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Option Pricing and Stable Trading Strategies in the Presence of Information Asymmetry

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OPTION PRICING AND STABLE TRADING STRATEGIES IN THE PRESENCE OF INFORMATION ASYMMETRY

by

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Pricing derivatives is one of the central issues in mathematical finance. The seminal work of Black, Scholes and Merton has been the cornerstone of option pricing since their introduction in 1973. Their work influenced the pricing theory of other derivatives as well.

This derivative pricing theory has two primary shortcomings. Firstly, the theoretical pricing in such theories are not accompanied by a stable trading strategy. Secondly, they often assume that the market agents use a uniform model for the underlying instrument and that the market prices of the derivatives reveal all the information about the underlying instrument.

Theoreticians like Grossman and Stiglitz have pointed out that market equilibrium models without considering the role of information dissemination, are often incomplete. On the other hand, traders like Soros, presented an empirical theory, called the theory of reflection, where he conjectures that a swing between a boom and a bust is the market norm.

The aim of this thesis is to develop the theoretical framework and conduct two carefully designed tests to demonstrate that the prevailing pricing techniques are too general to provide guidance for investment practice.

In the first part we provide evidence using a simple and well known trend tracking tool, that there is indeed inefficiencies in the option market that one can take advantage of. We also show that trading strategies that are stable under small

model perturbations are those of pure positions like buying a call, writing a covered call, a vertical spread or pure stock.

In the other part, we focus on the class of optionable biomedical companies with small market capitalization and narrow product focus, which are known for having price jumps whose timings are predictable. We present evidence using an alternative model that it is possible to extract more accurate information on the price movement of the stock from the option prices.

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Chapter 1

Introduction

Numerous financial products are traded in the marketplace everyday. These products range from tangibles to intangibles to equities as well as debts. Given virtually any asset that can be traded, there is a market for its derivatives. A derivative asset is a theoretical asset whose payout depends on the value of the underlying asset. Examples are stock futures, call or put options on a stock, collaterized debt obligations (CDO), credit default swaps (CDS), interest rate swaps and currency swaps to name a few. Despite the long history and wide applications, systematic trading of financial derivatives is relatively recent. In the early 1970's CBOE became the first organized exchange that traded standard option contracts. Since then, financial derivatives have experienced exponential growth in both variety and trading volume. Pricing these assets has become one of the central issues in mathematical finance.

One of the most important and influential paradigms in this area is the method of pricing derivatives using self-financing dynamic hedging portfolios, which has its roots in the Nobel Economic Prize winning works of Black, Scholes [BS73] and Merton [Mer73]. The basic idea is to trade continuously to maintain a self-financed portfolio of the underlying investment instruments such that the payoff of this portfolio replicates the payoff of the derivative in any economic condition. Then the cost of this portfolio would determine the 'fair' price of the derivative at hand. If the market price deviates from this theoretical fair price, then there is a theoretical arbitrage opportunity. Hence, the arbitrageurs will act upon this opportunity to bring the price back to its fair value. It is possible to think of the replicating portfolio as the one that maximizes a certain generic utility function.

There are two parts to the Black-Scholes-Merton pricing method. One involves determining the price and the other is finding and maintaining a replicating portfolio. The pricing part can be derived by the method of risk-neutral measures introduced by Cox, Ross and Rubinstein [CR76, CRR79]. One can view the Cox-Ross-Rubinstein approach as solving the problem dual to that of the utility maximization in the absence of any arbitrage opportunity.

In risk-neutral pricing, one finds a measure that makes the underlying asset price processes martingales. Then any new contingent claim (derivative) can be priced by simply taking the expectation under the martingale (risk-neutral) measure. Cox-Ross-Rubinstein's idea of a martingale measure was further developed into more general settings in the decades to follow [HP81, DS94],

The Black-Scholes-Merton pricing theory and its risk-neutral or martingale measure counterpart have had a profound effect in pricing theories of almost every asset and derivative. It often turns out that the risk-neutral approach is computationally simpler, if one is interested only in the pricing problem and not the accompanying trading strategy.

In practice these pricing methods are also frequently regarded as a mechanism of relating the price dynamics of the underlying asset to the prices of its derivatives. In this way, the market prices of the derivatives are used to estimate the price dynamics of the underlying asset and indirectly used to price other derivatives that are not traded in an exchange.

Although mathematically fascinating and overwhelmingly popular among both theoreticians and practitioners, the history of applying risk-neutral pricing theory has been less glorious. There are several limitations to the Black-Scholes-Merton pricing method. It assumes infinite liquidity for the underlying assets, zero transaction costs, and the ability to trade continuously. These assumptions clearly are never realized in a real market. In practice, approximate portfolios are used which are discretely updated at different frequencies.

Approximations of the 'fair' price that is given by an exact replicating portfolio is fundamentally different from the 'fair' price itself. By the very construction, an exact replicating portfolio guarantees an arbitrage opportunity when the market price deviates from the theoretical value. But, an approximate replicating portfolio is intrinsically risky.

It is important to note that any pricing mechanism implies a corresponding trading strategy designed to derive benefits from that pricing technique. If the market price deviates from the theoretical risk-neutral price, then one can construct a replicating portfolio to gather arbitrage profits. So, any pricing mechanism must go hand in hand with a corresponding trading strategy. This is a feature that is certainly lacking in pricing complex derivatives especially credit derivatives.

There is another inherent assumption in these pricing theories, especially when they are used to imply the price dynamics of the underlying from the market derivative prices. They often assume that all the market participants are using the same model to evaluate the price of the underlying financial instrument and the price of the instrument and its derivatives reflect all the information about the price process. However, uniformity and completeness of knowledge across the market agents is never a reality. If there is even a small disparity in the information held by market agents, they will react differently to the events of the market and thus making the price deviate from any model prediction.

Thus, any imperfection of information across the market agents, will result in a perturbed version of their model. It can be shown that even a small model perturbation can cause an exact replicating portfolio to become excessively risky. It can turn a win-win situation to a loss-loss scenario [Zhu08]. Thus, implementation of a replicating portfolio based pricing and its corresponding arbitrage trading strategy in real financial markets is difficult and such implementations may yield undesired results [Off99]. The dreaded 2008 financial crisis, which is not over yet, was one of the more devastating ramifications of the imperfections in existing pricing theory.

The shortcomings of pricing theories have drawn attention from financial researchers, economists and practitioners. On the theoretical side, Akerlof [Ake70], Grossman, Stiglitz and others [GS76, GS80] have considered information to be an integral part of the price dynamics in any market. They argue that the there is indeed a lack of uniformity in the information held by different market agents. It is apparent that agents holding different information, or beliefs will price assets differently and thus influencing the price dynamics.

We demonstrate the effect of information on a pricing process using an example. In the United States, when automobile insurance is sold to a driver, the underwriter usually prices it using a few demographic variables such as past behavior exhibited by the driver, and engineering features of the automobile in question.

Now, one can ask if it is possible to price the asset without collecting this information. Of course it is possible. But, then the price will be uniform across the clientele. And almost certainly, a professor in a university with three kids will not be happy to pay the same premium as a freshman student. He or she will immediately switch to another insurer who actually uses more information and offers a better rate. In this case the cost paid by an insurer in gathering information is compensated by a larger clientele.

On the other hand, an insurer uses information on only a handful of variables to price the insurance. It is apparent that, not everyone in the same pricing group is equally likely to get into an accident. In theory gathering more detailed information about the client, the insurer can price its policies even better. But, doing so is more expensive and gathering too much information could take the competitive edge away that is gained from a better pricing.

Using a simple model, Grossman and Stiglitz [GS76] showed that the process of a market reaching an equilibrium is through the dissemination of information. They also showed that a similar model also applied to equity trading [GS80]. However, it is practically impossible to distinguish the action of an informed trader from an uninformed one. Thus, directly validating the Grossman and Stiglitz model is difficult. However, we do observe that markets go through cycle of boom and bust. If perfect information were true, this would not happen. Some researchers show using market data that momentum based trading can be illusory in nature and its apparent advantage can be explained by high transaction costs involved [LZ04]. On the other hand some scholars argue that a regime switching model is indeed a better representation of reality [YZZ07].

Successful practitioner George Soros [SorOl] has even formed a theory based on his trading practices which conjectures that the swing between boom and bust is the market norm. This is consistent with the Grossman and Stiglitz model since as a piece of favorable (unfavorable) information disseminates from the informed to the uninformed the price should trend up (down).

Now that the practitioners observations indeed provide indirect evidence to Grossman and Stiglitz' theory of information asymmetry, it is natural to ask what is the implication of asymmetric information to the problem of pricing derivatives?

Our goal in this thesis is to examine two particular situations carefully, where asymmetric information is known to exist. We design a theoretical set up and test it using historical data to provide evidence that the prevailing pricing mechanisms are indeed too general in guiding an investment practice. We summarize our results in Chapters 3, 4 and 5 following preliminaries in Chapter 2.

First, we demonstrate that even the prices of extremely liquid options for S&P-500 index are not information efficient in Chapters 3 and 4. In Chapter 3, we investigate whether one can detect and take advantage of a trend in the price of an equity, using a call option on it. We look for strategies that will take advantage of a trend yet will be relatively stable under small model perturbations. Our analysis yields that, given any risk-reward perspective of an investor, there are two critical thresholds in the premium horizon of a call option on a stock in the up trend. A pure long call position performs the best below a lower threshold, writing a pure covered call yields the best results above the upper threshold and the pure stock position prevails in the middle. We tested our pricing mechanism and the associated trading strategy on the historical data of S&P-500 index. We tested our strategy using a rather simple and traditional moving average crossing trend tracking technique. The results show that using detectable trends in the market one can take advantage of the market option prices. Our analysis yields an alternative pricing mechanism paired with a trading strategy that is consistent with the theory of information asymmetry and reflection theory. This alternative pricing and the corresponding trading strategy is indeed robust and performs better than the market price.

In Chapter 4 we extend the idea of pricing and trading one option on the stock to involve multiple options. The strategy that is stable under small model perturbations is similar to that in the one option case. It is a position that involves one of the following: pure stock, pure long call, pure writing a covered call and a vertical call spread. We conducted tests using market data in the case of using two options. These tests show that the information advantage is more pronounced.

We then show that how information advantage could help in deriving more accurate price distribution of the underlying stocks from option prices. In Chapter 5, we looked at stocks of a particular class of companies - biomedical firms with small market capitalization. Such a biomedical firm usually has a narrow potential medicine or treatment method in its pipeline. Depending on the outcome of a key trial result or an important FDA decision on its potential product, the stock price will jumps up or down abruptly. Close to the jump, the option prices of the stocks of such firms are often highly elevated anticipating the volatility of the price movement. However, existing stock price models such as the standard diffusion model (Black-Scholes [BS73]) or a jump diffusion model (Merton [Mer76b]) do not reflect such behavior well. We use an alternative model involving two diffusion processes with jumps to represent the agents who believe that the stock price will jump up and down, respectively. It turns out such a model anticipates the jump well using the market option prices.

Many of the problems of using traditional derivative pricing methods have been exposed through several financial crises of different magnitudes in the past several decades. There have already been many studies on alternative pricing methods for financial derivatives [CGM01, Con06, HHS07b, HHS07a, IJS05, MZ04, Sch05, Sch07, SZ05]. The two case studies here adds additional systematical evidence for legitimacy of alternative mechanisms in pricing derivatives. We focus on the pricing and trading of options due to the accessibility of data. The ideas could also be useful in dealing with other more complicate financial derivatives. In our investigation we combine methods in variational analysis, in particular, convex analysis with traditional stochastic financial models which we believe will also be helpful in dealing with similar problems for other financial derivatives.

Chapter 2

Preliminaries

In this chapter we introduce the basic framework of derivative pricing. Most of the definitions and theorems are valid for discrete period and continuous period economies. We note where these situations differ significantly. For detailed discussions on these topics please consult Hull [Hul06], Roman [Rom04] and Karatzas and Shreve [KS01, KS08]. Hull [Hul06] gives an excellent overview of market practices and procedures regarding derivatives trading. Roman [Rom04] gives a description of discrete time period economy, while Karatzas and Shreve [KS01] develop the continuous time model carefully. For a discussion on stochastic caluculus please refer to Karatzas and Shreve [KS08].

2.1 Economy, asset and its derivatives

Definition 2.1.1. *(Economy) An economy with time horizon* **[0,T]** *is a probability space (f2, P) together with a filtration {^t}o<t<T> which satisfies the usual condition* [1].

Definition 2.1.2. *(Economy, discrete time) An economy with time horizon* **{0,1,..., T}** *is a probability space together with a filtration* {J^T}O<T<T

We can think of (fi, J*", *P)* as the set of all possible economic scenarios together with the chances of them happening. And for we can think of it as a representation

[1] **A filtration is said to satisfy the usual condition if it is right continuous and J^o contains all P-negligible subsets of &.**

of information available up to time t.

Definition 2.1.3. *(Asset) An asset is a semimartingale* $\{S_t\}_{0\le t\le T}$ *on an economy. This means that,* $\{S_t\}_{0\le t\le T}$ *is a progressively measurable stochastic process which is a sum of a local martingale and a bounded variation process.*

Definition 2.1.4. *(Asset, discrete time) An asset in a discrete time economy is an adapted stochastic process* $\{S_T\}_{0\le t\le T}$ *such that,* S_t *is in* L^1 *for all* t.

An asset is completely characterized by its price process. A semimartingale is a process which can be expressed as a sum of a local martingale and a bounded variation process. One can think of the bounded variation part as the potential for value appreciation or depreciation, whereas the local martingale part represents random noise. Technically, the reason we take asset processes to be semimartingale is that we want to be able to integrate with respect to that process.

Definition 2.1.5. *(Derivative) A derivative of an asset* $\{S_t\}_{0\le t\le T}$ *is another asset whose value/payoff* $p(T)$ *at a time* T *is given by*

$$P(T) = f(\{S_t\}_{0\le t\le T}),$$

where f *is some function.*

Example 2.1.6. *(Call option) A call option with the strike price* K, *and maturity (or expiration)* T, *on a stock with price process* $\{S_t\}_{0\le T\le T}$, *IS the right without any obligation, to buy the stock at time* T *at price* K *from the seller of the option. The payoff of this option at time* T *is given by:* $p(T) = (S_T - K)^+$.

The central question in option pricing theory is to find a fair value/price for this right (option) at the present time $t < T$.

Example 2.1.7. *(Put option) The put option is the right without obligation to sell an asset at a given price on a given date.*

These are the most common equity derivatives and also known as vanilla options.

Definition 2.1.8. *(Moneyness) An option with strike price equal to the asset price at the time of trading is called* at the money. *If the strike price is below the asset price at the time of trading, it is called* in the money. *If the strike price is above the asset price, it is called* out of the money.

It is obvious that any option can change its moneyness any time between the start of its trading and its expiry.

2.2 Trading strategy

Definition 2.2.1. *(Trading strategy) Let, $S^1, S^2, \ldots, S^n$ be n assets in an economy* { (f W *e assume that S^0 represents the risk-free asset or cash, and hence is a constant process with value 1. A trading strategy ©, ©$_t$ = (9°, 9}, 9^2,..., is a left continuous, locally bounded predictable (^t--measurable) process. 9^l_t signifies the amount of the asset S^l held at time t.*

Definition 2.2.2. *(Trading strategy, discrete time) Let, $S^l, S^2, \ldots, S^n$ be n assets in a discrete time economy* {(f2, J^", P), *We assume that S^0 represents the risk-free asset or cash, and hence is a constant process with value 1. A trading strategy 0, O_t = (9®, Q\, 9\,..., 6™) is a predictable -measurable) process. 6^l_t signifies the amount of the asset $S^{\%}$ held at time t.*

A trading strategy is often called a portfolio process.

Definition 2.2.3. *(Gain from a trading strategy) The terminal gain from a trading strategy © is denoted by G(Q) and defined as:*

$$G(Q) = \int_0^T (Q_u dS_t).$$

Definition 2.2.4. *(Gain from a trading strategy, discrete time) The terminal gain from a trading strategy © in a discrete time economy is denoted by G(Q) and defined as:*

$$G(0) = \sum_{t=1}^{T} \quad - \$_-i)>$$

Definition 2.2.5. *(Self-financing strategy) A trading strategy © is called self-financing, if*

$$d(Q_t, s_t) = (e_t, ds_t).$$

Definition 2.2.6. *(Self-financing strategy, discrete time) A trading strategy 0 in a discrete time economy is called self-financing, if*

$$(@_t, S_t) = (Q_{t+u} S_t).$$

This means that a self-financing trading strategy is one where you do not take any cash out of it or put cash in, during the trading process. For our purpose, it is adequate to look only at self-financing strategies.

Definition 2.2.7. *(Contingent claim) A contingent claim on an economy is an measurable random variable.*

If we are interested in pricing just one option we can think of *T* being the time of maturity of the option and the payoff from the option is then a contingent claim.

2.3 Arbitrage

Definition 2.3.1. *(Arbitrage trading strategy) An trading strategy © is an arbitrage if G(Q) is non-negative a.e. P and* _P[G(@) > 0] > 0.

Definition 2.3.2. *(Arbitrage contingent claim) A contingent claim f is called an arbitrage if f* >0 *a.e. P and P[f* > 0] > 0.

So arbitrage is essentially a way to generate risk-free profit.

