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Lie Loops Associated With $GL(\mathfrak{H})$, \mathfrak{H} A Separable Infinite Dimensional Hilbert Space

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LIE LOOPS ASSOCIATED WITH $GL(\mathcal{H})$, \mathcal{H} A SEPARABLE INFINITE
DIMENSIONAL HILBERT SPACE

by
Alper Bulut

A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
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Alper Bulut, Ph.D.

Western Michigan University, 2014

We investigate Lie loops as twisted semi-direct products of Lie groups for the finite dimensional and the infinite dimensional cases and examine the twisted semi-direct products of Lie algebras, as a possible candidate for the Akinis algebra of the twisted semi-direct product of Lie groups, showing that the twisted semi-direct product of Lie algebras is a Lie algebra.

There has been growing interest in K-loops for the last two decades after A. A. Ungar's discovery in (1988). Ungar showed that Einstein's addition of relativistically admissible velocities, \oplus , is neither commutative nor associative. H. Wefelscheid (1994) recognized that the set of admissible velocities together with the Einstein velocity addition, \oplus , form a K-loop. Kiechle (1998) provided examples of K-loops from classical groups over the ordered fields.

We investigate K-loops from real reductive connected Lie groups for the finite dimensional case, and extend our work to infinite dimensional cases, namely (i) K-loops from $GL(\infty, \mathcal{H})$, classical groups as subgroups of $GL(\mathcal{H})$, \mathcal{H} a separable Hilbert space, and (ii) K-loops from classical subgroups of $GL(\mathcal{H})$. We examine the left inner mapping group of $L_G(\infty, \mathcal{H})$ and show that $\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}})) \cong \text{PSO}(\infty, \mathcal{H}_{\mathbb{R}})$ if $G \in \{GL(\infty, \mathcal{H}_{\mathbb{R}}), SL(\infty, \mathcal{H}_{\mathbb{R}})\}$, and $\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{C}})) \cong \text{PSU}(\infty, \mathcal{H}_{\mathbb{C}})$ if $G \in \{GL(\infty, \mathcal{H}_{\mathbb{C}}), SL(\infty, \mathcal{H}_{\mathbb{C}})\}$.

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Alper Bulut

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0.1 Introduction

A Bol loop with the automorphic inverse property is called a K-loop. Karzel introduced the notion of near-domain in [16],[17], (F, \oplus, \cdot) , which is a generalization of a near-field where the additive structure of a near-domain is not necessarily associative. Each finite near-domain is a near-field, but the existence of a near-domain which is not a near-field still not known. Kerby and Wefelsheid investigated the additive structure of a near-domain, (F, \oplus) , and called such a loop a K-loop [14]. They could not find a proper example of a K-loop, hence no theoretical investigation was done. This period continued until Ungar's famous example of K-loop on $\mathbb{R}_c^3 := \{v \in \mathbb{R}^3 : |v| < c\}$ the set of admissible velocities. Ungar, [28], showed that \mathbb{R}_c^3 with the relativistic velocity addition, \oplus , is nonassociative and noncommutative. Wefelsheid recognized that (\mathbb{R}_c^3, \oplus) is a K-loop. Following Ungar's example, Kreuzer and Wefelsheid, [20] provided an axiomatic way to construct the K-loops from group transversals. Kiechle, [13], provided many examples of K-loops from classical groups over the ordered field. Kiechle remarked that K-loops can be also constructed from subgroups of $GL(\mathcal{H})$ via the Polar Decomposition Theorem. This remark was the main motivation behind this dissertation work. One of the axioms provided by Kreuzer and Wefelsheid is the exact decomposition, $G = KA$, where G is a group, K is a subset of G with the neutral element 1_G , and A is a subgroup of G . The exact decomposition provides another binary operation, \oplus , on K induced by group multiplication. Therefore, it is natural to study the struc-

ture of (K, \oplus) . For instance, if $G = GL(n, \mathbb{C})$, then $G = Pos(n, \mathbb{C})U(n, \mathbb{C})$ where $Pos(n, \mathbb{C})$ is the set of positive definite Hermitian matrices and $U(n, \mathbb{C})$ is the group of Unitary matrices. Given $A, B \in Pos(n, \mathbb{C})$ there exists unique $A \oplus B \in Pos(n, \mathbb{C})$ and $d_{A,B} \in U(n, \mathbb{C})$ such that $AB = (A \oplus B)d_{A,B}$, hence $A \oplus B = ABd_{A,B}^{-1}$. The spectral theorem for the classical groups is analogue to the polar decomposition theorem for the bounded linear operators on separable Hilbert space (or Cartan decomposition theorem in the Lie perspective).

Chapter 1 will be devoted to preliminaries that provide a background to the readers who are not familiar with the concept of loops, Lie groups, operators and related theories which are the elements of almost each chapters. The historical developments of loops and K-loops will be discussed, moreover we will introduce some of the classical complex Banach Lie groups of bounded linear operators on separable Hilbert space. Those complex Banach Lie groups also give examples of infinite dimensional K-loops which has not been discussed in the Literature. Cartan decomposition provides another way to look at the K-loop structures from the Lie perspective. Therefore there is a small section for Cartan decomposition theorem. We want readers to see the results without reading all chapters. The list of main results provides an idea about the essence of the work. Therefore, the Chapter 1 starts with a list of main results. This also guides the readers to read only associated chapters according to their interests. We also point out that for any mapping f on a set S , $(x)f$ or x^f , $x \in S$, are used to denote the function evaluation, that is f is acting on the right instead of acting on the left as usual.

Chapter 2 is a summary of the results, in the Lie perspective, that has been presented in 2010 at the Fall central AMS section meeting at Notre Dame University. In this chapter we twisted the usual semi-direct product and called it twisted semi-direct product and investigated its properties. The semi-direct product of Lie groups is a Lie group, but this is not necessarily true for the twisted semi-direct product. Examples of the finite and the infinite dimensional Lie Loops as twisted semi-direct product of Lie groups introduced. Chapter 2 also motivated us to study the Akiwis algebra of Lie loops in Chapter 5, since $Lie : G \rightarrow Lie(G)$ is functorial and semi-direct product of Lie groups is preserved under Lie . It has been shown in [10] that for every finite dimensional Akiwis algebra over \mathbb{R} there is at least one local Lie loop which has the given Akiwis algebra as tangent algebra.

Chapter 3 provides many examples of infinite dimensional K-loops from subgroups of $GL(\infty, \mathcal{H}_{\mathbb{F}}) := \bigcup_{n=2}^{\infty} GL(n, \mathcal{H}_{\mathbb{F}})$, where $\mathcal{H}_{\mathbb{F}}$ is separable Hilbert space on $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and $GL(n, \mathcal{H}_{\mathbb{F}}) := \{diag(A, 1, \dots) : A \in GL(n, \mathbb{C})\}$. We investigated the examples given by Kiechle in [13] and transformed them to the examples of infinite dimensional K-loops. We looked at them as real reductive connected Lie groups. We also showed that those K-loops can be also obtained from $Sp(P, Q, \infty), SO^*(\infty)$, where $P := P(n)$ and $Q := Q(n)$ be sequences of positive integers such that P is constant, Q is strictly increasing.

In Chapter 4, we discussed the examples of infinite dimensional Lie K-loops from orthogonal and symplectic complex Banach-Lie groups. Kiechle in [13] remarked that if $G = GL(\mathcal{H})$, then $G = Pos(\mathcal{H})U(\mathcal{H})$ is the exact decompo-

sition by the Polar Decomposition Theorem such that $Pos(\mathcal{H})$, positive self adjoint operators, is a K-loop whose binary operation induced by composition of operators. We extended this remark to $O(\mathcal{H}, J_{\mathbb{R}})$ and $Sp(\mathcal{H}, J_{\mathbb{Q}})$. Note that both $O(\mathcal{H}, J_{\mathbb{R}})$ and $Sp(\mathcal{H}, J_{\mathbb{Q}})$ are closed under taking adjoints. The Cartan decomposition at the level of Lie group allowed us to provide more examples of K-loops that were not given in [13]. Those K-loops can be also extended to the examples of infinite dimensional K-loops in $Pos(\mathcal{H})$.

Finally, Chapter 6 is a discussion of future projects such that we were not able to investigate them because of the time restriction.

Chapter 1

Preliminaries

The goal of this chapter is to construct some background for the reader. Our work considers mainly K-loops, a subclass of Lie loops, therefore the needed Lie theory will be discussed. In addition, we need some facts from Functional Analysis as $GL(\mathcal{H})$ is the group of invertible operators inside the space of bounded linear operators $\mathcal{L}(\mathcal{H})$, where \mathcal{H} is an infinite dimensional separable Hilbert space over \mathbb{C} .

1.1 Main Results

1.1.1 Twisted Semi-direct Product Lie Loop

Theorem 1.1.1. *Let H and K be Lie groups with $H \leq \text{Aut}(K)$ such that each $h \in H$ is smooth with smooth inverse. Let $\mathcal{L} := K \overline{\rtimes} H$. Then*

1. (\mathcal{L}, \oplus) is a Lie group if and only if H is an abelian Lie group.

2. If H is not abelian then (L, \oplus) is a Lie loop, which is not a Lie group.

Let H be a Lie subgroup of $GL(\mathcal{H})$. The Affine transformation group of H over \mathcal{H} is denoted by $Aff_{\mathcal{H}}(H)$ and it is defined by

$$Aff_{\mathcal{H}}(H) = \{\Phi_{(h,T)} : \mathcal{H} \rightarrow \mathcal{H} : x\Phi_{(h,T)} = h + x^T, x, h \in \mathcal{H}, T \in H\}.$$

We remark that $Aff_{\mathcal{H}}(H) = \mathcal{H} \rtimes H$ is a Lie group as semi-direct product of Banach Lie groups.

Theorem 1.1.2. *Let H be a Lie subgroup of $GL(\mathcal{H})$ and let $\mathcal{L} = \mathcal{H} \overline{\rtimes} H$. Then*

1. \mathcal{L} is a Lie loop.
2. \mathcal{L} is a left Bol loop.
3. $Lmlt(\mathcal{L}) \subseteq Aff_H(\mathcal{H}) \times H$.
4. $N(\mathcal{L}) = \{0\} \times Z(H)$.
5. $Z(\mathcal{L}) \cong Fix_{\mathcal{H}}(H) \cap Z(H)$.

Corollary 1.1.3. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and let P_1, P_2 and P_3 be the elements of Lie groups given below. Let $\mathcal{L}_m := \mathcal{K}_m \overline{\rtimes} P_m$ for $K_1 = \mathbb{F}^n$, $K_2 = \mathcal{H}_{\mathbb{F}}$, and $K_3 = \mathcal{H}$.*

- $P_1 \in \{GL(n, \mathbb{F}), SL(n, \mathbb{F}), U(n, \mathbb{F}), Sp(2n, \mathbb{F})\}$.
- $P_2 \in \{GL(\infty, \mathcal{H}_{\mathbb{F}}), SL(\infty, \mathcal{H}_{\mathbb{F}}), U(\infty, \mathcal{H}_{\mathbb{F}}), Sp(\infty, \mathcal{H}_{\mathbb{F}})\}$.
- $P_3 \in \{GL(\mathcal{H}), U(\mathcal{H}), Sp(\mathcal{H})\}$.

Then for $m \in \{1, 2, 3\}$

1. \mathcal{L}_m is a Lie loop.
2. \mathcal{L}_m is a left Bol loop.
3. $Lmlt(\mathcal{L}_m) \subseteq Aff_{\mathcal{K}_m}(P_m) \times P_m$.
4. $N(\mathcal{L}_m) \cong Z(P_m)$.
5. $Z(P_m) \cong Fix_{\mathcal{K}_m}(P_m) \cap Z(P_m)$.

1.1.2 K-Loops from $GL(\infty, \mathcal{H})$ Classical Groups as subgroups of $GL(\mathcal{H})$, \mathcal{H} a separable infinite dimensional Hilbert Space

Theorem 1.1.4. *Let $G \leq GL(\infty, \mathcal{H}_{\mathbb{C}})$ with $G = L_G(\infty, \mathcal{H}_{\mathbb{C}})U_G(\infty, \mathcal{H}_{\mathbb{C}})$ where $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ and $U_G(\infty, \mathcal{H}_{\mathbb{C}}) = U(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$. Then for any $A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$, there are $A \oplus B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A,B} \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, $(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K-loop.*

Theorem 1.1.5. *Let $G \in \{SL(\infty, \mathcal{H}_{\mathbb{R}}), GL(\infty, \mathcal{H}_{\mathbb{R}}), SL(\infty, \mathcal{H}_{\mathbb{C}}), GL(\infty, \mathcal{H}_{\mathbb{C}})\}$ and let $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ and $U_G(\infty, \mathcal{H}_{\mathbb{C}}) = U(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$. Then the following assertions hold.*

1. *For all $A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ there exists unique $A \oplus B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A,B} \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$ such that $AB = (A \oplus B)d_{A,B}$. Moreover $(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$*

is a K -loop. On the other hand the closure $(\overline{L_G(\infty, \mathcal{H}_{\mathbb{C}})}, \oplus)$ is also a K -loop in $\text{Pos}(\mathcal{H})$

2. Let $\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{K}}))$ be the left inner mapping group of $L_G(\infty, \mathcal{H}_{\mathbb{K}})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then

$$\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}})) \cong \text{PSO}(\infty, \mathcal{H}_{\mathbb{R}}), \text{ if } G \in \{SL(\infty, \mathcal{H}_{\mathbb{R}}), GL(\infty, \mathcal{H}_{\mathbb{R}})\}.$$

$$\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{C}})) \cong \text{PSU}(\infty, \mathcal{H}_{\mathbb{C}}), \text{ if } G \in \{SL(\infty, \mathcal{H}_{\mathbb{C}}), GL(\infty, \mathcal{H}_{\mathbb{C}})\}.$$

Theorem 1.1.6. Let $\mathbb{H} := \mathbb{R}(i, j)$ be the quaternions over \mathbb{R} . Let $G \in \{SL(\infty, \mathbb{H}), GL(\infty, \mathbb{H})\}$, then $\tilde{L}_G = \tilde{L}(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of G/\tilde{U}_G and is therefore a K -loop.

Theorem 1.1.7. If $G \in \{Sp(\infty, \mathcal{H}_{\mathbb{R}}), Sp(\infty, \mathcal{H}_{\mathbb{C}})\}$, then $\tilde{L}_G(\infty, \mathcal{H}_{\mathbb{C}}) = \tilde{L}(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of G/\tilde{U}_G and is therefore a K -loop.

Theorem 1.1.8. If $G \in \{O(P, Q, \infty), U(P, Q, \infty), Sp(P, Q, \infty), SO^*(\infty, \mathcal{H}_{\mathbb{C}})\}$, then $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of $G/U_G(\infty, \mathcal{H}_{\mathbb{C}})$ and is therefore a K -loop.

Theorem 1.1.9. If $G \in \{SO_o(P, Q, \infty), SO_o(\infty, \mathcal{H}_{\mathbb{C}})\}$, then $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of $G/U_G(\infty, \mathcal{H}_{\mathbb{C}})$ and is therefore a K -loop.

1.1.3 K -Loops from Classical subgroups of $GL(\mathcal{H})$, \mathcal{H} a separable infinite dimensional Hilbert Space

Theorem 1.1.10. Let G be a connected reductive real Lie group. Let $\Theta : G \rightarrow G$, $g \mapsto (\bar{g}^\top)^{-1}$, so that $d\Theta|_1 := \theta$ and let $\exp : \mathfrak{g} \rightarrow G$ be the exponential map.

Let $K := \{g \in G : \Theta(g) = g\}$. Then

1. For all $A, B \in \exp \mathfrak{p}$ there exist unique $A * B \in \exp \mathfrak{p}$ and $d_{A,B} \in K$ such that $AB = (A * B)d_{A,B}$ where \mathfrak{p} is the eigenspace of -1 with respect to θ ;
2. $(\exp(\mathfrak{p}), *)$ is a Lie K -loop.

Corollary 1.1.11. Let $G \in \{Sp(m, n), SO^*(2n), SO_{\circ}(n, \mathbb{C}), SO_{\circ}(m, n)\}$. Then

1. Given $A, B \in \exp(\mathfrak{p})$, there exists unique $A * B \in \exp(\mathfrak{p})$ and

$$d_{A,B} \in \begin{cases} Sp(m) \times Sp(n) & \text{if } G = Sp(m, n) \\ U(n) & \text{if } G = SO^*(2n) \\ SO(n) & \text{if } G = SO_{\circ}(n, \mathbb{C}) \\ SO(m) \times SO(n) & \text{if } G = SO_{\circ}(m, n) \end{cases}$$

such that $AB = (A * B)d_{A,B}$.

2. $(\exp(\mathfrak{p}), *)$ is a K -loop.

Theorem 1.1.12. Let $G(\mathcal{H})$ be one of the complex Banach Lie groups in $\{O(\mathcal{H}, J_{\mathbb{R}}), Sp(\mathcal{H}, J_{\mathbb{Q}})\}$, and let $Pos(\mathcal{H})$ and $U(\mathcal{H})$ are collection of positive self-adjoint operators and unitary operators respectively over \mathbb{C} . Let $P_G := G(\mathcal{H}) \cap Pos(\mathcal{H})$, and $U_G := G(\mathcal{H}) \cap U(\mathcal{H})$. Then for all $A, B \in P_G$ there exist unique $A \oplus B \in P_G$ and $d_{A,B} \in U_G$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, (P_G, \oplus) is a K -loop.

1.1.4 The Algebras Associated with Twisted Semi-direct product Lie Loops

Theorem 1.1.13. *If the loop L is the twisted semi-direct product of Lie groups K and H , $K^* := K \times 1$ and $H^* := 1 \times H$, then K^* and H^* are subgroups of the loop L such that K^* is normal, and $L = K^*H^*$*

Theorem 1.1.14. *Let \mathfrak{h} and \mathfrak{k} be two Lie algebras over the field \mathbb{F} , and let $\rho : \mathfrak{h} \rightarrow \text{Der}_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism. Then*

1. $\mathfrak{l} = \overline{\mathfrak{k}} \overline{\times} \mathfrak{h}$ is an Akivis algebra with the bracket and the trilinear operation given below respectively. Moreover, \mathfrak{l} is a Lie algebra.

$$\begin{aligned} \langle (k_1, h_1)(k_2, h_2), (k_3, h_3) \rangle &= [[(k_1, h_1), (k_2, h_2)], (k_3, h_3)] - [(k_1, h_1), [(k_2, h_2), (k_3, h_3)]] \\ [(k_1, h_1), (k_2, h_2)] &= ([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho, [h_1, h_2]) \end{aligned}$$

2. $\overline{\mathfrak{k}} = \{(k, 0) : k \in \mathfrak{k}\} \cong \mathfrak{k}$ is an ideal of \mathfrak{l} , i.e., $[\overline{\mathfrak{k}}, \mathfrak{l}] \subseteq \overline{\mathfrak{k}}$.
3. $\overline{\mathfrak{h}} = \{(0, h) : h \in \mathfrak{h}\} \cong \mathfrak{h}$ is a subalgebra of \mathfrak{l} , i.e., $[\overline{\mathfrak{h}}, \overline{\mathfrak{h}}] \subseteq \overline{\mathfrak{h}}$.

1.2 Loops

The foundation of quasigroups and loops dates back to the 1920's. Anton Kazimirovich Suschkewitsch (1889-1961) was a Russian mathematician who embarked upon the first systematic study of semigroups [12]. His doctoral

dissertation, *The theory of operations as the general theory of groups*, discussed a number of special classes of *magmas* that are associative structures which we call semigroups, and certain types of algebraic structures, namely "groups" without associativity. This is the first glimmering of quasigroup and loop structures. Suschkewitsch also realized that in the proof of Lagrange's theorem there is no assumption of associativity. Therefore, he naturally considered the existence of nonassociative finite algebraic structures that also satisfy Lagrange's theorem. This question motivated Suschkewitsch to study nonassociative systems.

The name *Quasigroup* is used for the first time by Ruth Moufang a German mathematician, in her famous paper, *Zur Struktur von Alternativkoerpern*, published in 1935. In this paper, Moufang defined an algebraic structure (Q^*, \cdot) which she called Quasigroup and that is now known as a Moufang loop.

When we look back to the period between 1930 and 1940 there were several papers published regarding quasigroups [8], [22], [5], [24]; (i) *Theory of Quasi-Groups* by B.A. Hausmann and Oystein Ore, 1937. (ii) *Quasi-Groups Which Satisfy certain Generalized Associative Laws* by D.C. Murdoch, 1939. (iii) *Quasi-Groups* by G.N Garrison, 1940. All these authors used the term "Quasigroup" in a broader sense than as we use today and all these papers do not assume the existence of a neutral element as opposed to the definition of quasigroup by Moufang in 1935. It was around this time that the terminology "quasigroup" went to historical chance since it was necessary to distinguish two classes of quasigroups, quasigroups which have a neutral element and the

quasigroup which do not have the neutral element. There was growing interest in non-associative algebra at the University of Chicago. Albert published his first paper about quasigroups, *Quasigroups I* in 1943, and he used the term *loop* for the first time as the quasigroup with identity element [1].

Note that we only focus the loop structures rather than quasigroups. Therefore the preliminaries will be based on the Loop theory. The formal definition of loop can be given as follow.

Definition 1.2.1. *Let L be a nonempty set with a binary operation \oplus .*

- i. $a \oplus x = b$ has a unique solution $x := a \backslash b \in L$ for given any $a, b \in L$,*
- ii. $y \oplus a = b$ has a unique solution $y := b / a \in L$ for given any $a, b \in L$,*
- iii. There exist $e \in L$ such that for any $a \in L$, $a \oplus e = e \oplus a = a$.*

*(L, \oplus) is called a **right loop** if (i) and (iii) are valid. (L, \oplus) is called a **left loop** if (ii) and (iii) are valid. (L, \oplus) is called a **loop** if (i), (ii), and (iii) are valid.*

Definition 1.2.2. *Let (L, \oplus) be a loop and let $L_x : L \rightarrow L$ and $R_x : L \rightarrow L$ be the two maps defined by $aR_x := a \oplus x$, $aL_x := x \oplus a$. The maps L_x and R_x are called the left and the right translation maps for x .*

It is well know that if (L, \oplus) is a loop, then the left and the right translation maps are bijective.

Definition 1.2.3. *Let (Q, \odot) be a loop. Then the left, right and middle nuclei of Q , subgroups of Q , can be defined respectively as follows:*

$$1. N_l = \{a \in Q | (a \odot x) \odot y = a \odot (x \odot y); \forall x, y \in Q\}$$

$$2. N_r = \{a \in Q | (x \odot y) \odot a = x \odot (y \odot a); \forall x, y \in Q\}$$

$$3. N_m = \{a \in Q | (x \odot a) \odot y = x \odot (a \odot y); \forall x, y \in Q\}$$

Definition 1.2.4. Let (Q, \odot) be a loop. The nucleus of (Q, \odot) is the intersection of the right, left and middle nuclei; $N(Q) = N_l \cap N_r \cap N_m$. The centrum of (Q, \odot) is the collection of elements which commute with each element of Q , and is denoted by $C(Q)$; $C(Q) = \{x \in Q | x \odot y = y \odot x \forall y \in Q\}$. The center of Q is denoted by $Z(Q)$ and is the intersection of its nucleus and its centrum: $Z(Q) = C(Q) \cap N(Q)$.

It is well known that the nuclei and the center of Q are subgroups of Q .

Definition 1.2.5. A loop (Q, \odot) is said to have the left inverse property (and is called an L.I.P. loop), if there exists a bijection $J_\alpha : Q \rightarrow Q$, $a \mapsto a^\alpha$ such that $a^\alpha \odot (a \odot x) = x$ for every $x \in Q$. Similarly, a loop (Q, \odot) is said to have the right inverse property (R.I.P.) if there exists a bijection $J_\beta : Q \rightarrow Q$, $a \mapsto a^\beta$ such that $(x \odot a) \odot a^\beta = x$ for every $x \in Q$.

Next we define some groups acting on Q .

Definition 1.2.6. Let (Q, \oplus) be a loop with the neutral element $e \in Q$. The right multiplication group $Rmlt(Q)$ of Q is the permutation group generated by all right translations of Q . The left multiplication group $Lmlt(Q)$ of Q is the permutation group generated by all left translations of Q . The multiplication group $Mlt(Q)$ of Q is the permutation group generated by all translations of

Q . $Mlt(Q) = \langle L_a, R_b : a, b \in Q \rangle$. The inner mapping group of Q , $Inn(Q)$, is the subgroup of $Mlt(Q)$ defined by $\{\phi \in Mlt(Q) : \phi(e) = e\}$.

