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Third Order Degree Regular Graphs

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THE CARL AND WINIFRED LEE HONORS COLLEGE

CERTIFICATE OF ORAL EXAMINATION

Leslie Hayes, having been admitted to the Carl and Winifred Lee Honors College in 1991, successfully presented the Lee Honors College Thesis on April 12, 1994.

The title of the paper is:

"Third Order Degree Regular Graphs"

Great Job!

Dr. Yousef Alavi
Mathematics and Statistics

Dr. Allen Schwenk
Mathematics and Statistics

Dr. Arthur Falk
Philosophy

Third Order Degree Regular Graphs

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Abstract

A graph G is regular of degree d if for every vertex v in G there exist exactly d vertices at distance 1 from v. A graph G is kth order regular of degree d if for every vertex v in G, there exist exactly d vertices at distance k from v. In this paper, third order regular graphs of degree 1 with small order are characterized.

1 Definitions and Examples

We denote the distance between two vertices u and v in a connected graph G of order p by $d(u,v)$. For an integer k with $1 \leq k \leq p-1$, the *k-neighborhood* $N_k(v)$ is defined by

$$N_k(v) = \{u | u \in V(G) \text{and } d(u,v) = k\}$$

and the *closed k-neighborhood* by

$$N_k[v] = N_k(v) \cup \{v\}.$$

Definition The *kth order degree of v* is $deg_k v = |N_k(v)|$.

Definition A graph G is *kth order regular of degree d* if $deg_k(v) = d$ for every vertex v of G. Consequently, first order regularity of degree d is synonomous with regularity of degree d.

For example, the graph G_1 of Figure 1 is second order regular of degree 4, while G_2 is first order regular of degree 4, second order regular of degree 5, and third order regular of degree 2.

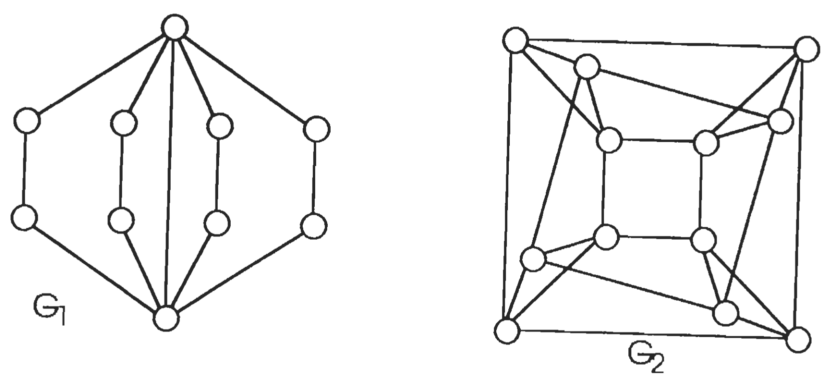


Figure 1

2 kth Order Regular Graphs of Degree 1

Alavi, Lick and Zou [1] conjectured that for $k \geq 2$, every connected, *kth* order regular graph of degree 1 is either a path of length $2k - 1$ or has diameter k. This conjecture is verified in our first result.

Theorem 1 *For $k \geq 2$, every connected, kth order regular graph of degree 1 is either a path of length $2k - 1$ or has diameter k.*

PROOF. I still have to write this one up! □

3 Third Order Regular Graphs of Degree 1

Let $\overline{nK_2}$ denote the complete n-partite graph in which each partite set contains exactly 2 vertices. In [1] it is shown that a connected graph G is second order regular of degree 1 if and only if G is either a path of length 3 or G is $\overline{nK_2}$ for some $n \geq 2$. We present some similar results for graphs that are third order regular of degree 1. We first introduce the following definitions.

Definition A kth order regular graph G of degree d is *maximal* if for every pair u, v of nonadjacent vertices the graph $G + uv$ is not kth order regular of degree d.

For example, for $k \geq 2$, the graphs P_{2k} and C_{2k} are kth order regular of degree 1 and C_{2k} is maximal, while P_{2k} is not.

