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Modular Monochromatic Colorings, Spectra and Frames in Graphs

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MODULAR MONOCHROMATIC COLORINGS, SPECTRA AND FRAMES IN GRAPHS

by

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Historically, a number of problems and puzzles introduced over a period of decades initially appeared to have no connection to graph colorings but, upon further analysis, suggested graph coloring concepts and problems. One of these problems is the well-known combinatorial problem called the Lights Out Puzzle, which can be represented by a graph coloring problem which we describe in this work. For a nontrivial connected graph G and an integer $k \geq 2$, let $c : V(G) \to \mathbb{Z}_k$ be a vertex coloring of G where $c(v) \neq 0$ for at least one vertex v of G. Then the coloring c induces a new coloring $\sigma : V(G) \to \mathbb{Z}_k$ of G defined by $\sigma(v) = \sum_{u \in N[v]} c(u)$ where $N[v]$ is the closed neighborhood of v and addition is performed in $\mathbb{Z}_k$. If $\sigma(u) = \sigma(v) = t \in \mathbb{Z}_k$ for every two vertices u and v in G, then the coloring c is called a modular monochromatic (k,t)-coloring of G. Several results dealing with modular monochromatic $(k,0)$-colorings are presented, particularly the case where $k = 2$. The modular monochromatic $(2,1)$-coloring and $(2,0)$- colorings are not only closely related to the Lights Out Puzzle but also related to some well-known studied domination parameters, namely odd and even dominations in graphs.

In a modular monochromatic $(2,0)$-coloring of a graph G, we have $\sigma(v) = 0 \in \mathbb{Z}_2$ for every vertex v in G. A graph G having a modular monochromatic $(2,0)$-coloring is a $(2,0)$-colorable graph. The minimum number of vertices colored 1 in a modular monochromatic $(2,0)$-coloring of G is the $(2,0)$-chromatic number $\chi_{(2,0)}(G)$ of G. Thus $2 \leq \chi_{(2,0)}(G) \leq n$ and $\chi_{(2,0)}(G)$ is even for every $(2,0)$-colorable graph G of order n. A monochromatic $(2,0)$-colorable graph G of order n is $(2,0)$-extremal if $\chi_{(2,0)}(G) = n$. It is known that a tree T is $(2,0)$-extremal if and only if every vertex of T has odd degree. We characterize all trees of order n having $(2,0)$-chromatic number $n-1, n-2$ or $n-3$ and investigate the structures of connected graphs having the large $(2,0)$-chromatic numbers in general.

A dominating set S of a graph G is an even dominating set of G if every vertex of G is dominated by an even number of vertices in S and the minimum number of vertices in an even dominating set of G is the even domination number $\gamma_e(G)$ of G. We study the structures of $(2,0)$-colorable graphs with prescribed order and $(2,0)$-chromatic number and the relationship between modular monochromatic $(2,0)$-colorings and even dominating sets in graphs. It is shown that for each pair a, b of even integers with $2 \leq a \leq b$, there is a connected graph G such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$. A triple (a, b, n) of positive integers is realizable if there is a connected graph G of order n such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$. Realizable triples are determined.

For a $(2,0)$-colorable graph G, the monochromatic $(2,0)$-spectrum $S_{(2,0)}(G)$ of G is the set of all positive integers k for which exactly k vertices of G can be colored 1 in a monochromatic $(2,0)$-coloring of G. Monochromatic $(2,0)$-spectra are determined for several well-known classes of graphs. If G is a connected graph of order $n \geq 2$ and $a \in S_{(2,0)}(G)$, then a is even and $1 \leq |S_{(2,0)}(G)| \leq \lfloor n/2 \rfloor$. It is shown that for every pair k, n of integers with $1 \leq k \leq \lfloor n/2 \rfloor$, there is a connected graph G of order n such that $|S_{(2,0)}(G)| = k$. A set S of positive even integers is $(2,0)$-realizable if S is the monochromatic $(2,0)$-spectrum of some connected graph. Although there are infinitely many non-$(2,0)$-realizable sets, it is shown that every set of positive even integers is a subset of some $(2,0)$-realizable set. Other results and questions are also presented on $(2,0)$-realizable sets in graphs.

A graph G is called an odd-degree graph if every vertex of G has odd degree. For a nontrivial odd-degree graph H, a connected graph $G \neq H$ is a monochromatic $(2,0)$-frame of H if G has a minimum monochromatic $(2,0)$-coloring c such that the subgraph induced by the vertices colored 1 by c is H. The monochromatic frame number of H is defined as $fn(H) = \min\{|V(G) - V(H)|\}$ where the minimum is taken over all monochromatic $(2,0)$-frames G of H. Recall that a monochromatic $(2,0)$-colorable graph G of order n is a $(2,0)$-extremal graph if $\chi_{(2,0)}(G) = n$. It is shown that $fn(H) \leq 2$ for every connected $(2,0)$-extremal graph H and the monochromatic frame numbers are determined for several well-known classes of $(2,0)$-extremal graphs. Furthermore, it is shown that if H is the union of $k \geq 2$ connected $(2,0)$-extremal graphs, then $fn(H) = k-1$. Other results and questions are also presented on monochromatic $(2,0)$-frames in graphs.

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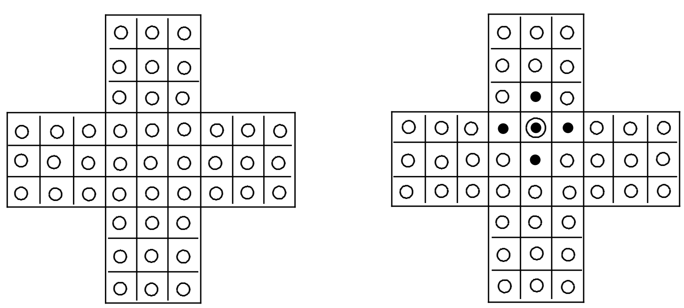
Introduction

1.1 Lights Out Puzzle

Historically, a number of problems and puzzles introduced over a period of decades initially appeared to have no connection to graph colorings but, upon further analysis, suggested graph coloring concepts and problems. One of these problems is the well-known combinatorial problem called the Lights Out Puzzle, which concerns the electronic game of "Lights Out" consisting of a cube, each of whose six faces contains 9 squares in 3 rows and 3 columns. Thus there are 54 squares in all. Figure 1.1(a) shows the "front" of the cube as well as the faces on the top, bottom, left and right. The back of the cube is not shown.



(a) (b)

Figure 1.1: Lights Out puzzle

A button is located on each square of a Lights Out cube. The button controls a light on the square which is either on or off. When the button is pushed, the light on that square changes from on to off or from off to on. Moreover, not only is the light on that square reversed when its button is pushed but the lights on its four neighboring squares (top, bottom, left, right) are reversed as well. The four neighboring squares of the middle square of a face lie on the same face as the middle square. Only three neighboring squares of a "side square" (top middle, bottom middle, left middle, right middle) lie on the same face of such a square, with the remaining neighboring square lying on an adjacent face as the middle square. Only two neighboring squares of a "corner" square lie on the same face as that square; the other two neighboring squares lie on two other faces. For example, if all 54 lights are initially on and the button on the top middle square on the front face is pushed, then this light goes off as well as the lights on its four neighboring squares (see Figure 1.1(b)).

One version of the Lights Out Puzzle is to begin with such a cube where all lights are on and to locate a set of buttons such that when each is pressed, all lights are turned out. Two observations are immediate: (1) It is never necessary to press any button more than once. (2) The order in which buttons are pressed does not matter.

As it has been observed elsewhere (see [6] for example), this game has a setting in graph theory. Suppose that each square of a Lights Out cube is a vertex and two vertices are joined by an edge if they correspond to neighboring squares. This results in a 4-regular graph G of order 54. The goal is then to find a collection S of vertices of G, which correspond to the buttons to be pushed, such that every vertex of G is in the closed neighborhood of an odd number of vertices of S. If the buttons corresponding the vertices of S are pushed, then each square will have its light reversed an odd number of times, resulting in all lights on the cube being turned out.

1.2 Graphical Models

The Lights Out Puzzle can in fact be played on any connected graph G on which there is a light at each vertex of G. The game has a solution for the graph G if all lights are initially on and if there exists a collection S of vertices which, when the button on each vertex of S is pressed, all lights of G will be out. This problem has another interpretation. A vertex v of a graph G *dominates* a vertex u if u is in the closed neighborhood $N[v]$ of v (consisting of v and the vertices in the open neighborhood $N(v)$ of v). The two books by Haynes, Hedetniemi and Slater [8, 9] give a detailed description of domination theory. The Lights Out Puzzle has a solution for a given graph G if and only if G contains a set S of vertices such that every vertex of G is dominated by an odd number of vertices of S. Such a dominating set S in a graph G has been called an *odd dominating set* of G. In [15] Sutner showed that every graph has an odd dominating set and so this version of the Lights Out Puzzle is solvable on every graph.

Theorem 1.2.1 (Sutner's Theorem) *The Lights Out Puzzle is solvable on every graph.*

As described in [6], the Lights Out Puzzle is equivalent to beginning with a connected graph G where every vertex is initially assigned the color 1 in $\mathbb{Z}_2 = \{0, 1\}$ (the set of integers modulo 2) or we begin with an initial coloring $c_0 : V(G) \to \mathbb{Z}_2$ such that $c_0(v) = 1$ for each vertex v of G, corresponding to the lights, at all vertices of G being on, and finding a set $S \subseteq V(G)$ and a switching operation $c : V(G) \to \mathbb{Z}_2$ such that

$$c(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S. \end{cases}$$

Then c can also be considered as a vertex coloring of G. A new coloring $\sigma : V(G) \to \mathbb{Z}_2$ is then defined by

$$\sigma(v) = 1 + \sum_{u \in N[v]} c(u).$$

In summary, we begin with a connected graph G in which every vertex light is on (or every vertex v of G is assigned the color $c_0(v) = 1$). A set S of vertices of G is sought such that only the button on each vertex of S is to be pressed (or every vertex v of S is assigned the color $c(v) = 1$ with all other vertices of G colored 0) and finally, after the button on each vertex in S is pressed, all lights of G are turned off (or each vertex of G is assigned the color $\sigma(v) = 0$). Consequently, the goal of the Lights Out Puzzle is therefore to have $\sigma(v) = 0$ in $\mathbb{Z}_2$ for all $v \in V(G)$.

The Lights Out Puzzle can be generalized to one where each on-off light is replaced by a light that can be bright (high beam), dim (low beam) or off; that is, $\mathbb{Z}_2$ can be replaced by $\mathbb{Z}_3$ or, even more generally, $\mathbb{Z}_2$ can be replaced by $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ (the set of integers modulo k) for some integer $k \geq 3$ or perhaps replaced by any set of colors. This gives rise to new graph coloring problems.

1.3 A Graph Coloring Problem

For an integer $k \geq 2$ and a nontrivial connected graph G, let $c_0 : V(G) \to \mathbb{Z}_k$ be an *initial coloring* of G. Define a *switching operation* $c : V(G) \to \mathbb{Z}_k$ where $c(v) \neq 0$ for at least one vertex v of G. Thus c may also be considered a (*switching*) *coloring* of G, where adjacent vertices may be assigned the same color. Then the initial coloring c_0 and the switching coloring c induce a new coloring $\sigma(c_0, c) = \sigma : V(G) \to \mathbb{Z}_k$ of the graph G defined by

$$\sigma(v) = c_0(v) + \sum_{u \in N[v]} c(u) \tag{1.1}$$

where addition is performed in $\mathbb{Z}_k$. In general, the initial coloring c_0 has the property that $c_0(v) \in \{0, 1, 2, \ldots, k-1\}$ for every vertex v of G. Hence $c_0(v) = 0$ means the light is initially off at v while the values $1, 2, \ldots, k-1$ indicate degrees of brightness of the light at v. For the switching coloring c, if $c(v) = 0$, then c contributes no change to the brightness of the light at v or to that of a neighbor of v; while if $c(v) = 1$, then c contributes an increase in one step to the brightness of the light at v and to that of each neighbor of v. More generally, if $c(v) = \ell \geq 1$, then c contributes an increase in ℓ steps to the brightness of the light at v and to that of each neighbor of v (in this case, the button at v will be pressed ℓ times). Consequently, this is equivalent to each light at every vertex of G being a k-way light. The induced coloring σ indicates the final color of the vertices of G. If $\sigma(u) = \sigma(v)$ for every two vertices u and v in G, then the coloring c is called a (*modular*) *monochromatic k-coloring*. If we start with the case where $c_0(v) = 0$ for all $v \in V(G)$, then

$$\sigma(v) = \sum_{u \in N[v]} c(u) \tag{1.2}$$

for each vertex v of G. Hence $\sigma(v) = \sigma_c(v)$ is called the *color sum of a vertex v* with respect to the coloring c. Throughout this work, we will use $\sigma(v)$ for $\sigma_c(v)$ if the coloring c under consideration is clear from the context.

To illustrate these concepts, Figure 1.2 shows two vertex colorings c' and c'' of a graph G, where the color of a vertex assigned by c' or c'' is placed within the vertex and the color sum of a vertex is placed next to the vertex. The coloring c' is a monochromatic 2-coloring since $\sigma_{c'}(v) = 1$ for each $v \in V(G)$ while c'' is a monochromatic 3-coloring since $\sigma_{c''}(v) = 0$ for each $v \in V(G)$. These two examples also illustrate a useful observation.

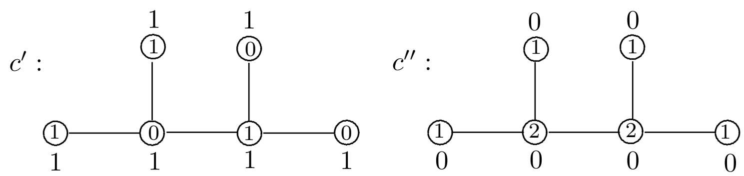


Figure 1.2: Illustrating monochromatic colorings

Observation 1.3.1 *Let u and v be two nonadjacent vertices of a connected graph G such that $N(u) = N(v)$. If c is a modular monochromatic k-coloring of G for some integer $k \geq 2$, then necessarily $c(u) = c(v)$.*

While the most studied colorings in graph theory are the proper colorings (in which every two adjacent vertices are required to be colored differently) and another popular type of colorings are rainbow colorings (in which every two different vertices in a graph, or in a certain kind of subgraph, are required to be colored differently), here we are interested in so-called monochromatic colorings which induce another coloring in some manner where every two vertices are colored the same.

We refer to the books [4, 7] for graph-theoretical notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

Chapter 2

Modular Monochromatic Colorings

2.1 Introduction

In Chapter 1, we mentioned that Sutner showed that every graph has an odd dominating set and so the Lights Out Puzzle is solvable on every graph (namely Sutner's theorem). We begin this chapter by showing that Sutner's theorem implies that every connected graph has a modular monochromatic 2-coloring.

Proposition 2.1.1 *Every connected graph G has a modular monochromatic 2-coloring c such that $\sigma(v) = 1$ for each $v \in V(G)$.*

Proof. We saw in Section 1.1 that solving the Lights Out Puzzle on a connected graph G is equivalent to finding a set S of vertices of G and a coloring $c : V(G) \to \mathbb{Z}_2$ defined by $c(v) = 1$ if $v \in S$ and $c(v) = 0$ if $v \in V(G) - S$ such that $1 + \sum_{u \in N[v]} c(u) = 0$ in $\mathbb{Z}_2$ for every vertex $v \in V(G)$ or $\sigma(v) = \sum_{u \in N[v]} c(u) = 1$ in $\mathbb{Z}_2$ for every vertex $v \in V(G)$. It then follows by Sutner's theorem (Theorem 1.2.1) that such a set S exists and so G has a modular monochromatic 2-coloring c such that $\sigma(v) = 1$ for each $v \in V(G)$. ■

Proposition 2.1.1 gives rise to other questions. For example, for a given connected graph G and an integer $k \geq 2$, does there exist a modular monochromatic k-coloring c of G such that the color sum $\sigma(v)$ of each vertex v is 0? Of course, this is always the case for the (trivial) modular monochromatic k-coloring c where $c(v) = 0$ for each vertex v of G. Our interest, however, lies with whether there is a nontrivial monochromatic k-coloring c, where then $c(v) \neq 0$ for at least one vertex v of G. In this case, we have a k-way light at each vertex of a connected graph and initially the light is off at each vertex. The problem that interests us here is whether it is possible to locate a collection S of vertices of G so that if each vertex of S is pressed an appropriate number of times, then all lights of G will once again be out. More generally, for a given connected graph G and integers $k \geq 2$ and t with $0 \leq t \leq k-1$ does there exist a modular monochromatic k-coloring of G such that the color sum of each vertex is t? This suggests another concept. A monochromatic k-coloring c of a graph G is said to be of *type* t if the induced vertex coloring σ has the property that $\sigma(v) = t$ for each vertex v of G. Such a coloring is also referred to as a (modular) *monochromatic* (k,t)-*coloring* of G. We are particularly interested in (nontrivial) monochromatic $(k,0)$-colorings of graphs, which, as we mentioned, corresponds to all lights being off at the end.

2.2 Some Well-known Classes of Graphs

In this section we study modular monochromatic $(k,0)$-colorings in several well-known classes of graphs, namely, cycles, paths, stars, double stars, complete graphs and complete bipartite graphs. First, we determine, for a given integer $k \geq 2$, all cycles C_n of order $n \geq 3$ that have a modular monochromatic $(k,0)$-coloring. Let $\mathbb{Z}_k^* = \mathbb{Z}_k - \{0\}$ be the set of nonzero elements of $\mathbb{Z}_k$.

Proposition 2.2.1 *For integers k and n with $k \geq 2$ and $n \geq 3$, the cycle C_n of order n has a modular monochromatic $(k, 0)$-coloring if and only if either $n \equiv 0 \pmod 3$ or $k \equiv 0 \pmod 3$. In particular, C_n has a modular monochromatic $(2, 0)$-coloring if and only if $n \equiv 0 \pmod 3$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. It suffices to verify the following.

(a) If $n \equiv 0 \pmod 3$, then C_n has a modular monochromatic $(k, 0)$-coloring;

(b) If $n \equiv 1, 2 \pmod 3$, then C_n has a modular monochromatic $(k, 0)$-coloring if and only if $k \equiv 0 \pmod 3$.

To verify (a), suppose that $n \geq 3$ and $n \equiv 0 \pmod 3$. For each $a \in \mathbb{Z}_k^*$, the coloring $c : V(C_n) \to \mathbb{Z}_k$ defined by

$$c(v_i) = \begin{cases} a & \text{if } i \equiv 1 \pmod 3 \\ k - a & \text{if } i \equiv 2 \pmod 3 \\ 0 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

is a modular monochromatic $(k, 0)$-coloring.

Next, we verify (b). First, suppose that $k \equiv 0 \pmod 3$ and so $k = 3\ell$ for some positive integer ℓ. The coloring that assigns ℓ to each vertex of C_n is a modular monochromatic $(k, 0)$-coloring. For the converse, suppose that $n \geq 3$ and $n \equiv 1, 2 \pmod 3$ and C_n has a modular monochromatic $(k, 0)$-coloring $c : V(C_n) \to \mathbb{Z}_k$ for some integer $k \geq 2$. We show that $k \equiv 0 \pmod 3$. Since $c(v) \in \mathbb{Z}_k^*$ for some vertex v of C_n, we may assume that $c(v_1) = a \neq 0$. Suppose that $c(v_2) = b$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = k - a - b$ (or $c(v_3) = 2k - a - b = k - a - b$ in $\mathbb{Z}_k$). Since $\sigma(v_3) = 0$, we have $c(v_4) = a$. Now, $\sigma(v_4) = 0$

and so $c(v_5) = b$. Continuing in this manner, we have

$$c(v_i) = \begin{cases} a & \text{if } i \equiv 1 \pmod 3 \\ b & \text{if } i \equiv 2 \pmod 3 \\ k-a-b & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

Note that $\sigma(v_1) = c(v_n) + c(v_1) + c(v_2)$ and $\sigma(v_n) = c(v_{n-1}) + c(v_n) + c(v_1)$. First, suppose that $n \equiv 1 \pmod 3$. Then $c(v_1) = c(v_n) = a$, $c(v_{n-1}) = k - a - b$ and $c(v_2) = b$. Hence $\sigma(v_1) = 2a + b \equiv 0 \pmod k$ and $\sigma(v_n) = (k-a-b)+2a = k+a-b \equiv 0 \pmod k$. Thus $a \equiv b \pmod k$. Since $a \in \mathbb{Z}_k^*$, it follows that $k \nmid a$. On the other hand, since $\sigma(v_1) = 2a + b \equiv 0 \pmod k$ and $a \equiv b \pmod k$, we have $3a \equiv 0 \pmod k$ and so $k \mid 3a$. Therefore, $3a = kx$ for some integer x and so $3 \mid kx$. Since 3 is a prime, either $3 \mid k$ or $3 \mid x$. If $3 \mid k$, then $k \equiv 0 \pmod 3$, as desired; while if $3 \nmid k$, then $3 \mid x$ and so $x = 3y$ for some integer y. Thus $3a = k(3y)$ and so $k \mid a$, which is a contradiction. Thus $k \equiv 0 \pmod 3$ in this case. Next, suppose that $n \equiv 2 \pmod 3$. Then $c(v_1) = c(v_{n-1}) = a$ and $c(v_n) = c(v_2) = b$. Hence $\sigma(v_1) = a + 2b \equiv 0 \pmod k$ and $\sigma(v_n) = 2a + b \equiv 0 \pmod k$. This implies $a - b \equiv 0 \pmod k$ and $a \equiv b \pmod k$. Again, $3a \equiv 0 \pmod k$ and, using the same argument as when $n \equiv 1 \pmod 3$, we arrive at $k \equiv 0 \pmod 3$ here as well. ■

By the proof of Proposition 2.2.1, if $k \geq 3$ and $n \geq 4$ are integers with $k \equiv 0 \pmod 3$ and $n \equiv 1, 2 \pmod 3$, then a modular monochromatic $(k, 0)$-coloring of the cycle C_n of order $n \geq 4$ either assigns the color $k/3$ to each vertex of C_n or assigns the color $2k/3$ to each vertex of C_n.

Proposition 2.2.2 *For integers $k, n \geq 2$, the path P_n of order n has a modular monochromatic $(k, 0)$-coloring if and only if $n \equiv 2 \pmod 3$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $n \geq 2$. First, suppose that $n \equiv 2 \pmod 3$. Let $k \geq 2$ and $a \in \mathbb{Z}_k$ such that $a \neq 0$. Define the coloring $c^* : V(P_n) \to \mathbb{Z}_k$ by

$$c^*(v_i) = \begin{cases} a & \text{if } i \equiv 1 \pmod 3 \\ k-a & \text{if } i \equiv 2 \pmod 3 \\ 0 & \text{if } i \equiv 0 \pmod 3. \end{cases} \tag{2.1}$$

Then $\sigma_{c^*}(v_i) = a + (k-a) = k = 0 \in \mathbb{Z}_k$ for $1 \leq i \leq n$ and so c^* is a modular monochromatic $(k,0)$-coloring of P_n.

For the converse, assume, to the contrary, that there are integers k and n where $k \geq 2$, $n \geq 3$ and $n \equiv 0, 1 \pmod 3$ such that P_n has a modular monochromatic $(k,0)$-coloring $c : V(P_n) \to \mathbb{Z}_k$. Suppose that $c(v_1) = a \in \mathbb{Z}_k$. First, assume that $a = 0$. Since $c(v_1) = \sigma(v_1) = 0$, it follows that $c(v_2) = 0$. In general, suppose that $c(v_i) = 0$ for $i = 1, 2, \ldots, t$ where $t < n$. Since $\sigma(v_t) = 0$ and $c(v_{t-1}) = c(v_t) = 0$, it follows that $c(v_{t+1}) = 0$. Thus, $c(v) = 0$ for every vertex v of P_n, which is a contradiction. Hence we may assume that $a \neq 0$. Since $c(v_1) = a$ and $\sigma(v_1) = 0$, it follows that $c(v_2) = k - a$. Now $c(v_1) = a$, $c(v_2) = k - a$ and $\sigma(v_2) = 0$ imply that $c(v_3) = 0$. Since $c(v_2) = k - a$, $c(v_3) = 0$ and $\sigma(v_3) = 0$, it follows that $c(v_4) = a$. Continuing this procedure, we see that c must be the coloring defined in (2.1). However then, $\sigma(v_n) = k - a \neq 0$ if $n \equiv 0 \pmod 3$ and $\sigma(v_n) = a \neq 0$ if $n \equiv 1 \pmod 3$. In either case, a contradiction is produced. ■

By Proposition 2.2.2, if $n \equiv 0, 1 \pmod 3$, then P_n has no modular monochromatic $(k,0)$-coloring for every integer $k \geq 2$. On the other hand, for each integer $k \geq 2$, the coloring $c^* : V(P_n) \to \mathbb{Z}_k$ defined in (2.1) has the property that $\sigma_{c^*}(v_i) = 0 \in \mathbb{Z}_k$ for $1 \leq i \leq n-1$ *except one vertex*, namely v_n where $\sigma_{c^*}(v_n) = k - a$ if $n \equiv 0 \pmod 3$ and $\sigma_{c^*}(v_n) = a$ if $n \equiv 1 \pmod 3$.

Proposition 2.2.3 *For integers k and n with $k \geq 2$ and $n \geq 3$, the star $K_{1,n-1}$ of order n has a modular monochromatic $(k, 0)$-coloring if and only if there exists $a \in \mathbb{Z}_k^*$ such that $a(2-n) \equiv 0 \pmod{k}$. In particular, $K_{1,n-1}$ has a modular monochromatic $(2, 0)$-coloring if and only if n is even.*

Proof. Let $G = K_{1,n-1}$ where $V(G) = \{u, v_1, v_2, \ldots, v_{n-1}\}$ and u is the central vertex of G. First, suppose that there is $a \in \mathbb{Z}_k^*$ such that $a(2-n) \equiv 0 \pmod{k}$. Define the coloring $c^* : V(G) \to \mathbb{Z}_k$ by $c^*(u) = a$ and $c^*(v_i) = k - a$ for $1 \leq i \leq n-1$. Since $a(2-n) \equiv 0 \pmod{k}$, it follows that $\sigma_{c^*}(u) = \sigma_{c^*}(v_i) = 0$ for $1 \leq i \leq n-1$ and so c^* is a modular $(k, 0)$-monochromatic coloring of G.

For the converse, suppose that G has a modular monochromatic $(k, 0)$-coloring $c : V(G) \to \mathbb{Z}_k$ where $k \geq 2$ and $n \geq 3$. Suppose further that $c(u) = a \in \mathbb{Z}_k$. Since $c(u) = a$ and $\sigma(v_i) = 0$ for $1 \leq i \leq n-1$, it follows that $c(v_i) = k - a$ and so $a \neq 0$. Furthermore, $\sigma(u) = a + (n-1)(k-a) = (n-1)k + a(2-n) = 0$ in $\mathbb{Z}_k$, which implies that $a(2-n) \equiv 0 \pmod{k}$. In the case where $k = 2$, it follows that $a = 1$ and so n must be even. ■

In the proof of Proposition 2.2.3, the coloring c^* of the star $K_{1,n-1}$ (defined by $c^*(u) = a$ if u is the central vertex and $c^*(v) = k - a$ if $v \in V(K_{1,n-1}) - \{u\}$) has the property that $\sigma_{c^*}(v) = 0$ for every vertex v of $K_{1,n-1}$ *except possible one vertex*, namely its central vertex u. Furthermore, any modular monochromatic $(k, 0)$-coloring of $K_{1,n-1}$ must assign the same color to each end-vertex of $K_{1,n-1}$ by Observation 1.3.1. A *double star* is a tree of diameter 3. Thus each double star has exactly two non-end-vertices, which are referred to as the *central vertices* of the double star.

Proposition 2.2.4 *Let G be a double star of order at least 5 with central vertices u and v. Suppose that u is adjacent to exactly s end-vertices $u_1, u_2, \ldots, u_s$ and v is adjacent to* *exactly t end-vertices $v_1, v_2, \ldots, v_t$ where $s, t \geq 1$ and $s + t \geq 3$. For an integer $k \geq 2$, the double star G has a modular monochromatic $(k, 0)$-coloring if and only if there exist $a, b \in \mathbb{Z}_k^*$ such that $a + b(1 - t) \equiv 0 \pmod{k}$ and $b + a(1 - s) \equiv 0 \pmod{k}$. In particular, G has a modular monochromatic $(2, 0)$-coloring if and only if s and t are both even.*

Proof. First, suppose that there exist $a, b \in \mathbb{Z}_k^*$ and $a + b(1 - t) \equiv 0 \pmod{k}$ and $b + a(1 - s) \equiv 0 \pmod{k}$. Define a coloring $c : V(G) \to \mathbb{Z}_k$ by $c(u) = a$, $c(u_i) = k - a$ $(1 \leq i \leq s)$, $c(v) = b$ and $c(v_j) = k - b$ $(1 \leq j \leq t)$. Since $\sigma(u) = b + a(1 - s) \equiv 0 \pmod{k}$, $\sigma(u_i) = a + (k - a) \equiv 0 \pmod{k}$ for $1 \leq i \leq s$, $\sigma(v) = a + b(1 - t) \equiv 0 \pmod{k}$ and $\sigma(v_j) = b + (k - b) \equiv 0 \pmod{k}$ for $1 \leq j \leq t$, it follows that c is a modular monochromatic $(k, 0)$-coloring of G.

For the converse, suppose that G has a modular monochromatic $(k, 0)$-coloring c for some integer $k \geq 2$. If $c(u) = c(v) = 0$, then $c(x) = 0$ for all $x \in V(G)$, which is impossible. If exactly one of $c(u)$ and $c(v)$ is not zero, say $c(u) = a \in \mathbb{Z}_k^*$ and $c(v) = 0$, then $c(v_i) = 0$ for $1 \leq i \leq t$, which implies that $\sigma(v) = c(u) = a \neq 0$, a contradiction. Thus, $c(v) \neq 0$ in $\mathbb{Z}_k$. Suppose that $c(v) = b \in \mathbb{Z}_k^*$. Since $c(u) = a$ and $c(v) = b$, it follows that $c(u_i) = k - a$ for $1 \leq i \leq s$ and $c(v_j) = k - b$ for $1 \leq j \leq t$. Then $\sigma_c(u) = a + b + s(k - a) = b + a(1 - s) \equiv 0 \pmod{k}$, $\sigma_c(v) = a + b + t(k - b) = a + b(1 - t) \equiv 0 \pmod{k}$ and $\sigma(u_i) = \sigma(v_j) = 0$ for all i, j with $1 \leq i \leq s$ and $1 \leq j \leq t$. In the case when $k = 2$, it follows that $a = b = 1$ and so s and t must be both even. ■

For complete graphs, modular monochromatic $(k, 0)$-colorings exist for every integer $k \geq 2$.

Proposition 2.2.5 *For integers k and n with $k \geq 2$ and $n \geq 3$, the complete graph K_n of order n has a modular monochromatic $(k, 0)$-coloring.*

Proof. Let u and v be two distinct vertices of K_n. Then the coloring $c : V(K_n) \to \mathbb{Z}_k$ defined by $c(u) = 1$, $c(v) = k - 1$ and $c(x) = 0$ for each $x \in V(K_n) - \{u, v\}$ is a modular monochromatic $(k, 0)$-coloring. ■

Proposition 2.2.6 *For an integer $k \geq 2$, the complete bipartite graph $K_{r,s}$ where $r, s \geq 2$ has a modular monochromatic $(k, 0)$-coloring if and only if there are $a, b \in \mathbb{Z}_k^*$ such that $a + bs \equiv 0 \pmod{k}$ and $b + ar \equiv 0 \pmod{k}$.*

Proof. Let $G = K_{r,s}$ with partite sets U and V where $|U| = r$ and $|V| = s$. First, suppose that c is a modular monochromatic $(k, 0)$-coloring of G. By Observation 1.3.1, $c(x) = c(y)$ if x and y belong to a same partite set of G. We may assume, without loss of generality, that $c(u) = a$ for each $u \in U$ and $c(v) = b$ for each $v \in V$. Hence $\sigma(u) = a + bs$ for each $u \in U$ and $\sigma(v) = b + ar$ for each $v \in V$. This implies that $a, b \in \mathbb{Z}_k^*$ and $a + bs \equiv 0 \pmod{k}$ and $b + ar \equiv 0 \pmod{k}$. For the converse, if there are $a, b \in \mathbb{Z}_k^*$ such that $a + bs \equiv 0 \pmod{k}$ and $b + ar \equiv 0 \pmod{k}$, then the coloring c described above is in fact a modular monochromatic $(k, 0)$-coloring of G. ■

With the aid of Proposition 2.2.6, we are able to determine some complete bipartite graphs that have a modular monochromatic $(k, 0)$-coloring.

Proposition 2.2.7 *For integers $k, r, s \geq 2$, if $r \in \mathbb{Z}_k^*$ such that k and r are relatively prime, then there is a unique $s \in \mathbb{Z}_k^*$ such that $K_{r,s}$ has a modular monochromatic $(k, 0)$-coloring.*

Proof. Suppose that $k \geq 2$ and $r \in \mathbb{Z}_k^*$ such that k and r are relatively prime. It is well-known that r has a unique inverse s in the group $\mathbb{Z}_k$. Thus $rs \equiv 1 \pmod{k}$ and so $k \mid (rs - 1)$. Let U and W be the partite sets of $K_{r,s}$, where $|U| = r \geq 2$ and $|W| = s \geq 2$.

Let a be a fixed nonzero element in $\mathbb{Z}_k^*$. Define a coloring $c : V(K_{r,s}) \to \mathbb{Z}_k$ by $c(u) = a$ if $u \in U$ and $c(w) = -ar$ if $w \in W$. Then $\sigma(u) = a + (-ar)s = a(1 - rs) \equiv 0 \pmod{k}$ for each $u \in U$ and $\sigma(w) = -ar + ar = 0$ in $\mathbb{Z}_k$ for each $w \in W$. Then c is a modular monochromatic $(k, 0)$-coloring and $K_{r,s}$. ■

By Proposition 2.2.7, for each integer $k \geq 2$, if $r, s \geq 2$ are integers such that either $r, s \equiv 1 \pmod{k}$ or $r, s \equiv k - 1 \pmod{k}$, then $K_{r,s}$ has a modular monochromatic $(k, 0)$-coloring. Furthermore, the following is an immediate consequence of Proposition 2.2.7.

Corollary 2.2.8 *If $k \geq 2$ is a prime, then for each integer $r \in \mathbb{Z}_k^*$, there is a unique integer $s \in \mathbb{Z}_k^*$ such that $K_{r,s}$ has a modular monochromatic $(k, 0)$-coloring.*

For each prime k with $2 \leq k \leq 17$, the table in Figure 2.1 shows all complete bipartite graphs $K_{r,s}$ that have a modular monochromatic $(k, 0)$-coloring. For each such graph $K_{r,s}$ having partite sets U and W with $|U| = r$ and $|W| = s$ (where $r = i$ and $s = j$ indicate $r \equiv i \pmod{k}$ and $s \equiv j \pmod{k}$, respectively), a modular monochromatic $(k, 0)$-coloring of $K_{r,s}$ is also given in the table of Figure 2.1 that assigns the color a to each vertex in U and the color b to each vertex in W.

k	r	s	a	b
2	1	1	1	1
3	1	1	1	2
	2	2	1	1
5	1	1	1	4
	2	3	1	3
	4	4	1	1
7	1	1	1	6
	2	4	1	5
	3	5	1	4
	6	6	1	1

k	r	s	a	b
11	1	1	1	10
	2	6	1	9
	3	4	1	8
	5	9	1	6
	7	8	1	4
	10	10	1	1
13	1	1	1	12
	2	7	1	11
	3	9	1	10
	4	10	1	9
	5	8	1	8
	6	11	1	7
	12	12	1	1

k	r	s	a	b
17	1	1	1	16
	2	9	1	15
	3	6	1	14
	4	13	1	13
	5	7	1	12
	8	15	1	9
	10	12	1	7
	11	14	1	6
	16	16	1	1

Figure 2.1: Modular monochromatic $(k, 0)$ colorings of $K_{r,s}$

By Proposition 2.2.6, the complete bipartite graph G has a modular monochromatic $(2,0)$-coloring if and only if each partite set of G contains an odd number of vertices. This result can be extended to complete multipartite graphs in general.

Proposition 2.2.9 *A complete multigraph G has a modular monochromatic $(2,0)$-coloring if and only if at least two partite sets of G contain an odd number of vertices of G.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph for some integer $t \geq 2$. Suppose that the partite sets of G are $U_1, U_2, \ldots, U_t$ where $|U_i| = n_i$ for $1 \leq i \leq t$. First, assume that at least two of $n_1, n_2, \ldots, n_t$ are odd, say n_1 and n_2 are odd. Then the coloring that assigns color 1 to each vertex in $U_1 \cup U_2$ and color 0 to the remaining vertices of G is a modular monochromatic $(2,0)$-coloring of G.

For the converse, suppose that at most one of $n_1, n_2, \ldots, n_t$ is odd and assume, to the contrary, that G has a modular monochromatic $(2,0)$-coloring $c : V(G) \to \mathbb{Z}_2$. Then $c(x) = 1$ for at least one vertex x of G and $c(x) = c(y)$ if x and y belong to a same partite set of G by Observation 1.3.1. If all integers $n_1, n_2, \ldots, n_t$ are even, then $\sigma(x) = 1 \neq 0$ in $\mathbb{Z}_2$ for every vertex x for which $c(x) = 1$. Thus, we may assume that exactly one of $n_1, n_2, \ldots, n_t$ is odd, say n_1 is odd. If $c(x) = 1$ for some $x \in U_1$, then $\sigma(x) = 1 \neq 0$, which is impossible. Thus $c(x) = 0$ for all $x \in U_1$ and so $c(y) = 1$ for each vertex $y \in U_i$ for some i with $2 \leq i \leq t$. However then, $\sigma(y) = 1$ for all $y \in U_i$, which again is impossible. ■

The following observation will be useful to us. For a subgraph H of a graph G and a coloring $c : V(G) \to \mathbb{Z}_k$ where $k \geq 2$, the *restriction c_H of c to H* is the coloring of H defined by $c_H(v) = c(v)$ for each $v \in V(H)$.

Observation 2.2.10 *Let G be a connected graph and let c be a modular monochromatic (k,t)-coloring for some integers k and t with $k \geq 2$ and $0 \leq t \leq k-1$.*

(a) *If G' is a graph obtained by attaching a connected graph at any vertex of G that is colored 0 by c, then G' also has a modular monochromatic (k,t)-coloring.*

(b) *If S is any set of vertices of G that are colored 0 by c, then the restriction of c to $G-S$ is a modular monochromatic (k,t)-coloring of $G-S$ (where it is possible that the restriction of c assigns the color 0 to every vertex of some component of $G-S$).*

With the aid of Observation 2.2.10, we can obtain certain sufficient conditions for a connected graph to have a modular monochromatic $(k,0)$-coloring for every integer $k \geq 2$. For example, if a connected graph G contains a path $P_\ell = (v_1, v_2, \ldots, v_\ell)$ of order $\ell \geq 5$ with $\ell \equiv 2 \pmod 3$ such that $\deg_G v_1 = \deg_G v_\ell = 1$ and $\deg_G v_i = 2$ for each i with $i \equiv 1, 2 \pmod 3$ and $2 \leq i \leq \ell - 1$, then G has a modular monochromatic $(k,0)$-coloring.

2.3 Modular Monochromatic $(2,0)$-Colorings

A modular monochromatic 2-coloring of a graph G assigns the colors in $\mathbb{Z}_2$ to the vertices of G and the color sum of each vertex is computed in $\mathbb{Z}_2$. By Proposition 2.1.1, every connected graph has a modular monochromatic $(2,1)$-coloring. On the other hand, we have seen that not all graphs have modular monochromatic $(2,0)$-colorings (see Proposition 2.2.1 for example). A graph G is called *modular monochromatic* $(2,0)$*-colorable* or simply $(2,0)$-*colorable* if G has a modular monochromatic $(2,0)$-coloring. First, we present a class of connected $(2,0)$-colorable graphs. A graph H is called an *odd-degree graph* if every vertex of H has odd degree. Thus, every odd-degree graph has even order. Since the coloring that assigns the color 1 to each vertex of an odd-degree graph is a modular monochromatic $(2,0)$-coloring, we have the following observation.

Observation 2.3.1 *Every odd-degree graph is* $(2,0)$*-colorable.*

Similarly, a graph H is an *even-degree graph* if every vertex of H has even degree. (Therefore, a connected graph H is an even-degree graph if and only if H is Eulerian.) As we saw in Proposition 2.2.1, an even-degree graph need not be $(2,0)$-colorable. With the aid of the following observation, we present a necessary and sufficient condition for an even-degree graph to be $(2,0)$-colorable.

Observation 2.3.2 *If a connected* $(2,0)$*-colorable graph* G *is not an odd-degree graph, then every modular monochromatic* $(2,0)$*-coloring of* G *must assign the color* 0 *to at least one vertex of* G*.*

Recall that an odd dominating set in a graph G is a dominating set S with the property that every vertex of G is dominated by an odd number of vertices of S.

Proposition 2.3.3 *An even-degree graph* G *is* $(2,0)$*-colorable if and only if* G *contains an odd dominating set* S *such that* $S \neq V(G)$*.*

Proof. First, suppose that G is $(2,0)$-colorable. Then there exists a modular monochromatic $(2,0)$-coloring c of G. Let S be the set of all vertices that are colored 0 by c. Then $S \neq \emptyset$ by Observation 2.3.2. Furthermore, since c is a modular monochromatic $(2,0)$-coloring, $S \neq V(G)$. We show that S is an odd dominating set of G. Let $v \in V(G)$. First, assume that $c(v) = 0$ and so $v \in S$. Since $\sigma(v) = 0$, it follows that v is adjacent to an even number of vertices that are colored 1 by c, where perhaps v is adjacent to no vertex colored 1 by c. Since $\deg v$ is even, v is adjacent to an even number of vertices in S and so v is dominated by an odd number of vertices of S. Next, assume that $c(v) = 1$ and so $v \notin S$. Since $\sigma(v) = 0$, it follows that v is adjacent to an odd number of vertices that are colored 1 by c. Since $\deg v$ is even, v is adjacent to (and therefore dominated by) an odd number of vertices in S. Thus S is an odd dominating set of G.

For the converse, assume that G contains an odd dominating set S such that $S \neq V(G)$. Then $V(G) - S \neq \emptyset$. Define a coloring $c^* : V(G) \to \mathbb{Z}_2$ of G by $c^*(v) = 0$ if $v \in S$ and $c^*(v) = 1$ if $v \in V(G) - S$. It remains to show that c^* is a modular monochromatic $(2, 0)$-coloring of G. Let $v \in V(G)$. First, assume that $c^*(v) = 0$ and so $v \in S$. Since S is an odd dominating set, v is adjacent to an even number of vertices in S. Because $\deg v$ is even, v is adjacent to an even number of vertices not in S and so is adjacent to an even number of vertices colored 1 by c^*. Thus $\sigma_{c^*}(v) = 0$. Next, assume that $c^*(v) = 1$ and so $v \notin S$. Since S is an odd dominating set, v is adjacent to an odd number of vertices in S. Because $\deg v$ is even, v is also adjacent to an odd number of vertices not in S (and colored 1 by c^*) and so $\sigma_{c^*}(v) = 0$. Therefore, c^* is a modular monochromatic $(2, 0)$-coloring of G and G is $(2, 0)$-colorable. ■

Suppose that G is a connected $(2, 0)$-colorable graph that is not an odd-degree graph and let a modular monochromatic $(2, 0)$-coloring of G be given. Then some vertices are colored 1 and some vertices are colored 0. Color each edge of G joining two vertices colored 1 with the color blue (B), color each edge of G joining two vertices of different colors with the color purple (P) and color each edge of G joining two vertices colored 0 with the color red (R). Let F_X be the subgraph of G induced by these edges colored X for each $X \in \{B, P, R\}$. Then $\{F_B, F_P, F_R\}$ is a decomposition of G. We now make some observations concerning the structure of the subgraphs F_B, F_P and F_R and the relationship among these three subgraphs.

(a) The degree of a vertex v in F_B is denoted by $\deg_{F_B} v$. Since $\sigma(v) = c(v) + \deg_{F_B} v = 1 + \deg_{F_B} v \equiv 0 \pmod 2$, it follows that $\deg_{F_B} v$ is odd and so F_B is an odd-degree graph and F_B has even order.

(b) The subgraph F_P is bipartite. For each vertex v in F_P, either $c(v) = 0$ or $c(v) = 1$. If $c(v) = 0$, then $\sigma(v) = \deg_{F_P} v \equiv 0 \pmod 2$ and so $\deg_{F_P} v$ is even, or equivalently, if $\deg_{F_P} v$ is odd, then $c(v) = 1$. Note that the converse is not true, that is, if $c(v) = 1$, then $\deg_{F_P} v$ can be either odd or even.

(c) Since $c(v) = 0$ for each vertex v in F_R, it follows that v is adjacent to an even number of vertices in F_B (where v may be adjacent to no vertices in F_B). Therefore, each vertex colored 0 in F_P has even degree in F_P, as observed in (b).

The observations listed in (a)–(c) give rise to the following two results.

Proposition 2.3.4 *If c is a modular monochromatic $(2,0)$-coloring of a connected graph, then the number of vertices colored 1 by c is even.*

Proof. We show that $|V(F_B)|$ is the number of vertices colored 1 by c. If x is a vertex of G with $c(x) = 1$, then since $\sigma(x) = 0$, it follows that x is adjacent to at least one vertex y colored 1. Thus $xy \in E(F_B)$ and so $x \in V(F_B)$ (and $y \in V(F_B)$ as well). On the other hand, every vertex of F_B is colored 1 by c and so $|V(F_B)|$ is the number of vertices colored 1 by c. Since F_B is an odd-degree graph, F_B has even order and the result follows. ∎

Theorem 2.3.5 *A nontrivial connected graph G is $(2,0)$-colorable if and only if G contains a set S of vertices such that $G-S$ contains at least one odd-degree component and every vertex of S is adjacent to an even number of vertices that belong to odd-degree components of $G-S$.*

Proof. First, suppose that G is $(2,0)$-colorable. Then there exists a modular monochromatic $(2,0)$-coloring c of G. Let F_B be the blue subgraph of G induced by c. Then F_B is an odd-degree graph each of whose components is an odd-degree component. Let $S = V(G) - V(F_B)$ (where it is possible that $S = \emptyset$) and so $G - S = F_B$. Let $v \in S$. Since $c(v) = 0$ and $\sigma(v) = 0$, it follows that v is adjacent to an even number of vertices of F_B (where it is possible that v is adjacent to no vertex of F_B).