2.4 Market completeness and replicating portfolio

Definition 2.4.1. *(Replicating portfolio) A trading strategy* **O** *is called a replicating portfolio for a contingent claim h (or replicate h) if,*

$$(©r, ST) = h, \quad a.e.$$

This idea of replicating portfolio is at the heart of Black-Scholes-Merton option pricing theory. If there is a derivative contingent claim that needs to be priced and one can come up with a replicating portfolio for it, then the value of that portfolio at any time should be the price of the derivative at that time. If the market price of the derivative deviates from the replicating portfolio price, then one can take advantage of the situation and create an arbitrage. Suppose, the theoretical price of a derivative is higher than replicating portfolio price. Then one can buy a replicating portfolio (and keep updating it with the trading strategy) and short the derivative. The cost of this position is negative, which is a profit. The payoffs from the replicating portfolio and the derivative cancel out in the end leaving the investor with the profit he/she made from the initial position.

Definition 2.4.2. *(Complete market) A market is called complete with respect to a set of assets, if any contingent claim can be replicated using a portfolio of those assets.*

In a complete market, every contigent claim is replicable. Therefore, any contingent derivative can be priced by the value of its replicating portfolio.

2.5 Risk-neutral measure

The idea of a replicating portfolio in the previous section gave us a way to price derivatives. However, in practice it often proves to be a difficult task to actually assemble a replicating portfolio. Cox and Ross [CR76] introduced the idea of a risk-neutral measure which can be easier to compute at times and arrives at the same pricing in a complete market. •

Definition 2.5.1. *A risk-neutral measure for a vector of assets (S^1, S^2,..., S^n) on an economy {(f2, &, P), {^t}} is a measure Q on (fi, which is equivalent to the measure P, and each S^l is a martingale under the measure Q, with respect to the filtration \}. We denote the set of risk-neutral measure by P)*

One of the reasons the equivalence of the measures are needed in the above definition, is that it makes the usual condition for the filtration identical under both the measures.

We state a characterization for existence of risk-neutral measures. We present a simpler case first, which is, of course, stronger than the most general case.

Theorem 2.5.2. *(Fundamental Theorem of Arbitrage Free Pricing, discrete time) [HP81] In an economy if the time horizon is discrete and finite or the underlying* *state space Q is finite, then: there is no arbitrage with respect to a set of assets, if and only if there is a risk-neutral measure for that set of assets.*

However, the assumption of no arbitrage turns out to be a weak condition in the general setting. Instead, a little stronger condition can be used.

Definition 2.5.3. *(No free lunch at vanishing risk /NFLVR]) An asset vector S is said to satisfy NFLVR, if there is no sequence {©"} of trading strategies and an extended real valued arbitrage contingent claim f, such that:*

$$\left(\int_0^T \Theta_t^n dS_t\right)^- \xrightarrow{L^\infty} 0$$

and

$$\int_0^T \Theta_t^n dS_t \xrightarrow{a.e.} f.$$

Theorem 2.5.4. *(Fundamental Theorem of Arbitrage Free Pricing)[DS94] There is a risk-neutral measure for an asset vector S if and only S satisfies NFLVR.*

Corollary 2.5.5. *If S is a vector of assets, and f is a contingent claim, then to have NFLVR, the price p_t of f at time t must satisfy:*

$$p_t \quad E \quad \{Efi(f\backslash\&_t): \quad Q \quad E \quad Jt\{S,P)\}.$$

Such a price is called a risk-neutral price.

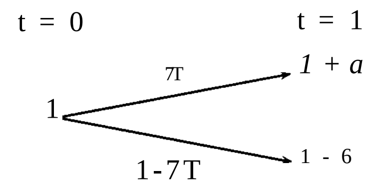
Here is one of the most interesting characterization of complete markets.

Theorem 2.5.6. *Under NFLVR, the market is complete with respect to a set of assets if and only if the class of risk-neutral measures is singleton.*

2.6 Examples of risk-neutral measures

2.6.1 A discrete example

Suppose, we have a one period economy with one risky asset with price 1. Assume that the asset can have only two possible values:



Then it is easy to see that the risk-neutral measure will be given by a number ***IT*** such that:

$$\pi(1+a)+(1-\pi)(1-b)=1, \text{ i.e. } \pi=\frac{b}{a+b}.$$

We also note that this measure is unique making this simple situation an example of a complete market. Also, observe that, if the asset had three possible values in the future, then the market would have been incomplete.

2.6.2 A continuous example

We are assuming that the rate of risk-free return is zero. Suppose, we have just one stock whose price St follows the stochastic differential equation:

$$dS_t = fiS_t dt + aS_t dW_u$$

where W_t is a standard Wiener process.

By Girsanov's Theorem, the unique risk-neutral measure is the one that makes

$$W_t = W_t + \frac{}{a} t$$

a standard Wiener process.

With this transformation the original SDE becomes:

$$dS_t = aS_t dW_t.$$

Solving this stochastic differential equation yields [KS08]:

$$St = sQ \exp \ \backslash aW_t - y\ t\ j\ .$$

So, under the risk-neutral measure, the original drift parameter // does not appear in the description of the process. Hence, Black-Scholes pricing is drift invariant.

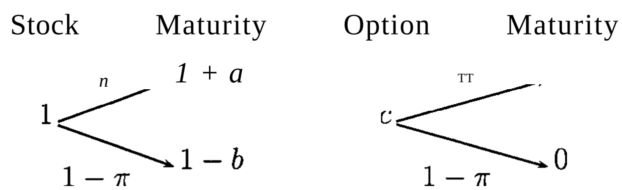
Chapter 3

Stable trading strategy using one option

In this chapter we investigate if there is any detectable trends in the market that can be taken advantage of using trading strategies involving one option on a stock. We look at trading strategies that are stable under small model perturbations. We find that a stable strategy is a switch between pure buying call pure writing a covered call or a pure stock position. We see that given any risk and reward preference of an investor there is an interval of premium for each position, where it performs the best among all competing positions. We also test this strategy on the S&P-500 index during the time period from 1996 to 2008.

3.1 Introductory analysis

We look at a simple example from [Zhu08] to illustrate some of the shortcomings of the replicating portfolio strategy. We consider a one period model with trading being allowed only at the beginning and end of the period. We consider one stock whose price has been standardized to 1. Suppose at the end of the period the stock price can only take two possible values. We further assume, that the stock has positive expected return. We look at the at-the-money call option on the stock maturing at the end of the period. We summarize this model in the following diagram where the numbers along the arrows signifying the probabilities corresponding to the path.

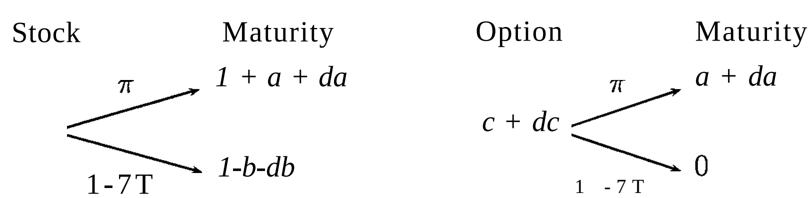


The replicating portfolio pricing theory constructs a portfolio of stock and cash such that it will replicate the payoff of the call option in any situation. It is easy to see that such a portfolio in this case would consist of ^ shares of stock and cash. Hence the theoretical fair price of the call option should be the cost of buying this portfolio which is

$$\frac{a}{a+b} \qquad \frac{a(b-1)}{a+b} \qquad \frac{ab}{a+b}$$

Now the market price for this call option need not match the theoretical price. Suppose the market premium for the at-the-money call option is given by c + *dc*. If *dc* > 0 then one can buy the portfolio at cost *c* and sell the call option at price *c* + *dc* and thus earning *dc* without any risk. Clearly *dc* is an arbitrage profit. If *dc* < 0 then one can buy the call option and sell the replicating portfolio to earn similar arbitrage profit. So theoretically, one should always assume this arbitrage strategy and possibly leverage it when the theoretical fair price and market price do not agree.

In practice, however, it is not possible to predict an exact model. Suppose, one computes the theoretical fair price based on the aforementioned model. But, the actual model of the market was:



Suppose, *dc* > 0. Then, according the arbitrage strategy described above, one should buy the replicating portfolio and sell the option. We can ignore the cash part since the payoff is zero no matter what happens. The arbitrage position will then be by ^ shares of stock and sell one call option. The diagram below describes the cost and payoff of such a position.

Cost	Payoff	Percentage gain
7T	$\frac{a(1-b)+ada}{a+b}$	$\frac{ada+(a+b)dc}{a(1-b)-(a+b)dc}$
^ jj-	$\frac{a(1-b)-adb}{a+b}$	$\frac{-adb+(a+b)dc}{a(1-b)-(a+b)dc}$

If the assumed model is accurate, *i.e.* $da - db = 0$, then this position makes a relative gain of a^^-ta+fydc regardless of what happens. However in real market situation, the $|da|$ and $|db|$ are usually larger than $|dc|$. We can see that, such a situation may result in a loss in both cases.

We can thus see that the replicating portfolio strategy is susceptible to model perturbations. To investigate this further, we look at a general portfolio of a stock and a call option on it consisting of a shares of stock and @ call option. Let us assume that the expected return of the stock is positive. That is, e = ~na — (1 — ir)b > 0. Now, under the perturbed model, the cost and payoff of the portfolio is following:

Cost	**Payoff**	**Relative gain**
a + *(3{c + dc)*	*a(a + ' a a \ +' (a + 0)da - pdc* mj	*a(a+0 •^)+(a+0)da-0dc* / *a+0(c+dc)*
	-Kot + fi^-adb-pdc	***atill+dc)***

We observe that the replicating strategy corresponds to a + = 0. While using this strategy, we can see that the relative returns are dominated by the model fluctuations *da* and *db* which are usually much larger than *dc* in absolute value. This results in the instability of replicating portfolio strategy as we observed before.

Now, we try to assess the worst case scenario of such a general portfolio. We assume that, max(|cZa|, $|db|$) < 5 and *e » S > dc.* Now we try to find a portfolio that performs best in the worst case. The worst case cost and payoff are given by:

Cost	Payoff	Relative gain
a + P(c + dc)	n ^ a{a + p^r_b) + \a + p\6-pdc, mx riA	a(a+P-^)+\a+p\S~pdc / a+0(c+dc)

We observe that the relative gains are homogeneous in the variables *a* and *(3.* We also note that changing the relative gain/loss proportionally results in equivalent portfolios. We now turn to the optimization problem of maximizing expected return under worst case scenario. So the problem takes the form:

$$\text{Maximize } f(a, /?) := e(a + (3\frac{\wedge}{a+b}r) - (\mathrm{tt}|a + (3\backslash + (1 - 7r)|a|)5 - (3dc,$$

subject to:

$$\backslash a\backslash + \backslash p\backslash = l.$$

Since / is piecewise linear, the possible solution points are the corner point of the set jck| + \(5\ = 1 and those satisfy a + /? = 0 or a = 0. It is now easy to check that the maximum is attained at one of the following points:

We note that, the respective portfolios correspond to either holding the stock, buying a call option or writing a call option.

3.2 The theoretical model and notations

We consider a one period economy with with one asset *(e.g.* a particular stock), whose price is standardized to be 1 at the beginning of the period. Suppose, the price of the asset is 1 + X at the end of the period. We assume that X is a random variable in L^1, that is, the expected return E(X) is finite. We further assume that X represents the real return, that is the return relative to the risk-free asset.

We also assume that the asset is a profitable one, in the sense that E(X) > 0.

We now consider a European option on the asset at the end of the period with strike price $1 - a$. Now, $a = 0$ stands for an at-the-money option. Note that $a < 0$ and $a > 0$ stand for out-of-the-money and in-the-money option respectively. Let p be the premium of such a call option relative to the spot price of the asset at the beginning of the period. We then look at the the relative return of the two associated investment systems namely, buying the call option and writing the call option. These returns are given by:

$$c(X, a, p) = \frac{(X + a)^+ - p}{p}$$

and

$$w(X, a, p) = \frac{p - a - (X + a)^-}{1 - p},$$

where c and w stand for the buying a call option and writing a covered call option with strike price $1 - a$ and premium p respectively.

We use a risk-reward function defined below to characterize the investor's attitude towards their wealth risk tolerance.

Definition 3.2.1. ***(Risk-reward, function)*** *Let* (f *b e a probability space signifying the possible economic scenarios. A function*

$$f : \quad \wedge \quad \mathbb{R} \cup \{-\infty\}$$

is called a risk-reward function if it is of the form

$$f = E^p \text{ OB-lOp}$$

where,

$u : [0, \infty) \longrightarrow \mathbb{R}^*$, *is an upper semicontinuous utility function satisfying:*

(U1) u is strictly concave. (Risk averse)

(U2) u is strictly increasing and $\lim_{t \to +\infty} u(t) = +\infty$. *(Profit seeking)*

(U3) $\lim_{t \to 0^+} u(t) = -\infty$. *(Bankruptcy forbidding)*

(U4) u(l) = 0 *andu'(* 1) *exists.* *(Standardized)*

p : *P)* —> M U {00} *is a lower semicontinuous convex risk measure satisfying:*

(Rl) *X* < *Y,* *a.s.* *p(X)* > *p(Y).* *(Monotonicity)*

(R2) *p(X* + *a)* = *p(X)* - *a,* Va G M. *(Cash-invariance)*

(R3) *p{XX* + (1 - A)Y) < A*p(X)* + (1 - *X)p(Y),* VA G [0,1], *X,Y e* L^l. *(Convexity)*

and

1 is an increasing lower semicontinuous convex function such that,

(II) *to* p(l) = 0 *and* (*1 o* p)'(l) *exists.*

Definition 3.2.2. *For a given investment system X* G L^1 *and a risk-reward function f, we define the effectiveness index of the investment system as:*

$$K(X) := \sup_{sG[0,oo)} /(I + sX).$$

This is a generalization of the Kelly Criterion, where *u* is taken to be the natural logarithm and *t* is constant 0 [Kel56]. Zhu [Zhu08] has already considered a similar effectiveness index where only the utility component is present.

Although, in theory a utility function represents the risk-aversion of an investor, the maximizing leverage could be too high at times to be practical. The introduction of an explicit risk component allows the investor a more direct control over the risk of a position.

We present some fundamental properties of *K* in the following results. These results are noted in [Zhu08] for a simple random variable when only the utility component is present.

Theorem 3.2.3. *Suppose* **(f2,** J^", *P)* *is a probability space. Suppose, X* G $\mathbf{L}^1$**(Q,** j^", *P*). *Then, the effectiveness of the X, given by K(X) has the following properties:*

(i) *(Non-negativity)* *K(X)* > 0, VX.

(ii) *(Arbitrage characterization)* $K(X) = \infty$ *if and only if* $X \geq 0$ *a.s. and* $E(X) > 0$.

(iii) *Suppose, ρ is a coherent risk measure, i.e. in addition to being convex, it satisfies $\rho(sX) = s\rho(X)$. Also assume that, $\rho(X) \geq E(-X)$.*

Then, $K(X) = 0$, *if* $E(X) \leq 0$.

Moreover, if and $L'(-1) = 0$, *the the converse is also true.*

(See Remark 3.2.4 (pp. 22) for a discussion about these conditions.)

(iv) *(Scale invariance)* $\forall r > 0$, $K(rX) = K(X)$.

Proof. (i) We just observe that, for $s = 0$, $f(1+sX) = f(1) = 0$. Hence, $K(X) \geq 0$.

(ii) Suppose, $X \geq 0$ a.s. with distribution F and $E(X) > 0$. Let, $f = E \circ u - t\circ\rho$ is a risk-reward function as defined above.

Let, $b > 0$ be such that, $F(b) < 1$. Then,

$E(u(1 + sX)) = \int_0^\infty u(1 + sx)dF(x) = \int_0^b u(1 + sx)dF(x) + \int_b^\infty u(1 + sx)dF(x) \geq u(1 + sb)(1-F(b))$.

Also, for such an X, $r > s \Rightarrow rX \geq sX$, a.e.

Since, $X \geq 0$, a.s., we have, for $s > 0$,

$$-t(\rho(1 + sX)) \geq t(\rho(1)) = 0.$$

Taking limit as $s \to \infty$, we get, $K(X) = \infty$.

Let, $K(X) = \infty$. We shall show, $X \geq 0$ a.s. and $E(X) > 0$. It is clear that if $K(X) = \infty$, then the supremum is not achieved. It must be "attained" at ∞. Suppose, X is not an arbitrage. That is, $E(X^-) > 0$. It is obvious that X is essentially bounded below. Otherwise, $E(u(1 + sX))$ will be $-\infty$ for all non-zero s. Suppose, essinf$(X) = a$. Then, $E(u(1 + sX)) = -\infty$, $\forall s > -K$ Hence, $f(1 + sX) = -\infty$, $\forall s > -\frac{1}{a}$ implying that $K(X) \neq \infty$. Therefore, essinf$(X) \geq 0$. Remains to show that $E(X) > 0$. If $E(X) = 0$, then, $X = 0$ a.e. This implies that $K(X) = 0$. This is a contradiction.

(iii) Suppose, $E(X) \leq 0$. Suppose, $s \in [0, \infty)$ is such that, $E(u(1 + sX)) > -\infty$. Then, using Jensen's inequality and concavity of u, we have, $E(u(1 + sX)) \leq u(1 + sE(X)) \leq u(1) = 0$. Also, for such an s,

$$\ell(\rho(1 + sX)) = \ell(s\rho(X) - 1) \geq (\ell(-sE(X) - 1)) \geq \ell(-1) = 0.$$

Hence, $K(X) = 0$.

For the other direction, let $K(X) = 0$. Then,

$s \downarrow 0$ s

Hence, $E(X) \leq 0$.

