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# Phantom Maps, Decomposability, and Spaces Meeting Particular Finiteness Conditions

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PHANTOM MAPS, DECOMPOSABILITY, AND SPACES MEETING  
PARTICULAR FINITENESS CONDITIONS

by

James Schwass

A dissertation submitted to the Graduate College  
in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy  
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PARTICULAR FINITENESS CONDITIONS

James Schwass, Ph.D.

Western Michigan University, 2015

The purpose of this dissertation is to extend principles for detecting the existence of essential phantom maps into spaces meeting particular finiteness conditions. Zabrodsky shows that a space  $Y$  having the homotopy type of a finite CW complex is the target of essential phantom maps if and only if  $Y$  has a nontrivial rational homology group. We show this observation holds on the collection of finite generalized CW complexes. Similarly, Iriye shows a finite-type, simply connected suspension space is the target of essential phantom maps if and only if it has a nontrivial rational homology group. We show this observation holds on a large class of simply connected, finite-type co-H-spaces, and begin investigating extensions to the collection of spaces having finite LS category.

To locate phantom maps into finite generalized CW complexes we study the Gray index of phantom maps and make use of a highly natural filtration on the set of phantom maps between two spaces studied by Hà and Strom. To locate phantom maps into co-H-spaces, we develop decomposition methods in phantom map theory and make use of geometric realizations of natural decompositions of tensor algebras discovered by Selick, Grbić, Theriault, and Wu. We also study the Gray index of phantom maps into co-H-spaces.

Our observations on the Gray index of phantom maps lead to the definition and study of a new homotopy invariant of spaces: the Minimal Inbound Gray Index.

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James Schwass

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# Chapter 1

## Preliminaries

### 1.1 Introduction

The stage was set for the discovery of phantom maps in the 1960s by Milnor's axiomatic study of homology theories [32] in conjunction with the advent of extraordinary cohomology theories in algebraic topology. Milnor was interested in demonstrating that for each Abelian group  $G$  there is essentially only one ordinary cohomology theory with coefficient group  $G$ . One of the pieces of this argument produces exact sequences, commonly known as Milnor exact sequences, which can be thought of as the birthplace of phantom maps.

Specifically, suppose  $E^*$  is a cohomology theory. Then each  $E^m$  is representable by Brown's theorem, i.e. there is a space  $E_m$  and there are natural isomorphisms  $E^m(X) \cong [X, E_m]$  for each CW complex  $X$ . Milnor studied additive cohomology theories  $E^*$  and was led to consider short exact sequences

$$(1.1) \quad 0 \rightarrow \text{Ph}(X, E_m) \longrightarrow E^m(X) \xrightarrow{\xi} \varprojlim E^m(X) \rightarrow 0$$

where  $\varprojlim E^*(X) = \lim_{\leftarrow} E^*(X_i)$ ,  $X_i$  is the  $i$ -skeleton of  $X$ , and  $\text{Ph}(X, E_m)$  denotes the set of homotopy classes of phantom maps  $X \rightarrow E_m$ . We describe these sequences in their original form and give a bit more detail in Section 1.4.1. The structure of these short exact sequences can be seen to give rise to the definition of a phantom map, which will be given shortly.

Now, in case  $E^* = H^*(-; G)$  is ordinary cohomology with coefficients in a group  $G$ , the map  $\xi$  is a natural isomorphism. This is representative of the fact that the spaces representing ordinary cohomology  $H^n(-; G)$ , namely the Eilenberg-MacLane spaces  $K(G, n)$ , are not the targets of essential phantom maps for any group  $G$  or any integer  $n$ . This also illustrates the importance of extraor-



dinary cohomology theories in the discovery of interesting phantom maps.

In 1968, Anderson and Hodgkin [2] discovered many interesting phantom maps  $K(\mathbb{Z}, n) \rightarrow BU$  for every odd  $n \geq 3$  by studying Milnor exact sequences associated with the extraordinary cohomology theory  $K^*$  known as complex  $K$ -theory. In particular, Anderson and Hodgkin show

$$\tilde{K}_{\leftarrow}^0(K(\mathbb{Z}, n)) = 0$$

which then implies

$$[K(\mathbb{Z}, n), BU] \cong \text{Ph}(K(\mathbb{Z}, n), BU).$$

Of course, one also has

$$\tilde{K}^0(K(\mathbb{Z}, n)) \cong [K(\mathbb{Z}, n), BU] \neq 0$$

for each odd  $n \geq 3$ , thus demonstrating that

$$\text{Ph}(K(\mathbb{Z}, n), BU) \cong [K(\mathbb{Z}, n), BU] \neq 0.$$

While these were not the first phantom maps discovered, they will be of importance to us in Section 4.1. The first published account of phantom maps came four years prior, in the note [1] of Adams and Walker, who were responding to an unpublished question of Paul Olum; we suspect this question was formulated out of an interest in the Milnor exact sequences.

Concretely, phantom maps are a wonderful source for examples in homotopy theory. We note that if  $X \xrightarrow{f} Y$  is a phantom map, then  $H_*(f) = 0$  and  $\pi_*(f) = 0$ , but there are many essential phantom maps  $f \not\approx *$ . So phantom maps can be used to answer negatively a question that occurs to many students beginning their study of algebraic topology. Phantom maps can also be used to answer negatively the “Same  $n$ -Type” question posed by J.H.C. Whitehead; Ghienne [14] shows that phantom maps can be used to construct spaces  $X$  and  $Y$  which are not homotopy equivalent, but whose Postnikov approximations  $X^{(n)}$  and  $Y^{(n)}$  are homotopy equivalent for all  $n$ .

More abstractly, phantom maps constitute a part of the homotopy category of topological spaces that is completely overlooked by homology and homotopy groups. Since homotopy and homology groups are our primary means of modeling the homotopy category, we see that to fully understand the homotopy category, one must understand the collection of phantom maps. The value of forming such an understanding is tangible in the work of Roitberg [35]. Using knowledge of

phantom maps and exploiting a connection between  $\text{Ph}(X, X)$  and a naturally occurring normal subgroup of the homotopy automorphism group  $\text{Aut}(X)$  of a space  $X$ , Roitberg shows  $\text{Aut}(K(\mathbb{Z}, 2) \times S^3)$  is a semidirect product of  $\mathbb{R}$  and  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . Homotopy automorphism groups are notoriously difficult to compute.

Phantom maps can be problematic beasts; the Milnor exact sequences illustrate that  $\text{Ph}(X, E_m)$  is the obstruction to computing  $E^m(X)$  from the often easier-to-compute  $E^{\leftarrow m}(X)$ . In the same way phantom maps form an obstruction to the convergence of many spectral sequences, and to carrying out particular homotopy theoretic constructions. This leads many mathematicians to seek criteria which ensure all phantom maps vanish in a specific context. For example, in his thesis [15], Gray shows that if  $Y$  is a rationally trivial target, i.e.  $\pi_*(Y) \otimes \mathbb{Q} = 0$ , then  $Y$  is not the target of essential phantom maps. We will call this Gray's vanishing criterion.

On the other hand, phantom maps are abundant, but difficult to locate. This leads us to develop detection principles, which help identify contexts in which one must exercise care in carrying out a particular construction, and lead us to more interesting examples of phantom maps to study in the hopes of forming a better understanding of phantom phenomenon. The most satisfying detection principles take the form of a partial converse to Gray's vanishing criterion, and are obtained by restricting to a collection of spaces meeting particular finiteness conditions. Here are two examples which will be central to our work; we refer to these as Zabrodsky's and Iriye's detection principles, respectively.

**Theorem 1.1.** [42] *If the connected space  $Y$  has the homotopy type of a finite CW complex, then  $Y$  is the target of essential phantom maps from finite type domains if and only if  $Y$  is rationally nontrivial.*

**Theorem 1.2.** [23] *If  $Y \simeq \Sigma A$  is a finite type suspension space, then  $Y$  is the target of essential phantom maps from finite type domains if and only if  $Y$  is rationally nontrivial.*

The primary purpose of this dissertation is to extend these detection principles by relaxing the finiteness conditions imposed on the space  $Y$ . In Section 4.1 we show that we can replace the finite complex in Zabrodsky's detection principle with a finite generalized CW complex. In Section 4.2 we show that Iriye's detection principle holds if we replace the suspension space with almost any simply connected co-H-space. In both cases our method is to use old examples of phantom maps to locate new ones, though the mechanisms used to accomplish this feat differ. In extending Zabrodsky's detection principle we rely on the machinery of Hà and Strom [21], described in Section 1.4.4. We endeavored to extend Iriye's detection principle in a similar way, and were successful in special cases, but in

generality we encountered an insurmountable obstacle. To obtain our generalization of Iriye's detection principle we rely on new bootstrapping mechanisms, developed in Chapter 3, which center on homotopy decompositions. Of particular interest is Proposition 3.1, which relates the hunt for examples of phantom maps to homotopy decompositions of loop spaces; such homotopy decompositions have been of wide interest due to their utility in gaining information about homotopy groups, among many other applications, and so there is an ever growing library of loop space decompositions to which we can apply this result.

Both Iriye's detection principle and our generalization are statements about spaces meeting a particular finiteness condition; a noncontractible suspension space has cone length one, while a noncontractible co-H-space has Lusternik-Schnirelmann category one (both invariants to be described shortly). We are led to wonder what, if anything, is special about the number one in this phenomenon; specifically, can we replace the spaces of cone length or category one in these detection principles with spaces of cone length or category  $n$  for some  $n \geq 2$ ? We take a first step in studying generalizations of these detection principles in the direction of spaces of finite cone length or Lusternik-Schnirelmann category in Section 4.3 by describing principles for detecting the existence of phantom maps into certain prototypes for such spaces.

The secondary purpose of this dissertation is to study the Gray index of phantom maps. The Gray index is a numerical (homotopy) invariant of phantom maps, introduced by Gray in [15]. Loosely speaking, this invariant quantifies the degree to which a phantom map is nearly trivial. Along with being one of the few pieces of concrete information we can obtain about phantom maps, the Gray index is of value in locating new examples of phantom maps, as evidenced in Section 4.1. Hà and Strom have shown that the Gray indices attainable by phantom maps into a space  $Y$  are limited to those integers  $k \geq 1$  for which  $\pi_{k+1}(Y) \otimes \mathbb{Q} \neq 0$  - this is a part of what we will call Gray's principle, introduced in detail in Section 1.4.4. Recently Tsakanikas [40] has shown that in the presence of finiteness conditions, each of these attainable Gray indices is, in fact, *attained*.

**Theorem 1.3.** [40] *If  $Y$  is a finite complex and  $\pi_{k+1}(Y) \otimes \mathbb{Q} \neq 0$ , then there are phantom maps into  $Y$  of Gray index  $k$ .*

In Section 4.1 we show this result holds if one replaces  $Y$  with a finite generalized CW complex whose rationalization is a co-H-space. On the other hand, in Section 4.2 it is seen that the conclusion of Theorem 1.3 fails if we replace  $Y$  with a particular suspension space. This leads to the definition of a new homotopy invariant of spaces in Section 4.4.

## 1.2 Notation and Conventions

All spaces will be implicitly based and maps are assumed to preserve base points, which will be denoted as  $*$ . Spaces are assumed to have the homotopy types of CW-complexes.

Given a space  $X$ , by  $X_n \rightarrow X$  we mean the  $n$ -skeleton of an arbitrary, but fixed CW structure for  $X$  or a CW approximation thereof, unless otherwise noted. By  $X^{(n)}$  we mean the  $n$ th Postnikov approximation of  $X$ , and  $X\langle n \rangle$  denotes the  $n$ -connected cover of  $X$ . For brevity, we will often write  $\Omega Y^{(n)}$  for  $\Omega(Y^{(n)})$ , and  $\Omega A \wedge \Omega B$  for  $(\Omega A) \wedge (\Omega B)$ .

One of our primary bookkeeping tools will be the notion of connectivity.

**Definition.** A space  $X$  is  $(n - 1)$ -**connected** if  $\pi_k(X) = 0$  for  $k < n$ . The **connectivity**  $\text{conn}(X)$  of a space  $X$  is defined by  $\text{conn}(X) = n - 1$  if  $X$  is  $(n - 1)$ -connected but not  $n$ -connected. In other words,  $\pi_k(X) = 0$  for  $k < n$  but  $\pi_n(X) \neq 0$ .

Similarly, we say a space  $X$  is **rationally**  $(n - 1)$ -**connected** if  $\pi_k(X) \otimes \mathbb{Q} = 0$  for  $k < n$ . One then obtains the definition for the **rational connectivity**  $\text{conn}_{\mathbb{Q}}(X)$  of a space  $X$  as above.

Keeping track of connectivity is worthwhile in light of the Hurewicz theorem and its rational analog, which we refer to frequently throughout.

**Theorem 1.4** (Hurewicz Theorem). *Suppose  $n \geq 2$ . If  $\text{conn}(X) = n - 1$  then the Hurewicz map  $\pi_n(X) \rightarrow H_n(X; \mathbb{Z})$  is an isomorphism of groups.*

**Theorem 1.5** (Rational Hurewicz Theorem). *Suppose  $n \geq 2$ . If  $\text{conn}_{\mathbb{Q}}(X) = n - 1$  then the rational Hurewicz map  $\pi_k(X) \otimes \mathbb{Q} \rightarrow H_k(X; \mathbb{Q})$  is an isomorphism for  $n \leq k \leq 2n$ .*

By the Hurewicz theorem, a connected space  $X$  is  $(n - 1)$ -connected if and only if  $\tilde{H}_k(X; \mathbb{Z}) = 0$  for  $k < n$ . A similar statement holds rationally.

**Definition.** A space  $X$  is said to be a **finite type domain** if  $\tilde{H}_n(X; \mathbb{Z})$  is a finitely generated Abelian group for every  $n \geq 1$ . Dually, we say  $X$  is a **finite type target** if  $\pi_n(X)$  is a finitely generated group for every  $n \geq 1$ .

In case  $Y$  is simply connected, Serre's work [39] shows that there is no distinction between finite type domains and targets, and so if  $Y$  is simply connected we will say  $Y$  **has finite type (over  $\mathbb{Z}$ )** to mean  $Y$  is a finite type domain and target.

## 1.3 Localization

Following Ponto and May [26], we will specialize to the collection of nilpotent spaces.

**Definition.** A space  $X$  is **nilpotent** if  $\pi_1(X)$  is nilpotent and acts nilpotently on  $\pi_n(X)$  for  $n \geq 2$ . In particular, every simply connected space is nilpotent.

Ponto and May describe how to construct well-behaved localization functors on the full subcategory of nilpotent spaces in the category of reasonable nice topological spaces. In this section we recall the nomenclature and basic properties of localization of nilpotent groups and topological spaces. We refer the reader to [26] for an in depth account of localization. For brevity, our exposition will largely focus on simply connected spaces and Abelian groups; we leave the extensions to nilpotent groups and spaces to Ponto and May.

Localization provides a means to adapt some arguments, crafted with simply connected spaces in mind, to be applied to nilpotent, finite type spaces with finite fundamental groups; if  $Y$  is such a space, then localized at a large enough (or well chosen) prime,  $Y_{(p)}$  is simply connected. On the other hand, according to Arkowitz [3] if  $X$  is a co-H-space, then  $\pi_1(X)$  is a free group. So if  $X$  is a co-H-space with nontrivial fundamental group, the localizations of  $X$  will not be simply connected, and so we will often specialize to simply connected co-H-spaces.

More importantly, there are strong connections between the homotopy theory of  $p$ -local spaces and the theory of modules over the  $p$ -local integers  $\mathbb{Z}_{(p)}$ . We will say more about this in Section 1.7; these connections will be central to the development of new detection principles for phantom maps in Section 4.2.

For the moment we fix a (possibly empty) subset  $T$  of the set  $P$  of prime integers. Set  $S = P \setminus T$  and write  $\mathbb{Z}_T = \mathbb{Z}[S^{-1}]$  for the ring of  $T$ -local integers in the sense of Ponto and May. When  $T = \{p\}$  is a singleton set we have  $\mathbb{Z}_T = \mathbb{Z}_{(p)}$ , the ring of  $p$ -local integers, while if  $T = \emptyset$  then  $\mathbb{Z}_T = \mathbb{Q}$ , and if  $T = P$  then  $\mathbb{Z}_T = \mathbb{Z}$ .

**Definition.** Suppose  $A$  is an Abelian group.

- (1) We say  $A$  is  **$T$ -local** if it admits a  $\mathbb{Z}_T$ -module structure.
- (2) A map  $A \xrightarrow{\varphi} A_T$  of Abelian groups is called a (algebraic)  **$T$ -localization** if
  - $A_T$  is  $T$ -local,

- $\varphi$  is universal among maps from  $A \xrightarrow{\psi} B$  with  $B$  a  $T$ -local group. Diagrammatically, each extension problem in the category of Abelian groups

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A_T \\ & \searrow \psi & \swarrow \eta \\ & & B \end{array}$$

with  $B$  a  $T$ -local group is uniquely solvable.

An example of a  $T$ -localization is the map  $A \rightarrow A \otimes \mathbb{Z}_T$  given by  $a \mapsto a \otimes 1$ . Since  $\mathbb{Z}_T$  is a subring of the rationals, this ring is torsion free, and hence the underlying Abelian group is a flat  $\mathbb{Z}$ -module. So, the functor  $- \otimes \mathbb{Z}_T$  taking Abelian groups to  $T$ -local Abelian groups (i.e.  $\mathbb{Z}_T$ -modules) is an exact functor, which implies (algebraic) localization is an exact functor.

We now turn to topological localization, which will be defined in terms of a universal property for maps into local spaces in parallel with the definition of algebraic localization above.

**Definition.** (1) A  $T$ -**equivalence** is a map  $f : X \rightarrow Y$  inducing an isomorphism on  $H_*(-; \mathbb{Z}_T)$ . When  $T = \emptyset$  we call a  $T$ -equivalence a **rational equivalence**.

(2) A space  $Z$  is  $T$ -**local** if

$$[Y, Z] \xrightarrow{\xi^*} [X, Z]$$

is an isomorphism for all  $T$ -equivalences  $X \xrightarrow{\xi} Y$ . When  $T = \emptyset$  we call a  $T$ -local space a **rational space**.

(3) A map  $X \xrightarrow{\varphi} X_T$  is (topological)  $T$ -**localization** if  $\varphi$  is universal among maps  $X \rightarrow Y$  with  $Y$  a  $T$ -local space. Diagrammatically, each extension problem in the homotopy category of topological spaces

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X_T \\ & \searrow \psi & \swarrow \eta \\ & & Y \end{array}$$

with  $Y$  a  $T$ -local space has a unique solution. When  $T = \emptyset$  we call  $T$ -localization **rationalization**.

(4) We say a connected space  $Y$  is **rationally nontrivial** if  $\tilde{H}_*(Y; \mathbb{Q}) \neq 0$  or, equivalently, if  $\pi_*(Y) \otimes \mathbb{Q} \neq 0$ .

We record here some recognition principles for  $T$ -local spaces and a few alternate characterizations of localization maps.

**Theorem 1.6.** [26, pg. 111] *For a nilpotent space  $Z$  the following are equivalent,*

- (1)  $Z$  is  $T$ -local,
- (2) for every  $n \geq 1$  the group  $\pi_n(Z)$  is  $T$ -local,
- (3) for every  $n \geq 1$  the group  $\tilde{H}_n(Z; \mathbb{Z})$  is  $T$ -local.

**Theorem 1.7.** [26] *Every nilpotent space admits a localization  $X \rightarrow X_T$  at each  $T \subseteq P$ . This localization is unique up to homotopy, being defined by a universal property in the homotopy category. Suppose  $X \xrightarrow{\varphi} Y$  is a map from a nilpotent space  $X$  to a  $T$ -local space  $Y$ ; the following are equivalent*

- (1)  $\varphi$  is topological  $T$ -localization,
- (2)  $\varphi$  is a  $\mathbb{Z}_T$ -equivalence,
- (3)  $\pi_n(\varphi)$  is algebraic  $T$ -localization  $\pi_n(X) \rightarrow \pi_n(Y)$  for all  $n \geq 1$ ,
- (4)  $\tilde{H}_n(\varphi; \mathbb{Z})$  is algebraic  $T$ -localization for all  $n \geq 1$ .

These last two results help us to formulate a finite type condition for  $T$ -local spaces analogous to the definition of a finite type space given above. This is important to us because the  $T$ -local integers  $\mathbb{Z}_T$  need not be finitely generated as an Abelian group. Indeed, by definition the group  $\mathbb{Z}_T$  is not finitely generated as a  $\mathbb{Z}$ -algebra whenever  $T$  is a finite collection of primes, and so cannot be a finitely generated  $\mathbb{Z}$ -module. So, the  $T$ -localization of a finite type space need not be of finite type in the sense of Definition 1.2. On the other hand, the  $T$ -localization of a finite type space is of finite type over  $\mathbb{Z}_T$  in the following sense, in light of Theorem 1.6.

**Definition.** Suppose  $Y$  is a  $T$ -local space. We say  $Y$  is a **finite type domain (resp. target) over  $\mathbb{Z}_T$** , or  $Y$  is a **finite  $T$ -type domain (resp. target)**, if for every  $n \geq 1$  the group  $\tilde{H}_n(Y; \mathbb{Z})$  (resp.  $\pi_n(Y)$ ) is a finitely generated  $\mathbb{Z}_T$ -module (or a finitely  $T$ -generated nilpotent group in the case of  $\pi_1(Y)$ .)

In case  $Y$  is simply connected, Serre's work [39] shows that there is no distinction between finite type domains and targets, and so if  $Y$  is simply connected we will say  $Y$  **has finite type over  $\mathbb{Z}_T$**  to mean  $Y$  is a finite  $T$ -type domain and target.