For the converse, suppose that G contains a set S of vertices such that $G-S$ contains at least one odd-degree component and every vertex of S is adjacent to an even number of vertices that belong to odd-degree components of $G-S$. Define a coloring $c : V(G) \to \mathbb{Z}_2$ by assigning the color 1 to each vertex that belongs to an odd-degree component of $G-S$ and the color 0 to the remaining vertices of G. We show that c is a modular monochromatic $(2,0)$-coloring of G. Let $x \in V(G)$. First, suppose that $c(x) = 1$. Then x belongs to an odd-degree component G_0 of $G-S$. Thus x is adjacent only to vertices in G_0 or to vertices in S. Thus $\sigma(x) = 1 + \deg_{G_0} x = 0$ in $\mathbb{Z}_2$. Next, suppose that $c(x) = 0$. Thus either (i) $x \in S$ or (ii) $x \notin S$ and so x belongs to a component of $G-S$ that is not an odd-degree component.

- If $x \in S$, then x is adjacent to an even number of vertices belonging to odd-degree components of $G-S$ and so $\sigma(x) = 0$ in $\mathbb{Z}_2$.

- If $x \notin S$ and x belongs to a component of $G-S$ that is not an odd-degree component, then x is not adjacent to any vertex colored 1 and so $\sigma(x) = 0$.

Therefore, c is a modular monochromatic $(2, 0)$-coloring of G, as claimed. ■

The following is a consequence of Theorem 2.3.5.

Corollary 2.3.6 *A nontrivial connected graph G is $(2, 0)$-colorable if and only if G contains an odd-degree subgraph H such that every vertex of G not in H is adjacent to an even number of vertices of H.*

Proof. First, assume that G is $(2, 0)$-colorable. Then there is a modular monochromatic $(2, 0)$-coloring c of G. Then the blue subgraph F_B of G induced by c has the desired property. For the converse, suppose that G has an odd-degree subgraph H such that every vertex of G not in H is adjacent to an even number of vertices of H. Then $S = V(G) - V(H)$ has the property described in Theorem 2.3.5 and the result follows. ■

2.4 Modular Monochromatic $(2, 0)$-Colorings of Trees

By Proposition 2.1.1, every connected graph has a modular monochromatic $(2, 1)$-coloring and so every tree does as well. As with connected graphs, not every tree has a modular monochromatic $(2, 0)$-coloring. For example, it follows by Propositions 2.2.2, 2.2.3 and 2.2.4 that

(a) a nontrivial path of order n is $(2, 0)$-colorable if and only if $n \equiv 2 \pmod 3$;

(b) a star of order at least 4 is $(2, 0)$-colorable if and only if the central vertex of the star is odd;

(c) a double star of order at least 5 is $(2,0)$-colorable if and only if the two central vertices of the double star are odd.

For a graph G, let H be a subgraph of G and let v be a vertex of G not belonging to H. The vertex v is *adjacent to H* if v is adjacent to some vertex of H. If T is a nontrivial tree and F_B is the blue subgraph of T induced by a modular monochromatic $(2,0)$-coloring of T, then each vertex $v \in V(T) - V(F_B)$ is adjacent to at most one vertex in any odd-degree component of F_B and so v is adjacent to an even number of odd-degree components of F_B. Thus, the following is a consequence of Theorem 2.3.5.

Corollary 2.4.1 *A nontrivial tree T is $(2,0)$-colorable if and only if T contains a set S of vertices such that $T - S$ contains at least one odd-degree component and every vertex of S is adjacent to an even number of odd-degree components of $T - S$.*

It can be shown that if T is a path, a star or a double star and T is $(2,0)$-colorable, then every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to each end-vertex of T. This, however, is not the case for trees in general. For example, let T be the tree obtained from the path $P_5 = (v_1, v_2, v_3, v_4, v_5)$ by adding a tree T' of order at least 3 and identifying a non-end-vertex of T' with the vertex v_3. Then the coloring $c : V(T) \to \mathbb{Z}_2$ defined by $c(v_i) = 1$ for $i = 1, 2, 4, 5$ and $c(x) = 0$ for each $x \in V(T) - \{v_1, v_2, v_4, v_5\}$ is a modular monochromatic $(2,0)$-coloring of T. Hence, in this case, a modular monochromatic $(2,0)$-coloring of T could assign the color 0 to end-vertices of T. On the other hand, every modular monochromatic $(2,0)$-coloring of a tree must assign the color 1 to at least two end-vertices of the tree, as we show next.

Theorem 2.4.2 *If c is a modular monochromatic $(2,0)$-coloring of a tree, then c assigns the color 1 to at least two end-vertices of the tree.*

Proof. Suppose that the statement is false. Among those trees having a modular monochromatic $(2,0)$-coloring in which at most one end-vertex is colored 1, let T be one of minimum order and let c be a modular monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most one end-vertex of T. Since every modular monochromatic $(2,0)$-coloring of a path or a star must assign the color 1 to every end-vertex, it follows that T is neither a path nor a star. Because T has at least two end-vertices, there are end-vertices of T that are colored 0 by c. Suppose that $x_1, x_2, \ldots, x_k$ are the end-vertices of T that are colored 0 by c. Then either $x_1, x_2, \ldots, x_k$ are the only end-vertices of T or T has $k+1$ end-vertices. Let y_i be the neighbor x_i in T for $1 \le i \le k$. The vertices $y_1, y_2, \ldots, y_k$ may not be distinct. By assumption, $c(x_i) = 0$ for $1 \le i \le k$. Since $\sigma_c(x_i) = c(y_i)$, it follows that $c(y_i) = 0$ for $1 \le i \le k$. Let $T' = T - \{x_1, x_2, \ldots, x_k\}$. Since T is not a star, T' is a nontrivial tree. Then each end-vertex of T' belongs to the set $\{y_1, y_2, \ldots, y_k\}$ with at most one exception (namely, the end-vertex of T that is colored 1 by c if it exists). Define the coloring $c' : V(T') \to \mathbb{Z}_2$ by $c'(v) = c(v)$ for each $v \in V(T')$. Since $\sigma_{c'}(v) = \sigma_c(v)$ for each $v \in V(T')$ and c is a modular monochromatic $(2,0)$-coloring, it follows that c' is a modular monochromatic $(2,0)$-coloring of T'. Since c' assigns the color 0 to each end-vertex of T' (with at most one exception) and the order of T' is smaller than the order of T, this contradicts the defining property of T. ■

Theorem 2.4.2 is best possible since for every integer $k \ge 3$, there exists a tree T with k end-vertices such that every modular monochromatic $(2,0)$-coloring of T assigns the color 1 to exactly two end-vertices of T. For example, let T_k be the tree obtained from the path $P_5 = (v_1, v_2, v_3, v_4, v_5)$ by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ and joining u_i to

v_3 for $1 \leq i \leq k-2$. The only modular monochromatic $(2,0)$-coloring of T_k assigns the color 1 to v_1, v_2, v_4, v_5 and the color 0 to the remaining vertices of T_k and so every modular monochromatic $(2,0)$-coloring of T_k assigns the color 1 to exactly two end-vertices of T_k.

While every modular monochromatic $(2,0)$-coloring of a tree must assign the color 1 to at least two vertices of the tree, there are trees T for which every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to *every* vertex of T. We now characterize such trees.

Theorem 2.4.3 *Suppose that T is a $(2,0)$-colorable tree. Then every modular monochromatic $(2,0)$-coloring of T assigns the color 1 to every vertex of T if and only if T is an odd-degree tree.*

Proof. By Observation 2.3.2, if T is not an odd-degree tree, then *every* modular monochromatic $(2,0)$-coloring of T must assign the color 0 to at least one vertex of T. For the converse, assume, to the contrary, that there is an odd-degree tree T such that T has a modular monochromatic $(2,0)$-coloring c that assigns the color 0 to some vertex of T. By Theorem 2.4.2 then, c must assign the color 1 to some end-vertex of T. Thus there are adjacent vertices v_1 and v_2 of T such that $c(v_1) = 1$ and $c(v_2) = 0$. Since $\sigma(v_2) = 0$ in $\mathbb{Z}_2$ and v_2 is adjacent to v_1 that is colored 1, it follows that there is $v_3 \in N(v_2) - \{v_1\}$ such that $c(v_3) = 1$. Since $\sigma(v_3) = 0$ in $\mathbb{Z}_2$ and $c(v_3) = 1$, there is an odd number of vertices in $N(v_3)$ that are colored 1. Because $|N(v_3) - \{v_2\}| = \deg_T v_3 - 1$ is even, there exists $v_4 \in N(v_3) - \{v_2\}$ that is colored 0. Continuing in this manner, we arrive at a path $(v_1, v_2, \ldots, v_k)$ in T for some integer $k \geq 2$ such that v_k is an end-vertex of T and $c(v_{k-1}) \neq c(v_k)$, which is impossible. ■

While according to Theorem 2.4.3 every modular monochromatic $(2,0)$-coloring of an odd-degree tree T must assign the color 1 to every vertex of T, this is not the case for odd-degree connected graphs in general. In fact, there are (infinitely many) odd-degree connected graphs G having a modular monochromatic $(2,0)$-coloring that assigns the color 0 to some vertices of G. For example, if G is a complete graph of even order $n \geq 4$, the Petersen graph P or the cube Q_3, then G is a regular odd-degree graph and in each case G has a modular monochromatic $(2,0)$-coloring that assigns the color 0 to some vertices of G. As another example, if G is the graph obtained from $K_n - e$, where $n \geq 4$ is even and $e = uv$, by adding two new vertices u' and v' to $K_n - e$ and joining u' to u and v' to v, then G is not regular and the coloring that assigns the color 1 to u, u', v, v' and the color 0 to the remaining vertices of G is a modular monochromatic $(2,0)$-coloring. Figure 2.2 shows such a modular monochromatic $(2,0)$-coloring for K_4, P, Q_3, and a non-regular graph G, where the solid vertices indicate the vertices that are colored 1 in each case.

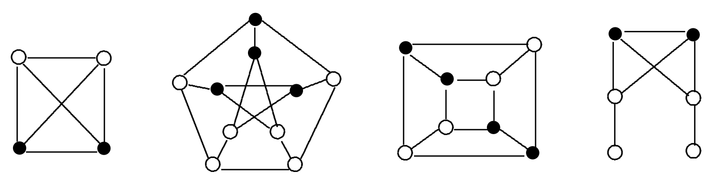


Figure 2.2: Modular monochromatic $(2,0)$-colorings
of four odd-degree graphs

A $(2,0)$-colorable tree T is called $(2,0)$-*minimal* if for every end-vertex v of T, the subtree $T - v$ is not $(2,0)$-colorable. For example, every $(2,0)$-colorable path, star and double star is $(2,0)$-minimal. We describe another class of $(2,0)$-minimal trees.

Theorem 2.4.4 *Every odd-degree tree is* $(2,0)$*-minimal.*

Proof. Assume, to the contrary, that there is an odd-degree tree T that is not $(2,0)$-minimal. Thus T contains an end-vertex v such that $T' = T - v$ has a modular monochromatic $(2,0)$-coloring. Suppose that v is adjacent to the vertex u in T and so u is the only even vertex of T'. Let c be a modular monochromatic $(2,0)$-coloring of T'. Since T' contains an even vertex, c must assign the color 0 to at least one vertex of T'. On the other hand, c must assign the color 1 to some vertex of T'. Let w be a vertex in T' for which $c(u) \neq c(w)$ and $d_{T'}(u,w)$ is minimum among all vertices of T' whose colors are different from $c(u)$ and let P be the $u - w$ path in T'.

First, suppose that $c(u) = 0$ and $c(w) = 1$. It then follows by the defining property of w that w is the only vertex colored 1 in P and so w is adjacent to a vertex x colored 0 in P (where the vertex x may be u). Since $\sigma(w) = 0$ and $c(w) = 1$, there is an odd number of vertices in $N(w) - \{x\}$ that are colored 1. Because $|N(w) - \{x\}|$ is even, there is a vertex $w_1 \in N(w) - \{x\}$ that is colored 0. Since $c(w_1) = \sigma(w_1) = 0$ and w_1 is adjacent to w which is colored 1, it follows that w_1 must be adjacent to a vertex w_2 colored 1. Continuing this procedure, we arrive at a path $(w = w_0, w_1, w_2, \ldots, w_k)$ in T' for some positive integer k such that w_k is an end-vertex of T' and $c(w_{k-1}) \neq c(w_k)$, which is impossible. Next, suppose that $c(u) = 1$ and $c(w) = 0$. Then w is the only vertex colored 0 in P and so w is adjacent to a vertex colored 1 in P. A similar argument as above shows that there are two adjacent vertices w_ℓ and $w_{\ell-1}$ in T' where $\ell \geq 1$ such that w_ℓ is an end-vertex of T' and $c(w_{\ell-1}) \neq c(w_\ell)$, which is impossible. ■

The following is a consequence of Theorem 2.4.4.

Corollary 2.4.5 *If T is a tree having exactly one even vertex, then T is not $(2,0)$-colorable.*

Proof. Suppose that T is a tree having exactly one even vertex u. Let T' be the tree obtained from T by adding a new vertex v and joining v to u. Then T' is an odd-degree tree and v is an end-vertex of T'. It then follows by Proposition 2.4.4 that T' is $(2,0)$-minimal and so $T = T' - v$ is not $(2,0)$-colorable. ■

Next, we investigate the structure of $(2,0)$-colorable trees T for which every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to *every end-vertex* of T.

Theorem 2.4.6 *If T is a $(2,0)$-colorable tree for which every modular monochromatic $(2,0)$-coloring of T assigns the color 1 to each end-vertex of T, then either T is an odd-degree tree or T consists of $k \geq 2$ pairwise disjoint odd-degree trees and $k-1$ vertices of degree 2, each adjacent to two of these odd-degree trees.*

Proof. Suppose that T is a $(2,0)$-colorable tree for which every modular monochromatic $(2,0)$-coloring of T assigns the color 1 to each end-vertex of T. If T is an odd-degree tree, then we have the desired result. Thus we may assume that T is not an odd-degree tree and so T contains at least one even vertex. It then follows by Corollary 2.4.5 that T contains at least two even vertices. We show that T consists of $k \geq 2$ pairwise disjoint odd-degree trees and $k-1$ vertices of degree 2, each adjacent to two of these odd-degree trees.

By Theorem 2.4.3, every modular monochromatic $(2,0)$-coloring must assign the color 0 to at least one vertex of T. Let c be any modular monochromatic $(2,0)$-coloring of T. First, we claim that no two adjacent vertices can be colored 0 by c; for otherwise, suppose that $uv \in E(T)$ where $c(u) = c(v) = 0$. Then $T - uv$ consists of two components T_1 and T_2. Note that either T_1 or T_2 has a vertex colored 1 by c, say the former. Since the restriction c_1 of c to T_1 is a (nontrivial) modular monochromatic $(2,0)$-coloring of T_1, it follows that c_1 can be extended to a modular monochromatic $(2,0)$-coloring c' of T by assigning the color 0 to all vertices of T_2. However then c' must assign the color 0 to at least one end-vertex of T, which is a contradiction.

Let S be the set of vertices colored 0 by c. By the claim above, S is an independent set of vertices of T. Let $v \in S$. We show that $\deg_T v = 2$. Since $\sigma(v) = 0$ and $c(v) = 0$, it follows that v is adjacent to an even number of vertices that are colored 1 and so $\deg_T v = \ell \geq 2$ is even. If $\ell \geq 4$, then, since T is not an odd-degree tree and $T - v$ consists of ℓ nontrivial components, all vertices in $\ell - 2$ (≥ 2) of these components can be recolored 0 producing a modular monochromatic $(2, 0)$-coloring in which at least $\ell - 2$ end-vertices are colored 0, which is a contradiction. Therefore, S is an independent set of vertices of degree 2 in T.

Now $T - S = F_B$ is the blue subgraph induced by c. We saw that F_B is an odd-degree forest. Since T contains at least two even vertices, $|S| \geq 1$ and so F_B has two or more components. Thus F_B consists of k odd-degree trees T_1, T_2, $\ldots$, T_k for some integer $k \geq 2$. Since T is a tree, $T - S$ consists of k components and each vertex of S is adjacent to exactly two components of $T - S$, it follows that $|S| = k - 1$. ■

The class of trees described in Theorem 2.4.6 gives rise to the following definition. Let T' be a tree of order $k \geq 1$, where $V(T') = \{v_1, v_2, \ldots, v_k\}$ and $E(T') = \{e_1, e_2, \ldots, e_{k-1}\}$. The *subdivision graph* $S(T')$ of T' is the tree of order $2k - 1$ obtained from T' by replacing each edge e_i $(1 \leq i \leq k - 1)$ by the vertex u_i which is joined to the two vertices of T' incident with e_i. A tree T is an *odd-degree subdivision tree* if the vertices $v_1, v_2, \ldots, v_k$ of some subdivision graph $S(T')$ of a tree T' of order k correspond to k pairwise disjoint odd-degree trees $T_1, T_2, \ldots, T_k$ in T and $V(T) - \cup_{i=1}^{k} V(T_i)$ consists of an independent set of $k-1$ vertices of T each adjacent to exactly two of the trees $T_1, T_2, \ldots, T_k$. In particular, if $k = 1$, then an odd-degree subdivision tree is an odd-degree tree. With the aid of Theorem 2.4.4, Corollary 2.4.5 and Theorem 2.4.6, we now present a characterization of $(2,0)$-minimal trees.

Theorem 2.4.7 *A tree T is $(2,0)$-minimal if and only if T is an odd-degree subdivision tree.*

Proof. First, suppose that T is a $(2,0)$-minimal tree. Then every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to every end-vertex of T by Observation 2.2.10(b). It then follows by Theorem 2.4.6 that T is an odd-degree subdivision tree.

Suppose that the converse is false. Then there exist odd-degree subdivision trees that are not $(2,0)$-minimal. Among all such trees, let T be one having the minimum number of vertices of degree 2. By Theorem 2.4.4, T is not an odd-degree tree. Thus, suppose that T is constructed from a tree T' of order $k \geq 2$ with $V(T') = \{v_1, v_2, \ldots, v_k\}$, where T consists of k odd-degree trees T_1, T_2, $\ldots$, T_k (such that T_i corresponds to v_i for $1 \leq i \leq k$) and $k-1$ vertices $u_1, u_2, \ldots, u_{k-1}$ of degree 2 in T, each adjacent to exactly two of these k odd-degree trees. Since the coloring that assigns the color 0 to each vertex u_i $(1 \leq i \leq k-1)$ and the color 1 to the remaining vertices of T is a modular monochromatic $(2,0)$-coloring of T, it follows that T is $(2,0)$-colorable. Because T is not $(2,0)$-minimal, there is an end-vertex v of T such that $T - v$ is $(2,0)$-colorable.

Since T' has at least two end-vertices, we may assume, without loss of generality, that v_k is an end-vertex of T' and $v \notin V(T_k)$. Furthermore, we may assume that u_{k-1} is adjacent to T_{k-1} and T_k. Then the tree $T^* = T - (V(T_k) \cup \{u_{k-1}\})$ is an odd-degree subdivision tree constructed from the tree $T' - v_k$ of order $k - 1 \geq 1$, the $k-1$ odd-degree trees T_1, T_2, $\ldots$, T_{k-1} and $k - 2 \geq 0$ vertices $u_1, u_2, \ldots, u_{k-2}$ of degree 2 where u_i is adjacent to exactly two of these $k-1$ odd-degree trees in T^*. Note that v is an end-vertex of T^*. Since T^* is an odd-degree subdivision tree with exactly $k-2$ vertices of degree 2, it then follows by the defining property of T that T^* is $(2,0)$-minimal. On the other hand, we claim, for the end-vertex v of T^*, the tree

$$T^* - v = (T - v) - (V(T_k) \cup \{u_{k-1}\}) \text{ is } (2,0)\text{-colorable.} \tag{2.2}$$

The statement in (2.2) provides a contradiction and therefore verifies the converse.

To verify the claim in (2.2), suppose that u_{k-1} is adjacent to x and y in T where $x \in V(T_k)$ and $y \in V(T_{k-1})$. Let c be a modular monochromatic $(2,0)$-coloring of $T-v$. Since $\sigma(u_{k-1}) = 0$, either c assigns the color 1 to no vertex in $\{x, y, u_{k-1}\}$ or to exactly two vertices in $\{x, y, u_{k-1}\}$. Furthermore, since T_k is an odd-degree tree, x is adjacent to $x_1, x_2, \ldots, x_p$ in T_k for some odd positive integer p. We consider two cases, according to whether $c(x) = 0$ or $c(x) = 1$.

Case 1. $c(x) = 0$. First, suppose that $c(y) = c(u_{k-1}) = 0$. Since the component of $T - yu_{k-1}$ that contains u_{k-1} has exactly one even vertex (namely x), it then follows by Corollary 2.4.5 that the restriction of c to this component must assign the color 0 to each of its vertices. Thus the restriction of c to $T^* - v$ is a (nontrivial) modular monochromatic $(2,0)$-coloring of $T^* - v$ and $T^* - v$ is $(2,0)$-colorable. Next, suppose that $c(y) = c(u_{k-1}) = 1$. Since $\sigma(x) = c(x) = 0$, it follows that x must be adjacent to at least one vertex in T_k that is colored 1, say $c(x_1) = 1$. Let Q be the component of $T_k - x$ that contains x_1. Since $c(x) = 0$, the restriction of c to Q is a modular monochromatic $(2,0)$-coloring of Q. On the other hand, Q contains exactly one even vertex (namely x_1) and so Q is not $(2,0)$-colorable by Corollary 2.4.5, which is a contradiction. Thus the statement in (2.2) holds in this case.

Case 2. $c(x) = 1$. We claim that $c(x_j) = 1$ for each integer j with $1 \le j \le p$. Assume this is not the case, where say $c(x_1) = 0$. Again, let Q be the component of $T_k - x$ that contains x_1. Since $\sigma(x_1) = 0$ and $c(x) = 1$, it follows that x_1 is adjacent to a vertex x_1' in Q that is colored 1. Let Q' be the component of $Q - x_1$ that contains x_1'. Then Q' is a nontrivial tree having exactly one even vertex x_1'. Since $c(x_1') = 1$ and $c(x_1) = 0$, the restriction of c to Q' is a modular monochromatic $(2,0)$-coloring of Q', which is impossible by Corollary 2.4.5. Thus, as claimed, $c(x_j) = 1$ for each integer j with $1 \leq j \leq p$. Since p is odd and $\sigma(x) = 0$, it follows that $c(u_{k-1}) = 0$. Since $\sigma(u_{k-1}) = 0$ and $c(x) = 1$, it follows that $c(y) = 1$ and so c assigns the color 1 to at least one vertex of $T^* - v$. Thus the restriction of c to the tree $T^* - v$ is a modular monochromatic $(2,0)$-coloring of $T^* - v$ and so $T^* - v$ is $(2,0)$-colorable. Thus the statement in (2.2) holds in this case as well. ∎

Equivalently, Theorem 2.4.7 can be restated as follows.

> A tree T is $(2,0)$-minimal if and only if T contains an independent set of vertices of degree 2 whose removal results in an odd-degree forest.

By Observation 2.2.10(b), if T is a $(2,0)$-minimal tree, then every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to every end-vertex of T. Therefore, the following is a consequence of Theorems 2.4.6 and 2.4.7.

Corollary 2.4.8 *For a nontrivial tree T, the following* (1), (2) *and* (3) *are equivalent:*

(1) *T is $(2,0)$-minimal,*

(2) *T is $(2,0)$-colorable and every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to each end-vertex of T,*

(3) *T is an odd-degree subdivision tree.*

Chapter 3

On (2, 0)-Chromatic Numbers

Recall that for a nontrivial connected $(2,0)$-colorable graph G, the *modular monochromatic* $(2,0)$*-chromatic number* or simply $(2,0)$*-chromatic number* of G is the minimum number of vertices colored 1 in a modular monochromatic $(2,0)$-coloring and is denoted by $\chi_{(2,0)}(G)$. A monochromatic $(2,0)$-coloring of G having $\chi_{(2,0)}(G)$ vertices colored 1 is a *minimum monochromatic* $(2,0)$*-coloring*. By Proposition 2.3.4, $\chi_{(2,0)}(G)$ is a positive even integer for each $(2,0)$-colorable graph G. Furthermore, if G is a connected $(2,0)$-colorable graph of order n, then $2 \le \chi_{(2,0)}(G) \le n$.

3.1 On $(2,0)$-Extremal Graphs

We have seen that if G is a connected $(2,0)$-colorable graph of order n, then $2 \le \chi_{(2,0)}(G) \le n$ and $\chi_{(2,0)}(G)$ is even. First, we describe the structure of $(2,0)$-colorable graphs G having $\chi_{(2,0)}(G) = 2$.

Observation 3.1.1 *Let G be a connected $(2,0)$-colorable graph. Then $\chi_{(2,0)}(G) = 2$ if and only if G contains two adjacent vertices u and v such that for each $w \in V(G) - \{u, v\}$, either both u and v belong to $N(w)$ or neither u nor v belong to $N(w)$.*

A $(2,0)$-colorable graph G of order n is called $(2,0)$-*extremal* if $\chi_{(2,0)}(G) = n$. We have seen in Theorem 2.4.3 that odd-degree trees are the only $(2,0)$-extremal trees; that is, for a nontrivial $(2,0)$-colorable tree T of order n, then $\chi_{(2,0)}(T) = n$ if and only if T is an odd-degree tree. Also, we have seen that Theorem 2.4.3 is not true in general. For example, $\chi_{(2,0)}(K_n) = 2$ for each even integer $n \geq 4$ and $\chi_{(2,0)}(P) = \chi_{(2,0)}(Q_3) = 4$ for the Petersen graph P and the cube Q_3. On the other hand, if G is a connected $(2,0)$-colorable graph of order n that is not an odd-degree graph, every modular monochromatic $(2,0)$-coloring of G must assign the color 0 to at least one vertex of G and so then $\chi_{(2,0)}(G) \leq n-1$. Thus if G is a connected graph of order n with $\chi_{(2,0)}(G) = n$, then G is an odd-degree graph. Next, we determine the $(2,0)$-chromatic numbers of two classes of connected odd-degree graphs (namely, the Cartesian products $C_n \square K_2$ of a cycle C_n and K_2 and the coronas of C_n for $n \geq 3$). We will see that each of these two classes of odd-degree graphs contains both $(2,0)$-extremal and non-$(2,0)$-extremal graphs. First, we introduce an additional definition and some preliminary results. For a modular monochromatic k-coloring $c : V(G) \to \mathbb{Z}_k$, $k \geq 2$, define its *complementary coloring* $\overline{c} : V(G) \to \mathbb{Z}_k$ by $\overline{c}(v) = k - 1 - c(v)$. In particular, the complementary coloring $\overline{c}$ of a monochromatic $(2,0)$-coloring is defined by $\overline{c}(v) = 1 - c(v)$ for each $v \in V(G)$. Thus $c(v) = 0$ if and only if $\overline{c}(v) = 1$ and $c(v) = 1$ if and only if $\overline{c}(v) = 0$.

Proposition 3.1.2 *If G is a connected odd-degree graph and c is a monochromatic $(2,0)$-coloring, then either $\overline{c}$ is a trivial coloring that assigns the color 0 to every vertex of G or $\overline{c}$ is a monochromatic $(2,0)$-coloring of G.*

Proof. Suppose that c is a monochromatic $(2,0)$-coloring of an odd-degree graph. If c assigns the color 1 to every vertex of G, then $\overline{c}$ is a trivial coloring that assigns the color 0 to every vertex of G. Thus, we may assume that c assigns the color 0 to some vertices of

G. For $v \in V(G)$, let $N(v) = N_0(v) \cup N_1(v)$ where $N_i(v)$ consists of vertices in $N(v)$ that are colored i by c for $i = 0, 1$. Then $|N(v)| = \deg_G v$ is odd and so $|N_0(v)|$ and $|N_1(v)|$ are of opposite parity. First suppose that $c(v) = 0$. Since $\sigma_c(v) = |N_1(v)| = 0$ in $\mathbb{Z}_2$, it follows that $|N_1(v)|$ is even and so $|N_0(v)|$ is odd. Then $\overline{c}(v) = 1$ and $\sigma_{\overline{c}}(v) = 1 + |N_0(v)| = 0$ in $\mathbb{Z}_2$. Next, suppose that $c(v) = 1$. Since $\sigma_c(v) = 1 + |N_1(v)| = 0$ in $\mathbb{Z}_2$, it follows that $|N_1(v)|$ is odd and so $|N_0(v)|$ is even. Then $\overline{c}(v) = 0$ and $\sigma_{\overline{c}}(v) = |N_0(v)| = 0$ in $\mathbb{Z}_2$. Therefore, $\overline{c}$ is a monochromatic $(2, 0)$-coloring of G. ■

The following is a consequence of Proposition 3.1.2.

Corollary 3.1.3 *If G is a connected odd-degree graph of order n that is not $(2, 0)$-extremal, then $\chi_{(2,0)}(G) \leq n/2$.*

Proof. Let c be a monochromatic $(2, 0)$-coloring of G. Let X_0 and X_1 be the color classes of c, where each vertex in X_i is colored i for $i = 0, 1$ and $|X_0| + |X_1| = n$. Since G is not $(2, 0)$-extremal, $X_0 \neq \emptyset$ and $X_1 \neq \emptyset$. Hence $|X_0| \leq n/2$ or $|X_1| \leq n/2$. By Proposition 3.1.2, $\overline{c}$ is a monochromatic $(2, 0)$-coloring of G. Furthermore, $\overline{c}$ assigns the color 1 to each vertex in X_0 and the color 0 to each vertex in X_1. Therefore, $\chi_{(2,0)}(G) \leq \min\{|X_0|, |X_1|\} \leq n/2$. ■

The upper bound $n/2$ in Corollary 3.1.3 is sharp as we will see in the next two results. We are now prepared to determine the $(2, 0)$-chromatic numbers of the Cartesian products $C_n \,\square\, K_2$ of a cycle C_n and K_2 and the coronas of C_n for $n \geq 3$.

Theorem 3.1.4 *For each integer $n \geq 3$,*

$$\chi_{(2,0)}(C_n \,\square\, K_2) = \begin{cases} n & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $G = C_n \,\square\, K_2$ be the Cartesian product of the n-cycle C_n and K_2 where the two copies of C_n in G are $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ and u_i is adjacent to v_i for $1 \leq i \leq n$. Since G is an odd-degree graph, G is $(2, 0)$-colorable. First, we verify the following claim.

> For a monochromatic $(2, 0)$-coloring c of G, let F_R be the subgraph of G induced by the vertices of colored 0 by c and F_B the subgraph of G induced by the vertices of colored 1 by c. Then neither F_R nor F_B contains a block $P_k \,\square\, K_2$ of length $k \geq 2$.

First, we show that F_R cannot contain a block $P_k \,\square\, K_2$ of length $k \geq 2$. If this is not the case, let $B_k = P_k \,\square\, K_2$ be a maximal block in F_R. We may assume, without loss of generality, that $V(B_k) = \{u_i, v_i : 1 \leq i \leq k\}$, where either $c(u_{k+1}) = 1$ or $c(v_{k+1}) = 1$, say the former. However then, $\sigma(u_k) = 1$, which is impossible. Thus F_R cannot contain a block $P_k \,\square\, K_2$ of length $k \geq 2$. Since the complementary coloring $\overline{c}$ is also a monochromatic $(2, 0)$-coloring of G and $\overline{c}(v) = 1$ if and only if $c(v) = 0$ for each $v \in V(G)$, it follows that F_B cannot contain a block $P_k \,\square\, K_2$ of length $k \geq 2$. Thus the claim holds.

We begin by consider the case when $n \geq 4$ is even. Since the coloring that assigns the color 1 to each u_i and v_i if i is odd and $1 \leq i \leq n - 1$ and the color 0 to the remaining vertices of G is a monochromatic $(2, 0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \leq n$. To show that $\chi_{(2,0)}(G) \geq n$, let c be any monochromatic $(2, 0)$-coloring of G and let F_B be the odd-degree subgraph of G induced by the vertices of colored 1 by c. Thus each component of F_B is an odd-degree graph of even order. We consider two cases, according to the orders of components of F_B.

Case 1. *Some component in F_B is K_2.* We may assume, without loss of generality, that $K_2 = (v_1, v_2)$ or $K_2 = (u_2, v_2)$. First, suppose that $K_2 = (v_1, v_2)$. Then $c(u_1) = c(u_2) = c(v_3) = 0$. Since $\sigma(u_2) = 0$, it follows that $c(u_3) = 1$ and so $c(u_4) = 1$. Then the fact that $c(v_3) = 0$ implies that $c(v_4) = 0$ and $c(u_5) = 0$. Replacing v_1, v_2, v_3, u_1, u_2 by u_3, u_4, u_5, v_3, v_4 and applying the same argument above, we obtain that $c(v_5) = c(v_6) = 1$ and $c(u_5) = c(u_6) = c(v_7) = 0$. Continuing in this manner, we obtain the following: for each integer j with $1 \leq j \leq n$,

- $c(u_j) = 1$ if $j \equiv 0, 3 \pmod 4$ and $c(v_i) = 1$ if $i \equiv 1, 2 \pmod 4$ and
- $c(u_j) = 0$ if $j \equiv 1, 2 \pmod 4$ and $c(v_i) = 0$ if $i \equiv 0, 3 \pmod 4$.

Since $n \geq 4$ is even, either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. If $n \equiv 0 \pmod 4$, then c is in fact a monochromatic $(2, 0)$-coloring of G that assigns the color 1 to exactly n vertices of G and so $\chi_{(2,0)}(G) = n$. If $n \equiv 2 \pmod 4$, then (a) and (b) imply that $c(u_1) = c(u_2) = c(u_{n-1}) = c(u_n) = 0$ and $c(v_1) = c(v_2) = c(v_{n-1}) = c(v_n) = 1$. However then $\sigma(u_n) = 1$ in $\mathbb{Z}_2$ for example and so c is not a monochromatic $(2, 0)$-coloring of G, which is a contradiction.

Next, suppose that $K_2 = (u_2, v_2)$. Then $c(u_1) = c(u_3) = c(v_1) = c(v_3) = 0$. Since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 1$. Continuing in this manner, we obtain that $c(u_i) = c(v_i) = 1$ if i is even and $c(u_i) = c(v_i) = 0$ if i is odd. This implies that c assigns the color 1 to exactly n vertices of G and so $\chi_{(2,0)}(G) = n$.

Case 2. *Every component in F_B has order* 4 *or more.* Let F_R be the subgraph of G induced by the vertices of colored 0 by c. We claim that c must assign the color 1 to at least 4 vertices in each block $B_4 = P_4 \square K_2$ of length 4 in G. For otherwise, we may assume that c assigns the color 1 to at most 3 vertices in the block B_4 where $V(B_4) = \{u_i, v_i : 1 \leq i \leq 4\}$. Thus $c(u_i) = c(v_i) = 0$ for some $i \in \{1, 2, 3, 4\}$. Assume, without loss of generality that

$i = 1$ or $i = 2$. First suppose that $c(u_1) = c(v_1) = 0$. Since F_R does not contain $P_2 \,\square\, K_2$, it follows that $c(u_2) = 1$ or $c(v_2) = 1$. Since every component in F_B has order 4 of more, B_4 contains at least 4 vertices colored 1 by c, a contradiction. Next, $c(u_2) = c(v_2) = 0$. Since F_R does not contain $P_2 \,\square\, K_2$, it follows that c must assign the color 1 to at least one vertex in $\{u_1, v_1\}$ and at least one vertex in $\{u_3, v_3\}$. We may assume that $c(u_3) = 1$. Since every component in F_B has order 4 of more, at least 3 vertices in $\{u_3, u_4, v_3, v_4\}$ must be colored 1 and so B_4 contains at least 4 vertices colored 1 by c, a contradiction. Thus, as claimed, each clock B_4 of length 4 contains at least 4 vertices colored 1 by c.

If $n \equiv 0 \pmod 4$, then c must assign the color 1 to at least n vertices in G and so $\chi_{(2,0)}(G) \geq n$. Thus, we may assume that $n \equiv 2 \pmod 4$, say $n = 4\ell + 2$ for some positive integer ℓ. By the arguments above, c must assign the color 1 to at least 4ℓ vertices in the set $\{u_i, v_i : \ 1 \leq i \leq 4\ell\}$ and to at least one vertex in the set $\{u_{n-1}, u_n, v_{n-1}, v_n\}$. Hence c must assign the color 1 to at least $4\ell + 1$ vertices in G and so $\chi_{(2,0)}(G) \geq 4\ell + 1$. Since $\chi_{(2,0)}(G)$ is an even integer, $\chi_{(2,0)}(G) \geq 4\ell + 2 = n$. Therefore, $\chi_{(2,0)}(G) = n$ in this case as well.

We now consider the case when $n \geq 3$ is odd. Since G is an odd-degree graph of order $2n$, it follows that G is $(2,0)$-colorable and $\chi_{(2,0)}(G) \leq 2n$. Let c be a minimum monochromatic $(2,0)$-coloring of G. Observe that the argument presented in Case 1 (where n is even) shows that the subgraph F_B can contain a component of order 2 only when n is even; that is, if n is odd, then F_B cannot contain a component of order 2. Thus each component of F_B has order at least 4. We claim that $F_B = G$. If this is not the case, then $4 \leq |V(F)| \leq 2n - 2$ for each component F of F_B. Let F' be a component of smallest order in F_B. First suppose that $6 \leq |V(F')| \leq 2n - 2$. Since F' is an odd-degree subgraph in G, each of whose vertices has degree 1 or 3, it follows that F' must contain two adjacent vertices of degree 3 and so F' contains $P_2 \square K_2$, as a subgraph, which is impossible. Next, suppose that $|V(F')| = 4$. Since the subgraph F_B cannot contain a block $P_2 \square K_2$, it follows that $F' = K_{1,3}$, where say $V(F') = \{u_2, v_1, v_2, v_3\}$ and v_2 is the central vertex of F'. This implies that $c(u_1) = c(u_3) = 0$. Since $\sigma(x) = 0$ for each $x \in \{u_1, v_1, u_3, v_3\}$, it follows that $c(y) = 0$ for each $y \in \{u_4, v_4, u_n, v_n\}$. Furthermore, since $\sigma(u_4) = \sigma(v_4) = 0$, it follows that $c(u_5) = 0$ and $c(v_5) = 1$. (See Figure 3.1.) For each $i = 1, 2, \ldots, k$, where $k = \lfloor n/4 \rfloor$, let $B_i = P_4 \square K_2$ be the block of length 4 in G with $V(B_i) = \{u_{4(i-1)+p}, v_{4(i-1)+p} : p = 1, 2, 3, 4\}$. As we saw, the vertices in the block B_1 are colored as shown in Figure 3.1.

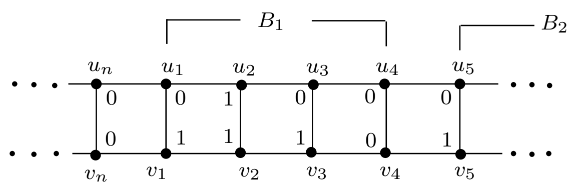


Figure 3.1: Illustrating the colors of the vertices of B_1

Since $c(u_4) = c(u_n)$, $c(v_4) = c(v_n)$, $c(u_5) = c(u_1)$ and $c(v_5) = c(v_1)$, we can apply this argument to B_2, B_3 and so on, concluding that the vertices in each B_i $(1 \le i \le k)$ must be colored the same as B_1; that is, for each integer j with $1 \le j \le n$,

- $c(u_j) = 1$ if $j \equiv 2 \pmod 4$ and $c(u_j) = 0$ if $j \equiv 0, 1, 3 \pmod 4$;
- $c(v_j) = 1$ if $j \equiv 1, 2, 3 \pmod 4$ and $c(v_j) = 0$ if $j \equiv 0 \pmod 4$.

Since n is odd, either $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$. In each case, $c(v_n) = 1$, which contradicts the fact that $c(v_n) = 0$. Therefore, $F_B = G$ and $\chi_{(2,0)}(G) = 2n$. ■

For a graph H, the *corona* $\mathrm{cor}(H)$ of H is that graph obtained from H by adding a pendant edge to each vertex of H. The following observation will be useful to us in determining the $(2, 0)$-chromatic number of the corona of a cycle.

Observation 3.1.5 *If uv is a pendant edge in a $(2,0)$-colorable graph G, then $c(u) = c(v)$ for every modular monochromatic $(2,0)$-coloring c of G.*

Theorem 3.1.6 *For each integer $n \geq 3$,*

$$\chi_{(2,0)}(\mathrm{cor}(C_n)) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ where $n \geq 3$ and let $G = \mathrm{cor}(C_n)$ be the corona of C_n obtained from C_n by adding the pendant edge u_iv_i at v_i for each i with $1 \leq i \leq n$. First, suppose that $n \geq 3$ is odd. Since G is an odd-degree graph of order $2n$, it follows that G is $(2,0)$-colorable and $\chi_{(2,0)}(G) \leq 2n$. Next, we show that every modular monochromatic $(2,0)$-coloring must assign the color 1 to each vertex of G. Assume, to the contrary, that there is a modular monochromatic $(2,0)$-coloring c of G that assigns the color 0 to some vertex v of G. Since c is a modular monochromatic $(2,0)$-coloring, c must assign the color 1 to at least one vertex of G. We claim that c cannot assign the same color to two or more consecutive vertices of C_n. If this is not the case, then we may assume that $c(v_1) = c(v_2) = \cdots = c(v_p) = a$ and $c(v_{p+1}) = b$ where $p \geq 2$ and $a \neq b$ in $\mathbb{Z}_2$. It then follows by Observation 3.1.5 that $c(u_1) = c(u_2) = \cdots = c(u_p) = a$ and $c(u_{p+1}) = b$. If $a = 0$ and $b = 1$, then $\sigma(v_p) = 0+0+0+1 \neq 0$; while if $a = 1$ and $b = 0$, then $\sigma(v_p) = 1+1+1+0 \neq 0$, which is a contradiction in each case. Hence, as claimed, c cannot assign the same color to two or more consecutive vertices of C_n. This implies that the colors $c(v_1), c(v_2), \ldots, c(v_n)$ alternate 0 and 1. However, this is impossible since n is odd. Therefore, $\chi_{(2,0)}(G) = 2n$ for all odd integers n.

Next, suppose that $n \geq 4$ is even. By the argument above, if c^* is a modular monochromatic $(2,0)$-coloring of G that assigns the color 0 to at least one vertex of G, then the colors $c^*(v_1), c^*(v_2), \ldots, c^*(v_n)$ alternate 0 and 1 and $c^*(v_i) = c^*(u_i)$ for $1 \leq i \leq n$, which implies that $\chi_{(2,0)}(G) \geq n$. Furthermore, the coloring that assigns the color 0 to both u_i and v_i for each even i with $1 \leq i \leq n$ is a modular monochromatic $(2,0)$-coloring and so $\chi_{(2,0)}(G) \leq n$. Thus $\chi_{(2,0)}(G) = n$ for all even integers n. ■

By Theorems 3.1.4 and 3.1.6, the upper bound $n/2$ in Corollary 3.1.3 is sharp as we mentioned before. The graph $\mathrm{cor}(C_n)$, $n \geq 3$, contains cut-vertices.

3.2 Some Non-$(2,0)$-Extremal 2-Connected Graphs

In this section, we show that there are infinitely many odd-degree 2-connected graphs that are not $(2,0)$-extremal. In order to do this, we first present the following result.

Proposition 3.2.1 *If c is a monochromatic $(2,0)$-coloring of a connected graph G, then c must assign the color 1 to an even number of even vertices of G. In particular, if G has exactly one even vertex x, then $c(x) = 0$.*

Proof. For a monochromatic $(2,0)$-coloring c of G, let $\sigma_c(G) = \sum_{v \in V(G)} \sigma(v)$. Since $\sigma(v) = 0$ in $\mathbb{Z}_2$ for each $v \in V(G)$, it follows that $\sigma_c(G) = 0$ in $\mathbb{Z}_2$ and $\sigma_c(G)$ is even. Observe that a vertex colored 0 contributes 0 to $\sigma_c(G)$; while each vertex v colored 1 contributes $1 + \deg_G v$ to $\sigma_c(G)$ (namely, $c(v)$ contributes 1 to each color sum $\sigma(u)$ for every $u \in N[v]$). Let V_1 be the set of odd vertices of G and V_2 the set of even vertices of G such that each vertex in $V_1 \cup V_2$ is colored 1 by c. Then

$$\sigma_c(G) = \sum_{v \in V_1} (1 + \deg_G v) + \sum_{v \in V_2} (1 + \deg_G v).$$

Since $\sigma_c(G)$ and $\sum_{v \in V_1} (1 + \deg_G v)$ are both even, it follows that $\sum_{v \in V_2} (1 + \deg_G v)$ is even and so $|V_2|$ is even. ■

For example, let G be the graph obtained from $K_4 - e$ by adding a pendant edge at a vertex degree 2 in $K_4 - e$. Then G has exactly one vertex of even degree. By Proposition 3.2.1, every monochromatic $(2,0)$-coloring must assign the color 0 to the vertex of even degree in G. Since the coloring that assigns the color 1 only to the two vertices of degree 3 in $K_4 - e$ is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) = 2$ and G is $(2,0)$-colorable.

Theorem 3.2.2 *For each integer $n \geq 3$, the wheel $W_n = C_n \vee K_1$ is $(2,0)$-colorable if and only if $n \not\equiv 2, 4 \pmod 6$ and*

$$\chi_{(2,0)}(C_n \vee K_1) = \begin{cases} \frac{2n}{3} & \textit{if } n \equiv 0 \pmod 6 \\ n+1 & \textit{if } n \equiv 1, 5 \pmod 6 \\ \frac{n}{3}+1 & \textit{if } n \equiv 3 \pmod 6. \end{cases} \tag{3.1}$$

Proof. For an integer $n \geq 3$, let $G = C_n \vee K_1$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $V(K_1) = \{v\}$. First, suppose that G is $(2,0)$-colorable. Assume, to the contrary, that $n \equiv 2, 4 \pmod 6$. Let c be a modular monochromatic $(2,0)$-coloring of G. Since n is even, v is the only even vertex of G and so $c(v) = 0$ by Proposition 3.2.1. However then, the restriction of c to the cycle $C_n = G - v$ is a modular monochromatic $(2,0)$-coloring of C_n, which contradicts to Proposition 2.2.1 since $n \not\equiv 0 \pmod 3$.

For the converse, assume that $n \not\equiv 2, 4 \pmod 6$. If n is odd, then G is an odd-degree graph and so G is $(2,0)$-colorable. Thus, we may assume that n is even and so $n \equiv 0 \pmod 6$. Since the coloring $c : V(G) \to \mathbb{Z}_2$ defined by

$$c(x) = \begin{cases} 1 & \text{if } x = v_i \text{ where } i \equiv 1, 2 \ (\text{mod } 3) \text{ and } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

is a modular monochromatic $(2,0)$-coloring of G, it follows that G is $(2,0)$-colorable. It remains to verify (6.2). First, suppose that $n \equiv 0 \pmod 6$. Since the modular monochromatic $(2,0)$-coloring described in (3.2) assigns the color 1 to exactly $2n/3$ vertices of G, it follows that $\chi_{(2,0)}(G) \le 2n/3$.