(iv) This follows from the definition. •

Remark 3.2.4. *The homogeneity assumption on top of the convexity assumption of the risk measure implies that it is a coherent risk measure. We know that a coherent risk measure can be dually represented as:*

$$\rho(X) = \sup_{Q \in \mathcal{P}} E_Q(-X),$$

where is a class of measures for which $E(X)$ *is well defined. The assumption* $\rho(X) \geq E(-X)$ *ensures that the model probability measure is included in the risk calculation and not ignored completely.*

The assumption on the derivative of L *ensures that there is no penalty for investing in negative risk assets.*

3.3 Option pricing interval and trading strategy

We have seen in our introductory analysis that a trading position that is stable under model perturbations, is one of pure stock, pure long call or pure writing covered call positions. It is clear that the performances of these positions depend on the premium of the call option in consideration. In this section, we analyze the performances of these positions for a varying premium. These results are generalizations of results in [Zhu08] that include the explicit risk component and that allow any L^1 random variable as the return on the stock. In a nutshell, there should be and p^ defining an option pricing interval that naturally lead to a suitable trading strategy.

3.3.1 Option pricing interval

Theorem 3.3.1. *Suppose, X is an L^1 random variable. Suppose, f satisfies the conditions of 3.2.3 (pp. 20) part (Hi). Suppose, $a \in -\text{conv}(supp(X))$* [1]*. Define,*

$$u_a = E[(X + a)^+] \tag{3.3.1}$$

and

$$l_a = a + E[(X + a)^-]. \tag{3.3.2}$$

Then, for $p > u_a$, $K(c(X, a,p)) = 0$ and for $p < l_a$, $K(w(X, a,p)) = 0$. Moreover,

$$u_a - l_a = E(X).$$

Proof. It is easy to observe that,

and

$$\frac{1-p}{E(w(X,a,p))} \sim E[(X + a)^-]$$

Clearly, $E(c(X, a, u_a)) = E(w(X, a, l_a)) = 0$. The conclusion follows readily from Theorem 3.2.3 (pp. 20), part (iii). •

[1] **conv denotes the convex hull of a set.** *supp* **denotes the support of the measure on the real line induced by the random variable** *X, i.e.* **the smallest closed set such that the probability of** *X* **being in that set is 1. (X) = *A* 6 M if** *P(ui : X(io) £ A) =* **1,** *A* **is closed and for any closed** *B* **such that** *B* **C** *A* **implies P(w :** *X(u)* **G** *B) <* **1**

Theorem 3.3.2. *Suppose, f satisfies the conditions of 3.2.3 (pp. 20) part (Hi). Then,* V l G L^1^, &, P), *and* a G - conv((X)), 3\p* G (l_a, u_a), *such that*

$$K(c(X_{,a},p:)) = K(w(X,a,p*_a)).$$

Proof. Being the maximums of locally Lipschitz functions, both $K(c(X,a,p))$ and $K(w(X, a,p))$ are continuous functions in p. Since, $K(c, (X, a, l_a)) > K(w(X, a, l_a)) =$ 0 and $K(w(x,a,u_a)) > K(c(X,a,u_a)) = 0$, by intermediate value theorem, there exists $p^* € (l_a, u_a)$, such that,

$$K(c(X,a,pl)) — K(w(X,a,pD).$$

Also, it is easy to see that $K(c(X,a,p))$ and $K(w(X,a,p))$ are strictly monotone functions of p. Hence the uniqueness of $p*_a$ follows. •

Theorem 3.3.3. *Suppose, f is any risk-reward function. Then for any X £* L^l, a G conv((X)) *andp* G $[l_a,u_a]$,

$$min\{K(c(X, a,p)),K(w(X,a,p))\} < K(X).$$

Proof. Fix p. Let s_c and s_w be the optimal leverage that maximize /(I + $sc(X, a,p)$ and /(I + $sw(X, a,p)$) respectively. The maximum is attainable and s_c and s_w exist since / is upper semicontinuous. Then we have:

$$K(c(X, a,p)) = /(I + s_c{}^{(* + \circ)+\sim P})$$

and

$$K(w(X,a,p)) \hat{=} f\{l^f + s_w \frac{p — a — (X + a)}{1 - p}).$$

Let, $a = p/s_c$ and /? = (!— $p)/s_w$. The using concavity of u, we have:

$$\frac{a}{a + (3}K(c(X,a,p)) + \frac{}{a + 0}K(w(X,a,p))$$

$$\frac{a}{a + P}/(1 + s_c c(X, a,p)) + \frac{}{a + (3}/(1 + s_w w(X,a,p))$$

$$\leq f\left(1 + \frac{\alpha}{\alpha+\beta} s_c c(X,\alpha,p) + \frac{\beta}{\alpha+\beta} s_w w(X,\alpha,p)\right)$$

$$< K(X).$$

•

Theorem 3.3.4. *Let X be any L^1 random variable, and let, a* G — conv((X)), *then,* 3! Pa such that,

$$K(c(X,a,p^c_a)) = K(w(x,a,pf)) = K(X).$$

Consequently, forp < *we have K(c(X,a,p)) > K(X) and for p > pY we have, K(w(X,a,p)) > K(X).*

Proof. Suppose, *(ft,* *P)* is the underlying probability space. For any *u* G *Q* such that + a)$^+$ > 0, we have, $\lim_{p|0}$c(X(u/), *a,p)* = oo. By the assumption on a, *P{X{u) > -a)* > 0. Hence,

$$\lim_{p|0} i\hat{\ } cpT, a,p)) = oo.$$

Now, we know that *K(c(X,a,pl)) < K(X).* Since *K(c(X,a,-))* is a strictly decreasing function, 3! G (0,p*] such that, *K(c(X,a,p%)) = K(X).*

The proof for writing option is similar. •

3.3.2 The option replacement strategy

So, we observe that, given a subjective choice of reasonable risk-reward function, there are two critical option premium, such that, if the premium is below the lower critical threshold (p^c), it is possible to improve risk-reward value using a suitable cash and call combination than the best cash and stock combination. Similar situation arises with cash and writing call combination above the upper critical threshold (p^w).

Given an investor's personal risk-reward function and a given call option on a given stock, one should buy call options if the market premium for that option is below p^c. One should write call options if the premium rises above p^w. In between the thresholds, one should invest in pure stock.

From our motivating example it is clear that replicating portfolio pricing can be extremely unstable under model perturbations. As we know using Cox Ross and Rubinstein's binomial model [BS73, CRR79], that limit of such two outcome cases gives rise to the Black-Scholes pricing, it is imperative that instability exists at each step of such limiting process which in turn results in the instability of such pricing.

Also, the Black-Scholes pricing can be seen as the solution of equating Merton maximum utility and the option writer's maximum utility under the assumption of constant volatility and market completeness. This is also a special case of utility indifference pricing, where the condition of market completeness and constant volatility is removed [HN89, MZ04, REKOO, SZ05]. In all these approaches, the maximization is done over the class of admissible trading strategies and results in a trading strategy which is optimal in some sense but not necessarily robust or stable.

Cont, Schied and Hernandes-Hernandes approached this problem of instability by finding a robust maximum utility portfolio of options, where the model uncertainties are represented by different measures on the same measurable space consisting of possible economic conditions [Con06, HHS07b, Sch05]. Carr, Geman and Madan approached the issue of model inaccuracy representing the beliefs of different market agents by different probability measures. The price in such approaches is determined by its acceptability to the agents [CM01, CGM01].

Our approach looks at the model uncertainty from a different perspective. Also, our strategy is motivated by maximizing profit in the worst case scenario. Our pursuit is to find a strategy that will be stable under model perturbations as well as optimal in the worst case. We also ensure that the strategy is computationally tractable and easily implementable.

Thus, we look at the aspect of computing these thresholds for a given stock and a given option. We know that when the stock price can take only two possible values at the maturity, the thresholds collapse to a single point, which is the risk-neutral price of the option. Even, for the simple case of three possibles values of the stock at maturity, and a simple utility function like natural logarithm, the thresholds have to be evaluated using numerical methods. For the sake of finding a good starting point for these thresholds, we shall look into bounds for such thresholds using second order stochastic dominance.

3.4 Bounds for pricing intervals and stochastic dominance

Prom the definitions of the critical price thresholds, it is clear that one cannot expect any closed form solutions for computing these values. Also, the numerical evaluations are computation intensive. Our goal here is to present some bounds for these thresholds to help ease the computations in estimating the thresholds. Such bounds have been derived in [Zhu08] for simple random variables with investors risk-aversion represented by a utility function only. The central technique used in those results was to use the concept of vector majorization. However, majorization is not applicable to a general random variable. But, vector majorization translates to the concept of second order stochastic dominance in case of a general random variable. Also, second order stochastic dominance has an inherent relationship with utility functions. In this section we build on the existing results to derive these bounds in the general case. Levy [Lev85] calculated a sharper bound for the lower critical threshold, when the return is continuous and the return distribution is invertible. Levy also presented an upper bound for the lower threshold for the continuous and invertible distribution case. Zhu [Zhu08] demonstrated a lower bound and upper bound the lower threshold that hold for returns which are simple random variables with finitely many possibilities. The lower bound that Zhu obtained for the discrete case is a a little relaxed than the Levy bounds. In this section we show that the Zhu lower bounds hold for any L^l return. This also generalizes the Levy upper bounds to the general scenario.

Definition 3.4.1. *Suppose,* x,y **G** $M.^N$ *are two vectors. Suppose,* $x^\wedge$ *and* $y^\wedge$ *denote the same vectors where the co-ordinates are arranged in the descending order. Then, we say* x *majorizes* y*, denoted by* $x >z y$ *if*

$$k \qquad k$$

$$n=1 \qquad n=l$$

and

$$N \quad N$$

$$n=l \quad n=l$$

Definition 3.4.2. *Suppose, X and Y are two random variables. We say, X dominates Y stochastically in the second order, denoted by* $X >_{z_2} Y$, *if*

$$\int_{-\infty}^{a} (1 - F_X(t))\, dt \geq \int_{-\infty}^{a} (1 - F_Y(t))\, dt, \ \forall a \in \mathbb{R},$$

where Fx and Fy denote the distributions of X and Y respectively.

Suppose, we identify the vectors $x = (xi,\ldots,x_n)$ and $y = (yi,\ldots,y_n)$ with two random variables X and Y, uniform on the sets $\{x\backslash,\ldots,x_n\}$ and $\{yi,\ldots,y_n\}$ respectively. Then x h y if and only if X Y with $E(X) = E(Y)$.

One important characterization of second order stochastic dominance is the following:

Theorem 3.4.3. *[HL69] X if and only if* $E(h(X)) > \mathbf{E}(h(Y))$ *for any increasing concave function h.*

The above theorem also shows that, if a random variable stochastically dominates another one with the same expectation, then the dominating random variables has a smaller variance.

Theorem 3.4.4. *Suppose, X is any* L^1 *random variable. Suppose, u is an increasing concave function such that:*

$$E|u(X)| < oo.$$

Then, for any a£l,

$$E(u((X + a)^{+}-p))>E(u(X)),$$

i.e.

$$pc(X,a,p) \quad y_2 \quad X,$$

where p is such that,

$$E p O = E((X + a)^+ - p).$$

To prove this result, we shall make use of this theorem proved in the case of a discrete case and use appropriate limits to show it in the general case.

Lemma 3.4.5. *[Zhu08] Suppose, X is a simple random variable which is uniform on co-ordinates of the vector x G M.^{N} and a G (—(xi'*-)$^+$, and p > 0 is such that,*

$$E(X) = E(pc(X,a,p)).$$

Then,

$$pc(X,a,p) \quad y_2 \quad X.$$

That is, Theorem 3-4-4 (PP- 28) is true for such a random variable.

Lemma 3.4.6. *Theorem 3-4-4 (PP- 28) is true for any simple random variable X, i.e. any random variable X taking values g\ > <2 > • • • > 9n- with* P(X = *gi*) = *qi*

Proof. Suppose, {<&„} C Q is a sequence of non decreasing rational numbers such that, $q_{i>n} > 0$, V i = 2,3,..., N, n G N and,

$$\lim_{n \to \infty} q_{i,n} = q_h \quad Mi = 2,3,\ldots, N.$$

Define,

$$Q_{l,n} = 1 - \sum_{i=2}^{N} Q_{i,n}.$$

Let $\{X_n\}$ be a sequence of random variables such that,

$$P(X_n = g\wedge = q\wedge_n.$$

Let F_n be the distribution function of X_n and F be the distribution function of X. Then,

$$F_n\{x) = F\{x), \quad Vx \ i \ (g_N,gi).$$

Let, $x \in [q_i, q_{i+1})$. Then,

$$F_n(x) = \sum_{j=1}^{N}$$

Hence,

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \sum_{j=1}^{N} q_{jn} = \sum_{j=1}^{N} = F(x).$$

Therefore, $X_n \Rightarrow X$[2]. Actually, we showed that, $F_n(x) \to F(x)$, $\forall x \in R$. It now follows easily that, $(X_n + a)^{\pm} \to (X + a)^{\pm}$.

Define,

$$p_n = a + E(X_n + a)^{-}.$$

Since all the X_n's and X are compactly supported, we can apply Dominated Convergence Theorem to conclude that,

$$\lim_{n\to\infty} E(X_n + a)^{-} = E(X + a)^{-}.$$

and hence,

$$\lim_{n\to\infty} p_n = p.$$

Since, $\{p_n\}$ is just a sequence of real numbers, we observe that,

$$(X_n + a)^{+} - p_n \Rightarrow (X + a)^{+} - p.$$

Since all the random variables are compactly supported on $[q_N, q_1]$, U is bounded on the support of the random variables. This leads us to conclude,

2 "$\Rightarrow$" denotes convergence in distribution or weak convergence. A sequence of Borel probability measures P_n on a metric space S converges weakly to another Borel probability measure P if $E^{P_n}(f) \to E^{P}(f)$, for all bounded continuous real valued function f on S. A sequence of random variables X_n is said to converge weakly to X if the distributions of X_n converge to that of X. If F_n denotes the distribution X_n and F denotes the distribution of X, then $X_n \Rightarrow X$ if and only if $F_n(x) \to F(x)$, for all points of continuity x of F.

$$\lim_{n\to\infty} E[u((X_n + a)^+ - p)) = E(u((X + a)^+ - p)), \quad \text{and} \quad \lim_{n\to\infty} E(u(X_n)) = E(u(X)).$$

But, by Lemma 3.4.5 (pp. 29)

$$E(u(X_n)) \leq E(u((X_n + a)^+ - p_n)).$$

Hence, by taking limits on both sides of this inequality, we have the desired conclusion. ∎

Lemma 3.4.7. *Let, be a probability space. Let $X : \Omega \to \mathbb{R}$ be a compactly supported random variable. Let $K \subset \mathbb{R}$ be the support of X. Let $\text{conv}(K)$ be the convex hull of K. Then for any utility function u,*

$$E(u(X)) \leq E(u((X + a)^+ - p)),$$

where $a \in -\text{conv}(K)$ and $p \in \mathbb{R}$ is such that,

$$E(X) = E((X + a)^+ - p).$$

Proof. By the strong law of large numbers it is always possible to construct a sequence of simple random variables X_n such that $X_n \to X$ and X_n's and X have the same support. And then the same proof as lemma 3.4.6 (pp. 29) works. ∎

Proof. (Theorem 3.4.4 (pp. 28)) Suppose, 1_A denote the indicator of the set A. Define,

$$X_N = X 1_{(-N,N)} + N 1_{[N,\infty)} - N 1_{(-\infty,-N]}.$$

So, X_N is the truncation of the random variable X beyond a compact interval $[-N, N]$. We can pick N large enough so that, $a \in [-N, N]$. We shall only consider n such that, $n > N$.

Clearly, $X_n \to X$.

Since X_n and $u(X_n)$ are truncations of L^1 random varaibles, $\{X_n\}$ and $\{u(X_n)\}$ are uniformly integrable. Hence, a refinement of Dominated Convergence Theorem applies ([Bil99], pp-31) and we are back to the set up of Lemma 3.4.6 (pp. 29) and the result follows. •

We note a generalization of Theorem 5.7 of [Zhu08] below.

Theorem 3.4.8. *Suppose, X is an L^1 random variable. Let, p_a be defined as:*

Then,

$$w(X,a,p_a) \;\; y_2 \;\; X \qquad c(X, a, p_a).$$

This result is already proved in [Zhu08] for simple random variables with finite support. It can be generalized easily using a similar limiting argument as the proof of 3.4.4 (pp. 28).

This result allows us to note a sharper bound for pY under some mild conditions.

Definition 3.4.9. *A risk measure p is called law-invariant if $p(X) = p(Y)$ whenever, X and Y have the same distribution.*

Definition 3.4.10. *A coherent risk measure p is called SSD preserving, if*

$$Y \wedge p(X) < p(Y).$$

Now, we are ready to write down two immediate corollaries of Theorem 3.4.8 (pp. 32)

Corollary 3.4.11. *If p is a coherent SSD-preserving risk measure, then,*

$$K(c(X, a,p_a)) < K\{X) < K(w(X,a,p_a)).$$

Proof. The result follows easily from the characterization of stochastic dominance given in Theorem 3.4.3 (pp. 28) and using the SSD-preserving property of p. •

We now discuss the implications of the SSD-preserving condition on a risk measure. We discuss it in the case when *p* is restricted to *L*°°. Relationship between second order stochastic dominance and coherent risk measures have been studied already in the case of essentially bounded random variables. This is definitely a smaller class of random variables than L^1, but from a practical standpoint this is not a huge restriction, since it is not at all an unreasonable assumption for a return to be bounded. In some theoretical setting an unbounded random variable may yield simpler computation. But, one can always truncate an L^1 random variable outside a big enough interval to have a good approximation.