Bruck [3] showed that the inner mapping group is generated by three kinds of mappings that measure deviation from associativity and commutativity. The maps generating the inner mapping group can be written as certain compositions of left and right translation maps and their inverses as follows:

$$R_{a,b} = R_{b \oplus a}^{-1} R_a R_b \quad (1.1)$$

$$L_{a,b} = L_{a \oplus b}^{-1} L_a L_b \quad (1.2)$$

$$T_a = R_a^{-1} L_a \quad (1.3)$$

The group generated by the $R_{a,b}$ for all $a, b \in Q$ is called the right inner mapping group of Q , similarly the group generated by $L_{a,b}$ for all $a, b \in Q$ is called the left inner mapping group of Q . The generator $L_{a,b}$ is also called the precession map in [13] and denoted by $\delta_{a,b}$. It can be easily observed that for $x \in Q$, $a \oplus (b \oplus x) = (a \oplus b) \oplus \delta_{a,b}(x)$, so $\delta_{a,b}$ for $a, b \in Q$ is a measurement of deviation from associativity. It is clear that if $\delta_{a,b} = id$ for all $a, b \in Q$, then (Q, \oplus) is associative. If (Q, \oplus) is a right loop then the left translation map L_a is bijective. Therefore, the precession map $\delta_{a,b} = L_{a \oplus b}^{-1} L_a L_b$ is also bijective as a composition of bijective maps.

1.3 K-Loops

One of the goals for this dissertation is to look at the K-loop structures arising from subgroups of $GL(\mathcal{H})$ in the Lie perspective. There are several definitions of Lie loops in the literature [23], [18] and [9]. A topological loop is defined in [11] as a loop Q with a topology such that the multiplication, right and left division maps are all continuous. Recall that a topological space, X , is called *homogeneous* if and only if its group of homeomorphisms is transitive. This means for any pair of points $(x, y) \in X \times X$ we can find a homeomorphism $\beta : X \rightarrow X$ such that $\beta(x) = y$. If Q is a topological loop then for any $x, y \in Q$ there exists a unique $a \in Q$ such that $x \oplus a = y$, or we can write $R_a(x) = y$ where $a = x \setminus y$. Notice that all translation maps in a topological loop are homeomorphisms. Therefore, we obtain the following proposition.

Proposition 1.3.1. [11] *The underlying space of a topological loop is homogeneous.*

Definition 1.3.2. [23] *Let \mathcal{C} be the category of topological spaces, C^∞ -differentiable manifolds or analytical manifolds. A quasigroup Q is a \mathcal{C} -quasigroup if Q is an object in the category \mathcal{C} and the mappings $(x, y) \mapsto x \circ y$, $(x, y) \mapsto x \setminus y$, $(x, y) \mapsto y/x : Q \times Q \rightarrow Q$ are \mathcal{C} -morphisms.*

Definition 1.3.3. [9] *A (local) Lie loop is a real analytic manifold M with a base point e and three analytic functions $(x, y) \mapsto x \circ y$, $(x, y) \mapsto x \setminus y$, $(x, y) \mapsto y/x : M \times M \rightarrow M$ (respectively, $U \times U \rightarrow M$ for an open neighborhood U of e in M) such that the following conditions are satisfied: (i) $x \circ e = e \circ x = x$,*

(ii) $x \circ (x \backslash y) = x$, and (iii) $(x/y) \circ y = x$ for all $x, y \in M$. If the multiplication is associative, then M is (local) Lie group.

Definition 1.3.4. [18] *A homogeneous Lie loop is a homogeneous loop G that is also a C^∞ -differentiable manifold such that the loop multiplication $\mu : G \times G \rightarrow G$ is differentiable.*

All definitions given above agree that a Lie loop, L , has two structures: (i) a loop structure, and (ii) a differentiable Manifold structure, such that multiplication, right and left division maps are all smooth. We define a Lie loop as follows:

Definition 1.3.5. *A Lie loop, (L, \circ) , is a loop that is also a smooth manifold such that $\circ, \backslash, / : L \times L \rightarrow L$; $(x, y) \mapsto x \circ y$, $(x, y) \mapsto x \backslash y$, and $(x, y) \mapsto y/x$ are all smooth, i.e., infinitely differentiable.*

Definition 1.3.6. *A K-loop is a loop (Q, \oplus) which satisfies the left Bol identity and the automorphic inverse property given below respectively.*

$$a \oplus (b \oplus (a \oplus c)) = (a \oplus (b \oplus a)) \oplus c \quad (1.4)$$

$$(a \oplus b)^{-1} = a^{-1} \oplus b^{-1} \quad (1.5)$$

We remark that the Bol identity implies that each element of Q has a two-sided inverse. (See the discussion after Remark 1.3.12)

Kreuzer and Wefelscheid [20] undertook an axiomatic investigation and provided a new construction method for K-loops from groups following A.A.

Ungar who used the name *gyrogroups* for K-loops. A. A. Ungar [28] investigated the relativistic addition \oplus of the velocities belonging to the open ball \mathbb{R}_c^3 of the three dimensional Euclidean space \mathbb{R}^3 . Here $\mathbb{R}_c^3 = \{v \in \mathbb{R}^3 : \|v\| < c\}$, where c represents the vacuum speed of light. The Einstein velocity addition is a binary operation \oplus on the ball \mathbb{R}_c^3 that is defined by

$$u \oplus v = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left\{ u + v + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} (u \times (u \times v)) \right\}$$

where $u, v \in \mathbb{R}_c^3$. In this definition, \cdot and \times represent the usual dot product and cross product in \mathbb{R}^3 and γ_u is the *Lorenz-Factor* given by

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{\|u\|^2}{c^2}}}.$$

A.A.Ungar [28] showed that;

Theorem 1.3.7. [28] (\mathbb{R}_c^3, \oplus) is a non-associative and non-commutative loop.

Remark 1.3.8. H.Wefelscheid recognized that (\mathbb{R}_c^3, \oplus) is a K-loop [20].

Given a right loop (Q, \oplus) , the precession map $\delta_{a,b} : Q \rightarrow Q$ plays an important role for constructing K-loops, thus it might be useful to summarize some of the important properties. For the convenience of the reader we prove the following lemma.

Lemma 1.3.9. [20] Let (Q, \oplus) be a right loop. For all $a, b \in Q$ the precession map $\delta_{a,b} : Q \rightarrow Q$ satisfies

1. $\delta_{a,b}(e) = e$

$$2. \delta_{a,e} = \delta_{e,a} = id$$

Proof.

$$\delta_{a,b}(e) = L_{a\oplus b}^{-1}L_aL_b(e) \tag{1.6}$$

$$= L_{a\oplus b}^{-1}(a \oplus (b \oplus e)) \tag{1.7}$$

$$= L_{a\oplus b}^{-1}(a \oplus b). \tag{1.8}$$

If we let that $\delta_{a,b}(e) = L_{a\oplus b}^{-1}(a \oplus b) = x$, then $L_{a\oplus b}(x) = a \oplus b$ since (Q, \oplus) is a right loop, and therefore it is a bijection. On the other hand $L_{a\oplus b}(x) = a \oplus b$ is equivalent to $(a \oplus b) \oplus x = (a \oplus b) \oplus e$, and by the uniqueness of the solution $x = e$. To see that $\delta_{a,e} = \delta_{e,a} = id$, consider that $a \oplus x = a \oplus (e \oplus x) = (a \oplus e) \oplus \delta_{a,e}(x) = a \oplus \delta_{a,e}(x)$, and by the uniqueness of the solution we conclude that $\delta_{a,e} = id$. It can be proved similarly that $\delta_{e,a} = id$. \square

Lemma 1.3.10. [20] *Let (Q, \oplus) be a right loop. For any $a, b, x, y \in Q$,*

$$1. (a \oplus b) \oplus x = a \oplus (b \oplus \delta_{a,b}^{-1}(x))$$

$$2. \delta_{a,b}(x \oplus y) = \delta_{a,b}(x) \oplus \delta_{a\oplus b, \delta_{a,b}(x)}^{-1} \delta_{a, b \oplus x} \delta_{b,x}(y)$$

$$3. \text{ For any automorphism } \beta \text{ of } (Q, \oplus), \beta \delta_{a,b} \beta^{-1} = \delta_{\beta(a), \beta(b)}$$

Kreuzer and Wefelscheid [20] provided an axiomatic way to construct a K-loop as follows:

Theorem 1.3.11. *Let G be a group. Let A be a subgroup of G and let K be a subset of G such that:*

1. $G = KA$ is an exact decomposition, i.e., for every element $g \in G$ there are unique elements $k \in K$ and $a \in A$ such that $g = ka$.
2. If e is the neutral element of G , then $e \in K$.
3. For each $x \in K$, $xKx \subseteq K$.
4. For each $y \in A$, $yKy^{-1} \subseteq K$.
5. For each $k_1, k_2 \in K$ and $\alpha \in A$, if $k_1k_2\alpha \in K$, then there exists $\beta \in A$ such that $k_1k_2\alpha = \beta k_2k_1$.

Then for all $a, b \in K$ there exists unique $a \oplus b \in K$ and $d_{a,b} \in A$ such that $ab = (a \oplus b)d_{a,b}$. Moreover (K, \oplus) is a K -loop.

Remark 1.3.12. *The first condition, the exact decomposition of G , implies that for any $x, y \in K \subseteq G$ there exist unique $k \in K$ and $a \in A$ such that $xy = ka$. We define $x \oplus y := k$ and $d_{x,y} := a$. Therefore for any $x, y \in K$, $xy = (x \oplus y)d_{x,y}$, and this is equivalent to $x \oplus y = xyd_{x,y}^{-1}$. So $\oplus : K \times K \rightarrow K$ is a new binary operation in terms of group multiplication, and it can be shown that (K, \oplus) is a K -loop. Note that the conditions (1) and (2) guarantee that (K, \oplus) is a right loop. On the other hand, the condition (3) implies the Bol identity, and the condition (4) implies $K^{-1} \subseteq K$ and $|Kx \cap yA| = 1$, so that (K, \oplus) turns out also to be a left loop. Finally the condition (5) provides the automorphic inverse property. Thus (K, \oplus) is a K -loop.*

For a loop (Q, \oplus) the left inverse and the right inverse of an element in Q are not necessarily the same, but one of the interesting properties of K -loop

structures is to have the same right and the left inverses of any element. This can be easily verified by the Bol identity. There are two binary operations in the construction of K-loops, the group operation and the K-loop operation which induced by the group operation. It is natural to ask about the relation between the inverses of an element with respect to the K-loop operation and with respect to the group operation. The answer is that they are the same element. To see this we use the exactness of the decomposition $G = KA$. Let x^{-1} be the inverse of a with respect to the group operation, and let y^{-1} be the inverse of a with respect to the K-loop operation. Then $e = ax^{-1} = (a \oplus x^{-1})d_{a,x^{-1}} = (a \oplus y^{-1})d_{a,y^{-1}} = ay^{-1}$. The exactness implies that $a \oplus x^{-1} = a \oplus y^{-1}$, and the uniqueness of the solution gives $x^{-1} = y^{-1}$.

Kreuzer and Wefelscheid [20] pioneered an abstract way to construct a K-loop. In this construction G is an algebraic group which may also belong to one of the categories of topological groups or Lie groups. Kiechle [13] investigated many K-loop structures derived from classical groups, namely subgroups of $GL(n, \mathbb{C})$. The ultimate goal of this dissertation is to view $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ as real and complex Lie groups, respectively, and investigate the Lie K-loop structures derived from $G \leq GL(n, \mathbb{C})$ where G is a connected reductive Lie subgroup. Moreover, we extend the finite dimensional cases to the infinite dimensional cases such that $G \leq GL(\mathcal{H})$ and $G \leq GL(\infty, \mathcal{H}_{\mathbb{F}})$ where $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $\mathcal{H}_{\mathbb{F}}$ is an infinite dimensional separable Hilbert space over \mathbb{F} .

1.4 Hilbert Spaces

This section is a short introduction to *Hilbert space*. We mainly follow Retherford [25]. In this section let X be a linear space over \mathbb{C} , the complex numbers.

Definition 1.4.1. A norm, $\| \cdot \|$, is a function from X into the non-negative real numbers \mathbb{R}^+ satisfying

1. $\|x\| = 0$ if and only if $x = 0$. Note that the first zero is the real number, but the second zero is the zero vector of the linear space X .
2. $\|cx\| = |c| \|x\|$, for each $x \in X$ and for each $c \in \mathbb{C}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Notice that by definition $\|x\| \geq 0$. A vector space X endowed with a norm $\| \cdot \|$ is called a normed space.

Definition 1.4.2. let X be a linear space over \mathbb{C} . An inner product, $\langle \cdot, \cdot \rangle$ is a function $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$ that satisfies the following axioms.

1. $\langle x, x \rangle \geq 0$.
2. $\langle x, x \rangle = 0$ if and only if $x = 0$.
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$.
4. $\langle cx, y \rangle = c \langle x, y \rangle$ for all $x, y \in X$ and for all $c \in \mathbb{C}$.
5. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for any $x, y \in X$

Inner product space is a vector space endowed with an inner product. Notice that $\langle x, x \rangle = \overline{\langle x, x \rangle}$, thus $\langle x, x \rangle \in \mathbb{R}$ for any $x \in X$. On the other hand $\langle x, cy \rangle = \overline{\langle cy, x \rangle} = \overline{c \langle y, x \rangle} = \bar{c} \overline{\langle y, x \rangle} = \bar{c} \langle x, y \rangle$. Similarly it can be shown that an inner product is also conjugate linear with respect to second argument by using linearity and conjugate symmetry. Every inner product space X is a normed space with $\|x\| := \sqrt{\langle x, x \rangle}$. Similarly it is also a metric space with the metric on X defined by

$$d(x, y) := \|x - y\| = \sqrt{\langle x - y, x - y \rangle}.$$

Definition 1.4.3. Let X be a normed space, and let (x_n) be a sequence in X . We say that (x_n) is a Cauchy sequence with respect to the given norm if for every $\epsilon > 0$ there exists an integer N such that for any $m, n \geq N$, $\|x_n - x_m\| < \epsilon$. We say that the sequence (x_n) has a limit $x \in X$ with respect to $\| \cdot \|$, if for any ϵ there exists an integer N such that for all $n \geq N$, $\|x_n - x\| < \epsilon$. The normed space X is called complete, if for each Cauchy sequence converges in X .

Definition 1.4.4. A complete inner product space is called Hilbert Space, and complete normed space is called Banach space.

Example 1.4.5. Let $l_2 := \{(a_n) \mid \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$, the space of square summable sequences (a_n) of complex numbers. We can be define an inner product on l_2 . If $a = (a_n)$ and $b = (b_n)$ then define $\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n$. It can be shown that l_2 is an infinite dimensional separable Hilbert space.

1.4.1 Bounded Linear Operators on a Hilbert Space

Let \mathcal{H} be a Hilbert space over \mathbb{C} and $T : \mathcal{H} \rightarrow \mathcal{H}$ a linear operator; that is $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$ for all $x, y \in \mathcal{H}$ and $c \in \mathbb{C}$. The operator T is called **bounded** if there exists $M > 0$ such that for all $x \in \mathcal{H}$, $\|Tx\| \leq M \|x\|$.

Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators;

$$\mathcal{L}(\mathcal{H}) = \{T : \mathcal{H} \rightarrow \mathcal{H} \mid T \text{ is linear and bounded}\}$$

Note that the collection of bounded operators forms a linear space by defining $(T_1 + T_2)(x) := T_1(x) + T_2(x)$ and $(cT)(x) := cT(x)$. We can also define a norm which is called *operator norm* or *supremum norm* on $\mathcal{L}(\mathcal{H})$ by

$$\|T\| := \text{Sup} \{\|Tx\| : \|x\| \leq 1\}.$$

It can be easily shown that $(\mathcal{L}(\mathcal{H}), \|\cdot\|)$ is a Banach space. For a Banach space X let X^* be the linear space of all bounded linear operators from X to the scalar field \mathbb{C} . The space X^* is called the *dual space* of X .

Theorem 1.4.6. [25] *Let \mathcal{H} be a Hilbert space.*

1. *Let $f \in \mathcal{H}^*$, then there exists a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$. Moreover $\|f\| \leq \|y\|$.*
2. *Let $y \in \mathcal{H}$. Define $f_y(x) = \langle x, y \rangle$, then $f_y \in \mathcal{H}^*$.*

Let $T \in \mathcal{L}(\mathcal{H})$. Define for $y \in \mathcal{H}$, $T^*y \in \mathcal{H}^*$ as $(T^*y)(x) := \langle Tx, y \rangle$.

By the Riesz representation theorem there exist a unique $z \in \mathcal{H}$ such that

$(T^*y)(x) = \langle Tx, y \rangle = \langle x, z \rangle$. We write $z = T^*y$ under this identification now $T^*y \in \mathcal{H}$ and thus $\langle Tx, y \rangle = \langle x, T^*y \rangle$. The operator T^* is called the adjoint of T .

Definition 1.4.7. *The operator $T \in \mathcal{L}(\mathcal{H})$ is called self-adjoint, if $T = T^*$; that is, $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$.*

Notice that if T is self-adjoint then for any $x \in \mathcal{H}$, $\langle Tx, x \rangle \in \mathbb{R}$, and this fact allows us to define a partial ordering on the self-adjoint operators. We say that the self adjoint operator $T \geq 0$ if $\langle Tx, x \rangle \geq 0 \forall x \in \mathcal{H}$ and for any $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ we say that $T_1 \geq T_2$ if and only if $T_1 - T_2 \geq 0$.

1.4.2 The Polar Decomposition Theorem

Let \mathcal{H} be an infinite dimensional separable Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators from \mathcal{H} to \mathcal{H} , and denote the collection of invertible bounded linear operators by $GL(\mathcal{H})$. *The Polar Decomposition Theorem* is a highly important tool. This theorem asserts that for any $T \in \mathcal{L}(\mathcal{H})$, T can be decomposed as a composition of two bounded operators $T = UA$ such that U is a partial isometry and A is the positive square root of T^*T . Note that U is an isometry or unitary operator if T is an invertible operator. In this case we can form a K-loop by following the method of [20] and [13]. The following lemma plays an important role for the polar decomposition theorem.

Lemma 1.4.8. [25] *Every positive self-adjoint $T \in \mathcal{L}(\mathcal{H})$ has a unique positive square root.*

What do we mean by positive square root? Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint and $T \geq 0$. If there is $A \in \mathcal{L}(\mathcal{H})$ such that A is self-adjoint and $A^2 = T$, then A is called a square root of T . If $A \geq 0$, A is called a positive square root of T , and it is denoted by $A = T^{\frac{1}{2}}$.

Theorem 1.4.9. [25] *Let $T \in \mathcal{L}(\mathcal{H})$, then there exists $U \in \mathcal{L}(\mathcal{H})$ such that*

1. $T = UA$ where $A = (T^*T)^{1/2}$
2. $\|Ux\| = \|x\|$ for $x \in \overline{\mathcal{R}(A)}$, the closure of $\mathcal{R}(A)$ in $\mathcal{L}(\mathcal{H})$, where $\mathcal{R}(A)$ is the range of A .
3. $Ux = 0$ for $x \in \overline{\mathcal{R}(A)}^\perp$.

An operator U satisfying (b) and (c) is called a partial isometry.

Lemma 1.4.10. [25] *Let $T \in \mathcal{L}(\mathcal{H})$, then $TT^* \geq 0$ and $T^*T \geq 0$.*

Lemma 1.4.11. [25] *Let \mathcal{H} be a complex Hilbert space, and let $T \in \mathcal{L}(\mathcal{H})$ such that $\langle Tx, x \rangle = 0$ for any $x \in \mathcal{H}$, then $T = 0$.*

Definition 1.4.12. *An operator is called normal provided $T^*T = TT^*$, and is called unitary provided $T^{-1} \in \mathcal{L}(\mathcal{H})$ and $T^* = T^{-1}$.*

Lemma 1.4.13. [25]

Let $T \in \mathcal{L}(\mathcal{H})$. Then

1. $T \in \mathcal{L}(\mathcal{H})$ is unitary if and only if T is an isometric isomorphism.
2. If T is invertible isometric operator, then T is unitary.

1.4.3 The Spectrum of T for $T \in \mathcal{L}(\mathcal{H})$

Definition 1.4.14. *Let $T \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of T provided there is an $x \in \mathcal{H}$, $x \neq 0$ such that $Tx = \lambda x$. The vector x is called an eigenvector associated to λ . Clearly if λ is an eigenvalue of T then $T - \lambda I$ is not invertible. Here I is the identity operator on \mathcal{H} .*

Definition 1.4.15. *Let $T \in \mathcal{L}(\mathcal{H})$. The spectrum of T , written $\sigma(T)$ is defined as follows:*

$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$. *In particular, an $n \times n$ matrix T , $T - \lambda I$ is not invertible if and only if the determinant of $(T - \lambda I)$ is zero. Therefore, for the finite dimensional case $\sigma(T)$ is the set of just eigenvalues of T . The complement $\rho(T)$ of the spectrum of T , $\rho(T) = \mathbb{C} \setminus \sigma(T)$, is called the resolvent of T .*

Proposition 1.4.16. [25] *The spectrum $\sigma(T)$ of a bounded operator T is a nonempty compact subset of \mathbb{C} .*

1.5 The Cartan Decomposition

In this section we follow A.W. Knap [19].

Definition 1.5.1. *We say that the group G is a linear connected reductive real group, if G is a closed connected subgroup of the complex matrices that is stable under conjugate transpose.*

A closed subgroup of a Lie group is a Lie group, therefore any linear connected reductive real group G is a Lie group. There is an automorphism $\Theta : G \rightarrow G$ defined by $\Theta(g) = (\bar{g}^\tau)^{-1}$, called the Cartan involution [19]. Let $K := \{g \in G : \Theta g = g\}$, then K is a maximal compact subgroup of G . Note that the Cartan involution $\Theta : G \rightarrow G$ is a Lie group automorphism. We can define the Cartan involution at the level of the Lie algebra by differentiating it at the identity element. Let \mathfrak{g} be the Lie algebra of the Lie group G , and let θ be differential of Θ at 1, therefore $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ is given by $\theta(x) = -\bar{x}^\tau$ for any $x \in \mathfrak{g}$.

Consider the eigenvalues $+1$ and -1 of the automorphism θ . If we denote the corresponding eigenspaces in \mathfrak{g} by \mathfrak{k} and \mathfrak{p} respectively, then by definition of eigenspaces $\mathfrak{k} = \{v \in \mathfrak{g} : \theta(v) = v\} = \{v \in \mathfrak{g} : \bar{v}^\tau = -v\}$ and $\mathfrak{p} = \{v \in \mathfrak{g} : \theta(v) = -v\} = \{v \in \mathfrak{g} : \bar{v}^\tau = v\}$. Therefore, \mathfrak{k} is the collection of skew-hermitian matrices in \mathfrak{g} and \mathfrak{p} is the collection of hermitian matrices, or, it equivalently, the self-adjoint matrices. It is clear that $\mathfrak{p} \cap \mathfrak{k} = \{0\}$. Moreover for any $v \in \mathfrak{g}$; $v = \frac{v-\bar{v}^\tau}{2} + \frac{v+\bar{v}^\tau}{2}$ where $\frac{v-\bar{v}^\tau}{2} \in \mathfrak{k}$ and $\frac{v+\bar{v}^\tau}{2} \in \mathfrak{p}$. On the other hand if $v = k_1 + p_1 = k_2 + p_2$ then $k_1 - k_2 = p_2 - p_1 = 0$ implies that $k_1 = k_2$ and $p_1 = p_2$. Therefore, \mathfrak{g} has a Cartan decomposition as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. It can be easily verified that

- $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$
- $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$
- $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$

Here $[\cdot, \cdot]$ is the Lie bracket on \mathfrak{g} , $[X, Y] = XY - YX$. For instance let $v, w \in \mathfrak{p}$, then $[v, w] = vw - wv$ and observe that

$$-(vw - wv)^* = -((vw)^* - (wv)^*) \quad (1.9)$$

$$= v^*w^* - w^*v^* \quad (1.10)$$

$$= (-v)(-w) - (-w)(-v) \quad (1.11)$$

$$= vw - wv, \quad (1.12)$$

Note that for the convenience we used v^* for the conjugate transpose of v . We conclude that $[v, w] = vw - wv \in [\mathfrak{p}, \mathfrak{p}]$ is skew-hermitian, hence $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Notice that \mathfrak{k} is a Lie subalgebra of \mathfrak{g} , since the bracket operation is closed by $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$. The Cartan decomposition on the Lie group level takes the form $G = Kexp\mathfrak{p}$.