In light of the fundamental theorem of finitely generated modules over a PID, many proofs regarding spaces which are of finite type over  $\mathbb{Z}$  also apply for spaces of finite type over  $\mathbb{Z}_T$  for any  $T \subseteq P$ . Moreover,  $T$ -local groups can only possibly have torsion at the primes  $p \in T$  and so when we specialize to  $T = \{p\}$  we can take a “one-prime-at-a-time” look at homotopy theory. This can be convenient for computations, and is essential in formulating the connections between algebra and topology described in Section 1.7.

By Theorem 1.7, or from the construction of cocellular localizations in [26] it is evident that localization commutes with Postnikov approximations, in symbols

$$(Y_{(p)})^{(n)} \simeq (Y^{(n)})_{(p)}.$$

Localization commutes with many other constructions, as well. We record some here.

**Proposition 1.8.** *[26] Topological localization commutes with products, loops, and more generally with pullbacks. Dually, localization commutes with suspension, wedge sums, smash products, and more generally with pushouts.*

This observation can be used to show that localization commutes with the homotopy functors determined by Ganea’s construction, the topological join, and many other familiar homotopy theoretic constructions. Proposition 1.8 also shows that the property of being a co-H-space is preserved under localization.

**Corollary 1.9.** *The localization of a co-H-space (resp. H-space) is again a co-H-space (resp. an H-space).*

## 1.4 Phantom Maps

Our primary objects of study will be phantom maps between spaces having the homotopy type of finite type CW complexes. There are several different, though related, notions of a phantom map in the category of topological spaces in the literature. These notions coincide if we restrict our attention to phantom maps between finite type domains and targets. We focus on “phantom maps of the first kind” in the language of the survey article [27] of McGibbon, and refer the interested reader there for more on other notions of phantom maps. This gives the most convenient formulation of the notion of a phantom map for our purposes, as will become evident in our exposition of the Gray index of phantom maps in Section 1.4.3.



**Definition.** Suppose  $X$  is a CW complex. A map  $X \xrightarrow{f} Y$  is a **phantom map** if the restrictions

$$(1.2) \quad X_n \hookrightarrow X \xrightarrow{f} Y$$

of  $f$  to each  $n$ -skeleton of  $X$  are nullhomotopic.

We extend this definition to domains  $X$  that are not CW complexes by replacing the inclusion of the  $n$ -skeleton  $X_n \hookrightarrow X$  in (1.2) with the  $n$ -skeleton of an arbitrary CW approximation of  $X$ , though in this work most phantom maps will have a CW complex as their domain.

Our first example of a phantom map is the constant map  $X \xrightarrow{*} Y$ . Of more interest are **essential** phantom maps; a map is called essential if it is not nullhomotopic. We write  $\text{Ph}(X, Y)$  for the subset of  $[X, Y]$  consisting of homotopy classes of phantom maps. We remark here that  $\text{Ph}(X, Y)$  is functorial in both  $X$  and  $Y$ , which will be verified in Proposition 1.19.

We take an early opportunity to exposit the prevalence of Eckmann-Hilton duality in the theory of phantom maps by giving a dual characterization of phantom maps.

**Proposition 1.10.** *A map  $X \xrightarrow{f} Y$  is phantom if and only if the extension  $X \xrightarrow{f} Y \rightarrow Y^{(n)}$  to each  $n$ th Postnikov approximation of  $Y$  is nullhomotopic.*

To our knowledge there are precisely two (dual) classes of phantom maps that admit any sort of geometric description. It would be remiss of us to omit an account of these maps here. To each space  $X$ , McGibbon and Gray [17] associate a map  $\theta_X : X \rightarrow \bigvee_{n \geq 1} \Sigma X_n$ , called the **universal outbound phantom map** associated to  $X$ . This map is obtained as the homotopy cofiber of the folding map  $\bigvee_{n \geq 1} X_n \rightarrow X$ , and so must be a phantom map. Moreover, any phantom map  $X \xrightarrow{\varphi} Y$  must factor through  $\theta_X$  as in

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \theta_X & \nearrow \text{---} \\ & \bigvee_{n \geq 1} \Sigma X_n & \end{array}$$

which (largely) justifies the nomenclature. Dually we obtain the **universal inbound phantom map**  $\Gamma_Y : \prod_{n \geq 1} \Omega Y^{(n)} \rightarrow Y$  associated to a space  $Y$  as the homotopy fiber of the natural map  $Y \rightarrow \prod_{n \geq 1} Y^{(n)}$ .

### 1.4.1 Towers of Groups and the Milnor Exact Sequence

By a tower  $\{G_n\}$  of groups we mean a diagram

$$(1.3) \quad \dots \xrightarrow{p_{n+1}} G_n \xrightarrow{p_n} \dots \xrightarrow{p_3} G_2 \xrightarrow{p_2} G_1$$

in the category of groups. We mean something similar by a tower of Abelian groups, or a tower of sets, or really a tower of any sort of gadget - these are  $\mathbb{N}^{\text{op}}$ -shaped diagrams in various categories. By  $\lim G_n$  we mean the limit of the diagram (1.3) in the appropriate category.

It turns out the functor  $\lim$  taking towers of Abelian groups to Abelian groups is left exact, but not right exact. We write  $\lim^1$  for the first derived functor of  $\lim$ . This functor will be of central importance to our study of phantom maps, and so we take a moment to describe it in more concrete terms.

Suppose  $\{G_n\}$  is a tower of Abelian groups. Then according to [27]  $\lim G_n$  is the kernel and  $\lim^1 G_n$  is the cokernel of the map

$$\prod G_n \xrightarrow{\text{id-sh}} \prod G_n$$

given by

$$(a_1, a_2, \dots) \mapsto (a_1 - p_2(a_2), a_2 - p_3(a_3), \dots).$$

Bousfield and Kan [5, pgs 254–255] extend the definition of  $\lim^1$  to towers of arbitrary groups as follows: Given a tower  $\{G_n\}$  of groups let  $\prod G_n$  act on  $\prod G_n$  by

$$(g_n) \cdot (x_n) = (g_n x_n (p_{n+1}(g_{n+1})^{-1})),$$

where  $G_{n+1} \xrightarrow{p_{n+1}} G_n$  is the structure map in the tower  $\{G_n\}$ . Then  $\lim^1 G_n$  is the orbit space of this action. This is important to us because we will have occasion to refer to  $\lim^1 G_n$  where  $G_n$  is a tower of not necessarily Abelian groups. The towers we will be interested in are induced either by CW structures or by Postnikov towers; the Postnikov tower for a space  $Y$

$$\dots \rightarrow Y^{(n+1)} \rightarrow Y^{(n)} \rightarrow \dots$$

naturally gives rise to a tower of groups

$$\dots \rightarrow [X, \Omega Y^{(n+1)}] \rightarrow [X, \Omega Y^{(n)}] \rightarrow \dots$$

Dually, a CW structure on a space  $X$  gives rise to a tower of group  $\{[\Sigma X_n, Y]\}$ . The importance of the functor  $\lim^1$  in the theory of phantom maps is then evidenced by Theorem Bousfield and Kan. This result is an extension of some of

Milnor's findings in [32].

**Theorem 1.11.** [5] *For pointed CW complexes  $X$  and  $Y$  there are short exact sequences of pointed sets*

$$* \longrightarrow \lim^1[\Sigma X_n, Y] \longrightarrow [X, Y] \longrightarrow \lim[X_n, Y] \longrightarrow *,$$

and

$$* \longrightarrow \lim^1[X, \Omega Y^{(n)}] \longrightarrow [X, Y] \longrightarrow \lim[X, Y^{(n)}] \longrightarrow *.$$

We call these the Milnor exact sequences.

**Corollary 1.12.** [5] *For pointed complexes  $X$  and  $Y$ , we have bijections of pointed sets*

$$\lim^1_n[\Sigma X_n, Y] \cong \text{Ph}(X, Y) \cong \lim^1_n[X, \Omega Y^{(n)}].$$

In light of Corollary 1.12 and the identifications  $X \simeq \text{hocolim } X_n$  and  $Y \simeq \text{holim } Y^{(n)}$  we see that essential phantom maps witness the failure of the commutativity of the functors  $[-, Y]$  and  $\text{hocolim}$ , and dually  $[X, -]$  and  $\text{holim}$ .

We now study the functor  $\lim^1$  in greater detail. In particular, we wish to characterize those towers  $\{G_n\}$  of groups for which  $\lim^1 G_n$  vanishes.

**Definition.** Given a tower of gadgets (groups, sets, etc.)  $\{G_n\}$  let  $G_k^{(n)}$  denote the image in  $G_k$  of the composite of the structure maps

$$G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_k$$

when  $n \geq k$  and for  $n \leq k$  set  $G_k^{(n)} = 1$ . This defines, for each  $k \geq 1$  a subtower  $\{G_k^{(n)}\}$  indexed by  $n$  of the tower  $\{G_n\}$ . Notice the sequence of images  $G_k^{(n)}$  are nested; we say the tower  $\{G_n\}$  **satisfies the Mittag-Leffler condition** if all of the nested sequences  $G_k^{(n)}$  satisfy a descending chain condition: explicitly, for each  $k$  there is some  $N$  so that for all  $n \geq N$  one has  $G_k^{(n)} = G_k^{(N)}$ .

The value of the Mittag-Leffler condition to the computation of  $\lim^1 G_n$  for  $\{G_n\}$  a tower of groups is evident in the following result.

**Proposition 1.13.** [5] *If the tower of groups  $\{G_n\}$  satisfies the Mittag-Leffler condition, then  $\lim^1 G_n = *$ .*

In the context being considered in the present work, the next result of McGibbon and Møller shows that the Mittag-Leffler condition is even more powerful. For finite type spaces  $X$  and  $Y$  this reduces the task of determining if  $\text{Ph}(X, Y) = *$  to

the often more approachable problem of determining if the tower  $[X, \Omega Y^{(n)}]$  satisfies the Mittag-Leffler condition. We note that if  $X$  and  $Y$  are of finite type, then  $[\Sigma X_n, Y]$  and  $[X, \Omega Y^{(n)}]$  are countable. We give a proof of a similar statement in Proposition 3.7 that can be used to verify this claim.

**Theorem 1.14.** *[28, McGibbon, Møller] If  $\{G_n\}$  is a tower of countable groups, then  $\lim^1 G_n = *$  if and only if  $\{G_n\}$  is Mittag-Leffler. Moreover, if  $\lim^1 G_n \neq *$ , then this set is uncountable.*

That the vanishing of  $\lim^1$  is equivalent to the Mittag-Leffler condition on towers of countable groups had already been known to Dydak and Segal [10], who found applications in shape theory. The statement regarding the cardinality of  $\lim^1 G_n$  had been known to Gray [15] in case the tower in question consists of Abelian groups; this assertion serves to illustrate that if we can find a single essential phantom map between two spaces, then homotopically distinct examples of such maps are abundant.

This next result, which traces its roots to McGibbon and Roitberg in [29], states that the Mittag-Leffler condition is independent of group structures, or that this is more a property of towers of sets.

**Lemma 1.15.** *Suppose  $\{G_n\}$  and  $\{H_n\}$  are towers of sets, and there is a morphism  $f : \{G_n\} \rightarrow \{H_n\}$  with  $f_n$  surjective for each  $n$ . If  $\{G_n\}$  is Mittag-Leffler, then  $\{H_n\}$  is Mittag-Leffler.*

*Proof.* If  $\{G_n\}$  is Mittag-Leffler then for each  $k$  there is some  $N \in \mathbb{N}$  so that for  $n \geq N$  one has

$$G_k^{(N)} = G_k^{(n)}.$$

By hypothesis, there are surjections  $f_k : G_k \rightarrow H_k$  for all  $k$ . A quick diagram chase shows that these surjections induce surjections  $f_k^{(n)} : G_k^{(n)} \rightarrow H_k^{(n)}$ . In other words,

$$H_k^{(n)} = \{f(x) \mid x \in G_k^{(n)}\}.$$

But, for  $n \geq N$  we have  $G_k^{(n)} = G_k^{(N)}$  and so this shows  $H_k^{(n)} = H_k^{(N)}$ . Since  $k \in \mathbb{N}$  was chosen arbitrarily, we have shown the tower  $\{H_n\}$  is Mittag-Leffler.  $\square$

We will use this in Section 3.1 to show how splittings of  $\Omega Y$  can give a great deal of information about phantom maps into  $Y$ .

## 1.4.2 Phantom Maps and Rational Equivalences

The theory of phantom maps between finite type spaces is intrinsically linked to rational homotopy theory. In Gray's thesis we see a nascent form of what we are

calling Gray's principle (Proposition 1.23), which gives a test in terms of rational homotopy invariants of  $X$  and  $Y$  for determining when  $\text{Ph}(X, Y) = *$ . Around the same time, Zabrodsky and Meier were interested in calculating  $\text{Ph}(X, Y)$  in terms of rational homotopy invariants of  $X$  and  $Y$ . These methods were wildly successful in some special cases, but rapidly left the realm of approachability by existing algebraic techniques, and so fell by the wayside. But, the link to rational homotopy theory had been established, and in the 1990s this link gave rise to stunning developments in the theory of phantom maps. Our first example is the following.

**Theorem 1.16.** [30, C.A. McGibbon, J. Roitberg] *For a nilpotent CW complex  $X$  of finite (domain) type, the following are equivalent*

- (i)  $\text{Ph}(X, Y) = *$  for all finite type targets  $Y$ ,
- (ii)  $\text{Ph}(X, S^n) = *$  for all  $n$ , and
- (iii) there is a rational equivalence  $\Sigma X \rightarrow \bigvee S^{n_\alpha}$ .

*For a nilpotent CW complex  $Y$  of finite (target) type, the following are equivalent*

- (i')  $\text{Ph}(X, Y) = *$  for all finite type domains  $X$ ,
- (ii')  $\text{Ph}(K(\mathbb{Z}, m), Y) = *$  for all  $m$ , and
- (iii') there is a rational equivalence from some  $\prod_\alpha K(\mathbb{Z}, m_\alpha) \rightarrow \Omega Y$ .

In light of this result we think of Eilenberg-MacLane spaces as universal test *domains* for phantom maps and spheres as universal test *targets* for phantom maps, which is a departure from the usual roles of these spaces in the tradition of Eckmann-Hilton duality.

Generally speaking, (co)fiber sequences do not induce exact sequences of phantom sets, so a large part of the basic homotopy theorists' toolkit is rendered ineffective in studying phantom maps. McGibbon and Roitberg have developed a few tools that help us implement at least some of the rudiments of this toolkit.

**Theorem 1.17.** [30, C.A. McGibbon, J. Roitberg] *If  $A$  and  $B$  are finite type domains and  $\varphi : A \rightarrow B$  induces surjections on  $H^*(-; \mathbb{Q})$ , then*

$$\varphi^* : \text{Ph}(B, Y) \rightarrow \text{Ph}(A, Y)$$

*is surjective for all finite type  $Y$ .*

Dually, if  $A$  and  $B$  are finite type targets and  $\varphi : A \rightarrow B$  induces surjections on  $\pi_* \otimes \mathbb{Q}$ , then

$$\varphi_* : \text{Ph}(X, A) \rightarrow \text{Ph}(X, B)$$

is surjective for all finite type  $X$ .

The hypotheses of the theorem are quite restrictive. In practice, we use this theorem to bootstrap from knowledge of phantoms into a space  $B$ , for example, to knowledge of phantoms into the space  $A$  by way of a highly structured comparison map  $A \rightarrow B$ . We find in many cases that we are unable to produce such highly structured comparison maps. In Section 1.4.4 we present refinements of these results that can be applied in more general circumstances, at the expense of decidedly nontrivial computations.

### 1.4.3 The Gray Index of Phantom Maps

Suppose  $X \xrightarrow{f} Y$  is any map. We have seen that the following conditions are equivalent:

- $f$  is a phantom map,
- for each  $n$  the composite  $X_n \rightarrow X \xrightarrow{f} Y$  is nullhomotopic,
- for each  $n$  the composite  $X \xrightarrow{f} Y \rightarrow Y^{(n)}$  is nullhomotopic.

So, in some sense a phantom map is a map that appears trivial if we only consider its restrictions or extensions to particular truncations of its domain or target, respectively. One might say that phantom maps are nearly trivial. This leads us to endeavor to construct a means of measuring the degree to which a phantom map is nearly trivial. This measurement is formalized by the notion of the Gray index of a phantom map, introduced by Gray in [15]. We turn now to introducing this homotopy invariant.

From the cofiber and fiber sequences

$$X_n \rightarrow X \rightarrow X/X_n \quad \text{and} \quad Y\langle n \rangle \rightarrow Y \rightarrow Y^{(n)}$$

and Proposition 1.10 we infer  $f : X \rightarrow Y$  is phantom if and only if the extension and lifting problems

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \nearrow \bar{f} \\ & X/X_n & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & Y\langle n \rangle \\ & \nearrow \tilde{f} & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

are solvable up to homotopy for each  $n$ .

**Definition.** The **Gray index**  $G(f)$  of a map  $f : X \rightarrow Y$  is the least integer  $n$  so that  $\bar{f}$  cannot be chosen to be a phantom map.

We will see below that the Gray index of a phantom map does not depend on a CW structure for its domain. It is then clear from the definition that the Gray index is a homotopy invariant of phantom maps. Of course, the constant map  $X \xrightarrow{*} Y$  has  $G(*) = \infty$ . Since the Gray index defines, among other things, a function from the collection of homotopy classes of phantom maps to  $\mathbb{N} \cup \{\infty\}$  This implies that if  $\varphi$  is a phantom map with  $G(\varphi)$  finite, then  $\varphi$  is essential. This observation is crucial to our reasoning; we will often demonstrate a phantom map is essential by showing its Gray index is finite without further comment.

One might also define the “dual Gray index”  $G'(f)$  to be the least  $n$  so that  $\tilde{f}$  cannot be chosen to be a phantom map. Fortunately, Hà and Strom have shown us that  $G(f)$  and  $G'(f)$  are effectively encoding the same data about the phantom map  $f$ . Note Proposition 1.10 follows from this observation.

**Proposition 1.18.** [21, Ha, Strom] For a phantom map  $f : X \rightarrow Y$ ,

$$G'(f) = G(f) + 1.$$

Since the dual Gray index was defined without reference to any CW structure, we can see the Gray index of a phantom map is independent of the choice of a CW structure for its domain (or a CW approximation thereof) as a consequence of Proposition 1.18.

In trying to produce new examples of phantom maps from existing examples we are led, by Theorem 1.17, for example, to compose phantom maps with other maps. The next proposition shows these composites are necessarily phantom, but there is no reason to believe such a composite is nontrivial. In fact, if  $X \xrightarrow{\varphi} Y$  and  $Y \xrightarrow{\psi} Z$  are both phantom maps, then the composite  $\psi\varphi$  is trivial [17]. In many cases the inequalities established in the next proposition can be used to show a phantom map is essential.

**Proposition 1.19.** Suppose  $X \xrightarrow{f} Y$  is a phantom map, and  $Y \xrightarrow{\sigma} Z, W \xrightarrow{\tau} X$  are any maps. Then

- (1)  $f\tau$  and  $\sigma f$  are phantom, and
- (2) if  $G(f)$  is finite, then  $G(f\tau), G(\sigma f) \geq G(f)$ .

*Proof.* We prove the claims about  $\sigma f$  - the claims about  $f\tau$  are proved in the dual manner. For (1) note that the composite

$$X_n \xrightarrow{i_n} X \xrightarrow{f} Y \xrightarrow{\sigma} Z$$

is trivial for all  $n$  because  $f i_n$  is trivial for all  $n$  by virtue of the fact that  $f$  is phantom.

For the (2) note that in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\sigma} & Z \\ & \searrow & \nearrow \bar{f} & \nearrow \sigma \bar{f} & \\ & & X/X_n & & \end{array}$$

if  $\bar{f}$  is phantom, then  $\sigma \bar{f}$  gives a phantom factorization of  $\sigma \circ f$  through  $X/X_n$ .  $\square$

**Example 1.20.** As an application of Proposition 1.19 we compute the Gray index of  $\theta_{\mathbb{C}P^\infty}$ , the universal outbound phantom map associated with  $\mathbb{C}P^\infty$ . This answers a question that was posed to me at the Young Topologists Meeting at the Ecole Polytechnique Federale de Lausanne by Kristian Jonsson Moi of the University of Copenhagen. This application will be indicative the use of the proposition in the sequel.

In [31] it is shown that there are phantom maps  $\mathbb{C}P^\infty \rightarrow S^2 \vee S^2$  of every positive even Gray index, so there is a phantom map  $f : \mathbb{C}P^\infty \rightarrow S^2 \vee S^2$  with  $G(f) = 2$ . Since  $f = \bar{f} \circ \theta_{\mathbb{C}P^\infty}$ , we infer  $G(\theta_{\mathbb{C}P^\infty}) \leq G(f) = 2$ , but since  $\text{conn}(\mathbb{C}P^\infty) = 1$  we know if  $g : \mathbb{C}P^\infty \rightarrow Y$  is any phantom map, then  $G(g) \geq 2$ , so  $G(\theta_{\mathbb{C}P^\infty}) = 2$ .

#### 1.4.4 The Gray Filtration

In this section we describe how the Gray index gives rise to a natural filtration on  $\text{Ph}(X, Y)$ , following Hà and Strom [21]. This filtration gives a refinement of Theorem 1.17, which results in a robust computational framework in which to hunt for phantom maps. To use this framework, one must compute the Gray index of phantom maps in particular contexts. This computation can be highly nontrivial, as will be seen in Section 4.2.