For a more detailed account of this discussion please see [ADEH02, Lei05]

Definition 3.4.12. *For an* L^1 *random variable X, the conditional value at risk at level a is denoted by* $\mathrm{CV@R}_a(\mathrm{X})$ *and is defined by:*

$$\mathrm{CV@R}_a\mathrm{pO} = - E\{X\backslash X < q)$$

where $P(X < q) = a$.

Conditional value at risk is an example of a coherent risk measure. The reason for introducing this definition will be apparent soon. We introduce a characterization theorem similar to Theorem 3.4.3 (pp. 28)

Theorem 3.4.13. *[FS02] For X, Y e L°°, X* t_2 *Y if and only if,*

$$CV@R_a(X) < \mathrm{CV@R_Q(Y)}, \ \mathrm{Ma} \ e \ [0,1].$$

This result motivates the following characterization of SSD-preserving risk measures.

Theorem 3.4.14. *[FS02] A coherent risk measure p on L°° satisfying the Fatou property is SSD-preserving if and only if it admits a representation of the following form:*

$$\rho(X) = \sup_{\mu \in \mathscr{M}} \int_{[0,1]} \mathrm{CV@R}_\alpha(X)\mu(d\alpha), \tag{3.4.2}$$

where $\mathcal{M}$ *is a class of Borel probability measures on [0,1].*

We now note the relationship between SSD-preserving risk measures and law-invariant risk measures in the case of an atom-less probability measures.

Theorem 3.4.15. *Suppose the underlying probability space (U, P) is atomless and* ρ *is a coherent risk measure on* L^∞. *Then,* ρ *is law-invariant if and only if* ρ *admits a representation described in (3-4-2 (pp. 33)).*

Hence, for an atomless measure, ρ *is SSD-preserving if and only if it is law-invariant.*

The above characterization asserts that being SSD-preserving is a mild condition in case of continuous random variables as it only only requires law-invariance of the risk measure. Law-invariance is a reasonable assumption for a risk-measure in practice, as most of the times, only the distribution of the return can be observed and not the underlying measurable space.

We note the follwing summarizing theorem:

Theorem 3.4.16. *Suppose,* u_a, l_a, ρ_Y, ρ^*_a *and* ρ_a *are defined as in (3.3.1 (pp. 23)),(3.3.2 (pp. 23)), Theorem 3.3.4 (pp. 25), Theorem 3.3.2 (pp. 24) and (3.4.1 (pp. 32)). Then,*

(i) $l_a \leq \rho_a \leq \rho_a \leq \rho^*_a \leq u_a$. *Furthermore, if* ρ *is a coherent risk-measure preserving second-order stochastic dominance, then,*

$$\rho_Y \leq \rho_a \leq u_a.$$

(ii) $E(X) = E(c(X,a,\rho_a)] = E(w(X,a,\rho_a))$.

(iii) $w(X,a,\rho_a) \succeq_2 X \succeq_2 c(X,a,\rho_a)$.

(iv) $\rho \in [\rho_S, \rho_Y] \Rightarrow \max(K(c(X,a,\rho)), K(w(X,a,\rho)) \leq K(X)$.

(v) $\rho \in [l_a, \rho^*_a] \Rightarrow K(c(X,a,\rho)) \geq K(X) \geq K(w(X,a,\rho))$.

(vi) $\rho \in [\rho_Y, u_a] \Rightarrow K(c(X,a,\rho)) \leq K(X) \leq K(w(X,a,\rho))$.

Proof. (i) Follows from Theorems 3.3.1 (pp. 23), 3.3.2 (pp. 24), 3.3.4 (pp. 25), 3.4.11 (pp. 32).

(ii) Follows from definition.

(iii) Follows from Theorem 3.4.8 (pp. 32).

(iv) Follows from Theorem 3.3.4 (pp. 25).

(v) Follows from Theorems 3.3.3 (pp. 24) and 3.3.4 (pp. 25).

(vi) Follows from Theorems 3.3.3 (pp. 24) and 3.3.4 (pp. 25).

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3.5 Robustness of the option replacement strategy

Whereas the motivation of the option replacement strategy arises from the simple motivating example, the robustness of the strategy can formalized using this slight generalization of Theorem 6.1 in [Zhu08]. We show under mild assumptions that for any Lipschitz function *h, K o h* is locally Lipschitz near any "reasonable" *X*. Since *c(X,a,p)* and *w(X,a,p)* are Lipschitz functions of *X*, the maximum of *K(X),K(c(X,a,p)),K(w(X,a,p))* is locally Lipschitz near any reasonable *X*. This gives us the robustness of the strategy.

Theorem 3.5.1. *Let f be any risk-reward function satisfying the conditions of Theorem 3.2.3 (pp. 20), part (iii), and K be the corresponding effectiveness measure. Let h :* R —> R *be a Lipschitz function. Then K oh is locally Lipschitz near every X* G *L°°(Sl,&,P) satisfying K(h(X))* G (0,oo).

Proof. Since, *h* is globally Lipschitz, it is enough to show that *K* is locally Lipschitz near any *X* G *L°°* satisfying *K(X)* G (0, oo).

Suppose, / is any risk-reward function satisfying the conditions of part (iii) of Theorem 3.2.3 (pp. 20). Let, *X* G *L°°* be such that, *K{X)* G (0, oo). Since, *K(X)* > —oo, essinf(X) > —oo. Let, essinf(X) = *x* > —oo. Now, *K(X)* < oo implies that essinf(X) < 0. Hence, we have, *x* G (—oo,0).

Let, *sx* and *sy* denote the best leverage for the investment systems X and Y respectively, i.e.

$$K\{X) = /(I + s_xX) \quad \text{and} \quad K(Y) = /(I + S_yY).$$

We observe that, sx G (0, — a n d (1+sx^O G int(dom/). Being a concave function, / is locally Lipschitz on int(dom/). That is, 3 *6* > 0 and $L > 0$ such that,

$$\backslash\backslash f(Z)-f(Y)\backslash\backslash < L\backslash\backslash Z-Y\backslash\backslash,$$

for Y, Z G $B_S\{1 + s_xX)$.

Claim: Given, X G $L^{\circ\circ}(Q,$ J^", $P)$ such that essinf(X) — *x,3e* > 0, small enough, such that,

$$\backslash\backslash X\text{-}Y\backslash\backslash < e \implies \text{essinf}(y) <$$

Proof: Suppose, essinf(Y) > § and *P(X <* = *r).* We know that *rj* > 0. On the set {a; : *X(w)* < |*X(u)* - > Therefore, Now e = ^ does the required job.

Choose, e > 0, small enough such that — y|| < e gives essinf(Y) < | and

$$1 + syK, \quad l + s_xY., \quad l + s_xX \quad eB_s(l + s_xX).$$

Then, ||X-F|| < e

$$\begin{aligned} LS_y\|X-Y\| &> f(l + s_YY)-f(l + s_YX) \\ &> K(Y)-K(X) \\ &> f(l + s_xY)-f(l + s_xX) \\ &> -Ls_x\|X-Y\| \end{aligned}$$

Since, essinf(F),essinf(X) < |, we have *sx,sy* G (0, — H e n c e , K is Lipschitz on $\boldsymbol{B_e(X)}$. •

Remark 3.5.2. *We proved the above theorem for returns in the* $L^{\circ\circ}$ *space. We note* *that, such a result cannot be extended to an LP space when 1 < p < oo. We observed that it is a necessary condition for K(X) > —oo, that X is essentially bounded below. However, any neighborhood of an LP random variable contains an essentially unbounded below random variable, for p* £ [l,oo). *Hence,* int(dom /) *is empty in any W space, forp € [1, oo).*

3.6 Implementation

In this project we tested this strategy on market data. We tried to find whether our method yield different option pricing. We also looked at the practicalities of this stable strategy.

3.6.1 Choosing an investment system

We implemented this strategy starting with monthly returns of Standard & Poor-500 index and associated options. S&P-500 is a weighted average of shares of 500 representative companies from different economic sectors. The weight of each share used to be the market value of the respective company. In mid 2005, the index changed it's weights to the available shares of respective companies available for public trading [P0008]. S&P-500 thus captures the general trends of the economy and has a smoother price process compared to any given company.

Our goal is to test the option replacement strategy to see if it performs better than just investing in stock. We propose to use historical market data for this purpose.

While computing the monthly returns on the index we look at the prices of the index on the last trading day on or before the third Friday of each month. The rationale being that, the third Friday of each month is the day when option contracts expire. We considered call options expiring after one month. To simplify computations, considered the at-money options. In our notation this translates to $a = 0$.

3.6.2 Fixing a risk-reward function

To implement the option replacement strategy we need to choose a risk-reward function. Here we use a combination of a very common utility function, namely the logarithmic utility function together with a truncated conditional value at risk. We use a hard constraint on the conditional value at risk to prevent large draw downs.

Let CV@RQ, denote the conditional value at risk at level a. Suppose, t has the form:

$$M(X) = \begin{cases} 0 & : x < M \\ 00 & : x > M \end{cases}$$

So our risk-reward function / will have the following form:

$$f(X) = E(\log pO) - t_M(CV@R_Q PO). \tag{3.6.1}$$

If one uses f(X) = E(log(X)) as the risk-reward function, then this reduces to the well known Kelly Criterion [Kel56]. Whereas, Kelly criterion does maximize one's expected utility, it is rather aggressive for any practical purpose and exposes the investor to large risk. Also, sometimes, Kelly criterion demands leverages that may not be legally viable. Also, Kelly criterion maximizes one's utility in the long run, but the payoff distributions of any given stock is bound to change over time, and thus not presenting the opportunity to actually stay in the game for the long run.

Ziemba et. al. [MZB92] and Vince [Vin90] also discussed cutting back from the Kelly optimal *s* to control downside risk. Fractional Kelly criterion uses a fixed proportion of the Kelly-optimal portion. But our approach is different from theirs in focusing directly on stability and has a dynamic choice. Also, Kelly criterion focuses more on the long term wealth and independence of the bets. We are certainly deviating from the independence assumption by looking at trends. Although theoretically the concave shape of logarithm is supposed to provide the investor with risk-aversion, it does not do a very good job of doing so in reality. The idea of fractional Kelly is a way to cut back in order to get some protection against exorbitant leverages that Kelly suggests at times. We have found in our experiments with the data, that even a conservative 1/2-Kelly provides a bumpy ride and leaves the investor exposed to huge risks. This takes a severe toll when a bubble bursts in the market.

The main difference between using CV@R together with logarithm and using just logarithm is that the two components of the risk-reward function looks at two different aspect of the return distribution. Whereas the logarithm (or any utility function) focuses on the overall performance of a return, the CV@R function looks at just the lower tail of the distribution. Improving a return distribution without affecting the lower tail changes the way a utility function asses the risk of that distribution. It assigns a lower risk to that return. On the other hand, not changing the lower risk does not change the draw down scenario. And CV@R captures precisely that. Composing the risk function with an indicator function lets an investor use a hard constraint on the draw down risk.

Theorem 3.6.1. *Theorem 3.2.3 (pp. 20), part (iii) is also true for this risk-reward function. That is, when f is of the form 3.6.1 (pp. 38), then*

$$K\{X) > 0 \wedge E(X) > 0,$$

for any X e L°°{Sl,&,P).

3.6.3 Exponential moving average

Suppose, $\{x_t\}$ is a discrete time series. Exponential moving average is a moving average that puts a little more wait on the latest point in the time series rather than the ones in the past. The intention is to enhance any upward or downward trend that may be present in the data [Kau98]. The *N* point moving average at time point *t* is defined recursively as:

$$\delta_{t,N} = \frac{2}{JyXt\ H} \quad \overline{N}\ Jj\frac{2}{} \delta_{t-1,N} -$$

In practice one often computes each St,N separately with $S_t \sim N+i,N — Xt-N+i$ •

3.6.4 Identifying a trend in the market

Now an upward trend in the market can be captured by comparing a long term moving average with a short term one [0'N95, KK02], One common practice is to observe when the 40-day exponential moving average is greater than 200-day (40-week) exponential moving average. If this happens, the general feeling is that the market is going upwards and one should invest. If the inequality is reversed, then the market is perceived to be going down. This is called the trading signal. We implemented the aforesaid strategy with this trading signal. We looked at a filtered trading system, where one invests in the stock (or option) according to the 40-day vs. 200-day exponential moving average.

There are many ways of identifying a market trend. We deliberately choose a rather traditional method [0'N95] to illustrate identifiable trend to be an integral part of the market movement which persists despite being known to many players.

3.6.5 Interest adjustment

Prime interest rate for daily investments were obtained from Economic Research branch of Federal Reserve Bank of St. Louis.

(http://research.stlouisfed.org/fred2/data/PRIME.txt).

We also computed cumulative returns if one had to invest 1 USD on 1992-01-01. We used this cumulative returns to adjust monthly returns Suppose the closing price of SP-500 on date *d()* was p_0 and that on date *d* was *p*. Suppose the corresponding cumulative returns on 1 USD were Co and c\ respectively as obtained from the preprocessed prime interest rate data. Then we considered the adjusted return on SP-500 in the period *do* — *d* as — — — — 2.
Po c_0

3.6.6 Approximating the return distribution

Now we try to address the question of approximating the return distribution. We look at the filtered investment system with only investing when the market is going upwards according to the exponential moving average comparison. We approximated the distribution of a given month's filtered return by the empirical distribution of previous 60 months. That is, we approximate a month's return distribution by a uniform distribution over the months in last five years when the trend was up according to the exponential average signal.

3.6.7 Market premium

In a real market situation one will seldom find an exact at money option. Usually the strike prices are at Si, $5, $10 or similar increments. For S&P-500, the increments in the strike price are $5 or $10, where the price of the stock moves in the order of $1000. So, we look at the closest in-the-money and out-of-the-money options. We again remain conservative and buy the nearest in-the-money option and write nearest out-of-the-money option.

3.7 Results and discussion

We present the results of our experiments in the particular case of $M = 0.95$ and a = 0.05 (Figure 3.1 (pp. 41)), (Table 3.1 (pp. 43)).

Figure 3.1: One option based return

Comparison of one option based stable trading strategy with the raw return on the underlying S&P-500 index, both adjusted continuously for interest at the prime rate

- It is apparent that our option replacement strategy outperforms the buy and hold strategy by far. In the 150 months we tested initial wealth more than quadruples after adjusting for continuously compounded interest at the prime rate. On the other hand, the underlying index S&P-500 lost about 2% of its value when adjust for interest similarly.

- Even with the simple trend-tracking technique, we see that it is possible to take advantage of long term trends and guard against long and large draw downs.
- We looked at similar yet different trading signals for the same period. We repeated the above exercise comparing 30, 35, 45 and 50 day moving average with the 200-day moving average. The results are very similar on nature which suggests the robustness of our approach.
- As the returns suggest, our system recovers rather quickly after a loss.
- Even with this simple trend-tracking, we see that our investments were protected from the two big downturns of our time, namely, 2001 tech-bubble burst and the 2008 housing and credit market crunch. Between 2000 and 2002, our strategy return remained flat lined, whereas the index lost about 38% of its value in nominal terms. We only ran our test through the middle of 2008, when the trading signal prohibited any trading in our strategy. During that same year S&P-500 lost about 38% over the course of the year.
- The results indicate evidence contrary to the popular belief that option prices reflect all the information about the underlying stock. We observe that the market is indeed far from efficient. We used a rather simple trend tracking tool which is arguably one of the most well known technique. The fact that we found market inefficiency using a trend tracking tool on a liquid and widely used index, makes the results even more convincing. These findings reinforces the necessities of alternative option pricing methods and corresponding trading strategies.