Proposition 1.5.2. [19] *Let G be a linear connected reductive real group, then K is compact connected and is a maximal compact subgroup of G . Its Lie algebra is \mathfrak{k} . Moreover, the map $K \times \mathfrak{p} \rightarrow G$ given by $(k, X) \mapsto kexpX$ is a diffeomorphism, i.e., an isomorphism in the category of smooth manifolds, onto G .*

1.6 The Classical Complex Banach-Lie Groups of Bounded Operators on \mathcal{H}

In this section, we follow Pierre de la Harpe [7].

Let \mathcal{H} be an infinite dimensional separable Hilbert space over \mathbb{C} , and let J be a semi-linear operator on \mathcal{H} . This means that for any $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, (i) $J(x + y) = J(x) + J(y)$, and (ii) $J(\lambda x) = \lambda^\Theta J(x)$, where $\Theta \in \text{Aut}(\mathbb{C})$. The only continuous automorphisms of the complex numbers are the identity and complex conjugation, therefore (ii) can be replaced by $J(\lambda x) = \bar{\lambda}J(x)$, $\bar{\lambda}$ is the complex conjugation of λ . $J_{\mathbb{R}}$ is called a conjugation if $\langle Jx, Jy \rangle = \overline{\langle x, y \rangle}$ for any $x, y \in \mathcal{H}$ and $J^2 = Id$. A semi-linear operator J on \mathcal{H} is called an anti-conjugation if $\langle Jx, Jy \rangle = \overline{\langle x, y \rangle}$ and $J^2 = -Id$.

Examples of classical complex Banach-Lie groups of bounded operators are given in [7], e.g., $GL(\mathcal{H})$, $O(\mathcal{H}, J_{\mathbb{R}})$ and $Sp(\mathcal{H}, J_{\mathbb{Q}})$. Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on \mathcal{H} , and let $GL(\mathcal{H})$ be the group of invertible operators in $\mathcal{L}(\mathcal{H})$. The Orthogonal and Symplectic Banach-Lie groups consist of those elements in $GL(\mathcal{H})$ that leave invariant the following bilinear forms respectively: $\mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}; (x, y) \mapsto \langle x, J_{\mathbb{R}}y \rangle$ and $(x, y) \mapsto \langle x, J_{\mathbb{Q}}y \rangle$. Here is $J_{\mathbb{R}}$ is a conjugation, and $J_{\mathbb{Q}}$ an anti-conjugation. Therefore the orthogonal and symplectic complex Banach-Lie groups can be defined by

1. $O(\mathcal{H}, J_{\mathbb{R}}) := \{T \in GL(\mathcal{H}) : \langle Tx, J_{\mathbb{R}}Ty \rangle = \langle x, J_{\mathbb{R}}y \rangle\}$
2. $Sp(\mathcal{H}, J_{\mathbb{Q}}) := \{T \in GL(\mathcal{H}) : \langle Tx, J_{\mathbb{Q}}Ty \rangle = \langle x, J_{\mathbb{Q}}y \rangle\}$

Theorem 1.6.1. [7] *Let $G(\mathcal{H})$ be one of the classical complex Banach-Lie groups in $\{GL(\mathcal{H}), O(\mathcal{H}, J_{\mathbb{R}}), Sp(\mathcal{H}, J_{\mathbb{Q}})\}$ and let \mathfrak{g} be the Lie algebra of $G(\mathcal{H})$. Let $Sym(\mathcal{H})$ and $Pos(\mathcal{H})$ be the set of self-adjoint and positive self adjoint operators \mathcal{H} to \mathcal{H} . Then*

1. *The exponential map $exp : \mathfrak{g} \cap Sym(\mathcal{H}) \rightarrow G(\mathcal{H}) \cap Pos(\mathcal{H})$ is an analytic isomorphism.*
2. *The Polar Decomposition Theorem provides an analytic isomorphism between $G(\mathcal{H})$ and $[G(\mathcal{H}) \cap U(\mathcal{H})] \times [G(\mathcal{H}) \cap Pos(\mathcal{H})]$.*

1.7 Akivis Algebra

In this section we follow [9] and [4]. A (local) Lie loop is a real analytic manifold M with a base point e and three analytic functions $(x, y) \mapsto x \circ y$, $(x, y) \mapsto x \setminus y$, $(x, y) \mapsto y / x : M \times M \rightarrow M$ (respectively, $U \times U \rightarrow M$ for an open neighborhood U of e in M) such that the following conditions are satisfied:

1. $x \star e = e \star x = x$ for all $x \in M$.
2. $x \star (x \setminus y) = y$ for all $x, y \in M$.
3. $(x / y) \star y = x$ for all $x, y \in M$.

Let \mathfrak{m} be the tangent space of M at e , equipped with a skew-symmetric bilinear operation and a trilinear operation $(x, y) \mapsto [x, y], (x, y, z) \mapsto \langle x, y, z \rangle :$

$\mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ defined below. These are defined as follow: Let B be a convex symmetric open neighborhood of 0 in \mathfrak{m} such that the exponential function maps B diffeomorphically onto an open neighborhood V of e in M and transports the operation \star into \mathfrak{m} by

$X \star Y = (\exp|_B)^{-1}(\exp(X) \star \exp(Y))$ for X and Y in a neighborhood C of 0 in B such that $\exp(C) \star \exp(C) \subseteq V$. The left and the right division maps can be transported similarly. Then we set

1. $[X, Y] = \lim_{t \rightarrow 0} t^{-2}((tX \star tY)/(tY \star tX))$
2. $\langle X, Y, Z \rangle = \lim_{t \rightarrow 0} t^{-3}(((tX \star tY) \star tZ)/(tX \star (tY \star tX)))$

The skew-symmetric bilinear and trilinear operations are linked by the Akivis identity given below.

$$\sum_{g \in S_3} \langle X_{g(1)}, X_{g(2)}, X_{g(3)} \rangle = [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2],$$

where S_3 is the permutation group on $\{1, 2, 3\}$.

Definition 1.7.1. [9] *A vector space with a skew-symmetric bilinear and a trilinear operation which are linked by the Akivis identity is called an Akivis algebra.*

Note that if M is a Lie group, then the Akivis algebra \mathfrak{m} is the traditional Lie algebra.

Definition 1.7.2. [4] *Let \mathcal{A}_1 and \mathcal{A}_2 be two Akivis algebras. A homomorphism $\alpha : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a linear map such that $[X, Y]^\alpha = [X^\alpha, Y^\alpha]$ and $\langle X, Y, Z \rangle^\alpha = \langle X^\alpha, Y^\alpha, Z^\alpha \rangle$ for all X, Y , and $Z \in \mathcal{A}_1$.*

Chapter 2

The Twisted Semi-direct Product Lie Loops

2.1 Semi-direct Product

Let H and K be groups such that $H \leq \text{Aut}(K)$ and consider $G := K \times H$ as a set and define a binary operation \odot on G such that $(k_1, h_1) \odot (k_2, h_2) = (k_1 k_2^{h_1^{-1}}, h_1 h_2)$. Then (G, \odot) is a group with identity element $(1_K, 1_H)$. G is called the *normal* product of K by H or the **semi-direct product** of K by H in [6], and it is denoted by $G = K \rtimes H$.

2.2 Twisted Semi-direct Product

Definition 2.2.1. *Let H and K be groups such that $H \leq \text{Aut}(K)$. Let $G := K \times H$ and define a binary operation \oplus on G as*

$$(k_1, h_1) \oplus (k_2, h_2) = (k_1 k_2^{h_1}, h_1 h_2)$$

We will call G with this binary operation the twisted semi-direct product of K by H and denote it by $G = K \overline{\rtimes} H$.

In this definition each element of H is an automorphism from K to K and the notation $k_2^{h_1}$ is the image of k_2 under h_1 . It can be observed that the twisted semi-direct product is not necessarily associative. Let $(k_1, h_1), (k_2, h_2)$ and (k_3, h_3) be arbitrary elements of G , then observe that

$$((k_1, h_1) \odot (k_2, h_2)) \odot (k_3, h_3) = (k_1 k_2^{h_1}, h_1 h_2) \odot (k_3, h_3) \quad (2.1)$$

$$= ((k_1 k_2^{h_1}) k_3^{h_1 h_2}, (h_1 h_2) h_3) \quad (2.2)$$

$$(k_1, h_1) \odot ((k_2, h_2) \odot (k_3, h_3)) = (k_1, h_1) \odot (k_2 k_3^{h_2}, h_2 h_3) \quad (2.3)$$

$$= (k_1 (k_2 k_3^{h_2})^{h_1}, h_1 (h_2 h_3)) \quad (2.4)$$

$$= (k_1 (k_2^{h_1} k_3^{h_2 h_1}), (h_1 h_2) h_3) \quad (2.5)$$

$$= ((k_1 k_2^{h_1}) k_3^{h_2 h_1}, (h_1 h_2) h_3) \quad (2.6)$$

We conclude that G is associative if H is an abelian group. If H is not abelian there exists $h_1, h_2 \in H$ such that $h_1 h_2 \neq h_2 h_1$. Therefore there exists

a nonzero $k_3 \in K$ such that $k_3^{h_1 h_2} \neq k_3^{h_2 h_1}$. Then for nonzero $k_1, k_2 \in K$, $(k_1 k_2^{h_1}) k_3^{h_1 h_2} \neq (k_1 k_2^{h_1}) k_3^{h_2 h_1}$ which is equivalent to:

$$((k_1, h_1) \odot (k_2, h_2)) \odot (k_3, h_3) \neq (k_1, h_1) \odot ((k_2, h_2) \odot (k_3, h_3))$$

Therefore if H is not abelian, then G is not associative.

Theorem 2.2.2. *Let H and K be Lie groups with $H \leq \text{Aut}(K)$ such that each $h \in H$ is smooth with smooth inverse. Let $\mathcal{L} := K \overline{\rtimes} H$. Then*

1. (\mathcal{L}, \oplus) is a Lie group if and only if H is an abelian Lie group.
2. If H is not abelian, then (\mathcal{L}, \oplus) is a Lie loop, not a Lie group.

Proof. Let H and K be Lie groups such that H is abelian, and suppose $H \leq \text{Aut}(K)$ such that the elements of H are smooth with smooth inverses. (\mathcal{L}, \oplus) with the, twisted semi-direct product is a group with the neutral element (e_k, e_h) where e_h is the identity map on K and e_k is the neutral element of K . For each $(k, h) \in \mathcal{L}$, $(e_k, e_h)(k, h) = (e_k k^{e_h}, e_h h) = (k, h)$ and $(k, h)(e_k, e_h) = (k e_k^h, h e_h) = (k, h)$. On the other hand each element $(k, h) \in \mathcal{L}$ has a unique inverse such that

$$(k, h)^{-1} = ((k^{-1})^{h^{-1}}, h^{-1}).$$

$(k, h)((k^{-1})^{h^{-1}}, h^{-1}) = (k((k^{-1})^{h^{-1}})^h, h h^{-1}) = (k(k^{-1})^{e_h}, e_h) = (k k^{-1}, e_h) = (e_k, e_h)$. Similarly $((k^{-1})^{h^{-1}}, h^{-1})(k, h) = (e_k, e_h)$. To see that \oplus is associative

on \mathcal{L} . Let $(k_1, h_1), (k_2, h_2)$ and $(k_3, h_3) \in \mathcal{L}$. Then

$$(k_1, h_1)[(k_2, h_2)(k_3, h_3)] = (k_1, h_1)(k_2 k_3^{h_2}, h_2 h_3) \quad (2.7)$$

$$= (k_1(k_2 k_3^{h_2})^{h_1}, h_1(h_2 h_3)) \quad (2.8)$$

$$= (k_1(k_2^{h_1} k_3^{h_2 h_1}), (h_1 h_2) h_3) \quad (2.9)$$

$$= ((k_1 k_2^{h_1}) k_3^{h_1 h_2}, (h_1 h_2) h_3) \quad (2.10)$$

$$= (k_1 k_2^{h_1}, h_1 h_2)(k_3, h_3) \quad (2.11)$$

$$= [(k_1, h_1)(k_2, h_2)](k_3, h_3) \quad (2.12)$$

Therefore \mathcal{L} is a group. On the other hand multiplication and inversion maps on \mathcal{L} are both smooth as a composition of smooth maps, H and K are Lie groups, as follow

$$\mu_{\mathcal{L}}((k_1, h_1), (k_2, h_2)) = (k_1 k_2^{h_1}, h_1 h_2) \quad (2.13)$$

$$= (\mu_K(k_1, k_2^{h_1}), \mu_H(h_1, h_2)) \quad (2.14)$$

$$= (\mu_K(k_1^{e_h}, k_2^{h_1}), \mu_H(h_1, h_2)) \quad (2.15)$$

$$= ((\mu_K \circ (e_h \times h_1))(k_1, k_2), \mu_H(h_1, h_2)) \quad (2.16)$$

$$= ((\mu_K \circ (e_h \times h_1)) \times \mu_H)((k_1, k_2), (h_1, h_2)) \quad (2.17)$$

$i_{\mathcal{L}}(k, h) = ((k^{-1})^{h^{-1}}, h^{-1}) = ((h^{-1} \circ i_K) \times i_H)(k, h)$. Therefore we conclude that if H is an abelian Lie group and K is a Lie groups with $H \leq \text{Aut}(K)$ such that the elements of H are smooth with smooth inverses, then \mathcal{L} with twisted

semi-direct product is a Lie group. Conversely if \mathcal{L} with twisted semi-direct product is a Lie Loop, then \oplus is associative and this forces H to be abelian.

If H is not an abelian Lie group, then (\mathcal{L}, \oplus) is not associative so it is not a Lie Group, but for given any $(k_1, h_1), (k_2, h_2) \in \mathcal{L}$ the equation $(k_1, h_1) \oplus (x_1, x_2) = (k_2, h_2)$ has always a unique solution $(x_1, x_2) \in \mathcal{L}$ such that $k_1(x_1)^{h_1} = k_2$ and $h_1 x_2 = h_2$ where $x_1 = (k_1^{-1} k_2)^{h_1^{-1}}$ and $x_2 = h_1^{-1} h_2$. Similarly the equation $(y_1, y_2) \oplus (k_2, h_2) = (k_1, h_1)$ has always a unique solution $(y_1, y_2) = (k_1(k_2^{-1})^{(h_1 h_2^{-1})}, h_1 h_2^{-1})$. Therefore (\mathcal{L}, \oplus) is a loop. The multiplication, the twisted semi-direct product, on \mathcal{L} is smooth as we discussed on the first part of the Theorem. On the other hand $\backslash, / : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ the left and the right divisions are also smooth as a composition of smooth maps. The left and the right division maps can be formulated as follows

1. $(k_1, h_1) \backslash (k_2, h_2) = ((k_1 \backslash k_2)^{h_1^{-1}}, h_1 \backslash h_2)$
2. $(k_1, h_1) / (k_2, h_2) = (k_1(k_2^{-1})^{h_1/h_2}, h_1/h_2)$

Therefore, if H is an nonabelian Lie group then (\mathcal{L}, \oplus) is a Lie loop, not a Lie group. □

Note that in the definition of left and right division map if we set that $k_1 = k_2$ in (1) and $h_1 = h_2$ in (2) we will obtain the following Lemma.

Lemma 2.2.3. *Let H and K be groups such that $H \leq \text{Aut}(K)$ and let $\mathcal{L} := K \overline{\rtimes} L$. Then*

1. $(k_1, h_1) \backslash (k_2, h_2) = (e_k, h_1 \backslash h_2)$ if $k_1 = k_2$.

2. $(k_1, h_1)/(k_2, h_2) = (k_1/k_2, e_h)$ if $h_1 = h_2$.

Theorem 2.2.4. *Let \mathcal{H} be separable Hilbert space over \mathbb{C} , and let G be Lie subgroup of $GL(\mathcal{H})$ and let $\mathcal{L} = \mathcal{H} \overline{\rtimes} H$. Then*

1. \mathcal{L} is a Lie loop.
2. \mathcal{L} is a left Bol loop.
3. $Lmlt(\mathcal{L}) \subseteq Aff_H(\mathcal{H}) \times H$.
4. $N(\mathcal{L}) = \{0\} \times Z(H)$.
5. $Z(\mathcal{L}) \cong Fix_{\mathcal{H}}(H) \cap Z(H)$.

Proof. The infinite dimensional separable Hilbert space over \mathbb{C} is a Banach Lie group in the sense that $(\mathcal{H}, +)$ is a group and it is infinite dimensional manifold modeled to itself such that the operations $f : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}; (x, y) \mapsto x + y$ and $i : \mathcal{H} \rightarrow \mathcal{H}; x \mapsto -x$ are both smooth. On the other hand H is a Lie subgroup of $GL(\mathcal{H})$. Therefore, \mathcal{H} and H are both Lie groups.

1. Let $h_1, h_2 \in \mathcal{H}$ and $T_1, T_2 \in H$. The twisted semi-direct product on \mathcal{L} can be formulated as follow:

$$(h_1, T_1)(h_2, T_2) = (h_1 + h_2^{T_1}, T_1 T_2).$$

The equations $(h_1, T_1)(x_1, x_2) = (h_2, T_2)$ and $(y_1, y_2)(h_2, T_2) = (h_1, T_1)$

have unique solutions

$$(x_1, x_2) = ((-h_1)^{T_1^{-1}} + h_2^{T_1^{-1}}, T_1^{-1}T_2) \quad (2.18)$$

$$(y_1, y_2) = (h_1 - h_2^{T_1T_2^{-1}}, T_1T_2^{-1}) \quad (2.19)$$

Therefore, (\mathcal{L}, \oplus) is a Loop. On the other hand, \mathcal{L} has a manifold structure as cartesian product of manifolds. Finally, the maps $\mu, \backslash, / : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ are all smooth as composition of smooth maps:

$$\mu((h_1, T_1), (h_2, T_2)) = (h_1 + h_2^{T_1}, T_1T_2); \quad (2.20)$$

$$(h_1, T_1) \backslash (h_2, T_2) = ((-h_1)^{T_1^{-1}} + h_2^{T_1^{-1}}, T_1^{-1}T_2); \quad (2.21)$$

$$(h_1, T_1) / (h_2, T_2) = (h_1 - h_2^{T_1T_2^{-1}}, T_1T_2^{-1}). \quad (2.22)$$

We conclude that \mathcal{L} is a Lie loop.

2. Let (\mathcal{L}, \oplus) be a left Bol loop if and only if

$(h_1, T_1)[(h_2, T_2)[(h_1, T_1)(h_3, T_3)]] = [[(h_1, T_1)(h_2, T_2)](h_1, T_1)](h_3, T_3)$ for all elements $(h_1, T_1), (h_2, T_2)$, and (h_3, T_3) of \mathcal{L} . Note that we used juxtaposition for the binary operation, \oplus , for convenience. The left hand side gives

$$\begin{aligned} (h_1, T_1)[(h_2, T_2)[(h_1, T_1)(h_3, T_3)]] &= (h_1, T_1)[(h_2, T_2)(h_1 + h_3^{T_1}, T_1T_3)] \\ &= (h_1, T_1)(h_2 + h_1^{T_2} + h_3^{T_1T_2}, T_2(T_1T_3)) \\ &= (h_1 + h_2^{T_1} + h_1^{T_2T_1} + h_3^{T_1T_2T_1}, T_1(T_2(T_1T_3))). \end{aligned}$$

The right hand side: is

$$\begin{aligned}
[(h_1, T_1)[(h_2, T_2)(h_1, T_1)](h_3, T_3) &= [(h_1, T_1)(h_2 + h_1^{T_2}, T_2T_1)](h_3, T_3) \\
&= (h_1 + h_2^{T_1} + h_1^{T_2T_1}, T_1(T_2T_1))(h_3, T_3) \\
&= (h_1 + h_2^{T_1} + h_1^{T_2T_1} + h_3^{T_1(T_2T_1)}, (T_1(T_2T_1))T_3).
\end{aligned}$$

Note that $T_1(T_2(T_1T_3)) = (T_1(T_2T_1))T_3$ since H is a group. We conclude that (\mathcal{L}, \oplus) is a Bol loop.

3. Recall that $Aff_H(\mathcal{H}) = \{\Phi_{(x,A)} : \mathcal{H} \rightarrow \mathcal{H} : y\Phi_{(x,A)} = x + yA, \forall x, y \in \mathcal{H}, A \in H\}$.

The left multiplication group of \mathcal{L} is generated by all left translation maps. Notice that for any $L_{(r,A)} \in Lmlt(\mathcal{L})$ and for any $(x, X) \in \mathcal{L}$,

$$\begin{aligned}
(x, X)L_{(r,A)} &= (r, A) \oplus (x, X) \\
&= (r + x^A, AX) \\
&= (x\Phi_{(r,A)}, XL_A) \\
&= (x, X)(\Phi_{(r,A)} \times L_A).
\end{aligned}$$

This means that there exists a one to one correspondence, $L_{(r,A)} \leftrightarrow (\Phi_{(r,A)}, L_A)$, between generators of $Lmlt(\mathcal{L})$ and the elements of $Aff_H(\mathcal{H}) \times Lmlt(H)$. We must check that this correspondence preserves the group products. To see that, let $L_{(r,A)} \leftrightarrow (\Phi_{(r,A)}, L_A)$ and let $L_{(k,B)} \leftrightarrow (\Phi_{(k,B)}, L_B)$.

Then observe that

$$\begin{aligned}
(x, X)[L_{(r,A)}L_{(k,B)}] &= [(x, X)L_{(r,A)}]L_{(k,B)} = [(r, A)(x, X)]L_{(k,B)} \\
&= (k, B)[(r, A)(x, X)] \\
&= (k, B)(r + x^A, AX) \\
&= (k + (r + x^A)^B, BAX) \\
&= (k + r^B + x^{AB}, BAX) \\
&= (x\Phi_{(k+r^B, AB)}, XL_AL_B) \\
&= (x, X)(\Phi_{(k+r^B, AB)}, L_AL_B) \\
&= (x, X)(\Phi_{(r,A)}\Phi_{(k,B)}, L_AL_B) \\
&= (x, X)[(\Phi_{(r,A)}, L_A)(\Phi_{(k,B)}, L_B)].
\end{aligned}$$

We conclude that this correspondence preserves the group products, thus

$$Lmlt(\mathcal{L}) \subseteq Aff_H(\mathcal{H}) \times Lmlt(H).$$

4. First we will find the left, middle and right nucleus of \mathcal{L} .

$$\begin{aligned}
N_L(\mathcal{L}) &= \{(r, A) \in \mathcal{L} : [(r, A)(x, X)](y, Y) = (r, A)[(x, X)(y, Y)], \forall (x, X), (y, Y) \in \mathcal{L}\} \\
&= \{(r, A) \in \mathcal{L} : (r + x^A, AX)(y, Y) = (r, A)(x + y^X, XY)\} \\
&= \{(r, A) \in \mathcal{L} : (r + x^A + y^{AX}, AXY) = (r + x^A + y^{XA}, AXY)\} \\
&= \{(r, A) \in \mathcal{L} : y^{AX} = y^{XA}, \forall (x, X), (y, Y) \in \mathcal{L}\} \\
&\cong \mathcal{H} \times Z(H).
\end{aligned}$$

Since $y^{AX} = y^{XA}$ for all $y \in \mathcal{H}$ and for all $X \in H$, $A \in Z(H)$. We showed that \mathcal{L} is a left Bol loop, hence $N_L(\mathcal{L}) = N_M(\mathcal{L})$ since the left and middle nuclei of left Bol loops are same. For the properties of Bol loop please see [26]. The right nuclei of \mathcal{L} can be found as follow:

$$\begin{aligned}
N_R(\mathcal{L}) &= \{(r, A) \in \mathcal{L} : [(x, X)(y, Y)](r, A) = (x, X)[(y, Y)(r, A)], \forall (x, X), (y, Y) \in \mathcal{L}\} \\
&= \{(r, A) \in \mathcal{L} : (x + y^X, XY)(r, A) = (x, X)[(y + r^Y, YA)]\} \\
&= \{(r, A) \in \mathcal{L} : (x + y^X + r^{XY}, XYA) = (x + y^X + r^{YX}, XYA)\} \\
&= \{(r, A) \in \mathcal{L} : r^{XY} = r^{YX}, \forall (x, X), (y, Y) \in \mathcal{L}\}
\end{aligned}$$

$$N_R(\mathcal{L}) \cong \{0\} \times H \cong H.$$

$r^{XY} = r^{YX}$ for all $X, Y \in H$, and this forces that $r = 0 \in \mathcal{H}$. We just calculated left, right and middle nucleus of \mathcal{L} . The nucleus of \mathcal{L} is the intersection of left, right and middle nucleus. Therefore $N(\mathcal{L}) = \{(r, A) \in \mathcal{L} : r = 0, A \in Z(H)\} \cong Z(H)$.