We write  $\text{Ph}^k(X, Y)$  for the subset of  $\text{Ph}(X, Y)$  consisting of phantoms of Gray index at least  $k$ . Note that  $\text{Ph}(X, Y) = \text{Ph}^1(X, Y)$ . When  $X$  is connected this follows from the observation that we can give  $X$  a CW structure with a single zero cell, namely the basepoint of  $X$ , so  $X \rightarrow X/X_0$  is a homotopy equivalence. In case



$X$  is not connected we follow [21] and observe that if  $X \xrightarrow{f} Y$  is a phantom map, then  $f(X)$  must be contained in the basepoint component of  $Y$ , for if not  $\pi_0(f)$  is essential and  $f$  is not phantom. Also, the restriction of  $f$  to each component of  $X$  must also be phantom, and so by giving  $X$  a CW structure with a single zero cell for each connected component we see  $f$  factors through  $X \rightarrow X/X_0$  by a phantom map  $X/X_0 \rightarrow Y$ . We call the descending filtration  $\text{Ph}^k(X, Y)$  of  $\text{Ph}(X, Y)$  the **Gray filtration**.

Proposition 1.19 shows that this filtration by Gray index is natural in both  $X$  and  $Y$ . Hà and Strom [21] discover more structure in this filtration by demonstrating that the Gray filtration on  $\text{Ph}(X, Y)$  corresponds with a natural algebraic filtration of the set  $\lim^1 G_n$  with  $G_n = [X, \Omega Y^{(n)}]$  or  $[\Sigma X_n, Y]$ , and studying the structural properties of this algebraic filtration. This endows the Gray filtration with its computational utility.

We recall some notation from Section 1.4.1 for the reader's convenience. Given a tower  $\{G_n\}$  of groups, we defined for each  $k \geq 1$  a subtower  $\{G_k^{(n)}\}$  by setting

$$G_k^{(n)} = \begin{cases} 1 & \text{if } n < k \\ \text{Im}(G_n \rightarrow G_k) & \text{if } n \geq k. \end{cases}$$

This gives rise to commutative triangles of surjections of towers of groups

$$\begin{array}{ccc} \{G_n\} & \longrightarrow & \{G_{k+1}^{(n)}\} \\ & \searrow & \downarrow \\ & & \{G_k^{(n)}\}. \end{array}$$

Since  $\lim^1$  takes surjections to surjections, we obtain a triangle of surjections of pointed sets

$$(1.4) \quad \begin{array}{ccc} \lim^1 \{G_n\} & \xrightarrow{p_{k+1}} & \lim^1_n \{G_{k+1}^{(n)}\} \\ & \searrow p_k & \downarrow \\ & & \lim^1_n \{G_k^{(n)}\}. \end{array}$$

Hà and Strom give an alternative characterization of the Gray index in terms of this construction.

**Proposition 1.21.** [21] *Suppose  $G_n = [X, \Omega Y^{(n)}]$  or  $[\Sigma X_n, Y]$ , so  $\lim^1 G_n \cong \text{Ph}(X, Y)$ . We use the notation of (1.4). For a phantom map  $f$ ,  $G(f)$  is the greatest  $k$  for which  $p_k(f) = *$ .*

As a consequence, we obtain the following identification.

**Theorem 1.22.** [21] *For finite type spaces  $X$  and  $Y$*

$$\text{Ph}^k(X, Y) = \ker(p_k).$$

Since the construction of the maps  $p_k$  is natural, this shows that the filtration described at the beginning of the section coincides with a natural filtration on  $\lim^1\{G_n\}$ . Using this observation, Hà and Strom extend observations originating in Gray's thesis to give one of the most powerful tools for computing the Gray index of phantom maps. We will frequently refer to this result as **Gray's principle**.

**Theorem 1.23** (Gray's Principle). [21] *If  $X$  and  $Y$  are nilpotent of finite type, and there are phantom maps  $X \rightarrow Y$  of Gray index  $k$ , then both  $H^k(X; \mathbb{Q})$  and  $\pi_{k+1}(Y) \otimes \mathbb{Q}$  are nonzero.*

Another implication of the naturality of the Gray filtration is a specialized version of Theorem 1.16. This will be used in Sections 4.2 and 4.3 to gain information about the Gray indices attained by phantom maps into  $\Sigma K(\mathbb{Z}, n)$  and the spaces  $G_m(K(\mathbb{Z}, n))$  arising from Ganea's construction, respectively. This last collection of spaces will be described in Section 1.6.

**Theorem 1.24.** [21] *Let  $X$  be a nilpotent, finite type domain. Then each of the following implies the next.*

- (1)  $\text{Ph}(X, S^{k+1}) = *$ ,
- (2) *there is a map  $g : X \rightarrow \prod \Omega S^{k+1}$  inducing a surjection on  $H^k(-; \mathbb{Q})$ ,*
- (3)  $\text{Ph}^k(X, Y) = \text{Ph}^{k+1}(X, Y)$  *for all finite type, nilpotent  $Y$ .*

*Dually, if  $Y$  is a nilpotent, finite type target, then each of the following implies the next*

- (1')  $\text{Ph}(K(\mathbb{Z}, k), Y) = *$ ,
- (2') *there is a map  $g : \bigvee \Sigma K(\mathbb{Z}, k) \rightarrow Y$  inducing a surjection on  $\pi_{k+1} \otimes \mathbb{Q}$ ,*
- (3')  $\text{Ph}^k(X, Y) = \text{Ph}^{k+1}(X, Y)$  *for all finite type, nilpotent  $X$ .*

For reasons of rational homotopy, along with Gray's principle, we see that when  $k$  is even (3) implies (1) above, and when  $k$  is odd (3') implies (1').

Yet another consequence of the naturality of the Gray filtration is the aforementioned refined, albeit computationally more expensive version of Theorem

1.17. Following Hà and Strom we note that a natural way to study a filtered group is to study subquotients - while  $\text{Ph}(X, Y)$  in general need not have the structure of a group, we can use the tower theoretic interpretation of the Gray index to work out the appropriate notion of a subquotient for our filtered set. Specifically, let  $\text{Ph}^k(X, Y)/\text{Ph}^l(X, Y) = p_l(\text{Ph}^k(X, Y)) \subseteq \lim^1_n G_l^{(n)}$ , and note that this is isomorphic to the quotient group in case  $\text{Ph}(X, Y)$  has the structure of a group (which is known to be Abelian, if such a group structure exists, so the quotient above makes sense).

**Theorem 1.25.** [21] *Suppose  $A$  and  $B$  are nilpotent CW complexes of finite type. If  $\varphi : A \rightarrow B$  induces surjections on  $H^k(-; \mathbb{Q})$  with  $l \leq k \leq m$ , then*

$$p_{m+1}\varphi^*(\text{Ph}^l(B, Y)) = \text{Ph}^l(A, Y)/\text{Ph}^{m+1}(A, Y)$$

for all finite type  $Y$ .

Dually, if  $\varphi : A \rightarrow B$  induces surjections on  $\pi_m \otimes \mathbb{Q}$  with  $l \leq k \leq m$  then

$$p_m\varphi_*(\text{Ph}^{l-1}(X, A)) = \text{Ph}^{l-1}(X, B)/\text{Ph}^m(X, B)$$

for all finite type  $X$ .

As noted in [21] the theorem implies that if  $A \xrightarrow{f} Y$  is a phantom map with  $G(f) = k$  and  $\varphi : A \rightarrow B$  induces a surjection on  $H^k(-; \mathbb{Q})$ , then there is a phantom map  $A \xrightarrow{f'} Y$  with  $p_{k+1}(f') = p_{k+1}(f)$  and  $f'$  is the image of a phantom map  $B \xrightarrow{g} Y$  under  $\varphi^*$ . Since  $G(f) = k$  we have  $p_{k+1}(f) \neq *$  and so  $f' \not\cong *$ , since  $p_{k+1}$  is a pointed function;  $p_{k+1}(*) = *$ . It follows that  $\text{Ph}(B, Y) \neq *$ . This discussion and its dual exemplify our use of Theorem 1.25 in what follows. In particular, this result allows us to relax the hypotheses of Theorem 1.17 provided we are able to obtain information on the Gray indices attained by phantom maps between particular spaces. We present a specific example at the end of this section, which will subsequently be utilized in Section 4.1.

We will need one final technical result to ensure finite Gray indices are attained in cases where essential phantom maps exist. We have noted above that the Gray index of the constant map is infinite. The question of whether the converse of this statement holds has an interesting history. In his thesis [15] Gray appears to show that every phantom map of infinite Gray index must be homotopically trivial. However, a flaw was later discovered in this argument, and later still McGibbon and Strom [31] located examples of essential phantom maps of infinite Gray index. The targets of these maps are not of finite type, and so it was conjectured that a phantom map between finite type spaces having infinite Gray index must be homotopically trivial. A counterexample to this conjecture was constructed by

Iriye in [24]. Nonetheless, Corollary 1.27 shows that in some special cases the only phantom map of infinite Gray index is the constant map. The next theorem, from which we derive the corollary, will be crucial to many arguments in the sequel.

**Theorem 1.26.** *If  $X$  and  $Y$  are finite type and  $\text{Ph}(X, Y) = \text{Ph}^\infty(X, Y)$ , then  $\text{Ph}(X, Y) = *$ .*

**Corollary 1.27.** *Suppose for the integer  $k$  it happens that  $H^m(X; \mathbb{Q})$  or  $\pi_{m+1}(Y) \otimes \mathbb{Q}$  is zero for all  $m > k$ . Then  $\text{Ph}^k(X, Y) = *$ .*

**Example 1.28.** We show that for each  $n$  the space  $BU(n)$  is the target of essential phantom maps of every Gray index  $k$  for which  $\pi_{k+1}(BU(n)) \otimes \mathbb{Q} \neq 0$  to illustrate the use of the tools of this section in the sequel. Our starting point is an observation of Anderson and Hodgkin from 1967.

**Theorem 1.29.** [2] *For each odd  $m \geq 3$ ,  $\text{Ph}(K(\mathbb{Z}, m), BU) \neq *$ .*

For the rest of this example, we write  $K_m = K(\mathbb{Z}, m)$ . Applying Gray's principle in conjunction with the observation that  $\tilde{H}^*(K_m; \mathbb{Q})$  is concentrated in dimension  $m$  when  $m$  is odd we see the only possible Gray indices for phantom maps  $K_m \rightarrow BU$  are  $m$  or  $\infty$ . Theorem 1.26 then assures us that there are phantoms  $K_m \rightarrow BU$  of Gray index  $m$  for each odd  $m \geq 3$ .

Now, the natural map  $BU(n) \rightarrow BU$  is a  $(2n+1)$ -equivalence, which is to say this map induces isomorphisms on  $\pi_k$  for  $k \leq 2n$  and a surjection (onto the zero group) on  $\pi_{2n+1}$ . Theorem 1.25 shows that for each odd  $m < 2n$  the inclusion  $BU(n) \rightarrow BU$  induces surjections

$$\text{Ph}(K_m, BU(n)) / \text{Ph}^{2n}(K_m, BU(n)) \rightarrow \text{Ph}(K_m, BU) / \text{Ph}^{2n}(K_m, BU).$$

Now applying Gray's principle we have  $\text{Ph}(K_m, BU(n)) = \text{Ph}^m(K_m, BU(n))$  and  $\text{Ph}^{2n}(K_m, BU(n)) = \text{Ph}^\infty(K_m, BU(n))$  so we have shown

$$\text{Ph}^m(K_m, BU(n)) / \text{Ph}^\infty(K_m, BU(n)) \neq 0,$$

which is to say there are phantoms  $K_m \rightarrow BU(n)$  for each odd  $m$  with  $3 \leq m \leq 2n-1$ , each of which has Gray index  $m$ .

## 1.5 Cone Length

Here we describe an invariant known as cone length [9], or strong category [11],[12]. We prefer the first term. We should note that in the next definition  $Y_{(i)}$  does not

refer in any way to a localization of  $Y$  – this notation was chosen to avoid confusion with a CW structure for  $Y$ .

**Definition.** A **length  $n$  cone decomposition** of a space  $Y$  is a sequence of cofiber sequences

$$(1.5) \quad A_i \rightarrow Y_{(i)} \rightarrow Y_{(i+1)}, \quad i = 0, \dots, n$$

with  $Y_{(0)} \simeq *$  and  $Y_{(n)} \simeq Y$ . For  $n = \infty$  we replace this last condition with  $Y \simeq \operatorname{colim} Y_{(i)}$ , and remark that this agrees with the homotopy colimit, since we are considering a telescope of cofibrations.

Familiar examples of length  $n$  cone decompositions are CW structures for spaces having the homotopy type of  $n$ -dimensional CW complexes, and the homology decompositions of [6].

**Definition.** The **cone length**  $\operatorname{cl}(Y)$  of a space  $Y$  to be the least  $n$  for which  $Y$  admits a length  $n$ -cone decomposition, allowing for the possibility  $\operatorname{cl}(Y) = \infty$ .

In [4] Arkowitz, Stanley, and Strom introduce and study a variant of cone length. For a collection  $\mathcal{A}$  of spaces a length  $n$   $\mathcal{A}$ -cone decomposition is defined by requiring the spaces in (1.5) belong to the collection  $\mathcal{A}$ . The  $\mathcal{A}$ -cone length  $\operatorname{cl}_{\mathcal{A}}(Y)$  of a space  $Y$  is then the least  $n$  for which  $Y$  admits a length  $n$   $\mathcal{A}$ -cone decomposition.

Given a collection of spaces  $\mathcal{A}$  we say a space  $Y$  with  $\operatorname{cl}_{\mathcal{A}}(Y) < \infty$  is an  **$\mathcal{A}$ -finite space**. Of particular interest will be the  $\mathcal{F}$ -finite spaces, where  $\mathcal{F}$  denotes the collection of finite type wedges of spheres. The collection of  $\mathcal{F}$ -finite spaces includes the stages of the James construction on finite type spaces, and, more generally, any space with a finite length generalized CW structure.

In [22] Iriye makes some striking observations regarding the structure of  $\mathcal{F}$ -finite spaces, which will be of importance in Section 4.1.

**Theorem 1.30.** [22, Iriye] *Suppose  $Y$  is a  $p$ -local,  $\mathcal{F}$ -finite, and  $\pi_1(Y)$  is finite. If  $Y$  is not rationally trivial, then*

- (i) *if  $\operatorname{conn}_{\mathbb{Q}}(Y) = 2n - 1$  then there is a map  $Y \rightarrow BU(n)_{(p)}$  inducing a surjection on  $\pi_{2n} \otimes \mathbb{Q}$ , and*
- (ii) *if  $\operatorname{conn}_{\mathbb{Q}}(Y) = 2n$  then there is a rational equivalence  $Y \rightarrow Y'$  to a space  $Y'$  also of finite  $\mathcal{F}$ -cone length with  $S_{(p)}^{2n+1}$  a homotopy retract of  $Y'$ .*

## 1.6 Lusternik-Schnirelmann Category

The **Lusternik-Schnirelmann category** of a space  $Y$  is the least  $n$  for which  $Y$  can be covered by  $n + 1$  open subsets, each of which is contractible in  $Y$ . This invariant arose in dynamics; the theorem of Lusternik and Schnirelmann shows  $\text{cat}(M) + 1$  is a lower bound for the number of critical points of a smooth function on the smooth manifold  $M$ .

On the collection of connected CW complexes we have two alternate characterizations of Lusternik-Schnirelmann category, which we exposit here. The first variant we describe is due to G.W. Whitehead [41]. To each space  $X$  we associate the  $k$ -fold **fat wedge**  $T^k(X)$ , which is the subspace of  $\prod_1^k X$  of tuples  $(x_1, \dots, x_k)$  with at least one  $x_i = *$ . Whitehead shows that  $\text{cat}(X)$  is the least  $n$  for which the diagonal map  $\Delta : X \rightarrow \prod_1^{n+1} X$  lifts, up to homotopy, through the inclusion  $T^{n+1}(X) \rightarrow \prod_1^{n+1} X$ . We remark here that  $T^2(X) = X \vee X$ , so if  $\text{cat}_W(X) = 1$  then we can form a homotopy commutative diagram

$$\begin{array}{ccc} & & X \vee X \\ & \nearrow \varphi & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

The map  $\varphi$  is then seen to give  $X$  the structure of a co-H-space.

We now turn to describing our second alternate characterization of Lusternik-Schnirelmann category, due to Ganea. In [13] Ganea associates to each connected CW complex  $Y$  a sequence of fibrations  $G_m(Y) \xrightarrow{p_m} Y$ . We call the spaces  $G_m(Y)$  the spaces of Ganea, and the maps  $p_m$  the Ganea fibrations. The construction of these spaces and maps boils down to an inductive cofiber-fiber replacement process, which we now exposit in its original form.

We begin with the pathspace fiber sequence

$$F_0(Y) = \Omega Y \longrightarrow \mathcal{P}(Y) = G_0(Y) \longrightarrow Y,$$

and inductively produce the required family of fibrations. Given the fibration  $G_m(Y) \xrightarrow{p_m} Y$  and write  $F_m(Y) \xrightarrow{i_m} G_m(Y)$  for the fiber of  $p_m$ . Let  $C_{m+1}(Y) = G_m(Y) \cup_{i_m} CF_m(Y)$  be the mapping cone of  $i_m$  and let  $C_{m+1}(Y) \xrightarrow{r_{m+1}} Y$  be the map extending  $p_m$  by sending  $CF_m(Y)$  to  $*$  in  $Y$ . Convert  $r_{m+1}$  into a homotopy equivalent fibration  $G_{m+1}(Y) \xrightarrow{p_{m+1}} Y$ , where  $G_{m+1}(Y)$  is the pullback in the

diagram

$$\begin{array}{ccc}
 G_{m+1}(Y) & \longrightarrow & \mathcal{P}(Y) \\
 \downarrow & \text{pullback} & \downarrow \\
 C_{m+1} & \xrightarrow{r_{m+1}} & Y.
 \end{array}$$

Since  $G_{m+1}(Y)$  arises as a pullback in the preceding square, we obtain a map  $G_m(Y) \rightarrow G_{m+1}(Y)$  fitting in a commutative diagram

$$\begin{array}{ccc}
 G_m(Y) & \longrightarrow & G_{m+1}(Y) \\
 \searrow p_m & & \downarrow p_{m+1} \\
 & & Y.
 \end{array}$$

Ganea shows that  $\text{cat}(Y)$  is the least  $n$  for which  $p_n$  has a homotopy section.

A few pertinent facts about Ganea's construction are

- $G_1(Y) \simeq \Sigma\Omega Y$ ,
- $F_m(Y)$  is homotopy equivalent to the  $(m + 1)$ -fold join of  $\Omega Y$ ; in symbols  $F_m(Y) \simeq (\Omega Y)^{* (m+1)}$ ,
- In the long fiber sequence  $\cdots \rightarrow \Omega Y \xrightarrow{\partial} F_m(Y) \xrightarrow{i_m} G_m(Y) \xrightarrow{p_m} Y$  the map  $\partial$  is nullhomotopic if  $m \geq 1$ . Consequently, for each  $m \geq 1$  we have natural homotopy equivalences

$$\Omega G_m(Y) \simeq \Omega Y \times \Omega F_m(Y).$$

We note that  $\text{cl}(G_m(Y)) \leq m$  for any  $m$  and  $Y$ , since the construction of  $G_m(Y)$  gives an explicit  $m$ -cone decomposition for this space.

We record here an inequality relating the invariants  $\text{cl}$  and  $\text{cat}$ , which we will refer to in Section 4.2.

**Theorem 1.31.** [9] *For a space  $X$  one has*

$$\text{cl}(X) \leq \text{cat}(X) \leq \text{cl}(X) + 1$$

In conjunction with the following observation, we see that  $\mathcal{F}$ -finite spaces are examples of spaces having finite Lusternik-Schnirelmann category.

**Theorem 1.32.** [4] *For a space  $X$ ,  $\text{cl}(X) \leq \text{cl}_{\mathcal{A}}(X)$  for all collections  $\mathcal{A}$ .*

## 1.7 Geometric Realizations of Natural Coalgebra Decompositions of Tensor Algebras

For this section we fix a field of positive characteristic as our ground ring  $k$ . All homology in this section has  $k$  coefficients. We write  $T$  for the free graded tensor algebra functor taking  $k$ -modules to  $k$ -algebras. Recall that as  $k$ -modules  $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$  where  $V^{\otimes 0} = k$ . This identifies  $V$  as a submodule of  $T(V)$ . The algebra  $T(V)$  is equipped with a unit  $k \rightarrow T(V)$  and augmentation  $T(V) \rightarrow k$  defined by inclusion of and projection onto  $k = V^{\otimes 0}$ , respectively.

The tensor algebra  $T(V)$  is (uniquely) naturally endowed with the structure of a Hopf algebra by declaring the elements of  $V$  to be primitive. More explicitly, since  $T(V)$  is the free  $k$ -algebra on  $V$ , the  $k$ -module homomorphism  $V \rightarrow T(V) \otimes T(V)$  given by  $v \mapsto 1 \otimes v + v \otimes 1$  extends uniquely to a map of  $k$ -algebras  $\Delta : T(V) \rightarrow T(V) \otimes T(V)$ , giving a comultiplication on  $T(V)$ . One can check that the unit and augmentation are morphisms of coalgebras and algebras, respectively, and so we have given  $T(V)$  the structure of a Hopf algebra. This discussion serves to illustrate that we can think of the tensor algebra functor  $T$  as taking its values in the categories of  $k$ -algebras,  $k$ -coalgebras, or  $k$ -Hopf algebras.

We now briefly lay out some notation and nomenclature. For a Hopf algebra  $M$ , write  $IM$  for the augmentation ideal of  $M$ , and write  $QM = IM/(IM)^2$  for the module of indecomposables of  $M$ .

**Definition.** (1) A **natural coalgebra retract** of  $T$  is a functor  $A$  from  $k$ -modules to  $k$ -coalgebras equipped with natural transformations  $A \xrightarrow{I} T$  and  $T \xrightarrow{R} A$  so that  $RI$  is the identity natural transformation on  $A$ .

(2) A **natural coalgebra decomposition** of  $T$  is a pair of functors  $A, B$  from  $k$ -modules to  $k$ -coalgebras equipped with natural coalgebra isomorphisms  $T \cong A \otimes B$ . Since  $\otimes$  is the categorical product in the category of  $k$ -coalgebras, it follows that if  $T \cong A \otimes B$  is a natural coalgebra decomposition, then both  $A$  and  $B$  are natural coalgebra retracts of  $T$ .