Let c be a minimum modular monochromatic $(2,0)$-coloring of G. Since v is the only even vertex in G, it follows that $c(v) = 0$. We may assume, without loss of generality, that $c(v_1) = 1$. Since $\sigma(v_1) = 0$, exactly one of $c(v_n)$ and $c(v_2)$ is 0, say $c(v_n) = 0$ and $c(v_2) = 1$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 0$ and then $c(v_4) = 1$. Continuing in this procedure, we have $c(v_i) = 1$ if $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 0$ if $i \equiv 0 \pmod 3$. Thus c assigns the color 1 to at least $2n/3$ vertices of G and so $\chi_{(2,0)}(G) \ge 2n/3$. Hence $\chi_{(2,0)}(G) = 2n/3$ when $n \equiv 0 \pmod 6$.

Next, suppose that $n \equiv 1, 5 \pmod 6$ and we show that $\chi_{(2,0)}(G) = n + 1$. Assume, to the contrary, that $\chi_{(2,0)}(G) \le n$. Let c be a minimum modular monochromatic $(2,0)$-coloring of G. Since $n \not\equiv 0 \pmod 3$, $c(v) = 1$; for otherwise there is a monochromatic $(2,0)$-coloring of $C_n = G - v$ where $n \not\equiv 0 \pmod 3$, which is not possible by Proposition 2.2.1 and so $c(v_i) = 0$ for some i with $1 \le i \le n$, say $c(v_1) = 0$. Since $\sigma(v_1) = 0$, exactly one of $c(v_n)$ and $c(v_2)$ is 0, say $c(v_n) = 1$ and $c(v_2) = 0$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 1$ and then $c(v_4) = 0$. Continuing in this procedure, we have $c(v_i) = 0$ if $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 1$ if $i \equiv 0 \pmod 3$. Since $n \equiv 1, 5 \pmod 6$, it follows that $n \equiv 1, 2 \pmod 3$ and so $c(v_n) = 0$, which is a contradiction. Therefore, $\chi_{(2,0)}(G) = n + 1$ when $n \equiv 1, 5 \pmod 6$.

Finally, suppose that $n \equiv 3 \pmod 6$ and we show that $\chi_{(2,0)}(G) = 1 + n/3$. Since the coloring $c : V(G) \to \mathbb{Z}_2$ defined by

$$c(x) = \begin{cases} 1 & \text{if } x = v \text{ or } x = v_i \text{ where } i \equiv 1 \pmod 3 \text{ and } 1 \le i \le n \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

is a modular monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \le 1 + n/3$. To show that $\chi_{(2,0)}(G) \ge 1 + n/3$, let c be a minimum modular monochromatic $(2,0)$-coloring of G. First, suppose that $c(v) = 0$. Then the restriction of c to C_n is a monochromatic $(2, 0)$-coloring of C_n. Since $\chi_{(2,0)}(C_n) = 2n/3$ when $n \equiv 0 \pmod 3$, a contradiction is produced. Next, suppose that $c(v) = 1$. Since $\chi_{(2,0)}(G) \leq 1 + n/3$, it follows that $c(v_i) = 0$ for some i with $1 \leq i \leq n$, say $c(v_1) = 0$. Since $\sigma(v_1) = 0$, exactly one of $c(v_n)$ and $c(v_2)$ is 0, say $c(v_n) = 1$ and $c(v_2) = 0$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 1$. Since $\sigma(v_3) = 0$, this implies that $c(v_4) = 0$ and so $c(v_5) = 0$. Continuing in this procedure, we have $c(v_i) = 0$ if $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 1$ if $i \equiv 0 \pmod 3$. Hence c assigns the color 1 to at least $1+n/3$ vertices of G and so $\chi_{(2,0)}(G) \geq 1+n/3$. Therefore, $\chi_{(2,0)}(G) = 1+n/3$ when $n \equiv 3 \pmod 6$. ■

Next, we show that for each even integer $n \geq 4$ there exists a 2-connected graph G of order n such that $\chi_{(2,0)}(G) = n$. In order to show this, we now describe a class of $(2, 0)$-extremal graphs F_n of order n for each integer n where $n \equiv 4 \pmod 6$. Let $n = 6\ell + 4$ for some positive integer ℓ. Then F_n is the graph of order n obtained from $C_{2(2\ell+1)}=(w_1,\ z_1,\ w_2,\ z_2,\ \ldots,\ w_{2\ell+1},\ z_{2\ell+1}, w_1)$ and $2\ell + 2$ vertices $x, y_1, y_2, \ldots, y_{2\ell+1}$ by adding edges y_iw_i, y_iz_i and xy_i for each $1 \leq i \leq 2\ell + 1$.

Theorem 3.2.3 *If n is an integer with $n \geq 10$ and $n \equiv 4 \pmod 6$, then $\chi_{(2,0)}(F_n) = n$.*

Proof. Let $n = 6\ell + 4$ for some positive integer ℓ and let F_n be constructed as described above. Since F_n is an odd-degree graph, F_n is $(2, 0)$-colorable. Thus $\chi_{(2,0)}(F_n) \leq 6\ell+4$ since the order of F_n is $6\ell+4$. We claim that the only monochromatic $(2, 0)$-coloring of F_n is the coloring that assigns the color 1 to every vertex of F_n and so $\chi_{(2,0)}(F_n) = 6\ell + 4$. Assume, to the contrary that, this is not the case. Then there is a monochromatic $(2, 0)$-coloring c of F_n such that $c(u) = 0$ and $c(v) = 1$ for some vertices u and v of F_n. Since F_n is an odd-degree graph, the nontrivial complementary coloring $\bar{c}$ of c is also a monochromatic

$(2,0)$-coloring of F_n. Thus, without loss of generality, let $c(x) = 0$. Now we claim that, for each $1 \le i \le 2\ell+1$, $c(y_i) = 0$. To see this, first observe that, since $\deg x$ is odd and $c(x) = 0$ there is at least one vertex y_i such that $c(y_i) = 0$. If, for all $1 \le i \le 2\ell+1$, $c(y_i) = 0$ then the claim holds. Thus we may assume that there is one vertex y_j where $i \ne j$ such that $c(y_j) = 1$. It then follows that there are two consecutive numbers i and j where $j = i+1$ such that $c(y_i) = 0$ and $c(y_{i+1}) = 1$. Since $\sigma(y_i) = 0$, either $c(w_i) = c(z_i) = 0$ or $c(w_i) = c(z_i) = 1$. If $c(w_i) = c(z_i) = 0$ then since $\sigma(z_i) = 0$, it follows that $c(w_{i+1}) = 0$ and so $c(z_{i+1}) = 1$ since $\sigma(w_{i+1}) = 0$. Now since $\sigma(z_{i+1}) = 0$, $c(w_{i+2}) = 0$ and since $\sigma(w_{i+2}) = 0$, it follows that $c(y_{i+2})$ and $c(z_{i+2})$ are not the same. This, however, implies that $\sigma(y_{i+2}) \ne 0$, which is a contradiction. Hence we may assume $c(w_i) = c(z_i) = 1$. Then since $\sigma(z_i) = 0$, it follows that $c(w_{i+1}) = 0$. By using exact same argument as the case where $c(w_i) = c(z_i) = 0$, the contradiction is produced. Thus, as claimed, for each $1 \le i \le 2\ell+1$, $c(y_i) = 0$. Next, we will show that, for each $1 \le i \le 2\ell+1$, $c(w_i) = c(z_i) = 0$. Observe that since for each $1 \le i \le 2\ell+1$, $\sigma(y_i) = 0$, it follows that $c(w_i) = c(z_i)$. If, for each $1 \le i \le 2\ell+1$, $c(w_i) = c(z_i) = 1$ then $\sigma(w_i) = \sigma(z_i) = 1$, which is impossible. Now if there are two consecutive numbers i and j where $j = i+1$ such that $c(w_i) = c(z_i) = 0$ and $c(w_j) = c(z_j) = 1$ then $\sigma(z_i) = 1$, which is again a contradiction.

Thus c assigns the color 0 to every vertex of F_n, which is not possible. Hence the only nontrivial coloring of F_n is the coloring that assigns the color 1 to every vertex of F_n and so $\chi_{(2,0)}(F_n) \le 6\ell+4$. ∎

Theorem 3.2.4 *For each even integer $n \ge 4$, there exists a 2-connected graph G of order n such that $\chi_{(2,0)}(G) = n$ if and only if $n \ne 4$.*

Proof. First, let $n = 4$ and G a 2-connected graph of order 4. Since if $\chi_{(2,0)}(G) = n$ then G is an odd degree graph. Observe that there is only one odd degree 2-connected graph of order 4; namely K_4. However, $\chi_{(2,0)}(K_4) = 2$. Thus there is no 2-connected graph G of order 4 such that $\chi_{(2,0)}(G) = 4$.

For the converse, assume $n \neq 4$ and so $n \geq 6$. If $n \equiv 0, 2 \pmod 6$, then let G be the wheel $W_{n-1} = C_{n-1} \vee K_1$. Thus G is 2-connected and $\chi_{(2,0)}(G) = n$ by Theorem 3.2.2; while if $n \equiv 4 \pmod 6$, then let $G = F_n$ be the 2-connected graph defined in the proof of Theorem 3.2.3 and so $\chi_{(2,0)}(F_n) = n$. ■

We have seen that if G is a connected graph of order n with $\chi_{(2,0)}(G) = a$, then a is even and $2 \leq a \leq n$. Next, we show that every pair a, n of integers where a is even and $2 \leq a \leq n$ is realizable as the $(2,0)$-chromatic number and the order of some connected graph, respectively.

Theorem 3.2.5 *For each pair a, n of integers where a is even and $2 \leq a \leq n$, there is a connected graph G of order n such that $\chi_{(2,0)}(G) = a$.*

Proof. For $a = 2$, let G_2 be the graph obtained from $P_2 = (u, v)$ by adding $n - 2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ and joining each w_i $(1 \leq i \leq n-2)$ to both u and v. For $a = n$ which is even, let G_n be an odd-degree tree of order n. By Observation 3.1.1 and Theorem 2.4.3, the statement is true for $a = 2$ and $a = n$. Thus, we may assume that $4 \leq a \leq n - 1$ and $n \geq 5$. Consider the complete bipartite graph $K_{2,n-a+1}$ with partite set $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{n-a+1}\}$. Let G_a be the graph of order n obtained from $K_{2,n-a+1}$ by adding $a - 3$ new vertices $z_1, z_2, \ldots, z_{a-3}$ and joining each vertex z_i $(1 \leq i \leq a - 3)$ to y_1. It remains to show that $\chi_{(2,0)}(G_a) = a$.

Let $Z = \{z_1, z_2, \ldots, z_{a-3}\}$. Since the coloring that assigns the color 1 to each vertex in $X \cup \{y_1\} \cup Z$ and the color 0 to the remaining vertices of G_a is a modular monochromatic $(2,0)$-coloring, it follows that $\chi_{(2,0)}(G_a) \leq |X \cup \{y_1\} \cup Z| = a$. To show that $\chi_{(2,0)}(G_a) \geq a$, let c be a modular monochromatic $(2,0)$-coloring of G_a. By Observations 1.3.1 and 3.1.5 if $u, v \in X$, $u, v \in Y - \{y_1\}$ or $u, v \in Z \cup \{y_1\}$, then $c(u) = c(v)$. First, suppose that $c(y) = 1$ for all $y \in Y - \{y_1\}$. Since $\sigma_c(y) = 0$ for each $y \in Y - \{y_1\}$, it follows that $c(x) = 1$ for each $x \in X$. However then, $\sigma_c(y) = 1$ for each $y \in Y - \{y_1\}$, which is impossible. Hence $c(y) = 0$ for all $y \in Y - \{y_1\}$. First, suppose that $c(x) = 0$ for each $x \in X$. Then $c(y_1)$ must be 0 and so $c(z) = 0$ for all $z \in Z$, which is impossible as c must assign the color 1 to at least one vertex of G_a. Hence $c(x) = 1$ for each $x \in X$. This forces $c(y_1) = 1$ and then $c(z) = 1$ for all $z \in Z$. Therefore, c must assign the color 1 to each vertex in $X \cup \{y_1\} \cup Z$ and so $\chi_{(2,0)}(G_a) \geq |X \cup \{y_1\} \cup Z| = a$. ■

3.3 Comparing Two Numbers

In this section, we compare $(2,0)$-chromatic numbers with a well-studied domination concept, namely, even domination numbers in graphs. A dominating set S of a graph G is an *even dominating set* of G if every vertex of G is dominated by an even number of vertices in S. The minimum number of vertices in an even dominating set of G is the *even domination number* $\gamma_e(G)$ of G. In the proof of Proposition 3.2.5, the set of vertices colored 1 in each graph is in fact the even dominating set of the graph. On the other hand, let F_a be the graph of order n obtained from $G - \{y_3, y_4, \ldots, y_{n-a+1}\}$ and a connected graph H of order $n - a - 1$ by joining a vertex of H to the vertex y_2. Then an argument similar to the one in the proof of Proposition 3.2.5 show that $\chi_{(2,0)}(F_a) = a$. In this case, the set of vertices colored 1 in F_a is not an even dominating set of the graph for each integer a

with $4 \leq a \leq n-1$. The following result describes the relationship between $(2,0)$-colorable graphs and graphs having an even dominating set.

Proposition 3.3.1 *Let G be a connected graph.*

(a) *If G has a modular monochromatic $(2,0)$-coloring c such that the set S of vertices colored 1 by c is a dominating set of G, then S is an even dominating set of G.*

(b) *If G has an even dominating set D, then the coloring that assigns the color 1 to each vertex in D and the color 0 to the remaining vertices of G is a modular monochromatic $(2,0)$-coloring of G.*

Proof. To verify (a), we show that every vertex v in G is dominated by an even number of vertices in S. If $v \in S$, then since $c(v) = 1$ and $\sigma(v) = 0$ it follows that v is adjacent to an odd number of vertices of S and so v is dominated by an even number of vertices in S. If $v \notin S$, then $c(v) = \sigma(v) = 0$ and so v is adjacent to an even number of vertices in S. Since S is a dominating set of G, it follows that v must be adjacent to some vertex in S. Thus v is dominated by an even number of vertices in S.

To verify (b), let c' be the coloring that assigns the color 1 to each vertex in D and the color 0 to the remaining vertices of G. Let $v \in V(G)$. If $v \in D$, then $c(v) = 1$ and $\sigma(v) = 0$ since v is adjacent to an odd number of vertices of D. If $v \in V(G) - D$, then $c(v) = 0$ and $\sigma(v) = 0$ since v is adjacent to an even number of vertices of D. ■

By Proposition 3.3.1(b), if a connected graph G has an even dominating set, then G is $(2,0)$-colorable. However, the converse is not true. For example, let T be the tree obtained from a path $(v_1, v_2, v_3, v_4, v_5)$ of order 5 and a path (u_1, u_2, u_3) of order 3 by adding the edge u_3v_3. Then T is $(2,0)$-colorable where the coloring that assigns the color 1 to each vertex in

$\{v_1, v_2, v_4, v_5\}$ and the color 0 to the remaining vertices of T is a modular monochromatic $(2,0)$-coloring of T. Since this coloring is the unique modular monochromatic $(2,0)$-coloring of T and the set of vertices colored 1 is not an even dominating set of T, it follows by Proposition 3.3.1(b) that T has no even dominating set.

We have seen that if G is a connected graph that has an even dominating set, then $2 \leq \chi_{(2,0)}(G) \leq \gamma_e(G)$ by Proposition 3.3.1(b) and $\chi_{(2,0)}(G)$ and $\gamma_e(G)$ are both even integers. In fact, more can be said. First, we present two lemmas, the first of which is an observation.

Lemma 3.3.2 *If G is a graph having an even dominating set and uv is a pendant edge of G, then u and v belong to every even dominating set of G.*

Lemma 3.3.3 *If H is a $(2,0)$-colorable graph and G is the graph obtained from H by adding an even number of pendant edges at some vertex of H, then G is $(2,0)$-colorable and $\chi_{(2,0)}(H) \leq \chi_{(2,0)}(G)$.*

Proof. Suppose that G is obtained from H by adding k pendant edges vw_i $(1 \leq i \leq k)$ at the vertex v of H, where $k \geq 2$ is even. Let c be a modular monochromatic $(2,0)$-coloring of H. Then c can be extended to a modular monochromatic $(2,0)$-coloring of G by assigning the color $c(v)$ to each of the new vertices $w_1, w_2, \ldots, w_k$ in G. Thus G is $(2,0)$-colorable. It remains to show that $\chi_{(2,0)}(H) \leq \chi_{(2,0)}(G)$. Let c be a minimum modular monochromatic $(2,0)$-coloring of G. If $c(v) = 0$, then $c(w_i) = 0$ for each i with $1 \leq i \leq k$. Thus the restriction c_H of c to H is a modular monochromatic $(2,0)$-coloring of H that assigns the color 1 to $\chi_{(2,0)}(G)$ vertices of H and so $\chi_{(2,0)}(H) \leq \chi_{(2,0)}(G)$. Thus, we may assume that $c(v) = 1$. Thus $c(w_i) = 1$ for each i with $1 \leq i \leq k$. Since k is even, v must be adjacent to some vertex $u \in V(H)$ that is colored 1 by c. Since $\sum_{i=1}^{k} c(w_i) = 0$ in $\mathbb{Z}_2$, the restriction c_H of c to H is a modular monochromatic $(2,0)$-coloring that assigns the color 1 to $\chi_{(2,0)}(G) - k$ vertices of H and so $\chi_{(2,0)}(H) \le \chi_{(2,0)}(G)$. ■

We are now prepared to present the following realization result.

Theorem 3.3.4 *For each pair a, b of even integers with $2 \le a \le b$, there is a connected graph G such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$.*

Proof. If $a = b \ge 2$, then let G be any odd-degree tree of order a. By Theorem 2.4.3, $\chi_{(2,0)}(G) = a$. Since $V(G)$ is an even dominating set of G and $a = \chi_{(2,0)}(G) \le \gamma_e(G) \le a$, it follows that $\gamma_e(G) = a$ as well. Thus, we may assume that $2 \le a < b$ and a and b are both even. We consider two cases.

Case 1. $a = 2$. Let G be the graph of order $b + 1 \ge 5$ obtained from the triangle (v_1, v_2, v_3, v_1) by adding $b - 2 \ge 2$ pendant edges v_1w_i $(1 \le i \le b - 2)$ at v_1. Since G contains two adjacent vertices v_2 and v_3 such that for each $v \in V(G) - \{v_2, v_3\}$, either v is adjacent to both v_2 and v_3 or v is adjacent to neither of v_2 and v_3, it then follows by Observation 3.1.1 that $\chi_{(2,0)}(G) = 2$. Next, we show that $\gamma_e(G) = b$. Note that $V(G) - \{v_3\}$ is an even dominating set of G and so $\gamma_e(G) \le b$. Next, let S be an even dominating set of G with $|S| = \gamma_e(G)$. Since v_1w_i is a pendant edge, $v_1, w_i \in S$ by Lemma 3.3.2 for $1 \le i \le b-2$ and so $|S| \ge b - 1$. Furthermore, by Propositions 2.3.4 and 3.3.1(b), $|S|$ is even and so $|S| \ne b - 1$. Thus $\gamma_e(G) = |S| \ge b$ and so $\gamma_e(G) = b$.

Case 2. $4 \le a < b$. Let $b = a + k \ge 6$, where $k \ge 2$ is an even integer. We start with the Cartesian product $H = C_a \,\square\, K_2$ of C_a and K_2 where the two copies of C_a in H are $(u_1, u_2, \ldots, u_a, u_1)$ and $(v_1, v_2, \ldots, v_a, v_1)$ and u_i is adjacent to v_i for $1 \le i \le a$. Let $G_{a,k}$ be a graph obtained from H by adding k pendant edges $v_a w_j$ for $1 \le j \le k$. Since a is even, $\chi_{(2,0)}(H) = a$ by Theorem 3.1.4. It then follows by Lemma 3.3.3 that $\chi_{(2,0)}(G_{a,k}) \ge a$. Since the coloring that assigns the color 1 to each u_i and v_i if i is odd and $1 \le i \le a-1$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G_{a,k}) \le a$ and so $\chi_{(2,0)}(G_{a,k}) = a$.

Next, we show that $\gamma_e(G) = a + k = b$. Let

$$X = \{u_i, v_i : i \text{ is even and } 2 \le i \le a\} \text{ and } W = \{w_1, w_2, \ldots, w_k\}.$$

Since $X \cup W$ is an even dominating set of G and so $\gamma_e(G) \le |X \cup W| = a + k$. Let S be a minimum even dominating set of G and so $|S| = \gamma_e(G)$. Since $v_a w_j$ is a pendant edge in G for $1 \le j \le k$, it follows by Lemma 3.3.2 that $\{v_a\} \cup W \subseteq S$. We claim that S contains at least two vertices in each set $X_i = \{u_i, u_{i+1}, v_i, v_{i+1}\}$ where $1 \le i \le a$ and each subscript of a vertex is expressed as an integer modulo a. If this is not the case, then there is some X_i where $1 \le i \le a$ such that $|S \cap X_i| \le 1$. If $S \cap X_i = \emptyset$, then any vertex in X_i is dominated by at most one vertex of S, which is impossible. Thus, we may assume that $|S \cap X_i| = 1$. First, suppose that $u_i \in S$ or $u_{i+1} \in S$, say $u_i \in S$ and $u_{i+1} \notin S$, and so $v_i, v_{i+1} \notin S$. However then, v_{i+1} is dominated by at most one vertex of S, which is impossible. Next, suppose that $v_i \in S$ or $v_{i+1} \in S$. We may assume that $v_i \in S$ and $v_{i+1} \notin S$. However then, u_{i+1} is dominated by at most one vertex of S, which is impossible. Therefore, as claimed, S must contains at least two vertices in each set $X_i = \{u_i, u_{i+1}, v_i, v_{i+1}\}$ for $1 \le i \le a$. This implies that $\gamma_e(G) = |S| \ge a + k$ and so $\gamma_e(G) = a + k = b$. ■

Theorem 3.3.4 gives rise to the following question. For which triples (a, b, n) of integers where $2 \le a \le b \le n$ and a and b are even, does there exist a connected graph G of order n such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$? Define a triple (a, b, n) of integers where

$2 \leq a \leq b \leq n$ and a and b are even to be *realizable* if there is a connected graph G of order n such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$. As it turns out, there are infinitely many non-realizable triples. As an illustration, we show that $(2, 4, 4)$ is not realizable.

Proposition 3.3.5 *The triple* $(2, 4, 4)$ *is not realizable.*

Proof. By Observation 3.1.1, there are three connected graphs of order 4 having $(2, 0)$-chromatic number 2, namely $G_1 = K_4, G_2 = K_4 - e$ and G_3 is the graph obtained from K_3 by adding a pendant edge at some vertex K_3. Since $\gamma_e(G_1) = \gamma_e(G_2) = 2$ and $\gamma_e(G_3)$ does not exist, there is no connected graph G of order 4 with $\chi_{(2,0)}(G) = 2$ and $\gamma_e(G) = 4$. ■

In order to establish a characterization of all realizable triples, we first present the following result.

Theorem 3.3.6 *Let G be a connected $(2, 0)$-colorable graph of order n such that $\gamma_e(G)$ exists. If $\frac{\gamma_e(G)}{2} < \chi_{(2,0)}(G) < \gamma_e(G)$, then*

$$n \geq \frac{2\chi_{(2,0)}(G) + \gamma_e(G)}{2}. \tag{3.4}$$

Proof. Suppose that G is a connected graph of order n with $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$ where $b/2 < a < b$. We claim that $n > b$. If this is not the case, then G is an odd-degree graph by Proposition 3.3.1(b). It then follows by Corollary 3.1.3 that either G is $(2, 0)$-extremal (in which case $a = b$) or $a \leq n/2 = b/2$, which is a contradiction. Therefore, $n > b$, as claimed.

Let c be a minimum monochromatic $(2, 0)$-coloring of G, where X is the set of the vertices of G colored 1 by c, and let Y be a minimum even dominating set of G. Then $|X| = a$ and $|Y| = b$. Let $A = X - Y$, $B = X \cap Y$, $C = Y - X$ and $D = V(G) - (X \cup Y)$.

Then these four pairwise disjoint sets A, B, C, D form a partition of $V(G)$ (where it is possible that some of these sets are empty). Also, $a = |X| = |A| + |B|$, $b = |Y| = |B| + |C|$ and $n = |A| + |B| + |C| + |D|$. Define a new coloring $c^* : V(G) \to \mathbb{Z}_2$ by

$$c^*(v) = \begin{cases} 1 & \text{if } v \in A \cup C \\ 0 & \text{otherwise.} \end{cases}$$

We first show that c^* is a monochromatic $(2,0)$-coloring of G. For a set S of vertices of G and a vertex v of G, let $d_S v = |N(v) \cap S|$ be the number of neighbors of v that belong to S. Hence $\deg_G v = d_A v + d_B v + d_C v + d_D v$. Since (i) $X = A \cup B$ and $Y = B \cup C$ and (ii) c is a monochromatic $(2,0)$-coloring of G and Y is an even dominating set of G, the subgraphs $G[X] = G[A \cup B]$ and $G[Y] = G[B \cup C]$, induced by $A \cup B$ and $B \cup C$, respectively, are odd-degree graphs. Thus if $v \in X$, then $\deg_{G[X]} v = d_A v + d_B v$ is odd and if $v \in Y$, then $\deg_{G[Y]} v = d_B v + d_C v$ is odd. In particular,

- each vertex in X is adjacent to an odd number of vertices in X, while each vertex not in X is adjacent to an even number of vertices in X;
- each vertex in Y is adjacent to (or dominated by) an odd number of vertices in Y, while each vertex not in Y is adjacent to an even number of vertices in Y.

Therefore, we have the following observations:

(1) If $v \in X = A \cup B$, then $d_A v$ and $d_B v$ are of opposite parity. Furthermore,

(1.1) If $v \in A$, then v is adjacent to an even number of vertices in $Y = B \cup C$ and so $d_B v$ and $d_C v$ are the same parity,

(1.2) If $v \in B$, then v is adjacent to an odd number of vertices in $Y = B \cup C$ and so $d_B v$ and $d_C v$ are of opposite parity and so $d_A v$ and $d_C v$ are the same parity by (1).

(2) If $v \in Y = B \cup C$, then $d_B v$ and $d_C v$ are of opposite parity. Furthermore,

(2.1) If $v \in B$, then $d_A v$ and $d_C v$ are the same parity by (1.2).

(2.2) If $v \in C$, then v is adjacent to an even number of vertices in $X = A \cup B$ and so $d_A v$ and $d_B v$ are the same parity.

(3) If $v \in D$, then any two of $d_A v, d_B v$ and $d_C v$ are the same parity.

We now show that $\sigma_{c^*}(v) = 0$ in $\mathbb{Z}_2$ for each $v \in V(G)$. Since each vertex v is adjacent to exactly $d_A v + d_C v$ vertices colored 1 by c^*, it follows that

$$\sigma_{c^*}(v) = c^*(v) + d_A v + d_C v. \tag{3.5}$$

- If $v \in A$, then $c^*(v) = 1$. Since $d_A v$ and $d_B v$ are of opposite parity by (1) and $d_B v$ and $d_C v$ are the same parity by (1.1), it follows that $d_A v$ and $d_C v$ are of opposite parity and so $d_A v + d_C v$ is odd.
- If $v \in B$, then $c^*(v) = 0$ and $d_A v + d_C v$ is even by (2.1).
- If $v \in C$, then $c^*(v) = 1$. Since $d_B v$ and $d_C v$ are of opposite parity by (2) and $d_A v$ and $d_B v$ are the same parity by (2.2), it follows that $d_A v$ and $d_C v$ are of opposite parity and so $d_A v + d_C v$ is odd.
- If $v \in D$, then $c^*(v) = 0$ and $d_A v + d_C v$ is even by (3).

In each case, $\sigma_{c^*}(v) = 0$ in $\mathbb{Z}_2$ by (3.5). Hence c^* is a monochromatic $(2,0)$-coloring of G and so $\chi_{(2,0)}(G) = a \le |A| + |C|$. Let $n = b + \ell$ where $\ell \ge 1$. Since $a \le |A| + |C|$ and $a = |A| + |B|$, it follows that $|B| \le |C|$. Because $b = |B| + |C|$ and $|A| + |B| + |C| \le n = b + \ell$, we have $|A| \le \ell$. Now $b = |B| + |C| \ge 2|B| = 2(a - |A|) \ge 2(a - \ell)$ and so $\ell \ge a - \frac{1}{2}b$. Thus $n = b + \ell \ge b + (a - \frac{1}{2}b) = \frac{2a+b}{2}$, as desired. ■

The lower bound in (3.4) is sharp, as we will see soon. By Theorem 3.3.6, if a, b and n are positive integers, where a and b are even, $b/2 < a < b$ and $n < \frac{2a+b}{2}$, then (a, b, n) is not realizable. For example, $(6, 8, 8)$, $(6, 8, 9)$, $(8, 10, 10)$, $(8, 10, 11)$ or $(8, 10, 12)$ are not realizable. Next we characterize all realizable triples.

Theorem 3.3.7 *Let (a, b, n) be a triple of integers where $2 \leq a \leq b \leq n$ and a and b are even. Then (a, b, n) is realizable if and only if (i) $a = b$, (ii) $a \leq \frac{b}{2}$ and $(a, b, n) \neq (2, 4, 4)$ or (iii) $\frac{b}{2} < a < b$ and $n \geq \frac{2a+b}{2}$.*

Proof. First, suppose that (a, b, n) is realizable. Since $a \leq b$, either $a = b$ where (i) holds or $a < b$. If $a < b$, then either $a \leq \frac{b}{2}$ or $a > \frac{b}{2}$. If $a \leq \frac{b}{2}$, then $(a, b, n) \neq (2, 4, 4)$ by Proposition 3.3.5 and so (ii) holds. If $\frac{b}{2} < a < b$, then $n \geq \frac{2a+b}{2}$ by Theorem 3.3.6 and so (iii) holds. To verify the converse, we consider three cases.

Case 1. $a = b$. If $a = b = n$, then an odd-degree tree of order n has the desired property by Theorem 2.4.3. Thus, we may assume that $a = b < n$. For $a = b = 2$, let G be the graph of order n obtained from $K_{2,n-2}$ with partite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{n-2}\}$ by adding the edge x_1x_2. Then the coloring that assigns the color 1 to each vertex in X and the color 0 to the remaining vertices of G is a minimum monochromatic $(2, 0)$-coloring of G and X is a minimum even dominating set of G. For $a = b \geq 4$, let G be the graph of order n obtained from $K_{2,n-a+1}$ with partite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{n-a+1}\}$ by adding $a - 3$ pendant edges z_iy_1 $(1 \leq i \leq a - 3)$ at y_1. Then the coloring that assigns the color 1 to each vertex in $S = \{x_1, x_2, y_1, z_1, \ldots, z_{a-3}\}$ and the color 0 to the remaining vertices of G is a minimum monochromatic $(2, 0)$-coloring of G and S is a minimum even dominating set of G. Thus $\chi_{(2,0)}(G) = \gamma_e(G)$ in this case.

Case 2. *$a \le b/2$ and $(a, b, n) \ne (2, 4, 4)$.* Let $b = a + k$, where then $k \ge 2$ is even. We consider two subcases, according to whether $b = n$ or $b < n$.

Subcase 2.1. *$b = n$.* Since $(a, b, n) \ne (2, 4, 4)$ and $b \ge 2a$, it follows that $k \ge 4$. We begin with the graph $K_4 - e$ where $V(K_4 - e) = \{x_1, x_2, y_1, y_2\}$ and $\deg x_1 = \deg x_2 = 3$. Construct the graph G of order $n = b$ from $K_4 - e$ by adding (1) the $a - 2$ pendant edges $z_i x_1$ $(1 \le i \le a - 2)$ at x_1, (2) one pendant edge wy_1 at y_1 and (3) the $k - 3$ pendant edges $v_j y_2$ $(1 \le j \le k - 3)$ at y_2. The coloring that assigns the color 1 to each vertex in $\{x_1, x_2, z_1, z_2, \ldots, z_{a-2}\}$ and the color 0 to the remaining vertices of G is a minimum monochromatic $(2, 0)$-coloring of G and $V(G)$ is the only even dominating set of G. Thus $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$.

Subcase 2.2. *$b < n$.* We consider two subcases, according to whether $a = 2$ or $a \ge 4$.

Subcase 2.2.1. *$a = 2$.* For $b = 4$, let G be the graph of order $n \ge 5$ obtained from $2P_2 + K_1$ (where the two copies of P_2 are (x_1, x_2) and (y_1, y_2) and v is adjacent to every vertex of $\{x_1, x_2, y_1, y_2\}$) by adding $n-5 \ge 0$ new vertices $w_1, w_2, \ldots, w_{n-5}$ and joining each w_i $(1 \le i \le n-5)$ to both x_1 and x_2. (If $n = 5$, then $G = 2P_2 + K_1$.) Then the coloring that assigns the color 1 to x_1 and x_2 and the color 0 to the remaining vertices of G is a minimum monochromatic $(2, 0)$-coloring of G and $\{x_1, x_2, y_1, y_2\}$ is a minimum even dominating set of G. For $b \ge 6$, let $b = 2 + k$ for some even integer $k \ge 4$. Then $n - b = n - 2 - k \ge 2$. We start with the graph $H = K_{2,n-2-k} + e$ where the partite sets of $K_{2,n-2-k}$ are $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{n-2-k}\}$ and $e = x_1x_2$. Now construct the graph of G order n from H by

(1) adding the two pendant edges $y_{n-2-k}w_1$ and $y_{n-2-k}w_2$ at y_{n-2-k},

(2) adding one pendant w_1w_3 at w_1 and

(3) adding the $k-3$ pendant edges w_2w_i for $4 \leq i \leq k$ at w_2.

The coloring that assigns the color 1 to x_1 and x_2 and the color 0 to the remaining vertices of G is a minimum monochromatic $(2,0)$-coloring of G and the set $\{x_1, x_2, w_1, w_2, \ldots, w_k\}$ is a minimum even dominating set of G. In either case, $\chi_{(2,0)}(G) = 2$ and $\gamma_e(G) = b$.

Subcase 2.2.2. $a \geq 4$. Let $b = a + k$ where $k \geq a \geq 4$. We start with the graph $H = K_{2,n-b+1}$ with partite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{n-b+1}\}$ where $n - b + 1 \geq 2$. Let G be the graph of order n obtained from H in the following three steps:

(1) adding the $a-3$ pendant edges y_1z_i $(1 \leq i \leq a-3)$ at y_1,

(2) adding two pendant edges w_1y_{n-b+1} and w_2y_{n-b+1} at y_{n-b+1},

(3) adding one pendant edge w_1w_3 at w_1 and the $k-3$ pendant edges w_2w_i for $4 \leq i \leq k$ at w_2.

The coloring that assigns the color 1 to each vertex in $S = \{x_1, x_2, y_1, z_1, z_2, \ldots, z_{a-3}\}$ and the color 0 to the remaining vertices of G is a minimum monochromatic $(2,0)$-coloring of G and the set $S \cup \{w_1, w_2, \ldots, w_k\}$ is a minimum even dominating set of G. Therefore, $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = a + k = b$ in this case.

Case 3. $\frac{b}{2} < a < b$ *and* $n \geq \frac{2a+b}{2}$. Since $a > \frac{b}{2}$ and $n \geq \frac{2a+b}{2}$, it follows that $b \geq 6$ and $n > 7$. Write $n = b + \ell = b + \left(a - \frac{b}{2}\right) + \left[\ell - \left(a - \frac{b}{2}\right)\right]$. Let $s = a - \frac{b}{2} \geq 1$, $t = \frac{b}{2} - 1 \geq 2$ and let $p = \ell - s = \ell - \left(a - \frac{b}{2}\right) \geq 0$. We consider two subcases, according to whether $b \equiv 2 \pmod 4$ or $b \equiv 0 \pmod 4$.

Subcase 3.1. $b \equiv 2 \pmod 4$. Then $s = a - \frac{b}{2}$ is an odd integer and $t = \frac{b}{2} - 1$ is an even integer. Let G be the graph obtained from the graph $K_{2,s} + e$ with partite sets $\{v, w\}$ and $U = \{u_1, u_2, \ldots, u_s\}$ where $e = vw$ in the following three steps:

(1) add t new vertices $v_1, v_2, \ldots, v_t$ and joining each v_i to v for $1 \le i \le t$,

(2) add t new vertices $w_1, w_2, \ldots, w_t$ and joining each w_i to w for $1 \le i \le t$ and

(3) if $p \ge 1$, then add p new vertices $x_1, x_2, \ldots, x_p$ and joining each x_j to both w_1 and w_2 for $1 \le j \le p$.

Then the order of G is $n = 2 + s + 2t + p = 2 + s + 2t + (\ell - s) = 2 + 2t + \ell = b + \ell$. It remains to show that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$. Let $V = \{v, v_1, v_2, \ldots, v_t\}$ and $W = \{w, w_1, w_2, \ldots, w_t\}$. Thus $|V| = |W| = t + 1 = \frac{b}{2}$, $|U \cup V| = a$ and $|V \cup W| = b$. Each of the subgraphs of G induced by $U \cup V$, $V \cup W$ and $U \cup W$, respectively, is an odd-degree graph.

We first show that $\chi_{(2,0)}(G) = a$. Since the coloring that assigns the color 1 to each vertex in $U \cup V$ and the color 0 to remaining vertices of G is a monochromatic $(2, 0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \le |U \cup V| = a$. To show $\chi_{(2,0)}(G) \ge a$, let c be a minimum monochromatic $(2, 0)$-coloring of G. Let $X = \{x_1, x_2, \ldots, x_p\}$ where $X = \emptyset$ if $p = 0$. Since $c(w_1) = c(w_2)$ by Observation 1.3.1, it follow that $c(x) = 0$ for each $x \in X$. Hence the restriction of c to $H = G - X$ (also denoted by c) is a monochromatic $(2, 0)$-coloring of H. By Observations 1.3.1 and 4.1.4, all vertices in each of U, V or W, respectively, must be colored the same. Furthermore, at least one of v and w must be colored 1. We may assume, without loss of generality, that $c(v) = 1$ and so every vertex in V is colored 1. If $c(w) = 1$, then every vertex in W is colored 1 and so $\chi_{(2,0)}(G) \ge |V \cup W| = b$, which contradicts the fact that $\chi_{(2,0)}(G) \le a < b$. This implies that $c(w) = 0$ and so every vertex in W is colored 0. Since $\sigma(v) = 0$ and $\frac{b}{2}$ is odd, at least one vertex in U must be colored 1 and so every vertex in U is colored 1. Hence $\chi_{(2,0)}(G) \ge |U \cup V| = a$. Therefore, $\chi_{(2,0)}(G) = a$.

Next, we show that $\gamma_e(G) = b$. Since $V \cup W$ is an even dominating set of G, it follows that $\gamma_e(G) \leq |V \cup W| = b$. On the other hand, every even dominating set of G must contain $V \cup W$ by Lemma 3.3.2 and so $\gamma_e(G) \geq |V \cup W| = b$. Thus $\gamma_e(G) = b$.

Subcase 3.2. $b \equiv 0 \pmod 4$. Then $s = a - \frac{b}{2}$ is an even integer and $t = \frac{b}{2} - 1$ is an odd integer. Let $G' = G - e$ where G is the graph constructed in Case 1 and $e = vw$. Then the order of G' is n. An argument similar to one on Case 1 shows (1) the coloring that assigns the color 1 to each vertex in $U \cup V$ and the color 0 to the remaining vertices of G' is a minimum monochromatic $(2,0)$-coloring of G' and (2) the set $V \cup W$ is a minimum even dominating of G'. Therefore, $\chi_{(2,0)}(G') = a$ and $\gamma_e(G') = b$. ■

Chapter 4

On Nearly (2, 0)-Extremal Trees

Recall that a graph G is called an *odd-degree graph* if every vertex of G has odd degree and a $(2,0)$-colorable graph G of order n is $(2,0)$-*extremal* if $\chi_{(2,0)}(G) = n$. We have seen that a $(2,0)$-colorable tree is $(2,0)$-extremal if and only if T is an odd-degree tree. In this chapter, we characterize all trees of order n having $(2,0)$-chromatic number $n-1, n-2$ or $n-3$ and investigate the structures of connected graphs having the large $(2,0)$-chromatic numbers.

4.1 On a Class of Trees

In order to present characterizations of trees of order n having $(2,0)$-chromatic number $n-1, n-2$ or $n-3$, we first study a special family of trees and establish some preliminary results. It will be useful to recall some definitions and results appeared in Section 2.4.

A $(2,0)$-colorable tree T is called $(2,0)$-*minimal* if for every end-vertex v of T, the subtree $T - v$ is not $(2,0)$-colorable. For example, every $(2,0)$-colorable path, star and double star is $(2,0)$-minimal. We now describe a class of trees that are closed related to $(2,0)$-minimal trees. For a graph G, let H be a subgraph of G and let v be a vertex of G not belonging to H. The vertex v is *adjacent to H* if v is adjacent to some vertex of H. Let

T' be a tree of order $k \geq 1$, where $V(T') = \{v_1, v_2, \ldots, v_k\}$ and $E(T') = \{e_1, e_2, \ldots, e_{k-1}\}$. The *subdivision graph* $S(T')$ of T' is the tree of order $2k-1$ obtained from T' by replacing each edge e_i $(1 \leq i \leq k-1)$ by the vertex u_i which is joined to the two vertices of T' incident with e_i. A tree T is an *odd-degree subdivision tree* if the vertices $v_1, v_2, \ldots, v_k$ of some subdivision graph $S(T')$ of a tree T' of order k correspond to k pairwise disjoint odd-degree trees $T_1, T_2, \ldots, T_k$ in T and $V(T) - \cup_{i=1}^{k} V(T_i)$ consists of an independent set of $k-1$ vertices of T each adjacent to exactly two of the trees $T_1, T_2, \ldots, T_k$. In this case, T is referred to as an *odd-degree subdivision tree with respect to odd-degree trees* $T_1, T_2, \ldots, T_k$ and each of the vertices $u_1, u_2, \ldots, u_{k-1}$ is called a *subdividing vertex* of T. If F is the odd-degree forest consisting of odd-degree trees $T_1, T_2, \ldots, T_k$, then T is also referred to as an *odd-degree subdivision tree with respect to* F. In particular, if $k = 1$, then an odd-degree subdivision tree is an odd-degree tree. Among the results established on trees is the following.

Theorem 4.1.1 *For a nontrivial tree T, the following* (1), (2) *and* (3) *are equivalent:*

(1) *T is $(2,0)$-minimal,*

(2) *T is $(2,0)$-colorable and every modular monochromatic $(2,0)$-coloring of T must assign the color* 1 *to each end-vertex of T,*

(3) *T is an odd-degree subdivision tree.*

The following is a consequence of Theorem 4.1.1 (which we have seen in Chapter 2).

Corollary 4.1.2 *If T is a nontrivial tree having exactly one even vertex, then T is not $(2,0)$-colorable.*

We now determine the monochromatic $(2,0)$-coloring and the $(2,0)$-chromatic number of an arbitrary odd-degree subdivision tree with respect to an odd-degree forest.

Theorem 4.1.3 *Let F be an odd-degree forest. If T is an odd-degree subdivision tree with respect to F, then the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is the unique monochromatic $(2,0)$-coloring of T and $\chi_{(2,0)}(T) = |V(F)|$.*

Proof. Since the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a monochromatic $(2,0)$-coloring of T, it remains to show that this is the only monochromatic $(2,0)$-coloring of T. We proceed by induction on the number k of components of a forest. By Theorem 2.4.3, the result holds for $k = 1$. Suppose, for some integer $k \geq 2$, that the statement is true for all odd-degree forests having exactly $k-1$ components. Let F be an odd-degree forest of order $p \geq 2$ having exactly k components, say $T_1, T_2, \ldots, T_k$. Assume, to the contrary, that there is an odd-degree subdivision tree T with respect to F such that T has a monochromatic $(2,0)$-coloring c that assigns the color 0 to some vertex in F. Let S be the set of vertices of T colored 0 by c. We claim that S is an independent set of vertices of degree 2 in T. First, we show that S is an independent set of vertices of T; for otherwise, suppose that $uv \in E(T)$ where $c(u) = c(v) = 0$. Then $T - uv$ consists of two components Q_1 and Q_2. Note that either Q_1 or Q_2 has a vertex colored 1 by c, say the former. Since the restriction c_1 of c to Q_1 is a (nontrivial) modular monochromatic $(2,0)$-coloring of Q_1 by Observation 2.2.10(b), it follows that c_1 can be extended to a modular monochromatic $(2,0)$-coloring c' of T by assigning the color 0 to all vertices of Q_2. However then c' must assign the color 0 to at least one end-vertex of T, which is a contradiction by Theorem 4.1.1. Next we show that each vertex of S has degree 2 in T. Let $v \in S$. Since $\sigma(v) = 0$ and $c(v) = 0$, it follows that v is adjacent to an even number of vertices that are colored 1 and so $\deg_T v = d \geq 2$ is even. If $d \geq 4$, then, since T is not an odd-degree tree and $T - v$ consists of d nontrivial components, all vertices in $d-2$ of these components can be recolored 0 producing a modular monochromatic $(2,0)$-coloring in which at least $d-2$ end-vertices are colored 0, which is a contradiction by Theorem 4.1.1. Therefore, S is an independent set of vertices of degree 2 in T, as claimed.

Suppose that U is the set of subdividing vertices of T where then $|U| = k - 1$ and $\deg_T u = 2$ for each $u \in U$. We may assume, without loss of generality, that T_1 is an end-tree of T, that is, T_1 is adjacent to exactly one vertex $u \in U$. Suppose that u is adjacent to $w \in V(T_1)$ and $x \in V(T_j)$ where $j \neq 1$, say $j = 2$. We consider two cases, according to whether $c(w) = 0$ or $c(w) = 1$.

Case 1. $c(w) = 0$. Then $c(u) = 1$ and $\deg_T w = 2$. Since $w \in V(T_1)$ and $uw \in E(T)$, it follows that w is an end-vertex of T_1, say w is adjacent w' in T_1. Hence $T_1 - w$ is a tree with exactly one even vertex (namely, w'). By Corollary 4.1.2 then, $T_1 - w$ is not $(2,0)$-colorable. On the other hand, since $\sigma(w) = 0$ and $c(u) = 1$, it follows that $c(w') = 1$. Because $c(w) = 0$ and w is an end-vertex of T_1, the restriction of c to $T_1 - w$ is a nontrivial monochromatic $(2,0)$-coloring of $T_1 - w$ by Observation 2.2.10(b). Hence $T_1 - w$ is $(2,0)$-colorable, which is a contradiction.