5. Let (h, T) be in the center of \mathcal{L} , then (h, T) is in the nucleus of \mathcal{L} . Therefore $h = 0$ and $T \in Z(H)$, but $(0, T)$ is also in the centrum and this implies that $(0, T)(x, Y) = (x, Y)(0, T)$ for all $(x, Y) \in \mathcal{L}$. That is $(x^T, TY) = (x, YT)$ if and only if $x^T = x$ for all $x \in \mathcal{H}$ if and only if $x \in \text{Fix}_{\mathcal{H}}(H)$, where $\text{Fix}_{\mathcal{H}}(H) := \{x \in \mathcal{H} : x^T = x \forall T \in H\}$. We conclude that $Z(\mathcal{L}) \cong \text{Fix}_{\mathcal{H}}(H) \cap Z(H)$.

□

Let $\mathcal{H}_{\mathbb{F}}$ be a finite dimensional separable Hilbert space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then $GL(\mathcal{H}_{\mathbb{F}}) \cong GL(n, \mathbb{F})$ for some positive integer n . Therefore, the Theorem 2.2.4 can be also stated for finite dimensional classical groups or for infinite dimensional Lie subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ as given in Corollary 2.2.5. We also considered Lie subgroups of $GL(\infty, \mathcal{H}_{\mathbb{F}})$, please see the Chapter 3 for the definitions of infinity by infinity matrix groups in $GL(\mathcal{H}_{\mathbb{F}})$.

Corollary 2.2.5. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and let P_1, P_2 and P_3 be the elements of Lie groups below. Let $\mathcal{L}_m := \mathcal{K}_m \overline{\times} P_m$ for $K_1 = \mathbb{F}^n$, $K_2 = \mathcal{H}_{\mathbb{F}}$, and $K_3 = \mathcal{H}$.*

- $P_1 \in \{GL(n, \mathbb{F}), SL(n, \mathbb{F}), U(n, \mathbb{F}), Sp(2n, \mathbb{F})\}$.
- $P_2 \in \{GL(\infty, \mathcal{H}_{\mathbb{F}}), SL(\infty, \mathcal{H}_{\mathbb{F}}), U(\infty, \mathcal{H}_{\mathbb{F}}), Sp(\infty, \mathcal{H}_{\mathbb{F}})\}$.
- $P_3 \in \{GL(\mathcal{H}), U(\mathcal{H}), Sp(\mathcal{H})\}$.

Then for $m \in \{1, 2, 3\}$

1. \mathcal{L}_m is a Lie loop.
2. \mathcal{L}_m is a left Bol loop.
3. $Lmlt(\mathcal{L}_m) \subseteq Aff_{\mathcal{K}_m}(P_m) \times P_m$.
4. $N(\mathcal{L}_m) \cong Z(P_m)$.
5. $Z(P_m) \cong Fix_{\mathcal{K}_m}(P_m) \cap Z(P_m)$.

Remark 2.2.6. *We want to point out that $N(\mathcal{L})$ is not normal in \mathcal{L} . Otherwise for any $(h, T) \in \mathcal{L}$; $(h, T)N(\mathcal{L}) = N(\mathcal{L})(h, T)$. We showed that (\mathcal{L}, \oplus)*

is a left Bol loop, hence it is a left inverse property loop. Therefore, $N(\mathcal{L}) = ((-h)^{T^{-1}}, T^{-1})[N(\mathcal{L})(h, T)]$, but for any $(0, B) \in N(\mathcal{L})$,

$$\begin{aligned}
((-h)^{T^{-1}}, T^{-1})[(0, B)(h, T)] &= ((-h)^{T^{-1}}, T^{-1})(h^B, BT) \\
&= ((-h + h^B)^{T^{-1}}, T^{-1}BT) \\
&= ((-h + h^B)^{T^{-1}}, B[T^{-1}T]) \\
&= ((-h + h^B)^{T^{-1}}, B)
\end{aligned}$$

$(-h + h^B)^{T^{-1}} = 0$ if and only if $B = I$, but B can be chosen in the center different than I . Therefore, $N(\mathcal{L})$ is not normal.

Chapter 3

K-Loops from $GL(\infty, \mathcal{H})$

Classical Groups as subgroups of $GL(\mathcal{H})$, \mathcal{H} a separable infinite dimensional Hilbert Space

Let $(e_i)_{i=1}^{\infty}$ be a fixed orthonormal basis of the infinite dimensional separable Hilbert space over \mathbb{C} . Any operator in $GL(\mathcal{H})$ can be represented by an $\infty \times \infty$ matrix. We will only interested in certain types of $\infty \times \infty$ matrices that can be determined as given in Lemma 3.0.7.

Lemma 3.0.7. *Let $(e_i)_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}_{\mathbb{F}}$, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and let V_n be the n dimensional subspace of $\mathcal{H}_{\mathbb{F}}$ which generated by $\{e_1, e_2, \dots, e_n\}$. Let $G = \{T \in GL(\mathcal{H}_{\mathbb{F}}) : TV_n = V_n, T|_{V_n^{\perp}} = id\}$, and let $GL(n, \mathcal{H}_{\mathbb{F}})$ be the*

matrix representation of the operators given in G . Then

$$GL(n, \mathcal{H}_{\mathbb{F}}) := \left\{ \begin{bmatrix} A & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix} : A \in GL(n, \mathbb{F}) \right\}.$$

Proof. If $T \in G$ such that $TV_n = V_n$ and $T|_{V_n^\perp} = Id$, then $T(e_i) \in V_n$ for each $i \in \{1, 2, 3, \dots, n\}$, so there exists unique $(a_{ki})_{k=1}^\infty$ such that;

$$T(e_i) = a_{1i}e_1 + a_{2i}e_2 + \dots + a_{ni}e_n + 0e_{n+1} + 0e_{n+2} + \dots$$

V_n is finite dimensional vector space. Therefore, it is a closed subspace of $\mathcal{H}_{\mathbb{F}}$, hence $\mathcal{H}_{\mathbb{F}} = V_n \oplus V_n^\perp$ where $V_n^\perp = \{x \in \mathcal{H}_{\mathbb{F}} : \langle x, y \rangle = 0 \forall y \in V_n\}$. It is clear that $V_n^\perp = \text{span}\{e_{n+1}, e_{n+2}, \dots\}$. The condition $T|_{V_n^\perp} = Id$ implies that;

$$T(e_j) = 0e_1 + \dots + 0e_{j-1} + 1e_j + 0e_{j+1} + \dots \text{ for any } e_j \text{ for } j \geq n+1.$$

Moreover $GL(V_n) \cong GL(n, \mathbb{F})$. Therefore, the matrix representation of $T \in G$ is of the form $A_T = \begin{bmatrix} A & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix}$ such that $[A]_{ij} = a_{ij}$ where $1 \leq i, j \leq n$. We conclude that $GL(n, \mathcal{H}_{\mathbb{F}}) \cong \{T \in GL(\mathcal{H}_{\mathbb{F}}) : TV_n = V_n, T|_{V_n^\perp} = Id\}$. \square

3.1 Lie Groups in $GL(\mathcal{H})$

We will define some of the Lie groups closed under conjugate transpose in $GL(\mathcal{H}_{\mathbb{F}})$ by taking a Lie group, closed under conjugate transpose, in $GL(n, \mathbb{F})$ and adding the constant sequence $a_m = 1$ to its main diagonal and zeros elsewhere. For the list of Lie groups which are closed under conjugate transpose in $GL(n, \mathbb{F})$ please see [29].

Let $P := P(n)$ and $Q := Q(n)$ be sequences of positive integers such that P is constant and Q is strictly increasing, and let $S(n) := P(n) + Q(n)$, and let $J_{m,n} := \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}$, $Q_{m,n} := \begin{bmatrix} 0 & I_m \\ -I_n & 0 \end{bmatrix}$, and $K_{2n} := \text{diag}(\underbrace{P, \dots, P}_{n \text{ times}})$ where $P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We will define the following Lie groups, they are closed under conjugate transpose, and they are the finite dimensional subgroups of $GL(\mathcal{H}_{\mathbb{F}})$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

$$K_1: GL(n, \mathcal{H}_{\mathbb{F}}) := \{\text{diag}(A, 1, \dots) : A \in GL(n, \mathbb{F})\}, \text{ where } \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}.$$

$$K_2: SL(n, \mathcal{H}_{\mathbb{F}}) := \{\text{diag}(A, 1, \dots) : A \in SL(n, \mathbb{F})\}, \text{ where } \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}.$$

We remark that we use $\mathcal{H}_{\mathbb{H}}$ by abuse notation, we are actually considering $\mathcal{H}_{\mathbb{C}}$.

$$K_3: U(n, \mathcal{H}_{\mathbb{C}}) := \{\text{diag}(A, 1, \dots) : A \in U(n)\}.$$

$$K_4: O(n, \mathcal{H}_{\mathbb{R}}) := \{\text{diag}(A, 1, \dots) : A \in O(n)\}.$$

$$K_5: Sp(2n, \mathcal{H}_{\mathbb{F}}) := \{\text{diag}(A, 1, \dots) : A \in GL(2n, \mathbb{F}), A^* K_{2n} A = K_{2n}\}, \text{ where } \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}.$$

$$K_6: SO(n, \mathcal{H}_{\mathbb{C}}) := \{\text{diag}(A, 1, \dots) : A \in SL(n, \mathbb{C}), A^{\top} A = I_n\}.$$

$$K_7: SO^*(2n, \mathcal{H}_{\mathbb{C}}) := \{\text{diag}(A, 1, \dots) : A \in SL(2n, \mathbb{C}), A^{\top} K_{2n} A = K_{2n}\}.$$

$$K_8: U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}) := \{\text{diag}(A, 1, \dots) : A \in GL(S(n), \mathbb{C}), A^* J_{P(n), Q(n)} A = J_{P(n), Q(n)}\}.$$

$$K_9: O(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) := \{\text{diag}(A, 1, \dots) : A \in GL(S(n), \mathbb{R}), A^{\top} J_{P(n), Q(n)} A = J_{P(n), Q(n)}\}.$$

$$K_{10}: Sp(P(n), Q(n), \mathcal{H}_{\mathbb{H}}) := \{diag(A, 1, \dots) : A \in GL(2S(n), \mathbb{C}), A^* J_{2P(n), 2Q(n)} A = J_{2P(n), 2Q(n)}\}.$$

$$K_{11}: SO(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) := \{diag(A, 1, \dots) : A \in SL(S(n), \mathbb{R}), A^T J_{P(n), Q(n)} A = J_{P(n), Q(n)}\}.$$

We will also transform the set of positive definite Hermitian matrices, L_n , in $GL(n, \mathbb{C})$ into $GL(\mathcal{H}_{\mathbb{C}})$ by

$$L(n, \mathcal{H}_{\mathbb{C}}) := \{diag(A, 1, \dots) : A \in L_n\}.$$

The corresponding operator of a matrix in $L(n, \mathcal{H}_{\mathbb{C}})$ is contained in $Pos(\mathcal{H})$.

Proposition 3.1.1. *$GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$, and $GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(n+1, \mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$.*

Proof. Let $T_1, T_2 \in GL(n, \mathcal{H}_{\mathbb{F}})$, then $(T_1 T_2)V_n = T_1(T_2 V_n) = T_1 V_n = V_n$, and for any $x \in V_n^\perp$, $(T_1 T_2)(x) = T_1(T_2(x)) = T_1(x) = x = Id(x)$, so $(T_1 T_2)|_{V_n^\perp} = Id$. On the other hand for any $T \in GL(n, \mathcal{H}_{\mathbb{F}})$, $TV_n = V_n$ implies that $T^{-1}(TV_n) = T^{-1}V_n$ which is equivalent to $T^{-1}V_n = V_n$, moreover for any $x \in V_n^\perp$, $Tx = x$ and so $x = T^{-1}(x)$ which means $T^{-1}|_{V_n^\perp} = Id$. Therefore, $GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$.

To see that $GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(n+1, \mathcal{H}_{\mathbb{F}})$, we only need to show that $GL(n, \mathcal{H}_{\mathbb{F}})$ is contained in $GL(n+1, \mathcal{H}_{\mathbb{F}})$. Let $A \in GL(n, \mathcal{H}_{\mathbb{F}})$, so there exists $B \in GL(n, \mathbb{F})$ such that $A = diag(B, 1, \dots)$. Let $C = diag(B, 1)$, then $C^{-1} = diag(B^{-1}, 1)$, i.e., $C \in GL(n+1, \mathbb{F})$ and $A = (C, 1, \dots) \in GL(n+1, \mathcal{H}_{\mathbb{F}})$. We conclude that $GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(n+1, \mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$. \square

Lemma 3.1.2. *Let G be a Lie subgroup of $GL(n, \mathbb{F})$, closed under conjugate transpose, then $\sigma : G \rightarrow GL(\mathcal{H}_{\mathbb{F}}) : A \mapsto diag(A, 1, \dots)$ is a smooth monomor-*

phism and $\sigma(G)$ is a Lie subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ closed under conjugate transpose for $F \in \{\mathbb{R}, \mathbb{C}\}$.

Proof. If $\sigma(A) = \sigma(B)$, then $\text{diag}(A, 1, \dots) = \text{diag}(B, 1, \dots)$ iff $A = B$, so σ is injective. On the other hand, σ is a group homomorphism as follow:

$$\sigma(AB) = \text{diag}(AB, 1, \dots) \tag{3.1}$$

$$= \text{diag}(A, 1, \dots)\text{diag}(B, 1, \dots) \tag{3.2}$$

$$= \sigma(A)\sigma(B). \tag{3.3}$$

□

Lemma 3.1.3. $K_i, i \in \{1, \dots, 11\}$, is a Lie subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ closed under conjugate transpose for some $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Proof. We have already proved that $K_1 = GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$. Moreover, $K_i \leq GL(\mathcal{H}_{\mathbb{F}})$ for $i > 1$ by Lemma 3.1.2. □

Remark 3.1.4. Explicitly we see that for some $F \in \{\mathbb{R}, \mathbb{C}\}$:

1. $SL(n, \mathbb{F}) \cong K_2 \leq GL(\mathcal{H}_{\mathbb{F}})$.
2. $U(n, \mathbb{C}) \cong K_3 \leq U(\mathcal{H}_{\mathbb{C}})$.
3. $O(n, \mathbb{R}) \cong K_4 \leq O(\mathcal{H}_{\mathbb{R}})$.
4. $Sp(2n, \mathbb{F}) \cong K_5 \leq GL(\mathcal{H}_{\mathbb{F}})$.
5. $SO(n, \mathbb{C}) \cong K_6 \leq GL(\mathcal{H}_{\mathbb{C}})$.

$$6. SO^*(2n) \cong K_7 \leq GL(\mathcal{H}_{\mathbb{C}}).$$

$$7. U(P(n), Q(n)) \cong K_8 \leq GL(\mathcal{H}_{\mathbb{C}}).$$

$$8. O(P(n), Q(n)) \cong K_9 \leq GL(\mathcal{H}_{\mathbb{R}}).$$

$$9. Sp(P(n), Q(n)) \cong K_{10} \leq GL(\mathcal{H}_{\mathbb{C}}).$$

$$10. SO(P(n), Q(n)) \cong K_{11} \leq GL(\mathcal{H}_{\mathbb{R}}).$$

We will define the following Lie groups as the union of ascending chain of Lie groups which are closed under conjugate transpose in $GL(\mathcal{H}_{\mathbb{F}})$ for some $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ as given below.

$$R_1: GL(\infty, \mathcal{H}_{\mathbb{F}}) := \bigcup_{n=2}^{\infty} GL(n, \mathcal{H}_{\mathbb{F}}) \text{ in } GL(\mathcal{H}_{\mathbb{F}}).$$

$$R_2: SL(\infty, \mathcal{H}_{\mathbb{F}}) := \bigcup_{n=2}^{\infty} SL(n, \mathcal{H}_{\mathbb{F}}) \text{ in } GL(\mathcal{H}_{\mathbb{F}}).$$

$$R_3: U(\infty, \mathcal{H}_{\mathbb{C}}) := \bigcup_{n=2}^{\infty} U(n, \mathcal{H}_{\mathbb{C}}) \text{ in } U(\mathcal{H}_{\mathbb{C}}).$$

$$R_4: O(\infty, \mathcal{H}_{\mathbb{R}}) := \bigcup_{n=2}^{\infty} O(n, \mathcal{H}_{\mathbb{R}}) \text{ in } O(\mathcal{H}_{\mathbb{R}}).$$

$$R_5: Sp(\infty, \mathcal{H}_{\mathbb{F}}) := \bigcup_{n=2}^{\infty} Sp(2n, \mathcal{H}_{\mathbb{F}}) \text{ in } GL(\mathcal{H}_{\mathbb{F}}).$$

$$R_6: SO^*(\infty, \mathcal{H}_{\mathbb{C}}) := \bigcup_{n=2}^{\infty} SO^*(2n, \mathcal{H}_{\mathbb{C}}) \text{ in } GL(\mathcal{H}_{\mathbb{C}}).$$

$$R_7: SO(\infty, \mathcal{H}_{\mathbb{C}}) := \bigcup_{n=2}^{\infty} SO(n, \mathcal{H}_{\mathbb{C}}) \text{ in } GL(\mathcal{H}_{\mathbb{C}}).$$

$$R_8: U(P, Q, \infty) := \bigcup_{n=1}^{\infty} U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}) \text{ in } GL(\mathcal{H}_{\mathbb{C}}).$$

$$R_9: O(P, Q, \infty) := \bigcup_{n=1}^{\infty} O(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) \text{ in } GL(\mathcal{H}_{\mathbb{R}}).$$

R_{10} : $Sp(P, Q, \infty) := \bigcup_{n=1}^{\infty} Sp(P(n), Q(n), \mathcal{H}_{\mathbb{H}})$ in $GL(\mathcal{H}_{\mathbb{H}})$.

R_{11} : $SO(P, Q, \infty) := \bigcup_{n=1}^{\infty} SO(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ in $GL(\mathcal{H}_{\mathbb{R}})$.

We will also define $L(\infty, \mathcal{H}_{\mathbb{C}}) := \bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})$ that is contained in positive self-adjoint operator in $GL(\mathcal{H}_{\mathbb{C}})$. We remark that each R_i defined above is also closed under conjugate transpose. We will give the following lemma to verify that each R_i , $i \in \{1, \dots, 11\}$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ for some $F \in \{\mathbb{R}, \mathbb{C}\}$.

Lemma 3.1.5. *Let G_i be a subgroup of G for each $i \in \{1, 2, \dots\}$ such that*

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots$$

Then $G^ := \bigcup_{i=1}^{\infty} G_i$ is a subgroup of G .*

Proof. Let $x, y \in G^*$, then there are $n, m \in \mathbb{N}^+$ such that $x \in G_n$, and $y \in G_m$. If $n = m$, then $x, y \in G_n$, so $xy^{-1} \in G_n \subseteq G^*$ since G_n is a subgroup of G . Otherwise either $n < m$ or $n > m$, without loss of generality suppose that $n < m$. That implies that $G_n \subseteq G_m$, thus $x, y \in G_m$, so $xy^{-1} \in G_m \subseteq G^*$. We conclude that the union of ascending chain of subgroups is a subgroup. \square

Lemma 3.1.6. *If H is a subgroup of the topological group G , then the closure of H , \overline{H} , is also a subgroup of G .*

Proof. Let H be a subgroup of the topological group G . The multiplication and the inversion are both continuous maps, thus $f : H \times H \rightarrow H$, $(x, y) \mapsto xy^{-1}$ is a continuous map. The preimage of continuous function on a closed set is closed. Therefore $f^{-1}(\overline{H})$ is closed, and $H \times H \subseteq f^{-1}(\overline{H})$. Taking the closure

of the both hand sides gives that $\overline{H \times H} \subseteq f^{-1}(\overline{H})$, but $\overline{H} \times \overline{H} \subseteq \overline{H \times H}$, thus $f(\overline{H} \times \overline{H}) \subseteq \overline{H}$. That means for each $x, y \in \overline{H}$, $xy^{-1} \in \overline{H}$. We conclude that the closure of a subgroup of a topological group is again a subgroup. \square

Theorem 3.1.7 (Main Theorem). *Let $i \in \{1, \dots, 7\}$. Then*

1. R_i is a subgroup of $GL(\mathcal{H})$ such that R_i is closed under conjugate transpose.
2. For any element A of $L(\infty, \mathcal{H}_{\mathbb{C}})$, $A = A^*$ and $v^*Av > 0$, where v is any nonzero vector of \mathcal{H} .

Proof. 1. We have already proved in Proposition 3.1.1 that $GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$, and $GL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(n+1, \mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$. Therefore, by Lemma 3.1.5 $GL(\infty, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

$SL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$ by Lemma 3.1.6. If $A \in SL(n, \mathcal{H}_{\mathbb{F}})$, then there exists $B \in SL(n, \mathbb{F})$ such that $A = \text{diag}(B, 1, \dots)$. If we let $C = \text{diag}(B, 1)$, then $\det(C) = \det(B) = 1$, i.e., $C \in SL(n+1, \mathbb{F})$, so $A = \text{diag}(C, 1, \dots) \in SL(n+1, \mathcal{H}_{\mathbb{F}})$. We conclude that $SL(n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $SL(n+1, \mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$. Therefore, $SL(\infty, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ by Lemma 3.1.5 for $F \in \{\mathbb{R}, \mathbb{C}\}$.

$U(n, \mathcal{H}_{\mathbb{C}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{C}})$ for each $n \in \mathbb{N}$ by Lemma 3.1.6. If $A \in U(n, \mathcal{H}_{\mathbb{C}})$, then we can find $B \in U(n)$ such that $A = \text{diag}(B, 1, \dots)$.

Setting $C = \text{diag}(B, 1)$ gives that $C \in U(n+1)$ since

$$\begin{aligned}
\text{diag}(B, 1)(\text{diag}(B, 1))^* &= \text{diag}(B, 1)\text{diag}(B^*, 1) \\
&= \text{diag}(BB^*, 1) \\
&= \text{diag}(I_n, 1) \\
&= I_{n+1}.
\end{aligned}$$

We conclude that $A = \text{diag}(C, 1, \dots) \in U(n+1, \mathcal{H}_{\mathbb{C}})$, and $U(\infty, \mathcal{H}_{\mathbb{C}})$ is a subgroup $GL(\mathcal{H}_{\mathbb{C}})$ by Lemma 3.1.5. The case $O(\infty, \mathcal{H}_{\mathbb{R}})$ is similar.

$Sp(2n, \mathcal{H}_{\mathbb{F}})$, $F \in \{\mathbb{R}, \mathbb{C}\}$, is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$ by Lemma 3.1.6. To see $Sp(2n, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $Sp(2n+2, \mathcal{H}_{\mathbb{F}})$, let $A \in Sp(2n, \mathcal{H}_{\mathbb{F}})$ then $A = \text{diag}(B, 1, \dots)$ for some $B \in GL(2n, \mathbb{F})$ such that $B^*K_{2n}B = K_{2n}$. Let $C = \text{diag}(B, 1, 1)$, then observe that

$$C^*K_{2n+2}C = \text{diag}(B^*, I_2)\text{diag}(K_{2n}, K_2)\text{diag}(B, I_2) \quad (3.4)$$

$$= \text{diag}(B^*K_{2n}B, K_2) \quad (3.5)$$

$$= \text{diag}(K_{2n}, K_2) \quad (3.6)$$

$$= K_{2n+2} \quad (3.7)$$

This indicates that $A = \text{diag}(C, 1, \dots) \in Sp(2n+2, \mathcal{H}_{\mathbb{F}})$ and $Sp(2n, \mathcal{H}_{\mathbb{F}}) \leq Sp(2n+2, \mathcal{H}_{\mathbb{F}})$ for each $n \in \mathbb{N}$. Therefore, $Sp(\infty, \mathcal{H}_{\mathbb{F}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{F}})$ by Lemma 3.1.6. We can similarly show that $SO^*(2n, \mathcal{H}_{\mathbb{C}})$ is a subgroup of $SO^*(2n+2, \mathcal{H}_{\mathbb{C}})$ for each $n \in \mathbb{N}$ and $SO^*(\infty, \mathcal{H}_{\mathbb{C}})$ is a

subgroup of $GL(\mathcal{H}_{\mathbb{C}})$.

We will discuss the cases $U(P, Q, \mathcal{H}_{\mathbb{C}})$, $O(P, Q, \mathcal{H}_{\mathbb{R}})$, and $Sp(P, Q, \mathcal{H}_{\mathbb{H}})$ in section 3.2, and the case $SO(P, Q, \mathcal{H}_{\mathbb{R}})$ in section 3.3.