(3) A **natural sub-Hopf algebra** of  $T$  is a subfunctor  $B$  from  $k$ -modules to  $k$ -Hopf algebras. A natural sub-Hopf algebra is **coalgebra split** if  $B$  is a natural coalgebra retract of  $T$  when regarded as a functor into  $k$ -coalgebras.

Of course, a natural way to study mathematical objects is to break them into more tractable pieces. By seeking decompositions of functors in this way we obtain coherent decompositions of all tensor algebras. The power of these methods, and their connections to homotopy theory, will be seen in our study of phantom maps



into co-H-spaces in Section 4.2. We should note that in the sequel we only make use of the wide applicability of the forthcoming decomposition methods, and have no need for their naturality.

We will be interested in a particular coalgebra decomposition of the tensor algebra functor known as the minimal decomposition, which we now set about describing. Beginning with Cohen, there was an interest in studying the minimal functorial coalgebra retract  $A^{\min}(V)$  of  $T(V)$  containing  $V$ ; the functor  $A^{\min}$  from  $k$ -modules to  $k$ -coalgebras exists for theoretical reasons. Cohen conjectured that the primitives of  $T(V)$ , considered as a Hopf algebra, having tensor length not a power of  $p$  must lie in the complement of  $A^{\min}(V)$  in  $T(V)$ . This conjecture was confirmed by Selick and Wu in [38], who discovered the minimal decomposition and began studying its structural properties.

**Theorem 1.33.** [38] *There is a natural sub-Hopf algebra  $B^{\max}(V)$  of  $T(V)$  equipped with natural isomorphisms*

$$T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$$

where  $L_n(V) \subseteq B^{\max}(V)$  if  $n$  is not a power of  $p$ , the characteristic of the ground ring. Here  $L_n(V)$  denotes the submodule of homogeneous Lie elements of tensor length  $n$  in  $T(V)$ . The natural coalgebra decomposition

$$T \cong A^{\min} \otimes B^{\max}$$

is known as the **minimal decomposition**.

Being a subalgebra of a tensor algebra, it follows that  $B^{\max}(V)$  is also a tensor algebra. That is, there is an isomorphism of  $k$ -algebras

$$B^{\max}(V) \cong T \left( \bigoplus_{n \geq 2} Q_n B^{\max}(V) \right)$$

where  $Q_n B^{\max}(V)$  is the image of submodule

$$B_n^{\max}(V) = IB^{\max}(V) \cap V^{\otimes n} \subseteq T(V)$$

of  $B^{\max}(V)$  consisting of elements of tensor length  $n$  in  $T(V)$  lying in the augmentation ideal of  $B^{\max}(V)$  under the natural map  $B^{\max}(V) \rightarrow QB^{\max}(V)$ . We justify the indexing  $n \geq 2$  by noting that  $B_1^{\max}(V) = 0$  since  $V \subseteq A^{\min}(V)$ , the coalgebra complement of  $B^{\max}(V)$  in  $T(V)$ .

Next we turn to describing the connection with the preceding algebraic notions and natural decompositions of the loop space on a co-H-space. We begin

by discussing the loopspaces of suspension spaces. The classical Bott-Samelson theorem asserts that there is a natural isomorphism of  $k$ -algebras  $H_*(\Omega\Sigma X) \cong T(H_*(X))$ . In [38] Selick and Wu geometrically realize the minimal decomposition of  $T(H_*(X))$  as functorial decompositions of the functor  $\Omega$  taking the category of suspension spaces and suspension maps to the homotopy category of topological spaces. We will precisely state the nature of this geometric realization in its full generality in Theorem 1.34.

The generalized Bott-Samelson theorem implies that for a co-H-space  $Y$  there is a natural isomorphism of  $k$ -algebras

$$H_*(\Omega Y) \cong T(\Sigma^{-1}\tilde{H}_*(Y)),$$

and so ideally one can geometrically realize the decompositions of the tensor algebra functor as functorial decompositions of the functor  $\Omega$  taking the category of co-H-spaces and co-H-maps to the homotopy category of topological spaces. This was achieved in a series of papers [36], [37], [19], which ultimately hinged on a generalization of the James splitting  $\Sigma\Omega\Sigma X \simeq \Sigma\bigvee_{n \geq 1} X^{\wedge n}$  to a natural decomposition of  $\Sigma\Omega Y$  with  $Y$  any simply connected co-H-space in [20].

The isomorphism in the Bott-Samelson tends to not be an isomorphism of Hopf algebras, since the coalgebra structure in  $H_*(\Omega Y)$  can get quite complicated. Indeed, Grbić, Theriault, and Wu assert that this is an isomorphism of Hopf algebras only if  $Y \simeq \Sigma^2 X$  for some space  $X$ . On the other hand, according to Milnor and Moore, the Hopf algebra associated to the bigraded Hopf algebra  $E^0 H_*(\Omega Y)$  associated to the augmentation ideal filtration is isomorphic to  $T(\Sigma^{-1}\tilde{H}_*(Y))$  as Hopf algebras, which allows Grbić, Theriault, and Wu to geometrically realize coalgebra decompositions of tensor algebras as decompositions of looped co-H-spaces.

The preceding history lends credibility to our pursuit of a generalization of Theorem 1.2; if a particular statement holds for all suspension spaces, one may well ask if the same holds for all co-H-spaces.

For the rest of this section  $Y$  is a co-H-space and  $V = \Sigma^{-1}\tilde{H}_*(Y)$ . First we see that natural coalgebra retracts have geometric realizations. As an immediate consequence, we can geometrically realize natural coalgebra decompositions of  $T$ . We write  $\mathbf{CoH}$  for the category of co-H-spaces and co-H maps between them, and  $\mathcal{T}$  for the category of based topological spaces (i.e. nilpotent spaces of finite type over some subring of the rationals).

**Theorem 1.34.** [19] *If the functor  $A$  from  $k$ -modules to  $k$ -coalgebras is a coalgebra retract of  $T$ , then there is a functor  $A : \mathbf{CoH} \rightarrow \mathcal{T}$  that is a natural retract of  $\Omega$  for which*

$$H_*(A(Y)) \cong A(V).$$

**Corollary 1.35.** [19] *If  $T \cong A \otimes B$  is a natural coalgebra decomposition, then there are functors  $A, B : \mathbf{CoH} \rightarrow \mathcal{T}$  and natural homotopy equivalences*

$$\Omega Y \simeq A(Y) \times B(Y)$$

where

$$H_*(A(Y)) \cong A(V) \quad \text{and} \quad H_*(B(Y)) \cong B(V).$$

As noted in [19], since  $B^{\max}(V)$  is naturally a sub-Hopf algebra of  $T(V)$ , instead of simply a natural coalgebra retract, ideally one can geometrically realize this additional structure as well, which is the content of the next result.

**Theorem 1.36.** [19] *Suppose  $B$  is a natural coalgebra-split sub-Hopf algebra of  $T$ . There exist functors  $Q_n B : \mathbf{CoH} \rightarrow \mathcal{T}$  with*

- $\tilde{H}_*(Q_n B(Y)) \cong \Sigma Q_n B(V)$ ,
- $Q_n B(Y)$  is naturally a retract of an  $(n - 1)$ -fold desuspension of  $Y^{\wedge n}$
- $B(Y) \simeq \Omega (\bigvee_{n \geq 2} Q_n B(Y))$ .

The statement (2) requires some explanation - according to Theriault, and later Grbić, Theriault, and Wu, or was it Gray, it was shown that if  $X$  and  $Y$  are simply connected co-H-spaces, then  $X \wedge Y \simeq \Sigma Z$  is a suspension space; moreover,  $Z$  can be naturally endowed with a co-H-structure through this identification. Inductively, it follows that an  $n$ -fold smash product of co-H-spaces is an  $(n - 1)$ -fold suspension; symbolically, for simply connected co-H-spaces  $X_i, i = 1, \dots, n$

$$(1.6) \quad \bigwedge_{i=1}^n X_i \simeq \Sigma^{n-1} Z$$

for some space  $Z$ . There may, in fact, be many choices for the space  $Z$  - for example, the well known decomposition

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$$

and the failure of the identity

$$X \times Y \simeq X \vee Y \vee (X \wedge Y)$$

witnesses the failure of a cancellation property for  $\Sigma$ . But, since homology and cohomology have suspension isomorphisms, for all spaces  $Z$  fitting in (1.6),  $H_*(Z)$  and  $H_*(\bigwedge_{i=1}^n X_i)$  are closely related (by a grading shift), and all such spaces  $Z$

have the same homology (over any coefficients). In the particular case of Theorem 1.36 we have  $Q_n B(Y)$  naturally a retract of a space  $Z$  with  $\Sigma^{n-1} Z \simeq Y^{\wedge n}$ , the  $n$ th smash power of  $Y$ .

In Chapter 3 we develop methods that reduce the detection of phantom maps into a large collection of co-H-spaces  $Y$  to a cursory analysis of the module of indecomposables  $QB^{\max}(V) = \bigoplus_{n \geq 2} Q_n B^{\max}(V)$  of  $B^{\max}(V)$  where  $V = \Sigma^{-1} \widetilde{H}_*(Y)$ . This analysis is carried out in section 4.2.

# Chapter 2

## Localization and the Gray Index

To study phantom maps into  $\mathcal{F}$ -finite spaces or finite type co-H-spaces we will be led by Theorem 1.30 and the observations of Section 1.7, respectively, to localize spaces at a prime  $p$  to obtain a more manageable structure. Here we describe how one can draw inferences about phantom maps into a space  $Y$  from information about phantom maps into its  $p$ -localization.

We begin by remarking that this approach is not fool-proof in general. Many phantom maps vanish under localization, by which we mean  $X \xrightarrow{f} Y$  is a phantom map and  $X_{(p)} \xrightarrow{f_{(p)}} Y_{(p)}$  is nullhomotopic [27],[22]. In Chapter 4 we have the good fortune of working with spaces that are not so poorly conditioned.

Since the maps  $Y \rightarrow Y_{(p)}$  are rational equivalences for all primes  $p$ , by Theorem 1.17 the natural map  $Y \rightarrow \prod_p Y_{(p)}$  induces a surjection

$$\text{Ph}(X, Y) \rightarrow \prod_p \text{Ph}(X, Y_{(p)}).$$

So, if for any prime  $p$  the  $p$ -localization  $Y_{(p)}$  of  $Y$  is the target of essential phantom maps, then so too is  $Y$ . We refine this observation in the following sense.

**Theorem 2.1.** *If  $Y_{(p)}$  is the target of a phantom map of Gray index  $n$ , then so too is  $Y$ .*

For the proof we require a lemma, which will also be used to prove Proposition 3.1 and to make several Gray index computations in the sequel.

**Lemma 2.2.** *If  $Y$  is the target of phantom maps of Gray index  $n$ , then we can choose the domain  $X$  of said phantoms to have  $\text{conn}(X) = \text{conn}_{\mathbb{Q}}(X) = n - 1$ .*

*Proof.* This follows from the definition of the Gray index, Proposition 1.19, and Gray's principle. If  $X \xrightarrow{\varphi} Y$  is a phantom with  $G(\varphi) = n$  we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow q & \nearrow \bar{\varphi} \\ & X/X_{n-1} & \end{array}$$

with  $\bar{\varphi}$  phantom. By Gray's principle,  $H^n(X; \mathbb{Q}) \neq 0$  and so

$$\text{conn}(X/X_{n-1}) = \text{conn}_{\mathbb{Q}}(X/X_{n-1}) = n - 1.$$

Then we have

$$n = G(\varphi) \geq G(\bar{\varphi}) > n - 1,$$

the first inequality coming from Proposition 1.19 and the second from Gray's principle. So  $X/X_{n-1} \xrightarrow{\bar{\varphi}} Y$  is the required map.  $\square$

*Proof of Theorem 2.1.* By Lemma 2.2 there is a phantom map  $X \xrightarrow{\varphi} Y_{(p)}$  of Gray index  $n$ , where  $\text{conn}_{\mathbb{Q}}(X) = n - 1$ . Since  $Y \rightarrow Y_{(p)}$  is a rational equivalence, by Theorem 1.17 we can lift  $\varphi$  to a phantom map  $\psi$ , as in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ & \searrow \varphi & \downarrow \\ & & Y_{(p)}. \end{array}$$

Proposition 1.19 implies  $G(\psi) \leq G(\varphi) = n$ , while  $G(\psi) \geq \text{conn}_{\mathbb{Q}}(X) + 1 = n$ .  $\square$

In what follows we will use Theorem 2.1 to reduce the problem of producing phantom maps of a specified Gray index into a space  $Y$  to the marginally simpler task of producing phantom maps of said Gray index into the  $p$ -localization  $Y_{(p)}$  of  $Y$ .

# Chapter 3

## Decomposition Methods in Phantom Map Theory

### 3.1 Splittings of $\Omega Y$ and Phantom Maps into $Y$

In this section we see how homotopy decompositions of  $\Omega Y$  can give useful information about phantom maps into  $Y$ . Our main result is Proposition 3.1, which will be used to relate a particular type of decomposability with the existence of essential phantom maps in Section 3.2 and will also provide the bridge from the machinery of Section 1.7 to phantom map theory required to prove our results on phantom maps into co-H-spaces in 4.2. This same proposition will be used to study phantom maps into the spaces  $G_m(Y)$  of Ganea in Section 4.3. Many more applications should be found in the vast library of homotopy decompositions of loopspaces that have been discovered over the years.

Before stating our main result of this section, we require some terminology. Recall an H-space is a space  $X$  equipped with a (homotopy unital) multiplication  $X \times X \xrightarrow{\mu} X$ . If  $(X, \mu)$  and  $(Y, \eta)$  are H-spaces, a map  $X \xrightarrow{f} Y$  is called an H-map if  $f$  respects the H-space structures on  $X$  and  $Y$ , which is to say there is a homotopy commutative diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ f \times f \downarrow & & \downarrow f \\ Y \times Y & \xrightarrow{\eta} & Y. \end{array}$$

The H-spaces  $X$  and  $Y$  are said to be H-equivalent if there is an H-map between them that happens to be a homotopy equivalence.

**Proposition 3.1.** (1) If  $\Omega Y \simeq A \times \Omega B$ , and  $\text{Ph}(X, Y) = *$  for all finite type  $X$ , then  $\text{Ph}(X, B) = *$  for all finite type  $X$ .

(2) Assume  $Y$  is simply connected, and suppose  $\Omega Y$  is  $H$ -equivalent to  $A \times \Omega B$ . If  $Y$  is not the target of phantom maps of a particular Gray index  $n$ , then neither is  $B$ .

*Proof.* (1) Take  $X$  to be an arbitrary finite type domain and write

$$G_n = [X, \Omega Y^{(n)}] \quad \text{and} \quad H_n = [X, \Omega B^{(n)}].$$

We make use of the identification

$$\text{Ph}(X, Y) \cong \lim^1 G_n \quad \text{and} \quad \text{Ph}(X, B) \cong \lim^1 H_n.$$

By Theorem 1.14 if  $\text{Ph}(X, Y) = *$  then  $\{G_n\}$  is Mittag-Leffler. Since  $\Omega Y \simeq A \times \Omega B$  we have a natural projection  $\Omega Y \rightarrow \Omega B$  inducing surjections  $G_n \rightarrow H_n$  and so by Lemma 1.15 the tower  $\{H_n\}$  is Mittag-Leffler, and so  $\lim^1 H_n \cong \text{Ph}(X, B) = *$ .

(2) We assume  $B$  is the target of a phantom map of Gray index  $n$ , and prove the same is true of  $Y$ . By Lemma 2.2 we can find a phantom map  $X \xrightarrow{\varphi} B$  of Gray index  $n$  where  $\text{conn}_{\mathbb{Q}}(X) = n - 1$ . With  $G_n$  and  $H_n$  as above, we will produce a commutative square

$$(3.1) \quad \begin{array}{ccc} \lim^1 G_n & \xrightarrow{f} & \lim^1 H_n \\ p_k \downarrow & & \downarrow p_k \\ \lim^1_n G_k^{(n)} & \longrightarrow & \lim^1 H_k^{(n)}. \end{array}$$

Assume for the moment this has been done, and  $\varphi : X \rightarrow B$  is a phantom map of Gray index  $n$ . Then  $p_k(\varphi) \neq *$  for  $n > k$  by Theorem 2 of [21]. Now, since  $f$  is surjective,  $\varphi = f(\psi)$  for some phantom map  $\psi : X \rightarrow Y$ . From commutativity of the preceding diagram  $p_k(\psi) \neq *$  for  $k > n$  and so  $G(\psi) \leq n$ . But since the domain  $X$  of  $\psi$  has connectivity  $n - 1$  it follows that  $G(\psi) = n$ .

To produce the diagram (3.1) we note that an  $H$ -equivalence  $\Omega Y \rightarrow A \times \Omega B$  gives rise to  $H$ -equivalences  $(\Omega Y)^{(n)} \rightarrow (A \times \Omega B)^{(n)}$  for all  $n$  (since Postnikov towers of  $H$ -spaces can be constructed as towers of  $H$ -spaces and  $H$ -maps, etc.)

Using the identification  $(\Omega Y)^{(n)} \simeq \Omega(Y^{(n+1)})$  this gives homomorphisms  $G_n \rightarrow H_n$  for all  $n$ . These are seen to piece together to give a morphism of towers of groups  $\{G_n\} \rightarrow \{H_n\}$  by inspecting the structure maps for each of the towers.

A morphism of towers  $\{G_n\} \rightarrow \{H_n\}$  naturally gives rise to morphisms of



subtowers  $\{G_k^{(n)}\} \rightarrow \{H_k^{(n)}\}$  for each  $k$ , giving a commutative square of epimorphisms of towers of groups

$$\begin{array}{ccc} \{G_n\} & \longrightarrow & \{H_n\} \\ \downarrow & & \downarrow \\ \{G_k^{(n)}\} & \longrightarrow & \{H_k^{(n)}\}. \end{array}$$

Applying the functor  $\lim^1$  gives diagram (3.1) and completes the proof. □

Here we describe the dual of Proposition 3.1. The proof is similar. Our only use of this result herein will be to establish a dual the forthcoming Theorem 3.6, whose implications will be studied at another time.

**Proposition 3.2.** (1) *If  $\Sigma X \simeq A \vee \Sigma B$  and  $\text{Ph}(X, Y) = *$  for all finite type  $Y$ , then  $\text{Ph}(B, Y) = *$  for all finite type  $Y$ .*

(2) *Suppose  $\Sigma X$  is co- $H$ -equivalent to  $A \vee \Sigma B$ . If  $X$  is not the domain of phantom maps of Gray index  $n$ , then neither is  $B$ .*

### 3.1.1 Application: Phantom Maps into Moment Angle Complexes and Davis-Januszkiewicz Spaces

We have noted above that Proposition 3.1 has many applications to the study of phantom maps. Here we record another application. This discussion relates our topological findings to the algebraic theory of phantom maps in the derived category of a ring via the work of Christenson [8]. We will refrain from going into too much detail on algebraic phantom maps, but will say as much as this: the derived category of a ring is an example of a Monogenic Brown Category, in which one can formulate the appropriate notion of phantom map according to [8]. Below will describe a connection between the Davis-Januszkiewicz spaces  $DJ(K)$  associated to a simplicial complex  $K$  and the Stanley-Reisner ring of  $K$  (definitions forthcoming), which we hope convinces the reader of a plausible link between our findings on topological phantom maps into  $DJ(K)$  and algebraic phantom maps into the Stanley-Reisner ring associated to  $K$ . We will study the connections between topological and algebraic phantom maps further in subsequent works.

In [18] Grbic, Theriault, Panov, and Wu describe homotopy decompositions of the loopspaces of a large class of spaces arising in the rapidly evolving field of toric topology; we briefly follow these authors' exposition of the basic properties of the spaces of interest before describing our application.

**Definition.** Suppose  $K$  is a finite simplicial complex on the vertex set  $V = \{v_1, \dots, v_m\}$ . Given a finite set  $(\mathbf{X}, \mathbf{A}) = \{(X_i, A_i)\}_{i=1}^m$  of pairs of topological spaces with  $A_i \subseteq X_i$  for all  $i$ , and a simplex  $\sigma \in K$  we define

$$(\mathbf{X}, \mathbf{A})^\sigma = \{(x_1, \dots, x_m) \in \prod_{i=1}^m X_i \mid x_i \in A_i \text{ when } v_i \notin \sigma\},$$

which is a generalization of the fat wedge of Section 1.6. We define the **polyhedral product**  $(\mathbf{X}, \mathbf{A})^K$  of  $(\mathbf{X}, \mathbf{A})$  over  $K$  to be the following subset of  $\prod_1^m X_i$

$$(\mathbf{X}, \mathbf{A})^K = \cup_{\sigma \in K} (\mathbf{X}, \mathbf{A})^\sigma.$$

For notational convenience, if in  $(\mathbf{X}, \mathbf{A})$  every  $X_i = X$  and  $A_i = A$  for fixed spaces  $X$  and  $A$  we write  $(X, A)^K$  for  $(\mathbf{X}, \mathbf{A})^K$ .

**Definition.** • The **moment-angle complex**  $\mathcal{Z}_K$  associated with  $K$  is the polyhedral product  $(D^2, S^1)^K$ .

- The **Davis-Januszkiewicz space** is the polyhedral product  $DJ(K) = (\mathbb{C}P^\infty, *)^K$ .
- For a simplicial complex  $K$ , and a ring  $k$  the **Stanley-Reisner ring**  $k[K]$  of  $K$  is the quotient of  $k[v_1, \dots, v_m]$  by the ideal

$$I_K = (v_{i_1} \dots v_{i_k} \mid \{v_{i_1}, \dots, v_{i_k}\} \notin K)$$

generated by square free monomials representing the “nonfaces” of  $K$ . We specify a grading on this ring by setting  $\deg(v_i) = 2$  and assigning  $k$  to degree zero.