Case 2. $c(w) = 1$. We claim that $c(u) = 0$, for otherwise, suppose that $c(u) = 1$. Since $\sigma(u) = 0$ and $c(w) = 1$, it follows that $c(x) = 0$ and $\deg_T x = 2$. Let T^* be the tree obtained from T_1 by adding the pendant edge uw. Then T^* has exactly one even vertex (namely w) and so T^* is not $(2,0)$-colorable by Corollary 4.1.2. On the other hand, since $c(w) = 1$ and $c(u) = 1$, the restriction of c to T^* is a nontrivial monochromatic $(2,0)$-coloring of T^* by

Observation 2.2.10(b), which is a contradiction. Thus, $c(u) = 0$, as claimed. By (1) then, $c(x) = c(w) = 1$ and the restriction of c to T_1 is a nontrivial monochromatic $(2, 0)$-coloring of T_1. Since T_1 is an odd-degree tree, $c(z) = 1$ for all $z \in V(T_1)$ and so c does not assign the color 0 to any vertex of T_1. Hence c must assign the color 0 to some vertex in $V(F) - V(T_1)$, namely some vertex in T_i for some $i \in \{2, 3, \ldots, k\}$. Let $T' = T - (\{u\} \cup V(T_1))$. Then T' is an odd-degree subdivision tree with respect to the odd-degree forest F' having exactly $k-1$ components $T_2, T_3, \ldots, T_k$. By the induction hypothesis, the coloring that assigns the color 1 to each vertex of F' and the color 0 to the remaining vertices of T' is the unique monochromatic $(2, 0)$-coloring of T'. On the other hand, since $c(x) = 1$ and $c(u) = 0$, the restriction c' of c to T' is a nontrivial monochromatic $(2, 0)$-coloring of T'. However, c' assigns the color 0 to some vertex of F', which is a contradiction. Thus, Cases 1 and 2 cannot occur.

Therefore, the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is the unique monochromatic $(2, 0)$-coloring of T and so $\chi_{(2,0)}(T) = |V(F)|$. ■

We have seen in Theorem 4.1.1 that if T is an odd-degree subdivision tree with respect to an odd-degree forest, then every monochromatic $(2, 0)$-coloring of T must assign the color 1 to each of the end-vertices of T. Hence Theorem 4.1.3 improves this known result.

For an odd-degree forest F, let $ods(F)$ be the set of all odd-degree subdivision trees with respect to F and let $\mathcal{P}_{ods}(F)$ be the set of trees T such that either $T \in ods(F)$ or T is obtained from some $T_o \in ods(F)$ by adding pendant edges at one or more subdividing vertices of T_o. The following observation will be useful to us (which we have seen in Chapter 3).

Observation 4.1.4 *If uv is a pendant edge in a $(2,0)$-colorable graph G, then $c(u) = c(v)$ for every modular monochromatic $(2,0)$-coloring c of G.*

Theorem 4.1.5 *Let F be an odd-degree forest. If $T \in \mathcal{P}_{ods}(F)$, then the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is the unique monochromatic $(2,0)$-coloring of T and $\chi_{(2,0)}(T) = |V(F)|$.*

Proof. Since the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a monochromatic $(2,0)$-coloring of T, it remains to show that this is the only monochromatic $(2,0)$-coloring of T. We proceed by induction on the number k of components of a forest. By Theorem 2.4.3, the result holds for $k = 1$. Suppose, for some integer $k \geq 2$, that the statement is true for all odd-degree forests having exactly $k-1$ components. Let F be an odd-degree forest having exactly k components, say $T_1, T_2, \ldots, T_k$ and let $T \in \mathcal{P}_{ods}(F)$. We may assume, without loss of generality, that T_1 is an end-tree of T, that is, T_1 is adjacent to exactly one subdividing vertex u of T. Then either $\deg_T u = 2$ or there are ℓ pendant edges of T at u, say $uv_1, uv_2, \ldots, uv_\ell$ be the pendant edges of T at u. Suppose that u is adjacent to $w \in V(T_1)$ and $x \in V(T_j)$ where $j \neq 1$, say $j = 2$. Let c be a monochromatic $(2,0)$-coloring of T. We consider two cases.

Case 1. $c(u) = 0$. By Observation 4.1.4 then, $c(v_i) = 0$ for each i with $1 \leq i \leq \ell$. Let $X = \{u, v_1, v_2, \ldots, v_\ell\}$. Then $T - X$ has exactly two components, namely T_1 and $H = T - (V(T_1) \cup X)$. Since $c(u) = \sigma(u) = 0$, it follows that $c(w) = c(x)$. Furthermore, either the restriction c_{T_1} of c to T_1 or the restriction c_H of c to H is a nontrivial monochromatic $(2,0)$-coloring of T_1 or H, respectively. First, suppose that c_{T_1} is a nontrivial monochromatic $(2,0)$-coloring of T_1. Since T_1 is an odd-degree tree, $c(z) = 1$ for each $z \in V(T_1)$ by Theorem 2.4.3. Thus $c(w) = 1$ and $c(x) = 1$, which implies that c_H is a nontrivial monochromatic $(2,0)$-coloring of H as well. Let F' be the odd-degree forest consisting of the $k-1$ components $T_2, T_3, \ldots, T_k$. Then $H \in \mathcal{P}_{ods}(F')$. By the induction hypothesis, the coloring that assigns the color 1 to each vertex of F' and the color 0 to the remaining vertex of H is the unique monochromatic $(2,0)$-coloring of H. Hence c must assign the color 1 to at least $|V(F')|$ vertices of H. This implies that c must assign the color 1 to at least $|V(F')| + |V(T_1)| = |V(F)|$ vertices of T and so $\chi_{(2,0)}(T) = |V(F)|$. Furthermore, the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a unique monochromatic $(2,0)$-coloring of T. Next, suppose that c_H is a nontrivial monochromatic $(2,0)$-coloring of H. Again, by the induction hypothesis, $c(x) = 1$ and $c(w) = 1$. Now apply an argument similar to the one above shows that $\chi_{(2,0)}(T) = |V(F)|$ and the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a unique monochromatic $(2,0)$-coloring of T.

Case 2. $c(u) = 1$. In what follows, we show that this case is impossible. Then $c(v_i) = 1$ for $1 \leq i \leq \ell$ by Observation 4.1.4. Let $N_{T_1}(w) = \{w_1, w_2, \ldots, w_t\}$ be the set of neighbors of w in T_1, where t is an odd integer. Next, we consider two subcases, according to whether $c(w) = 0$ or $c(w) = 1$.

Subcase 2.1. $c(w) = 0$. Since $\sigma(w) = 0$ and $c(u) = 1$, it follows that $c(w_i) = 1$ for some $i \in \{1, 2, \ldots, t\}$, say $c(w_1) = 1$. Now $\sigma(w_1) = c(w) = 0$ implies that w_1 is not an end-vertex of T_1. Let Q_1 be the component of $T_1 - w$ that contains w_1. Then Q_1 is a nontrivial tree with exactly one even vertex (namely w_1) and so Q_1 is not $(2,0)$-colorable by Corollary 4.1.2. However, the restriction of c to Q_1 is a nontrivial monochromatic $(2,0)$-coloring of Q_1, which is impossible.

Subcase 2.2. $c(w) = 1$. Since $\sigma(w) = 0$, $c(u) = 1$ and $\deg_{T_1} w = t$ is odd, there is at least one $i \in \{1, 2, \ldots, t\}$ such that $c(w_i) = 0$, say $c(w_1) = 0$. Since $\sigma(w_1) = c(w_1) = 0$ and $c(w) = 1$, it follows that w_1 is adjacent to some vertex $w_{1,1}$ in T_1 for which $c(w_{1,1}) = 1$. Then $w_{1,1}$ cannot be an end-vertex of T_1. Let $Q_{1,1}$ be the component of $T_1 - w_1$ that contains $w_{1,1}$. Then $Q_{1,1}$ is a nontrivial tree with exactly one even vertex (namely $w_{1,1}$) and so $Q_{1,1}$ is not $(2,0)$-colorable. On the other hand, the restriction of c to $Q_{1,1}$ is a nontrivial monochromatic $(2,0)$-coloring of $Q_{1,1}$, which is impossible.

Hence, Case 2 is impossible and so only Case 1 can occur. Therefore, $\chi_{(2,0)}(T) = |V(F)|$ and the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a unique monochromatic $(2,0)$-coloring of T. ■

4.2 On Nearly $(2,0)$-Extremal Graphs

A $(2,0)$-colorable graph G of order n is said to be *nearly* $(2,0)$*-extremal* if $\chi_{(2,0)}(G) = n-1$. Thus, if G is a nearly $(2,0)$-extremal graph of order n, then n is odd. We first characterize all nearly $(2,0)$-extremal trees. To simplify the notation, if F is an odd-degree forest having exactly k components and T is an odd-degree subdivision tree with respect to F, then T is referred to as an *ods-k-tree with respect to F* (or simply an *ods-k-tree*). Thus, each ods-k-tree has exactly $k-1$ subdividing vertices. In particular, an ods-1-tree is an odd-degree tree and so has no subdividing vertices.

Theorem 4.2.1 *Let T be a $(2,0)$-colorable tree of order $n \geq 3$. Then $\chi_{(2,0)}(T) = n-1$ if and only if T is an ods-2-tree.*

Proof. First, suppose that T is a nontrivial tree of order n with $\chi_{(2,0)}(T) = n-1$. Let there be given a monochromatic $(2,0)$-coloring c of T such that $c(v) = 0$ and $c(x) = 1$ for all $x \in V(T) - \{v\}$. Then $\deg v$ is even. We claim that $\deg v = 2$. For otherwise, suppose that $\deg v = s \geq 4$ and let $N(v) = \{v_1, v_2, \ldots, v_s\}$. Let T_i $(1 \leq i \leq s)$ be the component of $T - v$ containing v_i. Then the coloring c' defined by $c'(x) = c(x)$ if $x \in V(T_1) \cup V(T_2)$ and $c'(x) = 0$ for the remaining vertices of T is a monochromatic $(2,0)$-coloring of T such that fewer than $n-1$ vertices of G are colored 1, which is impossible. Thus, $\deg v = 2$, as claimed. For each $x \neq v$ in T, since $c(x) = 1$ and $\sigma(x) = 0$, it follows that x is adjacent to an odd number of vertices colored 1. Thus $\deg v_1$ and $\deg v_2$ are even and $\deg x$ is odd for all $x \in V(T) - \{v, v_1, v_2\}$. Therefore, T_1 and T_2 are odd-degree trees.

For the converse, suppose that T consists of two disjoint odd-degree trees T_1 and T_2 and a vertex v of degree 2 that is adjacent to T_1 and T_2. Suppose that v is adjacent to $v_i \in V(T_i)$ for $i = 1, 2$. Note that the coloring that assigns the color 1 to each vertex in T_1 and T_2 and the color 0 to the vertex v is a monochromatic $(2,0)$-coloring of T, implying that $\chi_{(2,0)}(T) \leq n-1$. Assume, to the contrary, that $\chi_{(2,0)}(T) \leq n-2$. Let c be a minimum monochromatic $(2,0)$-coloring of T. Then c assigns the color 0 to at least two vertices of T and so there is $w \neq v$ such that $c(w) = 0$.

We claim that $c(v) = 0$. If this is not the case, then $c(v) = 1$. Since $\sigma(v) = 0$, either $c(v_1) = 1$ and $c(v_2) = 0$ or $c(v_1) = 0$ and $c(v_2) = 1$, say the former. Let $T' = T - V(T_2)$. Then T' is a nontrivial tree with exactly one even vertex, namely v_1. Thus T' is not $(2,0)$-colorable by Corollary 4.1.2. On the other hand, since $c(v_1) = 1$ and $c(v_2) = 0$, the restriction of c to T' is a monochromatic $(2,0)$-coloring of T' by Observation 2.2.10(b). However then, T' is $(2,0)$-colorable, which is a contradiction. Thus, as claimed, $c(v) = 0$.

By Observation 2.2.10(b) and Theorem 4.1.1, it can be shown that no two adjacent vertices can be colored 0. Hence $c(v_1) = c(v_2) = 1$. We may assume, without loss of generality, that $w \in V(T_1) - \{v_1\}$. Since $c(v) = 0$, the restriction c_1 of the coloring c to the subtree T_1 is a nontrivial monochromatic $(2,0)$-coloring of T_1 by Observation 2.2.10(b) and c_1 assigns the color 0 to the vertex w in T_1. This is impossible since $\chi_{(2,0)}(T_1) = |V(T_1)|$ by Theorem 2.4.3 and the only monochromatic $(2,0)$-coloring of T_1 must assign the color 1 to every vertex of T_1. Therefore, $\chi_{(2,0)}(T) \geq n-1$ and the result follows. ■

As is the case of $(2,0)$-extremal trees (Theorem 2.4.3), Theorem 4.2.1 is not true for connected graphs in general. By Proposition 2.3.4, if G is a connected graph of order $n \geq 3$ with $\chi_{(2,0)}(G) = n-1$, then n must be odd and there is a monochromatic $(2,0)$-coloring c of G such that $c(v) = 0$ for exactly one vertex v of G and $G - v$ is an odd-degree graph. Thus $\deg_G v$ is even and so G must contain at least three even vertices. By Theorem 4.2.1, if T is a $(2,0)$-colorable tree of order n with $\chi_{(2,0)}(T) = n-1$, then T has exactly three even vertices. However, this is not the case for connected graphs in general. In fact, there are connected $(2,0)$-colorable graphs G of order n with $\chi_{(2,0)}(G) = n-1$ such that G has a large number of even vertices. For example, for each pair k, ℓ of integers where $k \geq 1$ and $\ell \geq 0$, let G be the graph obtained from $H = K_{2,2k}$ with partite sets $\{u, v\}$ and $X = \{x_1, x_2, \ldots, x_{2k}\}$ by adding $2\ell+1$ pendant edges vv_i for $1 \leq i \leq 2\ell+1$ at the vertex v of H. Then G has $2k+1$ even vertices (namely each vertex in $\{u\} \cup X$). The order of G is $n = 2 + 2k + 2\ell + 1$. We claim that $\chi_{(2,0)}(G) = n-1$. Since the coloring that assigns the color 0 to u and the color 1 to all vertices in $V(G) - \{u\}$ is a monochromatic $(2,0)$-coloring, $\chi_{(2,0)}(G) \leq n-1$. Let c be a minimum monochromatic $(2,0)$-coloring of G. First, suppose that $c(v) = 0$. Then $c(v_i) = 0$ for $1 \leq i \leq 2\ell+1$ by Observation 4.1.4. Since c must assign the color 1 to at least two vertices of G by Proposition 2.3.4, at least one vertex in X is colored 1 by c, say $c(x_1) = 1$. Because $\sigma(x_1) = 0$ in $\mathbb{Z}_2$, it follows that $c(u) = 1$. Now, the fact that $\sigma(x_j) = 0$ for $2 \leq j \leq 2k$ and $c(v) = 0$ implies that $c(x_j) = 1$ for each j with $2 \leq j \leq 2k$. However then, $\sigma(u) = 1$ in $\mathbb{Z}_2$, which is impossible. Next, suppose that $c(v) = 1$. Thus $c(v_i) = 1$

for $1 \leq i \leq 2\ell+1$ by Observation 4.1.4. If $c(u) = 0$, then $c(x_j) = 1$ for $1 \leq j \leq 2k$ and so c must assign the color 1 to $n-1$ vertices of G, as desired. Hence we may assume that $c(u) = 1$. Since $\sigma(u) = 0$, it follows that $c(x_j) = 1$ for some j with $1 \leq j \leq 2k$. However then $\sigma(x_j) = 1$ in $\mathbb{Z}_2$, which is impossible. Therefore, the coloring that assigns the color 0 to u and the color 1 to all vertices in $V(G) - \{u\}$ is the only monochromatic $(2,0)$-coloring of G and so $\chi_{(2,0)}(G) = n-1$, as claimed.

There is another possible interesting feature of connected graphs G of order $n \geq 3$ with $\chi_{(2,0)}(G) = n-1$; that is, if c is a minimum modular monochromatic $(2,0)$-coloring of G such that $c(v) = 0$ for exactly one vertex v in G, then it is possible that $\chi_{(2,0)}(G-v)$ is significantly smaller than $\chi_{(2,0)}(G)$. In fact, it can be shown that if $H = C_n \,\Box\, K_2$ is the Cartesian product of the n-cycle C_n and K_2 where $n \equiv 2 \pmod 4$ and G is the graph of order $2n+1$ obtained from H by adding a new vertex v and joining v to two adjacent vertices on a copy of C_n in H, then $\chi_{(2,0)}(G) = 2n$ and $\chi_{(2,0)}(G-v) = n$. In order to show this, we first present some preliminary results. We have seen that in Theorem 3.1.4 that for each integer $n \geq 3$,

$$\chi_{(2,0)}(C_n \,\Box\, K_2) = \begin{cases} n & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

The following lemma is a consequence of the proof of Theorem 3.1.4.

Lemma 4.2.2 *For an even integer $n \geq 4$, let $G = C_n \,\Box\, K_2$ where the two copies of C_n in G are $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ and u_i is adjacent to v_i for $1 \leq i \leq n$. For a monochromatic $(2,0)$-coloring c of G, let F_B and F_R be the subgraphs of G induced by the vertices of colored 1 or 0 by c, respectively. If $F_B \neq G$, then*

(a) F_B cannot contain a block $P_k \,\Box\, K_2$ of length $k \geq 2$,

(*b*) *F_B can contain an edge on a copy of C_n as a component only when $n \equiv 0 \pmod 4$,*

(*c*) *F_R cannot contain a block $P_k \,\square\, K_2$ of length $k \geq 2$.*

Theorem 4.2.3 *For an even integer $n \geq 6$ with $n \equiv 2 \pmod 4$, let $H = C_n \,\square\, K_2$ and let G be the graph of order $2n+1$ obtained from H by adding a new vertex v and joining v to two adjacent vertices on a copy of C_n in H. Then $\chi_{(2,0)}(G) = 2n$ and $\chi_{(2,0)}(G - v) = n$.*

Proof. Let $H = C_n \,\square\, K_2$ be the Cartesian product of the n-cycle C_n and K_2 where the two copies of C_n in H are $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ and u_i is adjacent to v_i for $1 \leq i \leq n$ and let G be the graph obtained from H by adding a new vertex v and joining v to v_1 and v_2 in H. Since n is even, it follows by Theorem 3.1.4 that $\chi_{(2,0)}(G - v) = \chi_{(2,0)}(H) = n$.

It remains to show that $\chi_{(2,0)}(G) = 2n$. Since the coloring that assigns the color 0 to v and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \leq 2n$. To show that $\chi_{(2,0)}(G) \geq 2n$, let c be a minimum monochromatic $(2,0)$-coloring of G. We consider two cases, depending on $c(v) = 0$ or $c(v) = 1$.

Case 1. $c(v) = 0$. The restriction of c to H (also denoted by c) is a monochromatic $(2,0)$-coloring of H by Observation 2.2.10(b). Let $n = 4k+2$ for some positive integer k. For each $i = 1, 2, \ldots, k$, let $B_i = P_4 \,\square\, P_2$ be the block of length 4 in H with $V(B_i) = \{u_{4(i-1)+p}, v_{4(i-1)+p} : p = 1, 2, 3, 4\}$. Since $\sigma(v) = 0$, either $c(v_1) = c(v_2) = 0$ or $c(v_1) = c(v_2) = 1$. There are two subcases.

Subcase 1.1. $c(v_1) = c(v_2) = 0$. Since c must assign the color 1 to at least one vertex in any block $P_2 \,\square\, P_2$ in H by Lemma 4.2.2(a), at least one of u_1 and u_2 is colored 1, say

$c(u_2) = 1$. If $c(u_1) = 1$, then $c(u_n) = c(u_3) = 0$, which implies that F_B contains (u_1, u_2) as a component, a contradiction by Lemma 4.2.2(b). Thus $c(u_1) = 0$. Since $c(u_1) = c(v_2) = 0$, $c(u_2) = 1$ and $\sigma(u_2) = 0$, it follows that $c(u_3) = 1$. Now since $\sigma(v_2) = 0$ and $c(u_2) = 1$, we have $c(v_3) = 1$, which forces that $c(u_4) = 1$ and $c(v_4) = 0$. Now $\sigma(v_4) = \sigma(u_4) = 0$ implies that $c(v_5) = c(u_5) = 0$. Thus the vertices in the block B_1 are colored as shown in Figure 4.1. Now $\sigma(u_1) = \sigma(v_1) = 0$ implies that $c(u_n) = 1$ and $c(v_n) = 0$. Since $c(u_4) = c(u_n)$, $c(v_4) = c(v_n)$, $c(u_5) = c(u_1)$ and $c(v_5) = c(v_1)$, we can apply this argument to B_2, B_3 and so on and conclude that the vertices in each B_i $(1 \le i \le k)$ must be colored the same as B_1; That is, for each j with $1 \le j \le n$,

- $c(u_j) = 1$ if $j \equiv 0, 2, 3 \pmod 4$ and $c(u_j) = 0$ if $j \equiv 1 \pmod 4$ and
- $c(v_j) = 1$ if $j \equiv 3 \pmod 4$ and $c(v_j) = 0$ if $j \equiv 0, 1, 2 \pmod 4$.

However then, $\sigma_c(u_n) = c(u_{n-1}) + c(u_n) + c(v_n) + c(u_1) = 0 + 1 + 0 + 0 \neq 0$ in $\mathbb{Z}_2$, which is a contradiction.

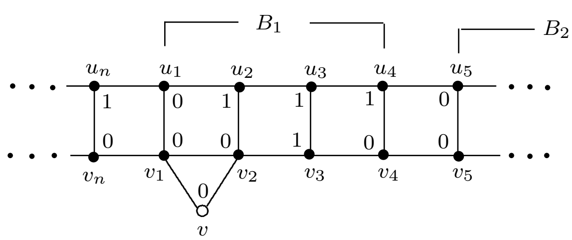


Figure 4.1: Illustrating the coloring c in Subcase 1.1

Subcase 1.2. $c(v_1) = c(v_2) = 1$. First, assume that $c(u_1) = c(u_2)$. If $c(u_1) = c(u_2) = 0$, then $c(v_n) = c(v_3) = 0$ since $\sigma(v_1) = \sigma(v_2) = 0$. However then, F_B contains a component (v_1, v_2), which is impossible. If $c(u_1) = c(u_2) = 1$, then there is a block $P_2 \,\square\, P_2$ each of whose vertices is colored 1. By Lemma 4.2.2(a), either this is impossible or all vertices of H

are colored 1, which implies that $\chi_{(2,0)}(G) \geq 2n$. Thus, we may assume that $c(u_1) \neq c(u_2)$, say $c(u_1) = 0$ and $c(u_2) = 1$. Since $\sigma(u_1) = \sigma(v_1) = 0$, we must have $c(u_n) = c(v_n) = 0$. Then $\sigma(v_2) = 0$ implies $c(v_3) = 1$ and $\sigma(u_2) = 0$ implies $c(u_3) = 0$. Now because $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 0$. Furthermore, $\sigma(u_4) = \sigma(v_4) = 0$ implies that $c(u_5) = 0$ and $c(v_5) = 1$. Thus the vertices in the block B_1 are colored as shown in Figure 4.2. As with Subcase 1.1, since $c(u_4) = c(u_n)$, $c(v_4) = c(v_n)$, $c(u_5) = c(u_1)$ and $c(v_5) = c(v_1)$, we can apply this argument to B_2, B_3 and so on and conclude that the vertices in each B_i $(1 \leq i \leq k)$ must be colored the same as B_1; That is, for each j with $1 \leq j \leq n$,

- $c(u_j) = 1$ if $j \equiv 2 \pmod 4$ and $c(u_j) = 0$ if $j \equiv 0, 1, 3 \pmod 4$ and
- $c(v_j) = 1$ if $j \equiv 1, 2, 3 \pmod 4$ and $c(v_j) = 0$ if $j \equiv 0 \pmod 4$.

Since $n \equiv 2 \pmod 4$, it follows that $c(v_n) = 1$, which contradicts the fact that $c(v_n) = 0$ (see Figure 4.2).

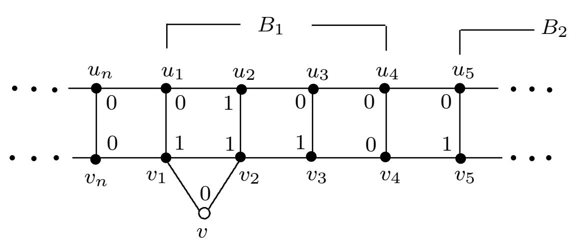


Figure 4.2: Illustrating the coloring c in Subcase 1.2

Case 2. $c(v) = 1$. Since $\sigma(v) = 0$, exactly one of v_1 and v_2 is colored 1, say $c(v_1) = 0$ and $c(v_2) = 1$. Because $\sigma(v_1) = \sigma(v_2) = 0$, it follow that $c(u_1) = c(v_n)$ and $c(u_2) = c(v_3)$.

First, suppose that $c(u_1) = c(v_n) = c(u_2) = c(v_3)$ beginning with $c(x) = 0$ for each $x \in \{u_1, u_2, v_3, v_n\}$. Since $\sigma(u_1) = 0$, it follows that $c(u_n) = 0$. However then, each vertex in $\{u_1, v_1, u_n, v_n\}$ is colored 0, which is impossible by Lemma 4.2.2(c). Now let $c(x) = 1$ for each $x \in \{u_1, u_2, v_3, v_n\}$. Since $\sigma(u_2) = 0$, it follows that $c(u_3) = 1$ and so each vertex in $\{u_2, v_2, u_3, v_3\}$ is colored 1. Since $\sigma(u_3) = \sigma(v_3) = 0$, we have $c(u_4) = c(v_4) = 1$. Recursively, for each i with $3 \leq i \leq n-2$, since $\sigma(u_i) = \sigma(v_i) = 0$, we have $c(u_{i+1}) = c(v_{i+1}) = 1$. In particular, $c(u_{n-1}) = c(v_{n-1}) = 1$. However then, $\sigma_c(u_n) = c(u_{n-1}) + c(u_n) + c(v_n) + c(u_1) = 1 + 0 + 1 + 1 \neq 0$ in $\mathbb{Z}_2$, which is a contradiction.

Next suppose that $c(u_1) = c(v_n) \neq c(u_2) = c(v_3)$. There are two subcases.

Subcase 2.1. *$c(u_1) = c(v_n) = 0$ and $c(u_2) = c(v_3) = 1$.* Since $\sigma(u_1) = \sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_n) = 1$, $c(u_3) = 0$ and $c(v_n) = 0$. Because $\sigma(u_3) = \sigma(v_3) = 0$, we have $c(u_4) = c(v_4) = 0$. First, suppose that $n = 6$. Now $\sigma(u_4) = \sigma(v_4) = 0$ implies that $c(u_5) = 0$ and $c(v_5) = 1$. Since $\sigma(v_5) = 0$, we must have $c(v_6) = 1$, which contradicts that $c(v_n) = 0$. We now suppose that $n \geq 10$ and let $n = 4k + 2$ for some integer $k \geq 2$. For each $i = 1, 2, \ldots, k-1$, let $B_i' = P_4 \square P_2$ be the block of length 4 in H with $V(B_i') = \{u_{4i+p}, v_{4i+p} : p = 0, 1, 2, 3\}$. In particular, the vertex set of B_1' is $\{u_i, v_i : 4 \leq i \leq 7\}$. An argument similar to the one in Subcase 1.2 (see Figure 4.2) shows that the vertices of B_1' are colored as shown in Figure 4.3. Furthermore, since $c(u_7) = c(u_3)$, $c(v_7) = c(v_3)$, $c(u_8) = c(u_4)$ and $c(v_8) = c(v_4)$, we can apply this argument to B_2', B_3' and so on. We then obtain that the vertices in each B_i' must be colored the same as B_1'; That is, for each j with $4 \leq j \leq n-3$,

- $c(u_j) = 1$ if $j \equiv 2 \pmod 4$ and $c(u_j) = 0$ if $j \equiv 0, 1, 3 \pmod 4$ and
- $c(v_j) = 1$ if $j \equiv 1, 2, 3 \pmod 4$ and $c(v_j) = 0$ if $j \equiv 0 \pmod 4$.

Since $\sigma(u_n) = \sigma(v_n) = 0$, we have $c(u_{n-1}) = c(v_{n-1}) = 1$. Furthermore, $\sigma(u_{n-1}) = \sigma(v_{n-1}) = 0$ implies that $c(u_{n-2}) = 1$ and $c(v_{n-2}) = 0$. However then, $\sigma_c(v_{n-2}) = c(v_{n-2}) +$

$c(v_{n-3}) + c(u_{n-2}) + c(v_{n-1}) = 0 + 1 + 1 + 1 \neq 0$ in $\mathbb{Z}_2$, which is a contradiction.

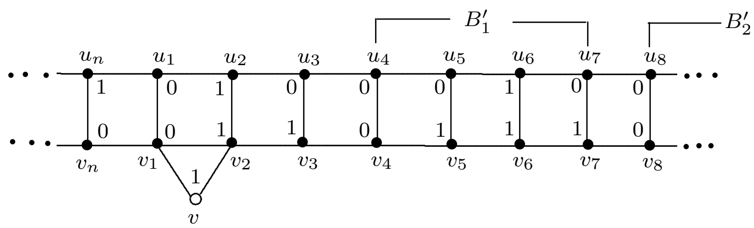


Figure 4.3: Illustrating the coloring c in Subcase 2.1

Subcase 2.2. $c(u_1) = c(v_n) = 1$ *and* $c(u_2) = c(v_3) = 0$. Since $\sigma(u_1) = \sigma(u_2) = 0$, it follows that $c(u_3) = 0$ and $c(u_n) = 1$. Let $B_1'' = P_4 \,\square\, P_2$ be a block of length 4 with vertex set $\{u_i, v_i : 3 \leq i \leq 6\}$. Then the vertices of B_1'' are colored as shown Figure 4.4. If $n = 6$, then a contradiction is produced as $c(u_6) = 0$. For $n \geq 10$, we apply an argument similar to one in Subcase 2.1 (where B_1'' corresponds to the block B_1' in Subcase 2.1) and conclude that the vertices u_i and v_i for $i = 7, 8$ are colored as shown in Figure 4.4. Since $c(u_7) = c(u_3)$, $c(v_7) = c(v_3)$, $c(u_8) = c(u_4)$ and $c(v_8) = c(v_4)$, we now apply the argument to the next block $B_2'' = P_4 \,\square\, P_2$ (see Figure 4.4) and so on. We then obtain for each j with $3 \leq j \leq n$,

- $c(u_j) = 1$ if $j \equiv 1 \pmod 4$ and $c(u_j) = 0$ if $j \equiv 0, 2, 3 \pmod 4$ and
- $c(v_j) = 1$ if $j \equiv 0, 1, 2 \pmod 4$ and $c(v_j) = 0$ if $j \equiv 3 \pmod 4$.

Since $n \equiv 2 \pmod 4$, it follows that $c(u_n) = 0$, which is impossible as $c(u_n) = 1$.

Therefore, $\chi_{(2,0)}(G) = 2n$ and the coloring that assigns the color 0 to v and the color 1 to the remaining vertices of G is the unique monochromatic $(2, 0)$-coloring of G. ■

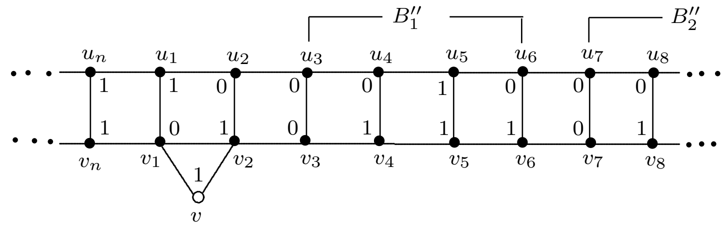


Figure 4.4: Illustrating the coloring c in Subcase 2.2

4.3 On Trees Having Large $(2, 0)$-Chromatic Numbers

In this section, we characterize all trees of order n having $(2, 0)$-chromatic number $n - 2$ or $n - 3$.

Theorem 4.3.1 *Let T be a $(2, 0)$-colorable tree of order $n \geq 4$. Then $\chi_{(2,0)}(T) = n - 2$ if and only if either T is an ods-3-tree or T is obtained from an ods-2-tree T' by adding a pendant edge at the subdividing vertex of T'.*

Proof. By Theorem 4.1.5, it remains to show that if T is a $(2, 0)$-colorable tree of order $n \geq 4$ with $\chi_{(2,0)}(T) = n - 2$, then either T is an ods-3-tree or T is obtained from an ods-2-tree T' by adding a pendant edge at the subdividing vertex of T'. Let c be a minimum monochromatic $(2, 0)$-coloring of T such that $c(u) = c(v) = 0$ for two distinct vertices u and v in T and $c(x) = 1$ for all $x \in V(T) - \{u, v\}$. Since $\sigma(u) = \sigma(v) = 0$, each of u and v is adjacent to an even number of vertices colored 1 by c (it is possible that u or v is adjacent to no vertex colored 1). First, we claim that each of u and v is adjacent to at most two vertices colored 1 by c. For otherwise, we may assume that u is adjacent to at least four vertices colored 1 by c. Let $\{u_1, u_2, \ldots, u_s\}$ be the set of the neighbors of u colored 1 by c, where then $s \geq 4$. Now let Q_j be the component of $T - u$ that contains u_j for $1 \leq j \leq s$. Then the coloring c' defined by $c'(x) = c(x)$ if $x \in V(Q_1) \cup V(Q_2)$ and $c'(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to fewer than $n-2$ vertices of T, which is a contradiction. Thus, as claimed, each of u and v is adjacent to at most two vertices colored 1 by c. Next, we consider two cases, according to the adjacency of u and v.

Case 1. $uv \notin E(T)$. Then every vertex adjacent to u or v is colored 1 and so $\deg_T u = \deg_T v = 2$ by the claim above. Since $uv \notin E(T)$, it follows that $T - \{u,v\}$ consists of three components, say T_1, T_2 and T_3, and each of u and v is adjacent to exactly two of T_1, T_2 and T_3. Furthermore, $F_B = T - \{u,v\}$ is an odd forest and so each T_i is an odd-degree tree for $i = 1,2,3$. Thus T is an ods-3-tree with respect to $T - \{u,v\}$.

Case 2. $uv \in E(T)$. Since each of u and v is adjacent to at most two vertices colored 1 by c, it follows that $1 \le \deg_T v, \deg_T u \le 3$. Because the order of T is at least 4, we may assume that $\deg_T u = 3$. We show that $\deg_T v = 1$, for otherwise, $T - uv$ consists of two nontrivial components Q_u and Q_v containing u and v, respectively. Then the coloring c^* defined by $c^*(x) = c(x)$ if $x \in V(Q_u)$ and $c^*(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to fewer than $n-2$ vertices of T, which is a contradiction. Therefore, $\deg_T v = 1$. Since $F_B = T - \{u,v\}$ is an odd-degree forest consisting of two components T_1 and T_2. Each T_i $(i = 1,2)$ is an odd-degree tree. Thus $T' = T - v$ is an ods-2-tree (with respect to $T - \{u,v\}$) and T is obtained by adding the pendant edge uv at the subdividing vertex u of T'. ■

Theorem 4.3.2 *Let T be a $(2,0)$-colorable tree of order $n \ge 5$. Then $\chi_{(2,0)}(T) = n-3$ if and only if T satisfies one of the following*

(1) *T is a subdivision graph $S(K_{1,3})$ of $K_{1,3}$,*

(2) *T is an ods-4-tree,*

(3) *T is obtained from an ods-3-tree T' by adding a pendant edge at one subdividing vertex of T',*

(4) *T is obtained from an ods-2-tree T' by adding two pendant edges at the subdividing vertex of T'.*

Proof. Since $\chi_{(2,0)}(S(K_{1,3})) = 4$ and Theorem 4.1.5, it remains to show that if T is a $(2,0)$-colorable tree of order $n \geq 5$ with $\chi_{(2,0)}(T) = n-3$, then T satisfies one of (1) – (4). We may assume that $T \neq S(K_{1,3})$. Let c be a minimum monochromatic $(2,0)$-coloring of T such that $c(u) = c(v) = c(w) = 0$ for three distinct vertices u, v and w in T and $c(x) = 1$ for all $x \in V(T) - \{u,v,w\}$. Since $\sigma(u) = \sigma(v) = \sigma(w) = 0$, each of u, v and w is adjacent to an even number of vertices colored 1 by c (it is possible that u or v is adjacent to no vertex colored 1). First, we claim that each of u, v and w is adjacent to at most two vertices colored 1 by c. For otherwise, we may assume that u is adjacent to at least four vertices colored 1 by c. Let $\{u_1, u_2, \ldots, u_s\}$ be the set of the neighbors of u colored 1 by c, where then $s \geq 4$. Now let Q_j be the component of $T - u$ that contains u_j for $1 \leq j \leq s$. For each j with $1 \leq j \leq s$, since $c(u) = 0$, $c(u_j) = 1$ and $\sigma(u_j) = 0$, each u_j must be adjacent to some vertex colored 1 in Q_j and so Q_j is a nontrivial tree. Thus $|V(Q_1) \cup V(Q_2)| \leq n-5$. The coloring c' defined by $c'(x) = c(x)$ if $x \in V(Q_1) \cup V(Q_2)$ and $c'(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most $n-5$ vertices of T, which is a contradiction. Thus, as claimed, each of u, v and w is adjacent to at most two vertices colored 1 by c. The subtree $T[\{u,v,w\}]$ induced by $\{u,v,w\}$ is either $\overline{K}_3$, $P_2 \cup K_1$ (the union of P_2 and K_1) or P_3. We consider these three cases.

Case 1. $T[\{u,v,w\}] = \overline{K}_3$. Then each vertex in $\{u,v,w\}$ is only adjacent to vertices colored 1 by c and so $\deg_T u = \deg_T v = \deg_T w = 2$ by the claim above. Since $\{u,v,w\}$ is an independent set, the forest $F = T - \{u, v, w\}$ consists of four components, say T_1, T_2, T_3 and T_4 and each vertex in $\{u, v, w\}$ is adjacent to exactly two of T_1, T_2, T_3 and T_4. Furthermore, $F_B = F$ is an odd-degree forest and so each T_i is an odd-degree tree for $i = 1, 2, 3, 4$. Thus T is an ods-4-tree with respect to the odd-degree forest F whose subdividing vertices are u, v, w.

Case 2. *$T[\{u, v, w\}] = P_2 \cup K_1$, say $uv \in E(T)$ and $uw, vw \notin E(T)$.* Since each of u, v and w is adjacent to at most two vertices colored 1 by c, it follows that (i) $1 \le \deg_T v, \deg_T u \le 3$ and $\deg_T v$ and $\deg_T u$ are odd and (ii) $\deg_T w = 2$. Because T is connected, at least one of $\deg_T v$ and $\deg_T u$ is 3, say $\deg_T u = 3$. Next, we claim that $\deg_T v = 1$, for otherwise, $\deg_T v = 3$. Let Q_u and Q_v be the two components of $T - uv$ that contains u and v, respectively. Note that each of Q_u and Q_v contains at least five vertices. The coloring c^* defined by $c^*(x) = c(x)$ if $x \in V(Q_u)$ and $c^*(x) = 0$ otherwise is a monochromatic $(2, 0)$-coloring of T that assigns the color 1 to at most $n - 6$ vertices of T, which is a contradiction. Therefore, $\deg_T v = 1$. Since (a) uv is a pendant edge of T, (b) $\deg_T u = 3$ and (c) $\deg_T w = 2$, it follows that $F_B = T - \{u, v, w\}$ is an odd-degree forest consisting of three components, say T_1, T_2 and T_3. Each T_i $(i = 1, 2, 3)$ is an odd-degree tree. Thus $T' = T - v$ is an ods-3-tree (with respect to $T - \{u, v, w\}$) having two subdividing vertices u and w and T is obtained by adding the pendant edge uv at the subdividing vertex u of T'.

Case 3. *$T[\{u, v, w\}] = (v, u, w)$.* Since each of u, v, w is adjacent to at most two vertices colored 1 by c, it follows that (i) $1 \le \deg_T v, \deg_T w \le 3$ and each of $\deg_T v$ and $\deg_T w$ is odd and (ii) $2 \le \deg_T u \le 4$ and $\deg_T u$ is even.

First, we claim that $\deg_T u = 4$, for otherwise, $\deg_T u = 2$. Since the order of T is at least 5, at least one $\deg_T v$ and $\deg_T w$ is 3, say $\deg_T v = 3$. This then will implies that $\deg_T w = 1$. To see this, suppose that $\deg_T w = 3$. Let Q_u and Q_v be the two components of $T - uv$ that contains u and v, respectively. Then Q_u contains at least six vertices. The coloring c^* defined by $c^*(x) = c(x)$ if $x \in V(Q_v)$ and $c^*(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most $n - 7$ vertices of T, which is a contradiction. Therefore, $\deg_T w = 1$. Let v_1 and v_2 be the two neighbors of v that are colored 1. Since $c(v_i) = 1$ for $i = 1, 2$, it follows that v_i is adjacent to an odd number of vertices colored 1. Let Q_1, Q_2 and Q_3 be the three components of $T - v$ where $v_i \in V(Q_i)$ for $i = 1, 2$ and $u \in V(Q_3)$. We may assume that $|V(Q_1)| \leq |V(Q_2)|$. If $T \neq S(K_{1,3})$, it follows that $|(V(Q_2)| \geq 4$. The coloring c^* defined by $c^*(x) = 1$ if $x \in V(Q_1) \cup \{u, w\}$ and $c(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most $n - 4$ vertices of T, which is a contradiction. Therefore, $\deg_T u = 4$, as claimed.

Next, we claim that $\deg_T v = \deg_T w = 1$. Let u_1 and u_2 be the two neighbors of u that are colored 1 by c. Furthermore, T_1, T_2, T_3, T_4 be the four components of $T - u$ where $u_i \in V(T_i)$ for $i = 1, 2$, $v \in V(T_3)$ and $w \in V(T_4)$. Since the restriction of c to T_i for $i = 1, 2$ is a nontrivial monochromatic $(2,0)$-coloring of T_i, it follows that T_i is an odd-degree tree. If $|V(T_3) \cup V(T_4)| \geq 3$, then the coloring c' defined by $c^*(x) = 1$ if $x \in V(T_1) \cup V(T_2)$ and $c(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most $n - 4$ vertices of T, which is a contradiction. Thus $T_3 = T_4 = K_1$ and so $\deg_T v = \deg_T w = 1$, as claimed.

Let $F = T - \{u, v, w\}$ be an odd-degree forest consisting of T_1 and T_2 and let T' be the ods-2-tree (with respect to F) with the subdividing vertex u. Then T is obtained by adding the two pendant edges uv and uw at the subdividing vertex u of T'. ■

According to Theorems 2.4.3 and 4.2.1, 4.3.1 and 4.3.2, for each $a \in \{0, 1, 2, 3\}$, if T is a $(2, 0)$-colorable tree of order n and $T \neq S(K_{1,3})$ such that $\chi_{(2,0)}(T) = n - a$, then $T \in \mathcal{P}_{ods}(F)$ for some odd-degree forest F having b components for some $b \in \{1, 2, 3, 4\}$. This, however, is not the case if $a \geq 4$. For example, let T' be an ods-2-tree consisting of two odd-degree trees T_1, T_2, and the subdividing vertex u and let $P = (v_1, v_2, v_3)$ be a path of order 3. If T is a tree of order n obtained from T' and P by adding the edge uv_1, which is shown in Figure 4.5(a), or T is the tree obtained from T' and P by adding the edge uv_2, which is shown in Figure 4.5(b), then $T \notin \mathcal{P}_{ods}(F)$ for any odd-degree forest F. However, the only monochromatic $(2, 0)$-coloring of T assigns the color 0 to each vertex in $\{u, v_1, v_2, v_3\}$ and the color 1 to the remaining vertices of T and so $\chi_{(2,0)}(T) = n - 4$.

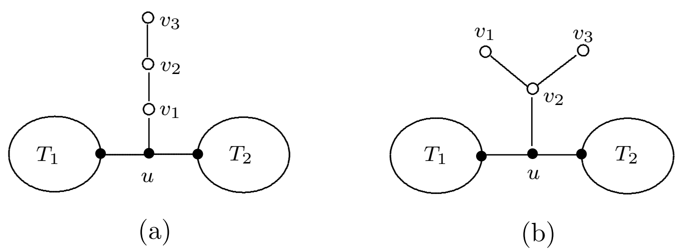


Figure 4.5: Trees T of order n with $\chi_{(2,0)}(T) = n - 4$

It is worthwhile to mention that if T is a tree obtained from an ods-k-tree by adding four or more vertices, then $\chi_{(2,0)}(T)$ can be relatively small. For example, let T' be an ods-2-tree consisting of two odd-degree trees T_1, T_2, and the subdividing vertex u and let $H = 2P_2$ where (x_1, x_2) and (y_1, y_2) be the two copies of P_2 in H. If T is obtained from T' and H by adding two edges ux_1 and uy_1, then $\chi_{(2,0)}(T) = 4$ (see Figure 4.6). In fact, the coloring that assigns the color 1 to each vertex of H and the color 0 to the remaining vertices of T is a minimum monochromatic $(2, 0)$-coloring of T.

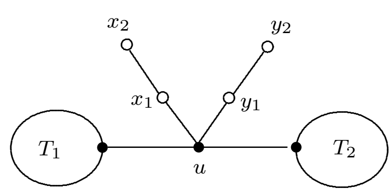


Figure 4.6: A tree T with $\chi_{(2,0)}(T) = 4$

Chapter 5

On Monochromatic $(2,0)$-Spectra

5.1 Introduction

In this chapter, we study the concept of the monochromatic $(2,0)$-spectrum of a graph. For a $(2,0)$-colorable graph G, the *monochromatic* $(2,0)$*-spectrum* $S_{(2,0)}(G)$ (or simply $(2,0)$-*spectrum*) of G is the set of all positive integers k for which exactly k vertices of G can be colored 1 in a monochromatic $(2,0)$-coloring of G. Thus if G is a $(2,0)$-colorable graph of order n and $a \in S_{(2,0)}(G)$, then $\chi_{(2,0)}(G) \leq a \leq n$. We have seen in By Proposition 2.3.4 that if c is a modular monochromatic $(2,0)$-coloring of a connected graph G, then the subgraph of G induced by the vertices colored 1 by c is an odd-degree graph and so the number of vertices colored 1 by c is even. Therefore, if G is a $(2,0)$-colorable graph of order n, then

$$S_{(2,0)}(G) \subseteq \left\{\chi_{(2,0)}(G), \chi_{(2,0)}(G)+2, \ldots, 2\lfloor n/2\rfloor\right\} \subseteq \{2,4,\ldots,2\lfloor n/2\rfloor\}. \tag{5.1}$$

5.2 On $(2,0)$-Spectra of Some Well-Known Graphs

In this section, we determine the $(2,0)$-spectra for some well-known classes of graphs. In order to do this, we first present several well-known $(2,0)$-colorable graphs which have been determined in previous chapters.