Definition Two vertices u and v are defined as *antipodal* in a graph G if $d(u, v) = \text{diam } G$. Since paths are not maximal, recall from theorem 1 that maximal kth order regular graphs of degree 1 have diameter k. Hence, if x is a vertex in a maximal kth order regular graph of degree 1, the *antipodal vertex of the vertex* x, denoted x', is the unique vertex at distance k from x. We refer to a vertex and its antipode as an *antipodal pair*.

Proposition 1 *Let x and y be distinct vertices in a connected, maximal third order regular graph G of degree 1. Then $xy \in E(G)$ if and only if $x'y' \in E(G)$.*

PROOF. Assume that $xy \in E(G)$, and suppose, to the contrary, that $x'y' \notin E(G)$. Then $d(x, x') = d(y, y') = 3$. Since P_6 is not a maximal third order regular graph of degree 1, it follows from Theorem 1 that G has diameter 3. Since x' is the unique vertex at distance 3 from x, we have that $d(x', y) \leq 2$. If $d(x', y) = 1$, then, since $d(y, x) = 1$, it follows that $d(x', x) \leq 2$, producing a contradiction. Thus $d(x', y) = 2$. Similarly, $d(y', x) = 2$.

Since $x'y' \notin E(G)$, it follows that $G + x'y'$ is not third order regular of degree 1. Hence, for some pair w, w' of antipodal vertices in G, there exists a $w - w'$ path of length at most 2 in $G + x'y'$. Since $d(x, x') = d(y, y') = 3$ in $G + x'y'$, there must exist $z, z' \in V(G)$, with $\{z, z'\} \cap \{x, x', y, y'\} = \emptyset$ such that there exist $z - x'$ and $z' - y'$ paths or $z - y'$ and $z' - x'$ paths. Then

$$d(z, x') + 1 + d(y', z') < 3$$

or

$$d(z', x') + 1 + d(y', z) < 3,$$

producing a contradiction. $\square$

Proposition 1 implies that there is an automorphism of a maximal third order regular graph of degree 1 that interchanges each vertex with its antipode.

Corollary 1 *Let G be a connected, maximal third order regular graph of degree 1. Let x, x' be a pair of antipodal vertices in G. Then for $v \in V(G)$ with $v \neq x, x'$, either $vx \in E(G)$ or $vx' \in E(G)$.*

PROOF. Let v be a vertex of G such that $v \neq x, x'$. Assume, to the contrary, that v is neither adjacent to x nor to x'. Since diam $G = 3$ and x' is the unique vertex at distance 3 from x, it follows that $d(v, x) = d(v, x') = 2$. By the previous proposition, $d(v', x) = d(v', x') = 2$.

Since G is maximal, $G + vx$ is not third order regular of degree 1. Since in $G + vx$, we know that $d(x, x') = d(v, v') = 3$, there must exist some pair z, z' for which $d(z, z') \leq 2$ in $G + vx$. Thus

$$d(z, v) + 1 + d(x, z') \leq 2$$

or

$$d(z', v) + 1 + d(x, z) \leq 2,$$

producing a contradiction.

Thus $vx \in E(G)$ and, by the symmetry shown in Proposition 1, 'we have that $v'x \in E(G)$. □

Corollary 2 follows immediately.

Corollary 2 *Let G be a connected, maximal third order regular graph of degree 1 of order p. Then G is regular of degree $(p-2)/2$.*

These results provide us with the following theorems that characterize third order regular graphs of degree 1 of small order.

Theorem 2 *The only connected, maximal third order regular graph of degree 1 and of order 6 is C_6.*

PROOF. From Corollary 2, a maximal third order regular graph of degree 1 of order p is regular of degree $(p-2)/2$. Let G be a connected, maximal third order regular graph of degree 1 and of order 6. Then G is regular of degree 2. Since G is connected, G must be a cycle with six vertices. □

Theorem 3 *The only connected and maximal third order regular graph of degree 1 and of order 8 is the cube.*

PROOF. Let G be a connected, maximal third order regular graph of degree 1 and order 8. From Corollary 2, G is regular of degree 3. If we delete two antipodal vertices x and x' from G, the resulting graph of order 6 is regular of degree 2. Only two such graphs are possible, namely G_3 and G_4, shown in Figure 2.

