  $W$  is a *cycle*.

A circuit in a graph  $G$  that contains every edge of  $G$  is an *Eulerian circuit*, while an open trail containing every edge of  $G$  is an *Eulerian trail*. A graph containing an Eulerian circuit is an *Eulerian graph* and a graph containing an Eulerian trail is a *traversable graph*. In 1736, Leonhard Euler established the following characterization of Eulerian graphs [15].

**Theorem 1.4.1 (Euler's Theorem)** A nontrivial connected graph  $G$  is Eulerian if and only if every vertex of  $G$  has even degree.

In 1912, Oswald Veblen [21] also obtained an interesting characterization of Eulerian graphs in terms of graph decompositions. A *decomposition*  $\mathcal{D}$  of a graph  $G$  is a collection  $\{H_1, H_2, \dots, H_t\}$  of nonempty subgraphs such that  $H_i = G[E_i]$  for some (nonempty) subset  $E_i$  of  $E(G)$ , where  $\{E_1, E_2, \dots, E_t\}$  is a partition of  $E(G)$ . Thus no subgraph  $H_i$  in a decomposition of  $G$  contains isolated vertices. If  $\mathcal{D}$  is a decomposition of  $G$ , then we say  $G$  is decomposed into the subgraphs  $H_1, H_2, \dots, H_t$ . If  $\mathcal{D} = \{H_1, H_2, \dots, H_t\}$  is a decomposition of a graph  $G$  such that  $H_i \cong H$  for some graph  $H$  for each  $i$  ( $1 \leq i \leq t$ ), then  $\mathcal{D}$  is an *H-decomposition* of  $G$  or an *isomorphic decomposition* of  $G$ . If there exists an  $H$ -decomposition of a

graph  $G$ , then  $G$  is said to be *H-decomposable*. If each  $H_i$  in a decomposition  $\mathcal{D} = \{H_1, H_2, \dots, H_t\}$  is a cycle, then  $\mathcal{D}$  is called a *cycle decomposition* or a *cyclic decomposition*. Similarly, if each  $H_i$  in  $\mathcal{D}$  is a path, then  $\mathcal{D}$  is called a *path decomposition*.

**Theorem 1.4.2 (Veblen's Theorem)** A nontrivial connected graph  $G$  is Eulerian if and only if  $G$  has a cycle decomposition.

An important corollary of Theorem 1.4.1 is the following characterization of traversable graphs.

**Corollary 1.4.3** *A nontrivial connected graph  $G$  is traversable if and only if  $G$  contains exactly two vertices of odd degree. Any Eulerian trail in  $G$  then begins at one of these vertices and terminates at the other.*

If a graph  $G$  has four or more odd vertices, then  $G$  contains neither an Eulerian circuit nor an Eulerian trail, which again explains why there was no journey about Königsberg that crossed each bridge exactly once. Even though these graphs do not contain Eulerian circuits or Eulerian trails, there are some interesting properties that these graphs possess. We saw in the discussion related to finding a solution to the Chinese Postman Problem in Section 1.2 that if a connected graph  $G$  contains  $2k$  odd vertices, then these  $2k$  odd vertices can be partitioned into  $k$  pairs resulting in  $k$  pairwise edge-disjoint paths in  $G$ , each connecting pairs of odd vertices. In fact, it is well-known that  $G$  itself can be decomposed to  $k$  open trails connecting odd vertices and more can be said.

**Theorem 1.4.4** *If  $G$  is a connected graph containing  $2k \geq 4$  odd vertices, then  $G$  can be decomposed into  $k$  open trails connecting odd vertices but no fewer.*

## Chapter 2

# Irregular Eulerian Walks

While the Chinese Postman Problem asks for the minimum length of a closed walk in a connected graph  $G$  such that every edge of  $G$  appears on the walk once or twice, another problem of interest is that of determining the minimum length of a closed walk in  $G$  in which no two edges of  $G$  appear the same number of times. Such walks in a graph  $G$  distinguish the edges of  $G$  by their occurrences on the walk, which gives rise to the concept of irregular Eulerian walks in graphs.

### 2.1 Eulerian Walks in Graphs

Let  $G$  be a nontrivial connected graph of size  $m$ . By an *Eulerian walk* in  $G$ , we mean a closed walk that contains every edge of  $G$ . Thus the length of an Eulerian walk  $W$  in  $G$  is  $m$  if and only if  $W$  is an Eulerian circuit. In general then, the minimum length of an Eulerian walk in  $G$  is  $m + d$  for some nonnegative integer  $d$ . We saw that if every edge of  $G$  is replaced by two parallel edges, then the resulting multigraph  $M$  is Eulerian and each Eulerian circuit in  $M$  gives rise to an

Eulerian walk in  $G$  that encounters every edge of  $G$  exactly twice. Hence if  $G$  is not Eulerian, then the minimum length of an Eulerian walk in  $G$  is more than  $m$  but not more than  $2m$  and every edge appears once or twice in such an Eulerian walk in  $G$ .

Let  $H$  be a weighted graph obtained by assigning weights (positive integers) to the edges of a connected graph  $G$ . Then the *degree*  $\deg_H v$  of a vertex  $v$  in  $H$  is the sum of the weights of the edges incident with  $v$ . Determining the minimum length of an Eulerian walk in  $G$  is then equivalent to determining an assignment of the weights 1 or 2 to the edges of  $G$  such that the sum of these weights is minimum and the degree of every vertex in  $H$  is even. The subgraph induced by the edges labeled 2 is the union of edge-disjoint paths in  $G$ . As we mentioned before, this problem is directly related to a well-known problem called the Chinese Postman Problem, which is the problem of determining the minimum length of a round trip that traverses every road in a mail route at least once.

## 2.2 Eulerian Irregularity

For every nontrivial connected graph  $G$  of size  $m$ , there is always an Eulerian walk in which each edge of  $G$  is encountered the same number of times. An *irregular Eulerian walk* in  $G$  is an Eulerian walk that encounters no two edges of  $G$  the same number of times. Thus the length of an irregular Eulerian walk in  $G$  is at least  $1+2+\dots+m = \binom{m+1}{2}$ . If  $E(G) = \{e_1, e_2, \dots, e_m\}$  and each edge  $e_i$  ( $1 \leq i \leq m$ ) of  $G$  is replaced by  $2i$  parallel edges, then the resulting multigraph  $M$  is Eulerian and

each Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk in which each edge  $e_i$  of  $G$  appears exactly  $2i$  times in the walk. Thus  $G$  contains an irregular Eulerian walk of length  $2 + 4 + 6 + \dots + 2m = 2\binom{m+1}{2} = m^2 + m$ . The length of a walk  $W$  is denoted by  $L(W)$ . If  $W$  is an irregular Eulerian walk of minimum length in a connected graph  $G$  of size  $m$ , then  $\binom{m+1}{2} \leq L(W) \leq 2\binom{m+1}{2}$ . A problem of interest here is that of determining the minimum length of an irregular Eulerian walk in  $G$ , which we refer to as the *Eulerian irregularity* of  $G$ , which is denoted by  $EI(G)$ . Therefore, if  $G$  is a connected graph of size  $m$ , then

$$\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}. \quad (2.1)$$

Both bounds in (2.1) are sharp. First, we show that the lower bound in (2.1) is sharp. For an odd integer  $n \geq 5$ , let  $G = C_n^2$  be the square of the  $n$ -cycle  $C_n$ . That is, if  $C_n = (v_1, v_2, \dots, v_n, v_1)$ , then

$$E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\} \cup \\ \{v_1v_3, v_3v_5, \dots, v_{n-2}v_n, v_nv_2, v_2v_4, \dots, v_{n-3}v_{n-1}, v_{n-1}v_1\}.$$

Thus,  $G$  is a 4-regular graph of size  $m = 2n$  and

$$C = (v_1, v_2, v_3, \dots, v_n, v_1, v_3, v_5, \dots, v_n, v_2, v_4, \dots, v_{n-3}, v_{n-1}, v_1)$$

is an Eulerian circuit of  $G$ . Suppose that  $C$  encounters the edges  $e_1, e_2, \dots, e_m$  in this order and each edge  $e_i$  ( $1 \leq i \leq m$ ) is replaced by  $i$  parallel edges. Then the resulting multigraph  $M$  is Eulerian and so each Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk in which each edge  $e_i$  of  $G$  appears exactly  $i$  times

in the walk. Thus  $EI(G) = \binom{m+1}{2}$  and so the lower bound in (2.1) is sharp. To see that the upper bound in (2.1) is sharp, we first state a theorem due to Mei-Ko Kwan [18].

**Kwan's Theorem** *Let  $G$  be a connected graph and let  $W$  be a closed walk of minimum length containing every edge of  $G$  at least once. Then  $W$  encounters no edge of  $G$  more than twice and no more than half of the edges in any cycle appear twice.*

**Theorem 2.2.1** *For a connected graph  $G$  of size  $m \geq 1$ ,*

$$EI(G) = 2\binom{m+1}{2} \text{ if and only if } G \text{ is a tree.}$$

**Proof.** Assume first that  $G$  contains a bridge  $uv$ . Let  $W$  be an Eulerian walk of  $G$  with initial vertex  $u$ . Then the first time that  $v$  is encountered on  $W$ , it is preceded by  $u$  and the next time that  $u$  is encountered on  $W$ , it is preceded by  $v$ . Therefore,  $uv$  occurs an even number of times on  $W$ . If  $G$  is a tree, then every edge of  $G$  is a bridge and so each edge of  $G$  is encountered an even number of times on  $W$ . Therefore,  $EI(G) \geq 2\binom{m+1}{2}$ . It then follows by (2.1) that  $EI(G) = 2\binom{m+1}{2}$ .

Suppose next that  $G$  is not a tree. Then  $G$  contains at least one cycle. By Kwan's theorem, there is an Eulerian walk  $W$  in which no edge of  $G$  occurs on  $W$  more than twice and some edges occur on  $W$  exactly once. Let  $e_1, e_2, \dots, e_k$  ( $k \geq 1$ ) be those edges occurring exactly once on  $W$  and let  $f_1, f_2, \dots, f_\ell$  be those edges occurring exactly twice on  $W$ . By assigning each edge  $e_i$  ( $1 \leq i \leq k$ ) the



weight  $2i - 1$  and each edge  $f_j$  ( $1 \leq j \leq \ell$ ) the weight  $2j$  if  $\ell \geq 1$ , we obtain a weighted graph in which every vertex is even. Thus there is an Eulerian walk in  $G$  where  $e_i$  appears  $2i - 1$  times and  $f_j$  appears  $2j$  times. Since there is an irregular Eulerian walk of length less than  $2\binom{m+1}{2}$ , it follows that  $EI(G) < 2\binom{m+1}{2}$ . ■

If we were to consider all sets  $S$  of  $m$  positive integers and label the edges of a connected graph  $G$  with distinct elements of  $S$  so that every vertex is even in the resulting weighted graph, then the minimum of the sums of the elements of all such sets  $S$  is  $EI(G)$ .

## 2.3 Optimal Irregular Eulerian Walks

An irregular Eulerian walk  $W$  in a connected graph  $G$  of size  $m$  is said to be *optimal* if  $L(W) = \binom{m+1}{2}$ . In this case, the edges of  $G$  can be ordered as  $e_1, e_2, \dots, e_m$  such that  $e_i$  ( $1 \leq i \leq m$ ) is encountered exactly  $i$  times in  $W$ . As we have seen, there are graphs that possess an optimal irregular Eulerian walk. A graph  $G$  is *optimal* if it contains an optimal irregular Eulerian walk. First, we present a characterization of such connected graphs.

**Theorem 2.3.1** *Let  $G$  be a connected graph of size  $m$ . Then  $G$  contains an optimal irregular Eulerian walk if and only if  $G$  contains a subgraph of size  $\lceil m/2 \rceil$ , every vertex of which is even.*

**Proof.** First, assume that  $G$  contains a subgraph  $F$  of size  $\lceil m/2 \rceil$  such that every vertex of  $F$  is even. Then

$$E(G) = \{e_1, e_2, \dots, e_{\lceil m/2 \rceil}\} \cup \{e'_1, e'_2, \dots, e'_{\lfloor m/2 \rfloor}\},$$

where  $E(F) = \{e_1, e_2, \dots, e_{\lceil m/2 \rceil}\}$ . We construct an Eulerian multigraph  $M$  by replacing each edge  $e_i$  where  $1 \leq i \leq \lceil m/2 \rceil$  by  $2i - 1$  parallel edges and replacing each edge  $e'_j$ , where  $1 \leq j \leq \lfloor m/2 \rfloor$ , by  $2j$  parallel edges. An Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk  $W$  in  $G$  such that each edge  $e_i$  of  $G$  appears exactly  $2i - 1$  times in  $W$ , where  $1 \leq i \leq \lceil m/2 \rceil$ , and each edge  $e'_j$  of  $G$  appears exactly  $2j$  times in  $W$  where  $1 \leq j \leq \lfloor m/2 \rfloor$ . Then the length of  $W$  is  $1 + 2 + 3 + \dots + m = \binom{m+1}{2}$  and  $W$  is an optimal irregular Eulerian walk in  $G$ .

For the converse, suppose that  $G$  contains an optimal irregular Eulerian walk  $W$ . We may assume that  $E(G) = \{f_1, f_2, \dots, f_m\}$ , where  $f_i$  appears exactly  $i$  times ( $1 \leq i \leq m$ ) on  $W$ . Let  $F$  be the subgraph of  $G$  of size of  $\lceil m/2 \rceil$  induced by the set  $\{f_1, f_3, \dots, f_{2\lfloor (m-1)/2 \rfloor + 1}\}$  and let  $F'$  be the subgraph of  $G$  induced by the set  $\{f_2, f_4, \dots, f_{2\lceil (m-1)/2 \rceil}\}$ . Thus  $\{F, F'\}$  is a decomposition of  $G$ . We claim that every vertex of  $F$  is even. Let  $M$  be the weighted graph obtained by assigning the weight  $i$  ( $1 \leq i \leq m$ ) to each edge  $f_i$  of  $G$ . Let  $H$  be the weighted subgraph of  $M$  induced by the edges of  $F$  and let  $H'$  be the weighted subgraph of  $M$  induced by the edges of  $F'$ . Since  $G$  has an Eulerian walk in which each edge  $f_i$  appears exactly  $i$  times, every vertex of  $M$  has even degree. Since  $\deg_M v = \deg_H v + \deg_{H'} v$  for every vertex  $v$  of  $G$  and  $\deg_M v$  and  $\deg_{H'}$  are both even, it follows that  $\deg_H v$  is even. Suppose that  $\deg_F v = k$ . Then  $v$  is incident with  $k$  edges in  $H$ , each of odd weight. Since  $\deg_H v$  is even,  $k$  is even and so  $v$  is an even vertex in  $F$ . ■

By Theorem 2.3.1, the graphs  $G_1$  and  $G_3$  of Figure 2.1 contain optimal irregular Eulerian walks while  $G_2$  and  $G_4$  do not. Since the Petersen graph  $P$  has size 15 and  $P$  contains an 8-cycle, it follows by Theorem 2.3.1 that  $P$  contains an optimal irregular Eulerian walk. On the other hand, by Theorem 2.3.1, no cycle contains an optimal irregular Eulerian walk. In fact,  $EI(C_m) = 1 + 3 + 5 + \dots + (2m - 1) = m^2$  for each  $m \geq 3$ . We have seen that if  $n \geq 5$  is odd, then  $C_n^2$  contains an optimal irregular Eulerian walk. Since the size of  $C_n^2$  is  $2n$  and  $C_n$  is a 2-regular graph of size  $n$  in  $C_n^2$ , it follows by Theorem 2.3.1 that  $C_n^2$  contains an optimal irregular Eulerian walk for each integer  $n \geq 4$ .

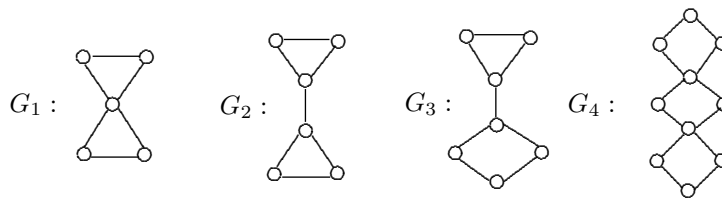


Figure 2.1: Illustrating Theorem 2.3.1

By Theorem 2.3.1, if  $G$  is a connected graph of size  $m$ , then  $EI(G) = \binom{m+1}{2}$  if and only if  $G$  contains a subgraph of size  $\lceil m/2 \rceil$ , every vertex of which is even. The following is also a consequence of Theorem 2.3.1.

**Corollary 2.3.2** *If  $G$  is a connected bipartite graph of size  $m \geq 1$  such that  $m \equiv 1 \pmod{4}$  or  $m \equiv 2 \pmod{4}$ , then  $G$  does not contain an optimal irregular Eulerian walk.*

**Proof.** If  $m \equiv 1 \pmod{4}$  or  $m \equiv 2 \pmod{4}$ , then  $\lceil m/2 \rceil$  is odd. If  $G$  contains an optimal irregular Eulerian walk, then by Theorem 2.3.1,  $G$  contains a subgraph

$H$  of size  $\lceil m/2 \rceil$ , each of whose vertices is even. Therefore,  $H$  is a bipartite graph of odd size, each vertex of which is even. This is impossible. ■

Next, we determine all those complete graphs and complete bipartite graphs containing an optimal irregular Eulerian walk. In order to this, we first present two well-known results about complete graphs (see [11, p. 424-426]).

**Theorem 2.3.3** *Let  $n \geq 3$  be an integer.*

- (1) *If  $n$  is odd, then  $K_n$  is Hamiltonian-factorable.*
- (2) *If  $n$  is even, then  $K_n$  can be factored into  $\frac{n}{2} - 1$  Hamiltonian cycles and a 1-factor.*

**Theorem 2.3.4** *For each integer  $n \geq 2$ , the complete graph  $K_n$  contains an optimal irregular Eulerian walk if and only if  $n \geq 4$ .*

**Proof.** Since neither  $K_2$  nor  $K_3$  has an optimal irregular Eulerian walk, it remains to show that  $K_n$  contains an optimal irregular Eulerian walk for  $n \geq 4$ . By Theorem 2.3.1, it suffices to show that  $K_n$  contains a subgraph of size  $\lceil m/2 \rceil$  where  $m = \binom{n}{2}$  such that each vertex of this subgraph is even. We consider the cases when  $n \equiv r \pmod{4}$  for  $r = 0, 1, 2, 3$ .

*Case 1.*  $n \equiv 0 \pmod{4}$ . Then  $n = 4k$  for some positive integer  $k$ . The size  $m$  of  $K_{4k}$  is  $\binom{4k}{2} = 2k(4k - 1) = 8k^2 - 2k$  and so  $m/2 = 4k^2 - k$ . Let  $\{H_1, H_2, \dots, H_{2k-1}\}$  be a Hamiltonian-factorization of  $K_{4k-1}$ . Then the subgraph

$H$  of  $K_{4k-1}$  with  $E(H) = E(H_1) \cup E(H_2) \cup \dots \cup E(H_k)$  is a  $2k$ -regular subgraph of size  $k(4k-1) = 4k^2 - k = m/2$  in  $K_{4k}$ .

*Case 2.*  $n \equiv 1 \pmod{4}$ . Then  $n = 4k + 1$  for some positive integer  $k$ . The size  $m$  of  $K_{4k+1}$  is  $\binom{4k+1}{2} = 2k(4k+1) = 8k^2 + 2k$  and so  $m/2 = 4k^2 + k$ . Let  $\{H_1, H_2, \dots, H_{2k}\}$  be a Hamiltonian-factorization of  $K_{4k+1}$ . Then the subgraph  $H$  of  $K_{4k+1}$  with  $E(H) = E(H_1) \cup E(H_2) \cup \dots \cup E(H_k)$  is a  $2k$ -regular subgraph of size  $k(4k+1) = m/2$  in  $K_{4k+1}$ .

*Case 3.*  $n \equiv 2 \pmod{4}$ . Then  $n = 4k + 2$  for some positive integer  $k$ . The size  $m$  of  $K_{4k+2}$  is  $\binom{4k+2}{2} = (2k+1)(4k+1) = 8k^2 + 6k + 1$  and so  $\lceil m/2 \rceil = 4k^2 + 3k + 1$ . For  $k = 1$ , the graph  $K_{2,4}$  is a subgraph of size 8 in  $K_6$ , each vertex of which is even. Thus, we may assume that  $k \geq 2$ . Let  $\{U, V\}$  be a partition of the vertex set of  $K_{4k+2}$ , where  $U = \{u_1, u_2, \dots, u_{2k+1}\}$  and  $V = \{v_1, v_2, \dots, v_{2k+1}\}$ , and let  $H$  be the complete subgraph of order  $2k+1$  with vertex set  $U$  and let  $F$  be the complete subgraph of order  $2k+1$  with vertex set  $V$ . Suppose first that  $k+1$  is even. Then  $k+1 = 2p$  for some integer  $p \geq 2$ . Then the subgraph consisting of  $H$ ,  $F$  and the  $(k+1)$ -cycle  $(u_1, v_1, u_2, v_2, \dots, u_p, v_p, u_1)$  has size  $2k(2k+1) + (k+1) = 4k^2 + 3k + 1 = m/2$ , each vertex of which is even. Suppose next that  $k+1 \geq 3$  is odd. Then  $k+4$  is even and so  $k+4 = 2q$  for some integer  $q \geq 3$ . Let  $H' = H - \{u_1u_2, u_2u_3, u_3u_1\}$ . Then the subgraph consisting of  $H'$ ,  $F$  and the  $(k+4)$ -cycle  $(u_1, v_1, u_2, v_2, \dots, u_q, v_q, u_1)$  has size  $2k(2k+1) - 3 + (k+4) = 4k^2 + 3k + 1 = m/2$ , each vertex of which is even.

*Case 4.*  $n \equiv 3 \pmod{4}$ . Then  $n = 4k + 3$  for some positive integer  $k$ . The size  $m$  of  $K_{4k+3}$  is  $\binom{4k+3}{2} = (2k+1)(4k+3) = 8k^2 + 10k + 3$  and so  $\lceil m/2 \rceil = 4k^2 + 5k + 2$ . For  $k = 1$ , the graph  $K_{1,1,5}$  is a subgraph of size 11 in  $K_7$ , each vertex of which is even. Thus, we may assume that  $k \geq 2$ . Let  $\{U, V, W\}$  be a partition of the vertex set of  $K_{4k+3}$ , where  $U = \{u_1, u_2, \dots, u_{2k+1}\}$ ,  $V = \{v_1, v_2, \dots, v_{2k+1}\}$  and  $|W| = 1$ . Let  $H$  be the complete subgraph of order  $2k+1$  with vertex set  $U$  and let  $F$  be the complete subgraph of order  $2k+1$  with vertex set  $V$ . If  $3k+2$  is even, so  $3k+2 = 2p$  for some integer  $p \geq 3$ , then the subgraph consisting of  $H$ ,  $F$  and the  $(3k+2)$ -cycle  $(u_1, v_1, u_2, v_2, \dots, u_p, v_p, u_1)$  has size  $2k(2k+1) + (3k+2) = 4k^2 + 5k + 2 = m/2$ , each vertex of which is even. If  $3k+2$  is odd, then  $k \geq 3$  and  $3k+5$  is even. Thus  $3k+5 = 2q$  for some integer  $q \geq 4$ . Let  $H' = H - \{u_1u_2, u_2u_3, u_3u_1\}$ . Then the subgraph consisting of  $H'$ ,  $F$  and the  $(3k+5)$ -cycle  $(u_1, v_1, u_2, v_2, \dots, u_q, v_q, u_1)$  has size  $2k(2k+1) - 3 + (3k+5) = 4k^2 + 5k + 2 = m/2$ , each vertex of which is even. In each case,  $K_n$  contains a subgraph of size  $\lceil m/2 \rceil$ , each vertex of which is even. By Theorem 2.3.1,  $K_n$  contains an optimal irregular Eulerian walk. ■

**Theorem 2.3.5** *For integers  $r$  and  $s$  with  $2 \leq r \leq s$ , the complete bipartite graph  $K_{r,s}$  contains an optimal irregular Eulerian walk if and only if*

(i)  $r$  and  $s$  are both even and  $(r, s) \neq (2, 4k+2)$  for any nonnegative integer  $k$

or

(ii) at least one of  $r$  and  $s$  is odd and  $rs \not\equiv 1, 2 \pmod{4}$ .

**Proof.** Let  $G = K_{r,s}$  whose partite sets are

$$U = \{u_1, u_2, \dots, u_r\} \text{ and } W = \{w_1, w_2, \dots, w_s\}.$$

The size of  $G$  is  $m = rs$ . First, assume that  $r$  and  $s$  are both even and  $(r, s) = (2, 4k + 2)$  for some nonnegative integer  $k$ . We claim that  $K_{2,4k+2}$  does not have an optimal irregular Eulerian walk; for otherwise, by Theorem 2.3.1, the graph  $K_{2,4k+2}$  contains a subgraph  $H$  of size  $4k + 2 = 2(2k + 1)$ , each vertex of which is even. Suppose that the partite sets of  $H$  are  $U'$  and  $W'$  where then  $U' \subseteq U$  and  $W' \subseteq W$ . Since  $\deg_G w = 2$  for each  $w \in W$  and each vertex of  $H$  is even, it follows that  $\deg_H w = 2$  for each  $w \in W'$  and so  $U' = U$ . Furthermore, the size of  $H$  is  $2|W'| = 2(2k + 1)$  and so  $|W'| = 2k + 1$ . However then,  $H = K_{2,2k+1}$  with partite sets  $U$  and  $W'$  and  $\deg_H u = 2k + 1$  for each  $u \in U$ , which is a contradiction. Therefore, as claimed,  $K_{2,4k+2}$  does not have an optimal irregular Eulerian walk. Next, assume that at least one of  $r$  and  $s$  is odd and  $rs \equiv 1, 2 \pmod{4}$ . It then follows by Corollary 2.3.2 that  $G$  does not have an optimal irregular Eulerian walk.

For the converse, consider three cases, according to the parity of  $r$  and  $s$ .

*Case 1.*  $r$  and  $s$  are both even and  $(r, s) \neq (2, 4k + 2)$  for any nonnegative integer  $k$ . Thus  $4 \leq r \leq s$ . First, suppose that at least one of  $r$  and  $s$  is congruent to 0 modulo 4. If  $r \equiv 0 \pmod{4}$ , then let  $H = K_{r/2,s}$ ; while if  $s \equiv 0 \pmod{4}$ , then let  $H = K_{r,s/2}$ . In each case,  $H$  is a subgraph of size  $\lceil m/2 \rceil = m/2 = rs/2$  in  $G$ , each vertex of which is even. Next, suppose that neither  $r$  nor  $s$  is congruent to 0 modulo 4. Thus each of  $r$  and  $s$  is congruent to 2 modulo 4 and so  $6 \leq r \leq s$ . Then  $r = 4a + 2$  and  $s = 4b + 2$  for some positive integers  $a$  and  $b$  where  $a \leq b$ . So  $m/2 = 2(2a + 1)(2b + 1)$ . Now let  $F = (u_1, w_1, u_2, w_2, \dots, u_{2a+1}, w_{2a+1}, u_1)$

be a cycle of order  $2(2a + 1)$  in  $G$  and let  $F' = K_{4a+2,2b}$  be the subgraph in  $G$  induced by  $U$  and  $W'$  where  $W' \subseteq W - \{w_1, w_2, \dots, w_{2a+1}\}$  with  $|W'| = 2b$ . Now consider the subgraph  $H$  consisting of  $F$  and  $F'$ , where the partite sets of  $H$  are  $U$  and  $W' \cup \{w_1, w_2, \dots, w_{2a+1}\}$  and  $E(H) = E(F) \cup E(F')$ . Then the size of  $H$  is  $(4a + 2)(2b) + 2(2a + 1) = 2(2a + 1)(2b + 1)$  and each vertex of  $H$  is even.

*Case 2.*  $r$  and  $s$  are both odd and  $rs \not\equiv 1, 2 \pmod{4}$ . If  $r$  and  $s$  are both congruent to 1 modulo 4 or  $r$  and  $s$  are both congruent to 3 modulo 4, then  $m = rs \equiv 1 \pmod{4}$ . Thus exactly one of  $r$  and  $s$  is congruent to 1 modulo 4 and the other is congruent to 3 modulo 4. There are two subcases.

*Subcase 2.1.*  $r \equiv 1 \pmod{4}$  and  $s \equiv 3 \pmod{4}$ . Then  $r = 4a + 1$  and  $s = 4b + 3$  for some positive integers  $a$  and  $b$  and so  $\lceil m/2 \rceil = 8ab + 6a + 2b + 2$ . Let  $b = a + k$ , where  $k \geq 0$ . For each  $i$  with  $0 \leq i \leq k$ , define the  $k + 1$  graphs  $G_0, G_1, G_2, \dots, G_k$  recursively such that  $G_i = K_{4a+1,4(a+i)+3}$  for  $0 \leq i \leq k$  and  $G_0 \subseteq G_1 \subseteq \dots \subseteq G_k$ . Let  $G_0 = K_{4a+1,4a+3}$  with partite sets  $U_0 = \{u_1, u_2, \dots, u_{4a+1}\}$  and  $W_0 = \{w_1, w_2, \dots, w_{4a+3}\}$ . For  $i \geq 1$ , let  $G_i = K_{4a+1,4(a+i)+3}$  with partite sets  $U_i = U_0$  and  $W_i = W_{i-1} \cup \{a_i, b_i, c_i, d_i\}$ . Denote the size of  $G_i$  by  $m_i$  for  $0 \leq i \leq k$ . Then

$$\lceil m_i/2 \rceil = 8a(a + i) + 6a + 2(a + i) + 2 = (8a^2 + 8a + 2) + 2(4a)i + 2i.$$

We now define the  $k + 1$  graphs  $H_0, H_1, H_2, \dots, H_k$  recursively such that  $H_i$  ( $0 \leq i \leq k$ ) is a subgraph of  $G_i$ , the size of  $H_i$  is  $m_{H_i} = \lceil m_i/2 \rceil$ , each vertex of  $H_i$  is even and  $H_0 \subseteq H_1 \subseteq \dots \subseteq H_k$ .



- Let  $H_0$  be the graph obtained from  $K_{4a,2a+2}$  with partite sets

$$U' = \{u_1, u_2, \dots, u_{4a}\} \text{ and } W' = \{w_1, w_2, \dots, w_{2a+2}\} \quad (2.2)$$

by (i) deleting the edge  $u_{4a}w_{2a+2}$  and (ii) adding the three edges  $u_{4a}w_{2a+3}$ ,  $w_{2a+3}u_{4a+1}$ ,  $u_{4a+1}w_{2a+2}$ . The size of  $H_0$  is  $m_{H_0} = 8a^2 + 8a + 2$  and each vertex of  $H_0$  is even.

- We now construct  $H_0 \subseteq H_1$  as follows. Let  $F_1$  be the subgraph of  $G_1$  constructed from the graph  $K_{4a,2}$  with partite sets  $U'$  as described in (2.2) and  $\{a_1, b_1\}$  by (i) deleting the edge  $u_{4a}a_1$  and (ii) adding the three edges  $u_{4a}c_1$ ,  $c_1u_{4a+1}$ ,  $u_{4a+1}a_1$ . Then the graph  $H_1$  consists of  $F_1$  and  $H_0$ , that is,  $V(H_1) = V(F_1) \cup V(H_0)$  and  $E(H_1) = E(F_1) \cup E(H_0)$ . The size of  $H_1$  is  $m_{H_1} = m_{H_0} + 2(4a) + 2 = (8a^2 + 8a + 2) + 2(4a) + 2$  and each vertex of  $H_1$  is even.

- Suppose that  $H_i$  has been constructed for some integer  $i$  with  $0 \leq i < k$  and  $H_i$  has the desired properties. We now construct  $H_i \subseteq H_{i+1}$  as follows. Let  $F_{i+1}$  be the subgraph of  $G_{i+1}$  constructed from the graph  $K_{4a,2}$  with partite sets  $U'$ , as described in (2.2), and  $\{a_{i+1}, b_{i+1}\}$  by (i) deleting the edge  $u_{4a}a_{i+1}$  and (ii) adding the three edges  $u_{4a}c_{i+1}$ ,  $c_{i+1}u_{4a+1}$ ,  $u_{4a+1}a_{i+1}$ . Then the graph  $H_{i+1}$  consists of  $F_{i+1}$  and  $H_i$ , that is, that is,  $V(H_{i+1}) = V(F_{i+1}) \cup V(H_i)$

and  $E(H_{i+1}) = E(F_{i+1}) \cup E(H_i)$ . The size of  $H_{i+1}$  is

$$\begin{aligned}
m_{H_{i+1}} &= m_{H_i} + 2(4a) + 2 \\
&= [(8a^2 + 8a + 2) + (2i)(4a) + 2i] + 2(4a) + 2 \\
&= (8a^2 + 8a + 2) + 2(i+1)(4a) + 2(i+1).
\end{aligned}$$

and each vertex of  $H_{i+1}$  is even.

By Theorem 2.3.1,  $G$  contains an optimal irregular Eulerian walk in this case.

*Subcase 2.2.*  $r \equiv 3 \pmod{4}$  and  $s \equiv 1 \pmod{4}$ . Then  $r = 4a + 3$  and  $s = 4b + 1$  for some positive integers  $a$  and  $b$  and so  $\lceil m/2 \rceil = 8ab + 2a + 6b + 2$ . In this case,  $a < b$ . Let  $b = a + k$ , where  $k \geq 1$ . For each  $i$  with  $0 \leq i \leq k$ , define the  $k$  graphs  $G_1, G_2, \dots, G_k$  recursively such that  $G_i = K_{4a+3, 4(a+i)+1}$  for  $1 \leq i \leq k$  and  $G_1 \subseteq G_2 \subseteq \dots \subseteq G_k$ . Let  $G_1 = K_{4a+3, 4(a+1)+1}$  with partite sets  $U_1 = \{u_1, u_2, \dots, u_{4a+3}\}$  and  $W_1 = \{w_1, w_2, \dots, w_{4(a+1)+1}\}$ . For  $i \geq 2$ , let  $G_i = K_{4a+3, 4(a+i)+1}$  with partite sets  $U_i = U_1$  and  $W_i = W_{i-1} \cup \{a_i, b_i, c_i, d_i\}$ . Denote the size of  $G_i$  by  $m_i$  for  $1 \leq i \leq k$ . Then

$$\begin{aligned}
\lceil m_i/2 \rceil &= 8a(a+i) + 2a + 6(a+i) + 2 \\
&= (8a^2 + 16a + 8) + 2(4a+2)(i-1) + 2(i-1).
\end{aligned}$$

We define the  $k$  graphs  $H_1, H_2, \dots, H_k$  recursively such that  $H_i$  ( $1 \leq i \leq k$ ) is a subgraph of  $G_i$ , the size of  $H_i$  is  $\lceil m_i/2 \rceil$ , each vertex of  $H_i$  is even and  $H_1 \subseteq H_2 \subseteq \dots \subseteq H_k$ .

- Let  $H_1$  be the graph consisting of  $F = K_{2a+4, 4a}$  with partite sets

$$\{u_1, u_2, \dots, u_{2a+4}\} \text{ and } \{w_1, w_2, \dots, w_{4a}\}$$

and  $F' = K_{2,4}$  with partite sets

$$\{u_{2a+5}, u_{2a+6}\} \text{ and } \{w_{4a+1}, w_{4a+2}, w_{4a+3}, w_{4a+4}\};$$

that is,  $V(H_1) = V(F) \cup V(F')$  and  $E(H_1) = E(F) \cup E(F')$ . Then the size of  $H_1$  is  $m_{H_1} = (2a + 4)(4a) + 8 = 8a^2 + 16a + 8$  and each vertex of  $H_1$  is even.

- We now construct  $H_2$  such that  $H_1 \subseteq H_2$  as follows. Let  $F_2$  be the subgraph of  $G_2$  constructed from the graph  $K_{4a+2,2}$  with partite sets  $\{u_1, u_2, \dots, u_{4a+2}\}$  and  $\{a_2, b_2\}$  by (i) deleting the edge  $u_{4a+2}a_2$  and (ii) adding the three edges  $u_{4a+2}c_2, c_2u_{4a+3}, u_{4a+3}a_2$ . Then the graph  $H_2$  consists of  $F_2$  and  $H_1$ , that is,  $V(H_2) = V(F_2) \cup V(H_1)$  and  $E(H_2) = E(F_2) \cup E(H_1)$ . The size of  $H_2$  is  $m_{H_2} = m_{H_1} + 2(4a + 2) + 2 = (8a^2 + 16a + 8) + 2(4a + 2) + 2$  and each vertex of  $H_2$  is even.
- Suppose that  $H_i$  has been constructed as desired for some integer  $i$  with  $1 \leq i < k$  and  $H_i$  has the desired properties. We now construct  $H_{i+1}$  such that  $H_i \subseteq H_{i+1}$  as follows. Let  $F_{i+1}$  be the subgraph of  $G_{i+1}$  constructed from the graph  $K_{4a+2,2}$  with partite sets  $\{u_1, u_2, \dots, u_{4a+2}\}$  and  $\{a_{i+1}, b_{i+1}\}$  by (i) deleting the edge  $u_{4a+2}a_{i+1}$  and (ii) adding the three edges  $u_{4a+2}c_{i+1}, c_{i+1}u_{4a+3}, u_{4a+3}a_{i+1}$ . Then the graph  $H_{i+1}$  consists of  $F_{i+1}$  and  $H_i$ .

The size of  $H_{i+1}$  is

$$\begin{aligned}
m_{H_{i+1}} &= m_{H_i} + 2(4a + 2) + 2 \\
&= [(8a^2 + 16a + 8) + 2(4a + 2)(i - 1) + \\
&\quad 2(i - 1)] + 2(4a + 2) + 2 \\
&= (8a^2 + 16a + 8) + 2(4a + 2)i + 2i.
\end{aligned}$$

and each vertex of  $H_{i+1}$  is even.

By Theorem 2.3.1,  $G$  contains an optimal irregular Eulerian walk in this case.

*Case 3. Exactly one of  $r$  and  $s$  is odd and  $rs \not\equiv 1, 2 \pmod{4}$ .* There are two subcases.

*Subcase 3.1.  $r$  is odd and  $s$  is even.* If  $s \equiv 2 \pmod{4}$ , then  $m = rs \equiv 2 \pmod{4}$ . Thus we assume that  $s \equiv 0 \pmod{4}$ . Then  $r = 2a + 1$  and  $s = 4b$  for some positive integers  $a$  and  $b$  and so  $\lceil m/2 \rceil = 2b(2a + 1)$ . Let  $U = \{u_1, u_2, \dots, u_{2a+1}\}$  and  $W = \{w_1, w_2, \dots, w_{4b}\}$  be the partite sets of  $K_{2a+1, 4b}$  and let  $F_1 = K_{2a, 2b}$  be the subgraph of  $K_{2a+1, 4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$ . If  $b$  is even, then let  $F_2 = K_{2, b}$  be the subgraph of  $K_{2a+1, 4b}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_{2b+1}, w_{2b+2}, \dots, w_{3b}\}$ . If  $b$  is odd, then let  $F_2$  be the subgraph of  $K_{2a+1, 4b}$  constructed from the graph  $K_{2, b-1}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_{2b+1}, w_{2b+2}, \dots, w_{3b-1}\}$  by (i) deleting the edge  $u_2 w_{3b-1}$  and (ii) adding the three edges  $u_2 w_{3b}, w_{3b} u_{2a+1}, u_{2a+1} w_{3b-1}$ . In each case, the graph  $H$  consists of  $F_1$  and  $F_2$ . Then  $H$  has size  $2b(2a + 1)$  and each vertex of  $H$  is even.

*Subcase 3.2.  $r$  is even and  $s$  is odd.* If  $r \equiv 2 \pmod{4}$ , then  $m = rs \equiv 2 \pmod{4}$ . Thus we assume that  $r \equiv 0 \pmod{4}$ . Then  $r = 4a$  and  $s = 2b + 1$  for some positive integers  $a$  and  $b$  and so  $\lceil m/2 \rceil = 2a(2b + 1)$ . Let  $U = \{u_1, u_2, \dots, u_{4a}\}$  and  $W = \{w_1, w_2, \dots, w_{2b+1}\}$  be the partite sets of  $K_{4a, 2b+1}$  and let  $F_1 = K_{2a, 2b}$  be the subgraph of  $K_{4a, 2b+1}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$ . If  $a$  is even, then let  $F_2 = K_{a, 2}$  be the subgraph of  $K_{4a, 2b+1}$  with partite sets  $\{u_{2a+1}, u_{2a+2}, \dots, u_{3a}\}$  and  $\{w_1, w_2\}$ . If  $a$  is odd, then let  $F_2$  be the subgraph of  $K_{4a, 2b+1}$  constructed from the graph  $K_{a-1, 2}$  with partite sets  $\{u_{2a+1}, u_{2a+2}, \dots, u_{3a-1}\}$  and  $\{w_1, w_2\}$  by (i) deleting the edge  $w_2u_{3a-1}$  and (ii) adding the three edges  $w_2u_{3a}, u_{3a}w_{2b+1}, w_{2b+1}u_{3a-1}$ . In each case, the graph  $H$  consists of  $F_1$  and  $F_2$ . Then  $H$  has size  $2a(2b + 1)$  and each vertex of  $H$  is even.

By Theorem 2.3.1,  $G$  contains an optimal irregular Eulerian walk in this case. ■

## 2.4 A Realization Result on the Eulerian Irregularity of a Graph

We have seen that if  $G$  is a nontrivial connected graph of size  $m$ , then  $\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}$ . This gives rise to the following question:

*For given positive integers  $k$  and  $m$  with  $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ , is there a connected graph  $G$  of size  $m$  such that  $EI(G) = k$ ?*

In this section, we present a necessary and sufficient condition for a pair  $k, m$  of positive integers such that there is a nontrivial connected graph  $G$  of size  $m$  with  $EI(G) = k$ . In order to do this, we first present some preliminary results.

Recall that a *weighted graph* is a graph in which each edge  $e$  is assigned a positive integer called the *weight* of the edge and denoted by  $w(e)$ . The *degree of a vertex*  $v$  in a weighted graph  $H$  is the sum of the weights of the edges incident with  $v$  and is denoted by  $\deg_H v$  (or  $\deg v$  if the weighted graph  $H$  under consideration is clear). A weighted graph  $H$  is *Eulerian* if  $H$  is connected and every vertex has even degree. For an Eulerian walk  $W$  of a connected graph  $G$ , let  $G_W$  be the weighted graph obtained from  $G$  by assigning to each edge  $uv$  of  $G$  the number of times  $uv$  is encountered on  $W$ . In this case,  $G_W$  is said to be *induced* by  $W$ . Consequently, the vertex set of  $G_W$  is  $V(G)$  and every vertex in  $G_W$  has even degree. Thus, the weighted graph  $G_W$  induced by an Eulerian walk  $W$  in  $G$  is Eulerian. Furthermore, for an Eulerian walk  $W$  of a connected graph  $G$ , let  $M$  be the multigraph obtained from  $G$  by replacing each edge  $uv$  of  $G$  by the number of parallel edges equal to the number of times  $uv$  is encountered on  $W$ . In this case,  $M$  is said to be *induced* by  $W$ . Consequently,  $M$  is an Eulerian multigraph whose vertex set is  $V(G)$ .