Proposition 5.2.1 *(a) Every nontrivial complete graph is* $(2,0)$*-colorable.*

(b) For positive integers r *and* s*, the complete bipartite graph* $K_{r,s}$ *of order* $r+s$ *is* $(2,0)$*-colorable if and only if* r *and* s *are both odd.*

(c) A nontrivial path P_n *of order* n *is* $(2,0)$*-colorable if and only if* $n \equiv 2 \pmod 3$*.*

(d) A cycle C_n *of order* $n \geq 3$ *is* $(2,0)$*-colorable if and only if* $n \equiv 0 \pmod 3$*.*

For each integer $n \geq 2$, let $[[n]] = \{2, 4, \ldots, 2\lfloor n/2 \rfloor\}$ be the set of all even integers between 2 and n. We begin with graphs with the largest possible $(2,0)$-Spectrum.

Proposition 5.2.2 *If* $n \geq 2$ *is an integer, then* $S_{(2,0)}(K_n) = [[n]]$*.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n and, for each i ($1 \leq i \leq \lfloor n/2 \rfloor$), let $U_i = \{v_1, v_2, \ldots, v_{2i}\}$ and so $|U_i| = 2i$. Now let c_i be a coloring that assigns the color 1 to each vertex in U_i and the color 0 to the remaining vertices of K_n. Since c_i is a modular monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $2i$ vertices of K_n for $1 \leq i \leq \lfloor n/2 \rfloor$, it follows that $S_{(2,0)}(K_n) = [[n]]$. ■

The complete graph K_n is not the only graph of order n having $(2,0)$-Spectrum $[[n]]$. In fact, more can be said. First, we present additional definitions. For two vertex-disjoint graphs G and H, let $G + H$ and $G \vee H$ denote the union and join of G and H, respectively. For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and n pairwise vertex-disjoint graphs $H_1, H_2, \ldots, H_n$, the *composition* $G[H_1, H_2, \ldots, H_n]$ of G and H_i ($1 \leq i \leq n$) is the graph $H_1 + H_2 + \cdots + H_n$ (the union of $H_1, H_2, \ldots, H_n$) together with the edges in the set

$\{xy : x \in V(H_i), y \in V(H_j), v_iv_j \in E(G)\}$. If there is a graph H such that $H_i \cong H$ for $1 \le i \le n$, then we write $G[H] = G[H_1, H_2, \ldots, H_n]$.

Proposition 5.2.3 *For each integer $n \ge 5$, there is a connected graph G that is not complete such that $S_{(2,0)}(G) = [[n]]$.*

Proof. For each even integer $n \ge 6$, let $G = P_{\frac{n}{2}}[K_2]$ be the composition of $P_{n/2}$ and K_2 where $(u_1, u_2, \ldots, u_{n/2})$ and $(v_1, v_2, \ldots, v_{n/2})$ are two copies of $P_{\frac{n}{2}}$ in G. For each i with $1 \le i \le n/2$, let $V_i = \{u_1, v_1, u_2, v_2, \ldots, u_i, v_i\}$ and let c_i be the coloring that assigns the color 1 to each vertex in V_i and the color 0 to the remaining vertices of G. Since c_i is a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $2i$ vertices of G for $1 \le i \le n/2$, it follows that $S_{(2,0)}(G) = [[n]]$.

For each odd integer $n \ge 5$, let $G = \left(\frac{n-1}{2}K_2\right) \vee K_1$ where (u_i, v_i) be a copy of K_2 for $1 \le i \le (n-1)/2$. For each i with $1 \le i \le (n-1)/2$, let $V_i = \{u_1, v_1, u_2, v_2, \ldots, u_i, v_i\}$ and let c_i be the coloring that assigns the color 1 to each vertex in V_i and the color 0 to the remaining vertices of G. Since c_i is a modular monochromatic $(2,0)$-coloring of G, it follows that $S_{(2,0)}(G) = [[n]]$. ■

At the other extreme, there is a class of $(2,0)$-colorable graphs having the smallest possible $(2,0)$-Spectrum, namely a singleton. Recall that a $(2,0)$-colorable graph G of order n is called $(2,0)$-*extremal* if $\chi_{(2,0)}(G) = n$.

Observation 5.2.4 *If G is a $(2,0)$-extremal graph of order n, then $S_{(2,0)}(G) = \{n\}$.*

It is not surprising that $(2,0)$-extremal graphs are not the only $(2,0)$-colorable graphs having a singleton $(2,0)$-Spectrum. In fact, if G is a $(2,0)$-colorable graph such that G has a unique monochromatic $(2,0)$-coloring, then $S_{(2,0)}(G)$ is a singleton. This is the case for each of $(2,0)$-colorable complete bipartite graphs, paths or cycles, as we show next.

Proposition 5.2.5 *If $K_{r,s}$ is a $(2,0)$-colorable complete bipartite graph for some positive integers r and s, then $S_{(2,0)}(K_{r,s}) = \{r+s\}$.*

Proof. By Proposition 5.2.1, if $K_{r,s}$ is $(2,0)$-colorable, then r and s are both odd. Let U and V be the two partite sets of $K_{r,s}$ and let c be a monochromatic $(2,0)$-coloring of $K_{r,s}$. Observe that if x and y are in the same partite set then $N(x) = N(y)$ and so $c(x) = c(y)$. Thus if c assigns the color 0 to at least one vertex x of $K_{r,s}$, say $x \in U$, then every vertex in U is assigned the color 0. Now since $\sigma(x) = 0$, it follows that every vertex in V is colored 0, as well since $|V|$ is odd and so c assigns the color 0 to every vertex of $K_{r,s}$ which contradicts the definition of c. This implies the only monochromatic $(2,0)$-coloring of $K_{r,s}$ is the coloring that assigns the color 1 to every vertex of $K_{r,s}$; that is, $K_{r,s}$ is $(2,0)$-extremal. Therefore, $\chi_{(2,0)}(K_{r,s}) = r+s$ and $S_{(2,0)}(K_{r,s}) = \{r+s\}$. ■

Proposition 5.2.6 *If P_n is a $(2,0)$-colorable path of order $n \geq 2$, then $S_{(2,0)}(P_n) = \{2(n+1)/3\}$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ be a $(2,0)$-colorable path of order n. By Proposition 5.2.1, $n \equiv 2 \pmod 3$. Let $n = 3\ell+2$ where $\ell \geq 0$ and c a minimum monochromatic $(2,0)$-coloring of P_n. We claim that $c(v_1) = 1$. If this is not the case, then $c(v_1) = 0$. Since $\sigma(v_i) = 0$ for $1 \leq i \leq n$, it follows that $c(v_i) = 0$, which contradicts the definition of c. Thus, as claimed, $c(v_1) = 1$. Since $\sigma(v_1) = 0$, it follows that $c(v_2) = 1$. Moreover, since $\sigma(v_2) = 0$, it forces $c(v_3) = 0$ and since $\sigma(v_3) = 0$, we have that $c(v_4) = 1$. Continuing this procedure, we obtain

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1,2 \pmod 3) \\ 0 & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

Observe that the color assigned to each vertex v_j $(2 \leq j \leq n)$ is forced and so c is the only monochromatic $(2,0)$-coloring of P_n. Thus $S_{(2,0)}(P_n) = \{2\ell + 2\} = \{2(n+1)/3\}$. ∎

Proposition 5.2.7 *If C_n is a $(2,0)$-colorable cycle of order $n \geq 3$, then $S_{(2,0)}(C_n) = \{2n/3\}$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a $(2,0)$-colorable cycle of order $n \geq 3$. By Proposition 5.2.1, $n \equiv 0 \pmod 3$. Let c be a minimum monochromatic $(2,0)$-coloring of C_n. Since c assigns the color 1 to at least one vertex, we may assume that $c(v_1) = 1$. Since $\sigma(v_1) = 0$, it follows that either $c(v_2) = 1$ and $c(v_n) = 0$ or $c(v_2) = 0$ and $c(v_n) = 1$, say the former. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 0$. Moreover, since $\sigma(v_3) = 0$, it forces $c(v_4) = 1$. Continuing this procedure, we obtain

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \pmod 3 \\ 0 & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

Observe that the color assigned to each vertex v_j $(3 \leq j \leq n-1)$ is forced and so c is the only monochromatic $(2,0)$-coloring of C_n. Thus, $S_{(2,0)}(C_n) = \{2n/3\}$. ∎

We have seen that if G has a unique $(2,0)$-coloring (up to isomorphism), then $S_{(2,0)}(G)$ is a singleton. However, the converse of this statement is not true in general; that is, there are $(2,0)$-colorable graphs G such that $S_{(2,0)}(G) = \{k\}$ for some even integer $k \geq 2$ but G has different monochromatic $(2,0)$-colorings. First, we make an observation for the case when $k = 2$. If c is a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly two vertices u and v, then u and v must be adjacent and so the subgraph induced by the vertices colored 1 is K_2. Thus, if G is a $(2,0)$-colorable graph such that $S_{(2,0)}(G) = \{2\}$, then every monochromatic $(2,0)$-coloring of G assigns the color 1 to exactly two adjacent vertices and so the subgraph induced by the vertices colored 1 by c is K_2. For the case when $k \geq 4$, we have the following.

Theorem 5.2.8 *For each even integer $k \geq 4$, there exists a $(2,0)$-colorable graph G such that (i) $S_{(2,0)}(G) = \{k\}$ and (ii) G has two monochromatic $(2,0)$-colorings c' and c'' for which the subgraphs induced by the vertices colored 1 by c' and c'', respectively, are non-isomorphic.*

Proof. Let $k \geq 4$ be an even integer and let $P = \left(v_1, v_2, \ldots, v_{(3k/2)-1}\right)$ be the path of order $(3k/2) - 1$. Since $k \geq 4$ is even, $k \equiv 0 \pmod 4$ or $k \equiv 2 \pmod 4$. We consider these two cases.

Case 1. $k \equiv 0 \pmod 4$. Then $k = 4\ell$ for some positive integer ℓ. Let G be a graph obtained from P and the $2\ell - 1$ pairwise nonadjacent new vertices $x_1, x_2, \ldots, x_{2\ell-1}$ by joining each x_j $(1 \leq j \leq 2\ell - 1)$ to both $v_{3\ell-1}$ and $v_{3\ell+1}$. Observe that the coloring c' of G defined by

$$c'(v) = \begin{cases} 1 & \text{if } v = v_i \text{ where } i \equiv 1, 2 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

is a monochromatic $(2,0)$-coloring of G. On the other hand, the coloring c'' defined by

$$c''(v) = \begin{cases} 1 & \text{if } v = v_i \text{ where } 1 \leq i \leq 3\ell - 1 \text{ and } i \equiv 1, 2 \pmod 3 \\ & \text{and } v = x_j \text{ where } 1 \leq j \leq 2\ell - 1 \text{ and } v = v_{3\ell} \\ 0 & \text{otherwise} \end{cases}$$

is also a monochromatic $(2,0)$-coloring of G. Each of c' and c'' assigns the color 1 to exactly $k = 4\ell$ vertices of G. Furthermore, the subgraph induced by the vertices colored 1 in c' is $(2\ell)K_2$; while the subgraph induced by the vertices colored 1 in c'' is $(\ell - 1)K_2 + K_{1,2\ell+1}$. Thus, it remains to show that $S_{(2,0)}(G) = \{k\}$; that is, every monochromatic $(2,0)$-coloring of G must assign the color 1 to exactly k vertices of G. Let c be any monochromatic $(2,0)$-coloring of G. It then follows by Observation 1.3.1 that $c(x_j) = c(x_s) = c(v_{3\ell}) \in \{0,1\}$ for each pair j, s of distinct integers with $1 \le j, s \le 2\ell - 1$, We consider two subcases.,

Subcase 1.1. $c(x_j) = c(x_s) = c(v_{3\ell}) = 0$. Then $c(v_{3\ell-1}) = c(v_{3\ell+1})$ since $\sigma(x_j) = 0$ for each $1 \le j \le 2\ell - 1$. If $c(v_{3\ell-1}) = c(v_{3\ell+1}) = 0$, then c must assign the color 0 to every vertex of G, which is impossible. Thus $c(v_{3\ell-1}) = c(v_{3\ell+1}) = 1$. Since $\sigma(v_{3\ell-1}) = \sigma(v_{3\ell+1}) = 0$, it follows that $c(v_{3\ell-2}) = c(v_{3\ell+2}) = 1$. Since $\sigma(v_{3\ell-2}) = \sigma(v_{3\ell+2}) = 0$, it follows that $c(v_{3\ell-3}) = c(v_{3\ell+3}) = 0$. Since $\sigma(v_{3\ell-3}) = \sigma(v_{3\ell+3}) = 0$, it follows that $c(v_{3\ell-4}) = c(v_{3\ell+4}) = 1$. Since $\sigma(v_{3\ell-4}) = \sigma(v_{3\ell+4}) = 0$, it follows that $c(v_{3\ell-5}) = c(v_{3\ell+5}) = 1$. Since $\sigma(v_{3\ell-5}) = \sigma(v_{3\ell+5}) = 0$, it follows that $c(v_{3\ell-6}) = c(v_{3\ell+6}) = 0$. Continue this procedure, we arrive at if $c(x_j) = c(x_s) = c(v_{3\ell}) = 0$, then c assigns the color 1 to exactly k vertices v_i where $i \equiv 1, 2 \pmod 3$ and $1 \le i \le 3k/2 - 1$; that is, c is the coloring c' described above.

Subcase 1.2. $c(x_j) = c(x_s) = c(v_{3\ell}) = 1$. Then $c(v_{3\ell-1}) \ne c(v_{3\ell+1})$ since $\sigma(x_j) = 0$ for each $1 \le j \le 2\ell - 1$. Without loss of generality, we assume $c(v_{3\ell-1}) = 1$ and $c(v_{3\ell+1}) = 0$. Since $\sigma(v_{3\ell+1}) = 0$ and 2ℓ is even, $c(v_{3\ell+2}) = 0$. Since $\sigma(v_{3\ell+2}) = 0$, $c(v_{3\ell+3}) = 0$. Continue this process, we have that c assigns the color 0 to v_i where $3\ell + 1 \le i \le 3k/2 - 1$. Moreover, since $\sigma(v_{3\ell-1}) = 0$, $c(v_{3\ell-2}) = 1$. Since $\sigma(v_{3\ell-2}) = 0$, we have that $c(v_{3\ell-3}) = 0$. Since $\sigma(v_{3\ell-3}) = 0$, it follows that $c(v_{3\ell-4}) = 1$. Now, since $\sigma(v_{3\ell-4}) = 0$, $c(v_{3\ell-5}) = 1$. Since $\sigma(v_{3\ell-5}) = 0$, we have that $c(v_{3\ell-6}) = 0$. Continue this procedure, we arrive at if $c(x_j) = c(x_s) = c(v_{3\ell}) = 1$, then c assigns the color 1 to exactly k vertices; namely v_i where $i \equiv 1, 2 \pmod 3$ and $1 \le i \le 3\ell - 1$, $v_{3\ell}$ and x_j where $1 \le j \le 2\ell - 1$; that is, c is the coloring c'' described above. Hence $S_{(2,0)}(G) = \{k\}$. The argument above also shows that G has exactly two distinct monochromatic $(2,0)$-colorings.

Case 2. $k \equiv 2 \pmod 4$. Then $k = 4\ell + 2$ for some positive integer ℓ. Let G be a graph obtained from P and the $2\ell + 1$ pairwise nonadjacent new vertices $x_1, x_2, \ldots, x_{2\ell+1}$ by joining each x_j $(1 \le j \le 2\ell + 1)$ to both $v_{3\ell+1}$ and $v_{3\ell+2}$. Observe that the coloring c' of G such that

$$c'(v) = \begin{cases} 1 & \text{if } v = v_i \text{ where } i \equiv 1, 2 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

is a monochromatic $(2, 0)$-coloring of G. On the other hand, the coloring c'' defined by

$$c''(v) = \begin{cases} 1 & \text{if } v = v_i \text{ where } 1 \le i \le 3\ell + 1 \text{ and } i \equiv 1, 2 \pmod 3 \\ & \text{and } v = x_j \text{ where } 1 \le j \le 2\ell + 1 \\ 0 & \text{otherwise} \end{cases}$$

is also a monochromatic $(2, 0)$-coloring of G. Each of c' and c'' assigns the color 1 to exactly $k = 4\ell + 2$ vertices of G. Furthermore, the subgraph induced by the vertices colored 1 in c' is $(2\ell + 1)K_2$; while the subgraph induced by the vertices colored 1 in c'' is $\ell K_2 + K_{1,2\ell+1}$. Thus, it remains to show that $S_{(2,0)}(G) = \{k\}$; that is, every monochromatic $(2, 0)$-coloring of G must assign the color 1 to exactly k vertices of G. Let c be any monochromatic $(2, 0)$-coloring of G. It then follows by Observation 1.3.1 that $c(x_j) = c(x_s) \in \{0, 1\}$ for each pair j, s of distinct integers with $1 \le j, s \le 2\ell + 1$. We consider two subcases.

Subcase 2.1. $c(x_j) = c(x_s) = 0$. Then $c(v_{3\ell+1}) = c(v_{3\ell+2})$ since $\sigma(x_j) = 0$ for each $1 \le j \le 2\ell+1$. If $c(v_{3\ell+1}) = c(v_{3\ell+2}) = 0$, then c must assign the color 0 to every vertex of G, which is impossible. Thus $c(v_{3\ell+1}) = c(v_{3\ell+2}) = 1$. Since $\sigma(v_{3\ell+1}) = \sigma(v_{3\ell+2}) = 0$, it follows that $c(v_{3\ell}) = c(v_{3\ell+3}) = 0$. Since $\sigma(v_{3\ell}) = \sigma(v_{3\ell+3}) = 0$, it follows that $c(v_{3\ell-1}) = c(v_{3\ell+4}) = 1$. Since $\sigma(v_{3\ell-1}) = \sigma(v_{3\ell+4}) = 0$, it follows that $c(v_{3\ell-2}) = c(v_{3\ell+5}) = 1$. Since $\sigma(v_{3\ell-2}) = \sigma(v_{3\ell+5}) = 0$, it follows that $c(v_{3\ell-3}) = c(v_{3\ell+6}) = 0$. Since $\sigma(v_{3\ell-3}) = \sigma(v_{3\ell+6}) = 0$, it follows that $c(v_{3\ell-4}) = c(v_{3\ell+7}) = 1$. Since $\sigma(v_{3\ell-4}) = \sigma(v_{3\ell+7}) = 0$, it follows that $c(v_{3\ell-5}) = c(v_{3\ell+8}) = 1$. Continue this procedure, we arrive at if $c(x_j) = c(x_s) = 0$, then c

assigns the color 1 to exactly k vertices v_i where $i \equiv 1, 2 \pmod 3$ and $1 \le i \le 3k/2 - 1$; that is, c is the coloring c' described above.

Subcase 2.2. $c(x_j) = c(x_s) = 1$. Then $c(v_{3\ell+1}) \neq c(v_{3\ell+2})$ since $\sigma(x_j) = 0$ for each $1 \le j \le 2\ell + 1$. Without loss of generality, we assume $c(v_{3\ell+1}) = 1$ and $c(v_{3\ell+2}) = 0$. Since $\sigma(v_{3\ell+2}) = 0$ and $2\ell + 1$ is odd, $c(v_{3\ell+3}) = 0$. Since $\sigma(v_{3\ell+3}) = 0$, $c(v_{3\ell+4}) = 0$. Continue this process, we have that c assigns the color 0 to v_i where $3\ell + 2 \le i \le 3k/2 - 1$. Moreover, since $\sigma(v_{3\ell+1}) = 0$, $c(v_{3\ell}) = 0$. Since $\sigma(v_{3\ell}) = 0$, we have that $c(v_{3\ell-1}) = 1$. Since $\sigma(v_{3\ell-1}) = 0$, it follows that $c(v_{3\ell-2}) = 1$. Now, since $\sigma(v_{3\ell-2}) = 0$, $c(v_{3\ell-3}) = 0$. Since $\sigma(v_{3\ell-3}) = 0$, we have that $c(v_{3\ell-4}) = 1$. Continue this procedure, we arrive at if $c(x_j) = c(x_s) = 1$, then c assigns the color 1 to exactly k vertices; namely v_i where $i \equiv 1, 2 \pmod 3$ and $1 \le i \le 3\ell + 1$, and x_j where $1 \le j \le 2\ell + 1$; that is, c is the coloring c'' described above. Hence $S_{(2,0)}(G) = \{k\}$. Again, the argument above also shows that G has exactly two distinct monochromatic $(2, 0)$-colorings. ■

We illustrate the proof of Theorem 5.2.8 for $k = 6$. Let G be the graph obtained from $P_8 = (v_1, v_2, \ldots, v_8)$ and three pairwise nonadjacent vertices x_1, x_2, x_3 by joining each x_i $(1 \le i \le 3)$to both v_4 and v_5. Observe that the coloring c' of G that assigns the color 1 to each vertex in the set $\{v_1, v_2, v_4, v_5, v_7, v_8\}$ and the color 0 to the remaining vertices of G is a monochromatic $(2, 0)$-coloring of G. On the other hand, the coloring c'' that assigns the color 1 to each vertex in the set $\{v_1, v_2, v_4, x_1, x_2, x_3\}$ and the color 0 to the remaining vertices of G is also a monochromatic $(2, 0)$-coloring of G. The subgraph induced by the vertices colored 1 in c' is $3K_2$ (the union of three copies of K_2); while the subgraph induced by the vertices colored 1 in c' is $K_2 + K_{1,3}$ (the union of K_2 and $K_{1,3}$) and so c' and c'' are different. We claim that $S_{(2,0)}(G) = \{6\}$; that is, every monochromatic $(2, 0)$-coloring of G must assign the color 1 to exactly 6 vertices of G. Let c be any monochromatic $(2, 0)$-

coloring of G. It then follows by Observation 1.3.1 that $c(x_1) = c(x_2) = c(x_3) \in \{0, 1\}$. If $c(x_1) = c(x_2) = c(x_3) = 0$, then $c(v_4) = c(v_5)$ since $\sigma(x_i) = 0$ for each $1 \leq i \leq 3$. If $c(v_4) = c(v_5) = 0$, then c must assign the color 0 to every vertex of G, which is impossible. Thus $c(v_4) = c(v_5) = 1$. Since $\sigma(v_4) = \sigma(v_5) = 0$, it follows that $c(v_3) = c(v_6) = 0$. Since $\sigma(v_3) = \sigma(v_6) = 0$, it follows that $c(v_2) = c(v_7) = 1$. Since $\sigma(v_2) = \sigma(v_7) = 0$, it follows that $c(v_1) = c(v_8) = 1$. Thus if $c(x_1) = c(x_2) = c(x_3) = 0$, then c assigns the color 1 to exactly 6 vertices v_1, v_2, v_4, v_5, v_7, and v_8; that is, c is the coloring c' described above. If $c(x_1) = c(x_2) = c(x_3) = 1$, then $c(v_4) \neq c(v_5)$ since, for each $1 \leq i \leq 3$, $\sigma(x_i) = 0$. Without loss of generality, we assume $c(v_4) = 1$ and $c(v_5) = 0$. Since $\sigma(v_4) = 0$, $c(v_3) = 0$. Since $\sigma(v_3) = 0$, we have that $c(v_2) = 1$. Since $\sigma(v_2) = 0$, it follows that $c(v_1) = 1$. Moreover, since $\sigma(v_5) = 0$, $c(v_6) = 0$. Since $\sigma(v_6) = 0$, we have that $c(v_7) = 0$. Since $\sigma(v_7) = 0$, it follows that $c(v_8) = 0$. Thus if $c(x_1) = c_(x_2) = c(x_3) = 1$ then c assigns the color 1 to exactly 6 vertices v_1, v_2, v_4, x_1, x_2, and x_3; that is, c is the coloring c'' described above. Hence $S_{(2,0)}(G) = \{6\}$ and G has exactly two distinct monochromatic $(2, 0)$-colorings.

For each graph G of order n we have considered so far, either $|S_{(2,0)}(G)| = 1$ or $|S_{(2,0)}(G)| = \lfloor n/2 \rfloor$. This, of course, is not the case in general. Next, we determine three classes of graphs constructed from the n-cycle C_n, namely the wheel $C_n \vee K_1$, the ladder $C_n \,\square\, K_2$ and the corona $\mathrm{cor}(C_n)$ of C_n. In order to do this, we first present some preliminary results.

For each integer $n \geq 3$, recall that the graph $C_n \vee K_1$ is $(2, 0)$-colorable if and only if $n \not\equiv 2, 4 \pmod 6$ and

$$\chi_{(2,0)}(C_n \vee K_1) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 6 \\ n + 1 & \text{if } n \equiv 1, 5 \pmod 6 \\ \frac{n}{3} + 1 & \text{if } n \equiv 3 \pmod 6. \end{cases} \tag{5.2}$$

Also, if $G \in \{C_n \,\square\, K_2, \text{cor}(C_n)\}$, then G is $(2,0)$-colorable and

$$\chi_{(2,0)}(G) = \begin{cases} n & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases} \tag{5.3}$$

For a modular monochromatic k-coloring $c : V(G) \to \mathbb{Z}_k$, $k \geq 2$, recall that the *complementary coloring* $\overline{c} : V(G) \to \mathbb{Z}_k$ of c is defined by $\overline{c}(v) = k - 1 - c(v)$. In particular, the complementary coloring $\overline{c}$ of a monochromatic $(2,0)$-coloring is defined by $\overline{c}(v) = 1 - c(v)$ for each $v \in V(G)$. Thus $c(v) = 0$ if and only if $\overline{c}(v) = 1$ and $c(v) = 1$ if and only if $\overline{c}(v) = 0$. We have seen in Proposition 3.1.2 that if G is a connected odd-degree graph and c is a monochromatic $(2,0)$-coloring, then either $\overline{c}$ is a trivial coloring that assigns the color 0 to every vertex of G or $\overline{c}$ is a monochromatic $(2,0)$-coloring of G.

Also, recall that if c is a monochromatic $(2,0)$-coloring of a connected graph G, then c must assign the color 1 to an even number of even vertices of G by Proposition 3.2.1. Hence, if G has exactly one even vertex x, then $c(x) = 0$.

Theorem 5.2.9 *Let $n \geq 3$ be an integer.*

(*a*) *If $C_n \vee K_1$ is $(2,0)$-colorable, then*

$$S_{(2,0)}(C_n \vee K_1) = \begin{cases} \{2n/3\} & \text{if } n \equiv 0 \pmod 6 \\ \{n+1\} & \text{if } n \equiv 1, 5 \pmod 6 \\ \{(n/3)+1, 2n/3, n+1\} & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

(*b*) *If $G \in \{C_n \,\square\, K_2, \text{cor}(C_n)\}$, then*

$$S_{(2,0)}(G) = \begin{cases} \{n, 2n\} & \text{if } n \text{ is even} \\ \{2n\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We first verify (a). Let $G = C_n \vee K_1$ where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ for some integer $n \geq 3$ and $V(K_1) = \{v\}$. If G is $(2,0)$-colorable, then $n \not\equiv 2, 4 \pmod 6$. First, suppose that

$n \equiv 0 \pmod 6$. Since $\chi_{2,0}(G) = 2n/3$, it follows that there is no monochromatic $(2,0)$-coloring of G that assigns the color 1 to ℓ vertices where $2 \le \ell \le 2n/3 - 2$. Now assume that there is a monochromatic $(2,0)$-coloring c of G that assigns the color 1 to k vertices where $2n/3 + 2 \le k \le n$. Since v is the only vertex that is even degree, $c(v) = 0$ by Proposition 3.2.1. Moreover, since $2n/3 + 2 \le k \le n$, there are three consecutive vertices of C_n that assigned the color 1 by c; without loss of generality, say $c(v_1) = c(v_2) = c(v_3) = 1$. This, however, implies that $\sigma(v_2) = 1$, which is impossible. Thus such a monochromatic $(2,0)$-coloring c of G does not exist. So $S_{(2,0)}(G) = \{2n/3\}$.

Next, suppose that $n \equiv 1, 5 \pmod 6$. Since $\chi_{(2,0)}(G) = n + 1$, it follows that the only $(2,0)$-coloring of G is the coloring that assigns the color 1 to every vertex of G and so $S_{(2,0)}(G) = \{n+1\}$. Finally, we let $n \equiv 3 \pmod 6$. Note that since G is an odd-degree graph, it follows that the coloring that assigns the color 1 to every vertex of G is a monochromatic $(2,0)$-coloring of G. Since $\chi_{(2,0)}(G) = (n/3)+1$, there is a monochromatic $(2,0)$-coloring of G that assigns the color 1 to exactly $(n/3)+1$ vertices of G. By Proposition 3.1.2 then, G has a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $(n+1)-(n/3+1) = 2n/3$ vertices of G. Furthermore, there is no $(2,0)$-coloring of G that assigns the color 1 to exactly ℓ vertices for each ℓ with $2 \le \ell \le n/3 - 1$ or $2n/3 + 2 \le \ell \le n - 1$. Now assume that there is a monochromatic $(2,0)$-coloring c of G that assigns the color 1 to t vertices where $n/3 + 3 \le t \le 2n/3 - 2$. If $c(v) = 0$, then the restriction of c to C_n is a monochromatic $(2,0)$-coloring of C_n that assigns the color 1 to t vertices of C_n, which is impossible since $\chi_{(2,0)}(C_n) = 2n/3$. If $c(v) = 1$, then since $\sigma(v) = 0$, there is a vertex v_i for some i $(1 \le i \le n)$ such that $c(v_i) = 1$; without loss of generality, assume that $c(v_1) = 1$. Since $\sigma(v_1) = 0$, it follows that c assigns the same color to v_2 and v_n. If $c(v_2) = c(v_n) = 1$, then it follows that every vertex of G is assigned the color 1, which is impossible. Now, if $c(v_2) = c(v_n) = 0$, then since $\sigma(v_2) = 0$, c assigns the color 0 to v_3. Since $\sigma(v_3) = 0$, $c(v_4) = 1$. Continuing this procedure, we have that

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 0 & \text{if } i \equiv 0, 2 \pmod 3 \end{cases}$$

So the number of vertices colored 1 by c is $(n/3) + 1$, which is not possible. Hence $S_{(2,0)}(G)$ $= \{(n/3) + 1, 2n/3, n + 1\}$.

Next, we verify (b). Let $G \in \{C_n \square K_2, \operatorname{cor}(C_n)\}$. If n is odd, then G is a $(2, 0)$-extremal graph of order $2n$. It then follows by Observation 5.2.4 that $S_{(2,0)}(G) = \{2n\}$. Thus, we may assume that n is even. Since G is an odd-degree graph, the coloring that assign the color 1 to each vertex of G is a monochromatic $(2, 0)$-coloring of G and so $2n \in S_{(2,0)}(G)$. Since $\chi_{(2,0)}(G) = n$, each monochromatic $(2, 0)$-coloring must assign the color 1 to at least n vertices of G. Suppose that c is a monochromatic $(2, 0)$-coloring of G that assigns the color 1 to ℓ vertices of G where $n \le \ell \le 2n-2$. By Proposition 3.1.2, $\bar{c}$ is a monochromatic $(2, 0)$-coloring of G that assigns the color 1 to $2n - \ell \le n$ vertices of G. This implies that $\ell = n$. Consequently, every monochromatic $(2, 0)$-coloring of G either assigns the color 1 to all vertices of G or to exactly n vertices of G. Therefore, $S_{(2,0)}(G) = \{n, 2n\}$. ■

If G is a nontrivial connected graph of order n , then $1 \le |S_{(2,0)}(G)| \le \lfloor n/2 \rfloor$. Next, we show that every pair k, n of integers with $1 \le k \le \lfloor n/2 \rfloor$ can be realized as the cardinality of the spectrum and the order of a connected graph, respectively. We have seen in Observation 4.1.4 that if uv is a pendant edge in a $(2, 0)$-colorable graph G, then $c(u) = c(v)$ for every modular monochromatic $(2, 0)$-coloring c of G. This observation will be useful for us.

Theorem 5.2.10 *For every pair k, n of integers with $1 \le k \le \lfloor n/2 \rfloor$, there is a connected graph G of order n such that $|S_{(2,0)}(G)| = k$.*

Proof. Let k and n be integers such that $1 \leq k \leq \lfloor n/2 \rfloor$. Since the result is true for $k = 1$ or $k = \lfloor n/2 \rfloor$, we may assume that $2 \leq k \leq \lfloor n/2 \rfloor - 1$. Thus $n \geq 6$. We consider two cases, according to whether n is odd or n is even.

Case 1. *n is odd.* Let $H = kK_2 \vee K_1$ be the join of kK_2 and K_1, where $V(K_1) = \{v\}$ and $E(kK_2) = \{v_{i1}v_{i2} : 1 \leq i \leq k\}$, and let $F = K_{1,n-2k-2}$ be the star of order $n - 2k - 1$, where $V(F) = \{u, u_1, u_2, \ldots, u_{n-2k-2}\}$ and u is the central vertex of F. The graph G of order n is obtained from H and F by adding the edge uv. We show that $|S_{(2,0)}(G)| = k$. For each j with $1 \leq j \leq k$, the coloring c_j that assigns the color 1 to the vertices in $V_j = \{v_{i1}, v_{i2} : 1 \leq i \leq j\}$ and the color 0 to the remaining vertices of G is a modular monochromatic $(2,0)$-coloring of G. Hence $\{2, 4, \ldots, 2k\} \subseteq S_{(2,0)}(G)$ and so $|S_{(2,0)}(G)| \geq k$.

Next, we show that $|S_{(2,0)}(G_k)| \leq k$. Assume, to the contrary, $|S_{(2,0)}(G_k)| > k$. Then there is a modular monochromatic $(2,0)$-coloring c of G such that c assigns the color 1 to $\ell \geq 2(k+1)$ vertices of G. First, suppose that $c(v) = 0$. Since $|V(H) - \{v\}| = 2k$, there is a vertex in $V(F)$ that is colored 1 by c. It then follows by Observation 4.1.4 that every vertex in $V(F)$ must be colored 1 by c. Since $c(v) = 0$, it follows that $c(v_{i1}) = c(v_{i2})$ for $1 \leq i \leq k$. However then, $\sigma(v) = 1$ in $\mathbb{Z}_2$ which is impossible. Next, suppose that $c(v) = 1$. Since $\sigma(u) = 0$ and $c(v) = 1$, it follows that every vertex in $V(F)$ must be colored 1 by c. Since n is odd, $n - 2k - 2$ is odd and so $\sigma(u) = 1$ in $\mathbb{Z}_2$, which is impossible. Therefore, $|S_{(2,0)}(G)| \leq k$ and so $|S_{(2,0)}(G)| = k$.

Case 2. *n is even.* Let H be defined in Case 1. The graph G of order n is obtained from H by (i) adding $n - 2k - 1$ new vertices u, u_1, u_2, $\ldots$, u_{n-2k-2} and (ii) joining u to v_{11} and v_{12} and joining each of u_i $(1 \leq i \leq n - 2k - 2)$ to the vertex v. We show that $|S_{(2,0)}(G)| = k$. For each j with $1 \leq j \leq k$, the coloring c_j that assigns the color 1 to the vertices in $V_j = \{v_{i1}, v_{i2} : 1 \leq i \leq j\}$ and the color 0 to the remaining vertices of G is a modular monochromatic $(2,0)$-coloring of G. Hence $\{2, 4, \ldots, 2k\} \subseteq S_{(2,0)}(G)$ and so $|S_{(2,0)}(G)| \geq k$.

Next, we show that $|S_{(2,0)}(G_k)| \leq k$. Assume, to the contrary, $|S_{(2,0)}(G_k)| > k$. Then there is a modular monochromatic $(2,0)$-coloring c of G such that c assigns the color 1 to $\ell \geq 2(k+1)$ vertices of G. First, suppose that $c(v) = 0$. Then $c(u_i) = 0$ for $1 \leq i \leq n-2k-2$ by Observation 4.1.4. Thus c assigns the color 1 to at most $2k+1$ vertices in G, which is a contradiction. Next, suppose that $c(v) = 1$. Since $\sigma(v_{11}) = c(v) + c(v_{11}) + c(v_{12}) + c(u) = 0$, it follows that either exactly one vertex in $\{v_{11}, v_{12}, u\}$ is colored 1 or every vertex in $\{v_{11}, v_{12}, u\}$ are colored 1. In either case, however, we have that $\sigma(u) = 1$, which is impossible. Hence $|S_{(2,0)}(G_k)| \leq k$ and so $|S_{(2,0)}(G_k)| = k$. ■

5.3 On $(2,0)$-Realizable Sets

In the proof of Theorem 5.2.10, we saw that for each pair k, n of positive integers with $2 \leq k \leq \lfloor n/2 \rfloor$, there is a connected graph G of order n such that $S_{(2,0)}(G) = \{2, 4, \ldots, 2k\}$. This gives rise to the following question: Let S be a set consisting of $k \geq 2$ positive even integers. Does there exist a connected graph G such that $S_{(2,0)}(G) = S$? If the answer is yes, then the set S is referred to as a $(2,0)$-*realizable set*; otherwise, S is a non-$(2,0)$-realizable. Thus, $\{2, 4, \ldots, 2k\}$ is realizable for each integer $k \geq 2$. In fact, for each even integer $s \geq 2$, the singleton $\{s\}$ is $(2,0)$-realizable. For example, if $s = 2$, let $G = K_3$ and so $S_{(2,0)}(G) = \{2\}$; while if $s \geq 4$, let G be an odd-degree tree of order s (a tree each of whose vertices has odd degree) and so G is $(2,0)$-colorable. Since the only monochromatic $(2,0)$-coloring of G assigns the color 1 to every vertex of G, it follows that $S_{(2,0)}(G) = \{s\}$. On the other hand, there are infinitely many sets of positive even integers that are not the

$(2,0)$-spectrum of any $(2,0)$-colorable graph. To show this, we first present an additional definition and a useful observation.

For two monochromatic (s,t)-colorings c and c' of a connected graph G and two integers x and y, define *the linear combination* $xc + yc' : V(G) \to \mathbb{Z}_s$ of c and c' by

$$(xc + yc')(v) = xc(v) + yc'(v) \text{ for each } v \in V(G).$$

Lemma 5.3.1 *If c and c' are monochromatic (s,t)-colorings of a connected graph G, then each linear combination $xc + yc'$ is a monochromatic $(s,(x+y)t)$-coloring of G for all integers x and y.*

Proof. For a vertex $v \in V(G)$, let $\sigma_c(v) = as + t$ and $\sigma_{c'}(v) = bs + t$, where $a, b \in \mathbb{Z}$. Let $c'' = xc + yc'$. Then $\sigma_{c''}(v) = x(as+t) + y(bs+t) = (xa+yb)s + (x+y)t$. Thus $\sigma_{c''}(v) = (x+y)t \in \mathbb{Z}_s$. ■

In fact, with the aid of mathematical induction and Lemma 5.3.1, we have the following result.

Corollary 5.3.2 *If $c_1, c_2, \ldots, c_k$ are $k(\geq 2)$ monochromatic $(s,0)$-colorings of a connected graph G for some integer $s \geq 2$, then each linear combination of $c_1, c_2, \ldots, c_k$ is also a monochromatic $(s,0)$-coloring of G.*

We are now prepared to show that there are infinitely many sets of positive even integers that are not the $(2,0)$-spectrum of any $(2,0)$-colorable graph.

Proposition 5.3.3 *For each integer $k \geq 32$, the set $\{6, 14, k\}$ is not the monochromatic $(2,0)$-spectrum for any connected graph.*

Proof. Assume, to the contrary, that $S = \{6, 14, k\}$ is the monochromatic $(2,0)$-spectrum for some connected graph G. For each $i \in \{6, 14, k\}$, let c_i be a monochromatic $(2,0)$-coloring of G that assigns the color 1 to exactly i vertices of G and let V_i be the set of vertices colored 1 by c_i. By Corollary 5.3.2, for each pair i, j of distinct integers where $i, j \in S$ and $i \neq j$, the linear combination $c_i + c_j$ of c_i and c_j is also a monochromatic $(2,0)$-coloring of G. Let $V_{i,j}$ be the set of vertices colored 1 by $c_i + c_j$ where $i, j \in S$ and $i \neq j$. Then

$$|V_{i,j}| = |V_i| + |V_j| - 2|V_i \cap V_j|. \tag{5.4}$$

Also, the linear combination $c_6 + c_{14} + c_k$ is a monochromatic $(2,0)$-coloring of G. Let $V_{6,14,k}$ be the set of vertices colored 1 by $c_6 + c_{14} + c_k$. Since S is the monochromatic $(2,0)$-spectrum G, it follows that $|V_{i,j}| \in S$ for each pair i, j of distinct integers where $i, j \in S$ and $|V_{6,14,k}| \in S$. In the Venn digram of $V_6 \cup V_{14} \cup V_k$ shown in Figure 5.1, let $W_1 = V_6 - (V_{14} \cup V_k)$, $W_2 = (V_6 \cap V_{14}) - V_k$, $W_3 = V_{14} - (V_6 \cup V_k)$, $W_4 = (V_6 \cap V_k) - V_{14}$, $W_5 = V_6 \cap V_{14} \cap V_k$, $W_6 = (V_{14} \cap V_k) - V_6$ and $W_7 = V_k - (V_6 \cup V_{14})$. Let $w_i = |W_i|$ for $1 \leq i \leq 7$. Since $0 \leq |V_6 \cap V_{14}| \leq 6$ and $|V_{6,14}| \in S$, it follows by (5.4) that $|V_{6,14}| \leq 20$, which implies that $|V_6 \cap V_{14}| = 3$ (and so $|V_{6,14}| = 14$). Similarly, $|V_6 \cap V_k| = 3$ (and so $|V_{6,k}| = k$); while $|V_{14} \cap V_k| = 7$ (and so $|V_{14,k}| = k$). Since $w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = |V_6 \cup V_{14}| = 17$, it follows that $w_7 = |V_k - (V_6 \cup V_{14})| = k - 17 \geq 15$, which implies that $|V_{14,k}| \geq 15$ and $|V_{6,14,k}| \geq 15$. Since $S = \{6, 14, k\}$ is the monochromatic $(2,0)$-spectrum of G, it follows that $|V_{14,k}| = k$ and $|V_{6,14,k}| = k$. Therefore, we have the following:

$$\begin{aligned}
|V_6| &= 6 = w_1 + w_2 + w_4 + w_5 \\
|V_{14}| &= 14 = w_2 + w_3 + w_5 + w_6 \\
|V_k| &= k = w_4 + w_5 + w_6 + w_7 \\
|V_6 \cap V_{14}| &= 3 = w_2 + w_5 \\
|V_6 \cap V_k| &= 3 = w_4 + w_5 \\
|V_{14} \cap V_k| &= 7 = w_5 + w_6 \\
|V_{14,k}| &= k = w_2 + w_3 + w_4 + w_7 \\
|V_{6,14,k}| &= k = w_1 + w_3 + w_5 + w_7.
\end{aligned}$$

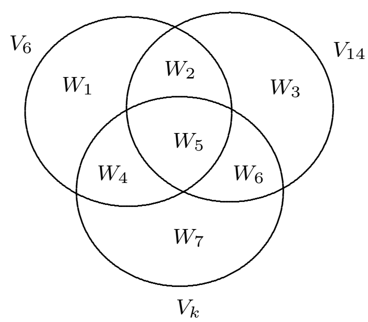


Figure 5.1: The Venn digram of $V_6 \cup V_{14} \cup V_k$

Since $w_2 + w_5 = w_4 + w_5 = 3$, it follows that $w_2 = w_4$. Now $w_2 + w_3 + w_4 + w_7 = w_1 + w_3 + w_5 + w_7 = k$ implies that $w_2 + w_4 = w_1 + w_5$. Because $w_1 + w_2 + w_4 + w_5 = 6$, it follows that $w_2 + w_4 = w_1 + w_5 = 3$. However then, $w_2 = w_4$ and $w_2 + w_4 = 3$, which is impossible. ∎

With the aid of an argument similar to the one used in the proof of Proposition 5.3.3, we can show that for any given integer $t \geq 3$, there is a set of t positive even integers that is not $(2, 0)$-realizable; that is, the following is a consequence of Proposition 5.3.3.

Corollary 5.3.4 *For each integer $t \geq 3$, there is a set S of positive even integers such that $|S| = t$ and S is not the $(2,0)$-spectrum of any $(2,0)$-colorable graph.*

Proof. Since the statement is true for $t = 3$ by Proposition 5.3.3, we may assume that $t \geq 4$. Let $k \geq 32$ be an integer and $T \subseteq \{k + 20 + 2i : i \in \mathbb{N}\}$ where $|T| = t - 3 \geq 1$. Now, let $S_t = S \cup T$ where $S = \{6, 14, k\}$ and so $|S_t| = t$. We claim that S_t is not $(2,0)$-realizable.

Assume, to the contrary, that the set S_t is the monochromatic $(2,0)$-spectrum for some connected graph G. For each integer $j \in S_t$, let c_j be a monochromatic $(2,0)$-coloring of G that assigns the color 1 to exactly j vertices of G and let V_j be the set of vertices colored 1 by c_j. By using mathematical induction, the result in Corollary 5.3.2 can be extended to any linear combination of (finite) monochromatic (s,0)-colorings of G. In particular, for each pair j, l of distinct integers where $j, l \in S$, the linear combination $c_j + c_l$ of c_j and c_l is a monochromatic $(2,0)$-coloring of G. Let $V_{j,l}$ be the set of vertices colored 1 by $c_j + c_l$ where $j, l \in S$ and $j \neq l$. Then

$$|V_{j,l}| = |V_j| + |V_l| - 2|V_j \cap V_l|. \tag{5.5}$$

Also, the linear combination $c_6 + c_{14} + c_k$ is a monochromatic $(2,0)$-coloring of G. Let $V_{6,14,k}$ be the set of vertices colored 1 by $c_6 + c_{14} + c_k$. Since S_t is the monochromatic $(2,0)$-spectrum G, it follows that $|V_{j,l}| \in S_t$ for each pair j, l of distinct integers where $j, l \in S$ and $|V_{6,14,k}| \in S_t$. In the Venn digram of $V_6 \cup V_{14} \cup V_k$ shown in Figure 5.1, for $1 \leq s \leq 7$, let W_s be defined as in the proof of Proposition 5.3.3 and let $w_s = |W_s|$. Observe that since $\sum_{s=1}^{7} w_s \leq 6 + 14 + k = k + 20 < k + 22$, it follows that $|V_{j,l}| \in S$ for each pair j, l of distinct integers where $j, l \in S$ and $|V_{6,14,k}| \in S$. By using exact same argument as we have done in Proposition 5.3.3, it follows that this is impossible. Thus S_t is not the $(2,0)$-spectrum of any $(2,0)$-colorable graph. ■

By Corollary 5.3.4, there are infinitely many sets of positive even integers that are not the $(2,0)$-spectrum of any $(2,0)$-colorable graph. However, every set of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph. In order to establish this fact, we first present two additional definitions and some preliminary results. For a monochromatic $(2,0)$-coloring c of a graph G, let $\mathcal{C}_1$ be the set of vertices colored 1 by c and $\mathcal{C}_0 = V(G) - \mathcal{C}_1$ be the set of vertices colored 0 by c. A vertex v is a *zero-vertex* of a $(2,0)$-colorable graph G if $c(v) = 0$ for every monochromatic $(2,0)$-coloring c of G. Similarly, a vertex v is a *one-vertex* of a $(2,0)$-colorable graph G if $c(v) = 1$ for every monochromatic $(2,0)$-coloring c of G.

