Edge Colorings of Graphs and Their Applications

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Edge Colorings of Graphs and Their Applications

by

Daniel Johnston

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Edge Colorings of Graphs and Their Applications

Daniel Johnston, Ph.D.
Western Michigan University, 2015

Edge colorings have appeared in a variety of contexts in graph theory. In this work, we study problems occurring in three separate settings of edge colorings.

For more than a quarter century, edge colorings have been studied that induce vertex colorings in some manner. One research topic we investigate concerns edge colorings belonging to this class of problems. By a twin edge coloring of a graph $G$ is meant a proper edge coloring of $G$ whose colors come from the integers modulo $k$ that induce a proper vertex coloring in which the color of a vertex is the sum of the colors of its incident edges. The minimum $k$ for which $G$ has a twin edge coloring is the twin chromatic index of $G$. Several results on this concept have been obtained as well as a conjecture.

A red-blue coloring of a graph $G$ is an edge coloring of $G$ in which every edge is colored red or blue. The Ramsey number of $F$ and $H$ is the smallest positive integer $n$ such that every red-blue coloring of the complete graph of order $n$ results in a red $F$ or a blue $H$. The related concept of bipartite Ramsey number has been defined and studied when $F$ and $H$ are bipartite. We introduce a new class of Ramsey numbers which extend these two well-studied concepts in the area of extremal graph theory and present results and problems on these new concepts.

Let $F$ be a graph of size 2 or more having a red-blue coloring in which there is at least one edge of each color. One blue edge is designated as the root of $F$. For such an edge-colored graph $F$, an $F$-coloring of a graph $G$ is a red-blue coloring of $G$ in which every blue edge is the root of some copy of $F$ in $G$. The $F$-chromatic index of $G$ is the minimum number of red edges in an $F$-coloring of $G$. In this setting, we provide a bichromatic view of two well-known concepts in graph theory, namely matchings and domination, and present results and problems in this area of research.
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Daniel Johnston
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Chapter 1

Introduction

In this chapter, we present basic definitions, notation and concepts in graph theory that are involved in our research. We refer to the books [15, 18] for graph theory notation and terminology not described in this work. All graphs under consideration are nontrivial connected graphs.

1.1 Edge Colorings

The subject of edge colorings is one of the major areas of graph theory. While there are many concepts and problems in graph theory dealing with edge colorings, the best known and most studied is that of proper edge colorings of a graph $G$ where each edge of $G$ is assigned one color from a given set of colors and adjacent edges are colored differently. That is, an edge coloring of $G$ is proper if the two edges in every copy of $P_3$ in $G$ are colored differently. The fundamental problem here is determining the minimum number of colors needed in a proper edge coloring of $G$. This number is called the chromatic index of $G$ and is denoted by $\chi'(G)$. The classic theorem in this connection is due to Vadim Vizing [52] who proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every nonempty graph $G$. A graph $G$ is said to be of Class 1 if $\chi'(G) = \Delta(G)$ and of Class 2 if $\chi'(G) = \Delta(G) + 1$. In particular, a regular graph $G$ is of Class 1 if and only if $G$ is 1-factorable. Determining which graphs belong to which class is a major problem of study in this area.

There are edge colorings of a graph $G$ where adjacent edges may be colored the same but containing a certain subgraph where no two edges are colored the same. A subgraph of $G$ all of whose edges are colored differently is called a rainbow subgraph of $G$. For a graph $F$ of order $p$ without isolated vertices and a given integer $n \geq p$, the rainbow number $rb_n(F)$ of $F$ is the smallest positive integer $k$ such that every edge coloring of
$K_n$ with $k$ colors in which each color is assigned to at least one edge results in a rainbow $F$. One result in this connection involves the Turán graph $T_{n,k-1}$, which is the $(k-1)$-partite graph of order $n$, the cardinalities of whose partite sets differ by at most 1. The size of $T_{n,k-1}$ is denoted by $t_{n,k-1}$. This Turán number $t_{n,k-1}$ is the maximum size of a graph of order $n$ containing no complete subgraph of order $k$. Montellano-Ballesteros and Neumann-Lara [47] proved for $3 \leq k < n$ that $rb_n(K_{k+1}) = t_{n,k-1} + 2$.

A rainbow coloring of a connected graph $G$ is an edge coloring of $G$ such that every two vertices of $G$ are connected by a rainbow path. The minimum number of colors used in a rainbow coloring of a connected graph $G$ is the rainbow connection number of $G$ and is denoted by $rc(G)$. It was shown in [11] for integers $s$ and $t$ with $2 \leq s \leq t$ that $rc(K_{s,t}) = \min \left\{ \left\lceil \sqrt{t} \right\rceil, 4 \right\}$. These concepts were first introduced in [11] and studied further by many (see [12, 16, 44] for example). There is now even a book [45] on this subject. The term “rainbow connection” was chosen as that is the name of a song sung by Kermit in The Muppet Movie.

Among the most famous problems in graph theory are those concerning edge colorings of complete graphs with two colors. By a red-blue coloring of a graph $G$ is meant an edge coloring of $G$ in which every edge is colored red or blue. For given graphs $F$ and $H$, the Ramsey number $R(F,H)$ of two graphs $F$ and $H$ is the minimum positive integer $n$ for which every red-blue coloring of $K_n$ results in either a red $F$ (a copy of $F$ where each edge is colored red) or a blue $H$. It is consequence of a theorem of Ramsey that the Ramsey number $R(F,H)$ exists for every pair $F, H$ of graphs.

A related Ramsey number is the rainbow Ramsey number $RR(F,H)$ of two graphs $F$ and $H$, defined as the minimum positive integer $n$ for which every edge coloring of $K_n$ using any number of colors results in either a monochromatic copy of $F$ (where all edges are colored the same) or a rainbow copy of $H$. As a consequence of a result of Erdős and Rado [23], the rainbow Ramsey number $RR(F,H)$ exists if and only if $F$ is a star or $H$ is a forest. On the other hand, if $H$ has size $m$ and $k$ is an integer with $k \geq m$, then for every pair $F, H$ of graphs, there is always a smallest positive integer $n$ such that any edge coloring of $K_n$ using no more than $k$ colors always results in a monochromatic $F$ or rainbow $H$ (see [17, p. 319]). This number is denoted by $RR_k(F,H)$. In particular, $RR_3(K_3,K_3) = 11$.

While proper edge colorings, monochromatic subgraphs, rainbow subgraphs and rainbow colorings have been the subject of many studies, there are also numerous other red-blue colorings of graphs whose definitions depend on a fixed graph $H$, certain red-blue colorings of $H$ and a specified blue edge of the resulting edge-colored graph $F$ of $H$. This
gives rise to the concepts of color frames $F$ of a given graph $H$ and red-blue colorings of graphs called $F$-colorings, which we will study in this work.

1.2 Domination and Stratification

An area of graph theory that has received increased attention during recent decades is that of domination. Two books [30, 31] by Haynes, Hedetniemi and Slater are devoted to this subject. Although initiated by Berge [5] and Ore [48] in 1958 and 1962, respectively, it was a paper by Cockayne and Hedetniemi [20] in 1977 that brought popularity to the subject and then led to a theory. This subject is based on a simple definition. A vertex $v$ dominates a vertex $u$ in a graph $G$ if either $u = v$ or $u$ is adjacent to $v$. A set $S$ of vertices in $G$ is called a dominating set for $G$ if every vertex of $G$ is dominated by some vertex in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set for $G$. If $S$ is a dominating set for $G$ with $|S| = \gamma(G)$, then $S$ is called a minimum dominating set or a $\gamma$-set. A dominating set $S$ for a graph $G$ is called a minimal dominating set if no proper subset of $S$ is a dominating set of $G$. Certainly, every minimum dominating set is minimal, but the converse is not true. The maximum cardinality of a minimal dominating set for $G$ is called the upper domination number of $G$ and is denoted by $\Gamma(G)$. The following theorem is due to Ore [48].

**Theorem 1.2.1** If $S$ is a minimal dominating set of a graph $G$ without isolated vertices, then $V(G) - S$ is a dominating set of $G$.

Over the years, many variations and generalizations of domination have been introduced (see [30, 31]). Each type of domination is based on a condition under which a vertex $v$ dominates a vertex $u$ in a graph. We list some of the best studied examples:

- A set $S$ of vertices in a graph $G$ is independent dominating set for $G$ if $S$ is both a dominating set and an independent set. The minimum cardinality of an independent dominating set for $G$ is the independent domination number $i(G)$.

- A set $S$ of vertices in a graph $G$ containing no isolated vertices is a total dominating set (or an open dominating set) for $G$ if every vertex of $G$ is adjacent to some vertex of $S$. The minimum cardinality of a total dominating set for $G$ is the total domination number $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ is called a minimum total dominating set or $\gamma_t$-set for $G$. 
A set $S$ of vertices in a graph $G$ is a $k$-step dominating set for some positive integer $k$ if for every vertex $v$ of $G$ there exists a $u - v$ path of length $k$ in $G$ some vertex $u \in S$. The minimum cardinality of a $k$-step dominating set for $G$ is the $k$-step domination number $\gamma^{(k)}(G)$. A $k$-step dominating set of cardinality $\gamma^{(k)}(G)$ is called a minimum $k$-step dominating set or $\gamma^{(k)}$-set for $G$.

A set $S$ of vertices in a graph $G$ is a restrained dominating set if every vertex of $G$ not in $S$ is adjacent to both a vertex in $S$ and a vertex not in $S$. The minimum cardinality of a restrained dominating set for $G$ is the restrained domination number $\gamma_r(G)$. A restrained dominating set of cardinality $\gamma_r(G)$ is called a minimum restrained dominating set or $\gamma_r$-set for $G$.

A set $S$ of vertices in a graph $G$ is a $k$-dominating set for some positive integer $k$ if every vertex not in $S$ is adjacent to at least $k$ vertices in $S$. The minimum cardinality of a $k$-dominating set for $G$ is the $k$-domination number $\gamma_k(G)$. A $k$-dominating set of cardinality $\gamma_k(G)$ is called a minimum $k$-dominating set or $\gamma_k$-set for $G$.

In 1999 a new way of looking at domination was introduced by Chartrand, Haynes, Henning and Zhang [9] that encompassed several of the best known domination parameters in the literature. In fact, this gave rise to an infinite class of domination parameters, each of which is defined for every graph. This new view of domination was based on a concept introduced by Naveed Sherwani in 1992 and first studied by Reza Rashidi [49] in 1994. A graph $G$ whose vertex set $V(G)$ is partitioned is a stratified graph. If $V(G)$ is partitioned into $k$ subsets, then $G$ is $k$-stratified. In particular, the vertex set of a 2-stratified graph is partitioned into two subsets. Typically, the vertices of one subset in a 2-stratified graph are considered to be colored red and those in the other subset are colored blue. In this context, a red-blue coloring of a graph $G$ is an assignment of colors to the vertices of $G$, where each vertex is colored either red or blue. In a red-blue coloring, all vertices of $G$ may be colored the same. A red-blue coloring in which at least one vertex is colored red and at least one vertex is colored blue thereby produces a 2-stratification of $G$. For example, Figure 1.1 shows 2-stratified graphs that result from the path $P_3$ of order 3, where each solid vertex is red.

We next describe how domination was defined in [9] with the aid of stratification. Let $F$ be a 2-stratified graph in which some blue vertex $\rho$ is designated as the root of $F$. The graph $F$ is then said to be rooted at $\rho$. Since $F$ is 2-stratified, $F$ contains at least two vertices, at least one of each color. There may be blue vertices in $F$ in addition to
the root. In the case of $P_3$, there are five different 2-stratified graphs, denoted by $F_i$ ($1 \leq i \leq 5$), all of which are shown in Figure 1.2.

By an $F$-coloring of a graph $G$, we mean a red-blue coloring of $G$ such that for every blue vertex $u$ of $G$, there is a copy of $F$ in $G$ with $\rho$ at $u$. Therefore, every blue vertex $u$ of $G$ belongs to a copy $F'$ of $F$ rooted at $u$. A red vertex $v$ in $G$ is said to $F$-dominate a vertex $u$ if $u = v$ or there exists a copy $F'$ of $F$ rooted at $u$ and containing the red vertex $v$. The set $S$ of red vertices in a red-blue coloring of $G$ is an $F$-dominating set of $G$ if every vertex of $G$ is $F$-dominated by some vertex of $S$, that is, this red-blue coloring of $G$ is an $F$-coloring. The minimum number of red vertices in an $F$-dominating set is called the $F$-domination number $\gamma_F(G)$ of $G$. An $F$-dominating set with $\gamma_F(G)$ vertices is a minimum $F$-dominating set. The $F$-domination number of every graph $G$ is defined since $V(G)$ is an $F$-dominating set. This concept provides a generalization of domination (see [18]) and has been studied in many articles (see [25, 32] and [33] - [37] for example). For the five different 2-stratified graphs $F_i$ ($1 \leq i \leq 5$) of $P_3$ shown in Figure 1.2, while the parameter $\gamma_{F_3}(G)$ is new, the other four $F$-domination numbers are well-known domination parameters. More precisely,

- If $G$ is a graph without isolated vertices, then $\gamma_{F_1}(G) = \gamma_t(G)$.
- If $G$ is a connected graph of order 3 or more, then $\gamma_{F_2}(G) = \gamma(G)$.
- If $G$ is a connected graph of order 3 or more, then $\gamma_{F_4}(G) = \gamma_r(G)$.
- If $G$ is a connected graph of order 3 or more, then $\gamma_{F_5}(G) = \gamma_2(G)$. 
1.3 Edge Domination

As with vertex colorings and edge colorings in graphs, there is also an edge-version of domination in graphs. An edge $e$ in a graph $G$ is said to dominate itself and all edges adjacent to $e$. A set $S$ of edges of $G$ is an edge dominating set of $G$ if every edge of $G$ is dominated by some edge in $S$. The minimum size of an edge dominating set of $G$ is the edge domination number of $G$ and is denoted by $\gamma'(G)$. Moreover, $\gamma'(G)$ is the domination number of the line graph of $G$. An edge dominating set of size $\gamma'(G)$ is called a minimum edge dominating set of $G$, while an edge dominating set $S$ of a graph $G$ is a minimal edge dominating set if no proper subset of $S$ is also an edge dominating set of $G$. While a minimum edge dominating set is minimal, the converse is not true. The maximum size of a minimal edge dominating set in $G$ is the upper edge domination number of $G$ and is denoted by $\gamma''(G)$. Our work also involves the following well-studied edge domination parameters that are analogous to the vertex versions of domination parameters:

- A total edge dominating set in a connected graph $G$ is a subset $S$ of $E(G)$ such that every edge of $G$ is adjacent to an edge of $S$. Thus a total edge dominating set contains no independent edges. If $G$ is a nonempty graph containing no component $K_2$, then $E(G)$ is a total edge dominating set and so every connected graph of order at least 3 has a total edge dominating set. The total edge domination number $\gamma'_t(G)$ is the minimum size of a total edge dominating set in $G$. A total edge dominating set of cardinality $\gamma'_t(G)$ is called a minimum total edge dominating set or $\gamma'_t$-set for $G$.

- A set $S$ of edges of a graph $G$ is a restrained edge dominating set if every edge not in $S$ is adjacent to an edge in $S$ and to an edge in $E(G) - S$. Every graph has a restrained edge dominating set since $E(G)$ is such a set. The restrained edge domination number $\gamma'_r(G)$ is the minimum size of a restrained edge dominating set of $G$. A restrained edge dominating set of size $\gamma'_r(G)$ is a minimum restrained edge dominating set of $G$. A restrained edge dominating set of cardinality $\gamma'_r(G)$ is called a minimum restrained edge dominating set or $\gamma'_r$-set for $G$.

- A set $S$ of edges of a graph $G$ is a $k$-edge dominating set of $G$ if every edge not in $S$ is dominated by at least $k$ edges in $S$. Since $E(G)$ is such a set, every graph $G$ has a $k$-edge dominating set. The minimum size of a $k$-edge dominating set of $G$ is the $k$-edge domination number of $G$ and is denoted by $\gamma'_k(G)$. A $k$-edge dominating set of cardinality $\gamma'_k(G)$ is called a minimum $k$-edge dominating set or $\gamma'_k$-set for $G$. 

1.4 Matching and Edge Independence

A central topic in graph theory is that of matchings. In fact, Lovász and Plummer have written a book [46] on the theory of matchings. A set of edges in a graph $G$ is independent if no two edges in the set are adjacent in $G$. The edges in an independent set of edges of $G$ form a matching in $G$. If $M$ is a matching in a graph $G$ with the property that every vertex of $G$ is incident with an edge of $M$, then $M$ is a perfect matching in $G$. Clearly, if $G$ has a perfect matching $M$, then $G$ has even order and the subgraph induced by $M$ is a 1-factor of $G$.

A matching of maximum size in $G$ is a maximum matching. Thus every perfect matching is a maximum matching but the converse is not true. In particular, if the order of $G$ is odd, then $G$ cannot have a perfect matching. The edge independence number $\alpha'(G)$ of $G$ is the number of edges in a maximum matching of $G$. The number $\alpha'(G)$ is also referred to as the matching number of $G$.

A matching $M$ in a graph $G$ is a maximal matching of $G$ if $M$ is not a proper subset of any other matching in $G$. While every maximum matching is maximal, a maximal matching need not be a maximum matching. The minimum number of edges in a maximal matching of $G$ is called the lower edge independence number or lower matching number $\alpha''(G)$ of $G$. Necessarily, $\alpha''(G) \leq \alpha'(G)$. It was shown in [40] that if $G$ is a graph and $k$ is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then $G$ contains a maximal matching with $k$ edges.
Chapter 2

Twin Edge Colorings of Graphs

2.1 Introduction

In 1968, Rosa [51] introduced a vertex labeling that induces an edge-distinguishing labeling defined by subtracting labels. In particular, for a graph $G$ of size $m$, a vertex labeling (an injective function) $f : V(G) \to \{0, 1, \ldots, m\}$ was called a $\beta$-valuation by Rosa if the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$ was bijective. In 1972 Golomb [27] called a $\beta$-valuation a graceful labeling and a graph possessing a graceful labeling a graceful graph. It is this terminology that has become standard. Much research has been done on graceful graphs. A popular conjecture in graph theory, due to Anton Kotzig and Gerhard Ringel, is the following.

The Graceful Tree Conjecture  Every nontrivial tree is graceful.

In 1991 Gnana Jothi [26] introduced a concept that, in a certain sense, reverses the roles of vertices and edges in graceful labelings (see also [24]). For a connected graph $G$ of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of $G$ that induces a bijective function $f' : V(G) \to \mathbb{Z}_n$ defined by $f'(v) = \sum_{e \in E_v} f(e)$ for each vertex $v$ of $G$, where $E_v$ is the set of edges of $G$ incident with a vertex $v$. Such a labeling $f$ is called a modular edge-graceful labeling, while a graph possessing such a labeling is called modular edge-graceful (see [42]). Verifying a conjecture by Gnana Jothi on trees, Jones, Kolasinski and Zhang [43] showed not only that every tree of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$ but a connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$. These concepts have been studied in greater detail by Jones [41].
Prior to Jothi’s paper, an edge labeling (with positive integers) of a connected graph $G$ was introduced in 1986 [10] for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called irregular. This concept was later looked at in another manner. For the set $\mathbb{N}$ of positive integers, an edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be vertex-distinguishing if the coloring $c' : V(G) \to \mathbb{N}$ induced by $c$ and defined by $c'(v) = \sum_{e \in E_v} c(e)$ has the property that $c'(x) \neq c'(y)$ for every two distinct vertices $x$ and $y$ of $G$. The research in [10] dealt with minimizing the largest color assigned to the edges of the graph to produce an irregular graph. Vertex-distinguishing edge colorings have received increased attention during the past 25 years (see [18, pp. 370-385]).

A neighbor-distinguishing coloring of a graph $G$ is a coloring in which every pair of adjacent vertices of $G$ are colored differently. Such a coloring is more commonly called a proper vertex coloring. The minimum number of colors needed in a proper vertex coloring of a graph $G$ is the chromatic number of $G$ and denoted by $\chi(G)$. While $\chi(G) \leq 1 + \Delta(G)$, one of the well-known theorems in graph colorings is due to Brooks [6] who proved that $\chi(G) \leq \Delta(G)$ for every connected graph $G$ that is not an odd cycle or a complete graph. A number of neighbor-distinguishing vertex colorings different from standard proper colorings have been introduced in the literature (see [18, pp. 383-391], for example).

In 2005 non-proper edge colorings of graphs were studied that induce a proper vertex coloring [2]. In particular, for $k \in \mathbb{N}$, let $c : E(G) \to \{1, 2, \ldots, k\}$ be an edge coloring of $G$ (where adjacent edges may be assigned the same color). A vertex coloring $c' : V(G) \to \mathbb{N}$ is defined where $c'(v)$ is the sum of the colors of the edges incident with $v$. If $c'$ is a proper vertex coloring of $G$, then $c$ is called a neighbor-distinguishing edge coloring of $G$ (see [18, pp. 385]). A major conjecture in this area is the following [50]:

**The 1-2-3 Conjecture** For every connected graph $G$ of order at least 3, there exists a neighbor-distinguishing edge coloring of $G$ using only the colors 1, 2, 3.

Among the various edge colorings studied in graph theory, the best known and most studied are proper edge colorings. In a proper edge coloring of a graph $G$, each edge of $G$ is assigned a color from a given set of colors where adjacent edges are colored differently. The minimum number of colors needed in a proper edge coloring of $G$ is called the chromatic index of $G$ and is denoted by $\chi'(G)$. The classic theorem in this connection is due to Vizing [52] who proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every nonempty
A related and also well-studied graph coloring is the so-called total coloring of a graph $G$ that assigns colors to both the vertices and edges of $G$ so that not only the vertex coloring and edge coloring are proper but no vertex and an incident edge are assigned the same color. The minimum number of colors required for a total coloring of $G$ is the total chromatic number of $G$, denoted by $\chi''(G)$. It then follows that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of $G$. A well-known conjecture in this area is due independently to Behzad and Vizing (see [18, pp. 282]).

**The Total Coloring Conjecture** For every graph $G$, $\chi''(G) \leq 2 + \Delta(G)$.

Inspired by the graph colorings described above, we introduce a proper edge coloring of a graph that induces a proper vertex coloring where the colors belong to $\mathbb{Z}_k$ for some integer $k \geq 2$. All graphs under consideration here are connected graphs of order at least 3.

### 2.2 Twin Chromatic Index

For a connected graph $G$ of order at least 3, a proper edge coloring $c : E(G) \to \mathbb{Z}_k$ for some integer $k \geq 2$ is sought for which the induced vertex coloring $c' : V(G) \to \mathbb{Z}_k$ defined by

$$c'(v) = \sum_{e \in E_v} c(e) \quad \text{in } \mathbb{Z}_k,$$

(where the indicated sum is computed in $\mathbb{Z}_k$) results in a proper vertex coloring of $G$. We refer to such a coloring as a twin edge $k$-coloring or simply a twin edge coloring of $G$. The minimum $k$ for which $G$ has a twin edge $k$-coloring is called the twin chromatic index of $G$ and is denoted by $\chi'_t(G)$. Since a twin edge coloring is not only a proper edge coloring of $G$ but induces a proper vertex coloring of $G$, it follows that

$$\chi'_t(G) \geq \max\{\chi(G), \chi'(G)\}. \quad (2.1)$$

Since $\max\{\chi(G), \chi'(G)\} = \chi'(G)$ except when $G$ is a complete graph of even order, we have $\chi'_t(G) \geq \chi'(G)$ except possibly when $G$ is a complete graph of even order. As an illustration, Figure 2.1 shows a twin edge 4-coloring $c$ of the Petersen graph $P$, where the color $c'(v)$ is placed inside each vertex $v$ of $P$ and so $\chi'_t(P) \leq 4$. Since $P$ is a 3-regular
graph and $P$ is not 1-factorable, $\chi'(P) = \Delta(P) + 1 = 4$. It then follows by (2.1) that $\chi'_t(P) = \chi'(P) = 4$.

While $\chi'_t(G)$ does not exist if $G$ is the connected graph of order 2, every connected graph of order at least 3 has a twin edge coloring. To see this, let $G$ be a connected graph of size $m \geq 2$. If $m = 2$, then assign the colors 1 and 2 in $\mathbb{Z}_2$ to the two edges of $G$. If $m \geq 3$, then assign the $m$ elements $0, 1, 2, 4, \ldots, 2^m - 2 \in \mathbb{Z}_{2^{m-1}}$ to the $m$ edges of $G$ in a one-to-one manner so that the color 0 is assigned to a pendant edge if $G$ has such an edge. Hence for every two adjacent vertices $u$ and $v$ in $G$, the sets of edges colored by nonzero elements in $\mathbb{Z}_{2^{m-1}}$ and incident with $u$ and $v$, respectively, are distinct. Since the base 2 representations of the colors of these vertices are different, it follows that adjacent vertices are assigned distinct colors in $\mathbb{Z}_{2^{m-1}}$. Thus, this coloring is a twin edge coloring. This observation yields the following.

**Proposition 2.2.1** If $G$ is a connected graph of order at least 3 and size $m$, then $\chi'_t(G)$ exists. Furthermore, $\chi'_t(G) \leq 2^{m-1}$ if $m \geq 3$.

To illustrate the concept of twin edge colorings, we determine the twin chromatic indexes of two familiar classes of graphs, namely paths and cycles. We begin with paths.

**Proposition 2.2.2** If $P_n$ is a path of order $n \geq 3$, then $\chi'_t(P_n) = 3$.

**Proof.** Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$ where $e_i = v_iv_{i+1}$ for $i = 1, 2, \ldots, n-1$. Since $\chi'(P_n) = 2$, it follows that $\chi'_t(P_n) \geq \chi'(P_n) = 2$. First, we show that $\chi'_t(P_n) \neq 2$. Let $c$ be a proper edge coloring of $P_n$ using the colors of $\mathbb{Z}_2$. Then
$c(e_i) = 1 \in \mathbb{Z}_2$ for some $i \in \{1, 2, \ldots, n-1\}$ and so $c(e_{i-1}) = 0$ if $i \geq 2$ and $c(e_{i+1}) = 0$ if $i \leq n-2$. However then, $c'(v_i) = c'(v_{i+1}) = 1$ and so $c$ is not a twin edge 2-coloring. Thus, as claimed, $\chi_t'(P_n) \geq 3$. It remains to show that $P_n$ has a twin edge 3-coloring. A coloring $c : E(P_n) \rightarrow \mathbb{Z}_3$ is defined as follows:

- For $n \equiv 0 \pmod{3}$ or $n \equiv 1 \pmod{3}$, let $c(e_j) = r$ if $j \equiv r \pmod{3}$ for $r = 0, 1, 2$. For example, if $n = 6$, then $(c(e_1), c(e_2), \ldots, c(e_5)) = (1, 2, 0, 1, 2)$; while if $n = 7$, then $(c(e_1), c(e_2), \ldots, c(e_6)) = (1, 2, 0, 1, 2, 0)$. If $n \equiv 0 \pmod{3}$, then for $1 \leq i \leq n$,

  $$c'(v_i) = \begin{cases} 
0 & \text{if } i \equiv 2 \pmod{3} \\
1 & \text{if } i \equiv 1 \pmod{3} \\
2 & \text{if } i \equiv 0 \pmod{3}.
\end{cases} \tag{2.2}$$

If $n \equiv 1 \pmod{3}$, then $c'(v_i)$ is given in (2.2) for $1 \leq i \leq n-1$ and $c'(v_n) = 0$. Hence $(c'(v_1), c'(v_2), \ldots, c'(v_6)) = (1, 0, 2, 1, 0, 2)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_7)) = (1, 0, 2, 1, 0, 2, 0)$.

- For $n \equiv 2 \pmod{3}$, let $c(e_j) = 2 + r$ if $j \equiv r \pmod{3}$ for $r = 0, 1, 2$. Then $c'(v_1) = c'(v_n) = 0$ and for $2 \leq i \leq n-1$,

  $$c'(v_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{3} \\
1 & \text{if } i \equiv 2 \pmod{3} \\
2 & \text{if } i \equiv 1 \pmod{3}.
\end{cases}$$

For example, if $n = 8$, then $(c(e_1), c(e_2), \ldots, c(e_7)) = (0, 1, 2, 0, 1, 2, 0)$ and $(c'(v_1), c'(v_2), \ldots, c'(v_8)) = (0, 1, 0, 2, 1, 0, 2, 0)$. Therefore, $\chi_t'(P_n) \geq 3$ and so $\chi_t'(P_n) = 3$ for $n \geq 3$.

To determine twin chromatic indexes of cycles, the following observation will be useful.

**Observation 2.2.3** If a connected graph $G$ contains two adjacent vertices of degree $\Delta(G)$, then $\chi_t'(G) \geq 1 + \Delta(G)$. In particular, if $G$ is a connected $r$-regular graph for some integer $r \geq 2$, then $\chi_t'(G) \geq 1 + r$.

**Proposition 2.2.4** If $C_n$ is a cycle of order $n \geq 3$, then

$$\chi_t'(C_n) = \begin{cases} 
3 & \text{if } n \equiv 0 \pmod{3} \\
4 & \text{if } n \not\equiv 0 \pmod{3} \text{ and } n \not= 5 \\
5 & \text{if } n = 5.
\end{cases}$$
Proof. Let $C_n = (v_1, v_2, \ldots , v_n , v_{n+1} = v_1)$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots , n$ and $e_{n+1} = e_1$. By Observation 2.2.3, $\chi_t' (C_n) \geq 3$. First, suppose that $n \equiv 0 \pmod{3}$ and so $n = 3k$ for some positive integer $k$. Define the coloring $c : E(C_n) \to \mathbb{Z}_3$ by $c(e_i) = 2 + r$ if $i \equiv r \pmod{3}$ for $r = 0, 1, 2$. Then for $1 \leq i \leq n$,

$$c'(v_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{3} \\
1 & \text{if } i \equiv 2 \pmod{3} \\
2 & \text{if } i \equiv 1 \pmod{3}. 
\end{cases}$$

For example, if $n = 6$, then $(c(e_1), c(e_2), \ldots , c(e_6)) = (0, 1, 2, 0, 1, 2)$ and $(c'(v_1), c'(v_2), \ldots , c'(v_6)) = (2, 1, 0, 2, 1, 0)$. Hence $\chi_t' (C_n) = 3$ when $n \equiv 0 \pmod{3}$.

Next, suppose that $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 5$. First, we make an observation, namely, if $c$ is a twin edge coloring of $C_n$ and $|i - j| = 2$, then $c(e_i) \neq c(e_j)$. Suppose, say, that $c(e_1) = c(e_3)$. However then, $c'(v_2) = c(e_1) + c(e_2) = c(e_2) + c(e_3) = c'(v_3)$, which is impossible. This implies that if $n \not\equiv 0 \pmod{3}$, then $\chi_t' (C_n) \geq 4$. To show that $\chi_t' (C_n) \leq 4$, define the coloring $c : E(C_n) \to \mathbb{Z}_4$ as follows:

- For $n \equiv 1 \pmod{3}$, let $c(e_i) = 2 + r$ if $i \equiv r \pmod{3}$ for $r = 0, 1, 2$ and $1 \leq i \leq n - 1$ and $c(e_n) = 3$. Then $c'(v_1) = 3, c'(v_n) = 1$ and for $2 \leq i \leq n - 1$,

$$c'(v_i) = \begin{cases} 
1 & \text{if } i \equiv 2 \pmod{3} \\
2 & \text{if } i \equiv 1 \pmod{3} \\
3 & \text{if } i \equiv 0 \pmod{3}. 
\end{cases} \tag{2.3}$$

(In particular, $c'(v_2) = 1$ and $c'(v_{n-1}) = 3$.) For example, if $n = 7$, then $(c(e_1), c(e_2), \ldots , c(e_7)) = (0, 1, 2, 0, 1, 2, 3)$ and $(c'(v_1), c'(v_2), \ldots , c'(v_7)) = (3, 1, 3, 2, 1, 3, 1)$. Hence $\chi_t' (C_n) = 4$ when $n \equiv 1 \pmod{3}$.

- Let $n \equiv 2 \pmod{3}$ and $n \geq 8$. If $n = 8$, let $(c(e_1), c(e_2), \ldots , c(e_8)) = (0, 1, 2, 3, 0, 1, 2, 3)$ while if $n \geq 11$, let $c(e_i) = 2 + r$ if $i \equiv r \pmod{3}$ for $r = 0, 1, 2$ and $1 \leq i \leq n - 9$ and let $(c(e_n-8), c(e_n-7), \ldots , c(e_n)) = (0, 1, 2, 3, 0, 1, 2, 3)$. Consequently, if $n = 8$, then $(c'(v_1), c'(v_2), \ldots , c'(v_8)) = (3, 1, 3, 1, 3, 1, 3, 1)$; while if $n \geq 11$, then $c'(v_1) = 3, c'(v_i)$ is the same as in (2.3) for $2 \leq i \leq n - 9$ and $(c'(v_{n-8}), c'(v_{n-7}), \ldots , c'(v_n)) = (3, 1, 3, 1, 3, 1, 3, 1)$. For example, if $n = 11$, then $(c(e_1), c(e_2), \ldots , c(e_{11})) = (0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3)$ and $(c'(v_1), c'(v_2), \ldots , c'(v_{11})) = (3, 1, 3, 2, 1, 3, 1, 3, 1, 3, 1, 3, 1)$. Hence $\chi_t' (C_n) = 4$ when $n \equiv 2 \pmod{3}$ and $n \geq 8$. 


Finally, we show that \( \chi'_r(C_5) = 5 \). We have already observed that \( \chi'_r(C_5) \geq 3 \). Let \( C_5 = (v_0, v_1, v_2, v_3, v_4, v_5 = v_0) \) and let \( c : E(C_5) \to \mathbb{Z}_5 \) be defined by \( c(v_i, v_{i+1}) = i \) for \( 0 \leq i \leq 4 \). Since \( c'(v_0) = 4, c'(v_1) = 1, c'(v_2) = 3, c'(v_3) = 0 \) and \( c'(v_4) = 2 \), it follows that \( c \) is a twin edge 5-coloring of \( C_5 \) and so \( \chi'_r(C_5) \leq 5 \). We now show that \( \chi'_r(C_5) = 5 \).

Suppose that there is a twin edge \( k \)-coloring where \( k = 3 \) or \( k = 4 \). Then some element \( a \in \mathbb{Z}_k \) must be used twice, say \( c(v_0v_1) = c(v_2v_3) = a \). Suppose that \( c(v_1v_2) = b \), where \( b \neq a \). Then \( c'(v_1) = c'(v_2) = a + b \), which is a contradiction. Thus, \( \chi'_r(C_5) = 5 \).

### 2.3 Complete Graphs

We now investigate twin edge colorings of complete graphs \( K_n \) beginning with the case when \( n \) is odd. The following observation will be useful to us.

**Observation 2.3.1** Let \( n \geq 2 \) be an integer. If \( n \) is odd, then \( \binom{n}{2} = 0 \) in \( \mathbb{Z}_n \) and if \( n \) is even, then \( \binom{n}{2} = \frac{n}{2} \) in \( \mathbb{Z}_n \).

**Theorem 2.3.2** If \( n \geq 3 \) is an odd integer, then \( \chi'_r(K_n) = n \).

**Proof.** By Observation 2.3.3, \( \chi'_r(K_n) \geq 1 + \Delta(K_n) = n \). To show that \( \chi'_r(K_n) \leq n \), let \( V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\} \) and arrange the vertices \( v_0, v_1, \ldots, v_{n-1} \) consecutively in a regular \( n \)-gon and join every two vertices by a straight line segment, producing \( K_n \). For each \( i \) (\( 0 \leq i \leq n - 1 \)), assign to \( v_{i-1}v_{i+1} \) and those edges parallel to \( v_{i-1}v_{i+1} \) the color \( i \). Then \( v_i \) has the color \( \binom{n}{2} - i \), resulting in a proper vertex coloring of \( K_n \). Thus \( \chi'_r(K_n) = n \).

When \( n \geq 4 \) is even, however, \( \chi'_r(K_n) \neq n \).

**Theorem 2.3.3** If \( n \geq 4 \) is an even integer, then \( \chi'_r(K_n) \geq n + 1 \).

**Proof.** Since \( \chi'_r(K_n) \geq 1 + \Delta(K_n) = n \) by Observation 2.3.3, it remains to show that \( \chi'_r(K_n) \neq n \). Assume, to the contrary, that \( \chi'_r(K_n) = n \). Then there is a proper edge coloring of \( K_n \) using the colors in \( \mathbb{Z}_n \) that results in a proper vertex coloring of \( K_n \). Since every vertex of \( K_n \) has degree \( n - 1 \), the edges incident with each vertex of \( K_n \) are colored with an \((n - 1)\)-element subset of \( \mathbb{Z}_n \). For example, if \( v \) is a vertex of \( K_n \), then there is exactly one element \( a \in \mathbb{Z}_n \) that is not used in coloring the edges incident with \( v \). Consequently, at most \( \frac{n}{2} - 1 \) edges of \( K_n \) are colored \( a \), implying that there exists some other vertex \( u \) of \( K_n \) none of whose incident edges are colored \( a \). However then,
Proof. By Theorem 2.3.3, it suffices to show that $\chi'_t(K_n) \geq n + 1$. Thus $\chi'_t(K_n) \geq n + 1$.

If $n \geq 4$ is an even integer, then either $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$. We consider these two situations, beginning with $n \equiv 0 \pmod{4}$.

**Theorem 2.3.4** If $n \geq 4$ is an integer with $n \equiv 0 \pmod{4}$, then $\chi'_t(K_n) = n + 1$.

**Proof.** By Theorem 2.3.3, it suffices to show that $K_n$ has a twin edge $(n+1)$-coloring. Let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and arrange the vertices $v_0, v_1, \ldots, v_{n-1}$ consecutively in a regular $n$-gon and join every two vertices by a straight line segment, thereby producing $K_n$.

Since $n \equiv 0 \pmod{4}$ and $n \geq 4$, it follows that $n = 4k$ for some positive integer $k$. First, let $M_0, M_1, \ldots, M_{2k-1}$ be $2k$ pairwise edge-disjoint matchings of size $2k-1$ in $K_{4k}$ where each matching $M_i$ ($0 \leq i \leq 2k-1$) consists of those $2k-1$ edges perpendicular to $v_i v_{2k+i}$. Then $H = K_{4k} - \left( \bigcup_{i=0}^{2k-1} M_i \right)$ is therefore a $(2k)$-regular graph. The graph $H$ has a 1-factorization $\{F_1, F_2, \ldots, F_{2k}\}$ where $F_i$ ($1 \leq i \leq 2k$) consists of the edge $v_i v_{i+1}$ and those edges parallel to $v_i v_{i+1}$. For $n = 8$ and $k = 2$, this is illustrated in Figure 2.2.

Let $X_1 = \{v_0 v_{2k-1}, v_1 v_{2k-2}, \ldots, v_{k-1} v_k\}$ and $X'_1 = \{v_{2k} v_{4k-1}, v_{2k+1} v_{4k-2}, \ldots, v_{3k-1} v_{3k}\}$. Thus $|X_1| = |X'_1| = k$ and $E(F_{k-1}) = X_1 \cup X'_1$. In particular, if $k = 2$, then $E(F_1) = X_1 \cup X'_1$ where $X_1 = \{v_0 v_3, v_1 v_2\}$ and $X'_1 = \{v_4 v_7, v_5 v_6\}$; while if $k = 3$, then $E(F_2) = X_1 \cup X'_1$ where $X_1 = \{v_0 v_5, v_1 v_4, v_2 v_3\}$ and $X'_1 = \{v_6 v_{11}, v_7 v_{10}, v_8 v_9\}$.
Define a coloring $c : E(K_{4k}) \to \mathbb{Z}_{4k+1}$ as follows. If $k = 2$, let

$$c(e) = \begin{cases} 
0 & \text{if } e \in X'_1 \\
 i - 1 & \text{if } e \in E(F_i) \text{ where } 2 \leq i \leq 2k \\
 2k & \text{if } e \in X_1 \\
 2k + j + 1 & \text{if } e \in M_j \text{ where } 0 \leq j \leq 2k - 1.
\end{cases}$$

If $k \geq 3$, let

$$c(e) = \begin{cases} 
0 & \text{if } e \in X'_1 \\
 i & \text{if } e \in E(F_i) \text{ where } 1 \leq i \leq k - 2 \\
 i - 1 & \text{if } e \in E(F_i) \text{ where } k \leq i \leq 2k \\
 2k & \text{if } e \in X_1 \\
 2k + j + 1 & \text{if } e \in M_j \text{ where } 0 \leq j \leq 2k - 1.
\end{cases} \tag{2.4}$$

For $n = 12$ and $k = 3$, the matchings $M_0$, $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ and 1-factors $F_1$, $F_2$, $F_3$, $F_4$, $F_5$, $F_6$ are illustrated in Figure 2.3. As we saw, $E(F_2) = X_1 \cup X'_1$ where $X_1 = \{v_0v_5, v_1v_4, v_2v_3\}$ and $X'_1 = \{v_6v_{11}, v_7v_{10}, v_8v_9\}$. The coloring $c$ defined in (2.4) is also illustrated in Figure 2.3.

![Graph illustration](image)

Figure 2.3: Illustrating $M_0, M_1, M_2, M_3, M_4, M_5$ and $F_1, F_2, F_3, F_4, F_5, F_6$ for $K_{12}$

Then $c$ is a proper edge coloring. For $0 \leq i \leq 2k - 1$, 
Since \( c \leq 4 \) and \( n = 6 \) and \( n \equiv 2 \pmod{4} \), it suffices to show that \( \chi'_t(K_n) = n + 1 \).  

**Theorem 2.3.5** If \( n \geq 6 \) is an integer with \( n \equiv 2 \pmod{4} \), then \( \chi'_t(K_n) = n + 1 \).

**Proof.** Since \( \chi'_t(K_n) \geq n + 1 \) by Theorem 2.3.3, it suffices to show that \( K_n \) has a twin edge \((n + 1)\)-coloring when \( n \geq 6 \) with \( n \equiv 2 \pmod{4} \). Let \( n = 4k + 2 \) for some positive integer \( k \) and let \( V(K_{4k+2}) = \{v_0, v_1, \ldots, v_{4k+1}\} \). Arrange the vertices \( v_1, v_2, \ldots, v_{4k+1} \) consecutively in a regular \((4k + 1)\)-gon, place \( v_0 \) in the center of the \((4k + 1)\)-gon and then join every two vertices by a straight line segment, thereby producing \( K_{4k+2} \).

Let \( \mathcal{F} = \{F_1, F_2, \ldots, F_{4k+1}\} \) be the 1-factorization of \( K_{4k+2} \), in which \( F_i \) is the 1-factor of \( K_{4k+2} \) that consists of the edge \( v_0v_{2k+1+i} \) and the \( 2k \) edges perpendicular to \( v_0v_{2k+1+i} \) when \( 1 \leq i \leq 2k \) and \( F_i \) consists of the edge \( v_0v_{-2k+i} \) and the \( 2k \) edges perpendicular to \( v_0v_{-2k+i} \) where \( 2k+1 \leq i \leq 4k+1 \). Also, let \( M_i = E(F_i) \) (\( 1 \leq i \leq 4k+1 \)) denote the perfect matching of \( K_{4k+2} \) resulting from \( F_i \). Observe that the edge \( v_1v_{i+1} \) belongs to \( M_i \) for \( 1 \leq i \leq 4k \) and \( v_{4k+1}v_1 \in M_{4k+1} \). Figure 2.4 shows the case when \( n = 6 \) and \( k = 1 \).

![Figure 2.4: The 1-factorization \( \mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\} \) of \( K_6 \)](image-url)
We will now define an edge coloring $c_1$ described below that assigns the $4k + 1$ colors in $\mathbb{Z}_{4k+3} - \{0, 1\}$ to the $4k + 1$ matchings $M_1, M_2, \ldots, M_{4k+1}$ such that (i) $c_1$ assigns exactly one color to all edges in $M_i$ for each $i$ ($1 \leq i \leq 4k + 1$) and (ii) $c_1(e) \neq c_1(f)$ if $e \in M_i$, $f \in M_j$ where $i \neq j$.

- For an even integer $i$ with $2 \leq i \leq 4k$, let
  \[
  c_1(e) = \begin{cases} 
  (2k + 3) - i & \text{if } e \in M_i \text{ and } 2 \leq i \leq 2k \\
  i - 2k & \text{if } e \in M_i \text{ and } 2k + 2 \leq i \leq 4k.
  \end{cases}
  \]

- For $i = 1$ or $i = 2k + 1$, let
  \[
  c_1(e) = \begin{cases} 
  2k + 3 & \text{if } e \in M_1 \\
  2k + 2 & \text{if } e \in M_{2k+1}.
  \end{cases}
  \]

- For the remaining $2k - 1$ matchings $M_3, M_5, \ldots, M_{2k-1}$ and $M_{2k+3}, M_{2k+5}, \ldots, M_{4k+1}$, the coloring $c_1$ assigns the remaining $2k - 1$ colors $2k + 4, 2k + 5, \ldots, 4k + 2$ to these matchings in an arbitrary way such that distinct colors are assigned to the edges in distinct matchings.

For example, if $n = 6$, then the coloring $c_1 : E(K_6) \to \mathbb{Z}_7$ is shown as follows:

<table>
<thead>
<tr>
<th>$e \in { M_1, M_2, M_3, M_4, M_5 }$</th>
<th>$c_1(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\in M_1$</td>
<td>5</td>
</tr>
<tr>
<td>$\in M_2$</td>
<td>3</td>
</tr>
<tr>
<td>$\in M_3$</td>
<td>4</td>
</tr>
<tr>
<td>$\in M_4$</td>
<td>2</td>
</tr>
<tr>
<td>$\in M_5$</td>
<td>6</td>
</tr>
</tbody>
</table>

For $n = 10$, the coloring $c_1 : E(K_{10}) \to \mathbb{Z}_{11}$ assigns the remaining $2k - 1 = 3$ colors 8, 9, 10 to the matchings $M_3, M_7$ and $M_9$ in an arbitrary way. Thus a possible choice of the coloring $c_1$ is shown as follows:

<table>
<thead>
<tr>
<th>$e \in { M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9 }$</th>
<th>$c_1(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\in M_1$</td>
<td>7</td>
</tr>
<tr>
<td>$\in M_2$</td>
<td>5</td>
</tr>
<tr>
<td>$\in M_3$</td>
<td>8</td>
</tr>
<tr>
<td>$\in M_4$</td>
<td>3</td>
</tr>
<tr>
<td>$\in M_5$</td>
<td>6</td>
</tr>
<tr>
<td>$\in M_6$</td>
<td>2</td>
</tr>
<tr>
<td>$\in M_7$</td>
<td>9</td>
</tr>
<tr>
<td>$\in M_8$</td>
<td>4</td>
</tr>
<tr>
<td>$\in M_9$</td>
<td>10</td>
</tr>
</tbody>
</table>

In each case, the two colors 0 and 1 are not used.

Hence, the $2k$ colors 2, 3, \ldots, $2k + 1$ in $\mathbb{Z}_{4k+3}$ are used to color the edges in the $2k$ matchings $M_2, M_4, \ldots, M_{4k}$; while the $2k + 1$ colors $2k + 2, 2k + 3, \ldots, 4k + 2$ in $\mathbb{Z}_{4k+3}$ are used to color the edges in $2k + 1$ matchings $M_1, M_3, \ldots, M_{4k+1}$. Therefore, $c_1$ is a proper edge coloring of $K_{4k+2}$. Since the colors 0 and 1 are not used,
\[ c'_1(v) = 2 + 3 + \cdots + (4k + 2) = \binom{4k + 3}{2} - 0 - 1 \]

for each \( v \in V(K_{4k+2}) \).

Next, we define a new edge coloring \( c : E(K_{4k+2}) \to \mathbb{Z}_{4k+3} \) from the coloring \( c_1 \) as follows: First, we partition \( M_{2k+1} \) into two sets \( X \) and \( Y \) where

\[
X = \{v_i v_{4k+3-i} : \text{\( i \) is odd and } 3 \leq i \leq 2k + 1\} \\
Y = \{v_0 v_1 \} \cup \{v_i v_{4k+3-i} : \text{\( i \) is even and } 2 \leq i \leq 2k\}.
\]

For each \( e \in E(K_{4k+2}) \), let

\[
c(e) = \begin{cases} 
0 & \text{if } e \in \{v_i v_{i+1} : \text{\( i \) is even and } 0 \leq i \leq 4k\} \\
1 & \text{if } e \in \{v_1 v_2\} \cup X \\
c_1(e) & \text{otherwise.}
\end{cases}
\] (2.5)

Figure 2.5 shows how to obtain the coloring \( c \) from the coloring \( c_1 \) for \( n \in \{6, 10, 14\} \), where only the edges \( e \) with \( c(e) \in \{0, 1\} \) are drawn. An edge \( e \) is labeled by \( (c_1(e), c(e)) \) to indicate that the color \( c_1(e) \) of \( e \) is changed by \( c(e) \in \{0, 1\} \) as described in (2.5).

More precisely, if \( e \) is drawn by a solid line in Figure 2.5, then \( c(e) = 0 \); while if \( e \) is drawn by a dashed line, then \( c(e) = 1 \) (see Figure 2.7 for \( K_{4k+2} \) in general).

Let \( b = \binom{4k+3}{2} - 1 \), where then \( b = -1 \) in \( \mathbb{Z}_{4k+3} \) and let \( c' : V(K_{4k+2}) \to \mathbb{Z}_{4k+3} \) be the vertex coloring induced by \( c \). Then

- For \( i = 0, 1, 2 \),
  \[
  c'(v_0) = b - (2k + 2) \\
  c'(v_1) = b - (2k + 2) - (2k + 3) + 1 = b - 1 \\
  c'(v_2) = b - (2k + 3) - (2k + 1) + 1 = b - 0.
  \]

- For \( 3 \leq i \leq 2k + 1 \),
  \[
  c'(v_i) = \begin{cases} 
  b - (2k + 3 - i) & \text{if } i \text{ is even} \\
  b - (2k + 2) - (2k + 4 - i) + 1 = b - (2 - i) & \text{if } i \text{ is odd.}
  \end{cases}
  \]

- For \( 2k + 2 \leq i \leq 4k + 1 \),
  \[
  c'(v_i) = \begin{cases} 
  b - (2k + 2) - (i - 2k) + 1 = b - (i + 1) & \text{if } i \text{ is even} \\
  b - (i - 1 - 2k) & \text{if } i \text{ is odd.}
  \end{cases}
  \]
Figure 2.5: Illustrating the coloring $c$ for $K_6$, $K_{10}$ and $K_{14}$.

Figure 2.6 shows the coloring $c'$ for $K_{10}$ where $b = \binom{11}{2} - 1$. In this case, $n = 10$, $k = 3$ and the color $b - 7 = b - (2k + 1) \in \mathbb{Z}_{11}$ is not used by the coloring $c'$.

For each $i$ with $0 \leq i \leq 4k + 1$, let $c'(v_i) = b - a_i$ for $0 \leq i \leq 4k + 1$. If $s_{c'} = (a_0, a_1, \ldots, a_{4k+1})$ (where $a_i = b - c'(v_i)$ for $0 \leq i \leq 4k + 1$), then

$$s_{c'} = (2k + 2, 1, 0, 4k + 2, 2k - 1, 4k, 2k - 3, \ldots, 2k + 8, 5, 2k + 6, 3, 2k + 4 = b - c'(v_{2k+1}), 2k + 3, 2, 2k + 5, 4, \ldots, 4k - 1, 2k - 2, 4k + 1, 2k).$$

For example, the sequences $s_{c'}$ for $n = 6, 10, 14$ are the following:

- $(4, 1, 0, 6, 5, 2)$ for $n = 6$ and $k = 1$
- $(6, 1, 0, 10, 3, 8, 7, 2, 9, 4)$ for $n = 10$ and $k = 2$
- $(8, 1, 0, 14, 5, 12, 3, 10, 9, 2, 11, 4, 13, 6)$ for $n = 14$ and $k = 3$.

In conclusion, we observe that $\{c'(v) : v \in V(K_{4k+2})\} = \{b-i : 0 \leq i \leq 4k+2, i \neq 2k+1\}$ and so $c$ is a twin edge $(4k + 3)$-coloring of $K_{4k+2}$.
In summary, we have the following.