Table 3.1: One option results

Return from stable trading strategy involving one option (nearest to the money) on S&P-500 index during the period January 1996 to May 2008. Legends: IRR-Raw return on the index, ICR-Interest adjusted cumulative return on the index, TS-trading signal, Lev-leverage, CR-cumulative return on our stable strategy

Trading date	IRR	ICR	TS	Pos	Lev	CR
19-Jan-1996	0.059085	1.000000	Trade	Write	1.911000	1.030335
16-Feb-1996	-0.010108	1.052311	Trade	Write	2.332000	1.043226
15-Mar-1996	0.005675	1.035103	Trade	Write	2.508000	1.102786
19-Apr-1996	0.036957	1.032775	Trade	Write	1.886000	1.140132
17-May-1996	-0.003095	1.064188	Trade	Write	2.776000	1.184638
21-Jun-1996	-0.042154	1.052536	Trade	Write	2.596000	1.102646
19-Jul-1996	0.041457	1.001808	Trade	Write	2.143000	1.148946
16-Aug-1996	0.032787	1.036758	Trade	Write	1.919000	1.197547
20-Sep-1996	0.034642	1.062314	Trade	Write	1.813000	1.235668
18-Oct-1996	0.037703	1.092182	Trade	Write	1.807000	1.278074
15-Nov-1996	0.015252	1.126211	Trade	Write	2.072000	1.331148
20-Dec-1996	0.036455	1.134379	Trade	Write	2.511000	1.406924
17-Jan-1997	0.032982	1.168316	Trade	Write	2.403000	1.492295
21-Feb-1997	-0.022039	1.197341	Trade	Write	2.475000	1.486828
21-Mar-1997	-0.022650	1.163567	Trade	Write	2.640000	1.483599
18-Apr-1997	0.082744	1.129860	Trade	Write	2.103000	1.552038
16-May-1997	0.083097	1.215400	Trade	Write	3.506000	1.700952
20-Jun-1997	0.018471	1.305711	Trade	Write	2.443000	1.792622
18-Jul-1997	-0.015831	1.321187	Trade	Write	2.641000	1.821522
15-Aug-1997	0.055173	1.291822	Trade	Write	3.842000	2.053253

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Table 3.1 — continued

Date	IRR	ICR	TS	Pos	Lev	CR
19-Sep-1997	-0.006681	1.352031	Trade	Write	2.585000	2.132635
17-Oct-1997	0.020050	1.334271	Trade	Write	3.557000	2.336084
21-Nov-1997	-0.016935	1.349976	Trade	Write	3.848000	2.423622
19-Dec-1997	0.015558	1.318490	Trade	Write	4.518000	2.778945
16-Jan-1998	0.075610	1.330301	Trade	Write	2.658000	3.000001
20-Feb-1998	0.062802	1.419271	Trade	Write	2.196000	3.148288
20-Mar-1998	0.021435	1.498601	Trade	Write	2.455000	3.303835
17-Apr-1998	-0.012461	1.520776	Trade	Write	2.543000	3.366385
15-May-1998	-0.007288	1.492066	Trade	Write	3.118000	3.538980
19-Jun-1998	0.078227	1.469170	Trade	Write	2.336000	3.760713
17-Jul-1998	-0.088957	1.573804	Trade	Write	2.389000	3.120282
21-Aug-1998	-0.056503	1.422165	Trade	Write	1.620000	2.989181
18-Sep-1998	0.035615	1.333088	Stay	-	-	2.989181
16-Oct-1998	0.101409	1.371744	Stay	-	-	2.989181
20-Nov-1998	0.021039	1.499327	Trade	Write	1.037000	3.064594
18-Dec-1998	0.046489	1.521798	Trade	Write	1.161000	3.156617
15-Jan-1999	-0.003274	1.583105	Trade	Write	1.329000	3.268688
19-Feb-1999	0.048499	1.566241	Trade	Write	1.380000	3.414048
19-Mar-1999	0.015170	1.632470	Trade	Write	1.205000	3.522754
16-Apr-1999	0.008560	1.647412	Trade	Write	1.360000	3.656283
21-May-1999	0.009434	1.649212	Trade	Write	1.203000	3.785473
18-Jun-1999	0.056552	1.654904	Trade	Write	1.142000	3.893501
16-Jul-1999	-0.057916	1.737950	Trade	Write	1.194000	3.720763
20-Aug-1999	-0.000890	1.624784	Trade	Write	1.082000	3.808580
17-Sep-1999	-0.065904	1.613153	Trade	Write	1.104000	3.626648
15-Oct-1999	0.139962	1.497334	Trade	Write	1.272000	3.791261
19-Nov-1999	-0.000668	1.693432	Trade	Write	0.963000	3.861532
17-Dec-1999	0.014292	1.681303	Trade	Write	1.123000	3.995005

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Table 3.1 — continued

Date	IRR	ICR	TS	Pos	Lev	CR
21-Jan-2000	-0.066097	1.691491	Trade	Write	0.971000	3.814208
18-Feb-2000	0.087944	1.569261	Trade	Write	1.094000	3.950020
17-Mar-2000	-0.104604	1.695848	Trade	Write	1.035000	3.623478
21-Apr-2000	0.072959	1.646972	Stay	-	-	3.623478
19-May-2000	0.040876	1.604140	Trade	Write	0.811000	3.718688
16-Jun-2000	0.010741	1.657588	Trade	Write	0.778000	3.808635
21-Jul-2000	0.007790	1.660202	Trade	Call	0.050000	3.692530
18-Aug-2000	-0.017369	1.660987	Trade	Stock	0.581000	3.655537
15-Sep-2000	-0.046991	1.620287.	Trade	Write	0.742000	3.581709
20-0ct-2000	-0.020910	1.530147	Stay	-	-	3.581709
17-Nov-2000	-0.040630	1.487274	Stay	-	-	3.581709
15-Dec-2000	0.023168	1.416488	Stay	-	-	3.581709
19-Jan-2001	-0.030554	1.436459	Stay	-	-	3.581709
16-Feb-2001	-0.116017	1.383274	Stay	-	-	3.581709
16-Mar-2001	0.080354	1.214844	Stay	-	-	3.581709
20-Apr-2001	0.039405	1.302362	Stay	-	-	3.581709
18-May-2001	-0.060064	1.345953	Stay	-	-	3.581709
15-Jun-2001	-0.002890	1.258336	Stay	-	-	3.581709
20-Jul-2001	-0.040368	1.246493	Stay	-	-	3.581709
17-Aug-2001	-0.168825	1.189997	Stay	-	-	3.581709
21-Sep-2001	0.111493	0.982957	Stay	-	-	3.581709
19-0ct-2001	0.060709	1.087772	Stay	-	-	3.581709
16-Nov-2001	0.005480	1.149093	Stay	-	-	3.581709
21-Dec-2001	-0.015119	1.149936	Stay	-	-	3.581709
18-Jan-2002	-0.020752	1.128430	Stay	-	-	3.581709
15-Feb-2002	0.056132	1.100994	Stay	-	-	3.581709
15-Mar-2002	-0.035150	1.158566	Stay	-	-	3.581709
19-Apr-2002	-0.016513	1.112763	Stay	-	-	3.581709

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Table 3.1 — continued

Date	IRR	ICR	TS	Pos	Lev	CR
17-May-2002	-0.106137	1.090408	Stay	-	-	3.581709
21-Jun-2002	-0.142932	0.970246	Stay	-	-	3.581709
19-Jul-2002	0.095558	0.828542	Stay	-	-	3.581709
16-Aug-2002	-0.089775	0.904415	Stay	-	-	3.581709
20-Sep-2002	0.046133	0.819480	Stay	-	-	3.581709
18-0ct-2002	0.028766	0.854167	Stay	-	-	3.581709
15-Nov-2002	-0.015475	0.875638	Stay	-	-	3.581709
20-Dec-2002	0.006732	0.858581	Stay	-	-	3.581709
17-Jan-2003	-0.059449	0.861547	Stay	-	-	3.581709
21-Feb-2003	0.056144	0.807034	Stay	-	-	3.581709
21-Mar-2003	-0.002467	0.849570	Stay	-	-	3.581709
18-Apr-2003	0.056760	0.844716	Stay	-	-	3.581709
16-May-2003	0.054421	0.889757	Trade	Call	0.041000	3.805892
20-Jun-2003	-0.002380	0.934363	Trade	Call	0.050000	3.616180
18-Jul-2003	-0.002668	0.929239	Trade	Call	0.050000	3.442098
15-Aug-2003	0.046060	0.923920	Trade	Write	0.740000	3.501730
19-Sep-2003	0.002914	0.962776	Trade	Call	0.050000	3.368043
17-0ct-2003	-0.003887	0.962624	Trade	Stock	0.589000	3.360355
21-Nov-2003	0.051571	0.955211	Trade	Call	0.050000	3.653872
19-Dec-2003	0.046993	1.001395	Trade	Call	0.050000	3.985496
16-Jan-2004	0.003755	1.045241	Trade	Call	0.050000	3.871404
20-Feb-2004	-0.030006	1.045150	Trade	Call	0.050000	3.678426
19-Mar-2004	0.022374	1.010683	Trade	Call	0.050000	3.735404
16-Apr-2004	-0.036180	1.030131	Trade	Call	0.050000	3.549206
21-May-2004	0.037913	0.989060	Trade	Call	0.050000	3.734305
18-Jun-2004	-0.029629	1.023413	Trade	Call	0.050000	3.548362
16-Jul-2004	-0.002760	0.989946	Trade	Call	0.046000	3.385700
20-Aug-2004	0.027496	0.983138	Stay	-	-	3.385700

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Table 3.1 — continued

Date	IRR	ICR	TS	Pos	Lev	CR
17-Sep-2004	-0.018032	1.006689	Stay	-	-	3.385700
15-0ct-2004	0.056073	0.984968	Trade	Call	0.050000	3.729555
19-Nov-2004	0.020404	1.035408	Trade	Call	0.050000	3.808356
17-Dec-2004	-0.022065	1.052476	Trade	Call	0.050000	3.618725
21-Jan-2005	0.028873	1.024085	Trade	Call	0.050000	3.785470
18-Feb-2005	-0.009937	1.049304	Trade	Call	0.050000	3.597026
18-Mar-2005	-0.039533	1.034503	Trade	Call	0.050000	3.418185
15-Apr-2005	0.040836	0.989261	Trade	Write	1.817000	3.559392
20-May-2005	0.023275	1.023878	Trade	Call	0.050000	3.696769
17-Jun-2005	0.009006	1.042898	Trade	Call	0.050000	3.674523
15-Jul-2005	-0.006686	1.047351	Trade	Call	0.050000	3.491710
19-Aug-2005	0.014922	1.034061	Trade	Write	1.692000	3.576096
16-Sep-2005	-0.047112	1.044271	Trade	Call	0.050000	3.398237
21-0ct-2005	0.058224	0.988682	Stay	-	-	3.398237
18-Nov-2005	0.015261	1.040722	Trade	Call	0.050000	3.461981
16-Dec-2005	-0.004600	1.050925	Trade	Stock	1.160000	3.443613
20-Jan-2006	0.020412	1.038844	Trade	Write	1.648000	3.535223
17-Feb-2006	0.015545	1.054048	Trade	Call	0.050000	3.586740
17-Mar-2006	0.003083	1.064293	Trade	Write	1.677000	3.682968
21-Apr-2006	-0.033746	1.059750	Trade	Call	0.050000	3.499946
19-May-2006	-0.012225	1.017856	Trade	Write	1.843000	3.527732
16-Jun-2006	-0.008989	0.999262	Trade	Write	1.889000	3.582912
21-Jul-2006	0.049996	0.982565	Stay	-	-	3.582912
18-Aug-2006	0.013492	1.025182	Trade	Write	1.580000	3.662349
15-Sep-2006	0.036920	1.032459	Trade	Write	1.795000	3.767596
20-0ct-2006	0.023820	1.062142	Trade	Call	0.050000	3.956868
17-Nov-2006	0.018477	1.080583	Trade	Call	0.050000	4.073368
15-Dec-2006	0.002389	1.093607	Trade	Write	1.609000	4.169075

Continued on next page...

Table 3.1 — continued

Date	IRR	ICR	TS	Pos	Lev	CR
19-Jan-2007	0.017504	1.087583	Trade	Call	0.050000	4.284517
16-Feb-2007	-0.047123	1.099640	Trade	Call	0.050000	4.071979
16-Mar-2007	0.070226	1.041211	Trade	Write	1.947000	4.244281
20-Apr-2007	0.025870	1.105552	Trade	Stock	1.128000	4.367159
18-May-2007	0.006672	1.126998	Trade	Stock	1.128000	4.399820
15-Jun-2007	0.000776	1.127361	Trade	Write	1.770000	4.536837
20-Jul-2007	-0.057467	1.119347	Trade	Write	1.950000	4.195184
17-Aug-2007	0.055196	1.048366	Trade	Write	2.647000	4.594004
21-Sep-2007	-0.016464	1.097561	Trade	Write	1.510000	4.603576
19-0ct-2007	-0.027915	1.073092	Trade	Write	1.727000	4.556294
16-Nov-2007	0.017632	1.037068	Trade	Write	2.317000	4.884503
21-Dec-2007	-0.107292	1.047863	Trade	Write	1.706000	4.158887
18-Jan-2008	0.018714	0.930249	Stay	-	-	4.158887
15-Feb-2008	-0.015170	0.943074	Stay	-	-	4.158887
21-Mar-2008	0.045746	0.923496	Stay	-	-	4.158887
18-Apr-2008	0.025188	0.961861	Stay	-	-	4.158887
16-May-2008	-0.075364	0.982233	Stay	-	-	4.158887

Chapter 4

Stable trading strategy using multiple options

In the last chapter we have dealt with a stock which is expected to make profit and a call option on it. We looked at trading strategies that are stable under small model perturbations. For most stocks that are optionable, usually there are options with multiple strike prices available on that stock. Hence, it is a natural question to ask about a stable trading strategy when one have more than one call option available at hand, all with the same maturity.

We intend to carry out a preliminary analysis similar to the one option case and look for a stable strategy.

4.1 Notations

As before, we assume that the return on the stock we are considering is given by a random variable X, such that $E(X) > 0$. We shall consider call options with strike prices $1 - a_1, 1 - a_2, \cdots, 1 - a_n$, where,

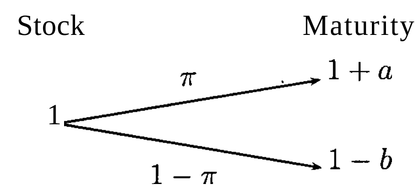
$$a_1 > a_2 > \cdots > a_n$$

We shall denote the market prices of such options by $p_1, p_2, \cdots, p_n$ respectively. We observe that to avoid arbitrage, we must have

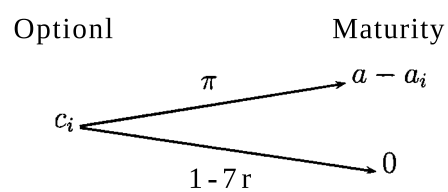
$$p_1 > p_2 > \cdots > p_n.$$

4.2 Worst case analysis

We look at the simple model that we used in the one option case. We look at a one period model with one stock that is expected to make profit.



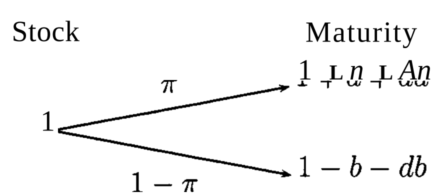
We now look at n call options with strike $(1 - a_i)$, $i = 1, \dots, n$, with $a_1 > a_2 > \cdots > a_n$. Suppose, c_i denotes the theoretical risk-neutral price of the call with strike $(1 - a_i)$.



It is easy to compute that the risk-neutral prices are the following:

$$c_i = \frac{b(a - a_i)}{a + b}$$

Now, like before, we look at the perturbed model:



and

Optionl Maturity

π $a — a; + da$

$(k + dci$

0

1 - 7T

Now, we look at a general portfolio of a shares of the underlying stock, fa amount of option *i*. So, under the perturbed model, the payoff can be expressed as:

Price of position Maturity

π i=i

$a + \sum_{i=1} (Ate + da))$

1 - 7 T

- M « + E A ST5) ~ a d b
»=I

+ E f t
i=i

We assumed that the expected return on the stock is positive, i.e. $e = na — (1 — 7r)6 < 0$. We also assume that the model perturbations of the stock price *da* and *db* are bounded in absolute value by some positive number 5 which is bigger than the perturbations in the option price and is very small compared to the expected return. That is, max(|o?a|, |d6|) < 6, and e >> 5 > max $(\backslash dci\backslash)$.

Using these assumptions, the worst case payoff for the perturbed model becomes:

Price of position Maturity

$i=1$ **a + E f t** »=i

EA a + dc)

i = l

π

a + E (A(Q + dci)) t=1

$1-\pi$

6 « + E A a+6 |a|<5 i=i

i = l

Hence, the expected return in the worst case scenario becomes, as a function of a and P^s:

$f\{a,/3_1,...,P_n)$ = $ela+[J2\&$ $a+b$ + tr a — (1 — 7r)|a| 5
$i=1$ j=i

ea;

E (a (a + b $dc.$
Z=1

Using the homogeneity of / in the variables a and $ft's$, it is sufficient to study the function on the L^1 sphere:

n

$i=1$ '

Using the piecewise linearity of /, we also observe that the maxima will occur at the corner points of the set $|a| + \sum_{t=i}^{n} |A| = 1>$ and those which satisfy either $a + \sum_{i=1}^{n} A = 0$ o r a = 0. Evaluation of those points yield that the maximum can occur at one of the following points:

- e_0
- $1 < i < n$
- $i(e_0 - ei), 1 < i < n$
- $\backslash\{e.i - ej), 1 < i < j < n,$

where, {eo, e i , . . . , e„} is the canonical basis for R™+1.

It is easy to observe that the first position corresponds to a pure stock position. Second type of possibilities correspond to the pure long call positions. Third type of points correspond to pure write position in one of the calls. The last type yields a pure vertical spread position.

Therefore, when dealing with *n* call options on a stock, the strategy that is stable under model perturbations is one of the following:

- Pure stock position
- Buying one of the calls
- Writing one of the calls
- One of the vertical spreads.

It is interesting to observe that more exotic positions like butterfly, strip, strap, straddle, strangle etc. do not appear in the list.