For a connected graph  $G$  of size  $m$  with edge set  $\{e_1, e_2, \dots, e_m\}$  and an Eulerian walk  $W$ , let  $a_i$  be the number of times that  $e_i$  is encountered in  $W$  for  $1 \leq i \leq m$ . If  $W$  is an Eulerian walk of minimum length, then  $a_i \in \{1, 2\}$ , while if  $W$  is an irregular Eulerian walk of minimum length, then  $a_i \in \{1, 2, \dots, 2m\}$  and  $a_i \neq a_j$  for all  $i, j$  with  $1 \leq i \neq j \leq m$ . In general, a multiset  $S = \{a_1, a_2, \dots, a_m\}$  of positive integers is *Eulerian realizable* if there is a connected graph  $G$  of size  $m$ , an ordering  $e_1, e_2, \dots, e_m$  of the edges of  $G$  and an Eulerian walk  $W$  in  $G$  such that  $e_i$  is encountered exactly  $a_i$  times in  $W$  for  $1 \leq i \leq m$ . We now present a necessary

and sufficient conditions for a multiset  $S$  of  $m \geq 3$  positive integers to be Eulerian realizable.

**Theorem 2.4.1** *For an integer  $m \geq 3$ , a multiset  $S = \{a_1, a_2, \dots, a_m\}$  of positive integers is Eulerian realizable if and only if either (i) no element in  $S$  is odd or (ii) at least three elements in  $S$  are odd.*

**Proof.** First, suppose that exactly one or exactly two elements of  $S$  are odd. For any connected graph  $F$  of size  $m$  and any ordering  $f_1, f_2, \dots, f_m$  of the edges of  $F$ , let  $H$  be the weighted graph obtained from  $F$  by assigning the weight  $a_i$  to  $f_i$  for  $1 \leq i \leq m$ . Since either exactly one edge of  $F$  is assigned an odd weight or exactly two edges of  $F$  are assigned odd weights, it follows that  $H$  must have at least two vertices of odd degree. Hence  $F$  cannot have a closed walk in which  $f_i$  is encountered  $a_i$  times for  $i = 1, 2, \dots, m$ .

To verify the converse, first suppose that no element in  $S$  is odd. Let  $G$  be any connected graph of size  $m$  with  $E(G) = \{e_1, e_2, \dots, e_m\}$  and let  $H$  be the weighted graph obtained by assigning the weight  $a_i$  to  $e_i$  for  $1 \leq i \leq m$ . Since every element in  $S$  is even, each vertex of  $H$  has even degree and so  $G$  has a closed walk in which  $e_i$  is encountered  $a_i$  times for  $i = 1, 2, \dots, m$ . Next, suppose that exactly  $k \geq 3$  elements in  $S$  are odd. If  $k = m$ , then let  $G = C_m$ ; while if  $k < m$ , then let  $G$  be the graph obtained from the  $k$ -cycle  $C_k$  of order  $k$  and the path  $P_{m-k}$  of order  $m - k$  by joining an end-vertex of  $P_{m-k}$  to a vertex of  $C_k$ . Then the size of  $G$  is  $m$ . Let  $H$  be the weighted graph obtained by assigning the  $k$  odd weights to the  $k$  edges of  $C_k$  and the  $m - k$  even weights to the remaining  $m - k$  edges of  $G$ .

Then each vertex of  $H$  is even and so there is an ordering  $e_1, e_2, \dots, e_m$  of edges of  $G$  and a closed walk  $W$  in  $G$  such that  $e_i$  is encountered  $a_i$  times in  $W$  for  $i = 1, 2, \dots, m$ . ■

In the problem of finding an Eulerian walk  $W$  of minimum length in  $G$ , we minimize the number of edges that are encountered exactly twice in  $W$ . In the problem of finding an irregular Eulerian walk  $W$  of minimum length in  $G$ , we have a different situation. For an Eulerian walk  $W$  in  $G$ , let  $m_1 = m_1(W)$  be the number of edges that are encountered exactly once in  $W$  and  $m_2 = m_2(W)$  the number of edges that are encountered exactly twice in  $W$ , where then  $m = m_1 + m_2$ . Let  $e_1, e_2, \dots, e_{m_1}$  be those edges occurring exactly once on  $W$  and let  $f_1, f_2, \dots, f_{m_2}$  be those edges occurring exactly twice on  $W$ . We construct an Eulerian multigraph  $M$  by replacing each edge  $e_i$  ( $1 \leq i \leq m_1$ ) by  $2i - 1$  parallel edges and replacing each edge  $f_j$  ( $1 \leq j \leq m_2$ ) by  $2j$  parallel edges. An Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk  $W^*$  in  $G$  such that  $e_i$  ( $1 \leq i \leq m_1$ ) appears exactly  $2i - 1$  times in  $W^*$  and  $f_j$  ( $1 \leq j \leq m_2$ ) appears exactly  $2j$  times in  $W^*$ . Thus, the length of  $W^*$  is

$$[1 + 3 + \dots + (2m_1 - 1)] + [2 + 4 + \dots + 2m_2] = m_1^2 + m_2(m_2 + 1)$$

where  $m = m_1 + m_2$ . Therefore, in the problem of finding an irregular Eulerian walk of minimum length in  $G$ , we investigate those connected graphs  $G$  that minimize  $|m_1(W) - m_2(W)|$  over all Eulerian walks  $W$  in  $G$ . In the view of this observation, we present the following lemma.



**Lemma 2.4.2** *Let  $G$  be a nontrivial connected graph of size  $m$ . If  $G$  contains an even subgraph  $F$  of size  $x$ , then there is an irregular Eulerian walk of length  $x^2 + (m - x)(m - x + 1)$  in  $G$  and so  $EI(G) \leq x^2 + (m - x)(m - x + 1)$ .*

**Proof.** Let  $F$  be an even subgraph of size  $x$  in  $G$  and let

$$E(G) = \{e_1, e_2, \dots, e_x\} \cup \{f_1, f_2, \dots, f_{m-x}\},$$

where  $E(F) = \{e_1, e_2, \dots, e_x\}$ . We construct an Eulerian multigraph  $M$  by replacing each edge  $e_i$  where  $1 \leq i \leq x$  by  $2i - 1$  parallel edges and replacing each edge  $f_j$  where  $1 \leq j \leq m - x$  by  $2j$  parallel edges. An Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk  $W$  in  $G$  such that each edge  $e_i$  of  $G$  appears exactly  $2i - 1$  times in  $W$  where  $1 \leq i \leq x$  and each edge  $f_j$  of  $G$  appears exactly  $2j$  times in  $W$  where  $1 \leq j \leq m - x$ . Then the length of  $W$  is  $x^2 + (m - x)(m - x + 1)$  and so  $EI(G) \leq L(W) = x^2 + (m - x)(m - x + 1)$ . ■

With the aid of Lemma 2.4.2, we determine the Eulerian irregularity of a special class of connected graphs. A graph  $G$  is *unicyclic* if  $G$  is connected and contains exactly one cycle. The next result provide the Eulerian irregularity of a unicyclic graph in terms of its size and the size of its unique cycle.

**Proposition 2.4.3** *If  $G$  is a unicyclic graph of size  $m \geq 3$  and the unique cycle in  $G$  is a  $k$ -cycle for some integer  $k \geq 3$ , then*

$$EI(G) = k^2 + (m - k)(m - k + 1).$$

*In particular, if  $G = C_n$ , then  $EI(C_n) = n^2$ .*

**Proof.** Since  $G$  contains an even subgraph of size  $k$ , namely  $C_k$ , it follows by Lemma 2.4.2 that  $EI(G) \leq k^2 + (m - k)(m - k + 1)$ . Now, let  $C_k = (v_1, v_2, \dots, v_k, v_{k+1} = v_1)$  be the unique cycle in  $G$  and let  $E(G) - E(C_k) = \{f_1, f_2, \dots, f_{m-k}\}$ . Let  $W$  be an irregular Eulerian walk of minimum length in  $G$ . Since each edge  $f_j \in E(G) - E(C_k)$  is a bridge in  $G$  for  $1 \leq j \leq m - k$ , it follows that  $f_j$  must be encountered an even number of times on  $W$ . Furthermore, either every edge on  $C_k$  is encountered an odd number of times on  $W$  or every edge on  $C_k$  is encountered an even number of times on  $W$ . Thus

$$\begin{aligned} EI(G) &= L(W) \geq [1 + 3 + \dots + (2k - 1)] + [2 + 4 + \dots + 2(m - k)] \\ &= k^2 + (m - k)(m - k + 1), \end{aligned}$$

giving the desired result. In particular, if  $G = C_n$ , then an irregular Eulerian walk of minimum length encounters each edge of  $C_n$  an odd number of times and so  $EI(C_n) = 1 + 3 + \dots + (2n - 1) = n^2$ . ■

If  $W$  is an irregular Eulerian walk of minimum length in a nontrivial connected graph  $G$ , then the set of occurrences of edges of  $G$  in  $W$  satisfies certain conditions, which are described in the next result.

**Lemma 2.4.4** *Let  $G$  be a nontrivial connected graph  $G$  of size  $m$  and let  $W$  be an irregular Eulerian walk of minimum length in  $G$ . If there are  $x$  edges of  $G$  that are encountered an odd number of times in  $W$  and there are  $m - x$  edges of  $G$  that are encountered an even number of times in  $W$ , then the numbers of times of the edges of  $G$  encountered in  $W$  are  $1, 3, \dots, 2x - 1, 2, 4, \dots, 2(m - x)$  and so  $EI(G) = x^2 + (m - x)(m - x + 1)$ .*

**Proof.** Let  $W$  be an irregular Eulerian walk of minimum length in  $G$ , where then  $L(W) = EI(G)$ . For each edge  $e$  of  $G$ , let  $w(e)$  be the number of times that  $e$  is encountered in  $W$ . Let  $\{e_1, e_2, \dots, e_x\}$  be the set of edges of  $G$  that are encountered an odd number of times in  $W$  and  $\{f_1, f_2, \dots, f_y\}$  the set of edges that are encountered an even number of times in  $W$ , where  $y = m - x$ . We may assume that  $w(e_1) < w(e_2) < \dots < w(e_x)$  and  $w(f_1) < w(f_2) < \dots < w(f_y)$ . Thus  $w(e_i) \geq 2i - 1$  for  $1 \leq i \leq x$  and  $w(f_j) \geq 2j$  for  $1 \leq j \leq y$ , which implies that  $L(W) \geq x^2 + y(y + 1)$ . Now consider the Eulerian multigraph  $M$  obtained from  $G$  by replacing each edge  $e_i$  ( $1 \leq i \leq x$ ) by  $2i - 1$  parallel edges and each edge  $f_j$  ( $1 \leq j \leq y$ ) by  $2j$  parallel edges. An Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk  $W^*$  in  $G$  such that  $e_i$  ( $1 \leq i \leq x$ ) appears exactly  $2i - 1$  times in  $W^*$  and  $f_j$  ( $1 \leq j \leq y$ ) appears exactly  $2j$  times in  $W^*$ . Thus, the length of  $W^* = x^2 + y(y + 1)$ . Since  $W$  is an irregular Eulerian walk of minimum length,  $L(W) \leq L(W^*) = x^2 + y(y + 1)$ . Therefore,  $L(W) = x^2 + y(y + 1)$  and so  $w(e_i) = 2i - 1$  for  $1 \leq i \leq x$  and  $w(f_j) = 2j$  for  $1 \leq j \leq y$ . Therefore, the numbers of times of the edges of  $G$  encountered in  $W$  are  $1, 3, \dots, 2x - 1, 2, 4, \dots, 2(m - x)$  and so  $EI(G) = x^2 + (m - x)(m - x + 1)$ . ■

We are now prepared to present the following realization result.

**Theorem 2.4.5** *Let  $k$  and  $m$  be positive integers with  $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ . Then there exists a nontrivial connected graph  $G$  of size  $m$  with  $EI(G) = k$  if and only if there exists an integer  $x$  with  $0 \leq x \leq m$  and  $x \neq 1, 2$  such that  $x^2 + (m - x)(m - x + 1) = k$ .*

**Proof.** First, suppose that  $G$  is a nontrivial connected graph of size  $m$  such that  $EI(G) = k$ . Let  $W$  be an irregular Eulerian walk of length  $EI(G)$  in  $G$ . Suppose that there are  $x \geq 0$  edges of  $G$  that are encountered an odd number of times in  $W$  and  $m - x$  edges that are encountered an even number of times in  $W$ . It then follows by Lemma 2.4.4 that  $L(W) = x^2 + (m - x)(m - x + 1)$ . Furthermore,  $x \neq 1, 2$  by Theorem 2.4.1.

For the converse, let  $k$  and  $m$  be positive integers with  $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$  and let  $x$  be an integer such that  $0 \leq x \leq m$ ,  $x \neq 1, 2$ , and  $x^2 + (m - x)(m - x + 1) = k$ . By Theorem 5.1.1, we may assume that  $\binom{m+1}{2} < k < 2\binom{m+1}{2}$ . Thus  $x > 0$  and so  $x \geq 3$ . Let  $G$  be a unicyclic graph of size  $m$  that contains the cycle  $C_x$  of order  $x$ . It then follows by Proposition 2.4.3 that  $EI(G) = x^2 + (m - x)(m - x + 1) = k$ . ■

By Theorem 2.4.5, a pair  $k, m$  of positive integers with  $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$  can be realized as the Eulerian irregularity and the size of some nontrivial connected graph if and only if there exists an integer  $x$  with  $0 \leq x \leq m$  and  $x \neq 1, 2$  such that  $x^2 + (m - x)(m - x + 1) = k$ . To determine the possible values of such integers  $x$ , we consider the real-valued function

$$L(x) = x^2 + (m - x)(m - x + 1) = 2x^2 - (2m + 1)x + m^2 + m. \quad (2.3)$$

Since  $L(x)$  is a concave-up parabola which has the minimum value at  $x_0 = \frac{2m+1}{4}$ , it follows that the closer  $x$  is to  $x_0$ , the closer  $L(x)$  is to  $L(x_0)$ . For a positive integer  $m$ , let  $[0..m]$  be the set of all integers  $x$  with  $0 \leq x \leq m$ . We list the elements of  $[0..m]$  as an ordered sequence  $s$  of length  $m + 1$  where

$$s = (x_1, x_2, \dots, x_{m+1}) \quad (2.4)$$

such that

$$L(x_1) \leq L(x_2) \leq \cdots \leq L(x_{m+1}), \quad (2.5)$$

where then

$$L(x_1) = \binom{m+1}{2}, L(x_2) = \binom{m+1}{2} + 1, L(x_3) = \binom{m+1}{2} + 3, \dots, L(x_{m+1}) = 2\binom{m+1}{2}.$$

The sequence  $s$  in (2.4) that satisfies (2.3) and (2.5) is referred to as the *Eulerian irregular sequence of  $m$* . We now state a useful observation on Eulerian irregular sequences.

**Observation 2.4.6** *Let  $m$  be a positive integer.*

- *If  $m$  is even, then the Eulerian irregular sequence of  $m$  is*

$$\left(\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{m}{2} \right\rceil + 1, \left\lceil \frac{m}{2} \right\rceil - 1, \left\lceil \frac{m}{2} \right\rceil + 2, \left\lceil \frac{m}{2} \right\rceil - 2, \dots, \left\lceil \frac{m}{2} \right\rceil + \left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right), \left\lceil \frac{m}{2} \right\rceil - \left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right), m, 0\right); \quad (2.6)$$

- *If  $m$  is odd, then the Eulerian irregular sequence of  $m$  is*

$$\left(\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{m}{2} \right\rceil - 1, \left\lceil \frac{m}{2} \right\rceil + 1, \left\lceil \frac{m}{2} \right\rceil - 2, \left\lceil \frac{m}{2} \right\rceil + 2, \dots, \left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor, \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor = m, 0\right) \quad (2.7)$$

We next present a formula for the Eulerian irregularity  $EI(G)$  of a graph  $G$  in terms of the size of  $G$  and the size of a certain even subgraph of  $G$ .

**Theorem 2.4.7** *Let  $G$  be a nontrivial connected graph of size  $m$  and  $(x_1, x_2, \dots, x_{m+1})$  the Eulerian irregular sequence of  $m$ . If*

$$\alpha = \min\{i : G \text{ contains an even subgraph } F \text{ of size } x_i, 1 \leq i \leq m+1\},$$

*then  $EI(G) = x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$ .*

**Proof.** By Theorem 5.1.1, we may assume that  $G$  is not a tree. Since  $G$  contains an even subgraph of size  $x_\alpha$ , it follows by Lemma 2.4.2 that

$$EI(G) \leq x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1).$$

Let  $W$  be an irregular Eulerian walk of length  $EI(G)$  in  $G$ . Let  $E'$  be the set of edges of  $G$  that are encountered an odd number of times in  $W$  and let  $E''$  be the set of edges of  $G$  that are encountered an even number of times in  $W$ . Since  $G$  is not a tree, it follows by Kwan's Theorem that  $E' \neq \emptyset$ . Let  $F'$  be the subgraph induced by  $E'$  and  $F''$  the subgraph induced by  $E''$ . We claim that every vertex of  $F'$  is even. Let  $M$  be the weighted graph obtained by assigning the weight  $w(e)$  to each edge  $e$  of  $G$ , where  $w(e)$  is the number of times that  $e$  is encountered in  $W$ . Let  $H'$  be the weighted subgraph of  $M$  induced by the edges of  $F'$  and let  $H''$  be the weighted subgraph of  $M$  induced by the edges of  $F''$ . Since  $G$  has an Eulerian walk in which each edge  $e$  appears exactly  $w(e)$  times, every vertex of  $M$  has even degree. Since  $\deg_M v = \deg_{H'} v + \deg_{H''} v$  for every vertex  $v$  of  $G$  and  $\deg_M v$  and  $\deg_{H''} v$  are both even, it follows that  $\deg_{H'} v$  is even. Suppose that  $\deg_{F'} v = k$ . Then  $v$  is incident with  $k$  edges in  $G$ , each of odd weight. Since  $\deg_{H'} v$  is even,  $k$  is even and so  $v$  is an even vertex in  $F'$ . Therefore,  $F'$  is an even subgraph. Suppose that the size of  $F'$  is  $x$ , where then  $1 \leq x \leq m$ . It then follows by Lemma 2.4.4 that  $EI(G) = L(W) = x^2 + (m - x)(m - x + 1)$ . By the defining property of  $x_\alpha$  and Observation 2.4.6, it follows that  $x = x_\alpha$  and so  $EI(G) = x_\alpha^2 + (m - x_\alpha)(m - x_\alpha + 1)$ . ■

## 2.5 Eulerian Irregularities of Complete Bipartite Graphs

To illustrate the results obtained in Section 2.4, we determine Eulerian irregularities of two classes of graphs, namely the complete graphs and complete bipartite graphs. We have seen in Theorems 2.3.4 and 2.3.5 that (1) for each integer  $n \geq 2$ , the complete graph  $K_n$  is optimal if and only if  $n \geq 4$  and (2) for integers  $r$  and  $s$  with  $2 \leq r \leq s$ , the complete bipartite graph  $K_{r,s}$  is optimal if and only if (i)  $r$  and  $s$  are both even and  $(r, s) \neq (2, 4k + 2)$  for any nonnegative integer  $k$  or (ii) at least one of  $r$  and  $s$  is odd and  $rs \not\equiv 1, 2 \pmod{4}$ .

Since  $K_2$  is a tree and  $K_3$  is a cycle, the Eulerian irregularities of complete graphs can be determined by Theorem 5.1.1, Proposition 2.4.3 and Theorem 2.3.4, which we state as follows.

**Theorem 2.5.1** *For each integer  $n \geq 2$ ,*

$$EI(K_n) = \begin{cases} 2 & \text{if } n = 2 \\ 9 & \text{if } n = 3 \\ \binom{n}{2} + 1 & \text{if } n \geq 4. \end{cases}$$

We now determine the Eulerian irregularity of a complete bipartite graph.

**Theorem 2.5.2** *If the complete bipartite graph  $K_{r,s}$  is not optimal where  $2 \leq r \leq s$ , then*

$$EI(K_{r,s}) = \begin{cases} \binom{rs+1}{2} + 6 & \text{if } r \text{ and } s \text{ are both even} \\ \binom{rs+1}{2} + 1 & \text{if at least one of } r \text{ and } s \text{ is odd.} \end{cases}$$

**Proof.** Suppose that  $G = K_{r,s}$  is not optimal where  $2 \leq r \leq s$ . By Theorem 2.3.5, either

- $r$  and  $s$  are both even and  $(r, s) = (2, 4k + 2)$  for some  $k \geq 0$  or
- at least one of  $r$  and  $s$  is odd and  $rs = 1, 2 \pmod{4}$ .

Let  $m = rs$  be the size of  $G$ .

First, suppose that  $r$  and  $s$  are both even and  $(r, s) = (2, 4k + 2)$  for some  $k \geq 0$ . Then  $m = rs = 8k + 4$  and so  $\frac{m}{2} = 4k + 2$ . Since  $G$  contains the even subgraph  $H = K_{2,2k+2}$  of size  $4k + 4 = \frac{m}{2} + 2$ , it follows by Lemma 2.4.2 that

$$EI(G) \leq (4k + 4)^2 + (4k)(4k + 1) = \binom{rs + 1}{2} + 6.$$

Since  $G$  contains neither even subgraph of odd size  $4k + 3$  nor even subgraph of odd size  $4k + 1$ , it follows by Theorem 2.4.5, Observation 2.4.6 and Theorem 2.4.7 that  $EI(G) \geq \binom{rs+1}{2} + 6$  and so  $EI(G) = \binom{rs+1}{2} + 6$ .

Next, suppose that at least one of  $r$  and  $s$  is odd and  $rs = 1, 2 \pmod{4}$ . Denote the partite sets of  $G$  by

$$U = \{u_1, u_2, \dots, u_r\} \text{ and } W = \{w_1, w_2, \dots, w_s\}.$$

We consider three cases, according to the parity of  $r$  and  $s$ .

*Case 1.  $r$  is odd and  $s$  is even.* Since  $rs \equiv 2 \pmod{4}$  and  $r \leq s$ , it follows that  $r = 2a + 1$  and  $s = 4b + 2$ , where  $a, b \geq 1$  and  $a \leq 2b$ . Since  $G$  is not optimal by Theorem 2.3.5 and  $m = rs$  is even, it follows by Observation 2.4.6 that



$EI(G) \geq L(\frac{m}{2} + 1) = \binom{m+1}{2} + 1$  where  $L(x)$  is defined in (2.3) for an integer  $x$ .

That is,

$$EI(G) \geq \left(\frac{m}{2} + 1\right)^2 + \left(\frac{m}{2} - 1\right) \frac{m}{2}.$$

By Lemma 2.4.2, it remains to show that  $G$  contains an even subgraph of size  $\frac{m}{2} + 1$ . Observe that

$$\frac{m}{2} + 1 = (2a + 1)(2b + 1) + 1 = 4ab + 2a + 2b + 2.$$

We consider two subcases, according to whether  $a + b$  is odd or  $a + b$  is even.

*Subcase 1.1.  $a + b$  is odd.* First, suppose that  $a \leq b$ . Then  $3b + a + 1 \leq 4b + 2$  and  $\frac{m}{2} + 1 = 4ab + 2(a + b + 1)$ . Let  $F_1 = K_{2a, 2b}$  be the subgraph of  $G$  induced by  $\{u_1, u_2, \dots, u_{2a}\} \cup \{w_1, w_2, \dots, w_{2b}\}$  and let  $F_2 = K_{2, a+b+1}$  the subgraph of  $G$  induced by  $\{u_1, u_2\} \cup \{w_{2b+1}, w_{2b+2}, \dots, w_{3b+a+1}\}$ . Then let  $H$  be the even subgraph consisting of  $F_1$  and  $F_2$  whose vertex set is  $V(F_1) \cup V(F_2)$  and whose edge set  $E(F_1) \cup E(F_2)$ . Then the size of  $H$  is  $\frac{m}{2} + 1$ .

Next, suppose that  $b < a \leq 2b$ . If  $a$  is even and  $b$  is odd, then  $\frac{m}{2} + 1 = a(4b + 2) + 2(b + 1)$ . Let  $F_1 = K_{a, 4b+2}$  with partite sets  $\{u_1, u_2, \dots, u_a\}$  and  $W$  and let  $F_2 = K_{2, b+1}$  with partite sets  $\{u_{a+1}, u_{a+2}\}$  and  $\{w_1, w_2, \dots, w_{b+1}\}$ . Then let  $H$  be the even subgraph consisting of  $F_1$  and  $F_2$ . If  $a$  is odd and  $b$  is even, then  $\frac{m}{2} + 1 = (a - 1)(4b) + 6b + 2(a + 1)$ . Let  $F'_1 = K_{a-1, 4b}$  with partite sets  $\{u_1, u_2, \dots, u_{a-1}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F'_2 = K_{2, 3b}$  with partite sets  $\{u_a, u_{a+1}\}$  and  $\{w_1, w_2, \dots, w_{3b}\}$  and let  $F'_3 = K_{a+1, 2}$  with partite sets  $\{u_1, u_2, \dots, u_{a+1}\}$  and  $\{w_{4b+1}, w_{4b+2}\}$ . Then let  $H$  be the even subgraph consisting of  $F'_1, F'_2$  and  $F'_3$  and the size of  $H$  is  $\frac{m}{2} + 1$ .

*Subcase 1.2.  $a + b$  is even.* Then  $a$  and  $b$  are of the same parity. First, suppose that  $a$  and  $b$  are both odd, say  $a = 2p + 1$  and  $b = 2q + 1$  for some integers  $p, q \geq 0$ . Then  $\frac{m}{2} + 1 = (2a)(2b) + 2(2q) + 2(2p) + 6$ . If  $b = 1$ , then  $a = 1$  (since  $r \leq s$ ) and so  $G = K_{3,6}$ . The even subgraph of  $G$  consisting of  $C_4 = (u_1, w_1, u_2, w_2, u_1)$  and  $C_6 = (u_1, w_3, u_2, w_4, u_3, w_5, u_1)$  has size 10. Thus, we may assume that  $b \geq 2$  and so  $3b + 4 \leq 4b + 2$ . Let  $F_1 = K_{2a,2b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$ , let  $F_2 = K_{2,2q}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_{2b+1}, w_{2b+2}, \dots, w_{3b-1}\}$ , let  $F_3 = K_{2p,2}$  with partite sets  $\{u_1, u_2, \dots, u_{2p}\}$  and  $\{w_{3b}, w_{3b+1}\}$  and let

$$F_4 = C_6 = (u_1, w_{3b+2}, u_2, w_{3b+3}, u_3, w_{3b+4}, u_1).$$

Then let  $H$  be the even subgraph consisting of  $F_1, F_2, F_3$  and  $F_4$  and the size of  $H$  is  $\frac{m}{2} + 1$ .

Next, suppose that  $a$  and  $b$  are both even, say  $a = 2p$  and  $b = 2q$  for some integers  $p, q \geq 1$ . Then  $\frac{m}{2} + 1 = (2a)(2b) + 2[2(p-1) + 2q] + 6$ . If  $b = 2$ , then  $a = 2$  or  $a = 4$ . Thus  $G = K_{5,10}$  or  $G = K_{9,10}$ . For  $K_{5,10}$ , it follows that  $\frac{m}{2} + 1 = 26$  and let the even subgraph of  $G$  be consisted of  $K_{2,10}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_1, w_2, \dots, w_{10}\}$  and  $C_6 = (u_3, w_1, u_4, w_2, u_5, w_3, u_3)$ . For  $K_{9,10}$ , it follows that  $\frac{m}{2} + 1 = 46$  and let the even subgraph of  $G$  be consist of  $K_{4,10}$  with partite sets  $\{u_1, u_2, u_3, u_4\}$  and  $\{w_1, w_2, \dots, w_{10}\}$  and  $C_6 = (u_5, w_1, u_6, w_2, u_7, w_3, u_5)$ . In each case,  $G$  has an even subgraph of size  $\frac{m}{2} + 1$ .

We now assume that  $b \geq 4$  and so  $3b + 5 \leq 4b + 2$ . Let  $F_1 = K_{2a,2b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$ , let  $F_2 = K_{2(p-1),2}$  with partite sets  $\{u_1, u_2, \dots, u_{2(p-1)}\}$  and  $\{w_{2b+1}, w_{2b+2}\}$ , let  $F_3 = K_{2,2q}$  with partite sets  $\{u_1, u_2\}$  and

$\{w_{2b+3}, w_{2b+4}, \dots, w_{3b+2}\}$  and let  $F_4 = C_6 = (u_1, w_{3b+3}, u_2, w_{3b+4}, u_3, w_{3b+5}, u_1)$ . Then let  $H$  be the even subgraph consisting of  $F_1, F_2, F_3$  and  $F_4$  and the size of  $H$  is  $\frac{m}{2} + 1$ .

*Case 2.  $r$  is even and  $s$  is odd.* In this case, the size  $m = rs$  is even. Furthermore,  $r = 4a + 2$  and  $s = 2b + 1$ , where  $a \geq 0$  and  $b \geq 1$ . Then

$$\frac{m}{2} + 1 = (2a)(2b) + 2(a + b + 1).$$

Since  $r \leq s$ , it follows that  $a < b$  and so  $a + b + 1 \leq 2b$ . First, suppose that  $a + b$  is odd. Let  $F_1 = K_{2a, 2b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$  and let  $F_2 = K_{2, a+b+1}$  with partite sets  $\{u_{2a+1}, u_{2a+2}\}$  and  $\{w_1, w_2, \dots, w_{a+b+1}\}$ . Then let  $H$  be the even subgraph consisting of  $F_1$  and  $F_2$  and the size of  $H$  is  $\frac{m}{2} + 1$ .

Next, suppose that  $a + b$  is even. First, assume that  $a$  and  $b$  are both odd, say  $a = 2p + 1$  and  $b = 2q + 1$  for some integers  $p, q \geq 0$ . If  $a = 1$ , then  $G = K_{6, 2b+1}$ , where  $b \geq 3$ , and  $\frac{m}{2} + 1 = 4b + 2(b + 2) = 4b + 2(b - 1) + 6$ . Let  $F_1 = K_{4, b}$  with partite sets  $\{u_1, u_2, u_3, u_4\}$  and  $\{w_1, w_2, \dots, w_b\}$ , let  $F_2 = K_{2, b-1}$  with partite sets  $\{u_5, u_6\}$  and  $\{w_1, w_2, \dots, w_{b-1}\}$  and let

$$F_3 = C_6 = (u_1, w_{b+1}, u_2, w_{b+2}, u_3, w_{b+3}, u_1).$$

Then let  $H$  be the even subgraph consisting of  $F_1, F_2$  and  $F_3$  and the size of  $H$  is  $\frac{m}{2} + 1$ . Thus, we may assume that  $a \geq 3$  and so  $3a + 4 \leq 4a + 2$ . Then  $\frac{m}{2} + 1 = (2a)(2b) + 2(2p) + 2(2q) + 6$ . Let  $F'_1 = K_{2a, 2b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$ , let  $F'_2 = K_{2p, 2}$  with partite sets  $\{u_{2a+1}, u_{2a+2}, u_{3a-1}\}$  and  $\{w_1, w_2\}$ , let  $F'_3 = K_{2, 2q}$  whose partite sets  $\{u_{3a}, u_{3a+1}\}$  and  $\{w_1, w_2, \dots, w_{2q}\}$  and let

$$F'_4 = C_6 = (u_{3a+2}, w_1, u_{3a+3}, w_2, u_{3a+4}, w_3, u_{3a+2}).$$

Then let  $H$  be the even subgraph consisting of  $F'_1, F'_2, F'_3$  and  $F'_4$  and the size of  $H$  is  $\frac{m}{2} + 1$ .

Next, suppose that  $a$  and  $b$  are both even, say  $a = 2p$  and  $b = 2q$  for some integers  $p, q \geq 1$ . Then  $\frac{m}{2} + 1 = (2a)(2b) + 2[2(p-1) + 2q] + 6$ . Let  $F_1 = K_{2a,2b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{2b}\}$ , let  $F_2 = K_{2(p-1),2}$  whose partite sets  $\{u_{2a+1}, u_{2a+2}, \dots, u_{3a-2}\}$  and  $\{w_1, w_2\}$ , let  $F_3 = K_{2,2q}$  with partite sets  $\{u_{3a-1}, u_{3a}\}$  and  $\{w_1, w_2, \dots, w_{2q}\}$  and let

$$F_4 = C_6 = (u_{3a+1}, w_1, u_{3a+2}, w_2, u_{3a+3}, w_3, u_{3a+1}).$$

Then let  $H$  be the even subgraph consisting of  $F_1, F_2, F_3$  and  $F_4$  and the size of  $H$  is  $\frac{m}{2} + 1$ .

Case 3.  $r$  and  $s$  are both odd. Since  $m = rs$  is odd, it follows by Observation 2.4.6 that  $EI(G) \geq L(\lceil \frac{m}{2} \rceil - 1) = \binom{m+1}{2} + 1$ . By Lemma 2.4.2, it remains to show that  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$ . Since  $rs \equiv 1 \pmod{4}$ , either  $r, s \equiv 3 \pmod{4}$  or  $r, s \equiv 1 \pmod{4}$ . We consider these two subcases.

*Subcase 3.1.*  $r, s \equiv 3 \pmod{4}$ . Then  $r = 4a+3$  and  $s = 4b+3$ , where  $0 \leq a \leq b$ .

Thus

$$\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2} = 8ab + 6a + 6b + 4 = (2a)(4b) + 2(3a + 3b) + 4.$$

First, suppose that  $a$  and  $b$  are both even. If  $a = b = 0$ , then  $G = K_{3,3}$  and  $\frac{m-1}{2} = 4$ . Let  $H = C_4 = (u_1, w_1, u_2, w_2, u_1)$ . Thus, we now assume that

$b \geq a \geq 2$ . Let  $F_1 = K_{2a,4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F_2 = K_{3a,2}$  with partite sets  $\{u_1, u_2, \dots, u_{3a}\}$  and  $\{w_{4b+1}, w_{4b+2}\}$ , let  $F_3 = K_{2,3b}$  with partite sets  $\{u_{3a+1}, u_{3a+2}\}$  and  $\{w_1, w_2, \dots, w_{3b}\}$  and let

$$F_4 = C_4 = (u_{3a+3}, w_1, u_{3a+4}, w_2, u_{3a+3}).$$

Then let  $H$  be the even subgraph consisting of  $F_1, F_2, F_3$  and  $F_4$  and the size of  $H$  is  $\frac{m-1}{2}$ .

Next, suppose that exactly one of  $a$  and  $b$  is odd. First, assume that  $a$  is even and  $b$  is odd, where then  $a \geq 0$  and  $b \geq 1$ . If  $a = 0$ , then  $G = K_{3,4b+3}$  and  $\frac{m-1}{2} = 6b + 4 = 6(b-1) + 10$ . Let  $F_1 = K_{2,3(b-1)}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_1, w_2, \dots, w_{3(b-1)}\}$ , let  $F_2 = C_4 = (u_1, w_{3b-2}, u_2, w_{3b-1}, u_1)$  and  $F_3 = C_6 = (u_1, w_{3b}, u_2, w_{3b+1}, u_3, w_{3b+2}, u_1)$ . Then let  $H$  be the even subgraph consisting of  $F_1, F_2$  and  $F_3$  and the size of  $H$  is  $\frac{m-1}{2}$ . If  $a = 2$ , then  $G = K_{11,4b+3}$  where  $b \geq 3$  and  $\frac{m-1}{2} = 4(4b) + 6(b+1) + 10$ . Let  $F'_1 = K_{4,4b}$  with partite sets  $\{u_1, u_2, u_3, u_4\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F'_2 = K_{2,3(b+1)}$  with partite sets  $\{u_5, u_6\}$  and  $\{w_1, w_2, \dots, w_{3(b+1)}\}$ ,

$$F'_3 = C_{10} = (u_7, w_1, u_8, w_2, u_9, w_3, u_{10}, w_4, u_{11}, w_5, u_7).$$

Then let  $H$  be the even subgraph consisting of  $F'_1, F'_2$  and  $F'_3$  and the size of  $H$  is  $\frac{m-1}{2}$ . We now assume  $a \geq 4$  and  $3a + 7 \leq 4a + 3$ . Then

$$\frac{m-1}{2} = (2a)(4b) + 2(3a) + 2[3(b-1)] + 10.$$

Let  $F''_1 = K_{2a,4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F''_2 = K_{3a,2}$  with partite sets  $\{u_1, u_2, \dots, u_{3a}\}$  and  $\{w_{4b+1}, w_{4b+2}\}$ , let  $F''_3 = K_{2,3(b-1)}$  with

partite sets  $\{u_{3a+1}, u_{3a+2}\}$  and  $\{w_1, w_2, \dots, w_{3b-3}\}$  and let  $F_4'' = C_{10}$  be a cycle of order 10 in the subgraph  $K_{5,5}$  of  $G$  with partite sets  $\{u_{3a+3}, u_{3a+4}, \dots, u_{3a+7}\}$  and  $\{w_1, w_2, \dots, w_5\}$ . Then let  $H$  be the even subgraph consisting of  $F_1'', F_2'', F_3''$  and  $F_4''$  and the size of  $H$  is  $\frac{m-1}{2}$ .

Next, assume that  $a$  is odd and  $b$  is even, where then  $1 \leq a < b$ . If  $a = 1$ , then  $G = K_{7,4b+3}$  and  $\frac{m-1}{2} = 2(4b) + 2(3b+2) + 6$ . Let  $F_1 = K_{2,4b}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F_2 = K_{2,3b+2}$  with partite sets  $\{u_3, u_4\}$  and  $\{w_1, w_2, \dots, w_{3b+2}\}$ , let  $F_3 = C_6 = (u_5, w_1, u_6, w_2, u_7, w_3, u_5)$ . Then let  $H$  be the even subgraph consisting of  $F_1, F_2$  and  $F_3$  and the size of  $H$  is  $\frac{m-1}{2}$ . Thus, we now assume that  $a \geq 3$ . Then  $\frac{m-1}{2} = (2a)(4b) + 6(a-1) + 6b + 10$ . Let  $F_1' = K_{2a,4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F_2' = K_{3(a-1),2}$  with partite sets  $\{u_1, u_2, \dots, u_{3a-3}\}$  and  $\{w_{4b+1}, w_{4b+2}\}$ , let  $F_3' = K_{2,3b}$  with partite sets  $\{u_{3a-2}, u_{3a-1}\}$  and  $\{w_1, w_2, \dots, w_{3b}\}$ , let  $F_4' = C_4 = (u_{3a}, w_1, u_{3a+1}, w_2, u_{3a})$  and let  $F_5' = C_6 = (u_{4a+1}, w_1, u_{4a+2}, w_2, u_{4a+3}, w_3, u_{4a+1})$ . Then let  $H$  be the even subgraph consisting of  $F_1', F_2', F_3', F_4'$  and  $F_5'$  and the size of  $H$  is  $\frac{m-1}{2}$ .

Final, suppose that  $a$  and  $b$  are both odd. Let  $a = 2p + 1$  and  $b = 2q + 1$  for some integers  $p, q \geq 0$ . Then  $\frac{m-1}{2} = (2a)(4b) + 6(a-1) + 6(b-1) + 16$ . Let  $F_1 = K_{2a,4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F_2 = K_{3(a-1),2}$  with partite sets  $\{u_1, u_2, \dots, u_{3a-3}\}$  and  $\{w_{4b+1}, w_{4b+2}\}$ , let  $F_3 = K_{2,3(b-1)}$  with partite sets  $\{u_{3a-2}, u_{3a-1}\}$  and  $\{w_1, w_2, \dots, w_{3b-3}\}$  and let  $F_4 = K_{4,4}$  with partite sets  $\{u_{3a}, u_{3a+1}, u_{3a+2}, u_{3a+3}\}$  and  $\{w_1, w_2, w_3, w_4\}$ . Then let  $H$  be the even subgraph consisting of  $F_1, F_2, F_3$  and  $F_4$  and the size of  $H$  is  $\frac{m-1}{2}$ .

*Subcase 3.2.*  $r, s \equiv 1 \pmod{4}$ . Then  $r = 4a + 1$  and  $b = 4b + 1$  where  $1 \leq a \leq b$ .

Thus

$$\left\lceil \frac{m}{2} \right\rceil - 1 = \frac{m-1}{2} = 8ab + 2a + 2b.$$

First, suppose that  $a + b$  is even. Then  $\frac{m-1}{2} = (2a)(4b) + 2(a + b)$ . Let  $F_1 = K_{2a,4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$  and let  $F_2 = K_{2,a+b}$  with partite sets  $\{u_{2a+1}, u_{2a+2}\}$  and  $\{w_1, w_2, \dots, w_{a+b}\}$ . Then let  $H$  be the even subgraph consisting of  $F_1$  and  $F_2$  and the size of  $H$  is  $\frac{m-1}{2}$ .

Next, suppose that  $a + b$  is odd and so  $a + b \geq 3$ . If  $a + b = 3$ , then  $G = K_{5,9}$  and  $\frac{m-1}{2} = 22$ . Let  $F_1 = K_{2,8}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_1, w_2, \dots, w_8\}$  and  $F_2 = C_6 = (u_3, w_1, u_4, w_2, u_5, w_3, u_3)$ . Then let  $H$  be the even subgraph of size 22 consisting of  $F_1$  and  $F_2$  and the size of  $H$  is  $\frac{m-1}{2}$ . Thus we may assume that  $a + b \geq 5$ . If  $a = 1$ , then  $b \geq 4$  and  $G = K_{5,4b+1}$ . Now  $\frac{m-1}{2} = 2(4b) + 2(b-2) + 6$ . Let  $F'_1 = K_{2,4b}$  with partite sets  $\{u_1, u_2\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ ,  $F'_2 = K_{2,b-2}$  with partite sets  $\{u_3, u_4\}$  and  $\{w_1, w_2, \dots, w_{b-2}\}$  and  $F'_3 = C_6 = (u_3, w_{b-1}, u_4, w_b, u_5, w_{b+1}, u_3)$ . Then let  $H$  be the even subgraph consisting of  $F'_1$ ,  $F'_2$  and  $F'_3$  and the size of  $H$  is  $\frac{m-1}{2}$ . Now assume that  $a \geq 2$ . Then  $\frac{m-1}{2} = (2a)(4b) + 2(a + b - 3) + 6$ . Let  $F''_1 = K_{2a,4b}$  with partite sets  $\{u_1, u_2, \dots, u_{2a}\}$  and  $\{w_1, w_2, \dots, w_{4b}\}$ , let  $F''_2 = K_{2,a+b-3}$  with partite sets  $\{u_{2a+1}, u_{2a+2}\}$  and  $\{w_1, w_2, \dots, w_{a+b-3}\}$  and let  $F''_3 = C_6 = (u_{2a+3}, w_1, u_{2a+4}, w_2, u_{2a+5}, w_3, u_{2a+3})$ . Then let  $H$  be the even subgraph consisting of  $F''_1$ ,  $F''_2$  and  $F''_3$  and the size of  $H$  is  $\frac{m-1}{2}$ . ■

## Chapter 3

# Eulerian Irregularity in Graphs

In this chapter, we study the Eulerian irregularities of some well-known classes of graphs, namely graphs of cycle rank 2, prisms, grids, powers of cycles and circulants, beginning with graphs of cycle rank 2.

### 3.1 Graphs of Cycle Rank 2

For a connected graph  $G$  of order  $n$  and size  $m$ , the number of edges that must be deleted from  $G$  to obtain a spanning tree of  $G$  is  $m - n + 1$ . The number  $m - n + 1$  is called the *cycle rank* of  $G$ . Thus the cycle rank of a tree is 0 and the cycle rank of a *unicyclic* graph is 1. The cycle rank of a connected graph of order  $n$  and size  $m = n + 1$  is therefore 2. In this section, we study the Eulerian irregularity of graphs of cycle rank 2.

Let  $G$  be a connected graph of order  $n \geq 5$  and cycle rank 2. Then  $G$  contains one of the following three graphs of Figure 3.1 as a subgraph. If  $G$  contains two edge-disjoint cycles, then we say that  $G$  is of *type I*; otherwise,  $G$  is of *type II*.