Davis-Januszkiewicz spaces can be thought of as geometric realizations of Stanley-Reisner rings, in the following sense.

**Proposition 3.3.** [7] *There is an isomorphism of graded commutative algebras*

$$H^*(DJ(K); k) \cong k[K]$$

for any ring  $k$ .

The moment angle complex and Davis-Januszkiewicz spaces associated to a particular simplicial complex are related by a fiber sequence, which endows  $\mathcal{Z}_K$  with a torus action. Here we are more interested in the decompositions this fibration helps to locate.

**Proposition 3.4.** [7] *There is a homotopy fibration sequence*

$$\mathcal{Z}_K \rightarrow DJ(K) \rightarrow (\mathbb{C}P^\infty)^m,$$

*which splits after looping, giving natural homotopy decompositions*

$$\Omega DJ(K) \simeq \Omega \mathcal{Z}_K \times T^m$$

*where  $T^m$  denotes the  $m$ -torus.*

Proposition 3.1, the results of Chapter 2 and the following can be used to show that in special cases  $DJ(K)$  and  $\mathcal{Z}_K$  are the targets of essential phantom maps from finite type domains, and to make inferences about the Gray indices of these phantom maps. Before stating the result, we need a definition.

**Definition.** Suppose  $K$  is a simplicial complex on the vertex set  $V = \{v_1, \dots, v_m\}$ . A subset  $W \subseteq V$  is a **missing face** of  $K$  if every proper subset of  $W$  lies in  $K$ , but  $W$  does not. We say  $K$  is **flag** if every missing face of  $K$ , has exactly two vertices.

**Proposition 3.5.** [18] *If  $K$  is a flag complex, then localized at an odd prime  $\Omega DJ(K)$  and  $\Omega \mathcal{Z}_K$  decompose as products of spheres and loops on spheres.*

Consequently if  $H^{2n+1}(\Omega Y; k) \neq 0$  for some  $n$ , then  $Y$  is the target of essential phantom maps when  $Y = DJ(K)$  or  $\mathcal{Z}_K$  with  $K$  a flag complex.

If  $K$  is a flag complex which is Golod over some  $k$ , which is to say if multiplication in the Tor-algebra  $H^*(\mathcal{Z}_K) = \text{Tor}_{k[v_1, \dots, v_n]}(k[K], k)$  is trivial, along with all other higher Massey products, then  $\mathcal{Z}_K$  is homotopy equivalent to a wedge of spheres.

The requirement that  $K$  be a flag complex above may not be required – according to [18] all known examples of Golod complexes have associated moment angle complexes which are homotopy equivalent to rationally nontrivial, finite type suspensions, and hence fall under the purview of Iriye’s theorem, recorded here as Theorem 1.2. However, a proof of this claim remains elusive. In these cases, Proposition 3.1 can be used to witness essential phantom maps from finite type domains into  $DJ(K)$ .

## 3.2 Phantom Maps into Wedge Sums

In this section we relate a particular type of decomposability, which we call rationally nontrivial wedge decomposability, to the existence of essential phantom maps into a space. Note that wedge decomposability is usually a topic of discussion reserved for spaces being considered as domains in the tradition of Eckmann-Hilton duality. This continues a well-documented trend of a strange reversal of Eckmann-Hilton duality in phantom map theory.

**Definition.** We will say a space  $Y$  is **wedge decomposable** if there is a homotopy equivalence  $Y \simeq A \vee B$  with neither  $A$  nor  $B$  a contractible space. This decomposition is **rationally nontrivial** if both  $A$  and  $B$  are rationally nontrivial.

Our main result gives a test for detecting phantom maps into wedge decomposable spaces. This result will be used in our study of phantom maps into co-H-spaces in Section 4.2.

**Theorem 3.6.** *Assume  $Y$  is simply connected, and has finite type over  $\mathbb{Z}$  or is  $p$ -local and has finite type over  $\mathbb{Z}_{(p)}$ . Suppose  $Y$  has a wedge decomposition  $Y \simeq A \vee B$ . If either*

- (1)  *$A$  or  $B$  is the target of essential phantom maps, or*
- (2) *the aforementioned decomposition is rationally nontrivial,*

*then  $Y$  is the target of essential phantom maps from finite type domains.*

For the proof of Theorem 3.6 in the case  $Y$  is  $p$ -local, we will need a refined version of a result of Iriye from [23], which we develop now. Iriye's Corollary 1.5 gives a test for detecting phantom maps into the suspensions of  $p$ -localizations of finite type complexes. We extend these results to the suspensions of  $p$ -local spaces that may not be the  $p$ -localization of finite type complexes, but are of finite type over  $\mathbb{Z}_{(p)}$ . The first step in such a generalization is this next observation, which assures us that the vanishing of phantom maps into such spaces is entirely characterized by the Mittag-Leffler property of an associated tower of groups. The proof of this proposition can also be used to show that if  $X$  and  $Y$  are of finite type, then  $[\Sigma X_n, Y]$  and  $[X, \Omega Y^{(n)}]$  are countable groups.

**Proposition 3.7.** *If  $Y$  is  $p$ -local and of finite type over  $\mathbb{Z}_{(p)}$ , then  $[X, \Omega Y^{(n)}]$  is countable for all  $n$ .*

*Proof.* Suppose  $\text{conn}(Y) = m - 1$  so  $\Omega Y^{(k)} \simeq *$  for  $k < m$  and  $\Omega Y^{(m)} \not\simeq *$ . We induct up a Postnikov system for  $\Omega Y$ . The base case is  $\Omega Y^{(m)} \simeq K(\pi_m(Y), m - 1)$ , in which case

$$[X, \Omega Y^{(m)}] \cong H^{m-1}(X; \pi_m(Y))$$

is a finitely generated  $\mathbb{Z}_{(p)}$ -module, hence a countable group.

Next, assume  $[X, \Omega Y^{(n)}]$  is countable for some  $n$  and consider the fibration sequence

$$\cdots \rightarrow K(\pi_{n+1}(Y), n - 1) \rightarrow \Omega Y^{(n+1)} \rightarrow \Omega Y^{(n)} \rightarrow K(\pi_{n+1}(Y), n).$$

This induces an exact sequence of groups

$$H^{n-1}(X; \pi_{n+1}(Y)) \rightarrow [X, \Omega Y^{(n+1)}] \rightarrow [X, \Omega Y^{(n)}] \rightarrow H^n(X; \pi_{n+1}(Y))$$

which gives rise to a short exact sequence

$$1 \rightarrow K \rightarrow [X, \Omega Y^{(n+1)}] \rightarrow I \rightarrow 1$$

where  $K$  is a subquotient of  $H^{n-1}(X; \pi_{n+1}(Y))$  and  $I$  is subgroup of  $[X, \Omega Y^{(n)}]$ . In particular, both  $K$  and  $I$  are countable. Then the result follows from standard observations in group theory; since  $K$  is countable, if  $[X, \Omega Y^{(n+1)}]$  is uncountable, then so too is  $I$ , a contradiction.  $\square$

In particular, if  $Y$  is  $p$ -local and of finite type over  $\mathbb{Z}_{(p)}$  and  $X$  is a finite type space, then  $\text{Ph}(X, Y) = *$  if and only if the tower  $\{[X, \Omega Y^{(n)}]\}$  is Mittag-Leffler. This is the main point required to complete the construction of the rational equivalence  $\prod K(\mathbb{Z}_{(p)}, m_\beta) \rightarrow \Omega Y$  as given by McGibbon and Roitberg. This gives the following partial refinement of the work of McGibbon and Roitberg from [30].

**Proposition 3.8.** *If  $Y$  is a  $p$ -local space of finite type over  $\mathbb{Z}_{(p)}$ , and  $\text{Ph}(X, Y) = *$  for all finite type domains  $X$ , then there is a rational equivalence*

$$\prod K(\mathbb{Z}_{(p)}, m_\beta) \rightarrow \Omega Y.$$

The converse of this statement could feasibly hold, but we have not yet had occasion to check this. Indeed, if conjugacy classes in  $[X, \Omega Y^{(n)}]$  are of finite cardinality, then the converse of Proposition 3.8 can be established using the proof of Theorem 1.16 given by McGibbon and Roitberg in [30]. With Proposition 3.8 at hand we arrive at a fine-tuned version of Iriye's Corollary 1.5:

**Theorem 3.9.** *Suppose  $Y$  is a  $p$ -local space of finite type over  $\mathbb{Z}_{(p)}$ . If either*

- (1) there is some  $\alpha \in \pi_{2n+1}(Y)$  of infinite order whose image under the Hurewicz map is also of infinite order, or
- (2) there is some  $v \in H^{2n}(Y; \mathbb{Z})$  of infinite order whose square  $v^2$  is also of infinite order,

then  $\Sigma Y$  is the target of essential phantom maps from finite type domains.

Iriye's proof of Corollary 1.5 in [23] can be used to establish this result, simply replacing Iriye's Theorem 2.1 with Proposition 3.8. We furnish this proof in Appendix A. We are now equipped to prove Theorem 3.6.

*Proof of Theorem 3.6.* First we assume condition (1) holds. Since  $A$  is a retract of  $Y$  we can find maps  $i : A \rightarrow Y, r : Y \rightarrow A$  so that  $ri \simeq \text{id}_A$ . If  $\varphi : K \rightarrow A$  is an essential phantom map from a finite type domain, then  $i\varphi : K \rightarrow Y$  is a phantom map from a finite type domain. This composite must be nontrivial, since  $ri\varphi \simeq \varphi$ .

For (2) we note that since  $Y$  is simply connected, so too are  $A$  and  $B$ . In the long fiber sequence induced by the inclusion  $i : A \vee B \rightarrow A \times B$

$$\dots \longrightarrow \Omega F \xrightarrow{\Omega f} \Omega(A \vee B) \xrightarrow{\Omega i} \Omega A \times \Omega B \xrightarrow{\partial} F \xrightarrow{f} A \vee B \xrightarrow{i} A \times B$$

we can identify  $F \simeq (\Omega A) * (\Omega B)$ , where  $X * Y$  denotes the join of topological spaces  $X$  and  $Y$ , and we find that  $\partial \simeq *$ . It follows that  $\Omega i$  has a section, and  $\Omega f$  has a retraction, which gives a natural homotopy equivalence

$$(3.2) \quad \Omega(A \vee B) \simeq \Omega A \times \Omega B \times \Omega((\Omega A) * (\Omega B)).$$

For a more complete account of this discussion we refer the reader to the work of Gray [16]. We now proceed by cases.

**Case I:** Suppose  $Y$  has finite type. Then so too do  $A$  and  $B$ . Now, if both  $A$  and  $B$  are rationally nontrivial, then  $(\Omega A) * (\Omega B)$  is the suspension of a connected, rationally nontrivial space, hence is the target of essential phantom maps from finite type domains by Iriye's theorem, recorded here as Theorem 1.2. Applying Proposition 3.1 to the splitting (3.2) then implies  $A \vee B$  is the target of essential phantom maps.

**Case II:** In case  $Y$  is  $p$ -local and of finite type over  $\mathbb{Z}_{(p)}$  our goal will be, as above, to show that  $\Omega A * \Omega B$  is the target of phantoms. But, since  $\Omega A * \Omega B$  is not of finite type, we must make use of Theorem 3.9. To do so we need to discover more about  $\Omega A \wedge \Omega B$ . Suppose  $\text{conn}_{\mathbb{Q}}(A) = n$  and  $\text{conn}_{\mathbb{Q}}(B) = m$ . Choose  $a \in H^n(\Omega A; \mathbb{Z}), b \in H^m(\Omega B; \mathbb{Z})$  of infinite order. We proceed by cases.

**Case A:** If  $n$  and  $m$  are both even, then  $a^2, b^2$  can be seen to be of infinite order, since  $H^*(\Omega A; \mathbb{Q})$  contains  $\mathbb{Q}[\bar{a}]$  as a subalgebra, where  $\bar{a}$  is the image of  $a$  under rationalization, and similarly  $\mathbb{Q}[\bar{b}]$  is a subalgebra of  $H^*(\Omega B; \mathbb{Q})$ . Then  $(a \otimes b)^2$  has infinite order in  $H^*(\Omega A \wedge \Omega B; \mathbb{Z})$ , since  $(\bar{a} \otimes \bar{b})^2$  is nonzero in  $H^*(\Omega A \wedge \Omega B; \mathbb{Q})$  and Theorem 3.9 part (2) applies. Here we use the Künneth theorem to embed  $H^*(\Omega A; \mathbb{Z}) \otimes H^*(\Omega B; \mathbb{Z})$  in  $H^*(\Omega A \wedge \Omega B; \mathbb{Z})$  as a submodule.

**Case B:** If  $n$  is even and  $m$  is odd, then  $\text{conn}_{\mathbb{Q}}(\Omega A \wedge \Omega B) = n + m - 1$  and by the Hurewicz theorem  $\pi_{n+m}(\Omega A \wedge \Omega B) \rightarrow H_{n+m}(\Omega A \wedge \Omega B)$  is an isomorphism, with  $n + m$  odd, so part (1) of Theorem 3.9 applies.

**Case C:** Suppose  $n$  and  $m$  are both odd, and without loss of generality assume  $n \leq m$ . Since  $\text{conn}_{\mathbb{Q}}(\Omega A \wedge \Omega B) = n + m - 1$  the rational Hurewicz homomorphism  $\pi_{2n+m} \otimes \mathbb{Q} \rightarrow H_{2n+m}(-; \mathbb{Q})$  is an isomorphism by the rational Hurewicz theorem. Since  $n$  and  $m$  are odd,  $2n + m$  is odd, while  $\pi_{2n+m}(\Omega A \wedge \Omega B) \otimes \mathbb{Q} \neq 0$ , and so part (1) of Theorem 3.9 applies. □

We note that if the equivalence (3.2) can be chosen to be an equivalence of H-spaces (which need not be the case in general) we can actually apply the full force of part (2) of Proposition 3.1 to make inferences on the Gray indices attained by phantom maps into  $Y \simeq A \vee B$  based on similar knowledge about  $A, B$  and  $(\Omega A) * (\Omega B)$ .

Here we record the dual to Theorem 3.6, whose proof is dual to that given above. We intend to make use of this result to pursue a dual program of study in a future work.

**Theorem 3.10.** *Suppose  $X$  has a product decomposition  $X \simeq A \times B$  where either*

- (1)  *$A$  or  $B$  is the domain of essential phantom maps to finite type targets, or*
- (2) *the aforementioned decomposition is rationally nontrivial,*

*then  $X$  is the domain of essential phantom maps to finite type targets.*

# Chapter 4

## Phantom Maps into Spaces Meeting Finiteness Conditions

### 4.1 Phantoms Maps into $\mathcal{F}$ -Finite Spaces

In this section  $\mathcal{F}$  denotes the class of finite type wedges of spheres. We study  $\mathcal{F}$ -finite spaces  $Y$ . One way to view  $\mathcal{F}$ -finite spaces is as finite generalized CW complexes; in other words an  $\mathcal{F}$ -finite space  $Y$  can be built from a point in finitely many stages by attaching cells of possibly different dimensions at each stage.

We prove the following generalization of Theorem 1.1 and partial generalization of Theorem 1.3.

**Theorem 4.1.** (1) *If  $Y$  is  $\mathcal{F}$ -finite and  $\pi_1(Y)$  is finite, then  $Y$  is the target of essential phantom maps if and only if  $Y$  is rationally nontrivial. Moreover, if  $\text{conn}_{\mathbb{Q}}(Y) = n$ , then  $Y$  is the target of phantom maps of Gray index  $n$ .*

(2) *If, in addition, the rationalization of  $Y$  is a co- $H$ -space, then  $Y$  is the target of phantom maps of Gray index  $k$  if and only if  $\pi_{k+1}(Y) \otimes \mathbb{Q} \neq 0$ .*

*Proof.* According to Iriye [22], if  $Y$  is  $\mathcal{F}$ -finite, then so too is  $Y_{(p)}$ . Also, since  $Y \rightarrow Y_{(p)}$  is a rational equivalence,  $\text{conn}_{\mathbb{Q}}(Y) = \text{conn}_{\mathbb{Q}}(Y_{(p)})$ . So, in light of Theorem 2.1 we can replace  $Y$  with its  $p$ -localization, though we will not burden the notation with this assumption. For brevity, we will write  $K_n$  for  $K(\mathbb{Z}, n)$ .

(1) **Case I:** Assume  $\text{conn}_{\mathbb{Q}}(Y) = 2n$ . By Theorem 1.30 we have a rational equivalence  $Y \rightarrow Y'$ , and maps  $S^{2n+1} \rightarrow Y', Y' \rightarrow S^{2n+1}$  whose composite  $S^{2n+1} \rightarrow S^{2n+1}$  is a rational equivalence. It follows that the retraction  $Y' \rightarrow S^{2n+1}$  induces a surjection on  $\pi_* \otimes \mathbb{Q}$ . So, Theorem 1.17 implies  $\text{Ph}(K_{2n}, Y') \rightarrow \text{Ph}(K_{2n}, S^{2n+1}) \neq *$



is a surjection, and hence  $\text{Ph}(K_{2n}, Y') \neq *$ . This same theorem and the existence of a weak equivalence  $Y \rightarrow Y'$  implies  $\text{Ph}(K_{2n}, Y) \neq *$ . What's more, if  $\varphi \in \text{Ph}(K_{2n}, Y)$  maps to  $\psi$  in  $\text{Ph}(K_{2n}, S^{2n+1})$ , then  $G(\varphi) \leq G(\psi)$  by Proposition 1.19, while  $G(\psi) = 2n$ , since  $\pi_*(S^{2n+1}) \otimes \mathbb{Q}$  is concentrated in dimension  $2n + 1$ . Finally, since  $\text{conn}_{\mathbb{Q}}(Y) = 2n$  we infer  $G(\varphi) \geq 2n$  and so we must have  $G(\varphi) = 2n$ .

**Case II:** For the case  $\text{conn}_{\mathbb{Q}}(Y) = 2n - 1$  we take a map  $Y \xrightarrow{\xi} BU(n)$  inducing a surjection on  $\pi_{2n} \otimes \mathbb{Q}$  guaranteed to exist by Theorem 1.30. Then by Theorem 1.25,  $\xi$  induces a surjection

$$\begin{array}{c} \text{Ph}^{2n-1}(K_{2n-1}, Y) / \text{Ph}^{2n}(K_{2n-1}, Y) \\ \downarrow \\ \text{Ph}^{2n-1}(K_{2n-1}, BU(n)) / \text{Ph}^{2n}(K_{2n-1}, BU(n)), \end{array}$$

and by Theorem 1.29 the target of this map is not the one-point set. So,  $Y$  is the target of phantom maps of Gray index  $2n - 1$ .

(2) We construct comparison map  $Y \rightarrow Z$  with the following properties:

- $Y \rightarrow Z$  induces a surjection on  $\pi_{k+1} \otimes \mathbb{Q}$ , and
- $Z$  is  $\mathcal{F}$ -finite, and  $\text{conn}_{\mathbb{Q}}(Z) = k$  (in particular  $\pi_{k+1}(Z) \otimes \mathbb{Q} \neq 0$ ).

Then, by part (1) and Theorem 1.25 we obtain the result. We now set about constructing the space  $Z$  and the map  $Y \rightarrow Z$ . A first step is found in the theory of homology decompositions as introduced in [6] and as implemented in [21].

Decompose  $Y = \text{colim } L_k$  where each  $L_k$  is a subcomplex of  $Y$  with  $L_k \rightarrow Y$  a  $k$ -skeleton and  $L_k$  of dimension  $k$ . Such a decomposition exists by the theory of homology decompositions. From the same theory we know we can find subcomplexes  $L_{k-1} \subseteq M_k \subseteq L_k$  with  $M_k$  a  $k$ -dimensional subcomplex of  $Y$  and the inclusion  $M_k \rightarrow Y$  is a rational  $k$ -skeleton. Examining the long exact sequence in homology induced by the cofiber sequence  $M_k \rightarrow Y \rightarrow Y/M_k = Z$  reveals that the map  $Y \rightarrow Z$  induces isomorphisms on rational homology above dimension  $k$ . We claim that this implies  $Y \rightarrow Z$  induces a surjection on  $\pi_{k+1} \otimes \mathbb{Q}$ .

To verify, consider the diagram

$$\begin{array}{ccc} \pi_{k+1}(Y) \otimes \mathbb{Q} & \xrightarrow{\text{Hur}} & H_{k+1}(Y; \mathbb{Q}) \\ \downarrow & & \downarrow \\ \pi_{k+1}(Z) \otimes \mathbb{Q} & \xrightarrow{\text{Hur}} & H_{k+1}(Z; \mathbb{Q}). \end{array}$$

The right map is an isomorphism by the preceding discussion, while the bottom map is an isomorphism by the rational Hurewicz theorem. To show the left map is surjective, it suffices to show the top map is surjective. This follows from the observation that  $Y$  is rationally a wedge of spheres, so we can easily see that the generators of  $H_{k+1}(Y; \mathbb{Q})$  are in the image of the Hurewicz map. In fact, the rational Hurewicz homomorphism  $\pi_*(Y) \otimes \mathbb{Q} \rightarrow H_*(Y; \mathbb{Q})$  is a surjection if and only if the rationalization of  $Y$  is a co-H-space.  $\square$

**Remark 4.2.** *We can prove (2) without the hypothesis regarding a co-H-structure on the rationalization of  $Y$  for low Gray indices using the rational Hurewicz theorem. Suppose  $Y$  is any  $\mathcal{F}$ -finite space with  $\text{conn}_{\mathbb{Q}}(Y) = n - 1$ . Then  $\text{Hur} : \pi_{k+1}(Y) \otimes \mathbb{Q} \rightarrow H_{k+1}(Y; \mathbb{Q})$  is surjective for  $k \leq 2n$  by the rational Hurewicz theorem, and so for each  $k$  in the range  $n \leq k \leq 2n$  for which  $\pi_{k+1}(Y) \otimes \mathbb{Q} \neq 0$  we can see  $Y$  is the target of phantom maps of Gray index  $k$  by the methods of the proof just given.*

## 4.2 Phantom Maps to Co-H-Spaces

Iriye's theorem, Theorem 1.2, on phantom maps into suspension spaces leads us to wonder:

**Question 4.3.** *If  $\text{cat}(Y) = 1$ , is  $Y$  the target of essential phantom maps if and only if  $Y$  is not rationally trivial?*

As evidenced in Section 4.1, knowledge of the Gray indices attained by phantom maps between spaces can be valuable in computations. So, we are also led to wonder

**Question 4.4.** *What Gray indices are attainable by a phantom map into a space  $Y$  with  $\text{cl}(Y) = 1$ , or  $\text{cat}(Y) = 1$ ?*

Our main results give a largely positive answer to Question 4.3. We also begin to study Question 4.4. Our discoveries show that we cannot expect Tsakanikas's theorem, Theorem 1.3, to hold if we replace  $Y$  with a co-H-space, and motivate the definition of a "new" homotopy invariant of spaces in Section 4.4.