We saw in Observation 1.3.1 that if u and v be two nonadjacent vertices of a connected graph G such that $N(u) = N(v)$ and c is a modular monochromatic k-coloring of G for some integer $k \geq 2$, then necessarily $c(u) = c(v)$. With the aid of Observation 1.3.1, we have the following theorem.

Theorem 5.3.5 *For each integer $i = 1, 2$, let G_i be a $(2,0)$-colorable graph having a zero-vertex and let $S_{(2,0)}(G_i) = S_i$. Then the following set is a $(2,0)$-realizable set:*

$$S_1 \cup S_2 \cup \{s_1 + s_2 : \ s_i \in S_i \text{ for } i = 1, 2\}. \tag{5.6}$$

Proof. Let S be the set described in (5.6). For each integer $i = 1, 2$, let v_i be a zero-vertex of G_i. Now, let G be the graph obtained from G_i $(i = 1, 2)$ by adding two new vertices x and y and joining each of x and y to both v_1 and v_2. First, we make the following three observations:

(a) If c' is a $(2,0)$-coloring of G_1, then c' can be extended to a (2,0)-coloring c of G by assigning the color 1 to vertices in $\mathcal{C}_1'$ and the color 0 to the remaining vertices of G

and so $s_1 \in S_{(2,0)}(G)$.

(b) If c'' is a $(2,0)$-coloring of G_2 then c'' can be extended to a (2,0)-coloring c of G by assigning the color 1 to vertices in $\mathcal{C}_1''$ and the color 0 to the remaining vertices of G and so $s_2 \in S_{(2,0)}(G)$.

(c) If c' and c'' are $(2,0)$-colorings of G_1 and G_2, respectively then a coloring of G that assigns the color 1 to $\mathcal{C}_1' \cup \mathcal{C}_1''$ and the color 0 to the remaining vertices is a $(2,0)$-coloring of G and so $s_1 + s_2 \in S_{(2,0)}(G)$ where $s_i \in S_i$ for $i = 1, 2$.

By (a) – (c), it follows that G is $(2,0)$-colorable and $S \subseteq S_{(2,0)}(G)$. To show that $S_{(2,0)}(G) \subseteq S$, we let c be a $(2,0)$-coloring of G. Then $c(x) = c(y)$ by Observation 1.3.1. We claim that $c(x) = c(y) = 0$; for if $c(x) = c(y) = 1$ then since $\sigma(x) = 0$, exactly one of v_1 and v_2 is assigned the color 1 by c. If $c(v_1) = 1$, then it can be shown that the restriction c_{G_1} of c to G_1 is a $(2,0)$-coloring of G_1 such that $c_{G_1}(v_1) = 1$, which is impossible; while if $v(v_2) = 1$, then it also can be shown that the restriction c_{G_2} of c to G_2 is a $(2,0)$-coloring of G_2 such that $c_{G_2}(v_2) = 1$, which is impossible. Hence $c(x) = c(y) = 0$, as claimed and so the restrictions c_{G_1} and c_{G_2} of c to G_1 and G_2 are $(2,0)$-colorings of G_1 and G_2, respectively. Observe that at most one of c_{G_1} and c_{G_2} is trivial; for otherwise c is a trivial $(2,0)$-coloring of G, which is impossible. Hence $S_{(2,0)}(G) \subseteq S$ and so $S_{(2,0)}(G) = S$. Therefore, S is a $(2,0)$-realizable set. ■

We are now prepared to present the following result.

Theorem 5.3.6 *Every set of positive even integers is a subset of the* $(2,0)$*-spectrum of some* $(2,0)$*-colorable graph.*

Proof. Let S be a set of positive even integers. We proceed by the induction on the cardinality of a set S to prove the following strong result:

> Every set S of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph containing a zero-vertex.

First, suppose that $S = \{s\}$ for some positive even integer s. For $s = 2$, let $G = K_{1,3} + e$ and so $S_{(2,0)}(G) = \{2\}$ and the end-vertex (and the vertex of degree 3) is a zero-vertex of G. For $s \geq 4$, let G be a path of order $3s/2 - 1$. Since $3s/2 - 1 \equiv 2 \pmod 3$, it follows by Proposition 5.2.1 that G is $(2,0)$-colorable. Moreover, $S_{(2,0)}(G) = \{s\}$ and G has zero-vertices by Proposition 5.2.6. Suppose for some integer $k \geq 2$ and every integer ℓ with $1 \leq \ell \leq k-1$, each ℓ-element set of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph containing a zero-vertex. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a set consisting of k positive even integers. Consider $S_1 = S - \{s_k\}$ and $S_2 = \{s_k\}$. By induction hypothesis, there are $(2,0)$-colorable graphs G_1 and G_2 such that $S_i \subseteq S_{(2,0)}(G_i)$ for $i = 1, 2$ and each graph G_i contains a zero-vertex. Let v_i be a zero-vertex of G_i for $i = 1, 2$ and let G be the graph obtained from G_1 and G_2 by adding two new vertices x and y and joining each of x and y to both v_1 and v_2. It then follows by Theorem 5.3.5 that $S_{(2,0)}(G) = S_1 \cup S_2 \cup \{s_1 + s_k : s_1 \in S_1\}$ (so $S \subseteq S_{(2,0)}(G)$) and x is a zero-vertex of G. ■

By Theorem 5.3.6, every set of positive even integers is a subset of a $(2,0)$-realizable set. Consequently, we obtain the following result.

Corollary 5.3.7 *For each set S of positive even integers, there is an infinite sequence $S_1, S_2, \ldots, S_n, \ldots$ of $(2,0)$-realizable sets such that*

$$S \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots \subsetneq S_n \subsetneq \ldots.$$

Theorem 5.3.8 *For each integer $i = 1, 2$, let G_i be a $(2, 0)$-colorable graph having a one-vertex and let $S_{(2,0)}(G_i) = S_i$. Then the following set is a $(2, 0)$-realizable set:*

$$\{s_1 + 2, s_2 + 2, s_1 + s_2 : \ s_i \in S_i \text{ for } i = 1, 2\}. \tag{5.7}$$

Proof. Let S be the set described in (5.7). For each integer $i = 1, 2$, let v_i be a one-vertex of G_i. Now, let G be the graph obtained from G_i $(i = 1, 2)$ by adding two new vertices x and y and joining each of x and y to both v_1 and v_2. First, we make the following three observations:

(a) If c' is a $(2, 0)$-coloring of G_1, then c' can be extended to a (2,0)-coloring c of G by assigning the color 1 to vertices in $\mathcal{C}_1' \cup \{x, y\}$ and the color 0 to the remaining vertices of G and so $s_1 + 2 \in S_{(2,0)}(G)$.

(b) If c'' is a $(2, 0)$-coloring of G_2 then c'' can be extended to a (2,0)-coloring c of G by assigning the color 1 to vertices in $\mathcal{C}_1'' \cup \{x, y\}$ and the color 0 to the remaining vertices of G and so $s_2 + 2 \in S_{(2,0)}(G)$.

(c) If c' and c'' are $(2, 0)$-colorings of G_1 and G_2, respectively then a coloring of G that assigns the color 1 to $\mathcal{C}_1' \cup \mathcal{C}_1''$ and the color 0 to the remaining vertices is a $(2, 0)$-coloring of G and so $s_1 + s_2 \in S_{(2,0)}(G)$ where $s_i \in S_i$ for $i = 1, 2$.

By (a) – (c), it follows that G is $(2, 0)$-colorable and $S \subseteq S_{(2,0)}(G)$. To show that $S_{(2,0)}(G) \subseteq S$, we let c be a $(2, 0)$-coloring of G. Then $c(x) = c(y)$ by Observation 1.3.1. If $c(x) = c(y) = 0$ then $c(v_1) = c(v_2) = 1$; for otherwise there is a $(2, 0)$-coloring of G_i assigning the coloring 0 to v_i where $i = 1, 2$, which is not possible. Since the restriction of c to G_i is a nontrivial $(2, 0)$-coloring of G_i, it follows that $|\mathcal{C}_1| \in \{s_1 + s_2 : \ s_i \in S_i \text{ for } i = 1, 2\}$. Now if $c(x) = c(y) = 1$ then since $\sigma(x) = 0$, exactly one of v_1 and v_2 is colored 1 by c. If

$c(v_1) = 1$, then the restriction c_{G_2} of c to G_2 assigns the color 0 to every vertex of G_2; for otherwise there is a $(2,0)$-coloring of G_2 assigns the color 0 to v_2, which is impossible and so $|\mathcal{C}_1| \in \{s_1 + 2 : \; s_1 \in S_1\}$; while if $c(v_2) = 1$, then, by using the similar argument as the case where $c(v_1) = 1$, we have that $|\mathcal{C}_1| \in \{s_2 + 2 : \; s_2 \in S_2\}$. Hence $S_{(2,0)}(G) \subseteq S$ and so $S_{(2,0)}(G) = S$. Therefore, S is a $(2,0)$-realizable set. ■

Theorem 5.3.9 *For each integer $i = 1, 2$, let G_i be a $(2,0)$-colorable graph and let $S_{(2,0)}(G_i) = S_i$. If G_1 contain a zero-vertex and G_2 contains a one-vertex, then the following set is a $(2,0)$-realizable set:*

$$S_1 \cup \{s_2 + 2, s_1 + s_2 + 2 : s_i \in S_i \text{ for } i = 1, 2\}. \tag{5.8}$$

Proof. Let S be the set described in (5.8), let v_1 be a zero-vertex of G_1 and v_2 a one-vertex of G_2. Now, let G be the graph obtained from G_i by adding two new vertices x and y and joining each of x and y to both v_1 and v_2. First, we make the following three observations:

(a) If c' is a $(2,0)$-coloring of G_1, then c' can be extended to a (2,0)-coloring c of G by assigning the color 1 to vertices in $\mathcal{C}_1'$ and the color 0 to the remaining vertices of G and so $s_1 \in S_{(2,0)}(G)$.

(b) If c'' is a $(2,0)$-coloring of G_2 then c'' can be extended to a (2,0)-coloring c of G by assigning the color 1 to vertices in $\mathcal{C}_1'' \cup \{x, y\}$ and the color 0 to the remaining vertices of G and so $s_2 + 2 \in S_{(2,0)}(G)$.

(c) If c' and c'' are $(2,0)$-colorings of G_1 and G_2, respectively then a coloring of G that assigns the color 1 to $\mathcal{C}_1' \cup (\mathcal{C}_1'' \cup \{x, y\})$ and the color 0 to the remaining vertices is a $(2,0)$-coloring of G and so $s_1 + s_2 + 2 \in S_{(2,0)}(G)$ where $s_i \in S_i$ for $i = 1, 2$.

By (a) – (c), it follows that G is $(2,0)$-colorable and $S \subseteq S_{(2,0)}(G)$. To show that $S_{(2,0)}(G) \subseteq S$, we let c be a $(2,0)$-coloring of G. Then $c(x) = c(y)$ by Observation 1.3.1. If $c(x) = c(y) = 0$ then since $\sigma(x) = 0$, $c(v_1) = c(v_2)$. If $c(v_1) = c(v_2) = 1$ then the restriction c_{G_1} of c to G_1 is a $(2,0)$-coloring of G_1 such that $c_{G_1}(v_1) = 1$, which is impossible since v_1 is a zero-vertex of G_1. Thus $c(v_1) = c(v_2) = 0$. Since the restriction c_{G_2} of c to G_2 is a $(2,0)$-coloring of G_2 and v_2 is a one-vertex G_2, it follows that c_{G_2} is trivial; for otherwise there is a nontrivial $(2,0)$-coloring of G_2 assigns the color 0 to v_2, which is impossible and necessarily the restriction c_{G_1} of c to G_1 is a nontrivial $(2,0)$-coloring of G_1. Thus $|\mathcal{C}_1| \in S_1$. Now if $c(x) = c(y) = 1$ then , again since $\sigma(x) = 0$, exactly one of v_1 and v_2 is colored 1 by c. It is impossible for c to assign the color 1 to v_1 because v_1 is a zero-vertex of G_1. Thus $c(v_1) = 0$ and $c(v_2) = 1$. Observe that the restrictions c_{G_1} and $c_{G_2 \cup \{x,y\}}$ of c to G_1 and $G_2 \cup \{x, y\}$ are $(2,0)$-colorings of G_1 and $G_2 \cup \{x, y\}$, respectively. If c_{G_1} is trivial then $|\mathcal{C}_1| \in \{s_2 + 2; s_2 \in S_2\}$; while if c_{G_1} is nontrivial then $|\mathcal{C}_1| \in \{s_1 + s_2 + 2 : s_i \in S_i \text{ for i} = 1, 2\}$. Thus $S_{(2,0)}(G) \subseteq S$ and so $S_{(2,0)}(G) = S$. Therefore, S is a $(2,0)$-realizable set. ∎

Proposition 5.3.10 *If c' and c'' are two monochromatic $(2,0)$-colorings of a graph G such that $\mathcal{C}_1' \cap \mathcal{C}_1'' = \emptyset$, then the coloring c that assigns the color 1 to $\mathcal{C}_1' \cup \mathcal{C}_1''$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G.*

Proof. For each $v \in V(G)$, we show that $\sigma_c(v) = 0$. First, suppose that $c(v) = 1$. Then $v \in \mathcal{C}_1 = \mathcal{C}_1' \cup \mathcal{C}_1''$ and so $v \in \mathcal{C}_1'$ or $v \in \mathcal{C}_1''$, without loss of generality say the former. Since $\mathcal{C}_1' \cap \mathcal{C}_1'' = \emptyset$, it follows that $c''(v) = 0$ and v is adjacent to even number of vertices in $\mathcal{C}_1''$. (It is possible that v is adjacent to no vertex in $\mathcal{C}_1''$.) On the other hand, since $v \in \mathcal{C}_1'$, it follows that $c'(v) = 1$ and so v is adjacent to an odd number of vertices in $\mathcal{C}_1'$. Hence v is adjacent to an odd number of vertices in $\mathcal{C}_1$ and so $\sigma_c(v) = 0$. Next, suppose that $c(v) = 0$.

Thus $v \notin \mathcal{C}_1$ and so $c'(v) = c''(v) = 0$. Since c' and c'' are monochromatic $(2,0)$-colorings of a graph G, it follows that v is adjacent to an even number of vertices in $\mathcal{C}_1'$ and in $\mathcal{C}_1''$. Hence v is adjacent to an even number of vertices in $\mathcal{C}_1$ and so $\sigma_c(v) = 0$. ■

Proposition 5.3.11 *For a positive integer t and each integer i with $1 \le i \le t$, let G_i be a $(2,0)$-colorable graph containing a zero-vertex v_i. If G is the graph obtained from $G_1, G_2, \ldots, G_t$ by identifying all vertices v_i $(1 \le i \le t)$ and labeling the identified vertex by v, then G is a $(2,0)$-colorable graph and v is a zero-vertex of G.*

Proof. First, we show that a $(2,0)$-coloring of the graph G_i $(1 \le i \le t)$ can be extended a $(2,0)$-coloring of G. We may assume that c_1 is a $(2,0)$-coloring of the graph G_1. The coloring c of a graph G obtained by $c(x) = c_1(x)$ if $x \in V(G_1)$ and $c(x) = 0$ if $x \in V(G) - V(G_1)$ is a $(2,0)$-coloring of G. Hence G is $(2,0)$-colorable. It remains to show that v is a zero-vertex of G. If this were not the case, then there exists a $(2,0)$-coloring c of G such that $c(v) = 1$. Since $\sigma_c(v) = 0$, there are odd neighbors of v that are colored 1 by c. This implies that there is $j \in \{1, 2, \ldots, t\}$ such that the number of neighbors of v colored 1 by c in G_j is odd and so the number of neighbors of v colored 1 by c in $N^*(v) = N(v) - N_{G_j}(v)$ is even. Let c_{G_j} be the restriction of c to G_j. We show that c_{G_j} is a $(2,0)$-coloring of G_j. If $x \in V(G_j) - \{v\}$, then $\sigma_{c_{G_j}}(x) = \sigma_c(x) = 0$; while if $x = v$, then $\sigma_{c_{G_j}}(x) = \sigma_c(v) - \sum_{u \in N^*(v)} c(u) = \sigma_c(v) = 0$ in $\mathbb{Z}_2$, which implies that c_{G_j} is a monochromatic $(2,0)$-coloring of G_j. Since c_{G_j} assigns the color 1 to v_j, which is v in G, this is a contradiction. ■

5.4 Graphs with Prescribed Order and $(2,0)$-Spectrum

Lemma 5.4.1 *If v is a zero-vertex of a $(2,0)$-colorable graph G and G' is a graph obtained from G by adding an even number of pendant edges at v, then $S_{(2,0)}(G) = S_{(2,0)}(G')$.*

Proof. Let $vv_1, vv_2, \ldots, vv_t \in E(G') - E(G)$ are the pendant edges at v where t is even. If c is a monochromatic $(2,0)$-coloring of G, then the coloring c' defined by $c'(x) = c(x)$ if $x \in V(G)$ and $c'(x) = 0$ if $x \notin V(G)$ is a monochromatic $(2,0)$-coloring of G' and so $S_{(2,0)}(G) \subseteq S_{(2,0)}(G')$. Thus, it remains to show that $S_{(2,0)}(G') \subseteq S_{(2,0)}(G)$. Let c' be a monochromatic $(2,0)$-coloring of G'. It then follows by Observation 4.1.4 that c' assigns the same color to all vertices in $\{v, v_1, v_2, \ldots, v_t\}$. Let c'_G be the restriction of the coloring c' to G. If $x \in V(G) - \{v\}$, then $\sigma_{c'_G}(x) = \sigma_{c'}(x) = 0$; while

$$\sigma_{c'_G}(v) = \sigma_{c'}(v) - \textstyle\sum_{i=1}^{t} c'(v_i) = \sigma_{c'}(v) = 0 \text{ in } \mathbb{Z}_2$$

(since t is even and c' is a monochromatic $(2,0)$-coloring of G'). Hence c'_G is a monochromatic $(2,0)$-coloring of G. By the defining property of v, it follows that $c'_G(v) = c'(v) = 0$ and so $c'(v_i) = 0$ for each i with $1 \leq i \leq t$ by Observation 4.1.4. Thus, the numbers of vertices assigned the color 1 by c' and c'_G are the same. Hence $S_{(2,0)}(G') \subseteq S_{(2,0)}(G)$ and the result follows. ■

The condition that a number of pendant edges at v is even is necessary in Lemma 5.4.1. To see this, we consider a wheel $G = C_n \vee K_1$ where $n \equiv 0 \pmod 6$. It is shown by Theorem 5.2.9 that $S_{(2,0)}(G) = \{2n/3\}$. Observe that the only even vertex of G is the vertex v of K_1 and so v is assigned the color 0 by every $(2,0)$-coloring of G. Thus v is a zero-vertex of G. The graph G' obtained from G and a single vertex u by adding an edge uv is an odd-degree graph and so $n+2 \in S_{(2,0)}(G')$ but $n+2 \notin S_{(2,0)}(G)$. Thus $S_{(2,0)}(G) \neq S_{(2,0)}(G')$.

Proposition 5.4.2 *For a positive integer $\ell \geq 2$, there is a graph G of order $2\ell + 2$ such that $S_{(2,0)}(G) = \{2, 2\ell\}$.*

Proof. We start with the graph $H = K_{1,2\ell} + e$ where $V(H) = \{v, v_1, v_2, \ldots, v_{2\ell}\}$ and v is the central vertex of $K_{1,2\ell}$ and $e = v_1v_2$. Let G be the graph of order $2\ell + 2$ obtained from H by adding a new vertex w and joining w to both v_3 and v_4. We now show that $S_{(2,0)}(G) = \{2, 2\ell\}$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{v_1, v_2\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $2 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except v_2 and w is a $(2,0)$-coloring of G. Thus $2\ell \in S_{(2,0)}(G)$.

By (a) – (b), it follows that G is $(2,0)$-colorable and $\{2, 2\ell\} \subseteq S_{(2,0)}(G)$. To show that $S_{(2,0)}(G) \subseteq \{2, 2\ell\}$. Let c be a monochromatic $(2,0)$-coloring of G. We consider two cases depending on the color of v assigned by c.

Case 1. $c(v) = 0$. Then $c(v_i) = 0$ for $5 \leq i \leq 2\ell$, $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$. We claim that $c(v_3) = c(v_4) = 0$. If this is not the case then $c(v_3) = c(v_4) = 1$. Since $\sigma(v_3) = 0$, $c(w) = 1$. This, however, implies that $\sigma(w) = 1$, which is a contradiction and so, as claimed, $c(v_3) = c(v_4) = 0$ and so c assigns the color 0 to w. Now, $c(v_1) = c(v_2) = 1$; for otherwise, c assigns the color 0 to every vertex of G, which is not possible. Thus if $c(v) = 0$ then c assigns the color 1 to exactly two vertices, namely v_1 and v_2, and the color 0 to the remaining vertices of G. Hence $|\mathcal{C}_1| \in \{2, 2\ell\}$.

Case 2. $c(v) = 1$. Then $c(v_i) = 1$ for $5 \leq i \leq 2\ell$. Since v_3 and v_4 are nonadjacent and they have the same neighbors, $c(v_3) = c(v_4)$. If $c(v_3) = c(v_4) = 0$, then since $\sigma(v_3) = 0$, $c(w) = 1$. This would implies that $\sigma(w) = 1$, which is a contradiction. Thus $c(v_3) = c(v_4) = 1$ and so c assigns the color 0 to w. Now since $\sigma(v_1) = 0$, $c(v_1) \neq c(v_2)$. Thus c assigns the color 1 to each vertex in the set $\{v, v_1, v_3, v_4, \ldots, v_{2\ell}\}$ or each vertex in the set $\{v, v_2, v_3, v_4, \ldots, v_{2\ell}\}$ and the color 0 to the remaining vertex of G. Hence $|\mathcal{C}_1| \in \{2, 2\ell\}$.

Thus $\{2, 2\ell\} = S_{(2,0)}(G)$. ■

It is worthwhile to note that w is a zero-vertex of a graph G in Proposition 5.4.2.

Proposition 5.4.3 *For a positive integer $\ell \geq 2$, there is a graph G of order $2\ell + 3$ such that $S_{(2,0)}(G) = \{2, 2\ell\}$.*

Proof. We start with the graph $H = K_{1,2\ell} + e$ where $V(H) = \{v, v_1, v_2, \ldots, v_{2\ell}\}$ and v is the central vertex of $K_{1,2\ell}$ and $e = v_1v_2$. Let G be the graph of order $2\ell + 3$ obtained from H by adding two new vertices w_1 and w_2 and joining w_1 and w_2 to both v_3 and v_4. We now show that $S_{(2,0)}(G) = \{2, 2\ell\}$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{v_1, v_2\}$ and the color 0 to the remaining vertices of G is a $(2, 0)$-coloring of G. Thus $2 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{v_2, w_1, w_2\}$ is a $(2, 0)$-coloring of G. Thus $2\ell \in S_{(2,0)}(G)$.

By (a) – (b), it follows that G is $(2, 0)$-colorable and $\{2, 2\ell\} \subseteq S_{(2,0)}(G)$. To show that $S_{(2,0)}(G) \subseteq \{2, 2\ell\}$. Let c be a monochromatic $(2, 0)$-coloring of G. We note that since v_3 and v_4 are nonadjacent and they have the same neighbors, $c(v_3) = c(v_4)$. By similar reason, $c(w_1) = c(w_2)$. We consider two cases depending on the color of v assigned by c.

Case 1. $c(v) = 0$. Then $c(v_i) = 0$ for $5 \leq i \leq 2\ell$, $c(v_1) = c(v_2)$. We claim that $c(v_3) = c(v_4) = 0$. If this is not the case then $c(v_3) = c(v_4) = 1$. Either $c(w_1) = c(w_2) = 0$ or $c(w_1) = c(w_2) = 1$ produces a contradiction and so, as claimed, $c(v_3) = c(v_4) = 0$. Thus

c assigns the color 0 to w_1 and w_2. Now, $c(v_1) = c(v_2) = 1$; for otherwise, c assigns the color 0 to every vertex of G, which is not possible. Hence if $c(v) = 0$ then c assigns the color 1 to exactly two vertices, namely v_1 and v_2, and the color 0 to the remaining vertices of G. Hence $|\mathcal{C}_1| \in \{2, 2\ell\}$.

Case 2. $c(v) = 1$. Then $c(v_i) = 1$ for $5 \leq i \leq 2\ell$. If $c(v_3) = c(v_4) = 0$, then either $c(w_1) = c(w_2) = 0$ or $c(w_1) = c(w_2) = 1$ results in $\sigma(v_3) = 1$, which is impossible and so $c(v_3) = c(v_4) = 1$. Thus $c(w_1) = c(w_2) = 0$. Now since $\sigma(v_1) = 0$, $c(v_1) \neq c(v_2)$. Thus c assigns the color 1 to each vertex in the set $\{v, v_1, v_3, v_4, \ldots, v_{2\ell}\}$ or each vertex in the set $\{v, v_2, v_3, v_4, \ldots, v_{2\ell}\}$ and the color 0 to the remaining vertex of G. Hence $|\mathcal{C}_1| \in \{2, 2\ell\}$.

Thus $\{2, 2\ell\} = S_{(2,0)}(G)$. ■

It is worthwhile to note that w_1 and w_2 are zero-vertices of a graph G in Proposition 5.4.3.

Corollary 5.4.4 *For each pair ℓ, n of integers where $\ell \geq 2$ and $n \geq 2\ell + 2$, there is a graph G of order n such that $S_{(2,0)}(G) = \{2, 2\ell\}$.*

Proof. By Propositions 5.4.2 and 5.4.3, there are graphs G_1 and G_2 containing zero-vertices of order $2\ell + 2$ and $2\ell + 3$, respectively, such that $S_{(2,0)}(G_1) = S_{(2,0)}(G_2) = \{2, 2\ell\}$. If n is even then $s = n - (2\ell + 2)$ is even. Let G_1' be the graph obtained from G_1 by adding s new vertices and joining those new vertices to a zero-vertex of G_1. If n is odd then $t = n - (2\ell + 3)$ is even. Let G_2' be the graph obtained from G_2 by adding t new vertices and joining those new vertices to a zero-vertex of G_2. Then it is shown in Lemma 5.4.1 that $S_{(2,0)}(G_i) = S_{(2,0)}(G_i')$ where $i = 1, 2$ and thus the result follows. ■

Proposition 5.4.5 *For a positive integer $\ell \geq 3$, there is a graph G of order $2\ell + 4$ such that $S_{(2,0)}(G) = \{4, 2\ell\}$.*

Proof. Let $H = 2P_2$ where the two copies of P_2 are (u, v) and (x, y). Let G be the graph obtained from H and 2ℓ vertices $v_1, v_2, \ldots, v_{2\ell-2}, w_1, w_2$ by

- joining v_i $(1 \leq i \leq 2\ell - 2)$ to both v and x
- joining w_s $(s = 1, 2)$ to both u and to v.

We will show that $S_{(2,0)}(G) = \{4, 2\ell\}$ where $\ell \geq 3$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{u, v, x, y\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $4 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{u, v, w_1, w_2\}$ is a $(2,0)$-coloring of G. Thus $2\ell \in S_{(2,0)}(G)$.

By (a) – (b), it follows that G is $(2,0)$-colorable and $\{4, 2\ell\} \subseteq S_{(2,0)}(G)$. To show that $S_{(2,0)}(G) \subseteq \{2, 2\ell\}$. Let c be a $(2,0)$-coloring of G. Observe that, for every i and j such that $1 \leq i \neq j \leq 2\ell - 2$, since v_i and v_j are nonadjacent and they all have the same neighbors, $c(v_i) = c(v_j)$. By similar reason, $c(w_1) = c(w_2)$. We consider two cases.

Case 1. $c(v_i) = 0$, for each i $(1 \leq i \leq 2\ell - 2)$. Thus $c(v) = c(x)$. If $c(v) = c(x) = 0$ then $c(y) = 0$ and $c(u) = c(w_s)$ since $\sigma(w_s) = 0$ where $s = 1, 2$. If $c(u) = c(w_s) = 0$ then c assigns the color 0 to every vertex of G, which is not possible and so $c(u) = c(w_s) = 1$. This, however, implies that $\sigma(u) = 1$, which is impossible. Now, if $c(v) = c(x) = 1$ then $c(y) = 1$ since $\sigma(y) = 0$ and either $c(u) = 0$ and $c(w_s) = 1$ or $c(u) = 1$ and $c(w_s) = 0$ since $\sigma(w_s) = 0$. If $c(u) = 0$ and $c(w_s) = 1$ then $\sigma(v) = 1$, which is not possible and so

$c(u) = 1$ and $c(w_s) = 0$. Therefore if, for each i $(1 \leq i \leq 2\ell - 2)$, $c(v_i) = 0$ then c assigns the color 1 to each vertex in $\{u, v, x, y\}$ and the color 0 to the remaining vertices of G. Thus $|\mathcal{C}_1| \in \{4, 2\ell\}$.

Case 2. $c(v_i) = 1$, for each i $(1 \leq i \leq 2\ell - 2)$. Thus either $c(v) = 0$ and $c(x) = 1$ or $c(v) = 1$ and $c(x) = 0$. We, first, assume the former. Now $c(y) = 1$ because $\sigma(y) = 0$. Since $\sigma(w_s) = 0$, $c(u) = c(w_s)$ and as we have seen in *Case* 1, $c(u) = c(w_s) \neq 1$. Thus $c(u) = c(w_s) = 0$. Therefore, in the former case, c assigns the color 1 to 2ℓ vertices in the set $\{x, y, v_1, v_2, \ldots, v_{2\ell-2}\}$ and the color 0 to the remaining vertices of G. Next we assume that $c(v) = 1$ and $c(x) = 0$ and so $c(y) = 0$. Since $\sigma(w_s) = 0$, $c(u) \neq c(w_s)$. If $c(u) = 0$ and $c(w_s) = 1$ then $\sigma(u) = 1$, which is impossible and so $c(u) = 1$ and $c(w_s) = 0$. Thus, in the later case, c assigns the color 1 to 2ℓ vertices in the set $\{u, v, v_1, v_2, \ldots, v_{2\ell-2}\}$ and the color 0 to the remaining vertices of G. Hence $|\mathcal{C}_1| \in \{4, 2\ell\}$.

Thus $S_{(2,0)}(G) = \{4, 2\ell\}$. ■

Note that w_1 and w_1 of a graph G in Proposition 5.4.5 are zero-vertices.

Proposition 5.4.6 *For a positive integer $\ell \geq 3$, there is a graph G of order $2\ell + 5$ such that $S_{(2,0)}(G) = \{4, 2\ell\}$.*

Proof. Let $H = 2P_2$ where the two copies of P_2 are (u, v) and (x, y). Let G be the graph obtained from H and $2\ell + 1$ vertices $v_1, v_2, \ldots, v_{2\ell-2}, w, w_1, w_2$ by

- joining v_i $(1 \leq i \leq 2\ell - 2)$ to both v and x
- joining w_s $(s = 1, 2)$ to both u and to v
- joining w to w_2.

We will show that $S_{(2,0)}(G) = \{4, 2\ell\}$ where $\ell \geq 3$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{u, v, x, y\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $4 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{u, v, w, w_1, w_2\}$ is a $(2,0)$-coloring of G. Thus $2\ell \in S_{(2,0)}(G)$.

By (a) – (b), it follows that G is $(2,0)$-colorable and $\{4, 2\ell\} \subseteq S_{(2,0)}(G)$. Let c be a $(2,0)$-coloring of G. Note that, for every i and j such that $1 \leq i \neq j \leq 2\ell - 2$, $c(v_i) = c(v_j)$, $c(w) = c(w_2)$. We consider two cases.

Case 1. $c(v_i) = 0$, for each i $(1 \leq i \leq 2\ell - 2)$. Thus $c(v) = c(x)$. If $c(v) = c(x) = 0$ then $c(y) = 0$ and $c(u) = c(w_1)$ since $\sigma(w_1) = 0$. If $c(u) = c(w_1) = 0$ then c assigns the color 0 to w and w_2; for otherwise $\sigma(v) = 1$, which is impossible and so c assigns the color 0 to every vertex of G, which contradicts to the definition of c. If $c(u) = c(w_1) = 1$ then $c(w) = c(w_2) = 0$ since $\sigma(u) = 0$. However, then $\sigma(w_2) = 1$, which is impossible. Now, if $c(v) = c(x) = 1$ then $c(y) = 1$ since $\sigma(y) = 0$ and either $c(u) = 0$ and $c(w_1) = 1$ or $c(u) = 1$ and $c(w_1) = 0$ since $\sigma(w_1) = 0$. If $c(u) = 0$ and $c(w_1) = 1$ then $c(w) = c(w_2) = 0$ since $\sigma(v) = 0$. This, however, implies that $\sigma(w_2) = 1$, which is impossible. If $c(u) = 1$ and $c(w_1) = 0$, then $c(w) = c(w_2) = 0$; for otherwise $\sigma(v) = 1$, which is not possible. Hence if, for each i $(1 \leq i \leq 2\ell - 2)$, $c(v_i) = 0$ then c assigns the color 1 to four vertices in the set $\{u, v, x, y\}$ and the color 0 to the remaining vertices of G. Hence $|\mathcal{C}_1| \in \{4, 2\ell\}$.

Case 2. $c(v_i) = 1$, for each i $(1 \leq i \leq 2\ell - 2)$. Then either $c(v) = 0$ and $c(x) = 1$ or $c(v) = 1$ and $c(x) = 0$. We, first, assume the former. Now $c(y) = 1$ because $\sigma(y) = 0$. Since $\sigma(w_1) = 0$, $c(u) = c(w_1)$ and as we have seen in *Case* 1, $c(u) = c(w_1) \neq 1$. Thus $c(u) = c(w_s) = 0$ and so $c(w) = c(w_2) = 0$. Therefore, in the former case, c assigns the color

1 to 2ℓ vertices in the set $\{x, y, v_1, v_2, \ldots, v_{2\ell-2}\}$ and the color 0 to the remaining vertices of G. Next, we assume that $c(v) = 1$ and $c(x) = 0$ and so $c(y) = 0$. Since $\sigma(w_1) = 0$, $c(u) \neq c(w_1)$. If $c(u) = 0$ and $c(w_1) = 1$ then either $c(w) = c(w_2) = 0$ or $c(w) = c(w_2) = 1$ produces a contradiction. Thus we may assume $c(u) = 1$ and $c(w_1) = 0$. It then follows that $c(w) = c(w_2) = 0$; for otherwise $\sigma(v) = 1$, which is impossible. Hence, in the later case, c assigns the color 1 to 2ℓ vertices in the set $\{u,\ v,\ v_1,\ v_2,\ \ldots,\ v_{2\ell-2}\}$ and the color 0 to the remaining vertices of G. Hence $|\mathcal{C}_1| \in \{4, 2\ell\}$.

Thus $S_{(2,0)}(G) = \{4, 2\ell\}$. ■

Note that w, w_1 and w_2 of a graph G in Proposition 5.4.6 are zero-vertices.

Corollary 5.4.7 *For each pair ℓ, n of integers where $\ell \geq 3$ and $n \geq 2\ell + 4$, there is a graph G of order n such that $S_{(2,0)}(G) = \{4, 2\ell\}$.*

Proof. By Propositions 5.4.5 and 5.4.6, there are graphs G_1 and G_2 containing zero-vertices of order $2\ell+4$ and $2\ell+5$, respectively, such that $S_{(2,0)}(G_1) = S_{(2,0)}(G_2) = \{4, 2\ell\}$. If n is even then $s = n - (2\ell + 4)$ is even. Let G_1' be the graph obtained from G_1 by adding s new vertices and joining those new vertices to a zero-vertex of G_1. If n is odd then $t = n - (2\ell + 5)$ is even. Let G_2' be the graph obtained from G_2 by adding t new vertices and joining those new vertices to a zero-vertex of G_2. Then it is shown in Lemma 5.4.1 that $S_{(2,0)}(G_i) = S_{(2,0)}(G_i')$ where $i = 1, 2$ and thus the result follows. ■

Proposition 5.4.8 *For a positive integer $\ell \geq 3$, there is a graph G of order $2\ell + 6$ such that $S_{(2,0)}(G) = \{6, 2(\ell + 1)\}$.*

Proof. Let $H = K_{1,2\ell} + e$ with $V(H) = \{v, v_1, v_2, \ldots, v_{2\ell}\}$ where v is the central vertex of $K_{1,2\ell}$ and $e = v_1v_2$. Let G be the graph of order $2\ell + 6$ obtained from H and 5 vertices

u_1, u_2, u_3, u_4, w by joining u_1 and u_2 to v_1, u_3 and u_4 to v_2 and w to both v_3 and v_4. We will show that $S_{(2,0)}(G) = \{6, 2(\ell + 1)\}$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{v_1, v_2, u_1, u_2, u_3, u_4\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $6 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{u_1, u_2, v_1, w\}$ is a $(2,0)$-coloring of G. Thus $2(\ell + 1) \in S_{(2,0)}(G)$.

By (a) – (b), it follows that G is $(2,0)$-colorable and $\{6, 2(\ell + 1)\} \subseteq S_{(2,0)}(G)$. Let c be a $(2,0)$-coloring of G. Note that, for every i and j such that $5 \leq i \neq j \leq 2\ell$, $c(v) = c(v_i) = c(v_j)$, $c(v_3) = c(v_4)$, $c(v_1) = c(u_1) = c(u_2)$ and $c(v_2) = c(u_3) = c(u_4)$. We consider two cases, according to whether $c(v) = 0$ or $c(v) = 1$.

Case 1. $c(v) = 0$. Then $c(v_i) = 0$ for $5 \leq i \leq 2\ell$, $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$. We claim that $c(v_3) = c(v_4) = 0$. If this is not the case, then $c(v_3) = c(v_4) = 1$. Since $\sigma(v_3) = 0$, it follows that $c(w) = 1$. This, however, implies that $\sigma(w) = 1$, which is a contradiction. Thus, as claimed, $c(v_3) = c(v_4) = 0$ and so c assigns the color 0 to w. Now, $c(v_1) = c(v_2) = 1$; for otherwise, c assigns the color 0 to every vertex of G which is not possible. Hence $c(u_1) = c(u_2) = c(u_3) = c(u_4) = 1$. Thus if $c(v) = 0$ then c assigns the color 1 to six vertices in the set $\{u_1,\ u_2,\ u_3,\ u_4,\ v_1,\ v_2\}$, and the color 0 to the remaining vertices of G. Thus $|\mathcal{C}_1| \in \{6, 2(\ell + 1)\}$.

Case 2. $c(v) = 1$. Then $c(v_i) = 1$ for $5 \leq i \leq 2\ell$. If $c(v_3) = c(v_4) = 0$, then since $\sigma(v_3) = 0$, $c(w) = 1$. This would imply that $\sigma(w) = 1$, which is impossible. Thus $c(v_3) = c(v_4) = 1$ and so c assigns the color 0 to w. Now, observe that $c(v_1) \neq c(v_2)$. If $c(v_1) = 1$ and $c(v_2) = 0$, then $c(u_1) = c(u_2) = 1$ and $c(u_3) = c(u_4) = 0$. Thus c

assigns the color 1 to each vertex in the set $\{u_1,\ u_2,\ v,\ v_1,\ v_3,\ v_4, \ldots,\ v_{2\ell}\}$ and the color 0 to the remaining vertex of G. If $c(v_1) = 0$ and $c(v_2) = 1$, then $c(u_1) = c(u_2) = 0$ and $c(u_3) = c(u_4) = 1$. So c assigns the color 1 to each vertex in the set $\{u_1,\ u_2,\ v,\ v_2,\ v_3,\ v_4, \ldots,\ v_{2\ell}\}$ and the color 0 to the remaining vertex of G. Thus $|\mathcal{C}_1| \in \{6, 2(\ell+1)\}$. Thus $S_{(2,0)}(G) = \{6, 2(\ell+1)\}$. ■

Note that a vertex w in Proposition 5.4.8 is a zero-vertex.

Proposition 5.4.9 *For a positive integer $\ell \geq 3$, there is a graph G of order $2\ell+7$ such that $S_{(2,0)}(G) = \{6, 2(\ell+1)\}$.*

Proof. Let $H = K_{1,2\ell} + e$ with $V(H) = \{v, v_1, v_2, \ldots, v_{2\ell}\}$ where v is the central vertex of $K_{1,2\ell}$ and $e = v_1v_2$. Let G be the graph of order $2\ell+7$ obtained from H and 6 vertices $u_1,\ u_2,\ u_3,\ u_4,\ w_1, w_2$ by joining u_1 and u_2 to v_1, u_3 and u_4 to v_2, w_1 and w_2 to both v_3 and v_4. We show that $S_{(2,0)}(G) = \{6, 2(\ell+1)\}$. First observe that

(*a*) A coloring of G that assigns the color 1 to each vertex in $\{v_1, v_2, u_1, u_2, u_3, u_4\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $6 \in S_{(2,0)}(G)$.

(*b*) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{u_1, u_2, v_1, w_1, w_2\}$ is a $(2,0)$-coloring of G. Thus $2(\ell+1) \in S_{(2,0)}(G)$.

By (a) – (b), it follows that G is $(2,0)$-colorable and $\{6, 2(\ell+1)\} \subseteq S_{(2,0)}(G)$. Let c be a $(2,0)$-coloring of G. Note that, for every i and j such that $5 \leq i \neq j \leq 2\ell$, $c(v) = c(v_i) = c(v_j)$, $c(v_3) = c(v_4)$, $c(v_1) = c(u_1) = c(u_2)$, $c(v_2) = c(u_3) = c(u_4)$, and $c(w_1) = c(w_2)$. We consider two cases, according to whether $c(v) = 0$ or $c(v) = 1$.

Case 1. $c(v) = 0$. Then $c(v_i) = 0$ for $5 \leq i \leq 2\ell$, $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$. We claim that $c(v_3) = c(v_4) = 0$. If this is not the case, then $c(v_3) = c(v_4) = 1$. Either

$c(w_1) = c(w_2) = 0$ or $c(w_1) = c(w_2) = 1$ produces $\sigma(v_3) = 1$, which is a contradiction. Thus $c(v_3) = c(v_4) = 0$ and so c assigns the color 0 to w_1 and w_2. Now, $c(v_1) = c(v_2) = 1$; for otherwise, c assigns the color 0 to every vertex of G which is not possible. Hence c assigns the color 1 to u_1, u_2, u_3 and u_4. Thus if $c(u) = 0$ then c assigns the color 1 to six vertices in the set $\{u_1, u_2, u_3, u_4, v_1, v_2\}$, and the color 0 to the remaining vertices of G. Thus $|\mathcal{C}_1| \in \{6, 2(\ell + 1)\}$.

Case 2. $c(v) = 1$. Then $c(v_i) = 1$ for $5 \leq i \leq 2\ell$. If $c(v_3) = c(v_4) = 0$, either $c(w_1) = c(w_2) = 0$ or $c(w_1) = c(w_2) = 1$ produces $\sigma(v_3) = 1$, which is a contradiction. Thus $c(v_3) = c(v_4) = 1$ and so c assigns the color 0 to w_1 and w_2. Now, observe that $c(v_1) \neq c(v_2)$. If $c(v_1) = 1$ and $c(v_2) = 0$, then c assigns the color 1 to u_1 and u_2. Hence c assigns the color 1 to each vertex in the set $\{u_1, u_2, v, v_1, v_3, v_4, \ldots, v_{2\ell}\}$ and the color 0 to the remaining vertex of G. If $c(v_1) = 0$ and $c(v_2) = 1$, then c assigns the color 1 to u_3 and u_4 and the color 0 to u_1 and u_2. Hence c assigns the color 1 to each vertex in the set $\{u_3, u_4, v, v_2, v_3, v_4, \ldots, v_{2\ell}\}$ and the color 0 to the remaining vertex of G. Thus $|\mathcal{C}_1| \in \{6, 2(\ell + 1)\}$. Thus $S_{(2,0)}(G) = \{6, 2(\ell + 1)\}$. ■

Note that w_1 and w_2 of G in Proposition 5.4.9 are zero-vertices.

Corollary 5.4.10 *For each pair ℓ, n of integers where $\ell \geq 3$ and $n \geq 2\ell + 6$, there is a graph G of order n such that $S_{(2,0)}(G) = \{6, 2(\ell + 1)\}$.*

Proof. By Propositions 5.4.8 and 5.4.9, there are graphs G_1 and G_2 containing zero-vertices of order $2\ell + 6$ and $2\ell + 7$, respectively, such that $S_{(2,0)}(G_1) = S_{(2,0)}(G_2) = \{6, 2(\ell + 1)\}$. If n is even then $s = n - (2\ell + 6)$ is even. Let G_1' be the graph obtained from G_1 by adding s new vertices and joining those new vertices to a zero-vertex of G_1. If n is odd then

$t = n - (2\ell + 7)$ is even. Let G_2' be the graph obtained from G_2 by adding t new vertices and joining those new vertices to a zero-vertex of G_2. Then it is shown in Lemma 5.4.1 that $S_{(2,0)}(G_i) = S_{(2,0)}(G_i')$ where $i = 1, 2$ and thus the result follows. ■

Proposition 5.4.11 *For a positive integer $\ell \geq 3$, there is a graph G of order $2\ell + 4$ such that $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell + 1)\}$.*

Proof. Let $H = K_{1,2\ell} + e$ with $V(H) = \{v, v_1, v_2, \ldots, v_{2\ell}\}$ where v is the central vertex of $K_{1,2\ell}$ and $e = v_1v_2$. Let G be the graph of order $2\ell + 4$ obtained from H and 3 new vertices u_1, u_2 and w by joining vertices u_1 and u_2 to v_1, w to v_3 and v_4. We will show that $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell + 1)\}$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{u_1, u_2, v_1, v_2\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $4 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{u_1, u_2, v_1, w\}$ is a $(2,0)$-coloring of G. Thus $2\ell \in S_{(2,0)}(G)$.