If  $G$  is of type I, then  $G$  contains a subgraph  $H_1$  obtained from two edge-disjoint cycles  $C_{k_1}$  and  $C_{k_2}$  by either identifying a vertex of  $C_{k_1}$  with a vertex of  $C_{k_2}$  or by connecting a vertex of  $C_{k_1}$  and a vertex of  $C_{k_2}$  by a path as shown in Figure 3.1(a) and (b). In this case,  $H_1$  is called a  $(k_1, k_2)$ -subgraph of  $G$ . If  $G$  is of type II, then  $G$  contains a subgraph  $H_2$  obtained from three internally disjoint  $u - v$  paths  $P_{k_1+1}, P_{k_2+1}, P_{k_3+1}$  of lengths  $k_1, k_2, k_3$ , respectively, as shown in Figure 3.1(c). In this case,  $H_2$  is called a  $(k_1, k_2, k_3)$ -subgraph of  $G$ .

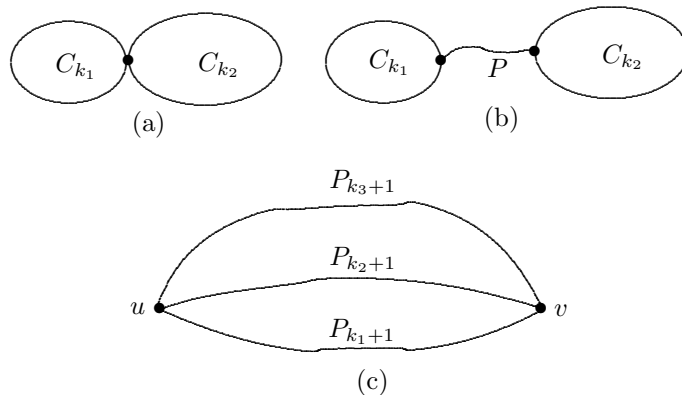


Figure 3.1: Subgraphs in a graph of cycle rank 2

**Theorem 3.1.1** *Let  $G$  be a graph of order  $n \geq 5$ , size  $m$  and cycle rank 2. If  $G$  is of type I and contains a  $(k_1, k_2)$ -subgraph, where  $3 \leq k_1 \leq k_2$ , then*

$$EI(G) = \begin{cases} (k_1 + k_2)^2 + 2 \binom{m - k_1 - k_2 + 1}{2} & \text{if } m - (k_1 + k_2) \geq k_2 \\ k_2^2 + 2 \binom{m - k_2 + 1}{2} & \text{if } m - (k_1 + k_2) < k_2. \end{cases}$$

**Proof.** Since  $G$  is of cycle rank 2,  $m = n + 1$ . First, we make an observation. Since each bridge of  $G$  is encountered an even number of times in an irregular Eulerian walk  $W$ , it follows that either all edges on  $C_{k_1}$  are encountered an odd

number of times in  $W$  or all edges on  $C_{k_1}$  are encountered an even number of times in  $W$ . Similarly, this is the case for all edges on  $C_{k_2}$ . Divide the edge set  $E(G)$  into three sets  $E_1$ ,  $E_2$  and  $E_3$ , where  $E_i = E(C_{k_i})$  for  $i = 1, 2$  and  $E_3 = E(G) - (E_1 \cup E_2)$  is the set of all bridges of  $G$ . Thus  $\{E_1, E_2, E_3\}$  is a partition of  $E(G)$  if  $E_3 \neq \emptyset$ . Let  $W$  be an irregular Eulerian walk of minimum length in  $G$ . Now let  $E'(W)$  be the set of edges of  $G$  that are encountered an odd number of times in  $W$  and  $E''(W)$  the set of edges that are encountered an even number of times in  $W$ . As we indicated above,  $E_3 \subseteq E''(W)$ . Since  $W$  is an irregular Eulerian walk of minimum length in  $G$ , we may assume that  $E'(W) = \{e_1, e_2, \dots, e_a\}$  and  $E''(W) = \{f_1, f_2, \dots, f_b\}$  for some nonnegative integers  $a$  and  $b$  such that  $e_i$  ( $1 \leq i \leq a$ ) appears exactly  $2i - 1$  times in  $W$  and  $f_j$  ( $1 \leq j \leq b$ ) appears exactly  $2j$  times in  $W$ . Therefore,

$$\begin{aligned} L(W) &= [1 + 3 + \dots + (2a - 1)] + (2 + 4 + \dots + 2b) \\ &= a^2 + b(b + 1). \end{aligned}$$

Let  $p = m - (k_1 + k_2)$ . We consider two cases.

*Case 1.*  $p \geq k_2$ . There are four possibilities for  $W$ , according to the sets  $E'(W)$  and  $E''(W)$ .

- If  $E'(W) = E_1 \cup E_2$  and  $E''(W) = E_3$ , then  $L(W) = (k_1 + k_2)^2 + p(p + 1)$ .
- If  $E'(W) = E_1$  and  $E''(W) = E_2 \cup E_3$ , then, since  $p \geq k_2 \geq k_1$ ,

$$\begin{aligned} L(W) &= k_1^2 + (k_2 + p)(k_2 + p + 1) = k_1^2 + k_2^2 + 2pk_2 + p^2 + k_2 + p \\ &\geq k_1^2 + 2k_1k_2 + k_2^2 + p(p + 1) = (k_1 + k_2)^2 + p(p + 1). \end{aligned}$$

- If  $E'(W) = E_2$  and  $E''(W) = E_1 \cup E_3$ , then, since  $p \geq k_2$ ,

$$\begin{aligned} L(W) &= k_2^2 + (k_1 + p)(k_1 + p + 1) = k_1^2 + k_2^2 + 2pk_1 + p^2 + k_1 + p \\ &\geq k_1^2 + 2k_1k_2 + k_2^2 + p(p + 1) = (k_1 + k_2)^2 + p(p + 1). \end{aligned}$$

- If  $E''(W) = E_1 \cup E_2 \cup E_3$ , then

$$L(W) = (k_1 + k_2 + p)(k_1 + k_2 + p + 1) \geq (k_1 + k_2)^2 + p(p + 1).$$

Thus  $L(W) = (k_1 + k_2)^2 + p(p + 1)$  is minimum when  $E'(W) = E_1 \cup E_2$  and  $E''(W) = E_3$ , in which case, the difference between  $|E'(W)|$  and  $|E''(W)|$  in absolute value is the minimum. Therefore,  $EI(G) = (k_1 + k_2)^2 + p(p + 1)$ .

*Case 2.*  $p < k_2$ . Again, there are four possibilities for  $W$ , according to the sets  $E'(W)$  and  $E''(W)$ .

- If  $E'(W) = E_2$  and  $E''(W) = E_1 \cup E_3$ , then  $L(W) = k_2^2 + (k_1 + p)(k_1 + p + 1)$ .
- If  $E'(W) = E_1$  and  $E''(W) = E_2 \cup E_3$ , then, since  $k_2 \geq k_1$ ,

$$\begin{aligned} L(W) &= k_1^2 + (k_2 + p)(k_2 + p + 1) = k_1^2 + k_2^2 + 2pk_2 + p^2 + k_2 + p \\ &\geq k_2^2 + k_1^2 + 2pk_1 + p^2 + k_1 + p \\ &= k_2^2 + (k_1 + p)^2 + k_1 + p = k_2^2 + (k_1 + p)(k_1 + p + 1). \end{aligned}$$

- If  $E'(W) = E_1 \cup E_2$  and  $E''(W) = E_3$ , then, since  $k_1 \geq 3$  and so  $2k_1 > k_1$ ,

$$\begin{aligned}
L(W) &= (k_1 + k_2)^2 + p(p + 1) = k_1^2 + 2k_1k_2 + k_2^2 + p^2 + p \\
&\geq k_2^2 + k_1^2 + 2k_1(p + 1) + p^2 + p \quad \text{since } p + 1 \leq k_2 \\
&= k_2^2 + (k_1 + p)^2 + 2k_1 + p \\
&> k_2^2 + (k_1 + p)^2 + k_1 + p = k_2^2 + (k_1 + p)(k_1 + p + 1).
\end{aligned}$$

- If  $E''(W) = E_1 \cup E_2 \cup E_3$ , then

$$\begin{aligned}
L(W) &= (k_1 + k_2 + p)(k_1 + k_2 + p + 1) \\
&= (k_1 + k_2 + p)^2 + (k_1 + k_2 + p) \\
&= (k_1 + k_2)^2 + 2(k_1 + k_2)p + p^2 + k_1 + k_2 + p \\
&\geq (k_1 + k_2)^2 + k_1 + p(p + 1) > k_2^2 + (k_1 + p)^2 + k_1 + p \\
&= k_2^2 + (k_1 + p)(k_1 + p + 1)
\end{aligned}$$

Thus  $L(W) = k_2^2 + (k_1 + p)(k_1 + p + 1)$  is minimum when  $E'(W) = E_2$  and  $E''(W) = E_1 \cup E_3$ . Therefore,  $EI(G) = k_2^2 + (k_1 + p)(k_1 + p + 1)$ .  $\blacksquare$

**Theorem 3.1.2** *Let  $G$  be a graph of order  $n \geq 4$ , size  $m$  and cycle rank 2. Suppose that  $G$  is of type II and contains a  $(k_1, k_2, k_3)$ -subgraph, where  $1 \leq k_1 \leq k_2 \leq k_3$ . Let*

$$M = \min \left\{ \left| (k_i + k_j) - \left\lceil \frac{m}{2} \right\rceil \right| : i, j \in \{1, 2, 3\}, i \neq j \right\}. \quad (3.1)$$

- (1) *If  $M = 0$ , then  $EI(G) = \binom{m+1}{2}$ ;*

(2) For  $M \geq 1$ ,

- if there exist at least two distinct pairs  $(i, j)$  where  $i, j \in \{1, 2, 3\}$  such that  $|(k_i + k_j) - \lceil \frac{m}{2} \rceil| = M$  and  $(k_i + k_j) - \lceil \frac{m}{2} \rceil$  are different in signs; that is, there are at least two pairs  $(r, s)$  and  $(\ell, t)$ , where  $r, s, \ell, t \in \{1, 2, 3\}$  and  $(r, s) \neq (\ell, t)$ , such that

$$(k_r + k_s) - \lceil \frac{m}{2} \rceil = M \text{ and } \lceil \frac{m}{2} \rceil - (k_\ell + k_t) = M,$$

then

$$EI(G) = \begin{cases} (k_r + k_s)^2 + (m - k_r - k_s)(m - k_r - k_s + 1) & \text{if } m \text{ is even} \\ (k_\ell + k_t)^2 + (m - k_\ell - k_t)(m - k_\ell - k_t + 1) & \text{if } m \text{ is odd.} \end{cases} \quad (3.2)$$

- if for all pairs  $(i, j)$  where  $i, j \in \{1, 2, 3\}$  such that  $|(k_i + k_j) - \lceil \frac{m}{2} \rceil| = M$  either  $k_i + k_j > \lceil \frac{m}{2} \rceil$  for all such pairs  $(i, j)$  or  $k_i + k_j < \lceil \frac{m}{2} \rceil$  for all such pairs  $(i, j)$  and  $(r, s)$  is one of such pairs, then

$$EI(G) = (k_r + k_s)^2 + (m - k_r - k_s)(m - k_r - k_s + 1). \quad (3.3)$$

**Proof.** If  $M = 0$ , then there is an even subgraph  $C_{k_i+k_j}$  of order  $k_i + k_j$  of  $G$ , where  $i, j \in \{1, 2, 3\}$ . Since the size of  $C_{k_i+k_j}$  is  $k_i + k_j = \lceil \frac{m}{2} \rceil$ , it follows that  $EI(G) = \binom{m+1}{2}$ .

Now let  $M \geq 1$ . First assume that there exist at least two distinct pairs  $(r, s)$  and  $(\ell, t)$  such that  $(k_r + k_s) - \lceil \frac{m}{2} \rceil = M$  and  $\lceil \frac{m}{2} \rceil - (k_\ell + k_t) = M$ . Thus  $k_r + k_s = \lceil \frac{m}{2} \rceil + M$  and  $k_\ell + k_t = \lceil \frac{m}{2} \rceil - M$ . Let  $H_1 = C_{k_r+k_s}$  and  $H_2 = C_{k_\ell+k_t}$ . Then  $H_1$  and  $H_2$  are even subgraphs of size  $k_r + k_s$  and  $k_\ell + k_t$ , respectively. By Observation 2.4.6, it follows that (3.2) holds.

Next, suppose that for all pairs  $(i, j)$  such that  $|(k_i + k_j) - \lceil \frac{m}{2} \rceil| = M$  either  $k_i + k_j > \lceil \frac{m}{2} \rceil$  for all such pairs  $(i, j)$  or  $k_i + k_j < \lceil \frac{m}{2} \rceil$  for all such pairs  $(i, j)$ . Let  $(r, s)$  be one of pairs. Then  $H = C_{k_r+k_s}$  is an even subgraph of size  $k_r + k_s$  in  $G$ . By Observation 2.4.6, it follows that (3.3) holds. ■

## 3.2 Prisms and Grids

The *Cartesian product*  $G \square H$  of two graphs  $G$  and  $H$  has vertex set  $V(G) = V(G) \times V(H)$  and two distinct vertices  $(u, v)$  and  $(x, y)$  of  $G \square H$  are adjacent if either (1)  $u = x$  and  $vy \in E(H)$  or (2)  $v = y$  and  $ux \in E(G)$ . The graph  $C_n \square K_2$  where  $n \geq 3$  is called a *prism* while  $P_n \square P_q$  where  $n \geq q \geq 2$  is called a *grid*. In this section, we determine the Eulerian irregularities of all prisms and grids. In order to do this, we first recall two useful lemmas in Chapter 2, the first of which is a consequence of Theorem 2.3.1 (also see Corollary 2.3.2), while the second one is a consequence of the proof of Theorem 2.4.5 (also see Lemma 2.4.2).

**Lemma 3.2.1** *If  $G$  is a connected bipartite graph of size  $m \geq 1$  such that  $m \equiv 1 \pmod{4}$  or  $m \equiv 2 \pmod{4}$ , then  $G$  is not optimal.*

**Lemma 3.2.2** *Let  $G$  be a nontrivial connected graph of size  $m$ . If  $G$  contains an even subgraph of size  $x$ , then there is an irregular Eulerian walk of length  $x^2 + (m - x)(m - x + 1)$  in  $G$  and so  $EI(G) \leq x^2 + (m - x)(m - x + 1)$ .*

**Theorem 3.2.3** *For each integer  $n \geq 3$ , the prism  $C_n \square K_2$  is optimal if and only if  $n \not\equiv 2 \pmod{4}$ . Furthermore, if  $C_n \square K_2$  is not optimal, then  $EI(C_n \square K_2) = \binom{3n+1}{2} + 1$ .*

**Proof.** For  $G = C_n \square K_2$ , let  $(u_1, u_2, \dots, u_n, u_1)$  and  $(v_1, v_2, \dots, v_n, v_1)$  be two disjoint copies of  $C_n$  in  $G$  such that  $u_i v_i \in E(G)$  for  $1 \leq i \leq n$ . If  $n \equiv 2 \pmod{4}$ , then  $G$  is a cubic bipartite graph of size  $m = 3n$ . Since  $m$  is congruent to 2 modular 4, it follows by Lemma 3.2.1 that  $G$  is not optimal.

For the converse, suppose that  $n \not\equiv 2 \pmod{4}$ . Then  $n \equiv r \pmod{4}$ , where  $r = 0, 1, 3$ . We consider these three cases. In each case, we show that  $G$  contains a 2-regular subgraph of size  $\lceil m/2 \rceil$ , where  $m$  is the size of  $G$ .

*Case 1.*  $n \equiv 0 \pmod{4}$ . Then  $n = 4k$  for some positive integer  $k$  and so the size of  $G$  is  $m = 3n = 12k$ . Thus  $m/2 = 6k$ . The  $6k$ -cycle

$$C_{6k} = (v_1, v_2, \dots, v_{3k}, u_{3k}, u_{3k-1}, \dots, u_1, v_1).$$

is a 2-regular subgraph of size  $6k$  in  $G$ .

*Case 2.*  $n \equiv 1 \pmod{4}$ . Then  $n = 4k + 1$  for some positive integer  $k$  and so the size of  $G$  is  $m = 3n = 12k + 3$ . Thus  $\lceil m/2 \rceil = 6k + 2$ . The  $(6k + 2)$ -cycle

$$C_{6k+2} = (u_1, u_2, \dots, u_{3k+1}, v_{3k+1}, v_{3k}, \dots, v_1, u_1)$$

is a 2-regular subgraph of size  $6k + 2$  in  $G$ .

*Case 3.*  $n \equiv 3 \pmod{4}$ . Then  $n = 4k + 3$  for some integer  $k \geq 0$  and the size of  $G$  is  $m = 3n = 12k + 9$  and so  $\lceil m/2 \rceil = 6k + 5$ . The  $(6k + 5)$ -cycle

$$\begin{aligned} C_{6k+5} = & (v_1, v_2, \dots, v_{2k+1}, v_{2k+2}, u_{2k+2}, u_{2k+3}, v_{2k+3}, \\ & v_{2k+4}, u_{2k+4}, u_{2k+5}, v_{2k+5}, v_{2k+6}, u_{2k+6}, u_{2k+7}, \dots, \\ & u_{4k+1}, v_{4k+2}, u_{4k+2}, u_{4k+3}, v_{4k+3}, v_1) \end{aligned}$$

is a 2-regular subgraph of size  $6k + 5$  in  $G$ . In each case,  $G$  contains a 2-regular subgraph of size  $\lceil m/2 \rceil$ , By Theorem 2.3.1,  $G$  is optimal.

We now assume that  $n \equiv 2 \pmod{4}$  and so  $EI(G) \geq \binom{m+1}{2} + 1$ . Since  $m$  is even, to show that  $EI(G) \leq \binom{m+1}{2} + 1$ , it suffices to show that  $G$  contains an even subgraph of size  $\frac{m}{2} + 1$ . Let  $n = 4k + 2$  for some positive integer  $k$  and so  $\frac{m}{2} + 1 = 6k + 4$ . Note that  $G$  contains vertex-disjoint  $H_1 = C_4$  and  $H_2 = C_{6k}$  as subgraphs. Thus,  $G$  has an even subgraph of size  $\frac{m}{2} + 1 = 6k + 4$  and so  $EI(G) \leq \binom{m+1}{2} + 1$  by Lemma 3.2.2, giving the desired result.  $\blacksquare$

We next determine the Eulerian irregularities of all grids  $P_n \square P_q$  where  $n \geq q \geq 2$ , beginning with the case when  $q = 2$ .

**Theorem 3.2.4** *For each integer  $n \geq 3$ , the Cartesian product  $P_n \square K_2$  of  $P_n$  and  $K_2$  is optimal if and only if  $n \equiv 2, 3 \pmod{4}$ . Furthermore, if  $P_n \square K_2$  is not optimal, then  $EI(P_n \square K_2) = \binom{3n-1}{2} + 1$ .*

**Proof.** If  $n \equiv 0, 1 \pmod{4}$ , then the size  $m = 3n - 2$  of  $P_n \square K_2$  is congruent to 2 or 1 modulo 4. It then follows by Lemma 3.2.1 that  $P_n \square K_2$  is not optimal. For the converse, suppose that  $n \equiv 2, 3 \pmod{4}$ . Let  $G = P_n \square K_2$ , where  $(u_1, u_2, \dots, u_n)$  and  $(v_1, v_2, \dots, v_n)$  are the two copies of  $P_n$  in  $G$  such that  $u_i v_i \in E(G)$  for  $1 \leq i \leq n$ . The size  $m$  of  $G$  is  $3n - 2$ . For  $n \equiv 2 \pmod{4}$ , let  $n = 4k + 2$  for some positive integer  $k$  and  $\lceil m/2 \rceil = 6k + 2$ . The cycle

$$C_{6k+2} = (u_1, u_2, \dots, u_{3k+1}, v_{3k+1}, v_{3k-1}, \dots, v_2, v_1, u_1)$$



is a 2-regular subgraph of size  $\lceil m/2 \rceil$  in  $G$ . For  $n \equiv 3 \pmod{4}$ , let  $n = 4k + 3$  for some nonnegative integer  $k$  and  $\lceil m/2 \rceil = 6k + 4$ . The cycle

$$C_{6k+4} = (u_1, u_2, \dots, u_{3k+2}, v_{3k+2}, v_{3k+1}, \dots, v_2, v_1, u_1)$$

is a 2-regular subgraph of size  $\lceil m/2 \rceil$  in  $G$ . By Theorem 2.3.1,  $G$  is optimal if  $n \equiv 2, 3 \pmod{4}$ .

We now assume that  $G$  is not optimal. Then either  $n \equiv 0 \pmod{4}$  or  $n \equiv 1 \pmod{4}$  and  $EI(G) \geq \binom{m+1}{2} + 1$ . It remains to show that  $EI(G) \leq \binom{m+1}{2} + 1$ . For  $n \equiv 0 \pmod{4}$ , let  $n = 4k$  for some positive integer  $k$ . Since  $m = 3n - 2$  is even, it suffices to show that  $G$  contains an even subgraph of size  $\frac{m}{2} + 1 = 6k$ . This is true as  $G$  contains  $C_{6k}$  as a subgraph and so  $EI(G) = \binom{m+1}{2} + 1$  if  $n \equiv 0 \pmod{4}$ . For  $n \equiv 1 \pmod{4}$ , let  $n = 4k + 1$  for some positive integer  $k$ . Since  $m = 3n - 2 = 12k + 1$  is odd, it suffices to show that  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2} = 6k$ . This is true as  $G$  contains  $C_{6k}$  as a subgraph and so  $EI(G) = \binom{m+1}{2} + 1$  if  $n \equiv 1 \pmod{4}$ . ■

We now consider grids  $P_n \square P_q$  for  $n \geq q \geq 3$  in general.

**Theorem 3.2.5** *For each pair  $(n, q)$  of integers with  $n \geq q \geq 3$ , the grid  $P_n \square P_q$  is optimal if and only if  $(n, q)$  satisfies one of the following conditions:*

- (i) *If  $n$  and  $q$  are even, then either both  $n$  and  $q$  are congruent to 0 modulo 4 or both  $n$  and  $q$  are congruent to 2 modulo 4;*
- (ii) *If  $n$  is even and  $q$  is odd, then  $n \equiv 0 \pmod{4}$  and  $q \equiv 1 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and  $q \equiv 3 \pmod{4}$ ;*

(iii) If  $n$  is odd and  $q$  is even, then  $n \equiv 1 \pmod{4}$  and  $q \equiv 0 \pmod{4}$  or  $n \equiv 3 \pmod{4}$  and  $q \equiv 2 \pmod{4}$ .

**Proof.** Suppose that  $G = P_n \square P_q$  consists of  $q$  paths of order  $n$ , which we denote by  $P_{n,i} = (v_{1,i}, v_{2,i}, \dots, v_{n,i})$  for  $1 \leq i \leq q$  such that  $v_{t,i}$  is adjacent to  $v_{t,j}$  ( $1 \leq t \leq n$ ) when  $|i - j| = 1$ . The size of  $G$  is  $m = n(q - 1) + (n - 1)q$  and  $G$  is a bipartite graph. Write  $n = 4k + r_n$  and  $q = 4\ell + r_q$ , where  $r_n, r_q \in \{0, 1, 2, 3\}$ . Let  $G' = P_{4k} \square P_{4\ell}$  be the induced subgraph of  $G$  with

$$V(G') = \{v_{a,b} : 1 \leq a \leq 4k, 1 \leq b \leq 4\ell\}. \quad (3.4)$$

That is,  $G'$  is the induced subgraph in  $G$  consisting of the  $4\ell$  paths  $P_{4k,i}$  of order  $4k$  where

$$P_{4k,i} = (v_{1,i}, v_{2,i}, \dots, v_{4k,i}) \text{ for } 1 \leq i \leq 4\ell \quad (3.5)$$

such that  $v_{t,i}$  is adjacent to  $v_{t,j}$  ( $1 \leq t \leq 4k$ ) when  $|i - j| = 1$ . Then  $G'$  contains  $k\ell$  vertex-disjoint copies (or blocks) of  $P_4 \square P_4$ , denoted by  $B_1, B_2, \dots, B_{k\ell}$  as shown in Figure 3.2.

$B_1$	$B_{k+1}$	$\cdots$	$B_{(\ell-1)k+1}$
$B_2$	$B_{k+2}$	$\cdots$	$B_{(\ell-1)k+2}$
$B_3$	$B_{k+3}$	$\cdots$	$B_{(\ell-1)k+3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_k$	$B_{2k}$	$\cdots$	$B_{k\ell}$

Figure 3.2: The subgraph  $G' = P_{4k} \square P_{4\ell}$  in  $G$

In particular, for  $1 \leq i \leq k$ , the vertices of  $B_i$  appear in the way as shown in Figure 3.3. Note that  $B_i = P_4 \square P_4$  contains each of the even cycles

$$C_4, C_6, C_8, C_{10}, C_{12}, C_{14}, C_{16}$$

as a subgraph.

$v_{4i-3,1}$	$v_{4i-3,2}$	$v_{4i-3,3}$	$v_{4i-3,4}$
$v_{4i-2,1}$	$v_{4i-2,2}$	$v_{4i-2,3}$	$v_{4i-2,4}$
$v_{4i-1,1}$	$v_{4i-1,2}$	$v_{4i-1,3}$	$v_{4i-1,4}$
$v_{4i,1}$	$v_{4i,2}$	$v_{4i,3}$	$v_{4i,4}$

Figure 3.3: The block  $B_i = P_4 \square P_4$  in  $G'$  for  $1 \leq i \leq k$

We consider three cases, according to the parities of  $n$  and  $q$ .

*Case 1.  $n$  and  $q$  are even.* If one of  $n$  and  $q$  is congruent to 0 modulo 4 and the other is congruent to 2 modulo 4, then  $m \equiv 2 \pmod{4}$  and so  $G$  is not optimal by Lemma 3.2.1. For the converse, suppose that both  $n$  and  $q$  are congruent to 0 modulo 4 or both  $n$  and  $q$  are congruent to 2 modulo 4. We consider two subcases. In each subcase, we construct an even subgraph  $H$  of size  $\lceil m/2 \rceil$ .

*Subcase 1.1.  $n \equiv 0 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ .* Then  $n = 4k$  and  $q = 4\ell$  for some positive integers  $k$  and  $\ell$  with  $k \geq \ell$ . In this case, the size  $m_H$  of a graph  $H$  with the desired properties is

$$m_H = \frac{m}{2} = 16k\ell - 2k - 2\ell.$$

The graph  $G' = P_{4k} \square P_{4\ell}$  contains  $k\ell$  vertex-disjoint copies  $B_1, B_2, \dots, B_{k\ell}$  of  $P_4 \square P_4$  as shown in Figure 3.2.

- For  $\ell = 1$ , there are  $k$  vertex-disjoint blocks  $B_1, B_2, \dots, B_k$  of  $P_4 \square P_4$  in  $G$ .

Let  $H_1 = C_{12}$  in  $B_1$  and  $H_i = C_{14}$  in  $B_i$  for  $2 \leq i \leq k$ . Now let  $H$  be the

union of these vertex-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k$ ) in  $G$ . Then  $H$  is a 2-regular graph of the size  $m_H = 12 + 14(k - 1) = 14k - 2$ .

- For  $\ell = 2$ , there are  $2k$  vertex-disjoint blocks  $B_1, B_2, \dots, B_{2k}$  of  $P_4 \square P_4$  in  $G$ . If  $k = 2$ , let  $H_1 = C_8$  in  $B_1$  and  $H_i = C_{16}$  in  $B_i$  for  $2 \leq i \leq 4$ . Now let  $H$  be the union of these vertex-disjoint subgraphs  $H_i$  ( $1 \leq i \leq 4$ ) in  $G$ . Then  $H$  is a 2-regular graph of the size  $m_H = 3 \cdot 16 + 8 = 56 = m/2$ . If  $k \geq 3$ , let  $H_i = C_{16}$  in  $B_i$  for  $1 \leq i \leq k - 2$  and let  $H_i = C_{14}$  in  $B_i$  for  $k - 1 \leq i \leq 2k$ . Now let  $H$  be the union of these vertex-disjoint subgraphs  $H_i$  ( $1 \leq i \leq 2k$ ) in  $G$ . Then  $H$  is a 2-regular graph of the size  $m_H = 16(k - 2) + 14(k + 2) = 30k - 4$ .
- For  $\ell \geq 3$ , let  $H_i = C_{16}$  in  $B_i$  for  $1 \leq i \leq k\ell - k - \ell$  and let  $H_i = C_{14}$  in  $B_i$  for  $k\ell - k - \ell + 1 \leq i \leq k\ell$ . Now let  $H$  be the union of these vertex-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k\ell$ ) in  $G$ . Then  $H$  is a 2-regular graph of the size  $m_H = 16(k\ell - k - \ell) + 14(k + \ell) = 16k\ell - 2k - 2\ell$ .

*Subcase 1.2.*  $n \equiv 2 \pmod{4}$  and  $q \equiv 2 \pmod{4}$ . Then  $n = 4k + 2$  and  $q = 4\ell + 2$  for some integers  $k$  and  $\ell$  with  $k \geq \ell \geq 1$ . In this case, the size  $m_H$  of a graph  $H$  with the desired properties is

$$\begin{aligned} m_H &= \frac{m}{2} = (4k + 2)(4\ell + 2) - (2k + 1) - (2\ell + 1) \\ &= (4k + 2)(4\ell + 2) - 2(k + \ell + 1) = 16k\ell + 6k + 6\ell + 2. \end{aligned}$$

Let  $G' = P_{4k} \square P_{4\ell}$  be the induced subgraph of  $G$  as defined in (3.4) or (3.5) that contains the  $k\ell$  vertex-disjoint blocks  $B_1, B_2, \dots, B_{k\ell}$  of  $P_4 \square P_4$  as shown in

Figure 3.2. Let  $H_i = C_{16}$  in  $B_i$  for  $1 \leq i \leq k\ell$  and let  $C = C_{6k}$  and  $C' = C_{6\ell+2}$  be two vertex-disjoint cycles of orders  $6k$  and  $6\ell + 2$  respectively in  $G - E(G')$ , where

$$\begin{aligned} C &= (v_{1,4\ell+1}, v_{2,4\ell+1}, \dots, v_{3k,4\ell+1}, v_{3k,4\ell+2}, v_{3k-1,4\ell+2}, \dots, \\ &\quad v_{1,4\ell+2}, v_{1,4\ell+1}) \\ C' &= (v_{4k+1,1}, v_{4k+1,2}, \dots, v_{4k+1,3\ell+1}, v_{4k+2,3\ell+1}, v_{4k+2,3\ell}, \dots, \\ &\quad v_{4k+2,1}, v_{4k+1,1}). \end{aligned}$$

Now let  $H$  be the union of these vertex-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k\ell$ ),  $C$  and  $C'$  in  $G$ . Then  $H$  is a 2-regular graph of the size  $m_H = 16k\ell + 6k + 6\ell + 2$ .

*Case 2.  $n$  is even and  $q$  is odd.* If  $n \equiv 0 \pmod{4}$  and  $q \equiv 3 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and  $q \equiv 1 \pmod{4}$ , then  $m \equiv 1 \pmod{4}$  and so  $G$  is not optimal by Lemma 3.2.1. For the converse, suppose that either  $n \equiv 0 \pmod{4}$  and  $q \equiv 1 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and  $q \equiv 3 \pmod{4}$ . We consider two subcases. In each subcase, we construct an even subgraph  $H$  of size  $\lceil m/2 \rceil$  in  $G$ .

*Subcase 2.1.  $n \equiv 0 \pmod{4}$  and  $q \equiv 1 \pmod{4}$ .* Then  $n = 4k$  and  $q = 4\ell + 1$  for some positive integers  $k$  and  $\ell$  with  $k \geq \ell + 1$ . In this case, the size  $m_H$  of a graph  $H$  with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = 4k(4\ell + 1) - 2k - 2\ell = 16k\ell + 2k - 2\ell.$$

For each  $i$  with  $1 \leq i \leq k\ell$ , let  $H_i = C_{16}$  in  $B_i$  as shown in Figure 3.4(a), where the edges not belonging to  $C_{16}$  are not drawn; while for each  $j$  with  $1 \leq j \leq k - 1$ , let  $F_j = C_4$  lying between  $B_j$  and  $B_{j+1}$  as shown in Figure 3.4(b). Then  $H_i$  ( $1 \leq i \leq k\ell$ ) and  $F_j$  ( $1 \leq j \leq k - 1$ ) are edge-disjoint subgraphs of  $G$ .

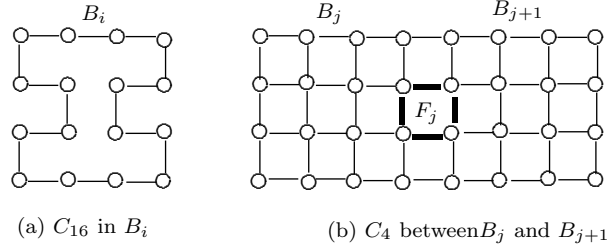


Figure 3.4: The cycles  $C_{16}$  and  $C_4$

- If  $k - \ell$  is even, then  $k - \ell = 2p$  for some integer  $p \geq 1$ . Let  $H_i = C_{16}$  in  $B_i$  for  $1 \leq i \leq k\ell$ , where  $H_i$  is shown as in Figure 3.4(a). For each  $j$  with  $1 \leq j \leq p \leq k - 1$ , let  $F_j = C_4$  as defined in Figure 3.4(b) lying between  $B_j$  and  $B_{j+1}$ . Now let  $H$  consist of these edge-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k\ell$ ) and  $F_j$  ( $1 \leq j \leq p$ ). That is,

$$\begin{aligned}
 V(H) &= \left( \bigcup_{i=1}^{k\ell} V(H_i) \right) \cup \left( \bigcup_{j=1}^p V(F_j) \right) \\
 E(H) &= \left( \bigcup_{i=1}^{k\ell} E(H_i) \right) \cup \left( \bigcup_{j=1}^p E(F_j) \right)
 \end{aligned}$$

Then  $H$  is a graph of the size  $m_H = 16k\ell + 4p = 16k\ell + 2(k - \ell)$  and each vertex of  $H$  has degree 2 or 4.

- If  $k - \ell$  is odd, then  $k - \ell = 2p + 1$  for some integer  $p \geq 0$ . Then  $p + 1 \leq k - 1$ . Let  $H_i = C_{16}$  in  $B_i$  as shown in Figure 3.4(a) for  $1 \leq i \leq k\ell - 1$  and  $H_{k\ell} = C_{14}$  in  $B_{k\ell}$ . For each  $j$  with  $1 \leq j \leq p + 1 \leq k - 1$ , let  $F_j = C_4$  that lies between  $B_j$  and  $B_{j+1}$  as shown in Figure 3.4(b). Now let  $H$  consist of these edge-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k\ell$ ) and  $F_j$  ( $1 \leq j \leq p + 1$ ). Then  $H$  is a graph of the size  $m_H = 16(k\ell - 1) + 14 + 4(p + 1) = 16k\ell + 2(k - \ell)$  and each vertex of  $H$  has degree 2 or 4.

*Subcase 2.2.*  $n \equiv 2 \pmod{4}$  and  $q \equiv 3 \pmod{4}$ . Then  $n = 4k+2$  and  $q = 4\ell+3$  for some positive integers  $k$  and  $\ell$  with  $k \geq \ell + 1$ . In this case, the size  $m_H$  of a graph  $H$  with the desired properties is

$$\begin{aligned} m_H &= \left\lceil \frac{m}{2} \right\rceil = (4k+2)(4\ell+3) - (2k+1) - (2\ell+1) \\ &= 16k\ell + 10k + 6\ell + 4. \end{aligned}$$

For each  $i$  with  $1 \leq i \leq k\ell$ , let  $H_i = C_{16}$  in  $B_i$  where for  $1 \leq i \leq k$ , the graphs  $B_i$  and  $H_i$  are defined as shown in Figure 3.4(a). For each  $j$  with  $1 \leq j \leq k-1$ , let  $F_j = C_4$  between  $B_j$  and  $B_{j+1}$  are defined in Figure 3.4(b). Furthermore, let  $C = C_{6\ell}$  and  $C' = C_{6k+8}$  where

$$\begin{aligned} C &= (v_{4k+1,1}, v_{4k+1,2}, \dots, v_{4k+1,3\ell}, v_{4k+2,3\ell}, v_{4k+2,3\ell-1}, \dots, \\ &\quad v_{4k+2,1}, v_{4k+1,1}), \\ C' &= (v_{1,4\ell+2}, v_{2,4\ell+2}, \dots, v_{3k+4,4\ell+2}, v_{3k+4,4\ell+3}, v_{3k+3,4\ell+3}, \dots, \\ &\quad v_{1,4\ell+3}, v_{1,4\ell+2}). \end{aligned}$$

Since  $k \geq \ell + 1$ , it follows that  $3k+4 \leq 4k+2$  and so such a cycle  $C'$  of order  $6k+8$  exists. Now let  $H$  consist of these edge-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k\ell$ ),  $F_j$  ( $1 \leq j \leq k-1$ ),  $C$  and  $C'$ . Then  $H$  is a graph of the size

$$m_H = 16k\ell + 4(k-1) + 6\ell + 6k + 8 = 16k\ell + 10k + 6\ell + 4$$

and each vertex of  $H$  has degree 2 or 4.

*Case 3.*  $n$  is odd and  $q$  is even. If  $n \equiv 1 \pmod{4}$  and  $q \equiv 2 \pmod{4}$  or  $n \equiv 3 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ , then  $m \equiv 1 \pmod{4}$  and so  $G$  is not optimal by

Lemma 3.2.1. For the converse, suppose that either  $n \equiv 1 \pmod{4}$  and  $q \equiv 0 \pmod{4}$  or  $n \equiv 3 \pmod{4}$  and  $q \equiv 2 \pmod{4}$ . We consider these two subcases. In each subcase, we construct an even subgraph  $H$  of size  $\lceil m/2 \rceil$  in  $G$ . Let  $G' = P_{4k} \square P_{4\ell}$  be the induced subgraph in  $G$  consisting of the  $4\ell$  paths of order  $4k$  as defined in (3.5) and let  $B_1, B_2, \dots, B_{k\ell}$  are the  $k\ell$  vertex-disjoint blocks of  $P_4 \square P_4$  in  $G'$  as shown in Figure 3.2.

*Subcase 3.1.*  $n \equiv 1 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ . Then  $n = 4k + 1$  and  $q = 4\ell$  for some positive integers  $k$  and  $\ell$  with  $k \geq \ell$ . In this case, the size  $m_H$  of a graph  $H$  with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = (4k + 1)4\ell - 2k - 2\ell = 16k\ell - 2k + 2\ell.$$

Let  $H_i = C_{14}$  in  $B_i$  if  $1 \leq i \leq k - \ell$  and let  $H_i = C_{16}$  in  $B_i$  if  $k - \ell + 1 \leq i \leq k\ell$ . Let  $H$  be the union of these vertex-disjoint subgraphs  $H_i$  for  $1 \leq i \leq k\ell$ . Then  $H$  is a 2-regular subgraph of  $G$  and the size of  $H$  is

$$14(k - \ell) + 16[k\ell - (k - \ell)] = 16k\ell - 2k + 2\ell.$$

*Subcase 3.2.*  $n \equiv 3 \pmod{4}$  and  $q \equiv 2 \pmod{4}$ . Then  $n = 4k + 3$  and  $q = 4\ell + 2$  for some positive integers  $k$  and  $\ell$  with  $k \geq \ell$ . In this case, the size  $m_H$  of a graph  $H$  with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = (4k + 3)(4\ell + 2) - (2k + 1) - (2\ell + 1) = 16k\ell + 6k + 10\ell + 4.$$

Let  $H_i = C_{16}$  in  $B_i$  if  $1 \leq i \leq k\ell$  which are defined as shown in Figure 3.4(a) and for each  $j$  with  $1 \leq j \leq \ell - 1 \leq k - 1$ , let  $F_j = C_4$  between  $B_j$  and  $B_{j+1}$  as defined



in Figure 3.4(b). Furthermore, let  $C = C_{6\ell}$  and  $C' = C_{6k+8}$  where

$$\begin{aligned} C &= (v_{4k+2,1}, v_{4k+2,2}, \dots, v_{4k+2,3\ell}, v_{4k+3,3\ell}, v_{4k+3,3\ell-1}, \dots, \\ &\quad v_{4k+3,1}, v_{4k+2,1}), \\ C' &= (v_{1,4\ell+1}, v_{2,4\ell+1}, \dots, v_{3k+4,4\ell+1}, v_{3k+4,4\ell+2}, v_{3k+3,4\ell+2}, \dots, \\ &\quad v_{1,4\ell+2}, v_{1,4\ell+1}). \end{aligned}$$

Since  $3k + 4 \leq 4k + 3$ , such a cycle  $C'$  of order  $6k + 8$  exists. Now let  $H$  consist of these edge-disjoint subgraphs  $H_i$  ( $1 \leq i \leq k\ell$ ),  $F_j$  ( $1 \leq j \leq \ell - 1$ ),  $C$  and  $C'$ . Then  $H$  is a graph of the size

$$m_H = 16k\ell + 4(\ell - 1) + 6\ell + 6k + 8 = 16k\ell + 10\ell + 6k + 4$$

and each vertex of  $H$  has degree 2 or 4. ■

**Theorem 3.2.6** *For integers  $n, p$  with  $n \geq p \geq 3$ , if  $P_n \square P_q$  is not optimal, then*

$$EI(P_n \square P_q) = \binom{n(q-1) + (n-1)q + 1}{2} + 1.$$

**Proof.** By Theorem 3.2.5, if  $P_n \square P_q$  is not optimal, then  $n$  and  $q$  satisfy one of the following:

- (i) If  $n$  and  $q$  are even, then either  $n \equiv 0 \pmod{4}$  and  $q \equiv 2 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ ;
- (ii) If  $n$  is even and  $q$  is odd, then either  $n \equiv 0 \pmod{4}$  and  $q \equiv 3 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and  $q \equiv 1 \pmod{4}$ ;

(iii) If  $n$  is odd and  $q$  is even, then  $n \equiv 1 \pmod{4}$  and  $q \equiv 2 \pmod{4}$  or  $n \equiv 3 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ .

Suppose that  $G = P_n \square P_q$  consists of  $q$  paths of order  $n$ , which we denote by

$$P_{n,i} = (v_{1,i}, v_{2,i}, \dots, v_{n,i}) \text{ for } 1 \leq i \leq q \quad (3.6)$$

such that  $v_{t,i}$  is adjacent to  $v_{t,j}$  ( $1 \leq t \leq n$ ) when  $|i - j| = 1$ . The size of  $G$  is  $m = n(q-1) + (n-1)q$ . Then  $n = 4k + r_n$  and  $q = 4\ell + r_q$ , where  $r_n, r_q \in \{0, 1, 2, 3\}$ . Let  $G' = P_{4k} \square P_{4\ell}$  be the subgraph of  $G$  with

$$V(G') = \{v_{a,b} : 1 \leq a \leq 4k, 1 \leq b \leq 4\ell\}. \quad (3.7)$$

The graph  $G'$  is also defined in (3.4) and (3.5). Then  $G'$  contains  $k\ell$  vertex-disjoint copies (or blocks) of  $P_4 \square P_4$ , denoted by  $B_{i,j}$  where  $1 \leq i \leq k$  and  $1 \leq j \leq \ell$ . These blocks  $B_{i,j}$  appear in  $G'$  in the way as shown in Figure 3.5.

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$\cdots$	$B_{1,\ell}$
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$\cdots$	$B_{2,\ell}$
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$\cdots$	$B_{3,\ell}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{k,1}$	$B_{k,2}$	$B_{k,3}$	$\cdots$	$B_{k,\ell}$

Figure 3.5: The subgraph  $G' = P_{4k} \square P_{4\ell}$  in  $G$

In each  $B_{i,j} = P_4 \square P_4$ , the vertices of  $B_{i,j} = P_4 \square P_4$  appear in the way as shown in Figure 3.6.