First we study rationally nontrivial co-H-spaces  $Y$  that do not have the rational homotopy type of a single sphere; another way to characterize such spaces is by requiring  $\dim_{\mathbb{Q}} \tilde{H}_*(Y; \mathbb{Q}) \geq 2$ . We use the machinery of Section 1.7 to decompose such spaces in such a way that our observations from Chapter 3 can be applied.

**Theorem 4.5.** *If  $Y$  is a simply connected, finite type  $\dim_{\mathbb{Q}} \tilde{H}_*(Y; \mathbb{Q}) \geq 2$ , then  $Y$  is the target of essential phantom maps from finite type domains.*

*Proof.* We begin by choosing a suitable prime at which to localize; we need to localize for two reasons - to obtain the geometric realizations of the minimal decomposition of tensor algebras of Section 1.7, and we will also need to kill off torsion in homology below a certain point. This second feat is approachable because  $H_{\leq N}(Y; \mathbb{Z})$  is known to be a finitely generated Abelian group for each  $N$ , and so has torsion at only finitely many primes. Since we can kill off torsion below any level  $N$  we like, we do not feel compelled to be particularly efficient about this; we may remove more torsion than is necessary from the homology of  $Y$ , but by the observations in Chapter 2 show that this will not adversely affect our search for phantom maps.

Let  $m = \text{conn}_{\mathbb{Q}}(Y)$  and choose  $n$  by

- $n = m$  if  $\dim H_m(Y; \mathbb{Q}) \geq 2$ , or
- $n = \text{conn}_{\mathbb{Q}}(Y/Y_m)$  otherwise.

Choose a prime  $p \geq 5$  so  $H_{\leq 2m+n+3}(Y; \mathbb{Z})$  has no  $p$ -torsion; this choice will be justified shortly. We replace  $Y$  with its  $p$ -localization for two reasons; this will not negatively impact our hunt for essential phantom maps in light of the natural surjection  $\text{Ph}(X, Y) \rightarrow \prod_p \text{Ph}(X, Y_{(p)})$ , and we want to leave localization out of the notation.

We examine the geometric realization of the minimal decomposition,

$$\begin{aligned} \Omega Y &\simeq A \times B \\ &\simeq A \times \left( \bigvee_{n \geq 2} Q_n \right) \end{aligned}$$

where  $A = A^{\min}(Y)$ ,  $B = B^{\max}(Y)$  and  $Q_i = Q_i B^{\max}(Y)$ . The notation is from Section 1.7.

Our goal at this point is to demonstrate that  $Q_2$  and  $Q_3$  are rationally non-trivial. Recall that by Theorem 1.36, the space  $\Sigma^{i-1}Q_i$  is naturally a retract of  $Y^{\wedge i}$  for each  $i$ . So, since  $H_{\leq 2m+n+3}(Y; \mathbb{Z})$  is torsion free, so too are

$$H_{\leq 2m+n+2}(Q_2; \mathbb{Z}) \quad \text{and} \quad H_{\leq 2m+n+1}(Q_3; \mathbb{Z}).$$

So, if we can show  $H_{\leq 2m+n+2}(Q_2; \mathbb{F}_p) \neq 0$  and  $H_{\leq 2m+n+1}(Q_3; \mathbb{F}_p)$  are nonzero, it will follow that  $Q_2$  and  $Q_3$  are rationally nontrivial.

Choose  $x \in H_{m+1}(Y; \mathbb{Z})$  where  $\text{conn}_{\mathbb{Q}}(Y) = m$  and  $y \in H_{n+1}(Y; \mathbb{Z})$  with  $n$  and  $m$  as above; if  $n = m$  choose  $x$  and  $y$  to be linearly independent in  $H_{m+1}(Y; \mathbb{Z})$ , otherwise  $x$  and  $y$  are linearly independent for dimension reasons. Write  $a, b$  for the desuspensions of  $x$  and  $y$  in  $\Sigma^{-1}(Y; \mathbb{Z}) = W$ . We can naturally identify  $a \in H_m(\Omega Y; \mathbb{Z})$  and  $b \in H_n(\Omega Y; \mathbb{Z})$  through the inclusion

$$\Sigma^{-1}\tilde{H}_*(Y; \mathbb{Z}) \rightarrow H_*(\Omega Y; \mathbb{Z}).$$

Now  $[a, b]$  and  $[[b, a], a]$  are nontrivial in  $T(W)$  since  $a$  and  $b$  are linearly independent in  $W$ . Recall the Pontryagin product

$$H_*(\Omega Y) \otimes H_*(\Omega Y) \rightarrow H_*(\Omega Y).$$

is induced by the loop multiplication  $\Omega Y \times \Omega Y \rightarrow \Omega Y$ , and so

$$[a, b] \in H_{m+n}(\Omega Y; \mathbb{Z}), \quad [[b, a], a] \in H_{2m+n}(\Omega Y; \mathbb{Z}).$$

Since  $H_{\leq 2m+n}(\Omega Y; \mathbb{Z})$  is torsion free by our choice of  $p$ , we can be sure  $[a, b]$  and  $[[b, a], a]$  are of infinite (additive) order. (We could also see this by appealing

to representation theory, but since an algebraic-topological proof is available, we choose this one!)

Write  $\bar{a}, \bar{b}$  for the mod  $p$ -reductions of  $a$  and  $b$  in

$$V = \Sigma^{-1} \tilde{H}_*(Y; \mathbb{F}_p) \subseteq H_*(\Omega Y; \mathbb{F}_p).$$

We can assume  $\bar{a}, \bar{b} \neq 0$ , for if  $a$  and  $b$  are in the image of the degree  $p$  map  $H_*(\Omega Y) \rightarrow H_*(\Omega Y)$  we can just replace  $a$  with a different element of the infinite group  $H_m(\Omega Y; \mathbb{Z})$  (which contains a copy of  $\mathbb{Z}_{(p)}$  by our choice of  $m$ ) and similarly for  $b$ .

We note that since neither 2 nor 3 is a power of  $p \geq 5$ , by Theorem 1.33

$$L_2(V), L_3(V) \subseteq B^{\max}(V) \cong H_*(B; \mathbb{F}_p),$$

and so

$$[\bar{a}, \bar{b}] \in H_{n+m}(B; \mathbb{F}_p) \quad \text{and} \quad [[\bar{b}, \bar{a}], \bar{a}] \in H_{2m+n}(B; \mathbb{F}_p).$$

Finally, recall that in the minimal decomposition  $T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$  the algebra generators  $V$  of  $T(V)$  are contained entirely in  $A^{\min}(V)$ . For tensor length reasons, then, we can see that  $[\bar{a}, \bar{b}]$  is irreducible in  $B^{\max}(V)$ ;  $B^{\max}(V)$  is generated as an algebra by elements of  $\otimes$ -length at least two, and the product in  $B^{\max}(V)$  is inherited from the product in  $T(V)$  - i.e. multiplication in  $B^{\max}(V)$  is concatenation of tensors. In particular, multiplication in  $B^{\max}(V)$  is a graded product with respect to the grading on  $B^{\max}(V)$  induced by the tensor length grading on  $T(V)$ . Then since  $[\bar{a}, \bar{b}]$  has  $\otimes$ -length 2 in  $T(V)$ , if  $[\bar{a}, \bar{b}] = rc$  for some  $r, c \in T(V)$  we must have  $|r|_{\otimes} + |c|_{\otimes} = 2$  which implies, without any real loss of generality, that  $|r|_{\otimes} = 0$  and  $|c|_{\otimes} = 2$ . Here  $|\cdot|_{\otimes}$  is standing for the tensor length. So  $r \neq 0 \in \mathbb{F}_p$  since  $[\bar{a}, \bar{b}] \neq 0 \in T(V)$  and it follows that  $r$  is a unit in  $\mathbb{F}_p$  and  $[\bar{a}, \bar{b}]$  and  $c$  are associates. In particular,  $[\bar{a}, \bar{b}]$  is not in the image of  $IB^{\max}(V) \otimes IB^{\max}(V) \rightarrow IB^{\max}(V)$  where  $IM$  denotes the augmentation ideal of the Hopf-algebra  $M$ . It follows that the image of  $[\bar{a}, \bar{b}]$  in  $Q_2 B^{\max}(V)$  is nonzero. But then  $[\bar{a}, \bar{b}] \in Q_2 B^{\max}(V) \cong \Sigma^{-1} \tilde{H}_*(Q_2; \mathbb{F}_p)$ , and so

$$H_{m+n+1}(Q_2; \mathbb{F}_p) \neq 0.$$

By a similar argument,

$$H_{2m+n+1}(Q_3; \mathbb{F}_p) \neq 0.$$

It follows from our choice of  $p$  above that  $Q_2$  and  $Q_3$  are rationally nontrivial spaces.

Consequently  $\bigvee_{n \geq 2} Q_i$  is the target of essential phantom maps by Theorem 3.6. It follows that  $Y$  is also the target of essential phantom maps by Proposition 3.1, since

$$\Omega Y \simeq A \times \Omega \left( \bigvee_{n \geq 2} Q_i \right).$$

□

In case  $\dim_{\mathbb{Q}} \widetilde{H}_*(Y; \mathbb{Q}) = 1$  we are unable to produce the nontrivial commutators required to appeal to Theorem 1.33, and so the proof given above cannot be applied.

Our next main result detects phantoms to co-H-spaces rationally equivalent to a single even dimensional sphere, and gives computations of Gray indices attained by phantom maps into co-H-spaces based on their rational cohomology groups.

**Theorem 4.6.** *Suppose  $Y$  is a co-H-space of finite type over  $\mathbb{Z}$  or  $\mathbb{Z}_{(p)}$  for some prime  $p$  with  $H^{2n}(Y; \mathbb{Q}) \neq 0$  for some  $n \geq 1$ . Then  $Y$  is the target of a phantom map of Gray index  $4n - 2$  from a finite type domain.*

We will derive Theorem 4.6 as a consequence of the following computation.

**Proposition 4.7.** *For  $n$  odd or  $n = 2$ ,  $\Sigma K(\mathbb{Z}, n)$  is the target of phantom maps of Gray index  $k$  if and only if  $\pi_{k+1}(\Sigma K(\mathbb{Z}, n)) \otimes \mathbb{Q} \neq 0$  and  $k > n$ .*

Before turning to the proof of Proposition 4.7 we establish two lemmas. First we study adjoints of phantom maps which will be crucial to the proof of Proposition 4.7 in the case  $n = 2$ .

**Lemma 4.8.** *Suppose  $f : \Sigma X \rightarrow Y$  and  $\widehat{f} : X \rightarrow \Omega Y$  are adjoint. Then*

- (1)  *$f$  is phantom if and only if  $\widehat{f}$  is phantom, and*
- (2)  *$G(f)$  is finite if and only if  $G(\widehat{f})$  is finite, in which case  $G(f) = G(\widehat{f}) - 1$ .*

*Proof.* For (1) suppose  $f : \Sigma X \rightarrow Y$  is phantom. Then  $\Sigma X \xrightarrow{f} Y \rightarrow Y^{(n)}$  is trivial for all  $n$ . But then the adjoint  $X \xrightarrow{\widehat{f}} \Omega Y \rightarrow \Omega Y^{(n)}$  is trivial, since  $[\Sigma X, Y^{(n)}] \cong [X, \Omega Y^{(n)}]$  is an isomorphism. Hence  $\widehat{f}$  is phantom. Dually, if  $\widehat{f}$  is phantom, then

$$X_n \hookrightarrow X \xrightarrow{\widehat{f}} \Omega Y$$

is trivial for all  $n$ , and so too is

$$\Sigma X_n \hookrightarrow \Sigma X \xrightarrow{f} Y,$$

so  $f$  is phantom.

For (2) consider the diagrams

$$\begin{array}{ccc}
 & & Y\langle n \rangle \\
 & \nearrow \varphi & \downarrow \\
 \Sigma X & \xrightarrow{f} & Y
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & & \Omega(Y\langle n \rangle) \\
 & \nearrow \hat{\varphi} & \downarrow \\
 X & \xrightarrow{\hat{f}} & \Omega Y.
 \end{array}$$

By (1)  $\varphi$  is phantom if and only if  $\hat{\varphi}$  is phantom. The result then follows from the observation that  $\Omega(Y\langle n \rangle) \simeq (\Omega Y)\langle n - 1 \rangle$ .  $\square$

This next result shows that while  $\pi_{m+1}(\Sigma K(\mathbb{Z}, m)) \otimes \mathbb{Q} \neq 0$ , there are no phantom maps into  $\Sigma K(\mathbb{Z}, m)$  of Gray index  $m$ . In particular, we cannot expect an extension of Theorem 1.3 to allow for  $Y$  to be a co-H-space, or even a suspension space.

**Lemma 4.9.** *Every phantom map into  $\Sigma K(\mathbb{Z}, m)$  has Gray index at least  $2m$ .*

*Proof of Lemma 4.9.* This is a straightforward application of Theorem 1.24 and Gray's principle. Since  $\Sigma K(\mathbb{Z}, m) \xrightarrow{\text{id}} \Sigma K(\mathbb{Z}, m)$  is surjective on  $\pi_{m+1}$  we infer  $\Sigma K(\mathbb{Z}, m)$  is not the target of essential phantom maps of Gray index  $m$ . The result then follows from the observation that the second nonzero rational homotopy group of  $\Sigma K(\mathbb{Z}, n)$  is  $\pi_{2n+1}$ .  $\square$

*Proof of Proposition 4.7.* In the case  $n$  is odd the proof of Proposition 4.7 is incredibly straightforward. For, in this case  $\Sigma K(\mathbb{Z}, n)$  is rationally equivalent to  $S^{n+1}$  which implies  $\pi_*(\Sigma K(\mathbb{Z}, n)) \otimes \mathbb{Q}$  is concentrated in dimensions  $n + 1$  and  $2n + 1$ . By Gray's principle the only finite Gray index permissible to phantom maps into  $\Sigma K(\mathbb{Z}, n)$  is then  $2n$ . The result then follows from Theorem 1.26.

For the case  $n = 2$  there is more work to be done. We will make use of a rational equivalence  $\Omega \Sigma \mathbb{C}P^\infty \xrightarrow{\xi} BU$  extending the natural map  $\mathbb{C}P^\infty = BU(1) \hookrightarrow BU$ . In Example 1.28 we noted that for each odd  $m \geq 3$  there is a phantom map  $K(\mathbb{Z}, m) \rightarrow BU$  of Gray index  $m$ . By Theorem 1.17 such a map lifts through the rational equivalence  $\xi$  and so  $\text{Ph}(K(\mathbb{Z}, m), \Omega \Sigma \mathbb{C}P^\infty) \neq *$ . By Theorem 1.26 it follows that there are phantom maps  $K(\mathbb{Z}, m) \xrightarrow{\varphi} \Omega \Sigma \mathbb{C}P^\infty$  of Gray index  $m$ . The result then follows from the fact that the adjoints  $\Sigma K(\mathbb{Z}, m) \xrightarrow{\hat{\varphi}} \Sigma \mathbb{C}P^\infty$  of the maps  $\varphi$  are phantom maps of Gray index  $m + 1$ .  $\square$

*Proof of Theorem 4.6.* Let  $Y \xrightarrow{g} K(\mathbb{Z}, 2n)$  represent an element of  $H^{2n}(Y; \mathbb{Z})$  of

infinite order. Since  $\text{cat}(Y) = 1$  there is a lift  $\lambda$  in the diagram

$$\begin{array}{ccc} & & \Sigma K(\mathbb{Z}, 2n-1) \\ & \nearrow \lambda & \downarrow p \\ Y & \xrightarrow{g} & K(\mathbb{Z}, 2n). \end{array}$$

Since  $g$  induces a surjection on  $\pi_{2n} \otimes \mathbb{Q}$  and  $p$  induces an isomorphism on  $\pi_{2n}$  we can be sure  $\pi_{2n}(\lambda) \otimes \mathbb{Q}$  is surjective. Since  $\Sigma K(\mathbb{Z}, 2n-1)$  is rationally equivalent to  $S^{2n}$  we have an isomorphism of vector spaces

$$(4.1) \quad \pi_*(\Sigma K(\mathbb{Z}, 2n-1)) \otimes \mathbb{Q} \cong \mathbb{Q} \cdot \alpha \oplus \mathbb{Q} \cdot [\alpha, \alpha],$$

where  $\alpha \in \pi_{2n}(\Sigma K(\mathbb{Z}, 2n-1)) \otimes \mathbb{Q}$  is a nonzero element and  $[-, -]$  denotes the Whitehead product. Since  $\alpha$  is in the image of  $\pi_{2n}(\lambda)$ , it follows from the naturality of the Whitehead product that  $\pi_*(\lambda) \otimes \mathbb{Q}$  is surjective.

By Proposition 4.7 and Lemma 2.2 we can find a space  $X$  with  $\text{conn}_{\mathbb{Q}}(X) = 4n-3$  and a phantom map

$$X \xrightarrow{\varphi} \Sigma K(\mathbb{Z}, 2n-1)$$

of Gray index  $4n-2$ . Since  $\pi_*(\lambda) \otimes \mathbb{Q}$  is surjective, by Theorem 1.17 there is a phantom map  $\psi$  fitting in a homotopy commutative diagram

$$\begin{array}{ccc} & & Y \\ & \nearrow \psi & \downarrow \lambda \\ X & \xrightarrow{\varphi} & \Sigma K(\mathbb{Z}, 2n-1). \end{array}$$

Now,  $G(\psi) \leq 4n-2$  by Proposition 1.19. Since  $\text{conn}_{\mathbb{Q}}(X) = 4n-3$ , we have  $G(\psi) \geq 4n-2$  by Gray's principle, which completes the proof.  $\square$

Theorem 4.6 says nothing about co-H-space  $Y$  with  $\tilde{H}^*(Y; \mathbb{Q})$  concentrated in odd dimensions. This is a consequence of the complexity of  $\Sigma K(\mathbb{Z}, 2n)$  relative to that of  $\Sigma K(\mathbb{Z}, 2n-1)$ . In particular  $\Sigma K(\mathbb{Z}, 2n)$  is rationally equivalent to  $\bigvee_{k \geq 1} S^{2nk+1}$ , which can be seen via the James splitting, and so a map into  $\Sigma K(\mathbb{Z}, 2n)$  inducing a surjection on  $\pi_{2n+1} \otimes \mathbb{Q}$  need not induce a surjection on  $\pi_* \otimes \mathbb{Q}$ . As such, we cannot make use of Theorem 1.17. One might hope to use Theorem 1.25 and the maps  $Y \rightarrow \Sigma K(\mathbb{Z}, 2n)$  lifting representatives of cohomology classes  $Y \rightarrow K(\mathbb{Z}, 2n+1)$ , but in general the connectivity of these lifts is not high enough relative to the Gray indices attained by phantom maps into  $\Sigma K(\mathbb{Z}, 2n)$ ,



in light of Lemma 4.9.

We are left to wonder the following.

**Question 4.10.** *Is a finite type co-H-space  $Y$  which is rationally equivalent to  $S^{2n+1}$  necessarily the target of essential phantom maps from finite type domains?*

A positive answer to this question would confirm the conjecture we made at the outset of this project.

**Conjecture 4.11.** A finite type co-H-space  $Y$  is the target of essential phantom maps from finite type domains if and only if  $Y$  is rationally nontrivial.

In light of the forthcoming Proposition 4.12, Theorem 4.1 and Iriye's theorem (Theorem 1.2) we see that a rationally nontrivial, simply connected finite type co-H-space  $Y$  that is not the target of essential phantom maps must satisfy the following properties:

- $\text{cl}_{\mathcal{F}}(Y) = \infty$ ,
- $\text{cl}(Y) = 2$ ,
- $Y \sim_{\mathbb{Q}} S^{2n+1}$  for some  $n \geq 1$ ,
- $H_*(Y; \mathbb{Z})$  has torsion at each prime  $p$ .

**Proposition 4.12.** *If  $Y \sim_{\mathbb{Q}} S^{2n+1}$  is a co-H-space and  $H_*(Y; \mathbb{Z})$  has no  $p$ -torsion for some prime  $p$ , then  $Y$  is the target of essential phantom maps of Gray index  $2n$  from finite type domains.*

*Proof.* Localizing at  $p$  we find  $Y_{(p)} \simeq S_{(p)}^{2n+1} \vee T$  where  $T$  is some  $p$ -local torsion space. The result then follows from Theorem 3.6, Proposition 1.19 and the observation that  $S_{(p)}^{2n+1}$  is the target of essential phantom maps of Gray index  $2n$ .  $\square$

### 4.3 Phantom Maps into the Spaces of Ganea

The results of this section should be thought of as a first step along the way to studying phantom maps into spaces of finite LS category. Here  $G_m(Y)$  denotes the  $m$ th space of Ganea associated with the space  $Y$ , as introduced in Section 1.6. We recall that the spaces  $G_m(Y)$  can be thought of as prototypes for spaces having LS category  $\leq m$ .