**Theorem 2.3.6** For each integer \( n \geq 3 \),

\[
\chi_t'(K_n) = \begin{cases} 
  n & \text{if } n \text{ is odd} \\ 
  n + 1 & \text{if } n \text{ is even.}
\end{cases}
\]

### 2.4 Complete Bipartite Graphs

In this section we determine twin chromatic indexes of the complete bipartite graphs \( K_{a,b} \) where \( 1 \leq a \leq b \), beginning with stars \( K_{1,b} \) for \( b \geq 2 \).

**Proposition 2.4.1** If \( K_{1,b} \) is a star of order \( b \geq 2 \), then

\[
\chi_t'(K_{1,b}) = \begin{cases} 
  b + 1 & \text{if } b \not\equiv 1 \pmod{4} \\ 
  b + 2 & \text{if } b \equiv 1 \pmod{4}
\end{cases}
\]

**Proof.** Let \( v \) be the central vertex of \( K_{1,b} \). First, note that \( \chi_t'(K_{1,b}) \geq \chi'(K_{1,b}) = b \) and each twin edge coloring \( c \) of \( K_{1,b} \) must assign \( b \) distinct colors to the edges of \( K_{1,b} \). Thus, \( c' \) must assign the same color to each end-vertex that \( c \) assigns to its incident edge. Since the color of \( v \) must be distinct from each end-vertex, it follows that \( \chi_t'(K_{1,b}) \geq b + 1 \).

First, we show that \( \chi_t'(K_{1,b}) = b + 1 \) if \( b \not\equiv 1 \pmod{4} \). Suppose, initially, that \( b \) is even. Assign the colors \( 1, 2, \ldots, b \) of \( \mathbb{Z}_{b+1} \) to the edges of \( K_{1,b} \). Then the end-vertices...
are assigned these colors as well. The central vertex $v$ of $K_{1,b}$ is assigned the color $\sum_{i=1}^{b} i = \binom{b+1}{2}$ in $\mathbb{Z}_{b+1}$. Since $b+1$ is odd, $\binom{b+1}{2} = 0$ in $\mathbb{Z}_{b+1}$ by Observation 2.3.1 and so this is a proper vertex coloring of $K_{1,b}$. Next, suppose that $b$ is odd. Since $b \neq 1 \pmod{4}$, it follows that $b \equiv 3 \pmod{4}$ or $b+1 \equiv 0 \pmod{4}$. Then $b+1 = 4k$ for some positive integer $k$. Assign the $b = 4k - 1$ edges of $K_{1,b}$ the colors $0, 1, 2, \ldots, 4k - 1$ excluding the color $k$. The end-vertices of $K_{1,b}$ are therefore assigned the same colors. The vertex $v$ is then assigned the color

$$\left(\sum_{i=0}^{4k-1} i\right) - k = \binom{4k}{2} - k = 2k(4k - 1) - k = 8k^2 - 3k$$

which is the color $k$ in $\mathbb{Z}_{b+1}$. Since this is a proper vertex coloring of $K_{1,b}$, it follows that $\chi'_t(K_{1,b}) = b + 1$ when $b \equiv 3 \pmod{4}$.

It remains to show that $\chi'_t(K_{1,b}) = b + 2$ when $b \equiv 1 \pmod{4}$. Thus $b + 1 = 4k + 2$ for some positive integer $k$. First, assume to the contrary, that there exists a twin edge coloring of $K_{1,b}$ using the colors of $\mathbb{Z}_{b+1}$. Then the edges of $K_{1,b}$ can be colored with $0, 1, 2, \ldots, 4k + 1$ except for one element, say $a$, of $\mathbb{Z}_{b+1}$. Then the color of $v$ must be $a$. 

Figure 2.7: Constructing the coloring $c$ from the coloring $c_1$ in the proof of Theorem 2.3.5.
Since the color of \( v \) is \( \left( \sum_{i=0}^{4k+1} i \right) - a = \left( \frac{4k+2}{2} \right) - a = (2k+1)(4k+1) - a \), it follows that \( (2k+1)(4k+1) - a \equiv a \pmod{4k+2} \) and so \( 4k+2 \) divides \( (2k+1)(4k+1) - 2a \). This, however, is impossible since \( (2k+1)(4k+1) - 2a \) is odd. Therefore, \( \chi'_t(K_{1,b}) \geq b + 2 \).

Now let \( c \) be the edge coloring of \( K_{1,b} \) that assigns the colors \( 0, 2, 3, \ldots, b-1, b+1 \in \mathbb{Z}_{b+2} \) to the \( b \) edges of \( K_{1,b} \) (and so 1 and \( b \) are not used). Then the induced vertex coloring \( c' \) assigns the colors \( 0, 2, 3, \ldots, b-1, b+1 \) of \( \mathbb{Z}_{b+2} \) to the \( b \) end-vertices of \( K_{1,b} \) and the color 1 to the central vertex of \( K_{1,b} \). This produces a proper vertex coloring of \( K_{1,b} \) and so \( \chi'_t(K_{1,b}) = b + 2 \).

We now determine \( \chi'_t(K_{a,b}) \) where \( 2 \leq a \leq b \) and \( b \in \{a, a+1\} \).

**Theorem 2.4.2** If \( a \geq 2 \) and \( b \) are integers with \( b \in \{a, a+1\} \), then \( \chi'_t(K_{a,b}) = a + 2 \).

**Proof.** Let \( U = \{u_1, u_2, \ldots, u_a\} \) and \( V = \{v_1, v_2, \ldots, v_b\} \) be the partite sets of \( K_{a,b} \).

We consider two cases, according to whether \( b = a \) or \( b = a + 1 \).

**Case 1.** \( b = a \). By Observation 2.2.3, \( \chi'_t(K_{a,a}) \geq a + 1 \). First, we show that \( \chi'_t(K_{a,a}) \neq a + 1 \). Suppose that \( \chi'_t(K_{a,a}) = a + 1 \). Then there is a twin edge \((a+1)\)-coloring \( c \) of \( K_{a,a} \) using the colors in \( \mathbb{Z}_{a+1} \). Hence \( c \) assigns exactly \( a \) colors to the \( a \) incident edges of each vertex of \( K_{a,a} \). Consider \( u_1 \) and let \( t \in \mathbb{Z}_{a+1} \) such that \( c \) assigns the colors in \( \mathbb{Z}_{a+1} - \{t\} \) to the edges incident with \( u_1 \) (and so no edge incident with \( u_1 \) is colored \( t \)). We claim that for each vertex \( v_j \) (\( 1 \leq j \leq a \)), there is an edge incident with \( v_j \) that is colored \( t \); for otherwise, we may assume that no edge incident with \( v_1 \) is colored \( t \). However then, \( c \) assigns the colors in \( \mathbb{Z}_{a+1} - \{t\} \) to the edges incident with \( v_1 \) and so \( c'(v_1) = c'(v_1) \), which is impossible. Thus, as claimed, there is an edge incident with \( v_j \) that is colored \( t \) for \( j = 1, 2, \ldots, a \). Hence there are at least \( a \) edges of \( K_{a,a} \) that are colored \( t \). Since no edge incident with \( u_1 \) is colored \( t \), it follows that at least two edges colored \( t \) are incident with the same vertex in \( U \), which is a contradiction. Therefore, \( \chi'_t(K_{a,a}) \neq a + 1 \) and so \( \chi'_t(K_{a,a}) \geq a + 2 \).

Next, we show that \( K_{a,a} \) has a twin edge \((a+2)\)-coloring. Since \( K_{a,a} \) is bipartite and \( a \)-regular, \( K_{a,a} \) is \( 1 \)-factorable. Let \( \{F_0, F_1, \ldots, F_{a-1}\} \) be a \( 1 \)-factorization of \( K_{a,a} \) where

\[ E(F_i) = \{u_jv_{j+i} : 1 \leq j \leq a \} \text{ for } 0 \leq i \leq a - 1 \]

(all subscripts are expressed as integers modulo \( a \)). For example, \( E(F_0) = \{u_jv_j : 1 \leq j \leq a\} \), \( E(F_1) = \{u_jv_{j+1} : 1 \leq j \leq a \} \) and \( E(F_{a-1}) = \{u_jv_{j+(a-1)} : 1 \leq j \leq a\} \). This is shown in Figure 2.8 for \( K_{4,4} \).
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This is shown for $c$ thus $|\chi| = F_a + 1$.

We consider two cases, according to whether $a$ is odd or $a$ is even.

Subcase 1.1 $a$ is odd. Then $a = 2k + 1$ for some positive integer $k$. Let $M_a$ and $M_{a+1}$ be the following matchings in $K_{a,a}$:

$$M_a = \{u_1v_1, u_3v_2, u_4v_4, u_6v_6, \ldots, u_{2k}v_{2k}\}$$

$$M_{a+1} = \{u_1v_2, u_3v_3, u_4v_5, u_6v_7, \ldots, u_{2k}v_{2k+1}\}.$$ 

Thus $|M_a| = |M_{a+1}| = k+1$. For each $i$ with $0 \leq i \leq a-1$, let $M_i = E(F_i) - (M_a \cup M_{a+1})$.

This is shown for $K_{5,5}$ in Figure 2.9. Define a proper edge coloring $c : E(K_{a,a}) \to \mathbb{Z}_{a+2}$ by $c(e) = i$ if $e \in M_i$ for $0 \leq i \leq a+1$. Since $c'(u) = \binom{a}{2}$ or $c'(u) = \binom{a}{2} - 4$ if $u \in U$ and $c'(v) = \binom{a}{2} - 1$ or $c'(v) = \binom{a}{2} - 2$ if $v \in V$, it follows that $c$ is a twin edge $(a+2)$-coloring.

Therefore, $\chi'_t(K_{a,a}) = a + 2$.

Figure 2.8: A 1-factorization $\{F_0, F_1, F_2, F_3\}$ of $K_{4,4}$

Figure 2.9: Illustrating $F_0, F_1, F_2, F_3, F_4$ and $M_5, M_6$ for $K_{5,5}$ where $M_i = E(F_i) - (M_5 \cup M_6)$ for $0 \leq i \leq 4$
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Subcase 1.2. $a$ is even. Then $a = 2k \geq 2$ for some positive integer $k$. Let $M_a$ and $M_{a+1}$ be the following matchings in $K_{a,a}$:

$$M_a = \{u_1v_1, u_3v_3, \ldots, u_{2k-1}v_{2k-1}\}$$
$$M_{a+1} = \{u_1v_2, u_3v_4, \ldots, u_{2k-1}v_{2k}\}.$$ 

Thus $|M_a| = |M_{a+1}| = k$. For each $i$ with $0 \leq i \leq a - 1$, let $M_i = E(F_i) - (M_a \cup M_{a+1})$. This is shown for $K_{4,4}$ in Figure 2.10. Define a proper edge coloring $c : E(K_{a,a}) \rightarrow \mathbb{Z}_{a+2}$ by $c(e) = i$ if $e \in M_i$ for $0 \leq i \leq a + 1$. Since $c'(u) = \binom{a}{2}$ or $c'(u) = \binom{a}{2} - 4$ if $u \in U$ and $c'(v) = \binom{a}{2} - 2$ if $v \in V$, it follows that $c$ is a twin edge $(a + 2)$-coloring. Therefore, $\chi'_t(K_{a,a}) = a + 2$.

![Diagram](image)

Figure 2.10: Illustrating $F_0, F_1, F_2, F_3$ and $M_1, M_2$ for $K_{4,4}$ where $M_i = E(F_i) - (M_4 \cup M_5)$ for $0 \leq i \leq 3$

Case 2. $b = a + 1$. Since $\Delta(K_{a,a+1}) = a + 1$, it follows that $\chi'_t(K_{a,a+1}) \geq a + 1$. First, we show that $\chi'_t(K_{a,a+1}) \neq a + 1$. Suppose that $\chi'_t(K_{a,a+1}) = a + 1$. Then there is a twin edge $(a + 1)$-coloring $c$ of $K_{a,a+1}$ using colors in $\mathbb{Z}_{a+1}$. Since $\deg u_i = a + 1$ for $1 \leq i \leq a$, it follows that $c$ assigns all colors in $\mathbb{Z}_{a+1}$ to the $a + 1$ edges incident with each vertex $u_i$. Thus, a edges in $K_{a,a+1}$ are colored 0. Since $|V| = a + 1$, some vertex in $V$ is not incident with any edge colored 0, say $v_1$. Consequently, $c$ assigns the $a$ colors in $\mathbb{Z}_{a+1} - \{0\}$ to the $a$ edges incident with $v_1$. However then, $c'(v_1) = c'(u_i) = \binom{a+1}{2}$ for $1 \leq i \leq a$, which is impossible. Therefore, $\chi'_t(K_{a,a+1}) \neq a + 1$ and $\chi'_t(K_{a,a+1}) \geq a + 2$.

Next, we show that $K_{a,a+1}$ has a twin edge $(a + 2)$-coloring. Define a proper edge coloring $c : E(K_{a,a+1}) \rightarrow \mathbb{Z}_{a+2}$ using only the colors in $\mathbb{Z}_{a+2} - \{0\}$ as follows: For each $i$ with $1 \leq i \leq a$, let $c(u_i,v_{i+j}) = j + 1$ for each $j$ with $0 \leq j \leq a$. In particular, $c(u_i,v_i) = 1$ for $1 \leq i \leq a$. Thus, $c'(u_i) = \binom{a+2}{2}$ for $1 \leq i \leq a$. Furthermore, $c'(v_j) = \binom{a+2}{2} - (j + 1)$ for $1 \leq j \leq a$ and $c'(v_{a+1}) = \binom{a+2}{2} - 1$. Since $c'(v_j) \neq \binom{a+2}{2}$ in $\mathbb{Z}_{a+2}$ for $1 \leq j \leq a + 1$, it follows that $c'$ is a proper vertex coloring of $K_{a,a+1}$. Therefore, $\chi'_t(K_{a,a+1}) = a + 2$. ■
Finally, we determine $\chi'_t(K_{a,b})$ for all integers $a$ and $b$ with $a \geq 2$ and $b \geq a + 2$.

**Theorem 2.4.3** If $a \geq 2$ and $b$ are integers with $b \geq a + 2$, then $\chi'_t(K_{a,b}) = b$.

**Proof.** Since $\chi'_t(K_{a,b}) \geq \chi'(K_{a,b}) = \Delta(K_{a,b}) = b$, it suffices to show that $K_{a,b}$ has a twin edge $b$-coloring. Let $U = \{u_1, u_2, \ldots, u_a\}$ and $V = \{v_1, v_2, \ldots, v_b\}$ be the partite sets of $K_{a,b}$. Suppose that

$$\sum_{i=0}^{a-1} i = \left(\frac{a}{2}\right) \equiv k \pmod{b}$$

and

$$\sum_{i=0}^{b-1} i = \left(\frac{b}{2}\right) \equiv \ell \pmod{b},$$

where $0 \leq k, \ell \leq b - 1$. We consider two cases, depending on whether $a$ and $b$ are relatively prime.

**Case 1.** $a$ and $b$ are not relatively prime. Then $d = \gcd(a, b) \geq 2$ and $b = pd$ for some $p \in \mathbb{N}$. For $0 \leq i \leq d - 1$, let $X_i = \{i, i + a, i + 2a, \ldots, i + (p - 1)a\}$ be a subset of $Z_b$. In fact, $X_0, X_1, \ldots, X_{d-1}$ are the cosets of the subgroup $X_0 = \{0, 2a, \ldots, (p-1)a\}$ in the group $Z_b$. Hence $X = \{X_0, X_1, \ldots, X_{d-1}\}$ is a partition of $Z_b$. Next, let $X, X' \in X$ such that $k \in X$ and $\ell \in X'$. We define a coloring $c : E(K_{a,b}) \to Z_b$, according to whether $X \neq X'$ or $X = X'$.

**Subcase 1.1.** $X \neq X'$. For $1 \leq i \leq a$ and $1 \leq j \leq b$, define $c(u_iv_j) = i + j - 2 \in Z_b$. Then $c'(u_i) = \ell \in X'$ for $1 \leq i \leq a$ and $c'(v_j) = k + (j - 1)a \in X$ for $1 \leq j \leq b$. Since $X \cap X' = \emptyset$, it follows that $c'(u_i) \neq c'(v_j)$ for all $i, j$ with $1 \leq i \leq a$ and $1 \leq j \leq b$. Thus $c'$ is a proper vertex coloring.

**Subcase 1.2.** $X = X'$. Since $d \geq 2$, it follows that $k + 1 \notin X$, say $k + 1 \in X'' \in X$. For $1 \leq i \leq a - 1$ and $1 \leq j \leq b$, define $c(u_iv_j) = i + j - 2 \in Z_b$. Furthermore, define $c(u_av_j) = a + j - 1 \in Z_b$ for $1 \leq j \leq b$. Then $c'(u_i) = \ell \in X$ for $1 \leq i \leq a$ and $c'(v_j) = (k + 1) + (j - 1)a \in X''$ for $1 \leq j \leq b$. Since $X \cap X'' = \emptyset$, it follows that $c'(u_i) \neq c'(v_j)$ for all $i, j$ with $1 \leq i \leq a$ and $1 \leq j \leq b$. Thus $c'$ is a proper vertex coloring.

**Case 2.** $a$ and $b$ are relatively prime. Let $Z_b = \{\ell, \ell + a, \ldots, \ell + (b - 1)a\}$. We start with a proper edge coloring $c_1 : E(K_{a,b}) \to Z_b$ defined by $c_1(u_iv_j) = i + j - 2$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. Then $c'_1(u_i) = \ell$ for $1 \leq i \leq a$ and $c'_1(v_j) = k + (j - 1)a \in X$ for $1 \leq j \leq b$. Since $a$ and $b$ are relatively prime, $\{c'_1(v_j) : 1 \leq j \leq b\} = Z_b$. Therefore,
there exists exactly one integer \( t \) with \( 1 \leq t \leq b \) such that \( c'_1(v_t) = \ell \). Thus \( c'_1 \) is not a proper vertex coloring. We now produce a twin edge \( b \)-coloring \( c \) from \( c_1 \) as follows: Let \( r = \lfloor a/2 \rfloor \) and \( s = t + \lfloor (b - 1)/2 \rfloor \) in \( \mathbb{Z}_b \), where then \( 1 \leq s \leq b \) and \( s \neq t \), and let \( c \) be the coloring obtained from \( c_1 \) by interchanging the colors of the edges \( u_r v_t \) and \( u_r v_s \) in \( c_1 \); that is,

\[
c(e) = \begin{cases} 
  c_1(e) & \text{if } e \in E(K_{a,b}) - \{u_r v_t, u_r v_s\} \\
  c_1(u_r v_s) & \text{if } e = u_r v_t \\
  c_1(u_r v_t) & \text{if } e = u_r v_s.
\end{cases}
\]

We show that \( c'(u_i) = \ell \) for \( 1 \leq i \leq a \) and \( c'(v_j) \neq \ell \) for \( 1 \leq j \leq b \).

By the defining property of \( c \), it follows that \( c'(u_i) = c'_1(u_i) = \ell \) and \( c'(v_j) = c'_1(v_j) \neq \ell \) for \( 1 \leq j \leq b \) and \( j \neq s, t \). Thus, it remains to show that \( c'(v_t) \neq \ell \) and \( c'(v_s) \neq \ell \).

Since \( \ell = k + (t - 1)a \) and \( s = t + \lfloor (b - 1)/2 \rfloor \), it follows that

\[
c'(v_t) = c'_1(v_t) - c_1(u_r v_t) + c_1(u_r v_s) = \ell - (r + t - 2) + (r + s - 2) \\
   = \ell - t + s = \ell - t + [t + \lfloor (b - 1)/2 \rfloor] = \ell + \lfloor (b - 1)/2 \rfloor
\]

\[
c'(v_s) = c'_1(v_s) - c_1(u_r v_s) + c_1(u_r v_t) = [k + (s - 1)a] - (r + s - 2) + (r + t - 2) \\
   = [k + (s - 1)a] - s + t = [k + (s - 1)a] - \lfloor (b - 1)/2 \rfloor \\
   = k + (t + \lfloor (b - 1)/2 \rfloor - 1)a - \lfloor (b - 1)/2 \rfloor = \ell + (a - 1)\lfloor (b - 1)/2 \rfloor.
\]

We consider two cases, according to whether \( b \) is odd or \( b \) is even.

**Subcase 2.1. \( b \) is odd.** Then \( \lfloor (b - 1)/2 \rfloor = \frac{b - 1}{2} \). We claim that

\[
c'(v_t) = \ell + \frac{b - 1}{2} \neq \ell \text{ in } \mathbb{Z}_b
\]

(2.6)

\[
c'(v_s) = \ell + (a - 1)\frac{b - 1}{2} \neq \ell \text{ in } \mathbb{Z}_b
\]

(2.7)

Since \( b \) is odd, \( \ell = 0 \) in \( \mathbb{Z}_b \) by Observation 2.3.1, while \( \frac{b - 1}{2} \neq 0 \) in \( \mathbb{Z}_b \), which implies that (2.6) holds. To verify (2.7), we show that \( \frac{b - 1}{2}(a - 1) \neq 0 \mod b \). If this were not the case, then \( \frac{b - 1}{2}(a - 1) = bx \) for some integer \( x \). This implies that \( 2bx = (b - 1)(a - 1) = a - 1 \) in \( \mathbb{Z}_b \) or \( a - 1 \equiv 0 \mod b \). However then, \( b \mid (a - 1) \), which is impossible.

**Subcase 2.2. \( b \) is even.** Then \( \lfloor (b - 1)/2 \rfloor = \frac{b - 1}{2} - 1 \) and \( \ell = \frac{b}{2} \) in \( \mathbb{Z}_b \) by Observation 2.3.1. Since \( a \) and \( b \) are relatively prime, it follows that \( a \geq 3 \) is odd and so \( b \geq a + 3 \geq 6 \). We
claim that
\[ c'(v_t) = \ell + \left( \frac{b}{2} - 1 \right) = b - 1 \neq \ell \text{ in } \mathbb{Z}_b \quad (2.8) \]
\[ c'(v_s) = \ell + \left( \frac{b}{2} - 1 \right) (a - 1) \neq \ell \text{ in } \mathbb{Z}_b . \quad (2.9) \]

Since \( \ell = \frac{b}{2} \) in \( \mathbb{Z}_b \) and \( b - 1 \neq \frac{b}{2} \) in \( \mathbb{Z}_b \), it follows that (2.8) holds. To verify (2.9), we show that \( (\frac{b}{2} - 1)(a - 1) \neq 0 \) (mod \( b \)). If this were not the case, then \( \frac{b-2}{2}(a-1) = bx \) for some positive integer \( x \). Since \( b \) is even, \( b = 2y \) for some integer \( y \geq 3 \). Then \( a = 2\frac{xy}{y-1} + 1 \).

Since \( a \) is an integer and \( y \geq 3 \), it follows that \( (y - 1) \mid y \) and so \( (y - 1) \mid x \). Let \( x = (y - 1)z \) for some positive integer \( z \). However then, \( a = 2yz + 1 = bz + 1 \), which is impossible.

Thus \( c' \) is a proper vertex coloring of \( K_{a,b} \) and so \( \chi'_1(K_{a,b}) = b \). \[ \blacksquare \]

We now illustrate the proof of Theorem 2.4.3 for the graphs \( K_{4,6} \) and \( K_{5,8} \).

- For \( K_{4,6} \) (where \( a = 4 \) and \( b = 6 \) are not relatively prime), let

\[ k = \sum_{i=0}^{3} i = 6 \equiv 0 \text{ (mod 6) and } \ell = \sum_{i=0}^{5} i = 15 \equiv 3 \text{ (mod 6).} \]

Since \( \ell \not\in \{0, 2, 4\} \), the two cosets are \( \{0, 2, 4\} \) and \( \{1, 3, 5\} \). Let \( c \) be the twin edge 6-coloring of \( K_{4,6} \) using the colors in \( \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} \) (which is described in the proof of Theorem 2.4.3). For each \( i \) with \( 0 \leq i \leq 5 \), let \( M_i \) be the set of edges colored \( i \) by \( c \), which are shown in Figure 2.11.

![Figure 2.11: The color classes of the edge coloring \( c \)](image)

Then the induced vertex coloring \( c' \) of \( c \) assigns the colors in \( \mathbb{Z}_6 \) to the vertices of \( K_{4,6} \) as follows:
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\[ c'(u_i) = \sum_{i=0}^{5} i = 15 \equiv 3 \mod 6 \] for \( 1 \leq i \leq 4; \]
\[ c'(v_1) = 0 + 1 + 2 + 3 = 6 \equiv 0 \mod 6, \]
\[ c'(v_2) = 1 + 2 + 3 + 4 = 10 \equiv 4 \mod 6, \]
\[ c'(v_3) = 2 + 3 + 4 + 5 = 14 \equiv 2 \mod 6, \]
\[ c'(v_4) = 3 + 4 + 5 + 0 = 12 \equiv 0 \mod 6, \]
\[ c'(v_5) = 4 + 5 + 0 + 1 = 10 \equiv 4 \mod 6, \]
\[ c'(v_6) = 5 + 0 + 1 + 2 = 8 \equiv 2 \mod 6. \]

- For \( K_{5,8} \) (where \( a = 5 \) and \( b = 8 \) are relatively prime), let \( r = \lfloor 5/2 \rfloor = 3 \). In this case, \( t = 3 \) and \( s = t + \lfloor (b - 1)/2 \rfloor = t + 3 = 6 \). Let \( c \) be the twin edge 8-coloring of \( K_{5,8} \) using the colors in \( \mathbb{Z}_8 \) (which is obtained from the coloring \( c_1 \) as described in the proof of Theorem 2.4.3). For each \( i \) with \( 0 \leq i \leq 7 \), let \( M_i \) be the set of edges colored \( i \) by \( c \), which are shown in Figure 2.12. Observe that (i) \( c_1(u_3v_3) = 4 \) and \( c(u_3v_3) = 7 \) and (ii) \( c_1(u_3v_6) = 7 \) and \( c(u_3v_6) = 4 \), which are indicated in \( M_4 \) and \( M_7 \) in Figure 2.12.

![Figure 2.12: The color classes of the edge coloring c](image)

Then the induced vertex coloring \( c' \) of \( c \) assigns the colors in \( \mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\} \) to the vertices of \( K_{5,8} \) as follows:
In summary, we have the following.

**Theorem 2.4.4** For positive integers \(a\) and \(b\) with \(a \leq b\),

\[
\chi'_t(K_{a,b}) = \begin{cases} 
  b & \text{if } b \geq a + 2 \text{ and } a \geq 2 \\
  b + 1 & \text{if either } a = 1 \text{ and } b \not\equiv 1 \pmod{4} \text{ or } b = a + 1 \geq 3 \\
  b + 2 & \text{if either } a = 1 \text{ and } b \equiv 1 \pmod{4} \text{ or } b = a \geq 2.
\end{cases}
\]

2.5 An Upper Bound

In this section, we establish an upper bound for the twin chromatic index of a connected graph of order at least 3. The idea for this proof was suggested by Noga Alon.

**Theorem 2.5.1** For every connected graph \(G\) of order at least 3,

\[
\chi'_t(G) \leq 4\Delta(G) - 3.
\]

**Proof.** Let \(G\) be a connected graph of size \(m\), let \(e_1, e_2, \ldots, e_m\) be an ordering of the edges of \(G\) where \(e_i = u_iv_i\) for \(1 \leq i \leq m\) and let \(k = 4\Delta(G) - 3\). We define \(m+1\) colorings \(c_0, c_1, c_2, \ldots, c_m\) inductively such that \(c_i : E(G) \to \mathbb{Z}_k\) for \(0 \leq i \leq m\) and \(c_m\) is a twin edge \(k\)-coloring of \(G\).

First, define \(c_0(e_i) = 0 \in \mathbb{Z}_k\) for \(1 \leq i \leq m\). Thus \(c'_0(v) = 0\) for each \(v \in V(G)\).

Suppose that \(c_{i-1} : E(G) \to \mathbb{Z}_k\) has been defined for some positive integer \(i\) with \(1 \leq i \leq m\).
Thus, assume, to the contrary, that edges of $G$. In particular, $A_1 = B_1 = D_1 = \{0\}$. Notice that

$$|A_i| \leq (\deg u_i - 1) + (\deg v_i - 1) \leq 2\Delta(G) - 2$$
$$|B_i| \leq \deg u_i - 1 \leq \Delta(G) - 1$$
$$|D_i| \leq \deg v_i - 1 \leq \Delta(G) - 1.$$

Since $k = 4\Delta(G) - 3$, it follows that

$$|Z_k - (A_i \cup B_i \cup D_i)| \geq (4\Delta(G) - 3) - [(2\Delta(G) - 2) + 2(\Delta(G) - 1)] = 1$$

and so $Z_k - (A_i \cup B_i \cup D_i) \neq \emptyset$. For each $i$ with $1 \leq i \leq m$, the coloring $c_i : E(G) \to Z_k$ is defined by

$$c_i(e_j) = \begin{cases} c_{i-1}(e_j) & \text{if } i \neq j \\ \min\{Z_k - (A_i \cup B_i \cup D_i)\} & \text{if } i = j. \end{cases} \quad (2.10)$$

In particular, $c_1(e_1) = 1$ and $c_1(e_j) = 0$ for $2 \leq j \leq m$. We show that $c = c_m : E(G) \to Z_k$ is a twin edge coloring of $G$.

First, we show that $c$ is a proper edge coloring of $G$. Let $e$ and $f$ be any two adjacent edges of $G$, say $e = e_s$ and $f = e_t$ where $1 \leq s < t \leq m$. We show that $c(e_s) \neq c(e_t)$. Assume, to the contrary, that $c(e_s) = c(e_t)$. By the inductive definition of the coloring $c$, it follows that

$$c_{t-1}(e_s) = c_t(e_s) = c_{t+1}(e_s) = \cdots = c_{m-1}(e_s) = c_m(e_s) = c(e_s)$$
$$c_{t-1}(e_t) = c_t(e_t) = c_{t+1}(e_t) = \cdots = c_{m-1}(e_t) = c_m(e_t) = c(e_t).$$

Thus, $c_{t-1}(e_s) = c_t(e_t)$. Since $e_s$ and $e_t$ are adjacent, it follows that

$$c_{t-1}(e_s) \in A_t = \{c_{t-1}(e) : e \text{ is adjacent to } e_t\}.$$

Since $c_t(e_t) \notin A_t \cup B_t \cup D_t$ by (2.10), this contradicts the definition of the edge coloring $c_t$. Therefore, $c(e) \neq c(f)$ for every two adjacent edges $e$ and $f$ of $G$ and so $c$ is a proper edge coloring of $G$. 

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Next, we show that the induced vertex coloring $c'$ is a proper vertex coloring of $G$. Assume, to the contrary, that there are adjacent vertices $u$ and $v$ in $G$ such that $c'(u) = c'(v)$. Since $G$ is a connected graph of order $n \geq 3$, it follows that $|E_u \cup E_v| \geq 2$. Among all edges in $E_u \cup E_v$, let $e_t$ be one having the largest subscript $t$ where then $2 \leq t \leq m$. We consider two cases, according to whether $e_t \neq uv$ or $e_t = uv$.

**Case 1.** $e_t \neq uv$. Since $e_t \in E_u \cup E_v$, it follows that $e_t \in E_u$ or $e_t \in E_v$, say $e_t \in E_u$. Then $e_t = utv_t$ where $u_t = u$ and $v_t \neq v$. In what follows, we show that

$$c'_{t-1}(v) = c'_{t-1}(u_t) + c_t(e_t). \quad (2.11)$$

Since the subscript of each edge $e \in E_v$ is less than $t$, it follows that

$$c(e) = c_m(e) = c_{m-1}(e) = \cdots = c_t(e) = c_{t-1}(e). \quad (2.12)$$

Thus

$$c'(v) = \sum_{e \in E_v} c(e) = \sum_{e \in E_v} c_{t-1}(e) = c'_{t-1}(v). \quad (2.13)$$

Similarly, since the subscript of each $e \in E_{ut} - \{e_t\}$ is less than $t$, it follows that $e$ also satisfies (2.12). Furthermore,

$$c(e_t) = c_m(e_t) = c_{m-1}(e_t) = \cdots = c_{t+1}(e_t) = c_t(e_t) \quad (2.14)$$

and

$$c_{t-1}(e_t) = 0. \quad (2.15)$$

On the other hand, it follows by (2.12), (2.14) and (2.15) that

$$c'(u_t) = \sum_{e \in E_{ut}} c(e) = \left( \sum_{e \in E_{ut} - \{e_t\}} c_t(e) \right) + c_t(e_t)
= \left( \sum_{e \in E_{ut} - \{e_t\}} c_{t-1}(e) \right) + c_{t-1}(e_t) + c_t(e_t)
= \left( \sum_{e \in E_{ut}} c_{t-1}(e) \right) + c_t(e_t) = c'_{t-1}(u_t) + c_t(e_t).$$

Since $c'(u_t) = c'(v)$ and $v \in N(u_t) - \{v_t\}$, it follows by (2.13) that
\[ c'_t(v) = c'(v) = c'(u_t) = c'_{t-1}(u_t) + c_t(e_t). \]

Thus (2.11) holds and so \( c_t(e_t) \in B_t \). This, however, contradicts the defining property of the edge coloring of \( c_t \).

**Case 2.** \( e_t = uv \). Then \( u = u_t \) and \( v = v_t \). We first show that

\[ c'_{t-1}(u_t) = c'_{t-1}(v_t). \] (2.16)

Since the subscript of each \( e \in (E_{u_t} \cup E_{v_t}) - \{e_t\} \) is less than \( t \), it follows that

\[ c(e) = c_m(e) = c_{m-1}(e) = \cdots = c_t(e) = c_{t-1}(e). \] (2.17)

Also,

\[ c(e_t) = c_m(e_t) = c_{m-1}(e_t) = \cdots = c_t(e_t) \] (2.18)

and

\[ c_{t-1}(e_t) = 0. \] (2.19)

By (2.17)–(2.19), an argument similar to the one in Case 1 shows that

\[ c'(u_t) = \sum_{e \in E_{u_t}} c(e) = \sum_{e \in E_{u_t}} c_t(e) = \left( \sum_{e \in E_{u_t} - \{e_t\}} c_t(e) \right) + c_t(e_t) \]

\[ = \left( \sum_{e \in E_{u_t} - \{e_t\}} c_{t-1}(e) \right) + c_{t-1}(u_t) + c_t(e_t) \]

\[ = \left( \sum_{e \in E_{u_t}} c_{t-1}(e) \right) + c_t(e_t) = c'_{t-1}(u_t) + c_t(e_t) \]

and

\[ c'(v_t) = \sum_{e \in E_{v_t}} c(e) = \sum_{e \in E_{v_t}} c_t(e) = \left( \sum_{e \in E_{v_t} - \{e_t\}} c_t(e) \right) + c_t(e_t) \]

\[ = \left( \sum_{e \in E_{v_t} - \{e_t\}} c_{t-1}(e) \right) + c_{t-1}(e_t) + c_t(e_t) \]

\[ = \left( \sum_{e \in E_{v_t}} c_{t-1}(e) \right) + c_t(e_t) = c'_{t-1}(v_t) + c_t(e_t). \]
Therefore,
\[ c'(u_t) = c'_{t-1}(u_t) + c_t(e_t) \quad \text{and} \quad c'(v_t) = c'_{t-1}(v_t) + c_t(e_t). \]

Since \( c'(u_t) = c'(v_t) \), it follows that \( c'_{t-1}(u_t) = c'_{t-1}(v_t) \) and so (2.16) holds.

Let \( e_s \in (E_{u_t} \cup E_{v_t}) - \{e_t\} \) such that the subscript \( s \) is the largest among all edges in \( (E_{u_t} \cup E_{v_t}) - \{e_t\} \). Then \( 1 \leq s < t \). Next we show that
\[ c'_{t-1}(v_t) = c'_{s-1}(u_s) + c_s(e_s). \quad (2.20) \]

Let \( E'_{u_t} = E_{u_t} - \{e_t\} \) and \( E'_{v_t} = E_{v_t} - \{e_t\} \). Thus
\[ E'_{u_t} \cup E'_{v_t} = (E_{u_t} \cup E_{v_t}) - \{e_t\}. \]

We may assume that \( e_s \in E'_{u_t} \). Then \( e_s = u_sv_s \) where \( u_s = u_t \) and \( v_s \neq v_t \). Since the subscript of each \( e \in E'_{v_t} \) is less than \( s \) and \( s < t \), it follows that
\[ c_{t-1}(e) = c_{t-2}(e) = \cdots = c_s(e) = c_{s-1}(e). \quad (2.21) \]

Similarly, the subscript of each \( e \in E'_{u_t} - \{e_s\} \) is less than \( s \) and so \( e \) also satisfies (2.21). Furthermore,
\[ c_{t-1}(e_s) = c_{t-2}(e_s) = \cdots = c_s(e_s), \quad (2.22) \]
and
\[ c_{s-1}(e_s) = 0 \quad \text{and} \quad c_i(e_t) = 0 \quad \text{for each} \quad i \quad \text{with} \quad 0 \leq i \leq t - 1. \quad (2.23) \]

By (2.21), it follows that
\[ c'_{t-1}(v_t) = \sum_{e \in E_{v_t}} c_{t-1}(e) = \sum_{e \in E_{v_t}} c_{s-1}(e) = c'_{s-1}(v_t). \quad (2.24) \]
By (2.21)–(2.23), it follows that
\[ c'_{t-1}(u_t) = \sum_{e \in E_{u_t}} c_{t-1}(e) \]
\[ = \left( \sum_{e \in E_{u_t} - \{e_s, e_t\}} c_{t-1}(e) \right) + c_{t-1}(e_t) + c_{t-1}(e_s) \]
\[ = \left( \sum_{e \in E_{u_t} - \{e_s, e_t\}} c_{s-1}(e) \right) + [c_{s-1}(e_t) + c_{s-1}(e_s)] + c_s(e_s) \]
\[ = \left( \sum_{e \in E_{u_t}} c_{s-1}(e) \right) + c_s(e_s) = c'_{s-1}(u_t) + c_s(e_s). \]

Since \( u_t = u_s \), it follows that
\[ c'_{t-1}(u_t) = c'_{s-1}(u_s) + c_s(e_s). \quad (2.25) \]

Because \( v_t \in N(u_s) - \{v_s\} \), it then follows by (2.16), (2.24) and (2.25) that
\[ c'_{s-1}(v_t) = c'_{t-1}(v_t) = c'_{t-1}(u_t) = c'_{s-1}(u_t) + c_s(e_s). \]

Hence (2.20) holds and so \( c_s(e_s) \in B_s \), which is a contradiction.

In each case, \( c'(u) \neq c'(v) \) and so \( c' \) is a proper vertex coloring of \( G \).

Observe that if \( G = C_5 \), then \( \chi'_t(G) = 5 = 4\Delta(G) - 3 \) and the upper bound in Theorem 2.5.1 is attainable for this graph.
Chapter 3

Twin Edge Coloring Conjecture

3.1 Introduction

Recall that for a connected graph \( G \) of order at least 3, let \( c : E(G) \to \mathbb{Z}_k \) be a proper edge coloring of \( G \) for some integer \( k \geq 2 \). A vertex coloring \( c' : V(G) \to \mathbb{Z}_k \) is then defined by

\[
c'(v) = \sum_{e \in E_v} c(e) \quad \text{in} \quad \mathbb{Z}_k,
\]

where \( E_v \) is the set of edges of \( G \) incident with a vertex \( v \) and the indicated sum is computed in \( \mathbb{Z}_k \). If the induced vertex coloring \( c' \) is a proper vertex coloring of \( G \), then \( c \) is referred to as a twin edge \( k \)-coloring or simply a twin edge coloring of \( G \). The minimum \( k \) for which \( G \) has a twin edge \( k \)-coloring is called the twin chromatic index of \( G \) and is denoted by \( \chi'_t(G) \). Since a twin edge coloring is not only a proper edge coloring of \( G \) but induces a proper vertex coloring of \( G \), it follows that

\[
\chi'_t(G) \geq \max\{\chi(G), \chi'(G)\} \geq \Delta(G).
\]

For every connected graph \( G \) for which the twin chromatic index has been determined, we have seen that \( \chi'_t(G) = \Delta(G) + i \) for some \( i \in \{0, 1, 2, 3\} \). This leads us to the following problem: Is \( \chi'_t(G) \leq \Delta(G) + 3 \) for every connected graph \( G \) of order at least 3?

We have seen that \( \chi'_t(C_5) = 5 = \Delta(C_5) + 3 \). For each connected graph that is not the 5-cycle, the following conjecture analogous to the Total Coloring Conjecture.

**Conjecture 3.1.1** If \( G \) is a connected graph of order at least 3 that is not a 5-cycle, then \( \chi'_t(G) \leq \Delta(G) + 2 \).
Chapter 3. Twin Edge Coloring Conjecture

Conjecture 3.1.1 was verified in Chapter 2 for several well-known classes of graphs, namely paths, cycles, complete graphs and complete bipartite graphs, which we state next.

Theorem 3.1.2 If $n, a, b$ are integers with $n \geq 3$, $1 \leq a \leq b$ and $b \geq 2$, then

\[
\chi'_t(P_n) = 3
\]
\[
\chi'_t(C_n) = \begin{cases} 
  3 & \text{if } n \equiv 0 \pmod{3} \\
  4 & \text{if } n \not\equiv 0 \pmod{3} \text{ and } n \neq 5 \\
  5 & \text{if } n = 5
\end{cases}
\]
\[
\chi'_t(K_n) = \begin{cases} 
  n & \text{if } n \text{ is odd} \\
  n + 1 & \text{if } n \text{ is even}
\end{cases}
\]
\[
\chi'_t(K_{a,b}) = \begin{cases} 
  b & \text{if } b \geq a + 2 \text{ and } a \geq 2 \\
  b + 1 & \text{if either } a = 1 \text{ and } b \not\equiv 1 \pmod{4} \\
  & \text{or } b = a + 1 \geq 3 \\
  b + 2 & \text{if either } a = 1 \text{ and } b \equiv 1 \pmod{4} \\
  & \text{or } b = a \geq 2.
\end{cases}
\]

In this chapter, we verify Conjecture 3.1.1 for several other well-known classes of cubic graphs and graphs with small maximum degrees, namely all permutation graphs of the 5-cycle $C_5$, prisms, grids as well as some classes of trees.

3.2 The Permutation Graphs of the 5-Cycle

The Cartesian product $G \square H$ of two graphs $G$ and $H$ has vertex set $V(G \square H) = V(G) \times V(H)$ and two distinct vertices $(u, v)$ and $(x, y)$ of $G \square H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. The Cartesian product $G \square K_2$ of a graph $G$ and $K_2$ is a special case of a more general class of graphs. Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\alpha$ be a permutation of the set $S = \{1, 2, \ldots, n\}$. The permutation graph $P_\alpha(G)$ of a graph $G$ is the graph of order $2n$ obtained from two copies of $G$, where the second copy of $G$ is denoted by $G'$ and the vertex $v_i$ in $G$ is denoted by $u_i$ in $G'$ and $v_i$ is joined to the vertex $u_{\alpha(i)}$ in $G'$. The edges $v_iu_{\alpha(i)}$ are called the permutation edges of $P_\alpha(G)$. This concept was first introduced by Chartrand and Harary [7]. Therefore, if $\alpha$ is the identity map on $S$, then $P_\alpha(G) = G \square K_2$. Figure 3.1 shows the four permutation graphs of the 5-cycle $C_5$ where the graph in Figure 3.1(a) is $C_5 \square K_2$. 
and the graph in Figure 3.1(d) is the Petersen graph \( P \). Each of these permutation graphs is a 3-regular (or cubic) graph. Indeed, these four graphs appear on the cover of the book \textit{Graph Theory} by Harary [28]. In this section, we determine the twin chromatic indexes of these four permutation graphs. The following observation will be useful.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figures.png}
\caption{The permutation graphs of \( C_5 \)}
\end{figure}

\textbf{Observation 3.2.1} If a connected graph \( G \) contains two adjacent vertices of degree \( \Delta(G) \), then \( \chi'_t(G) \geq 1 + \Delta(G) \). In particular, if \( G \) is a connected \( r \)-regular graph for some integer \( r \geq 2 \), then \( \chi'_t(G) \geq 1 + r \).

As a consequence of Observation 3.2.1, each permutation graph of \( C_5 \) has twin chromatic index at least 4. Neither of the twin chromatic indexes of the two permutation graphs of \( C_5 \) in Figures 3.1(a) and (b) is 4, as we show next.

\textbf{Proposition 3.2.2} For the graph \( G = C_5 \square K_2, \chi'_t(G) = 5 \).

\textbf{Proof.} Suppose that the two copies of \( C_5 \) in \( G \) are \((u_1, u_2, u_3, u_4, u_5, u_1)\) and \((v_1, v_2, v_3, v_4, v_5, v_1)\) and \( u_i \) is adjacent to \( v_i \) for \( 1 \leq i \leq 5 \). By Observation 3.2.1, \( \chi'_t(G) \geq 4 \). We show that \( \chi'_t(G) \neq 4 \). Assume, to the contrary, that \( c : E(G) \to \mathbb{Z}_4 \) is a twin edge 4-coloring of \( G \). Then \( c \) must assign the same color to two edges in the 5-cycle \((u_1, u_2, u_3, u_4, u_5, u_1)\). We may assume that \( c(u_1u_2) = c(u_3u_4) = a \). Suppose that
c(u_2u_3) = b, c(u_2v_2) = c and c(u_3v_3) = d. Since \( c'(u_2) \neq c'(u_3) \), it follows that \( c \neq d \). Thus \{a, b, c, d\} = \( \mathbb{Z}_4 \) and \( c(v_2v_3) \in \{a, b\} \). In either case, \( c(v_1v_2) = d \) and \( c(v_3v_4) = c \). However then, \( c'(v_2) = c'(v_3) \), a contradiction. Therefore, \( \chi'_i(G) \geq 5 \).

Next, we show that \( G \) has a twin edge 5-coloring. For each \( i \) with \( 1 \leq i \leq 5 \), let \( e_i = u_i u_{i+1} \) (where \( u_5u_6 = u_5u_4 \)), \( f_i = u_iv_i \) and \( e'_i = v_i v_{i+1} \) (where \( v_5v_6 = v_5v_1 \)). Define the coloring \( c : E(G) \to \mathbb{Z}_5 \) by

\[
(c(e_1), c(e_2), \ldots, c(e_5)) = (0, 1, 2, 3, 4) \\
(c(e'_1), c(e'_2), \ldots, c(e'_5)) = (1, 2, 3, 4, 0) \\
(c(f_1), c(f_2), \ldots, c(f_5)) = (2, 3, 4, 0, 1).
\]

Then \( (c'(u_1), \ldots, c'(u_5)) = (1, 4, 2, 0, 3) \) and \( (c'(v_1), \ldots, c'(v_5)) = (3, 1, 4, 2, 0) \). Since \( c' \) is a proper vertex coloring, \( c \) is a twin edge 5-coloring. Therefore, \( \chi'_i(G) = 5 \).

**Proposition 3.2.3** For the graph \( G \) in Figure 3.1(b), \( \chi'_i(G) = 5 \).

**Proof.** By Observation 3.2.1, \( \chi'_i(G) \geq 4 \). We show that \( \chi'_i(G) \neq 4 \). Assume, to the contrary, that \( G \) has a twin edge 4-coloring \( c : E(G) \to \mathbb{Z}_4 \). Thus, the induced vertex coloring \( c' \) is a proper 4-coloring using colors in \( \mathbb{Z}_4 \). For each \( i \) with \( 0 \leq i \leq 3 \), let \( C'_i \) be the color class of \( c' \) consisting of vertices colored \( i \) by \( c' \). Suppose that \( |C'_i| = t_i \geq 0 \) for \( 0 \leq i \leq 3 \). We claim that each \( t_i \) \( (0 \leq i \leq 3) \) is even. First, we make an observation.

Let \( v \in V(G) \). Since \( G \) is 3-regular and \( c' \) is a proper coloring, if \( c'(v) = i \in \mathbb{Z}_4 \), then there is exactly one element \( j_i \in \mathbb{Z}_4 \) such that no edge incident with \( v \) is colored \( j_i \) by \( c \). (In fact, \( j_0 = 2, j_1 = 1, j_2 = 0 \) and \( j_3 = 3 \).) Furthermore, if \( u \in V(G) \) with \( c'(u) \neq i \), then \( u \) is incident with exactly one edge colored \( j_i \) by \( c \).

Assume, to the contrary, that \( t_i \) is odd for some \( i \) with \( 0 \leq i \leq 3 \). Then we may assume that \( t_0 \) is odd (since the proof for the case when \( t_i \) is odd for \( i \neq 0 \) uses the same argument). Since the order of \( G \) is 10, there are \( 10 - t_0 \) vertices that are not colored 0 by \( c' \). It then follows by the observation above that each of these \( 10 - t_0 \) vertices is incident with exactly one edge colored 2 by \( c \). Thus, there is an odd number (namely \( 10 - t_0 \)) of vertices incident with edges colored 2. Since \( c \) is a proper edge coloring, the set of edges colored 2 is independent and so the number of vertices incident with edges colored 2 must be even, which is a contradiction. Therefore, as claimed, each \( t_i \) \( (0 \leq i \leq 3) \) is even and so \( t_i \geq 4 \) for some \( i \in \mathbb{Z}_4 \). We may assume, without loss of generality, that \( t_0 \geq 4 \) (since the proof for the case when \( t_i \geq 4 \) for \( i \neq 0 \) uses the same argument).
CHAPTER 3. TWIN EDGE COLORING CONJECTURE

We now label the vertices of $G$ as shown Figure 3.2. Let $C_u = (u_1, u_2, u_3, u_4, u_5, u_1)$ be the outer 5-cycle of $G$ and $C_v = (v_1, v_2, v_3, v_5, v_4, v_1)$ the inner 5-cycle. Since $c'$ is a proper coloring of $G$, at most two vertices in each of the two 5-cycles $C_u$ and $C_v$ can be colored the same. Hence $t_0 = 4$; that is, exactly four vertices of $G$ are colored 0 by $c'$. Furthermore, exactly two vertices in each of $C_u$ and $C_v$ are colored 0 by $c'$. By the symmetry of the graph $G$, we may assume that $\{u_1, u_3\} \subseteq C_0'$, $\{u_2, u_4\} \subseteq C_0'$ or $\{u_3, u_5\} \subseteq C_0'$. We consider these three cases.

![Figure 3.2: The labeling of the vertices of the graph in Figure 3.1(b)](image)

**Case 1.** $u_1$ and $u_3$ are colored 0 by $c'$. Thus no vertex in $\{u_2, u_4, u_5, v_1, v_3\}$ can be colored 0 and so $v_2, v_4, v_5$ are the only possible vertices in the inner cycle $C_v$ that can be colored 0 by $c'$. Since $c'(u_1) = c'(u_3) = 0$, it then follows by the observation that no edge incident with $u_1$ or $u_3$ can be colored 2 by $c$. Hence $c(u_2u_1) \neq 2$ and $c(u_2u_3) \neq 2$. On the other hand, since $c'(u_2) \neq 0$, there is exactly one edge incident with $u_2$ that is colored 2 by $c$. Hence $c(u_2v_2) = 2$. This implies that $c'(v_2) \neq 0$. However then, $c'(v_4) = c'(v_5) = 0$, which is impossible since $v_4v_5 \in E(G)$.

**Case 2.** $u_2$ and $u_4$ are colored 0 by $c'$. Thus no vertex in $\{u_1, u_3, u_5, v_2, v_4\}$ can be colored 0 and so $v_1, v_3, v_5$ are the only possible vertices in the inner cycle $C_v$ that can be colored 0 by $c'$. Since $v_3v_5 \in E(G)$, at most one of $v_3$ and $v_5$ can be colored 0 and so $c'(v_1) = 0$. First, suppose that $c'(v_5) = 0$. Then $v_4$ is adjacent to three vertices that are colored 0 by $c'$ (namely $u_4, v_1, v_5$) and none of the edges $v_4u_4, v_4v_1, v_5v_5$ can be colored 2. On the other hand, since $c'(v_4) \neq 0$, exactly one of the edges $v_4u_4, v_4v_1, v_5v_5$ is colored 2, which is impossible. Next, suppose that $c'(v_3) = 0$. Thus no edge incident with $v_3$ is colored 2. In particular, $c(v_2v_3) \neq 2$. Since $c'(v_2) \neq 0$ and $c'(u_2) = c'(v_1) = 0$, it follows that $c(u_2v_2) \neq 2$ and $c(v_1v_2) \neq 2$ and so $c(v_2v_3) = 2$, which is impossible.