$v_{4i-3,4j-3}$	$v_{4i-3,4j-2}$	$v_{4i-3,4j-1}$	$v_{4i-3,4j}$
$v_{4i-2,4j-3}$	$v_{4i-2,4j-2}$	$v_{4i-2,4j-1}$	$v_{4i-2,4j}$
$v_{4i-1,4j-3}$	$v_{4i-1,4j-2}$	$v_{4i-1,4j-1}$	$v_{4i-1,4j}$
$v_{4i,4j-3}$	$v_{4i,4j-2}$	$v_{4i,4j-1}$	$v_{4i,4j}$

Figure 3.6: The block  $B_{i,j} = P_4 \square P_4$  in  $G'$

Note that  $B_{i,j}$  contains five edge-disjoint copies of  $C_4$ , namely

$$Q_1 = (v_{4i-3,4j-3}, v_{4i-3,4j-2}, v_{4i-2,4j-2}, v_{4i-2,4j-3}, v_{4i-3,4j-3})$$

$$Q_2 = (v_{4i-3,4j-1}, v_{4i-3,4j}, v_{4i-2,4j}, v_{4i-2,4j-1}, v_{4i-3,4j-1})$$

$$Q_3 = (v_{4i-1,4j-1}, v_{4i-1,4j}, v_{4i,4j}, v_{4i,4j-1}, v_{4i-1,4j-1})$$

$$Q_4 = (v_{4i-1,4j-3}, v_{4i-1,4j-2}, v_{4i,4j-2}, v_{4i,4j-3}, v_{4i-1,4j-3})$$

$$Q_5 = (v_{4i-2,4j-2}, v_{4i-2,4j-1}, v_{4i-1,4j-1}, v_{4i-1,4j-2}, v_{4i-2,4j-2})$$

where  $Q_5$  is at the center of  $B_{i,j}$  and surrounded clockwise by  $Q_1, Q_2, Q_3, Q_4$ . For each pair  $i, j$  with  $1 \leq i \leq k$  and  $1 \leq j \leq \ell$ , let  $F_{i,j}$  be the even subgraph of  $B_{i,j}$  consisting of the five edge-disjoint subgraphs  $Q_1, Q_2, Q_3, Q_4, Q_5$ , each of which is a copy of  $C_4$  and let  $F'_{i,j}$  be the even subgraph of  $B_{i,j}$  consisting of the four edge-disjoint subgraphs  $Q_1, Q_2, Q_3, Q_4$ . Thus, the size of  $F_{i,j}$  is 20 and the size of  $F'_{i,j}$  is 16 for all  $i, j$  with  $1 \leq i \leq k$  and  $1 \leq j \leq \ell$ . We consider three cases.

*Case 1.  $n$  and  $q$  are even.* Since  $m$  is even, it suffices to show that  $G$  has an even subgraph of size  $\frac{m}{2} + 1$ . There are two subcases.

*Subcase 1.1.  $n \equiv 0 \pmod{4}$  and  $q \equiv 2 \pmod{4}$ .* Let  $n = 4k$  and  $q = 4\ell + 2$ , where then  $k > \ell \geq 1$ . Note that  $\frac{m}{2} + 1 = 16k\ell + 4k + 2(k - \ell)$ . Let  $G' = P_{4k} \square P_{4\ell}$

be the subgraph of  $G$  as described in (3.7) and  $G^* = P_n \square P_2$  be the subgraph of  $G$  which is the Cartesian product of  $P_{n,4\ell+1}$  and  $P_{n,4\ell+2}$  as described in (3.6).

- If  $k = \ell + 1$ , then for  $1 \leq i \leq k - 1$  and  $j = 1$ , let  $H_{i,1} = F_{i,1}$  and  $H_{k,1} = F'_{k,1}$ , while for  $1 \leq i \leq k$  and  $2 \leq j \leq \ell$ , let  $H_{i,j} = F'_{i,j}$ .
- If  $k \geq \ell + 2$ , then for  $1 \leq i \leq k$  and  $j = 1$ , let  $H_{i,1} = F_{i,1}$ , for  $1 \leq i \leq k$  and  $2 \leq j \leq \ell$ , let  $H_{i,j} = F'_{i,j}$  and let  $H_{k,\ell+1} = C_{2(k-\ell)}$  be a cycle of order  $2(k - \ell)$  in  $G^*$  (which is possible since  $2 \leq k - \ell \leq 4\ell + 1$ ).

In each case, let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_{i,j}$  and then the size of  $m_H$  is  $\frac{m}{2} + 1$ .

*Subcase 1.2.*  $n \equiv 2 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ . Let  $n = 4k + 2$  and  $q = 4\ell$ , where then  $k \geq \ell \geq 1$ . Note that  $\frac{m}{2} + 1 = 16k\ell - 2k + 6\ell$ . In this case, we consider the subgraph  $G'' = P_{4k+1} \square P_{4\ell}$  of  $G$  with vertex set

$$V(G'') = \{v_{a,b} : 1 \leq a \leq 4k + 1, 1 \leq b \leq 4\ell\}. \quad (3.8)$$

Then  $G''$  contains  $(k - 1)\ell$  vertex-disjoint copies (or blocks)  $P_4 \square P_4$ , which are denoted by  $B_{i,j}$  where  $1 \leq i \leq k - 1$  and  $1 \leq j \leq \ell$  and  $\ell$  vertex-disjoint copies of  $P_5 \square P_4$ , which are denoted by  $B'_j$  where  $1 \leq j \leq \ell$ . These blocks  $B_{i,j}$  and  $B'_j$  appear in  $G''$  in the way as shown in Figure 3.7.

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$\cdots$	$B_{1,\ell}$
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$\cdots$	$B_{2,\ell}$
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$\cdots$	$B_{3,\ell}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{(k-1),1}$	$B_{(k-1),2}$	$B_{(k-1),3}$	$\cdots$	$B_{(k-1),\ell}$
$B'_1$	$B'_2$	$B'_3$	$\cdots$	$B'_\ell$

Figure 3.7: The subgraph  $G'' = P_{4k+1} \square P_{4\ell}$  in  $G$

For each  $j$  with  $1 \leq j \leq \ell$ , the vertices of  $B'_j = P_5 \square P_4$  appear in the way as shown in Figure 3.8.

$v_{4k-3,4j-3}$	$v_{4k-3,4j-2}$	$v_{4k-3,4j-1}$	$v_{4k-3,4j}$
$v_{4k-2,4j-3}$	$v_{4k-2,4j-2}$	$v_{4k-2,4j-1}$	$v_{4k-2,4j}$
$v_{4k-1,4j-3}$	$v_{4k-1,4j-2}$	$v_{4k-1,4j-1}$	$v_{4k-1,4j}$
$v_{4k,4j-3}$	$v_{4k,4j-2}$	$v_{4k,4j-1}$	$v_{4k,4j}$
$v_{4k+1,4j-3}$	$v_{4k+1,4j-2}$	$v_{4k+1,4j-1}$	$v_{4k+1,4j}$

Figure 3.8: The block  $B'_j = P_5 \square P_4$  in  $G''$

Note that each  $B'_j$  ( $1 \leq j \leq \ell$ ) contains each of  $C_{14}$  and  $C_{18}$  as a subgraph. For  $i = 1$  and  $1 \leq j \leq \ell$ , let  $H_{1,j} = F_{1,j}$  in  $B_{1,j}$ , for each pair  $i, j$  with  $2 \leq i \leq k-1$  and  $1 \leq j \leq \ell-1$ , let  $H_{i,j} = F'_{i,j}$  in  $B_{i,j}$ , for  $2 \leq i \leq k$  and  $j = \ell$ , let  $H_{i,\ell} = C_{14}$  (in  $B_{i,\ell}$  if  $1 \leq i \leq k-1$  and in  $B'_k$  if  $i = k$ ) and for  $i = k$  and  $1 \leq j \leq \ell-1$ , let  $H_{k,j} = C_{18}$  in  $B'_j$ . Let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_{i,j}$  and then the size of  $m_H$  is  $\frac{m}{2} + 1$ .

*Case 2.  $n$  is even and  $q$  is odd.* Since  $m$  is odd, it suffices to show that  $G$  has an even subgraph of size  $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$ . Let  $G' = P_{4k} \square P_{4\ell}$  be the subgraph of  $G$  as described in (3.7). There are two cases.

*Subcase 2.1.*  $n \equiv 0 \pmod{4}$  and  $q \equiv 3 \pmod{4}$ . Let  $n = 4k$  and  $q = 4\ell + 3$ , where then  $k > \ell \geq 1$ . Note that  $\frac{m-1}{2} = 16k\ell + 4k + 2(3k - \ell - 1)$  and  $2 \leq 3k - \ell - 1 \leq 4k$ . Let  $G^* = P_n \square P_2$  be the subgraph of  $G$  which is the Cartesian product of  $P_{n,4\ell+1}$  and  $P_{n,4\ell+2}$  as described in Subcase 1.1. For  $1 \leq i \leq k$  and  $j = 1$ , let  $H_{i,1} = F_{i,1}$  in  $B_{i,1}$ , for  $1 \leq i \leq k$  and  $2 \leq j \leq \ell$ , let  $H_{i,j} = F'_{i,j}$  in  $B_{i,j}$  and let  $H_{k,\ell+1} = C_{2(3k-\ell-1)}$  be a subgraph in  $G^*$ . Let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_{i,j}$  and then the size of  $H_m$  is  $\frac{m-1}{2}$ .

*Subcase 2.2.*  $n \equiv 2 \pmod{4}$  and  $q \equiv 1 \pmod{4}$ . Let  $n = 4k + 2$  and  $q = 4\ell + 1$ , where then  $k \geq \ell \geq 1$ . Note that  $\frac{m-1}{2} = 16k\ell + 2(k + 3\ell)$  and  $2 \leq k + 3\ell \leq 4\ell + 1 = q$ . Let  $F^* = P_2 \square P_q$  be the subgraph of  $G$  which is the Cartesian product of the two paths

$$(v_{n-1,1}, v_{n-1,2}, \dots, v_{n-1,q}) \text{ and } (v_{n,1}, v_{n,2}, \dots, v_{n,q}). \quad (3.9)$$

For each pair  $i, j$  with  $1 \leq i \leq k$  and  $1 \leq j \leq \ell$ , let  $H_{i,j} = F'_{i,j}$  in  $B_{i,j}$  and let  $H_{k,\ell+1} = C_{2(k+3\ell)}$  be a subgraph in  $F^*$ . Let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_{i,j}$  and then the size of  $H_m$  is  $\frac{m-1}{2}$ .

*Case 3.*  $n$  is odd and  $q$  is even. Since  $m$  is odd, we are seeking for an even subgraph of size  $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$  in  $G$ . Let  $G' = P_{4k} \square P_{4\ell}$  and  $G'' = P_{4k+1} \square P_{4\ell}$  be the subgraphs of  $G$  as described in (3.7) and (3.8), respectively. There are two cases.

*Subcase 3.1.*  $n \equiv 1 \pmod{4}$  and  $q \equiv 2 \pmod{4}$ . Let  $n = 4k + 1$  and  $q = 4\ell + 2$ , where then  $k > \ell \geq 1$ . Note that  $\frac{m-1}{2} = 16k\ell + 4k + 2(k + \ell)$  and  $2 \leq k + \ell \leq 4k$ . Let  $G^* = P_n \square P_2$  be the subgraph of  $G$  as described in Subcase 1.1. For  $1 \leq i \leq k$

and  $j = 1$ , let  $H_{i,1} = F_{i,1}$  in  $B_{i,1}$ , for  $1 \leq i \leq k$  and  $2 \leq j \leq \ell$ , let  $H_{i,j} = F'_{i,j}$  in  $B_{i,j}$  and let  $H_{k,\ell+1} = C_{2(k+\ell)}$  be a subgraph in  $G^*$ . Let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_{i,j}$  and then the size of  $H_m$  is  $\frac{m-1}{2}$ .

*Subcase 3.2.*  $n \equiv 3 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ . Let  $n = 4k + 3$  and  $q = 4\ell$ , where then  $k \geq \ell \geq 1$ . Let  $F^* = P_2 \square P_q$  be the subgraph of  $G$  which is the Cartesian product of the two paths described in (3.9).

- If  $\ell = 1$ , then  $G = P_{4k+3} \square P_4$  and  $\frac{m-1}{2} = 14k + 8$ . For  $1 \leq i \leq k$ , let  $H_i = C_{14}$  in  $B_{i,1}$  and let  $H_{k+1}$  be a cycle  $C_8$  of order 8 where

$$H_{k+1} = (v_{4k+1,1}, v_{4k+1,2}, v_{4k+1,3}, v_{4k+1,4}, v_{4k+2,4}, v_{4k+2,3}, v_{4k+2,2}, v_{4k+2,1}, v_{4k+1,1}).$$

Let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_i$  and then the size of  $H_m$  is  $\frac{m-1}{2}$ .

- If  $\ell \geq 2$ , then  $\frac{m-1}{2} = 16k\ell - 2k + 6\ell + 2(2\ell - 1)$ . Let  $H_1$  be the even subgraph of size  $16k\ell - 2k + 6\ell$  in  $P_{4k+1} \square P_{4\ell}$  (which is described in Subcase 1.2) and let  $H_2 = C_{2(2\ell-1)}$  be a subgraph of  $F^*$ . Let  $H$  be the even subgraph of  $G$  consisting of edge-disjoint subgraphs  $H_1$  and  $H_2$  and then the size of  $H_m$  is  $\frac{m-1}{2}$ . ■

### 3.3 Optimal Powers of Cycles

For a connected graph  $G$  and a positive integer  $k$ , the  $k$ th power  $G^k$  of  $G$  is that graph whose vertex set is  $V(G)$  such that  $uv$  is an edge of  $G^k$  if  $1 \leq d_G(u, v) \leq k$ .

The graph  $G^2$  is called the *square* of  $G$  and  $G^3$  is the *cube* of  $G$ . If  $k \geq \text{diam}(G)$ , then  $G^k$  is a complete graph. We have seen in Theorem 2.3.4 that all complete graphs of order at least 4 are optimal.

By Theorem 2.3.1, the  $n$ -cycle  $C_n$  is not optimal; while by Theorem 2.3.4, the complete graph  $K_n$  is. Thus if  $k = 1$ , then  $C_n^1 = C_n$  is not optimal; while if  $k \geq \lfloor n/2 \rfloor$ , then  $C_n^k$  is. We show, in fact, that  $C_n^k$  is optimal for each integer  $k \geq 2$ . In order to do this, we introduce an additional definition. For a positive integer  $t$ , the  $t$ -step  $G^{[t]}$  of a connected graph  $G$  is that graph whose vertex set is  $V(G)$  such that  $uv$  is an edge of  $G^{[t]}$  if  $d_G(u, v) = t$ . In particular,  $G^{[1]} = G$ . Furthermore, if  $t \leq k$ , then  $G^{[t]}$  is a subgraph of  $G^k$  and

$$E(G^k) = E(G^{[1]}) \cup E(G^{[2]}) \cup \dots \cup E(G^{[k]}).$$

For the  $n$ -cycle  $C_n = (v_1, v_2, \dots, v_n, v_{n+1} = v_1)$  where  $n \geq 3$  and each integer  $i$  with  $1 \leq i \leq n$ , the vertex  $v_i$  is adjacent to  $v_{i+t}$  and  $v_{i-t}$  in  $G^{[t]}$ , where the subscripts are expressed as integers modulo  $n$ . Thus  $C_n^{[t]}$  is a 2-regular graph of order  $n$  if  $t \neq n/2$  and  $C_n^{[t]} = \frac{n}{2}K_2$  if  $t = n/2$  where then  $n$  is even. The  $k$ th power  $C_n^k$  of  $C_n$  is then a  $2k$ -regular graph of order  $n$  and size  $kn$  if  $k < n/2$ .

**Theorem 3.3.1** *For each pair  $k, n$  of integers, where  $2 \leq k \leq \lfloor n/2 \rfloor$  and  $n \geq 4$ , the  $k$ th power  $C_n^k$  of the  $n$ -cycle is optimal.*

**Proof.** If  $k = \lfloor n/2 \rfloor$ , then  $C_n^k = K_n$ , which is optimal by Theorem 2.3.4. Thus, we now assume that  $k < \lfloor n/2 \rfloor$ . Let  $C_n = (v_1, v_2, \dots, v_n, v_{n+1} = v_1)$  where  $n \geq 4$ . The size of  $C_n^k$  is  $m = kn$ . If  $k$  is even, say  $k = 2p$  for some positive integer  $p$ ,



then the subgraph  $C_n^p$  is a  $(2p)$ -regular graph of size  $pn = \lceil m/2 \rceil$ . It then follows by Theorem 2.3.1 that  $C_n^k$  is optimal if  $k$  is even. Thus, it remains to consider the case when  $k \geq 3$  is odd. Since  $k < \lfloor n/2 \rfloor$ , it follows that  $n \geq 8$ . We show that  $C_n^k$  contains a subgraph  $H_k$  of size  $\lceil m/2 \rceil$ , each of whose vertex is even. We begin with the cube  $C_n^3$  of  $C_n$ . There are two cases, according to whether  $n$  is even or  $n$  is odd.

*Case 1.  $n$  is even.* Let  $C^* = (v_1, v_3, v_5, \dots, v_{n-1}, v_1)$  be the cycle of order  $n/2$  in  $C_n^3$  and let  $H_3$  be the spanning subgraph of  $G$  with  $E(H_3) = E(C_n) \cup E(C^*)$ . Then the size of  $H_3$  is  $3n/2$  and each vertex of  $H_3$  has degree 2 or 4. By Theorem 2.3.1,  $C_n^3$  is optimal if  $n$  is even.

*Case 2.  $n$  is odd.* Let  $n = 2\ell + 1$  for some integer  $\ell \geq 4$ . Then  $\lceil m/2 \rceil = \lceil 3n/2 \rceil = 3\ell + 2$ . First, suppose that  $\ell$  is even. Let  $C'$  be the cycle of order  $n - 4$  in  $G$  defined by

$$C' = (v_2, v_3, \dots, v_{\ell-1}, v_{\ell+2}, v_{\ell+1}, v_{\ell+4}, v_{\ell+5}, \dots, v_{n-1}, v_2)$$

and let  $C''$  be the circuit in  $G$  defined by

$$C'' = (v_1, v_3, \dots, v_{\ell-1}, v_\ell, v_{\ell+2}, v_{\ell+3}, v_{\ell+1}, v_\ell, v_{\ell+3}, v_{\ell+4}, v_{\ell+6}, \dots, v_{n-1}, v_1).$$

Figure 3.9(a) shows  $C'$  and  $C''$  for  $n = 9$  and  $n = 13$ , where the edges of  $C'$  are drawn in solid lines and the edges of  $C''$  are drawn in dashed lines. Let  $H_3$  be the subgraph of  $G$  induced by  $E(C') \cup E(C'')$ . Then the size of  $H_3$  is  $|E(C')| + |E(C'')| = (n - 4) + 7 + (n - 5)/2 = 3\ell + 2 = \lceil 3n/2 \rceil$  and each vertex of  $H_3$  has degree 2 or 4.

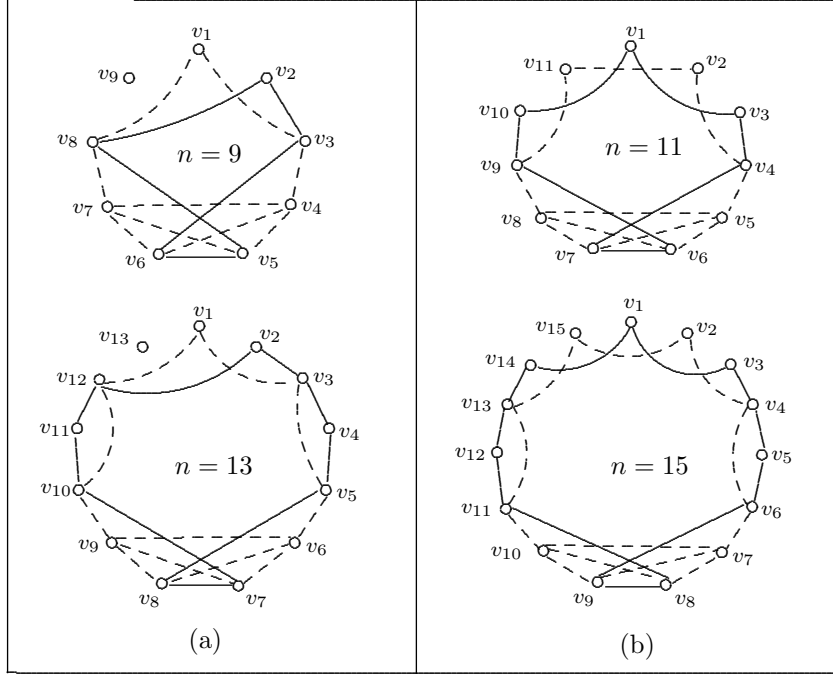


Figure 3.9: Subgraphs  $C'$  and  $C''$  in  $C_n^3$  for  $n = 9, 11, 13, 15$

Next suppose that  $\ell$  is odd. Let  $C'$  be the cycle of order  $n - 4$  in  $G$  defined by

$$C' = (v_1, v_3, v_4, \dots, v_{\ell-1}, v_{\ell+2}, v_{\ell+1}, v_{\ell+4}, v_{\ell+5}, \dots, v_{n-1}, v_1)$$

and let  $C''$  be the circuit in  $G$  defined by

$$C'' = (v_2, v_4, \dots, v_{\ell-1}, v_\ell, v_{\ell+2}, v_{\ell+3}, v_{\ell+1}, v_\ell, v_{\ell+3}, v_{\ell+4}, v_{\ell+6}, \dots, v_n, v_2).$$

Figure 3.9(b) shows  $C'$  and  $C''$  for  $n = 11$  and  $n = 15$ , where the edges of  $C'$  are drawn in solid lines and the edges of  $C''$  are drawn in dashed lines. Let  $H_3$  be the subgraph of  $G$  induced by  $E(C') \cup E(C'')$ . Then the size of  $H_3$  is  $|E(C')| + |E(C'')| = (n - 4) + 7 + (n - 5)/2 = 3\ell + 2 = \lceil 3n/2 \rceil$  and each vertex of  $H_3$  has degree 2 or 4.

In general, if  $k \geq 5$  is odd and  $n = 2\ell + 1$ , then  $\lceil m/2 \rceil = \lceil kn/2 \rceil = k\ell + \lceil k/2 \rceil$ . For  $k = 5$ , let  $H_5$  consists of  $H_3$  and  $C_n^{[4]}$ . Since each vertex in  $H_3$  and  $C_n^{[4]}$  is even, every vertex of  $H_5$  is even and the size of  $H_5$  is  $|E(H_3)| + n = (3\ell + 2) + (2\ell + 1) = 5\ell + 3 = 5\ell + \lceil 5/2 \rceil$ . More generally then, for an odd integer  $k \geq 7$  with  $k \leq \lfloor n/2 \rfloor - 3$ , the subgraph  $H_{k+2}$  consists of  $H_k$  and  $C_n^{[k+1]}$ , where every vertex of  $H_k$  and  $C_n^{[k+1]}$  is even and the size of  $H_k$  is  $k\ell + \lceil k/2 \rceil$ . Hence, every vertex of  $H_{k+2}$  is even and the size of  $H_{k+2}$  is  $|E(H_k)| + n = (k\ell + \lceil k/2 \rceil) + (2\ell + 1) = (k + 2)\ell + \lceil (k + 2)/2 \rceil$ .

Therefore,  $C_n^k$  is optimal for each integer odd integer  $k$  with  $3 \leq k \leq \lfloor n/2 \rfloor$ . ■

### 3.4 Circulants

We saw that the  $n$ -cycle  $C_n$  is not optimal for each  $n \geq 3$ . On the other hand, the  $k$ th power  $C_n^k$  of the  $n$ -cycles is optimal for each  $k \geq 2$ . The  $k$ th power of  $C_n$  is a special case of a more general class of graphs. For each integer  $n \geq 3$  and  $k \geq 1$  distinct integers  $n_1, n_2, \dots, n_k$  where  $1 \leq n_1 < n_2 < \dots < n_k \leq \lfloor n/2 \rfloor$ , the  $k$ -circulant  $C_n(n_1, n_2, \dots, n_k)$  is that graph with  $n$  vertices  $v_1, v_2, \dots, v_n$  such that  $v_i$  ( $1 \leq i \leq k$ ) is adjacent to  $v_{i \pm n_j \pmod n}$  for each  $j$  with  $1 \leq j \leq k$ . The integers  $n_i$  ( $1 \leq i \leq k$ ) are called the *jump sizes* of the circulant. The circulants  $C_{10}(1, 3)$ ,  $C_{10}(1, 5)$  and  $C_{12}(1, 2, 5)$  are shown in Figure 3.10. In particular,  $C_n(1) = C_n$  and  $C_n(1, 2, \dots, k)$  is the  $k$ th power  $C_n^k$  of  $C_n$ . Furthermore,  $C_n^k = K_n$  for all  $k \geq \lfloor \frac{n}{2} \rfloor$ .

For a circulant  $C_n(n_1, n_2, \dots, n_k)$ , if  $n_k < n/2$ , then  $C_n(n_1, n_2, \dots, n_k)$  is  $2k$ -regular and so Eulerian, while if  $n_k = n/2$ , then  $n$  is even and  $C_n(n_1, n_2, \dots, n_k)$  is  $(2k - 1)$ -regular. Thus circulants  $C_n(n_1, n_2, \dots, n_k)$  are symmetric classes of regular

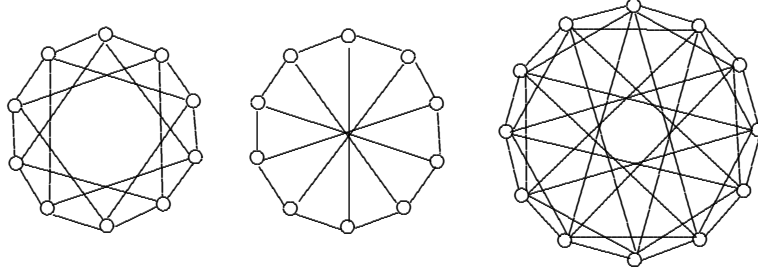


Figure 3.10: The circulants  $C_{10}(1, 3)$ ,  $C_{10}(1, 5)$  and  $C_{12}(1, 2, 5)$

graphs. In this section, we investigate the problem concerning which  $k$ -circulants are optimal.

### 3.4.1 $k$ -Circulants

In this section, we establish some general results on  $k$ -circulants.

**Theorem 3.4.1** *For integers  $n, k, t$  where  $n \geq 4$ ,  $k \geq 2$  and  $t \geq 0$ , let  $n_1, n_2, \dots, n_{k+2t}$  be  $k + 2t$  distinct integers such that  $1 \leq n_i \leq n/2$  for  $1 \leq i \leq k$  and  $1 \leq n_i < n/2$  for  $k + 1 \leq i \leq k + 2t$ . If the  $k$ -circulant  $C_n(n_1, n_2, \dots, n_k)$  is optimal, then the  $(k + 2t)$ -circulant  $C_n(n_1, n_2, \dots, n_{k+2t})$  is optimal.*

**Proof.** Let  $G_0 = C_n(n_1, n_2, \dots, n_k)$  and  $G = C_n(n_1, n_2, \dots, n_{k+2t})$ . Then  $G_0$  is a subgraph of  $G$ . If  $t = 0$ , then  $G_0 = G$  is optimal. Thus, we may assume that  $t \geq 1$ . Suppose that the size of  $G_0$  is  $m_0$  and the size of  $G$  is  $m$ . Since  $G_0$  is optimal, it follows by Theorem 2.3.1 that  $G_0$  has an even subgraph  $H_0$  of size  $\lceil \frac{m_0}{2} \rceil$ . Since  $1 \leq n_i < n/2$  for  $k + 1 \leq i \leq k + 2t$ , it follows that  $m = m_0 + 2tn$ . Let  $H$  be the even subgraph consisting of the edge-disjoint subgraphs  $H_0$  and

$H_1 = C_n(n_{k+1}, n_{k+2}, \dots, n_{k+t})$  of  $G$ , that is,  $E(H) = E(H_0) \cup E(H_1)$ . Then the size of  $H = \lceil \frac{m_0}{2} \rceil + nt = \lceil \frac{m_0 + 2nt}{2} \rceil = \lceil \frac{m}{2} \rceil$ . Thus  $G$  is optimal by Theorem 2.3.1. ■

### 3.4.2 Eulerian Circulants

We first consider Eulerian  $k$ -circulants  $C_n(n_1, n_2, \dots, n_k)$ , where then  $n_i < n/2$  for each  $i = 1, 2, \dots, k$ . With the aid of Theorem 3.4.1, we are able to extend the result on  $k$ th powers of cycles in Theorem 3.3.1 to certain Eulerian circulants.

**Proposition 3.4.2** *Let  $G = C_n(n_1, n_2, \dots, n_k)$  where  $k \geq 2$  and  $1 \leq n_1 < n_2 < \dots < n_k < n/2$ . If  $k$  is even or  $k$  is odd and there exist three distinct elements  $n_r, n_s, n_t \in \{n_1, n_2, \dots, n_k\}$  such that  $C_n(n_r, n_s, n_t)$  is optimal, then  $G$  is optimal.*

**Proof.** The graph  $G$  is a  $2k$ -regular graph of order  $n$  and size  $m = nk$ . First, suppose that  $k \geq 2$  is even and consider  $G_0 = C_n(n_1, n_2)$ . Since  $1 \leq n_1 < n_2 < n/2$ , the subgraph  $H = C_n(n_1)$  is an even subgraph of size  $n = \lceil \frac{m}{2} \rceil$  in  $G_0$  and so  $G_0$  is optimal by Theorem 2.3.1. Thus  $G$  is optimal by Theorem 3.4.1. Next, suppose that  $k \geq 3$  is odd and there exist three distinct elements  $n_r, n_s, n_t \in \{n_1, n_2, \dots, n_k\}$  such that  $C_n(n_r, n_s, n_t)$  is optimal. Since  $1 \leq n_1 < n_2 < \dots < n_k < n/2$ , it follows by Theorem 3.4.1 that  $G$  is optimal. ■

In order to determine other Eulerian optimal circulants, we present a lemma.

**Lemma 3.4.3** *For each pair  $a, n$  of integers, where  $2 \leq a < n/2$ , if  $d = \gcd((a, n))$  and  $p = n/d$ , then  $C_n(a)$  can be decomposed into  $d$   $p$ -cycles.*

**Proof.** Let  $C_n = (v_0, v_1, \dots, v_{n-1}, v_n = v_0)$  where  $n \geq 3$ . Since  $d = \gcd(a, n)$  and  $d \mid a$ , it follows that  $a/d$  is an integer and so  $n \mid pa$ . Hence  $i \equiv i+pa \pmod{n}$  for all integers  $i$ . Furthermore, since  $p = n/d$  and  $d = \gcd(a, n)$ , if  $n \mid xa$  for any positive integer  $x$ , then  $p \mid x$ . Thus  $p$  is the smallest positive integer  $x$  such that  $i \equiv i+xa \pmod{n}$ . Moreover,  $p = n/d \geq 3$  as  $d \leq a < n/2$ . Now for each integer  $i$  with  $0 \leq i \leq d-1$ , define a  $p$ -cycle  $Q_i$  in  $C_n(a)$  as  $Q_i = (v_i, v_{i+a}, v_{i+2a}, \dots, v_{i+pa}, v_i)$ . We claim that  $V(Q_i) \cap V(Q_j) = \emptyset$  for all pairs  $i, j$  with  $0 \leq i \neq j \leq d-1$ . Assume, to the contrary, that  $V(Q_i) \cap V(Q_j) \neq \emptyset$  for some pair  $i, j$ , say  $i > j$ . Then there exists  $u \in V(Q_i) \cap V(Q_j)$  such that  $u = v_{i+\ell a} = v_{j+ta}$ . Hence  $i + \ell a \equiv j + ta \pmod{n}$  and so  $(i - j) + (\ell - t)a = nq$  for some integer  $q$ . Since  $d \mid a$  and  $d \mid n$ , it follows that  $d \mid (i - j)$ , which is impossible as  $1 \leq i - j < d - 1$ . Therefore,  $Q_0, Q_2, \dots, Q_{d-1}$  are vertex-disjoint (and edge-disjoint)  $p$ -cycles in  $C_n(a)$ . Since  $n = dp$  is the size of  $C_n(a)$ , it follows that  $\{Q_0, Q_2, \dots, Q_{d-1}\}$  is a decomposition of  $C_n(a)$  into  $p$ -cycles. Figure 3.11 shows such a cycle decomposition of  $C_{15}(6)$  into three 5-cycles. ■

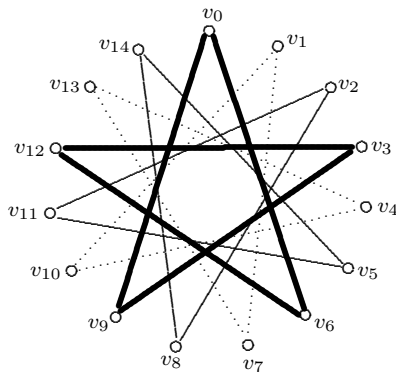


Figure 3.11: A cycle decomposition of  $C_{15}(6)$  into three 5-cycles

**Proposition 3.4.4** *Let  $G = C_n(n_1, n_2, \dots, n_k)$  where  $n$  is even,  $k \geq 2$  and  $1 \leq n_1 < n_2 < \dots < n_k < n/2$ . If there is  $i \in \{1, 2, \dots, k\}$  such that  $n_i$  is even, then  $G$  is optimal.*

**Proof.** By Proposition 3.4.2, we may assume that  $k \geq 3$  is odd. Furthermore, by Theorem 3.4.1, it suffices to consider  $k = 3$  and  $G_0 = C_n(n_1, n_2, n_3)$ . Assume, without loss of generality, that  $n_1$  is even and  $d = \gcd(n, n_1) = 2t$  for some positive integer  $t$ . By Lemma 3.4.3,  $C_n(n_1)$  can be decomposed into  $2t$  cycles  $Q_1, Q_2, \dots, Q_{2t}$  of length  $n/2t$ . Thus  $G_0$  can be decomposed into  $Q_1, Q_2, \dots, Q_{2t}, C_n(n_2), C_n(n_3)$ . The even subgraph  $H$  consisting of  $C_n(n_2)$  and  $Q_i$  ( $1 \leq i \leq t$ ) has size  $n + \frac{n}{2} = \lceil \frac{m}{2} \rceil$ . It then follows by Theorem 2.3.1 that  $G_0$  is optimal and so is  $G$  by Theorem 3.4.1. ■

### 3.4.3 Non-Eulerian Circulants

By Proposition 3.4.2, for all integers  $n, n_1, n_2$  where  $n \geq 4$  and  $1 \leq n_1 < n_2 < n/2$ , the circulant  $C_n(n_1, n_2)$  is optimal. This, however, is not the case when  $n_2 = n/2$ . In general, if  $1 \leq n_1 < n_2 < \dots < n_k = n/2$ , where  $k \geq 2$ , then  $n$  is even and  $C_n(n_1, n_2, \dots, n_k)$  is a  $(2k - 1)$ -regular graph of order  $n$  and size  $m = n(2k - 1)/2$ . Thus these circulants are not Eulerian. In particular,  $C_n(1, n/2)$  is a 3-regular graph of order  $n$  and size  $m = 3n/2$ . If  $n = 4$ , then  $C_4(1, 2) = K_4$ ; while if  $n = 6$ , then  $C_6(1, 3) = K_{3,3}$ . Recall that all optimal complete graphs and complete bipartite graphs have determined in Theorems 2.3.4 and 2.3.5. which we state next.

- The complete graph  $K_n$  of order  $n$  is optimal if and only if  $n \geq 4$ .
- For integers  $r$  and  $s$  with  $2 \leq r \leq s$ , the complete bipartite graph  $K_{r,s}$  is optimal if and only if (i)  $r$  and  $s$  are both even and  $(r, s) \neq (2, 4k + 2)$  for any nonnegative integer  $k$  or (ii) at least one of  $r$  and  $s$  is odd and  $rs \not\equiv 1, 2 \pmod{4}$ .

By Theorems 2.3.4 and 2.3.5, the graph  $C_4(1, 2)$  is optimal while  $C_6(1, 3)$  is not. In fact, more can be said. Also, recall that if  $G$  is a connected bipartite graph of size  $m \geq 1$  such that  $m \equiv 1 \pmod{4}$  or  $m \equiv 2 \pmod{4}$ , then  $G$  is not optimal.

**Proposition 3.4.5** *For each even integer  $n \geq 4$ , the graph  $C_n(1, n/2)$  is optimal if and only if  $n \not\equiv 6 \pmod{8}$ . Furthermore, if  $n \equiv 6 \pmod{8}$ , then  $EI(G) = \binom{m+1}{2} + 1$  where  $m = 3n/2$  is the size of  $C_n(1, n/2)$ .*

**Proof.** Since the statement is true when  $n = 4$  or  $n = 6$ , we assume that  $n \geq 8$  is even. Let  $C_n = (v_1, v_2, \dots, v_n, v_1)$ . Then  $C_n(1, n/2)$  is a 3-regular graph of order  $n$  and size  $m = 3n/2$ . First, assume that  $n \equiv 6 \pmod{8}$  and so  $n = 8p + 6$  for some positive integer  $p$ . For each  $i$  with  $1 \leq i \leq n$ , the vertex  $v_i$  is adjacent to  $v_{i-1}$ ,  $v_{i+1}$  and  $v_{i+n/2}$  in  $C_n(1, n/2)$ , where the subscript of each vertex is expressed as an integer modular  $n$ . Since  $n/2 = 4p + 3$  is odd, the integers  $i - 1, i + 1, i + n/2$  are of the same parity and  $i$  and  $j$  where  $j \in \{i - 1, i + 1, i + n/2\}$  are of the opposite parity. Thus  $C_n(1, n/2)$  is a bipartite graph whose partite sets are the same as those of  $C_n$ . Since  $m = 12p + 9 = 4(3p + 2) + 1$ , it follows by Lemma 3.2.1 that  $C_n(1, n/2)$  is not optimal.



For the converse, assume that  $n \equiv 0, 2, 4 \pmod{8}$ . For  $n \equiv 0 \pmod{8}$ , let  $n = 8p$  for some positive integer  $p$ . Then  $m = 12p$  and  $m/2 = 6p = (6p - 2) + 2 = 2(3p - 1) + 2$ . The cycle  $(v_1, v_2, \dots, v_{3p}, v_{7p}, v_{7p-1}, \dots, v_{4p+1}, v_1)$  is a cycle of size  $6p$  in  $C_n(1, n/2)$ . For  $n \equiv 2 \pmod{8}$ , let  $n = 8p + 2$  for some positive integer  $p$ . Then  $m = 12p + 3$  and  $\lceil m/2 \rceil = 6p + 2$ . The cycle  $(v_1, v_2, \dots, v_{3p+1}, v_{7p+2}, v_{7p+1}, \dots, v_{4p+2}, v_1)$  is a cycle of size  $6p + 2$  in  $C_n(1, n/2)$ . For  $n \equiv 4 \pmod{8}$ , let  $n = 8p + 4$  for some positive integer  $p$ . Then  $m = 12p + 6$  and  $\lceil m/2 \rceil = m/2 = 6p + 3 = 3(2p + 1)$ . Let  $d = n/2 = 4p + 2$  be the diameter of  $C_n$ . The cycle  $(v_1, v_2, \dots, v_{2p+2}, v_{2p+2+d}, v_{2p+2+d+1}, v_{2p+2+2d+1}, v_{2p+2+2d+2}, \dots, v_{n-2}, v_{n-1}, v_{n-1+d}, v_{n-1+d+1}, v_{n-1+d+2}, v_1)$  is a cycle of size  $6p + 3$  in  $C_n(1, n/2)$ , where the subscript of each vertex is expressed as an integer modular  $n$ . Figure 3.12 shows such cycles in the circulants  $C_n(1, n/2)$  for  $n = 12, 20, 28$ . Therefore,  $C_n(1, n/2)$  is optimal by Theorem 2.3.1 if  $n \equiv 0, 2, 4 \pmod{8}$ .

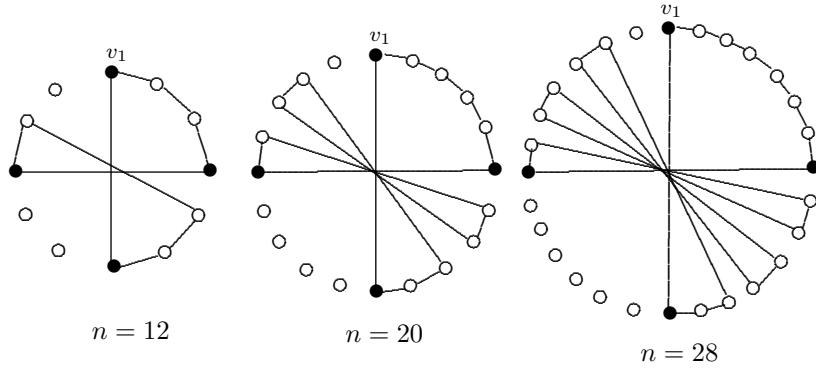


Figure 3.12: Circulants  $C_n(1, n/2)$  are optimal  $n = 12, 20, 28$

Now, suppose that  $n \equiv 6 \pmod{8}$ . Since  $m = 3n/2$  is odd, it suffices to show that  $G$  contains an even subgraph of size  $\frac{m-1}{2}$ . Let  $n = 8k + 6$  for some integer

$k \geq 0$ . Then  $\frac{m-1}{2} = 6k + 4$ . Note that  $G$  contains  $C_{6k+4}$  as a subgraph. For example,  $(v_1, v_2, \dots, v_{3k+2}, v_{7k+5}, v_{7k+4}, v_{7k+3}, \dots, v_{4k+4}, v_1)$  is such a cycle of order  $6k + 4$ , where each subscript is expressed as an integer module  $n$ . Thus  $EI(G) = \binom{m+1}{2} + 1$ . ■

By Proposition 3.4.2, if  $k \geq 2$  is even, then all Eulerian  $k$ -circulants  $C_n(n_1, n_2, \dots, n_k)$  are optimal. For non-Eulerian  $k$ -circulants when  $k$  is even, we have the following, which is a consequence of Theorem 3.4.1 and Proposition 3.4.5.

**Corollary 3.4.6** *If  $n$  and  $k$  are even integers with  $n, k \geq 2$  and  $n \not\equiv 6 \pmod{8}$ , then the graph  $C_n(1, n_2, n_3, \dots, n_k)$  is optimal for all integers  $n_2, n_3, \dots, n_k$  with  $2 \leq n_2 < n_3 < \dots < n_k = n/2$ .*

### 3.4.4 Circulants $C_n(n_1, n_2, n_3)$

We saw that 3-circulants play an important role in determining which  $k$ -circulants are optimal when  $k \geq 3$  is odd. In this section we investigate 3-circulants  $C_n(n_1, n_2, n_3)$  for small values of  $n_1, n_2, n_3$ , namely at least two of  $n_1, n_2, n_3$  are 1, 2 or 3. First, we establish some definitions and notation. Let  $G = C_n(n_1, n_2, n_3)$  where  $C_n = (v_1, v_2, \dots, v_n = v_0, v_{n+1} = v_1)$ . In what follows, all subscripts are expressed as integers modulo  $n$ . For a positive integer  $b$ , a *block  $B$  of order  $b$*  in  $C_n$  is an ordered set of  $b$  consecutive vertices  $v_{i+1}, v_{i+2}, \dots, v_{i+b}$  of  $C_n$  for some  $i$  with  $1 \leq i \leq n$  and is denoted by  $B = [v_{i+1}, v_{i+b}]$  or  $B = (v_{i+1}, v_{i+2}, \dots, v_{i+b})$ . Two blocks  $B$  and  $B'$  of order  $b$  and  $b'$ , respectively, are *consecutive* in  $C_n$  if  $B = [v_{i+1}, v_{i+b}]$  and  $B' = [v_{j+1}, v_{j+b'}]$  for some integers  $i$  and  $j$  where  $j \in \{i + b - 1, i + b\}$  and

$1 \leq i, j \leq n$ . Thus two consecutive blocks of  $C_n$  have at most one vertex in common. If  $B_1, B_2, \dots, B_s$  ( $s \geq 2$ ) are pairwise disjoint consecutive blocks of  $C_n$  such that  $B_1 \cup B_2 \cup \dots \cup B_s = V(C_n)$ , then  $\{B_1, B_2, \dots, B_s\}$  is referred to as a *partition* of  $C_n$ . For a block  $B$  of  $C_n$ , let  $G[B]$  denote the subgraph of  $G$  induced by the vertices in  $B$ .