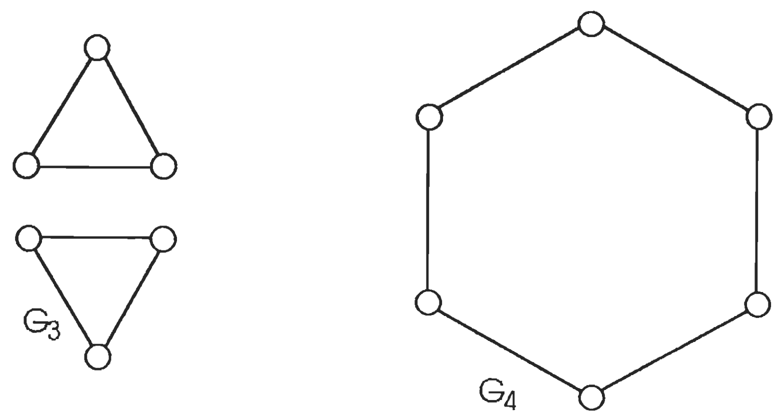


Figure 2

We now construct G from these graphs. Since, in adding two vertices to G_3, the resulting graph must be connected, we always have G_4 as a subgraph of G. Thus we need only examine the possible graphs that can be constructed from G_4. Since no pair of antipodal vertices are adjacent to the same vertex, there are two such graphs possible, namely G_5 and G_6 shown in Figure 3.

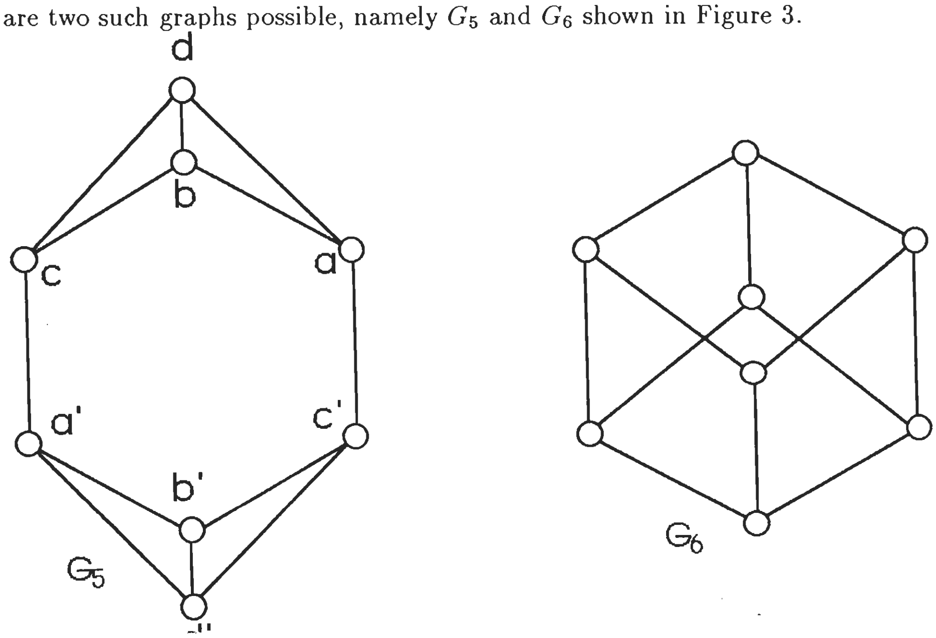
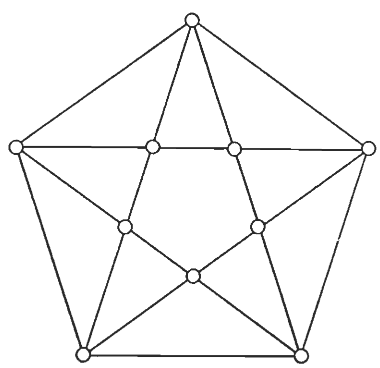