2. Let $A \in L(\infty, \mathcal{H}_{\mathbb{C}})$. Then there exists $n \in \mathbb{N}$ such that $A = \text{diag}(B, 1, \dots)$ with an B , n by n positive definite Hermitian matrix in $M(n, \mathbb{C})$. That means $B = B^*$, and $v^*Bv > 0$ for any nonzero vector $v \in \mathcal{H}_{\mathbb{C}}$. Therefore,

$$A^* = \begin{bmatrix} B & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix}^* = \begin{bmatrix} B^* & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix} = A.$$

This implies that for any $A \in L(\infty, \mathcal{H}_{\mathbb{C}})$, $A = A^*$.

To see the positive definiteness of the elements of $L(\infty, \mathcal{H}_{\mathbb{C}})$, let $v = (v_n)_{n=1}^{\infty} \in \mathcal{H}_{\mathbb{C}}$ such that $w_1 = (v_1, \dots, v_n)$ and $w_2 = (v_{n+1}, v_{n+2}, \dots)$. If v is a nonzero vector then either w_1 or w_2 are nonzero, then

$$v^*Av = w_1^*Bw_1 + \sum_{i=n+1}^{\infty} |v_i|^2 > 0.$$

We conclude that for any element A of $L(\infty, \mathcal{H}_{\mathbb{C}})$, $v^*Av > 0$, where v is any nonzero vector of \mathcal{H} .

□

Lemma 3.1.8. *Let (G, \oplus) be a topological K -loop, and let H be a subspace of G such that (H, \oplus) is also a K -loop. Then (\overline{H}, \oplus) is a K -loop in G .*

Proof. \overline{H} is a K -loop in G , if \overline{H} is a loop and the left Bol identity and the automorphic inverse property both hold. For non-closure points H is already a

K-loop, therefore we only need to verify the conditions for the closure points. First of all the left Bol identity and the automorphic inverse properties are trivially true, since for given closure points a, b and c of H , a, b and c are elements of G which is a K-loop, therefore the automorphic inverse property and the left Bol identity are satisfied. Moreover, given any closure points a, b of H the equations $a \oplus x = b$ and $y \oplus a = b$ always have unique solutions $x, y \in G$. We claim that $x, y \in \overline{H}$. Let (a_n) and (b_n) are two sequences in H such that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$, then $x = (\lim_{n \rightarrow \infty} a_n) \setminus (\lim_{n \rightarrow \infty} b_n) = \lim_{n \rightarrow \infty} (a_n \setminus b_n)$ since the left division is continuous. For each $n \in \mathbb{N}$, $a_n \setminus b_n \in H$, therefore the sequence $(a_n \setminus b_n)$ is a sequence in H . Therefore the limit of this sequence, x , should stay in the closure of H . We can similarly show that y is in the closure of H . We conclude that (\overline{H}, \oplus) is also a K-loop in G . \square

Lemma 3.1.9. [13] *Let τ be an involutory automorphism of $GL(n, \mathbb{C})$ which commutes with $*$, conjugate transpose, and assume that τ induces a map $\mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}$ such that $\text{diag}(\alpha_1, \dots, \alpha_n)^\tau = \text{diag}(\alpha_1^\tau, \dots, \alpha_n^\tau)$, and $\alpha > 0 \Leftrightarrow \alpha^\tau > 0$. Let $J \in GL(n, \mathbb{C})$ be such that $J^* = \lambda J^{-1}$ for some $\lambda \in \mathbb{R}$, and let $G = GL(n, \tau, J) := \{A \in GL(n, \mathbb{C}) : A^\tau = JAJ^{-1}\}$. Then for $A \in G$,*

1. $A^* \in G$
2. *There exists $B \in L_G$ with $AA^* = B^2$, hence L_G is a transversal of G/Ω_G and (L_G, \oplus) is a K-loop.*

Lemma 3.1.10. [13] *Let $A, B \in L(\infty, \mathcal{H}_{\mathbb{C}})$, then $A \in BU(\infty, \mathcal{H}_{\mathbb{C}})$ if and only if $A = B$.*

Proof. Let $A, B \in L(\infty, \mathcal{H}_{\mathbb{C}})$ such that $A = \text{diag}(\lambda_1, \dots, \lambda_n, 1, 1, \dots)$ for $\lambda_i \in \mathbb{R}^+$, $i \in \{1, 2, \dots, n\}$, and let $B^{-1} = (c_1, c_2, \dots, c_n, e_{n+1}, e_{n+2}, \dots)$, where $c_k \in \mathcal{H}_{\mathbb{C}}$ are the column vectors of B^{-1} . Note that $B^{-1} \in L(\infty, \mathcal{H}_{\mathbb{C}})$ and $B^{-1}A = (\lambda_1 c_1, \dots, \lambda_n c_n, e_{n+1}, e_{n+2}, \dots) \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$. Notice that;

$$(\lambda_k c_k)^*(\lambda_j c_j) = (\lambda_k \lambda_j)(c_k^* c_j) = 0 \text{ for } k \neq j.$$

Therefore B^{-1} has orthogonal columns. On the other hand $B^{-1} = (B^{-1})^*$ since $B^{-1} \in L(\infty, \mathcal{H}_{\mathbb{C}})$, hence $(B^{-1})^2 = (B^{-1})^* B^{-1} = \text{diag}(c_1^* c_1, \dots, c_n^* c_n, 1, 1, \dots)$.

The map $\kappa : L(\infty, \mathcal{H}_{\mathbb{C}}) \rightarrow L(\infty, \mathcal{H}_{\mathbb{C}}); X \mapsto X^2$ is injective and this implies that $B^{-1} = \text{diag}(\mu_1, \dots, \mu_n, e_{n+1}, e_{n+2}, \dots)$ where $\mu_k \in \mathbb{R}^+$. $B^{-1}A = \text{diag}(\lambda_1 \mu_1, \dots, \lambda_n \mu_n, 1, 1, \dots) \in U(\infty, \mathcal{H}_{\mathbb{C}})$. Each $\lambda_k \mu_k > 0$, so $B^{-1}A = I$; i.e, $A = B$. Note that if A is not diagonal, then by the spectral theorem the proof can be reduced to the proof of the diagonal case. \square

Theorem 3.1.11. *Let $G \leq GL(\infty, \mathcal{H}_{\mathbb{C}})$ with $G = L_G(\infty, \mathcal{H}_{\mathbb{C}})U_G(\infty, \mathcal{H}_{\mathbb{C}})$ where $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ and $U_G(\infty, \mathcal{H}_{\mathbb{C}}) = U(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$. Then for any $A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$, there are $A \oplus B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A,B} \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, $(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K-loop.*

Proof. $(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K-loop if the following assertions hold for all $A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ and all $W \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$.

1. $AU_G(\infty, \mathcal{H}_{\mathbb{C}}) = BU_G(\infty, \mathcal{H}_{\mathbb{C}})$ if and only if $A = B$.
2. $WL_G(\infty, \mathcal{H}_{\mathbb{C}})W^{-1} = L_G(\infty, \mathcal{H}_{\mathbb{C}})$.
3. $AL_G(\infty, \mathcal{H}_{\mathbb{C}})A = L_G(\infty, \mathcal{H}_{\mathbb{C}})$.

4. $ABW \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ if and only if $W^*BA \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$, then $ABW = W^*BA$.

We verify these assertions.

1. This is a result of the Lemma 3.1.10.
2. First suppose that $A \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ and $W \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$. Then $(WAW^{-1})^* = (W^{-1})^*A^*W^* = (W^*)^*AW^{-1} = WAW^{-1}$. Note that $A^* = A$ by Theorem 3.1.7(2). On the other hand for any nonzero $v \in \mathcal{H}_{\mathbb{C}}$; $v^*(WAW^{-1})v = (W^*v)^*A(W^{-1}v) = (W^{-1}v)^*A(W^{-1}v) > 0$ by Theorem 3.1.7(2).
3. Let $A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$, then $(ABA)^* = A^*B^*A^* = ABA$, and for any nonzero $v \in \mathcal{H}_{\mathbb{C}}$; $v^*(ABA)v = (Av)^*B(Av) > 0$ by Theorem 3.1.7(2).
4. If $ABW \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$, then by Theorem 3.1.7(2) $(ABW)^* = ABW$, $A^* = A$, and $B^* = B$. Therefore, $(ABW)^* = ABW = W^*B^*A^* = W^*BA$.

□

Theorem 3.1.12. *Let $G \in \{SL(\infty, \mathcal{H}_{\mathbb{R}}), GL(\infty, \mathcal{H}_{\mathbb{R}}), SL(\infty, \mathcal{H}_{\mathbb{C}}), GL(\infty, \mathcal{H}_{\mathbb{C}})\}$ and let $L_G(\infty, \mathcal{H}_{\mathbb{K}}) = L(\infty, \mathcal{H}_{\mathbb{K}}) \cap G$ and $U_G(\infty, \mathcal{H}_{\mathbb{K}}) = U(\infty, \mathcal{H}_{\mathbb{K}}) \cap G$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the following assertions hold.*

1. *For all $A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ there exists unique $A \oplus B \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A,B} \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$ such that $AB = (A \oplus B)d_{A,B}$. Moreover $(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K -loop. On the other hand, the closure $(\overline{L_G(\infty, \mathcal{H}_{\mathbb{C}})}, \oplus)$ is also a K -loop in $Pos(\mathcal{H})$.*

2. Let $\text{linn}(L(\infty, \mathcal{H}_K))$ be the left inner mapping group of $L(\infty, \mathcal{H}_K)$ for $K \in \{\mathbb{R}, \mathbb{C}\}$, then

$$\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}})) \cong \text{PSO}(\infty, \mathcal{H}_{\mathbb{R}}), \text{ if } G \in \{SL(\infty, \mathcal{H}_{\mathbb{R}}), GL(\infty, \mathcal{H}_{\mathbb{R}})\}.$$

$$\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{C}})) \cong \text{PSU}(\infty, \mathcal{H}_{\mathbb{C}}), \text{ if } G \in \{SL(\infty, \mathcal{H}_{\mathbb{C}}), GL(\infty, \mathcal{H}_{\mathbb{C}})\}.$$

Proof. 1. We will prove the assertion for the case that $G = GL(\infty, \mathcal{H}_{\mathbb{C}})$. We want to show that $GL(\infty, \mathcal{H}_{\mathbb{C}}) = L_G(\infty, \mathcal{H}_{\mathbb{C}})U_G(\infty, \mathcal{H}_{\mathbb{C}})$. First consider that $A \in \bigcup_{n=2}^{\infty} GL(n, \mathcal{H}_{\mathbb{C}})$, so there exist $n \in \mathbb{N}$, $n \geq 2$ and there exist $B_A \in GL(n, \mathbb{C})$ such that

$$A = \begin{bmatrix} B_A & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix}.$$

It can be shown that B_A has a unique decomposition $B_A = PU$ where P is positive definite Hermitian matrix and U is a unitary matrix. Now $B_A B_A^*$ is positive definite and Hermitian, and by the spectral theorem, $B_A B_A^* = U \text{diag}(\lambda_1, \dots, \lambda_n) U^*$ where U is unitary matrix whose columns consists of corresponding eigenvectors of λ_i for $1 \leq i \leq n$ of $B_A B_A^*$. Since positive definite Hermitian matrices have a unique positive square root we can define $P := (B_A B_A^*)^{1/2}$ such that P is also positive definite and Hermitian, and define $U := P^{-1} B_A$. It can be also shown that U is a unitary matrix, hence $B_A = P(P^{-1} B_A) = PU$. Kiechle [13] proved that if $G \in \{SL(n, \mathbb{R}), SL(n, \mathbb{C}), GL(n, \mathbb{R}), GL(n, \mathbb{C})\}$, then $G = L_G \Omega_G$ is an exact decomposition and (L_G, \oplus) is a K-loop, where $L_G = L \cap G$

and $\Omega_G = \Omega \cap G$ such that L is the set of positive definite Hermitian matrices in $\mathbb{C}^{n \times n}$, and Ω is the group of unitary matrices.

Now P and U can be thought of as belonging to $L(n, \mathcal{H}_{\mathbb{C}})$ and $U(n, \mathcal{H}_{\mathbb{C}})$ respectively such that

$$A = \begin{bmatrix} [B_A]_{n \times n} & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix} = \begin{bmatrix} P_{n \times n} & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix} \begin{bmatrix} U_{n \times n} & 0 \\ 0 & I_{\infty \times \infty} \end{bmatrix}.$$

This implies that

$$GL(n, \mathcal{H}_{\mathbb{C}}) = L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}}) \text{ for each } n \geq 2.$$

Taking the unions on both sides preserves the equality, thus

$$\bigcup_{n=2}^{\infty} GL(n, \mathcal{H}_{\mathbb{C}}) = \bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}}).$$

We claim that $\bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}}) = \bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})\bigcup_{n=2}^{\infty} U(n, \mathcal{H}_{\mathbb{C}})$

(Note that this argument also works for the cases $G \in \{GL(n, \mathbb{R}), SL(n, \mathbb{R}), SL(n, \mathbb{C})\}$).

Clearly if $S = \bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})$ and $T = \bigcup_{n=2}^{\infty} U(n, \mathcal{H}_{\mathbb{C}})$, then $ST \subseteq$

$\bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}})$ since $\bigcup_{n=2}^{\infty} GL(n, \mathcal{H}_{\mathbb{C}})$ is a group. On the other

hand $L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}}) \subseteq ST$ for $n \geq 2$. Hence $\bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}}) \subseteq$

ST . Therefore,

$$\bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}})U(n, \mathcal{H}_{\mathbb{C}}) = \bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}}) \bigcup_{n=2}^{\infty} U(n, \mathcal{H}_{\mathbb{C}}) \quad (3.8)$$

$$\bigcup_{n=2}^{\infty} GL(n, \mathcal{H}_{\mathbb{C}}) = \bigcup_{n=2}^{\infty} L(n, \mathcal{H}_{\mathbb{C}}) \bigcup_{n=2}^{\infty} U(n, \mathcal{H}_{\mathbb{C}}) \quad (3.9)$$

$$GL(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}})U(\infty, \mathcal{H}_{\mathbb{C}}) \quad (3.10)$$

The decomposition, $GL(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}})U(\infty, \mathcal{H}_{\mathbb{C}})$, is exact since $GL(\infty, \mathcal{H}_{\mathbb{C}})$ and $U(\infty, \mathcal{H}_{\mathbb{C}})$ are both subgroups of $GL(\mathcal{H})$ by Proposition 3.1.1 and Lemma 3.1.5. Therefore, for any $A, B \in L(\infty, \mathcal{H}_{\mathbb{C}})$, $AB \in GL(\infty, \mathcal{H}_{\mathbb{C}})$. The exact decomposition implies that there exist $A \oplus B \in L(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A,B} \in U(\infty, \mathcal{H}_{\mathbb{C}})$ such that $AB = (A \oplus B)d_{A,B}$. Note that the juxtaposition AB is the usual matrix multiplication in $GL(\infty, \mathcal{H}_{\mathbb{C}})$. Moreover, $(L(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K-loop by Theorem 3.1.11, and $(\overline{L(\infty, \mathcal{H}_{\mathbb{C}})}, \oplus)$ is also a K-loop by Lemma 3.1.8.

2. We remark that $wL_G(\infty, \mathcal{H}_{\mathbb{C}})w^{-1} \subseteq L_G(\infty, \mathcal{H}_{\mathbb{C}})$ for all $w \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$ and this allows us to define the following group homomorphism

$$\theta : U_G(\infty, \mathcal{H}_{\mathbb{C}}) \rightarrow \text{Aut}(L_G(\infty, \mathcal{H}_{\mathbb{C}})); \theta(A) := \theta_A \text{ such that } \theta_A(X) = AXA^{-1} \text{ for all } A \in U_G(\infty, \mathcal{H}_{\mathbb{C}}) \text{ and for all } X \in L_G(\infty, \mathcal{H}_{\mathbb{C}}).$$

To see that θ is indeed a group homomorphism let $A, B \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$,

then $\theta(AB) = \theta_{AB}$. Then for arbitrary $X \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$,

$$\theta_{AB}(X) = ABXB^{-1}A^{-1} \quad (3.11)$$

$$= \theta_A(BXB^{-1}) \quad (3.12)$$

$$= \theta_A(\theta_B(X)) \quad (3.13)$$

$$= (\theta_A\theta_B)(X). \quad (3.14)$$

We conclude that $\theta(AB) = \theta(A)\theta(B)$, hence θ is a group homomorphism. On the other hand θ_A , for all $A \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$, is an automorphism of $(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$. Before seeing this we remark that $X \oplus Y = XY^2X$ for all $X, Y \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$. Notice that

$$\theta_A(X \oplus Y) = A(X \oplus Y)A^{-1} \quad (3.15)$$

$$= AXY^2XA^{-1} \quad (3.16)$$

$$= AX(A^{-1}A)Y(A^{-1}A)Y(A^{-1}A)XA^{-1} \quad (3.17)$$

$$= AXA^{-1}(AYA^{-1})^2AXA^{-1} \quad (3.18)$$

$$= AXA^{-1} \oplus AYA^{-1} \quad (3.19)$$

$$= \theta_A(X) \oplus \theta_A(Y). \quad (3.20)$$

We conclude that $\theta_A \in (L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$. We denote the kernel of θ by $\text{Ker}\theta$, a normal subgroup of $U_G(\infty, \mathcal{H}_{\mathbb{C}})$. Let $A \in \text{Ker}\theta$, then $\theta(A) = 1$; i.e, the identity map. Thus $\theta_A(X) = X$ for all $X \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$, which equivalent to $AX = XA$ for all $X \in L_G(\infty, \mathcal{H}_{\mathbb{C}})$. We conclude that

$$\text{Ker}\theta = C_{U_G(\infty, \mathcal{H}_{\mathbb{C}})}(L_G(\infty, \mathcal{H}_{\mathbb{C}})).$$

Consider the subset $D_G = \langle d_{A,B} : A, B \in L_G(\infty, \mathcal{H}_{\mathbb{R}}) \rangle$ of $U_G(\infty, \mathcal{H}_{\mathbb{R}})$.

Then

$$\theta(d_{A,B}) = \theta_{d_{A,B}} \text{ and } \theta_{d_{A,B}}(X) = d_{A,B}Xd_{A,B}^{-1}.$$

We claim that $d_{A,B}Xd_{A,B}^{-1} = \delta_{A,B}(X)$, $\forall X \in L_G(\infty, \mathcal{H}_{\mathbb{R}})$ where $\delta_{A,B} \in \text{innl}(L_G(\infty, \mathcal{H}_{\mathbb{R}})) = \langle \delta_{A,B} : A, B \in L_G(\infty, \mathcal{H}_{\mathbb{R}}) \rangle$ such that $A \oplus (B \oplus X) = (A \oplus B) \oplus \delta_{A,B}(X)$. Notice that

$$(A \oplus B) \oplus d_{A,B}Xd_{A,B}^{-1} = (ABd_{A,B}^{-1}) \oplus d_{A,B}Xd_{A,B}^{-1} \quad (3.21)$$

$$= ABd_{A,B}^{-1}d_{A,B}Xd_{A,B}^{-1}d_{A \oplus B, d_{A,B}Xd_{A,B}^{-1}}^{-1} \quad (3.22)$$

$$= ABXd_{A,B}^{-1}d_{A \oplus B, d_{A,B}Xd_{A,B}^{-1}}^{-1} \quad (3.23)$$

This implies that $ABX = [(A \oplus B) \oplus d_{A,B}Xd_{A,B}^{-1}]d_{A \oplus B, d_{A,B}Xd_{A,B}^{-1}}d_{A,B}$.

On the other hand $A \oplus (B \oplus X) = A \oplus (BX d_{B,X}^{-1}) = ABXd_{B,X}^{-1}d_{A, B \oplus X}^{-1}$.

We can solve ABX as $ABX = [A \oplus (B \oplus X)]d_{A, B \oplus X}d_{B,X}$. Therefore

$$[(A \oplus B) \oplus d_{A,B}Xd_{A,B}^{-1}]d_{A \oplus B, d_{A,B}Xd_{A,B}^{-1}}d_{A,B} = [A \oplus (B \oplus X)]d_{A, B \oplus X}d_{B,X},$$

and uniqueness of the decomposition implies that

$$[(A \oplus B) \oplus d_{A,B}Xd_{A,B}^{-1}] = [A \oplus (B \oplus X)] = (A \oplus B) \oplus \delta_{A,B}(X).$$

We can use the same argument to conclude that $d_{A,B}Xd_{A,B}^{-1} = \delta_{A,B}(X)$. What we showed is that θ is sending the generators of D_G into the generators of $\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}}))$. Moreover, A, B chosen arbitrarily, hence we cover the all generator of $\text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}}))$. In other words the restriction of θ to D_G provides a surjective group homomorphism. Therefore, we can use the first isomorphism theorem for groups to conclude that:

$$D_G/\text{Ker}\theta \cong \text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}})).$$

Let $G = SL(\infty, \mathcal{H}_{\mathbb{R}})$. Then $U_G(\infty, \mathcal{H}_{\mathbb{R}}) = SO(\infty, \mathcal{H}_{\mathbb{R}})$. If we show that $D_G = SO(\infty, \mathcal{H}_{\mathbb{R}})$, then $SO(\infty, \mathcal{H}_{\mathbb{R}})/\text{Ker}\theta \cong \text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}}))$, i.e, $PSO(\infty, \mathcal{H}_{\mathbb{R}}) \cong \text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}}))$ which completes the proof.

The first inclusion $D_G = \langle d_{A,B} : A, B \in L_G(\infty, \mathcal{H}_{\mathbb{R}}) \rangle \subseteq SO(\infty, \mathcal{H}_{\mathbb{R}})$ is clear since $d_{A,B} \in U_G(\infty, \mathcal{H}_{\mathbb{R}}) = SO(\infty, \mathcal{H}_{\mathbb{R}})$. Therefore, we only need to show that $SO(\infty, \mathcal{H}_{\mathbb{R}}) \subseteq D_G$. Let $U \in SO(n, \mathcal{H}_{\mathbb{R}})$, then there is $U_1 \in SO(n, \mathbb{R})$ such that $U = \text{diag}(U_1, I_{\infty \times \infty})$. It has been proved in [14],[15] that there is $d_{A,B} \in D_{SL(n, \mathbb{R})}$ such that $U_1 = d_{A,B}$. Let $\overline{d_{A,B}} = \text{diag}(d_{A,B}, I_{\infty \times \infty})$, then $U = \overline{d_{A,B}}$. Therefore, $SO(n, \mathcal{H}_{\mathbb{R}}) \subseteq D_{SL(n, \mathcal{H}_{\mathbb{R}})}$, then

$$\bigcup_{n=2}^{\infty} SO(n, \mathcal{H}_{\mathbb{R}}) \subseteq \bigcup_{n=2}^{\infty} D_{SL(n, \mathcal{H}_{\mathbb{R}})}.$$

That implies that $SO(\infty, \mathcal{H}_{\mathbb{R}}) \subseteq D_{SL(\infty, \mathcal{H}_{\mathbb{R}})}$, hence $SO(\infty, \mathcal{H}_{\mathbb{R}})/\text{Ker}\theta \cong \text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}}))$ and $PSO(\infty, \mathcal{H}_{\mathbb{R}}) \cong \text{linn}(L_G(\infty, \mathcal{H}_{\mathbb{R}}))$.

Let $G = SL(\infty, \mathcal{H}_{\mathbb{C}})$. Then $U_G(\infty, \mathcal{H}_{\mathbb{C}}) = U(\infty, \mathcal{H}_{\mathbb{C}}) \cap SL(\infty, \mathcal{H}_{\mathbb{C}}) = SU(\infty, \mathcal{H}_{\mathbb{C}})$. Therefore,

$\theta : SU(\infty, \mathcal{H}_{\mathbb{C}}) \rightarrow Aut(L_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$, and $D_G = \langle d_{A,B} : A, B \in L_G(\infty, \mathcal{H}_{\mathbb{C}}) \rangle$.