**Case 3.** $u_3$ and $u_5$ are colored 0 by $c'$. Hence $v_1, v_2, v_4$ are the only possible vertices in the inner cycle $C_v$ that can be colored 0 by $c'$. Since $v_1$ is adjacent to $v_2$ and $v_4$, it...
follows that \( c'(v_2) = c'(v_4) = 0 \). Since \( c'(u_4) \neq 0 \) and \( u_4 \) is adjacent to three vertices colored 0 by \( c \) (namely \( u_3, u_5, v_4 \)), this is impossible as we saw in Case 2.

From this, we now know that \( \chi'_t(G) \geq 5 \). Next, we show that \( G \) has a twin edge 5-coloring. For each \( i \) with \( 1 \leq i \leq 5 \), let \( e_i = u_iu_{i+1} \) (where \( u_5u_6 = u_5u_1 \)) and \( f_i = u_iv_i \). Furthermore, let \( e'_1 = v_1v_2, e'_2 = v_2v_3, e'_3 = v_3v_5, e'_4 = v_5v_4, e'_5 = v_4v_1 \). Define the coloring \( c : E(G) \to \mathbb{Z}_5 \) by

\[
(c(e_1), c(e_2), \ldots, c(e_5)) = (1, 2, 3, 4, 0) \\
(c(e'_1), c(e'_2), \ldots, c(e'_5)) = (2, 3, 4, 1, 3) \\
(c(f_1), c(f_2), \ldots, c(f_5)) = (4, 0, 1, 2, 2).
\]

Then \((c'(u_1), \ldots, c'(u_5)) = (0, 3, 1, 4, 1)\) and \((c'(v_1), \ldots, c'(v_5)) = (4, 0, 3, 1, 2)\). Since \( c' \) is a proper vertex coloring, \( c \) is a twin edge 5-coloring. Therefore, \( \chi'_t(G) = 5 \).

We saw by Observation 3.2.1 that each permutation graph of \( C_5 \) has twin chromatic index at least 4. Figure 3.3 shows a twin edge 4-coloring \( c \) for the graph in Figure 3.1(c) and the Petersen graph \( P \) in Figure 3.1(d), where the color \( c(e) \) is placed next to the edge \( e \) and the color \( c'(v) \) is placed inside each vertex \( v \). Thus the twin chromatic index of each of these two graphs is 4, as we state below.

![Diagram](image-url)

Figure 3.3: Twin edge 4-colorings of two permutation graphs of \( C_5 \)

**Proposition 3.2.4** The twin chromatic indexes of the graphs in Figures 3.1(c) and (d) are both 4.
3.3 Grids and Prisms

For integers \( n \) and \( q \) with \( n, q \geq 2 \), the graph \( P_n \boxtimes P_q \) is called a grid while the graph \( C_n \boxtimes K_2 \) where \( n \geq 3 \) is called a prism. Thus, if \( n \geq 3 \), the maximum degree of a prism and grid is 3 and 4, respectively. In this section, we determine twin chromatic indexes of these graphs, beginning with \( C_n \boxtimes K_2 \). By Proposition 3.2.2, \( \chi'_t(C_5 \boxtimes K_2) = 5 \). In fact, \( C_5 \boxtimes K_2 \) is the unique exception among prisms, as we show next.

**Theorem 3.3.1** If \( n \geq 3 \) is an integer with \( n \neq 5 \), then \( \chi'_t(C_n \boxtimes K_2) = 4 \).

**Proof.** Let \( G = C_n \boxtimes K_2 \) where the two copies of \( C_n \) in \( G \) are \( (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \) and \( (v_1, v_2, \ldots, v_n, v_{n+1} = v_1) \) and \( u_i \) is adjacent to \( v_i \) for \( 1 \leq i \leq n \). By Observation 3.2.1, \( \chi'_t(G) \geq 4 \). Thus, it remains to show that \( G \) has a twin edge 4-coloring \( c \).

For each \( i \) with \( 1 \leq i \leq n \), let \( e_i = u_iu_{i+1} \), \( f_i = u_iv_i \) and \( e'_i = v_iv_{i+1} \). Let \( E_u = \{e_i : 1 \leq i \leq n\} \), \( E_{uv} = \{f_i : 1 \leq i \leq n\} \) and \( E_v = \{e'_i : 1 \leq i \leq n\} \). We now define the coloring \( c : V(G) \to \mathbb{Z}_4 \), depending on whether \( n = 0 \) (mod 3), \( n = 1 \) (mod 3) or \( n = 2 \) (mod 3). In each of these cases, we define \( c(e) \) for each \( e \in E_u \cup E_v \) such that for each \( f \in E_{uv} \) the coloring \( c \) assigns exactly three colors in \( \mathbb{Z}_4 \) to the four edges incident with \( f \) (and so \( c(f) \) is uniquely determined). For a twin edge 4-coloring \( c \) of \( G \), define

\[
\begin{align*}
se &= (c(e_1), c(e_2), \ldots, c(e_n)) \\
se' &= (c(e'_1), c(e'_2), \ldots, c(e'_n)) \\
sf &= (c(f_1), c(f_2), \ldots, c(f_n)) \\
su &= (c'(u_1), c'(u_2), \ldots, c'(u_n)) \\
sv &= (c'(v_1), c'(v_2), \ldots, c'(v_n))
\end{align*}
\]

- For \( n \equiv 0 \) (mod 3) and \( 1 \leq i, j \leq n \), let

\[
c(e) = \begin{cases} 
0 & \text{if } e = e_i \text{ and } i \equiv 0 \text{ (mod 3)} \text{ or } e = e'_j \text{ and } j \equiv 1 \text{ (mod 3)} \\
1 & \text{if } e = e_i \text{ and } i \equiv 1 \text{ (mod 3)} \text{ or } e = e'_j \text{ and } j \equiv 2 \text{ (mod 3)} \\
2 & \text{if } e = e_i \text{ and } i \equiv 2 \text{ (mod 3)} \text{ or } e = e'_j \text{ and } j \equiv 0 \text{ (mod 3)}.
\end{cases}
\]

For example, if \( n = 6 \), then \( s_e = (1, 2, 0, 1, 2, 0) \) and \( s_{e'} = (0, 1, 2, 0, 1, 2) \); Then \( c(f_i) = 3 \) for \( 1 \leq i \leq n \). Let \( s_1 = (0, 2, 1) \) and \( s_2 = (1, 0, 2) \). Thus \( su = (s_1, s_1, \ldots, s_1) \) and \( sv = (s_2, s_2, \ldots, s_2) \), that is, \( su = (0, 2, 1, 0, 2, 1, \ldots, 0, 2, 1) \) and \( sv = (1, 0, 2, 1, 0, 2, \ldots, 1, 0, 2) \).
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For \( n \equiv 1 \pmod{3} \), \( 1 \leq i \leq n - 1 \) and \( 2 \leq j \leq n \), let

\[
c(e) = \begin{cases} 
0 & \text{if } e = e_n \text{ or } e = e'_1 \\
1 & \text{if } e = e_i \text{ and } i \equiv 1 \pmod{3} \text{ or } e = e'_j \text{ and } j \equiv 2 \pmod{3} \\
2 & \text{if } e = e_i \text{ and } i \equiv 2 \pmod{3} \text{ or } e = e'_j \text{ and } j \equiv 0 \pmod{3} \\
3 & \text{if } e = e_i \text{ and } i \equiv 0 \pmod{3} \text{ or } e = e'_j \text{ and } j \equiv 1 \pmod{3}.
\end{cases}
\]

Then \((c(f_n), c(f_1), c(f_2)) = (1, 2, 3)\) and \(c(f_i) = 0\) for \( 3 \leq i \leq n - 1 \). For example, if \( n = 4 \), then \( s_e = (1, 2, 3, 0) \) and \( s_{e'} = (0, 1, 2, 3) \); if \( n = 7 \), then \( s_e = (1, 2, 3, 1, 2, 3, 0) \) and \( s_{e'} = (0, 1, 2, 3, 1, 2, 3) \). Let \( s_1 = (1, 0, 3) \) and \( s_2 = (3, 1, 0) \). Suppose that \( n = 4 + 3k \) for some integer \( k \geq 0 \). Then \( s_u = (3, 2, s_1, s_1, \ldots, s_1, 1, 0) \) and \( s_v = (1, 0, s_2, s_2, \ldots, s_2, 3, 2) \) where there are exactly \( k \) subsequences \( s_1 \) or \( s_2 \) in \( s_u \) or \( s_v \), respectively.

For \( n \equiv 2 \pmod{3} \), let \( n = 3k + 8 \) for some integer \( k \geq 0 \) and \( s = (1, 2, 3) \). Now let

\[
s_e = (s, s, \ldots, s, 1, 2, 3, 0, 1, 2, 3, 0) \\
s_{e'} = (0, 1, 2, 3, s, s, \ldots, s, 0, 1, 2, 3),
\]

where there are exactly \( k \) subsequences \( s \) in \( s_e \) and \( s_{e'} \), respectively. Then

\[
s_f = (2, 3, 0, 0, \ldots, 0, 0, 1, 2, 3, 0, 1)
\]

is uniquely determined by \( s_e \) and \( s_{e'} \). In particular, if \( n = 8 \), then

\[
s_e = (1, 2, 3, 0, 1, 2, 3, 0) \\
s_{e'} = (0, 1, 2, 3, 0, 1, 2, 3) \\
s_f = (2, 3, 0, 1, 2, 3, 0, 1).
\]

Let \( s_1 = (3, 1, 0) \) and \( s_2 = (1, 0, 3) \). Then \( s_u = (3, 2, 1, 0, s_1, s_1, \ldots, s_1, 3, 2, 1, 0) \) and \( s_v = (s_2, s_2, \ldots, s_2, 2, 1, 0, 3, 2) \). More precisely, if we write \( n = 8 + 3k \) for some integer \( k \geq 0 \), then there are exactly \( k \) subsequences \( s_1 \) in \( s_u \); while if we write \( n = 5 + 3\ell \) for some \( \ell \geq 1 \), then there are exactly \( \ell \) subsequences \( s_2 \) preceding \( 2, 1, 0, 3, 2 \) in \( s_v \).

In each case, \( G \) has a twin edge 4-coloring and so \( \chi'_t(G) = 4 \).

Next we determine the twin chromatic indexes of grids \( P_n \square P_q \) for all integers \( n, q \geq 2 \). We begin with the case when \( q = 2 \).
Theorem 3.3.2 If \( n \geq 2 \) is an integer, then \( \chi'_1(P_n \boxtimes K_2) = 4 \).

Proof. Since \( P_2 \boxtimes K_2 = C_4 \) and \( \chi'_1(C_4) = 4 \), we may assume that \( n \geq 3 \). Let \( G = P_n \boxtimes K_2 \) where the two copies of \( P_n \) in \( G \) are \((u_1, u_2, \ldots, u_n)\) and \((v_1, v_2, \ldots, v_n)\) and \( u_i \) is adjacent to \( v_i \) for \( 1 \leq i \leq n \). By Observation 3.2.1, \( \chi'_1(G) \geq 4 \). Thus, it remains to show that \( G \) has a twin edge 4-coloring \( c \). For each \( i \) with \( 1 \leq i \leq n - 1 \), let 
\[
e_i = u_i u_{i+1} \quad \text{and} \quad e'_i = v_i v_{i+1}
\]
and for each \( j \) with \( 1 \leq j \leq n \), let \( f_j = u_j v_j \). For a twin edge 4-coloring \( c \) of \( G \), define
\[
s_e = (c(e_1), c(e_2), \ldots, c(e_{n-1}))
\]
\[
s_{e'} = (c(e_1'), c(e_2'), \ldots, c(e_{n-1}'))
\]
\[
s_f = (c(f_1), c(f_2), \ldots, c(f_n))
\]
\[
s_u = (c'(u_1), c'(u_2), \ldots, c'(u_n))
\]
\[
s_v = (c'(v_1), c'(v_2), \ldots, c'(v_n)).
\]

- For \( n \equiv 0 \pmod{3} \) and \( 1 \leq i, j \leq n - 1 \), let
\[
c(e) = \begin{cases} 
0 & \text{if } e = f_i \text{ and } 1 \leq i \leq n \\
1 & \text{if } e = e_i \text{ and } i \equiv 1 \pmod{3} \text{ or } e = e'_i \text{ and } j \equiv 0 \pmod{3} \\
2 & \text{if } e = e_i \text{ and } i \equiv 2 \pmod{3} \text{ or } e = e'_i \text{ and } j \equiv 1 \pmod{3} \\
3 & \text{if } e = e_i \text{ and } i \equiv 0 \pmod{3} \text{ or } e = e'_i \text{ and } j \equiv 2 \pmod{3}.
\end{cases}
\]

For example, if \( n = 3 \), then \( s_e = (1,2) \) and \( s_{e'} = (2,3) \); if \( n = 6 \), then \( s_e = (1,2,3,1,2) \) and \( s_{e'} = (2,3,1,2,3) \); while if \( n = 9 \), then \( s_e = (1,2,3,1,2,3,1,2,3) \) and \( s_{e'} = (2,3,1,2,3,1,2,3) \) (where \( s_f = (0,0,\ldots,0) \) in each case). Hence, \( s_u = (1,3,2) \) and \( s_v = (2,1,3) \) for \( n = 3 \). For \( n \geq 6 \), let \( s_1 = (3,1,0) \) and \( s_2 = (1,0,3) \). If \( n = 6 \), then \( s_u = (1,s_1,3,2) \) and \( s_v = (2,s_2,1,3) \); while if \( n \geq 9 \), then \( s_u = (1,s_1,s_1,\ldots,s_1,3,2) \) and \( s_v = (2,s_2,s_2,\ldots,s_2,1,3) \).

- For \( n \equiv 1 \pmod{3} \) and \( 1 \leq i, j \leq n - 1 \), let
\[
c(e) = \begin{cases} 
0 & \text{if } e = e_i \text{ and } i \equiv 0 \pmod{3} \text{ or } e = e'_j \text{ and } j \equiv 1 \pmod{3} \\
1 & \text{if } e = e_i \text{ and } i \equiv 1 \pmod{3} \text{ or } e = e'_j \text{ and } j \equiv 2 \pmod{3} \\
2 & \text{if } e = e_i \text{ and } i \equiv 2 \pmod{3} \text{ or } e = e'_j \text{ and } j \equiv 0 \pmod{3} \\
3 & \text{if } e = f_i \text{ and } 1 \leq i \leq n.
\end{cases}
\]

For example, if \( n = 4 \), then \( s_e = (1,2,0) \) and \( s_{e'} = (0,1,2) \); while if \( n = 7 \), then \( s_e = (1,2,0,1,2,0) \) and \( s_{e'} = (0,1,2,0,1,2) \) (where \( s_f = (3,3,\ldots,3) \) in each case). Let \( s = (0,2,1) \). Then \( s_u = (s,s,\ldots,s,3) \) and \( s_v = (3,s,s,\ldots,s) \).
• For \( n \equiv 2 \pmod{3} \) and \( 1 \leq i, j \leq n - 1 \), let

\[
\begin{align*}
c(e) &= \begin{cases} 
0 & \text{if } e = e_i \text{ and } i \equiv 0 \pmod{3} \text{ or } e = e_j' \text{ and } j \equiv 1 \pmod{3} \\
1 & \text{if } e = e_i \text{ and } i \equiv 1 \pmod{3} \text{ or } e = e_j' \text{ and } j \equiv 2 \pmod{3} \\
2 & \text{if } e = e_i \text{ and } i \equiv 2 \pmod{3} \text{ or } e = e_j' \text{ and } j \equiv 0 \pmod{3} \\
or e = f_n \\
3 & \text{if } e = f_i \text{ and } 1 \leq i \leq n - 1.
\end{cases}
\end{align*}
\]

For example, if \( n = 5 \), then \( s_e = (1, 2, 0, 1) \) and \( s_{e'} = (0, 1, 2, 0) \); while if \( n = 8 \), then \( s_e = (1, 2, 0, 1, 2, 0, 1) \) and \( s_{e'} = (0, 1, 2, 0, 1, 2, 0) \) (where \( s_f = (3, 3, \ldots, 3, 2) \) in each case). Let \( s = (0, 2, 1) \). Then \( s_u = (s, s, \ldots, s, 0, 3) \) and \( s_v = (3, s, s, \ldots, s, 2) \).

Therefore, \( G \) has a twin edge 4-coloring and so \( \chi'_t(G) = 4 \).

In order to determine the twin chromatic indexes of all grids \( P_n \boxplus P_q \) for all integers \( n \) and \( q \) with \( n, q \geq 3 \), we first present an additional definition and three lemmas. A twin edge coloring \( c \) of a graph \( G \) is a nowhere-zero coloring if \( c(e) \neq 0 \) for each edge \( e \) of \( G \).

**Lemma 3.3.3** For each integer \( n \geq 3 \), the path \( P_n \) has a nowhere-zero twin edge 5-coloring.

**Proof.** Let \( P_n = (x_1, x_2, \ldots, x_n) \) and let \( e_i = x_ix_{i+1} \) for \( 1 \leq i \leq n - 1 \). Define a proper edge coloring \( c : E(P_n) \to \mathbb{Z}_5 - \{0\} \) by

\[
c(e_i) = \begin{cases} 
4 & \text{if } i \equiv 1 \pmod{3} \\
3 & \text{if } i \equiv 2 \pmod{3} \\
2 & \text{if } i \equiv 0 \pmod{3}.
\end{cases}
\]

If \( s_n = (c(e_1), c(e_2), \ldots, c(e_{n-1})) \), then \( s_3 = (4, 3) \), \( s_4 = (4, 3, 2) \), \( s_5 = (4, 3, 2, 4) \), \( s_6 = (4, 3, 2, 4, 3) \) and \( s_7 = (4, 3, 2, 4, 3, 2) \). For the induced vertex coloring \( c' \), we have \( c'(x_1) = 4 \), \( c'(x_i) = r \) if \( i \equiv r \pmod{3} \) and \( 2 \leq i \leq n - 1 \) and

\[
c'(x_n) = \begin{cases} 
3 & \text{if } n \equiv 0 \pmod{3} \\
2 & \text{if } n \equiv 1 \pmod{3} \\
4 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

If \( s'_n = (c(x_1), c(x_2), \ldots, c(x_n)) \), then \( s'_3 = (4, 2, 3) \), \( s_4 = (4, 2, 0, 2) \), \( s_5 = (4, 2, 0, 1, 4) \), \( s_6 = (4, 2, 0, 1, 2, 3) \) and \( s_7 = (4, 2, 0, 1, 2, 0, 2) \). Since \( c' \) is a proper vertex coloring, \( c \) is a nowhere-zero twin edge 5-coloring. \( \blacksquare \)
Lemma 3.3.4  For each integer \( n \geq 3 \), the graph \( P_n \square K_2 \) has a nowhere-zero twin edge 5-coloring.

**Proof.** Let \( G = P_n \square K_2 \) where the two copies of \( P_n \) in \( G \) are \((u_1, u_2, \ldots, u_n)\) and \((v_1, v_2, \ldots, v_n)\) and \( u_i \) is adjacent to \( v_i \) for \( 1 \leq i \leq n \). For each \( i \) with \( 1 \leq i \leq n - 1 \), let \( e_i = u_i u_{i+1} \) and \( e'_i = v_i v_{i+1} \) and for each \( j \) with \( 1 \leq j \leq n \), let \( f_j = u_j v_j \). For a twin edge coloring \( c \) of \( G \), define the five sequences \( s_e, s_{e'}, s_f, s_u \) and \( s_v \) as defined in (3.1) in the proof of Theorem 3.3.2. We now define a proper edge 5-coloring \( c : E(G) \to \mathbb{Z}_5 - \{0\} \) as follows: First, let \( c(f_i) = 4 \) for \( 1 \leq i \leq n \) and so \( s_f = (4,4,\ldots,4) \). To define the colors of the remaining edges of \( G \), we consider three cases, according to whether \( n \) is congruent to 0, 1 or 2 modulo 3.

- For \( n \equiv 0 \pmod{3} \), define the proper edge 5-coloring \( c \) such that
  
  \[
  s_e = (2,3,1,2,3,1,\ldots,2,3,1,2,3) \\
  s_{e'} = (1,2,3,1,2,3,\ldots,1,2,3,1,2).
  \]

  That is, if \( n = 3k \) for some positive integer \( k \), then there are \( k - 1 \) subsequences \((2,3,1)\) in \( s_e \) and \( k - 1 \) subsequences \((1,2,3)\) in \( s_{e'} \). For example, if \( n = 3 \), then \( s_e = (2,3) \) and \( s_{e'} = (1,2) \); if \( n = 6 \), then \( s_e = (2,3,1,2,3) \) and \( s_{e'} = (1,2,3,1,2) \); while if \( n = 9 \), then \( s_e = (2,3,1,2,3,1,2,3) \) and \( s_{e'} = (1,2,3,1,2,3,1,2) \). Then the induced vertex coloring \( c' \) satisfies

  \[
  s_u = (1,4,3,2,4,3,2,\ldots,4,3,2,4,2) \\
  s_v = (0,2,4,3,2,4,3,\ldots,2,4,3,2,1).
  \]

  That is, there are \( k - 1 \) subsequences \( s_1 = (4,3,2) \) in \( s_u \) and \( k - 1 \) subsequences \( s_2 = (2,4,3) \) in \( s_v \). For example, if \( n = 3 \), then \( s_u = (1,4,2) \) and \( s_v = (0,2,1) \); while if \( n \geq 6 \), then \( s_u = (1,s_1,s_1,\ldots,s_1,4,2) \) and \( s_v = (0,s_2,s_2,\ldots,s_2,2,1) \).

- For \( n \equiv 1 \pmod{3} \), define the proper edge 5-coloring \( c \) such that

  \[
  s_e = (3,1,2,3,1,2,\ldots,3,1,2) \\
  s_{e'} = (1,2,3,1,2,3,\ldots,1,2,3).
  \]

  That is, if \( n = 3k + 1 \) for some positive integer \( k \), then there are \( k \) subsequences \((3,1,2)\) in \( s_e \) and \( k \) subsequences \((1,2,3)\) in \( s_{e'} \). For example, if \( n = 4 \), then
Lemma 3.3.5 For an integer $n \geq 3$, let $c_1$ be the nowhere-zero twin edge 5-coloring of $P_n = (x_1, x_2, \ldots, x_n)$ as defined in Lemma 3.3.3 and let $c_2$ be the nowhere-zero twin edge 5-coloring of $P_n \sqcap P_2$ as defined in Lemma 3.3.4 where the two copies of $P_n$ in $P_n \sqcap P_2$ are $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ where $u_i$ is adjacent to $v_i$ for $1 \leq i \leq n$. Then $c_1'(x_i) \neq c_2'(u_i)$ for $1 \leq i \leq n$.

We are now prepared to determine the twin chromatic index of the grids $P_n \sqcap P_q$ for all integers $n$ and $q$ with $n, q \geq 3$. For two disjoint sets $U$ and $W$ of vertices of a graph $G$, let $[U, W]$ denote the set of edges of $G$ that join a vertex in $U$ and and a vertex in $W$. 
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**Theorem 3.3.6** If \( n \) and \( q \) are integers with \( n, q \geq 3 \), then \( \chi_t'(P_n \square P_q) = 5 \).

**Proof.** Let \( G = P_n \square P_q \) where \( n, q \geq 3 \). By Observation 3.2.1, \( \chi_t'(G) \geq 5 \). Thus, it remains to show that \( G \) has a twin edge 5-coloring. Suppose that \( G \) consists of \( q \) copies of the path \( P_n \) of order \( n \), which we denote by \( P_{n,i} = (v_{1,i}, v_{2,i}, \ldots, v_{n,i}) \) for \( 1 \leq i \leq q \) such that \( v_{t,i} \) is adjacent to \( v_{t,j} \) (\( 1 \leq t \leq n \)) when \( |i - j| = 1 \). Let \( V_i = V(P_{n,i}) \) for \( 1 \leq i \leq q \).

For an integer \( q \geq 3 \), we consider two cases, according to whether \( q \) is even or \( q \) is odd.

**Case 1.** \( q \) is even. Then \( q = 2k \) for some integer \( k \geq 2 \). For each integer \( t \) with \( 1 \leq t \leq k \), let \( B_t = G[V_{2t-1} \cup V_{2t}] \cong P_n \square K_2 \) be the subgraph of \( G = P_n \square P_q \) induced by \( V_{2t-1} \cup V_{2t} \) and let \( c_t \) be the nowhere-zero twin edge 5-coloring of \( B_t \) as defined in Lemma 3.3.4 (where \( V_{2t-1} \) corresponds to the set \( \{u_1, u_2, \ldots, u_n\} \) of vertices of \( P_n \square K_2 \) in Lemma 3.3.4). Define a proper edge coloring \( c : E(P_n \square P_q) \to \mathbb{Z}_5 \) by \( c(e) = c_t(e) \) if \( e \in E(B_t) \) for \( 1 \leq i \leq k \) and \( c(e) = 0 \) if \( e \in [V(B_t), V(B_{t+1})] \) for \( 1 \leq t \leq k - 1 \). By applying Lemma 3.3.5 repeatedly, we see that \( c \) is a twin edge 5-coloring of \( G \).

**Case 2.** \( q \) is odd. Then \( q = 2k + 1 \) for some positive integer \( k \). We consider two subcases.

**Subcase 2.1.** \( q = 3 \). We show that \( \chi_t'(P_n \square P_3) = 5 \) for each integer \( n \geq 3 \). Let \( B_1 = P_{n,1} \) and \( B_2 = G[V_2 \cup V_3] = P_n \square K_2 \) be the subgraph of \( G = P_n \square P_3 \) induced by \( V_2 \cup V_3 \). Furthermore, let \( c_1 \) be the nowhere-zero twin edge 5-coloring of \( B_1 \) defined in Lemma 3.3.3 and let \( c_2 \) be the nowhere-zero twin edge 5-coloring of \( B_2 \) defined in Lemma 3.3.4 (where \( V_2 \) corresponds to the set \( \{u_1, u_2, \ldots, u_n\} \) of vertices of \( P_n \square K_2 \) in Lemma 3.3.4). Define a proper edge coloring \( c : E(P_n \square P_3) \to \mathbb{Z}_5 \) by \( c(e) = c_t(e) \) if \( e \in E(B_i) \) for \( i = 1, 2 \) and \( c(e) = 0 \) if \( e \in [V(B_1), V(B_2)] \). Since \( c'(v) = c'_t(v) \) if \( v \in V(B_i) \) for \( i = 1, 2 \), it then follows by Lemma 3.3.5 that \( c \) is a twin edge 5-coloring of \( P_n \square P_3 \) and so \( \chi_t'(P_n \square P_3) = 5 \).

**Subcase 2.2.** \( q \geq 5 \). Let \( B_1 = G[V_1 \cup V_2 \cup V_3] = P_n \square P_3 \) and let \( c_1 \) be the twin edge 5-coloring of \( P_n \square P_3 \) defined in Subcase 2.1 when \( q = 3 \). For each integer \( t \) with \( 2 \leq t \leq k \), let \( B_t = G[V_{2t} \cup V_{2t+1}] = P_n \square K_2 \) and let \( c_t \) be the nowhere-zero twin edge 5-coloring of \( B_t \) defined in Lemma 3.3.4 (where \( V_{2t} \) corresponds to the set \( \{u_1, u_2, \ldots, u_n\} \) of vertices of \( P_n \square K_2 \) in Lemma 3.3.4). Define a proper edge coloring \( c : E(P_n \square P_q) \to \mathbb{Z}_5 \) by \( c(e) = c_t(e) \) if \( e \in E(B_t) \) for \( 1 \leq i \leq k \) and \( c(e) = 0 \) if \( e \in [V(B_t), V(B_{t+1})] \) for \( 1 \leq t \leq k - 1 \). By applying Lemma 3.3.5 repeatedly, we see that \( c \) is a twin edge 5-coloring of \( G \).
3.4 Trees with Small Maximum Degree

The twin chromatic indexes of all stars were determined in Chapter 2, which we restate below.

Proposition 3.4.1 If $K_{1,b}$ is a star of size $b \geq 2$, then

$$
\chi_t'(K_{1,b}) = \begin{cases} 
  b + 1 & \text{if } b \not\equiv 1 \pmod{4} \\
  b + 2 & \text{if } b \equiv 1 \pmod{4}
\end{cases}
$$

By Theorem 3.1.2 and Proposition 3.4.1, if $T$ is a path or a star, then $\chi_t'(T) \leq 2 + \Delta(T)$. Furthermore, if $T = K_{1,b}$, where $b \equiv 1 \pmod{4}$, then $\chi_t'(T) = 2 + \Delta(T)$ and so the bound $2 + \Delta(T)$ is sharp for stars. Indeed, if $T$ is a tree having maximum degree at most 6, then $\chi_t'(T) \leq 2 + \Delta(T)$. To describe the technique used to show this, we first present a useful observation.

Observation 3.4.2 Let $T$ be a tree that is not a star, let $v$ be a vertex of $T$ such that exactly one neighbor of $v$ is not a leaf and let $T^*$ be the tree obtained from $T$ by adding a pendant edge $uv$ at $v$. If $T$ has a twin $k$-coloring $c$ for which $c'(v) \not= 0$ and $c(vw) \not= 0$ for each $w \in N_T(v)$, then the coloring $c^*$ of $T^*$ defined by $c^*(e) = c(e)$ if $e \in E(T)$ and $c^*(uv) = 0$ is a twin edge $k$-coloring of $G$.

Theorem 3.4.3 Every tree $T$ having maximum degree at most 6 has a twin edge $(2 + \Delta(T))$-coloring and so $\chi_t'(T) \leq 2 + \Delta(T)$.

Proof. By Proposition 3.1.2, we may assume that $\Delta(T) \geq 3$ for a tree $T$. We verify this theorem only when $\Delta(T) = 3$. The proofs when $4 \leq \Delta(T) \leq 6$, although more complex, employ similar techniques. Let $T$ be a tree with $\Delta(T) = 3$. If $T = K_{1,3}$, then the three edges of $T$ can be colored 0, 1, 2 from $\mathbb{Z}_5$ and so the induced color of the vertex of degree 3 in $T$ is 3 and the induced colors of the leaves are 0, 1, 2. This is a twin edge 5-coloring of $T$ and so $\chi_t'(T) \leq 5$. Indeed, it was shown in Proposition 3.4.1 that $\chi_t'(T) = 4$. Hence we may assume that $T \not= K_{1,3}$.

Let $v \in V(T)$ such that $\deg_T v = 3$ and let $N_T(v) = \{v_1, v_2, v_3\}$. Since $T \not= K_{1,3}$, we may assume that $v_1$ is not a leaf. For a nonnegative integer $i$, let $A_i = \{u \in V(T) : d(v,u) = i\}$. Hence $A_0 = \{v\}$, $A_1 = \{v_1, v_2, v_3\}$ and $A_i \not= \emptyset$ for some $i \geq 2$. Furthermore,
if \( A_k \neq \emptyset \) for some \( k \geq 3 \), then \( A_{k-1} \neq \emptyset \). Indeed, for each vertex \( x \in A_k, k \geq 1 \), there exists exactly one vertex \( y \in A_{k-1} \) such that \( xy \in E(T) \).

We now define a twin edge 5-coloring \( c \) of \( T \). First, define \( c(vv_i) = i \) for \( i = 1, 2, 3 \). Thus \( c'(v) = 1 \). Furthermore, if \( v_i \) is a leaf \((i = 2, 3)\) of \( T \), then \( c'(v) \neq c(vv_i) = c'(v_i) \). Assume for a positive integer \( k \) that (1) a proper edge 5-coloring \( c \) has been defined for all edges \( wz \) of \( T \) for which \( w, z \in A_0 \cup A_1 \cup \cdots \cup A_k \) such that \( c(wz) \neq 0 \) (with the only possible exception if \( wz \) is a pendant edge), (2) the induced vertex 5-coloring \( c' \) for all vertices in \( A_0 \cup A_1 \cup \cdots \cup A_{k-1} \) is a proper vertex coloring and (3) if \( wz \in E(T) \) such that \( w \in A_{k-1}, z \in A_k \) and \( z \) is a leaf, then \( c'(w) \neq c(wz) = c'(z) \). We next describe how \( c(xw) \) can be defined for each edge \( xw \) where \( x \in A_k \) and \( w \in A_{k+1} \). Let \( y \) be the unique vertex in \( A_{k-1} \) that is adjacent to \( x \). Then \( c'(y) \in \{0, 1, 2, 3, 4\} \) and \( c(xy) \) has been defined. We consider five cases, according to the value of \( c'(y) \). In particular, we denote by Case \( i \) \((i = 0, 1, 2, 3, 4)\) the situation where \( c'(y) = i \). We begin with Case 0.

**Case 0.** \( c'(y) = 0 \). First, suppose that \( \deg_T x = 2 \) and \( x \) is adjacent to \( w \in A_{k+1} \). If \( c(xy) \in \{1, 4\} \), then define \( c(xw) = 2 \); while if \( c(xy) \in \{2, 3\} \), then define \( c(xw) = 1 \). Then \( c'(x) \neq c'(y) \). Furthermore, \( c'(x) \neq c(xw) \). Next, suppose that \( \deg_T x = 3 \) and \( x \) is adjacent to the two vertices \( w \) and \( z \) in \( A_{k+1} \). If \( c(xy) \in \{1, 4\} \), then define \( \{c(xw), c(xz)\} = \{2, 3\} \); while if \( c(xy) \in \{2, 3\} \), then define \( \{c(xw), c(xz)\} = \{1, 4\} \). Then \( c'(y) \neq c'(x) \). Furthermore, \( c'(x) \neq c(xw) \) and \( c'(x) \neq c(xz) \).

**Case 1.** \( c'(y) = 1 \). First, suppose that \( \deg_T x = 2 \) and \( x \) is adjacent to \( w \in A_{k+1} \). If \( c(xy) = 1 \), then define \( c(xw) = 2 \); while if \( c(xy) \in \{2, 3, 4\} \), then define \( c(xw) = 1 \). Thus \( c'(x) \neq c'(y) \) and \( c'(x) \neq c(xw) \). Next, suppose that \( \deg_T x = 3 \) and \( x \) is adjacent to the two vertices \( w \) and \( z \) in \( A_{k+1} \). If \( c(xy) = 1 \), then define (i) \( c(xw) = 2 \) and \( c(xz) = 4 \) when \( w \) is not a leaf and (ii) \( c(xw) = 0 \) and \( c(xz) = 2 \) when \( w \) is a leaf. If \( c(xy) \in \{2, 3\} \), then define \( \{c(xw), c(xz)\} = \{1, 4\} \); while if \( c(xy) = 4 \), then define (i) \( c(xw) = 1 \) and \( c(xz) = 2 \) when \( z \) is not a leaf and (ii) \( c(xw) = 3 \) and \( c(xz) = 0 \) when \( z \) is a leaf. Then \( c'(y) \neq c'(x) \). Furthermore, \( c'(x) \neq c(xw) \) and \( c'(x) \neq c(xz) \) when \( xw \) and \( xz \) are not pendant edges.

**Case 2.** \( c'(y) = 2 \). First, suppose that \( \deg_T x = 2 \) and \( x \) is adjacent to \( w \in A_{k+1} \). If \( c(xy) = 1 \), then define \( c(xw) = 2 \); while if \( c(xy) \in \{2, 3, 4\} \), then define \( c(xw) = 1 \). Thus \( c'(x) \neq c'(y) \) and \( c'(x) \neq c(xw) \). Next, suppose that \( \deg_T x = 3 \) and \( x \) is adjacent to the two vertices \( w \) and \( z \) in \( A_{k+1} \). If \( c(xy) \in \{1, 4\} \), then define \( \{c(xw), c(xz)\} = \{2, 3\} \). If \( c(xy) = 2 \), then define (i) \( c(xw) = 1 \) and \( c(xz) = 3 \) when \( w \) is not a leaf and (ii) \( c(xw) = 0 \) and \( c(xz) = 1 \) when \( w \) is a leaf. If \( c(xy) = 3 \), then define \( \{c(xw), c(xz)\} = \{1, 4\} \). Then
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$c'(y) \neq c'(x)$. Furthermore, $c'(x) \neq c(xw)$ and $c'(x) \neq c(xz)$ when $xw$ and $xz$ are not pendant edges.

Case 3. $c'(y) = 3$. First, suppose that $\deg_T x = 2$ and $x$ is adjacent to $w \in A_{k+1}$. If $c(xy) \in \{1, 2\}$, then define $c(xw) = 3$; while if $c(xy) \in \{3, 4\}$, then define $c(xw) = 1$. Thus $c'(x) \neq c'(y)$ and $c'(x) \neq c(xw)$. Next, suppose that $\deg_T x = 3$ and $x$ is adjacent to the two vertices $w$ and $z$ in $A_{k+1}$. If $c(xy) \in \{1, 4\}$, then define $\{c(xw), c(xz)\} = \{2, 3\}$. If $c(xy) = 2$, then define (i) $c(xw) = 1$ and $c(xz) = 3$ when $w$ is not a leaf and (ii) $c(xw) = 4$ and $c(xz) = 1$ when $w$ is a leaf. If $c(xy) = 3$, then define (i) $c(xw) = 1$ and $c(xz) = 2$ when $w$ is not a leaf and (ii) $c(xw) = 0$ and $c(xz) = 1$ when $w$ is a leaf. Then $c'(y) \neq c'(x)$. Furthermore, $c'(x) \neq c(xw)$ and $c'(x) \neq c(xz)$ when $xw$ and $xz$ are not pendant edges.

Case 4. $c'(y) = 4$. First, suppose that $\deg_T x = 2$ and $x$ is adjacent to $w \in A_{k+1}$. If $c(xy) \in \{1, 3\}$, then define $c(xw) = 2$; while if $c(xy) \in \{2, 4\}$, then define $c(xw) = 1$. Thus $c'(x) \neq c'(y)$ and $c'(x) \neq c(xw)$. Next, suppose that $\deg_T x = 3$ and $x$ is adjacent to the two vertices $w$ and $z$ in $A_{k+1}$. If $c(xy) = 1$, then define $\{c(xw), c(xz)\} = \{2, 3\}$; while if $c(xy) \in \{2, 3\}$, then define $\{c(xw), c(xz)\} = \{1, 4\}$. If $c(xy) = 4$, then define (i) $c(xw) = 2$ and $c(xz) = 1$ when $w$ is not a leaf and (ii) $c(xw) = 0$ and $c(xz) = 2$ when $w$ is a leaf. Then $c'(y) \neq c'(x)$. Furthermore, $c'(x) \neq c(xw)$ and $c'(x) \neq c(xz)$ when $xw$ and $xz$ are not pendant edges.

Since $\chi'_t(K_{1,5}) = 7$ by Proposition 3.4.1, the upper bound of 7 for $\chi'_t(T)$ when $T$ is a tree of maximum degree 5 cannot be improved.

3.5 Brooms, Double Stars and Regular Trees

In this section, we verify Conjecture 3.1.1 for several classes of trees, namely brooms, double stars and regular trees. The following two definitions will be useful to us. For integers $a$ and $b$ with $a < b$, let

$$[a..b] = \{a, a+1, \ldots, b\}$$

be the set of integers between $a$ and $b$ and let $\sigma(a, b)$ denote the sum of integers between $a$ and $b$, that is,

$$\sigma(a, b) = \sum_{i=a}^{b} i = a + (a + 1) + \cdots + b.$$  \hfill (3.2)


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3.5.1 Brooms and Double Stars

By Theorem 3.1.2, \( \chi'_t(P_n) = 3 \) for each integer \( n \geq 3 \) and for each integer \( r \geq 2 \),

\[
\chi'_t(K_{1,r}) = \begin{cases} 
  r + 1 & \text{if } r \not\equiv 1 \pmod{4} \\
  r + 2 & \text{if } r \equiv 1 \pmod{4}
\end{cases}
\]

Paths and stars belong to a special class of trees, namely brooms. A \textit{broom} is a tree obtained from a path by adding pendant edges at exactly one of the end-vertices of the path. We first verify Conjecture 3.1.1 for brooms. In fact, more can be said.

**Theorem 3.5.1** If \( T \) is a broom that is not a star, then \( T \) has a twin edge \((\Delta(T) + 1)\)-coloring and so \( \chi'_t(T) \leq \Delta(T) + 1 \).

**Proof.** By Theorem 3.1.2, we may assume that \( T \) is not a path and so \( \Delta(T) = \Delta \geq 3 \). Suppose that \( T \) is obtained from the path \( P_\ell = (v_1, v_2, \ldots, v_\ell) \) by adding \( \Delta - 1 \) pendant edges \( u_i v \) \((1 \leq i \leq \Delta - 1)\) at the end-vertex \( v \). We consider two cases, according to whether \( \Delta \) is even or \( \Delta \) is odd.

\textit{Case 1. \( \Delta \) is even.} Let \( \Delta = 2t \) for some integer \( t \geq 2 \). Define a proper edge coloring \( c : E(T) \to \mathbb{Z}_{2t+1} \) such that (i)

\[
\{ c(u_i v) : 1 \leq i \leq 2t - 1 \} = \{0, 2, 3, \ldots, 2t - 1\} = [0..2t-1] - \{1\}
\]

and (ii) \( c(v_j v_{j+1}) = r \) if \( j \equiv r \pmod{3} \) where \( r \in \{1, 2, 3\} \) and \( 1 \leq j \leq \ell - 1 \). Then the induced vertex coloring \( c' \) satisfies that \( c'(u_i) = c(u_i v) \neq 1 \) for \( 1 \leq i \leq 2t - 1 \) and \( c'(v) = \sigma(0, 2t - 1) = 1 \) in \( \mathbb{Z}_{2t+1} \), where \( \sigma(0, 2t - 1) \) is the sum of integers between 0 and \( 2t - 1 \), as described in (3.2). Let

\[
s'_\ell = (c'(v_2), c'(v_3), \ldots, c'(v_\ell)). \quad (3.3)
\]

If \( \ell = 3 \), then \( s'_\ell = (3, 2) \), if \( \ell = 4 \), then \( s'_\ell = (3, 5, 3) \), if \( \ell = 5 \), then \( s'_\ell = (3, 5, 4, 1) \) and if \( \ell \geq 6 \), then

\[
s'_\ell = \begin{cases} 
  (3, 5, 4, \ldots, 3, 5, 4, 3, 5, 3) & \text{if } \ell \equiv 1 \pmod{3} \\
  (3, 5, 4, \ldots, 3, 5, 4, 3, 5, 4, 1) & \text{if } \ell \equiv 2 \pmod{3} \\
  (3, 5, 4, \ldots, 3, 5, 4, 3, 5, 4, 3, 2) & \text{if } \ell \equiv 0 \pmod{3}.
\end{cases} \quad (3.4)
\]

Note that when \( t = 2 \), each entry 5 of the sequences in (3.4) is 0 in \( \mathbb{Z}_{2t+1} \). Since \( c' \) is a proper vertex coloring, \( c \) is a twin edge \((\Delta + 1)\)-coloring.
CHAPTER 3. TWIN EDGE COLORING CONJECTURE

Case 2. $\Delta$ is odd. Let $\Delta = 2t + 1$ for some integer $t \geq 1$. We consider two subcases, according to whether $t = 1$ or $t \geq 2$.

Subcase 2.1. $t = 1$. If $\ell = 3$, then define $c : E(T) \to \mathbb{Z}_4$ by $c(u_1v_1) = 0$, $c(u_2v_1) = 2$, $c(v_1v_2) = 1$ and $c(v_2v_3) = 0$. Hence $c'(u_1) = 0$, $c'(u_2) = 2$, $c'(v_1) = 3$, $c'(v_2) = 1$ and $c'(v_3) = 0$. Thus $c$ is a twin edge 4-coloring of $T$. If $\ell \geq 4$, then define $c : E(T) \to \mathbb{Z}_4$ by $c(u_1v_1) = 1$, $c(u_2v_1) = 2$, $c(v_1v_2) = 0$ and $c(v_jv_{j+1}) = r$ if $j - 1 \equiv r \pmod{3}$ and $2 \leq j < \ell - 1$. That is, $(c(v_2v_3), c(v_3v_4), \ldots, c(v_{\ell-1}v_\ell)) = (1, 2, 3, 1, 2, 3, \ldots)$, which ends at $r$ if $\ell - 1 \equiv r \pmod{3}$ for $r \in \{1, 2, 3\}$. Then $c'(u_1) = 1$, $c'(u_2) = 2$, $c'(v_1) = 3$ and $c'(v_2) = 1$. Furthermore, let $s_\ell$ be defined as in (3.3). If $\ell = 4$, then $s_\ell = (1, 3, 2)$, if $\ell = 5$, then $s_\ell = (1, 3, 5, 3)$, if $\ell = 6$, then $s_\ell = (1, 3, 5, 4, 1)$ and if $\ell \geq 7$, then

$$s_\ell' = \begin{cases} (1, 3, 5, 4, \ldots, 3, 5, 4, 3, 5, 3) & \text{if } \ell \equiv 2 \pmod{3} \\ (1, 3, 5, 4, \ldots, 3, 5, 4, 3, 5, 4, 1) & \text{if } \ell \equiv 0 \pmod{3} \\ (1, 3, 5, 4, \ldots, 3, 5, 4, 3, 5, 4, 3, 2) & \text{if } \ell \equiv 1 \pmod{3}. \end{cases}$$ (3.5)

Note that each entry 5 of the sequences in (3.5) is 1 in $\mathbb{Z}_{2t+2} = \mathbb{Z}_4$. Thus $c$ is a twin edge 4-coloring of $T$.

Subcase 2.2. $t \geq 2$. Define $c : E(T) \to \mathbb{Z}_{2t+2}$ such that (i)

$$\{c(u_iv) : 1 \leq i \leq 2t\} = \{1, \ldots, 2t + 1\} - \{t + 1\} = [1..2t + 1] - \{t + 1\}$$

(ii) $c(v_1v_2) = t + 1$ and (iii) $c(v_jv_{j+1}) = r$ if $j - 1 \equiv r \pmod{3}$ where $r \in \{1, 2, 3\}$ and $2 \leq j \leq \ell - 1$. Then the induced vertex coloring $c'$ satisfies that $c'(u_i) = c(u_iv) \neq t + 1$ for $1 \leq i \leq 2t$, $c'(v) = \sigma(0, 2t + 1) = t + 1$ in $\mathbb{Z}_{2t+2}$ and $c'(v_2) = t + 2$. Note that $1, t + 1, t + 2$ are distinct in $\mathbb{Z}_{2t+2}$. Furthermore, if $\ell = 3$, then $s_\ell' = (t + 2, 1)$, if $\ell = 4$, then $s_\ell' = (t + 2, 3, 2)$, if $\ell = 5$, then $s_\ell' = (t + 2, 3, 5, 3)$, if $\ell = 6$, then $s_\ell' = (t + 2, 3, 5, 4, 1)$ and if $\ell \geq 7$, then

$$s_\ell' = \begin{cases} (t + 2, 3, 5, 4, \ldots, 3, 5, 4, 3, 5, 3) & \text{if } \ell \equiv 2 \pmod{3} \\ (t + 2, 3, 5, 4, \ldots, 3, 5, 4, 3, 5, 4, 1) & \text{if } \ell \equiv 0 \pmod{3} \\ (t + 2, 3, 5, 4, \ldots, 3, 5, 4, 3, 5, 4, 3, 2) & \text{if } \ell \equiv 1 \pmod{3}. \end{cases}$$ (3.6)

Since $t \geq 2$, it follows that $t + 2 \neq 3$ in (3.6) and so $c'$ is a proper vertex coloring. Thus $c$ is a twin edge $(\Delta + 1)$-coloring.

A double star is a tree of diameter 3. Thus each double star has exactly two non-end-vertices called the central vertices of the double star. If the central vertices of a double
star have degree $a$ and $b$, respectively, then it is denoted by $S_{a,b}$ where the order of $S_{a,b}$ is $a+b$. If $a = b$, then $S_{a,b}$ is referred to as a regular double star; while if $a \neq b$, then $S_{a,b}$ is irregular (that is, every two non-end-vertices have different degrees). In particular, $S_{2,b}$ is a broom for each integer $b \geq 2$. In order to determine the twin chromatic indexes of all double stars, we first present some preliminary results on regular graphs or regular trees in general.

For an integer $r \geq 2$, a tree $T$ is $r$-regular if each non-end-vertex of $T$ has degree $r$. Thus, the degree set of an $r$-regular tree is $\{1, r\}$. In particular, a path $P_n$ order $n \geq 3$ is 2-regular and a star $K_{1,r}$ is $r$-regular for $r \geq 2$. We first show that if $T$ is an $r$-regular tree for some integer $r \geq 5$ such that $r \equiv 1 \pmod{4}$, then $\chi'_t(T) \geq r + 2$. The following lemma is a useful observation.

**Lemma 3.5.2** Let $r \geq 5$ be an integer such that $r \equiv 1 \pmod{4}$. Then

$$\sigma(0, r) - j \not\equiv j \pmod{r + 1}$$

for each integer $j \in [0..r]$.

**Theorem 3.5.3** If $T$ is a regular tree of order at least 6 such that $\Delta(T) \equiv 1 \pmod{4}$, then

$$\chi'_t(T) \geq \Delta(T) + 2.$$  

**Proof.** Suppose that $T$ is an $r$-regular tree for some integer $r \geq 5$ and $r \equiv 1 \pmod{4}$. Then $\Delta(T) = r$. By Observation 3.2.1, it follows that $\chi'_t(T) \geq r + 1$. We show that $\chi'_t(T) \not\equiv r + 1$. Assume, to the contrary, that $\chi'_t(T) = r + 1$. Let $c : E(T) \rightarrow \mathbb{Z}_{r+1}$ be a twin edge ($r+1$)-coloring of $T$. Let $v_1 \in V(T)$ such that $\deg v_1 = r$. Then there is exactly one color in $\mathbb{Z}_{r+1}$ that is not assigned to any edge incident with $v_1$ by $c$. Suppose that $\{c(v_1w) : w \in N(v_1)\} = \mathbb{Z}_{r+1} - \{j_1\}$ for some integer $j_1$; that is, $j_1$ is the only color that is not assigned to any edge incident with $v_1$. Since $c'(v_1) = \sigma(0, r) - j_1$ and $r \equiv 1 \pmod{4}$, it follows by Lemma 3.5.2 that $c'(v_1) \neq j_1$ and so $c'(v_1) \in \mathbb{Z}_{r+1} - \{j_1\}$. Hence $c'(v_1) = c(v_1w)$ for some $w \in N(v_1)$. Let $c'(v_1) = c(v_1v_2)$, where $v_2 \in N(v_1)$. If $v_2$ is an end-vertex of $T$, then $c'(v_2) = c(v_1v_2) = c'(v_1)$, which is impossible. Thus $v_2$ is not an end-vertex of $T$ and so $\deg v_2 = r$. Suppose that $\{c(v_2w) : w \in N(v_2)\} = \mathbb{Z}_{r+1} - \{j_2\}$ for some integer $j_2$, where then $c(v_1v_2) \in \mathbb{Z}_{r+1} - \{j_2\}$. By Lemma 3.5.2 again, $c'(v_2) \neq j_2$ and so $c'(v_2) = c(v_2v_3)$ for some $v_3 \in N(v_2)$. Since $c$ is a twin edge ($r+1$)-coloring of $T$, it follows that $c'(v_2) \neq c'(v_1) = c(v_1v_2)$; which implies that $v_3 \neq v_1$. A similar
argument shows that \( v_3 \) is not an end-vertex and so \( \deg v_3 = r \). Continuing in this manner, we arrive at a sequence \( v_1, v_2, \ldots, v_k \) of distinct \( k \geq 2 \) vertices of degree \( r \) in \( T \) such that (1) \( v_iv_{i+1} \in E(T) \) for \( 1 \leq i \leq k \) and (2) \( v_{k-1} \) is the only non-end-vertex to which \( v_k \) is adjacent and so \( v_k \) is adjacent to exactly \( r - 1 \) end-vertices of \( T \). Suppose that \( \{c(v_kw) : w \in N(v_k)\} = \mathbb{Z}_{r+1} - \{j_k\} \) for some integer \( j_k \), where then \( c(v_{k-1}v_k) \in \mathbb{Z}_{r+1} - \{j_k\} \). It then follows by Lemma 3.5.2 that \( c'(v_k) \neq j_k \). Hence \( c'(v_k) = c(v_kv_{k+1}) \) for some \( v_{k+1} \in N(v_k) - \{v_{k-1}\} \). Since \( v_{k+1} \) is an end-vertex of \( T \), it follows that \( c'(v_{k+1}) = c(v_kv_{k+1}) = c(v_k) \), which is impossible. Therefore, \( \chi'_t(T) \neq r + 1 \) and so \( \chi'_t(T) \geq r + 2 \). 