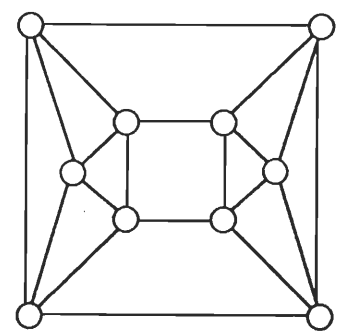
Figure 3

The graph G_5 is not third order regular of degree 1. Hence G_6, which is isomorphic to the cube, is the only maximal third order regular graph on eight vertices. □

Theorem 4 *The only connected, maximal third order regular graphs of degree 1 and of order 10 are* $\overline{K_2 \times K_5}$, *the antiprism on 10 vertices, and the cube with two pyramids. The latter two are shown in Figure 4.*



the antiprism on 10 vertices



the cube with two pyramids

Figure 4

PROOF. Let G be a connected, maximal third order regular graph of degree 1 and order 10. From Corollary 2, the graph G is regular of degree 4. If we delete a pair of antipodal vertices, u and u', from G, the resulting graph on eight vertices is regular of degree 3. The three such graphs possible are G_5, G_6, and G_7 (see Figure 5).

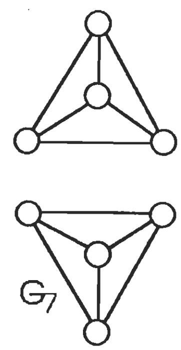


Figure 5

We now construct G from $G - uv$. We begin with G_7. Since G is connected, in adding two vertices to G_7 we create G_5 as a subgraph. Hence, we need only examine the possible graphs that can be constructed from G_5 and G_6.

Consider G_5. Without loss of generality, let $ud \in E(G)$. This forces $u'd' \in E(G)$. Since $d(u,b) = d(b,b') = 3$, we must have $u'b \in E(G)$, and thus $ub' \in E(G)$. Since antipodal vertices cannot both be adjacent to u or to u', we now have two possible graphs, namely the graphs G_8 and G_9 shown in Figure 6.

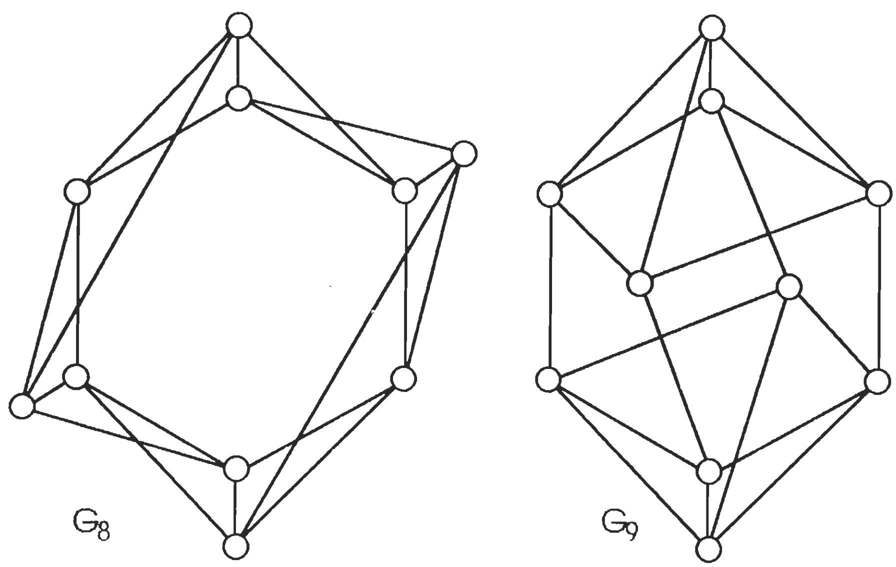


Figure 6

Both G_8 and G_9 are maximal third order regular of degree 1. The graph G_8 is isomorphic to the antiprism on ten vertices, and G_9 is isomorphic to the cube with two pyramids.