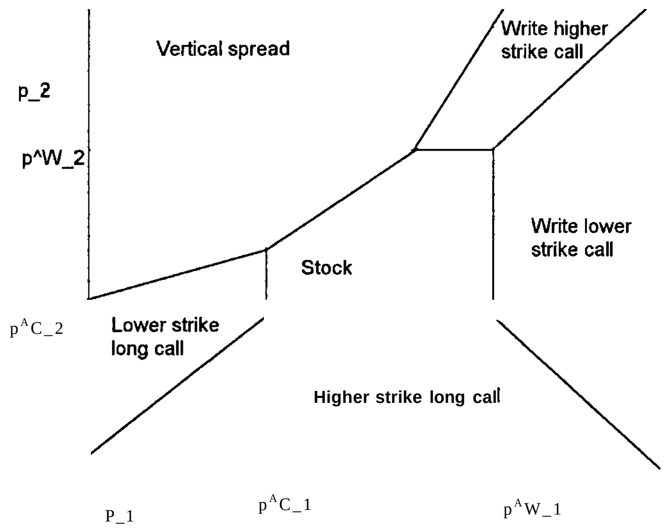


Figure 4.1: Premia horizon for two options
Schematic diagram showing the regions of the premia horizon where different trading positions have competitive edge

4.3 The strategy

Let, $\{S_j\}_{j\in I}$ denote the set of competing positions. It is apparent that the performance of an individual position depends on the prevailing market premiums for the options $\{p_i\}_{i=1,\dots,n}$. The strategy is to go with S_k when:

$$K(S_k) = \max K(S_j).$$

4.4 Two options

We now focus our attention to trading strategies involving two options on a profitable stock. We analyze the performances of available positions depending on the premium structure.

An example of the premium horizon, pointing out how strategies perform depending on the premiums is given in 4.1 (pp. 54). An explanation of the picture is given in Remark 4.4.2 (pp. 56) which follows from Theorem 4.4.1 (pp. 54).

Theorem 4.4.1. *Suppose, X denotes the return on the underlying stock with* $E(X) >$

0. *Suppose, Pi ,P2 ,pYiP™ denote the threshold premiums of the two competing options. Suppose,* $V(p\,p_2)$ *denote the return on the vertical spread withpi andp*$_2$ *being the premiums of the two options involved. Then*

(i) $K(V(p?,p^c_2)) < K\{X)$.

(ii) $K(V(pY,pY)) < K\{X)$.

Proof, (i) We know that:

$$= \frac{\{X + Qi)^+ — (X + a_2)^+ — \{pi — P2)}{Pi \sim P2} \wedge$$

Observe that: $K(V(pf,pC)) =$ $- p\$)V(_pc ,\$))$ and

$$pfc(X, o_1)_P f) = (pf -p\%)V\{pC,p°) +p\%c\{X, a_2,p_2).$$

Let,

$$Ci = pfc(X, ai.pf), \quad C_2 = p_2c(X, a_2,p_2)$$

and,

Then, we have:

$$C_x = v + c_2.$$

Let, Si be the leverage that maximizes /(I + sV), and let s_2 be the leverage that maximizes /(I + sC_2).

Observe that:

$$K(X) = K(C_2) = /(I + s_2C_2), \quad \text{and} \quad K\{V) = /(I + s_xV).$$

Define,

$$a = \frac{}{si} \quad \text{and} \quad \{\} = \frac{\&}{s_2}.$$

Then, we have,

$$\frac{\alpha}{\alpha+\beta}K(V) + \frac{\beta}{\alpha+\beta}K(X)$$

$$\frac{\alpha}{\alpha+\beta}K(V) + \frac{\beta}{\alpha+\beta}K(C_2)$$

$$\alpha + p \qquad \alpha + p$$

- f(l++ db S2Cy

$$\le K(C_1)$$

$$= K(X)$$

The result now follows.

The proof for part (ii) is similar. •

Remark 4.4.2. *This theorem proves that given a non-degenerate* L^1 *return, the premia horizon accommodates all the competing positions. And as we shall see later in the results, that each of these competing positions do show up as the preferable one.*

Without the vertical spread being a competing position, we known from the one option case that there is an area in the premia horizon, such that the stock performs better than buying any call option or writing any covered call. This will be the rectangular area with vertices (p f , p^7), {p\ ^2)1 (PiiPY)i (pYIPY)- *Observe that, the set:*

$$\{(p_1,p_2) : K(V(p_1,p_2)) = K(X)\}$$

is a straight line with slope 1, since, the premium for the vertical spread is just the difference between the two premia. Now, the theorem above shows that the line $\{(p_1,p_2) : V(p_1,p_2) = K(X)\}$ *lies above the diagonal of this rectangle formed by joining* (p_1^f,p_2) *and* (P\ iPY)- *Thus, the existence of all the competing positions is guaranteed for some combinations of the premia.*

In the figure demonstrating the existence of all the competing strategies, the line

$$\{(p_u p_2) \quad : \quad K(V(p_l, p_2)) \quad = \quad K(X)\}$$

passes below the upper left corner point (Pi,pY)• However, it is not obvious whether it is always the case. All the numerical examples we tried gave rise to that scenario, although we could not find a theoretical justification for that yet.

4.5 Results and discussion

We looked at the S&P-500 index with the same time horizon and same model as the one option case in the previous chapter. We used two options on the index each month. We used several combination of call options for testing our strategy. We present some representative results from those experiments in the next section. For the sake of comparison, we present results using the same parameters for the risk-reward function as used in the one option case. Just to refresh, we used:

$$f(X) = E(\log(X)) - {}_{tM}(CV@R_a(X)),$$

where $M = 0.95$ and $a = 0.05$.

We present the results in figures 4.2 (pp. 59), 4.3 (pp. 60) and 4.4 (pp. 61) and we discuss it below.

- We experimented with different combinations of call options that were available in the market. We started with two options nearest to the money to be able to compare with the one option case. The performance of the strategy involving two call options nearest to the money (spread $5) almost mimics the performance of the one option strategy although outperforming it marginally overall.

- In the next step, we looked at combination of options with the nearest out of the money option and another option with $20-$25 dollars into the money. We find that the the performance of the strategy with a larger ($20 to $25) spread performs significantly better than that of the $5 spread options (figure 4.2 (pp.

59)). We also note that, all of the possible trading positions show up in the trading strategy.

- We then looked into the effect of changing the spread between options when one option is the nearest out of the money and the other is into the money. We looked into changing the spread from $20 to $40 range. Although the performance is marginally better at the end, the $40 spread strategy do not perform consistently better over the $20 spread strategy (figure 4.3 (pp. 60)). We found that pure vertical spread positions disappear from the strategy in the larger spread. Going deeper into the money makes call options very expensive making it less profitable. Also, the relative advantage of writing a call diminishes as we go deeper into the money.

- Then we turned on to the strategies involving option combinations which spreads out of the money (figure4.4 (pp. 61)). However, there does not appear to be a gain in using options out of the money. Going far out of the money makes call options very poor positions as they are rarely exercisable. On the other hand, writing a covered call deeper out of the money is a more expensive position than writing near the money.

- When looking at symmetric spreads around the money, we found that they do not perform as well as spreads into the money. The performance is similar at stages but the inclusion of options out of the money tends to bring the edge of the strategy down.

- Summarily, including more than one option in the strategy appear to give an advantage over one option based strategies. For this particular index we looked at, the $20 spread into the money seemed to work the best. This allows us to look back to the one option case and suggest that using strike price into the money could yield better results than using the ones nearest to the money.

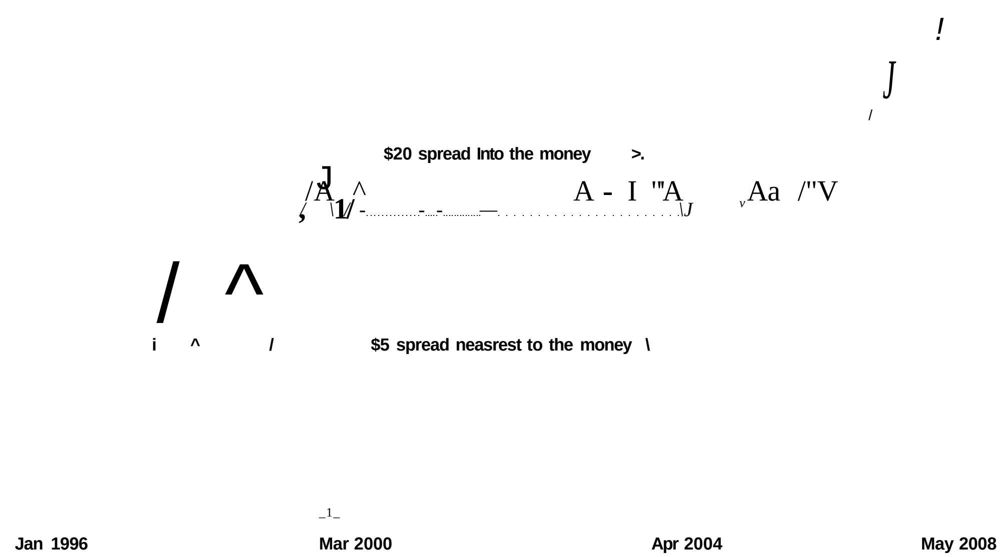


Figure 4.2: Performance with spread into the money
Comparison of two option based stable trading strategies with spreads $20 into the money and nearest to the money

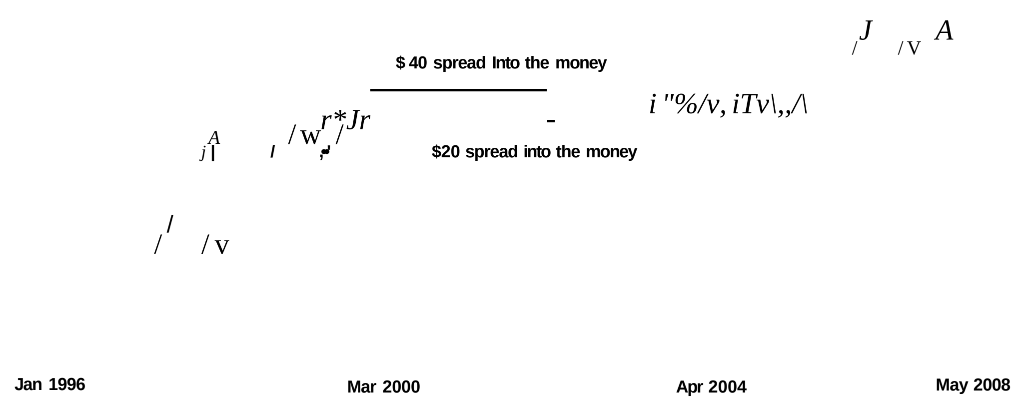


Figure 4.3: Comparison between spreads into the money
Comparison of two option based stable trading strategies with spreads $20 and $40 into the money

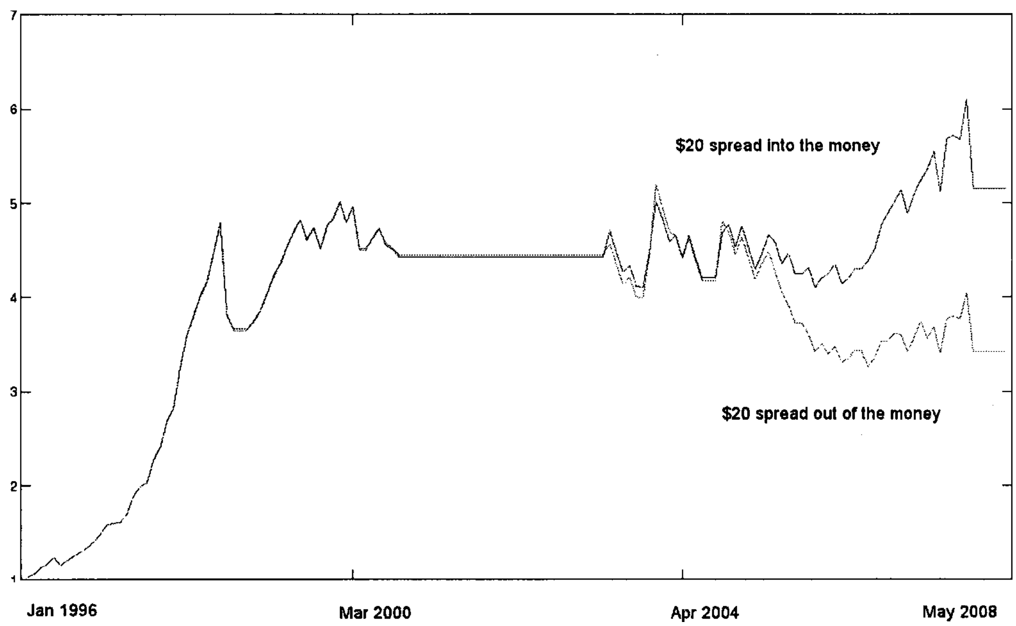


Figure 4.4: Performance with spread out of the money
Comparison of two option based stable trading strategies with spreads $20 out of the money and nearest to the money

Chapter 5

Option pricing for stocks with predictable price jumps

5.1 Motivating example

Dendreon Corporation is a small biomedical firm with a market capitalization of $284m. It has only one main drug candidate Provenge that is promising for treating a certain cancer. The company is in the stage of seeking FDA approval of the drug. Interesting stock price move (Jumps) happened in the period 3/30/2007—5/09/2007. Prior to 3/30 people knew that a FDA decision on DNDN's drug Provenge will come out in middle May which are likely to causing a jump in its stock price [Feu07]. So the premium of near the money option maturing in May have already in the 20-30% range early in March. The price of the stock indeed jumped: first up on 3/30 (the drug was recommended by an advisory committee) [Reu07, Den07] then down on 5/9 (FDA asked DNDN for additional data so that a decision on the drug is unlikely in the near future) [Her07] .Two years later the price of Dendreon stocks jumped again on 4/14/2009.

Dendreon is not alone in experiencing such 'anticipated' stock and option price jumps. In fact, this seems to be typical among small biomedical firms with a narrow line of drug or treatment candidates. Typically the abnormally high optional premium begin to appear when trial results or FDA decisions for their products are anticipated in the near future. Usually, the actual news will result in a big jump in the stock price as the survival and prosperity of these firms usually hinges on the success of these products. There are several other cases listed below that can help to understand the mechanism alluded to above.

Table 5.1: Price jumps

Abrupt price jumps in a day's trading for some biomedical companies

Ticker	Date of price jump	Price before jump	Price after jump
ALTH	30-Apr-2004	4.78	2.55
DNDN	30-Mar-2007	5.22	12.97
DNDN	09-May-2007	17.74	6.33
NABI	01-Nov-2005	12.85	3.03
NBIX	16-May-2006	54.63	20.76

Stock price jumps at the release of an important information is common. Small biomedical firms demonstrate jumps of unusual magnitude making them a good point to start our investigation.

5.2 Problems

Our observation is that definitive news on pending announcement of FDA decisions or drug clinical trial results on small biomedical companies with a narrow line of products will elevate their option premium for options covering the period when the news will be announced. This is because such news will indicate whether the company will survive and prosper or fail and as a result causing big stock price jumps either up or down. The question is whether this observation is supported by facts and if so can we quantify this observation. More precisely, what information about the pending price jump can be extracted from the behavior of the option prices.

5.3 Stock price model and option pricing

The news of a pending key decision is accompanied by by high option prices. Traditional Black-Scholes model [BS73] proves inadequate in explaining the situation as it only points to a high volatility of the underlying stock price. On the other hand, Merton's jump diffusion model [Mer76b] while allowing for jumps in the process, requires the jumps to arrive at a random time together with a randomly distributed jump magnitude. In the case of small biomedical firms, the arrival of the news of a pending decision is somewhat unpredictable. But, once the news arrives, the date of the information release is mostly known.

5.3.1 Model for the stock price process

Use r to denote the time when news breaks out that a FDA decision is pending or an important trial result will be announced at r + c. Our observation is that options with maturity T > r + c show unusual increase in their premium after r, while options with maturity T < r + c do not have similar premium increase. This is, of course, due to the expectation that the FDA decision or the trial results are being announced at **T+C** will cause a big jump in the stock price. While the actual jump of the stock price usually happens at r + c, the investors' opinions on the underlying company will be a mix of optimistic and pessimistic outlooks prior to the final verdict. The information at r about pending important announcement serves as a catalyst to prompt investors into forming this mixed view on the company's prospect after **T + C**. Since market agents incorporate future events into the price of a company's stock, we hypothesize that agents will price the stock base on this mixed view of what will be announced at $T + C$ after r. This will cause a major change in their view of the value of the stock at r. Thus, we model the stock's price St evolving to r with a jump diffusion model

$$d \log S_t = fidt + adB_t + rjd\{N_t - \mathrm{A}t). \qquad (5.3.1)$$

Here N_t is a Poisson process marking the number of jumps up to time t with intensity A and r) is a constant jump size to be specified and explained later. Since the Poisson process determines the arrival of r, we have r = inf{£ > 0 such that N_t = 1}.

Next we model the price process after r. We will use two stock price processes and Sj^{2}^ to represent the optimistic and pessimistic views, respectively. The optimistic view is that a good news is coming at r + c and, therefore, the stock worth much more than its current price so that *S^* = *q\S_T-* with > 1. On the other hand, the pessimistic view value the stock's price at r on a much lower level = $g_2$5V_ with < 1 in anticipation of a bad news coming at r + c. The stock price processes *S^*, *i* = 1,2 are described by two diffusion processes

$$d\backslash og(sP) = twit + (ndB_u z = 1,2, \qquad (5.3.2)$$

representing the two opposing point of views until r + c. At r + *c* the anticipated FDA decision or the trial result is announced and it becomes clear which process *(i* = 1 or 2) prevails. Between r and r + c, the two opposing views will coexist and the effect can be modeled by placing a probability 0 (or 1 — 0) as to how likely the process S^{1}^ (or the process S ^) will eventually be materialized.