We are now prepared to determine the twin chromatic indexes of all double stars.

**Theorem 3.5.4** If \( T \) is a regular double star, then

\[
\chi'_t(T) = \begin{cases} 
\Delta(T) + 1 & \text{if } \Delta(T) \not\equiv 1 \pmod{4} \\
\Delta(T) + 2 & \text{if } \Delta(T) \equiv 1 \pmod{4}
\end{cases}
\]

**Proof.** By Theorem 3.1.2, we may assume that \( T \) is not a path. Let \( T = S_{r,r} \) for some integer \( r \), where then \( \Delta(T) = r \geq 3 \). Suppose that the central vertices are \( u \) and \( v \), where \( \deg u = \deg v = r \). Let \( u_1, u_2, \ldots, u_{r-1} \) be the end-vertices of \( T \) that are adjacent to \( u \) and let \( v_1, v_2, \ldots, v_{r-1} \) be the end-vertices of \( T \) that are adjacent to \( v \). First, suppose that \( r \equiv 1 \pmod{4} \). By Observation 3.2.1, it follows that \( \chi'_t(T) \geq r + 1 \). It remains to show that \( T \) has a twin edge \( (r + 1) \)-coloring \( c : E(T) \to \mathbb{Z}_{r+1} \). We consider two cases, according to whether \( r \) is even or \( r \) is odd.

**Case 1.** \( r \geq 4 \) is even. Then \( \sigma(1,r) = 0 \) in \( \mathbb{Z}_{r+1} \). Define \( c(uv) = r \) and

\[
\{c(uu_i) : 1 \leq i \leq r - 1\} = [1..r - 1] \\
\{c(vv_i) : 1 \leq i \leq r - 1\} = [0..r - 1] - \{1\}.
\]

Then \( c \) is a proper edge coloring. Observe that \( c'(u) = \sigma(1,r) = 0 \) in \( \mathbb{Z}_{r+1} \) and \( c'(v) = \sigma(1,r) - 1 = r \) in \( \mathbb{Z}_{r+1} \). Thus \( c'(u) \neq c'(v) \). Furthermore, \( c'(u_i) \neq 0 = c'(u) \) and \( c'(u_i) \neq r = c'(v) \) for \( 1 \leq i \leq r - 1 \). Hence, \( c' \) is a proper vertex coloring and so \( c \) is a twin edge \( (r + 1) \)-coloring of \( T \).

**Case 2.** \( r \geq 3 \) is odd. Since \( r \equiv 1 \pmod{4} \), it follows that \( r \equiv 3 \pmod{4} \) and so \( r = 4t + 3 \) for some integer \( t \geq 0 \). First, suppose that \( t = 0 \). Define \( c : E(T) \to \mathbb{Z}_4 \)
by \( c(uv) = 2 \), \( \{c(uu_i), c(uw_2)\} = \{0, 1\} \) and \( \{c(vv_1), c(vv_2)\} = \{1, 3\} \). Then \( c'(u) = 3 \neq c'(u_i) \) in \( \mathbb{Z}_4 \) and \( c'(v) = 2 \neq c'(v_i) \) in \( \mathbb{Z}_4 \) for \( i = 1, 2 \). Also, \( c'(u) \neq c'(v) \). Therefore, \( c \) is a twin edge 4-coloring of \( T \).

Next, suppose that \( t \geq 1 \). Observe that \( \sigma(0, 4t + 3) = 2t + 2 \) in \( \mathbb{Z}_{r+1} = \mathbb{Z}_{4t+4} \) and so

\[ \sigma(0, 4t + 3) - (t + 1) = t + 1 \text{ in } \mathbb{Z}_{4t+4}. \] (3.7)

Define \( c(uv) = 2t + 2 \) and

\[
\begin{align*}
\{c(uu_i) : 1 \leq i \leq 4t + 2\} &= [0..4t + 3] - \{t + 1, 2t + 2\} \\
\{c(vv_i) : 1 \leq i \leq 4t + 2\} &= [1..4t + 3] - \{2t + 2\}.
\end{align*}
\]

Then \( c \) is a proper edge coloring of \( T \). By (3.7),

\[
\begin{align*}
c'(u) &= \sigma(1, 4t + 3) - (t + 1) = t + 1 \text{ in } \mathbb{Z}_{4t+4} \\
c'(v) &= \sigma(1, 4t + 3) = 2t + 2 \text{ in } \mathbb{Z}_{4t+4}.
\end{align*}
\]

Thus \( c'(u) \neq c'(v) \). Furthermore, \( c'(u_i) \neq t + 1 = c'(u) \) in \( \mathbb{Z}_{4t+4} \) and \( c'(v_i) \neq 2t + 2 = c'(v) \) in \( \mathbb{Z}_{4t+4} \) for \( 1 \leq i \leq 4t + 2 \). Hence, \( c' \) is a proper vertex coloring and so \( c \) is a twin edge \((r + 1)\)-coloring of \( T \).

Next, suppose that \( r \geq 5 \) and \( r \equiv 1 \) (mod 4). By Theorem 3.5.3, it follows that \( \chi'_t(T) \geq r + 2 \). Thus, it remains to show that \( T \) has a twin edge \((r + 2)\)-coloring \( c : E(T) \to \mathbb{Z}_{r+2} \). Let \( r = 4t + 1 \) for some integer \( t \geq 1 \). Then \( \sigma(0, 4t + 2) = 0 \) in \( \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+3} \) and so

\[
\begin{align*}
\sigma(0, 4t + 2) - (2t + 1) - 1 &= 2t + 1 \text{ in } \mathbb{Z}_{4t+3} \\
\sigma(0, 4t + 2) - (2t) - 3 &= 2t \text{ in } \mathbb{Z}_{4t+3}
\end{align*}
\]

Define \( c(uv) = 2t + 2 \) and

\[
\begin{align*}
\{c(uu_i) : 1 \leq i \leq 4t\} &= [0..4t + 2] - \{1, 2t + 1, 2t + 2\} \\
\{c(vv_i) : 1 \leq i \leq 4t\} &= [0..4t + 2] - \{3, 2t, 2t + 2\}.
\end{align*}
\]

Then \( c \) is a proper edge coloring of \( T \). Observe that

\[
\begin{align*}
c'(u) &= \sigma(0, 4t + 2) - (2t + 1) - 1 = 2t + 1 \text{ in } \mathbb{Z}_{4t+3} \\
c'(v) &= \sigma(0, 4t + 2) - 2t - 3 = 2t \text{ in } \mathbb{Z}_{4t+3}.
\end{align*}
\]

Thus \( c'(u) \neq c'(v) \). Furthermore, \( c'(u_i) \neq 2t + 1 = c'(u) \) and \( c'(v_i) \neq 2t = c'(v) \) for \( 1 \leq i \leq 4t \). The colorings \( c \) and \( c' \) are shown for \( r = 5 \) in Figure 3.4. Hence, \( c' \) is a proper vertex coloring and so \( c \) is a twin edge \((r + 2)\)-coloring of \( T \).
We have seen in Theorem 3.5.1 that if $T$ is a broom, then $\chi_1'(T) \leq \Delta(T) + 1$. This is also the case for irregular double stars, as we show next.

**Theorem 3.5.5** If $T$ is an irregular double star, then $T$ has a twin edge $(\Delta(T) + 1)$-coloring and so $\chi_1'(T) \leq \Delta(T) + 1$.

**Proof.** Let $T = S_{a,b}$ for some integers $a$ and $b$ with $2 \leq a < b$. By Theorem 3.5.1, we may assume that $a \geq 3$ and so $b \geq 4$. Suppose that the central vertices are $u$ and $v$, where $\text{deg} u = a$ and $\text{deg} v = b = \Delta(T)$. Let $u_1, u_2, \ldots, u_{a-1}$ be the end-vertices of $T$ that are adjacent to $u$ and let $v_1, v_2, \ldots, v_{b-1}$ be the end-vertices of $T$ that are adjacent to $v$. To construct a twin edge $(b+1)$-coloring of $T$, we consider two cases, according to whether $b$ is even or $b$ is odd.

- **Case 1. $b$ is even.** Let $b = 2t$ for some integer $t \geq 2$. There are two subcases, according to (i) $a = 2t - 1$ or $a = 2t - 2 \geq 4$ and (ii) $3 \leq a \leq 2t - 3$.

- **Subcase 1.1. $a = 2t - 1$ or $a = 2t - 2$.** First, suppose that $a = 2t - 1$. Define a proper edge coloring $c : E(T) \rightarrow \mathbb{Z}_{2t+1} = \{0, 1, \ldots, 2t\}$ by $c(uv) = 1$ such that

$$
\begin{align*}
U_c & = \{c(uu_i) : 1 \leq i \leq a - 1\} \\
& = \{0, 3, 4, \ldots, 2t - 1\} = [0..2t - 1] - \{1, 2\} \\
V_c & = \{c(vv_j) : 1 \leq j \leq b - 1\} \\
& = \{0, 2, 3, \ldots, 2t - 1\} = [0..2t - 1] - \{1\}.
\end{align*}
$$

Figure 3.5 shows an example of such a twin edge $(b+1)$-coloring of $T$.

Then the induced vertex coloring $c'$ satisfies that

$$
\begin{align*}
c'(u) & = 1 + 3 + 4 + \cdots + (2t - 1) = \sigma(0, 2t) - 2 - (2t) = 2t \text{ in } \mathbb{Z}_{2t+1} \\
c'(v) & = 1 + 2 + 3 + \cdots + (2t - 1) = \sigma(0, 2t) - (2t) = 1 \text{ in } \mathbb{Z}_{2t+1}.
\end{align*}
$$

Figure 3.4: A twin edge 7-coloring of $S_{5,5}$
For each end-vertex \( u_i \) \((1 \leq i \leq 2t - 2)\) or \( v_j \) \((1 \leq j \leq 2t - 1)\) of \( T \), it follows that
\[
\{c'(u_i) : 1 \leq i \leq 2t - 2\} = [0..2t-1] - \{1, 2\}
\]
\[
\{c'(v_j) : 1 \leq j \leq 2t - 1\} = [0..2t-1] - \{1\}.
\]

It follows that \( c'(u_i) \neq c'(u) = 2t \) for \( 1 \leq i \leq 2t - 2 \) and \( c'(v_j) \neq c'(v) = 1 \) for \( 1 \leq j \leq 2t - 1 \) in \( \mathbb{Z}_{2t+1} \). Hence \( c' \) is a proper vertex coloring and \( c \) is a twin edge \((2t+1)\)-coloring of \( T \).

For \( a = 2t - 2 \geq 4 \), let \( S_{2t-2,b} \) be obtained from \( S_{2t-1,b} \) by deleting the edge \( uu_1 \) where \( c(uu_1) = 0 \) in the case \( a = 2t - 1 \) (see Figure 3.5). Then the twin edge \((2t+1)\)-coloring \( c \) for \( S_{2t-1,b} \), as described above, gives rise to a twin edge \((2t+1)\)-coloring of \( S_{2t-2,b} \).

**Subcase 1.2.** \( 3 \leq a \leq 2t - 3 \). Define a proper edge coloring \( c : E(T) \to \mathbb{Z}_{2t+1} \) by \( c(uv) = 1 \) such that \( V_c = \{0, 2, 3, \ldots, 2t - 1\} = [0..2t-1] - \{1\} \) (which is the same as in Subcase 1.1 and is shown in Figure 3.5). Hence \( c'(v) = 1 \) in \( \mathbb{Z}_{2t+1} \) and \( c'(v_j) \neq c'(v) = 1 \) for \( 1 \leq j \leq 2t - 1 \). It remains to define the color \( c(uu_i) \) for \( 1 \leq i \leq a - 1 \). We consider two situations when \( a \geq 3 \) is odd or \( a \geq 4 \) is even.

First, suppose that \( a \) is odd. Let \( a - 1 = 2k \) for some positive integer \( k \). If \( k = 1 \), then define \( \{c(uu_1), c(uu_2)\} = \{0, 2t - 1\} \); while if \( k \geq 2 \), then define
\[
U_c = \{0, 2t - 1, 3, 2t - 2, \ldots, k + 1, 2t - k\}.
\]

Figure 3.6 shows an example of a possible coloring of each edge \( uu_i \) for \( 1 \leq i \leq a - 1 \).

Since \( c(uv) = 1 \), it follows that \( c'(u) = 1 + (2t - 1) = 2t \) in \( \mathbb{Z}_{2t+1} \). Since \( a - 1 \leq 2k \), it follows that \( k \leq t - 2 \) and so \( k + 1 \leq t - 1 \). Hence \( c'(uu_i) \neq 2t \) for \( 1 \leq i \leq a - 1 \). Therefore, \( c' \) is a proper vertex coloring and \( c \) is a twin edge \((2t+1)\)-coloring. Next, suppose that \( a \geq 4 \) is even. Let \( c \) be a twin edge \((2t+1)\)-coloring of \( S_{a+1,b} \) as described above for the odd integer \( a + 1 \geq 5 \). We may assume, without loss generality, that
Figure 3.6: A possible coloring of $uu_i$ for $1 \leq i \leq a - 1$ in Subcase 1.2

c(uu_1) = 0$ (see Figure 3.6). Now let $S_{a,b}$ be obtained from $S_{a+1,b}$ by deleting the edge $uu_1$. Then the twin edge $(2t + 1)$-coloring $c$ for $S_{a+1,b}$, as described above, gives rise to a twin edge $(2t + 1)$-coloring of $S_{a,b}$.

**Case 2. b is odd.** Let $b = 2t + 1$ for some integer $t \geq 2$. There are two subcases, according to (i) $a = 2t$ or $a = 2t - 1 \geq 3$ and (ii) $3 \leq a \leq 2t - 2$.

**Subcase 2.1.** $a = 2t$ or $a = 2t - 1$. First, suppose that $a = 2t$. Define a proper edge coloring $c : E(T) \to \mathbb{Z}_{2t+2}$ by $c(uv) = t$ such that

$$
U_c = \{c(uu_i) : 1 \leq i \leq a - 1\} \\
= \{0, 1, \ldots, 2t + 1\} - \{t - 1, t, t + 2\} = [0..2t + 1] - \{t - 1, t, t + 2\}
$$

$$
V_c = \{c(vv_j) : 1 \leq j \leq b - 1\} \\
= \{0, 2, 3, \ldots, t - 1, t + 1, \ldots, 2t + 1\} = [0..2t + 1] - \{1, t\}.
$$

Figure 3.7 shows an example of such a twin edge $(b + 1)$-coloring of $T$.

Figure 3.7: A twin edge $(b + 1)$-coloring of $T$ in Subcase 2.1
Then the induced vertex coloring \( c' \) satisfies that
\[
c'(u) = \sigma(0, 2t + 1) - (t - 1) - (t + 2) = t(2t + 1) = t + 2 \text{ in } \mathbb{Z}_{2t+2},
\]
\[
c'(v) = \sigma(0, 2t + 1) - 1 = (2t + 1)(t + 1) - 1 = t \text{ in } \mathbb{Z}_{2t+2}.
\]
For each end-vertex \( u_i \) \((1 \leq i \leq 2t - 1)\) or \( v_j \) \((1 \leq j \leq 2t)\) of \( T \), it follows that
\[
\{c'(u_i) : 1 \leq i \leq 2t - 1\} = [0..2t + 1] - \{t - 1, t, t + 2\}
\]
\[
\{c'(v_j) : 1 \leq j \leq 2t\} = [0..2t + 1] - \{1, t\}.
\]
It follows that \( c'(u_i) \neq c'(u) = t + 2 \text{ for } 1 \leq i \leq 2t - 1 \text{ and } c'(v_j) \neq c'(v) = t \text{ for } 1 \leq j \leq 2t \text{ in } \mathbb{Z}_{2t+2}. \) Hence \( c' \) is a proper vertex coloring and \( c \) is a twin edge \((2t + 2)\)-coloring of \( T \).

For \( a = 2t - 1 \geq 3 \), let \( S_{2t-1,b} \) be obtained from \( S_{2t,b} \) by deleting the edge \( uu_1 \) where \( c(uu_1) = 0 \) in the case \( a = 2t \) (see Figure 3.7). Then the twin edge \((2t + 2)\)-coloring \( c \) for \( S_{2t,b} \) gives rise to a twin edge \((2t + 2)\)-coloring of \( S_{2t-1,b} \).

**Subcase 2.2.** \( 3 \leq a \leq 2t - 2 \). Define a proper edge coloring \( c : E(T) \to \mathbb{Z}_{2t+2} \) by \( c(uv) = t \) such that \( V_c = \{0, 2, 3, \ldots, t - 1, t + 1, 2t + 1\} = [0..2t + 1] - \{1, t\} \) (which is the same as in Subcase 2.1 and is shown in Figure 3.7). Hence \( c'(v) = t \in \mathbb{Z}_{2t+2} \) and \( c'(v_j) \neq c'(v) = t \text{ for } 1 \leq j \leq 2t \). It remains to define the color \( c(uu_i) \) for \( 1 \leq i \leq a - 1 \).

We consider two situations when \( a \geq 3 \) is odd or \( a \geq 4 \) is even.

First, suppose that \( a \geq 4 \) is even. Let \( a - 1 = 2k + 1 \) for some positive integer \( k \). For \( k = 1 \), define \( \{c(uu_1), c(uu_2), c(uu_3)\} = \{0, t + 1, t + 3\} \). Then \( c'(u) = t + (t + 1) + (t + 3) = t + 2 \in \mathbb{Z}_{2t+2} \) and so \( c'(u) \neq c'(u_i) \) for \( i = 1, 2, 3 \). For \( k \geq 2 \) and so \( 2k + 1 \geq 5 \), define
\[
U_c = \{0, t + 1, t + 3, 1, 2t + 1, 2, 2t, \ldots, k - 1, 2t - k + 3\}.
\]
Figure 3.8 shows an example of a possible coloring of each edge \( uu_i \) for \( 1 \leq i \leq a - 1 \).

Again, \( c'(u) = t + (t + 1) + (t + 3) = t + 2 \in \mathbb{Z}_{2t+2} \). Since \( k \leq t - 1 \), it follows that \( 2t - k + 3 \geq t + 4 \). Thus \( c'(u) \neq c'(u_i) \) for \( 1 \leq i \leq a - 1 \). Therefore, \( c' \) is a proper vertex coloring and \( c \) is a twin edge \((2t + 2)\)-coloring. Next, suppose that \( a \geq 3 \) is odd. Let \( a - 1 = 2k \) for some positive integer \( k \). Let \( c \) be a twin edge \((2t + 2)\)-coloring of \( S_{a+1,b} \) as described above for the even integer \( a + 1 \geq 4 \). We may assume, without loss of generality, that \( c(uu_1) = 0 \) (see Figure 3.8). Now let \( S_{a,b} \) be obtained from \( S_{a+1,b} \) by deleting the edge \( uu_1 \). Then the twin edge \((2t + 2)\)-coloring \( c \) for \( S_{a+1,b} \), as described above, gives rise to a twin edge \((2t + 2)\)-coloring of \( S_{a,b} \).

The following is a consequence of Theorems 3.5.3, 3.5.4 and 3.5.5.
Then c

Let T

Proof. Since T

Recall that a tree T is r-regular for an integer r ≥ 2 if each non-end-vertex of T has degree r. In this section, we verify Conjecture 3.1.1 for regular trees of order at least 3. More precisely, we show that if T is a regular tree of order at least 3, then T has a twin edge (Δ(T) + 2)-coloring and so χ′_1(T) ≤ Δ(T) + 2. We begin with a lemma concerning stars only.

Lemma 3.5.7 For each integer r ≥ 3, the star K_{1,r} has a twin edge (r + 2)-coloring.

Corollary 3.5.6 A double star T has χ′_1(T) = Δ(T) + 2 if and only if T is an r-regular tree for some integer r ≥ 5 with r ≡ 1 (mod 4).

3.5.2 Regular Trees

Recall that a tree T is r-regular for an integer r ≥ 2 if each non-end-vertex of T has degree r. In this section, we verify Conjecture 3.1.1 for regular trees of order at least 3. More precisely, we show that if T is a regular tree of order at least 3, then T has a twin edge (Δ(T) + 2)-coloring and so χ′_1(T) ≤ Δ(T) + 2. We begin with a lemma concerning stars only.

Lemma 3.5.7 For each integer r ≥ 3, the star K_{1,r} has a twin edge (r + 2)-coloring.

Proof. Since χ′_1(K_{1,r}) = r + 2 for r ≡ 1 (mod 4), we may assume that r ≠ 1 (mod 4). Let T = K_{1,r} where V(T) = \{v, v_1, v_2, \ldots, v_r\} and deg v = r. We consider three cases.

Case 1. r ≡ 0 (mod 4). Let r = 4t for some integer t ≥ 1. Observe that

\[\sigma(1, r + 1) = \sigma(1, 4t + 1) = 2t + 1 \text{ in } \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+2}.\]

Define the edge coloring c : E(T) → \mathbb{Z}_{4t+2} such that

\[\{c(vv_i) : 1 \leq i \leq 4t\} = [1..4t + 1] - \{2t + 1\}.\]

Then c'(v) = σ(1, 4t + 1) - (2t + 1) = 0 ≠ c(v_i) for 1 ≤ i ≤ r = 4t. Hence c is a twin edge (r + 2)-coloring of T.

Case 2. r ≡ 2 (mod 4). Let r = 4t + 2 for some integer t ≥ 1. Observe that
\[ \sigma(1, r + 1) = \sigma(1, 4t + 3) = 2t + 2 \text{ in } \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+4}. \]

Define the edge coloring \( c : E(T) \to \mathbb{Z}_{4t+4} \) such that
\[
\{c(vv_i) : 1 \leq i \leq 4t + 2\} = [1..4t + 3] - \{2t + 2\}.
\]

Then \( c'(v) = \sigma(1, 4t + 3) - (2t + 2) = 0 \neq c(v_i) \) for \( 1 \leq i \leq r = 4t + 2 \). Hence \( c \) is a twin edge \((r + 2)\)-coloring of \( T \).

\textbf{Case 3.} \( r \equiv 3 \pmod{4} \). Let \( r = 4t + 3 \) for some integer \( t \geq 0 \). Observe that
\[ \sigma(0, r) = \sigma(0, 4t + 3) = 1 \text{ in } \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+5}. \]

Define the edge coloring \( c : E(T) \to \mathbb{Z}_{4t+5} \) such that
\[
\{c(vv_i) : 1 \leq i \leq 4t + 3\} = [0..4t + 3] - \{2\}.
\]

Then \( c'(v) = \sigma(0, 4t + 3) - 2 = -1 = 4t + 4 \neq c(v_i) \) for \( 1 \leq i \leq r = 4t + 3 \). Hence \( c \) is a twin edge \((r + 2)\)-coloring of \( T \).

\textbf{Theorem 3.5.8} If \( T \) is a regular tree of order at least 3, then
\[ \chi'_t(T) \leq \Delta(T) + 2. \]

\textbf{Proof.} By Theorem 3.4.3, we may assume that \( \Delta(T) \geq 7 \). For a given integer \( r \geq 7 \), we proceed by induction on the number of vertices of degree \( r \) in an \( r \)-regular tree to show that every \( r \)-regular tree has a twin edge \((r + 2)\)-coloring. The star \( K_{1,r} \) is the only \( r \)-regular tree that has exactly one vertex of degree \( r \) and \( K_{1,r} \) has a twin edge \((r + 2)\)-coloring by Lemma 3.5.7. Assume that if \( T^* \) is an \( r \)-regular tree having exactly \( k-1 \) vertices of degree \( r \) for some integer \( k \geq 2 \), then \( T^* \) has a twin edge \((r+2)\)-coloring.

Now, let \( T \) be an \( r \)-regular tree having exactly \( k \) vertices of degree \( r \). Then \( T \) contains a vertex \( v \) of degree \( r \) such that \( v \) is adjacent to exactly \( r-1 \) end-vertices and exactly one non-end-vertex. Let \( w \in V(T) \) for which \( vw \in E(T) \) and \( \deg w = r \). Next, let \( T' = T - (N(v) - \{w\}) \) be the tree that is obtained from \( T \) by removing the \( r-1 \) end-vertices of \( T \) that are adjacent to \( v \). Then \( T' \) is an \( r \)-regular tree having exactly \( k-1 \) vertices of degree \( r \). Furthermore, \( v \) is an end-vertex in \( T' \) and \( w \) is the only vertex that is adjacent to \( v \) in \( T' \). By the induction hypothesis, \( T' \) has a twin edge \((r+2)\)-coloring \( c_0 : E(T') \to \mathbb{Z}_{r+2} \). Next, we extend the coloring \( c_0 \) to a twin edge coloring.
CHAPTER 3. TWIN EDGE COLORING CONJECTURE

$c : E(T) \to \mathbb{Z}_{r+2}$ of $T$ such that $c(e) = c_0(e)$ for each $e \in E(T')$ (and so $c'(x) = c_0(x)$ for all $x \in V(T') - \{v\}$). First, we verify the following claim.

**Claim.** For each $r \geq 7$, there are six distinct elements

\[ \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}_{r+2} \]

such that

\[
\begin{align*}
\sigma(0, r+1) - \alpha_1 - \alpha_2 &= \alpha_1 \quad \text{in } \mathbb{Z}_{r+2} \\
\sigma(0, r+1) - \beta_1 - \beta_2 &= \beta_1 \quad \text{in } \mathbb{Z}_{r+2} \\
\sigma(0, r+1) - \gamma_1 - \gamma_2 &= \gamma_1 \quad \text{in } \mathbb{Z}_{r+2}. 
\end{align*}
\]

(3.8) \hspace{1cm} (3.9) \hspace{1cm} (3.10)

To verify this claim, we consider four cases, according to whether $r$ is congruent to 0, 1, 2 or 3 modulo 4.

**Case 0.** $r \equiv 0 \pmod{4}$. Let $r = 4t$ for some integer $t \geq 2$. Then $\sigma(0, 4t+1) = 2t + 1$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+2}$. If $t = 3$, then $4t + 2 = 14$ and

\[
\begin{align*}
\sigma(0, 4t+1) - 3 - 1 &= 3 \quad \text{in } \mathbb{Z}_{14} \\
\sigma(0, 4t+1) - 7 &= 0 \quad \text{in } \mathbb{Z}_{14} \\
\sigma(0, 4t+1) - 8 - 5 &= 8 \quad \text{in } \mathbb{Z}_{14}.
\end{align*}
\]

If $t = 7$, then $4t + 2 = 30$ and

\[
\begin{align*}
\sigma(0, 4t+1) - 7 - 1 &= 7 \quad \text{in } \mathbb{Z}_{30} \\
\sigma(0, 4t+1) - 21 - 3 &= 21 \quad \text{in } \mathbb{Z}_{30} \\
\sigma(0, 4t+1) - 20 - 5 &= 20 \quad \text{in } \mathbb{Z}_{30}.
\end{align*}
\]

If $t \geq 2$ and $t \neq 3, 7$, then

\[
\sigma(0, 4t+1) - t - 1 = t \quad \text{in } \mathbb{Z}_{4t+2} \\
\sigma(0, 4t+1) - 3t - 3 = 3t \quad \text{in } \mathbb{Z}_{4t+2} \\
\sigma(0, 4t+1) - (3t - 2) - 7 = 3t - 2 \quad \text{in } \mathbb{Z}_{4t+2}.
\]

**Case 1.** $r \equiv 1 \pmod{4}$. Let $r = 4t + 1$ for some integer $t \geq 2$. Since $\sigma(1, 4t + 2) = 0$ in $\mathbb{Z}_{4t+3}$, it follows that

\[
\begin{align*}
\sigma(0, 4t+2) - (2t + 2) - (4t + 2) &= 2t + 2 \quad \text{in } \mathbb{Z}_{4t+3} \\
\sigma(0, 4t+2) - (2t + 1) - 1 &= 2t + 1 \quad \text{in } \mathbb{Z}_{4t+3} \\
\sigma(0, 4t+2) - (2t) - 3 &= 2t \quad \text{in } \mathbb{Z}_{4t+3}.
\end{align*}
\]
Case 2. $r \equiv 2 \pmod{4}$. Let $r = 4t+2$ for some integer $t \geq 2$. Then $\sigma(0, 4t+3) = 2t+2$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+4}$. If $t = 2$, then $4t + 4 = 12$ and

\[
\begin{align*}
\sigma(0, 4t+3) - 3 - 0 & = 3 \quad \text{in } \mathbb{Z}_{12} \\
\sigma(0, 4t+3) - 4 - 10 & = 4 \quad \text{in } \mathbb{Z}_{12} \\
\sigma(0, 4t+3) - 5 - 8 & = 5 \quad \text{in } \mathbb{Z}_{12}.
\end{align*}
\]

If $t = 6$, then $4t + 4 = 28$ and

\[
\begin{align*}
\sigma(0, 4t+3) - 6 - 2 & = 6 \quad \text{in } \mathbb{Z}_{28} \\
\sigma(0, 4t+3) - 5 - 4 & = 5 \quad \text{in } \mathbb{Z}_{28} \\
\sigma(0, 4t+3) - 3 - 8 & = 3 \quad \text{in } \mathbb{Z}_{28}.
\end{align*}
\]

If $t \geq 3$ and $t \neq 6$, then

\[
\begin{align*}
\sigma(0, 4t+3) - t - 2 & = t \quad \text{in } \mathbb{Z}_{4t+4} \\
\sigma(0, 4t+3) - 3t - 6 & = 3t \quad \text{in } \mathbb{Z}_{4t+4} \\
\sigma(0, 4t+3) - (3t + 4) - (4t + 2) & = 3t + 4 \quad \text{in } \mathbb{Z}_{4t+4}.
\end{align*}
\]

Case 3. $r \equiv 3 \pmod{4}$. Let $r = 4t+3$ for some integer $t \geq 1$. Then $\sigma(0, 4t + 4) = 0$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+5}$. If $t = 5$, then $4t + 5 = 25$ and

\[
\begin{align*}
\sigma(0, 4t+4) - 10 - 5 & = 10 \quad \text{in } \mathbb{Z}_{25} \\
\sigma(0, 4t+4) - 11 - 3 & = 11 \quad \text{in } \mathbb{Z}_{25} \\
\sigma(0, 4t+4) - 2 - 21 & = 2 \quad \text{in } \mathbb{Z}_{25}.
\end{align*}
\]

If $t = 6$, then $4t + 5 = 29$ and

\[
\begin{align*}
\sigma(0, 4t+4) - 10 - 9 & = 10 \quad \text{in } \mathbb{Z}_{29} \\
\sigma(0, 4t+4) - 11 - 7 & = 11 \quad \text{in } \mathbb{Z}_{29} \\
\sigma(0, 4t+4) - 12 - 5 & = 12 \quad \text{in } \mathbb{Z}_{29}.
\end{align*}
\]

If $t \geq 1$ and $t \neq 5, 6$, then

\[
\begin{align*}
\sigma(0, 4t+4) - 2t - 5 & = 2t \quad \text{in } \mathbb{Z}_{4t+5} \\
\sigma(0, 4t+4) - 4t - 10 & = 4t \quad \text{in } \mathbb{Z}_{4t+5} \\
\sigma(0, 4t+4) - 2(t - 1) - 9 & = 2(t - 1) \quad \text{in } \mathbb{Z}_{4t+5}.
\end{align*}
\]
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Therefore, the claim holds; that is, for each integer \( r \geq 7 \), there are six distinct elements \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}_{r+2} \) that satisfy (3.8), (3.9) and (3.10), respectively.

We are now prepared to extend the coloring \( c_0 \) of \( T' \) to a twin edge coloring \( c : E(T) \to \mathbb{Z}_{r+2} \) of \( T \) such that \( c(e) = c_0(e) \) for each \( e \in E(T') \). Note that \( E(T) - E(T') = E_v - \{vw\} \), where \( E_v \) is the set of edges incident with \( v \) in \( T \) and \( |E_v - \{vw\}| = r - 1 \).

Let \( X = \{\alpha_1, \alpha_2\} \), \( Y = \{\beta_1, \beta_2\} \) and \( Z = \{\gamma_1, \gamma_2\} \) where \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}_{r+2} \) are described in (3.8), (3.9) and (3.10), respectively. Since \( |X| = |Y| = |Z| = 2 \) and \( X, Y \) and \( Z \) are pairwise disjoint, there is at least one of \( X, Y \) and \( Z \) that is disjoint from the set \( \{c'_0(w), c_0(vw)\} \). We may assume, without loss of generality, that \( X \cap \{c'_0(w), c_0(vw)\} = \emptyset \).

Define

\[
\{c(e) : e \in E_v\} = [0..r+1] - X = [0..r+1] - \{\alpha_1, \alpha_2\}
\]

where \( c(wv) = c_0(wv) \). Then \( c \) is a proper edge coloring of \( G \). By (3.8), it follows that \( c'(v) = \alpha_1 \neq c'_0(w) = c'(w) \) and so \( c' \) is a proper vertex coloring of \( G \). Therefore, \( c \) is a twin edge \((r+2)\)-coloring of \( G \).

By Theorems 3.5.3 and 3.5.8, if \( T \) is a regular tree of order at least 6 such that \( \Delta(T) \equiv 1 \pmod{4} \), then \( \chi'_t(T) = \Delta(T) + 2 \). Furthermore, we saw in Corollary 3.5.6 that if \( T \) is a double star, then \( \chi'_t(T) = \Delta(T) + 2 \) if and only if \( T \) is an \( r \)-regular tree for some integer \( r \geq 5 \) with \( r \equiv 1 \pmod{4} \) where then \( r = \Delta(T) \). From the examples we are aware of, it suggests that Corollary 3.5.6 is true for all trees in general. In any case, this problem appears to be worthy of further study.
Chapter 4

On $k$-Ramsey Numbers of Graphs

4.1 Introduction

One of the best known areas of study within Extremal Graph Theory concerns the investigation of the Ramsey number $R(F,H)$ of two graphs $F$ and $H$, defined as the smallest positive integer $n$ for which every assignment of colors red and blue to the edges (a red-blue coloring) of the complete graph $K_n$ of order $n$ results in a subgraph isomorphic to $F$ all of whose edges are colored red (a red $F$) or a subgraph isomorphic to $H$ all of whose edges are colored blue (a blue $H$). It is known that the Ramsey number exists for every two graphs although the exact value of this number is known for relatively few pairs of graphs.

A related concept is the bipartite Ramsey number $BR(F,H)$ of two bipartite graphs $F$ and $H$, defined (see [4]) as the smallest positive integer $r$ such that every red-blue coloring of the $r$-regular complete bipartite graph $K_{r,r}$ results in a red $F$ or a blue $H$. In [29], it was shown that the bipartite Ramsey number exists for every two bipartite graphs. Consequently, if $BR(F,H) = r$ for bipartite graphs $F$ and $H$, then every red-blue coloring of $K_{r,r}$ results in a red $F$ or a blue $H$ while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red $F$ nor a blue $H$. This led the authors in [3] to consider what might occur for red-blue colorings of the intermediate graph $K_{r-1,r}$ and to introduce a more general concept.

For two bipartite graphs $F$ and $H$ and an integer $k$ with $2 \leq k \leq R(F,H)$, the $k$-Ramsey number $R_k(F,H)$ of $F$ and $H$ is the smallest order of a balanced complete $k$-partite graph $G$ (in which the sizes of every two partite sets differ by at most 1) such that every red-blue coloring of $G$ results in a red $F$ or a blue $H$. The spanning subgraphs $G_R$ and $G_B$ of $G$ consist of the red and blue edges, respectively, of $G$. For example, it
is known that $R(C_4, C_4) = 6$ (see [15]) and so it suffices to investigate $R_k(C_4, C_4)$ where $2 \leq k \leq 5$. In fact, the following is established in [3].

**Theorem 4.1.1** For every integer $k$ with $2 \leq k \leq 6 = R(C_4, C_4)$,

$$R_k(C_4, C_4) = 12 - k.$$  

By Theorem 4.1.1, for each integer $k$ with $2 \leq k \leq 6$, the smallest order of a balanced complete $k$-partite graph for which every red-blue coloring produces a monochromatic $C_4$ is $12 - k$. A formula for $R_k(F, H)$ was obtained in [3] when $F$ and $H$ are both stars.

**Theorem 4.1.2** Let $k, s$ and $t$ be integers with $3 \leq k < R(K_{1,s}, K_{1,t})$ and $s + t \geq 5$.

(a) If $s + t - 2 = (k - 1)q$ for some positive integer $q$, then

$$R_k(K_{1,s}, K_{1,t}) = \begin{cases} 
  kq & \text{if } k \text{ and } q \text{ are odd and } s \text{ and } t \text{ are even} \\
  kq + 1 & \text{otherwise}.
\end{cases}$$

(b) If $s + t - 2 = (k - 1)q + r$ for integers $q$ and $r$ where $q \geq 1$ and $1 \leq r \leq k - 2$, then

$$R_k(K_{1,s}, K_{1,t}) = \begin{cases} 
  kq + r & \text{if } (k - r)q \text{ is odd and } s \text{ and } t \text{ are of opposite parity} \\
  kq + r + 1 & \text{otherwise}.
\end{cases}$$

By Theorem 4.1.2, when $3 \leq k < R(K_{1,s}, K_{1,t})$ and $s + t \geq 5$, it follows that

$$R_k(K_{1,s}, K_{1,t}) \text{ is either } s + t - 2 + \left\lfloor \frac{s + t - 2}{k - 1} \right\rfloor \text{ or } s + t - 1 + \left\lfloor \frac{s + t - 2}{k - 1} \right\rfloor,$$

depending on the values of $k, s$ and $t$.

In this work, we continue the study of $k$-Ramsey numbers. First, we present some preliminary results that will be useful to us.

**Proposition 4.1.3** Let $F$ and $H$ be two bipartite graphs. For every integer $k$ with $2 \leq k \leq R(F, H)$,

$$R_k(F, H) \geq R(F, H).$$
**Proposition 4.1.4** Let \( F \) and \( H \) be two bipartite graphs. For positive integers \( k \) and \( \ell \) with \( k \geq 2 \),

\[
R_{\ell k}(F, H) \leq R_k(F, H).
\]

**Proof.** Let \( R_k(F, H) = p \), let \( G_k \) be a balanced complete \( k \)-partite graph of order \( p \) and let \( G_{\ell k} \) be a balanced complete \((\ell k)\)-partite graph of order \( p \). Then \( G_k \) is a subgraph of \( G_{\ell k} \). Let there be given a red-blue coloring of \( G_{\ell k} \). This coloring then gives rise to a red-blue coloring of the subgraph \( G_k \) of \( G_{\ell k} \). Since \( R_k(F, H) = p \), it follows that \( G_k \) contains a red \( F \) or a blue \( H \) and so does \( G_{\ell k} \). Therefore, \( R_{\ell k}(F, H) \leq p = R_k(F, H) \).

**Proposition 4.1.5** Let \( F \) and \( H \) be two bipartite graphs. If \( k \) is an integer with \( k \leq R(F, H) \) for which \( R_k(F, H) = R(F, H) \) and \( \frac{R_k(F, H) - 1}{k} \leq 2 \), then \( R_\ell(F, H) = R_k(F, H) \) for each integer \( \ell \) with \( k \leq \ell \leq R(F, H) \).

**Proof.** Suppose that \( R_k(F, H) = R(F, H) = p \) and \((p-1)/k \leq 2 \). Thus \( p \leq 2k + 1 \). Let \( \ell \) be an integer with \( k \leq \ell \leq p \). We show that \( R_\ell(F, H) = R_k(F, H) \). Since \( R_\ell(F, H) = p \) if \( \ell = k \) or \( \ell = p \), we may assume that \( k < \ell < p \). By Proposition 4.1.3, \( R_\ell(F, H) \geq p \). Thus it remains to show that \( R_\ell(F, H) \leq p \). Let \( G \) be a balanced complete \( \ell \)-partite graph of order \( p \) and let \( G' \) be a balanced complete \( k \)-partite graph of order \( p \). Then \( G' \) is isomorphic to a subgraph of \( G \). Let there be given a red-blue coloring of \( G \) which then induces a red-blue coloring of \( G' \). Because \( R_k(F, H) = p \), every red-blue coloring of \( G' \) results in a red \( F \) or a blue \( H \) in \( G' \) and therefore in \( G \) as well. Hence \( R_\ell(F, H) \leq p \) and so \( R_\ell(F, H) = p \).
For a complete $k$-partite graph $G$ in which $k_i$ partite sets consist of $n_i$ vertices, where $1 \leq i \leq p$ for some positive integer $p$, and $n_1 > n_2 > \cdots > n_p$, we write $G = K_{k_1(n_1), k_2(n_2), \ldots, k_p(n_p)}$ where $k_i(n_i)$ is simply denoted by $n_i$ if $k_i = 1$ and $k = k_1 + k_2 + \ldots + k_p$.

4.2 On $k$-Ramsey Numbers $R_k(F, H)$ of Stripes

In this section, we investigate the $k$-Ramsey numbers $R_k(F, H)$ for certain stripes $F$ and $H$ (1-regular graphs) and for certain values of $k$. The stripe of size $r$ is denoted by $rK_2$ and consists of $r$ copies of the complete graph $K_2$, whose edges therefore form a matching of size $r$. The Ramsey number and bipartite Ramsey number of two stripes are known and were determined in [21] and [19], respectively.

**Theorem 4.2.1** [21] For integers $s$ and $t$ with $2 \leq s \leq t$,

$$R(sK_2, tK_2) = s + 2t - 1.$$  

**Theorem 4.2.2** [19] For integers $s$ and $t$ with $2 \leq s \leq t$,

$$BR(sK_2, tK_2) = s + t - 1.$$  

Another useful fact concerns the matching number (the largest size of a matching) in a balanced complete multipartite graph (see [46], for example).

**Theorem 4.2.3** The matching number of a balanced complete $k$-partite graph, $k \geq 2$, of order $n$ is $\lfloor n/2 \rfloor$.

By Theorem 4.2.1 and Proposition 4.1.3, for integers $k$, $s$ and $t$ with $2 \leq s \leq t$ and $2 \leq k \leq R(sK_2, tK_2)$, it follows that

$$R_k(sK_2, tK_2) \geq s + 2t - 1. \quad (4.1)$$

If the bipartite Ramsey number $BR(F, H)$ of two bipartite graphs $F$ and $H$ is $r$, then every red-blue coloring of $K_{r,r}$ produces a red $F$ or a blue $H$. Furthermore, there exists a red-blue coloring of $K_{r-1,r-1}$ that produces neither. Which of these two situations occurs for the graph $K_{r-1,r}$ depends on the graphs $F$ and $H$. That is, $R_2(F, H) = 2r$ or $R_2(F, H) = 2r - 1$. In the case of stripes, $R_2(sK_2, tK_2) = 2BR(sK_2, tK_2)$. 
Proposition 4.2.4  For integers \( s \) and \( t \) with \( 2 \leq s \leq t \),
\[
R_2(sK_2, tK_2) = 2s + 2t - 2.
\]

Proof.  By Proposition 4.1.3, \( R_2(sK_2, tK_2) = 2s + 2t - 2 \) or \( R_2(sK_2, tK_2) = 2s + 2t - 3 \). Let \( H = K_{s+t-2,s+t-1} \) and let \( U = \{ u_1, u_2, \ldots, u_{s+t-2} \} \) be the partite set of \( H \) with \( s + t - 2 \) vertices. By assigning the color red to every edge incident with one of \( u_1, u_2, \ldots, u_{s-1} \) and coloring all other edges of \( H \) blue, we see that \( H \) has no red \( sK_2 \) and no blue \( tK_2 \). Consequently, \( R_2(sK_2, tK_2) > 2s + 2t - 3 \) and so \( R_2(sK_2, tK_2) = 2s + 2t - 2 \). \( \blacksquare \)

4.2.1  The \( k \)-Ramsey Number \( R_k(2K_2, tK_2) \)

We first determine the value of \( R_k(2K_2, tK_2) \) for all integers \( k \) and \( t \) for which \( 2 \leq k \leq R(2K_2, tK_2) \). By Theorem 4.2.1, \( R(2K_2, tK_2) = 2t + 1 \). Hence we present the value of \( R_k(2K_2, tK_2) \) for \( 2 \leq k \leq 2t + 1 \). By Proposition 4.2.4, \( R_2(2K_2, tK_2) = 2t + 2 \) and since \( R_{2t+1}(2K_2, tK_2) = R(2K_2, tK_2) \), it remains to determine \( R_k(2K_2, tK_2) \) for \( 3 \leq k \leq 2t \).

Theorem 4.2.5  For every two integers \( t \) and \( k \) such that \( t \geq 2 \) and \( 3 \leq k \leq 2t \),
\[
R_k(2K_2, tK_2) = 2t + 1.
\]

Proof.  By Proposition 4.1.3, \( R_k(2K_2, tK_2) \geq R(2K_2, tK_2) = 2t + 1 \) and so it suffices to show that every red-blue coloring of the balanced complete \( k \)-partite graph \( G \) of order \( 2t + 1 \) results in either a red \( 2K_2 \) or a blue \( tK_2 \).

Let there be given a red-blue coloring of \( G \) for which there is no red \( 2K_2 \). Since the matching number of \( G \) is \( t \), there is a maximum matching \( M \) of size \( t \). Let \( w \) be the vertex of \( G \) that is incident with no edge in \( M \). If no edge of \( M \) is colored red, then \( G \) contains a blue \( tK_2 \). Otherwise, \( M \) contains exactly one red edge, say \( uw \). At least one of \( uw \) and \( vw \) is an edge of \( G \). If either \( uw \) or \( vw \) is an edge of \( G \) and is colored blue, then \( uv \) can be replaced in \( M \) by such an edge to produce a blue \( tK_2 \) in \( G \). Hence, if \( G \) contains \( uw \) or \( vw \), then this edge must be colored red.

First, suppose that both \( uw \) and \( vw \) are red and so \( G_R = K_3 \). Let \( u'v' \) be a blue edge of \( M \). We may assume that \( u \) and \( u' \) belong to different partite sets, so do \( v \) and \( v' \). If either \( uu' \) or \( vv' \) is red, then \( G \) contains a red \( 2K_2 \). Consequently, \( uu' \) and \( vv' \) are blue edges. Replacing \( u'v' \) and \( uw \) in \( M \) by \( uu' \) and \( vv' \) produces a blue \( tK_2 \).

Hence we may assume that \( u \) or \( v \) is in the same partite set \( V_1 \) as \( w \), say \( v \in V_1 \). Since \( G \) is a balanced complete \( k \)-partite graph where \( k \geq 3 \), there must be a blue edge
Corollary 4.2.6 For integers \( k \) and \( t \) with \( 2 \leq k \leq R(2K_2, tK_2) \) and \( t \geq 2 \),

\[
R_k(2K_2, tK_2) = \begin{cases} 
2t + 2 & \text{if } k = 2 \\
2t + 1 & \text{otherwise.}
\end{cases}
\]

4.2.2 The 3-Ramsey Number \( R_3(sK_2, tK_2) \)

We now establish the value of \( R_k(sK_2, tK_2) \) for \( k = 3 \) and all integers \( s \) and \( t \) with \( 2 \leq s \leq t \). We begin with the following lemma.

Lemma 4.2.7 Let \( s, t \) and \( k \) be integers with \( 2 \leq s \leq t \) and \( 3 \leq k \leq s + 2t - 4 \). If \( R_k((s-1)K_2, (t-1)K_2) = s+2t-4 \), then every red-blue coloring of the balanced complete \( k \)-partite graph \( G \) of order \( s+2t-1 \) produces either (i) a red \( sK_2 \) or a blue \( tK_2 \) or (ii) a monochromatic subgraph induced by three partite sets of largest order in \( G \).

Proof. Let \( G \) be a balanced complete \( k \)-partite graph of order \( s+2t-1 \) and let there be given a red-blue coloring of \( G \) in which there is neither a red \( sK_2 \) nor a blue \( tK_2 \). Furthermore, let \( V_1, V_2, V_3 \) be three largest partite sets of \( G \). Necessarily, \(|V_i| \geq 2\) for \( i = 1, 2, 3 \). Let \( F = G[V_1 \cup V_2 \cup V_3] \) be the complete 3-partite subgraph of \( G \) induced by \( V_1 \cup V_2 \cup V_3 \). If \( F \) is a blue subgraph, then (ii) occurs. Thus, we may assume that \( F \) contains a red edge, say \( v_1v_2 \) is a red edge, where \( v_i \in V_i \) for \( i = 1, 2 \). Let \( V_3 = \{w_1, w_2, \ldots, w_p\} \) and let \( G_i = G - \{v_1, v_2, w_i\} \) for \( i = 1, 2, \ldots, p \). Then \( G_i \) is a balanced complete \( k \)-partite graph of order \( s+2t-4 \). Since \( R_k((s-1)K_2, (t-1)K_2) = s+2t-4 \), it follows that \( G_i \) contains a red \((s-1)K_2\) or a blue \((t-1)K_2\). Because \( v_1v_2 \) is red and \( G \) contains no red \( sK_2 \), it follows that each \( G_i \) contains a blue \((t-1)K_2\). This implies that \( v_1w_i \) and \( v_2w_i \) are red edges for \( i = 1, 2, \ldots, p \). Since \( v_1w_i \) and \( v_2w_i \) are red for each \( i \) \((1 \leq i \leq p)\), we can proceed as above to show that every edge in \( F \) is red. In any case, \( F \) is a monochromatic subgraph of \( G \).
Theorem 4.2.8  For each integer $s \geq 2$,

$$R_3(sK_2, tK_2) = s + 2t - 1$$

for all integers $t$ with $t \geq s$.

Proof. We proceed by induction on $s$. By Corollary 4.2.6, the result is true for $s = 2$. Assume for an integer $s - 1 \geq 2$ that $R_3((s - 1)K_2, rK_2) = (s - 1) + 2r - 1$ for all integers $r$ with $r \geq s - 1$. We show that $R_3(sK_2, tK_2) = s + 2t - 1$ for all integers $t$ with $t \geq s$.

By Theorem 4.2.1 and Proposition 4.1.3, $R_3(sK_2, tK_2) \geq R(sK_2, tK_2) = s + 2t - 1$. It remains to show that $R_3(sK_2, tK_2) \leq s + 2t - 1$. By assumption, $R_3((s - 1)K_2, (t - 1)K_2) = s + 2t - 4$ since $t - 1 \geq s - 1 \geq 2$. Let there be given a red-blue coloring of the balanced complete 3-partite graph $G$ of order $s + 2t - 1$. By Lemma 4.2.7, either (i) there is a red $sK_2$ or a blue $tK_2$ in $G$ or (ii) every edge of $G$ has the same color. Should (i) occur, the proof is complete. Hence we may assume that (ii) occurs. Since the matching number of $G$ is $\lceil(s + 2t - 1)/2\rceil = t + \lceil(s - 1)/2\rceil \geq t$, it follows that $G$ has a monochromatic $tK_2$. Because $s \leq t$, it follows that $G$ has a red $sK_2$ or a blue $tK_2$ and so $R_3(sK_2, tK_2) \leq s + 2t - 1$. Therefore, $R_3(sK_2, tK_2) = s + 2t - 1$ for every two integers $s$ and $t$ with $2 \leq s \leq t$.

4.2.3 The 4-Ramsey Number $R_4(sK_2, tK_2)$

Next, we establish the value of $R_4(sK_2, tK_2)$ for all integers $s$ and $t$ with $2 \leq s \leq t$. For two partite sets $U$ and $W$ of a complete multipartite graph $G$, we denote the set of edges in $G$ joining $U$ and $W$ by $[U, W]$.

Theorem 4.2.9  For each integer $s \geq 2$,

$$R_4(sK_2, tK_2) = s + 2t - 1$$

for all integers $t$ with $t \geq s$.