We want to show that $D_G = SU(\infty, \mathcal{H}_{\mathbb{C}})$. Let $d_{A,B} \in D_G$, then $AB = (A \oplus B)d_{A,B} \in L_G(\infty, \mathcal{H}_{\mathbb{C}})U_G(\infty, \mathcal{H}_{\mathbb{C}}) = L_G(\infty, \mathcal{H}_{\mathbb{C}})SU(\infty, \mathcal{H}_{\mathbb{C}})$, hence $d_{A,B} \in SU(\infty, \mathcal{H}_{\mathbb{C}})$, and this indicates that $D_G \subseteq SU(\infty, \mathcal{H}_{\mathbb{C}})$. The finite case for the second containment was proved in [14],[15]. Therefore, $SU(n, \mathbb{C}) \subseteq D_{SL(n, \mathbb{C})}$ for each $n \in \mathbb{N}^+$, but $SU(n, \mathbb{C}) \cong SU(n, \mathcal{H}_{\mathbb{C}})$ and $SL(n, \mathbb{C}) \cong SL(n, \mathcal{H}_{\mathbb{C}})$. Therefore, for each $n \in \mathbb{N}$ and $n \geq 2$, $SU(n, \mathcal{H}_{\mathbb{C}}) \subseteq D_{SL(n, \mathcal{H}_{\mathbb{C}})}$. The last containment is preserved by taking infinite unions, i.e.,

$$\bigcup_{n=2}^{\infty} SU(n, \mathcal{H}_{\mathbb{C}}) \subseteq \bigcup_{n=2}^{\infty} D_{SL(n, \mathcal{H}_{\mathbb{C}})},$$

We conclude that $SU(\infty, \mathcal{H}_{\mathbb{C}}) \subseteq D_{SL(\infty, \mathcal{H}_{\mathbb{C}})}$, and hence $SU(\infty, \mathcal{H}_{\mathbb{C}}) = D_{SL(\infty, \mathcal{H}_{\mathbb{C}})}$ and $SU(\infty, \mathcal{H}_{\mathbb{C}})/Ker\theta \cong innl(L_G(\infty, \mathcal{H}_{\mathbb{C}}))$. The kernel of θ is $C_{SU(\infty, \mathcal{H}_{\mathbb{C}})}(L_G(\infty, \mathcal{H}_{\mathbb{C}}))$ that is the center of $SU(\infty, \mathcal{H}_{\mathbb{C}})$, therefore $PSU(\infty, \mathcal{H}_{\mathbb{C}}) \cong linn(L_G(\infty, \mathcal{H}_{\mathbb{C}}))$.

□

Theorem 3.1.13. *Let $\mathbb{H} := \mathbb{R}(i, j)$ be the quaternions over \mathbb{R} . Let $G \in \{SL(\infty, \mathbb{H}), GL(\infty, \mathbb{H})\}$, then $\tilde{L}_G = \tilde{L}(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of G/\tilde{U}_G and is therefore a K -loop.*

Proof. We will first consider the case that $G = GL(\infty, \mathbb{H}) := \bigcup_{n=1}^{\infty} GL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}})$, where $GL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}})$. Similarly $SL(\infty, \mathbb{H}) := \bigcup_{n=1}^{\infty} SL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}})$.

Let $q = a + bi + cj + dk$ be a quaternion, then $q = a + bi + (c + di)j = z + wj$. That means any quaternion can be embedded in \mathbb{C}^2 by setting $q = (z, w)$. Let $\gamma : \mathbb{H} \rightarrow M(2, \mathbb{C})$ defined by $\gamma(q) = \begin{bmatrix} z & -w \\ \bar{w} & \bar{z} \end{bmatrix}$, this map allows us to define the map $\phi : M(n, \mathbb{H}) \rightarrow M(2n, \mathbb{C})$ such that $[\phi(A)]_{r,s} = \gamma(a_{r,s})$, where $a_{r,s}$ is the entry at the intersection of r^{th} row and s^{th} column of the matrix A . The map ϕ is an injective map that carries the algebraic structure of $M(n, \mathbb{H})$ into $M(2n, \mathbb{C})$. That is $\phi(A+B) = \phi(A) + \phi(B)$, $\phi(AB) = \phi(A)\phi(B)$, and $\phi(A^*) = (\phi(A))^*$. For any $q = a + bi + (c + di)j$, the map $q \mapsto jqj^{-1}$ is an isomorphism of \mathbb{H} , and this map can be extended to an isomorphism of $M(n, \mathbb{H})$ by $A \mapsto KAK^{-1}$ where $K = \text{diag}(j, j, \dots, j)_{n \times n}$, applying ϕ to KAK^{-1} gives that $\phi(KAK^{-1}) = \phi(K)\phi(A)\phi(K^{-1}) = JAJ^{-1} = \bar{A}$ where $J = \text{diag}(P, P, \dots, P)$ and $P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that the copy of A in $M(2n, \mathbb{C})$ is denoted by A again. What we showed is that

$$\phi(M(n, \mathbb{H})) \subseteq \{A \in M(2n, \mathbb{C}) : JAJ^{-1} = \bar{A}\}.$$

Since the other inclusion is trivial we conclude that

$$M(n, \mathbb{H}) \cong \{A \in M(2n, \mathbb{C}) : JAJ^{-1} = \bar{A}\}.$$

Restricting $M(n, \mathbb{H})$ into $GL(n, \mathbb{H})$ gives that

$$GL(n, \mathbb{H}) \cong \{A \in GL(2n, \mathbb{C}) : JAJ^{-1} = \bar{A}\}.$$

Let $G(2n, \tau, J) := \{A \in GL(2n, \mathbb{C}) : JAJ^{-1} = \bar{A}\}$, where $\tau(A) = \bar{A}$, then $GL(n, \mathbb{H}) \cong G(2n, \tau, J)$. It was proved in Lemma 3.1.9 that if $G = G(2n, \tau, J)$,

then (L_G, \oplus) is a K-loop. $GL(n, \mathcal{H}_{\mathbb{H}}) \cong GL(n, \mathbb{H})$. We will associate any matrix in $GL(n, \mathcal{H}_{\mathbb{H}})$ with a matrix such that each entry is complex, and we will denote the collection of those matrices with $GL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}})$. That means $GL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}}) := \{diag(A, I_{\infty \times \infty}) : A \in GL(2n, \mathbb{C}) \text{ and } JAJ^{-1} = \overline{A}\}$. For convenience let $G_{2n} := GL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}})$, $\bigcup_{n=1}^{\infty} L(2n, \mathcal{H}_{\mathbb{C}}) := \tilde{L}(\infty, \mathcal{H}_{\mathbb{C}})$, and let $\bigcup_{n=1}^{\infty} U(2n, \mathcal{H}_{\mathbb{C}}) := \tilde{U}(\infty, \mathcal{H}_{\mathbb{C}})$.

The exact decomposition also hold for G_{2n} . Therefore,

$G(2n) = L_{G(2n)}(2n, \mathcal{H}_{\mathbb{C}})U_{G(2n)}(2n, \mathcal{H}_{\mathbb{C}})$ for each $n \geq 1$, and this implies that

$$\begin{aligned} \bigcup_{n=1}^{\infty} GL(2n, \mathbb{H}, \mathcal{H}_{\mathbb{C}}) &= \bigcup_{n=1}^{\infty} L_{G(2n)}(2n, \mathcal{H}_{\mathbb{C}})U_{G(2n)}(2n, \mathcal{H}_{\mathbb{C}}) \\ GL(\infty, \mathbb{H}) &= \bigcup_{n=1}^{\infty} L_{G(2n)}(2n, \mathcal{H}_{\mathbb{C}}) \bigcup_{n=1}^{\infty} U_{G(2n)}(2n, \mathcal{H}_{\mathbb{C}}) \\ GL(\infty, \mathbb{H}) &= \bigcup_{n=1}^{\infty} (L(2n, \mathcal{H}_{\mathbb{C}}) \cap G(2n)) \bigcup_{n=1}^{\infty} (U(2n, \mathcal{H}_{\mathbb{C}}) \cap G(2n)) \\ GL(\infty, \mathbb{H}) &= \tilde{L}_G(\infty, \mathcal{H}_{\mathbb{C}}) \tilde{U}_G(\infty, \mathcal{H}_{\mathbb{C}}) \end{aligned}$$

This decomposition is also exact since G and $\tilde{U}_G(\infty, \mathcal{H}_{\mathbb{C}})$ are both subgroups of $GL(\mathcal{H}_{\mathbb{C}})$ by Proposition 3.1.1 and Lemma 3.1.5. Therefore, given $A, B \in \tilde{L}(\infty, \mathcal{H}_{\mathbb{C}})$ there exist $A \oplus B \in \tilde{L}_G(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A,B} \in U_G(\infty, \mathcal{H}_{\mathbb{C}})$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, $(\tilde{L}_G(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K-loop by Theorem 3.1.11, and $(\overline{\tilde{L}_G(\infty, \mathcal{H}_{\mathbb{C}})}, \oplus)$ is also a K-loop in $Pos(\mathcal{H}_{\mathbb{C}})$ by Lemma 3.1.8. The proof is similar for the case $G = SL(\infty, \mathbb{H})$. \square

Theorem 3.1.14. *If $G \in \{Sp(\infty, \mathcal{H}_{\mathbb{R}}), Sp(\infty, \mathcal{H}_{\mathbb{C}})\}$, then $\tilde{L}_G(\infty, \mathcal{H}_{\mathbb{C}}) = \tilde{L}(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of G/\tilde{U}_G and is therefore a K-loop.*

Proof. Finite case is proven in [13] by Lemma 3.1.9. Let $\tau : GL(2n, \mathbb{C}) \rightarrow$

$GL(2n, \mathbb{C})$; $A \mapsto (A^{-1})^\top$, and $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, $\lambda = 1$, then $G(2n, \tau, J) = \{A \in GL(2n, \mathbb{C}) : A^\tau = JAJ^{-1}\}$, and $A^\tau = JAJ^{-1} \Leftrightarrow (A^\top)^{-1} = JAJ^{-1} \Leftrightarrow A^\top = JA^{-1}J^{-1} \Leftrightarrow A^\top J = JA^{-1} \Leftrightarrow A^\top JA = J$. Therefore, $G(2n, \tau, J) = \{A \in GL(2n, \mathbb{C}) : A^\top JA = J\}$ which is equal to $Sp(2n, \mathbb{C})$, and this indicates that (L_G, \oplus) is a K-loop for $G = Sp(2n, \mathbb{C})$. The case $G = Sp(2n, \mathbb{R})$ can be shown similarly.

It is obvious that $Sp(2n, \mathbb{C}) \cong Sp(2n, \mathcal{H}_\mathbb{C})$. Let $G = Sp(2n, \mathcal{H}_\mathbb{C})$, then

$$Sp(2n, \mathcal{H}_\mathbb{C}) = L_G(2n, \mathcal{H}_\mathbb{C})U_G(2n, \mathcal{H}_\mathbb{C}) \quad (3.24)$$

$$\bigcup_{n=2}^{\infty} Sp(2n, \mathcal{H}_\mathbb{C}) = \bigcup_{n=2}^{\infty} (L_G(2n, \mathcal{H}_\mathbb{C})U_G(2n, \mathcal{H}_\mathbb{C})) \quad (3.25)$$

$$= \bigcup_{n=2}^{\infty} L_G(2n, \mathcal{H}_\mathbb{C}) \bigcup_{n=2}^{\infty} U_G(2n, \mathcal{H}_\mathbb{C}) \quad (3.26)$$

$$Sp(\infty, \mathcal{H}_\mathbb{C}) = \tilde{L}_G(\infty, \mathcal{H}_\mathbb{C})\tilde{U}_G(\infty, \mathcal{H}_\mathbb{C}). \quad (3.27)$$

This decomposition is exact since $Sp(\infty, \mathcal{H}_\mathbb{C})$ and $\tilde{U}_G(\infty, \mathcal{H}_\mathbb{C})$ are both subgroups of $GL(\mathcal{H}_\mathbb{C})$ by Proposition 3.1.1 and Lemma 3.1.5. Therefore, given $A, B \in \tilde{L}_G(\infty, \mathcal{H}_\mathbb{C})$, $AB \in Sp(\infty, \mathcal{H}_\mathbb{C})$. Exact decomposition implies that there exist $A \oplus B \in \tilde{L}_G(\infty, \mathcal{H}_\mathbb{C})$ and $d_{A,B} \in \tilde{U}(\infty, \mathcal{H}_\mathbb{C})$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, $(\tilde{L}_G(\infty, \mathcal{H}_\mathbb{C}), \oplus)$ is a K-loop by Theorem 3.1.11, and $(\overline{\tilde{L}_G(\infty, \mathcal{H}_\mathbb{C})}, \oplus)$ is also a K-loop in $Pos(\mathcal{H}_\mathbb{C})$ by Lemma 3.1.8. \square

3.2 Pseudo Orthogonal and Pseudo Unitary Groups

Let $p, q \in \mathbb{N}$ with $0 < p \leq q$, and let $J_{p,q} = \text{diag}(I_p, -I_q)$. The pseudo unitary group is defined by

$$U(p, q) = \{A \in GL(p + q, \mathbb{C}) : A^* J_{p,q} A = J_{p,q}\}.$$

The pseudo orthogonal group is denoted by $O(p, q)$ such that

$$O(p, q) = U(p, q) \cap GL(p + q, \mathbb{R}).$$

Let $P := P(n)$ and $Q := Q(n)$ be two sequences of positive integers such that P is constant and Q is strictly increasing. Let $S(n) := P(n) + Q(n)$, then $U(P(n), Q(n)) := \{A \in GL(S(n), \mathbb{C}) : A^* J_{P(n), Q(n)} A = J_{P(n), Q(n)}\}$, and $O(P(n), Q(n)) := U(P(n), Q(n)) \cap GL(S(n), \mathbb{R})$. Lemma 3.1.6 indicates that $U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$ and $O(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ are subgroups of $U(\mathcal{H}_{\mathbb{C}})$ and $O(\mathcal{H}_{\mathbb{R}})$ respectively. On the other hand, two groups satisfy the ascending chain of subgroups conditions as follow:

Lemma 3.2.1. *$U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$ is a subgroup of $U(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{C}})$ for each $n \in \mathbb{N}$. Similarly, $O(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ is a subgroup of $O(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{R}})$ for each $n \in \mathbb{N}$.*

Proof. Let $A \in U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$, so we can find $B \in U(P(n), Q(n))$ such that $A = \text{diag}(B, 1, \dots)$. Consider the diagonal matrix $C = \text{diag}(B, \underbrace{1, \dots, 1}_{Q(n+1)-Q(n)})$.

We claim that $C \in U(P(n+1), Q(n+1))$. Observe that

$$\begin{aligned}
C^* J_{P(n+1), Q(n+1)} C &= \text{diag}(B^*, I_{Q(n+1)-Q(n)}) \text{diag}(I_{P(n+1)}, -I_{Q(n+1)}) B \\
&= \text{diag}(B^*, I_{Q(n+1)-Q(n)}) \text{diag}(I_{P(n)}, -I_{Q(n)}, -I_{Q(n+1)-Q(n)}) C \\
&= \text{diag}(B^*, I_{Q(n+1)-Q(n)}) \underbrace{\text{diag}(I_{P(n)}, -I_{Q(n)}, -I_{Q(n+1)-Q(n)})}_{J_{P(n), Q(n)}} C \\
&= \text{diag}(B^* J_{P(n), Q(n)}, I_{Q(n+1)-Q(n)} (-I_{Q(n+1)-Q(n)})) \text{diag}(B, I_{Q(n+1)-Q(n)}) \\
&= \text{diag}(B^* J_{P(n), Q(n)} B, I_{Q(n+1)-Q(n)} (-I_{Q(n+1)-Q(n)}) (I_{Q(n+1)-Q(n)})) \\
&= \text{diag}(J_{P(n), Q(n)}, -I_{Q(n+1)-Q(n)}) \\
&= \text{diag}(I_{P(n+1)}, -I_{Q(n)}, -I_{Q(n+1)-Q(n)}) \\
&= \text{diag}(I_{P(n+1)}, -I_{Q(n+1)}) \\
&= J_{P(n+1), Q(n+1)}
\end{aligned}$$

We showed that $C \in U(P(n+1), Q(n+1))$, i.e., $A = \text{diag}(C, 1, \dots) = \text{diag}(B, \underbrace{1, \dots, 1}_{Q(n+1)-Q(n)}, 1, \dots) \in U(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{C}})$. \square

We can similarly show that $O(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ is a subgroup of $O(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{R}})$ for each $n \in \mathbb{N}$. Therefore, pseudo unitary and pseudo orthogonal groups in $U(\mathcal{H}_{\mathbb{C}})$ and $O(\mathcal{H}_{\mathbb{R}})$ satisfy the ascending chain of subgroup conditions as shown below.

$$\begin{aligned}
\dots &\leq U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}) \leq U(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{C}}) \leq \dots \leq GL(\mathcal{H}_{\mathbb{C}}). \\
\dots &\leq O(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) \leq O(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{R}}) \leq \dots \leq GL(\mathcal{H}_{\mathbb{R}}).
\end{aligned}$$

We will define $U(P, Q, \infty)$ and $O(P, Q, \infty)$ as follow:

- $U(P, Q, \infty) := \bigcup_{n=1}^{\infty} U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$.
- $O(P, Q, \infty) := \bigcup_{n=1}^{\infty} O(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$.

Lemma 3.2.2. *Let $P := P(n)$ be a constant sequence and $Q := Q(n)$ be a strictly increasing sequence of positive integers. Then*

1. $U(P, Q, \infty)$ and $O(P, Q, \infty)$ are subgroups of $U(\mathcal{H}_{\mathbb{C}})$ and $O(\mathcal{H}_{\mathbb{R}})$ respectively.
2. $U(P, Q, \infty)$ and $O(P, Q, \infty)$ are closed under conjugate transpose.

Proof. $U(P, Q, \infty)$ and $O(P, Q, \infty)$ are subgroups of $U(\mathcal{H}_{\mathbb{C}})$ and $O(\mathcal{H}_{\mathbb{R}})$ respectively by Lemma 3.1.6, and they are subgroups of $U(\mathcal{H}_{\mathbb{C}})$ and $O(\mathcal{H}_{\mathbb{R}})$ respectively by Lemma 3.1.5.

1. Let $A \in U(P, Q, \mathcal{H}_{\mathbb{C}})$, then there exists $n \in \mathbb{N}$ such that $A \in U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$, i.e., $A = \text{diag}(B, 1, \dots)$ where $B = U(P(n), Q(n))$. We want to show that $A^* = \text{diag}(B^*, 1, \dots) \in U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$, and to see this it is enough to verify that if $B^* \in U(P(n), Q(n))$. First, notice that $J_{P(n), Q(n)}^{-1} = J_{P(n), Q(n)}$ and

$$B^* J_{P(n), Q(n)} B = J_{P(n), Q(n)} \quad (3.28)$$

$$J_{P(n), Q(n)} = (B^*)^{-1} J_{P(n), Q(n)} B^{-1} \quad (3.29)$$

$$J_{P(n), Q(n)}^{-1} = (B J_{P(n), Q(n)} B^*)^{-1} \quad (3.30)$$

$$J_{P(n), Q(n)} = B J_{P(n), Q(n)} B^* \quad (3.31)$$

$$J_{P(n), Q(n)} = (B^*)^* J_{P(n), Q(n)} B^*. \quad (3.32)$$

Therefore, $U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$ is closed under conjugate transpose. We can similarly show that $O(P, Q, \mathcal{H}_{\mathbb{R}})$ is closed under taking transpose, i.e., if $A \in O(P, Q, \mathcal{H}_{\mathbb{R}})$, then $A^{\top} \in O(P, Q, \mathcal{H}_{\mathbb{R}})$. \square

Lemma 3.2.3. $Sp(P(n), Q(n), \mathcal{H}_{\mathbb{H}})$ is a subgroup of $Sp(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{H}})$ for each $n \in \mathbb{N}$, and $Sp(P, Q, \infty)$ is a subgroups of $GL(\mathcal{H}_{\mathbb{H}})$.

Proof. $Sp(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$ is a subgroup of $GL(\mathcal{H}_{\mathbb{C}})$ for each $n \in \mathbb{N}$ by Lemma 3.1.6, and $Sp(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$ is a subgroup of $Sp(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{C}})$, let $A \in Sp(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$. Then $A = \text{diag}(B, 1, \dots)$ such that $B \in Sp(P(n), Q(n))$, i.e., $B^* J_{2P(n), 2Q(n)} B = J_{2P(n), 2Q(n)}$. If we set $C = \text{diag}(B, \underbrace{1, \dots, 1}_{2(Q(n+1)-Q(n))})$, then $C \in Sp(P(n+1), Q(n+1))$ as given below.

$$\begin{aligned}
C^* J_{2P(n+1), 2Q(n+1)} C &= \text{diag}(B^*, I_{2(Q(n+1)-Q(n))}) \text{diag}(I_{2P(n)}, -I_{2Q(n)}, -I_{2(Q(n+1)-Q(n))}) C \\
&= \text{diag}(B^* J_{2P(n), 2Q(n)} B, -I_{2(Q(n+1)-Q(n))}) \\
&= \text{diag}(J_{2P(n), 2Q(n)}, -I_{2(Q(n+1)-Q(n))}) \\
&= \text{diag}(I_{2P(n)}, -I_{2Q(n)}, -I_{2(Q(n+1)-Q(n))}) \\
&= \text{diag}(I_{2P(n+1)}, -I_{2Q(n+1)}) = J_{2P(n+1), 2Q(n+1)}
\end{aligned}$$

Please recall that P is a constant sequence then $P(n+1) = P(n)$. Therefore, $A = \text{diag}(C, 1, \dots) \in Sp(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{C}})$, and $Sp(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$ satisfies the ascending chain of subgroups condition, so $Sp(P, Q, \infty)$ is a subgroup of $GL(\mathcal{H}_{\mathbb{C}})$ by Lemma 3.1.5. \square

Theorem 3.2.4. If $G \in \{O(P, Q, \infty), U(P, Q, \infty), Sp(P, Q, \infty), SO^*(\infty, \mathcal{H}_{\mathbb{C}})\}$,

then $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of $G/U_G(\infty, \mathcal{H}_{\mathbb{C}})$ and is therefore a K-loop.

Proof. Let $\tau : GL(S(n), \mathbb{C}) \rightarrow GL(S(n), \mathbb{C})$ defined by $A \mapsto (A^{-1})^*$, and let $G = U(P(n), Q(n), \infty)$. Given $A \in G$, $\exists n \in \mathbb{N}$ such that $A \in U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})$. Then, it can be shown $G(S(n), \tau, J_{P(n), Q(n)}) = U(P(n), Q(n))$ as follow:

$$\begin{aligned}
G = G(S(n), \tau, J_{P(n), Q(n)}) &= \left\{ A \in GL(S(n), \mathbb{C}) : A^\tau = J_{P(n), Q(n)} A J_{P(n), Q(n)}^{-1} \right\}, \\
&= \left\{ A \in GL(S(n), \mathbb{C}) : (A^{-1})^* = J_{P(n), Q(n)} A J_{P(n), Q(n)}^{-1} \right\}, \\
&= \left\{ A \in GL(S(n), \mathbb{C}) : (A^*)^{-1} = J_{P(n), Q(n)} A J_{P(n), Q(n)}^{-1} \right\}, \\
&= \left\{ A \in GL(S(n), \mathbb{C}) : A^*(A^*)^{-1} = A^* J_{P(n), Q(n)} A J_{P(n), Q(n)}^{-1} \right\}, \\
&= \left\{ A \in GL(S(n), \mathbb{C}) : I_{S(n)} = A^* J_{P(n), Q(n)} A J_{P(n), Q(n)}^{-1} \right\}, \\
&= \left\{ A \in GL(S(n), \mathbb{C}) : J_{P(n), Q(n)} = A^* J_{P(n), Q(n)} A J_{P(n), Q(n)}^{-1} \right\}, \\
&= U(P(n), Q(n)) \cong U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}).
\end{aligned}$$

Therefore, $G(S(n), \tau, J_{P(n), Q(n)}) = U(P(n), Q(n))$ and by Lemma 3.1.9, $(L_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})}, \oplus)$ is a K-loop, i.e., $U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}) = L_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})} \Omega_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})}$ is an exact decomposition for each $n \geq 1$, where $\Omega = U(\mathcal{H}_{\mathbb{C}})$ and $L = L(S(n), \mathcal{H}_{\mathbb{C}})$.