### **Circulants** $C_n(1, 2, n_3)$

We first show that all 3-circulants  $C_n(1, 2, n_3)$  are optimal for each integer  $n_3 \geq 3$ .

**Theorem 3.4.7** *For each integer  $n \geq 8$ , the graph  $C_n(1, 2, n_3)$  is optimal for all integers  $n_3$  with  $3 \leq n_3 \leq n/2$ .*

**Proof.** Let  $G = C_n(1, 2, n_3)$  where  $C_n = (v_1, v_2, \dots, v_n, v_{n+1} = v_1)$ . By Proposition 3.4.4, if  $n_3 < n/2$ , then  $G$  is optimal. Thus, we may assume that  $n_3 = n/2$  and so  $n \geq 8$  is even. Let  $n = 2k$  for some integer  $k \geq 4$ . The size of  $G$  is  $m = 2n + \frac{n}{2} = 5k$ . The subgraph  $C_n(1, 2)$  of  $G$  contains  $k$  edge-disjoint triangles  $T_1, T_2, \dots, T_k$  defined by

$$T_i = (v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i-1}) \quad \text{for } 1 \leq i \leq k, \quad (3.10)$$

where each subscript is expressed as an integer modulo  $n$ . We consider three cases.

*Case 1.*  $\lceil \frac{m}{2} \rceil \equiv 0 \pmod{3}$ . Then  $\lceil \frac{m}{2} \rceil = 3q$  for some integer  $q \geq 4$ . If  $k$  is even, then  $\lceil \frac{m}{2} \rceil = \frac{5k}{2} = 3q$  and so  $q = \frac{5k}{6} < k$ . If  $k$  is odd, then  $\lceil \frac{m}{2} \rceil = \frac{5k+1}{2} = 3q$  and so  $q = \frac{5k+1}{6} < k$ . The even subgraph consisting of  $q$  edge-disjoint triangles  $T_1, T_2, \dots, T_q$  as defined in (3.10) has size  $\lceil \frac{m}{2} \rceil = 3q$ . Thus  $G$  is optimal.

*Case 2.*  $\lceil \frac{m}{2} \rceil \equiv 1 \pmod{3}$ . Then  $\lceil \frac{m}{2} \rceil = 3q + 1 = 3(q - 1) + 4$  for some integer  $q \geq 4$ . If  $k$  is even, then  $\lceil \frac{m}{2} \rceil = \frac{5k}{2} = 3q + 1$  and so  $q = \frac{5k-2}{6} < k$ . If  $k$  is odd, then  $\lceil \frac{m}{2} \rceil = \frac{5k+1}{2} = 3q + 1$  and so  $q = \frac{5k-1}{6} < k$ . Since  $q \leq k - 1$ , it follows that  $2q - 1 \leq 2k - 3$  and so  $2q + 2 \leq 2k$ . Thus the 4-cycle  $Q = (v_{2q-1}, v_{2q+1}, v_{2q+2}, v_{2q}, v_{2q-1})$  and the first  $q - 1$  triangles  $T_1, T_2, \dots, T_{q-1} = (v_{2q-3}, v_{2q-2}, v_{2q-1}, v_{2q-3})$  as defined in (3.10) are edge-disjoint. The even subgraph consisting of  $T_1, T_2, \dots, T_{q-1}$  and the 4-cycle  $Q$  has size  $\lceil \frac{m}{2} \rceil = 3q + 1$  and so  $G$  is optimal.

*Case 3.*  $\lceil \frac{m}{2} \rceil \equiv 2 \pmod{3}$ . Then  $\lceil \frac{m}{2} \rceil = 3q + 2$  for some integer  $q \geq 3$ . If  $k = 4$ , then  $G = C_8(1, 2, 4)$  and  $\lceil \frac{m}{2} \rceil = 10$ . The even subgraph of  $G$  consisting of  $T_1, T_2$  and the 4-cycle  $(v_5, v_7, v_8, v_6, v_5)$  has size 10. If  $k = 5$ , then  $G = C_{10}(1, 2, 5)$  and  $\lceil \frac{m}{2} \rceil = 13$ . The even subgraph of  $G$  consisting of  $T_1, T_2, T_3$  and the 4-cycle  $(v_7, v_9, v_{10}, v_8, v_7)$  has size 13. We now assume that  $k \geq 6$  and so  $\lceil \frac{m}{2} \rceil = 3(q-2) + 8$ . If  $k$  is even, then  $\lceil \frac{m}{2} \rceil = \frac{5k}{2} = 3q + 2$  and so  $q = \frac{5k-4}{6}$ . If  $k$  is odd, then  $\lceil \frac{m}{2} \rceil = \frac{5k+1}{2} = 3q + 2$  and so  $q = \frac{5k-3}{6}$ . In either case,  $2q + 3 \leq 2k$ . Then the two edge-disjoint 4-cycles  $Q_1 = (v_{2q-3}, v_{2q-1}, v_{2q}, v_{2q-2}, v_{2q-3})$  and  $Q_2 = (v_{2q}, v_{2q+2}, v_{2q+3}, v_{2q+1}, v_{2q})$  and the first  $q - 2$  triangles  $T_1, T_2, \dots, T_{q-2} = (v_{2q-5}, v_{2q-4}, v_{2q-3}, v_{2q-5})$  as defined in (3.10) are edge-disjoint. The even subgraph consisting of  $T_1, T_2, \dots, T_{q-2}$  and the 4-cycles  $Q_1$  and  $Q_2$  has size  $\lceil \frac{m}{2} \rceil = 3q + 2$ . Therefore,  $G$  is optimal. ■

The following is a consequence of Theorem 3.4.1, Proposition 3.4.2 and Theorem 3.4.7.

**Corollary 3.4.8** *For integers  $n$  and  $k$  where  $n \geq 8$  and  $k \geq 2$ ,  $C_n(1, 2, n_3, n_4, \dots, n_k)$  is optimal for all integers  $n_3, n_4, \dots, n_k$  with  $3 \leq n_3 < n_4 < \dots < n_k \leq n/2$ .*

**Circulants**  $C_n(1, 3, n_3)$

We first consider Eulerian circulants  $C_n(1, 3, n_3)$  where then  $4 \leq n_3 < n/2$ .

**Theorem 3.4.9** *For each integer  $n \geq 8$ , the graph  $C_n(1, 3, n_3)$  where  $4 \leq n_3 < n/2$  is optimal if and only if either  $n \not\equiv 2 \pmod{4}$  or  $n_3$  is even. Furthermore, if  $C_n(1, 3, n_3)$  is not optimal, then*

$$EI(C_n(1, 3, n_3)) = \binom{3n+1}{2} + 1.$$

**Proof.** Let  $G = C_n(1, 3, n_3)$  where  $4 \leq n_3 < n/2$  and  $C_n = (v_1, v_2, \dots, v_n = v_0, v_{n+1} = v_1)$ . The size of  $G$  is  $m = 3n$ . First, suppose that  $n \equiv 2 \pmod{4}$  and  $n_3$  is odd. Then  $G$  is a bipartite graph of size  $m \equiv 2 \pmod{4}$ . It then follows by Lemma 3.2.1 that  $G$  is not optimal and so  $EI(G) \geq \binom{3n+1}{2} + 1$ . To show that  $EI(G) \leq \binom{3n+1}{2} + 1$ , it suffices to show that  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil + 1$  (since  $m$  is even). Let  $n = 4k + 2$  for some integer  $k \geq 2$  and so  $\lceil \frac{m}{2} \rceil + 1 = 6k + 4 = (4k + 2) + (2k + 2)$ .

- First, suppose that  $k$  is even, say  $k = 2s$  for some positive integer  $s$ . Then  $2k + 2 = 4s + 2 = 4(s - 1) + 6$  and  $n = 8s + 2$ . Partition the cycle  $C_n$  into  $s$  consecutive blocks  $B_1, B_2, \dots, B_s$ , where each  $B_i$  has order 8 ( $1 \leq i \leq s - 1$ ) and  $B_s$  has size 10. For each  $i$  with  $1 \leq i \leq s - 1$ , let  $Q_i$  be a 4-cycle in  $G[B_i]$  and let  $Q_s$  be a 6-cycle in  $G[B_s]$ , each of which uses only edges in the subgraph  $C_n(1, 3)$  of  $G$ . Hence all subgraphs  $Q_i$  ( $1 \leq i \leq s$ ) and  $C_n(n_3)$  are pairwise edge-disjoint. Thus the subgraph consisting of  $Q_i$  ( $1 \leq i \leq s$ ) and  $C_n(n_3)$  is even and has size  $(4k + 2) + (2k + 2) = \lceil \frac{m}{2} \rceil + 1$  in  $G$ .

- Next, suppose that  $k$  is odd, say  $k = 2s + 1$  for some positive integer  $s$ . Then  $2k + 2 = 4s + 4 = 4(s + 1)$  and  $n = 8s + 6$ . Partition the cycle  $C_n$  into  $s + 1$  consecutive blocks  $B_1, B_2, \dots, B_{s+1}$  where each  $B_i$  has order 8 ( $1 \leq i \leq s$ ) and  $B_{s+1}$  has size 6. For each  $i$  with  $1 \leq i \leq s + 1$ , let  $Q_i$  be a 4-cycle in  $G[B_i]$  using only edges in the subgraph  $C_n(1, 3)$  of  $G$ . Then all subgraphs  $Q_i$  ( $1 \leq i \leq s + 1$ ) and  $C_n(n_3)$  are pairwise edge-disjoint and the subgraph consisting of  $Q_i$  ( $1 \leq i \leq s + 1$ ) and  $C_n(n_3)$  is even and has size  $(4k + 2) + (2k + 2) = \lceil \frac{m}{2} \rceil + 1$  in  $G$ .

For the converse, assume that if  $n \not\equiv 2 \pmod{4}$  or  $n_3$  is even. We show that  $G$  is optimal. By Theorem 2.3.1, it suffices to show that  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil$ . Suppose that  $n \equiv i \pmod{4}$  for  $i = 0, 1, 2, 3$ . We consider these four cases.

*Case 1.*  $n \equiv 0 \pmod{4}$ . Let  $n = 4k$  for some integer  $k \geq 2$ . Then  $m = 12k$  and  $\lceil \frac{m}{2} \rceil = 6k$ . Consider the  $(2k)$ -cycle  $C$  defined by

$$C = (v_0, v_1, v_4, v_5, v_8, \dots, v_{4i}, v_{4i+1}, v_{4i+4}, \dots, v_{4(k-1)}, v_{4(k-1)+1}, v_{4(k-1)+4} = v_0, v_1),$$

where the distances between two consecutive vertices of  $C$  alternate 1 and 3 in  $G$ . Note that  $C_n(n_3)$  and  $C$  are edge-disjoint. Hence the subgraph consisting of  $C_n(n_3)$  and  $C$  is even and size  $n + 2k = 6k = \lceil \frac{m}{2} \rceil$ .

*Case 2.*  $n \equiv 1 \pmod{4}$ . Let  $n = 4k + 1$  for some integer  $k \geq 2$ . Then  $m = 12k + 3$  and  $\lceil \frac{m}{2} \rceil = 6k + 2 = (4k + 1) + (2k + 1)$ . Consider the  $(2k + 1)$ -cycle

defined by

$$C = (v_0, v_1, v_2, \dots, v_{k+1}, v_{(k+1)+3}, v_{(k+1)+6}, \dots, \\ v_{(k+1)+3i}, v_{(k+1)+3(i+1)}, \dots, v_{(k+1)+3k} = v_0).$$

Then  $C_n(n_3)$  and  $C$  are edge-disjoint and the subgraph consisting of  $C_n(n_3)$  and  $C$  is even and has size  $n + (2k + 1) = \lceil \frac{m}{2} \rceil$ .

*Case 3.*  $n \equiv 2 \pmod{4}$ . Then  $n_3$  is even and so  $G$  is optimal by Proposition 3.4.4.

*Case 4.*  $n \equiv 3 \pmod{4}$ . Let  $n = 4k + 3$  for some integer  $k \geq 2$ . Then  $m = 12k + 9$  and  $\lceil \frac{m}{2} \rceil = 6k + 5 = (4k + 3) + (2k + 2)$ .

- First, assume that  $k$  is odd and so  $k = 2t + 1$  for some integer  $t \geq 1$ . For each  $i$  with  $1 \leq i \leq t + 1$ , let  $Q_i$  be a 4-cycle in  $G$  defined by

$$Q_i = (v_{3(i-1)+1}, v_{3(i-1)+2}, v_{3(i-1)+3}, v_{3(i-1)+4}, v_{3(i-1)+1}). \quad (3.11)$$

In particular,  $Q_1 = (v_1, v_2, v_3, v_4, v_1)$  and

$$Q_{t+1} = (v_{3t+1}, v_{3t+2}, v_{3t+3}, v_{3t+4}, v_{3t+1}).$$

Note that all 4-cycles  $Q_i$  ( $1 \leq i \leq t + 1$ ) and  $C_n(n_3)$  are pairwise edge-disjoint.

Hence the subgraph consisting of  $C_n(n_3)$  and  $Q_i$  ( $1 \leq i \leq t + 1$ ) is even and has size  $n + 4(t + 1) = n + 2k + 2 = \lceil \frac{m}{2} \rceil$ .

- Next, assume that  $k$  is even and so  $k = 2t$  for some integer  $t \geq 1$ . For each  $i$  with  $1 \leq i \leq t$ , let  $Q_i$  be a 4-cycle in  $G$  described in (3.11) and let  $Q$  be a 6-cycle defined by

$$Q = (v_{3t+1}, v_{3t+2}, v_{3t+3}, v_{3t+6}, v_{3t+5}, v_{3t+4}, v_{3t+1}).$$

Then all 4-cycles  $Q_i$  ( $1 \leq i \leq t$ ),  $Q$  and  $C_n(n_3)$  are pairwise edge-disjoint. Hence the subgraph consisting of  $C_n(n_3)$ ,  $Q_i$  ( $1 \leq i \leq t$ ) and  $Q$  is even and size  $n + 4(t - 1) + 6 = n + 2k + 2 = \lceil \frac{m}{2} \rceil$ . ■

We now consider non-Eulerian circulants  $C_n(1, 3, n/2)$  where then  $n \geq 8$  is even.

**Theorem 3.4.10** *For each even integer  $n \geq 8$ , the graph  $C_n(1, 3, n/2)$  is optimal if and only if  $n \not\equiv 2 \pmod{8}$ . Furthermore, if  $n \equiv 2 \pmod{8}$ , then*

$$EI(C_n(1, 3, n/2)) = \binom{m+1}{2} + 1$$

where  $m = \frac{5n}{2}$  is the size of  $C_n(1, 3, n/2)$ .

**Proof.** Let  $n = 2k$  for some integer  $k \geq 4$  and let  $G = C_{2k}(1, 3, k)$ , where

$$C_{2k} = (v_1, v_2, \dots, v_{2k}, v_{2k+1} = v_1).$$

We consider two cases, according to whether  $k$  is odd or  $k$  is even.

*Case 1.  $k$  is odd.* We claim that  $G$  is a bipartite graph with partite sets

$$U = \{v_i : i \text{ is odd and } 1 \leq i \leq 2k - 1\}$$

$$W = \{v_i : i \text{ is even and } 2 \leq i \leq 2k\}.$$

To see this, let  $e \in E(G)$  be any edge of  $G$  and we show that  $e$  joins a vertex in  $U$  and a vertex in  $W$ . Assume, without loss of generality, that  $e = v_1v_t$  where then



$v_1 \in U$  and  $t = 2, 3, \dots, 2k$ . It then follows by the defining property of  $G$  that either  $t \in \{2, 2k\}$ , or  $t \equiv 1 \pm 3 \pmod{2k}$  or  $t = 1 + k$ . In each case,  $t$  is even and so  $v_t \in W$ . Therefore,  $G$  is a bipartite graph with partite sets  $U$  and  $W$ , as claimed.

First, we define two subgraphs  $C_6 \star C_4$  and  $C_4$  in  $G$  in a block of size 8 and size 4, respectively.

- (i) For the block  $[v_{i+1}, v_{i+8}] = (v_{i+1}, v_{i+2}, \dots, v_{i+8})$  of order 8 on  $C_{2k}$ , let  $C_6 \star C_4$  denote the subgraph of  $G[v_{i+1}, v_{i+8}]$  that consists of the 6-cycle  $(v_{i+1}, v_{i+2}, v_{i+5}, v_{i+8}, v_{i+7}, v_{i+4}, v_{i+1})$  and the 4-cycle  $C_4 = (v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+3})$ . The graph  $C_6 \star C_4$  is shown in Figure 3.13(a) where the edges of  $C_4$  are drawn in dashed lines. Hence  $C_6 \star C_4$  consists of two edge-disjoint cycles  $C_6$  and  $C_4$  and the size of  $C_6 \star C_4$  is 10.
- (ii) For the block  $[v_{i+1}, v_{i+4}] = (v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$  of order 4 on  $C_{2k}$ , let  $C_4 = (v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+1})$  be a 4-cycle in  $G[v_{i+1}, v_{i+4}]$ . Note that if  $B$  is a block of order  $4 + 3q$  for some nonnegative integer  $q$ , then the subgraph  $G[B]$  induced by the vertices of  $B$  contains  $q + 1$  edge-disjoint copies of the 4-cycle  $C_4$ .

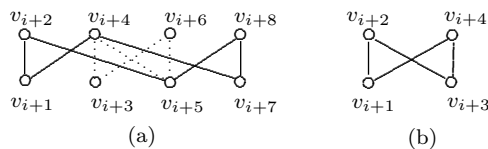


Figure 3.13: The subgraphs  $C_6 \star C_4$  and  $C_4$

Since  $k$  is odd, either  $k \equiv 1 \pmod{4}$  or  $k \equiv 3 \pmod{4}$ . We consider these two subcases.

*Subcase 1.1.*  $k \equiv 1 \pmod{4}$ . Since  $m = 5k \equiv 1 \pmod{4}$ , it follows by Lemma 3.2.1 that  $G$  is not optimal. Thus  $EI(G) \geq \binom{m+1}{2} + 1 = \binom{5k+1}{2} + 1$ . Next, we show that  $EI(G) \leq \binom{m+1}{2} + 1$ . Since  $m$  is odd, it suffices to show  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$ . Let  $k = 4s + 1$  for some positive integer  $s$  and so  $n = 2k = 8(s - 1) + 10$ . Then  $\frac{m-1}{2} = 10s + 2$ . Consider the  $s$  consecutive blocks  $B_1, B_2, \dots, B_s$  of  $C_n$  defined by  $B_i = [v_{7(i-1)+1}, v_{7i+1}]$  for  $1 \leq i \leq s - 1$  and  $s \geq 2$  and  $B_s = [v_{7s-6}, v_n]$  of  $C_{2k}$ . Then the order of  $B_i$  is 8 for  $1 \leq i \leq s - 1$  and the order of  $B_s$  is at least 10. For each  $i$  with  $1 \leq i \leq s - 1$ , let  $H_i = C_6 \star C_4$  be the subgraph of  $G[B_i]$  as defined in (i). Since the order of  $B_s$  is greater than 10, there are three edge-disjoint copies of  $C_4$  in  $G[B_s]$ , which we denoted by  $Q_1, Q_2$  and  $Q_3$  by (ii). Then the even subgraph  $H$  consisting of  $H_i$  ( $1 \leq i \leq s - 1$ ),  $Q_1, Q_2$  and  $Q_3$  has size  $10(s - 1) + 12 = 10s + 2$ .

*Subcase 1.2.*  $k \equiv 3 \pmod{4}$ . Let  $k = 4s + 3$  for some positive integer  $s$  and so  $n = 2k = 8s + 6$  and  $m = 5k = 20s + 15$ . We show that  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil = 10s + 8$ . Consider the  $s + 1$  consecutive blocks  $B_1, B_2, \dots, B_{s+1}$  where  $B_i = [v_{7(i-1)+1}, v_{7i+1}]$  for  $1 \leq i \leq s$  and  $B_{s+1} = [v_{7s+1}, v_n]$  of  $C_{2k}$ . Then the order of  $B_i$  is 8 for  $1 \leq i \leq s$  and the order of  $B_{s+1}$  is at least 7. For each  $i$  with  $1 \leq i \leq s$ , let  $H_i = C_6 \star C_4$  be the subgraph of  $G[B_i]$  as defined in (i). Since the order of  $B_{s+1}$  is at least 7, there are two edge-disjoint copies of  $C_4$  in  $G[B_{s+1}]$ , which we denoted by  $Q_1$  and  $Q_2$  by (ii). Then the even subgraph  $H$  consisting of  $H_i$  ( $1 \leq i \leq s$ ),  $Q_1$  and  $Q_2$  has size  $10s + 8$ .

Case 2.  $k$  is even. First, we define eight subgraphs  $A^*, B^*, C^*, D^*, E^*, F^*, G^*, H^*$  in  $G$ . To simplify notation, let  $B = (u_1, u_2, \dots, u_b) = (v_{i+1}, v_{i+2}, \dots, v_{i+b})$  be a block of order  $b \geq 4$  in  $C_n$  as shown in Figure 3.14.

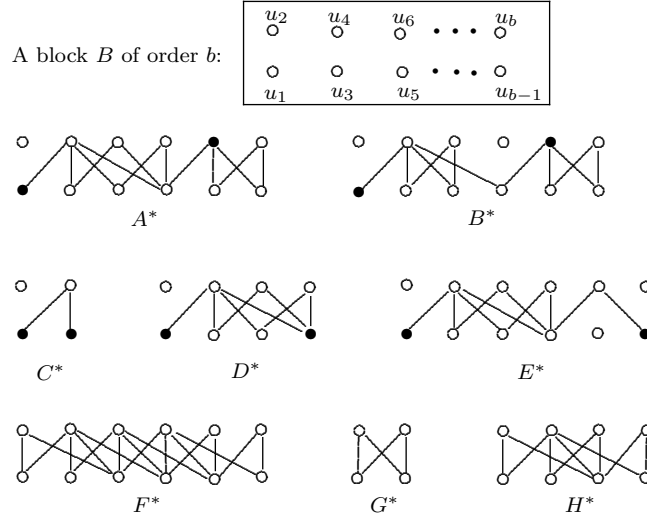


Figure 3.14: The eight subgraphs

- In a block  $B$  of order 12, let  $A^*$  be the subgraph of  $G[B]$  consisting of

$$C_6 = (u_3, u_4, u_5, u_8, u_7, u_6, u_3),$$

$$C_4 = (u_9, u_{10}, u_{11}, u_{12}, u_9)$$

and

$$P_4 = (u_1, u_4, u_7, u_{10}).$$

That is,  $E(A^*) = E(C_6) \cup E(C_4) \cup E(P_4)$ . There are two odd vertices in  $A^*$ , namely  $u_1$  and  $u_{10}$ . The vertex  $u_1$  is called the initial vertex of  $A^*$  and the vertex  $u_{10}$  is called the terminal vertex of  $A^*$ . Then the size of  $A^*$  is 13.

- In a block  $B$  of order 12, let  $B^*$  be the subgraph of  $G[B]$  consisting of

$$C_4=(u_3, u_4, u_5, u_6, u_3),$$

$$C_4 = (u_9, u_{10}, u_{11}, u_{12}, u_9)$$

and  $P_4 = (u_1, u_4, u_7, u_{10})$ . The two odd vertices in  $B^*$  are  $u_1$  and  $u_{10}$ . The vertex  $u_1$  is called the initial vertex of  $B^*$  and the vertex  $u_{10}$  is called the terminal vertex of  $B^*$ . Then the size of  $B^*$  is 11.

- In a block  $B$  of order 4, let  $C^* = P_3 = (u_1, u_4, u_3)$  The vertex  $u_1$  is called the initial vertex of  $C^*$  and the vertex  $u_3$  is called the terminal vertex of  $C^*$ . Then the size of  $C^*$  is 2.

- In a block  $B$  of order 8, let  $D^*$  be the subgraph of  $G[B]$  consisting of

$$C_6=(u_3, u_4, u_5, u_8, u_7, u_6, u_3) \text{ and } P_3 = (u_1, u_4, u_7).$$

The two odd vertices in  $D^*$  are  $u_1$  and  $u_7$ . The vertex  $u_1$  is called the initial vertex of  $D^*$  and the vertex  $u_7$  is called the terminal vertex of  $D^*$ . Then the size of  $D^*$  is 8.

- In a block  $B$  of order 12, let  $E^*$  be the subgraph of  $G[B]$  consisting of

$$C_6=(u_3, u_4, u_5, u_8, u_7, u_6, u_3) \text{ and } P_5 = (u_1, u_4, u_7, u_{10}, u_{11}).$$

The two odd vertices in  $E^*$  are  $u_1$  and  $u_{11}$ . The vertex  $u_1$  is called the initial vertex of  $E^*$  and the vertex  $u_{11}$  is called the terminal vertex of  $E^*$ . Then the size of  $E^*$  is 10.

- In a block  $B$  of order 12, let  $F^*$  be the subgraph of  $G[B]$  consisting of

$$C_4 = (u_3, u_4, u_5, u_6, u_3),$$

$$C_4 = (u_7, u_8, u_9, u_{10}, u_7) \text{ and}$$

$$C_{10} = (u_1, u_2, u_5, u_8, u_{11}, u_{12}, u_9, u_6, u_7, u_4, u_1).$$

Then  $F^*$  is an even subgraph of size 18.

- In a block  $B$  of order 4, let  $G^* = C_4 = (u_1, u_2, u_3, u_4, u_1)$ .
- In a block  $B$  of order 8, let  $H^*$  be the subgraph of  $G[B]$  consisting of

$$C_6 = (u_1, u_2, u_5, u_8, u_7, u_4, u_1) \text{ and } C_4 = (u_3, u_4, u_5, u_6, u_3).$$

Then  $H^*$  is an even subgraph of size 10.

These eight subgraphs are shown in Figure 3.14 where the odd vertices in each subgraph are drawn in solid vertices. Note that the subgraphs  $G^*$  and  $H^*$  are also shown in Figure 3.13 and we include these two subgraphs in Figure 3.14 for completion.

Since either  $k \equiv 0 \pmod{4}$  or  $k \equiv 2 \pmod{4}$ , we consider these two subcases.

*Subcase 2.1.*  $k \equiv 0 \pmod{4}$ . Let  $k = 4s$  for some positive integer  $s$  and so  $n = 2k = 8s$  and  $m = 5k = 20s$ . We show that  $G$  contains an even subgraph of size  $\lceil \frac{m}{2} \rceil = 10s$ . Partition  $C_n$  into  $s$  blocks  $B_1, B_2, \dots, B_s$ , each of which has order 8. For each  $i$  with  $1 \leq i \leq s$ , let  $H_i = H^*$  be the subgraph of  $G[B_i]$ , each

of which has size 10. Then the even subgraph consisting of  $H_i$  ( $1 \leq i \leq s$ ) has size  $10s$ .

*Subcase 2.2.*  $k \equiv 2 \pmod{4}$ . Let  $k = 4s + 2$  for some positive integer  $s$  and so  $n = 2k = 8s + 4$  and  $m = 5k = 20s + 10$ . Suppose that  $s \equiv i \pmod{3}$  for  $i = 0, 1, 2$ , where then  $s = 3t + i$  for  $i = 0, 1, 2$  for some nonnegative integer  $t$ . First, suppose that  $t = 0$ . Then  $i = 1, 2$ .

- If  $i = 1$ , then  $s = 1$  and so  $n = 12$  and  $m = 30$ . Hence  $\lceil \frac{m}{2} \rceil = 15$ . Consider the block  $B = [v_7, v_2]$  and let  $H^*$  be the subgraph of  $G[B]$  of size 10 in  $B$ , where  $H^*$  is shown in Figure 3.14. Let  $H$  be the subgraph consisting of  $C_5 = (v_1, v_4, v_5, v_6, v_7, v_1)$  and  $H^*$ . Then  $H$  is an even subgraph of  $G$  and has size 15, as shown in Figure 3.15(a).

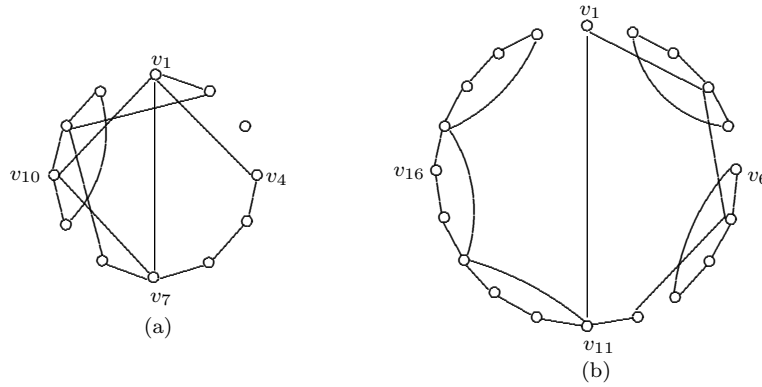


Figure 3.15: Two subgraphs in the proof of Subcase 2.2

- If  $i = 2$ , then  $s = 2$  and so  $n = 20$  and  $m = 50$ . Hence  $\lceil \frac{m}{2} \rceil = 25$ . Let  $B_1 = [v_2, v_5]$ ,  $B_2 = [v_6, v_9]$ ,  $B_3 = [v_{11}, v_{14}]$ ,  $B_4 = [v_{14}, v_{17}]$  and  $B_5 = [v_{17}, v_{20}]$ . For each  $i$  with  $1 \leq i \leq 5$ , let  $H_i = C_4$  in  $G[B_i]$ . The even subgraph  $H$  of

$G$  consists of  $H_i$  ( $1 \leq i \leq 5$ ) and  $C_5 = (v_1, v_4, v_7, v_{10}, v_{11}, v_1)$  has size 25, as shown Figure 3.15(b).

Thus, we may assume that  $t \geq 1$ . We consider three subcases.

*Subcase 2.2.1.*  $s = 3t$  for some integer  $t \geq 1$ . In terms of  $t$  then,  $k = 12t + 2$ ,  $n = 2k = 24t + 4$  and  $\lceil \frac{m}{2} \rceil = 30t + 5$ . Partition  $C_n$  into  $2t + 1$  consecutive blocks  $B_1, B_2, \dots, B_{2t+1}$ , where  $|B_i| = 12$  for  $1 \leq i \leq 2t + 1$  and  $i \neq t + 1$  and  $|B_{t+1}| = 4$ . For each  $i$  with  $1 \leq i \leq 2t + 1$ , let  $H_i$  be the subgraph of  $G[B_i]$  defined by

$$H_i \cong \begin{cases} A^* & \text{if } i = 1 \\ B^* & \text{if } 2 \leq i \leq t \\ C^* & \text{if } i = t + 1 \\ F^* & \text{if } t + 2 \leq i \leq 2t + 1. \end{cases}$$

Let  $H$  be the subgraph consisting of  $H_1, H_2, \dots, H_{2t+1}$  by joining (1) the terminal vertex of  $H_i$  to the initial vertex of  $H_{i+1}$  for  $1 \leq i \leq t$  and (2) the terminal vertex of  $H_{t+1}$  to the initial vertex of  $H_1$ . Then  $H$  is an even subgraph of  $G$  and has size  $30t + 5$ .

*Subcase 2.2.2.*  $s = 3t + 1$  for some integer  $t \geq 1$ . In terms of  $t$  then,  $k = 12t + 6$ ,  $n = 2k = 24t + 12$  and  $\lceil \frac{m}{2} \rceil = 30t + 15$ . Partition  $C_n$  into  $2t + 2$  consecutive blocks  $B_1, B_2, \dots, B_{2t+2}$ , where  $|B_i| = 12$  for  $1 \leq i \leq t$ ,  $|B_{t+1}| = 8$ ,  $|B_i| = 12$  for  $t + 2 \leq i \leq 2t + 1$  and  $|B_{2t+2}| = 4$ . For each  $i$  with  $1 \leq i \leq 2t + 2$ , let  $H_i$  be the

subgraph of  $G[B_i]$  defined by

$$H_i \cong \begin{cases} A^* & \text{if } i = 1 \\ B^* & \text{if } 2 \leq i \leq t \\ D^* & \text{if } i = t + 1 \\ F^* & \text{if } t + 2 \leq i \leq 2t + 1 \\ G^* & \text{if } t = 2t + 2. \end{cases}$$

Let  $H$  be the subgraph consisting of  $H_1, H_2, \dots, H_{2t+2}$  by joining (1) the terminal vertex of  $H_i$  to the initial vertex of  $H_{i+1}$  for  $1 \leq i \leq t$  and (2) the terminal vertex of  $H_{t+1}$  to the initial vertex of  $H_1$ . Then  $H$  is an even subgraph of  $G$  and has size  $30t + 15$ .

*Subcase 2.2.3.*  $s = 3t + 2$  for some integer  $t \geq 1$ . In terms of  $t$  then,  $k = 12t + 10$ ,  $n = 2k = 24t + 20$  and  $\lceil \frac{m}{2} \rceil = 30t + 25$ . Partition  $C_n$  into  $2t + 1$  consecutive blocks  $B_1, B_2, \dots, B_{2t+1}$ , where  $|B_i| = 12$  for  $1 \leq i \leq 2t + 1$  and  $|B_{2t+2}| = 8$ . For each  $i$  with  $1 \leq i \leq 2t + 2$ , let  $H_i$  be the subgraph of  $G[B_i]$  defined by

$$H_i \cong \begin{cases} A^* & \text{if } i = 1, 2 \\ B^* & \text{if } 3 \leq i \leq t \\ E^* & \text{if } i = t + 1 \\ F^* & \text{if } t + 2 \leq i \leq 2t + 1 \\ H^* & \text{if } t = 2t + 2 \end{cases}$$

Let  $H$  be the subgraph consisting of  $H_1, H_2, \dots, H_{2t+2}$  by joining (1) the terminal vertex of  $H_i$  to the initial vertex of  $H_{i+1}$  for  $1 \leq i \leq t$  and (2) the terminal vertex of  $H_{t+1}$  to the initial vertex of  $H_1$ . Then  $H$  is an even subgraph of  $G$  and has size  $30t + 25$ . ■



The following is a consequence of Theorem 3.4.1, Proposition 3.4.2 and Theorems 3.4.9 and 3.4.10.

**Corollary 3.4.11** *Let  $G = C_n(1, 3, n_3, n_4, \dots, n_k)$  where*

$$4 \leq n_3 < n_4 < \dots < n_k \leq n/2 \text{ and } k \geq 3.$$

- *For  $n_k < n/2$ , if  $n \not\equiv 2 \pmod{4}$  or  $n_i$  is even for some  $i$  with  $3 \leq i \leq k$ , then  $G$  is optimal.*
- *For  $n_k = n/2$ , if  $n \not\equiv 2 \pmod{8}$ , then  $G$  is optimal.*

**Circulants  $C_n(2, 3, n_3)$**

As with 3-circulants  $C_n(1, 2, n_3)$ , all 3-circulants  $C_n(2, 3, n_3)$  are optimal for each  $n_3 \geq 4$ .

**Theorem 3.4.12** *For each integer  $n \geq 8$ , the graph  $C_n(2, 3, n_3)$  is optimal for all integers  $n_3$  with  $4 \leq n_3 \leq n/2$ .*

**Proof.** Let  $G = C_n(2, 3, n_3)$  where  $C_n = (v_1, v_2, \dots, v_n, v_{n+1} = v_1)$ . We consider two cases, according to whether  $n_3 < n/2$  or  $n_3 = n/2$ .

*Case 1.*  $4 \leq n_3 < n/2$ . If  $n$  is even, then  $G$  is optimal by Proposition 3.4.4. Thus, we may assume that  $n$  is odd. Let  $n = 2t + 1$  for some integer  $t \geq 4$ . Then the size of  $G$  is  $m = 3n = 6t + 3$  and so  $\lceil \frac{m}{2} \rceil = 3t + 2 = (2t + 1) + (t + 1)$ . First, we make an observation. Let  $B = (u_1, u_2, \dots, u_b)$  be a block of order  $b$  (that is,  $u_1, u_2, \dots, u_b$  are  $b$  consecutive vertices in  $C_n$ ), which is shown in Figure 3.14.

(i) If  $b \geq 8$ , then  $G[B]$  contains a  $k$ -cycle for  $k = 4, 5$  using only edges in  $C_n(2, 3)$ . For example,

$$C_4 = (u_1, u_3, u_6, u_4, u_1) \text{ and } C_5 = (u_1, u_3, u_5, u_7, u_4, u_1).$$

(ii) if  $b \geq 10$ , then  $G[B]$  contains a  $k$ -cycle for  $k = 6, 7$  or  $2C_4$  using only edges in  $C_n(2, 3)$ . For example,

$$C_6 = (u_1, u_3, u_6, u_9, u_7, u_4, u_1), C_7 = (u_1, u_3, u_5, u_7, u_9, u_6, u_4, u_1)$$

and  $2C_4$  consists of

$$(u_1, u_3, u_6, u_4, u_1) \text{ and } (u_5, u_7, u_{10}, u_8, u_5).$$

Let  $t = 4s + r$  for some positive integer  $s$  where  $r = 0, 1, 2, 3$ . Then  $n = 8s + 2r + 1$  and  $t + 1 = 4s + r + 1$ . Partition  $C_n$  into  $s$  consecutive blocks  $B_0, B_1, \dots, B_{s-1}$  defined by  $B_i = [v_{8i+1}, v_{8(i+1)}]$  for  $0 \leq i \leq s-2$  and  $s \geq 2$  and  $B_{s-1} = [v_{8(s-1)+1}, v_n]$ . By (i), the subgraph  $G[B_i]$  induced by  $B_i$  contains  $C_4$  as a subgraph using only edges in  $C_n(2, 3)$  and if  $r = 0$ , then  $|B_{s-1}| = 9$  and so  $G[B_{s-1}]$  contains  $C_5$  as a subgraph using only edges in  $C_n(2, 3)$ . By (ii), if  $r \neq 0$ , then  $G[B_{s-1}]$  contains  $C_6, C_7$  or  $2C_4$  as a subgraph using only edges in  $C_n(2, 3)$ . We now define a subgraph  $H_i$  of  $G[B_i]$  using only edges in  $C_n(2, 3)$  define as follows: For each  $i$  with  $0 \leq i \leq s-2$ , let  $H_i \cong C_4$  (when  $s \geq 2$ ) and

$$H_{s-1} \cong \begin{cases} C_5 & \text{if } r = 0 \\ C_6 & \text{if } r = 1 \\ C_7 & \text{if } r = 2 \\ 2C_4 & \text{if } r = 3. \end{cases}$$

Hence all subgraphs  $H_i$  ( $0 \leq i \leq s-1$ ) and  $C_n(n_3)$  are pairwise edge-disjoint. Therefore, the subgraph consisting of  $H_i$  ( $0 \leq i \leq s-1$ ) and  $C_n(n_3)$  is an even subgraph of size  $(2t+1) + (t+1) = \lceil \frac{m}{2} \rceil$ .

*Case 2.*  $n_3 = n/2$ . Then  $n$  is even and so  $n = 2t$  for some integer  $t \geq 4$ . Let  $t = 4s + r$  for some positive integer  $s$  where  $r = 0, 1, 2, 3$ . Then  $n = 8s + 2r$  and  $m = 20s + 5r$ . Hence

$$\lceil \frac{m}{2} \rceil = \begin{cases} 10s & \text{if } r = 0 \\ 10s + 3 & \text{if } r = 1 \\ 10s + 5 & \text{if } r = 2 \\ 10s + 8 & \text{if } r = 3. \end{cases} \quad (3.12)$$

First, we make an observation. Let  $B = (u_1, u_2, \dots, u_b)$  be a block of order  $b$  as shown in Figure 3.14.

- (i) If  $b \geq 8$ , then  $G[B]$  contains an even subgraph  $F$  of size 10 using only edges in  $C_n(2, 3)$ . For example, let  $F$  consist of two edge-disjoint 5-cycles

$$(u_1, u_3, u_5, u_7, u_4, u_1) \text{ and } (u_2, u_4, u_6, u_8, u_5, u_2).$$

- (ii) If  $b \geq 10$ , then  $G[B]$  contains an even subgraph  $X$  of size 13 using only edges in  $C_n(2, 3)$ . For example, let  $X$  consist of the 7-cycle

$$(u_1, u_3, u_5, u_7, u_9, u_6, u_4, u_1)$$

and the 6-cycle

$$(u_2, u_4, u_7, u_{10}, u_8, u_5, u_2).$$

(iii) If  $b \geq 12$ , then  $G[B]$  contains an even subgraph  $Y$  of size 15 using only edges in  $C_n(2, 3)$ . For example, let  $Y$  consist of the 9-cycle

$$(u_1, u_3, u_5, u_7, u_9, u_{11}, u_8, u_6, u_4, u_1)$$

and the 6-cycle

$$(u_2, u_4, u_7, u_{10}, u_8, u_5, u_2)$$

as defined in (ii).

(iv) If  $b \geq 14$ , then  $G[B]$  contains an even subgraph  $Z$  of size 18 using only edges in  $C_n(2, 3)$ . For example, let  $Z$  consist of the even subgraph  $F$  defined in (i) and the two edge-disjoint 4-cycles

$$(u_7, u_9, u_{12}, u_{10}, u_7) \text{ and } (u_8, u_{11}, u_{13}, u_{10}, u_8).$$

These four subgraphs  $F, X, Y, Z$  are shown in Figure 3.16.

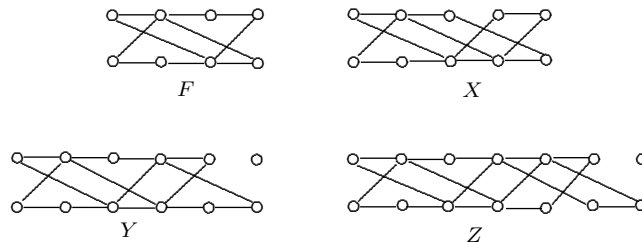


Figure 3.16: The four subgraphs  $F, X, Y, Z$

Partition  $C_n$  into  $s$  consecutive blocks  $B_0, B_1, \dots, B_{s-1}$  defined by  $B_i = [v_{8i+1}, v_{8(i+1)}]$  for  $0 \leq i \leq s-2$  and  $s \geq 2$  and  $B_{s-1} = [v_{8(s-1)+1}, v_n]$ . By (i), the subgraph  $G[B_i]$

induced by  $B_i$  contains an even subgraph  $F$  of size 10 using only edges in  $C_n(2, 3)$ . For each  $i$  with  $0 \leq i \leq s - 2$  and  $s \geq 2$ , let  $H_i \cong F$  as defined in (i). For  $r = 0$ , let  $H_{s-1} \cong F$  as defined in (i); for  $r = 1$ , let  $H_{s-1} \cong X$  as defined in (ii); for  $r = 2$ , let  $H_{s-1} \cong Y$  as defined in (iii) and for  $r = 3$ , let  $H_{s-1} \cong Z$  as defined in (iv). Therefore, the subgraph consisting of  $H_i$  ( $0 \leq i \leq s - 1$ ) is an even subgraph of size  $\lceil \frac{m}{2} \rceil$  as described in (3.12). ■

The following are consequences of Theorem 3.4.1, Proposition 3.4.2 and Theorem 3.4.12.