Proof. We proceed by induction on $s$. By Corollary 4.2.6, the result is true for $s = 2$. Assume for an integer $s - 1 \geq 2$ that $R_4((s - 1)K_2, rK_2) = (s - 1) + 2r - 1$ for all integers $r$ with $r \geq s - 1$. We show that $R_4(sK_2, tK_2) = s + 2t - 1$ for all integers $t$ with $t \geq s$.

By Theorem 4.2.1 and Proposition 4.1.3, $R_4(sK_2, tK_2) \geq R(sK_2, tK_2) = s + 2t - 1$. It remains to show that $R_4(sK_2, tK_2) \leq s + 2t - 1$. By assumption, $R_4((s - 1)K_2, (t - 1)K_2) = s + 2t - 4$ since $t - 1 \geq s - 1 \geq 2$. Let there be given a red-blue coloring of
the balanced complete 4-partite graph $G$ of order $s + 2t - 1$. By Lemma 4.2.7, either
(i) there is a red $sK_2$ or a blue $tK_2$ in $G$ or (ii) the subgraph induced by three partite
sets of largest order in $G$ is a monochromatic subgraph. Assume that there is neither a
red $sK_2$ nor a blue $tK_2$ and so (ii) occurs. Let $V_1, V_2, V_3, V_4$ be the four partite sets of
$G$ where $|V_1| \geq |V_2| \geq |V_3| \geq |V_4|$ and let $s + 2t - 1 = 4q + \ell$, where $\ell = 0, 1, 2, 3$. We
consider two cases, according to where $\ell \neq 3$ or $\ell = 3$.

**Case 1.** $\ell \neq 3$. Then $|V_3| = |V_4|$. Thus $F_1 = G[V_1 \cup V_2 \cup V_3]$ and $F_2 = G[V_1 \cup V_2 \cup V_4]$ are
monochromatic. Since $[V_1, V_2]$ belongs to both $E(F_1)$ and $E(F_2)$, it follows that $F_1$
and $F_2$ have the same color. Thus $G - [V_3, V_4]$ is monochromatic.

**Subcase 1.1.** $\ell = 0$. Then $|V_i| = q$ for $i = 1, 2, 3, 4$. Let $M_1$ be a maximum matching
between $[V_1, V_3]$ and $M_2$ a maximum matching between $[V_2, V_4]$. Then $M = M_1 \cup M_2$
is a maximum matching of size $2q = 4q/2 = (s + 2t - 1)/2 = t + (s - 1)/2 \geq t \geq s$. Since
$M$ is monochromatic, $G$ contains a red $sK_2$ or a blue $tK_2$.

**Subcase 1.2.** $\ell = 1$. Then $|V_i| = q + 1$ and $|V_i| = q$ for $i = 2, 3, 4$. Let $M_1$ be
a maximum matching between $[V_1, V_3]$ and $M_2$ a maximum matching between $[V_2, V_4]$. Then $M = M_1 \cup M_2$ is a maximum matching of size $2q$. Since $2q = 4q/2 = (s + 2t - 2)/2 =
t + (s - 2)/2 \geq t \geq s$ and $M$ is monochromatic, $G$ contains a red $sK_2$ or a blue $tK_2$.

**Subcase 1.3.** $\ell = 2$. Then $|V_i| = |V_i| = q + 1$ and $|V_3| = |V_4| = q$. Let $v_i \in V_i$ for
$i = 1, 2$. Let $M_1$ be a maximum matching between $[V_1 - \{v_1\}, V_3]$ and $M_2$ a maximum
matching between $[V_2 - \{v_2\}, V_4]$. Then $M = M_1 \cup M_2 \cup \{v_1v_2\}$ is a maximum matching
of size $2q + 1$. Since $2q + 1 = (4q + 2)/2 = (s + 2t - 1)/2 = t + (s - 1)/2 \geq t \geq s$ and $M$
is monochromatic, $G$ contains a red $sK_2$ or a blue $tK_2$.

**Case 2.** $\ell = 3$. Then $|V_i| = q + 1$ for $i = 1, 2, 3$ and $|V_4| = q$. In this case, $F = G[V_1 \cup V_2 \cup V_3]$ is monochromatic. First, we show that $F$ is blue. If this is not the
case, then $F$ contains a red matching of size $[(3q + 3)/2]$. First, suppose that $3q + 3$ is
even and so $[(3q + 3)/2] = (3q + 3)/2$. Since $12q + 12 = 3(4q + 3) + 3 = 3s + 6t \geq 9s > 8s$,
it follows that $3q + 3 > 2s$ and so $G$ has a red $sK_2$, a contradiction. Next, suppose that $3q + 3$ is odd and so $[(3q + 3)/2] = (3q + 2)/2$. Since $12q + 8 = 3(4q + 3) - 1 = 3s + 6t - 4 \geq 9s - 4 = 8s + (s - 4)$, it follows that $3q + 2 \geq 2s + \frac{s - 4}{2}$. Since $s \geq 3$ and $3q + 2$ is an
integer, $3q + 2 \geq 2s$. Thus, $G$ has a red $sK_2$, a contradiction. Therefore, $F$ is blue, as
claimed.

Suppose that $|V(F)| = |V_1 \cup V_2 \cup V_3| \geq 2t$. Since the matching number of $F$ is $[|V(F)|/2]$, it follows that $F$ contains a matching of size at least $t$ and so $G$ contains
a blue $tK_2$, which is a contradiction. Next, suppose that $|V(F)| < 2t$. Since the order of $G$ is $s + 2t - 1$, it follows that $|V_4| = q \geq s$. Let $M_1$ be a maximum matching in the subgraph $K_{q,q+1}$ of $G$ having partite sets $V_3$ and $V_4$. Then $|M_1| = q$. Let $p$ be the number of red edges in $M_1$. Since $G$ contain no red $sK_2$, it follows that $p \leq s - 1$.

Let

$$V_1 = \{u_1, u_2, \ldots, u_{q+1}\}, \quad V_2 = \{v_1, v_2, \ldots, v_{q+1}\},$$
$$V_3 = \{w_1, w_2, \ldots, w_{q+1}\} \text{ and } V_4 = \{x_1, x_2, \ldots, x_q\},$$

where $M_1 = \{w_ix_i : 1 \leq i \leq q\}$ and the edges $w_ix_i$ are red for $1 \leq i \leq p$. Consider the matching $M_2 = \{u_iv_i : 1 \leq i \leq q + 1\}$ between $V_1$ and $V_2$. There are two subcases.

Subcase 2.1. $p$ is even. Let $p = 2a$ for some positive integer $a$. For each integer $i$ with $1 \leq i \leq a$, replace the edge $u_iv_i$ in $M_2$ by the two edges $u_iw_{2i-1}$ and $v_iw_{2i}$, getting the blue matching

$$M_3 = \{u_iv_i : a + 1 \leq i \leq q + 1\} \cup \{u_iw_{2i-1}, v_iw_{2i} : 1 \leq i \leq a\}.$$  

Thus $|M_3| = q + 1 + a = q + 1 + \frac{p}{2}$. With the blue edges $w_ix_i$ $(p + 1 \leq i \leq q)$ in $M_1$, we obtain a blue matching

$$M_4 = M_3 \cup \{w_ix_i : p + 1 \leq i \leq q\}$$

of size $q + 1 + \frac{p}{2} + (q - p) = 2q - \frac{p}{2} + 1$. Since $p \leq s - 1$, it follows that

$$4q - p + 2 = (4q + 3) - p - 1 = s + 2t - p - 2 \geq 2t - 1.$$  

Thus $|M_4| = 2q - \frac{p}{2} + 1 \geq \frac{2t - 1}{2} = t - \frac{1}{2}$. Because $|M_4|$ is an integer, $|M_4| \geq t$ and so $G$ contains a blue $tK_2$, which is impossible.

Subcase 2.2. $p$ is odd. Let $p = 2a + 1$ for some positive integer $a$. For each $i$ with $1 \leq i \leq a$, replace the edge $u_iv_i$ in $M_2$ by the two edges $u_iw_{2i-1}$ and $v_iw_{2i}$ and replace the edge $u_{a+1}v_{a+1}$ by the two edges $u_{a+1}w_{2a+1}$ and $v_{a+1}w_{q+1}$, getting the blue matching

$$M_3 = \{u_iv_i : a + 2 \leq i \leq q + 1\} \cup \{u_iw_{2i-1}, v_iw_{2i} : 1 \leq i \leq a\} \cup \{u_{a+1}w_{2a+1}, v_{a+1}w_{q+1}\}.$$  

Thus $|M_3| = q + 2 + a = q + 2 + \frac{p-1}{2}$. With the blue edges $w_ix_i$ $(p + 1 \leq i \leq q)$ in $M_1$, we obtain a blue matching

$$M_4 = M_3 \cup \{w_ix_i : p + 1 \leq i \leq q\}$$

of size $q + 2 + a + \frac{p}{2} + 1 = 2q - \frac{p-1}{2} + 1$. Since $p \leq s - 1$, it follows that

$$4q - p + 2 = (4q + 3) - p - 1 = s + 2t - p - 2 \geq 2t - 1.$$  

Thus $|M_4| = 2q - \frac{p-1}{2} + 1 \geq \frac{2t - 1}{2} = t - \frac{1}{2}$. Because $|M_4|$ is an integer, $|M_4| \geq t$ and so $G$ contains a blue $tK_2$, which is impossible.
Hence of size $q$.

Therefore, $R_4(sK_2, tK_2) \leq s + 2t - 1$ and so $R_4(sK_2, tK_2) = s + 2t - 1$.

We close this section with a conjecture on the value of the $k$-Ramsey number of two stripes.
Conjecture 4.2.10 For integers $k$, $s$ and $t$ with $2 \leq s \leq t$. If $3 \leq k \leq R(sK_2,tK_2)$, then

$$R_k(sK_2,tK_2) = s + 2t - 1.$$ 

We have seen in (4.1) that $R_k(sK_2,tK_2) \geq s + 2t - 1$ for all integers $k$ with $3 \leq k \leq R(sK_2,tK_2)$. Thus, by Proposition 4.1.4 and Theorems 4.2.8 and 4.2.9, to verify Conjecture 4.2.10, it suffices to establish the conjecture for primes $k$ with $k \geq 5$.

4.3 Remarks on $k$-Ramsey Numbers of Non-bipartite Graphs

While the $k$-Ramsey number $R_k(F,H)$ exists for every two bipartite graphs $F$ and $H$ when $2 \leq k \leq R(F,H)$, such is not the case when $F$ and $H$ are not bipartite. For graphs $F$ and $H$ that are not bipartite, not only does $R_2(F,H)$ fail to exist but $R_3(F,H)$ and $R_4(F,H)$ also do not exist. To see this, let $G$ be any balanced complete 3-partite graph with partite sets $V_1, V_2$ and $V_3$. Assigning the color red to every edge of $[V_1, V_2]$ and blue to all other edges of $G$ results in $G_R$ and $G_B$ both being bipartite. Similarly, if $G$ is a balanced complete 4-partite graph with partite sets $V_1, V_2, V_3$ and $V_4$ and the color red is assigned to every edge of $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4]$ and blue to all other edges of $G$, then $G_R$ and $G_B$ are both bipartite. Indeed, even if $\chi(F) = \chi(H) = 3$, $R_5(F,H)$ need not exist. For example, $R_5(K_3,K_3)$ does not exist. To see this, let $G$ be a balanced complete 5-partite graph with partite sets $V_i$ for $1 \leq i \leq 5$. If the edges in $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4] \cup [V_4, V_5] \cup [V_5, V_1]$ are colored red and all other edges are colored blue, then $G$ does not contain a monochromatic $K_3$. Consequently, $R_k(K_3,K_3)$ exists only when $k = R(K_3,K_3) = 6$. On the other hand, $R_5(F,H)$ can exist when $\chi(F) = \chi(H) = 3$.

The observation that for every graph $G$ of order 5, either $G$ or its complement $\overline{G}$ contains a triangle or a 5-cycle leads to the following observation.

Observation 4.3.1 Every red-blue coloring of $K_5$ produces either a monochromatic $C_3$ or a monochromatic $C_5$.

From our preceding discussion, the $k$-Ramsey number of two odd cycles does not exist when $k = 2, 3, 4$. Such may not be the case when $k = 5$, however.

Theorem 4.3.2 For integers $k$ and $\ell$ with $k \geq \ell \geq 2$, the 5-Ramsey number $R_5(C_{2\ell+1},C_{2k+1})$ exists.
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Proof. As we have seen, the bipartite Ramsey number \( BR(K_{k,k}, K_{k,k}) \) exists, say \( BR(K_{k,k}, K_{k,k}) = p_1 \). Proceeding recursively, we let

\[
BR(K_{p_i,p_i}, K_{p_i,p_i}) = p_{i+1} \text{ for } i = 1, 2, 3, 4
\]

and let \( G = K_{5(p_5)} \) with partite sets \( U_5, W_5, X_5, Y_5, Z_5 \). Let there be given a red-blue coloring of \( G \). We use the 10 pairs of elements of \( S = \{U, W, X, Y, Z\} \) as a guide to prove the theorem. We consider the partition \( \{S_1, S_2, S_3, S_4, S_5\} \) of the 2-element subsets of \( S \) as follows and use this to show that \( G \) contains a red \( C_{2k+1} \) or a blue \( C_{2k+1} \):

\[
S_1 = \{\{U, W\}, \{X, Y\}\}, \quad S_2 = \{\{U, X\}, \{Y, Z\}\},
\]

\[
S_3 = \{\{W, Y\}, \{X, Z\}\}, \quad S_4 = \{\{U, Y\}, \{W, Z\}\},
\]

\[
S_5 = \{\{W, X\}, \{U, Z\}\}.
\]

First, we employ the two pairs in \( S_1 \). Since \( BR(K_{p_4,p_4}, K_{p_4,p_4}) = p_5 \) and \( |U_5| = |W_5| = p_5 \) and \( |X_5| = |Y_5| = p_5 \), the complete bipartite graph \( K_{p_5,p_5} \) with partite sets \( U_5 \) and \( W_5 \) and the complete bipartite graph \( K_{p_5,p_5} \) with partite sets \( X_5 \) and \( Y_5 \) both contain a monochromatic subgraph \( K_{p_4,p_4} \) with partite sets \( U_4 \subseteq U_5 \) and \( W_4 \subseteq W_5 \) in the first graph and \( X_4 \subseteq X_5 \) and \( Y_4 \subseteq Y_5 \) in the second graph. Furthermore, let \( Z_4 \subseteq Z_5 \) such that \( |Z_4| = p_4 \).

Next, we employ the two pairs in \( S_2 \). Since \( BR(K_{p_3,p_3}, K_{p_3,p_3}) = p_4 \) and \( |U_4| = |X_4| = p_4 \) and \( |Y_4| = |Z_4| = p_4 \), the complete bipartite graph \( K_{p_4,p_4} \) with partite sets \( U_4 \) and \( X_4 \) and the complete bipartite graph \( K_{p_4,p_4} \) with partite sets \( Y_4 \) and \( Z_4 \) both contain a monochromatic subgraph \( K_{p_3,p_3} \) with partite sets \( U_3 \subseteq U_4 \) and \( X_3 \subseteq X_4 \) in the first graph and \( Y_3 \subseteq Y_4 \) and \( Z_3 \subseteq Z_4 \) in the second graph. Furthermore, let \( W_3 \subseteq W_4 \) such that \( |W_3| = p_3 \).

Continuing in this manner, we arrive at a subgraph \( H = K_{5(k)} \) of \( G \) with partite sets \( U_1, W_1, X_1, Y_1, Z_1 \) such that (i) \( U_1 \subseteq U_5, W_1 \subseteq W_5, X_1 \subseteq X_5, Y_1 \subseteq Y_5, Z_1 \subseteq Z_5 \) and \( |U_1| = |W_1| = |X_1| = |Y_1| = |Z_1| = k \) and (ii) for each pair among the sets \( U_1, W_1, X_1, Y_1, Z_1 \), the complete bipartite graph \( K_{k,k} \) with these partite sets is monochromatic.

By Observation 4.3.1, every red-blue coloring of \( K_5 \) has either a monochromatic \( C_3 \) or a monochromatic \( C_5 \). This implies that \( H \) has a monochromatic \( K_{3(k)} \) or a monochromatic \( C_{5(k)} \) (a graph obtained from the 5-cycle \( (v_1, v_2, v_3, v_4, v_5, v_1) \) by replacing each vertex \( v_i \) with a set \( V_i \) of \( k \) vertices for \( 1 \leq i \leq 5 \)) We consider these two cases.
Case 1. $H$ has a monochromatic $K_{3(k)}$, say with partite sets $X_1, Y_1, Z_1$. Let $x_1 \in X_1$, $Y_1 = \{y_1, y_2, \ldots, y_k\}$ and $Z_1 = \{z_1, z_2, \ldots, z_k\}$. Then

$C_{2k+1} = (x_1, y_1, z_1, y_2, z_2, \ldots, y_k, z_k, x_1)$

is a monochromatic $(2k+1)$-cycle and

$C_{2\ell+1} = (x_1, y_1, z_1, y_2, z_2, \ldots, y_\ell, z_\ell, x_1)$

is a monochromatic $(2\ell+1)$-cycle in $G$.

Case 2. $H$ has a monochromatic $C_{5(k)}$, say with edge set

$[U_1, W_1] \cup [W_1, X_1] \cup [X_1, Y_1] \cup [Y_1, Z_1] \cup [Z_1, X_1]$. 

Let $u_1 \in U_1, w_1 \in W_1, x_1 \in X_1$ and let $Y_1 = \{y_1, y_2, \ldots, y_k\}$ and $Z_1 = \{z_1, z_2, \ldots, z_k\}$. Then

$C_{2k+1} = (u_1, w_1, x_1, y_1, z_1, y_2, z_2, \ldots, y_k-1, z_k-1, u_1)$

is a monochromatic $(2k+1)$-cycle and

$C_{2\ell+1} = (u_1, w_1, x_1, y_1, z_1, y_2, z_2, \ldots, y_{\ell-1}, z_{\ell-1}, u_1)$

is a monochromatic $(2\ell+1)$-cycle in $G$. 

4.4 On $k$-Ramsey Numbers of Unicyclic-Star Graphs

We have seen that the $k$-Ramsey number $R_k(F, H)$ was determined when $F = H = C_4$ (Theorem 4.1.1) and when $F$ and $H$ are stars (Theorem 4.1.2). In Section 4.2, $R_k(F, H)$ was determined for certain stripes (1-regular graphs) $F$ and $H$ and for certain values of $k$. While for bipartite graphs $F$ and $H$, $R(F, H)$ always exists, as does $BR(F, H)$, such is not the case when $F$ and $H$ are not bipartite. Of course, $R(F, H)$ exists, regardless of whether $F$ and $H$ are bipartite. It was observed in Section 4.3, however, that if $F$ and $H$ are not bipartite, then $R_k(F, H)$ is not defined for $2 \leq k \leq 4$ and possibly for other values of $k$ as well. Thus far, no class of non-bipartite graphs has been investigated in this area of research. In this section, we consider one such class. A connected graph $G$ is unicyclic if $G$ contains exactly one cycle. For an integer $t \geq 3$, the unicyclic-star graph
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$U_t$ is the unicyclic graph containing the star $K_{1,t}$ as a spanning subgraph. Consequently, $U_t$, $t \geq 3$, is a connected graph of order and size $t + 1$, containing a single cycle, namely a triangle, two vertices of which have degree 2 in $U_t$ and the third vertex has degree $t$. The unicyclic-stars $U_t$ are shown in Figure 4.3 for $t = 3, 4, 5$. We show that the smallest value of $k$ for which the $k$-Ramsey number $R_k(U_t) = R_k(U_t, U_t)$ exists is 6 and determine $R_6(U_t)$ for each $t \geq 3$. We also determine the $k$-Ramsey numbers $R_k(U_t)$ for many other pairs $k, t$ of positive integers.

![Figure 4.3: The unicyclic-stars $U_t$ for $t = 3, 4, 5$](image)

We begin with the Ramsey numbers of the unicyclic-star graphs $U_t$ for $t \geq 3$. As indicated above, for a graph $F$, we write $R(F)$ to denote the Ramsey number $R(F, F)$ and write $R_k(F)$ to denote the $k$-Ramsey number $R_k(F, F)$.

**Proposition 4.4.1** For each integer $t \geq 3$, $R(U_t) = 2t + 1$.

**Proof.** First, we show that $R(U_t) \geq 2t + 1$. Let $H = K_{2t}$ be a complete graph of order $2t$. Partition the vertex set of $H$ into two sets $X$ and $Y$ with $|X| = |Y| = t$. The red-blue coloring that assigns the color red to each edge in $G[X]$ and $G[Y]$ and the color blue to each edge in $G[X, Y]$ produces neither a red $U_t$ nor a blue $U_t$ and so $R(U_t) \geq 2t + 1$.

Next, we show that $R(U_t) \leq 2t + 1$. Let $G = K_{2t+1}$ be a complete graph of order $2t+1$ and let there be given a red-blue coloring of $G$. For a vertex $v$ of $G$, either $\deg_{GR} v \geq t$ or $\deg_{GB} v \geq t$, say the former. Then $|N_R(v)| = \deg_{GR} v \geq t$. If there is a red edge in $G[N_R(v)]$, then there is a red $U_t$ in $G$. Thus we may assume each edge in $G[N_R(v)]$ is blue. If $|N_R(v)| \geq t + 1$, then there is a blue $U_t$ in $G[N_R(v)]$ and so in $G$ as well. Hence we may assume that $|N_R(v)| = |N_B(v)| = t$ and that each edge in $G[N_B(v)]$ is red. Now, $G[N_R(v)] \cong K_t$ and $G[N_B(v)] \cong K_t$. Let $x \in N_R(v)$ and $y \in N_B(v)$. If $xy$ is red, then $G[N_B(v)]$ and $xy$ produce a red $U_t$; while if $xy$ is blue, then $G[N_R(v)]$ and $xy$ produce a blue $U_t$. In either case, there is a monochromatic $U_t$ in $G$ and so $R(U_t) \leq 2t + 1$. 

Since each graph $U_t$ is not bipartite, $R_k(U_t)$ does not exist for $k = 2, 3, 4$ and for $t \geq 3$. We next show that $R_5(U_t)$ also does not exist.
Proposition 4.4.2  For each integer $t \geq 3$, $R_5(U_t)$ does not exist.

Proof. Let $G$ be a balanced complete 5-partite graph of arbitrarily large order and let $V_1, V_2, \ldots, V_5$ be the partite sets of $G$. Define a red-blue coloring of $G$ by assigning the color red to all edges in $[V_i, V_{i+1}]$ where $1 \leq i \leq 5$ and $V_6 = V_1$ and the color blue to the remaining edges of $G$. This is illustrated in Figure 4.4, where a solid line between $V_i$ and $V_{i+1}$ indicates that all edges in $[V_i, V_{i+1}]$ are red, while a dashed line between two sets indicates that all edges between these two sets are blue. Since there is neither a red $K_3$ nor a blue $K_3$, there is no monochromatic $U_t$. Thus $R_5(U_t)$ does not exist.

![Figure 4.4: The red-blue coloring in the proof of Proposition 4.4.2](image)

Since $R(U_3) = 7$ by Proposition 4.4.1, it follows that $R_7(U_3) = 7$. Consequently, by Proposition 4.4.2, $R_6(U_3)$ is the only $k$-Ramsey number of $U_3$ whose value is not known. In fact, $R_6(U_3) = 7$, which we now verify. First, we show that $R_6(U_3) \leq 7$. Let there be given a red-blue coloring of the balanced complete 6-partite graph $G$ of order 7. Since $G$ contains a complete subgraph of order 6 and $R(K_3) = 6$, it follows that there is a red $K_3$ or a blue $K_3$ in $G$, say the former. Denote the vertex set of this red $K_3$ by $X$ and let $Y = V(G) - X$. Thus the subgraph $G[Y]$ of $G$ induced by $Y$ is either $K_4$ or $K_4 - e$ for some edge $e$. If there is a red edge in $[X, Y]$, then there is a red $U_3$. Thus we may assume that all edges in $[X, Y]$ are blue. If there is a blue edge in $G[Y]$, then there is a blue $U_3$; while if all edges in $G[Y]$ are red, then there is a red $U_3$. In either case, there is a monochromatic $U_3$ and so $R_6(U_3) \leq 7$. Next, we show that $R_6(U_3) \geq 7$. Let $H$ be the balanced complete 6-partite graph of order 6 and so $H = K_6$. The red-blue coloring of $H$ whose red subgraph is $2K_3$ and whose blue subgraph is $K_{3,3}$ contains no monochromatic $U_3$. Thus, $R_6(U_3) \geq 7$ and so $R_6(U_3) = 7$. Therefore, $R_k(U_3) = 7$ for $k = 6, 7$, the two values of $k$ for which $R_k(U_3)$ exists.
In this section, all \(k\)-Ramsey numbers \(R_k(U_t)\) are determined for \(t = 4, 5\) and \(6 \leq k \leq 2t + 1\). In addition, for \(t \geq 6\), we establish the following two results.

**Theorem 4.4.3** For integers \(k\) and \(t\) with \(t \leq k \leq R(U_t)\) and \(t \geq 6\),

\[
R_k(U_t) = \begin{cases} 
2t + 4 & \text{if } k = t = 7 \\
2t + 3 & \text{if (i) } t \text{ is odd and } k = t + 1 \text{ or (ii) } k = t \neq 7 \\
2t + 2 & \text{if } t \text{ is odd and } t + 2 \leq k \leq \lceil 3t/2 \rceil \\
2t + 1 & \text{otherwise.}
\end{cases}
\]

**Theorem 4.4.4** For each integer \(t \geq 6\),

\[
R_6(U_t) = \begin{cases} 
3t - 3 & \text{if } t \text{ is even} \\
3t - 2 & \text{if } t \text{ is odd.}
\end{cases}
\]

While the \(k\)-Ramsey numbers \(R_k(U_t)\) do not exist for \(2 \leq k \leq 5\) and \(t \geq 3\), Theorem 4.4.4 implies that \(R_k(U_t)\) exists for all integers \(k\) and \(t\) with \(6 \leq k \leq R(U_t)\) and \(t \geq 3\). To see this, let \(p = \lceil R_6(U_t)/6 \rceil\). Then every red-blue coloring of the regular complete 6-partite graph \(K_{6(p)}\) produces a monochromatic \(U_t\). For each integer \(k > 6\), the regular complete \(k\)-partite graph \(K_{k(p)}\) contains \(K_{6(p)}\) as a subgraph and so every red-blue coloring of \(K_{k(p)}\) produces a monochromatic \(U_t\). Therefore, \(R_k(U_t) \leq k[R_6(U_t)/6]\) and \(R_k(U_t)\) exists. As indicated in Theorem 4.4.3, for \(k > 6\), the \(k\)-Ramsey number \(R_k(U_t)\) is considerably smaller than \(k[R_6(U_t)/6]\).

**4.4.1 The \(k\)-Ramsey Numbers \(R_k(U_t)\) for \(t + 1 \leq k \leq R(U_t)\)**

We first determine \(R_k(U_t)\) for those integers \(k\) that exceed \(t\), namely all values of \(k\) with \(\max\{6, t + 1\} \leq k \leq R(U_t)\).

We begin with the \((t + 1)\)-Ramsey number \(R_{t+1}(U_t)\). By Proposition 4.4.2, we may assume that \(t \geq 5\). For the unicyclic-star graph \(U_t\) where \(t \geq 3\), the central vertex of the star \(K_{1,t}\) in \(U_t\) is also referred to as the central vertex (or the center) of \(U_t\).

**Theorem 4.4.5** For each integer \(t \geq 5\),

\[
R_{t+1}(U_t) = \begin{cases} 
2t + 3 & \text{if } t \text{ is odd} \\
2t + 1 & \text{if } t \text{ is even.}
\end{cases}
\]
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Proof. We first consider the case when $t$ is odd.

Case 1. $t \geq 5$ is odd. First, we show that $R_{t+1}(U_t) \geq 2t + 3$. Let $H = K_{(t+1)(2)}$ be the balanced complete $(t+1)$-partite graph of order $2t + 2$ each of whose partite sets $V_1, V_2, \ldots, V_{t+1}$ has two vertices. Let $X$ be the union of the $(t+1)/2$ partite sets $V_i$ for $1 \leq i \leq (t+1)/2$ and let $Y = V(H) - X$. Therefore, $|X| = |Y| = t + 1$ and $H[X] \cong H[Y] \cong K_{(t+1)/2}$. Define a red-blue coloring of $H$ by assigning the color red to each edge in $[X,Y]$ and the color blue to the remaining edges of $H$. Since the red subgraph $H_R$ is a bipartite graph with partite sets $X$ and $Y$, there is no red $U_t$. The blue subgraph $H_B = H[X] + H[Y]$ and so $\Delta(H_B) = t - 1$. Thus there is no blue $U_t$. Therefore, $R_{t+1}(U_t) \geq 2t + 3$.

Next, we show that $R_{t+1}(U_t) \leq 2t + 3$. Let $G$ be the balanced complete $(t+1)$-partite graph of order $2t + 3$ and so $G = K_{3t(2)}$. Let $W = \{w, w', w''\}$ be the partite set of cardinality 3 in $G$. Suppose that there is a red-blue coloring of $G$ containing neither a red $U_t$ nor a blue $U_t$. Since $\deg w = 2t$, either $|N_R(w)| = \deg G_R w \geq t$ or $|N_B(w)| = \deg G_B w \geq t$, say the former. If there is a red edge in $G[N_R(w)]$, then there is a red $U_t$ in $G$. Thus, each edge in $G[N_R(w)]$ is blue and so $G[N_R(w)] \subseteq G_B$.

First, suppose that $|N_R(w)| \geq t + 1$. Then either (i) $G[N_R(w)]$ contains a blue $U_t$ or (ii) $\deg_{G[N_R(w)]} u = t - 1$ for every $u \in N_R(w)$. Since, by assumption, (i) does not occur, it follows that (ii) occurs. Thus $G[N_R(w)] \cong K_{t+1}(2)$ and $G[N_B(w)] \cong K_{t+1}(2)$. So each vertex in $G[N_R(w)]$ is the central vertex of a subgraph $U_{t-1}$ in $G[N_R(w)]$. If there is a blue edge $e$ in $[N_R(w), \{w', w''\} \cup N_B(w)]$, then $G[N_R(w)]$ and $e$ produces a blue $U_t$. Thus each edge in $[N_R(w), W \cup N_B(w)]$ is red. Since $|W \cup N_B(w)| = t + 2$, it follows that each edge in $G[W \cup N_B(w)]$ is blue. Observe that $G[W \cup N_B(w)] \cong K_{3t-1}(2)$ and so contains a blue $U_t$, which is impossible.

Next, suppose that $|N_R(w)| = t$ and so $|N_B(w)| = t$. Then each edge in $G[N_R(w)]$ is colored blue and each edge in $G[N_B(w)]$ is colored red. Since $t$ is odd, there is a vertex $u \in N_R(w)$ such that $\deg_{G[N_R(w)]} u = t - 1$. Hence $u$ is the central vertex of a subgraph $U_{t-1}$ in $G[N_R(w)]$. If $u$ is joined to a vertex in $N_B(w)$ by a blue edge $e$, then $G[N_R(w)]$ and $e$ produce a blue $U_t$, a contradiction. Thus, we may assume that each edge in $[\{u\}, N_B(w)]$ is red and $u$ is adjacent to $t - 1$ vertices in $N_B(w)$ by red edges (since there is a vertex $u' \in N_B(w)$ such that $\{u, u'\}$ is a partite set of $G$). Since $uw$ is red and $G[N_B(w)]$ is a red subgraph, there is a red $U_t$ in $G[N_B(w) \cup \{u, w\}]$ and so in $G$, which is impossible. Hence $R_{t+1}(U_t) \leq 2t + 3$ and so $R_{t+1}(U_t) = 2t + 3$ when $t \geq 5$ is odd.
Case 2. $t \geq 6$ is even. Since $R(U_t) = 2t + 1$ by Proposition 4.4.1, it suffices to show $R_{t+1}(U_t) \leq 2t + 1$ by Proposition 4.1.3. Let $G$ be a balanced complete $(t+1)$-partite graph of order $2t + 1$ and so $G = K_{t(t+1)}$. Suppose that there exists a red-blue coloring of $G$ containing neither a red $U_t$ nor a blue $U_t$. Let $w$ be the vertex of $G$ with $\deg w = 2t$.

Either $|N_R(w)| = \deg_{G_R} w \geq t$ or $|N_B(w)| = \deg_{G_B} w \geq t$, say the former. Thus each edge in $G[N_R(w)]$ is blue.

First, suppose that $|N_R(w)| \geq t + 1$. Since $t + 1$ is odd, there is a vertex $u \in N_R(w)$ such that $u$ is adjacent to at least $t$ vertices in $G[N_R(w)]$. This, however, implies that there is a blue $U_t$ in $G[N_R(w)]$, which is impossible. Next, suppose that $|N_R(w)| = t$ and so $|N_B(w)| = t$. Then $\delta(G[N_R(w)]) \geq t - 2$ and $\delta(G[N_B(w)]) \geq t - 2$. Furthermore, each edge in $G[N_R(w)]$ is colored blue (as we indicated above) and each edge in $G[N_B(w)]$ is colored red. For each $u \in N_R(w)$, there is at most one vertex $v \in N_B(w)$ such that $wv$ is blue; for otherwise, $G[N_R(w)]$ and two blue edges in $\{u\}, N_B(w)$ produce a blue $U_t$, a contradiction. This implies that there are vertices $v^* \in N_B(w)$ and $u_1, u_2 \in N_R(w)$ where $u_1 \neq u_2$ such that $u_1v^*$ and $u_2v^*$ are red. However then, $G[N_B(w)]$ and the red edges $u_1v^*$ and $u_2v^*$ produce a red $U_t$, a contradiction. Hence $R_{t+1}(U_t) \leq 2t + 1$ and so $R_{t+1}(U_t) = 2t + 1$ when $t \geq 6$ is even.

Next, we determine the $(t + 2)$-Ramsey number of $U_t$ for even integers $t \geq 4$.

Theorem 4.4.6 For each even integer $t \geq 4$, $R_{t+2}(U_t) = 2t + 1$.

Proof. Since $R(U_t) = 2t + 1$, it suffices to show $R_{t+2}(U_t) \leq 2t + 1$ by Proposition 4.1.3. Let $G = K_{t(t-1)(2),3(1)}$ be the balanced complete $(t+2)$-partite graph of order $2t + 1$.

Suppose that there exists a red-blue coloring of $G$ containing neither a red $U_t$ nor a blue $U_t$. Let $w$ be a vertex of $G$ with $\deg w = 2t$. Hence either $|N_R(w)| = \deg_{G_R} w \geq t$ or $|N_B(w)| = \deg_{G_B} w \geq t$, say the former. If there is a red edge in $G[N_R(w)]$, then there is a red $U_t$. This implies that each edge in $G[N_R(w)]$ is blue.

First, suppose that $|N_R(w)| \geq t + 1$. Since there is no blue $U_t$ in $G[N_R(w)]$, it follows that $\deg_{G[N_R(w)]} u = t - 1$ for each $u \in N_R(w)$. This implies that $|N_R(w)| = t + 1$. However then, $G[N_R(w)]$ is a $(t - 1)$-regular graph of order $t + 1$. Since $t$ is even, this is impossible. Thus we may assume $|N_R(w)| = t$ and so $|N_B(w)| = t$. Hence each edge in $G[N_R(w)]$ is blue and each edge in $G[N_B(w)]$ is red. Let $x$ be a vertex of $G$ such that $\deg x = 2t$ and $x \neq w$. Thus, either $x \in N_R(w)$ or $x \in N_B(w)$, say the former. Since $\deg_{G[N_R(w)]} x = t - 1$, it follows that $x$ is the center of some blue $U_{t-1}$ in $G[N_R(w)]$; which implies that each edge in $\{x\}, N_B(w)$ is red. However then, $G[N_B(w) \cup \{x\}]$ contains a red $U_t$, which is contradiction. Thus $R_{t+2}(U_t) \leq 2t + 1$ and so $R_{t+2}(U_t) = 2t + 1$.  

\[\blacksquare\]
Since \( R(U_t) = 2t + 1 \) for each integer \( t \geq 4 \) by Proposition 4.4.1, the following result is a consequence of Proposition 4.1.5 and Theorems 4.4.5 and 4.4.6. [Note that Propositions 4.1.3 and 4.1.5 also hold when \( F \) and \( H \) are not bipartite graphs.]

**Corollary 4.4.7** If \( k \) and \( t \) are integers where \( t \geq 4 \) is even and \( \max\{6, t + 1\} \leq k \leq R(U_t) \), then \( R_k(U_t) = 2t + 1 \).

Next, we determine the \( k \)-Ramsey numbers of \( R_k(U_t) \) where \( k = t + a \) for some odd integer \( t \geq 5 \) and some integer \( a \) with \( 2 \leq a \leq t \).

**Theorem 4.4.8** For an odd integer \( t \geq 5 \)

\[
R_k(U_t) = \begin{cases} 
2t + 2 & \text{if } t + 2 \leq k \leq (3t + 1)/2 \\
2t + 1 & \text{if } (3t + 3)/2 \leq k \leq 2t.
\end{cases}
\]

**Proof.** Let \( k = t + a \) where \( 2 \leq a \leq t \). We consider two cases, according to whether \( 2 \leq a \leq (t + 1)/2 \) or \( (t + 3)/2 \leq a \leq t \).

**Case 1.** \( 2 \leq a \leq (t + 1)/2 \). Then \( t \geq 2a - 1 \). First, we show that \( R_{t+a}(U_t) \geq 2t + 2 \). Let \( H = K_{(t-a+1)(2), (2a-1)(1)} \) be the balanced complete \((t + a)\)-partite graph of order \( 2t + 1 \) whose partite sets are \( V_1, V_2, \ldots, V_{t+a} \) where \( |V_i| = 2 \) for \( 1 \leq i \leq t - a + 1 \) and \( |V_j| = 1 \) for \( t - a + 2 \leq j \leq t + a \). Let \( X \) be the union of \((t + 1)/2\) partite sets \( V_i \) for \( 1 \leq i \leq (t + 1)/2 \leq t - a + 1 \) and let \( Y = V(H) - X \). Then \( H[X] \cong K_{a+1,2} \) and \( H[Y] \cong K_{(t-a+1)(2), (2a-1)(1)} \). Define a red-blue coloring by assigning the color red to each edge in \([X, Y]\) and the color blue to the remaining edges of \( H \). Since the red subgraph \( H_R \) is a bipartite graph with partite sets \( X \) and \( Y \), there is no red \( U_t \). The blue subgraph \( H_B = H[X] + H[Y] \) has \( \Delta(H_B) = t - 1 \) and contains no blue \( U_t \). Therefore, \( R_{t+a}(U_t) \geq 2t + 2 \).

Next, we show that \( R_{t+a}(U_t) \leq 2t + 2 \). Let \( G = K_{(t-a+2)(2), (2a-2)(1)} \) be the balanced complete \((t + a)\)-partite graph of order \( 2t + 2 \). Thus \( G \) has \( 2(t - a + 2) \) vertices of degree \( 2t \) and \( 2a - 2 \) vertices of degree \( 2t + 1 \). Let there be given a red-blue coloring of \( G \) containing neither a red \( U_t \) nor a blue \( U_t \) and let \( w \) be a vertex of \( G \) with \( \operatorname{deg}_G w = 2t + 1 \). Thus, either \( |N_R(w)| \geq t + 1 \) or \( |N_B(w)| \geq t + 1 \), say the former. If there is a red edge in \( G[N_R(w)] \), then there is a red \( U_t \) in \( G \). Thus, each edge in \( G[N_R(w)] \) is blue and so \( G[N_R(w)] \) is a subgraph of the blue subgraph \( G_B \) of \( G \). Observe that each vertex in \( G[N_R(w)] \) has degree at least \( t - 1 \) in \( G[N_R(w)] \). Since \( G[N_R(w)] \) contains no blue \( U_t \) by assumption and each
vertex in \(G[N_R(w)]\) lies on a triangle in \(G[N_R(w)]\), it follows that \(\deg_{G[N_R(w)]} u = t - 1\) for each \(u \in N_R(w)\). Thus \(G[N_R(w)] \cong K_{\frac{t+1}{2}}(2)\) and \(G[N_B(w)] \cong K_{\frac{t-2a+3}{2},(2a-3)(1)}\). Each vertex in \(G[N_R(w)]\) is the central vertex of a subgraph \(U_{t-1}\) in \(G[N_R(w)]\). If there is a blue edge \(e\) in \([N_R(w), N_B(w)]\), then \(G[N_R(w)]\) and \(e\) produce a blue \(U_t\). Thus each edge in \([N_R(w), N_B(w)]\) is red. Since \(|N_B(w)| = t\), it follows that each edge in \(G[N_B(w)]\) is blue. However then, \(G([w] \cup N_B(w)) \cong K_{\frac{t-2a+3}{2}(2), (2a-3)(1)}\) and \(G\) contains a blue \(U_t\), a contradiction. Hence \(R_{t+a}(U_t) \leq 2t + 2\) and so \(R_{t+a}(U_t) = 2t + 2\) when \(t \geq 5\) is odd and \(2 \leq a \leq (t+1)/2\).

Case 2. \((t+3)/2 \leq a \leq t\). Then \(5 \leq t \leq 2a - 3\). We show that \(R_{t+a}(U_t) = 2t + 1\). Since \(R_{t+a}(U_t) \geq R(U_t) = 2t + 1\) for each \(a \geq 2\) by Proposition 4.1.3, it suffices to show that \(R_{t+a}(U_t) \leq 2t + 1\). Let \(G = K_{(t-a+1)(2), (2a-1)(1)}\) be the balanced complete \((t+a)\)-partite graph of order \(2t + 1\). Thus \(G\) has \(2(t-a+1)\) vertices of degree \(2t-1\) and \(2a-1\) vertices of degree \(2t\). Let there be given a red-blue coloring of \(G\) containing neither a red \(U_t\) nor a blue \(U_t\) and let \(w\) be a vertex of \(G\) with \(\deg_G w = 2t\). Thus, either \(|N_R(w)| \geq t\) or \(|N_B(w)| \geq t\), say the former. If there is a red edge in \(G[N_R(w)]\), then there is a red \(U_t\) in \(G\). Thus, each edge in \(G[N_R(w)]\) is blue and so \(G[N_R(w)]\) is a subgraph of the blue subgraph \(G_B\) of \(G\).

First, suppose that \(|N_R(w)| \geq t + 1\). Since \(t \leq 2a - 3 < 2a - 2\), it follows that \(2(t-a+1) = t + (t-2a+2) \leq t+1 \leq |N_R(w)|\) and so there is a vertex \(u \in N_R(w)\) such that \(\deg_G u = 2t\). Hence \(u\) is adjacent to at least \(t\) vertices in \(N_R(w)\). Hence \(G[N_R(w)]\) contains a blue \(U_t\) centered at \(u\), a contradiction. Hence we may assume \(|N_R(w)| = t\) and \(|N_B(w)| = t\). Thus, each edge in \(G[N_R(w)]\) is blue and each edge in \(G[N_B(w)]\) is red. Now \(\delta(G[N_R(w)]) \geq t - 2\) and \(\delta(G[N_B(w)]) \geq t - 2\). Hence each vertex \(u \in N_R(w)\) is joined to at most one vertex \(v \in N_B(w)\) by a blue edge; for otherwise, \(G[N_R(w)]\) (together with these two blue edges in \([u, N_B(w)]\)) produces a blue \(U_t\), a contradiction. This implies that there is \(v \in N_B(w)\) and \(u_1, u_2 \in N_R(w)\) where \(u_1 \neq u_2\) such that \(u_1v\) and \(u_2v\) are red. Then \(G[N_B(w)]\) (together with \(u_1v\) and \(u_2v\)) produces a red \(U_t\), a contradiction. Hence \(R_{t+a}(U_t) \leq 2t + 1\) and so \(R_{t+a}(U_t) = 2t + 1\) when \(t \geq 5\) is odd and \((t+3)/2 \leq a \leq t\).

By Proposition 4.4.1, \(R(U_t) = 2t + 1\) for each integer \(t \geq 4\). The following result is therefore a consequence of Proposition 4.1.5, Theorem 4.4.5, Corollary 4.4.7 and Theorem 4.4.8.
Theorem 4.4.9  For integers $k$ and $t$ with $t \geq 4$ and $\max\{6, t + 1\} \leq k \leq R(U_t)$,

$$R_k(U_t) = \begin{cases} 
2t + 3 & \text{if } t \text{ is odd and } k = t + 1 \\
2t + 2 & \text{if } t \text{ is odd and } t + 2 \leq k \leq (3t + 1)/2 \\
2t + 1 & \text{otherwise.}
\end{cases}$$

Therefore, for $t = 3, 4, 5$ and $6 \leq k \leq R(U_t)$, it follows that

- $R_k(U_3) = R(U_3) = 7$ for $k = 6, 7$,
- $R_k(U_4) = R(U_4) = 9$ for $k = 6, 7, 8, 9$ and
- $R_6(U_5) = 13$, $R_k(U_5) = 12$ for $k = 7, 8$ and $R_k(U_5) = R(U_5) = 11$ for $k = 9, 10, 11$.

Hence in what follows, we assume that $6 \leq k \leq t$. The $k$-Ramsey numbers $R_k(U_t)$ in which $k = t$ are referred to as the diagonal $k$-Ramsey numbers.

4.4.2 The Diagonal $k$-Ramsey Numbers $R_k(U_k)$

In this subsection, we determine the diagonal $k$-Ramsey numbers $R_k(U_k)$ of the graph $U_k$ when $k \geq 6$. More precisely, we show for each integer $k \geq 6$ that $R_k(U_k) = 2k + 3$ if $k \neq 7$ and $R_k(U_k) = 2k + 4 = 18$ if $k = 7$. We begin with $R_7(U_7)$.

Proposition 4.4.10  $R_7(U_7) = 18$.

**Proof.**  First, we show that $R_7(U_7) \geq 18$. Let $H = K_{3(3), 4(2)}$ be the balanced complete 7-partite graph $H$ of order 17 having the partite sets $V_1, V_2, \ldots, V_7$, where $|V_i| = 3$ for $i = 1, 2, 3$ and and $|V_i| = 2$ for $i = 4, 5, 6, 7$. Let $X = V_1 \cup V_2 \cup V_3$ and let $Y = V(H) - X$. Then $H[X] \cong K_{3(3)}$ and $H[Y] \cong K_{4(2)}$. Define a red-blue coloring of $H$ by assigning the color red to each edge in $H[X]$ and each edge in $H[Y]$ and the color blue to the remaining edges of $H$. This is illustrated in Figure 4.5, where a solid line between two sets indicates that all edges between the sets are red, while dashed lines between $X$ and $Y$ indicate that all edges in the set $[X, Y]$ are blue. Then $H_R = H[X] + H[Y] \cong K_{3(3)} + K_{4(2)}$, the union of the graphs $H[X]$ and $H[Y]$, and $H_B = H - E(H_R) \cong K_{9, 8}$. Since $\Delta(H_R) = 6$ and $H_B$ is a bipartite graph with partite sets $X$ and $Y$, there is neither a red $U_7$ nor a blue $U_7$ in $H$. Hence $R_7(U_7) \geq 18$. 
Next, we show that \( R_7(U_7) \leq 18 \). Let \( G = K_{4(3),3(2)} \) be the balanced complete 7-partite graph of order 18 with partite sets \( W_1, W_2, \ldots, W_7 \), where \( |W_i| = 3 \) for \( i = 1, 2, 3, 4 \) and \( |W_i| = 2 \) for \( i = 5, 6, 7 \). Suppose that there exists a red-blue coloring of \( G \) containing neither a red \( U_7 \) nor a blue \( U_7 \). We may assume that \( \Delta(G_R) \geq \Delta(G_B) \). Since \( \Delta(G) = 16 \), it follows that \( \Delta(G_R) \geq 8 \). Let \( v \) be a vertex of \( G \) with \( \deg_{G_R} v = \Delta(G_R) \). Since \( G \) contains no red \( U_7 \), it follows that the subgraph \( G[N_R(v)] \) of \( G \) induced by \( N_R(v) \) is a subgraph of \( G_B \). If \( \Delta(G_R) \geq 10 \), then \( G[N_R(v)] \) contains \( U_7 \) as a subgraph and so \( G \) has a blue \( U_7 \), which is impossible. Thus either \( \Delta(G_R) = 9 \) or \( \Delta(G_R) = 8 \). We consider these two cases.

**Case 1.** \( \Delta(G_R) = 9 \). Since \( G[N_R(v)] \) has order 9 and \( G[N_R(v)] \) contains no subgraph isomorphic to \( U_7 \), it follows that \( G[N_R(v)] \cong K_{3(3)} \), which is a 6-regular graph, and so each vertex of \( G[N_R(v)] \) is the center of some \( U_6 \) in \( G[N_R(v)] \). If there is a blue edge \( e \) joining a vertex of \( N_R(v) \) and a vertex of \( W = V(G) - (N_R(v) \cup \{v\}) \), then \( G[N_R(v)] \) together with \( e \) produces a blue \( U_7 \), which is impossible. Thus each edge in \( [N_R(v), W] \) is red and so each edge in \( G[W] \) is blue. This implies that \( G[W \cup \{v\}] \) is a blue subgraph of order 9 in \( G \). Since \( G[N_R(v)] \cong K_{3(3)} \), it follows that \( G[W \cup \{v\}] \not\cong K_{3(3)} \). However then, there is a blue \( U_7 \) in \( G[W \cup \{v\}] \) and so in \( G \) as well, which again is impossible.

**Case 2.** \( \Delta(G_R) = 8 \). In this case, we can assume that \( \deg_G v = \Delta(G) \). (To see this, let \( x \) be a vertex of \( G \) such that \( \deg_G x = \Delta(G) = 16 \). Since \( \Delta(G_R) = 8 \geq \Delta(G_B) \), it follows that \( x \) is incident with at most 8 blue edges and so \( x \) is incident with exactly 8 red edges.) Then \( \deg_{G_B} v = \Delta(G_B) = 8 \) and so \( G[N_B(v)] \subseteq G_R \). Since \( G \) contains neither a red \( U_7 \) nor a blue \( U_7 \), it follows that \( \Delta(G[N_R(v)]) = \Delta(G[N_B(v)]) = 6 \). Thus, each of \( G[N_R(v)] \) and \( G[N_B(v)] \) is a subgraph of order 8 having maximum degree 6 in \( G \). Since \( G = K_{4(3),3(2)} \), each of \( G[N_R(v)] \) and \( G[N_B(v)] \) is isomorphic to either \( K_{2(3),2} \)
or \( K_{4(2)} \) and it is impossible that both \( G[N_R(v)] \) and \( G[N_B(v)] \) are isomorphic to \( K_{4(2)} \). Thus, we may assume that \( G[N_R(v)] \cong K_{2(3),2} \). Let \( F_1 \cong U_6 \) be a subgraph of \( G[N_R(v)] \) whose center is \( x \) and let \( F_2 \cong U_6 \) be a subgraph of \( G[N_B(v)] \) whose center is \( y \) such that \( xy \) is an edge of \( G \). If \( xy \) is red, then \( xy \) and \( F_2 \) produce a red \( U_7 \) in \( G \), while if \( xy \) is blue, then \( xy \) and \( F_1 \) produce a blue \( U_7 \) in \( G \). In either case, a contradiction is produced.

Next, we determine the \( k \)-Ramsey numbers \( R_k(U_k) \) for \( k \geq 6 \) and \( k \neq 7 \).

**Theorem 4.4.11** For each integer \( k \geq 6 \) and \( k \neq 7 \), \( R_k(U_k) = 2k + 3 \).

**Proof.** First, we show that \( R_k(U_k) \geq 2k + 3 \). Let \( H = K_{2(3),(k-2)(2)} \) be the balanced complete \( k \)-partite graph of order \( 2k + 2 \), where the partite sets of \( H \) are \( V_1, V_2, \ldots, V_k \) with \( |V_i| = |V_j| = 3 \) and \( |V_i| = 2 \) for \( 3 \leq i \leq k \). We consider two cases.