Taking the infinite union of both sides preserves the equality as follow:

$$\begin{aligned}
U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}) &= L_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})} \Omega_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})}, \\
\bigcup_{n=1}^{\infty} U(P(n), Q(n), \mathcal{H}_{\mathbb{C}}) &= \bigcup_{n=1}^{\infty} L_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})} \Omega_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})}, \\
U(P, Q, \infty) &= \bigcup_{n=1}^{\infty} L_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})} \bigcup_{n=1}^{\infty} \Omega_{U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})}, \\
&= L_{\bigcup_{n=1}^{\infty} U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})} \Omega_{\bigcup_{n=1}^{\infty} U(P(n), Q(n), \mathcal{H}_{\mathbb{C}})}, \\
&= L_{U(P, Q, \infty)} \Omega_{U(P, Q, \infty)}.
\end{aligned}$$

Therefore, $U(P, Q, \infty) = L_{U(P, Q, \infty)} \Omega_{U(P, Q, \infty)}$, where $L = L(\infty, \mathcal{H}_{\mathbb{C}})$. We showed in Lemma 3.2.2 that $U(P, Q, \infty)$ is a subgroup of $U(\mathcal{H}_{\mathbb{C}})$. Moreover, $\Omega_{U(P, Q, \infty)}$ is a subgroup of $O(\mathcal{H}_{\mathbb{R}})$ as the intersection of two subgroups. Therefore, given any $A, B \in L_{U(P(n), Q(n), \infty)} \subseteq U(P(n), Q(n), \infty)$, $AB \in U(P, Q, \infty)$, and there exist unique $A \oplus B \in L_{U(P, Q, \infty)}$ and $d_{A, B} \in \Omega_{U(P, Q, \infty)}$ such that $AB = (A \oplus B)d_{A, B}$. On the other hand $(L_{U(P(n), Q(n), \infty)}, \oplus)$ is a K-loop by Theorem 3.1.11, and the closure of $L_{U(P(n), Q(n), \infty)}$ is also a K-loop in $Pos(\mathcal{H})$ by Lemma 3.1.8. The case $G = O(P, Q, \infty)$ is verbatim, except $\tau : GL(S(n), \mathbb{R}) \rightarrow GL(S(n), \mathbb{R})$ such that $A \mapsto (A^{-1})^{\top}$.

If $G = Sp(P, Q, \infty)$, the proof is similar except that $\tau : GL(S(n), \mathcal{H}_{\mathbb{H}}) \rightarrow GL(S(n), \mathcal{H}_{\mathbb{H}}): A \mapsto (A^{-1})^*$ such that $G(S(n), \tau, J_{P(n), Q(n)}) = Sp(P(n), Q(n))$. If $G = SO^*(\infty)$, then let $\tau : GL(2n, \mathbb{C}) \rightarrow GL(2n, \mathbb{C}): A \mapsto (A^{-1})^{\top}$. This gives $G(2n, \tau, K_{n, n}) = SO^*(2n)$. Therefore, the result follows by carrying $Sp(P(n), Q(n))$ and $SO^*(2n)$ into $GL(\mathcal{H}_{\mathbb{C}})$. \square

3.3 $SO_{\circ}(P, Q, \infty)$ and $SO_{\circ}(\infty, \mathcal{H}_{\mathbb{C}})$

Consider the bilinear form (\cdot, \cdot) on \mathbb{C}^n defined by $(x, y) = \sum_{k=1}^n x_k y_k$. Note that this bilinear form is not an inner product on \mathbb{C}^n since it is not conjugate symmetric. The set of all $n \times n$ complex matrices which preserve the given bilinear form is called complex orthogonal group $O(n, \mathbb{C})$. Complex orthogonal group is a subgroup of $GL(n, \mathbb{C})$. If $A \in O(n, \mathbb{C})$, then $(Ax, Ay) = (x, y)$ for all $x, y \in \mathbb{C}^n$ is equivalent to $A^{\top} A = I_n$. The determinant of A is ± 1 . We define $SO(n, \mathbb{C})$ as a set of matrices in $O(n, \mathbb{C})$ whose determinant is 1. Therefore,

$$SO(n, \mathbb{C}) = \{A \in SL(n, \mathbb{C}) : A^{\top} A = I_n\}.$$

We also recall that $SO(p, q) = O(p, q) \cap SL(p+q, \mathbb{R})$. Therefore,

$$SO(p, q) = \{A \in SL(p+q, \mathbb{R}) : A^{\top} J_{P(n), Q(n)} A = J_{P(n), Q(n)}\}.$$

We will use $SO_{\circ}(n, \mathbb{C})$ and $SO_{\circ}(p, q)$ to denote the connected components of the identity for $SO(n, \mathbb{C})$ and $SO(p, q)$ respectively. The connected components of $SO(n, \mathbb{C})$ and $SO(p, q)$ have exact decompositions by Theorem 4.0.8 as follow:

$SO_{\circ}(n, \mathbb{C}) = \exp(\mathfrak{p}_1)SO(n)$ and $SO_{\circ}(p, q) = \exp(\mathfrak{p}_2)(SO(p) \times SO(q))$, where \mathfrak{p}_1 is the eigenspace of -1 for $\theta : \mathfrak{so}(n, \mathbb{C}) \rightarrow \mathfrak{so}(n, \mathbb{C}) : A \mapsto -A^{\top}$, and \mathfrak{p}_2 is the eigenspace of -1 for θ defined on $\mathfrak{so}(p, q)$.

Lemma 3.3.1. *$SO(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ satisfies ascending chain condition in $GL(\mathcal{H}_{\mathbb{R}})$, i.e., $SO(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ is a subgroup of $SO(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{R}})$ for each $n \in \mathbb{N}$.*

Proof. Any matrix $A = \text{diag}(B, 1, \dots) \in SO(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ can be considered as an element of $SO(P(n+1), Q(n+1), \mathcal{H}_{\mathbb{R}})$ by setting $A = \text{diag}(C, 1, \dots)$, where $C := \text{diag}(B, \underbrace{1, \dots, 1}_{Q(n+1)-Q(n)})$. The proof is similar to the case $O(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$, therefore we reader might want to see the details in Lemma 3.2.1. \square

Let $P := P(n)$ be a constant sequence and let $Q = Q(n)$ be a strictly increasing sequence of positive integers. We define $SO_{\circ}(P, Q, \infty)$ and $SO_{\circ}(\infty, \mathcal{H}_{\mathbb{C}})$ as infinite union of ascending chain of matrix Lie groups in $GL(\mathcal{H}_{\mathbb{R}})$ and $GL(\mathcal{H}_{\mathbb{C}})$ respectively as follow:

1. $SO_{\circ}(\infty, \mathcal{H}_{\mathbb{C}}) := \bigcup_{n=2}^{\infty} SO_{\circ}(n, \mathcal{H}_{\mathbb{C}})$ in $GL(\mathcal{H}_{\mathbb{C}})$.
2. $SO_{\circ}(P, Q, \infty) := \bigcup_{n=1}^{\infty} SO_{\circ}(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ in $GL(\mathcal{H}_{\mathbb{R}})$.

$SO(P, Q, \infty)$ is a subgroup of $GL(\mathcal{H}_{\mathbb{R}})$ by Lemma 3.1.6, 3.1.5, and 3.3.1.

Theorem 3.3.2. *If $G \in \{SO_{\circ}(P, Q, \infty), SO_{\circ}(\infty, \mathcal{H}_{\mathbb{C}})\}$, then $L_G(\infty, \mathcal{H}_{\mathbb{C}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap G$ is a transversal of $G/U_G(\infty, \mathcal{H}_{\mathbb{C}})$ and is therefore a K-loop.*

Proof. We will prove the case $G = SO_{\circ}(P, Q, \infty)$. There exists an exact decomposition for each integer $n \geq 1$ by Theorem 4.0.8 as follow:

$SO_{\circ}(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) = \exp(\mathfrak{p}_n)(SO(P(n), \mathcal{H}_{\mathbb{R}}) \times SO(Q(n), \mathcal{H}_{\mathbb{R}}))$, where $\exp : \mathfrak{so}(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) \rightarrow SO(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$. The subalgebra \mathfrak{p}_n of $\mathfrak{so}(P(n), Q(n))$ is the eigenspace of -1 , i.e. $\mathfrak{p}_n = \{A \in \mathfrak{so}(P(n), Q(n)) : A = A^{\top}\}$. If $A \in \mathfrak{p}_n$, then $\exp(A) = \exp(A^{\top}) = (\exp(A))^{\top}$. On the other hand $\exp(A)$ is positive, so $\exp(\mathfrak{p}_n)$ is the set of positive definite symmetric matrices in $SO_{\circ}(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$. That means $\exp(\mathfrak{p}_n) = L(S(n), \mathcal{H}_{\mathbb{C}}) \cap SO_{\circ}(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) = L(\infty, \mathcal{H}_{\mathbb{C}}) \cap$

$SO_o(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$. On the other hand, if $G = SO_o(P, Q, \infty)$ then observe that

$$\begin{aligned} \bigcup_{n=1}^{\infty} (L(\infty, \mathcal{H}_{\mathbb{C}}) \cap SO_o(P(n), Q(n), \mathcal{H}_{\mathbb{R}})) &= L(\infty, \mathcal{H}_{\mathbb{C}}) \cap \bigcup_{n=1}^{\infty} SO_o(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) \\ &= L(\infty, \mathcal{H}_{\mathbb{C}}) \cap SO_o(P, Q, \mathcal{H}_{\mathbb{R}}) \\ &= L_G(\infty, \mathcal{H}_{\mathbb{C}}) \end{aligned}$$

The exact decomposition hold for each $n \geq 1$, thus taking the infinite unions of both sides preserve the equality and the exactness, since $\bigcup_{n=1}^{\infty} SO_o(P(n), Q(n), \mathcal{H}_{\mathbb{R}})$ and $\bigcup_{n=1}^{\infty} (SO(P(n), \mathcal{H}_{\mathbb{R}}) \times SO(Q(n), \mathcal{H}_{\mathbb{R}}))$ are both groups in $GL(\mathcal{H}_{\mathbb{R}})$. Therefore,

$$\begin{aligned} \bigcup_{n=1}^{\infty} SO_o(P(n), Q(n), \mathcal{H}_{\mathbb{R}}) &= \bigcup_{n=1}^{\infty} [\exp(\mathfrak{p}_n)(SO(P(n), \mathcal{H}_{\mathbb{R}}) \times SO(Q(n), \mathcal{H}_{\mathbb{R}}))], \\ SO_o(P, Q, \infty) &= \left[\bigcup_{n=1}^{\infty} \exp(\mathfrak{p}_n) \right] \left[\bigcup_{n=1}^{\infty} SO(P(n), \mathcal{H}_{\mathbb{R}}) \times SO(Q(n), \mathcal{H}_{\mathbb{R}}) \right], \\ &= L_{SO_o(P, Q, \infty)}(\infty, \mathcal{H}_{\mathbb{C}}) \left[\bigcup_{n=1}^{\infty} SO(P(n), \mathcal{H}_{\mathbb{R}}) \times \bigcup_{n=1}^{\infty} SO(Q(n), \mathcal{H}_{\mathbb{R}}) \right], \\ &= L_{SO_o(P, Q, \infty)}(\infty, \mathcal{H}_{\mathbb{C}})(SO(P, \infty) \times SO(Q, \infty)). \end{aligned}$$

We remark that $SO(n)$ is connected, and we can associate any pair $(A, B) \in SO(p) \times SO(q)$ with $diag(A, B) \in SO_o(p, q)$ by $(A, B) \mapsto diag(A, B)$. This map is well defined since $SO(p)$ and $SO(q)$ are connected, so there exists $\gamma_1 : [0, 1] \rightarrow SO(p)$ and $\gamma_2 : [0, 1] \rightarrow SO(q)$ such that $\gamma_1(0) = I_p$, $\gamma_1(1) = A$, and $\gamma_2(0) = I_q$, $\gamma_2(1) = B$. If we consider $\gamma := \gamma_1 \times \gamma_2$, then $\gamma(0) =$

$(A, B) \cong \text{diag}(A, B)$ and $\gamma(1) = (I_p, I_q) = I_{p+q}$, i.e. γ is a path from $\text{diag}(A, B)$ to identity of $SO(p, q)$. Therefore, $\text{diag}(A, B)$ is in the identity component of $SO(p, q)$. On the other hand, $SO(p, \infty) \times SO(q, \infty) = O(p, q, \infty) \cap SO_o(p, q, \infty)$.

Exact decomposition implies that for any $A, B \in SO_o(P, Q, \infty)$, there exists unique $A \oplus B \in L_{SO_o(P, Q, \infty)}(\infty, \mathcal{H}_{\mathbb{C}})$ and $d_{A, B} \in SO(P, \infty) \times SO(Q, \infty)$ such that $AB = (A \oplus B)d_{A, B}$. On the other hand, $(L_{SO_o(P, Q, \infty)}(\infty, \mathcal{H}_{\mathbb{C}}), \oplus)$ is a K-loop by Theorem 3.1.11, and the closure of $L_{SO_o(P, Q, \infty)}(\infty, \mathcal{H}_{\mathbb{C}})$ is also a K-loop in $Pos(\mathcal{H}_{\mathbb{R}})$ by Lemma 3.1.8. The case $SO_o(\infty, \mathcal{H}_{\mathbb{C}})$ can be proved similarly. □

Chapter 4

K-Loops from Classical subgroups of $GL(\mathcal{H})$, \mathcal{H} a separable infinite dimensional Hilbert Space

The orthogonal and symplectic complex Banach-Lie groups were defined in the first chapter as follows:

1. $O(\mathcal{H}, J_{\mathbb{R}}) := \{T \in GL(\mathcal{H}) : \langle Tx, J_{\mathbb{R}}Ty \rangle = \langle x, J_{\mathbb{R}}y \rangle\}$
2. $Sp(\mathcal{H}, J_{\mathbb{Q}}) := \{T \in GL(\mathcal{H}) : \langle Tx, J_{\mathbb{Q}}Ty \rangle = \langle x, J_{\mathbb{Q}}y \rangle\}$

Recall that $J_{\mathbb{R}}$ and $J_{\mathbb{Q}}$ are semilinear operators on \mathcal{H} and they are called conjugation and anticonjugation respectively. For all $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, $J_{\mathbb{R}}$

satisfies the given properties below.

1. $J_{\mathbb{R}}(x + y) = J_{\mathbb{R}}(x) + J_{\mathbb{R}}(y)$
2. $J_{\mathbb{R}}(\lambda x) = \bar{\lambda}J_{\mathbb{R}}(x)$
3. $\langle J_{\mathbb{R}}x, J_{\mathbb{R}}y \rangle = \overline{\langle x, y \rangle}$
4. $J_{\mathbb{R}}^2 = Id.$

Note that $J_{\mathbb{Q}}$ has the same properties except 4, that is $J_{\mathbb{Q}}^2 = -Id.$

Lemma 4.0.3. *For each $T \in O(\mathcal{H}, J_{\mathbb{R}})$, T satisfies that $J_{\mathbb{R}}TJ_{\mathbb{R}} = (T^*)^{-1}.$*

Proof. $T \in O(\mathcal{H}, J_{\mathbb{R}})$, so $\langle x, J_{\mathbb{R}}y \rangle = \langle Tx, J_{\mathbb{R}}Ty \rangle$ for each $x, y \in \mathcal{H}$. Let $y = J_{\mathbb{R}}x$, then

$$\langle x, J_{\mathbb{R}}^2x \rangle = \langle Tx, J_{\mathbb{R}}TJ_{\mathbb{R}}x \rangle \quad (4.1)$$

$$\langle x, x \rangle - \langle x, T^*J_{\mathbb{R}}TJ_{\mathbb{R}}x \rangle = 0 \quad (4.2)$$

$$\langle x, (I - T^*J_{\mathbb{R}}TJ_{\mathbb{R}})x \rangle = 0 \quad (4.3)$$

For each $x \in \mathbb{H}$ $\langle x, (I - T^*J_{\mathbb{R}}TJ_{\mathbb{R}})x \rangle = 0$, and this implies that $T^*J_{\mathbb{R}}TJ_{\mathbb{R}} = I$, so $J_{\mathbb{R}}TJ_{\mathbb{R}} = (T^*)^{-1}$. \square

Lemma 4.0.4. *For each $T \in Sp(\mathcal{H}, J_{\mathbb{Q}})$, T satisfies that $J_{\mathbb{Q}}TJ_{\mathbb{Q}} = -(T^*)^{-1}.$*

Proof. $T \in O(\mathcal{H}, J_{\mathbb{Q}})$, so $\langle x, J_{\mathbb{Q}}y \rangle = \langle Tx, J_{\mathbb{Q}}Ty \rangle$ for each $x, y \in \mathcal{H}$. Let $y =$

$J_{\mathbb{Q}}x$, then

$$\langle x, J_{\mathbb{Q}}^2 x \rangle = \langle Tx, J_{\mathbb{Q}} T J_{\mathbb{Q}} x \rangle \quad (4.4)$$

$$\langle x, -x \rangle - \langle x, T^* J_{\mathbb{Q}} T J_{\mathbb{Q}} x \rangle = 0 \quad (4.5)$$

$$\langle x, (-I - T^* J_{\mathbb{Q}} T J_{\mathbb{Q}}) x \rangle = 0 \quad (4.6)$$

For each $x \in \mathbb{H}$ $\langle x, (-I - T^* J_{\mathbb{Q}} T J_{\mathbb{Q}}) x \rangle = 0$, and this implies that $T^* J_{\mathbb{Q}} T J_{\mathbb{Q}} = -I$, so $J_{\mathbb{Q}} T J_{\mathbb{Q}} = -(T^*)^{-1}$. \square

Proposition 4.0.5. *If $T \in O(\mathcal{H}, J_{\mathbb{R}})$, then $T^* \in O(\mathcal{H}, J_{\mathbb{R}})$.*

Proof. Let $T \in O(\mathcal{H}, J_{\mathbb{R}})$. Then notice that

$$\langle T^* x, J_{\mathbb{R}} T^* y \rangle = \langle x, T J_{\mathbb{R}} T^* y \rangle \quad (4.7)$$

$$= \langle x, (J_{\mathbb{R}} J_{\mathbb{R}}) T J_{\mathbb{R}} T^* y \rangle \quad (4.8)$$

$$= \langle x, J_{\mathbb{R}} (J_{\mathbb{R}} T J_{\mathbb{R}}) T^* y \rangle \quad (4.9)$$

$$= \langle x, J_{\mathbb{R}} (T^*)^{-1} T^* y \rangle \quad (4.10)$$

$$= \langle x, J_{\mathbb{R}} y \rangle. \quad (4.11)$$

Therefore, $T^* \in O(\mathcal{H}, J_{\mathbb{R}})$, so $O(\mathcal{H}, J_{\mathbb{R}})$ is closed under adjoints. \square

Proposition 4.0.6. *If $T \in Sp(\mathcal{H}, J_{\mathbb{Q}})$, then $T^* \in Sp(\mathcal{H}, J_{\mathbb{Q}})$.*

The following lemma is well known.

Lemma 4.0.7. *Let $T \in GL(\mathcal{H})$ and $U \in U(\mathcal{H})$, then $T^* \in GL(\mathcal{H})$ and $U^* \in U(\mathcal{H})$.*

Proof. $T^* \in GL(\mathcal{H})$ if T^* is invertible. Notice that $I = (TT^{-1})^* = (T^{-1})^*T^*$, but this means T^* has a left inverse $(T^{-1})^*$. The right inverse, equal to the left inverse, can be found similarly. We conclude that $GL(\mathcal{H})$ is closed under taking adjoints. On the other hand $U^* \in U(\mathcal{H})$ if U^* preserves the inner product; that is, $\langle U^*(x), U^*(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$. We know $U \in U(\mathcal{H})$, hence $\langle U(x), U(y) \rangle = \langle x, y \rangle = \langle x, (U^*U)(y) \rangle$. Then $\langle U^*(x), U^*(y) \rangle = \langle x, ((U^*)^*U^*)(y) \rangle = \langle x, (UU^*)(y) \rangle = \langle x, y \rangle$ since U is unitary. Therefore, $U(\mathcal{H})$ is also closed under taking adjoints. \square

Theorem 4.0.8. *Let G be a connected reductive real Lie group. Let $\Theta : G \rightarrow G$, $g \mapsto (\bar{g}^\top)^{-1}$, so that $d\Theta|_1 := \theta$ and let $\exp : \mathfrak{g} \rightarrow G$ be the exponential map. Let $K := \{g \in G : \Theta(g) = g\}$. Then*

1. *For all $A, B \in \exp \mathfrak{p}$ there exist unique $A * B \in \exp \mathfrak{p}$ and $d_{A,B} \in K$ such that $AB = (A * B)d_{A,B}$ where \mathfrak{p} is the eigenspace of -1 with respect to θ ;*
2. *$(\exp(\mathfrak{p}), *)$ is a Lie K -loop.*

Proof. 1. Since G is a connected reductive real group, G has an exact decomposition $G = \exp(\mathfrak{p})K$, \mathfrak{p} is the eigenspace of -1 in \mathfrak{g} by Proposition 1.5.2. Therefore for all $A, B \in \exp(\mathfrak{p})$ there exist unique $A * B \in \exp(\mathfrak{p})$ and $d_{A,B} \in K$ such that $AB = (A * B)d_{A,B}$.

2. The eigenspace of -1 of θ , \mathfrak{p} in \mathfrak{g} , is given below

$$\mathfrak{p} = \{v \in \mathfrak{g} : v = \bar{v}^\top\}.$$

The zero vector, 0, belongs to \mathfrak{p} and $\exp(0) = I \in \exp\mathfrak{p}$.

3. Notice that \mathfrak{p} is the set of symmetric matrices. The exponential map is an analytic isomorphism from the set of symmetric matrices in \mathfrak{g} onto the set of positive symmetric operators in the real case, and from the set of conjugate symmetric matrices in \mathfrak{g} onto the set of positive self-adjoint operators in the complex case by Theorem 1.6.1.

For all $a \in \exp\mathfrak{p}$, $[a\exp(\mathfrak{p})a]^* = a^*[\exp(\mathfrak{p})]^*a^* = a\exp(\mathfrak{p})a$ since

$$\exp(x)^* = \exp(x^*) = \exp(x) \text{ for each } x \in \mathfrak{p}.$$

On the other hand $\langle (a\exp(x)a)y, y \rangle = \langle \exp(x)(ay), a^*y \rangle = \langle \exp(x)(ay), ay \rangle \geq 0$ for all $y \in \mathcal{H}$. We conclude that $a\exp(\mathfrak{p})a = \mathfrak{p}$ for all $a \in \mathfrak{p}$.

4. Let $\alpha \in K$, and let $x \in \mathfrak{p}$, then $\alpha\exp(x)\alpha^{-1} = \exp(\alpha x \alpha^{-1}) = \exp(\text{Ad}_\alpha(x))$, where $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$. Notice that $(\alpha x \alpha^{-1})^* = (\alpha^{-1})^* x^* \alpha^* = \alpha x \alpha^{-1}$, since $\alpha \in K$, i.e., $(\alpha^{-1})^* = \alpha$ and $x \in \mathfrak{p}$. Moreover, $\alpha\exp(x)\alpha^{-1} = \exp(\text{Ad}_\alpha(x))$ is positive definite by Theorem 1.6.1.

We conclude that $(\exp(\mathfrak{p}), *)$ is a K-loop by Theorem 1.3.11 such that $AB = (A * B)d_{A,B}^{-1}$. □

Corollary 4.0.9. *Let $G \in \{Sp(m, n), SO^*(2n), SO_\circ(n, \mathbb{C}), SO_\circ(m, n)\}$. Then*

1. *Given $A, B \in \exp(\mathfrak{p})$, there exists unique $A * B \in \exp(\mathfrak{p})$ and*

$$d_{A,B} \in \begin{cases} Sp(m) \times Sp(n) & \text{if } G = Sp(m, n) \\ U(n) & \text{if } G = SO^*(2n) \\ SO(n) & \text{if } G = SO_\circ(n, \mathbb{C}) \\ SO(m) \times SO(n) & \text{if } G = SO_\circ(m, n) \end{cases}$$

*such that $AB = (A * B)d_{A,B}$.*

2. *$(\exp(\mathfrak{p}), *)$ is a K -loop.*

Remark 4.0.10. *If $G = SO(m, n)$, then $\mathfrak{g} = \mathfrak{so}(m, n)$. The eigenspace of -1 for $\theta : \mathfrak{so}(m, n) \rightarrow \mathfrak{so}(m, n)$ is $\mathfrak{p} = \{v \in \mathfrak{so}(m, n) : v^\top = v\}$. Therefore, \mathfrak{p} is the set of symmetric matrices in $\mathfrak{so}(m, n)$. The exponential map takes any $v \in \mathfrak{so}(m, n)$, and sends into $SO(m, n) \cap Pos(m+n, \mathbb{R})$, where $Pos(m+n, \mathbb{R})$ is the set of positive definite matrices in $\mathbb{R}^{(m+n) \times (m+n)}$.*

Remark 4.0.11. *Kiechle [13] remarked that if $GL(\mathcal{H})$ is the unit group of the Banach algebra of bounded operators $\mathcal{H} \rightarrow \mathcal{H}$, then by the polar decomposition theorem, $GL(\mathcal{H}) = Pos(\mathcal{H})U(\mathcal{H})$ where $Pos(\mathcal{H})$ and $U(\mathcal{H})$ are the positive self-adjoint operators and unitary operators, respectively. This decomposition is exact, that is for all $A, B \in Pos(\mathcal{H})$ there exist unique elements $A \oplus B \in Pos(\mathcal{H})$ and $d_{A,B} \in U(\mathcal{H})$ such that $AB = (A \oplus B)d_{A,B}$. It can be shown by Kreuzer and Wefelscheid's method, $(Pos(\mathcal{H}), \oplus)$ is a K -loop. We showed that*

a similar construction can be made by taking G as the orthogonal or symplectic complex Banach Lie groups as given in the following theorem.