**Corollary 3.4.13** *For each integer  $n \geq 8$ , the graph  $C_n(2, 3, n_3, n_4, \dots, n_k)$  is optimal for all  $4 \leq n_3 < n_4 < \dots < n_k \leq n/2$ .*

### 3.4.5 Summary

We now summarize what we have obtained. Let  $G = C_n(n_1, n_2, \dots, n_k)$  where  $n_1, n_2, \dots, n_k$  are  $k$  distinct integers with  $1 \leq n_i \leq n/2$  for  $1 \leq i \leq k$ .

#### I. $k \geq 2$ is even and $n \geq 6$

- The Eulerian circulant  $G$  is optimal if  $1 \leq n_i < n/2$  for all  $i$  with  $1 \leq i \leq k$ .
- The non-Eulerian circulant  $C_n(1, n/2)$  is optimal if and only if  $n \not\equiv 6 \pmod{8}$ .
- If  $n \not\equiv 6 \pmod{8}$ , then the non-Eulerian circulant  $C_n(1, n_2, n_3, \dots, n_k)$  is optimal for all  $k \geq 2$  and  $2 \leq n_2 < n_3 < \dots < n_{k-1} < n_k = n/2$ . Note that the converse of this statement is not true. For example, the non-Eulerian circulant  $C_{22}(1, 3, 11)$  is optimal by Theorem 3.4.10, while  $n = 22 \equiv 6 \pmod{8}$ .

**II.  $k \geq 3$  is odd and  $n \geq 8$**

- If there exist  $n_r, n_s, n_t \in \{n_1, n_2, \dots, n_k\}$  such that  $C_n(n_r, n_s, n_t)$  is optimal and  $n_i < n/2$  for each  $i \in \{1, 2, \dots, k\} - \{r, s, t\}$ , then  $G$  is optimal.
- The graph  $C_n(1, 2, n_3, n_4, \dots, n_k)$  is optimal for all  $3 \leq n_3 < n_4 < \dots < n_k \leq n/2$ .
- The graph  $C_n(2, 3, n_3, n_4, \dots, n_k)$  is optimal for all  $4 \leq n_3 < n_4 < \dots < n_k \leq n/2$ .
- The graph  $C_n(1, 3, n_3)$  ( $n_3 < n/2$ ) is optimal if and only if  $n \not\equiv 2 \pmod{4}$  or  $n_3$  is even; while  $C_n(1, 3, n/2)$  is optimal if and only if  $n \not\equiv 2 \pmod{8}$ .

# Chapter 4

## Decompositions in Graphs

### 4.1 Trail, Circuit and Cycle Decompositions

Recall that in his famous 1736 paper [15], Leonhard Euler not only solved the Königsberg Bridge Problem, he described a generalization of this problem and solved this problem as well (although the proof was only completed in 1873 in a paper by Hierholzer [16]). This led to the class of graphs called *Eulerian graphs*, namely those graphs containing an Eulerian circuit. In terms of graphs, what Euler proved was that a connected graph  $G$  is Eulerian if and only if every vertex of  $G$  is even. Euler also showed that a connected graph  $G$  has an Eulerian trail if and only if  $G$  contains exactly two odd vertices, in which case any Eulerian trail begins at one of the odd vertices and ends at the other. Therefore, if a connected graph  $G$  has more than two odd vertices, then  $G$  has neither an Eulerian circuit nor an Eulerian trail. On the other hand, by Theorem 1.4.4, if  $G$  is a connected graph containing  $2k$  odd vertices for some positive integer  $k$ , then  $G$  can be decomposed into  $k$  open trails but no fewer. In 1973, Chartrand, Polimeni and Stewart [12]

proved the following.

**Theorem 4.1.1** *If  $G$  is a connected graph containing  $2k$  odd vertices for some positive integer  $k$ , then  $G$  can be decomposed into  $k$  open trails, at most one of which has odd length.*

We now present a generalization of Theorem 4.1.1. In order to do this, we present an additional definition. The *distance between two subgraphs  $F$  and  $H$*  in a connected graph  $G$  is

$$d(F, H) = \min\{d(u, v) : u \in V(F), v \in V(H)\}.$$

**Theorem 4.1.2** *Let  $G$  be a connected graph of size  $m$  containing  $2k$  odd vertices ( $k \geq 1$ ). Among all decompositions of  $G$  into  $k$  open trails, let  $s$  be the maximum number of such trails of odd length.*

- (a) *If  $m$  is even, then  $s$  is even and for every even integer  $a$  such that  $0 \leq a \leq s$ , there exists a decomposition of  $G$  into  $k$  open trails, exactly  $a$  of which have odd length.*
- (b) *If  $m$  is odd, then  $s$  is odd and for every odd integer  $b$  such that  $1 \leq b \leq s$ , there exists a decomposition of  $G$  into  $k$  open trails, exactly  $b$  of which have odd length.*

**Proof.** We only verify (a) as the proof of (b) is similar. Since the size of  $G$  is even,  $s$  is even and  $0 \leq s \leq k$ . If  $s = 0$ , then the result is true trivially. Thus we may assume that  $s \geq 2$ . It suffices to show that there exists a decomposition of  $G$  into



$k$  open trails, exactly  $s - 2$  of which have odd length. Among all decompositions of  $G$  into  $k$  open trails, consider those decompositions containing exactly  $s$  trails of odd length; and, among those, consider one, say  $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ , where the distance between some pair  $T_i, T_j$  of trails of odd length is minimum. We may assume that  $T_r$  is a  $u_r - v_r$  trail for  $1 \leq r \leq k$ . We claim that this minimum distance is 0. Assume that this is not the case. Suppose that  $P$  is a path of minimum length connecting a vertex  $w_i$  in  $T_i$  and a vertex  $w_j$  in  $T_j$ , and let  $w_i x$  be the edge of  $P$  incident with  $w_i$  (where it is possible that  $x = w_j$ ). Then  $w_i x$  belongs to a trail  $T_p$  among  $T_1, T_2, \dots, T_k$ . Necessarily,  $T_p$  has even length, for otherwise, the distance between  $T_i$  and  $T_p$  is 0, producing a contradiction. Since  $T_i$  and  $T_p$  have the vertex  $w_i$  in common,  $T_i$  and  $T_p$  may be replaced by trails  $T_i^*$  and  $T_p^*$  connecting odd vertices such that  $T_i^*$  has even length,  $T_p^*$  has odd length,

$$E(T_i) \cup E(T_p) = E(T_i^*) \cup E(T_p^*)$$

and  $w_i x$  belongs to  $T_p^*$ . Hence the distance between  $T_j$  and  $T_p^*$  is smaller than the distance between  $T_i$  and  $T_j$ , a contradiction. Thus, as claimed, the distance between  $T_i$  and  $T_j$  is 0 and so these two trails have a vertex  $w$  in common. Either the  $u_i - w$  subtrail  $T_i'$  or the  $w - v_i$  subtrail  $T_i''$  has odd length, say the former. We may also assume that the  $u_j - w$  subtrail  $T_j'$  of  $T_j$  has odd length and the  $w - v_j$  subtrail  $T_j''$  has even length. Then the  $u_i - u_j$  trail  $T_{ij}'$  formed from  $T_i'$  and  $T_j'$  and the  $v_i - v_j$  trail  $T_{ij}''$  formed from  $T_i''$  and  $T_j''$  both have even length. Then  $(\mathcal{T} - \{T_i, T_j\}) \cup \{T_{ij}', T_{ij}''\}$  is a decomposition of  $G$  into  $k$  open trails, exactly  $s - 2$  of which have odd length. ■

Theorem 4.1.1 is then a consequence of Theorem 4.1.2. In [10] a circuit decomposition theorem (similar to Theorem 4.1.2) on an Eulerian graph was established, as we state next. For completion, we also include a proof of this theorem presented in [10].

**Theorem 4.1.3** *For an Eulerian graph  $G$  of size  $m$ , let  $s$  be the maximum number of circuits of odd length in a circuit decomposition of  $G$ .*

- (a) *If  $m$  is even, then  $s$  is even and for every even integer  $a$  such that  $0 \leq a \leq s$ , there exists a circuit decomposition of  $G$ , exactly  $a$  of which have odd length.*
- (b) *If  $m$  is odd, then  $s$  is odd and for every odd integer  $b$  such that  $1 \leq b \leq s$ , there exists a circuit decomposition of  $G$ , exactly  $b$  of which have odd length.*

**Proof.** We only verify (a) as the proof of (b) is similar. Since the size of  $G$  is even,  $s$  is even. If  $s = 0$ , then the result is true trivially. Thus we may assume that  $s \geq 2$ . It suffices to show that there exists a circuit decomposition of  $G$ , exactly  $s - 2$  of which have odd length. Among all circuit decompositions of  $G$ , consider those circuit decompositions containing exactly  $s$  circuits of odd length; and, among those, consider one, say  $\mathcal{D} = \{C_1, C_2, \dots, C_k\}$  for some positive integer  $k$ , where the distance between some pair  $C_i, C_j$  of circuits of odd length is minimum. We claim that this minimum distance is 0. Assume that this is not the case. Suppose that  $P$  is a path of minimum length connecting a vertex  $w_i$  in  $C_i$  and a vertex  $w_j$  in  $C_j$ , and let  $w_i x$  be the edge of  $P$  incident with  $w_i$  (where it is possible that  $x = w_j$ ). Then  $w_i x$  belongs to a circuit  $C_p$  among  $C_1, C_2, \dots, C_k$ . Necessarily,  $C_p$  has even length,

for otherwise, the distance between  $C_i$  and  $C_p$  is 0, producing a contradiction. Since  $C_i$  and  $C_p$  have the vertex  $w_i$  in common,  $C_i$  and  $C_p$  may be replaced by the circuit  $C'$  consisting of  $C_i$  and  $C_p$  (that is,  $E(C') = E(C_i) \cup E(C_p)$ ) and  $C'$  has odd length. However then, the circuit decomposition  $\mathcal{D}' = (\{C_1, C_2, \dots, C_k\} - \{C_i, C_p\}) \cup \{C'\}$  has exactly  $s$  circuits of odd length and the distance between  $C_j$  and  $C'$  in  $\mathcal{D}'$  is smaller than the distance between  $C_i$  and  $C_j$  in  $\mathcal{D}$ , which contradicts the defining property of  $\mathcal{D}$ . Thus, as claimed, the distance between  $C_i$  and  $C_j$  is 0 and so  $C_i$  and  $C_j$  have a vertex in common. Hence the circuit  $C^*$  consisting of  $C_i$  and  $C_j$  has even length. Then  $(\{C_1, C_2, \dots, C_k\} - \{C_i, C_j\}) \cup \{C^*\}$  is a circuit decomposition of  $G$ , exactly  $s - 2$  of which have odd length. ■

In 1912 Oswald Veblen [21], one of the important early figures in topology, presented another characterization of Eulerian graphs (Theorem 1.4.2) which we restate as follows.

**Veblen's Theorem** *A connected graph  $G$  is Eulerian if and only if  $G$  has a cycle decomposition.*

Over the years there has been a host of theorems and conjectures dealing with the characteristics of cycles in a cycle decomposition of Eulerian graphs. Certainly one of the best known is due to Thomas Kirkman [17] and concerns decompositions of Eulerian complete graphs into 3-cycles (Steiner triple systems).

**Kirkman's Theorem** *The complete graph  $K_n$  with  $n \geq 3$  has a  $C_3$ -decomposition if and only if  $n \equiv 1 \pmod{6}$  or  $n \equiv 3 \pmod{6}$ .*

At the other extreme is the following theorem by Walecki (see [3]).

**Walecki's Theorem** *The complete graph  $K_n$  with  $n \geq 3$  is  $C_n$ -decomposable (Hamiltonian factorable) if and only if  $n$  is odd.*

A well-known conjecture in this area was made by Brian Alspach [2] in 1981.

**Alspach's Conjecture** *Suppose that  $n \geq 3$  is an odd integer and that  $m_1, m_2, \dots, m_t$  are integers such that  $3 \leq m_i \leq n$  for each  $i$  ( $1 \leq i \leq t$ ) and  $m_1 + m_2 + \dots + m_t = \binom{n}{2}$ . Then  $K_n$  can be decomposed into the cycles  $C_{m_1}, C_{m_2}, \dots, C_{m_t}$ . Furthermore, for every even integer  $m \geq 4$  and integers  $m_1, m_2, \dots, m_t$  such that  $3 \leq m_i \leq n$  for each  $i$  ( $1 \leq i \leq t$ ) with  $m_1 + m_2 + \dots + m_t = (n^2 - 2n)/2$ , there is a decomposition of  $K_n$  into a 1-factor and the cycles  $C_{m_1}, C_{m_2}, \dots, C_{m_t}$ .*

Alspach's Conjecture was verified in its entirety by Bryant, Horsley and Pettersson [7] in 2012. The results mentioned above led to another conjecture involving cycle decompositions of Eulerian graphs, introduced in [10].

**The Eulerian Cycle Decomposition Conjecture (ECDC)** *Let  $G$  be an Eulerian graph of size  $m$ , where  $a$  is the minimum number of odd cycles in a cycle decomposition of  $G$  and  $b$  is the maximum number of odd cycles in a cycle decomposition of  $G$ . For every integer  $\ell$  such that  $a \leq \ell \leq b$  and  $\ell$  and  $m$  are of the same parity, there exists a cycle decomposition of  $G$  containing exactly  $\ell$  odd cycles.*

It is therefore a consequence of the theorem by Bryant, Horsley and Pettersson that the ECDC is true for all Eulerian complete graphs. This observation was made in [10]. For positive integers  $n \geq 2$  and  $r$ , let  $K_{n(r)}$  be the regular complete

$n$ -partite graph each of whose partite sets consists of  $r$  vertices. Since removing the edges of a 1-factor from  $K_{2n}$ , where  $n \geq 2$ , produces the graph  $K_{n(2)}$ , it follows that the ECDC holds for  $K_{n(2)}$  for every integer  $n \geq 2$ . This conjecture was verified in [10] for Eulerian  $k$ th powers of cycles for  $k = 2, 3, 4$  and all Eulerian complete 3-partite graphs.

In this work we study, for graphs  $G$  and  $H$ , decompositions of  $G$  into  $k + 1 \geq 1$  subgraphs,  $k$  of which are isomorphic to  $H$  and where the remaining subgraph contains no subgraph isomorphic to  $H$ .

## 4.2 On $H$ -Maximal Decompositions

A graph  $H$  is said to *divide* a graph  $G$ , often expressed by writing  $H \mid G$ , if  $G$  is  $H$ -decomposable, that is, if  $G$  has a decomposition  $\{H_1, H_2, \dots, H_k\}$ , where  $H_i \cong H$  for  $i = 1, 2, \dots, k$ . If  $G$  has size  $m$ ,  $H$  has size  $m'$  and  $H \mid G$ , then certainly  $m' \mid m$ . On the other hand, if  $H \nmid G$ , then either  $G$  does not contain a subgraph isomorphic to  $H$  or  $G$  contains a decomposition of  $\mathcal{D} = \{H_1, H_2, \dots, H_k, R\}$  where  $H_i \cong H$  for each  $i$  ( $1 \leq i \leq k$ ) and  $R$  is a nonempty subgraph of  $G$  containing no subgraph isomorphic to  $H$ . The subgraph  $R$  may be referred to as the *remainder subgraph* for this decomposition. This observation may be considered as a graph theory analogue of the famous Division Algorithm for integers, where if the positive integer  $b$  is divided by the positive integer  $a$ , then there exist integers  $q$  and  $r$  with  $0 \leq r < a$  such that  $b = aq + r$ . Unlike the Division Algorithm for integers where  $q$  and  $r$  are unique, in this so-called Division Algorithm for graphs  $G$  and

$H$ , resulting in a decomposition  $\mathcal{D}$  (above) of  $G$  in terms of  $H$ , the integer  $k$  and remainder graph  $R$  need not be unique. This observation leads us to the subject of this chapter, namely that of determining all graphs  $H$  such that for every graph  $G$  the integers  $k$  in such decompositions consist of a set of consecutive integers.

For two graphs  $H$  and  $G$ , a decomposition  $\mathcal{D} = \{H_1, H_2, \dots, H_k, R\}$  of  $G$  is called *H-maximal* or an *H-maximal k-decomposition* if  $H_i \cong H$  for  $1 \leq i \leq k$  and  $R$  contains no subgraph isomorphic to  $H$ . If  $G$  contains no subgraph isomorphic to  $H$ , then  $k = 0$  and  $R = G$ . For graphs  $H$  and  $G$ , let

$$\text{Min}(G, H) = \min\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}$$

$$\text{Max}(G, H) = \max\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}.$$

Obviously,  $\text{Min}(G, H) \leq \text{Max}(G, H)$ . Throughout this section, we assume that  $H$  is a graph without isolated vertices. A graph  $H$  is said to possess the *intermediate decomposition property* (IDP) and  $H$  is called an *ID-graph* if for each graph  $G$  and each integer  $k$  with  $\text{Min}(G, H) \leq k \leq \text{Max}(G, H)$ , there exists an  $H$ -maximal  $k$ -decomposition of  $G$ . Trivially, the graph  $K_2$  is an ID-graph. On the other hand, neither the claw  $K_{1,3}$  nor the triangle  $K_3$  is an ID-graph. For example, the graph  $G$  of Figure 4.1 has a  $K_{1,3}$ -maximal 1-decomposition and a  $K_{1,3}$ -maximal 3-decomposition but has no  $K_{1,3}$ -maximal 2-decomposition. Similarly, the graph  $F$  of Figure 4.1 has a  $K_3$ -maximal 1-decomposition and a  $K_3$ -maximal 3-decomposition but has no  $K_3$ -maximal 2-decomposition.

These observations lead to the following problem.

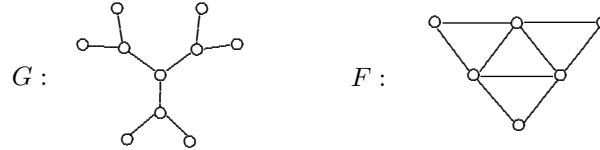


Figure 4.1: Illustrating that  $K_{1,3}$  and  $K_3$  are not ID-graphs

### The Intermediate Value Problem for $H$ -Maximal Decompositions

*Which graphs (without isolated vertices) are ID-graphs?*

We first show that each of the two graphs of size 2 (shown in Figure 4.2), namely the path  $P_3$  of order 3 and  $2K_2$  consisting of two components of order 2, is an ID-graph.

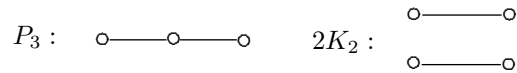


Figure 4.2: The two graphs of size 2

Recall the following result of Chartrand, Polimeni and Stewart [12] dealing with paths  $P_3$  of order 3 will be useful to us (also see Theorem 1.3.1).

**Theorem 4.2.1** *A nontrivial connected graph  $G$  is  $P_3$ -decomposable if and only if  $G$  has even size.*

For two edges  $e = u_1u_2$  and  $f = v_1v_2$  in a nontrivial connected graph  $G$ , an  $e - f$  path has  $e$  as its initial edge and  $f$  as its terminal edge. The *distance*  $d(e, f)$  between  $e$  and  $f$  is defined as

$$\min\{d(u, v) : u \in \{u_1, u_2\} \text{ and } v \in \{v_1, v_2\}\}.$$

**Proposition 4.2.2** *The graph  $P_3$  is an ID-graph.*

**Proof.** Let  $a = \text{Min}(G, P_3)$  and suppose that  $\mathcal{D}_a = \{H_1, H_2, \dots, H_a, R\}$  is a  $P_3$ -maximal  $a$ -decomposition of  $G$  where  $H_i = P_3$  for  $1 \leq i \leq a$  and  $R = rK_2$  for some integer  $r \geq 0$ . If  $r = 0$  or  $r = 1$ , then  $\text{Max}(G, P_3) = a$  and the result follows. Thus, we may assume that  $r \geq 2$  and show that  $G$  has a  $P_3$ -maximal  $(a + 1)$ -decomposition. Let  $e_1$  and  $e_2$  be two distinct edges in  $R$  such that

$$d(e_1, e_2) = \min\{d(e, f) : e, f \in R\}. \quad (4.1)$$

Necessarily,  $d(e_1, e_2) \geq 1$ . Let  $P$  be an  $e_1 - e_2$  path of shortest length in  $G$ . It then follows by (4.1) that each  $e \in E(P) - \{e_1, e_2\}$  belongs to  $H_i$  for some  $i$  with  $1 \leq i \leq a$ . Let

$$W = \{i : H_i \cap E(P) \neq \emptyset \text{ and } 1 \leq i \leq a\} \quad (4.2)$$

and let  $F$  be the subgraph induced by the set  $\{e_1, e_2\} \cup (\bigcup_{i \in W} E(H_i))$  of edges. We may assume, without loss of generality, that  $W = \{1, 2, \dots, t\}$  for some positive integer  $t$ . Since  $F$  is a connected graph of size  $2(t + 1)$ , it follows by Theorem 4.2.1 that  $F$  is  $P_3$ -decomposable. Suppose that  $\{H'_1, H'_2, \dots, H'_{t+1}\}$  is a  $P_3$ -decomposition of  $F$  and  $R' = R - \{e_1, e_2\}$ . Then  $\mathcal{D}_{a+1} = \{H'_1, H'_2, \dots, H'_{t+1}, H_{t+1}, \dots, H_a, R'\}$  is a  $P_3$ -maximal  $(a + 1)$ -decomposition of  $G$ . Continuing in this manner, we see that for each integer  $k$  with  $a \leq k \leq \text{Max}(G, P_3)$ , there exists a  $P_3$ -maximal  $k$ -decomposition of  $G$ . ■

**Proposition 4.2.3** *The graph  $2K_2$  is an ID-graph.*



**Proof.** Assume, to the contrary, that  $2K_2$  is not an ID-graph. Then there exists an integer  $a$  with  $\text{Min}(G, 2K_2) \leq a < a + 1 < b = \text{Max}(G, 2K_2)$  such that  $G$  has a  $2K_2$ -maximal  $a$ -decomposition  $\mathcal{D}_a = \{H_1, H_2, \dots, H_a, R\}$  but no  $2K_2$ -maximal  $(a+1)$ -decomposition. Since  $G$  has a  $2K_2$ -maximal  $b$ -decomposition and  $R$  contains no subgraph isomorphic to  $2K_2$ , it follows that  $R = K_{1,t}$  for some integer  $t \geq 4$ . Hence the size of  $G$  is  $m = 2a + t$ . First, we verify the following:

For each integer  $i$  with  $1 \leq i \leq a$ , exactly one edge in  $H_i$  is adjacent to every edge in  $R$ ; that is, for each integer  $i$  with  $1 \leq i \leq a$ , there is exactly one edge in  $H_i$  such that this edge and the edges of  $R$  form a star  $K_{1,t+1}$ .

Suppose that this statement is false for some subgraph  $H_i$  in  $\mathcal{D}_a$ , say  $H_i = H_1$  where  $E(H_1) = \{e_1, e_2\}$ . First, since  $H_1 = 2K_2$ , it is impossible that each of the two edges in  $H_1$  is adjacent to every edge of  $R$ . Thus neither of the two edges  $e_1$  and  $e_2$  of  $H_i$  is adjacent to all edges of  $R$ . Then  $e_1$  and  $e_2$  can be adjacent to at most two edges of  $R$ . Let  $E(R) = \{f_1, f_2, \dots, f_t\}$ . Since  $t \geq 4$ , there are two edges of  $R$ , say  $f_1$  and  $f_2$ , such that neither  $e_1$  nor  $e_2$  is adjacent to  $f_j$  for  $j = 1, 2$ . Let  $H'_j$  consist of  $e_j$  and  $f_j$  for  $j = 1, 2$  and let  $R' = R - \{f_1, f_2\}$ . Then  $\mathcal{D} = \{H'_1, H'_2, H_2, \dots, H_a, R'\}$  is a  $2K_2$ -maximal  $(a+1)$ -decomposition of  $G$ , which is impossible. Thus, as claimed, for each integer  $i$  with  $1 \leq i \leq a$ , exactly one edge in  $H_i$  is adjacent to every edge in  $R$ .

Next, for each  $i$  with  $1 \leq i \leq a$ , let  $f'_i \in E(H_i)$  such that  $f'_i$  and  $E(R)$  form a  $K_{1,t+1}$  in  $G$ . Thus  $G$  contains  $K_{1,t+a}$  as a subgraph. Since the size of  $G$  is  $2a + t$ ,

there are at most  $a$  sets of two nonadjacent edges of  $G$ . Therefore,  $G$  cannot have a  $2K_2$ -maximal  $b$ -decomposition for  $b \geq a + 2$ , which is a contradiction. ■

While the graph  $2K_2$  is an ID-graph, the graph  $3K_2$  is not. Let  $G = K_3 \square K_2$  be the Cartesian product of  $K_3$  and  $K_2$  whose edges are labeled as shown in Figure 4.3. The graph  $G$  has a  $3K_2$ -maximal 1-decomposition and a  $3K_2$ -maximal 3-decomposition. For example, let  $D_1 = \{H_1, R_1\}$  where  $H_1 = G[\{e_1, e_2, e_3\}]$  and  $D_3 = \{L_1, L_2, L_3, R_3\}$  where  $L_i = G[\{e_i, f_i, g_i\}]$  and  $R_3$  is an empty graph. Since the size of  $G$  is 9, it follows that  $\text{Min}(G, 3K_2) = 1$  and  $\text{Max}(G, 3K_2) = 3$ .

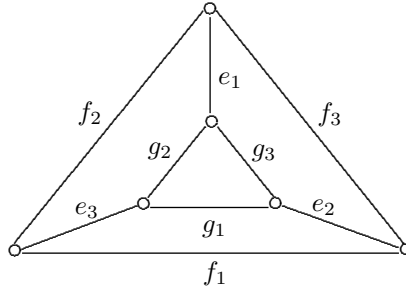


Figure 4.3: Showing that  $3K_2$  is not an ID-graph

It remains to show that  $G$  does not have a  $3K_2$ -maximal 2-decomposition. Assume, to the contrary, that  $G$  has a  $3K_2$ -maximal 2-decomposition  $\mathcal{D}_2 = \{F_1, F_2, R_2\}$  where  $F_1 \cong F_2 \cong 3K_2$  and  $R_2$  contains no subgraph isomorphic to  $3K_2$ . First, we observe that if  $F \in \{F_1, F_2\}$  such that  $F$  contains an edge in  $\{f_i, g_i\}$  where  $i = 1, 2, 3$ , then  $E(F) = \{e_i, f_i, g_i\}$ . At least one of  $F_1$  and  $F_2$  contains an edge in  $\{f_i, g_i : 1 \leq i \leq 3\}$ , say  $f_1 \in F_1$  and so  $E(F_1) = \{e_1, f_1, g_1\}$ . Since  $F_1$  and  $F_2$  are edge-disjoint and  $F_2 = 3K_2$ , it follows that  $F_2$  must contain an edge in  $\{f_i, g_i : i = 2, 3\}$ . By symmetry, we may assume that  $f_2 \in E(F_2)$ . However then,

$E(F_2) = \{e_2, f_2, g_2\}$  and so  $R_2 = 3K_2$  with edge set  $E(R_2) = \{e_3, f_3, g_3\}$ , which is a contradiction.

### 4.3 The ID-Graphs of Size 3

We have seen that none of the three graphs  $K_{1,3}, K_3, 3K_2$  of size 3 is an ID-graph. In this section, we show that the two remaining graphs of size 3 (without isolated vertices) are both ID-graphs. To do this, we introduce a decomposition concept involving sets of graphs without isolated vertices. For a set  $S$  of graphs and a graph  $G$ , a decomposition  $\mathcal{D} = \{H_1, H_2, \dots, H_k, R\}$  of  $G$  is called *S-maximal* or an *S-maximal k-decomposition* if  $H_i \cong H$  for some  $H \in S$  for each integer  $i$  with  $1 \leq i \leq k$  and  $R$  contains no subgraph isomorphic to any subgraph in  $S$ . For a set  $S$  of graphs without isolated vertices and a graph  $G$ , let

$$\begin{aligned} \text{Min}(G, S) &= \min\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\} \\ \text{Max}(G, S) &= \max\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\}. \end{aligned}$$

A set  $S$  of graphs without isolated vertices is said to possess the *intermediate decomposition property* (IDP) or  $S$  is called an *ID-set* if for every graph  $G$  and each integer  $k$  with  $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$ , there exists an  $S$ -maximal  $k$ -decomposition of  $G$ . By Propositions 4.2.2 and 4.2.3, if  $S = \{P_3\}$  or  $S = \{2K_2\}$ , then  $S$  is an ID-set. On the other hand, the set  $S = \{K_{1,3}, K_3\}$  is not an ID-set. For example, the graph  $G$  of Figure 4.1 has an  $S$ -maximal 1-decomposition and an  $S$ -maximal 3-decomposition but has no  $S$ -maximal 2-decomposition. (On the other hand, the graph  $F$  of Figure 4.1 has an  $S$ -maximal  $k$ -decomposition for

$k = 1, 2, 3$ .)

Observe that if  $S$  is the set of all graphs (connected or disconnected) of the same size  $m$ , then  $S$  is an ID-set. To see this, let  $G$  be a graph,  $a = \text{Min}(G, S)$  and let  $\mathcal{D} = \{H_1, H_2, \dots, H_a, R\}$  be any  $S$ -maximal  $a$ -decomposition of  $G$ . Since  $R$  contains no subgraph that is isomorphic to any graph in  $S$ , it follows that  $0 \leq |E(R)| \leq m - 1$ . Thus  $\text{Min}(G, S) = \text{Max}(G, S) = a$ . We state this observation below.

**Observation 4.3.1** *For each positive integer  $m$ , the set  $S_m$  of all graphs (connected or disconnected) of size  $m$  is an ID-set.*

By Observation 4.3.1, the sets  $S_2 = \{P_3, 2K_2\}$  of all graphs of size 2 and  $S_3 = \{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  of all graphs of size 3 are ID-sets, where  $P_3 + K_2$  is the union of  $P_3$  and  $K_2$ . It is the following problem, however, that is of more general interest to us.

### The Intermediate Value Problem for $S$ -Maximal Decompositions

*Which sets of graphs (without isolated vertices) are ID-sets?*

For a set  $S$  of graphs, a graph  $G$  is said to have the *intermediate decomposition property with respect to  $S$*  (IDP- $S$ ) if for each integer  $k$  with  $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$ , there exists an  $S$ -maximal  $k$ -decomposition of  $G$ . In this case, the graph  $G$  is referred to as an *IDP- $S$*  graph; otherwise,  $G$  is a *non-IDP- $S$*  graph. Of course, if every graph is an IDP- $S$  graph, then  $S$  is an ID-set.

The following theorem concerning sets of graphs that are not ID-sets will be useful in showing that both  $P_4$  and  $P_3 + K_2$  are ID-graphs.

**Theorem 4.3.2** *Let  $S$  be a set of graphs without isolated vertices that is not an ID-set and let  $\mathcal{F}_S$  be the set of all non-IDP- $S$  graphs, where  $G$  is a graph of minimum size in  $\mathcal{F}_S$ . Moreover, let  $a$  and  $b$  be the smallest integers with  $1 \leq a < b-1$  such that (i)  $G$  has an  $S$ -maximal  $a$ -decomposition  $\mathcal{D}_a = \{H_1, H_2, \dots, H_a, R_a\}$  and an  $S$ -maximal  $b$ -decomposition  $\mathcal{D}_b = \{L_1, L_2, \dots, L_b, R_b\}$  but (ii)  $G$  has no  $S$ -maximal  $k$ -decomposition for every integer  $k$  with  $a < k < b$ .*

(I) *If  $\mathcal{D}_c$  is an  $S$ -maximal  $c$ -decomposition of  $G$  where  $c \geq b$ , then  $H_i \notin \mathcal{D}_c$  for all  $i$  with  $1 \leq i \leq a$ .*

(II) *For all pairs  $i, j$  where  $i \in \{1, 2, \dots, a\}$  and  $j \in \{1, 2, \dots, b\}$ , it follows that  $E(H_i) \cap E(L_j) \neq \emptyset$ .*

(III) *The number  $b$  satisfies  $b \leq \min\{|E(H_i)| : 1 \leq i \leq a\}$ .*

**Proof.** To verify (I), we first assume that  $a = 1$ . Then  $\mathcal{D}_a = \{H_1, R_1\}$ . Since  $R_1$  contains no subgraph isomorphic to any graph in  $S$ , the statement is true for  $a = 1$ . Next, suppose that  $a \geq 2$ . Assume, to the contrary, that  $G$  has an  $S$ -maximal  $c$ -decomposition  $\mathcal{D}_c$  for some integer  $c \geq b \geq a + 2$  such that  $H_i \in \mathcal{D}_c$  for some  $i$  ( $1 \leq i \leq a$ ), say  $H_1 \in \mathcal{D}_c$ . Let  $G' = G - E(H_1)$ . Then  $\mathcal{D}_a - \{H_1\}$  is an  $S$ -maximal  $(a - 1)$ -decomposition and  $\mathcal{D}_c - \{H_1\}$  is an  $S$ -maximal  $(c - 1)$ -decomposition of  $G'$ . Since  $1 \leq a - 1 < b - 2 < b - 1 \leq c - 1$  and  $G'$  does not have an  $S$ -maximal  $k'$ -decomposition for each  $k'$  with  $a - 1 < k' < b - 1$ , it follows that

this decomposition together with  $H_1$  forms an  $S$ -maximal  $(k' + 1)$ -decomposition of  $G$  where  $a < k' + 1 < b$ . This, however, is impossible by property (ii) possessed by  $a$  and  $b$ . Thus  $G' \in \mathcal{F}_S$ . Since the size of  $G'$  is smaller than that of  $G$ , this contradicts the defining property of  $G$  and so (I) holds.

Suppose that statement (II) is false. Then there exist  $i, j$  where  $i \in \{1, 2, \dots, a\}$  and  $j \in \{1, 2, \dots, b\}$  such that  $E(H_i) \cap E(L_j) = \emptyset$ , say  $E(H_1) \cap E(L_1) = \emptyset$ . Let  $G^* = G - E(L_1)$ . Thus,  $G^*$  contains  $H_1$  as a subgraph. Hence  $G^*$  has an  $S$ -maximal  $k$ -decomposition  $\mathcal{D}^*$  for some positive integer  $k$  such that  $H_1 \in \mathcal{D}^*$ . We claim that  $k \leq a - 1$ . Suppose that this is not the case. Then  $k \geq a$  and so  $k + 1 > a$ . Observe that (1)  $\mathcal{D}^* \cup \{L_1\}$  is an  $S$ -maximal  $(k + 1)$ -decomposition of  $G$ , where, necessarily,  $k + 1 \geq b$ , (2)  $\mathcal{D}^* \cup \{L_1\}$  contains  $H_1$  and (3)  $H_1 \in \mathcal{D}_a$ . This contradicts (I). Therefore,  $k \leq a - 1$  as claimed.

Since  $1 \leq k \leq a - 1$ , it follows that  $a \geq 2$ . Observe that  $\mathcal{D}_b - \{L_1\}$  is an  $S$ -maximal  $(b - 1)$ -decomposition of  $G^*$ . We now claim that  $G^*$  has no  $S$ -maximal  $k^*$ -decomposition for each integer  $k^*$  with  $a - 1 < k^* < b - 1$ . If this is not the case, then there exists an  $S$ -maximal  $k^*$ -decomposition. However, this decomposition together with  $L_1$  forms an  $S$ -maximal  $(k^* + 1)$ -decomposition of  $G$ . Since  $a < k^* + 1 < b$ , this is impossible by (ii). Thus  $G^* \in \mathcal{F}_S$ . However, the size of  $G^*$  is smaller than the size of  $G$ , which contradicts the defining property of  $G$  and so (II) holds.

It remains to verify (III). Observe that if  $i \in \{1, 2, \dots, a\}$ . It then follows by (II) that  $E(H_i) \cap E(L_j) \neq \emptyset$  for every  $j \in \{1, 2, \dots, b\}$ . Since  $L_1, L_1, \dots, L_b$  are

pairwise edge-disjoint,  $|E(H_i)| \geq b$  for  $1 \leq i \leq a$  and (III) holds. ■

By Theorem 4.3.2, every graph of size 2 is an ID-graph. Therefore, Propositions 4.2.2 and 4.2.3 are consequences of Theorem 4.3.2.

For a set  $S$  of graphs without isolated vertices that is not an ID-set, a graph  $G$  of minimum size that is not an IDP- $S$  graph (as described in Theorem 4.3.2) is referred to as a *minimum non-IDP- $S$  graph*. If  $S = \{H\}$  consists of a single graph  $H$ , then a minimum non-IDP- $S$  graph is also referred to as a *minimum non-IDP- $H$  graph*.

**Theorem 4.3.3** *The graph  $P_4$  is an ID-graph.*

**Proof.** Assume, to the contrary, that  $P_4$  is not an ID-graph. Then there exists a graph  $G$  of minimum size that is not an IDP- $P_4$  graph. Then there are smallest integers  $a$  and  $b$  where  $1 \leq a \leq b - 2$  such that  $G$  has a  $P_4$ -maximal  $a$ -decomposition  $\mathcal{D}_a$  and a  $P_4$ -maximal  $b$ -decomposition  $\mathcal{D}_b$  but  $G$  has no  $P_4$ -maximal  $k$ -decomposition for every integer  $k$  with  $a < k < b$ . By Theorem 4.3.2(III),  $b \leq 3$ , which implies that  $a = 1$  and  $b = 3$ . Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1 \cong P_4$ ,  $L_i \cong P_4$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains a subgraph isomorphic to  $P_4$ . By assumption,  $G$  has no  $P_4$ -maximal 2-decomposition.

Let  $H_1 = (v_1, v_2, v_3, v_4)$  where  $e_i = v_i v_{i+1}$  for  $i = 1, 2, 3$ . Since, by Theorem 4.3.2(II), each subgraph  $L_i$  must contain an edge of  $H_1$ , we may assume that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$ . In particular,  $e_2 = v_2 v_3 \in E(L_2)$  and so at least one of  $v_2$  and  $v_3$  is an interior vertex of  $L_2$ , say  $v_2$  is an interior vertex of  $L_2$ . Then

$(u, v_2, v_3)$  is a subpath of  $L_2$  for some vertex  $u$  of  $G$ . Thus, either

$$L_2 = (u, v_2, v_3, w) \quad \text{or} \quad L_2 = (w, u, v_2, v_3) \quad (4.3)$$

for some vertex  $w$  of  $G$ . Next, we claim that either  $v_1$  is an interior vertex of  $L_1$  or  $v_4$  is an interior vertex of  $L_3$ . If this were not the case, then

$$L_1 = (v_1, v_2, x, y) \quad \text{and} \quad L_3 = (v_4, v_3, x', y')$$

for some vertices  $x, y, x', y'$  of  $G$ . We consider two cases, according to the two possible choices of  $L_2$  described in (4.3).

*Case 1.*  $L_2 = (u, v_2, v_3, w)$ . Since  $L_1$  and  $L_2$  are edge-disjoint,  $x \neq u$ . We now show that  $y = u$ . Suppose that  $y \neq u$ . Then  $(u, v_2, x, y) = P_4$  is a subgraph of  $R_1$ , which is impossible. Consequently,  $y = u$ . Since  $L_1$  and  $L_3$  are edge-disjoint,  $x' \neq w$ . Next, we show that  $y' = w$ . Suppose that  $y' \neq w$ . Then  $(w, v_3, x', y')$  is a subgraph of  $R_1$ , again impossible. Therefore,  $y' = w$ . Hence  $G$  contains the subgraph shown in Figure 4.4, where the bold edges are the edges of  $H_1$ . Let  $F_1 = (v_1, v_2, x, y)$  and  $F_2 = (u, v_2, v_3, v_4)$ . Then  $F_1 \cong F_2 \cong P_4$  and  $F_1$  and  $F_2$  are edge-disjoint. Since  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ , it follows that  $R_2 = G - [E(F_1) \cup E(F_2)]$  is a subgraph of  $R_1$  and so  $R_2$  contains no subgraph isomorphic to  $P_4$ . Therefore,  $\{F_1, F_2, R_2\}$  is a  $P_4$ -maximal 2-decomposition of  $G$ , which produces a contradiction.

*Case 2.*  $L_2 = (w, u, v_2, v_3)$ . Since  $L_1$  and  $L_2$  are edge-disjoint,  $x \neq u$ . We show that  $x = w$ . Suppose that  $x \neq w$ . Then  $(x, v_2, u, w) = P_4$  is a subgraph of  $R_1$ , which is impossible. Consequently,  $x = w$ . Next, we show that  $y = u$ . If  $y \neq u$ , then  $(y, x, v_2, u) = P_4$  is a subgraph of  $R_1$ , which is impossible. Hence  $y = u$ . Since  $x = w$  and  $y = u$ , it follows that  $wu = xy \in E(L_1) \cap E(L_2)$ , which is impossible.



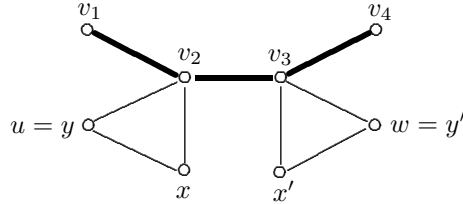


Figure 4.4: A step in the proof of Theorem 4.3.3

Therefore, as claimed, either  $v_1$  is an interior vertex of  $L_1$  or  $v_4$  is an interior vertex of  $L_3$ , say the former. Then  $(z, v_1, v_2)$  is a subpath of  $L_1$  for some vertex  $z$  of  $G$ . Define a decomposition  $\mathcal{D}_2 = \{F_1, F_2, R_2\}$  of  $G$  where  $F_1 = (z, v_1, v_2, v_3)$ ,  $F_2 = L_3$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Then  $F_1 \cong F_2 \cong P_4$  and  $E(F_1) \cap E(F_2) = \emptyset$ . Since  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ , it follows that  $R_2$  is a subgraph of  $R$  and so  $R_2$  contains no subgraph isomorphic to  $P_4$ . Hence  $\mathcal{D}_2$  is a  $P_4$ -maximal 2-decomposition, which is a contradiction. ■

We next show that  $P_3 + K_2$ , the remaining graph of size 3, is also an ID-graph.

**Theorem 4.3.4** *The graph  $P_3 + K_2$  is an ID-graph.*

**Proof.** Assume, to the contrary, that  $P_3 + K_2$  is not an ID-graph. Let  $G$  be a graph of minimum size that is not an IDP- $P_4$  graph. By Theorem 4.3.2 then,  $G$  has a  $(P_3 + K_2)$ -maximal 1-decomposition  $\mathcal{D}_1$  and a  $(P_3 + K_2)$ -maximal 3-decomposition  $\mathcal{D}_3$  but  $G$  has no  $(P_3 + K_2)$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1 \cong P_3 + K_2$ ,  $L_i \cong P_3 + K_2$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains a subgraph isomorphic to  $P_3 + K_2$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$  where  $e_1$  and  $e_2$  are adjacent edges in  $H_1$  (see Figure 4.5). We may assume, without

loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| = t \geq 6$ .

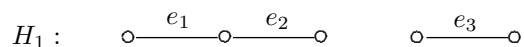


Figure 4.5: The graph  $H_1$  in the proof of Theorem 4.3.4

We claim that  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$  or  $R_1 = K_4$  (with possibly additional isolated vertices, in which case we consider the subgraph of  $R_1$  that consists of all nontrivial components of  $R_1$ ). First, suppose that  $R_1$  contains two or more nontrivial components. Since  $R_1$  does not contain  $P_3 + K_2$  as a subgraph, each nontrivial component is  $K_2$  and so  $R_1 = tK_2$ . Next, suppose that  $R_1$  is connected. Again, because  $R_1$  does not contain  $P_3 + K_2$  as a subgraph and the size of  $R_1$  is at least 6, it follows that  $R_1$  cannot contain vertex-disjoint copies of  $P_3$  and  $K_2$  and so either  $R_1 = K_{1,t}$  or  $R_1 = K_4$ . Consequently,  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$ , or  $R_1 = K_4$ . We consider these three cases.