**Case 1.** \( k \geq 6 \) is even. So \( k = 2p \) for some integer \( p \geq 3 \). Let

\[
X = \bigcup \{ V_i : 1 \leq i \leq k - 1 \text{ and } i \text{ is odd} \}
\]

\[
Y = \bigcup \{ V_i : 2 \leq i \leq k \text{ and } i \text{ is even} \}
\]

Thus \( H[X] \cong H[Y] = K_{3,(p-1)(2)} \). Define a red-blue coloring of \( H \) by assigning the color red to each edge in \( H[X] \) and each edge in \( H[Y] \) and the color blue to the remaining edges of \( H \). Then \( H_R = H[X] + H[Y] \) and \( H_B = H - E(H_R) \). Since \( \Delta(H_R) = 2p - 1 = k - 1 \) and \( H_B \) is a bipartite graph with partite sets \( X \) and \( Y \), there is neither a red \( U_k \) nor a blue \( U_k \). Hence \( R_k(U_k) \geq 2k + 3 \) when \( k \) is even.

**Case 2.** \( k \geq 9 \) is odd. So \( k = 2p + 1 \) for some integer \( p \geq 4 \). Let

\[
X = \bigcup \{ V_i : \text{ either } i = 1, 2 \text{ or } 6 \leq i \leq k - 1 \text{ and } i \text{ is even} \}
\]

\[
Y = \bigcup \{ V_i : \text{ either } i = 3, 4, 5 \text{ or } 7 \leq i \leq k \text{ and } i \text{ is odd} \}
\]

Thus \( H[X] = K_{2(3),(p-2)(2)} \) and \( H[Y] = K_{(p+1)(2)} \). Define a red-blue coloring of \( H \) by assigning the color red to each edge in \( H[X] \) and each edge in \( H[Y] \) and the color blue to the remaining edges of \( H \). Then \( H_R = H[X] + H[Y] \) and \( H_B = H - E(H_R) \). Since \( \Delta(H_R) = 2p = k - 1 \) and \( H_B \) is a bipartite graph with partite sets \( X \) and \( Y \), there is neither a red \( U_k \) nor a blue \( U_k \). Hence \( R_k(U_k) \geq 2k + 3 \) when \( k \) is odd.
Next, we show that \( R_k(U_k) \leq 2k + 3 \). Let \( G = K_{3(3), (k-3)(2)} \) be the balanced complete \( k \)-partite graph of order \( 2k + 3 \) where the partite sets of \( G \) are \( W_1, W_2, \ldots, W_k \) with \( |W_1| = |W_2| = |W_3| = 3 \) and \( |W_i| = 2 \) for \( 4 \leq i \leq k \). Then \( \deg w = 2k \) if \( w \in W_i \) for \( i = 1, 2, 3 \) and \( \deg w = 2k + 1 \) if \( w \in W_i \) for \( 4 \leq i \leq k \). Let there be given a red-blue coloring of \( G \) containing neither a red \( U_k \) nor a blue \( U_k \). We may assume that \( \Delta(G_R) \geq \Delta(G_B) \). Since \( \Delta(G) = 2k + 1 \), it follows that \( \Delta(G_R) \geq k + 1 \). Let \( v \) be a vertex of \( G \) with \( \deg_{G_R} v = \Delta(G_R) \). Since \( G \) contains no red \( U_k \), it follows that \( G[N_R(v)] \subseteq G_B \). Because \( k + 1 \geq 7 \), every vertex of \( G[N_R(v)] \) lies on a triangle in \( G \).

First, suppose that \( \Delta(G_R) \geq k + 2 \). If \( k = 6 \), then \( \Delta(G_R) \geq 8 \) and so there is a vertex \( u \) in \( G[N_R(v)] \) such that \( \deg_{G[N_R(v)]} u \geq 6 \); while if \( k \geq 8 \), then \( \Delta(G_R) \geq 10 \) and so \( G[N_R(v)] \not\cong K_{3(3)} \), which implies that there is a vertex \( u \) in \( G[N_R(v)] \) such that \( \deg_{G[N_R(v)]} u \geq k \). Because \( u \) lies on a triangle in \( G[N_R(v)] \), it follows that \( G[N_R(v)] \) contains a blue \( U_k \) centered at \( u \), a contradiction. Hence, we may assume that \( \Delta(G_R) = k + 1 \). Since there is no blue \( U_k \), no vertex of \( G[N_R(v)] \) has degree \( k \) or more. Let \( W = V(G) - (N_R(v) \cup \{v\}) \).

Assume first that \( G[N_R(v)] \cong K_{3(3)} \), which is \( 6 \)-regular and so \( k = 8 \). Each vertex \( u \) of \( G[N_R(v)] \) is the central vertex of a subgraph \( U_6 \) in \( G[N_R(v)] \) and so \( u \) is joined to at most one vertex in \( W \) by a blue edge (since \( G \) contains no blue \( U_8 \)). Hence there are vertices \( v^* \in N_B(v) \) and \( u_1, u_2 \in N_R(v) \) where \( u_1 \neq u_2 \) such that \( u_1 v^* \) and \( u_2 v^* \) are both red. Since \( G[N_B(v)] \cong K_{4(2)} \) is \( 6 \)-regular and each edge in \( G[N_B(v)] \) is red (for otherwise, \( G[N_B(v) \cup \{v\}] \) contains a blue \( U_8 \), which is impossible), it follows that \( G[N_B(v)] \) and \( u_1 v^* \) and \( u_2 v^* \) produce a red \( U_8 \) in \( G \), which is a contradiction. Next, assume that \( G[N_R(v)] \not\cong K_{3(3)} \). Thus there is a vertex \( u \) of \( G[N_R(v)] \) such that \( \deg_{G[N_R(v)]} u = k - 1 \). Hence each edge in \( \{u\}, W \) is red. Since \( |W| = k + 1 \), it follows that each edge in \( G[W] \) is blue. Thus \( G[W \cup \{v\}] \) is a subgraph of order \( k + 2 \geq 10 \), each of whose edges is colored blue, and so \( G[W \cup \{v\}] \) contains a blue \( U_k \), a contradiction. \( \blacksquare \)

As a consequence of Proposition 4.4.10 and Theorem 4.4.11, all diagonal \( k \)-Ramsey numbers where \( k \geq 6 \) are given in the next result.

**Theorem 4.4.12** For each integer \( k \geq 6 \),

\[
R_k(U_k) = \begin{cases} 
2k + 3 & \text{if } k \neq 7 \\
2k + 4 & \text{if } k = 7.
\end{cases}
\]

By Theorems 4.4.9 and 4.4.12, it remains to consider \( R_k(U_t) \) when \( 6 \leq k \leq t - 1 \).
4.4.3 The $k$-Ramsey Numbers $R_6(U_t)$

We have seen that

$$R_6(U_3) = 7, R_6(U_4) = 9, R_6(U_5) = 13 \text{ and } R_6(U_6) = 15.$$  

In this subsection, we determine the value of the 6-Ramsey numbers $R_6(U_t)$ for all integers $t \geq 7$.

**Theorem 4.4.13** For each integer $t \geq 7$,

$$R_6(U_t) = \begin{cases} 3t - 3 & \text{if } t \text{ is even} \\ 3t - 2 & \text{if } t \text{ is odd}. \end{cases}$$

**Proof.** First, we show that $R_6(U_t) \geq 3t - 3$ if $t$ is even and $R_6(U_t) \geq 3t - 2$ if $t$ is odd.

We consider these two cases.

**Case 1.** $t \geq 8$ is even. So $t = 2p$ for some integer $p \geq 4$. We show that $R_6(U_t) \geq 3t - 4 = 6p - 4$. Let $H = K_{2(p),4(p-1)}$ be the balanced complete 6-partite graph of order $6p - 4$ where the partite sets of $H$ are $V_1, V_2, \ldots, V_6$ with $|V_1| = |V_2| = p$ and $|V_i| = p - 1$ for $i = 3, 4, 5, 6$. Let $X = V_1 \cup V_3 \cup V_5$ and $Y = V_2 \cup V_4 \cup V_6$. Thus $H[X] \cong H[Y] = K_{p,2(p-1)}$. Define a red-blue coloring of $H$ by assigning the color red to each edge in $H[X]$ and each edge in $H[Y]$ and the color blue to the remaining edges of $H$ (see Figure 4.6, where, as expected, a solid line between two sets indicates that all edges between the sets are red, while dashed lines between $X$ and $Y$ indicate that all edges in the set $[X,Y]$ are blue). Then $H_R = H[X] + H[Y]$ and $H_B = H - E(H_R)$. Since $\Delta(H_R) = 2p - 1 = t - 1$ and $H_B$ is a bipartite graph with partite sets $X$ and $Y$, there is neither a red $U_t$ nor a blue $U_t$ in $H$. Hence $R_6(U_t) \geq 3t - 3$ when $t$ is even.

**Case 2.** $t \geq 7$ is odd. So $t = 2p + 1$ for some integer $p \geq 3$. We show that $R_6(U_t) \geq 3t - 3 = 6p$. Let $H = K_{6(p)}$ be the balanced complete 6-partite graph of order $6p$ where the partite sets of $H$ are $V_1, V_2, \ldots, V_6$ with $|V_i| = p$ for $1 \leq i \leq 6$. Let $X = V_1 \cup V_2 \cup V_3$ and $Y = V_4 \cup V_5 \cup V_6$. Thus $H[X] \cong H[Y] = K_{3(p)}$, which is $(2p)$-regular. Define a red-blue coloring of $H$ by assigning the color red to each edge in $H[X]$ and each edge in $H[Y]$ and the color blue to the remaining edges of $H$ (see Figure 4.7). Then $H_R = H[X] + H[Y]$ and $H_B = H - E(H_R)$. Since $\Delta(H_R) = 2p = t - 1$ and $H_B$ is a bipartite graph with partite sets $X$ and $Y$, there is neither a red $U_t$ nor a blue $U_t$. Hence $R_6(U_t) \geq 3t - 2$ when $t$ is odd.
Furthermore, each vertex in $U$.

Observe that if $\Delta(G) = 1$ for some integer $a \geq 2$. Thus, we show that $R_6(U_4) \leq 12a - 3$ in this subcase. Let $G = K_{3(2a),3(2a-1)}$ be the balanced complete 6-partite graph of order $12a - 3$ with partite sets $V_1, V_2, \ldots, V_6$ such that $|V_i| = 2a$ for $i = 1, 2, 3$ and $|V_i| = 2a - 1$ for $i = 4, 5, 6$. Suppose that there is a red-blue coloring of $G$ containing no monochromatic $U_{4a}$. We may assume that $\Delta(G_R) \geq \Delta(G_B)$. Let $v$ be a vertex of $G$ with $\deg_{G_R} v = \Delta(G_R)$. Since $\Delta(G) = 10a - 2$, it follows that $\Delta(G_R) \geq 5a - 1$. Because $G_R$ contains no red $U_{4a}$, it follows that $G[N_R(v)] \subseteq G_B$.

Observe that if $\Delta(G_R) \geq 6a - 1 = (2a - 1) + 4a$, then the blue graph $G[N_R(v)]$ contains $U_{4a}$ as a subgraph, a contradiction. Hence we may assume that $5a - 1 \leq \Delta(G_R) \leq 6a - 2$. Furthermore, each vertex in $N_R(v)$ lies on a triangle in $G[N_R(v)]$.

First, suppose that $5a - 1 \leq \Delta(G_R) \leq 6a - 3$. Let $u \in N_R(v)$. Then $\deg_G u = 10a - 2$ or $\deg_G u = 10a - 3$. Since $u$ lies on a triangle in the blue graph $G[N_R(v)]$, it follows

---

**Figure 4.6**: The red-blue coloring of $H = K_{2(p),4(p-1)}$ in Case 1

**Figure 4.7**: The red-blue coloring of $H = K_{6(p)}$ in Case 2
that \( \deg_{G_B} u \leq 4a - 1 \). However then,

\[
\deg_{G_R} u = \deg_G u - \deg_{G_B} u \geq (10a - 3) - (4a - 1) = 6a + 2 > \Delta(G_R),
\]

which is a contradiction.

Next, suppose that \( \Delta(G_R) = 6a - 2 \). Note that \( G[N_R(v)] \) is a complete 3-partite graph. Then each vertex in \( N_R(v) \) has degree \( 4a - 2 \) or \( 4a - 1 \) in \( G[N_R(v)] \) and there exists a vertex \( u \) such that \( \deg_{G[N_R(v)]} u = 4a - 1 \). In fact, \( G[N_R(v)] \cong K_{2a,2(2a-1)} \). Hence, each edge in \( \{u\}, V(G) - N_R(v) \) is red. Assume first that \( \deg_G u = \Delta(G) = 10a - 2 \) for some \( u \in N_R(v) \) such that \( \deg_{G[N_R(v)]} u = 4a - 1 \). Then \( u \) is adjacent to every vertex in \( V(G) - N_R(v) \). Since \( G_R \) contains no \( U_{4a} \) as a subgraph, it follows that \( G[V(G) - N_R(v)] \) is a blue subgraph of order \( 6a - 1 \). However then, \( G[V(G) - N_R(v)] \) contains a blue \( U_{4a} \), a contradiction. Thus, we may assume that if \( u \in N_R(v) \) with \( \deg_{G[N_R(v)]} u = 4a - 1 \), then \( \deg_G u = \Delta(G) = 10a - 3 \). This implies that \( G[N_R(v)] \cong K_{2a,2(2a-1)} \) and \( N_R(v) \subseteq V_1 \cup V_2 \cup V_3 \). In fact, we can assume that the partite sets of \( G[N_R(v)] \) are \( V_1, V_2 = \{x\} \) and \( V_3 = \{y\} \), where \( x \in V_2 \) and \( y \in V_3 \), and \( u \in V_2 - \{x\} \). Hence \( \deg_{G_R} u = |N_R(u)| = 6a - 2 = \Delta(G_R) \) and \( G[N_R(u)] \cong K_{1,3(2a-1)} \) is a blue subgraph of order \( 6a - 2 \). Note that the partite sets of \( G[N_R(u)] \) are \( \{y\}, V_4, V_5 \) and \( V_6 \). However then, \( G[N_R(u)] \) contains a blue \( U_{4a} \), a contradiction.

**Subcase 1.2.** \( t \equiv 2 \mod 4 \). Then \( t = 4a + 2 \) for some integer \( a \geq 2 \). We show that \( R_6(U_{4a+2}) \leq 12a + 3 \). Let \( G = K_{3(2a+1),3(2a)} \) be the balanced complete 6-partite graph of order \( 12a + 3 \) with partite sets \( V_1, V_2, \ldots, V_6 \) such that \( |V_i| = 2a + 1 \) for \( i = 1, 2, 3 \) and \( |V_i| = 2a \) for \( i = 4, 5, 6 \). Suppose that there is a red-blue coloring of \( G \) containing no monochromatic \( U_{4a+2} \). We may assume that \( \Delta(G_R) \geq \Delta(G_B) \). Let \( v \) be a vertex of \( G \) with \( \deg_{G_B} v = \Delta(G_B) \). Since \( \Delta(G) = 10a+3 \), it follows that \( \Delta(G_R) \geq 5a+2 \). Because \( G \) contains no red \( U_{4a+2} \), it follows that \( G[N_R(v)] \subseteq G_B \). If \( \Delta(G_R) \geq 6a+2 = 2a + (4a + 2) \), then the blue graph \( G[N_R(v)] \) contains \( U_{4a+2} \) as a subgraph, a contradiction. Thus \( 5a + 2 \leq \Delta(G_R) \leq 6a + 1 \) and each vertex in \( N_R(v) \) lies on a triangle in \( G[N_R(v)] \).

First, assume that \( 5a + 2 \leq \Delta(G_R) \leq 6a \). Let \( u \in N_R(v) \). Then \( \deg_G u = 10a + 3 \) or \( \deg_G u = 10a + 2 \). Since \( u \) lies on a triangle in the blue graph \( G[N_R(v)] \), it follows that \( \deg_{G_B} u \leq 4a + 1 \). However then,

\[
\deg_{G_R} u = \deg_G u - \deg_{G_B} u \geq (10a + 2) - (4a + 1) = 6a + 1 > \Delta(G_R),
\]

which is a contradiction.

Next, suppose that \( \Delta(G_R) = 6a + 1 \). Then \( G[N_R(v)] \) is a complete 3-partite graph. Thus \( G[N_R(v)] \) contains a vertex \( u \) such that \( \deg_{G[N_R(v)]} u = 4a + 1 \). In fact, \( G[N_R(v)] \cong
$K_{2a+1,2(2a)}$. If $\deg_G u = \Delta(G) = 10a + 3$ for some $u \in N_R(v)$ such that $\deg_{G[N_R(v)]} u = 4a + 1$, then $u$ is adjacent to every vertex in $V(G) - N_R(v)$. Each edge in $\{u\}, V(G) - N_R(v)$ is red and $G[V(G) - N_R(v)]$ is a blue subgraph of order $6a + 2$. However, then, $G[V(G) - N_R(v)]$ contains a blue $U_{4a+2}$, a contradiction. Thus, we may assume that if $u \in N_R(v)$ with $\deg_{G[N_R(v)]} u = 4a + 1$, then $\deg_G u = \Delta(G) - 1 = 10a + 2$. This implies that $G[N_R(v)] \cong K_{2a+1,2(2a)}$ and $N_R(v) \subseteq V_1 \cup V_2 \cup V_3$. In fact, we can assume that the partite sets of $G[N_R(v)]$ are $V_1, V_2 - \{x\}$ and $V_3 - \{y\}$, where $x \in V_2$ and $y \in V_3$, and $u \in V_2 - \{x\}$. Since $\deg_G u = 10a + 2$ and $\deg_{G[N_R(v)]} u = 4a + 1$, then there is exactly one vertex $x \in V(G) - N_R(v)$ such that $u$ and $x$ belong to the same partite set. Hence, $N_R(u) = V(G) - (N_R(v) \cup \{x\})$ and $|N_R(u)| = 6a + 1$. Observe that $\deg_{G_R} u = 6a + 1 = \Delta(G_R)$ and $G[N_R(u)] \cong K_{1,3(2a)}$ is a blue subgraph of order $6a + 1$. Note that the partite sets of $G[N_R(u)]$ are $\{y\}, V_4, V_5$ and $V_6$. However then, $G[N_R(v)]$ contains a blue $U_{4a+2}$, a contradiction.

**Case 2.** $t \geq 7$ is odd. We consider two subcases, depending on whether $t \equiv 1 \pmod{4}$ or $t \equiv 3 \pmod{4}$.

**Subcase 2.1.** $t \equiv 1 \pmod{4}$. Then $t = 4a + 1$ for some integer $a \geq 2$. Thus, we show that $R_6(U_{4a+1}) \leq 12a + 1$. Let $G = K_{2a+1,5(2a)}$ be the balanced complete 6-partite graph of order $12a + 1$. Suppose that there is a red-blue coloring of $G$ containing no monochromatic $U_{4a+1}$. We may assume that $\Delta(G_R) \geq \Delta(G_B)$. Let $v$ be a vertex of $G$ with $\deg_{G_R} v = \Delta(G_R)$. Since $\Delta(G) = 10a + 1$, it follows that $\Delta(G_R) \geq 5a + 1$. Because $G$ contains no red $U_{4a+1}$, it follows that $G[N_R(v)] \subseteq G_B$. Observe that if $\Delta(G_R) \geq 6a + 1 = 2a + (4a + 1)$,

then $G[N_R(v)]$ contains a blue $U_{4a+1}$ as a subgraph, a contradiction. Thus $5a + 1 \leq \Delta(G_R) \leq 6a$ and each vertex in $N_R(v)$ lies on a triangle in the blue graph $G[N_R(v)]$.

First, suppose that $\Delta(G_R) = 6a$. Hence $G[N_R(v)] \cong K_{3(2a)}$; for otherwise, $G[N_R(v)]$ would contain a blue $U_{4a+1}$. Since there are at most $2a + 1$ vertices having degree $10a = \Delta(G) - 1$ in $G$, it follows that $G[N_R(v)]$ contains a vertex $u$ such that $\deg_G u = \Delta(G) = 10a + 1$ and $\deg_{G[N_R(v)]} u = 4a$. Hence $u$ is adjacent to every vertex in $V(G) - N_R(v)$. Since $G_B$ contains no $U_{4a+1}$, each edge in $\{u\}, V(G) - N_R(v)$ is red and so $G[V(G) - N_R(v)]$ is a blue subgraph of order $6a + 1$. However then, $G[V(G) - N_R(v)]$ contains a blue $U_{4a+1}$, a contradiction.

We may therefore assume that $5a + 1 \leq \Delta(G_R) \leq 6a - 1$. Let $u \in N_R(v)$. Then $\deg_G u = 10a$ or $\deg_G u = 10a + 1$. Since $u$ lies on a triangle in the blue graph $G[N_R(v)]$, it follows that $\deg_{G_B} u \leq 4a$. However then,
which is a contradiction.

Subcase 2.2. \(t \equiv 3 \pmod{4}\). Hence \(t = 4a + 3\) for some integer \(a \geq 1\). We show here that \(R_6(U_{4a+3}) \leq 12a + 7\). Let \(G = K_{2a+2,5(2a+1)}\) be the balanced complete 6-partite graph of order \(12a + 7\). Suppose that there is a red-blue coloring of \(G\) containing no monochromatic \(U_{4a+3}\). We may assume that \(\Delta(G) \geq \Delta(G_B)\). Let \(v\) be a vertex of \(G\) with \(\deg_{G_B} v = \Delta(G)\). Since \(\Delta(G) = 10a + 6\), it follows that \(\Delta(G) \geq 5a + 3\). Because \(G\) contains no red \(U_{4a+3}\), it follows that \(G[N_R(v)] \subseteq G_B\). Observe that if

\[
\Delta(G) \geq 6a + 4 = (2a + 1) + (4a + 3),
\]

then \(G[N_R(v)]\) contains a blue \(U_{4a+3}\) as a subgraph, a contradiction. Thus \(5a + 3 \leq \Delta(G) \leq 6a + 3\) and each vertex in \(N_R(v)\) lies on a triangle in the blue graph \(G[N_R(v)]\).

First, suppose that \(\Delta(G) = 6a + 3\). Hence \(G[N_R(v)] \cong K_{3(2a+1)}\); for otherwise, \(G[N_R(v)]\) would contain a blue \(U_{4a+3}\). Thus \(G[N_R(v)]\) contains a vertex \(u\) such that \(\deg_{G} u = \Delta(G) = 10a + 6\) and \(\deg_{G[N_R(v)]} u = 4a + 2\). Hence \(u\) is adjacent to every vertex in \(V(G) - N_R(v)\). Since \(G_B\) contains no \(U_{4a+3}\), each edge in \([u, V(G) - N_R(v)]\) is red and so \(G[V(G) - N_R(v)]\) is a blue subgraph of order \(6a + 4\). However then, \(G[V(G) - N_R(v)]\) contains a blue \(U_{4a+3}\), a contradiction.

Hence, we may assume that \(5a + 3 \leq \Delta(G) \leq 6a + 2\). Let \(u \in N_R(v)\). Then \(\deg_{G} u = 10a + 6\) or \(\deg_{G} u = 10a + 5\). Since \(u\) lies on a triangle in the blue graph \(G[N_R(v)]\), it follows that \(\deg_{G_B} u \leq 4a + 2\). However then,

\[
\deg_{G} u = \deg_{G} u - \deg_{G_B} u \geq (10a + 5) - (4a + 2) = 6a + 3 > \Delta(G),
\]

which is a contradiction.  \(\blacksquare\)
4.4.4 Conclusions

Figure 4.8 summarizes the results we have obtained on the $k$-Ramsey numbers $R_k(U_t)$ of the graph $U_t$ for $6 \leq k \leq R(U_t)$ and $t \geq 6$. The $(k,t)$-entry in row $k$ and column $t$ is the number $R_k(U_t)$ and the symbol * indicates that this number has been determined. Where there is no $(k,t)$-entry, this indicates that $R_k(U_t)$ is unknown.

![Figure 4.8: Known and unknown $k$-Ramsey numbers $R_k(U_t)$](image)

4.5 On the 2-Ramsey Numbers of 4-Cycles

Ramsey numbers have also been defined for three or more graphs. In particular, for three graphs $F_1, F_2$ and $F_3$, the Ramsey number $R(F_1, F_2, F_3)$ of $F_1, F_2$ and $F_3$ is the smallest positive integer $n$ for which every red-blue-green coloring (in which every edge is colored red, blue or green) of the complete graph $K_n$ of order $n$ results in a red $F_1$, a blue $F_2$ or a green $F_3$. This gives rise to the concept of $k$-Ramsey number of three (or more) graphs. For three graphs $F_1, F_2$ and $F_3$ and an integer $k$ with $2 \leq k \leq R(F_1, F_2, F_3)$, the $k$-Ramsey number $R_k(F_1, F_2, F_3)$ of $F_1, F_2$ and $F_3$, if it exists, is the smallest order of a balanced complete $k$-partite graph $G$ for which every red-blue-green coloring of the edges of $G$ results in a red $F_1$, a blue $F_2$ or a green $F_3$. In particular, if $k = 2$ and $F_i \cong F$ for some graph $F$ where $i = 1, 2, 3$, then the 2-Ramsey number $R_2(F, F, F)$ is the smallest order of a balanced complete bipartite graph $G$ for which every red-blue-green coloring of the edges of $G$ results in a monochromatic $F$ (all of whose edges are colored the same). In general, it is very challenging to determine the value of $R_2(F, F, F)$ even when $F$ is a graph of smallest size. In this section, we show that $20 \leq R_2(C_4, C_4, C_4) \leq 21$. 
4.5.1 \( R_2(C_4, C_4, C_4) \geq 20 \)

First, we show that \( R_2(C_4, C_4, C_4) \) is at least 20.

**Theorem 4.5.1** \( R_2(C_4, C_4, C_4) \geq 20. \)

**Proof.** We describe a red-blue-green coloring \( c \) of \( K_{9,10} \) that avoids a monochromatic \( C_4 \), which implies that \( R_2(C_4, C_4, C_4) \geq 20. \) Let \( U = \{ u_1, u_2, \ldots, u_9 \} \) and \( V = \{ v_1, v_2, \ldots, v_{10} \} \) be the partite sets of \( K_{9,10} \). The edge coloring \( c \) of \( K_{9,10} \) is defined by the following table, where the \( u_i - v_j \) entry in row \( u_i \) and column \( v_j \) indicates the color of the edge \( u_i v_j \) for \( 1 \leq i \leq 9 \) and \( 1 \leq j \leq 10 \).

\[
\begin{array}{|c|cccccccccc|}
\hline
K_{9,10} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \\
\hline
u_1 & r & r & r & r & g & g & b & b & g & b \\
u_2 & b & b & g & r & r & r & g & g & b & b \\
u_3 & r & g & g & b & b & g & r & r & r & b \\
u_4 & b & r & b & b & g & r & g & b & r & g \\
u_5 & g & g & b & r & b & b & g & r & g & r \\
u_6 & g & b & r & b & r & b & b & g & r & g \\
u_7 & g & r & g & g & g & b & r & b & b & r \\
u_8 & b & g & r & g & b & r & b & r & b & g \\
u_9 & r & b & b & g & r & g & g & b & r & r \\
\hline
\end{array}
\]

A red-blue-green coloring of \( K_{9,10} \)

Next, we describe the structures of the red, blue and green subgraphs \( G_r, G_b \) and \( G_g \) isomorphic to \( K_{9,10} \) produced by this edge coloring \( c \). Figure 4.9 shows a spanning subgraph \( G \) of size 30 in \( K_{9,10} \), where each solid vertex is a vertex in \( U \) and each empty vertex is a vertex in \( V \). In fact, each of the resulting red, blue and green subgraphs \( G_r, G_b \) and \( G_g \) is isomorphic to the graph \( G \) of Figure 4.9. To illustrate this fact, we label a vertex \( u \in U \) by a triple \((u_p, u_q, u_s)\), \( 1 \leq p, q, s \leq 9 \), and a vertex \( v \in V \) by a triple \((v_p, v_q, v_s)\), \( 1 \leq p, q, s \leq 10 \), such that (1) the label \((u_p, u_q, u_s)\) of \( u \in U \) indicates that \( u = u_p \) in \( G_r \), \( u = u_q \) in \( G_b \) and \( u = u_s \) in \( G_g \) and (2) the label \((v_p, v_q, v_s)\) of \( v \in V \) indicates that \( v = v_p \) in \( G_r \), \( v = v_q \) in \( G_b \) and \( v = v_s \) in \( G_g \). Furthermore, if a vertex \( u \in U \) is labeled by \( u_p \), then \( u = u_p \) in each of \( G_r, G_b, G_g \). Similarly, a vertex \( v \in V \) labeled \( v_p \) indicates that \( v = v_p \) in each of \( G_r, G_b, G_g \).

The table below lists the red-neighborhood, blue-neighborhood and green-neighborhood \( N_R(u) \) of each vertex \( u \in U \). Observe that \( N_R(u) \cup N_B(u) \cup N_G(u) = V \) for each \( u \in U \).
Since no two vertices in $U$ have two common neighbors in $G$, it follows that $G$ is $C_4$-free and so $G_r$, $G_b$ and $G_g$ are $C_4$-free. Therefore, there is no monochromatic $C_4$ in this edge-colored $K_{9,10}$ and so $R_2(C_4,C_4,C_4) \geq 20$.

4.5.2 $R_2(C_4,C_4,C_4) \leq 21$

Next, we show that $R_2(C_4,C_4,C_4)$ is at most 21.

**Theorem 4.5.2** $R_2(C_4,C_4,C_4) \leq 21$.

**Proof.** We show that every red-blue-green coloring of $K_{10,11}$ results in a monochromatic $C_4$, which implies that $R_2(C_4,C_4,C_4) \leq 21$. Let

---

**Figure 4.9:** A spanning subgraph $G$ of size 30 in $K_{9,10}$

<table>
<thead>
<tr>
<th>$N_R(u_1)$</th>
<th>$N_B(u_1)$</th>
<th>$N_G(u_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${v_1,v_2,v_3,v_4}$</td>
<td>${v_7,v_8,v_{10}}$</td>
<td>${v_5,v_6,v_9}$</td>
</tr>
<tr>
<td>$N_R(u_2)$</td>
<td>$N_B(u_2)$</td>
<td>$N_G(u_2)$</td>
</tr>
<tr>
<td>${v_4,v_5,v_6,v_7}$</td>
<td>${v_{1},v_2,v_{10}}$</td>
<td>${v_3,v_8,v_9}$</td>
</tr>
<tr>
<td>$N_R(u_3)$</td>
<td>$N_B(u_3)$</td>
<td>$N_G(u_3)$</td>
</tr>
<tr>
<td>${v_7,v_8,v_9,v_1}$</td>
<td>${v_4,v_5,v_{10}}$</td>
<td>${v_2,v_3,v_6}$</td>
</tr>
<tr>
<td>$N_R(u_4)$</td>
<td>$N_B(u_4)$</td>
<td>$N_G(u_4)$</td>
</tr>
<tr>
<td>${v_2,v_6,v_9}$</td>
<td>${v_{1},v_3,v_4,v_8}$</td>
<td>${v_5,v_7,v_{10}}$</td>
</tr>
<tr>
<td>$N_R(u_5)$</td>
<td>$N_B(u_5)$</td>
<td>$N_G(u_5)$</td>
</tr>
<tr>
<td>${v_4,v_5,v_{10}}$</td>
<td>${v_3,v_5,v_6}$</td>
<td>${v_1,v_2,v_7,v_9}$</td>
</tr>
<tr>
<td>$N_R(u_6)$</td>
<td>$N_B(u_6)$</td>
<td>$N_G(u_6)$</td>
</tr>
<tr>
<td>${v_3,v_5,v_9}$</td>
<td>${v_{2},v_4,v_6,v_7}$</td>
<td>${v_1,v_8,v_{10}}$</td>
</tr>
<tr>
<td>$N_R(u_7)$</td>
<td>$N_B(u_7)$</td>
<td>$N_G(u_7)$</td>
</tr>
<tr>
<td>${v_2,v_7,v_{10}}$</td>
<td>${v_6,v_8,v_9}$</td>
<td>${v_1,v_3,v_4,v_5}$</td>
</tr>
<tr>
<td>$N_R(u_8)$</td>
<td>$N_B(u_8)$</td>
<td>$N_G(u_8)$</td>
</tr>
<tr>
<td>${v_3,v_6,v_8}$</td>
<td>${v_{1},v_5,v_7,v_9}$</td>
<td>${v_2,v_4,v_{10}}$</td>
</tr>
<tr>
<td>$N_R(u_9)$</td>
<td>$N_B(u_9)$</td>
<td>$N_G(u_9)$</td>
</tr>
<tr>
<td>${v_1,v_5,v_{10}}$</td>
<td>${v_2,v_3,v_9}$</td>
<td>${v_4,v_6,v_7,v_8}$</td>
</tr>
</tbody>
</table>
$U = \{u_1, u_2, \ldots, u_{11}\}$ and $V = \{v_1, v_2, \ldots, v_{10}\}$

be partite sets of $K_{11,10}$. Assume, to the contrary, that there is a red-blue-green coloring $c$ of $K_{10,11}$ that avoids a monochromatic $C_4$. Let $m_r$, $m_b$ and $m_g$ be the sizes of the resulting red, blue and green subgraphs $G_r$, $G_b$ and $G_g$, respectively, where say $m_r \geq m_b \geq m_g$.

Thus

$$m_r = \sum_{i=1}^{11} \deg_{G_r} u_i \geq \left\lceil \frac{110}{3} \right\rceil = 37.$$

In what follows, we show that $G_r$ contains $C_4$ as a subgraph, producing a contradiction.

Suppose, without loss of generality, that $\deg_{G_r} u_1 \geq \deg_{G_r} u_2 \geq \cdots \geq \deg_{G_r} u_{11}$.

Thus, $\deg_{G_r} u_1 \geq \left\lceil \frac{37}{11} \right\rceil = 4$. Since there is no red $C_4$ in $G_r$, it follows that

$$|N_R(u_i) \cap N_R(u_j)| \leq 1 \text{ for } 1 \leq i < j \leq 11. \quad (4.2)$$

This implies that

$$\sum_{i=1}^{3} \deg_{G_r} u_i \leq |V| + 3 = 13 \quad (4.3)$$

and

$$\sum_{i=1}^{4} \deg_{G_r} u_i \leq |V| + \binom{4}{2} = 16. \quad (4.4)$$

Since $\left\lceil \frac{37-16}{4} \right\rceil = 3$, it follows that $\deg_{G_r} u_5 \geq 3$ and so

$$\deg_{G_r} u_i \geq 3 \text{ for } 2 \leq i \leq 5. \quad (4.5)$$

If $\deg_{G_r} u_1 \geq 8$, then $\sum_{i=1}^{3} \deg_{G_r} u_i \geq 8 + 3 + 3 = 14$ by (4.5), which contradicts (4.3). Thus $\deg_{G_r} u_1 = 4, 5, 6, 7$ and so there are four cases to consider. First, we make an observation. If $\sum_{i=1}^{3} \deg_{G_r} u_i = 13$, then $N_R(u_1) \cup N_R(u_2) \cup N_R(u_3) = V$ and each of the following conditions (i), (ii) and (iii) hold in $G_r$:

(i) Since $|V| = 10$, it follows by (4.2) that $|N(u_i) \cap N(u_j)| = 1$ for $1 \leq i < j \leq 3$.

(ii) If $\deg_{G_r} u_3 = 3$, then $\deg_{G_r} u_i = 3$ for $4 \leq i \leq 11$, as $m_r \geq 37$.

(iii) No vertex of degree 3 or more in $G_r$ is adjacent to the vertex in $N(u_i) \cap N(u_j)$ for $1 \leq i < j \leq 3$. To see this, let $u \in U$ such that $N_R(u)$ contains the vertex $v \in N(u_1) \cap N(u_2)$ say. Thus, $(N_R(u) - \{v\}) \cap N_R(u_i) = \emptyset$ for $i = 1, 2$ by (4.2). Since $N_R(u)$ contains at most one vertex in $N_R(u_3) = V - [N_R(u_1) \cup N_R(u_2)]$, it follows that $\deg_{G_r} u \leq 2$. 

CHAPTER 4. ON K-RAMSEY NUMBERS OF GRAPHS

We are now prepared to consider these four cases.

Case 1. $\deg_G, u_1 = 7$. Then $\deg_G, u_2 = \deg_G, u_3 = 3$ by (4.3) and (4.5) and so $\sum_{i=1}^3 \deg_G, u_i = 13$. Hence $\deg_G, u_i = 3$ for $2 \leq i \leq 11$ by (ii). We may assume, without loss generality, that $N_R(u_1) = \{v_1, v_2, \ldots, v_7\}$, $N_R(u_2) = \{v_7, v_8, v_9\}$ and $N_R(u_3) = \{v_9, v_{10}, v_1\}$. Since $\deg_G, u_4 = \deg_G, u_5 = 3$, it follows by (4.2) that each of $u_4$ and $u_5$ is adjacent in $G_r$ to exactly one vertex in $N_R(u_i)$ for each $i = 1, 2, 3$ but not adjacent to any of $v_1, v_7, v_9$ in $G_r$. This implies that each of $u_4$ and $u_5$ is adjacent in $G_r$ to $v_8 \in N_R(u_2)$ and is adjacent to $v_{10} \in N_R(u_3)$. However then, $\{v_8, v_{10}\} \subseteq N_R(u_4) \cap N_R(u_5)$ and so there is a contradiction.

Case 2. $\deg_G, u_1 = 6$. Since $\deg_G, v_2 \geq \lceil \frac{37-6}{10} \rceil = 4$, it follows that $\deg_G, u_i = 3$ for $i = 3, 4, 5$ by (4.3) and (4.5) and $\deg_G, u_2 = 4$. We may assume, without loss generality, that $N_R(u_1) = \{v_1, v_2, \ldots, v_6\}$, $N_R(u_2) = \{v_5, v_6, v_7, v_8, v_9\}$ and $N_R(u_3) = \{v_9, v_{10}, v_1\}$. Since $\deg_G, u_i = 3$ for $i = 4, 5, 6$, it follows by (4.2) that each of $u_4, u_5, u_6$ is adjacent in $G_r$ to exactly one vertex in $N_R(u_i)$ in $G_r$ for $i = 1, 2, 3$ but not adjacent in $G_r$ to any of $v_1, v_6, v_9$ in $G_r$. This implies that at least two of $u_4, u_5$ and $u_6$ are both adjacent to $v_7$ or both adjacent to $v_8$ in $G_r$, say $u_4$ and $u_5$ are adjacent to $v_7$, and each of $u_4, u_5$ and $u_6$ is adjacent to $v_{10}$ in $G_r$. However then, $v_7, v_{10} \in N_R(u_4) \cap N_R(u_5)$ and so there is a red $C_4$ in $G_r$, which is a contradiction.

Case 3. $\deg_G, u_1 = 5$. Since $\deg_G, v_2 \geq \lceil \frac{37-5}{10} \rceil = 4$, it follows that $\deg_G, u_2 = 4, 5$. We consider these two subcases.

Subcase 3.1. $\deg_G, u_2 = 5$. Thus $\deg_G, u_3 = 3$ by (4.3) and (4.5). Since $m_r \geq 37$, it follows that $\deg_G, u_i = 3$ for $3 \leq i \leq 11$. We may assume, without loss generality, that $N_R(u_1) = \{v_1, v_2, v_3, v_4, v_5\}$, $N_R(u_2) = \{v_5, v_6, v_7, v_8, v_9\}$ and $N_R(u_3) = \{v_9, v_{10}, v_1\}$. Since $\deg_G, u_i = 3$ for $i = 4, 5, 6, 7$, each of $u_4, u_5, u_6, u_7$ is adjacent to exactly one vertex in $N_R(u_i)$ in $G_r$ for $i = 1, 2, 3$ but not to any of $v_1, v_5, v_9$ in $G_r$. Hence $u_i v_{10} \in E(G_r)$ for $i = 4, 5, 6, 7$. Furthermore, at least two vertices in $\{u_4, u_5, u_6, u_7\}$ are both adjacent to one of $v_6, v_7, v_8 \in N_R(u_2)$ in $G_r$, say $u_4$ and $u_5$ are adjacent to $v_7$ in $G_r$. However then, $v_7, v_{10} \in N_R(u_4) \cap N_R(u_5)$ and so there is a red $C_4$ in $G_r$, which is a contradiction.

Subcase 3.2. $\deg_G, v_2 = 4$. Since $\deg_G, u_3 \geq \lceil \frac{37-9}{9} \rceil = 4$, it follows that $\deg_G, u_3 = 4$. We may assume that $N_R(u_1) = \{v_1, v_2, v_3, v_4, v_5\}$, $N_R(u_2) = \{v_5, v_6, v_7, v_8\}$ and $N_R(u_3) = \{v_8, v_9, v_{10}, v_1\}$. If $\deg_G, u_4 = 4$, then $\sum_{i=1}^4 \deg_G, u_i = 17$, a contradiction.
Thus, $\deg_G, u_4 = 3$ and so $\deg_G, u_i = 3$ for $4 \leq i \leq 11$, as $m_r \geq 37$. Hence, each of $u_4, u_5, u_6, u_7, u_8$ is adjacent to exactly one vertex in $N_R(u_i)$ for each $i \in \{1, 2, 3\}$ but not adjacent to any vertex in $\{v_1, v_5, v_8\}$ in $G_r$. Therefore, each of five vertices $u_4, u_5, u_6, u_7, u_8$ is adjacent to exactly one of $v_6, v_7 \in N_R(u_2)$ and exactly one of $v_9, v_{10} \in N_R(u_3)$ in $G_r$. Since there are only four such possibilities, namely

$$A_1 = \{v_6, v_9\}, A_2 = \{v_6, v_{10}\}, A_3 = \{v_7, v_9\}, A_4 = \{v_7, v_{10}\},$$

there is $t \in \{1, 2, 3, 4\}$ such that $A_t \in N_R(u_i) \cap N_R(u_j)$ where $i, j \in \{4, 5, 6, 7, 8\}$ and $i \neq j$, which is a contradiction.

**Case 4.** $\deg_G, u_1 = 4$. Since $\deg_G, u_2 \geq \left\lceil \frac{37-4}{10} \right\rceil = 4$, $\deg_G, u_3 \geq \left\lceil \frac{37-8}{9} \right\rceil = 4$ and $\deg_G, u_4 \geq \left\lceil \frac{37-12}{8} \right\rceil = 4$, it follows that $\deg_G, u_i = 4$ for $1 \leq i \leq 4$. First, suppose that either $N_R(u_i) \cap N_R(u_j) = \emptyset$ for some pair $ij \in \{1, 2, 3\}$ and $i \neq j$ or $N_R(u_1) \cap N_R(u_1) \cap N_R(u_3) \neq \emptyset$. Then $N_R(u_1) \cup N_R(u_1) \cup N_R(u_3) = V$. Since $\deg_G, u_4 \geq 4$, it follows that $u_4$ must be adjacent to two vertices in one of the sets $N_R(u_1), N_R(u_2), N_R(u_3)$, resulting in a red $C_4$, which is a contradiction. Next, suppose that $|N_R(u_i) \cap N_R(u_j)| = 1$ for every pair $ij \in \{1, 2, 3\}$ and $i \neq j$ and $N_R(u_1) \cap N_R(u_1) \cap N_R(u_3) = \emptyset$. We may assume that $N_R(u_1) = \{v_1, v_2, v_3, v_4\}$, $N_R(u_2) = \{v_4, v_5, v_6, v_7\}$ and $N_R(u_3) = \{v_7, v_8, v_9, v_{10}\}$. Furthermore, $u_4$ is adjacent to exactly one vertex in each of the sets $\{v_2, v_3\}, \{v_5, v_6\}, \{v_8, v_9\}$ and is adjacent to $v_{10}$ in $G_r$. We may assume, without loss of generality, that $N_R(u_4) = \{v_2, v_5, v_8, v_{10}\}$. Since $\deg_G, u_5 \geq \left\lceil \frac{37-16}{7} \right\rceil = 3$, it follows that $\deg_G, u_5 = 3, 4$.

We consider these two subcases.

**Subcase 4.1.** $\deg_G, u_5 = 4$. Since (a) $u_5$ is adjacent to exactly one vertex in each of $\{v_2, v_3\}, \{v_5, v_6\}, \{v_8, v_9\}$ and is adjacent to $v_{10}$ and (b) $N_R(u_4) = \{v_2, v_5, v_8, v_{10}\}$, it follows that $N_R(u_5) = \{v_3, v_6, v_9, v_{10}\}$. Hence, each vertex $v \in V$ belongs to exactly two of the five sets $N_R(u_1), N_R(u_2), \ldots, N_R(u_5)$. Next, we consider $u_6$. Since $\deg_G, u_6 \geq \left\lceil \frac{37-20}{6} \right\rceil = 3$, it follows that $N_R(u_6)$ contains at least three vertices. Each of these three vertices, however, must belong to two of the red neighborhoods of $u_1, u_2, u_3, u_4$ and $u_5$. Therefore, at least one of $u_1, u_2, u_3, u_4$ and $u_5$ must share two red neighbors with $u_6$, which is impossible.

**Subcase 4.2.** $\deg_G, u_5 = 3$. Since $m_r \geq 37$, it follows that $\deg_G, u_i = 3$ for $5 \leq i \leq 11$. We now consider the possible 3-element sets for $N_R(u_i)$ for $5 \leq i \leq 11$. Let $S \in \{N_R(u_5), N_R(u_6), \ldots, N_R(u_{11})\}$ and let $i$ be the smallest integer $i \in \{1, 2, \ldots, 10\}$ such that $v_i \in S$. Hence $1 \leq i \leq 7$. 
In summary, there are only six possibilities for the seven sets $N_R(u_i)$ for $5 \leq i \leq 11$:

1. $i = 1$. Since $v_1 \in N_R(u_1) \cap N_R(u_3)$, it follows that $v_2, v_3, v_4, v_7, v_8, v_9 \notin S$. Furthermore, because $v_5, v_6 \in N_R(u_2)$ and $v_5, v_{10} \in N_R(u_4)$, we have $v_5 \notin S$ and so $S = \{v_1, v_6, v_{10}\}$.

2. $i = 2$. Since $v_2 \in N_R(u_1) \cap N_R(u_4)$, it follows that $v_3, v_4, v_5, v_8, v_{10} \notin S$. Furthermore, because $v_6, v_7 \in N_R(u_2)$ and $v_7, v_9 \in N_R(u_3)$, we have $v_7 \notin S$ and so $S = \{v_2, v_6, v_9\}$.

3. $i = 3$. Since $v_3 \in N_R(u_1)$, it follows that $v_4 \notin S$.

   (3.1) If $v_5 \in S$, then $v_4, v_6, v_7, v_8, v_{10} \notin S$ because $v_5 \in N_R(u_2) \cap N_R(u_4)$. Hence $S = \{v_3, v_5, v_9\}$.

   (3.2) If $v_5 \notin S$ but $v_6 \in S$, then $v_4, v_5, v_7 \notin S$ because $v_6 \in N_R(u_2)$. Hence $S$ is one of the three sets $\{v_3, v_6, v_8\}$, $\{v_3, v_6, v_9\}$ and $\{v_3, v_6, v_{10}\}$.

   (3.3) If $v_5, v_6 \notin S$ but $v_7 \in S$, then $v_8, v_9 \notin S$ because $v_7 \in N_R(u_3)$. Hence $S = \{v_3, v_7, v_{10}\}$.

   (3.4) If $v_5, v_6, v_7 \notin S$, then $u_8 \notin S$ because $v_8, v_9 \in N_R(u_3)$ and $v_8, v_{10} \in N_R(u_4)$. Hence $S = \{v_3, v_9, v_{10}\}$.

4. $i = 4$. Since $v_4 \in N_R(u_2)$, it follows that $v_5, v_6, v_7 \notin S$. Furthermore, because $v_8, v_9 \in N_R(u_3)$ and $v_8, v_{10} \in N_R(u_4)$, it follows that $u_8 \notin S$ and so $S = \{v_4, v_9, v_{10}\}$.

5. $i = 5$. Since $v_5 \in N_R(u_2) \cap N_R(u_4)$, it follows that $v_6, v_7, v_8, v_{10} \notin S$. However then, $|S| < 3$, which is impossible.

6. $i = 6$. Since $v_6 \in N_R(u_2)$, it follows that $v_7 \notin S$. Because $v_8, v_9 \in N_R(u_3)$ and $v_8, v_{10} \in N_R(u_4)$, it follows that $u_8 \notin S$ and so $S = \{v_6, v_9, v_{10}\}$.

7. $i = 7$. Hence, $S$ must contain at least one of $v_8$ and $v_9$, say $v_8 \in S$. However then, $|S \cap N_R(u_3)| = 2$, which is impossible.

In summary, there are only six possibilities for the seven sets $N_R(u_i)$ for $5 \leq i \leq 11$:

- $S_1 = \{v_1, v_6, v_{10}\}$
- $S_2 = \{v_2, v_6, v_9\}$
- $S_3 = \{v_3, v_5, v_9\}$
- $S_4 = \{v_3, v_7, v_{10}\}$
- $S_5 = \{v_3, v_6, v_8\}$, $\{v_3, v_6, v_9\}$, $\{v_3, v_6, v_{10}\}$
- $S_6 = \{v_3, v_9, v_{10}\}$, $\{v_4, v_9, v_{10}\}$, $\{v_6, v_9, v_{10}\}$
Therefore, there are two sets $N_R(u_i)$ and $N_R(u_j)$, where $i, j \in \{5, 6, \ldots, 11\}$ and $i \neq j$, such that $N_R(u_i)$ and $N_R(u_j)$ are chosen to be in the same set. This produces a red $C_4$, which is a contradiction.

By Theorems 4.5.1 and 4.5.2, it follows that $20 \leq R_2(C_4, C_4, C_4) \leq 21$. 

\n
Chapter 5

Bicoloring View of Matchings and Domination

5.1 Introduction

As we discussed in Chapter 1, although proper edge colorings, monochromatic subgraphs, rainbow subgraphs and rainbow colorings have been the subject of many studies, there are also numerous other red-blue colorings of graphs whose definitions depend on a fixed graph $H$, certain red-blue colorings of $H$ and a specified blue edge of the resulting edge-colored graph $F$ of $H$. This gives rise to the concepts of color frames $F$ of a given graph $H$ and red-blue colorings of graphs called $F$-colorings. Inspired by the concepts of domination and stratification in graphs, we study such $F$-colorings for all color frames $F$ of the paths of orders 3 and 4 and show that these $F$-colorings provide a new framework for edge independence and various types of edge domination in graphs.

5.2 On $F$-Colorings of Graphs

In a red-blue coloring of a graph $G$, every edge of $G$ is colored red or blue (where adjacent edges may be colored the same). Also, all edges of $G$ may be colored red or all edges may be colored blue. Let $F$ be a connected graph of size 2 or more with a red-blue coloring in which at least one edge is colored red and at least one edge of $F$ is colored blue. One of the blue edges of $F$ is designated as the root edge of $F$. The underlying graph of $F$ is the graph $H$ obtained by removing the colors assigned to the edges of $F$. In this case, $F$ is called a color frame of $H$. The simplest example of this is the unique color frame $F_0$ of the path $P_3$ of order 3 (shown in Figure 5.1). The five (distinct) color
frames $F_1, F_2, \ldots, F_5$ of the path $P_4$ of order 4 are also shown in Figure 5.1, where each root edge is indicated by a bold line.

\[
\begin{align*}
F_0 : & \quad \text{r} \quad \text{b} \\
F_1 : & \quad \text{b} \quad \text{r} \quad \text{b} \\
F_2 : & \quad \text{r} \quad \text{b} \quad \text{r} \\
F_3 : & \quad \text{r} \quad \text{r} \quad \text{b} \\
F_4 : & \quad \text{r} \quad \text{b} \quad \text{b} \\
F_5 : & \quad \text{r} \quad \text{b} \quad \text{b}
\end{align*}
\]

Figure 5.1: Color frames of $P_3$ and $P_4$

For a color frame $F$, an $F$-coloring of a graph $G$ is a red-blue coloring of $G$ in which every blue edge of $G$ is the root edge of a copy of $F$ in $G$. If $G$ contains no subgraph isomorphic to $F$, then the only $F$-coloring of $G$ is that in which every edge of $G$ is red. The $F$-chromatic index $\chi'_F(G)$ of $G$ is the minimum number of red edges in an $F$-coloring of $G$. Since the edge coloring of $G$ that assigns red to every edge is an $F$-coloring of $G$, the number $\chi'_F(G)$ exists for every color frame $F$ and every graph $G$. An $F$-coloring of $G$ having exactly $\chi'_F(G)$ red edges is called a minimum $F$-coloring of $G$. For a given color frame $F$ and a nonempty graph $G$ of size $m$, the coloring that assigns the color red to all edges of $G$ is vacuously an $F$-coloring. This implies that $\chi'_F(G)$ exists for every graph $G$ and that $1 \leq \chi'_F(G) \leq m$ where $m$ is the size of $G$.