(c) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{v_2, w\}$ is a $(2,0)$-coloring of G. Thus $2(\ell + 1) \in S_{(2,0)}(G)$.

By (a) – (c), it follows that G is $(2,0)$-colorable and $\{4, 2\ell, 2(\ell+1)\} \subseteq S_{(2,0)}(G)$. Let c be a monochromatic $(2,0)$-coloring of G. Note that, for every i and j such that $5 \leq i \neq j \leq 2\ell$, $c(v) = c(v_i) = c(v_j)$, $c(v_3) = c(v_4)$ and $c(v_1) = c(u_1) = c(u_2)$. We consider two cases, depending on the color of v assigned by c.

Case 1. $c(v) = 0$. Then $c(v_i) = 0$ for $5 \leq i \leq 2\ell$, $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$. We claim that $c(v_3) = c(v_4) = 0$. If this is not the case then $c(v_3) = c(v_4) = 1$ and so since

$\sigma(v_3) = 0$, $c(w) = 1$. This, however, implies that $\sigma(w) = 1$, which is a contradiction and so, as claimed, $c(v_3) = c(v_4) = 0$. Thus c assigns the color 0 to w. Now, $c(v_1) = c(v_2) = 1$; for otherwise, c assigns the color 0 to every vertex of G, which is not possible. Hence $c(u_1) = c(u_2) = 1$. Thus if $c(v) = 0$ then c assigns the color 1 to four vertices in the set $\{u_1, u_2, v_1, v_2\}$, and the color 0 to the remaining vertices of G. Thus $|\mathcal{C}_1| \in \{4, 2\ell, 2(\ell+1)\}$.

Case 2. $c(v) = 1$. Then $c(v_i) = 1$ for $5 \leq i \leq 2\ell$. If $c(v_3) = c(v_4) = 0$, then $c(w) = 1$ since $\sigma(v_3) = 0$. This would implies that $\sigma(w) = 1$, which is a contradiction. Thus $c(v_3) = c(v_4) = 1$ and so c assigns the color 0 to w. Now, observe that $c(v_1) \neq c(v_2)$ since $\sigma(v) = 0$. If $c(v_1) = 1$ and $c(v_2) = 0$, then $c(u_1) = c(u_2) = 1$. Thus c assigns the color 1 to each vertex in the set $\{v,\ u_1,\ u_2,\ v_1,\ v_3,\ v_4, \ldots,\ v_{2\ell}\}$ and the color 0 to the remaining vertex of G. If $c(v_1) = 0$ and $c(v_2) = 1$, then $c(u_1) = c(u_2) = 0$. So c assigns the color 1 to each vertex in the set $\{v,\ v_2,\ v_3,\ v_4, \ldots,\ v_{2\ell}\}$ and the color 0 to the remaining vertex of G. Thus $|\mathcal{C}_1| \in \{4, 2\ell, 2(\ell+1)\}$.

Hence $S_{(2,0)}(G) \subseteq \{4, 2\ell, 2(\ell+\ 1)\}$ and so $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell+\ 1)\}$. ■

Note that a vertex w of G in Proposition 5.4.11 is a zero-vertex.

Proposition 5.4.12 *For a positive integer $\ell \geq 3$, there is a graph G of order $2\ell+5$ such that $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell+\ 1)\}$.*

Proof. Let $H = K_{1,2\ell} + e$ with $V(H) = \{v, v_1, v_2, \ldots, v_{2\ell}\}$ where v is the central vertex of $K_{1,2\ell}$ and $e = v_1v_2$. Let G be the graph of order $2\ell+5$ obtained from H and 4 new vertices u_1, u_2, w_1, w_2 by joining the two vertices u_1 and u_2 to v_1, w_1 and w_2 to both v_3 and v_4. We will show that $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell+1)\}$. We will show that $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell+\ 1)\}$. First observe that

(a) A coloring of G that assigns the color 1 to each vertex in $\{u_1, u_2, v_1, v_2\}$ and the color 0 to the remaining vertices of G is a $(2,0)$-coloring of G. Thus $4 \in S_{(2,0)}(G)$.

(b) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{u_1, u_2, v_1, w_1, w_2\}$ is a $(2,0)$-coloring of G. Thus $2\ell \in S_{(2,0)}(G)$.

(c) A coloring of G that assigns the color 1 to every vertex of G except each vertex in $\{v_2, w_1, w_2\}$ is a $(2,0)$-coloring of G. Thus $2(\ell+1) \in S_{(2,0)}(G)$.

By (a) – (c), it follows that G is $(2,0)$-colorable and $\{4, 2\ell, 2(\ell+1)\} \subseteq S_{(2,0)}(G)$. Let c be a monochromatic $(2,0)$-coloring of G. Note that, for every i and j such that $5 \leq i \neq j \leq 2\ell$, $c(v) = c(v_i) = c(v_j)$, $c(v_3) = c(v_4)$, $c(w_1) = c(w_2)$ and $c(v_1) = c(u_1) = c(u_2)$. We consider two cases, depending on the color of v assigned by c.

Case 1. $c(v) = 0$. Then $c(v_i) = 0$ for $5 \leq i \leq 2\ell$, $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$. We claim that $c(v_3) = c(v_4) = 0$. If this is not the case, then $c(v_3) = c(v_4) = 1$. Either $c(w_1) = c(w_2) = 0$ or $c(w_1) = c(w_2) = 1$ produces $\sigma(v_3) = 1$, which is a contradiction and so, as claimed, $c(v_3) = c(v_4) = 0$. Thus $c(w_1) = c(w_2) = 0$. If $c(v_1) = c(v_2) = 0$ then c assigns the color 0 to every vertex of G, which is impossible. Hence $c(v_1) = c(v_2) = 1$ and so $c(u_1) = c(u_2) = 1$. Thus if $c(v) = 0$ then c assigns the color 1 to four vertices in the set $\{u_1, u_2, v_1, v_2\}$, and the color 0 to the remaining vertices of G and so $|\mathcal{C}_1| \in \{4, 2\ell, 2(\ell+1)\}$.

Case 2. $c(v) = 1$. Then $c(v_i) = 1$ for $5 \leq i \leq 2\ell$. If $c(v_3) = c(v_4) = 0$, then either $c(w_1) = c(w_2) = 0$ or $c(w_1) = c(w_2) = 1$ produces $\sigma(v_3) = 1$, which is a contradiction. Thus $c(v_3) = c(v_4) = 1$ and so c assigns the color 0 to w_1 and w_2. Now, observe that $c(v_1) \neq c(v_2)$ since $\sigma(v) = 0$. If $c(v_1) = 1$ and $c(v_2) = 0$, then $c(u_1) = c(u_2) = 1$. Hence c assigns the color 1 to each vertex in the set $\{v, u_1, u_2, v_1, v_3, v_4, \ldots, v_{2\ell}\}$ and the color 0 to the remaining vertex of G; while if $c(v_1) = 0$ and $c(v_2) = 1$, then $c(u_1) = c(u_2) = 0$. Thus c assigns the color 1 to each vertex in the set $\{v, v_2, v_3, v_4, \ldots, v_{2\ell}\}$ and the color 0 to the remaining vertex of G and so $|\mathcal{C}_1| \in \{4, 2\ell, 2(\ell+1)\}$.

Hence $S_{(2,0)}(G) \subseteq \{4, 2\ell, 2(\ell+1)\}$ and so $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell+1)\}$. ∎

Note that w_1 and w_2 of G in Proposition 5.4.12 are zero-vertices.

Corollary 5.4.13 *For each pair ℓ, n of integers where $\ell \geq 3$ and $n \geq 2\ell+4$, there is a graph G of order n such that $S_{(2,0)}(G) = \{4, 2\ell, 2(\ell+1)\}$.*

Proof. By Propositions 5.4.11 and 5.4.12, there are graphs G_1 and G_2 containing zero-vertices of order $2\ell+4$ and $2\ell+5$, respectively, such that $S_{(2,0)}(G_1) = S_{(2,0)}(G_2) = \{4, 2\ell, 2(\ell+1)\}$. If n is even then $s = n-(2\ell+4)$ is even. Let G_1' be the graph obtained from G_1 by adding s new vertices and joining those new vertices to a zero-vertex of G_1. If n is odd then $t = n-(2\ell+5)$ is even. Let G_2' be the graph obtained from G_2 by adding t new vertices and joining those new vertices to a zero-vertex of G_2. Then it is shown in Lemma 5.4.1 that $S_{(2,0)}(G_i) = S_{(2,0)}(G_i')$ where $i = 1, 2$ and thus the result follows. ∎

Chapter 6

On Monochromatic $(2,0)$-Frames

6.1 Introduction

We have seen in Proposition 2.3.4 that if c is a modular monochromatic $(2,0)$-coloring of a connected graph G, then the subgraph of G induced by the vertices colored 1 by c is an odd-degree graph. In particular, this is also the case for minimum monochromatic $(2,0)$-colorings of every $(2,0)$-colorable graph. By the results obtained in Chapter 3, it can be shown that if F is an odd-degree forest, there is a tree T having a minimum monochromatic $(2,0)$-coloring c such that the subgraph of T induced by the vertices colored 1 by c is F. In fact, this is true for all odd-degree graphs, as we show next.

Theorem 6.1.1 *For each odd-degree graph H, there is a connected graph G having a minimum monochromatic $(2,0)$-coloring c such that the subgraph F_B of G induced by the vertices colored 1 by c is H.*

Proof. Let H be an odd-degree graph of order n with $V(H) = \{v_1, v_2, \ldots, v_n\}$. Construct a connected graph G of order $n + n\binom{n}{2}$ from H as follows: For each pair i, j of integers where $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, add n new vertices to H and join these n vertices to both v_i and v_j. Thus, $V(G) - V(H)$ is an independent set of $n\binom{n}{2}$ vertices in G and each vertex in $V(G) - V(H)$ has degree 2 in G. We show that $\chi_{(2,0)}(G) = n$.

First, let c be the coloring defined by $c(x) = 1$ if $x \in V(H)$ and $c(x) = 0$ otherwise. If $x \in V(H)$, then $\sigma(x) = 1 + \deg_H x = 0$ in $\mathbb{Z}_2$; while if $x \in V(G) - V(H)$, then $c(x) = 0$, x is adjacent to exactly two vertices in H, and so $\sigma(x) = 0 + 2 = 0$ in $\mathbb{Z}_2$. Thus c is a monochromatic $(2,0)$-coloring of G and $\chi_{(2,0)}(G) \le n$. To show that $\chi_{(2,0)}(G) \ge n$, let c' be a minimum monochromatic $(2,0)$-coloring of G. Since $\deg_G x = 2$ for each vertex $x \in V(G) - V(H)$ and $V(G) - V(H)$ is independent, some vertex in H must be colored 1 by c'. We claim that $c'(x) = 1$ for every $x \in V(H)$. If this is not the case, then there are two vertices u and v in H such that $c'(u) = 0$ and $c'(v) = 1$. Let $W \subseteq V(G) - V(H)$ be the set of n independent vertices of degree 2 that are adjacent only to u and v in G. Since $\sigma_{c'}(w) = 0$ for each $w \in W$, it follows that $c'(w) = 1$ for every vertex in W. However then, c' must assign the color 1 to at least $n + 1$ vertices of G (namely $\{v\} \cup W$), which contradicts the fact that $\chi_{(2,0)}(G) \le n$. Hence, as claimed, $c'(x) = 1$ for each $x \in V(H)$ and so $\chi_{(2,0)}(G) \ge n$. Therefore, $\chi_{(2,0)}(G) = n$. Furthermore, the coloring c defined above that assigns the color 1 to each vertex of H and the color 0 to the remaining vertices of G is the only minimum monochromatic $(2,0)$-coloring of G and so $F_B = H$. ■

Theorem 6.1.1 suggests the following concept. For an odd-degree graph H (connected or not connected) of order p, a connected graph $G \neq H$ is a *monochromatic* $(2,0)$*-frame of* H or simply a $(2,0)$*-frame of* H if G has a minimum monochromatic $(2,0)$-coloring c such that the subgraph induced by the vertices colored 1 by c is H. By Theorem 6.1.1, every odd-degree graph H has a $(2,0)$-frame. The *monochromatic frame number* or simply the *frame number* $fn(H)$ of H is defined as

$$fn(H) = \min\{|V(G) - V(H)|\}$$

where the minimum is taken over all $(2,0)$-frames G of H. By the proof of Theorem 6.1.1, if H is an odd-degree graph of order p, then

$$1 \le fn(H) \le p\binom{p}{2}. \tag{6.1}$$

6.2 Frame Numbers of $(2,0)$-Extremal Graphs

Recall that a $(2,0)$-colorable graph G of order n is *a* $(2,0)$-*extremal graph* if $\chi_{(2,0)}(G) = n$. Thus every $(2,0)$-extremal graph is an odd-degree graph. We first show that the frame number of each $(2,0)$-extremal graph is at most 2.

Proposition 6.2.1 *If H is a connected $(2,0)$-extremal graph, then $fn(H) \le 2$.*

Proof. Let H be a connected $(2,0)$-extremal graph of order $p \ge 2$ and let xy be an edge of H. Let G be the graph obtained by adding two new vertices v_1 and v_2 and joining each v_i to x and y. The coloring that assign the color 0 to v_1 and v_2 and the color 1 to each vertex of H is a monochromatic $(2,0)$-coloring of G and so $\chi_{(2,0)}(G) \le p$. It remains to show that $\chi_{(2,0)}(G) \ge p$. Let c be a minimum monochromatic $(2,0)$-coloring of G. Thus $c(v_1) = c(v_2) \in \{0,1\}$ by Observation 1.3.1.

First, suppose that $c(v_1) = c(v_2) = 1$. Thus, exactly one of x and y must be colored 1, say $c(x) = 1$ and $c(y) = 0$. Let c' be the restriction of c to H. Since $\sigma_{c'}(v) = \sigma_c(v) = 0$ in $\mathbb{Z}_2$ for each $v \in V(H) - \{x,y\}$ and $\sigma_{c'}(v) = \sigma_c(v) - (c(v_1) + c(v_2)) = \sigma_c(v) - 2 = 0$ in $\mathbb{Z}_2$ for each $v \in \{x,y\}$, it follows that c' is a monochromatic $(2,0)$-coloring of H. However then, $c'(y) = 0$

and H is $(2,0)$-extremal, which is impossible. Next, suppose that $c(v_1) = c(v_2) = 0$. Then the restriction of c to H is a monochromatic $(2,0)$-coloring of H. Since H is $(2,0)$-extremal, $c(v) = 1$ for each $v \in V(H)$ and so $\chi_{(2,0)}(G) \geq p$. Hence $\chi_{(2,0)}(G) = p$. Thus G is a $(2,0)$-frame of H and so $fn(H) \leq 2$. ■

Proposition 6.2.2 *If T is an odd-degree tree, then $fn(T) = 1$.*

Proof. Let T be an odd-degree tree of order $p \geq 2$. Since K_3 is a frame of K_2, we may assume that $p \geq 3$. Let u and v be two vertices of T such that $d_T(u,v) = 2$. Now let G be the graph obtained from T by adding a new vertex w and joining w to u and v. Since the coloring that assigns the color 0 to w and the color 1 to each vertex of T is a monochromatic $(2,0)$-coloring, $\chi_{(2,0)}(G) \leq p$. It remains to show that $\chi_{(2,0)}(G) \geq p$. Let c be a minimum monochromatic $(2,0)$-coloring of G.

We claim that $c(w) = 0$. If this is not the case, then exactly one of u and v must be colored 1 by c, say $c(u) = 1$ and $c(v) = 0$. Since $\sigma(v) = 0$, it follows that v is adjacent to a vertex $w_1 \in V(T)$ such that $c(w_1) = 1$. Thus w_1 is adjacent to an odd number of vertices colored 1 in T. Since $\deg_T w_1$ is odd and $c(v) = 0$, the vertex w_1 is adjacent to a vertex $w_2 \in V(T)$ and $w_2 \neq v$ such that $c(w_2) = 0$. Continuing this procedure, we obtain a path $(w_1, w_2, \ldots, w_{k-1}, w_k)$ in T such that $c(w_i) \neq c(w_{i+1})$ for $1 \leq i \leq k-1$ (in particular, $c(w_{k-1}) \neq c(w_k)$) and w_k is an end-vertex of T (and an end-vertex of G), which is impossible. Thus, $c(w) = 0$, as claimed.

Since $c(w) = 0$, the restriction of c to T is a monochromatic $(2,0)$-coloring of T. Since T is an odd-degree tree, T is $(2,0)$-extremal and so $c(x) = 1$ for each $x \in V(T)$. Therefore, $\chi_{(2,0)}(G) \geq p$ and so $\chi_{(2,0)}(G) = p$. Furthermore, G has a unique monochromatic $(2,0)$-coloring which assigns the color 0 to w only. ■

For a graph H, the *corona* $\mathrm{cor}(H)$ of H is that graph obtained from H by adding a pendant edge to each vertex of H. We have seen in Theorem 3.1.6 that for each integer $p \geq 3$,

$$\chi_{(2,0)}(\mathrm{cor}(C_p)) = \begin{cases} 2p & \text{if } p \text{ is odd} \\ p & \text{if } p \text{ is even.} \end{cases}$$

Thus, the graph $\mathrm{cor}(C_p)$ is $(2,0)$-extremal if $p \geq 3$ is odd. Note that the corona of a p-cycle is an odd-degree graph. Next, we show that the frame number of every corona $\mathrm{cor}(C_p)$ of the p-cycle is 1 for each $p \geq 3$. Recall that If uv is a pendant edge in a $(2,0)$-colorable graph G, then $c(u) = c(v)$ for every modular monochromatic $(2,0)$-coloring c of G (see Observation 4.1.4).

Theorem 6.2.3 *For each integer $p \geq 3$, $fn(\mathrm{cor}(C_p)) = 1$.*

Proof. Since $\mathrm{cor}(C_p)$ is an odd-degree graph, $\mathrm{cor}(C_p)$ is $(2,0)$-colorable for each $p \geq 3$. For each integer $p \geq 3$, let $C_p = (v_1, v_2, \ldots, v_p, v_1)$ be the p-cycle and for each i with $1 \leq i \leq p$, let $u_i v_i$ be the pendant edge at a vertex v_i. Let G be the graph obtained from $\mathrm{cor}(C_p)$ and a single vertex w by joining w to both v_1 and v_2. Since the coloring that assigns the color 0 to w and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \leq 2p$. It remains to show that $\chi_{(2,0)}(G) \geq 2p$.

Let c be a minimum monochromatic $(2,0)$-coloring of G. Observe that G contains exactly three even vertices; namely v_1, v_2, w. By Proposition 3.2.1, exactly one or all of these three even vertices is color 0 by c and by Observation 4.1.4, $c(u_i) = c(v_i)$ for $1 \leq i \leq p$. If $c(v_1) = c(v_2) = c(w) = 0$, then c assigns the color 0 to every vertex of G; that is, c is a trivial monochromatic $(2,0)$-coloring of G, which is a contradiction. Thus exactly one of v_1, v_2 and w is colored 0 by c. By symmetry, we may assume, without loss of generality, that either (i) $c(v_1) = 0$ and $c(v_2) = c(w) = 1$ or (ii) $c(w) = 0$ and $c(v_1) = c(v_2) = 1$. If

(i) occurs, then $c(u_1) = 0$ and $c(u_2) = 1$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 1$ and so $c(u_3) = 1$. Now, since $\sigma(v_3) = 0$, $c(v_4) = 1$ and so $c(u_4) = 1$. Continue this procedure, we obtain that c assigns the color 1 to each vertex v_i and u_i where $2 \le i \le p$. This, however, is impossible since $\sigma(v_1) = 1$. If (ii) occurs, then $c(u_1) = c(u_2) = 1$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 1$ and so $c(u_3) = 1$. Again since $\sigma(v_3) = 0$, $c(v_4) = 1$ and so $c(u_4) = 1$. Continue this procedure, we obtain that c assigns the color 1 to each vertex in $V(G) - \{w\}$. In fact, this is a unique monochromatic $(2, 0)$-coloring of G. Hence $\chi_{(2,0)}(G) \ge 2p$ and so $\chi_{(2,0)}(G) = 2p$. Since the subgraph induced by the vertices colored 1 by c of G is $\mathrm{cor}(C_p)$, it follows that $fn(\mathrm{cor}(C_p)) = 1$. ■

Recall the following result in Chapter 3 (see Theorem 3.2.2).

Theorem 6.2.4 *Let $p \ge 3$ be an integer. The graph $C_p \vee K_1$ is $(2, 0)$-colorable if and only if $p \not\equiv 2, 4 \pmod 6$ and*

$$\chi_{(2,0)}(C_p \vee K_1) = \begin{cases} \frac{2p}{3} & \text{if } p \equiv 0 \pmod 6 \\ p+1 & \text{if } p \equiv 1, 5 \pmod 6 \\ \frac{p}{3}+1 & \text{if } p \equiv 3 \pmod 6. \end{cases} \tag{6.2}$$

By Theorem 6.2.4, the wheel $C_p \vee K_1$ of order $p + 1$ is $(2, 0)$-extremal only if $p \equiv 1, 5 \pmod 6$. Next, we show that the frame number of every $(2, 0)$-extremal wheel is 1.

Theorem 6.2.5 *For each integer $p \ge 3$ where $p \equiv 1, 5 \pmod 6$, $fn(C_p \vee K_1) = 1$.*

Proof. For each integer $p \ge 3$ where $p \equiv 1, 5 \pmod 6$, let $C_p = (v_1, v_2, \ldots, v_p, v_1)$ be the p-cycle and $V(K_1) = \{v\}$. Let G be the graph obtained from $C_p \vee K_1$ and a single vertex w by joining w to both v_1 and v_2. Since the coloring that assigns the color 0 to w and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G and so $\chi_{(2,0)}(G) \leq p+1$. It remains to show that $\chi_{(2,0)}(G) \geq p+1$.

Let c be a minimum monochromatic $(2,0)$-coloring of G. Observe that G contains exactly three even vertices; namely v_1, v_2, w. By Proposition 3.2.1, exactly one or all of these three even vertices is color 0 by c. If $c(v_1) = c(v_2) = c(w) = 0$, then the restriction of c to $C_p \vee K_1$ is a nontrivial $(2,0)$-coloring of $C_p \vee K_1$. This implies that $\chi_{(2,0)}(C_p \vee K_1) \leq p-1$, which contradicts Theorem 6.2.4. Thus exactly one of v_1, v_2 and w is colored 0 by c.

We claim that $c(w) = 0$ and $c(v_1) = c(v_2) = 1$. If this is not the case, then we may assume, without loss of generality, that $c(v_1) = 0$. Thus $c(w) = c(v_2) = 1$ and $c(v) = c(v_3)$ since $\sigma(v_2) = 0$. If $c(v) = c(v_3) = 0$ then since $\sigma(v_3) = 0$, it follows that $c(v_4) = 1$. Now, $c(v_5) = 1$ since $\sigma(v_4) = 0$. Since $\sigma(v_5) = 0$, $c(v_6) = 0$ and since $\sigma(v_6) = 0$, $c(v_7) = 1$. Continue this procedure, we obtain $c(v_i) = 1$ if $2 \leq i \leq p$ and $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 0$ if $3 \leq i \leq p$ and $i \equiv 0 \pmod 3$ which is impossible since this would imply that $\sigma(v_1) = 1$. If $c(v) = c(v_3) = 1$ then since $\sigma(v_3) = 0$, it follows that $c(v_4) = 1$. Now, $c(v_5) = 1$ since $\sigma(v_4) = 0$. Continue this procedure, we obtain that $c(v_i) = 1$ if $2 \leq i \leq p$ which is impossible since this would imply that $\sigma(v) = 1$. Thus the claim holds and so $c(v_1) = c(v_2) = 1$ and $c(w) = 0$. Hence $c(v) = c(v_3)$ since $\sigma(v_2) = 0$. If $c(v) = c(v_3) = 0$ then since $\sigma(v_3) = 0$, it follows that $c(v_4) = 1$. Now, $c(v_5) = 1$ since $\sigma(v_4) = 0$. Since $\sigma(v_5) = 0$, $c(v_6) = 0$ and since $\sigma(v_6) = 0$, $c(v_7) = 1$. Continue this procedure, we obtain $c(v_i) = 1$ if $2 \leq i \leq p$ and $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 0$ if $3 \leq i \leq p$ and $i \equiv 0 \pmod 3$ which is impossible since this would imply that $\sigma(v_1) = 1$. If $c(v) = c(v_3) = 1$ then since $\sigma(v_3) = 0$, it follows that $c(v_4) = 1$. Now, $c(v_5) = 1$ since $\sigma(v_4) = 0$. Continue this procedure, we obtain that $c(v_i) = 1$ if $2 \leq i \leq p$. Thus the coloring c that assigns that color 1 to every vertex of $V(G) - \{w\}$ is a unique monochromatic $(2,0)$-coloring of G. Hence $\chi_{(2,0)}(G) \geq p+1$ and so $\chi_{(2,0)}(G) = p+1$. Moreover, the subgraph induced by the vertices colored 1 by c of G is $C_p \vee K_1$ and so $fn(C_p \vee K_1) = 1$. ■

If $p \equiv 0, 3 \pmod 6$, then $C_p \vee K_1$ is also $(2,0)$-colorable but not $(2,0)$-extremal by Theorem 6.2.4. Since $C_p \vee K_1$ is not an odd-degree graph when $p \equiv 0 \pmod 6$, we consider the frame number of $C_p \vee K_1$ when $p \equiv 3 \pmod 6$. By Theorem 6.2.3, the frame number for every corona of a cycle is 1 regardless of being $(2,0)$-extremal or non-$(2,0)$-extremal. This is not the case for the wheels.

Theorem 6.2.6 *If $p \geq 3$ and $p \equiv 3 \pmod 6$, then $fn(C_p \vee K_1) = 2$.*

Proof. Let $H = C_p \vee K_1$, where $C_p = (v_1, v_2, v_3, \ldots, v_p, v_1)$ and $V(K_1) = \{u\}$. We first show that $fn(H) \geq 2$. Assume, to the contrary, that $fn(H) = 1$. Then H has a $(2,0)$-frame G of order $p+2$. Hence $\chi_{(2,0)}(G) = p+1$ and G has a minimum monochromatic $(2,0)$-coloring c for which the subgraph of G induced by the vertices colored 1 by c is H. Let $w \in V(G) - V(H)$. Thus w is adjacent to an even number of vertices of H, say w is adjacent to t vertices of H where $0 \leq t \leq p+1$ is even. Since G is connected, $t \neq 0$. On the other hand, if $t = p+1$, then the coloring of G that assigns the color 1 to each vertex in the set $\{u\} \cup \{v_i : i \equiv 1 \pmod 3 \text{ and } 1 \leq i \leq p\}$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G. This implies that $\chi_{(2,0)}(G) < p+1$, which is impossible. Thus $2 \leq t \leq (p+1) - 2$. Let

$$\begin{aligned}
S &= \{u\} \cup \{v_i : i \equiv 1 \pmod 3 \text{ and } 1 \leq i \leq p\} \\
T &= \{v_i : i \equiv 1, 2 \pmod 3 \text{ and } 1 \leq i \leq p\} \\
W &= \{u\} \cup \{v_i : \ i \equiv 2 \pmod 3 \text{ and } 1 \leq i \leq p\}
\end{aligned}$$

Then the colorings c_S, c_T and c_W of H that assign the color 1 to each vertex in the sets S, T and W, respectively, and the color 0 to the remaining vertices of H are monochromatic $(2,0)$-colorings of H. Also, note that

$$\begin{aligned} S \cap T &= \{v_i : i \equiv 1 \pmod 3 \text{ and } 1 \leq i \leq p\} \\ S \cap W &= \{u\} \\ T \cap W &= \{v_i : i \equiv 2 \pmod 3 \text{ and } 1 \leq i \leq p\}. \end{aligned}$$

First, assume that w is adjacent to an even number of vertices in at least one of the sets S, T and W. We may assume, without loss of generality, w is adjacent to an even number of vertices of S. Then the coloring c_S can be extended to a monochromatic $(2,0)$-coloring of G by assigning the color 0 to each vertex not in S. This implies that $\chi_{(2,0)}(G) \leq |S| < p+1$, which is impossible. Next, assume that w is adjacent to an odd number of vertices in each of the sets S, T and W. Let α be the number of vertices in $S \cap T$ that are adjacent to w, let β be the number of vertices in $S \cap W$ that are adjacent to w and let γ be the number of vertices in $T \cap W$ that are adjacent to w. Thus each of the integers $\alpha+\beta$, $\alpha+\gamma$ and $\beta+\gamma$ is odd. Since $\alpha+\beta$ is odd, we may assume that α is odd. Then β and γ are even. However, then $\beta+\gamma$ is even and so w is adjacent to an even number of vertices in W, which is impossible. Thus $fn(H) \geq 2$.

Next, we show that $fn(H) \leq 2$. Let F be the graph obtained from H and two new vertices x, y by joining x to u and v_1 and joining y to v_1 and v_2. We show that F is a $(2,0)$-frame of H. The coloring of F that assigns the color 0 to x and y and the color 1 to the remaining vertices of F is a monochromatic $(2,0)$-coloring of F. Thus $\chi_{(2,0)}(F) \leq p+1$. Let c be a monochromatic $(2,0)$-coloring of F. We consider four cases.

Case 1. $c(x) = c(y) = 0$. Then since $\sigma(x) = \sigma(y) = 0$, it follows that $c(u) = c(v_1) = c(v_2)$. If $c(u) = c(v_1) = c(v_2) = 0$ then c is a trivial $(2,0)$-coloring of F, which is impossible. If $c(u) = c(v_1) = c(v_2) = 1$ then c assigns the color 1 to each vertex of H.

Case 2. $c(x) = c(y) = 1$. Then $c(u) = c(v_2)$. If $c(u) = c(v_2) = 0$ then $c(v_1) = 1$. Since $\sigma(v_2) = 0$, $c(v_3) = 0$ and since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(v_4) = 0$, $c(v_5) = 0$. Continue this process, we have that, for each i with $2 \leq i \leq p$, $c(v_i) = 0$. This, however, implies that $\sigma(v_p) = 1$, which is impossible. If $c(u) = c(v_2) = 1$ then $c(v_1) = 0$. Since $\sigma(v_2) = 0$, $c(v_3) = 1$ and since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(v_4) = 0$, $c(v_5) = 1$. Continue this process, we have that, for each $i \neq 1$, $c(v_i) = 1$. This, however, implies that $\sigma(v_p) = 1$, which is impossible.

Case 3. $c(x) = 1$ and $c(y) = 0$. Then $c(v_1) = c(v_2)$. If $c(v_1) = c(v_2) = 0$ then $c(u) = 1$. Since $\sigma(v_2) = 0$, $c(v_3) = 1$ and since $\sigma(v_3) = 0$, $c(v_4) = 0$. Now, since $\sigma(v_4) = 0$, $c(v_5) = 0$. Since $\sigma(v_5) = 0$, $c(v_6) = 1$ and since $\sigma(v_6) = 0$, $c(v_7) = 0$. Continue this process, we have that, for each i with $2 \leq i \leq p$, $c(v_i) = 1$ where $i \equiv 0 \pmod 3$ and $c(v_i) = 0$ otherwise. This, however, implies that $\sigma(u) = 1$, which is impossible. If $c(v_1) = c(v_2) = 1$ then $c(u) = 0$. Since $\sigma(v_2) = 0$, $c(v_3) = 0$ and since $\sigma(v_3) = 0$, $c(v_4) = 1$. Now, since $\sigma(v_4) = 0$, $c(v_5) = 1$. Since $\sigma(v_5) = 0$, $c(v_6) = 0$ and since $\sigma(v_6) = 0$, $c(v_7) = 1$. Since $\sigma(v_7) = 0$, $c(v_8) = 1$. Continue this process, we have that, for each $1 \leq i \leq p$, $c(v_i) = 1$ where $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 0$, otherwise. This, however, implies that $\sigma(v_1) = 1$, which is impossible.

Case 4. $c(x) = 0$ and $c(y) = 1$. Then $c(v_1) \neq c(v_2)$. If $c(v_1) = 0$ and $c(v_2) = 1$ then $c(u) = 0$. Since $\sigma(v_2) = 0$, $c(v_3) = 0$ and since $\sigma(v_3) = 0$, $c(v_4) = 1$. Now, since $\sigma(v_4) = 0$, $c(v_5) = 1$. Since $\sigma(v_5) = 0$, $c(v_6) = 0$ and since $\sigma(v_6) = 0$, $c(v_7) = 1$. Since $\sigma(v_7) = 0$, $c(v_8) = 1$. Continue this process, we have that, for each $2 \leq i \leq p$, $c(v_i) = 1$ where $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 0$, otherwise. This, however, implies that $\sigma(v_p) = 1$, which is impossible. If $c(v_1) = 1$ and $c(v_2) = 0$ then $c(u) = 1$. Since $\sigma(v_2) = 0$, $c(v_3) = 1$ and since $\sigma(v_3) = 0$, $c(v_4) = 0$. Now, since $\sigma(v_4) = 0$, $c(v_5) = 0$. Since $\sigma(v_5) = 0$, $c(v_6) = 1$ and since $\sigma(v_6) = 0$, $c(v_7) = 0$. Since $\sigma(v_7) = 0$, $c(v_8) = 0$ and since $\sigma(v_8) = 0$, $c(v_9) = 1$. Continue this process, we have that, for each $2 \le i \le p$, $c(v_i) = 1$ where $i \equiv 0 \pmod 3$ and $c(v_i) = 0$, otherwise. This, however, implies that $\sigma(v_p) = 1$, which is impossible. Hence the only monochromatic $(2,0)$-coloring of F is the coloring that assigns the color 0 to x and y and the color 1 to the remaining vertices of F and so $\chi_{(2,0)}(F) = p + 1$. Observe that the subgraph of G induced by the vertices colored 1 by c is H. This shows $fn(H) \le 2$ and so $fn(H) = 2$. ■

It then follows by Theorems 6.2.5 and 6.2.6 that if $C_p \vee K_1$ is an odd-degree $(2,0)$-colorable wheel where $p \ge 3$, then

$$fn(C_p \vee K_1) = \begin{cases} 1 & \text{if } C_p \vee K_1 \text{ is } (2,0)\text{-extremal} \\ 2 & \text{otherwise.} \end{cases} \tag{6.3}$$

We include this section with the following question.

Problem 6.2.7 *Does every $(2,0)$-extremal graph have $(2,0)$-frame number 1?*

6.3 Frame Numbers of Unions of $(2,0)$-Extremal Graphs

Let T be a tree of order $k \ge 1$, where $V(T) = \{v_1, v_2, \ldots, v_k\}$ and $E(T) = \{e_1, e_2, \ldots, e_{k-1}\}$. The *subdivision graph* $S(T)$ of T is the tree of order $2k - 1$ obtained from T by replacing each edge e_i $(1 \le i \le k-1)$ by the vertex u_i which is joined to the two vertices of T incident with e_i. A graph G is an *odd-degree subdivision graph* if the vertices $v_1, v_2, \ldots, v_k$ of some subdivision graph $S(T)$ of a tree T of order k correspond to k pairwise disjoint odd-degree graphs $H_1, H_2, \ldots, H_k$ in G and $V(G) - \cup_{i=1}^{k} V(H_i)$ consists of an independent set of $k-1$ vertices of G each adjacent to exactly two of the graphs $H_1, H_2, \ldots, H_k$. In this case, G is referred to as an *odd-degree subdivision graph with respect to odd-degree graphs* $H_1, H_2, \ldots, H_k$ (or simply an *ods-graph with respect to* H, where $H = H_1 + H_2 + \cdots + H_k$ is the union of $H_1, H_2, \ldots, H_k$. The vertices $u_1, u_2, \ldots, u_{k-1}$ are called *subdividing vertices* of G.

Theorem 6.3.1 *Let H be an odd-degree graph consisting of $k \geq 1$ components H_1, H_2, $\ldots$, H_k such that each H_i is $(2,0)$-extremal. If G is the connected graph obtained from H and $k-1$ vertices $u_1, u_2, \ldots, u_{k-1}$ by joining each u_i to exactly two of $H_1, H_2, \ldots, H_k$, then $\chi_{(2,0)}(G) = |V(H)|$ and the coloring that assigns to color 1 to each vertex of H and the color 0 to each of $u_1, u_2, \ldots, u_{k-1}$ is the unique monochromatic $(2,0)$-coloring of G.*

Proof. Note that the graph G has a "tree-like" structure. That is, if we contract each component H_i to a vertex v_i $(1 \leq i \leq k)$, then the resulting graph is the subdivision graph of a tree of order k.

Since the coloring that assigns the color 1 to each vertex of H and the color 0 to each of $u_1, u_2, \ldots, u_{k-1}$ of G is a monochromatic $(2,0)$-coloring of G, it remains to show that this is the only monochromatic $(2,0)$-coloring of G. We proceed by induction on the number k of components of H. If $k = 1$, then $G = H$ is $(2,0)$-extremal and so the result holds for $k = 1$. Suppose, for some integer $k \geq 1$, that the statement is true for all odd-degree graphs having exactly k components, each of which is $(2,0)$-extemal. Let H be an odd-degree graph of order $p \geq 2$ having exactly $k+1$ components $H_1, H_2, \ldots, H_{k+1}$, each of which is $(2,0)$-extemal. Let G be the connected graph obtained from H and k vertices $u_1, u_2, \ldots, u_k$ by joining each u_i to exactly two of $H_1, H_2, \ldots, H_{k+1}$. Assume, without loss of generality, that H_1 is adjacent to exactly one of the vertices $u_1, u_2, \ldots, u_k$ (say H_1 is adjacent to u_1)

and u_1 is also adjacent to H_2. Let $x \in V(H_1)$ and $y \in V(H_2)$ such that u_1 is adjacent to x and y. Let c be a monochromatic $(2,0)$-coloring of G. We show that $c(u) = 0$ for each $u \in \{u_1, u_2, \ldots, u_k\}$ and $c(v) = 1$ f or each $v \in V(H)$.

First, we claim that $c(u_1) = 0$. If this is not the case, then exactly one of x and y is colored 1. First, suppose that $c(y) = 0$ and $c(x) = 1$. Let Q be the component of $G - y$ such that $V(Q) = V(H_1) \cup \{u_1\}$. The restriction c_Q of c to Q is a monochromatic $(2,0)$-coloring of Q. Since x is the only even vertex of Q and $c_Q(x) = c(x) = 1$, this contradicts Proposition 3.2.1. Next, suppose that $c(y) = 1$ and $c(x) = 0$. Since c is a monochromatic $(2,0)$-coloring of G, it follows that $\sigma_c(v) = 0$ in $\mathbb{Z}_2$ for each vertex $v \in V(H_1)$ and so $\sigma_c(H_1) = \sum_{v \in V(H_1)} \sigma_c(v)$ is even. If $v \in V(H_1)$ such that $c(v) = 0$, then v contributes 0 to $\sigma_c(H_1)$; while if $v \in V(H_1)$ such that $c(v) = 1$, then v contributes $1 + \deg_{H_1} v$ to $\sigma_c(H_1)$. Since H_1 is an odd-degree graph, each vertex $v \in V(H_1)$ contributes an even number to $\sigma_c(H_1)$. Furthermore, u_1 contributes 1 to $\sigma_c(H_1)$. Hence if we let V' be the set of vertices in H_1 that are colored 1 by c, then $\sigma_c(H_1) = 1 + \sum_{v \in V'}(1 + \deg_{H_1} v)$, which is odd. This is impossible. Therefore, $c(u_1) = 0$. as claimed.

Let Q_1 and Q_2 be the two components of $G - u_1$ where $x \in V(Q_1)$ and $y \in V(Q_2)$. For $i = 1, 2$, the restriction c_i of c to Q_i is either a monochromatic $(2,0)$-coloring of Q_i or a trivial coloring (that assigns 0 to each vertex). Since c must assign the color 1 to some vertex of G, either Q_1 or Q_2 has a vertex colored 1. First, suppose that Q_1 has a vertex colored 1. Since Q_1 is $(2,0)$-extremal, $c_1(v) = c(v) = 1$ for all $v \in V(Q_1)$. In particular, $c(x) = 1$ and so $c(y) = 1$ (as $\sigma(u_1) = 0$). Next, suppose that Q_2 has a vertex colored 1. By the induction hypothesis, c_2 must assign the color 1 to each vertex in $H_2, H_3, \ldots, H_{k+1}$ and the color 0 to $u_2, u_3, \ldots, u_{k+1}$. In particular $c(y) = 1$ and so $c(x) = 1$. Hence c assigns the color 1 to each vertex of H and the color 0 to each of $u_1, u_2, \ldots, u_{k+1}$. Therefore, c is the unique monochromatic $(2,0)$-coloring of G and $\chi_{(2,0)}(G) = |V(H)|$. ■

Theorem 6.3.2 *Let $H_1, H_2, \ldots, H_k$ be $k \geq 2$ connected $(2,0)$-extremal graphs and let $H = H_1 + H_2 + \cdots + H_k$ be the union of these k graphs. Then $fn(H) = k-1$.*

Proof. Suppose that the order of H is n. By Theorem 6.3.1, H has a $(2,0)$-frame of order $n+(k-1)$ and so $fn(H) \leq k-1$. It remains to show that $fn(H) \geq k-1$. Assume, to the contrary, that $fn(H) = t \leq k-2$. Then H has a $(2,0)$-frame G of order $n+t$ such that $\chi_{(2,0)}(G) = n$ and G has a minimum monochromatic $(2,0)$-coloring c for which

$$c(v) = 1 \text{ if } v \in V(H) \text{ and } c(v) = 0 \text{ if } v \notin V(H). \tag{6.4}$$

We claim that $\chi_{(2,0)}(G) < n$, which produces a contradiction. Let $V(G) - V(H) = \{u_1, u_2, \ldots, u_t\}$. For each i with $1 \leq i \leq k$, define a t-vector $\vec{\mathbf{a}}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,t})$ such that for each j with $1 \leq j \leq t$,

$$a_{i,j} = \begin{cases} 1 & \text{if } u_j \text{ is adjacent to an odd number of vertices of } H_i \\ 0 & \text{if } u_j \text{ is adjacent to an even number of vertices of } H_i. \end{cases}$$

If there is $i \in \{1, 2, \ldots, k\}$ such that $\vec{\mathbf{a}}_i$ is a zero vector, say $\vec{\mathbf{a}}_1$ is a zero vector (that is, each u_j is adjacent to an even number vertices of H_1 for $1 \leq j \leq t$), then the coloring that assigns the color 1 to each vertex in $V(H_1)$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring. This, however, implies that $\chi_{(2,0)}(G) \leq |V(H_1)| < |V(H)| = n$, a contradiction. Thus, we may assume that each $\vec{\mathbf{a}}_i$ is a nonzero vector for $1 \leq i \leq k$. Since c is a monochromatic $(2,0)$-coloring of G, it follows by (6.4) that

$$\sigma(u_j) = a_{1,j} + a_{2,j} + \cdots + a_{k,j} = 0 \text{ in } \mathbb{Z}_2 \text{ for } 1 \leq j \leq t.$$

Because $t \leq k-2$, the set $\mathcal{A} = \{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_k\}$ is linearly dependent in $\mathbb{R}^t$. Let p be the minimum cardinality of a linearly dependent subset of $\mathcal{A}$. Then $2 \leq p \leq t+1 < k$. We may assume, without loss of generality, that $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_p$ are linearly dependent vectors in $\mathbb{R}^t$. Because each entry of $\vec{\mathbf{a}}_j$ $(1 \leq j \leq p)$ is an element in $\mathbb{Z}_2$, there is a linear combination $\alpha_1\vec{\mathbf{a}}_1+\alpha_2\vec{\mathbf{a}}_2+\cdots+\alpha_p\vec{\mathbf{a}}_p = 0$ of these p vectors such that $\alpha_j \in \mathbb{Z}_2$ for $1 \leq j \leq p$. Furthermore, since p is the smallest number of linearly dependent vectors in $\mathcal{A}$, it follows that $\alpha_j = 1$ for $1 \leq j \leq p$ and so $\vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_2 + \cdots + \vec{\mathbf{a}}_p = 0$. Hence

$$a_{1,j} + a_{2,j} + \cdots + a_{p,j} = 0 \text{ in } \mathbb{Z}_2 \text{ for } 1 \leq j \leq t \tag{6.5}$$

Define a coloring $c' : V(G) \to \mathbb{Z}_2$ by

$$c'(v) = \begin{cases} 1 & \text{if } v \in V(H_1) \cup V(H_2) \cup \cdots \cup V(H_p) \\ 0 & \text{otherwise.} \end{cases}$$

We claim that c' is a monochromatic $(2,0)$-coloring of G. Observe that if $x \in V(H_i)$ and $y \in V(H_j)$ where $i \neq j$, then x and y are nonadjacent. Thus, if $v \in V(H_i)$ where $1 \leq i \leq p$, then $\sigma_{c'}(v) = 1 + \deg_{H_i} v$, while if $v \in V(H_i)$ where $p < i \leq k$, then $c'(v) = 0$ and v is adjacent to no vertex colored 1. In each case, if $v \in V(H)$, then $\sigma_{c'}(v) = 0$. For each j with $1 \leq j \leq t$, $\sigma_{c'}(u_j) = a_{1,j} + a_{2,j} + \cdots + a_{p,j} = 0$ in $\mathbb{Z}_2$ by (6.5). Hence c' is a monochromatic $(2,0)$-coloring of G. However then,

$$\chi_{(2,0)}(G) \leq |V(H_1) \cup V(H_2) \cup \cdots \cup V(H_p)| < |V(H)| = n,$$

which is a contradiction. Therefore, $fn(H) = k - 1$, as desired. ∎

Corollary 6.3.3 *If F is an odd-degree forest with $k \geq 2$ components, $fn(F) = k - 1$.*

6.4 Frame Numbers of Cubic Graphs

The *Cartesian product* $G \,\square\, H$ of two graphs G and H has vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \,\square\, H$ are adjacent if either (1) $u = x$

and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. The Cartesian product $G \,\square\, K_2$ of a graph G and K_2 is a special case of a more general class of graphs. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let α be a permutation of the set $S = \{1, 2, \ldots, n\}$. The *permutation graph* $P_\alpha(G)$ of a graph G is the graph of order $2n$ obtained from two copies of G, where the second copy of G is denoted by G' and the vertex v_i in G is denoted by u_i in G' and v_i is joined to the vertex $u_{\alpha(i)}$ in G'. The edges $v_i u_{\alpha(i)}$ are called the *permutation edges* of $P_\alpha(G)$. This concept was first introduced by Chartrand and Harary [3]. Therefore, if α is the identity map on S, then $P_\alpha(G) = G \,\square\, K_2$. Figure 6.1 shows the four permutation graphs of the 5-cycle C_5 where the graph G_a in Figure 6.1(a) is $C_5 \,\square K_2$ and the graph G_d in Figure 6.1(d) is the Petersen graph P (see [10]). We denote the graphs in Figures 6.1(b) and 6.1(c) by G_b and G_c, respectively. Each of these permutation graphs is a 3-regular (or cubic) graph.