Theorem 4.0.12. *Let $G(\mathcal{H})$ be one of the complex Banach Lie groups in $\{O(\mathcal{H}, J_{\mathbb{R}}), Sp(\mathcal{H}, J_{\mathbb{Q}})\}$, and let $Pos(\mathcal{H})$ and $U(\mathcal{H})$ are collection of positive self-adjoint operators and unitary operators respectively over \mathbb{C} . Let $P_G := G(\mathcal{H}) \cap Pos(\mathcal{H})$, and $U_G := G(\mathcal{H}) \cap U(\mathcal{H})$. Then for all $A, B \in P_G$ there exist unique $A \oplus B \in P_G$ and $d_{A,B} \in U_G$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, (P_G, \oplus) is a K-loop.*

Proof. Let $U_G = G(\mathcal{H}) \cap U(\mathcal{H})$ where $U(\mathcal{H})$ is the group of unitary operators. To see that (P_G, \oplus) is a K-loop, we follow Theorem 1.3.11.

1. $G(\mathcal{H}) = U_G P_G$ is an exact decomposition by Theorem 1.6.1.
2. $\langle Ix, x \rangle = \langle x, x \rangle = \|x\|^2 \geq 0$ for all $x \in \mathcal{H}$, so I is positive. On the other hand $\langle x, x \rangle = \langle Ix, x \rangle = \langle x, Ix \rangle = \langle x, I^*x \rangle$ for all $x \in \mathcal{H}$. The last equality indicates that $I = I^*$, thus I is also self-adjoint. We conclude that $I \in P_G$.
3. Let $P, Q \in P_G$. We want to show that $PQP \in P_G$. Notice that $\langle (PQP)(x), x \rangle = \langle Q(P(x)), P^*(x) \rangle = \langle Q(P(x)), P(x) \rangle \geq 0$ since Q is positive. On the other hand $(PQP)^* = (P^*)(Q^*)(P^*) = PQP$ that means PQP is also self-adjoint. Therefore, $PP_GP \subseteq P_G$ for all $P \in P_G$.
4. Let $T \in U_G$ and let $P \in P_G$. $T \in U_G$ implies that $T^* = T^{-1}$. We want to show that $TPT^{-1} \in P_G$. Observe that $\langle (TPT^{-1})(x), x \rangle =$

$\langle P(T^{-1}(x)), T^*(x) \rangle = \langle P(T^{-1}(x)), T^{-1}(x) \rangle \geq 0$ since $P \in P_G$. On the other hand it is easy to verify that: $(TPT^{-1})^* = (T^{-1})^*P^*T^* = (T^*)^*PT^{-1} = TPT^{-1}$, thus TPT^{-1} is also self-adjoint. Therefore, $TPT^{-1} \in P_G$ that is $TP_GT^{-1} \subseteq P_G$ for all $T \in U_G$.

5. Let $P, Q \in P_G$ and let $U \in U_G$. Notice that $U^* = U^{-1} \in U_G$ since U is unitary and U_G is a group. We want to show that if $PQU \in P_G$, then there exist $\beta \in U_G$ such that $PQU = \beta QP$. Assume that $PQU \in P_G$, so $(PQU)^* = PQU = U^*Q^*P^* = U^*QP$ where $U^* \in U_G$.

We conclude that (P_G, \oplus) is a K-loop. □

Chapter 5

The algebras associated with twisted semi-direct product Lie Loops and Lie K-loops

We can form new Lie algebras from old ones. Direct products and semi-direct products of Lie algebras are two different ways to obtain new Lie algebras. We will start with recalling the semi-direct product of Lie algebras. Let \mathfrak{h} and \mathfrak{k} be Lie algebras over the same ground field \mathbb{F} , and let $\rho : \mathfrak{h} \rightarrow \text{Der}_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism sending each $h \in \mathfrak{h}$ to a derivation of \mathfrak{k} such that

$$h \mapsto (h)\rho \text{ and } k \mapsto (k)(h)\rho.$$

We can form a new Lie algebra $\mathfrak{l} := \mathfrak{k} \times \mathfrak{h}$ such that \mathfrak{l} is a vector space with usual pointwise addition and scalar multiplication that are $(k_1, h_1) + (k_2, h_2) = (k_1 + k_2, h_1 + h_2)$ and $c(k_1, h_1) = (ck_1, ch_1)$ where $k_1, k_2 \in \mathfrak{k}$, $h_1, h_2 \in \mathfrak{h}$ and

$c \in \mathbb{F}$. On the other hand, the Lie product can be defined as,

$$[(k_1, h_1), (k_2, h_2)] = ([k_1, k_2] + (k_1)(h_2)\rho - (k_2)(h_1)\rho, [h_1, h_2]).$$

This product is indeed bilinear, anti-symmetric and it satisfies the Jacobi identity. The Jacobi identity can be verified using the the fact that $Der_{\mathbb{F}}(\mathfrak{k})$ is itself a Lie algebra with the Lie bracket is defined by $[f, g] = gf - fg$ for all $f, g \in Der_{\mathbb{F}}(\mathfrak{k})$. The new Lie algebra \mathfrak{l} is called the semi-direct product of the Lie algebra \mathfrak{k} by \mathfrak{h} and is denoted by $\mathfrak{l} = \mathfrak{k} \rtimes \mathfrak{h}$. The following result is well known; see, for example [30].

Theorem 5.0.13. *Let \mathfrak{h} and \mathfrak{k} be two Lie algebras over the field \mathbb{F} , and let $\rho : \mathfrak{h} \rightarrow Der_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism. Then*

1. *The semi-direct product $\mathfrak{l} = \mathfrak{k} \rtimes \mathfrak{h}$ is a Lie algebra.*
2. *$\bar{\mathfrak{k}} = \{(k, 0) : k \in \mathfrak{k}\} \cong \mathfrak{k}$ such that \mathfrak{k} is an ideal of \mathfrak{l} , i.e., $[\bar{\mathfrak{k}}, \mathfrak{l}] \subseteq \bar{\mathfrak{k}}$.*
3. *$\bar{\mathfrak{h}} = \{(0, h) : h \in \mathfrak{h}\} \cong \mathfrak{h}$ such that \mathfrak{h} is a subalgebra of \mathfrak{l} , i.e., $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subseteq \bar{\mathfrak{h}}$.*

Theorem 5.0.14. *If the loop L is the twisted semi-direct product of Lie groups K and H , $K^* := K \times 1$ and $H^* := 1 \times H$, then K^* and H^* are subgroups of the loop L such that K^* is normal, and $L = K^*H^*$*

Proof. We prove in Theorem 2.2.2 that L is a loop. H^* and K^* are both

contained in L . Let $(k, h) \in L$ and $(k_*, 1) \in K^*$, then

$$(k, h)^{-1}(k_*, 1)(k, h) = ((k^{-1})^{h^{-1}}, h^{-1})(k_*, 1)(k, h) \quad (5.1)$$

$$= ((k^{-1})^{h^{-1}}k_*^{h^{-1}}, h^{-1})(k, h) \quad (5.2)$$

$$= ((k^{-1}k_*)^{h^{-1}}, h^{-1})(k, h) \quad (5.3)$$

$$= ((k^{-1}k_*)^{h^{-1}}k^{h^{-1}}, h^{-1}h) \quad (5.4)$$

$$= ((k^{-1}k_*k)^{h^{-1}}, 1) \in K^* \quad (5.5)$$

Therefore, K^* is a normal subgroup of the loop L . On the other hand, K^* is also a Lie group since it is diffeomorphic to K .

On the other hand, given $(1, h_1), (1, h_2) \in H^*$ satisfies that $(1, h_1)(1, h_2)^{-1} = (1, h_1)(1, h_2^{-1}) = (1, h_1h_2^{-1}) \in H^*$, so H^* is a subgroup of the loop L . Moreover, H^* is a Lie group since it is diffeomorphic to the Lie group H .

It is clear that if $(k, 1) \in K^*$ and $(1, h) \in H^*$, then $(k, 1)(1, h) = (k, h) \in L$, i.e., $K^*H^* \subseteq L$. Conversely, if $(k, h) \in L$ then $(k, h) = (k, 1)(1, h) \in K^*H^*$, i.e., $L \subseteq K^*H^*$. We conclude that $L = K^*H^*$. \square

Corollary 5.0.15. [4] *The Akivis algebra of the Lie loop of the twisted semi-direct product of Lie groups is a Lie algebra.*

Hence, we investigate the twisted semi-direct product \mathfrak{k} (the Lie algebra of K) and \mathfrak{h} (the Lie algebra of H) as the natural candidate for the Akivis algebra of L .

In analogy to the twisted semi-direct product of groups, we make the following definition for Lie algebras.

Definition 5.0.16. *Let \mathfrak{h} and \mathfrak{k} be Lie algebras over the field \mathbb{F} , and let $\rho : \mathfrak{h} \rightarrow \text{Der}_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism. The twisted semi-direct product of Lie algebra \mathfrak{k} by \mathfrak{h} , $\mathfrak{k} \rtimes_{\rho} \mathfrak{h}$, is the cartesian product $\mathfrak{k} \times \mathfrak{h}$ equipped with the bracket $[(k_1, h_1), (k_2, h_2)] = ([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho, [h_1, h_2])$.*

We will show that the twisted semi-direct product of Lie algebras is also a Lie algebra. Moreover, this Lie algebra is an Akiwis algebra. An Akiwis algebra $(\mathcal{A}, [., .], \langle ., ., . \rangle)$ is a real vector space with a bilinear skew-symmetric map $(x, y) \mapsto [x, y] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, called the commutator map, and a trilinear map $(x, y, z) \mapsto \langle x, y, z \rangle : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, called the associator map, such that the following identity (called the Akiwis identity) holds [4]:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = \alpha - \beta \text{ where } \alpha \text{ and } \beta \text{ are given below.}$$

$$\alpha = \langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle \quad (5.6)$$

$$\beta = \langle x, z, y \rangle + \langle y, x, z \rangle + \langle z, y, x \rangle \quad (5.7)$$

If \mathcal{A} is a Lie algebra, then the left hand-side of the equality is zero by the Jacobi identity.

Lemma 5.0.17. *Any Lie algebra \mathfrak{l} , is an Akiwis algebra with the trilinear operation defined by $\langle x, y, z \rangle : \mathfrak{l} \times \mathfrak{l} \times \mathfrak{l} \rightarrow \mathfrak{l}; (x, y, z) \mapsto [[x, y], z] - [x, [y, z]]$.*

Proof. Let \mathfrak{l} be a Lie algebra, then there exists a bilinear skew-symmetric

operation, $[\cdot, \cdot] : \mathfrak{l} \times \mathfrak{l} \rightarrow \mathfrak{l}; (x, y) \mapsto [x, y]$. To see that \mathfrak{l} is indeed an Akivis algebra, we need to verify the Akivis identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = \alpha - \beta$ where α and β given below.

$$\alpha = \langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle \quad (5.8)$$

$$\beta = \langle x, z, y \rangle + \langle y, x, z \rangle + \langle z, y, x \rangle \quad (5.9)$$

Let $\gamma = [[x, y], z] + [[y, z], x] + [[z, x], y]$, then we want to show that $\gamma = \alpha - \beta$. \mathfrak{l} is a Lie algebra, thus it satisfies the Jacobi identity and this forces that $\gamma = 0$, hence we only need to show that $\alpha = \beta$. Notice that $\alpha = \langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle = [[x, y], z] - [x, [y, z]] + [[y, z], x] - [y, [z, x]] + [[z, x], y] - [z, [x, y]] = ([[x, y], z] + [[y, z], x] + [[z, x], y]) - ([x, [y, z]] + [y, [z, x]] + [z, [x, y]])$. Notice that $[[x, y], z] + [[y, z], x] + [[z, x], y] = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ because of the Jacobi identity. The same argument also implies that $\beta = 0$. Therefore, $\alpha = \beta$, and we conclude each Lie algebra is an Akivis algebra. \square

Theorem 5.0.18. *Let \mathfrak{h} and \mathfrak{k} be two Lie algebras over the field \mathbb{F} , and let $\rho : \mathfrak{h} \rightarrow \text{Der}_{\mathbb{F}}(\mathfrak{k})$ be a Lie algebra homomorphism. Then*

1. $\mathfrak{l} = \mathfrak{k} \overline{\times} \mathfrak{h}$ is an Akivis algebra with the bracket and the trilinear operation given below respectively. Moreover, \mathfrak{l} is a Lie algebra.

$$\begin{aligned} \langle (k_1, h_1)(k_2, h_2), (k_3, h_3) \rangle &= [[(k_1, h_1), (k_2, h_2)], (k_3, h_3)] - [(k_1, h_1), [(k_2, h_2), (k_3, h_3)]] \\ [(k_1, h_1), (k_2, h_2)] &= ([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho, [h_1, h_2]) \end{aligned}$$

2. $\bar{\mathfrak{k}} = \{(k, 0) : k \in \mathfrak{k}\} \cong \mathfrak{k}$ is an ideal of \mathfrak{l} , i.e., $[\bar{\mathfrak{k}}, \mathfrak{l}] \subseteq \bar{\mathfrak{k}}$.

3. $\bar{\mathfrak{h}} = \{(0, h) : h \in \mathfrak{h}\} \cong \mathfrak{h}$ is a subalgebra of \mathfrak{l} , i.e., $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subseteq \bar{\mathfrak{h}}$.

Proof. 1. The bracket on \mathfrak{l} is skew symmetric as follows:

$$[(k_1, h_1), (k_2, h_2)] = ([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho, [h_1, h_2]) \quad (5.10)$$

$$= (-[k_2, k_1] - ((k_1)(h_2)\rho - (k_2)(h_1)\rho), -[h_2, h_1]) \quad (5.11)$$

$$= -([k_2, k_1] + (k_1)(h_2)\rho - (k_2)(h_1)\rho, [h_2, h_1]) \quad (5.12)$$

$$= -[(k_2, h_2), (k_1, h_1)] \quad (5.13)$$

The bracket on \mathfrak{l} is bilinear since the bracket operations on \mathfrak{k} and \mathfrak{h} are bilinear. On the other hand $(h)\rho$ is linear for each $h \in \mathfrak{h}$. Therefore, we only need to verify the Jacobi identity on \mathfrak{l} , and this can be seen as follows: Let $x = [[(k_1, h_1), (k_2, h_2)], (k_3, h_3)]$, $y = [[(k_2, h_2), (k_3, h_3)], (k_1, h_1)]$, and $z = [[(k_3, h_3), (k_1, h_1)], (k_2, h_2)]$. We want to show that $x + y + z = 0$.

$$x = [([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho, [h_1, h_2]), (k_3, h_3)] = ([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho, k_3] + (k_3)([h_1, h_2])\rho - ([k_1, k_2] + (k_2)(h_1)\rho - (k_1)(h_2)\rho)(h_3)\rho, [[h_1, h_2], h_3]$$

For convenience we denote $(k)(h)\rho$ by k^h . Then x can be rewritten as

$$x = ([k_1, k_2] + k_2^{h_1} - k_1^{h_2}, k_3] + k_3^{[h_1, h_2]} - ([k_1, k_2] + k_2^{h_1} - k_1^{h_2})^{h_3}, [[h_1, h_2], h_3]).$$

Then using the linearity of the bracket operation in \mathfrak{k} gives that

$$x = ([k_1, k_2], k_3] + [k_2^{h_1}, k_3] - [k_1^{h_2}, k_3] + k_3^{[h_1, h_2]} - [k_1, k_2]^{h_3} - k_2^{h_1 h_3} + k_1^{h_2 h_3}, [[h_1, h_2], h_3]).$$

Notice that ρ is a homomorphism of Lie algebras, i.e., $[[h_1, h_3]]\rho = [(h_1)\rho, (h_3)\rho]$. On the other hand $Der_{\mathbb{F}}(\mathfrak{k})$ is also Lie algebra

with $[f, g] = gf - fg$ for each $f, g \in \text{Der}_{\mathbb{F}}(\mathfrak{k})$. Therefore, $([h_1, h_3])\rho = [(h_1)\rho, (h_2)\rho] = (h_2)\rho(h_1)\rho - (h_1)\rho(h_2)\rho$. Thus $k_3^{[h_1, h_2]} = k_3^{h_2h_1} - k_3^{h_1h_2}$. Moreover $(h)\rho$ is a derivation on \mathfrak{k} implies that $(h)\rho$ preserves the Leibniz rule and this gives that $[k_1, k_2]^{h_3} = [k_1^{h_3}, k_2] + [k_1, k_2^{h_3}]$. Then we conclude that,

$x = ([k_1, k_2], k_3) + [k_2^{h_1}, k_3] - [k_1^{h_2}, k_3] + k_3^{h_2h_1} - k_3^{h_1h_2} - [k_1^{h_3}, k_2] - [k_1, k_2^{h_3}] - k_2^{h_1h_3} + k_1^{h_2h_3}, [[h_1, h_2], h_3]$. We can similarly find y and z such that:

$$y = ([k_2, k_3], k_1) + [k_3^{h_2}, k_1] - [k_2^{h_3}, k_1] + k_1^{h_3h_2} - k_1^{h_2h_3} - [k_2^{h_1}, k_3] - [k_2, k_3^{h_1}] - k_3^{h_2h_1} + k_2^{h_3h_1}, [[h_2, h_3], h_1].$$

$$z = ([k_3, k_1], k_2) + [k_1^{h_3}, k_2] - [k_3^{h_1}, k_2] + k_2^{h_1h_3} - k_2^{h_3h_1} - [k_3^{h_2}, k_1] - [k_3, k_1^{h_2}] - k_1^{h_3h_2} + k_3^{h_1h_2}, [[h_3, h_1], h_2].$$

The second coordinate of \mathfrak{l} is $[[h_1, h_2], h_3] + [[h_2, h_3], h_1] + [[h_3, h_1], h_2]$ and this sum is zero since \mathfrak{h} is a Lie algebra. On the other hand, the sum of the first coordinate $[[k_2, k_3], k_1] + [[k_2, k_3], k_1] + [[k_3, k_1], k_2] = 0$ since \mathfrak{k} is also a Lie algebra, and the sum of the rest can be combined as

$$([k_2^{h_1}, k_3] - [k_2^{h_1}, k_3]) + (-[k_1^{h_2}, k_3] - [k_3, k_1^{h_2}]) + (k_3^{h_2h_1} - k_3^{h_2h_1}) + (-k_3^{h_1h_2} + k_3^{h_1h_2}) + (-[k_1^{h_3}, k_2] + [k_1^{h_3}, k_2]) + (-[k_1, k_2^{h_3}] - [k_2^{h_3}, k_1]) + (-k_2^{h_1h_3} + k_2^{h_1h_3}) + (k_1^{h_2h_3} - k_1^{h_2h_3}) + ([k_3^{h_2}, k_1] - [k_3^{h_2}, k_1]) + (k_1^{h_3h_2} - k_1^{h_3h_2}) + (k_2^{h_3h_1} - k_2^{h_3h_1}).$$

This sum is zero since the sums in each parentheses are zero. Therefore, $x + y + z = (0, 0)$, and we conclude that the twisted semi-direct product of Lie algebras is a Lie algebra. Moreover, \mathfrak{l} is also an Akivis Algebra by Lemma 5.0.17.

2. The first claim, $\mathfrak{k} \cong \bar{\mathfrak{k}}$, is clear. We will show that $[\bar{\mathfrak{k}}, \mathfrak{l}] \subseteq \bar{\mathfrak{k}}$. Let $(k, 0) \in \bar{\mathfrak{k}}$,

and let $(k^*, h) \in \mathfrak{l}$. Then

$[(k, 0), (k^*, h)] = ([k, k^*] + (k^*)(0)\rho - (k)(h)\rho, [0, h])$ where $(k^*)(0)\rho$ and $[0, h]$ are both zero. Therefore $[(k, 0), (k^*, h)] = ([k, k^*] - (k)(h)\rho, 0) = (k^{**}, 0) \in \bar{\mathfrak{k}}$.

We conclude that \mathfrak{k} is an ideal of the Lie algebra \mathfrak{l} .

3. The claim, $\mathfrak{h} \cong \bar{\mathfrak{h}}$, is again clear. To see that $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subseteq \bar{\mathfrak{h}}$, let $(0, h), (0, h^*) \in \bar{\mathfrak{h}}$. Then

$[(0, h), (0, h^*)] = ([0, 0] + (0)(h)\rho - (0)(h^*)\rho, [h, h^*]) = (0, h^{**}) \in \bar{\mathfrak{h}}$. Therefore, $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subseteq \bar{\mathfrak{h}}$, i.e., \mathfrak{h} is a subalgebra of the Lie algebra \mathfrak{l} . \square

Chapter 6

Future research

In Chapter 2, we discussed the twisted semi-direct product of Lie groups. We showed that any twisted semi-direct product of Lie groups $\mathcal{L} = K \overline{\rtimes} H$ is a Lie loop. It seems reasonable that the Akiwis algebra of \mathcal{L} is a Lie algebra.

In Chapter 3, we mainly discussed K-loops from $GL(\infty, \mathcal{H}_{\mathbb{F}})$ for $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Computing the Akiwis algebra of finite dimensional K-loops seems to be within reach, and we conjecture that they are in fact Lie algebras. We are also interested in computing the Akiwis algebras of infinite dimensional K-loops, and expect that they are also Lie algebras.

In Chapter 4, we discussed a general construction for K-loops from connected reductive real Lie groups. There are two questions we would like to answer in the future. Theorem 4.0.8 indicates that we can form a K-loop from a connected reductive real group G , via the involution map $\Theta : G \rightarrow G$, $g \mapsto (\bar{g}^{\top})^{-1}$. The Cartan decomposition at the level of Lie groups provide a

new binary operation, the K-loop operation, induced by group multiplication. The existence of a K-loop from a subgroup G of $GL(n, \mathbb{C})$, G is not connected reductive real Lie subgroup, needs to be investigated. If we can find such a group, then its properties might give a clue to extend the Theorem 4.0.8 to a larger class of K-Loops. On the other hand, the role of the involution map needs to be investigated too. If a different involution map yields a K-loop, then how this K-loop is related with the one constructed with $\Theta : G \rightarrow G$, $g \mapsto (\bar{g}^\top)^{-1}$.

For the infinite dimensional case, we investigated the examples of K-loops from a subgroup $GL(\mathcal{H})$, where \mathcal{H} is an infinite dimensional separable Hilbert space over \mathbb{C} . We showed that if $G(\mathcal{H})$ is one of the Banach Lie groups in $\{O(\mathcal{H}, J_{\mathbb{Q}}), Sp(\mathcal{H}, J_{\mathbb{R}})\}$, then (P_G, \oplus) is a K-loop, where $P_G = G(\mathcal{H}) \cap Pos(\mathcal{H})$ and $Pos(\mathcal{H})$ is the set of positive self-adjoint operators on \mathcal{H} . We can also form K-loops from classical complex or real Banach Lie groups of compact operators on \mathcal{H} since the polar decomposition provides an analytic isomorphism from $G(\mathcal{H}, C_p)$ to $(G(\mathcal{H}, C_p) \cap U(\mathcal{H}, C_p)) \times (G(\mathcal{H}, C_p) \cap Pos(\mathcal{H}, C_p))$ [7].

In Chapter 5, we investigated the Akivis algebras of the twisted semi-direct product of Lie algebras. We showed that the twisted semi-direct product of Lie algebras is a Lie algebra. We would like to also investigate whether or not the Akivis algebra of a twisted semi-direct Lie loop is the twisted semi-direct product of the Lie algebras of its Lie groups. That means if $\mathcal{L} = K \bar{\rtimes} H$, where K and H are Lie groups such that H is a subgroup of $Aut(K)$, and \mathfrak{k} and \mathfrak{h} are the Lie algebras of K and H respectively, do we have $\mathcal{A}(\mathfrak{l}) = \mathfrak{k} \bar{\rtimes} \mathfrak{h}$?

The fundamental theorem of Lie concerns the correspondence $Lie : G \rightarrow Lie(G)$, where $Lie(G)$ is the Lie algebra of the Lie group G . The map Lie is functorial. We would like to investigate the analogue of this correspondence between the category of Lie loops and the category of the Akinis algebras.

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