5.3.2 Risk neutral measure and related stock dynamics

In order to determine how the stock price and option prices behave, we need to impose a pricing kernel which in essence determines how the risk neutral stock price dynamic will look like. We assume a typical pricing kernel for jump-diffusion models [DPSOO] so that the stock price dynamic for 0 < *t* < r under the risk neutral equivalent martingale measure *Q* becomes

$$d \log S_t = (r - \wedge p j d t + adB_t + r)d\{N_t \sim \% \qquad (5.3.3)$$

Note that the jump intensity under measure *Q* (i.e., A) is allowed to be different from that under measure *P* (i.e., A). Since the jumps that indicated by q_x and q_2 are constants, it is unaffected by the measure transformation. However, the probability of which diffusion model will prevail after r may be subject to change due to the use of measure *Q*, and we denote the new probability by 4>.

In equilibrium, we note for *t* G [r, r + c)

$$St = 4>S\wedge + (1 — 4>)Sj;^{2}\backslash \qquad (5.3.4)$$

and for $t > r + c$

$$S_t = S_t^{(1)} \text{ or } S_t^{(2)}, \tag{5.3.5}$$

with probability ϕ or $1 - \phi$, respectively. In particular at $t = r$,

$$S_T = \quad + \quad = \quad + (1 - \tag{5.3.6}$$

which in turn implies $\eta = (S_T - S_{T-})/S_{T-} = \phi q_1 + (1 - \phi)q_2 - 1$. In short, equilibrium requires one of the parameters (q_1, q_2, η, ϕ) to be redundant. Under the risk neutral measure Q, the perceived stock price processes $S^{(i)}$ satisfy the diffusion equations

$$d\log S_t^{(i)} = (r - ?p)dt + *_i dB_t^{(i)}, \quad S_T^{(i)} = q_i S_{T-}, \ i = 1, 2, \tag{5.3.7}$$

where $\quad = 1,2$ are standard Q-Brownian motions.

In this model, the parameters needs to be estimated are q_1, q_2, ϕ, ϕ, A, A, μ, μ_1, μ_2, a, σ_1, σ_2 and η. We will back out these parameters from the actual stock and option data. To do so, we need to derive concrete option pricing formulae according to the model.

5.3.3 Option pricing formulae

Using $\mathcal{F}_t$ to denote the filtration representing information available up to time t, we note that $B_t, B_t^{(i)}, i = 1,2$ are Q-Brownian motions and N_t is a Q-Poisson process with intensity A relative to $\mathcal{F}_t$. Moreover, S_T is Markovian with respect to $\mathcal{F}_t$. We consider, at time t, the theoretical price of an European call option with strike price K and maturity date T. We calculate this price using the risk neutral measure Q by

$$p(S_T, T) = E_{\mathcal{F}_t}[e^{-r(T-t)}(S_T(R) - K)^+]. \tag{5.3.8}$$

It is worth emphasizing that in general the random variable S_T depends not only on T but also on r (if $r < T$) due to the jump in stock price at r. Thus, we write $S_T = S_T(T)$ to indicate explicitly this dependence. As discussed before $S_T(T)$ depends on the two random variables $S^{\wedge}(r), i = 1,2$. They are geometric Brownian motions with exactly one jump at r if r < T and can be explicitly represented. We summarize the representation in the following lemma.

Lemma 5.3.1. *Suppose that $t < \mathbf{r} < T$. Then*

$$4^{<\}}(r) = {}_{gi}S_t \exp((r - \quad - t) + a^{\wedge}r)B<?_{-t}^{\wedge}) , \tag{5.3.9}$$

where $B^{\wedge}$ are standard Q-Brownian motions and

$$*<(r) = \overline{\quad} \overline{\wedge \quad} .$$

Proof.

$$
\begin{aligned}
4(r) &= {}_{9i}5_{T-} \exp^{\wedge}(r - ^{\wedge})(T - r) + 07,41) \\
&= {}_{gi}S_t \exp ^{\wedge}(r - \tfrac{a^2}{y})(r - t) + aB_{r-t}J \exp [\{r - \quad af \quad - r) + ^{\wedge}4 - t \\
&= \quad \exp((r - ^{\wedge})(T - t) + aB_{r-t} + <7*4^{\wedge} ,
\end{aligned} \tag{5.3.10}
$$

where

$$<j = \frac{a^2(r-t) + a^2(T-r)}{T - t}$$

Since, B and $B^{\wedge}$ are both standard Q- Brownian motions, we can write,

$$aB_T-t + \quad =$$

where,

$$5?(r) = \frac{a^2(r-t) + a^2(T-r)}{T - t}$$

and is a standard Q- Brownian motion.

Hence, we can write

$$4V) = qiSt \exp((r - ^{\wedge})(T - *) + ^{\wedge}(t)4^{\wedge} , \tag{5.3.11}$$

where B^ is a standard Q-Brownian motion. •

Using the representation in Lemma 5.3.1 (pp. 67) we can calculate the theoretical option price $p(S_t, T)$. We will need the standard Black-Scholes European call option formula $CssiSt, K, s, r, a)$ and the parameters are spot price S_t, strike price K, time to maturity s, risk free rate r and volatility a. We also use the convention

$$C_{BS}(S_u K,s,r,a) = S_t\text{-}Ke\text{-}^{rs},$$

whenever, $K < 0$.

Now we can summarize the computation formula in the following theorem.

Theorem 5.3.2. ***Part 1:*** *Suppose, $t < \mathbf{r}$ and $T > t + c$. Then $p(St,T)$ given by:*

$$p(S_t,T) = El_{St)}[e^{\wedge T}\text{-}HST(r)\text{-}Ky] \quad (5.3.12)$$

has the following representation:

$$E^Q \; -r\{T\text{-}HSt(t) \sim K)^+]$$

$$\text{----} \; e^q \; [e^{\sim -r(T-}{}_{-HMT)} - K)^+ (l[t,T\text{-}c](T\sim) + l(T\text{-}c,T\backslash(r) + \mathbf{l(T,oo)(r))]}$$

$$= \; I_7 q \; [e^{\sim -r(T-}\text{-}HST(t) - K)^+ l_{[t^\wedge c]}\{r)\backslash$$

$$+ \; e^q \; [e^{\sim -r(T-} - K)^+ 1_{\{T^\wedge T\}}(t)]$$

$$+ \; A^{Q}(tA) \; [e^{\sim -r(T-}\text{-}HSt\{T) - K)^+ l_{iTt00)}(r)]$$

Let us denote the three integrals as pi, p_2 and $p^\wedge$ respectively. Then $p\backslash$, p_2, Pz can be written in computation friendly forms as following: (i)

$$Pi = \int^{T-c} [\&CBs(qiSt, K,T - t, r, <JI(t))$$

$$+ (1 - \wedge)C_{B5}(q_2 5_t, K,T\text{-}t, r, a_2(r))\}\backslash e\text{-}^{XT}dr \quad (5.3.13)$$

(ii)

$$p_2 = \frac{1}{j2n\{T - t)} J \int_{-oo}^{oo} \int_{T-c}^{T} \backslash e\text{-}^{XT} e\sim arf\wedge$$

$$\times CBS\wedge QISUK\wedge t, \ y), T — t, r, \ a_x(r))dTdy$$

$$\bullet v/27r(T - *) \int_{-oo}^{oo} \int_{T-c}^{T} \backslash e\sim{}^{\prime XT} e\sim\wedge\wedge$$

$$\times C_{BS}((l - \quad K_2(T, \ z), T - t, \ r, \ a_2(r))drdz, \qquad (5.3.14)$$

where

$$K_x(T, y) = K\text{-}(1 - (f>)Stq_2 exp \ ((r - \quad - i) + \wedge_2(r)y$$

and

$$AT_2(r, z) = AT - \wedge exp((r - \wedge)(T - t) + <7i(r)z$$

(iii)

$$P3 = CBs(St, K, T — t, r, c r. \qquad (5.3.15)$$

Part 2: *Suppose,* t < r and T < t + c. TTien $p(S_t,T)$ *given by:*

$$p(S_t,T) = 4_{5t)}[e\text{-}\wedge\text{-}^4)(5_T(r)\text{-}\wedge)^+] \qquad (5.3.16)$$

/ias *the following representation:*

$$p(S_t,T) = El_{St)}[e\text{-}<^T\text{-}HS_T(r)\text{-}K)^+]$$

$$= El_{St)}[\wedge\text{-}"(SHr) - \wedge)^+(l(t,T](r) + l(r,oo)(^r))] \qquad (5.3.17)$$

$$+ E\wedge_{St)}[e\wedge^T\text{-}HST(r)\text{-}K)^+ l_{\{T<oo)}(r)]$$

Let us denote the three integrals as and respectively. Then p± *and* p$ *can be written in computation friendly forms as following:*

(ii)

$$P4 = \frac{e^{-r(T-t)}}{\sqrt{\ldots(T-t)}} \int_{-\infty}^{\infty}\int_{t}^{T} \ldots \, x C_{BS}(\ldots S_t, K_1(r,y), T-t, r, \ldots) \, dr\, dy$$

$$\frac{\ldots}{(T-t)} \int_{-\infty}^{\infty}\int_{t}^{T} \ldots \, x C_{BS}((1 - \ldots \, f_2(r, \ldots T-t, r, a_2(r)) \, dr\, dz,$$

$$\frac{e^{-r(T-t)}}{\ldots(T-t)} \int_{-\infty}^{\infty}\int_{T}^{\ldots} \ldots$$

$$x \; \ldots S_t \ldots e^{r(\ldots T-\ldots)} \; 1 - \ldots \; f_1(r,y) \ldots$$

$$+ \; \ldots C_1(r,y) \; 1 - \ldots \; dr\, dy \qquad (5.3.18)$$

where K_1, K_2 and δ are as defined in part 1 of this theorem, (v)

$$P_5 = C_{BS}(S_t, K, T-t, r, \ldots(T)) e^{-\lambda(T-t)} \qquad (5.3.19)$$

Proof. Suppose $p(S_t, T)$ denotes the price of a call option with strike price K, maturity T and current stock price S_t.

We know,

$$p(S_t, T) = E^{Q}_{(t,S_t)}[e^{-r(T-t)}(S_t(T) - K)^{+}]$$

$$= E^{Q}_{(t,S_t)}[e^{-r(T-t)}(S_T(T) - K) \; (1_{[t,T-c]}(\tau) + 1_{(T-c,T]}(\tau) + 1_{(T,\infty)}(\tau))]$$

$$(5.3.20)$$

Case 1: $\tau < T - c$.

At T we would have known whether $S_T(T) = S^{1}(T)$ or $S_T(T) = S^{2}(t)$. However, since no one can know which case will be the reality before $T+C$, for pricing the option, we need to assign them the risk neutral probability Φ and $1 - \Phi$, respectively. Thus,

$$\mathbf{Pi} \;=\; E\, l \wedge l\, e - \wedge i\, S\, r \wedge - K\, y\, i \wedge T\,)\,] \qquad (5.3.21)$$

$$= EL_{ST)} \quad FE_{T)}[E^{-R(T-T)}(5T(R) - K)+l_{[ttT-c]}(T)\backslash T_T$$

$$= E^{Q} \; 1_{[t,T-c]}(r) Eg_{5T)}[e^{-\wedge T-<})(5_T(r) - K)^{+}]\backslash \quad \text{(Markovian property of } S_t)$$

$$= \int^{T-c} \sim\wedge_{-}^{XT} E l_{St)}$$

$$+(! - \wedge K \wedge J \wedge ' H S \wedge i r) - K) \quad dr$$

$$= * \int^{T-c} \wedge\sim^{XTE} l_{S_t)} \; E:^{Q}_{(T,<71S_{T-})1} \; r_a{-r\{T-t)(q}W\{S\$>\{T)-K)^{+}] \quad dr$$

$$+(1 \sim<t>) \int^{T-c} \sim\wedge\sim^{XT} E?_{t<St)} \; [<_{92}5_{T-})[e^{-r(T-t)}(4^{2)}(r) - K)+] \quad dr.$$

The two integrands remind us the Cox-Ross risk neutral option pricing. Indeed, they can be expressed in terms of Black-Scholes call option formulae and, thus, become more convenient in terms of computation.

We know from Lemma 5.3.1 (pp. 67),

$$S«(r) = {}_{gi}S_t \exp((r - \wedge)(T - t) + \quad \mathbf{W}),$$

where B^ are standard Q-Brownian motions. Thus, the two terms in p\ can be calculated as integrals of Black-Scholes call option formulae. More precisely,

$$\boldsymbol{Pi} = \int^{T-c} [4>C_{\boldsymbol{B}}\boldsymbol{s}(qiS_t, K, T - i, r, CTi(r))$$

$$+(1 - tfCBsfaSt, \; K, T - t, r, a_2\{r)))\backslash e^{-XT} dr. \qquad (5.3.22)$$

Case 2: r G (T - c,T],

Since at T whether the optimistic view or the pessimistic view will prevail is unresolved, we have

$$p_2 = E \wedge e - \wedge i S T i r) - K)^+ l_{iT} \wedge_{T]} (r)\} \quad (5.3.23)$$

$$= E \wedge J E ' \wedge W \wedge I R) + (1 - \$) S ? \backslash T) - \wedge] + 1_{(x-c, T1}(r)].$$

Using 5.3.11 (pp. 67) we can write p_2 as a triple integral

$$P2 \; \underset{T-c \; -oo \; -oo}{\overset{T \; oo \; oo}{HI}} e\text{-}'V\text{-}VXe\text{-} \; \backslash t \; \frac{e^{\;2(T-t)}}{y/27r(T - i)} \; \frac{e^{\;2(T-t)}}{V2tt(T - i)} \quad (5.3.24)$$

$$x \; 4{>}qiStexp \; (\{r - \quad - *) + \wedge(r)\wedge$$

$$+(1 - 4{>})q_2 S_t exp \; f(r - \quad -1) + \wedge(r)\wedge - \quad {}^+ dzdydr.$$

Denoting

$$K_1(T,y) = K\text{-}(l\text{-}4{>})q_2 S_t exp((r \quad)(T - t) + a_2(r)y$$

formula 5.3.24 (pp. 72) becomes

$$P2 = \underset{r-c \; -oo}{\overset{T \; oo}{J \; J}} \frac{e \; 5ir{=}iy}{J2tt(T - t)} \overset{oo}{J} e^{-r(r-f)} \frac{e}{x/2tt(T - t)} \quad (5.3.25)$$

$$x \; 4{>}qiS_t exp \; f(r - \quad -1) + \wedge(r)\wedge - \wedge(r, </)' \quad dzdydr.$$

We now recognize that the inner layer z-integral is

$$C_{BS}(HIS_U \; K_X(r, y), T - t, r, \wedge(r)).$$

Thus, P2 can be written in the following more concise and computation friendly form

$$P2 \quad \frac{1}{\wedge 2tt\{TT - t1)} \underset{T-c-oo}{\overset{T \; oo}{I \; I}} \backslash e\text{-}*^T e\text{-}*f{=}*C_B s(<hiSu \quad K,(T, \; y),T - t, \; r, \; dzdydr.$$

Symmetrically, we can also represent p_2 as

$$p_2 = \frac{1}{\sqrt{2\pi(T-t)}} \int_{T-c}^{T} \int_{-\infty}^{\infty} Xe^{-\lambda T} e^{-\wedge} C_{BS}((1-\phi) q 2 S_t, K_2(T,z), T-t, r, \sigma_2(r)) \, dz \, dT,$$

where

$$K_2(T, z) = K - FASTEXP \, (V - \quad - t) + \wedge(r) \wedge$$

Case 3: $T < r$.

In this case there is no jump before the maturity of the option. Thus, $S_T = S_T(T)$ is, in fact, independent of r. Since the continuous geometric Brownian motion stock price dynamic and the jump process are independent we have

$$= E I_{S_t)}[e^{-\wedge} - H S_T - K)^+] Q[r \wedge [t,T]\}$$

$$= C_{BS}(S_t, K, T-t, r, cj) e^{-\wedge T - t}$$

Part 2: Proof of part 2 is similar to that of part 1. We just observe that the computation of $P4$ is similar to that of p_2 and the computation of p_5 is similar to that of p_3 from part 1. •

Theorem 5.3.3.

Part 1: *For,* $r < t$ *and* $T > r + c$,

$$p(S_U T) = \phi C_{BS}(S?\backslash K, T - t, r, \sigma_1) + (1 - \phi) C_{BS}(SP, K, T - t, r, \sigma_2). \quad (5.3.26)$$

Part 2: *For* $r < t$ *and* $T < r + c$,

$$p(S_t,T) = \frac{1}{\sqrt{2N(L - T)}} \int_{-\infty}^{\infty} e^{-\wedge} CBstisP \wedge tiylT - t \wedge a j \, dy$$

$$\frac{1}{\sqrt{2\pi(i - t)}} \int_{-\infty}^{\infty} 2$$

where,

$$KM = K\text{-}\{ I - 0)S_t^{(2)}\exp ((r - \qquad - t) + a_2 y\hat{}j ,$$

and,

$$K_4(Z) = K - 0S_t^{(1)}\exp ((r - \qquad - t) + \qquad .$$

Proof. **Part 1:** $T > T + C$.
This is similar to Case 1 in the proof of Theorem 5.3.2 (pp. 68). Like before we have

$$0S_t^{(1)} + (1 - 4>)SP = St \tag{5.3.27}$$

and

$$
\begin{aligned}
p(S_t,T) &= E\hat{}_{St]}[e\text{-}\hat{}(S_T(r)\text{-}K)^+\} && (5.3.28)\\
&= \hat{}\ \hat{}\ [e - \hat{} - H S ? - K)^+] + (1 - 4>)E^* \qquad - K)\\
&= 0C_{BS}(S_t^{(1)},K, T — t,r, a,) + (1 - <t>)C_{BS}(S_t^{(2)}, K,T — t, r, a_2).
\end{aligned}
$$

We note that S_t is observable while S'f[1]' and $S\hat{}$ are not. Due to the constraint 5.3.27 (pp. 74) only one of them is a free parameter need to be estimated

Part 2: $T < T + C$.
This is similar to Case 2 in the proof of Theorem 5.3.2 (pp. 68). We know that,

$$0S_t^{(1)} + (1 - \qquad ^{(2)} - \tag{5.3.29}$$

and

$$
\begin{aligned}
p(S_t,T) &= {}^{J}e_{(t,s_t)}V\ \hat{} - H S T - K)\text{-} && (5.3.30)\\
&\approx E^Q_{(t,St)}\ {}^{-R(R_n T)}[05?^{)} + (1 - 0) 4^{2)} - K\backslash
\end{aligned}
$$

Expressing $S\hat{}p$ and $S^{(2)}$ as geometric Brownian motions, we can write $p(S_t, T)$ as a double integral

$$KS.D = \int\int -r(T-t)\_ \quad g \; 2(T-t) \quad g \; 2(T-t) \quad y/2ir(T - t) \; v/27r(r - *) \tag{5.3.31}$$

^ e x p l C r - ^ K T - t J + ^z

+(1 - }exp ((r - °j-)(T - t) + a2y)-K dzdy.