As an illustration, consider the five red-blue colorings $c_i$ ($1 \leq i \leq 5$) of $Q_3$ shown in Figure 5.2, where the solid lines are red edges and the dashed lines are blue edges. The coloring $c_1$ is an $F_1$-coloring for each color frame $F_i$ ($1 \leq i \leq 5$) shown in Figure 5.1. For each $j$ with $2 \leq j \leq 5$, the coloring $c_j$ is a minimum $F_j$-coloring of $Q_3$. Thus $\chi'_{F_2}(Q_3) = \chi'_{F_3}(Q_3) = 4$, $\chi'_{F_4}(Q_3) = 2$ and $\chi'_{F_5}(Q_3) = 3$. The coloring $c_5$ is a minimum $F_j$-coloring ($j = 0, 1$) and so $\chi'_{F_0}(Q_3) = \chi'_{F_1}(Q_3) = 3$.

These concepts were introduced by Chartrand and Zhang and first studied in [38] by Johnston, Kratky and Mashni. As we mentioned in Chapter 1, a vertex version of this concept was introduced in [9], which provided a generalization of the area of domination, and studied further by many (see [8, 25, 32] for example). Although these concepts are related through the line graph of a graph, this fact, as with proper colorings, has shown no benefit.
If $G$ is a disconnected graph with components $G_1$, $G_2$, ..., $G_k$ where $k \geq 2$, then
$$\chi_F'(G) = \chi_F'(G_1) + \chi_F'(G_2) + \cdots + \chi_F'(G_k).$$
Thus, it suffices to consider only connected graphs. For an $F$-coloring $c$ of a graph $G$, let $E_{c,r}$ denote the set of red edges of $G$ and $E_{c,b}$ the set of blue edges of $G$. (We also use $E_r$ and $E_b$ for $E_{c,r}$ and $E_{c,b}$, respectively, when the coloring $c$ under consideration is clear.) Thus $\{E_r, E_b\}$ is a partition of the edge set $E(G)$ of $G$. Furthermore, let $G_r = G[E_r]$ denote the red subgraph induced by $E_r$ and $G_b = G[E_b]$ the blue subgraph induced by $E_b$.

For a given color frame $F$, a minimal $F$-coloring of a graph $G$ is an $F$-coloring with the property that if any red edge of $G$ is re-colored blue, then the resulting red-blue coloring of $G$ is not an $F$-coloring of $G$. Obviously, every minimum $F$-coloring is minimal but the converse is not true in general (as we will soon see). The maximum number of red edges in a minimal $F$-coloring of $G$ is the upper $F$-chromatic index $\chi''_F(G)$ of $G$. Since every minimum $F$-coloring of $G$ is minimal, $\chi_F'(G) \leq \chi''_F(G)$. For example, consider the two red-blue colorings $c_2$ and $c_5$ of $Q_3$ shown in Figure 5.2. The coloring $c_5$ is a minimum $F_0$-coloring and so $c_5$ is also minimal. On the other hand, the coloring $c_2$ is a minimal $F_0$-coloring that is not minimum. In fact, $\chi''_{F_0}(Q_3) = 4$, while $\chi'_{F_0}(Q_3) = 3$ as we saw earlier.

In the next two sections, we show that $F_0$-colorings of graphs (and the $F_0$-chromatic index) have connections with two well-known concepts in graph theory.
5.3 A Bichromatic View of Matchings

A central topic in graph theory is that of matchings. In fact, Lovász and Plummer have written a book [46] on the theory of matchings. A set of edges in a graph $G$ is independent if no two edges in the set are adjacent in $G$. The edges in an independent set of edges of $G$ form a matching in $G$. If $M$ is a matching in a graph $G$ with the property that every vertex of $G$ is incident with an edge of $M$, then $M$ is a perfect matching in $G$. Clearly, if $G$ has a perfect matching $M$, then $G$ has even order and the subgraph induced by $M$ is a 1-factor of $G$.

A matching of maximum size in $G$ is a maximum matching. Thus every perfect matching is a maximum matching but the converse is not true. In particular, if the order of $G$ is odd, then $G$ cannot have a perfect matching. The edge independence number $\alpha'(G)$ of $G$ is the number of edges in a maximum matching of $G$. The number $\alpha'(G)$ is also referred to as the matching number of $G$.

A matching $M$ in a graph $G$ is a maximal matching of $G$ if $M$ is not a proper subset of any other matching in $G$. While every maximum matching is maximal, a maximal matching need not be a maximum matching. The minimum number of edges in a maximal matching of $G$ is called the lower edge independence number or lower matching number $\alpha''(G)$ of $G$. Necessarily, $\alpha''(G) \leq \alpha'(G)$. We will see that matchings in graphs can be looked at in terms of $F$-colorings for a specific color frame $F$. In particular, we show for the unique color frame $F_0$ of $P_3$ (shown in Figure 5.3, in which the only blue edge is the root edge of $F_0$) that the $F_0$-chromatic index $\chi'_{F_0}(G)$ is in fact the lower edge independence number of $G$. We begin with a lemma.

**Lemma 5.3.1** Let $G$ be a connected graph of size 2 or more and $F_0$ the color frame of $P_3$. If $G$ has a minimum $F_0$-coloring $c$ with $|E_{c,r}| = k$, then $G$ has a minimum $F_0$-coloring $c'$ with $|E_{c',r}| = k$ such that $E_{c',r}$ is a matching of $G$.

**Proof.** Let $F = F_0$. Among all minimum $F$-colorings of $G$ with exactly $k$ red edges, let $c'$ be one such that the red subgraph $G_r = G[E_{c',r}]$ has the maximum matching number. We claim that $E_{c',r}$ is a matching, for suppose that $E_{c',r}$ contains two adjacent edges, say $uv$ and $vw$ where $u, v, w \in V(G)$. If either $u$ or $w$ is an end-vertex of $G$, say the former,
then the red-blue coloring obtained from \(c'\) by changing the color of \(vw\) to blue is also an \(F\)-coloring of \(G\) with fewer red edges, which contradicts \(c'\) being a minimum \(F\)-coloring of \(G\). Thus neither \(u\) nor \(w\) is an end-vertex of \(G\). Suppose that \(w_1, w_2, \ldots, w_a\) are the vertices distinct from \(v\) that are adjacent to \(w\). If some edge \(ww_i\) \((1 \leq i \leq a)\) is red, then the red-blue coloring obtained from \(c'\) by changing the color of \(vw\) to blue is also an \(F\)-coloring of \(G\) with fewer red edges, a contradiction. Thus all edges \(ww_i\) \((1 \leq i \leq a)\) are blue. If there exists some vertex \(w_i\) \((1 \leq i \leq a)\) such that \(w_i\) is not incident to a red edge, then we can change the color of \(vw\) to blue and the color of \(ww_i\) to red, producing a minimum \(F\)-coloring in which the matching number of the red subgraph is larger than that of \(c'\). This contradicts the defining property of \(c'\). Thus every vertex \(w_i\) \((1 \leq i \leq a)\) is incident to at least one red edge. However then, the red-blue coloring obtained from \(c'\) by changing the color of \(vw\) to blue is again an \(F\)-coloring of \(G\) containing a smaller number of red edges, a contradiction. Therefore, \(E_{c',r}\) is a matching, as claimed.

\[\text{Theorem 5.3.2} \quad \text{Let} \ G \ \text{be a connected graph of size} \ 2 \ \text{or more. If} \ F_0 \ \text{is the color frame of} \ P_3, \ \text{then} \ \chi'_{F_0}(G) = \alpha''(G) \ \text{and} \ \chi''_{F_0}(G) \geq \alpha'(G).\]

\[\text{Proof.} \ \text{Let} \ F = F_0. \ \text{We first show that} \ \chi'_F(G) = \alpha''(G). \ \text{To verify that} \ \chi'_F(G) \leq \alpha''(G), \ \text{let} \ M \ \text{be a maximal matching of} \ G \ \text{with} \ |M| = \alpha''(G). \ \text{Since the red-blue coloring of} \ G \ \text{in which} \ M \ \text{is the set of red edges is an} \ F\text{-coloring of} \ G, \ \text{it follows that} \ \chi'_F(G) \leq |M| = \alpha''(G). \ \text{Next, we verify that} \ \alpha''(G) \leq \chi'_F(G). \ \text{By Lemma 5.3.1,} \ G \ \text{has a minimum} \ F\text{-coloring} \ c \ \text{such that} \ E_{c,r} \ \text{is a matching of} \ G. \ \text{We show that} \ E_{c,r} \ \text{is maximal. Let} \ e \in E(G) - E_{c,r} \ \text{be a blue edge. Since} \ c \ \text{is an} \ F\text{-coloring of} \ G, \ \text{it follows that} \ e \ \text{is adjacent to some edge in} \ E_{c,r}. \ \text{This implies that} \ E_{c,r} \cup \{e\} \ \text{is not a matching and so} \ E_{c,r} \ \text{is a maximal matching of} \ G. \ \text{Therefore,} \ \alpha''(G) \leq |E_{c,r}| = \chi'_F(G) \ \text{and so} \ \chi'_F(G) = \alpha''(G).\]

\[\text{To show that} \ \chi''_F(G) \geq \alpha'(G), \ \text{let} \ M \ \text{be a maximum matching of} \ G \ \text{and so} \ |M| = \alpha'(G). \ \text{Then the red-blue coloring} \ c \ \text{of} \ G \ \text{in which} \ E_{c,r} = M \ \text{is an} \ F\text{-coloring of} \ G. \ \text{It remains to show that} \ c \ \text{is a minimal} \ F\text{-coloring of} \ G. \ \text{Assume, to the contrary, that} \ c \ \text{is not minimal. Then there is a red edge} \ e \ \text{such that the red-blue coloring} \ c' \ \text{obtained from} \ c \ \text{by changing the color of} \ e \ \text{to blue is also an} \ F\text{-coloring of} \ G. \ \text{Since} \ E_{c,r} = M - \{e\} \ \text{and the blue edge} \ e \ \text{is not adjacent to any red edge in} \ E_{c',r}, \ \text{it follows that} \ c' \ \text{is not} \ F\text{-coloring of} \ G, \ \text{a contradiction. Therefore,} \ \chi''_F(G) \geq |M| = \alpha'(G).\]

\[\text{Although there are many connected graphs} \ G \ \text{for which} \ \chi''_{F_0}(G) = \alpha'(G), \ \text{it is also possible that} \ \chi''_{F_0}(G) > \alpha'(G). \ \text{For example, Figure 5.4 shows a connected graph} \ G \ \text{of order} \ 7 \ \text{such that} \ \chi_{F_0}(G) = 4 \ \text{and} \ \alpha'(G) = 3 \ \text{together with a minimal} \ F_0\text{-coloring with} \ 4 \ \text{red edges (indicated by bold lines).}\]
As a result of Theorem 5.3.2, we see that the lower matching number of $G$ is the minimum number of red edges in an $F_0$-coloring of $G$, where $F_0$ is the color frame of $P_3$. Theorem 5.3.2 therefore provides a new setting for the lower edge independence number (the lower matching number) $\alpha''(G)$ of a graph $G$. Furthermore, if $G$ is a connected graph of size 2 or more, then maximal matchings of $G$ and minimal $F_0$-colorings of $G$ are closely related, as we now show.

**Theorem 5.3.3** Let $G$ be a connected graph of size 2 or more, $F_0$ the color frame of $P_3$ and $M$ a matching of $G$. Then $M$ is a maximal matching of $G$ if and only if $G$ has a minimal $F_0$-coloring whose set of red edges is $M$.

**Proof.** First, suppose that $M$ is a maximal matching of $G$. As we saw in the proof of Theorem 5.3.2, the red-blue coloring having $M$ as the set of red edges is a minimal $F_0$-coloring of $G$. It remains to verify the converse. Assume that $c$ is a minimal $F_0$-coloring of $G$ such that $E_{c,r} = M$. We claim that $M$ is maximal, for otherwise, there is $e \notin M$ such that $M \cup \{e\}$ is a matching. This, however, implies that there is the blue edge $e$ that is not adjacent to any red edge in $M$ and so $c$ is not an $F_0$-coloring of $G$, which is a contradiction. 

It was shown in [40] that if $G$ is a graph and $k$ is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then $G$ contains a maximal matching with $k$ edges. The following is then a consequence of this result and Theorem 5.3.3.

**Corollary 5.3.4** Let $G$ be a connected graph of size 2 or more and $F_0$ the color frame of $P_3$. If $k$ is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then there is a minimal $F_0$-coloring of $G$ with exactly $k$ red edges.

We now describe the structure of the red subgraph produced by a minimal $F_0$-coloring of a graph. A graph $H$ is a galaxy if each component of $H$ is a star of order at least 2.
Theorem 5.3.5 Let $G$ be a connected graph of size 2 or more and $F_0$ the color frame of $P_3$. If $c$ is a minimal $F_0$-coloring of $G$, then the red subgraph induced by the set of red edges in $G$ is a galaxy.

Proof. Let $c$ be a minimal $F_0$-coloring of $G$ and let $S$ be a component of the corresponding red subgraph $G_r$ of $G$. If $S$ contains a path or triangle $(u,v,w,x)$ of length 3 as a subgraph, then the red-blue coloring obtained from $c$ by changing the color of $vw$ to blue is also an $F_0$-coloring of $G$ with fewer red edges, which contradicts that $c$ is minimal. Hence $S$ contains no cycle and the diameter of $S$ is at most 2, which implies that $S$ is a star. Therefore, $G_r$ is a galaxy.

5.4 A Bichromatic View of Edge Domination

An area of graph theory that has received increased attention during recent decades is that of domination. Two books [30, 31] by Haynes, Hedetniemi and Slater were devoted to this subject. An edge $e$ in a graph $G$ is said to dominate itself and all edges adjacent to $e$. A set $S$ of edges of $G$ is an edge dominating set of $G$ if every edge of $G$ is dominated by some edge in $S$. The minimum size of an edge dominating set of $G$ is the edge domination number of $G$ and is denoted by $\gamma'(G)$. Moreover, $\gamma'(G)$ is the domination number of the line graph of $G$. An edge dominating set of size $\gamma'(G)$ is called a minimum edge dominating set of $G$, while an edge dominating set $S$ of a graph $G$ is a minimal edge dominating set if no proper subset of $S$ is also an edge dominating set of $G$. While a minimum edge dominating set is minimal, the converse is not true. The maximum size of a minimal edge dominating set in $G$ is the upper edge domination number of $G$ and is denoted by $\gamma''(G)$.

Theorem 5.4.1 Let $G$ be a connected graph of size 2 or more. If $F_0$ is the color frame of $P_3$, then $\gamma'(G) = \chi'_{F_0}(G)$ and $\gamma''(G) = \chi''_{F_0}(G)$.

Proof. Let $F = F_0$. We first show that $\gamma'(G) = \chi'_{F}(G)$. Let $X$ be an edge dominating set with $|X| = \gamma'(G)$. Define a red-blue coloring $c$ of $G$ such that $E_{c,r} = X$. Let $e$ be a blue edge of $G$. Since $X$ is an edge dominating set of $G$, it follows that $e$ is adjacent to at least one red edge in $X = E_{c,r}$. Thus $c$ is an $F$-coloring of $G$ and so $\chi'_{F}(G) \leq |X| = \gamma'(G)$. Next, let $c'$ be an $F$-coloring of $G$ with $|E_{c',r}| = \chi'_{F}(G)$. Thus every edge not in $E_{c',r}$ is adjacent to at least one edge in $E_{c',r}$, which implies that $E_{c',r}$ is an edge dominating set of $G$. Thus $\gamma'(G) \leq |E_{c',r}| = \chi'_{F}(G)$.
Next, we show that $\gamma''(G) = \chi''_F(G)$. Let $Y$ be a minimal edge dominating set in $G$ with $|Y| = \gamma''(G)$. Define a red-blue coloring $c$ of $G$ such that $E_{c,r} = Y$. Let $e$ be a blue edge of $G$. Since $Y$ is an edge dominating set of $G$, it follows that $e$ is adjacent to at least one red edge in $Y = E_{c,r}$. Thus $c$ is an $F$-coloring of $G$. We claim that $c$ is minimal, for suppose that $c$ is not minimal. Then there is $e \in E_{c,r}$ such that the red-blue coloring $c_0$ obtained from $c$ by changing the color of $e$ to blue is also an $F$-coloring of $G$. This implies that each blue edge in the coloring $c_0$ is adjacent to at least one edge in $E_{c_0,r} = E_{c,r} - \{e\}$ and so the proper subset $E_{c_0,r}$ of $E_{c,r}$ is an edge dominating set of $G$, a contradiction. Thus, as claimed, $c$ is minimal and so $\chi''_F(G) \geq |Y| = \gamma''(G)$. Next, let $c'$ be a minimal $F$-coloring of $G$ with $|E_{c',r}| = \chi''_F(G)$. We claim that $E_{c',r}$ is a minimal edge dominating set in $G$. Since $c'$ is a $F$-coloring, every edge not in $E_{c',r}$ is adjacent to at least one edge in $E_{c',r}$ and so $E_{c',r}$ is an edge dominating set of $G$. Since $c'$ is minimal, if the color of any edge $e'$ in $E_{c',r}$ is changed to blue, then the resulting red-blue coloring is not an $F$-coloring. Thus some blue edge not in $E_{c',r} - \{e'\}$ is not adjacent to any edge in $E_{c',r} - \{e'\}$ and so $E_{c',r} - \{e'\}$ is not an edge dominating set of $G$. This implies that no proper subset of $E_{c',r}$ is an edge dominating set in $G$. Hence, $E_{c',r}$ is minimal. Thus $\gamma''(G) \geq |E_{c',r}| = \chi''_F(G)$ and so $\gamma''(G) = \chi''_F(G)$.

As a result of Theorem 5.4.1, the edge domination number of a connected graph $G$ of size 2 or more is the minimum number of red edges in an $F_0$-coloring of $G$ and the upper edge domination number is the maximum number of red edges in a minimal $F_0$-coloring of $G$. Therefore, Theorem 5.4.1 provides a new setting for the two edge domination numbers $\gamma'(G)$ and $\gamma''(G)$ of a graph $G$. Furthermore, the proof of Theorem 5.4.1 shows that (1) if $c$ is a minimal $F_0$-coloring of $G$, then $E_{c,r}$ is a minimal edge dominating set of $G$ and (2) if $S$ is a minimal edge dominating set of $G$, then the red-blue coloring $c'$ of $G$ with $E_{c',r} = S$ is a minimal $F_0$-coloring. Therefore, there is a one-to-one correspondence between the set of all minimal edge dominating sets of $G$ and the set of all minimal $F_0$-colorings of $G$. Hence the following is a consequence of the proof of Theorem 5.4.1.

**Corollary 5.4.2** Let $G$ be a connected graph of size 2 or more and $F_0$ the color frame of $P_3$. Then $S$ is a minimal edge dominating set of $G$ if and only if $G$ has a minimal $F_0$-coloring whose set of red edges is $S$.

In view of Theorems 5.3.2 and 5.4.1, $F_0$-colorings of graphs provide a new framework for both edge independence and edge domination and lead us to consider $F$-colorings of graphs for other choices of color frames $F$. 
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5.5 The Color Frames of $P_4$

In this section, we turn our attention to $F$-colorings of connected graphs of size at least 3, where $F$ is one of the five color frames $F_1, F_2, \ldots, F_5$ of $P_4$ shown in Figure 5.1 and the five parameters $\chi_{F_i}(G)$ ($1 \leq i \leq 5$) of a graph $G$. We begin with the color frame $F_1$, which we also refer to as the blue-red-blue color frame of $P_4$.

5.5.1 The Blue-Red-Blue Color Frame $F_1$ of $P_4$

Because of the symmetry of the blue-red-blue color frame $F_1$ of $P_4$ (shown in Figure 5.5), it doesn’t matter which of the two blue edges is chosen as the root edge of $F_1$. Since every $F_1$-coloring is also an $F_0$-coloring, it follows that $\chi_{F_0}(G) \leq \chi_{F_1}(G)$ for every connected graph $G$ of size at least 3. (5.1)

There are conditions under which equality in (5.1) holds. Let $\delta(G)$ denote the minimum degree of a graph $G$.

![Figure 5.5: The blue-red-blue color frame $F_1$ of $P_4$](image)

**Theorem 5.5.1** Let $G$ be a connected graph of size at least 3 and $F_1$ the blue-red-blue color frame of $P_4$. If $G$ is (i) triangle-free and $\delta(G) \geq 2$ or (ii) $\delta(G) \geq 3$, then $\chi_{F_1}(G) = \chi_{F_0}(G)$.

**Proof.** By (5.1), it remains only to show that $\chi_{F_1}(G) \leq \chi_{F_0}(G)$. Let $M$ be a maximal matching of $G$ such that $|M| = \alpha''(G)$. By Theorem 5.3.2, $\chi_{F_0}(G) = \alpha''(G)$. Let $c$ be a minimum $F_0$-coloring of $G$ such that $E_{c,r} = M$. We claim that $c$ is an $F_1$-coloring of $G$. Let $e_b$ be a blue edge of $G$. Since $E_{c,r}$ is a maximal matching, $E_{c,r} \cup \{e_b\}$ is not a matching and so $e_b$ is adjacent to a red edge $e_r$ in $E_{c,r}$. We may assume that $e_r = uv$ and $e_b = vw$. If $G$ is triangle-free and $\delta(G) \geq 2$ or $\delta(G) \geq 3$, then $u$ is adjacent to some vertex $u'$ distinct from $v$ and $w$. Since $e_r \in E_{c,r}$ and $E_{c,r}$ is a matching, $uu'$ cannot be red and so is blue. Thus $e_b$ belongs to the copy of $F$ with $E(F) = \{u'u, uv, vw\}$ rooted at $e_b$ in $G$ and so $c$ is an $F_1$-coloring of $G$, as claimed. Hence $\chi_{F_1}(G) \leq |E_{c,r}| = \alpha''(G)$. Therefore, $\chi_{F_1}(G) = \alpha''(G) = \chi_{F_0}(G)$.

The conditions (i) and (ii) in Theorem 5.5.1 are only sufficient for equality to hold in (5.1). For example, the graph $P_4$ in Figure 5.6 is triangle-free and has minimum
degree 1 but $\chi'_{F_0}(P_4) = \chi'_{F_1}(P_4) = 1$. Furthermore, the graphs $H_1$ and $H_2$, also shown in Figure 5.6, contain triangles but have minimum degree 1 and 2, respectively; yet $\chi'_{F_0}(H_1) = \chi'_{F_1}(H_1) = 2$ and $\chi'_{F_0}(H_2) = \chi'_{F_1}(H_2) = 1$.

![Graphs](image)

Figure 5.6: Graphs $G$ with $\chi'_{F_0}(G) = \chi'_{F_1}(G)$

There are graphs $G$ not satisfying the conditions stated in Theorem 5.5.1 such that $\chi'_{F_0}(G) \neq \chi'_{F_1}(G)$. For example, consider the graphs $G_1$, $G_2$ and $G_3$ in Figure 5.7.

![Graphs](image)

Figure 5.7: Graphs $G_i$ with $\chi'_{F_0}(G_i) < \chi'_{F_1}(G_i)$ ($1 \leq i \leq 3$)

The graph $G_1$ is triangle-free and $\delta(G_1) = 1$, $G_2$ contains a triangle and $\delta(G_2) = 1$ and $G_3$ contains a triangle and $\delta(G_3) = 2$. For each graph $G_i$ ($i = 1, 2, 3$), $\chi'_{F_0}(G_i) < \chi'_{F_1}(G_i)$. A minimum $F_0$-coloring and a minimum $F_1$-coloring for each of these graphs are shown in Figure 5.7. In each red-blue coloring, the solid edges are red edges and the dashed lines are blue edges. For $i = 1, 2, 3$, $\chi'_{F_1}(G_i) = \chi'_{F_0}(G_i) + 1$. The difference $\chi'_{F_1}(G) - \chi'_{F_0}(G)$ can not only be arbitrarily large, there is essentially no restrictions on the values of $\chi'_{F_0}(G)$ and $\chi'_{F_1}(G)$ of a graph $G$.

**Theorem 5.5.2** For each pair $a, b$ of positive integers $a \leq b$, there is a connected graph $G$ such that $\chi'_{F_0}(G) = a$ and $\chi'_{F_1}(G) = b$. 
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Proof. First, assume that \( a = b \). If \( a = b = 1 \), then \( \chi'_{F_0}(P_4) = \chi'_{F_1}(P_4) = 1 \); while if \( a = b \geq 2 \), then \( \chi'_{F_1}(K_{2a}) = \chi'_{F_0}(K_{2a}) = a \). Thus, we may assume that \( 1 \leq a < b \). For \( a = 1 \), the star \( K_{1,b} \) has \( \chi'_{F_0}(K_{1,b}) = 1 \) and \( \chi'_{F_1}(K_{1,b}) = b \) (as \( K_{1,b} \) contains no \( P_4 \) as a subgraph). Thus, we may assume that \( a \geq 2 \). First, let \( G_1, G_2, \ldots, G_a \) be \( a \) copies of the star \( K_{1,b-a+1} \) of order \( b - a + 2 \), where

\[
V(G_i) = \{v_i, v_{i,1}, v_{i,2}, \ldots, v_{i,b-a+1}\}
\]

and \( v_i \) is the central vertex of \( G_i \) for \( 1 \leq i \leq a \). The graph \( G \) is obtained from these \( a \) graphs by adding a new vertex \( v \) and joining \( v \) to each vertex \( v_i \) in \( G_i \) for \( 1 \leq i \leq a \). Then \( \chi'_{F_0}(G) = \alpha(G) = a \). It remains to show that \( \chi'_{F_1}(G) = b \). Since the red-blue coloring \( c \) with

\[
E_{c,r} = E(G) \cup \{vv_i : 2 \leq i \leq a\}
\]

is an \( F_1 \)-coloring of \( G \), it follows that \( \chi'_{F_1}(G) \leq |E_{c,r}| = b \). Next, we show that \( \chi'_{F_1}(G) \geq b \). Let \( c' \) be a minimum \( F_1 \)-coloring of \( G \). For each integer \( i \) with \( 1 \leq i \leq a \), if \( c' \) assigns blue to an edge in \( E(G_i) \), then \( c'(vv_i) \) must be red and some \( c'(vv_j) \) must be blue where \( j \neq i \). This in turn implies that \( c' \) must assign red to all edges in \( E(G_j) \). Furthermore, \( c' \) must assign blue to at least one edge in \( E(G_1) \cup E(G_2) \cup \cdots \cup E(G_a) \) (for otherwise, all edges in \( G \) must be colored red). We may assume that \( E(G_1) \cup \{vv_2\} \subset E_{c',r} \). Also, \( c' \) must assign red to at least one edge in \( E(G_i) \cup \{vv_i\} \) for \( 3 \leq i \leq a \), it follows that \( |E_{c',r}| \geq (b - a + 2) + (a - 2) = b \). Therefore, \( \chi'_{F_1}(G) = b \), as claimed. \( \blacksquare \)

5.5.2 The Red-Blue-Red Color Frame \( F_2 \) of \( P_4 \)

Consider the red-blue-red color frame \( F_2 \) of \( P_4 \) in Figure 5.8 in which the only blue edge is the root edge of \( F \). Since every \( F_2 \)-coloring is also an \( F_0 \)-coloring, \( \chi'_{F_0}(G) \leq \chi'_{F_2}(G) \) for every connected graph \( G \) of size at least 3.

![Figure 5.8: The red-blue-red color frame \( F_2 \) of \( P_4 \)](image)

A set \( S \) of edges of \( G \) is a \( k \)-edge dominating set of \( G \) if every edge in \( E(G) - S \) is dominated by at least \( k \) edges in \( S \). Since \( E(G) \) is such a set, every graph \( G \) has a \( k \)-edge dominating set. The minimum size of a \( k \)-edge dominating set of \( G \) is the \( k \)-edge domination number of \( G \) and is denoted by \( \gamma_k(G) \). A \( k \)-edge dominating set of size \( \gamma_k(G) \) is called a minimum \( k \)-edge dominating set of \( G \). Observe that if \( c \) is an \( F_2 \) coloring of a graph \( G \), then each blue edge is adjacent to two independent red edges. Thus \( E_{c,r} \) is
a 2-edge dominating set of $G$. This implies that
\[ \gamma'_2(G) \leq \chi'_{F_2}(G) \quad \text{for every connected graph } G \text{ of size at least 3.} \] (5.2)

Furthermore, no blue edges can be pendant edges; that is, every pendant edge must be colored red in any $F_2$-coloring of $G$. Hence if $G$ has $p$ pendant edges, then $\chi'_{F_2}(G) \geq p$. For example, if $G$ is a double star (a tree of diameter 3) of size $m \geq 5$, then $\chi'_{F_2}(G) = m - 1$. Since $\gamma'_2(G) = 3$, it follows that $\chi'_{F_2}(G) - \gamma'_2(G) = m - 4$ which can be arbitrarily large. In fact, more can be said.

**Proposition 5.5.3** For each pair $a, b$ of integers with $2 \leq a \leq b$, there is a connected graph $G$ such that $\gamma'_2(G) = a$ and $\chi'_{F_2}(G) = b$.

**Proof.** We consider two cases, according to $a = b$ or $a < b$.

**Case 1.** $a = b$. If $a$ is even, say $a = 2k$ where $k \geq 1$, then let $G$ be the graph obtained from $kP_3$ by adding a new vertex $v$ and joining $v$ to each end-vertex of $kP_3$. Then $\gamma'_2(G) = \chi'_{F_2}(G) = a$. If $a$ is odd, say $a = 2k + 1$ where $k \geq 1$, then let $G$ be the graph obtained from the union $kP_3 \cup P_2$ of $kP_3$ and $P_2$ by adding a new vertex $v$ and joining $v$ to each end-vertex of $kP_3 \cup P_2$. Then $\gamma'_2(G) = \chi'_{F_2}(G) = a$.

**Case 2.** $a < b$. For $a = 2$, let $G = K_{1,b}$. Then $\gamma'_2(G) = 2$ and $\chi'_{F_2}(G) = b$. For $a = 3$, let $G$ be a double of size $b + 1$. Then $\gamma'_2(G) = 3$ and $\chi'_{F_2}(G) = b$. We now assume $a \geq 4$. Let $G$ be the graph described in Case 1 such that $\gamma'_2(G) = \chi'_{F_2}(G) = a$. Let $v \in V(G)$ be the vertex of $G$ such that $\deg v = a$ if $a$ is even and $\deg v = a + 1$ if $a$ is odd. Let $H$ be the graph obtained from the graph $G$ in Case 1 by adding $b - a$ new vertices and joining each of these new vertices to the vertex $v$. Then $\gamma'_2(H) = a$ and $\chi'_{F_2}(H) = b$. ■

### 5.5.3 The Red-Red-Blue Color Frame $F_3$ of $P_4$

Consider the red-red-blue color frame $F_3$ of $P_4$ in Figure 5.9 in which the only blue edge is the root edge of $F$. Since every $F_3$-coloring is also an $F_0$-coloring, $\chi'_{F_0}(G) \leq \chi'_{F_3}(G)$ for every connected graph $G$ of size at least 3.

![Figure 5.9: The red-red-blue color frame $F_3$ of $P_4$](image)

A *total edge dominating set* in a connected graph $G$ is a subset $S$ of $E(G)$ such that every edge of $G$ is adjacent to an edge of $S$. Thus a total edge dominating set contains no
independent edges. If $G$ is a nonempty graph containing no component $K_2$, then $E(G)$ is a total edge dominating set and so every connected graph of order at least 3 has a total edge dominating set. The total edge domination number $\gamma'_t(G)$ is the minimum size of a total edge dominating set. A total edge dominating set of size $\gamma'_t(G)$ is a minimum total edge dominating set of $G$.

**Theorem 5.5.4** If $G$ is a connected graph of size at least 3, then

$$\gamma'_t(G) \leq \chi'_{F_3}(G).$$

**Proof.** First, we make an observation. If $G$ has a minimum $F_3$-coloring $c$ such that $E_{c,r}$ contains no independent edges, then each red edge is adjacent to a red edge and each blue edge is adjacent to a red edge, which implies that $E_{c,r}$ is a total edge dominating set of $G$ and so $\gamma'_t(G) \leq |E_{c,r}| = \chi'_{F_3}(G)$. Thus, it suffices to show that such a minimum $F_3$-coloring exists. Among all minimum $F_3$-colorings of $G$, let $c$ be one such that the number of independent edges in $E_{c,r}$ is minimum. If $E_{c,r}$ has no independent edges, then $c$ has the desired property. Thus, we may assume that $E_{c,r}$ contains an independent edge $f$. Thus $f$ does not belong to any copy of $F_3$ in this coloring $c$. Since $G$ is connected and $f$ is not adjacent to any red edge, $f$ is adjacent to a blue edge $e$. Because $c$ is an $F_3$-coloring, $e$ belongs to a copy of $F_3$. Suppose that $f = uv$, $e = vw$ and $e$ belongs to the copy $(v, w, x, y)$ of $F_3$ where then $wx$ and $xy$ are red edges.

Now the coloring $c'$ obtained from $c$ by interchanging the colors of $f$ and $e$ is a minimum $F_3$-coloring the number of whose independent edges is smaller than that of $c$. This contradicts the defining property of $c$. 

For each positive integer $\ell$,

$$\gamma'_t(P_{4\ell}) = \chi'_{F_3}(P_{4\ell}) = 2\ell \quad \text{and} \quad \gamma'_t(P_{4\ell+2}) = \chi'_{F_3}(P_{4\ell+2}) = 2\ell + 1.$$ 

Therefore, for each positive integer $k$, there is a connected graph $G$ such that $\gamma'_t(G) = \chi'_{F_3}(G) = k$. On the other hand, $\chi'_{F_3}(G) - \gamma'_t(G)$ can be arbitrarily large. In fact, more can be said. Two end-vertices of a graph are said to be similar if they are adjacent to a same vertex.

**Proposition 5.5.5** For each positive integer $k$, there is a connected graph $G_k$ such that

$$\chi'_{F_3}(G_k) - \gamma'_t(G_k) = k.$$
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Proof. We recursively construct a sequence $G_1, G_2, \ldots$ of graphs as follows. Let $G_1$ be the graph shown in Figure 5.10. The graph $G_1$ has two pairs of similar end-vertices, namely $\{u_1, u_2\}$ and $\{w, w_1\}$. The graph $G_2$ is obtained from $G_1$ and another copy of $G_1$ by identifying a pair of similar end-vertices in each graph (see Figure 5.10 where the solid vertices are identified vertices). Thus $G_2$ has two pairs of similar end-vertices. For each $k \geq 3$, the graph $G_k$ is obtained from $G_{k-1}$ and a copy of $G_1$ by identifying a pair of similar end-vertices in each graph. The graph $G_3$ is also shown in Figure 5.10.

For each integer $k \geq 1$, let $S_k$ be the set of bridges that are not pendant edges in $G_k$, where then the subgraph $G_k[S_k]$ induced by $S_k$ is $kP_3$, and let $X_k$ be the set of pendant edges each of which is incident to the center vertex of some component $P_3$ in $G_k[S_k]$. For example, $S_1 = \{uw, vw\}$ and $X_1 = \{vx\}$. Thus $S_k \cap X_k = \emptyset$, $|X_k| = k$ and the subgraph $G_k[S_k \cup X_k]$ induced by $S_k \cup X_k$ is $kK_{1,3}$. Since $S_k$ is a minimum total edge dominating set of $G_k$ and the red-blue coloring $c$ with $E_{c,r} = S_k \cup X_k$ is a minimum $F_3$-coloring of $G_k$, it follows that

$$\chi'_{F_3}(G_k) = |S_k \cup X_k| = |S_k| + |X_k| = \gamma'_t(G_k) + k$$

for $k \geq 1$. \hfill $\blacksquare$

![Figure 5.10: The graphs $G_1$, $G_2$ and $G_3$](image)
The two red-blue-blue color frames $F_4$ and $F_5$ of $P_4$ are shown in Figure 5.11 in which one of the two blue edges is the root edge of the color frame. Again, we indicate the root edge in each of $F_4$ and $F_5$ by a bold edge. It appears that $\chi'_{F_5}(G)$ is not related to any known edge domination parameters.

![Figure 5.11: The two red-blue-blue color frames $F_4$ and $F_5$ of $P_4$](image)

We consider the $F_5$-colorings of a connected graph of size at least 3. A set $S \subseteq E(G)$ is a \textit{restrained edge dominating set} if every edge not in $S$ is adjacent to an edge in $S$ and to an edge in $E(G) - S$. Every graph has a restrained edge dominating set since $E(G)$ is such a set. The \textit{restrained edge domination number} $\gamma'_r(G)$ is the minimum size of a restrained edge dominating set of $G$. A restrained edge dominating set of size $\gamma'_r(G)$ is a \textit{minimum restrained edge dominating set} of $G$. If $c$ is an $F_5$-coloring of $G$, then every blue edge is adjacent to a red edge and a blue edge. Thus the set $E_{c,r}$ of red edges is a restrained edge dominating set of $G$ and so $\gamma'_r(G) \leq |E_{c,r}|$. Therefore,

$$\gamma'_r(G) \leq \chi'_{F_5}(G)$$

for every connected graph $G$ of size at least 3. (5.3)

**Proposition 5.5.6** Let $a$ and $b$ be positive integers with $a \leq b$. Then there is a connected graph $G$ of size at least 3 such that $\gamma'_r(G) = a$ and $\chi'_{F_5}(G) = b$ if and only if $(a, b) \neq (1, 1)$.

**Proof.** Assume, to the contrary, that there is a connected graph $G$ of size at least 3 such that $\gamma'_r(G) = \chi'_{F_5}(G) = 1$. Let $S = \{e\}$ be a minimum restrained edge dominating set of $G$. Then each edge $f$ distinct from $e$ in $G$ must be adjacent to $e$ and another edge $f'$ that is also adjacent to $e$. Since (1) $f$ and $f'$ are adjacent and (2) $f$ and $f'$ are both adjacent to $e$, it follows that $\{e, f, f'\}$ forms a triangle. This implies that $G - e$ is the complete bipartite graph $K_{2,k}$ for some positive integer $k$ and the edge $e$ joins the only two vertices in one of the partite sets of $G$. If $k = 1$, then $G = K_3$ and $\chi'_{F_5}(G) = 3$, a contradiction. Thus $k \geq 2$ and the graph $G$ is shown in Figure 5.12. Let $c$ be a minimum $F_5$-coloring of $G$ where then $|E_{c,r}| = 1$. First assume that $E_{c,r} = \{e\}$. If $F$ is a copy of $F_5$ in $G$, then $e$ is the middle edge of $F$ and so no blue edge belongs to a copy of $F_5$. Thus $E_{c,r} = \{f\}$ where $f \neq e$. However then, the edge $f''$ is not adjacent to $f$ and so does not belong to a copy of $F_5$ in $G$, a contradiction.
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For the converse, let \( a \) and \( b \) of positive integers with \( a \leq b \) such that \( (a, b) \neq (1, 1) \). First, assume that \( a = b \geq 2 \). Let \( G = S(K_{1,a}) \) be the subdivision of \( K_{1,a} \); that is, \( G \) is obtained from \( K_{1,a} \) by subdividing each edge exactly once. Let \( S \) be the set of pendant edges of \( G \). Then \( S \) is a minimum restrained edge dominating set of \( G \) and the red-blue coloring \( c \) with \( E_{c,r} = S \) is a minimum \( F_5 \)-coloring of \( G \). Thus \( \gamma_4'(G) = \chi_{F_5}'(G) = a \). Next, assume that \( a < b \). Let \( G \) be the graph obtained from \( S(K_{1,a}) \) by adding \( b - a \) pendant edges at a vertex of degree 2 in \( S(K_{1,a}) \). Then \( \gamma_4'(G) = a \) and \( \chi_{F_5}'(G) = b \).

There is no connected graph \( G \) of size 3 or more such that \( \chi_{F_5}'(G) = \chi_{F_5}'(G) = 1 \). On the other hand, since \( \chi_{F_4}'(P_{3\ell+2}) = \chi_{F_5}'(P_{3\ell+2}) = \ell + 1 \) for each \( \ell \geq 1 \), there is a connected graph \( G \) such that \( \chi_{F_4}'(G) = \chi_{F_5}'(G) = k \) for each integer \( k \geq 2 \).

**Proposition 5.5.7** For each positive integer \( k \), there is a connected graph \( G_k \) such that

\[
\chi_{F_5}'(G_k) - \chi_{F_4}'(G_k) = k.
\]

**Proof.** For each \( k \geq 1 \), let \( G_k = S(K_{1,k+4}) \) be the subdivision of \( K_{1,k+4} \) and let \( S \) be the set of pendant edges of \( G_k \). Since (1) every \( F_5 \)-coloring of \( G_k \) must assign red to each edge in \( S \) and (2) the red-blue coloring \( c \) of \( G_k \) such \( E_{c,r} = S \) is an \( F_5 \)-coloring, \( \chi_{F_5}'(G_k) = |S| = k + 4 \).

Next, we show that \( \chi_{F_4}'(G_k) = 4 \). Let

\[
V(G_k) = \{u, u_1, u_2, \ldots, u_{k+4}, v_1, v_2, \ldots, v_{k+4}\}
\]

where \( u \) is the central vertex of the subgraph \( K_{1,k+4} \) in \( G_k \), \( u \) is adjacent to \( u_i \) for \( 1 \leq i \leq k + 4 \) and \( u_i \) is adjacent to \( v_i \) for \( 1 \leq i \leq k + 4 \). The red-blue coloring \( c_0 \) with \( E_{c_0,r} = \{uu_1, u_1v_1, u_2v_2, u_3v_3\} \) is an \( F_4 \)-coloring of \( G \) and so \( \chi_{F_4}'(G_k) \leq 4 \). Assume, to the contrary, that \( \chi_{F_4}'(G_k) \leq 3 \). Let \( c' \) be a minimum \( F_4 \)-coloring of \( G_k \) where then \( |E_{c'r}| \leq 3 \). Note that \( c' \) must assign blue to at least one edge in \( \{uu_i : 1 \leq i \leq k + 4\} \), say \( uu_1 \) is blue. Since \( uu_1 \) belongs to a copy of \( F_4 \), it follows that \( c' \) must assign red to at least one pendant edge in \( G_k \), say \( u_nv_n \) is red and belongs to the same copy of \( F_4 \) as

![Figure 5.12: A graph G with \( \gamma_4'(G) = 1 \)](image-url)
Then $uu_s$, $s \neq 1$, is blue and must belong to a copy of $F_4$. It follows that $\ell'$ must assign red to at least one pendant edge that is not $u_sv_s$ in $G_k$, say $u_tv_t$ is red, where then $1 \leq s \neq t \leq k + 4$ and $uu_s$ and $uu_t$ are blue. Let $j \in \{1, 2, \ldots, k + 4\} - \{s, t\}$. Either $u_jv_j$ is red or $u_jv_j$ is blue. If $u_jv_j$ is blue, then some edge $uu_p$ must be red where $p \notin \{1, j, s, t\}$. Hence $|E_{\ell'}| = 3$ and either
\[(i) \ E_{\ell'} = \{u_sv_s, u_tv_t, u_jv_j\} \text{ or } (ii) \ E_{\ell'} = \{u_sv_s, u_tv_t, uu_p\}.
\]
If (i) occurs, then each blue edge $u_iv_i$ where $i \in \{1, 2, \ldots, k + 4\} - \{s, t, j\}$ does not belong to a copy of $F_4$; while if (ii) occurs, then the blue edge $u_uv_p$ does not belong to a copy of $F_4$. In any case, a contradiction is produced and so $\chi'_{F_4}(G_k) = 4$. Therefore, $\chi'_{F_5}(G_k) = k$.

Although there are graphs $G$ for which $\chi'_{F_4}(G) > \chi'_{F_5}(G)$ (see the graph $G$ in Figure 5.13), it is not known whether there is a connected graph $G_k$ such that $\chi'_{F_4}(G_k) = \chi'_{F_5}(G_k) = k$ for every positive integer $k$.

For the five color frames $F_1, F_2, \ldots, F_5$ of $P_4$, a 2-element set $\{i, j\}$, where $i, j \in \{1, 2, \ldots, 5\}$, is called a realizable set if there exist a graph $G$ such that $\chi'_{F_i}(G) < \chi'_{F_j}(G)$ and a graph $H$ such that $\chi'_{F_j}(H) < \chi'_{F_i}(H)$. We will see that every 2-element subset of $\{1, 2, \ldots, 5\}$ is realizable with one possible exception. In order to show this, we first present three examples. In each of the following figures, the solid lines are red edges and the dashed lines are blue edges.

- For the graph $G$ of Figure 5.13, $\chi'_{F_1}(G) = 1$, $\chi'_{F_i}(G) = 3$ for $i = 2, 3, 4$ and $\chi'_{F_5}(G) = 2$. For $1 \leq i \leq 5$, minimum $F_i$-colorings of $G$ are also shown in Figure 5.13.

![Figure 5.13: Minimum $F_i$-colorings of a graph $G$ for $1 \leq i \leq 5$](image)

- For the graph $G$ of Figure 5.14, $\chi'_{F_1}(G) = 3$, $\chi'_{F_2}(G) = 5$, $\chi'_{F_3}(G) = 2$ and $\chi'_{F_4}(G) = \chi'_{F_5}(G) = 4$. Minimum $F_i$-colorings of $G$ are shown in Figure 5.14 for $1 \leq i \leq 5$. 


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Figure 5.14: Minimum $F_i$-colorings of a graph $G$ for $1 \leq i \leq 5$

- For the graph $G$ of Figure 5.15, $\chi'_{F_1}(G) = 1$, $\chi'_{F_2}(G) = 3$, $\chi'_{F_3}(G) = 2$ and $\chi'_{F_4}(G) = \chi'_{F_5}(G) = 4$. Minimum $F_i$-colorings of $G$ are shown in Figure 5.14 for $1 \leq i \leq 5$.

Figure 5.15: Minimum $F_i$-colorings of a graph $G$ for $1 \leq i \leq 5$

We are now prepared to present the following.

**Theorem 5.5.8**  Every 2-element subset of $\{1, 2, \ldots, 5\}$ is realizable except possibly $\{1, 2\}$ and $\{1, 5\}$.

**Proof.** For each subset $\{i, j\} \subseteq \{1, 2, \ldots, 5\}$ and $\{i, j\} \neq \{1, 2\}$, where $i < j$, we construct two graphs $G_{i,j}$ and $H_{i,j}$ such that $\chi'_{F_1}(G_{i,j}) < \chi'_{F_2}(G_{i,j})$ and $\chi'_{F_3}(H_{i,j}) < \chi'_{F_5}(H_{i,j})$. Recall that $\chi'_{F_1}(Q_3) = 3$, $\chi'_{F_2}(Q_3) = \chi'_{F_3}(Q_3) = 4$, $\chi'_{F_4}(Q_3) = 2$ and $\chi'_{F_5}(Q_3) = 3$.

- For the set $\{1, 3\}$, let $G_{1,3} = P_1$ and $H_{1,3}$ be the graph in Figure 5.14. Then $1 = \chi'_{F_1}(P_1) < \chi'_{F_3}(P_1) = 2$ and $2 = \chi'_{F_3}(H_{1,3}) < \chi'_{F_5}(H_{1,3}) = 3$.
- For the set $\{1, 4\}$, let $G_{1,4}$ be the graph in Figure 5.13 and $H_{1,4} = Q_3$. Then $1 = \chi'_{F_1}(G_{1,4}) < \chi'_{F_4}(G_{1,4}) = 3$ and $2 = \chi'_{F_4}(Q_3) < \chi'_{F_1}(Q_3) = 3$.
- For the set $\{2, 3\}$, let $G_{2,3} = C_6$ and $H_{2,3} = P_5$. Then $3 = \chi'_{F_2}(C_6) < \chi'_{F_3}(C_6) = 4$ and $2 = \chi'_{F_3}(P_5) < \chi'_{F_2}(P_5) = 3$. 

• For the set \{2, 4\}, let \(G_{2,4}\) be the graph of Figure 5.15 and \(H_{2,4}\) be the graph of Figure 5.16. Then \(3 = \chi'_{F_2}(G_{2,4}) < \chi'_{F_1}(G_{2,4}) = 4\) and \(3 = \chi'_{F_4}(H_{2,4}) < \chi'_{F_2}(H_{2,4}) = 4\).

\[Figure 5.16: A graph G with 3 = \chi'_{F_1}(G) < \chi'_{F_2}(G) = 4\]

• For the set \{2, 5\}, let \(G_{2,5}\) be the graph of Figure 5.15 and \(H_{2,5} = Q_3\). Then \(3 = \chi'_{F_2}(G_{2,5}) < \chi'_{F_5}(G_{2,5}) = 4\) and \(3 = \chi'_{F_3}(Q_3) < \chi'_{F_2}(Q_3) = 4\).

• For the set \{3, 4\}, let \(G_{3,4}\) be the graph shown in Figure 5.17 and let \(H_{3,4} = Q_3\). Then \(2 = \chi'_{F_3}(G_{3,4}) < \chi'_{F_4}(G_{3,4}) = 3\) and \(2 = \chi'_{F_4}(Q_3) < \chi'_{F_3}(Q_3) = 4\).

\[Figure 5.17: A graph G with \chi'_{F_3}(G) = 2 and \chi'_{F_4}(G) = \chi'_{F_3}(G) = 3\]

• For the set \{3, 5\}, let \(G_{3,5}\) be the graph \(G\) shown in Figure 5.17 and let \(H_{3,5} = Q_3\). Then \(2 = \chi'_{F_3}(G_{3,5}) < \chi'_{F_5}(G_{3,5}) = 3\) and \(3 = \chi'_{F_3}(Q_3) < \chi'_{F_3}(Q_3) = 4\).

• For the set \{4, 5\}, let \(G_{4,5} = Q_3\) and let \(H_{4,5}\) be the graph \(G\) shown in Figure 5.13. Then \(2 = \chi'_{F_4}(Q_3) < \chi'_{F_5}(Q_3) = 3\) and \(2 = \chi'_{F_5}(H_{4,5}) < \chi'_{F_3}(H_{4,5}) = 3\).

For every connected graph \(G\) that we have encountered, \(\chi'_{F_1}(G) \leq \chi'_{F_2}(G)\). Furthermore, since \(\chi'_{F_0}(G) = \gamma'(G) \leq \gamma'_2(G) \leq \chi'_{F_2}(G)\), it follows by Theorem 5.5.1 that if \(\delta(G) \geq 3\) or if \(G\) is triangle-free and \(\delta(G) \geq 2\), then \(\chi'_{F_1}(G) \leq \chi'_{F_2}(G)\). This leads us to the following conjecture.

**Conjecture 5.5.9** For every connected graph \(G\) of size at least 3,

\[\chi'_{F_1}(G) \leq \chi'_{F_2}(G)\]

We saw that if \(G\) is a disconnected graph with components \(G_1, G_2, \ldots, G_k\) where \(k \geq 2\), then \(\chi'(G) = \chi'(G_1) + \chi'(G_2) + \cdots + \chi'(G_k)\). Thus there are graphs \(G\) consisting of two components such that the numbers \(\chi'_{F_i}(G), i = 1, 2, 3, 4, 5\), are distinct. Whether there is a connected graph with this property is not known.
Chapter 6

On Color Frames of Claws

6.1 Introduction

The graph $K_{1,3}$ is often referred to as a claw. There are two color frames of a claw, both shown in Figure 6.1. The color frame $Y_1$ of a claw has exactly one red edge while $Y_2$ has exactly two red edges. In $Y_1$, there are therefore two blue edges and in $Y_2$ only one blue edge. By symmetry, we can choose either of the two blue edges in $Y_1$ as the root edge, while in $Y_2$, the only blue edge is the root edge of $Y_2$.

$Y_1$: \[ b \quad b \quad r \]

$Y_2$: \[ r \quad r \quad b \]

Figure 6.1: The two color frames of the claw $K_{1,3}$

In the vertex version, where $F$ is a 2-stratified graph of a claw, $F$-colorings were studied by Chartrand, Haynes, Henning and Zhang in the paper [8]. Hence, in the case of edge colorings, we study $F$-colorings of graphs for color frames $F$ of a claw.