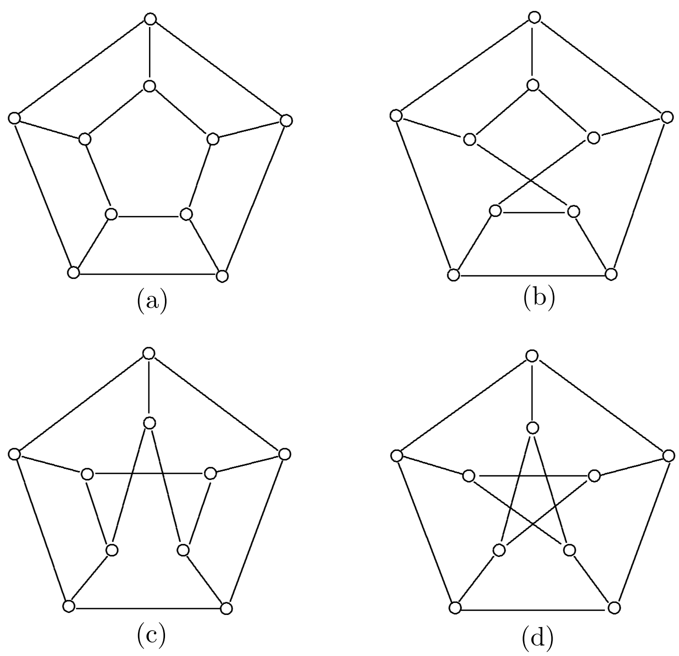


Figure 6.1: The permutation graphs of C_5

We have seen in Theorem 3.1.4 that for each integer $n \geq 3$,

$$\chi_{(2,0)}(C_n \,\square\, K_2) = \begin{cases} n & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 6.4.1 *For the four permutation graphs of C_5 in Figure 6.1,*

$$\chi_{(2,0)}(G_a) = \chi_{(2,0)}(G_b) = 10 \textit{ and } \chi_{(2,0)}(G_c) = \chi_{(2,0)}(G_d) = 4.$$

Proof. By Theorem 3.1.4, $\chi_{(2,0)}(G_a) = 10$ and we have seen that $\chi_{(2,0)}(P) = 4$ for the Petersen graph P. It remains to consider G_b and G_c.

We begin with the graph G_b. Let $(u_1, u_2, u_4, u_3, u_5, u_1)$ and $(v_1, v_2, v_3, v_4, v_5, v_1)$ be two 5-cycles of G_b and $u_iv_i \in E(G_b)$ for $1 \leq i \leq 5$. Since G_b is an odd degree graph, the coloring c that assigns the color 1 to every vertex of G_b is a $(2,0)$-coloring of G_b and so G_b is $(2,0)$-colorable. Assume, to the contrary, that $\chi_{(2,0)}(G_b) \leq 8$. Since G_b is an odd degree graph, it follows that either $\chi_{(2,0)}(G_b) = 2$ or $\chi_{(2,0)}(G_b) = 4$. We consider these two cases. Let c be a minimum monochromatic $(2,0)$-coloring of G_b in each case.

Case 1. $\chi_{(2,0)}(G_b) = 2$. Let x and y be two vertices of G_b such that $c(x) = c(y) = 1$. Since $\deg x = 3$, there is a vertex u of G_b such that $ux \in E(G_b)$. Necessarily, $uy \in E(G_b)$ since $\sigma(u) = 0$. This, however, implies that G_b contains a triangle, which is impossible.

Case 2. $\chi_{(2,0)}(G_b) = 4$. There are two subcases, according to the structure of the subgraph F_B induced by the vertices colored 1 by c.

Subcase 2.1. $F_B = K_{1,3}$. Let $\{x, y_1, y_2, y_3\}$ be the vertex set of $K_{1,3}$ where $\deg_{K_{1,3}} x = 3$. Since $\deg y_1 = 3$, let s_1 and s_2 be two neighbors of y_1. Since $\sigma(s_1) = 0$, it follows that s_1 is adjacent to one of y_2 and y_3. Without loss of generality, we assume that s_1 is adjacent to y_2. Now if s_2 is adjacent to y_2 then there is vertex t (which is a neighbor of y_3) of G_b such that $\sigma(w) = 1$, which is impossible. Thus s_2 is adjacent to y_3. Moreover, Since $\deg y_2 = 3$, let w neighbors of y_2. Since $\sigma(w) = 0$, it follows that w is adjacent to y_3. This, implies, x lies on three 4-cycles, which is impossible.

Subcase 2.2. $F_B = 2K_2$. Let (x_1, x_2) and (y_1, y_2) be two copies of K_2 of F_B. Since $\deg x_1 = 3$, let s_1 and s_2 be two neighbors of x_1. Since G_b does not contain a triangle, s_1 and s_2 are not adjacent to x_2. Moreover, since $\sigma(s_1) = 0$, it follows that s_1 is adjacent to one of y_1 and y_2. Without loss of generality, we assume that s_1 is adjacent to y_1.

If s_2 is adjacent to y_1 then if $t_1, t_2 \in N(x_2)$ then $t_1, t_2 \in N(y_2)$ since $\sigma(t_1) = \sigma(t_2) = 0$. This implies that G_b contains two disjoint 4-cycles $(x_1, s_1, y_1, s_2, x_1)$ and $(x_2, t_1, y_2, t_2, x_2)$ such that $x_1x_2, y_1y_2 \in E(G)$. Observe that G_b contains exactly three 4-cycles which are $(u_1, v_1, v_5, u_5, u_1)$, $(u_1, v_1, v_2, u_2, u_1)$ and $(u_3, u_4, v_4, v_3, u_3)$ and no two 4-cycles of G_b has such property, which is a contradiction.

If s_2 is adjacent to y_2 then if $t_1, t_2 \in N(x_2)$ then, without loss of generality, $t_1 \in N(y_1)$ and $t_2 \in N(y_2)$ since $\sigma(t_1) = \sigma(t_2) = 0$. This implies that there are two adjacent edges s_1x_1 and s_1y_1 lying on two 5-cycles and a 6-cycle which are $(y_1, s_1, x_1, x_2, t_1, y_1)$, $(y_1, s_1, x_1, s_2, y_2, y_1)$, and $(y_1, s_1, x_1, x_2, t_2, y_2, y_1)$. Observe that G_b contains exactly six 5-cycles which are

$$(v_1, v_2, v_3, v_4, v_5, v_1),\ (u_1, u_2, u_4, u_3, u_5, u_1),\ (u_2, v_2, v_3, v_4, u_4, u_2),$$

$$(u_3, v_3, v_4, v_5, u_5, u_3),\ (u_2, v_2, v_3, u_3, u_4, u_2) \text{ and } (u_3, u_4, v_4, v_5, u_5, u_3).$$

Moreover, no two adjacent edges of G_b has such property, which is a contradiction. Thus $\chi_{(2,0)}(G_b) = 10$, as claimed.

Finally, we consider the graph G_c. Let $(u_1, u_3, u_2, u_5, u_4, u_1)$ and $(v_1, v_2, v_3, v_4, v_5, v_1)$

be two 5-cycles of G_c and $u_iv_i \in E(G_c)$ for $1 \le i \le 5$. Observe that the coloring that assigns the color 1 to each vertex in the set $\{u_2, u_5, v_3, v_4\}$ and the color 0 to the remaining vertices of G_c is a $(2,0)$-coloring of G_c and so $\chi_{(2,0)}(G_c) \le 4$. Assume, to the contrary, that $\chi_{(2,0)}(G_c) < 4$. Since $\chi_{(2,0)}(G_c)$ is even, it follows that $\chi_{(2,0)}(G_c) = 2$. Let c be a minimum $(2,0)$-coloring of G and so c assigns the color 1 to exactly two vertices of G_c which are necessarily adjacent, say $c(u) = c(v) = 1$. Since G_c contains no triangle and $\deg(v) = 3$, there is $w \in N(v) - \{u\}$ and so $\sigma(w) = 1$, which is impossible. Thus $\chi_{(2,0)}(G_c) \ne 2$ and so $\chi_{(2,0)}(G_c) = 4$. ■

Next, we show that $fn(C_5 \,\square\, K_2) = 1$. In fact, this is true for all integers $n \ge 5$ with $n \equiv 1, 2, 3 \pmod 4$.

Theorem 6.4.2 *If $n \ge 5$ is an integer with $n \equiv 1, 2, 3 \pmod 4$, then $fn(C_n \,\square\, K_2) = 1$.*

Proof. First suppose that $n \equiv 1, 3 \pmod 4$. Let $(u_1, u_2, u_3, \ldots, u_n, u_1)$ and $(v_1,\ v_2,\ v_3,\ \ldots,\ v_n,\ v_1)$ be two copies of C_n in $C_n \,\square\, K_2$ where $u_iv_i \in E(C_n \,\square\, K_2)$ for each i $(1 \le i \le n)$. Let G be the graph obtained from $C_n \,\square\, K_2$ and a single vertex w by joining w to both v_1 and v_2. Since the coloring that assigns the color 0 to w and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \le 2n$. It remains to show that $\chi_{(2,0)}(G) \ge 2n$. Let c be a minimum monochromatic $(2,0)$-coloring of G. Observe that G contains exactly three even vertices; namely v_1, v_2, w. By Proposition 3.2.1, exactly one or all of these three even vertices is color 0 by c.

If $c(v_1) = c(v_2) = c(w) = 0$, then the restriction of c to $C_n \,\square\, K_2$ is a nontrivial $(2,0)$-coloring of $C_n \,\square\, K_2$. This implies that $\chi_{(2,0)}(C_n \,\square\, K_2) \le 2n - 2$, which contradicts to the fact that $\chi_{(2,0)}(C_n \,\square\, K_2) = 2n$. Thus exactly one of v_1, v_2 and w is colored 0 by c. We claim that $c(w) = 0$. If this is not the case then, without loss of generality, we may assume

$c(v_1) = 0$ and so $c(w) = c(v_2) = 1$. Since $\sigma(v_1) = \sigma(v_2) = 0$, it follows that $c(u_1) = c(v_n)$ and $c(u_2) = c(v_3)$. We consider four cases.

Case 1. $c(u_1) = c(v_n) = c(u_2) = c(v_3) = 0$. Then since $\sigma(u_1) = \sigma(u_2) = 0$, $c(u_n) = 0$ and $c(u_3) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 1$ and $c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$, forces $c(u_5) = 0$ and $c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 0$ and $c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 1$ and $c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 1$ and $c(v_8) = 0$.

Continuing this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 0, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1, 2 \pmod 4 \text{ and } i \geq 3 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 1, 2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0, 3 \pmod 4 \text{ and } i \geq 3. \end{cases}$$

This, however, implies that if $n \equiv 1 \pmod 4$ then $c(v_n) = 0$ and $c(v_n) = 1$, which is impossible and if $n \equiv 3 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is again impossible.

Case 2. $c(u_1) = c(v_n) = 0$ and $c(u_2) = c(v_3) = 1$. Then since $\sigma(u_1) = \sigma(u_2) = 0$, $c(u_n) = 1$ and $c(u_3) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$, forces $c(u_5) = 0$ and $c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 0$ and $c(v_7) = 1$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = c(v_8) = 0$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 0$ and $c(v_9) = 1$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = c(v_{10}) = 1$.

Continuing this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1, 2, 3 \pmod 4 \text{ and } i \geq 2 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0, 1, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0 \pmod 4 \text{ and } i \geq 2. \end{cases}$$

This, however, implies that if $n \equiv 1, 3 \pmod 4$ then $c(v_n) = 0$ and $c(v_n) = 1$, which is impossible.

Case 3. $c(u_1) = c(v_n) = 1$ and $c(u_2) = c(v_3) = 0$. Then since $\sigma(u_1) = \sigma(u_2) = 0$, $c(u_n) = 1$ and $c(u_3) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 0$ and $c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$, forces $c(u_5) = c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 0$ and $c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 0$ and $c(v_8) = 1$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = c(v_9) = 1$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = 0$ and $c(v_{10}) = 1$. Since $\sigma(u_{10}) = \sigma(v_{10}) = 0$, it follows that $c(u_{11}) = c(v_{11}) = 0$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 1 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0, 1, 2 \pmod 4 \text{ and } i \geq 2 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0, 2, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 3 \pmod 4 \text{ and } i \geq 2. \end{cases}$$

This, however, implies that if $n \equiv 1 \pmod 4$ then $\sigma(v_n) = c(u_n) + c(v_1) + c(v_{n-1}) + c(v_n) = 1 + 0 + 1 + 1 = 1$, which is impossible and if $n \equiv 3 \pmod 4$ then $c(v_n) = 0$ and $c(v_n) = 1$, which is again impossible.

Case 4. $c(u_1) = c(v_n) = c(u_2) = c(v_3) = 1$. Then since $\sigma(u_1) = \sigma(u_2) = 0$, $c(u_n) = 0$ and $c(u_3) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$, forces $c(u_5) = c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 1$.

Continue this process, we obtain that $c(x) = 1$ if $x \in V(G) - \{v_1\}$. This, however, implies that if $n \equiv 1, 3 \pmod 4$, then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

Hence $c(w) = 0$, as claimed and so $c(v_1) = c(v_2) = 1$. Now the restriction of c to $G - w$ is a monochromatic $(2, 0)$-coloring of $C_n \,\square\, K_2$. Since the only monochromatic $(2, 0)$-coloring of $C_n \,\square\, K_2$ is the coloring that assigns that color 1 to every vertex of $C_n \,\square\, K_2$, it follows that c assigns the color 1 to every vertex of G except the vertex w. Thus $\chi_{(2,0)}(G) \geq 2n$

and so $\chi_{(2,0)}(G) = 2n$. Furthermore, the subgraph induced by the vertices colored 1 by c is $C_n \square K_2$ and so $fn(C_n \square K_2) = 1$ where $n \equiv 1, 3 \pmod 4$.

Next, suppose that $n \geq 6$ with $n \equiv 2 \pmod 4$. Let $(u_1, u_2, u_3, \ldots, u_n, u_1)$ and $(v_1, v_2, v_3, \ldots, v_n, v_1)$ be two copies of C_n in the graph $C_n \square K_2$ such that $u_i v_i \in E(C_n \square K_2)$ for $1 \leq i \leq n$. Now let G be the graph obtained from $C_n \square K_2$ and a single vertex w by joining w to both v_1 and u_2.

Since the coloring that assigns the color 0 to w and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \leq 2n$. It remains to show that $\chi_{(2,0)}(G) \geq 2n$. Let c be a minimum monochromatic $(2,0)$-coloring of G. Observe that G contains exactly three even vertices; namely v_1, u_2, w. By Proposition 3.2.1, exactly one or all of these three even vertices is color 0 by c. We consider two cases.

Case 1. $c(u_2) = c(v_1) = c(w) = 0$. Since $\sigma(u_1) = \sigma(v_2) = 0$, it follows that $c(u_1) = c(u_n)$ and $c(v_2) = c(v_3)$.

If $c(u_1) = c(u_n) = c(v_2) = c(v_3) = 0$ then, by using the fact that the color sum of each vertex x in G is zero, it can be shown that c is a trivial $(2,0)$-coloring of G that is c assigns the color 0 to every vertex of G, which is impossible.

If $c(u_1) = c(u_n) = 0$ and $c(v_2) = c(v_3) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 0$ and $c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 0$ and $c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = c(v_7) = 1$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 0$ and $c(v_8) = 1$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = c(v_9) = 0$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = 0$ and $c(v_{10}) = 1$. Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0,2,3 \pmod 4 \text{ and } i \geq 2 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0,1,2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1 \pmod 4 \text{ and } i \geq 2. \end{cases}$$

This, however, implies that if $n \equiv 2 \pmod 4$ then $\sigma(u_n) = 1$, which is impossible.

If $c(u_1) = c(u_n) = 1$ and $c(v_2) = c(v_3) = 0$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = 1$ and $c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 0$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 1$ and $c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = c(v_8) = 1$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 1$ and $c(v_9) = 0$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = c(v_{10}) = 0$. Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 0,1,3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0 \pmod 4 \text{ and } i \geq 3 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1,2,3 \pmod 4 \text{ and } i \geq 3. \end{cases}$$

This, however, implies that if $n \equiv 2 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

If $c(u_1) = c(u_6) = c(v_2) = c(v_3) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 1$ and $c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = 1$ and $c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 0$ and $c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 0$ and $c(v_7) = 1$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 1$ and $c(v_8) = 0$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 1$ and $c(v_9) = 0$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 0,1 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 2,3 \pmod 4 \text{ and } i \geq 4 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 2,3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0,1 \pmod 4 \text{ and } i \geq 4. \end{cases}$$

This, however, implies that if $n \equiv 2 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

Case 2. Exactly one of u_2, v_1 and w is colored 0 by c. We claim that $c(w) = 0$. If this is not the case then, without loss of generality, we may assume $c(v_1) = 0$ and so $c(u_2) = c(w) = 1$. Then since $\sigma(u_1) = \sigma(v_2) = 0$, $c(u_1) \neq c(u_n)$ and $c(v_2) \neq c(v_3)$.

If $c(u_1) = c(v_2) = 0$ and $c(u_n) = c(v_3) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = 0$ and $c(v_n) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 0$ and $c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = 1$ and $c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 1$ and $c(v_6) = 0$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 0$ and $c(v_7) = 1$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 0$ and $c(v_8) = 1$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 1$ and $c(v_9) = 0$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = 1$ and $c(v_{10}) = 0$. Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 1, 2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0, 3 \pmod 4 \text{ and } i \geq 3 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1, 2 \pmod 4 \text{ and } i \geq 3. \end{cases}$$

This, however, implies that if $n \equiv 2 \pmod 4$ then $c(v_n) = 0$ and $c(v_n) = 1$, which is impossible.

If $c(u_n) = c(v_2) = 0$ and $c(u_1) = c(v_3) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = 1$ and $c(v_n) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 1$ and $c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 1$ and $c(v_6) = 0$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = c(v_7) = 1$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 1$ and $c(v_8) = 0$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = c(v_9) = 0$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = 1$ and $c(v_{10}) = 0$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 0, 2, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 3 \pmod 4 \text{ and } i \geq 3 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 1 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0, 1, 2 \pmod 4 \text{ and } i \geq 3. \end{cases}$$

If $n \equiv 2 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

If $c(u_1) = c(v_3) = 0$ and $c(u_n) = c(v_2) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = 1$ and $c(v_n) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = 1$ and $c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 1$ and $c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = c(v_8) = 0$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 1$ and $c(v_9) = 0$. Since $\sigma(u_9) = \sigma(v_9) = 0$, it follows that $c(u_{10}) = 1$ and $c(v_{10}) = 1$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 1, 2, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 2 \pmod 4 \text{ and } i \geq 3 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0, 1, 3 \pmod 4 \text{ and } i \geq 3. \end{cases}$$

If $n \equiv 2 \pmod 4$ then $c(v_n) = 0$ and $c(v_n) = 1$, which is impossible.

If $c(u_n) = c(v_3) = 0$ and $c(u_1) = c(v_2) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = 0$ and $c(v_n) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = c(v_5) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = c(v_8) = 1$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 0 \pmod 2 \text{ or } x = v_i \text{ where } i \equiv 0 \pmod 2 \text{ and } i \geq 2 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 1 \pmod 2 \text{ or } x = v_i \text{ where } i \equiv 1 \pmod 2 \text{ and } i \geq 3. \end{cases}$$

If $n \equiv 2 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

Thus, as claimed, $c(w) = 0$ and $c(u_2) = c(v_1) = 1$. Then since $\sigma(u_1) = \sigma(v_2) = 0$, $c(u_1) = c(u_n)$ and $c(v_2) = c(v_3)$.

If $c(u_1) = c(u_n) = c(v_2) = c(v_3) = 0$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 0$ and $c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = 0$ and $c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 1$ and $c(v_6) = 0$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 1$ and $c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 0$ and $c(v_8) = 1$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 0$ and $c(v_9) = 1$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 2, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0, 1 \pmod 4 \text{ and } i \geq 2 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0, 1 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 2, 3 \pmod 4 \text{ and } i \geq 2. \end{cases}$$

If $n \equiv 2 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

If $c(u_1) = c(u_n) = 0$ and $c(v_2) = c(v_3) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = 0$ and $c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 1$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = 0$ and $c(v_7) = 1$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = c(v_8) = 0$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = 0$ and $c(v_9) = 1$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1, 2, 3 \pmod 4 \text{ and } i \geq 2 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 0, 1, 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0 \pmod 4 \text{ and } i \geq 2. \end{cases}$$

If $n \equiv 2 \pmod 4$ then $c(u_n) = 0$ and $c(u_n) = 1$, which is impossible.

If $c(u_1) = c(u_n) = 1$ and $c(v_2) = c(v_3) = 0$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 0$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = 1$

and $c(v_4) = 0$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 1$ and $c(v_6) = 0$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = 1$ and $c(v_6) = 0$. Since $\sigma(u_6) = \sigma(v_6) = 0$, it follows that $c(u_7) = c(v_7) = 0$. Since $\sigma(u_7) = \sigma(v_7) = 0$, it follows that $c(u_8) = 1$ and $c(v_8) = 0$. Since $\sigma(u_8) = \sigma(v_8) = 0$, it follows that $c(u_9) = c(v_9) = 1$.

Continue this process, we obtain

$$c(x) = \begin{cases} 1 & \text{if } x = u_i \text{ where } i \equiv 0,1,2 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 1 \pmod 4 \text{ and } i \geq 3 \\ 0 & \text{if } x = u_i \text{ where } i \equiv 3 \pmod 4 \text{ or } x = v_i \text{ where } i \equiv 0,2,3 \pmod 4 \text{ and } i \geq 3. \end{cases}$$

If $n \equiv 2 \pmod 4$ then $\sigma(v_n) = c(u_n) + c(v_1) + c(v_{n-1}) + c(v_n) = 1 + 1 + 1 + 0 = 1$, which is impossible.

If $c(u_1) = c(u_6) = c(v_2) = c(v_3) = 1$ then since $\sigma(u_2) = \sigma(v_1) = 0$, it follows that $c(u_3) = c(v_n) = 1$. Moreover, since $\sigma(u_3) = \sigma(v_3) = 0$, it follows that $c(u_4) = c(v_4) = 1$. The fact that $\sigma(u_4) = \sigma(v_4) = 0$ forces $c(u_5) = c(v_5) = 1$. Since $\sigma(u_5) = \sigma(v_5) = 0$, it follows that $c(u_6) = c(v_6) = 1$. Continue this procedure, we obtain that $c(x) = 1$ where $x \in V(G) - w$.

Thus the only monochromatic $(2,0)$-coloring of G is the coloring that assigns the color 1 to every vertex of G except the vertex w. Hence $\chi_{(2,0)}(G) = 2n$ and since the subgraph induced by the vertices colored 1 by c is $C_n \,\square\, K_2$, $fn(C_n \,\square\, K_2) = 1$ where $n \equiv 2 \pmod 4$. ■

By Theorem 6.4.2, if $n \geq 5$ is an integer with $n \equiv 1,2,3 \pmod 4$, then $fn(C_n \,\square\, K_2) = 1$. This, however, is not the case when $n \equiv 0 \pmod 4$, as we show next.

Theorem 6.4.3 *If $n \geq 4$ with $n \equiv 0 \pmod 4$, then $fn(C_n \,\square\, K_2) \geq 2$.*

Proof. Let $(u_1, u_2, u_3, \ldots, u_n, u_1)$ and $(v_1, v_2, v_3, \ldots, v_n, v_1)$ be two copies of C_n in $C_n \,\square\, K_2$ and where $u_i v_i \in E(C_n \,\square\, K_2)$ for each i $(1 \leq i \leq n)$. Assume, to the contrary, that

$fn(C_n \,\square\, K_2) = 1$. Hence there is a $(2,0)$-colorable connected graph G of order $2n+1$ such that $\chi_{(2,0)}(G) = 2n$ and G has a minimum monochromatic $(2,0)$-coloring c for which the subgraph of G induced by the vertices colored 1 by c is $C_n \,\square\, K_2$. Let $w \in V(G) - V(C_n \,\square\, K_2)$. Thus w is adjacent to an even number of vertices of $C_n \,\square\, K_2$, say w is adjacent to t vertices of $C_n \,\square\, K_2$ where $0 \le t \le 2n$ is even.

Since G is connected, $t \neq 0$. On the other hand, if $t = 2n$, then the coloring of G that assigns the color 1 to each vertex in the set $\{u_i, v_i : i \text{ is odd and } 1 \le i \le 2n\}$ and the color 0 to the remaining vertices is a monochromatic $(2,0)$-coloring of G, which is impossible since $\chi_{(2,0)}(G) = 2n$. Thus if such t exists, $2 \le t \le 2n-2$. Let

$$\begin{aligned}
S &= \{x : x = u_i,\ i \equiv 1 \pmod 4 \text{ or } x = v_i,\ i \equiv 0, 1, 2 \pmod 4), 1 \le i \le n\} \\
T &= \{x : x = u_i,\ i \equiv 2 \pmod 4 \text{ or } x = v_i,\ i \equiv 1, 2, 3 \pmod 4), 1 \le i \le n\} \\
W &= \{x : x = u_i,\ i \equiv 1, 2 \pmod 4 \text{ or } x = v_i,\ i \equiv 0, 3 \pmod 4), 1 \le i \le n\}.
\end{aligned}$$

Observe that the colorings c_S, c_T and c_W of $C_n \,\square\, K_2$ that assign the color 1 to each vertex in the sets S, T and W, respectively, and the color 0 to the remaining vertices of $C_n \,\square\, K_2$ are $(2,0)$-colorings of $C_n \,\square\, K_2$. Also, note that

$$\begin{aligned}
S \cap T &= \{x : x = v_i,\ i \equiv 1, 2 \pmod 4)\} \\
S \cap W &= \{x : x = u_i,\ i \equiv 1 \pmod 4 \text{ or } x = v_i,\ i \equiv 0 \pmod 4), 1 \le i \le n\} \\
T \cap W &= \{x : x = u_i,\ i \equiv 2 \pmod 4 \text{ or } x = v_i,\ i \equiv 3 \pmod 4), 1 \le i \le n\}.
\end{aligned}$$

We consider two cases.

Case 1. w is adjacent to an even number of vertices in at least one set of S, T and W. Without loss of generality, let w be adjacent to an even number of vertices of S then the coloring c_S can be extended to a monochromatic $(2,0)$-coloring of G, which is impossible.

Case 2. w is adjacent to an odd number of vertices in each of the sets S, T and W. Let α be the number of vertices in $S \cap T$ that are adjacent to w, β the number of vertices in $S \cap W$ that are adjacent to w and γ the number of vertices in $T \cap W$ that are adjacent to w. Thus each of the integers $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$ is odd. Since $\alpha + \beta$ is odd, we may assume that α is odd. Then β and γ are even. However, then $\beta + \gamma$ is even and so w is adjacent to an even number of vertices in W, which is impossible. ■

We have the following conjecture for the frame number of $C_n \,\square\, K_2$ when $n \equiv 0 \pmod 4$.

Conjecture 6.4.4 *If $n \geq 4$ with $n \equiv 0 \pmod 4$, then $fn(C_n \,\square\, K_2) = 2$.*

Theorem 6.4.5 *For the graph G_b in* Figure 6.1(b), *$fn(G_b) = 1$.*

Proof. Let G be the graph obtained from G_b and a single vertex w by joining w to v_1 and v_2. Then we claim that G is a color frame of G_b. Since the coloring that assigns the color 0 to w and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, $\chi_{(2,0)}(G) \leq 10$. It remains to show that $\chi_{(2,0)}(G) \geq 10$.

Let c be a minimum monochromatic $(2,0)$-coloring of G. Observe that G contains exactly three even vertices; namely v_1, v_2, w. By Proposition 3.2.1, exactly one or all of these three even vertices is colored 0 by c. If $c(v_1) = c(v_2) = c(w) = 0$. Then the restriction of c to G_b is a nontrivial $(2,0)$-coloring of G_b, which is impossible since $\chi_{(2,0)}(G_b) = 10$.

We claim that $c(w) = 0$ and $c(v_1) = c(v_2) = 1$. If this is not the case, then we consider 2 cases.

Case 1. $c(v_1) = 0$ and $c(w) = c(v_2) = 1$. Since $\sigma(v_1) = \sigma(v_2) = 0$, $c(u_2) = c(v_3)$ and $c(u_1) = c(v_5)$.

If $c(u_2) = c(v_3) = c(u_1) = c(v_5) = 0$, then $c(u_4) = 1$ since $\sigma(u_2) = 0$ and $c(u_5) = 0$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 0$ since $\sigma(u_5) = 0$. This, however, implies that $\sigma(u_3) = 1$, which is impossible.

If $c(u_2) = c(v_3) = c(u_1) = c(v_5) = 1$, then $c(u_4) = 1$ since $\sigma(u_2) = 0$ and $c(u_5) = 0$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 0$ since $\sigma(u_5) = 0$ and $c(v_4) = 0$ since $\sigma(u_4) = 0$. This, however, implies that $\sigma(v_4) = 1$, which is impossible.

If $c(u_2) = c(v_3) = 0$ and $c(u_1) = c(v_5) = 1$, then $c(u_4) = 0$ since $\sigma(u_2) = 0$ and $c(u_5) = 1$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 1$ since $\sigma(u_5) = 0$ and $c(v_4) = 1$ since $\sigma(u_4) = 0$. This, however, implies that $\sigma(v_5) = 1$, which is impossible.

If $c(u_2) = c(v_3) = 1$ and $c(u_1) = c(v_5) = 0$, then $c(u_4) = 0$ since $\sigma(u_2) = 0$ and $c(u_5) = 1$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 1$ since $\sigma(u_5) = 0$ and $c(v_4) = 0$ since $\sigma(u_4) = 0$. This, however, implies that $\sigma(v_4) = 1$, which is impossible.

Case 2. $c(v_2) = 0$ and $c(w) = c(v_1) = 1$. Since $\sigma(v_1) = \sigma(v_2) = 0$, $c(u_2) = c(v_3)$ and $c(u_1) = c(v_5)$.

If $c(u_2) = c(v_3) = c(u_1) = c(v_5) = 0$, then $c(u_4) = 0$ since $\sigma(u_2) = 0$ and $c(u_5) = 1$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 1$ since $\sigma(u_5) = 0$ and $c(v_4) = 1$ since $\sigma(u_4) = 0$. This, however, implies that $\sigma(v_4) = 1$, which is impossible.

If $c(u_2) = c(v_3) = c(u_1) = c(v_5) = 1$, then $c(u_4) = 0$ since $\sigma(u_2) = 0$ and $c(u_5) = 1$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 1$ since $\sigma(u_5) = 0$ and $c(v_4) = 0$ since $\sigma(u_4) = 0$. This, however, implies that $\sigma(v_5) = 1$, which is impossible.

If $c(u_2) = c(v_3) = 0$ and $c(u_1) = c(v_5) = 1$, then $c(u_4) = 1$ since $\sigma(u_2) = 0$ and $c(u_5) = 0$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 0$ since $\sigma(u_5) = 0$. This, however, implies that $\sigma(u_3) = 1$, which is impossible.

If $c(u_2) = c(v_3) = 1$ and $c(u_1) = c(v_5) = 0$, then $c(u_4) = 1$ since $\sigma(u_2) = 0$ and $c(u_5) = 0$ since $\sigma(u_1) = 0$. Now, $c(u_3) = 0$ since $\sigma(u_5) = 0$ and $c(v_4) = 0$ since $\sigma(u_4) = 0$. This, however, implies that $\sigma(v_5) = 1$, which is impossible.

Thus, as claimed, $c(w) = 0$ and $c(v_1) = c(v_2) = 1$. Furthermore, the restriction of c to G_b is a monochromatic $(2, 0)$-coloring of G_b and since $\chi_{(2,0)}(G) = 10$ by Proposition **??**, it follows that c assigns the color 1 to every vertex of G_b. This implies that G is a color frame of G_b and so $fn(G_b) = 1$. ■

Theorem 6.4.6 *For the graph G_c in* Figure 6.1(c)*, $fn(G_c) = 2$.*

Proof. We first show that $fn(G_c) \geq 2$. Assume, to the contrary, that $fn(G_c) = 1$. Then there is a $(2, 0)$-colorable connected graph G of order 11 such that $\chi_{(2,0)}(G) = 10$ and G has a minimum monochromatic $(2, 0)$-coloring c for which the subgraph of G induced by the vertices colored 1 by c is G_c. Let $w \in V(G) - V(G_c)$. Thus w is adjacent to even number of vertices of G_c, say w is adjacent to t vertices of G_c where $0 \leq t \leq 10$ is even. Since G is connected, $t \neq 0$. On the other hand, if $t = 10$, then the coloring of G that assigns the color 1 to each vertex in the set $\{u_2, u_5, v_3, v_4\}$ and the color 0 to the remaining vertices is a $(2, 0)$-coloring of G. This implies $\chi_{(2,0)}(G) \leq 4$, which is impossible. Thus $2 \leq t \leq 8$. Let

$$S = \{u_2, u_5, v_3, v_4\},\ T = \{u_1, u_2, u_5, v_1\} \text{ and } W = \{u_1, v_1, v_3, v_4\}.$$

Then the colorings c_S, c_T and c_W of G_c that assign the color 1 to each vertex in the sets S, T and W, respectively, and the color 0 to the remaining vertices of G_c are monochromatic $(2, 0)$-colorings of G_c. Furthermore,

$$S \cap T = \{u_2, u_5\},\ S \cap W = \{v_3, v_4\} \text{ and } T \cap W = \{u_1, v_1\}.$$

First, assume that w is adjacent to even number of vertices in at least one of the sets S, T and W. We may assume, without loss of generality, that w is adjacent to an even number of vertices of S. Then the coloring c_S can be extended to a monochromatic $(2,0)$-coloring of G by assigning the color 0 to w. This implies that $\chi_{(2,0)}(G) \leq 4$, which is impossible. Next, assume that w is adjacent to an odd number of vertices in each of the sets S, T and W. Let α be the number of vertices in $S \cap T$ that are adjacent to w, let β the number of vertices in $S \cap W$ that are adjacent to w and let γ the number of vertices in $T \cap W$ that are adjacent to w. Thus each of the integers $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$ is odd. Since $\alpha + \beta$ is odd, we may assume that α is odd. Then β and γ are even. However, then $\beta + \gamma$ is even and so w is adjacent to an even number of vertices in W, which is impossible. Thus, $fn(G_c) \geq 2$.

To show that $fn(G_c) \leq 2$, let G be the graph obtained from G_c by adding two vertices x and y and joining (i) x to u_1 and u_2 and (ii) y to u_2 and v_3. We claim that G is a color frame of G_c. Since the coloring that assigns the color 0 to x and y and the color 1 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G, it follows that $\chi_{(2,0)}(G) \leq 10$. It remains to show that $\chi_{(2,0)}(G) \geq 10$. Let c be a minimum monochromatic $(2,0)$-coloring of G. We consider four cases.

Case 1. $c(x) = c(y) = 0$. Since $\sigma(x) = \sigma(y) = 0$, it follows that $c(u_1) = c(u_2) = c(v_3)$.

Subcase 1.1. $c(u_1) = c(u_2) = c(v_3) = 0$. Since $\sigma(u_3) = 0$, $c(u_3) = 0$ and since $\sigma(v_2) = 0$, $c(v_1) = c(v_2)$.

If $c(v_1) = c(v_2) = 0$, then c is a trivial $(2,0)$-coloring of G, which is impossible.

If $c(v_1) = c(v_2) = 1$, then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It then follows that $\sigma(u_5) = 1$, which is impossible.

Subcase 1.2. $c(u_1) = c(u_2) = c(v_3) = 1$. Since $\sigma(u_3) = 0$, $c(u_3) = 1$ and since $\sigma(v_2) = 0$, $c(v_1) = c(v_2)$.

If $c(v_1) = c(v_2) = 0$ then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

If $c(v_1) = c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$. So c assigns the color 1 to each vertex of G except x and y.

Case 2. $c(x) = c(y) = 1$. Since $\sigma(x) = \sigma(y) = 0$, it follows that $c(u_1) = c(v_3)$.

Subcase 2.1. $c(u_1) = c(v_3) = 0$. Then $c(u_2) = 1$ and since $\sigma(u_3) = 0$, $c(u_3) = 1$. Moreover, since $\sigma(v_2) = 0$, $c(v_1) \neq c(v_2)$.

If $c(v_1) = 1$ and $c(v_2) = 0$, then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

If $c(v_1) = 0$ and $c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_2) = 1$, which is impossible.

Subcase 2.2. $c(u_1) = c(v_3) = 1$. Then $c(u_2) = 0$ and since $\sigma(u_3) = 0$, $c(u_3) = 0$. Moreover, since $\sigma(v_2) = 0$, $c(v_1) \neq c(v_2)$.

If $c(v_1) = 1$ and $c(v_2) = 0$ then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_2) = 1$, which is impossible.

If $c(v_1) = 0$ and $c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

Case 3. $c(x) = 1$ and $c(y) = 0$. Then $c(u_1) \neq c(v_3)$.

Subcase 3.1. $c(u_1) = 1$ and $c(v_3) = 0$. Then $c(u_2) = 0$ and so $c(u_3) = 1$. Since $\sigma(v_2) = 0$, $c(v_1) = c(v_2)$.

If $c(v_1) = c(v_2) = 0$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_2) = 1$, which is impossible.

If $c(v_1) = c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

Subcase 3.2. $c(u_1) = 0$ and $c(v_3) = 1$. Then $c(u_2) = 0$ and so $c(u_3) = 0$. Since $\sigma(v_2) = 0$, $c(v_1) = c(v_2)$.

If $c(v_1) = c(v_2) = 0$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

If $c(v_1) = c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_2) = 1$, which is impossible.

Case 4. $c(x) = 0$ and $c(y) = 1$. Then $c(u_1) \neq c(v_3)$.

Subcase 4.1. $c(u_1) = 1$ and $c(v_3) = 0$. Then $c(u_2) = 1$ and so $c(u_3) = 0$. Since

$\sigma(v_2) = 0$, $c(v_1) \neq c(v_2)$.

If $c(v_1) = 1$ and $c(v_2) = 0$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

If $c(v_1) = 0$ and $c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 1$. This forces $c(u_5) = 1$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(v_1) = 1$, which is impossible.

Subcase 4.2. $c(u_1) = 0$ and $c(v_3) = 1$. Then $c(u_2) = 0$ and so $c(u_3) = 1$. Since $\sigma(v_2) = 0$, $c(v_1) \neq c(v_2)$.

If $c(v_1) = 1$ and $c(v_2) = 0$ then since $\sigma(v_3) = 0$, $c(v_4) = 1$. Since $\sigma(u_1) = 0$, $c(u_4) = 0$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(v_1) = 1$, which is impossible.

If $c(v_1) = 0$ and $c(v_2) = 1$ then since $\sigma(v_3) = 0$, $c(v_4) = 0$. Since $\sigma(u_1) = 0$, $c(u_4) = 1$ and since $\sigma(v_4) = 0$, $c(v_5) = 0$. This forces $c(u_5) = 0$ since $\sigma(v_5) = 0$. It, then, follows that $\sigma(u_5) = 1$, which is impossible.

Hence the only $(2, 0)$-coloring of G is the coloring that assigns the color 0 to x and y and the color 1 to the remaining vertices of G and so $\chi_{(2,0)}(G) = 10$. Observe that the subgraph of G induced by the vertices colored 1 by c is G_c. This shows $fn(G_c) \leq 2$ and so $fn(G_c) = 2$. ∎

Theorem 6.4.7 *If P is the Petersen graph, then $fn(P) \geq 2$.*

Proof. Let $(u_1, u_3, u_5, u_2, u_4, u_1)$ and $(v_1, v_2, v_3, v_4, v_5, v_1)$ be two 5-cycles of P and $u_iv_i \in E(P)$. Assume, to the contrary, that $fn(P) = 1$. Thus P has a $(2, 0)$-frame G of order 11 such that $\chi_{(2,0)}(G) = 10$ and G has a minimum monochromatic $(2,0)$-coloring c for which the subgraph of G induced by the vertices colored 1 by c is P. Let $w \in V(G) - V(P)$. Thus w is adjacent to an even number of vertices of P, say w is adjacent to t vertices of P where $0 \le t \le 10$ is even. Since G is connected, $t \neq 0$. On the other hand, if $t = 10$ then, the coloring of G that assigns the color 1 to each vertex in the set $\{u_2, u_4, v_1, v_5\}$ and the color 0 to the remaining vertices is a monochromatic $(2,0)$-coloring of G, which is impossible since $\chi_{(2,0)}(G) = 10$. Thus if such t exists, $2 \le t \le 8$.

Let $S = \{u_2, u_4, v_1, v_5\}$, $T = \{u_3, v_3, v_1, v_5\}$ and $W = \{u_2, u_3, u_4, v_3\}$. Observe that the colorings c_S, c_T and c_W of P that assign the color 1 to each vertex in the sets S, T and W, respectively, and the color 0 to the remaining vertices of P are $(2,0)$-colorings of P. Also, note that $S \cap T = \{v_1, v_5\}$, $S \cap W = \{u_2, u_4\}$ and $T \cap W = \{u_3, v_3\}$. We consider two cases.

Case 1. w is adjacent to an even number of vertices in at least one set of S, T and W. Without loss of generality, let w be adjacent to an even number of vertices of S then the coloring c_S can be extended to a monochromatic $(2,0)$-coloring c' of G, which is impossible.

Case 2. w is adjacent to an odd number of vertices in each of the sets S, T and W. Let α be the number of vertices in $S \cap T$ that are adjacent to w, β the number of vertices in $S \cap W$ that are adjacent to w and γ the number of vertices in $T \cap W$ that are adjacent to w. Thus each of the integers $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$ is odd. Since $\alpha + \beta$ is odd, we may assume that α is odd. Then β and γ are even. However, then $\beta + \gamma$ is even and so w is adjacent to an even number of vertices in W, which is impossible. ■

Again, we have the following conjecture for the frame number of the Petersen graph.

Conjecture 6.4.8 *If P is the Petersen graph, then $fn(P) = 2$.*

Chapter 7

Topics for Further Study

A problem in graph theory that has received increased attention during the past 35 years concerns studying methods of distinguishing the vertices of a connected graph from one another. At the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne, a weighting (or edge labeling with positive integers) of a connected graph G was introduced for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called *irregular*. This concept was looked at in another manner. In particular, let $\mathbb{N}$ denote the set of positive integers and let E_v denote the set of edges of G incident with a vertex v. An edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $c' : V(G) \to \mathbb{N}$ induced by c and defined by

$$c'(v) = \sum_{e \in E_v} c(e)$$

has the property that $c'(x) \neq c'(y)$ for every two distinct vertices x and y of G. A paper [2] on this concept was contained in the proceedings of this conference. The main emphasis of this research dealt with minimizing the largest color assigned to the edges of the graph to produce an irregular graph. This concept has been studied extensively (see Chapter 13 in [7] for example).

Many of these weighting concepts were later interpreted as coloring concepts with the resulting vertex-distinguishing coloring. As an example, we introduce one of such irregular colorings. A (proper) *coloring* of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G, where $\mathbb{N}$ is the set of positive integers. A *k-coloring* of G uses k colors. The *chromatic number* $\chi(G)$ of G is the minimum positive integer k for which there is a k-coloring of G. For a positive integer k and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of a graph G, the *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $\text{code}_c(v) = (a_0, a_1, \ldots, a_k)$, where a_0 is the color assigned to v (that is, $a_0 = c(v)$) and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i. Therefore, if $a_0 = i$, then $a_i = 0$, for $1 \leq i \leq k$ and $\sum_{i=1}^{k} a_i = \deg_G v$. The coloring c is called *irregular* if distinct vertices have distinct color codes and the *irregular chromatic number* $\chi_{ir}(G)$ of G is the minimum positive integer k for which G has an irregular k-coloring. An irregular k-coloring with $\chi_{ir}(G) = k$ is a *minimum irregular coloring*. Since every irregular coloring of a graph G is a coloring of G, it follows that $\chi(G) \leq \chi_{ir}(G)$. The concept of irregular coloring was introduced and studied in [12] and studied further in [13, 14], inspired by the problem in graph theory that concerns finding means to distinguish all the vertices of a connected graph. Irregular colorings are often referred to as *rainbow colorings*.

In view of the Lights Out Puzzle and irregular colorings, we introduce the concept of modular rainbow colorings in graphs. For an integer $k \geq 2$ and a connected graph G of order at least 3, let $c : V(G) \to \mathbb{Z}_k$ be a vertex coloring of G where adjacent vertices may be assigned the same color. For a vertex v of G, define the *color sum* $\sigma(v)$ of v by

$\sigma(v) = \sum_{u\in N[v]} c(u)$ where addition is performed in $\mathbb{Z}_k$. If $\sigma(u) \neq \sigma(v)$ for all pairs u, v of vertices in G, then the coloring c is called a *modular rainbow k-coloring.* A coloring c is a modular rainbow coloring if c is a modular rainbow k-coloring for some integer $k \geq 2$. Notice that if G contains two vertices u and v such that $N[u] = N[u]$, then G does not have a modular rainbow coloring. The minimum k for which G has a modular rainbow k-coloring is called the *modular rainbow number* $\mathrm{mr}(G)$. If G is a connected graph of order $n \geq 3$ such that $\mathrm{mr}(G)$ exists, then $\mathrm{mr}(G) \geq n$.

To illustrate these concepts, consider the coloring c of a graph G in Figure 7.1. where the color of a vertex is placed within the vertex and the color sum or the closed color sum of a vertex is placed next to the vertex. Since $\sigma(u) \neq \sigma(v)$ for all pairs u, v of vertices in G, the the coloring c is a closed modular rainbow 8-coloring of the graph G. Since the order of G is 8, it follows that $\mathrm{mr}(G) = 8$ for the graph G in Figure 7.1.

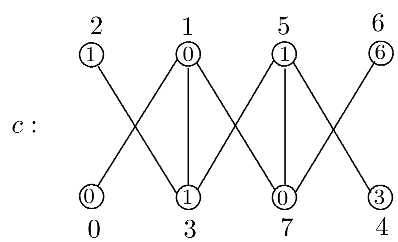


Figure 7.1: Illustrating the concept of modular rainbow coloring

We plan to investigate modular rainbow colorings in graphs and study the graphical structures of graphs with prescribed order, size and modular rainbow number. Furthermore, we plan to investigate the relationships between modular rainbow colorings and other known rainbow or irregular colorings in graphs.

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