Denoting

$$K_3(y) = K-(1 - 0)5_t^{(2)}\exp((r - \hat{})(T - 1) + a_2 y),$$

formula 5.3.31 (pp. 75) becomes

$$" \int e^{\sim 2(T-t)} \; y/2ir(T-t)\_ \; \int -r(r-t). \; e^{2(r-J)} \; VMT \; -1) \tag{5.3.32}$$

<£S$_t^{(i)}$ exp f(r)(T-t) + a_{lZ})-$K_3(y)$ dzdy.

We now recognize that the inner layer z-integral is

Thus, $p(S_t,T)$ can be written as an integral of a Black-Scholes call option formula

$$p(5_t,T) = \frac{}{VMT(T-1)} \int_{-w}^{oo} e^{-\hat{}C_B s(4>S?\backslash K_3(y),T - f, r, a_x)dy.}$$

As before, we can also symmetrically represent $p(St, T)$ as

$$p\{St,T) = \frac{1}{VMT \; -1)} \int e^{-Wr\hat{}C_{BS}((l - <P)S\hat{},K_4(z),T - t, r, a_2)dz,}$$

where

$$Kt(z) = K - 05_t^{(1)}\exp\ (V - \qquad - t) + \qquad .$$

•

5.4 Estimation of parameters

Let us recall that we used r to denote the time when the news of a pending important decision comes out. The time of such pending decision we denoted by r + c. We also noted that the option prices for options with maturity after $T + C$ show abnormal behavior indicating a fundamental change in the underlying price process.

Our goal in this investigation is to see if we can use option prices during (r, $T + C$) to predict any jumps that will happen in the future. From Theorem 5.3.3 (pp. 73) we see that such information is encoded in 6 where:

It is apparent that our parameters of interest are not directly observable parameters. But, nonetheless they do appear in the option prices formulae that we calculated in the previous section. So our goal is to estimate these parameters from the option prices. Before **T**, the stock price behavior should be similar to that of any other typical stock price.

We only concentrate on estimating these model parameters at a time after r. Usually, for these small companies, really long term options are not popular at all. They are rarely traded and hence do not have reliable price data, if any. The general trend is that the options maturing in next two months are the ones that are traded. There are exceptions to this general phenomenon. One situation where options are traded for a long period is when the maturity of the options are right after a key pending FDA decision. In this case also, the options generally start trading when news of such pending decisions are known. Although, theoretically, the options prices do carry information about all the model parameters, it is not a useful exercise from a practical point of view, for the reasons mentioned above.

Before starting the estimation process we note some obvious restrictions on the parameters of interest:

$$4> \in (0,1),\ <71, <72 > 0, \text{ and } 4>sl^{1]} + (1 - 0)S_t^{(2)} = S_t.$$

We observe that only one of the parameters and S^ are free. Let, ***yt;r,K*** denote the market price of an option at time *t,* with strike price *K* and maturing at *T.* Denote by *I,* the set of all triplets (t, T, K) such that ***jj(t,T,K)*** is observable.

Let, /(© : *t,T,K)* be the price of a call option given 0, at time *t* with strike K, maturing at T, as given in Theorem 5.3.3 (pp. 73). Then, we can estimate the parameters by solving a weighted least square problem:

$$\min_{\Theta} \sum_{(t,T,K)\in I} \left(\frac{f(\Theta : t, T, K) - y_{t,T,K}}{y_{t,T,K}} \right)^2 .$$

This particular choice of weights for the least squares make the problem that of minimizing sum of relative square error. The reason for using relative error is that investors are interested relative return on an investment.

This minimization problem is highly non-linear in nature and thus becoming a computational challenge. Also, the very nature of the pricing formula uses several layers of integration making it computationally expensive. After studying the function carefully, we decided to resort to the blunt force method of grid search. Conducting a grid search on four parameters can be very time consuming.

The pricing formula for options maturing before $T + C$ has a triple integral which cannot be further simplified. And this was the most expensive part of the computation. We observed that the same formula only depend on the value of fiS^. We used this observation to change the grid from parameters $\{0, a_i, a_2\}$ to ox, $er_2\}$. This was a significant savings on the computing time.

5.5 Results and discussion

5.5.1 Numerical tests

We looked at the companies classified as biomedical firms according to the Reuters. We further looked into a smaller class of companies that were optionable on the Chicago Board of Options Exchange during some part of the years 2001 to 2007.

There were one hundred and sixty six of those companies.

We collected the daily closing prices of all the options on all those companies on the Chicago Board of Options Exchange. Data were collected using the data service of Wharton Business School, University of Pennsylvania. The data collections were courtesy of the Risk Management Institute of National University of Singapore.

We concentrate on estimating the pending jump after r. Usually, for these small companies, the options are introduced in the market two months before their maturity. The time c between the news and the pending decision is in general longer than two months. Hence, it is rare for an option to be found such that it is introduces before r and will expire after $T + C$. For the same reason, options before r will carry little information about a future jump.

Ideally we need a good way to estimate the time when the news of a pending decision breaks out (r in our notation). However, we found it difficult to do a systematic news search as most of the reliable new sources were proprietary and subscription based.

We looked for a reasonable objective proxy that can be observed from the option data itself to tell us that $t > r$. We know that arrival of the news is accompanied by high premiums for options maturing after the pending decision ($T + C$ in our notation). We looked for nearest to the money options that had price at least 15% of the stock, price. Also, this high premium should be so high that it cannot be explained by the volatility of the underlying stock. So, we compared these high option premiums with Black-Scholes prices computed using historical volatility published Chicago Board of Options Exchange [0OEO8]. We selected the stock, if the market price of the nearest to the money options were more that 90% above the Black-Scholes premium estimated from the historical volatility.

We did not need to find out the exact date of the pending decision (r + c) as it has no bearing on the option price as long as it matures after that date.

After this screening we also looked at the trading volume of those nearest to the money two months prior to expiry. We considered them for our purpose only if they had a trading volume of at least 100 contracts[1]. This is to ensure that there is some liquidity in those options and there is interest about those options among the

[1]One contract of call options in Chicago Board of Options Exchange is a bundle of one hundred options with same strike and maturity

investors. Options on these small biomedical firms usually have a very low trading volume, no trading on most days. Checking for trading volume ensures to some extent the reliability of the price data.

We summarize our screening scheme below

- We looked at all the biomedical firms that were optionable on Chicago board of options exchange during some part of 2001 - 2007.

- We looked at nearest to the money options two months prior to expiry

- We checked for the premia of those nearest to the money options for being above 15% of the stock price.

- We compared those prices with Black-Scholes price computed with historical volatility published by Chicago Board of Options Exchange. Options were selected if they were priced more than 90% above the Black-Scholed price.

- We allowed options with trading volumes at least 100 contracts on the trading day.

For the estimation part we looked at the options estimated to mature after the pending decision. We fit our model using least squares on the trading date 5 weeks prior to the option's maturity. The rationale is that we can only estimate the time of price jump to be between the two option maturities.

We present the numerical results in the Table 5.2 (pp. 82). A sample graph is presented in Figure 5.1 (pp. 83) to the difference between the geometric Brownian motion estimate of the density of the future price and our estimate.

5.5.2 Discussion

We discuss our observations from the experiments below.

- and S^ are designed to predict the views of the optimistic and pessimistic camps of investors respectively. In most of the cases the estimated values of those parameters show a significant split in their opinions. They are usually quite far apart to indicate that there is a pending jump.

- The traditional geometric Brownian motion model (Black-Scholes-Merton model) can be seen as a special case of the model we considered. Thus fitting our model to the data implicitly compares it with the geometric Brownian motion model as well. However, in most of the cases we considered here, the estimates of our parameters are quite far from that of a single geometric Brownian motion process.

- In the table 5.2 (pp. 82) we presented an interval that we called the "anti-confidence" interval. This represented a region such that both of the optimistic and pessimistic processes put a very small probability for the future stock price to lie in that region. We can see that in all of the cases there was prediction of jump according to this anti-confidence interval, future price of the stock was outside this region.

- One of the challenges in dealing with this data set was the lack of trading volume on these options on most days. Chicago Board of Options Exchange lists the price quotes for all the options everyday they were available for trading. However, we noticed that often trading volume for such options are zero to negligible. Lack of trading volume certainly have implications about the reliability of the price quotes. Also, many of these companies' stock prices are very low to remain optionable throughout the period we focused on. Another key feature of options listed in Chicago Board of Options Exchange is that the strike prices of options are usually $5 apart. This restriction makes the available strikes of options on a cheap stock very sparse. For example, if the stock price of an asset is around $5-$6, the only available into the money option is that with a strike price $5. Also, the out of the money option that is nearest to the money has a strike of $10, which is almost double the strike price. Another issue we encountered was the unavailability of the precise information on r and $r + c$. All these factors contribute to limit the capabilities of the underlying model.

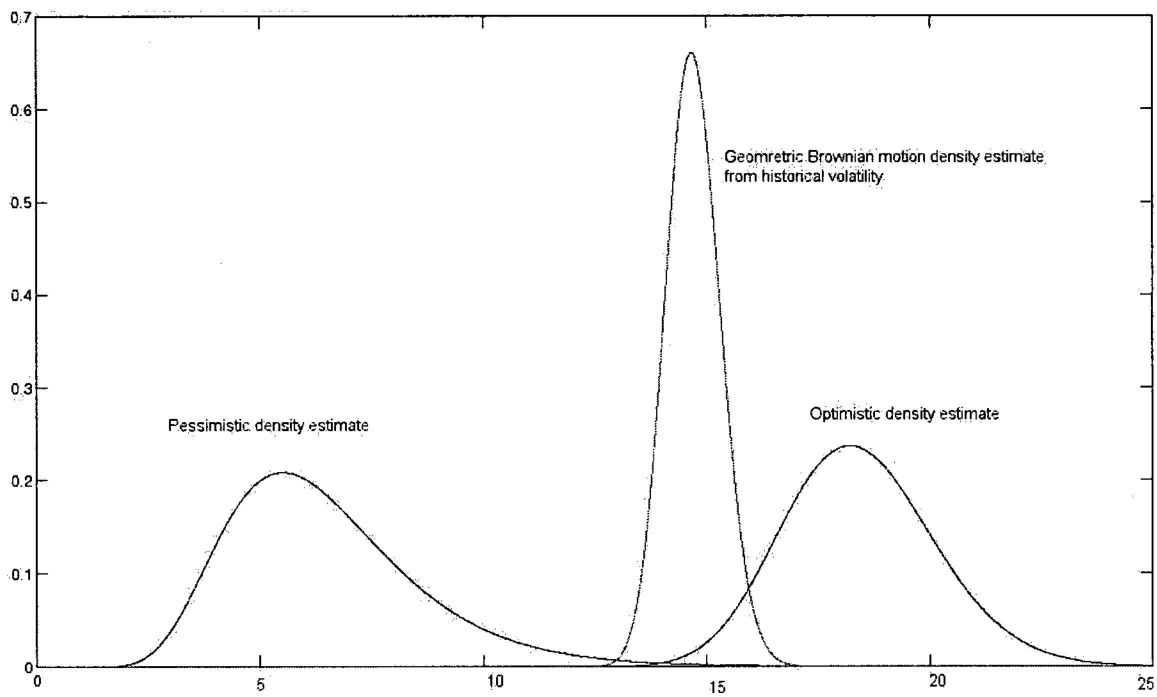
- We think that this model or some slight variation of this model can be used to model similar situation involving stocks of bigger corporations as well. For example, this model could be used to model the price movement of a company which is in talks about merger or is due to announce a key financial report. We anticipate that for big corporations, the data reliability issues will be much less severe. However, we cannot experiment using those scenarios at this time due our lack of access to data.

Table 5.2: Estimated parameters

Estimated model parameters for stocks with options showing abnormally high premia. TD: Trading date, OM: Option maturity date, HV: Historical volatility, Sj[1]^: Estimate of the optimistic process, : Estimate of the pessimistic process, <j\. Estimate of the optimistic process volatility, $<f_2$: Estimate of the pessimistic process volatility, FD: A future date, FP: Price at the future date, LNCI: Lognormal 95% confidence interval, ACI: An interval between the optimistic and pessimistic price estimates where both the processes put probability less than 5%

Ticker	TD	S_t	OM	HV	&	$s_t^{(2)}$		Ol	FD	FP	LNCI	ACI
AGEN	03/23/06	5.11	04/22/06	88.67%	8.93	4.44	160%	90%	04/19/06	2.17	4.42-5.91	5.02-7.17
ALTH	04/16/04	4.18	05/22/04	74.43%	4.66	3.07	120%	450%	05/19/04	1.93	3.70-4.72	-
BMRN	11/16/07	26.72	12/22/07	27.60%	30.23	21.46	30%	150%	12/19/07	34.67	25.54-27.95	26.35-29.01
CVTX	11/14/03	17.28	12/20/03	121.7%	19.84	13.48	90%	200%	12/17/03	13.50	14.17-21.08	-
DNDN	04/13/07	17.25	05/19/07	43.69%	27.12	10.67	90%	270%	05/16/07	5.99	16.06-18.53	15.45-23.97
HGSI	0210/06	10.49	03/18/06	44.09%	11.20	6.46	70%	270%	03/15/06	11.30	9.76-11.27	9.35-10.17
NPSP	05/17/02	23.31	06/22/02	54.17%	26.27	11.49	50%	330%	06/19/02	15.62	21.34-25.47	18.06-24.53
NPSP	02/10/06	13.21	06/18/06	45.74%	14.15	12.48	10%	170%	03/15/06	8.97	12.26-14.23	-
NFLD	12/15/06	14.79	01/20/07	49.01%	18.46	6.22	110%	390%	01/17/07	4.06	13.65-16.02	10.61-15.88
NBIX	05/12/06	52.74	06/17/06	33.73%	56.25	48.11	50%	70%	06/14/06	18.08	49.91-55.73	-
NABI	09/16/05	16.13	10/22/05	27.86%	16.13	9.55	70%	190%	10/19/05	12.10	15.42-16.88	12.39-14.65
NABI	10/14/05	12.17	11/19/05	45.72%	17.68	8.50	50%	170%	11/16/05	3.36	11.29-13.11	10.73-16.13
ITMN	08/16/02	21.59	09/21/02	87.66%	25.28	6.81	110%	290%	09/18/02	29.10	18.71-24.91	10.13-21.74

Figure 5.1: Comparison of future price densities
Estimate of probability densities of future (20-Jan-07) price of Northfield Laboratories (NFLD) based on 12-Dec-06 data



Chapter 6

Conclusion

We studied the presence of information asymmetry in the market. We looked at two scenarios and developed theoretical framework and conducted tests to find evidence for such asymmetry.

In the first two parts we investigated whether in the presence of trends one can improve on the performance of the underlying asset by using options. One of our goals was also to look into a strategy that will take advantage of such situations in a stable way, that is a strategy that will be relatively robust under small model perturbations. We find that even with a simple trend tracking technique one can take advantage of that knowledge to his/her own advantage. This is contrary to the traditional replicating portfolio pricing and trading strategies where the only advantage comes from arbitrage arising out of the anomaly between the theoretical price and the market price. We also see that a stable strategy consists of simple positions like stock, long call, writing a covered call or a vertical spread.

In the last part we looked at biomedical companies with small market capitalization and narrow product focus. These companies are well known for price jumps that come at a predictable time. They arise from the fact that their future prospects depend on key decisions by Government agencies. When the news of such a pending decision is known, investors having different information or belief act differently to impact the option prices in an atypical way. We modeled such situations and using market data we found evidence of such diverse information or beliefs among the investors.

We demonstrated that the idea of a uniform model across the market agents may not be true. In fact, it is possible in certain situations to find evidences of drastically different models being used among investors. This goes to show that information is an integral part of market behavior and play a significant role in market dynamics.

Of course the work presented in this thesis is not a comprehensive study of this vast topic of information asymmetry. We discuss a few possible future steps that can be taken from here.

From our experience in studying multiple-option strategies on a stock, we think that it may be possible to attack the problem of pricing of Credit Default Swaps (CDS). CDS can be thought of as a combination of several options with additional temporal structure on it. The ideas developed in studying the behavior of options may be useful to study this case.

We noted in the chapter dealing with biomedical firms with predictable price jumps, that a similar model may be used to investigate the price and option premium behaviors of even larger corporations when an important decision or result is pending.

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