*Case 1.*  $R_1 = tK_2$ . Since  $t \geq 6$  and  $H_1 \cong P_3 + K_2$ , there is at least one edge (say  $f_1$ ) in  $R_1$  that is not adjacent to any edge in  $H_1$  and at least one edge (say  $f_2$ ) in  $E(R_1) - \{f_1\}$  that is not adjacent to  $e_3$ . We claim, in fact, that

$$\text{no edge in } R_1 \text{ is adjacent to } e_3. \tag{4.4}$$

Suppose that there is an edge  $f_3$  in  $R_1$  that is adjacent to  $e_3$ . Let  $F_1 = G[\{e_1, e_2, f_1\}]$  and  $F_2 = G[\{e_3, f_2, f_3\}]$ . Then  $F_1 \cong F_2 \cong P_3 + K_2$  and  $E(F_1) \cap E(F_2) = \emptyset$ . Since  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ , it follows that  $R_2 = G - [E(F_1) \cup E(F_2)]$  is a subgraph

of  $R_1$ . Hence  $\{F_1, F_2, R_2\}$  is a  $(P_3 + K_2)$ -maximal 2-decomposition of  $G$ , which is impossible. Thus (4.4) holds.

Next we claim that

$$\text{there are two edges in } R_1, \text{ each of which is adjacent to } e_1 \text{ or } e_2. \quad (4.5)$$

Suppose this is not the case. Then there is at most one edge in  $R_1$  that is adjacent to  $e_1$  or  $e_2$ . It then follows by (4.4) that  $G$  is one of the graphs  $P_3 + (t + 1)K_2, P_4 + tK_2, K_{1,3} + tK_2$ . However then,  $G$  cannot have a  $(P_3 + K_2)$ -maximal 3-decomposition  $\mathcal{D}_3$ , which is a contradiction. Thus (4.5) holds. Let  $g_1$  and  $g_2$  be two distinct edges of  $R_1$  that are adjacent to  $e_1$  or  $e_2$ . Since  $g_1$  and  $g_2$  are nonadjacent, the subgraph  $G[\{e_1, e_2, g_1, g_2\}]$  induced by  $\{e_1, e_2, g_1, g_2\}$  is one of the two graphs shown in Figure 4.6, each of which can be decomposed into two copies  $T_1$  and  $T_2$  of  $P_3$ . Let  $F_1 = G[E(T_1) \cup \{f_1\}]$  (where  $f_1 \in E(R_1)$  is not adjacent to any edge in  $E(H_1) \cup E(R_1)$ ) and  $F_2 = G[E(T_2) \cup \{e_3\}]$  (where  $e_3$  is not adjacent to any edge in  $E(H_1) \cup E(R_1)$  by (4.4)). Then  $F_1 \cong F_2 \cong P_3 + K_2$  and  $F_1$  and  $F_2$  are edge-disjoint. Since  $E(H_1) \subseteq E(T_1) \cup E(T_2)$ , it follows that  $R_2 = G - (E(T_1) \cup E(T_2))$  is a subgraph of  $R_1$  and so contains no subgraph isomorphic to  $P_3 + K_2$ . Therefore,  $\{F_1, F_2, R_2\}$  is a  $(P_3 + K_2)$ -maximal 2-decomposition of  $G$ , which is a contradiction.

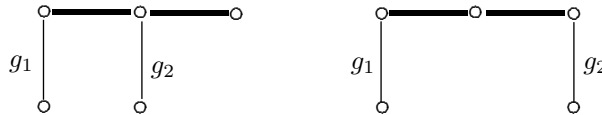


Figure 4.6: The subgraph  $G[\{e_1, e_2, g_1, g_2\}]$  in Case 1

*Case 2.*  $R_1 = K_{1,t}$ . Note that if an edge of  $H_1$  is adjacent with *all* edges in  $R_1$ ,

then this edge must be incident with the central vertex  $v$  of  $R_1$  (see Figure 4.7). Since  $e_3$  is not adjacent to  $e_1$  or  $e_2$ , it is impossible that both  $e_1$  and  $e_3$  (or both  $e_2$  and  $e_3$ ) are adjacent to all edges of  $R_1$ . Thus at most two of the three edges  $e_1, e_2$  and  $e_3$  can be adjacent to all edges of  $R_1$ . Hence there are three possibilities, namely (i) both  $e_1$  and  $e_2$  are adjacent to all edges of  $R_1$  (and  $e_3$  is not), (ii) exactly one of  $e_1, e_2$  and  $e_3$  is adjacent to all edges of  $R_1$  and (iii) none of  $e_1, e_2$  and  $e_3$  are adjacent to all edges of  $R_1$ . We consider these three situations.

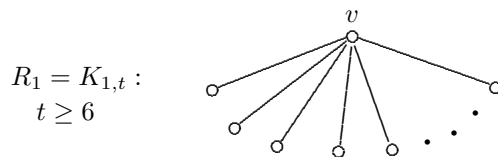


Figure 4.7: The graph  $R_1$  in Case 2

*Subcase 2.1.* Both  $e_1$  and  $e_2$  are adjacent to all edges of  $R_1$ . Then  $G - e_3 = K_{1,t+2}$ . However then,  $G$  cannot have a  $(P_3 + K_2)$ -maximal 3-decomposition  $\mathcal{D}_3$ , which is impossible.

*Subcase 2.2.* Exactly one of  $e_1, e_2$  and  $e_3$  is adjacent to all edges of  $R_1$ . Let  $e$  and  $f$  are the edges of  $H_1$  that are not adjacent to all edges of  $R_1$ . Then  $G - e - f = K_{1,t+1}$ . However then,  $G$  cannot have a  $(P_3 + K_2)$ -maximal 3-decomposition  $\mathcal{D}_3$ , which is impossible.

*Subcase 2.3.* None of  $e_1, e_2$  and  $e_3$  are adjacent to all edges of  $R_1$ . Thus none of  $e_1, e_2$  and  $e_3$  is incident with the central vertex  $v$  of  $R_1$ . Since the order of  $H_1$  is 5 and the size  $t$  of  $R_1$  is at least 6, there is an edge (say  $f_1$ ) in  $R_1$  that is not adjacent to any edge in  $H_1$ . Furthermore, there are at least two edges (say

$f_2$  and  $f_3$ ) that are not adjacent to  $e_3$ . Let  $F_1 = G[\{e_1, e_2, f_1\}] \cong P_3 + K_2$  and  $F_2 = G[\{e_3, f_2, f_3\}] \cong P_3 + K_2$ . Then  $R_2 = G - (E(F_1) \cup E(F_2))$  is a subgraph of  $R_1$ . Therefore,  $\{F_1, F_2, R_2\}$  is a  $(P_3 + K_2)$ -maximal 2-decomposition of  $G$ , which is a contradiction.

*Case 3.*  $R_1 = K_4$ . Observe that there exists  $f \in E(R_1)$  such that  $f$  is adjacent to neither  $e_1$  nor  $e_2$ . Let  $F_1 = G[\{e_1, e_2, f\}] \cong P_3 + K_2$ . The subgraph  $G[(E(R_1) - \{f\}) \cup \{e_3\}]$  is one of the three graphs in Figure 4.8. In each case, there is a subgraph  $P_3$  in  $R_1 - f$  that is vertex-disjoint from  $e_3$  (whose edges are drawn in bold in Figure 4.8). Let  $F_2 = G[E(P_3) \cup \{e_3\}] \cong P_3 + K_2$ . Then  $R_2 = G - (E(F_1) \cup E(F_2))$  is a subgraph of  $R_1$ . Therefore,  $\{F_1, F_2, R_2\}$  is a  $(P_3 + K_2)$ -maximal 2-decomposition of  $G$ , which is a contradiction. ■

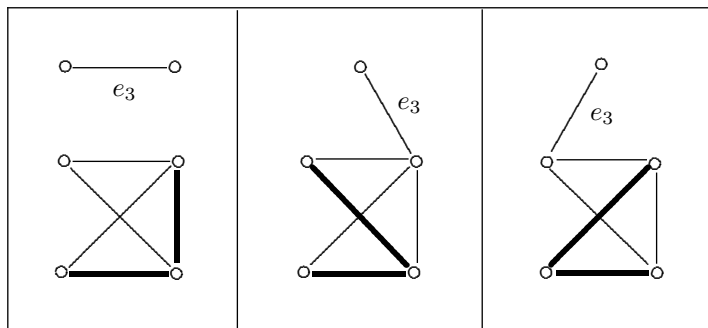


Figure 4.8: The subgraph  $G[(E(R_1) - \{f\}) \cup \{e_3\}]$  in Case 3

The following result summarizes what we have discovered for all graphs of size 2 or 3.

**Theorem 4.3.5** *A graph  $H$  of size 2 or 3 is an ID-graph unless  $H \in \{K_3, K_{1,3}, 3K_2\}$ .*

## 4.4 The ID-Sets of Graphs of Size 3

In this section, we turn to the problem of investigating whether certain sets of graphs of size 3 are ID-sets. Let  $S_3 = \{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  be the set of graphs of size 3 without isolated vertices. By Observation 4.3.1,  $S_3$  is an ID-set. Furthermore, by Theorem 4.3.5, we know all ID-sets consisting of exactly one graph of size 3. Therefore, we consider subsets  $S$  of  $S_3$  with  $2 \leq |S| \leq 4$ . There are 25 such sets. We begin with those sets consisting of two graphs of size 3 are ID-sets.

### 4.4.1 The 2-Element ID-Sets

While we already seen that  $\{K_{1,3}, K_3\}$  is not an ID-set, it turns out that there two other sets of two graphs of size 3 that are non ID-sets, namely  $\{3K_2, K_3\}$  and  $\{3K_2, K_{1,3}\}$ . For example, if  $S = \{3K_2, K_3\}$ , then the graph  $K_3 + 2K_{1,3}$  is a non-IDP- $S$  graph while if  $S = \{3K_2, K_{1,3}\}$ , then the graph  $2K_3 + K_{1,3}$  is a non-IDP- $S$  graph. Figure 4.9 provides a complete list of ID-sets and non-ID-sets consisting of one or two graphs of size 3.

	$P_4$	$P_3 + K_2$	$K_3$	$K_{1,3}$	$3K_2$
$P_4$	ID	ID	ID	ID	ID
$P_3 + K_2$	ID	ID	ID	ID	ID
$K_3$	ID	ID	NO	NO	NO
$K_{1,3}$	ID	ID	NO	NO	NO
$3K_2$	ID	ID	NO	NO	NO

Figure 4.9: ID-sets and non-ID sets of two graphs of size 3

We begin by verifying those 2-element ID-sets containing the graph  $P_3 + K_2$ .

The following observation will be useful, which is a consequence of Theorem 4.3.4.

**Observation 4.4.1** *Suppose that  $R$  is a graph without isolated vertices having size  $t \geq 6$ . If  $R$  does not contain  $P_3 + K_2$  as a subgraph, then  $R = tK_2$ ,  $R = K_{1,t}$  or  $R = K_4$ .*

**Theorem 4.4.2** *The set  $\{P_4, P_3 + K_2\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{P_4, P_3 + K_2\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1, L_i \in S$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains an subgraph isomorphic to any graph in  $S$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$ . We may assume, without loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| = t \geq 6$ . Since  $R_1$  does not contain  $P_4$  or  $P_3 + K_2$  as a subgraph,  $R_1 = K_{1,t}$  or  $R_1 = tK_2$ . We may assume  $H_1 - e_1 = P_3 = (u, v, w)$  where  $e_2 = uv$  and  $e_3 = vw$ .

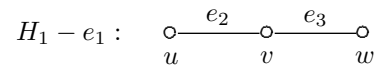


Figure 4.10: The graph  $H_1 - e_1$  in the proof of Theorem 4.4.2

Assume first that there is an edge  $e \in E(R_1) - E(L_1)$  such that  $e$  is adjacent to neither  $e_2$  nor  $e_3$ . Let  $F_1 = L_1$ ,  $F_2 = G[\{e_2, e_3, e\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal

2-decomposition, which is impossible. Hence we may assume that every edge in  $E(R_1) - E(L_1)$  is adjacent to  $e_2$  or  $e_3$ . Since  $|E(R_1) - E(L_1)| \geq 4$ , it is impossible if  $R_1 = tK_2$ . Hence  $R_1 = K_{1,t}$ . Let  $V(R_1) = \{x, x_1, x_2, \dots, x_t\}$  and  $x$  is the central vertex of  $R_1$  (see Figure 4.11). First, suppose that  $x \in \{u, v, w\}$ . If  $x = u$  or  $x = v$ , then  $G - \{e_1, e_3\} = K_{1,t+1}$ ; while if  $x = w$ , then  $G - \{e_1, e_2\} = K_{1,t+1}$ . In either case,  $G$  cannot have an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$ , a contradiction. Next, suppose that  $x \notin \{u, v, w\}$ . We may assume, without loss of generality, that  $xx_i \notin E(L_1)$  for  $1 \leq i \leq 4$ . Since each edge  $xx_i$  ( $1 \leq i \leq 4$ ) is incident with exactly one vertex in  $\{u, v, w\}$ , it follows that  $x_i \in \{u, v, w\}$  for  $1 \leq i \leq 4$ , this is impossible. ■

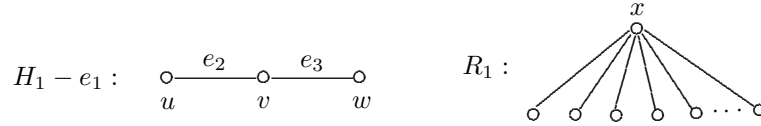


Figure 4.11: A step in the proof of Theorem 4.4.2

**Theorem 4.4.3** *The set  $\{K_3, P_3 + K_2\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{K_3, P_3 + K_2\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1, L_i \in S$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains an subgraph isomorphic to any graph in  $S$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$  and we may assume, without loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint



for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| = t \geq 6$ . Since  $R_1$  does not contain  $P_3 + K_2$  and  $K_3$  as a subgraph,  $R_1 = tK_2$  or  $R_1 = K_{1,t}$ . We may assume that  $H_1 - e_1 = P_3 = (u, v, w)$  where  $e_2 = uv$  and  $e_3 = vw$ .

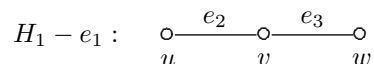


Figure 4.12: The graph  $H_1 - e_1$  in the proof of Theorem 4.4.3

First suppose that there is an edge  $e \in E(R_1) - E(L_1)$  such that  $e$  is adjacent to neither  $e_2$  nor  $e_3$ . Let  $F_1 = L_1$  and  $F_2 = G[\{e_2, e_3, e\}] \in \{P_3 + K_2, K_3\}$ . Then  $F_1$  and  $F_2$  are edge-disjoint and  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ . Since  $R_2 = G - (E(F_1) \cup E(F_2))$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible.

Next, suppose that each edge in  $E(R_1) - E(L_1)$  is incident with exactly one vertex in  $\{u, v, w\}$ . Since  $|E(R_1) - E(L_1)| = t - 2 \geq 4$ , this is impossible when  $R_1 = tK_2$ . Hence  $R_1 = K_{1,t}$ . Let  $V(R_1) = \{x, x_1, x_2, \dots, x_t\}$  and  $x$  is the central vertex of  $R_1$  (see Figure 4.11). First, suppose that  $x \in \{u, v, w\}$ , say  $x = u$  or  $x = v$ . Then  $G - \{e_1, e_3\} = K_{1,t+1}$ . (In fact,  $G$  is the graph obtained from  $K_3$  by adding  $t$  pendant edges at one vertex of  $K_3$ .) However then,  $G$  cannot have an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$ , a contradiction. Next, suppose that  $x \notin \{u, v, w\}$ . We may assume, without loss of generality, that  $xx_i \notin E(L_1)$  for  $1 \leq i \leq 4$ . Since each edge  $xx_i$  ( $1 \leq i \leq 4$ ) is incident with exactly one vertex in  $\{u, v, w\}$ , it follows that  $x_i \in \{u, v, w\}$  for  $1 \leq i \leq 4$ , this is impossible. ■

**Theorem 4.4.4** *The set  $\{K_{1,3}, P_3 + K_2\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{K_{1,3}, P_3 + K_2\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1, L_i \in S$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains a subgraph isomorphic to any graph in  $S$ .

Let  $E(H_1) = \{e_1, e_2, e_3\}$ . We may assume, without loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| = t \geq 6$ . Since  $R_1$  does not contain  $P_3 + K_2$  as a subgraph,  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$  or  $R_1 = K_4$ . Furthermore,  $R_1$  does not contain  $K_{1,3}$  as a subgraph and so  $R_1 = tK_2$ . Since each  $L_i - e_i$  is a subgraph of  $R_1$  for  $i = 1, 2, 3$  and  $R_1 = tK_2$ , it follows that  $L_i \cong P_3 + K_2$  for  $i = 1, 2, 3$ . Thus,  $\mathcal{D}_3$  is in fact a  $(P_3 + K_2)$ -maximal 3-decomposition.

First, suppose that  $H_1 \cong P_3 + K_2$ . Thus  $\mathcal{D}_1$  is a  $(P_3 + K_2)$ -maximal 1-decomposition. Since  $\mathcal{D}_3$  is a  $(P_3 + K_2)$ -maximal 3-decomposition and  $P_3 + K_2$  is an ID-graph, it follows that  $G$  has a  $(P_3 + K_2)$ -maximal 2-decomposition (and so an  $S$ -maximal 2-decomposition) which is impossible.

Next, suppose that  $H_1 \cong K_{1,3}$  (see Figure 4.13). We show that there is an edge in  $R_1$  that is adjacent to an edge in  $H_1$  and there are two edges in  $R_1$  that are not adjacent to any edge in  $H_1$ . If there were no edge in  $R_1$  that is adjacent to an edge

in  $H_1$ , then  $G = K_{1,3} + tK_2$ . However then,  $G$  has no  $S$ -maximal 3-decomposition, which is impossible. Furthermore, since  $R_1 = tK_2$  (where  $t \geq 6$ ) and  $H_1$  has exactly four vertices, at most four edges in  $R_1$  can be adjacent to an edge in  $H_1$ . Hence at least two edges in  $R_1$  are not adjacent to any edges in  $H_1$ .

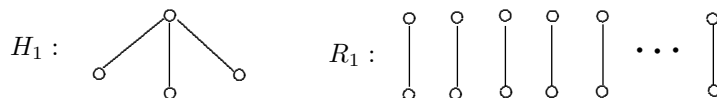


Figure 4.13: A step in the proof of Theorem 4.4.4

Let  $f_1, f_2, f_3 \in R_1$  such that  $f_1$  is adjacent an edge (say  $e_1$ ) in  $H_1$  and neither  $f_2$  nor  $f_3$  is adjacent to any edges in  $H_1$ . Let  $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$  and  $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$ . Since  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ , it follows that  $R_2 = G - E(F_1) \cup E(F_2)$  is a subgraph of  $R_1$ . Therefore,  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. ■

**Theorem 4.4.5** *The set  $\{3K_2, P_3 + K_2\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{3K_2, P_3 + K_2\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1, L_i \in S$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains a subgraph isomorphic to any graph in  $S$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$ . We may assume, without loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so

$|E(R_1)| = t \geq 6$ . Since  $R_1$  does not contain  $P_3 + K_2$  or  $3K_2$  as a subgraph,  $R_1 = K_{1,t}$  or  $R_1 = K_4$  (see Figure 4.14). We consider two cases, according to whether  $H_1 \cong 3K_2$  or  $H_1 \cong P_3 + K_2$ .

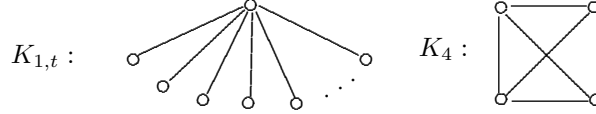


Figure 4.14: Two possible graphs for  $R_1$  in the proof of Theorem 4.4.5

*Case 1.*  $H_1 \cong 3K_2$ . First, suppose that there is an edge  $e \in E(R_1) - E(L_1)$  that is adjacent to neither  $e_2$  nor  $e_3$ . Then let  $F_1 = L_1$ ,  $F_2 = G[\{e_2, e_3, e\}] \cong 3K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Next, suppose that every edge in  $E(R_1) - E(L_1)$  is adjacent to  $e_2$  or  $e_3$ . Since  $|E(R_1) - E(L_1)| \geq 4$  and  $R_1 = K_{1,t}$  or  $R_1 = K_4$ , there is an edge  $f$  such that  $f$  is adjacent to exactly one of  $e_2$  and  $e_3$ . Let  $F_1 = L_1$ ,  $F_2 = G[\{e_2, e_3, f\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible.

*Case 2.*  $H_1 \cong P_3 + K_2$ . Let  $H_1 - e_1 \cong P_3 = (u, v, w)$  where  $e_2 = uv$  and  $e_3 = vw$ . First, suppose that there is an edge  $e \in E(R_1) - E(L_1)$  such that  $e$  is adjacent to neither  $e_2$  nor  $e_3$ . Let  $F_1 = L_1$ ,  $F_2 = G[\{e_2, e_3, e\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Hence we may assume that every edge in  $E(R_1) - E(L_1)$  is adjacent to  $e_2$  or  $e_3$ . Then  $R_1$  cannot be  $K_4$  and so  $R_1 = K_{1,t}$  where  $V(R_1) = \{x, x_1, x_2, \dots, x_t\}$  and  $x$  is the central

vertex of  $R_1$  (see Figure 4.15). First, suppose that  $x \in \{u, v, w\}$ , say  $x = u$  or  $x = v$ . Then  $G - \{e_1, e_3\} \cong K_{1,t+1}$ . However then,  $G$  cannot have an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$ , a contradiction. Next, suppose that  $x \notin \{u, v, w\}$ . We may assume, without loss of generality, that  $xx_i \notin E(L_1)$  for  $1 \leq i \leq 4$ . Since each edge  $xx_i$  ( $1 \leq i \leq 4$ ) is incident with exactly one vertex in  $\{u, v, w\}$ , it follows that  $x_i \in \{u, v, w\}$  for  $1 \leq i \leq 4$ , this is impossible.  $\blacksquare$

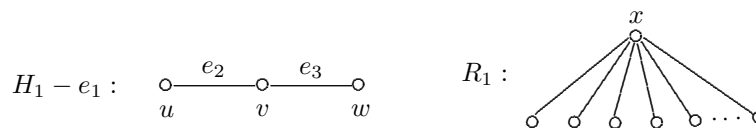


Figure 4.15: A step in the proof of Theorem 4.4.5

**Theorem 4.4.6** *Each of the sets  $\{K_{1,3}, P_4\}$  and  $\{K_3, P_4\}$  is an ID-set.*

**Proof.** Let  $S = \{K_{1,3}, P_4\}$  or  $S = \{K_3, P_4\}$ . Assume, to the contrary, that  $S$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2(III),  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but  $G$  has no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where each of  $H_1, L_1, L_2, L_3$  is isomorphic to some graph in  $S$  and  $R_1$  and  $R_3$  contain no subgraph isomorphic to any graph in  $S$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$  and we may assume that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| \geq 6$ .

We now construct an  $S$ -maximal 2-decomposition  $\mathcal{D}_2 = \{F_1, F_2, R_2\}$  of  $G$  as follows. Let  $e_1$  be an edge of  $H_1$  such that  $H_1 - e_1$  is connected. Then  $H_1 - e_1 = P_3$

(see Figure 4.16) is a component of size 2 in  $R_1$  at least one of whose two edges is adjacent to  $e_1$ . Let  $F_1 = L_1$ . To construct  $F_2$ , we consider two cases, according to whether  $S = \{K_{1,3}, P_4\}$  or  $S = \{K_3, P_4\}$ .

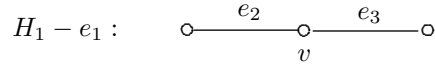


Figure 4.16: The graph  $H_1 - e_1$  in the proof of Theorem 4.4.6

*Case 1.*  $S = \{K_{1,3}, P_4\}$ . For  $H_1 - e_1 = P_3$ , there is an edge in  $L_2 - e_2$  (say  $f_2$ ) that is adjacent to  $e_2$  in  $H_1 - e_1$  and an edge in  $L_3 - e_3$  (say  $f_3$ ) that is adjacent to  $e_3$  in  $H_1 - e_1$ . If one of  $f_2$  and  $f_3$  is incident with the vertex  $v$  of degree 2 in  $H_1 - e_1$ , say  $f_2$  is incident with  $v$ , then let  $F_2 = G[\{e_2, e_3, f_2\}] \cong K_{1,3} \in S$ . Thus, we may assume each of  $f_2$  and  $f_3$  is incident with the two end-vertices of  $H_1 - e_1$ . Then at least one of  $f_2$  and  $f_3$ , say  $f_2$ , such that  $G[\{e_2, e_3, f_2\}] \neq K_3$ . Hence  $G[\{e_2, e_3, f_2\}] \cong P_4$  and let  $F_2 = G[\{e_2, e_3, f_2\}]$ .

*Case 2.*  $S = \{K_3, P_4\}$ . We show that there is an edge in  $E(R_1) - E(L_1)$  that is incident with an end-vertex of  $H_1 - e_1$ . If this is not the case, then neither  $L_2$  nor  $L_3$  is  $K_3$  and so  $L_2 \cong L_3 \cong P_4$ . Let  $E(L_2) - e_2 = \{e_4, e_5\}$  and  $E(L_3) - e_3 = \{e_6, e_7\}$ , where then no edge in  $\{e_4, e_5, e_6, e_7\}$  is incident with any end-vertex of  $H_1 - e_1$ . We may assume, without loss of generality, that  $L_2$  and  $L_3$  are the graphs shown in Figure 4.17. Since  $G[\{e_4, e_5, e_6, e_7\}] \cong P_5$  is a subgraph of  $R_1$ , it follows that  $R_1$  contains  $P_4$  as a subgraph, which is a contradiction. Therefore, there is an edge  $e \in E(R_1) - E(L_1)$  that is incident with an end-vertex of  $H_1 - e_1$ . Let  $F_2 = G[e_2, e_3, e]$ , which is either  $P_4$  or  $K_3$ .

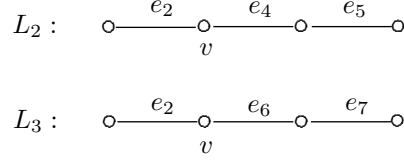


Figure 4.17: Graphs  $L_2$  and  $L_3$  in Case 2

In each case,  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$  and  $E(F_1) \cap E(F_2) = \emptyset$ . Hence  $R_2$  is a subgraph of  $R_1$  and so  $R_2$  contains no subgraph isomorphic to any graph in  $S$ . Therefore,  $\mathcal{D}_2 = \{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction.  $\blacksquare$

**Theorem 4.4.7** *The set  $\{3K_2, P_4\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{3K_2, P_4\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where  $H_1, L_i \in S$  ( $i = 1, 2, 3$ ) and neither  $R_1$  nor  $R_3$  contains an subgraph isomorphic to any graph in  $S$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$  and we may assume, without loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| = t \geq 6$ . We claim the following:

$$H_1 = 3K_2 \text{ and } L_i = P_4 \text{ for } i = 1, 2, 3. \quad (4.6)$$

We first show that  $H_1 = 3K_2$ . Assume, to the contrary, that  $H_1 = P_4 = (v_1, v_2, v_3, v_4)$  where  $e_i = v_i v_{i+1}$  for  $i = 1, 2, 3$ . We now show that  $L_i \cong P_4$  for

$i = 1, 2, 3$ . If this is not the case, then we may assume, without loss of generality, that  $L_1 \cong 3K_2$  or  $L_2 \cong 3K_2$ . Consider these two cases.

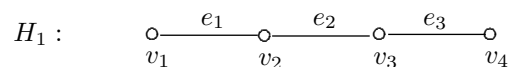


Figure 4.18: The graph  $H_1$  in the proof of Theorem 4.4.7

*Case 1.*  $L_1 \cong 3K_2$ . Let  $E(L_1) = \{e_1, f_1, f_2\}$  where  $e_1 = v_1v_2$ . Thus each  $f_i$  ( $i = 1, 2$ ) is incident with neither  $v_1$  nor  $v_2$ . We show that  $f_i$  ( $i = 1, 2$ ) is not incident with  $v_3$ . For otherwise, we may assume that  $f_1$  is incident with  $v_3$ . Let  $F_1 = G[\{e_1, e_2, f_1\}] \cong P_4$ ,  $F_2 = L_3$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Since  $f_1$  and  $f_2$  are nonadjacent, at most one of  $f_1$  and  $f_2$  can be incident with  $v_4$ . We may assume that  $f_1$  is not incident with  $v_4$ . Let  $F_1 = G[\{e_1, e_3, f_1\}] \cong 3K_2$ ,  $F_2 = L_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Thus,  $L_1 \cong P_4$ . Similarly,  $L_3 \cong P_4$ .

*Case 2.*  $L_2 \cong 3K_2$ . Let  $E(L_2) = \{e_2, g_1, g_2\}$  where  $e_2 = v_2v_3$ . Thus each  $g_i$  ( $i = 1, 2$ ) is incident with neither  $v_2$  nor  $v_3$ . We show that each  $g_i$  ( $i = 1, 2$ ) is incident with neither  $v_1$  nor  $v_4$ . For otherwise, we may assume that  $g_1$  is incident with  $v_4$ . Let  $F_1 = L_1$ ,  $F_2 = G[\{g_1, e_2, e_3\}] \cong P_4$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Hence neither  $g_1$  nor  $g_2$  in  $L_2$  is adjacent to any edge in  $\{e_1, e_2, e_3\}$ . Since  $L_1 \cong P_4$  (by Case 1), there is an edge  $f \in L_1 - \{e_1\}$



that is adjacent to  $e_1 = v_1v_2$  and so  $f$  is incident with exactly one of  $v_1$  and  $v_2$ . Thus  $G$  contains a subgraph  $F$  isomorphic to one of the graphs in Figure 4.19(a)–(e).

- If  $F$  is the graph in Figure 4.19(a), let  $F_1 = G[\{f, e_1, e_2\}] \cong P_4$  and  $F_2 = L_3$ .
- If  $F$  is the graph in Figure 4.19(b)–(d), let  $F_1 = G[\{e_1, f, e_3\}] \cong P_4$  and  $F_2 = L_2$ .
- If  $F$  is the graph in Figure 4.19(e), let  $F_1 = G[\{f, e_2, e_3\}] \cong P_4$  and  $F_2 = \{e_1, g_1, g_2\} \cong 3K_2$ .

In each case, let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Thus,  $L_2 \cong P_4$ .

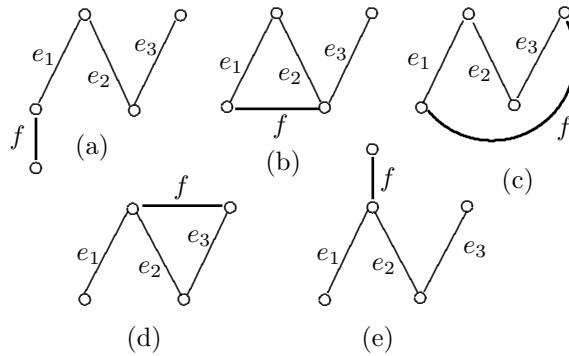


Figure 4.19: A step in the proof of Theorem 4.4.7

Therefore, if  $H_1 \cong P_4$ , then  $L_i \cong P_4$  for  $i = 1, 2, 3$ . Hence  $\mathcal{D}_1$  is a  $P_4$ -maximal 1-decomposition and  $\mathcal{D}_3$  is a  $P_4$ -maximal 3-decomposition. However then, since  $P_4$  is an ID-graph,  $G$  has a  $P_4$ -maximal 2-decomposition (and so an  $S$ -maximal 2-decomposition), which is impossible. Therefore, as claim,  $H_1 = 3K_2$ .

Next, we show that  $L_i = P_4$  for  $i = 1, 2, 3$ . We consider two cases.

*Case (i).* At least two of  $L_1, L_2$  and  $L_3$  are isomorphic to  $3K_2$ , say  $L_1 \cong L_2 \cong 3K_2$ . Let  $E(L_1) = \{e_1, f_1, f_2\}$  and  $E(L_2) = \{e_2, g_1, g_2\}$ . We show that each  $f_i$  ( $i = 1, 2$ ) is adjacent to both  $e_2$  and  $e_3$  and each  $g_i$  ( $i = 1, 2$ ) is adjacent to both  $e_1$  and  $e_3$ . If this is not the case, we may assume that  $f_1$  is not adjacent to  $e_2$ . Then let  $F_1 = G[\{e_1, e_2, f_1\}] \cong 3K_2$ , let  $F_2 = L_3$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Therefore, we may assume that  $G$  contains a subgraph of Figure 4.20. Let  $F_1 = G[\{g_1, e_1, g_2\}] \cong P_4$ ,  $F_2 = G[\{e_2, f_1, e_3\}] \cong P_4$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible.

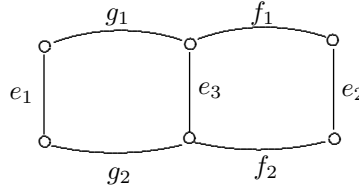


Figure 4.20: A step in Case (i) of the proof of Theorem 4.4.7

*Case (ii).* Exactly one of  $L_1, L_2$  and  $L_3$  is isomorphic to  $3K_2$ , say  $L_1 \cong 3K_2$  and  $L_i = P_4$  for  $i = 2, 3$ . Again, let  $E(L_1) = \{e_1, f_1, f_2\}$ . By the argument in Case (i), each of  $f_1$  and  $f_2$  is adjacent to  $e_2$  and  $e_3$ . Thus,  $G$  contains the graph of Figure 4.21(a) as a subgraph. We first show that no edge in  $L_2 - e_2$  can be adjacent to both  $e_1$  and  $e_2$  and no edge in  $L_3 - e_3$  can be adjacent to both  $e_1$  and  $e_3$ . If this is not case, we may assume that  $g \in E(L_2 - e_2)$  that is adjacent to both  $e_1$  and  $e_2$ . Let  $F_1 = G[\{e_1, g, e_2\}] \cong P_4$ ,  $F_2 = G[\{f_1, e_3, f_2\}] \cong P_4$  and

let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Suppose that  $E(L_2) = \{e_2, g_1, g_2\}$  and  $E(L_3) = \{e_3, h_1, h_2\}$  where  $g_1$  is adjacent to  $e_2$  and  $h_1$  is adjacent to  $e_3$ . Note that neither  $g_1$  nor  $h_1$  can be adjacent to both  $e_2$  and  $e_3$ ; for otherwise, we may assume that  $g_1$  is adjacent to both  $e_2$  and  $e_3$ . Then let  $F_1 = L_1$ ,  $F_2 = G[\{e_2, g_1, e_3\}] \cong P_4$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Hence  $G$  contains one of the graphs of Figure 4.21(b)–(d) as a subgraph.

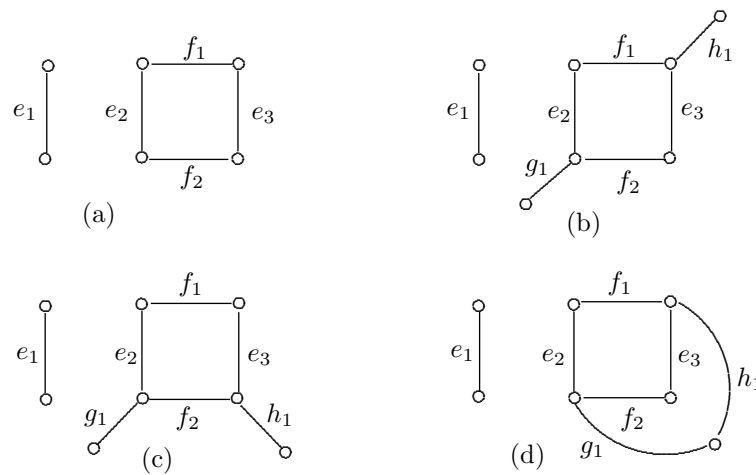


Figure 4.21: A step in Case (ii) of the proof of Theorem 4.4.7

First, suppose that  $G$  contains a subgraph isomorphic to the graph in Figure 4.21(b). Now let  $F_1 = G[\{e_1, g_1, h_1\}] \cong 3K_2$ ,  $F_2 = G[\{e_2, f_2, e_3\}] \cong P_4$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Next, suppose

that  $G$  contains a subgraph isomorphic to the graph in Figure 4.21(c) or in Figure 4.21(d). However then,  $R_1$  contains the subgraph  $G[\{g_1, f_2, h_1\}] \cong P_4$ , which is a contradiction.

Therefore,  $H_1 = 3K_2$  and  $L_i = P_4$  for  $i = 1, 2, 3$ , as we claimed in (4.6). We now show that if an edge in  $L_i - e_i$  that is adjacent to  $e_i$ , then this edge is not adjacent to any edges in  $E(H_1) - \{e_i\}$  for  $i = 1, 2, 3$ . If this is not the case, we may assume that  $f_1 \in E(L_1)$  is adjacent to  $e_1$  and  $e_2$ . Let  $F_1 = G[\{e_1, f_1, e_2\}] \cong P_4$ ,  $F_2 = L_3$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Then  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Next, we show that each  $e_i$  is the interior edge of  $L_i$  for  $i = 1, 2, 3$ . If this is not the case, we may assume that  $L_1 = (e_1, f_1, f_2)$ . If  $f_2$  is not adjacent to  $e_2$ , then let  $F_1 = G[\{e_1, f_2, e_2\}] \cong 3K_2$ ,  $F_2 = L_3$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Then  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. Similarly, if  $f_2$  is not adjacent to  $e_3$ , then there is an  $S$ -maximal 2-decomposition, which is impossible. Thus  $f_2$  is adjacent to both  $e_2$  and  $e_3$ . However then, since  $f_1$  is adjacent to  $f_2$ , either  $f_1$  and  $e_2$  are adjacent or  $f_1$  and  $e_3$  are adjacent, which is impossible. Therefore,  $e_i$  is the interior edge of  $L_i$  for  $i = 1, 2, 3$ , as claimed.

Let  $L_1 = (f_1, e_1, f_2)$ ,  $L_2 = (g_1, e_2, g_2)$  and  $L_3 = (h_1, e_3, h_2)$ . It then follows from the argument above that  $L_1$ ,  $L_2$  and  $L_3$  have the following properties:

- (a) No edge in  $L_i$  ( $i = 1, 2, 3$ ) is adjacent to any edges in  $\{e_1, e_2, e_3\} - \{e_i\}$ .
- (b) Since  $R_1$  contains no subgraph isomorphic to  $P_4$ , it follows that  $\{f_1, f_2\}$ ,  $\{g_1, g_2\}$  and  $\{h_1, h_2\}$  are sets of two independent edges.

Since  $R_1$  contains no  $3K_2$ , there are adjacent edges in  $\{f_1, f_2, g_1, g_2, h_1, h_2\}$ . By (a) and (b), we may assume that  $f_1$  and  $g_1$  are adjacent. Let  $F_1 = G[\{e_1, f_1, g_1\}] \cong P_3$ ,  $F_2 = \{f_2, e_2, e_3\} \cong 3K_2$  and let  $R_2 = G - (E(F_1) \cup E(F_2))$ . Then  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition, which is impossible. ■

#### 4.4.2 The 3-Element or 4-Element ID-Sets

By Observation 4.3.1, the set  $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  consisting of all graphs of size 3 is an ID-set. Also, by the table in Figure 4.9, we have the following:

- (1) A graph  $H$  of size 2 or 3 is an ID-graph unless  $H \in \{K_3, K_{1,3}, 3K_2\}$ .
- (2) Every 2-element subset  $S$  of  $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  is an ID-set unless  $S$  is a 2-element subset of  $\{K_3, K_{1,3}, 3K_2\}$ .

Therefore, it remains to determine the ID-sets and non-ID sets that are subsets  $S$  of  $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  with  $|S| = 3$  or  $|S| = 4$ .

We first show that neither of the sets  $\{K_3, 3K_2, P_4\}$  and  $\{K_{1,3}, 3K_2, P_4\}$  is an ID set. For  $\{K_3, 3K_2, P_4\}$ , let  $G = K_3 + 2K_{1,3}$  be the union of  $K_3$  and two copies of  $K_{1,3}$ . Then  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$ , where  $H_1 \cong K_3$  and  $R_1 \cong 2K_{1,3}$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$  where  $L_1 \cong L_2 \cong L_3 \cong 3K_2$  and  $R_3$  is an empty graph. However,  $G$  has no  $S$ -maximal 2-decomposition. For  $\{K_{1,3}, 3K_2, P_4\}$ , let  $G = 2K_3 + K_{1,3}$  be the union of two copies of  $K_3$  and  $K_{1,3}$ . Then  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$ , where  $H_1 \cong K_{1,3}$  and  $R_1 \cong 2K_3$  and an  $S$ -maximal 3-decomposition

$\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$  where  $L_1 \cong L_2 \cong L_3 \cong 3K_2$  and  $R_3$  is an empty graph. However,  $G$  has no  $S$ -maximal 2-decomposition. Hence neither set is an ID-set.

Next, we show that if  $S \subseteq \{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  with  $|S| \in \{3, 4\}$  such that  $S$  is neither  $\{K_3, 3K_2, P_4\}$  nor  $\{K_{1,3}, 3K_2, P_4\}$ , then  $S$  is an ID-set. In order to do this, we first present three useful observations (some of which we already saw earlier).

**Observation 4.4.8** *Let  $S$  be a non-ID-set of graphs of size 3. If  $G$  is a minimum non-IDP- $S$  graph having an  $S$ -maximal 1-decomposition  $\{H_1, R_1\}$ , then the size of  $R_1$  is at least 6.*

**Observation 4.4.9** *Suppose that  $R$  is a graph without isolated vertices having size  $t \geq 6$ .*

- (a) *If  $R$  does not contain  $P_3 + K_2$  as a subgraph, then  $R = tK_2$ ,  $R = K_{1,t}$  or  $R = K_4$ .*
- (b) *If  $R$  does not contain  $3K_2$  as a subgraph, then  $R$  has at most two components and  $R = K_{1,t}$ ,  $R = K_{1,r} + K_{1,s}$  where  $1 \leq r \leq s$  and  $r + s = t$ ,  $R = 2K_3$  or  $R = K_3 + K_{1,t-3}$ .*
- (c) *If  $R$  does not contain  $P_4$  as a subgraph, then each component of  $R$  is  $K_3$  or stars.*

**Observation 4.4.10** *Suppose that  $S$  is a non-ID-set of graphs and  $S$  contains an ID-subset  $S_0$ . If  $G$  is a non-IDP- $S$  graph such that  $G$  has an  $S$ -maximal  $a$ -decomposition  $\mathcal{D}_a = \{H_1, H_2, \dots, H_a, R_a\}$  and an  $S$ -maximal  $b$ -decomposition*

$\mathcal{D}_b = \{L_1, L_2, \dots, L_b, R_b\}$  but no  $S$ -maximal  $k$ -decomposition for every integer  $k$  with  $a < k < b$ , then either  $H_i \in S - S_0$  for some  $i \in \{1, 2, \dots, a\}$  or  $L_j \in S - S_0$  for some  $j \in \{1, 2, \dots, b\}$ .