**Theorem 4.13.** *For a finite type space  $Y$  the following are equivalent*

- (1)  $G_m(Y)$  is the target of essential phantom maps,
- (2)  $G_m(Y)$  is rationally nontrivial,
- (3)  $Y$  is rationally nontrivial.

*Proof.* We first note that  $G_m(Y)$  is rationally nontrivial if and only if  $Y$  is rationally nontrivial. Suppose  $Y$  is rationally nontrivial. Then  $F_m(Y) = (\Omega Y)^{*(m+1)}$  is a rationally nontrivial, finite type suspension space, hence is the target of essential phantom maps by Theorem 1.2. Then applying Proposition 3.1 to the splitting

$$\Omega G_m(Y) \simeq \Omega Y \times \Omega F_m(Y)$$

completes the proof. □

At the onset of this project we had hoped that we could use maps  $Y \rightarrow G_m(K(\mathbb{Z}, n+1))$  with  $\text{cat}(Y) = m$ ,  $\text{conn}_{\mathbb{Q}}(Y) = n$  and either Theorem 1.17 or Theorem 1.25 to locate essential phantom maps into  $Y$ . But, unlike in the case  $m = 1$ , a map into  $G_m(K(\mathbb{Z}, n+1))$  with  $m > 1$  that is onto on  $\pi_{n+1} \otimes \mathbb{Q}$  need not induce a surjection on  $\pi_* \otimes \mathbb{Q}$ , regardless of the parity of  $n$ . In other words, the algebra  $\pi_*(G_m(K(\mathbb{Z}, n+1))) \otimes \mathbb{Q}$  is not cyclic if  $m > 1$ . One might endeavor to implement the more finely-tuned Theorem 1.25, but in general the connectivity of the maps  $Y \rightarrow G_m(K(\mathbb{Z}, n+1))$  is not high enough relative to the Gray indices attained by phantom maps into this target space in light of the following generalization of Lemma 4.9. To see this as a generalization, we remind the reader  $G_1(K(\mathbb{Z}, n+1)) \simeq \Sigma K(\mathbb{Z}, n)$ .

**Lemma 4.14.** *Every phantom map to  $G_m(K(\mathbb{Z}, n+1))$  has Gray index at least  $mn + m + n - 1$ .*

*Proof.* The map  $\Sigma K(\mathbb{Z}, n) \simeq G_1(K(\mathbb{Z}, n+1)) \rightarrow G_m(K(\mathbb{Z}, n+1))$  is an isomorphism on  $\pi_{n+1} \otimes \mathbb{Q}$ , so by Theorem 1.24,  $G_m(K(\mathbb{Z}, n+1))$  is not the target of phantom maps of Gray index  $n$ . The result then follows from Gray's principle, as in the proof of Lemma 4.9. □

In many cases we are able to completely determine the list of Gray indices attained by phantom maps into  $G_m(K(\mathbb{Z}, n+1))$ .

**Proposition 4.15.** *For  $n$  odd  $G_m(K(\mathbb{Z}, n+1))$  is the target of phantom maps of Gray index  $k$  if and only if  $k > n+1$  and  $\pi_{k+1}(G_m(K(\mathbb{Z}, n+1))) \otimes \mathbb{Q} \neq 0$ .*

*Proof.* For odd  $n$  the rational homotopy of  $G_m(K(\mathbb{Z}, n+1))$  is concentrated in two dimensions;  $\Omega G_m(K(\mathbb{Z}, n+1)) \simeq K(\mathbb{Z}, n) \times \Omega \Sigma^m K(\mathbb{Z}, n)^{\wedge(m+1)}$ , and so when  $n$  is odd  $\Omega G_m(K(\mathbb{Z}, n+1))$  is rationally  $S^n \times \Omega S^{(m+1)n+m}$ . Note  $(m+1)n+m = m(n+1) + n$  is odd whenever  $n$  is odd. It follows that for all  $m$  and for  $n$  odd  $\pi_*(G_m(K(\mathbb{Z}, n+1))) \otimes \mathbb{Q}$  is concentrated in dimensions  $n+1$  and  $m(n+1) + n$ . The result then follows from Gray's principle.  $\square$

Further analysis of the existence of phantom maps into spaces of finite LS category is merited; the collection of such spaces includes many Lie groups, Grassmanians, Stiefel manifolds, and other interesting spaces. Many of the rationally nontrivial spaces of finite Lusternik-Schnirelmann category with which we are familiar, such as  $BU(n)$ , and real and complex Grassmanians, have been identified as the targets of essential phantom maps from finite type domains, and we have yet to encounter an example of such a space into which all phantom maps from finite type domains vanish, so we are led to ask the following.

**Question 4.16.** *Suppose  $\text{cat}(Y)$  is finite. If  $Y$  is not rationally trivial, is  $Y$  the target of essential phantom maps from finite type domains?*

## 4.4 The Homotopy Invariants MIG and MOG

Lemmas 4.9 and 4.14 motivate the definition of a new homotopy invariant of finite type spaces, which we call the **minimal inbound Gray index** of a space  $Y$ , written  $\text{MIG}(Y)$ . If  $Y$  is the target of essential phantom maps, then  $\text{MIG}(Y)$  is the minimal Gray index attained by a phantom map into  $Y$ . This is well defined by Theorem 1.26. If  $Y$  is not the target of essential phantom maps set  $\text{MIG}(Y) = \infty$ . Perhaps a cleaner way to say this is

$$\text{MIG}(Y) = \inf\{G(\varphi) \mid \varphi \text{ is a phantom map into } Y\}.$$

Since the Gray index is a homotopy invariant of a phantom map, it is clear that  $\text{MIG}$  is a homotopy invariant of a space  $Y$ . We could similarly define the minimal outbound Gray index  $\text{MOG}(Y)$  of a space  $Y$ .

As a consequence of Gray's principle, we derive the following inequalities.

$$(4.2) \quad \text{MIG}(Y) \geq \text{conn}_{\mathbb{Q}}(Y) \quad \text{and} \quad \text{MOG}(Y) \geq \text{conn}_{\mathbb{Q}}(Y) + 1.$$

We have shown in Section 4.1 that  $\text{MIG} = \text{conn}_{\mathbb{Q}}$  on the collection of  $\mathcal{F}$ -finite spaces. But, Lemma 4.9 shows the first inequality in (4.2) can be strict. One might wonder just how far  $\text{MIG}(Y)$  can deviate from  $\text{conn}_{\mathbb{Q}}(Y)$ . It appears as though the answer to this question depends on the LS category and rational connectivity of the space  $Y$ . Our main result on this topic is as follows.

**Theorem 4.17.** *If  $m \geq 2$  and  $\text{conn}_{\mathbb{Q}}(Y) = n$  then  $n \leq \text{MIG}(G_m(Y)) \leq nm + n + m - 1$ .*

This is a consequence of the fact that  $\text{MIG}$  stabilizes rapidly to rational connectivity, in the following sense.

**Theorem 4.18.** *For any  $m \geq 2$  and any finite type  $X$ ,*

$$\text{MIG}(\Sigma^2 X) = \text{conn}_{\mathbb{Q}}(\Sigma^2 X).$$

Before proving this theorem we establish a preliminary result.

**Lemma 4.19.** *For every  $n$ ,  $\text{MIG}(\Sigma^2 K(\mathbb{Z}, n)) = n + 1$ .*

*Proof.* We argue by the parity of  $n$ . For  $n$  odd  $\Sigma^2 K(\mathbb{Z}, n)$  is rationally equivalent to  $S^{n+2}$ , so  $\pi_*(\Sigma^2 K(\mathbb{Z}, n)) \otimes \mathbb{Q}$  is concentrated in dimension  $n + 2$ . By Iriye's theorem we know  $\Sigma^2 K(\mathbb{Z}, n)$  is the target of essential phantom maps, and by Theorem 1.26 we infer there are phantoms into  $\Sigma^2 K(\mathbb{Z}, n)$  of finite Gray index. By Gray's principle, these phantom maps have Gray index  $n + 1$ .

Now suppose  $n$  is even. Combining Gray's principle with Theorem 1.26 shows that it suffices to show

$$\text{Ph}(K(\mathbb{Z}, n+1), \Sigma^2 K(\mathbb{Z}, n)) \neq *.$$

By Theorem 1.24 if this is a singleton set, then there is a rational equivalence  $\Sigma K(\mathbb{Z}, n+1) \rightarrow \Sigma^2 K(\mathbb{Z}, n)$ , whose adjoint  $K(\mathbb{Z}, n+1) \rightarrow \Omega \Sigma^2 K(\mathbb{Z}, n)$  is nontrivial on  $\pi_{n+1} \otimes \mathbb{Q}$ . But, since  $\Omega \Sigma^2 K(\mathbb{Z}, n) = \Omega \Sigma K(\mathbb{Z}, 1) \wedge K(\mathbb{Z}, n)$  this contradicts Theorem 2.2 of [23] (also recorded herein as Proposition A.2), completing the proof.  $\square$

*Proof of Theorem 4.18.* If  $\text{conn}_{\mathbb{Q}}(X) = n - 1$  then there is a map  $X \rightarrow K(\mathbb{Z}, n)$  inducing a surjection on  $\pi_n \otimes \mathbb{Q}$ . It follows that  $\Sigma^2 X \xrightarrow{\xi} \Sigma^2 K(\mathbb{Z}, n)$  induces a surjection on  $\pi_{n+2} \otimes \mathbb{Q}$ .

For  $n$  odd this is a surjection on  $\pi_* \otimes \mathbb{Q}$  and so Theorem 1.17 does the job for us, in addition with our Gray index inequalities.

For  $n$  even we need to be more careful. We can find a phantom  $K(\mathbb{Z}, n+1) \rightarrow \Sigma^2 K(\mathbb{Z}, n)$  of Gray index  $n+1$ . By Theorem 1.25 this phantom map is in the image of

$$\xi_* : \text{Ph}(K(\mathbb{Z}, n+1), \Sigma^2 X) \rightarrow \text{Ph}(K(\mathbb{Z}, n+1), \Sigma^2 K(\mathbb{Z}, n))$$

modulo higher Gray index maps. In particular there is an essential phantom map  $\varphi : K(\mathbb{Z}, n+1) \rightarrow \Sigma^2 K(\mathbb{Z}, n)$  lifting through  $\xi$ , and  $\text{Ph}(K(\mathbb{Z}, n+1), \Sigma^2 X) \neq *$ . By Theorem 1.26 and Gray's principle we infer the existence of a phantom map  $K(\mathbb{Z}, n+1) \rightarrow \Sigma^2 X$  of Gray index  $n+1 = \text{conn}_{\mathbb{Q}}(\Sigma^2 X)$ , since  $\tilde{H}^*(K(\mathbb{Z}, n+1); \mathbb{Q})$  is concentrated in dimension  $n+1$ .  $\square$

*Proof of Theorem 4.17.* Suppose  $\text{conn}_{\mathbb{Q}}(X) = n$ . For each  $m$  we have  $\Omega G_m(X) \simeq \Omega X \times \Omega F_m(X)$  where  $F_m(X) \simeq (\Omega X)^{*(m+1)}$  is an  $m$ -fold suspension of rational connectivity  $nm + n + m - 1$ . The result then follows from Proposition 3.1.  $\square$

Before closing this section, we make a note on some interesting interplay between the invariant MIG studied above and its dual, MOG, defined at the beginning of the section. This gives additional motivation for the study of MOG, which will be carried out elsewhere.

**Theorem 4.20.** *Suppose  $\text{conn}_{\mathbb{Q}}(X) = 2n - 1$ . Then*

- (1)  $\text{MOG}(X) > 2n$  implies  $\text{MIG}(X) = 2n - 1$ , and
- (2)  $\text{MIG}(X) > 2n - 1$  implies  $\text{MOG}(X) = 2n$ .

*Proof.* (1) Suppose  $X$  is not the domain of phantom maps having Gray index  $2n$ , i.e.  $\text{Ph}^{2n}(X, Y) = \text{Ph}^{2n+1}(X, Y)$  for all finite type targets  $Y$ . Then by Theorem 1.24 there is a map  $X \rightarrow \prod \Omega S^{2n+1}$  inducing a surjection on  $H^n(-; \mathbb{Q})$ . By pinching off extra factors we can arrange for this to be an isomorphism on  $H^{2n}(-; \mathbb{Q})$ , and hence an isomorphism on  $\pi_{2n} \otimes \mathbb{Q}$  by the Hurewicz theorem. But then this is an isomorphism on  $\pi_* \otimes \mathbb{Q}$ , since the rational homotopy groups of an odd dimensional sphere are concentrated in a single dimension. By Tsakanikas's theorem (Theorem 1.3) there are phantom maps  $K \xrightarrow{\varphi} \prod \Omega S^{2n+1}$  of Gray index  $2n - 1$  with  $K$  of finite type. By Theorem 1.17,  $\varphi$  then lifts through the map  $X \rightarrow \prod \Omega S^{2n+1}$  by a phantom maps  $\psi$  as in the diagram

$$\begin{array}{ccc}
 & & X \\
 & \nearrow \psi & \downarrow \\
 K & \xrightarrow{\varphi} & \prod \Omega S^{2n+1}.
 \end{array}$$

Proposition 1.19 assures us  $G(\psi) \leq G(\varphi)$  and by Lemma 2.2 we can take  $\text{conn}_{\mathbb{Q}}(K) = 2n - 2$  so that  $G(\psi) = 2n - 1$ , and  $\text{MIG}(X) = 2n - 1$ .

(2) Now suppose  $X$  is not the target of phantom maps of Gray index  $2n - 1$ . Then by Theorem 1.24 there is a map  $\bigvee \Sigma K(\mathbb{Z}, 2n - 1) \rightarrow X$  inducing a surjection on  $\pi_{2n} \otimes \mathbb{Q}$ . Pinching off extra summands gives rise to an isomorphism on  $H^{2n}(-; \mathbb{Q})$ , via the Hurewicz theorem, which is then a surjection on  $H^*(-; \mathbb{Q})$  since  $\tilde{H}^*(\Sigma K(\mathbb{Z}, 2n - 1); \mathbb{Q})$  is concentrated in a single dimension. The claim then follows as above from Theorem 1.17, Proposition 1.19 and Lemma 2.2.  $\square$

# Appendix A

## Proof of Theorem 3.9

Here we prove Theorem 3.9, restated here as Theorem A.3, very nearly following Iriye's argument. The sole difference will be the use of a refined version of Iriye's Theorem 2.1, recorded above as Proposition 3.8, which we restate here as Theorem A.1 for the reader's convenience. For the rest of this appendix, and also for the next appendix, we fix a prime  $p$  and assume all spaces and groups such as  $\mathbb{Z}$  have been localized at  $p$ ; that is,  $\mathbb{Z}$  stands for  $\mathbb{Z}_{(p)}$ .

**Theorem A.1.** *If  $Y$  is a finite type  $p$ -local space, and  $\text{Ph}(X, Y) = *$  for all finite type domains  $X$ , then there is a rational equivalence  $\prod K(\mathbb{Z}, m_\beta) \rightarrow \Omega Y$ .*

The proof of Theorem A.3 will center on the following observation, which we generalize in Proposition B.2 to give an alternative proof of the existence of essential phantom maps to the (rationally nontrivial, finite type) spaces of Ganea.

**Proposition A.2.** *[23] Any map  $K(\mathbb{Z}, m) \rightarrow \Omega\Sigma(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, l))$  induces the trivial map on rational homotopy groups.*

**Theorem A.3.** *Suppose  $Y$  is a finite type  $p$ -local space. If either*

- (1) *there is some  $\alpha \in \pi_{2n+1}(Y)$  of infinite order whose image under the Hurewicz map is also of infinite order, or*
- (2) *there is some  $v \in H^{2n}(Y; \mathbb{Z})$  of infinite order whose square  $v^2$  is also of infinite order,*

*then  $\Sigma Y$  is the target of essential phantom maps from finite type domains.*

*Proof.* (1) In this case we have maps  $S^{2n+1} \xrightarrow{\alpha} Y$  and  $Y \xrightarrow{g} K(\mathbb{Z}, 2n+1)$  whose composite

$$S^{2n+1} \xrightarrow{\alpha} Y \xrightarrow{g} K(\mathbb{Z}, 2n+1)$$

is a rational equivalence. Then

$$\Omega S^{2n+2} \xrightarrow{\Omega\Sigma\alpha} \Omega\Sigma Y \xrightarrow{\Omega\Sigma g} \Omega\Sigma K(\mathbb{Z}, 2n+1)$$

is also a rational equivalence.

Suppose  $\text{Ph}(X, \Sigma Y) = *$  for all finite type domains  $X$ . Then by Theorem A.1 there is a rational homotopy equivalence  $\prod K(\mathbb{Z}, m_\beta) \rightarrow \Omega\Sigma Y$ .

Now, we have that

$$(A.1) \quad \begin{aligned} \pi_* \Omega S^{2n+2} \otimes \mathbb{Q} &\cong \pi_{*+1} S^{2n+2} \otimes \mathbb{Q} \\ &\cong \begin{cases} \mathbb{Q} & \text{if } *+1 = 2n+2, 4n+3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathbb{Q} & \text{if } * = 2n+1, 4n+2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So, we infer the existence of a map

$$K(\mathbb{Z}, 4n+2) \longrightarrow \Omega\Sigma Y \longrightarrow \Omega\Sigma K(\mathbb{Z}, 2n+1)$$

that is nontrivial on  $\pi_* \otimes \mathbb{Q}$ .

But we have the identity

$$\Omega\Sigma K(\mathbb{Z}, 2n+1) \simeq K(\mathbb{Z}, 2n+1) \times \Omega\Sigma(K(\mathbb{Z}, 2n+1) \wedge K(\mathbb{Z}, 2n+1)),$$

and so we must have a map

$$K(\mathbb{Z}, 4n+2) \rightarrow \Omega\Sigma(K(\mathbb{Z}, 2n+1) \wedge K(\mathbb{Z}, 2n+1))$$

inducing a nontrivial map on  $\pi_* \otimes \mathbb{Q}$ , contradicting Proposition A.2. The result follows.

For (2), take  $v \in H^{2n}(Y; \mathbb{Z})$  of infinite order with  $v^2$  also of infinite order. Then  $\Omega\Sigma v : \Omega\Sigma Y \rightarrow \Omega\Sigma K(\mathbb{Z}, 2n)$  induces a nontrivial map on  $\pi_{4n} \otimes \mathbb{Q}$ . If we assume  $\text{Ph}(X, \Sigma Y) = *$  for all finite type domains  $X$ , then we obtain a nontrivial map  $K(\mathbb{Z}, 4n) \rightarrow \Omega\Sigma K(\mathbb{Z}, 2n)$ . As above, this contradicts Proposition A.2.  $\square$



# Appendix B

## Alternative Proof of Detection Principles for Phantom Maps into the Spaces of Ganea

Here we present an alternative proof of our principle for detecting phantom maps to the spaces  $G_m(Y)$  of Ganea which parallels Iriye's proof of Theorem A.3. Some of the results contained herein, specifically Propositions B.2 and B.3, may be of independent interest, which motivated the inclusion of this appendix. We should note that these results are generalizations of observations of Iriye in [23] obtained largely by using the fact that Morava  $K$ -theories possess Künneth isomorphisms.

Our goal is to prove the following, with the equivalence of (2) and (3) being apparent.

**Theorem B.1.** *If  $Y$  is a finite type space and  $m \geq 1$  then the following are equivalent:*

- (1)  *$G_m(Y)$  is the target of essential phantom maps from finite type domains if and only if  $Y$  is rationally nontrivial,*
- (2)  *$G_m(Y)$  is rationally nontrivial,*
- (3)  *$Y$  is rationally nontrivial.*

The proof we give here is by no means the most efficient; our most efficient proof is presented in the body of the text. We will appeal to computations of the Morava  $K$ -theories of Eilenberg-MacLane spaces by Ravanel and Wilson, and as such will need to localize at some prime. As in the last section, we fix a prime  $p$  and assume all spaces and groups have been localized at  $p$ . We will continue to write  $\mathbb{Z}$  for  $\mathbb{Z}_{(p)}$ . We will require the following generalization of Proposition A.2.

**Proposition B.2.** *Suppose  $Y$  is a CW complex. Any map*

$$K(\mathbb{Z}, a) \rightarrow \Omega\Sigma(K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c) \wedge Y)$$

*is trivial on rational homotopy groups.*

Iriye's proof of Proposition A.2 hinges on an observation analogous to the following.

**Proposition B.3.** *Any map  $\Omega\Sigma(K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c) \wedge Y) \rightarrow K(\mathbb{Z}, a)$  is trivial on Morava  $K$ -theories.*

Before moving forward with proofs, we record a consequence of Proposition B.3. Recall  $\Omega\Sigma X$  is the free topological monoid on a CW complex  $X$ . According to Milnor [33] we can replace  $K(\mathbb{Z}, a)$  with a homotopy equivalent space that is a topological group, and in particular we can give  $K(\mathbb{Z}, a)$  the structure of a topological monoid. As a result, any map  $Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c) \rightarrow K(\mathbb{Z}, a)$  factors through the counit of the  $(\Sigma, \Omega)$ -adjunction as in

$$\begin{array}{ccc} Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c) & & \\ \downarrow & \searrow & \\ \Omega\Sigma(Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)) & \longrightarrow & K(\mathbb{Z}, a), \end{array}$$

and so we verify the following

**Corollary B.4.** *Every map*

$$Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c) \rightarrow K(\mathbb{Z}, a)$$

*is trivial on  $K(q)_*$ .*

Our proof of Proposition B.3 will largely follow Iriye's proof of a similar result in [23]. We make use of the Künneth isomorphisms in Morava  $K$ -theories to obtain our generalization. One major component of the proof is the computation of the Morava  $K$ -theories  $K(q)_*K(\mathbb{Z}, a)$  by Ravenel and Wilson in [34]. In particular, Ravenel and Wilson have shown that  $K(q)_*K(\mathbb{Z}, a) \neq 0$  for  $q \geq a - 1$ .