We now examine G_6. There are four ways in which two vertices u and u' can be added to G_6 such that antipodal vertices are not both adjacent to u or to u', as represented by G_{10}, G_{11}, G_{12}, and G_{13} of Figure 7.

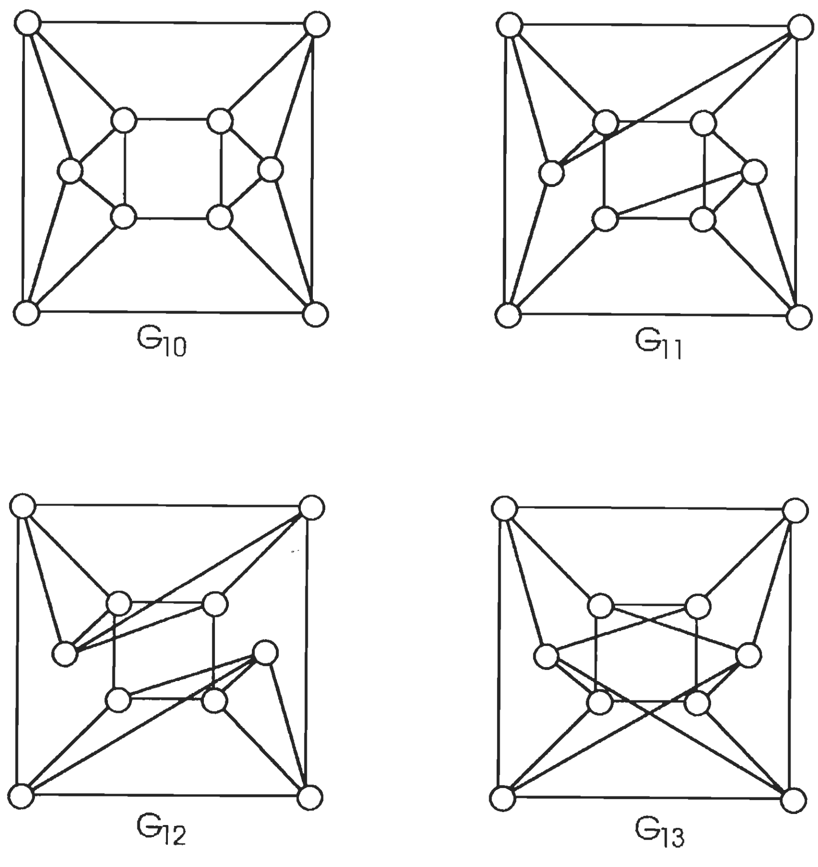


Figure 7

However, observe that $G_{10} \cong G_{12} \cong G_9$. Since $N[u] = N[v]$ in G_{11}, it is not third order regular of degree 1. Finally, $G_{13} \cong \overline{K_2 \times K_5}$ shown in Figure 8.

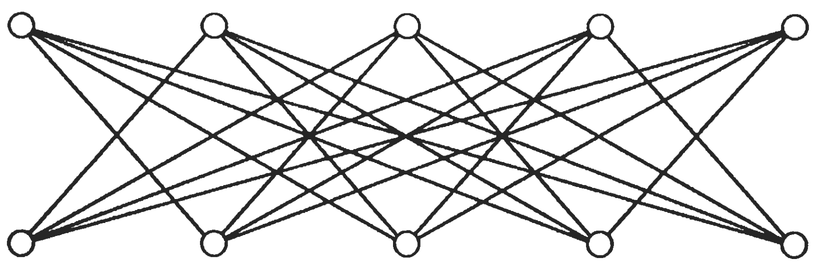


Figure 8. $\overline{K_2 \times K_5}$.

□

Acknowledgments

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[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs, Second Edition*, Wadsworth, Inc., Belmont, CA (1986).