**Proposition 4.4.11** *The set  $\{K_{1,3}, K_3, P_4\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{K_{1,3}, K_3, P_4\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. Since each graph in  $S$  has size 3, it follows by Theorem 4.3.2(III) that  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3$  but  $G$  has no  $S$ -maximal 2-decomposition. Let  $\mathcal{D}_1 = \{H_1, R_1\}$  and  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ , where each of  $H_1, L_1, L_2, L_3$  is isomorphic to some graph in  $S$  and  $R_1$  and  $R_3$  contain no subgraph isomorphic to any graph in  $S$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$ . We may assume, without loss of generality, that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  by Theorem 4.3.2(II). Since  $L_i$  and  $L_j$  are edge-disjoint for  $i \neq j$  and  $i, j \in \{1, 2, 3\}$ , it follows that  $L_i - e_i$  is a subgraph of  $R_1$  and so  $|E(R_1)| \geq 6$ . Furthermore, each component of  $R_1$  has size at most 2.

We now construct an  $S$ -maximal 2-decomposition  $\mathcal{D}_2 = \{F_1, F_2, R_2\}$  of  $G$  as follows. Let  $F_1 = L_1 \in S$ . Now, let  $e \in E(L_2) - \{e_2\}$  that is adjacent to  $e_2$ . Then the subgraph  $F_2 = G[\{e_2, e_3, e\}]$  induced by  $\{e_2, e_3, e\}$  is a connected subgraph of size 3 and so  $F_2 \in S$ . Furthermore,  $E(F_1) \cap E(F_2) = \emptyset$ . Since  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ , it follows that  $R_2$  is a subgraph of  $R_1$  and so  $R_2$  contains no subgraph isomorphic to any graph in  $S$ . Therefore,  $\mathcal{D}_2 = \{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction. ■

**Proposition 4.4.12** *Each of the following sets is an ID-set:*

$$\{K_{1,3}, K_3, 3K_2\}, \{K_{1,3}, K_3, 3K_2, P_4\} \text{ and } \{K_{1,3}, K_3, 3K_2, P_3 + K_2\}. \quad (4.7)$$

**Proof.** Let  $S$  be one of the sets in (4.7). Assume, to the contrary, that  $S$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  where  $|E(R_1)| = t \geq 6$  by Observation 4.4.8. Since  $R_1$  does not contain  $3K_2$  as a subgraph,  $R_1 = K_{1,t}$ ,  $R_1 = K_{1,r} + K_{1,s}$  where  $1 \leq r \leq s$  and  $r + s = t$ ,  $R_1 = 2K_3$  or  $R_1 = K_3 + K_{1,t-3}$  by Observation 4.4.9(b). Since  $R_1$  contains neither  $K_{1,3}$  nor  $K_3$  and  $|E(R_1)| = t \geq 6$ , this is impossible. ■

**Proposition 4.4.13** *Each of the following sets is an ID-set:*

$$\{K_{1,3}, 3K_2, P_3 + K_2\} \text{ and } \{K_{1,3}, 3K_2, P_3 + K_2, P_4\}. \quad (4.8)$$

**Proof.** Let  $S$  be one of the sets in (4.8). Assume, to the contrary, that  $S$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. By Theorem 4.3.2 then,  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  where  $|E(R_1)| = t \geq 6$  by Observation 4.4.8. Since  $R_1$  does not contain  $P_3 + K_2$  as a subgraph,  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$  or  $R_1 = K_4$  by Observation 4.4.9(a). Since  $R_1$  contains neither  $K_{1,3}$  nor  $3K_2$  as a subgraph and  $t \geq 6$ , this is impossible. ■

**Proposition 4.4.14** *The set  $\{K_{1,3}, K_3, P_3 + K_2\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{K_{1,3}, K_3, P_3 + K_2\}$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. Then  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$



but no  $S$ -maximal 2-decomposition. Then  $|E(R_1)| = t \geq 6$  by Observation 4.4.8. Since  $R_1$  does not contain  $P_3 + K_2$  as a subgraph,  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$  or  $R_1 = K_4$  by Observation 4.4.9(a). Since  $R_1$  contains neither  $K_{1,3}$  nor  $K_3$  as a subgraph and  $t \geq 6$ , it follows that  $R_1 = tK_2$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$ , where say  $e_i \in E(L_i)$  for  $i = 1, 2, 3$ , and so  $L_i - e_i$  is a subgraph of  $R_1$ . Since  $R_1 = tK_2$ , it follows that  $L_i - e_i = 2K_2$  and so  $L_i = P_3 + K_2$  for  $i = 1, 2, 3$ .

- If  $H_1 = K_{1,3}$ , then let  $S' = \{K_{1,3}, P_3 + K_2\}$ .
- If  $H_1 = K_3$ , then let  $S' = \{K_3, P_3 + K_2\}$ .
- If  $H_1 = P_3 + K_2$ , then let  $S' = \{P_3 + K_2\}$ .

In each case,  $S'$  is an ID-set. Since  $\mathcal{D}_1$  is an  $S'$ -maximal 1-decomposition and  $\mathcal{D}_3$  is an  $S'$ -maximal 3-decomposition, it follows that  $G$  has an  $S'$ -maximal 2-decomposition, which is a contradiction. ■

**Proposition 4.4.15** *Each of the following sets is an ID-set:*

$$\{K_{1,3}, P_4, P_3 + K_2\} \text{ and } \{K_{1,3}, K_3, P_4, P_3 + K_2\}. \quad (4.9)$$

**Proof.** Let  $S$  be one of the sets in (4.9). Assume, to the contrary, that  $S$  is not an ID-set. Let  $G$  be a minimum non-IDP- $S$  graph. Then  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$  but no  $S$ -maximal 2-decomposition. Then  $|E(R_1)| = t \geq 6$  by Observation 4.4.8. Since  $R_1$  does not contain  $P_3 + K_2$  as a subgraph,  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$  or  $R_1 = K_4$  by Observation 4.4.9(a). Since  $R_1$  contains neither  $K_{1,3}$  nor

$P_4$  as a subgraph and  $t \geq 6$ , it follows that  $R_1 = tK_2$ . Let  $E(H_1) = \{e_1, e_2, e_3\}$ , where say  $e_i \in E(L_i)$  for  $i = 1, 2, 3$ , and so  $L_i - e_i$  is a subgraph of  $R_1$ . We consider two cases.

*Case 1.*  $S = \{K_{1,3}, P_4, P_3 + K_2\}$ . Then  $L_i \in \{P_4, P_3 + K_2\}$ . Since  $\{P_4, P_3 + K_2\}$  is an ID-set, it follows that  $H_1 \cong K_{1,3}$ . Because  $\{K_{1,3}, P_4\}$  and  $\{K_{1,3}, P_3 + K_2\}$  are both ID-sets, at least one of  $L_i$  ( $1 \leq i \leq 3$ ) is  $P_4$  and at least one of  $L_i$  ( $1 \leq i \leq 3$ ) is  $P_3 + K_2$ . We may assume that  $L_1 \cong P_4 = (f_1, e_1, f_2)$  where  $e_1$  is the middle edge of  $L_1$  and  $L_2 \cong P_3 + K_2$  where  $e_2$  is adjacent an edge in  $L_2$ , say  $e_2$  is adjacent to  $g$  in  $L_2$ . This implies that  $G$  contains a subgraph isomorphic to one of the graphs in Figures 4.22(a) and 4.22(b), where the edges in  $L_1$  are drawn in bold. In each case, let  $F_1 = L_1$  and  $F_2 = G[\{e_2, e_3, g\}]$ . Thus  $F_2 \cong K_{1,3}$  or  $F_2 \cong P_4$ . Since  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ ,  $E(F_1) \cap E(F_2) = \emptyset$ , and  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction.

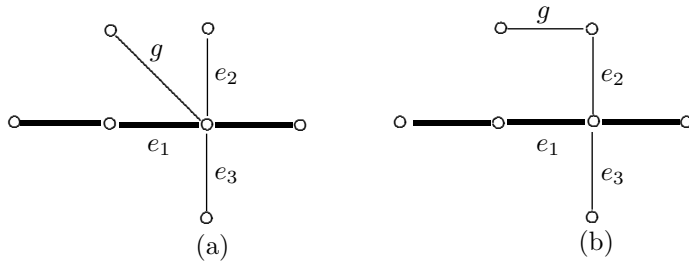


Figure 4.22: A step in the proof of Proposition 4.4.15

*Case 2.*  $S = \{K_{1,3}, K_3, P_4, P_3 + K_2\}$ . Then  $L_i \in \{P_4, P_3 + K_2\}$ , where  $i = 1, 2, 3$ . In particular,  $L_i \not\cong K_3$  for  $i = 1, 2, 3$ . Furthermore, we claim that  $H_1 \not\cong K_3$ . If

this were not the case, then observe that at least one edge of  $R_1$  is adjacent to some edge of  $H_1$ ; for otherwise,  $G = K_3 + tK_2$  and  $G$  cannot have an  $S$ -maximal 3-decomposition. On the other hand, since  $t \geq 6$ , at least two edges of  $R_1$  are not adjacent to any edge of  $H_1$ . Let  $f_1, f_2, f_3 \in E(R_1)$  such that  $f_1$  is adjacent to some edge of  $H_1$ , say  $f_1$  is adjacent to  $e_1$ , while neither  $f_2$  nor  $f_3$  is adjacent to any edge of  $H_1$ . Let  $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$ ,  $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ ,  $E(F_1) \cap E(F_2) = \emptyset$  and  $R_2$  is a subgraph of  $R_1$ , it follows that  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction. Thus,  $H_1 \not\cong K_3$ , as claimed. Hence  $H_1, L_i \in S' = \{K_{1,3}, P_4, P_3 + K_2\}$  for  $i = 1, 2, 3$ . Since  $S'$  is an ID-set by Case 1, it follows that  $G$  has an  $S$ -maximal 2-decomposition, which, again, is impossible. ■

In order to show that the remaining sets of graphs of size 3 are ID-sets, we first present a lemma.

**Lemma 4.4.16** *Let  $S = \{K_3, P_4, P_3 + K_2\}$ . If  $G$  is a minimum non-IDP- $S$  graph and  $\mathcal{D}_1 = \{H_1, R_1\}$  is an  $S$ -maximal 1-decomposition, then  $R_1 \neq tK_2$  where  $t \geq 6$ .*

**Proof.** Assume, to the contrary, that  $R_1 = tK_2$  where  $t \geq 6$ . Since  $G$  is a minimum non-IDP- $S$  graph,  $G$  also has an  $S$ -maximal 3-decomposition  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$  but no  $S$ -maximal 2-decomposition. Let  $E(H_1) = \{e_1, e_2, e_3\}$ , where say  $e_i \in E(L_i)$  for  $i = 1, 2, 3$ , and so  $L_i - e_i$  is a subgraph of  $R_1$ . Since  $R_1 = tK_2$ , each  $e_i$  ( $i = 1, 2, 3$ ) must be adjacent to some edge of  $R_1$ ; for otherwise, at least one of  $L_1, L_2, L_3$  is  $3K_2$ , which is impossible.

First, suppose that  $H_1 = K_3$  or  $H_1 = P_4$ . Then at least one edge of  $R_1$  is adjacent to some edge of  $H_1$ . Since  $R_1 = tK_2$  and  $t \geq 6$ , at least two edges of  $R_1$  are not adjacent to any edge of  $H_1$ . Let  $f_1, f_2, f_3 \in E(R_1)$  such that  $f_1$  is adjacent to some edge of  $H_1$ , say  $f_1$  is adjacent to  $e_1$ , while neither  $f_2$  nor  $f_3$  is adjacent to any edge of  $H_1$ . Let  $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$ ,  $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Since  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$  and  $E(F_1) \cap E(F_2) = \emptyset$ , it follows that  $R_2$  is a subgraph of  $R_1$  and  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction.

Next, suppose that  $H_1 = P_3 + K_2$ . We may assume that  $e_1$  and  $e_2$  are adjacent to edges in  $H_1$ . Since  $t \geq 6$ , there is an edge in  $R_1$  that is not adjacent to any edge of  $H_1$ . Furthermore, since  $G$  is connected, at least one of  $e_1$  and  $e_2$  is adjacent to some edge of  $R_1$  and  $e_3$  is adjacent to some edge of  $R_1$ . Let  $f_1, f_2, f_3 \in E(R_1)$  such that  $f_1$  is not adjacent to any edge of  $H_1$ ,  $f_2$  is adjacent to  $e_2$  and  $f_3$  is adjacent to  $e_3$ . First, suppose that  $f_2 \neq f_3$ . Let  $F_1 = G[\{e_2, f_1, f_2\}] \cong P_3 + K_2$ ,  $F_2 = G[\{e_1, e_3, f_3\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . Next, suppose that  $f_2 = f_3$ . Since  $t \geq 6$ , there is  $f_4 \in E(R_1) - \{f_1\}$  such that  $f_4$  is not adjacent to any edge of  $H_1$ . Let  $F_1 = G[\{e_1, e_2, f_4\}] \cong P_3 + K_2$ ,  $F_2 = G[\{e_3, f_1, f_3\}] \cong P_3 + K_2$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ . In either case,  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$  and  $E(F_1) \cap E(F_2) = \emptyset$ . Therefore,  $R_2$  is a subgraph of  $R_1$  and so  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction. ■

**Proposition 4.4.17** *Each of the following sets is an ID-set:*

$$\{K_3, 3K_2, P_3 + K_2\}, \{K_3, P_4, P_3 + K_2\},$$

$$\{3K_2, P_4, P_3 + K_2\}, \{K_3, 3K_2, P_4, P_3 + K_2\}.$$

**Proof.** Let  $S$  be one of the sets described above. Assume, to the contrary, that  $S$  is not an ID-set. Then  $G$  has an  $S$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  and an  $S$ -maximal 3-decomposition  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$  but no  $S$ -maximal 2-decomposition. Then  $|E(R_1)| = t \geq 6$  by Observation 4.4.8. Since  $R_1$  does not contain  $P_3 + K_2$  as a subgraph,  $R_1 = tK_2$ ,  $R_1 = K_{1,t}$  or  $R_1 = K_4$  by Observation 4.4.9(a). For each set  $S$  under consideration, it follows that (i) either  $3K_2 \in S$  or  $S = \{K_3, P_4, P_3 + K_2\}$  and (ii) either  $P_4 \in S$  or  $K_3 \in S$ . Hence  $R_1 \neq tK_2$  (by Lemma 4.4.16) and  $R_1 \neq K_4$ . Therefore,  $R_1 = K_{1,t}$ . Let  $E(R_1) = \{f_1, f_2, \dots, f_t\}$  where  $t \geq 6$  and let  $E(H_1) = \{e_1, e_2, e_3\}$  where, say,  $e_i \in E(L_i)$  for  $i = 1, 2, 3$ , and so  $L_i - e_i$  is a subgraph of  $R_1$ . First, we make an observation. For  $i = 1, 2, 3$ , since (a)  $L_i \in S$  and  $K_{1,3} \notin S$  and (b)  $L_i - e_i$  is a subgraph of  $R_1$  and  $R_1 = K_{1,t}$ , it follows that  $e_i$  is not incident with the center vertex of  $R_1$  (or  $e_i$  cannot be adjacent to all edges in  $R_1$ ). Since  $H_1 \in \{K_3, 3K_2, P_4, P_3 + K_2\}$ , we consider four cases.

*Case 1.*  $H_1 = K_3$ . Then at least three edges in  $R_1$  that are not adjacent to any edges in  $H_1$ , say  $f_1, f_2$  and  $f_3$  are three such edges in  $R_1$ . Let  $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2 \in S$ ,  $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2 \in S$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ .

*Case 2.*  $H_1 = 3K_2$ . Then  $\{3K_2, P_3 + K_2\} \subseteq S$ . We may assume that  $e_1$  is adjacent to  $f_1$  (and possibly to  $f_2$ ),  $e_2$  is adjacent to  $f_3$  (and possibly to  $f_4$ ) and  $e_3$  is adjacent to  $f_5$  (and possibly to  $f_6$ ). More precisely, neither  $e_1$  nor  $e_3$  is adjacent to  $f_3$  and  $e_2$  is not adjacent to  $f_1$  or  $f_2$ . Let  $F_1 = G[\{e_1, e_3, f_3\}] \cong 3K_2 \in S$ ,  $F_2 = G[\{e_2, f_1, f_2\}] \cong P_3 + K_2 \in S$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ .

*Case 3.*  $H_1 = P_4$ . As we observed earlier, no edge in  $H_1$  is incident with the center vertex of  $R_1$ . Thus at least two edges in  $R_1$  are not adjacent to any edge in  $H_1$ , say  $f_1$  and  $f_2$  are two such edges in  $R_1$ . We may assume that  $H_1 = (e_1, e_2, e_3)$ . Then there is  $f_3 \in E(R_1) - \{f_1, f_2\}$  such that  $f_3$  is not adjacent to  $e_1$ . Let  $F_1 = G[\{e_1, f_1, f_3\}] \cong P_3 + K_2 \in S$  where  $E(P_3) = \{f_1, f_3\}$ ,  $F_2 = G[\{e_2, e_3, f_2\}] \cong P_3 + K_2 \in S$ , where  $E(P_3) = \{e_2, e_3\}$ , and  $R_2 = G - (E(F_1) \cup E(F_2))$ .

*Case 4.*  $H_1 = P_3 + K_2$ . Let  $e_1$  and  $e_2$  be the two adjacent edges in  $H_1$ . We may assume (i)  $f_1$  is not adjacent to  $e_1$  or  $e_2$  and (ii)  $f_2$  and  $f_3$  are not adjacent to  $e_3$ . Let  $F_1 = G[\{e_1, e_2, f_1\}] \cong P_3 + K_2 \in S$ ,  $F_2 = G[\{e_3, f_2, f_3\}] \cong P_3 + K_2 \in S$  and  $R_2 = G - (E(F_1) \cup E(F_2))$ .

In each case,  $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$  and  $E(F_1) \cap E(F_2) = \emptyset$ . Hence  $R_2$  is a subgraph of  $R_1$  and so  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction. ■

In summary, we have the following.

**Theorem 4.4.18** *A subset  $S$  of the set  $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  of all graphs of size 3 without isolated vertices is an ID-set if and only if  $S$  is not one of the following eight sets:*

$$\begin{aligned} & \{3K_2\}, \{K_3\}, \{K_{1,3}\}, \{3K_2, K_3\}, \{3K_2, K_{1,3}\}, \\ & \{K_3, K_{1,3}\}, \{3K_2, K_3, P_4\}, \{3K_2, K_{1,3}, P_4\}. \end{aligned}$$

# Chapter 5

## Topics for Further Study

### 5.1 Eulerian Irregularity Spectrum

Let  $G$  be a nontrivial connected graph of size  $m$ . Recall that an *Eulerian walk* in  $G$  is a closed walk that contains every edge of  $G$ . An *irregular Eulerian walk* in  $G$  is an Eulerian walk that encounters no two edges of  $G$  the same number of times. The minimum length of an irregular Eulerian walk in  $G$ , is referred to as the *Eulerian irregularity* of  $G$ , which is denoted by  $EI(G)$ . For a set  $\mathcal{G}$  of connected graphs with a prescribed property, define the *Eulerian irregularity spectrum*  $S(\mathcal{G})$  of  $\mathcal{G}$  to be the set of all values of  $EI(G)$  where  $G \in \mathcal{G}$ ; that is,  $S(\mathcal{G}) = \{EI(G) : G \in \mathcal{G}\}$ . Recall the following result on the Eulerian irregularities of graphs in Chapter 2.

**Theorem 5.1.1** *If  $G$  is a nontrivial connected graph  $G$  of size  $m$ , then*

$$\binom{m+1}{2} \leq EI(G) \leq 2 \binom{m+1}{2}.$$

*Furthermore,*

- (a)  $EI(G) = 2 \binom{m+1}{2}$  if and only if  $G$  is a tree.

(b)  $EI(G) = \binom{m+1}{2}$  if and only if  $G$  contains a subgraph of size  $\lceil m/2 \rceil$ , every vertex of which is even.

For a positive integer  $m$ , let  $\mathcal{G}_m$  be the set of all connected graphs of size  $m$ . It then follows by Theorem 5.1.1 that both  $\binom{m+1}{2}$  and  $2\binom{m+1}{2}$  are elements of  $S(\mathcal{G}_m)$ . Also, recall the following realization result on the Eulerian irregularities of graphs (Theorem 2.4.5 in Chapter 2).

**Theorem 5.1.2** *Let  $k$  and  $m$  be positive integers with  $\binom{m+1}{2} \leq k \leq 2\binom{m+1}{2}$ . Then there exists a nontrivial connected graph  $G$  of size  $m$  with  $EI(G) = k$  if and only if there exists an integer  $x$  with  $0 \leq x \leq m$  and  $x \neq 1, 2$  such that  $x^2 + (m-x)(m-x+1) = k$ .*

By Theorem 5.1.1, if  $\mathcal{T}_n$  is the set of all trees of order  $n$ , then  $S(\mathcal{T}_n) = \{n(n-1)\}$ . For a nontrivial connected graph  $G$  that is not a tree, it follows by the proof of Theorem 5.1.1 that if  $e$  is a bridge of  $G$  and  $W$  is an Eulerian walk in  $G$ , then  $e$  must be encountered an even number of times on  $W$ . Thus every bridge of  $G$  must be encountered at least twice on any Eulerian walk in  $G$ . Also, recall the following theorem that appeared in Chapter 2.

**Kwan's Theorem** *Let  $G$  be a connected graph and let  $W$  be a closed walk of minimum length containing every edge of  $G$  at least once. Then  $W$  encounters no edge of  $G$  more than twice and no more than half of the edges in any cycle appear twice.*



We plan to study the Eulerian irregularity spectra for some classes of connected graphs that are not tree. In particular, we are interested in the Eulerian irregularity spectrum of the set of all 2-connected graphs having a fixed size.

## 5.2 Proper Eulerian Walks

An Eulerian walk  $W$  in a connected graphs  $G$  is *proper* if every two adjacent edges of  $G$  are encountered a different number of times in  $W$ . This is equivalent to assigning positive integer weights to the edges of  $G$  such that the resulting edge labeling is also a proper edge coloring of  $G$  and the degree of every vertex in the resulting weighted graph is even. This is shown in Figure 5.1. Since every connected graph has an irregular Eulerian walk and every irregular Eulerian walk is proper, it follows that every connected graph has a proper Eulerian walk.

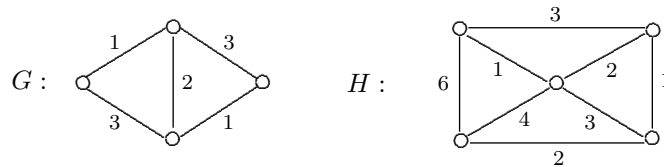


Figure 5.1: Proper Eulerian walks

The *Eulerian chromaticity* of  $G$  is the minimum length of a proper Eulerian walk  $W$  in  $G$  and is denoted by  $EC(G)$ . Since every irregular Eulerian walk is proper,  $EC(G) \leq EI(G)$  for every connected graph  $G$ . This is equivalent to minimizing the *sum* of the weights that can be assigned to the edges of  $G$  such that the resulting edge labeling is also a proper edge coloring of  $G$  and the degree of every vertex in the resulting weighted graph is even.

Let  $G$  be a connected graph of size  $m \geq 1$  with  $E(G) = \{e_1, e_2, \dots, e_m\}$ . If  $W$  is an Eulerian walk in  $G$  such that  $e_i$  is encountered exactly  $a_i$  times in  $W$  for  $i = 1, 2, \dots, m$ , then let  $C_W = \{a_1, a_2, \dots, a_m\}$ . Then  $C_M$  is in general a multiset of positive integers. The cardinality  $|C_W|$  of  $C_W$  is the number of distinct elements in  $C_W$ . If  $W$  is a proper Eulerian walk of  $G$ , then  $\chi'(G) \leq |C_W| \leq m$  and  $|C_W| = m$  if and only if  $W$  is irregular. This gives rise to two other concepts.

- The *chromatic Eulerian value* of  $G$  is the minimum positive integer  $p$  such that  $G$  has a proper Eulerian walk  $W$  in which no edge of  $G$  is encountered more than  $p$  times in  $W$ . If  $G$  is a connected graph of size  $m \geq 1$  with  $E(G) = \{e_1, e_2, \dots, e_m\}$  and each edge  $e_i$  ( $1 \leq i \leq m$ ) of  $G$  is replaced by  $2i$  parallel edges, then the resulting multigraph  $M$  is Eulerian and each Eulerian circuit in  $M$  gives rise to an irregular Eulerian walk in which each edge  $e_i$  of  $G$  appears exactly  $2i$  times in the walk. This irregular Eulerian walk is proper in which every edge of  $G$  is encountered at most  $2m$  times and so the chromatic Eulerian value of  $G$  is at most  $2m$ . This is equivalent to minimizing the *maximum value* of the weights that can be assigned to the edges of  $G$  such that the resulting edge labeling is also a proper edge coloring of  $G$  and the degree of every vertex in the resulting weighted graph is even.
- The *chromatic Eulerian index* of  $G$  is the minimum cardinality of  $C_W$  over all proper Eulerian walks  $W$  in  $G$  and is denoted by  $\chi'_E(G)$ . This is equivalent to minimizing the number of weights that can be assigned to the edges of  $G$  such that the resulting edge labeling is also a proper edge coloring of  $G$  and

the degree of every vertex in the resulting weighted graph is even.

### 5.3 Consecutive Eulerian Walks

An Eulerian walk  $W$  in a connected graphs  $G$  is *consecutive* if  $W$  has the property that every two consecutive edges in  $W$  are encountered a different number of times. In this case, the graph  $G$  cannot contain an end-vertex. Thus, we only consider connected graphs with minimum degree at least 2. If  $W$  is a proper Eulerian walk in  $G$ , then  $W$  is consecutive but the converse is not true in general. A problem here concerns minimizing the *maximum value* of the weights or minimizing the *sum* of the weights that can be assigned to the edges of  $G$  to produce a consecutive Eulerian walk in  $G$ .

### 5.4 On Maximal Decompositions in Graphs

First, we review some definitions and notation described in Chapter 4. For two graphs  $H$  and  $G$ , a decomposition  $\mathcal{D} = \{H_1, H_2, \dots, H_k, R\}$  of  $G$  is called an *H-maximal k-decomposition* if  $H_i \cong H$  for  $1 \leq i \leq k$  and  $R$  contains no subgraph isomorphic to  $H$ . If  $G$  contains no subgraph isomorphic to  $H$ , then  $k = 0$  and  $R = G$ . For graphs  $H$  and  $G$ , let

$$\text{Min}(G, H) = \min\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}$$

$$\text{Max}(G, H) = \max\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}.$$

A graph  $H$  is said to possess the *intermediate decomposition property* (IDP) and  $H$  is called an *ID-graph* if for each graph  $G$  and each integer  $k$  with  $\text{Min}(G, H) \leq$

$k \leq \text{Max}(G, H)$ , there exists an  $H$ -maximal  $k$ -decomposition of  $G$ .

For a set  $S$  of graphs and a graph  $G$ , a decomposition  $\mathcal{D} = \{H_1, H_2, \dots, H_k, R\}$  of  $G$  is called an  $S$ -maximal  $k$ -decomposition if  $H_i \cong H$  for some  $H \in S$  for each integer  $i$  with  $1 \leq i \leq k$  and  $R$  contains no subgraph isomorphic to any subgraph in  $S$ . For a set  $S$  of graphs without isolated vertices and a graph  $G$ , let

$$\begin{aligned} \text{Min}(G, S) &= \min\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\} \\ \text{Max}(G, S) &= \max\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\}. \end{aligned}$$

A set  $S$  of graphs without isolated vertices is said to possess the *intermediate decomposition property* (IDP) and  $S$  is called an *ID-set* if for every graph  $G$  and each integer  $k$  with  $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$ , there exists an  $S$ -maximal  $k$ -decomposition of  $G$ . As we have seen Theorem 4.4.18, a subset  $S$  of the set  $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$  of all graphs of size 3 without isolated vertices is an ID-set if and only if  $S$  is not one of the following eight sets:

$$\begin{aligned} &\{3K_2\}, \{K_3\}, \{K_{1,3}\}, \{3K_2, K_3\}, \{3K_2, K_{1,3}\}, \\ &\{K_3, K_{1,3}\}, \{3K_2, K_3, P_4\}, \{3K_2, K_{1,3}, P_4\}. \end{aligned}$$

For a set  $S$  of graphs, a graph  $G$  is said to have the *intermediate decomposition property with respect to  $S$*  (IDP-S) if for each integer  $k$  with  $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$ , there exists an  $S$ -maximal  $k$ -decomposition of  $G$ . In this case, the graph  $G$  is referred to as an *IDP-S* graph; otherwise,  $G$  is a *non-IDP-S* graph. Recall that for a set  $S$  of graphs without isolated vertices that is not an ID-set, a graph  $G$  of minimum size that is not an IDP-S graph (as described in

Theorem 4.3.2) is referred to as a *minimum non-IDP-S graph*. If  $S = \{H\}$  consists of a single graph  $H$ , then a minimum non-IDP- $S$  graph is also referred to as a *minimum non-IDP-H graph*.

### 5.4.1 Path-Cycle Sets of Graphs Having Size at Most 4

We plan to investigate the problem of determining which subsets of the set of all graphs of size at most 4 without isolated vertices are ID-sets. Since this is a challenging problem in general, we begin with those 2-element subsets consisting of paths and cycles, which we refer to as *path-cycle sets*. The following is a consequence of Theorem 4.3.2(III) and Theorem 4.4.18.

**Corollary 5.4.1** *Each of the sets  $\{P_3, C_3\}$ ,  $\{P_4, C_3\}$  and  $\{P_3, C_4\}$  is an ID-set.*

**Theorem 5.4.2** *The set  $\{P_4, C_4\}$  is an ID-set.*

**Proof.** Assume, to the contrary, that  $S = \{C_4, P_4\}$  is not an ID-set. Let  $G$  be a graph of minimum size that is not an IDP- $S$  graph. Then there are smallest integers  $a$  and  $b$  where  $1 \leq a \leq b - 2$  such that  $G$  has an  $S$ -maximal  $a$ -decomposition  $\mathcal{D}_a$  and an  $S$ -maximal  $b$ -decomposition  $\mathcal{D}_b$  but  $G$  has no  $S$ -maximal  $k$ -decomposition for every integer  $k$  with  $a < k < b$ . Since each graph in  $\mathcal{D}_a$  has size at most 4, it follows by Theorem 4.3.2(III) that  $b \leq 4$ . Hence there are three possibilities for  $a$  and  $b$ , namely (i)  $a = 1$  and  $b = 3$ , (ii)  $a = 1$  and  $b = 4$  or (iii)  $a = 2$  and  $b = 4$ . First, we show that

$$P_4 \notin \mathcal{D}_a \text{ for any } S\text{-maximal } a\text{-decomposition } \mathcal{D}_a \text{ of } G. \quad (5.1)$$

Assume, to the contrary, that there is an  $S$ -maximal  $a$ -decomposition  $\mathcal{D}_a$  of  $G$  such that  $P_4 \in \mathcal{D}_a$ . Since the size of  $P_4$  is 3, it follows by Theorem 4.3.2(III) that  $b = 3$  and so  $a = 1$ . Let  $\mathcal{D}_1 = \{H_1, R_1\}$  where  $H_1 = P_4$  and let  $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$  be an  $S$ -maximal 3-decomposition of  $G$ . Suppose that  $E(H_1) = \{e_1, e_2, e_3\}$ . By Theorem 4.3.2(II), we may assume that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$  and then  $L_i - e_i$  is a subgraph of  $R_1$ . If  $L_i = C_4$  for some  $i \in \{1, 2, 3\}$ , say  $L_1 = C_4$ , then  $L_1 - e_1 \cong P_4$  is a subgraph of  $R_1$ , which is impossible. Thus  $L_i = P_4$  for  $i = 1, 2, 3$ . Thus  $\mathcal{D}_1$  is a  $P_4$ -maximal 1-decomposition of  $G$  and  $\mathcal{D}_3$  is a  $P_4$ -maximal 3-decomposition of  $G$ . Since  $P_4$  is an ID-graph,  $G$  has a  $P_4$ -maximal 2-decomposition (and so an  $S$ -maximal 2-decomposition), which is impossible. Thus (5.1) holds. Next, we show that

$$C_4 \notin \mathcal{D}_b \text{ for any } S\text{-maximal } b\text{-decomposition } \mathcal{D}_b \text{ of } G. \quad (5.2)$$

Suppose that  $C_4$  belongs to some  $S$ -maximal  $b$ -decomposition

$$\mathcal{D}_b = \{L_1, L_2, \dots, L_b, R_b\}$$

of  $G$  where  $b = 3, 4$ . We may assume that  $L_1 = C_4$ . Let  $\mathcal{D}_a = \{H_1, \dots, R_1\}$  be an  $S$ -maximal  $a$ -decomposition of  $G$  where  $a = 1$  or  $a = 2$ . Then  $H_1 = C_4$  by (5.2). By Theorem 4.3.2(II),  $|E(L_1) \cap E(H_1)| = 1, 2$ . If  $L_1$  contains exactly one edge  $e$  of  $H_1$ , then  $L_1 - e = P_4 \in R_b$ , which is a contradiction. Thus  $L_1$  contains exactly two edges of  $H_1$ . This implies that the subgraph  $F = G[E(H_1) \cup E(L_1)]$  induced by  $E(H_1) \cup E(L_1)$  is one of the graphs shown in Figure 5.2, where  $L_1 = (v_1, v_2, v_3, v_4, v_1)$  and the bold edges are edges of  $H_1$ . In each case,  $F$  can be decomposed into two

edge-disjoint copies  $F_1$  and  $F_2$  of  $P_4$ . Since  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ , it follows that  $R_2 = G - E(F_1) \cup E(F_2)$  is a subgraph of  $R_1$ . Therefore,  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction. Consequently,  $L_i = P_4$  for each  $i = 1, 2, \dots, b$  and (5.2) holds.

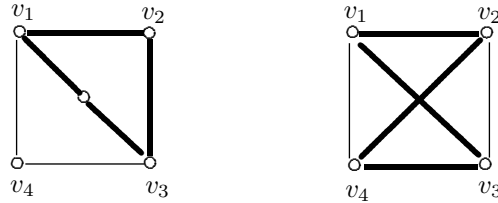


Figure 5.2: The subgraph  $F = G[E(H_1) \cup E(L_1)]$

We consider two cases, according to whether  $a = 1$  or  $a = 2$ .

*Case 1.*  $a = 1$ . By (5.1) and (5.2), let  $\mathcal{D}_1 = \{H_1, R_1\}$  where

$$H_1 \cong C_4 = (v_1, v_2, v_3, v_4, v_1)$$

and  $\mathcal{D}_b = \{L_1, L_2, \dots, L_b, R_b\}$  where  $L_i \cong P_4$  for  $1 \leq i \leq b$  and  $b = 3, 4$ . By Theorem 4.3.2(II), it follows that  $E(H_1) \cap E(L_i) \neq \emptyset$  for  $1 \leq i \leq b$ . Let  $E(H_1) = \{e_1, e_2, e_3, e_4\}$ . We may assume that  $e_i \in E(L_i)$  for  $i = 1, 2, 3$ . Since  $L_i \cong P_4$ , there is an edge  $f_i \in E(L_i)$  such that  $f_i$  is adjacent to  $e_i$  for  $i = 1, 2, 3$ . Hence  $G$  contains a connected subgraph  $F$  of size 6 that is isomorphic to one of the five graphs in Figure 5.3 and all of these five graphs contains  $C_4$  as a subgraph. In each case,  $F$  can be decomposed into two copies  $F_1$  and  $F_2$  of  $P_4$  (one of which is drawn in bold in Figure 5.3). Since  $E(H_1) \subseteq E(F_1) \cup E(F_2)$ , it follows that  $R_2 = G - E(F_1) \cup E(F_2)$  is a subgraph of  $R_1$ . Therefore,  $\{F_1, F_2, R_2\}$  is an  $S$ -maximal 2-decomposition of  $G$ , which is a contradiction.

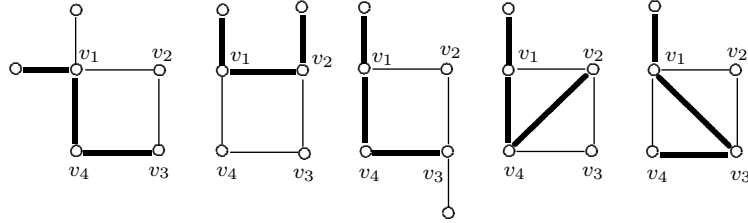


Figure 5.3: Subgraphs in Case 1

Case 2.  $a = 2$ . Let  $\mathcal{D}_2 = \{H_1, H_2, R_2\}$  where  $H_1 \cong H_2 \cong C_4$  and

$$\mathcal{D}_b = \{L_1, L_2, L_3, L_4, R_4\}$$

where  $L_i \cong P_4$  for  $1 \leq i \leq 4$ . Suppose that  $H_1 = (v_1, v_2, v_3, v_4, v_1)$ . As in Case 1,  $E(H_1) \cap E(L_i) \neq \emptyset$  for  $1 \leq i \leq 4$ . Let  $E(H_1) = \{e_1, e_2, e_3, e_4\}$ . We may assume that  $e_i \in E(L_i)$ . Since  $L_i \cong P_4$ , there is an edge  $f_i \in E(L_i)$  such that  $f_i$  is adjacent to  $e_i$  for  $i = 1, 2, 3, 4$ . This implies that  $G$  contains a subgraph  $F$  that is isomorphic to one of the graphs in Figure 5.3, each of which contains  $C_4$  as a subgraph. Let  $F_1 \cong F_2 \cong P_4$  defined in Case 1 and let  $F_3 = H_2$ . Then  $F_1, F_2, F_3$  are edge-disjoint and  $E(H_1) \cup E(H_2) \subseteq E(F_1) \cup E(F_2) \cup E(F_3)$ . Thus  $R_3 = G - [E(F_1) \cup E(F_2) \cup E(F_3)]$  is a subgraph of  $R_2$  and so  $R_3$  contains no subgraph isomorphic to any graph in  $\{P_4, C_4\}$ . Therefore,  $\{F_1, F_2, F_3, R_3\}$  is an  $S$ -maximal 3-decomposition of  $G$ , which is a contradiction. ■

By Corollary 5.4.1 and Theorem 5.4.2, it remains to consider the following questions.

1. Is  $\{P_5, C_3\}$  an ID-set?
2. Is  $\{P_5, C_4\}$  an ID-set?



We also plan to investigate those *path-star sets* or *cycle-star sets*, which are defined as expected.

### 5.4.2 Graphs of Larger Sizes

We have seen that neither  $K_3$  nor  $K_{1,3}$  is an ID-graph. In fact, for each integer  $n \geq 3$ , none of  $C_n, K_n, K_{1,n}$  are ID-graphs. To show this, we construct a non-ID-graph for each of  $C_n, K_n, K_{1,n}$ .

- For  $C_n$ , let  $F_0, F_1, \dots, F_n$  be the  $n + 1$  copies of  $C_n$  where

$$F_i = (v_{i,1}, v_{i,2}, \dots, v_{i,n}, v_{i,1})$$

for  $0 \leq i \leq n$ . The graph  $G$  is then obtained from  $F_0, F_1, \dots, F_n$  by identifying the edge  $v_{0,j}v_{0,j+1}$  in  $F_0$  with the edge  $v_{j,1}v_{j,2}$  for  $1 \leq j \leq n$  where  $v_{0,n}v_{0,n+1} = v_{0,n}v_{0,1}$ . The graph  $G$  is shown in Figure 4.1 for  $n = 3$  and in Figure 5.4(a) for  $n = 4$ . Then  $G$  has (1) a  $C_n$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  where  $H_1 = F_0$  and  $R_1$  is a cycle of order  $n^2 - n > n$  and (2) a  $C_n$ -maximal  $n$ -decomposition  $\mathcal{D}_n = \{L_1, L_2, \dots, L_n, R_n\}$  where  $L_i = F_i$  for  $1 \leq i \leq n$  and  $R_n$  is an empty graph. However,  $G$  does not have a  $C_n$ -maximal  $k$ -decomposition for each integer  $k$  with  $2 \leq k \leq n - 1$ . Thus  $C_n$  is not an ID-graph.

- For  $K_n$ , let  $F_0, F_1, \dots, F_n$  be the  $n+1$  copies of  $K_n$  where  $(v_{i,1}, v_{i,2}, \dots, v_{i,n}, v_{i,1})$  is a Hamiltonian cycle of  $F_i$  for  $0 \leq i \leq n$ . The graph  $G$  is then obtained from  $F_0, F_1, \dots, F_n$  by identifying the edge  $v_{0,j}v_{0,j+1}$  in  $F_0$  with the edge

$v_{j,1}v_{j,2}$  for  $1 \leq j \leq n$  where  $v_{0,n}v_{0,n+1} = v_{0,n}v_{0,1}$ . The graph  $G$  is shown in Figure 4.1 for  $n = 3$  and in Figure 5.4(b) for  $n = 4$ . Then  $G$  has a  $K_n$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  where  $H_1 = F_0$  and the clique number  $R_1$  is  $n - 1$  and so  $R_1$  contains no  $K_n$  as a subgraph and a  $K_n$ -maximal  $n$ -decomposition  $\mathcal{D}_n = \{L_1, L_2, \dots, L_n, R_n\}$  where  $L_i = F_i$  for  $1 \leq i \leq n$  and  $R_n$  is the graph  $K_n - C_n$ . However,  $G$  does not have a  $K_n$ -maximal  $k$ -decomposition for each integer  $k$  with  $2 \leq k \leq n - 1$ . Thus  $K_n$  is not an ID-graph.

- For  $K_{1,n}$ , let  $F_0, F_1, \dots, F_n$  be  $n + 1$  copies of  $K_{1,n}$  where

$$V(F_0) = \{u, u_1, u_2, \dots, u_n\}$$

and  $u$  is the central vertex of  $F_0$ . For each  $i$  with  $1 \leq i \leq n$ , let  $v_i$  be the central vertex of  $F_i$ . The graph  $G$  is then obtained from  $F_0, F_1, \dots, F_n$  by identifying the end-vertex  $u_i$  in  $F_0$  with the central vertex  $v_i$  in  $F_i$  for  $1 \leq i \leq n$ . The graph  $G$  is shown in Figure 4.1 for  $n = 3$  and in Figure 5.4(c) for  $n = 4$ . Then  $G$  has a  $K_{1,n}$ -maximal 1-decomposition  $\mathcal{D}_1 = \{H_1, R_1\}$  where  $H_1 = F_0$  and  $R_1 \cong nK_{1,n-1}$  and a  $K_{1,n}$ -maximal  $n$ -decomposition  $\mathcal{D}_n = \{L_1, L_2, \dots, L_n, R_n\}$  where  $L_i = F_i$  for  $1 \leq i \leq n$  and  $R_n$  is an empty graph. However,  $G$  does not have a  $K_{1,n}$ -maximal  $k$ -decomposition for any integer  $k$  with  $2 \leq k \leq n - 1$ . Thus  $K_{1,n}$  is not an ID-graph.

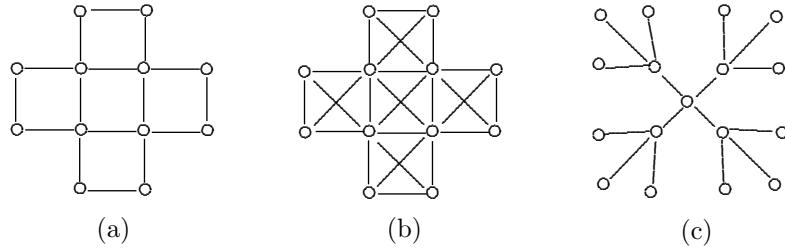


Figure 5.4: Illustrating that  $C_n, K_n, K_{1,n}$  are not ID-graphs for  $n = 4$

**Problem 5.4.3** For an integer  $n \geq 5$ , is  $P_n$  an ID-graph?

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