For each  $j \in \mathbb{N}$  the short exact sequence  $0 \rightarrow \mathbb{Z} \xrightarrow{p^j} \mathbb{Z} \rightarrow \mathbb{Z}/(p^j) \rightarrow 0$  of groups induces a fiber sequence

$$K(\mathbb{Z}/(p^j), k) \xrightarrow{\delta_j} K(\mathbb{Z}, k+1) \xrightarrow{p^j} K(\mathbb{Z}, k+1) \xrightarrow{\text{red}_j} K(\mathbb{Z}/(p^j), k+1).$$

**Theorem B.5.** [34] *There is an equivalence*

$$\operatorname{colim}_j K(q)_*K(\mathbb{Z}/(p^j), k) \cong K(q)_*K(\mathbb{Z}, k+1),$$

and  $p_*^j : K(q)_*K(\mathbb{Z}, k+1) \rightarrow K(q)_*K(\mathbb{Z}, k+1)$  is epimorphic (hence  $\operatorname{red}_j$  is trivial on  $K(q)_*$  for any  $j, q$ ).

**Lemma B.6.** *If  $Y$  is a finite type target with  $\operatorname{conn}_{\mathbb{Q}}(Y) = a-1$  and  $f : K(\mathbb{Z}, a) \rightarrow Y$  is nontrivial on  $\pi_* \otimes \mathbb{Q}$  then there is a map  $g : Y \rightarrow K(\mathbb{Z}, a)$  so that  $(gf)^* = p^j \in H^a(K(\mathbb{Z}, a); \mathbb{Z}) \cong \mathbb{Z}$  for some  $j > 0$ .*

*Proof.* Note  $f_* : \pi_*(K(\mathbb{Z}, a)) \otimes \mathbb{Q} \rightarrow \pi_*(Y) \otimes \mathbb{Q}$  is nontrivial if and only if  $f_* : \pi_a(K(\mathbb{Z}, a)) \otimes \mathbb{Q} \rightarrow \pi_a(Y) \otimes \mathbb{Q}$  is an injective map of  $\mathbb{Q}$ -vector spaces. If  $\operatorname{conn}_{\mathbb{Q}}(Y) = a-1$ , by the rational Hurewicz theorem  $f$  induces an injection on  $\pi_a$  if and only if  $f_* : H_a(K(\mathbb{Z}, a); \mathbb{Q}) \rightarrow H_a(Y; \mathbb{Q})$  is injective, which is equivalent to  $f$  inducing a surjection  $f^* : H^a(Y; \mathbb{Q}) \rightarrow H^a(K(\mathbb{Z}, a); \mathbb{Q})$ . Since  $H^a(Y; \mathbb{Q}) \cong [Y, K(\mathbb{Q}, a)]$  this implies we can find a map  $\tilde{g} : Y \rightarrow K(\mathbb{Q}, a)$  representing a class in  $H^a(Y; \mathbb{Q})$  so that  $\tilde{g} \circ f = 1 \in H^a(K(\mathbb{Z}, a); \mathbb{Q}) \cong \mathbb{Q}$ .

$$\begin{array}{ccc} K(\mathbb{Z}, a) & \xrightarrow{f} & Y \\ \operatorname{id} \downarrow & & \downarrow \tilde{g} \\ K(\mathbb{Z}, a) & \xrightarrow{r} & K(\mathbb{Q}, a), \end{array}$$

where  $r : K(\mathbb{Z}, a) \rightarrow K(\mathbb{Q}, a)$  is rationalization.

Now,  $\tilde{g} \in H^a(Y; \mathbb{Q}) \cong H_a(Y; \mathbb{Q}) \cong \pi_a(Y) \otimes \mathbb{Q} \cong \bigoplus_1^n \mathbb{Q}$ . So, there is some  $j \geq 0$  so that  $p^j \tilde{g} \in \bigoplus_1^n \mathbb{Z} \subseteq \bigoplus_1^n \mathbb{Q}$ , i.e.  $p^j \tilde{g}$  is an integral cohomology class, which is to say there is some  $g \in H^a(Y; \mathbb{Z})$  that rationalizes to  $p^j \tilde{g}$ , and hence  $g \circ f = p^j \in H^a(K(\mathbb{Z}, a); \mathbb{Z})$ . □

As a consequence of Theorem B.5 we infer that if  $g$  is as in Lemma B.6, then  $K(q)_*g$  is epimorphic, since  $K(q)_*p^j$  is epimorphic and  $K(q)_*p^j = K(q)_*g \circ K(q)_*f$ . In the next section, we will prove Proposition B.3. From this it follows that  $K(q)_*g = 0$ , which then implies  $K(q)_*K(\mathbb{Z}, a) = 0$  for all  $q$ , contradicting Ravenel and Wilson's computation of  $K(q)_*K(\mathbb{Z}, a) \neq 0$  for  $q \geq a-1$ , hence completing the proof of Proposion B.2.

## B.1 Proof of Proposition B.3

Let  $E_r^*(-)$  denote the Bockstein spectral sequence. Since  $E_1^*(-) = H^*(-; \mathbb{F}_p)$  has Künneth isomorphisms, we can show inductively that  $E_{r+1}^*(-) = H(E_r^*(-), \beta_r)$  has Künneth isomorphisms. In the following, we tend to write  $E_r$  for  $E_r^*$  when we have no need to refer to the grading on  $E_r$ .

Iriye has discovered the following relation between the Bockstein spectral sequence and the mod- $p$  (co)homology.

**Lemma B.7.** [23, Iriye] *Let  $f : X \rightarrow Y$  and  $n \in \mathbb{N}$ , and consider the conditions*

(i<sub>n</sub>)  $f_* : H_*(X; \mathbb{Z}) \rightarrow H_*(Y; \mathbb{Z})$  has a left inverse for  $* \leq n$ , i.e.  $f$  induces monomorphisms on  $H_*$  for  $* \leq n$ .

(ii<sub>n</sub>)  $f_* : H_*(X; \mathbb{Z}/(p)) \rightarrow H_*(Y; \mathbb{Z}/(p))$  induces monomorphisms  $f_* : E_r^*(X) \rightarrow E_r^*(Y)$  of the Bockstein spectral sequence for  $* \leq n$  for all  $r$ .

(iii<sub>n</sub>)  $f^* : H^*(Y; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$  has a right inverse for  $* \leq n$ , i.e.  $f$  induces epimorphisms on  $H^*$  for  $* \leq n$ .

(iv<sub>n</sub>)  $f^* : H^*(Y; \mathbb{Z}/(p)) \rightarrow H^*(X; \mathbb{Z}/(p))$  induces epimorphisms  $f^* : E_r^*(Y) \rightarrow E_r^*(X)$  for  $* \leq n$  for all  $r$ .

Then (i<sub>n</sub>)  $\Leftrightarrow$  (ii<sub>n</sub>)  $\Leftrightarrow$  (iv<sub>n</sub>), (i<sub>n</sub>)  $\Rightarrow$  (iii<sub>n</sub>), and (iii<sub>n+1</sub>)  $\Rightarrow$  i<sub>n</sub>.

In the proof of the following result, we will employ May's computation of the mod  $p$  Bockstein spectral sequence of Eilenberg-MacLane spaces in [25].

**Lemma B.8.** *For  $t$  sufficiently large, the map*

$$\begin{array}{c} \Omega\Sigma(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}, c)) \\ \downarrow \Omega\Sigma(1 \wedge 1 \wedge red) \\ \Omega\Sigma(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}/(p^t), c)) \end{array}$$

*induces an epimorphism on  $H^a(-; \mathbb{Z})$ .*

*Proof.* Take  $t > j$ . By Lemma B.7, it suffices to show  $\Omega\Sigma(1 \wedge 1 \wedge red)$  induces monomorphisms on  $E_*^r$  for  $* \leq 2p^{t+1-j}$  for all  $r$ . Now,  $E_*^r(\Omega\Sigma A) \cong T(\widetilde{E}_*^r(A))$ , where  $T(K)$  denotes the tensor algebra generated by a module  $K$  and  $\widetilde{E}^r$  is the Bockstein spectral sequence associated to  $\widetilde{H}(-; \mathbb{Z})$ . So, to show  $\Omega\Sigma(1 \wedge 1 \wedge red)$

induces monomorphisms on  $E_*^r$  for  $* \leq 2p^{t+1-j}$  it suffices to show  $1 \wedge 1 \wedge red$  induces monomorphisms

$$E_*^r(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}, c)) \rightarrow E_*^r(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}/(p^t), c))$$

for  $* \leq 2p^{t+1-j}$ .

Since the Bockstein spectral sequence has Kunneth isomorphisms, we can identify  $E^r(1 \wedge 1 \wedge red)$  with  $E^r(1) \otimes E^r(1) \otimes E^r(red)$ .

Assume for the moment that  $r+1 \leq t$ . Then according to [25],

$$E_{r+1}(K(\mathbb{Z}/(p^t), c)) \cong \mathbb{F}_p[y^{p^{r+1}} | y \in S] \otimes \Lambda[z(y)y^{p^{r+1}-p} | y \in S] \otimes A_{r+1}(c, t),$$

where

$$A_{r+1}(c, t) = \begin{cases} \mathbb{F}_p[i_c] \otimes \Lambda[\beta_t(i_c)] & \text{if } c \text{ is even} \\ \mathbb{F}_p[\beta_t(i_c)] \otimes \Lambda[i_c] & \text{if } c \text{ is odd,} \end{cases}$$

and

$$E_{r+1}(K(\mathbb{Z}, c)) \cong \mathbb{F}_p[y^{p^{r+1}} | y \in S] \otimes \Lambda[z(y)y^{p^{r+1}-p} | y \in S] \otimes A_{r+1}(c, \infty),$$

where

$$A_{r+1}(c, \infty) = \begin{cases} \mathbb{F}_p[i_c] & \text{if } c \text{ is even} \\ \Lambda[i_c] & \text{if } c \text{ is odd.} \end{cases}$$

For  $r+1 \leq t$ , it appears as though  $E_{r+1}(red)$  is surjective (in every degree). So, for  $r+1 \leq t$  we conclude that

$$1 \wedge 1 \wedge red_t : Y \wedge K(\mathbb{Z}/p^j, b-1) \wedge K(\mathbb{Z}, c) \rightarrow Y \wedge K(\mathbb{Z}/p^j, b-1) \wedge K(\mathbb{Z}/p^t, c)$$

induces a surjection on  $E_r$ .

Now suppose  $r+1 > t$ . We hope to show  $E_{r+1}(1 \wedge 1 \wedge red)$  is surjective in degrees  $* \leq 2p^{t-j+1}$ , where  $j \in \mathbb{N}$  is fixed and  $j < t$ . We'll achieve this by showing that  $E_{t+1}^*(K(\mathbb{Z}/p^j, b-1)) = 0$  for  $* \leq 2p^{t-j+1}$ . The result then follows, since  $(A \otimes B)_n = \sum_{i+j=n} A_i \otimes B_j$  for algebras  $A$  and  $B$ . Now,

$$E_{t+1}(K(\mathbb{Z}/p^j, b-1)) = \mathbb{F}_p[y^{p^{t+1}} | y \in S] \otimes \Lambda[z(y)y^{p^{t+1}-p} | y \in S] \otimes A_{t+1}(b-1, j),$$

where

$$A_{t+1}(b-1, j) = \begin{cases} \mathbb{F}_p[i_{b-1}^{p^{t+1-j}}] \otimes \Lambda[z(i_b)i_b^{p^{t+1-j}-p}] & \text{if } b \text{ is even} \\ \mathbb{F}_p & \text{if } b \text{ is odd.} \end{cases}$$

Generators for  $\mathbb{F}_p[y^{p^{t+1}}|y \in S]$  and  $\Lambda[z(y)y^{p^{t+1}-p}|y \in S]$  lie in dimensions at least  $p^{t+1} + p^{t+1} - p = 2p^{t+1} - p$ , and

$$2p^{t+1} - p - 2p^{t+1-j} = 2p^{t+1-j}(p^j - 1) - p > 0,$$

so we conclude  $E_{t+1}^*(K(\mathbb{Z}/p^j, b-1)) = 0$  for  $* \leq 2p^{t+1-j}$ .

We have shown our map induces surjections on  $E_{r+1}$  for  $r+1 \leq t$ , while  $E_{t+1}^*(\Omega\Sigma Y \wedge K(\mathbb{Z}/p^j, b-1) \wedge K(\mathbb{Z}, c)) = 0$  for  $* \leq 2p^{t+1-j}$ , hence our map induces surjections on  $E_{r+1}$  in dimensions  $* \leq 2p^{t+1-j}$  when  $r+1 > t$ . If we choose  $t > a+j-1$ , then Iriye's Lemma proves our map is epimorphic on  $H^a(-; \mathbb{F}_p)$ .  $\square$

Now we are equipped to prove Proposition B.3.

*Proof of Proposition B.3.* Suppose  $g : \Omega\Sigma(Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)) \rightarrow K(\mathbb{Z}, a)$ . In light of the equivalence

$$K(q)_*K(\mathbb{Z}, b) \cong \operatorname{colim}_j K(q)_*K(\mathbb{Z}/(p^j), b-1),$$

from Theorem B.5 and the fact that Morava  $K$ -theories possess Künneth isomorphisms, we make the identification

$$\begin{aligned} \operatorname{colim} K(q)_*(\Omega\Sigma^n(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}, c))) \\ \cong K(q)_*(\Omega\Sigma^n(Y \wedge K(\mathbb{Z}/, b) \wedge K(\mathbb{Z}, c))). \end{aligned}$$

For each  $j$ , define a map  $h_j = g \circ \Omega\Sigma^n(1 \wedge \delta_j \wedge 1)$

$$\Omega\Sigma^n(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}, c)) \xrightarrow{h_j} \Omega\Sigma^n(Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)),$$

so  $K(q)_*g = \operatorname{colim}_j K(q)_*h_j$ . By Lemma B.8 we can choose  $t$  sufficiently large so there is a map

$$h'_t : \Omega\Sigma^n(Y \wedge K(\mathbb{Z}/(p^j), b-1) \wedge K(\mathbb{Z}/(p^t), c)) \rightarrow K(\mathbb{Z}, a)$$

with  $h_j = h'_t \circ \Omega\Sigma^n(1 \wedge 1 \wedge \operatorname{red}_t)$ . It follows that  $h_j$  is trivial on  $K(q)_*$  for all  $j$  since  $\operatorname{red}_t$  is trivial on  $K(q)_*$  for all  $t$  and  $K(q)_*$  has Künneth isomorphisms. Hence  $K(q)_*g$  is trivial, which completes the proof of Proposition B.3, and hence proves Proposition B.2.  $\square$

## B.2 Phantom Maps into the Spaces of Ganea

Next, following Iriye's argument in the proof of Theorem 1.2, we use the Propositions B.2 and B.3 to prove Theorem B.1. The following lemma connects the discussion of the previous section with our study of phantom maps through the rational characterizations of McGibbon and Roitberg in [30].

**Lemma B.9.** *If  $\Omega X \simeq K \times \Omega \Sigma Z$  where  $Z \simeq Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)$  for any  $Y \not\simeq *$ , then  $X$  is the target of essential phantom maps from finite type domains.*

*Proof.* Suppose we have a splitting  $\Omega X \simeq K \times \Omega \Sigma Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)$ . If  $\text{Ph}(K, X) = *$ , for all finite type domains  $X$ , then by Theorem 1.16 there is a rational equivalence  $\prod_{\alpha} K(\mathbb{Z}, m_{\alpha}) \rightarrow \Omega X$ . The composite

$$\prod_{\alpha} K(\mathbb{Z}, m_{\alpha}) \rightarrow \Omega X \simeq K \times \Omega \Sigma Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c) \xrightarrow{\text{pr}_2} \Omega \Sigma Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)$$

induces a surjection on  $\pi_* \otimes \mathbb{Q}$ . Since  $\pi_*(\Omega \Sigma Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)) \otimes \mathbb{Q} \neq 0$ , this implies there is a map  $K(\mathbb{Z}, a) \rightarrow \Omega \Sigma Y \wedge K(\mathbb{Z}, b) \wedge K(\mathbb{Z}, c)$  that is nontrivial on  $\pi_* \otimes \mathbb{Q}$ , where  $a = \text{conn}_{\mathbb{Q}}(Y) + b + c$ . But, this contradicts Proposition B.2.  $\square$

**Lemma B.10.** *If  $\text{conn}_{\mathbb{Q}}(Y) = a - 1$  and  $g : Y \rightarrow K(\mathbb{Z}, a)$  is nontrivial on  $\pi_a \otimes \mathbb{Q}$ , then  $G_n(g)$  is nontrivial on  $\pi_{an+a-1} \otimes \mathbb{Q}$ .*

*Proof.* Recall  $\Omega G_n(Y) \simeq \Omega Y \times \Omega(\Omega Y^{*(n+1)})$  for any space  $Y$ . For brevity, write  $K_m = K(\mathbb{Z}, m)$  for a natural number  $m$ . Consider the diagram

$$\begin{array}{ccc} \Omega G_n(Y) & \xrightarrow{\Omega G_n(g)} & \Omega G_n(K_a) \\ \simeq \downarrow & & \downarrow \simeq \\ \Omega Y \times \Omega(\Omega Y^{*(n+1)}) & \xrightarrow{\Omega g \times \Omega(\Omega g^{*(n+1)})} & K_{a-1} \times \Omega(K_{a-1}^{*(n+1)}). \end{array}$$

Now if  $g$  is surjective on  $\pi_* \otimes \mathbb{Q}$ , then  $\Omega g$  is onto on  $\pi_* \otimes \mathbb{Q}$ . In this case

$$\bigotimes_1^{n+1} (\Omega g)_* : \bigotimes_1^{n+1} H_{a-1}(\Omega Y; \mathbb{Q}) \rightarrow \bigotimes_1^{n+1} H_{a-1}(K_{a-1}; \mathbb{Q})$$

is surjective, since by the Hurewicz theorem  $H_{a-1}(\Omega Y) \cong \pi_{a-1}(\Omega Y)$  and similarly for  $K_{a-1}$ , while the tensor product of surjective homomorphisms is again surjective. Now, using the Künneth isomorphisms we can identify the map  $\bigotimes_1^{n+1} (\Omega g)_*$  with the map

$$(\Omega g^{\wedge(n+1)})_* : H_{(a-1)(n+1)}(\Omega Y^{\wedge(n+1)}; \mathbb{Q}) \rightarrow H_{(a-1)(n+1)}(K_{a-1}^{\wedge(n+1)}; \mathbb{Q}),$$

hence  $\Omega g^{\wedge(n+1)}$  induces a surjection on  $H_{(a-1)(n+1)}(-; \mathbb{Q})$ . From the suspension isomorphisms in homology we infer

$$\Sigma^n(\Omega g^{\wedge(n+1)})_* : H_{an+a-1}(\Sigma^n(\Omega Y^{\wedge(n+1)}); \mathbb{Q}) \rightarrow H_{an+a-1}(\Sigma^n K_{a-1}^{\wedge(n+1)}; \mathbb{Q})$$

is surjective. But then, again by the Hurewicz theorem, this implies

$$\Sigma^n(\Omega g^{\wedge(n+1)})_* \pi_{an+a-1}(\Sigma^n(\Omega Y^{\wedge(n+1)})) \otimes \mathbb{Q} \rightarrow \pi_{an+a-1}(\Sigma^n K_{a-1}^{\wedge(n+1)}) \otimes \mathbb{Q}$$

is a surjection. Now,  $\Sigma^n(\Omega g^{\wedge(n+1)})$  is pointwise homotopy equivalent to  $\Omega g^{*(n+1)}$ , so we have shown that  $\Omega g^{*(n+1)}$  induces a surjection on  $\pi_{an+a-1} \otimes \mathbb{Q}$ .  $\square$

*Proof of Theorem B.1.* Assume  $\text{conn}_{\mathbb{Q}}(Y) = a - 1$ . We will show that  $G_n(Y)$  is the target of an essential phantom map. To this end, suppose to the contrary that  $\text{Ph}(-, G_n(Y)) \equiv *$ . Then by Theorem 1.16 there is a rational equivalence  $\prod K(\mathbb{Z}, m_\alpha) \rightarrow \Omega G_n(Y)$ . Now, from Lemma B.10 we infer the existence of a map  $g : Y \rightarrow K(\mathbb{Z}, a)$  so that  $G_n(g)$  induces a surjection on  $\pi_{an+a-1} \otimes \mathbb{Q}$ . In particular,  $\pi_{an+a-2}(\Omega G_n(Y)) \otimes \mathbb{Q} \neq 0$ , since

$$\pi_{an+a-2}(\Omega G_n(K(\mathbb{Z}, a)) \otimes \mathbb{Q}) \cong \pi_{an+a-2}(\Omega(K(\mathbb{Z}, a-1)^{*(n+1)}) \otimes \mathbb{Q}) \cong \mathbb{Q}.$$

So, there is a factor  $K(\mathbb{Z}, an + a - 2)$  in  $\prod K(\mathbb{Z}, m_\alpha)$ . But then the composite

$$\begin{array}{ccc} K(\mathbb{Z}, an + a - 2) \hookrightarrow \prod K(\mathbb{Z}, m_\alpha) & \longrightarrow & \Omega G_n(Y) \\ & & \Omega G_n(g) \downarrow \\ & & \Omega G_n(K(\mathbb{Z}, a)) \xrightarrow{\text{pr}_2} \Omega \Sigma^n K(\mathbb{Z}, a) \end{array}$$

is nontrivial on  $\pi_{an+a-2} \otimes \mathbb{Q}$ , contradicting Proposition B.2.  $\square$



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