6.2 Preliminary Results

Recall for an $F$-coloring $c$ of a graph $G$, that $E_{c,r}$ is the set of red edges of $G$ and $E_{c,b}$ the set of blue edges of $G$. (We also use $E_r$ and $E_b$ for $E_{c,r}$ and $E_{c,b}$, respectively, when the coloring $c$ under consideration is clear.) Thus $\{E_r, E_b\}$ is a partition of the edge set $E(G)$ of $G$ when $E_b \neq \emptyset$. Furthermore, let $G_r = G[E_r]$ denote the red subgraph induced by $E_r$ and $G_b = G[E_b]$ the blue subgraph induced by $E_b$. Thus $\{G_r, G_b\}$ is a decomposition of
CHAPTER 6. ON COLOR FRAMES OF CLAWS

If $G$ is a disconnected graph with components $G_1$, $G_2$, ..., $G_k$ where $k \geq 2$, then
\[
\chi'_F(G) = \chi'_F(G_1) + \chi'_F(G_2) + \cdots + \chi'_F(G_k).
\] (6.1)
Thus, it suffices to consider only connected graphs. We refer to the books [15, 18] for graph theory notation and terminology not described in this paper.

If $G$ is a connected graph with maximum degree $\Delta(G) \leq 2$ (and so $G$ is a path or cycle), then the only $Y_i$-coloring ($i = 1, 2$) of $G$ is the one that assigns the color red to every edge of $G$ and so $\chi'_{Y_i}(G)$ is the size of $G$. On the other hand, if $G$ contains a vertex $u$ with $\deg u \geq 3$ and $uv, uw \in E(G)$, then the red-blue coloring that assigns the color blue to the edges $uv$ and $uw$ and the color red to all other edges is a $Y_1$-coloring of $G$; while the red-blue coloring that assigns the color blue to $uv$ and the color red to all other edges is a $Y_2$-coloring of $G$. This leads to the following observation.

**Observation 6.2.1** If $G$ is a graph of size $m \geq 1$, then
\[
\chi'_{Y_1}(G) = \chi'_{Y_2}(G) = m \text{ if and only if } \Delta(G) \leq 2.
\]

It suffices therefore to consider only those connected graphs with maximum degree at least 3. For example, if $G$ is a star of size at least 3, then $\chi'_{Y_1}(G) = 1$ and $\chi'_{Y_2}(G) = 2$.

In the case where $G$ is a double star (a tree having diameter 3) of size at least 4, we have the following.

**Observation 6.2.2** If $G$ is a double star of size at least 4, then
\[
\chi'_{Y_1}(G) = \begin{cases} 2 & \text{if } G \text{ contains a vertex of degree 2} \\ 1 & \text{otherwise} \end{cases}
\]
and $\chi'_{Y_2}(G) = 3$.

As another example, consider the Petersen graph $P$. A $Y_i$-coloring $c_i$ of $P$ is shown in Figure 6.2 for $i = 1, 2$, where a red edge is indicated by a bold edge. Since $c_1$ produces three red edges while $c_2$ produces eight red edges, $\chi'_{Y_1}(P) \leq 3$ and $\chi'_{Y_2}(P) \leq 8$. In fact, we have equality in each case.

First, we show that $\chi'_{Y_1}(P) = 3$. Let $S_{3,3}$ be the double star whose two central vertices have degree 3. The graph $P$ is $S_{3,3}$-decomposable, that is, $P$ contains three edge-disjoint
copies of $S_{3,3}$ [15, p. 437]. Since $P$ is 3-regular, in any $Y_1$-coloring of $P$, there cannot be a blue copy of $S_{3,3}$ and so $|E_r| \geq 3$. That is, $\chi'_{Y_1}(P) \geq 3$ and so $\chi'_{Y_1}(P) = 3$.

Next, we show that $\chi'_{Y_2}(P) = 8$. Assume, to the contrary, that there is a $Y_2$-coloring $c$ of $P$ with $|E_{c,r}| = k \leq 7$. Let $(u_1, u_2, u_3, u_4, u_5, u_1)$ and $(v_1, v_3, v_5, v_2, v_4, v_1)$ be the outside and inside 5-cycles, respectively, in a dawning of $P$ such as in Figure 6.2, where $u_i v_i \in E(P)$ for $1 \leq i \leq 5$ and let $G_r$ be the red subgraph of order $n_r$ and size $m_r = k$ induced by $c$. Suppose, first, that $n_r < 10$ and let $u \in V(P) - V(G_r)$, say $u = u_1$. Since $u_1 u_2, u_1 v_1, u_1 u_5 \in E_{c,b}$, it follows that

$$X = \{u_2 v_2, u_2 u_3, v_1 v_3, v_1 v_4, u_5 v_5, u_5 u_4\} \subseteq E_{c,r}.$$ 

If $X = E_{c,r}$, then the blue edge $u_3 u_4$, for example, does not belong to a copy of $Y_2$. Thus $|E_{c,r}| = 7$ and $E_{c,r} = X \cup \{e\}$ for some edge $e \in E(P) - X$. If $e \in \{v_2 v_5, v_2 v_4, v_5 v_3\}$, then the blue edge $u_3 u_4$ does not belong to a copy of $Y_2$; while if $e \in \{u_3 u_4, u_3 v_3, u_4 v_4\}$, then the blue edge $v_2 v_5$ does not belong to a copy of $Y_2$. In either case, a contradiction is produced, which implies that $n_r = 10$.

Since $m_r = k \leq 7$, there are at least six vertices of degree 1 in $G_r$, say $u, v, w, x, y, z$ have degree 1 in $G_r$. Because the independence number of $P$ is 4, there are two vertices in $\{u, v, w, x, y, z\}$ that are adjacent in $P$, say $uv \in E(P)$. If $uv$ is blue, then $uv$ does not belong to a copy of $Y_2$. Thus $uv$ must be red. We may assume that $uv = u_1 v_1$. Since $\deg_{G_r} u_1 = \deg_{G_r} v_1 = 1$, it follows that for the four edges $u_1 u_2, u_1 u_5, v_1 v_3$ and $v_1 v_4$ are blue. Since each of these four blue edges must belong to a copy of $Y_2$, it follows that all of $u_2 v_2, u_2 u_3, u_5 v_5, u_5 u_4, v_3 v_5, v_3 u_3, v_4 u_4, v_4 v_2$ are red. However then, $|E_{c,r}| \geq 8$, which is a contradiction. Therefore, $\chi'_{Y_2}(P) = 8$.

If $G = C_{3k}$, $k \geq 1$, then $\chi'_{Y_1}(G) = 3k$ by Observation 6.2.1. Also $\alpha''(G) = k$. Consequently, $\chi'_{Y_1}(G) - \alpha''(G) = 2k$ can be arbitrarily large. On the other hand, if $G$ contains no vertices of degree 2, then this cannot occur.
Proposition 6.2.3  If $G$ is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) \leq \alpha'(G)$.

Proof. Let $M$ be a maximal matching of $G$ of size $\alpha'(G)$ and consider a red-blue coloring of $G$ such that $E_r = M$. Suppose that $e = uv$ is a blue edge. Since $M \cup \{e\}$ is not a matching in $G$, it follows that at least one of $u$ and $v$ is incident with exactly one red edge, say $u$ is incident with exactly one red edge. Since $\deg u \geq 3$, it follows that $u$ is incident with at least two blue edges. Hence $e$ belongs to a copy of $Y_1$. Thus, this is a $Y_1$-coloring of $G$ and so $\chi'_{Y_1}(G) \leq |M| = \alpha'(G)$. \hfill \qed

The $Y_1$- and $Y_2$-chromatic indexes can now be determined for graphs belonging to another familiar class.

Proposition 6.2.4  For each integer $n \geq 4$,

$$\chi'_{Y_1}(K_n) = \lfloor n/2 \rfloor \text{ and } \chi'_{Y_2}(K_n) = n - 1.$$  

Proof. Since every maximum matching in $K_n$ consists of $\lfloor n/2 \rfloor$ edges, it follows by Proposition 6.2.3 that $\chi'_{Y_1}(G) \leq \lfloor n/2 \rfloor$. Any red-blue coloring of $K_n$ having fewer than $\lfloor n/2 \rfloor$ red edges has a blue edge incident only with blue edges and so the coloring of $K_n$ is not a $Y_1$-coloring. Thus $\chi'_{Y_1}(G) \geq \lfloor n/2 \rfloor$ and so $\chi'_{Y_1}(K_n) = \lfloor n/2 \rfloor$.

Next, we show that $\chi'_{Y_2}(K_n) = n - 1$. Let there be given a minimum $Y_2$-coloring of $G = K_n$. Thus, if $uv$ is a blue edge, then at least one of $u$ and $v$ is incident with two or more red edges. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ with $0 \leq \deg_{G_r}(v_1) \leq \deg_{G_r}(v_2) \leq \cdots \leq \deg_{G_r}(v_n)$. If $\deg_{G_r}(v_2) \leq 1$, then $v_1v_2$ must be a red edge (otherwise, the blue edge $v_1v_2$ does not belong to any copy of $Y_2$) and so $\deg_{G_r}(v_1) = \deg_{G_r}(v_2) = 1$ while $\deg_{G_r}(v_i) \geq 2$ for $3 \leq i \leq n$. Otherwise, $\deg_{G_r}(v_i) \geq 2$ for $2 \leq i \leq n$. Hence, $|E_r| \geq n - 1$ in either case, that is, $\chi'_{Y_2}(K_n) = |E_r| \geq n - 1$. A red-blue coloring of $K_n$ inducing a red $C_{n-1}$ shows that $\chi'_{Y_2}(K_n) \leq n - 1$ and so $\chi'_{Y_2}(K_n) = n - 1$. \hfill \qed

6.3 Relationships Between the Two Color Frames of a Claw

An edge $e = uv$ in a graph $G$ is referred to as a non-claw edge if $e$ belongs to no claw in $G$. Thus if $e$ is a non-claw edge, then $\max\{\deg u, \deg v\} \leq 2$. Necessarily, every non-claw edge must be colored red in every $Y_i$-coloring $c$ of $G$ for $i = 1, 2$. Let $NC(G)$ be the set of all non-claw edges in a graph $G$. If $G$ is a path or a cycle, then $NC(G) = E(G)$ and
so $\chi'_1(G) = \chi'_2(G) = |NC(G)|$. We saw this earlier when considering the cycle $C_{3k}$.

On the other hand, if $G$ is a connected graph with maximum degree $\Delta(G) \geq 3$, then $NC(G) \neq E(G)$.

For a connected graph $G$, we now describe an upper bound for $\chi'_2(G)$ in terms of $\chi'_1(G)$ and the number of non-claw edges in $G$.

**Theorem 6.3.1** If $G$ is a nontrivial connected graph containing $\ell$ non-claw edges, then

$$\chi'_2(G) \leq 3\chi'_1(G) - 2\ell. \quad (6.2)$$

**Proof.** Suppose that $\chi'_1(G) = k$ and that $c_1$ is a minimum $Y_1$-coloring of $G$ resulting in the set $E_{c_1,r}$ of red edges in $G$. Let $E_{c_1,r} = X_0 \cup X_1 \cup X_2 \cup X_3$, where $X_0 = NC(G)$,

$$X_1 = \{e = uv \in E_{c_1,r} : \deg u = 1 \text{ and } \deg v \geq 3\},$$

$$X_2 = \{e = uv \in E_{c_1,r} : \deg u = 2 \text{ and } \deg v \geq 3\},$$

$$X_3 = \{e = uv \in E_{c_1,r} : \deg u \geq 3 \text{ and } \deg v \geq 3\}$$

such that $|X_i| = k_i \geq 0$ for $i = 1, 2, 3$. Since $|X_0| = \ell$, it follows that $k = \ell + k_1 + k_2 + k_3$. If any of the sets $X_1, X_2$ and $X_3$ are nonempty, let $X_1 = \{e_1, e_2, \ldots, e_{k_1}\}$, where $e_i = u_iv_i$, $\deg u_i = 1$ and $\deg v_i \geq 3$ for $1 \leq i \leq k_1$, $X_2 = \{f_1, f_2, \ldots, f_{k_2}\}$, where $f_j = w_jx_j$, $\deg w_j = 2$ and $\deg x_j \geq 3$ for $1 \leq j \leq k_2$ and $X_3 = \{h_1, h_2, \ldots, h_{k_3}\}$, where $h_t = y_tz_t$, $\deg y_t \geq 3$ and $\deg z_t \geq 3$ for $1 \leq t \leq k_3$. For each $i$ with $1 \leq i \leq k_1$, let $v_i' \neq u_i$ be a vertex adjacent to $v_i$; for each $j$ where $1 \leq j \leq k_2$, let $x_j' \neq w_j$ be a vertex adjacent to $x_j$; and for each $t$ where $1 \leq t \leq k_3$, let $y_t' \neq z_t$ be a vertex adjacent to $y_t$ and $z_t' \neq y_t$ be a vertex adjacent to $z_t$ such that $y_t' \neq z_t'$. We now define a new red-blue coloring of $G$, denoted by $c_2$, such that

$$E_{c_2,r} = E_{c_1,r} \cup \{v_iv_i' : 1 \leq i \leq k_1\} \cup \{x_jx_j' : 1 \leq j \leq k_2\} \cup \{y_ty_t', z_tz_t' : 1 \leq t \leq k_3\}.$$

[Note that the four sets on the right-hand side may not be pairwise disjoint.] Let $e$ be a blue edge in the coloring $c_2$. Then $e$ is also a blue edge in the coloring $c_1$. Since $c_1$ is a $Y_1$-coloring of $G$, it follows that $e$ belongs to a copy of $Y_1$ in the coloring $c_1$ of $G$ and so is adjacent to a red edge $e_r \in E_{c_1,r} \subseteq E_{c_2,r}$. Suppose that $e = uv$ and $e_r = vv'$ where $\deg v \geq 3$. It then follows by the definition of $c_2$ that there is a vertex $v' \neq w$ such that $v'$ is adjacent to $v$ and $vv'$ is a red edge in the coloring $c_2$. Thus $e$ belongs to a copy of $Y_2$ consisting of $\{e, e_r, vv'\}$. Hence $c_2$ is a $Y_2$-coloring with at most $\ell + 2k_1 + 2k_2 + 3k_3$
red edges. Since $k = \ell + k_1 + k_2 + k_3$, it follows that
\[
\chi'_Y(G) \leq \ell + 2k_1 + 2k_2 + 3k_3 = 3k - (2\ell + k_1 + k_2) \\
\leq 3k - 2\ell = 3\chi'_Y(G) - 2\ell
\]
and so (6.2) holds.

We noted that if $G$ is a path or cycle of size $m \geq 1$, then $\chi'_Y(G) = \chi'_Y(G) = m$. Since $\ell = m$ in this case, equality holds in (6.2). Equality in (6.2) can also hold for connected graphs with claws. For example, the graph $G_1$ of Figure 6.3 has one non-claw edge and $\chi'_Y(G_1) = 2$ while $\chi'_Y(G_1) = 4$. Therefore, $\chi'_Y(G_1) = 3\chi'_Y(G_1) - 2\ell$. For each integer $p \geq 2$, there is a connected graph $G_p$ having $p$ non-claw edges such that $\chi'_Y(G_p) = 2p$ and $\chi'_Y(G_p) = 4p$. For each $i$ with $1 \leq i \leq p$, let $S_i = S_{3,3}$ be the double star with central vertices $x_i$ and $v_i$ where $x_i$ is adjacent to the two end-vertices $w_i$ and $v_i$ is adjacent to the two end-vertices $y_i$ and $z_i$. Then $G_p$ be the graph obtained from the graphs $S_i$ ($1 \leq i \leq p$) by (1) joining $z_i$ and $y_{i+1}$ for $1 \leq i \leq p - 1$ and (2) adding a new vertex $y_{p+1}$ and joining $y_{p+1}$ to $z_p$. The graph $G_3$ is shown in Figure 6.3.

Then $G_p$ has the desired properties.

\begin{figure}
\centering
\begin{tabular}{c c}
\begin{tikzpicture}
\node (u1) at (0,0) [circle,draw] {}; \\
\node (u2) at (1,0) [circle,draw] {}; \\
\node (u3) at (2,0) [circle,draw] {}; \\
\node (v1) at (0,-1) [circle,draw] {}; \\
\node (v2) at (1,-1) [circle,draw] {}; \\
\node (v3) at (2,-1) [circle,draw] {}; \\
\node (w1) at (0,-2) [circle,draw] {}; \\
\node (w2) at (1,-2) [circle,draw] {}; \\
\node (w3) at (2,-2) [circle,draw] {}; \\
\node (x1) at (1.5,-2) [circle,draw] {}; \\
\node (x2) at (2.5,-2) [circle,draw] {}; \\
\node (x3) at (3.5,-2) [circle,draw] {}; \\
\node (y1) at (4.5,-2) [circle,draw] {}; \\
\node (y2) at (5.5,-2) [circle,draw] {}; \\
\node (y3) at (6.5,-2) [circle,draw] {}; \\
\node (z1) at (1.5,-3) [circle,draw] {}; \\
\node (z2) at (2.5,-3) [circle,draw] {}; \\
\node (z3) at (3.5,-3) [circle,draw] {}; \\
\draw (u1) -- (u2) -- (u3) -- (v1) -- (v2) -- (v3) -- (w1) -- (w2) -- (w3); \\
\draw (x1) -- (x2) -- (x3) -- (y1) -- (y2) -- (y3); \\
\draw (z1) -- (z2) -- (z3); \\
\end{tikzpicture} & \begin{tikzpicture}
\node (u1) at (0,0) [circle,draw] {}; \\
\node (u2) at (1,0) [circle,draw] {}; \\
\node (u3) at (2,0) [circle,draw] {}; \\
\node (v1) at (0,-1) [circle,draw] {}; \\
\node (v2) at (1,-1) [circle,draw] {}; \\
\node (v3) at (2,-1) [circle,draw] {}; \\
\node (w1) at (0,-2) [circle,draw] {}; \\
\node (w2) at (1,-2) [circle,draw] {}; \\
\node (w3) at (2,-2) [circle,draw] {}; \\
\node (x1) at (1.5,-2) [circle,draw] {}; \\
\node (x2) at (2.5,-2) [circle,draw] {}; \\
\node (x3) at (3.5,-2) [circle,draw] {}; \\
\node (y1) at (4.5,-2) [circle,draw] {}; \\
\node (y2) at (5.5,-2) [circle,draw] {}; \\
\node (y3) at (6.5,-2) [circle,draw] {}; \\
\node (z1) at (1.5,-3) [circle,draw] {}; \\
\node (z2) at (2.5,-3) [circle,draw] {}; \\
\node (z3) at (3.5,-3) [circle,draw] {}; \\
\draw (u1) -- (u2) -- (u3) -- (v1) -- (v2) -- (v3) -- (w1) -- (w2) -- (w3); \\
\draw (x1) -- (x2) -- (x3) -- (y1) -- (y2) -- (y3); \\
\draw (z1) -- (z2) -- (z3); \\
\end{tikzpicture}
\end{tabular}
\caption{Graphs $G$ for which $\chi'_Y(G) = 3\chi'_Y(G) - 2\ell$}
\end{figure}

For each connected graph $G$ with $\Delta(G) \geq 3$ that we have encountered thus far, we have seen that $\chi'_Y(G) < \chi'_Y(G)$. This is, in fact, true in general. In order to show this, we first establish some preliminary results. An edge-induced subgraph $F$ of a graph $G$ is called a $\Delta_k$-subgraph of $G$ if $\Delta(F) = k$. In particular, each component of a $\Delta_2$-subgraph is either a nontrivial path or a cycle. A $\Delta_k$-subgraph $F$ is maximal if $F$ is not a proper subgraph of a $\Delta_k$-subgraph.

**Theorem 6.3.2** Let $G$ be a connected graph of size 3 or more. If $F$ is a maximal $\Delta_2$-subgraph of minimum size in $G$, then $\chi'_Y(G) = |E(F)|$.

**Proof.** Since the red-blue coloring of $G$ that assigns red to each edge in $F$ and blue to the remaining edges of $G$ is a $Y_2$-coloring, it follows that $\chi'_Y(G) \leq |E(F)|$. 
It therefore remains to show that $\chi'_Y(G) \geq |E(F)|$. Among all minimum $Y_2$-colorings of $G$, let $c$ be one such that the sum of the degrees of the vertices of degree 3 or more in the resulting red subgraph is minimum. Let $F_{c,r} = G[E_{c,r}]$ be the red subgraph of $G$ induced by $E_{c,r}$. Then $\Delta(F_{c,r}) \geq 2$. We show that $\Delta(F_{c,r}) = 2$. Assume, to the contrary, that $F_{c,r}$ contains a vertex $v$ such that $\text{deg}_{F_{c,r}} v = k \geq 3$, where $vv_1, vv_2, \ldots, vv_k$ are the edges in $F_{c,r}$ incident with $v$. Since $c$ is a minimum $Y_2$-coloring, the red-blue coloring of $G$ in which the color of $vv_k$ is changed to blue is not a $Y_2$-coloring of $G$. This implies that there are edges $uv_k$ and $wv_k$, where $v \neq u, w$, such that (1) $uv_k$ is blue and $u$ is incident with at most one red edge in $F_{c,r}$ and (2) $wv_k$ is red and $\text{deg}_{F_{c,r}} v_k = 2$. The red-blue coloring $c'$ obtained from $c$ by interchanging the colors of $vv_k$ and $v_ku$ is also a minimum $Y_2$-coloring of $G$ (see Figure 6.4 where each red edge is indicated by a bold line).

In $c$:

```
\begin{figure}
\centering
\begin{tikzpicture}
  \node[vertex] (v1) at (0,1) {$v_1$};
  \node[vertex] (v2) at (0,0) {$v_2$};
  \node[vertex] (v) at (0,-1) {$v$};
  \node[vertex] (uk) at (1,-1) {$v_k$};
  \node[vertex] (u) at (1,-2) {$u$};
  \node[vertex] (w) at (2,-2) {$w$};
  \draw (v) -- (v1); \draw (v) -- (v2); \draw (v) -- (uk);
  \draw (uk) -- (u); \draw (uk) -- (w);
\end{tikzpicture}
\caption{In $c$}
\end{figure}
```

In $c'$:

```
\begin{figure}
\centering
\begin{tikzpicture}
  \node[vertex] (v1) at (0,1) {$v_1$};
  \node[vertex] (v2) at (0,0) {$v_2$};
  \node[vertex] (v) at (0,-1) {$v$};
  \node[vertex] (uk) at (1,-1) {$v_k$};
  \node[vertex] (u) at (1,-2) {$u$};
  \node[vertex] (w) at (2,-2) {$w$};
  \draw (v) -- (v1); \draw (v) -- (v2); \draw (v) -- (uk);
  \draw (uk) -- (u); \draw (uk) -- (w);
\end{tikzpicture}
\caption{In $c'$}
\end{figure}
```

In the red subgraph $F'_{c,r}$ of $G$ induced by $E'_{c,r}$, the degree of $v_k$ is 2, the degree of $u$ is at most 2 and the degree of $v$ is $k - 1$. Thus the number of vertices of degree 3 or more in $F'_{c,r}$ is at most that of $F_{c,r}$, while the sum of degrees of the vertices of degree 3 or more in $F'_{c,r}$ is less than that of $F_{c,r}$, which contradicts the defining property of $c$. Thus, $\Delta(F_{c,r}) = 2$. Therefore, $\chi'_Y(G) = |E(F_{c,r})| \geq |E(F)|$.\hfill\qed

From the proof of Theorem 6.3.2, we have the following corollary.

**Corollary 6.3.3** If $G$ is a connected graph of order 4 or more, then there is a minimum $Y_2$-coloring $c$ of $G$ such that the red subgraph of $G$ induced by $c$ is a maximal $\Delta_2$-subgraph of $G$.

We now show that with few exceptions, $\chi'_Y(G) < \chi'_Y(G)$ for a graph $G$.

**Theorem 6.3.4** If $G$ is a connected graph of order at least 4 that is neither a path nor a cycle, then $\chi'_Y(G) < \chi'_Y(G)$. 
Proof. By Corollary 6.3.3, there exists a minimum $Y_2$-coloring $c$ of $G$ such that the red subgraph $F = G[E_{c,r}]$ induced by the set $E_{c,r}$ of red edges is a maximal $\Delta_2$-subgraph of $G$. Consequently, every component of $F$ is a path or a cycle. We claim that there is a set $E'$ of edges of $G$ with $|E'| < |E_{c,r}|$ such that the red-blue coloring $c'$ of $G$ with $E_{c',r} = E'$ is a $Y_1$-coloring of $G$. We construct the set $E'$ as follows.

First, suppose that $F$ has a component $P$ that is a path. The set $E'$ contains all edges belonging to components of size 1 in $F$. If $P$ is a component of size 2 in $F$, say $P = (u, v, w)$, where $\deg_G v = 2$, then $E'$ contains both edges of $P$. If, on the other hand, $\deg_G v \geq 3$, then let $E'$ contain exactly one of $uv$ and $vw$. Next, consider a component $P$ of size $k \geq 3$, say $P = (v_1, v_2, \ldots, v_k)$. If $\deg_G v_i = 2$ for all $i$ with $2 \leq i \leq k$, then let $E(P) \subseteq E'$. If some vertex $v_i$ ($2 \leq i \leq k$) of $P$ has degree 3 or more in $G$, then let $v_i$ be the first vertex (after $v_1$) of $P$ that has degree 3 or more in $G$. Let $v_1v_2, \ldots, v_{i-1}v_i \in E'$ and $v_iv_{i+1} \notin E'$. If $i + 2 \leq k$, then repeat this procedure for $P^* = (v_{i+1}, v_{i+2}, \ldots, v_k)$; that is, if $\deg_G v_i = 2$ for $i + 2 \leq i \leq k$, then let $E(P^*) \subseteq E'$; otherwise, let $v_{i+2}$ be the first vertex (after $v_{i+1}$) on $P$ that has degree 3 or more in $G$. Let $v_{i+1}v_i+2, \ldots, v_{i+2}v_{i+1} \in E'$ and $v_{i+2}v_{i+1} \notin E'$. We repeat this procedure until we arrive at a vertex $v_{i+\ell}$ for some $\ell \geq 2$ such that either $\deg_G v_i = 2$ for $i + 2 \leq i \leq k$ or $i + 2 \geq k + 1$. This procedure is illustrated in Figure 6.5 for a possible component $P = (v_1, v_2, \ldots, v_k)$ of size $k = 17$ in $F$. In Figure 6.5, if an edge $e$ of $P$ is drawn in a bold line, then $e \in E'$; otherwise, $e \notin E'$. If an edge $f$ is drawn in a dashed line, then $f$ is not an edge of $P$ but $f$ is incident with an interior vertex of $P$ that has degree at least 3 in $G$.

![Figure 6.5: Selecting edges in $E'$ for components that are paths](image)

Second, suppose that $F$ has a component $C$ that is a cycle, say

$$C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1).$$

We consider three cases.

Case 1. $\deg_G v_i = 2$ for all $i$ with $1 \leq i \leq k$. Let $E(C) \subseteq E'$. 
Case 2. \( \deg_G v_i \geq 3 \) for all \( i \) with \( 1 \leq i \leq k \). If \( k \) is even, then let \( v_i v_{i+1} \in E' \) for each even integer \( i \) and \( v_i v_{i+1} \notin E' \) for each odd integer \( i \). Now suppose that \( k \) is odd. Suppose that \( vv_k \) is an edge of \( G \) that is not on \( C \), where \( v \) may or may not be on \( C \). Then \( vv_k \notin E(F) \). Let \( vv_k \in E' \) and \( v_i v_{i+1} \in E' \) for each odd integer \( i \) with \( 1 \leq i \leq k - 2 \) and the remaining edges of \( C \) do not belong to \( E' \). Thus for each cycle of size \( k \) in \( F \), there are \([k/2]\) edges added to the set \( E' \). This is illustrated in Figure 6.6 for \( C = C_3, C_4, C_5 \). In Figure 6.6, if an edge \( e \) is drawn in a bold line, then \( e \in E' \); otherwise, \( e \notin E' \). If an edge \( f \) is drawn in a dashed line, then \( f \) is not an edge of \( C \) but \( f \) is incident with a vertex of \( C \) that has degree at least 3 in \( G \).

![Figure 6.6: Selecting edges in \( E' \) for a cycle component in Case 2](image)

Case 3. There is at least one vertex of \( C \) having degree 2 and at least one vertex of \( G \) having degree 3 or more. We may assume that \( \deg_G v_1 = 2 \). Let \( v_{i_1} \) (\( i_1 \geq 2 \)) be the first vertex of \( C \) that has degree 3 or more in \( G \). Let \( v_1 v_2, \ldots, v_{i_1-1} v_{i_1} \in E' \) and \( v_1 v_{i_1+1} \notin E' \). If \( \deg_G v_i = 2 \) for \( i_1 + 2 \leq i \leq k \), then let \( v_i v_{i+1} \in E' \) for \( i_1 + 2 \leq i \leq k \); otherwise, let \( v_{i_2} \) (\( i_2 \geq i_1 + 2 \)) be the first vertex of \( C \) that has degree 3 or more in \( G \). Let \( v_{i_1+1} v_{i_1+2}, \ldots, v_{i_2-1} v_{i_2} \in E' \) and \( v_{i_2} v_{i_2+1} \notin E' \). We repeat this procedure until we arrive at a vertex \( v_{i_\ell} \) for some \( \ell \geq 2 \) such that either \( \deg_G v_i = 2 \) for \( i_\ell + 2 \leq i \leq k \) (and let \( v_i v_{i+1} \in E' \) for \( i_\ell + 2 \leq i \leq k \)) or \( i_\ell + 2 \geq k + 1 \). This procedure is illustrated in Figure 6.7 for a possible component \( C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1) \) of size \( k = 17 \) in \( F \). In Figure 6.7, if an edge \( e \) of \( C \) is drawn in a bold line, then \( e \in E' \); otherwise, \( e \notin E' \). If an edge \( f \) is drawn as a dashed line, then \( f \) is not an edge of \( C \) but \( f \) is incident with a vertex of \( C \) that has degree at least 3 in \( G \).

Since \( \Delta(G) \geq 3 \), there is either a component \( P \) in \( F \) that is a path one of whose interior vertices has degree at least 3 in \( G \) or a component \( C \) in \( F \) that is a cycle in which some vertex has degree at least 3 in \( G \).

- If \( z \) is an interior vertex of the path \( P \) such that \( \deg_G z \geq 3 \), then exactly one of the two edges incident with \( z \) on \( P \) belongs to \( E' \).
- If \( z \) is a vertex on the cycle \( C \) such that \( \deg_G z \geq 3 \), then either (1) exactly one of
Therefore, \(|E'| < |E_{c,r}|\).

We now show that the red-blue coloring \(c'\) of \(G\) with \(E_{c',r} = E'\) is a \(Y_1\)-coloring of \(G\). Let \(e = uv\) be a blue edge in the coloring \(c'\). First, suppose that \(e\) is a red edge in the coloring \(c\) and so either \(u\) or \(v\) has degree 3 or more in \(G\), say \(\deg_G v \geq 3\), and \(v\) is incident with a red edge different from \(e\) and at least one blue edge in the coloring \(c\). Thus \(e\) belongs to a copy of \(Y_1\) in the coloring \(c'\). Next, suppose that \(e\) is a blue edge in the coloring \(c\). Thus \(e\) belongs to a copy \(Y\) of \(Y_2\) in \(c\), say \(E(Y) = \{e = uv, ux, uy\}\), where \((x, u, y)\) is a path in a component of \(F\). By the definition of \(E'\), there are two possibilities.

1. Exactly one of \(ux\) and \(uy\) belongs to \(E'\) and so one of \(ux\) and \(uy\) is red and the other is blue in the coloring \(c'\). So \(e\) belongs to a copy of \(Y_1\).

2. Neither \(ux\) nor \(uy\) belongs to \(E'\) but there is an edge \(uu'\) not on \(C\) that belongs to \(E'\). Thus \(e\) is adjacent to at least one blue edge and the red edge \(uu'\) in the coloring \(c'\). Hence \(e\) belongs to a copy of \(Y_1\).

Therefore, \(c'\) is a \(Y_1\)-coloring of \(G\) and so

\[
\chi'_1(G) = |E'| < |E(F)| = \chi'_2(G),
\]

as desired.

By Observation 6.2.1, Theorems 6.3.1 and 6.3.4, we have the following.

Figure 6.7: Selecting edges in \(E'\) for a cycle component in Case 3

the two edges on \(C\) incident with \(z\) belongs to \(E'\) or (2) neither of these two edges belong to \(E'\) but there is an edge \(zz'\) not on \(C\) that belongs to \(E'\).
Corollary 6.3.5 If $G$ is a connected graph of order at least 4, then

$$\chi_{Y_1}'(G) \leq \chi_{Y_2}'(G) \leq 3\chi_{Y_1}'(G).$$

By Corollary 6.3.5, if $G$ is a connected graph of order at least 4 with $\chi_{Y_1}'(G) = a$ and $\chi_{Y_2}'(G) = b$, then $a \leq b \leq 3a$ and $b \geq 2$. We next show that every pair $a, b$ of positive integers with $a \leq b \leq 3a$ and $b \geq 2$ can be realized as $\chi_{Y_1}'(G)$ and $\chi_{Y_2}'(G)$, respectively, for some connected graph $G$ of order at least 4. To show this, we first present two lemmas.

Lemma 6.3.6 If $G$ is the corona of an $n$-cycle where $n \geq 3$, then

$$\chi_{Y_1}'(G) = \lceil n/2 \rceil \text{ and } \chi_{Y_2}'(G) = n.$$

Proof. Let $G = \text{cor}(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ for some integer $n \geq 3$. Suppose that $u_iv_i$ is the pendant edge of $G$ at $v_i$ for $1 \leq i \leq n$. First, we show that $\chi_{Y_1}'(G) = \lceil n/2 \rceil$. For each even integer $n \geq 4$, define a red-blue coloring $c$ of $G$ with

$$E_{c,r} = \{v_iv_{i+1} : i \text{ is odd, } 1 \leq i \leq n-1\};$$

while for each odd integer $n \geq 3$, define a red-blue coloring $c$ of $G$ with

$$E_{c,r} = \{v_iv_{i+1} : i \text{ is odd, } 1 \leq i \leq n-2\} \cup \{u_nv_n\}. \tag{6.4}$$

Since $c$ is a $Y_1$-coloring of $G$ in each case, $\chi_{Y_1}'(G) \leq |E_{c,r}| = \lceil n/2 \rceil$.

![Figure 6.8: Illustrate $Y_1$-colorings for $\text{cor}(C_8)$ and $\text{cor}(C_9)$](image)

To show that $\chi_{Y_1}'(G) \geq \lceil n/2 \rceil$, suppose that $G$ has a $Y_1$-coloring $c^*$ using at most $\lceil n/2 \rceil - 1$ red edges. Thus the size of the red subgraph $G_r^*$ induced by $c^*$ is at most $\lceil n/2 \rceil - 1$. Since $G_r^*$ contains no isolated vertices and $n/2 > \lceil n/2 \rceil - 1$, the order of $G_r^*$ is at most $n - 1$ and so there is at least one vertex $v$ of $C_n$ such that $v \notin V(G_r^*)$, say...
\[ \chi_{Y_1}(G) \geq \lceil n/2 \rceil \text{ and so } \chi'_{Y_1}(G) = \lfloor n/2 \rfloor. \]

Next, we show that \( \chi'_{Y_2}(G) = n \). The red-blue coloring that assigns the color red to each edge of \( C_n \) in \( G \) and the color blue to the remaining edges of \( G \) is a \( Y_2 \)-coloring of \( G \) with exactly \( n \) red edges. Hence \( \chi'_{Y_2}(G) \leq n \). To show that \( \chi'_{Y_2}(G) \geq n \), let \( c \) be a minimum \( Y_2 \)-coloring of \( G \). For each integer \( i \) with \( 1 \leq i \leq n \), at least one edge in \( \{u_i v_i, v_i v_{i+1}\} \) must be red; for otherwise, the blue edge \( u_i v_i \) does not belong to any copy of \( Y_2 \) in \( G \), which is a contradiction. Thus \( \chi'_{Y_2}(G) = |E_{c,r}| \geq n \). Therefore, \( \chi'_{Y_2}(G) = n \). \]

**Lemma 6.3.7** Let \( G_1 \) and \( G_2 \) be two vertex-disjoint graphs, where \( v_i \) is an end-vertex in \( G_i \) for \( i = 1, 2 \). If \( G \) is the graph obtained from \( G_1 \) and \( G_2 \) by identifying the vertex \( v_1 \) in \( G_1 \) with the vertex \( v_2 \) in \( G_2 \), then \( \chi'_Y(G) = \chi'_{Y_1}(G_1) + \chi'_{Y_1}(G_2) \) for \( i = 1, 2 \).

**Proof.** For a given \( Y \in \{Y_1, Y_2\} \), let \( \chi'_Y(G_i) = k_i \) for \( i = 1, 2 \). Suppose that \( v \) is the vertex of \( G \) obtained by identifying \( v_1 \) and \( v_2 \) in \( G_1 \) and \( G_2 \), respectively. Then \( V(G) = (V(G_1) - \{v_1\}) \cup (V(G_2) - \{v_2\}) \cup \{v\} \) and \( E(G) = E(G_1) \cup E(G_2) \) where \( E(G_1) \cap E(G_2) = \emptyset \). Let \( c_i \) be a minimum \( Y \)-coloring of \( G_i \) for \( i = 1, 2 \). Since the red-blue coloring \( c \) of \( G \) defined by \( c(e) = c_i(e) \) if \( e \in E(G_i) \) is a \( Y \)-coloring of \( G \) having exactly \( k_1 + k_2 \) red edges, it follows that \( \chi'_Y(G) \leq |E_{c,r}| = k_1 + k_2 \). Suppose that \( \chi'_Y(G) < k_1 + k_2 \). Let \( c^* \) be a minimum \( Y \)-coloring of \( G \) and let \( E_{c^*,r} = E_1 \cup E_2 \), where \( E_i \) is the set of red edges in \( G_i \) for \( i = 1, 2 \). Hence, either \( |E_1| < k_1 \) or \( |E_2| < k_2 \), say the former. Then the red-blue coloring \( c^*_i \) with \( E_{c^*_i,r} = E_1 \) is not a \( Y \)-coloring of \( G_1 \) and so there is a blue edge \( e \in E(G_1) \) that does not belong to any copy of \( Y \) in \( G_1 \). Since \( \deg_G v = 2 \), it follows that \( G \) contains no \( K_{1,3} \) having edges in both \( G_1 \) and \( G_2 \). Thus \( e \) does not belong to any copy of \( Y \) in \( G \), which is a contradiction. Thus \( \chi'_Y(G) = \chi'_{Y_1}(G_1) + \chi'_{Y_1}(G_2) \). \]

**Theorem 6.3.8** For a given pair \( a, b \) of positive integers, there exists a connected graph \( G \) of order at least 4 such that \( \chi'_{Y_1}(G) = a \) and \( \chi'_{Y_2}(G) = b \) if and only if \( a \leq b \leq 3a \) and \( b \geq 2 \).

**Proof.** We have seen that if \( G \) is a connected graph of order at least 4 with \( \chi'_{Y_1}(G) = a \) and \( \chi'_{Y_2}(G) = b \), then \( a \leq b \leq 3a \) and \( b \geq 2 \). It remains to verify the converse. If \( b = a \geq 2 \), then let \( G = P_{b+1} \) and so \( \chi'_{Y_1}(G) = \chi'_{Y_2}(G) = a \). If \( b = a + 1 \), let \( G \) be the graph obtained from \( P_{b+1} = (v_1, v_2, \ldots, v_{b+1}) \) by adding a new vertex \( v \) and joining \( v \) to the vertex \( v_2 \). Then \( \chi'_{Y_1}(G) = b - 1 = a \) and \( \chi'_{Y_2}(G) = b \). Thus, we may assume
that $b = a + k$ where $k \geq 2$. We consider two cases, according to whether $k \leq a$ or $a + 1 \leq k \leq 2a$.

**Case 1.** $k \leq a$. Let $a = k + \ell$ where $\ell \geq 0$. Then $b = a + k = (k + \ell) + k = 2k + \ell$. If $\ell = 0$, then let $G = cor(C_{2k})$. By Lemma 6.3.6, $\chi'_{Y_1}(G) = k = a$ and $\chi'_{Y_2}(G) = 2k = b$. If $\ell \geq 1$, then let $G$ be the graph obtained from the corona $cor(C_{2k})$ and the path $P_{\ell + 1} = (v_1, v_2, \ldots, v_{\ell + 1})$ by identifying the vertex $v_1$ of $P_{\ell + 1}$ with an end-vertex of $cor(C_{2k})$. Then $\chi'_{Y_1}(G) = k + \ell = a$ and $\chi'_{Y_2}(G) = 2k + \ell = b$ by Lemmas 6.3.6 and 6.3.7.

**Case 2.** $a + 1 \leq k \leq 2a$. Then $k = a + p$ for some $p$ with $1 \leq p \leq a$. Then $b = a + (a + p) = 2a + p$. We consider three subcases.

**Subcase 2.1.** $p = a$. Then $b = 3a$. Let $H_a$ be the tree obtained from the path $P_{3a+1} = (v_1, v_2, \ldots, v_{3a+1})$ by adding the pendant edge $u_i v_i$ at each vertex $v_i$ if $i \equiv 0, 2 \pmod{3}$ and $2 \leq i \leq 3a$. The graph $H_3$ is shown in Figure 6.9 for $a = 3$. By Observation 6.2.2 and successive applications of Lemma 6.3.7, it follows that $\chi'_{Y_1}(H_a) = a$ and $\chi'_{Y_2}(H_a) = 3a = b$.

![Figure 6.9: The graph $H_3$ with $\chi'_{Y_1}(H_3) = 3$ and $\chi'_{Y_2}(H_3) = 9$ in Subcase 2.1](image)

**Subcase 2.2.** $p = a - 1 \geq 1$. Then $b = 3a - 1$. Let $H_{a-1}$ be the tree constructed in Subcase 2.1 where $a$ is replaced by $a - 1$. Then $\chi'_{Y_1}(H_{a-1}) = a - 1$ and $\chi'_{Y_2}(H_{a-1}) = 3(a - 1)$. Let $x$ be a peripheral vertex of $H_{a-1}$ and let $H = K_{1,3}$. The graph $G$ is obtained from $H_{a-1}$ and $H$ by identifying the vertex $x$ in $H_{a-1}$ and an end-vertex of $H$. (The graph $G$ is shown in Figure 6.10 for $a = 3$.) Then $\chi'_{Y_1}(G) = (a - 1) + 1 = a$ and $\chi'_{Y_2}(G) = 3(a - 1) + 2 = 3a - 1$ by Lemma 6.3.7.

![Figure 6.10: The graph $G$ with $\chi'_{Y_1}(G) = 3$ and $\chi'_{Y_2}(G) = 8$ in Subcase 2.2](image)

**Subcase 2.3.** $1 \leq p \leq a - 2$. Then $a - p \geq 2$. Let $H_p$ be the tree in Subcase 2.1 (where $a$ is replaced by $p$) with $\chi'_{Y_1}(H_p) = p$ and $\chi'_{Y_2}(H_p) = 3p$. Let $H = cor(C_{2(a-p)})$ and so $\chi'_{Y_1}(H) = a - p$ and $\chi'_{Y_2}(H) = 2(a - p)$ by Lemma 6.3.6. Let $x$ be a peripheral
vertex of $H_p$. The graph $G$ is obtained from $H_p$ and $H$ by identifying the vertex $x$ in $H_p$ and an end-vertex of $H$. The graph $G$ is shown in Figure 6.11 for $p = 2$ and $a = 5$ where $H = \text{cor}(C_6)$. Then $\chi'_{Y_1}(G) = p + (a - p) + a$ and $\chi'_{Y_2}(G) = 3p + 2(a - p) = 2a + p = b$ by Lemmas 6.3.6 and 6.3.7.

![Figure 6.11: The graph $G$ with $\chi'_{Y_1}(G) = 5$ and $\chi'_{Y_2}(G) = 12$ in Subcase 2.3](image)

**6.4 Color Frames of Claws in Trees**

In this section, we first study $Y_1$-colorings in trees and show that if $T$ is a tree of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(T) = \alpha''(T)$. In order to show this, we first present an additional definition and a lemma. Let $C$ be a caterpillar of order at least 4 and let $(x_1, x_2, \ldots, x_d)$ be the spine of $C$. For each $i$ with $1 \leq i \leq d$, let $X_i$ be the set of end-vertices that are adjacent to $x_i$. Suppose that $|X_i| \geq 1$ for $1 \leq i \leq d$ and $d \geq 3$. Define a red-blue coloring $c$ of $C$ such that

1. $c(x_i, x_{i+1})$ is red for an odd integer $i$ with $1 \leq i \leq d - 1$ and $c(x_i, x_{i+1})$ is blue for an even integer $i$ with $2 \leq i \leq d - 1$ and

2. $c(x_i, x)$ is red for all $x \in X_i$ if $i$ is odd and $1 \leq i \leq d$ and $c(x_i, x)$ is blue for all $x \in X_i$ if $i$ is even and $2 \leq i \leq d$.

This edge-colored caterpillar $C$ is then called a red-blue caterpillar rooted at $x_1$.

![Figure 6.12: A red-blue caterpillar rooted at $x_1$ for $d = 5$](image)

**Lemma 6.4.1** Let $T$ be a tree of order at least 4 having no vertex of degree 2. If $c$ is a minimum $Y_1$-coloring of $T$ such that the red subgraph $G_{c,r}$ has the largest edge
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independence number among all minimum \(Y_1\)-colorings of \(T\), then every non-end-vertex of \(T\) is incident with at least two blue edges.

**Proof.** Assume, to the contrary, that there is a non-end-vertex \(u\) such that \(u\) is incident at most one blue edge. Suppose that \(N(u) = \{u_1, u_2, \ldots, u_a\}\) where \(a \geq 3\). We consider two cases.

Case 1. \(u\) is incident with exactly one blue edge. First, suppose that there is \(u_i\) (\(1 \leq i \leq a\)) such that \(u_i\) is an end-vertex. Note that \(uu_i\) cannot be colored blue, for otherwise, this blue edge does not belong to any copy of \(Y_1\). Thus \(uu_i\) must be red. However then, since \(u\) is incident with a blue edge, we can change the color of \(uu_i\) to blue and the resulting coloring is also a \(Y_1\)-coloring. This is impossible since \(c\) is a minimum \(Y_1\)-coloring. Therefore, no vertex \(u_i\) (\(1 \leq i \leq a\)) is an end-vertex of \(T\).

For each \(i\) with \(1 \leq i \leq a\), let \(N(u_i) = \{u, u_{i,1}, u_{i,2}, \ldots, u_{i,a_i}\}\) where then \(a_i \geq 2\). We may assume, without loss of generality, that \(uu_a\) is blue and \(uu_1\) and \(uu_2\) are red. If there is \(p\) (\(1 \leq p \leq a_1\)) such that \(u_1u_{1,p}\) is red, then we can change the color of \(uu_1\) to blue and the resulting coloring is also a \(Y_1\)-coloring. This is impossible since \(c\) is a minimum \(Y_1\)-coloring. Thus \(u_1u_{1,p}\) is blue for all \(p\) with \(1 \leq p \leq a_1\). Similarly, \(u_2u_{2,q}\) is blue for all \(q\) with \(1 \leq q \leq a_2\). If there is some \(p\) (\(1 \leq p \leq a_1\)) such that \(u_{1,p}\) is incident with no red edge, then we can interchange the colors of \(uu_1\) and \(u_1u_{1,p}\) to obtain a minimum \(Y_1\)-coloring whose red subgraph has a larger edge independence number, a contradiction. Thus each \(u_{1,p}\) (\(1 \leq p \leq a_1\)) must be incident with at least one red edge. (Similarly each \(u_{2,q}\) (\(1 \leq q \leq a_2\)) must be incident with at least one red edge.)

If every vertex \(u_{1,p}\) (\(1 \leq p \leq a_1\)) is incident with two or more blue edges, then we can change the color of \(uu_1\) to blue and the resulting coloring is also a \(Y_1\)-coloring with fewer red edges, which is impossible since \(c\) is a minimum \(Y_1\)-coloring. Thus, there is some \(u_{1,p}\) (say \(u_{1,1}\)) such that \(u_{1,1}\) is incident with exactly one blue edge (namely, the blue edge \(u_1u_{1,1}\)). We now have a red-blue caterpillar \(C\) rooted at \(u\) with the spine \((x_1 = u, x_2 = u_1, x_3 = u_{1,1})\) such that

(i) \(X_1 = \{u_2, \ldots, u_{a-1}\}\) where \(a \geq 3\) is the set of end-vertices adjacent to \(x_1\) in \(C\), \(X_2 = \{u_{1,2}, \ldots, u_{1,a_1}\}\) where \(a_1 \geq 2\) is the set of end-vertices adjacent to \(x_2\) in \(C\) and \(X_3 = N(u_{1,1}) - \{u_1\}\) is the set of end-vertices adjacent to \(x_3\) in \(C\);

(ii) each vertex in \(X_1 \cup X_2\) is not an end-vertex of \(T\), that is, each end-vertex of \(C\) that is adjacent to \(x_i\) (\(i = 1, 2\)) is not an end-vertex of \(T\);
(iii) each edge incident with \(x_1 = u\) in \(C\) is red, each edge incident with \(x_2 = u_1\) is blue (except for \(x_1 x_2\)) and each edge incident with \(x_3 = u_{1,1}\) is red (except for \(x_2 x_3\)).

If there is an end-vertex of \(C\) adjacent to \(x_3\) that is an end-vertex of \(T\), then this procedure stops. Otherwise, we consider \(u_{1,1}\) in the same way as we consider \(u\), that is, let \(u_{1,1} = v\) and let \(N(v) = \{u_1, v_1, v_2, \ldots, v_b\}\) where \(b \geq 2\). We may assume that \(vv_1\) and \(vv_2\) are red.

For each \(i\) with \(1 \leq i \leq b\), let \(N(v_i) = \{v, v_{i,1}, v_{i,2}, \ldots, v_{i,b_i}\}\) where \(b_i \geq 2\). Repeating the procedure as above, we may assume that \(v_1 v_{1,p}\) is blue for all \(p\) with \(1 \leq p \leq b_1\) and \(v_2 v_{2,q}\) is blue for all \(q\) with \(1 \leq q \leq v_2\). Furthermore, there is a vertex \(v_{1,p}\) (say \(v_{1,1}\)) that is incident with exactly one blue edge. We now have a red-blue caterpillar rooted at \(u\) with the spine \((u = x_1, u_1 = x_2, u_{1,1} = x_3, v_1 = x_4, v_{1,1} = x_5)\), which is also denoted by \(C\), such that no end-vertex of \(C\) adjacent to \(x_i\) \((1 \leq i \leq 4)\) is an end-vertex of \(T\). If there is an end-vertex of \(C\) adjacent to \(x_5\) that is an end-vertex of \(T\), then this procedure stops. Otherwise, we continue and until we obtain a red-blue caterpillar \(C\) rooted at \(u\) with the spine \((x_1, x_2, \ldots, x_d)\) (where \(x_1 = u, x_2 = u_1, x_3 = u_{1,1}\) and so on) and \(d \geq 3\) is odd. For each \(i\) with \(1 \leq i \leq d\), let \(X_i\) be the set of end-vertices of \(C\) that are adjacent to \(x_i\). Thus \(X_i\) contains no end-vertex of \(T\) for \(1 \leq i \leq d - 1\) and \(X_d\) contains at least one end-vertex of \(T\), say \(x \in X_d\) is an end-vertex of \(T\). Since the edge \(x_d x\) is red, we can change the color of \(x_d x\) to blue and the resulting coloring is also a \(Y_1\)-coloring with fewer red edges, a contradiction.

**Case 2. **\(u\) **is incident with no blue edge.** If \(u\) is adjacent to at least two end-vertices, say \(u_1\) and \(u_2\), then the coloring obtained from \(c\) by changing the color of \(uu_1\) and \(uu_2\) to blue is a \(Y_1\)-coloring with fewer red edges than \(c\), which is impossible. Thus \(u\) is adjacent to at most one end-vertex. First, suppose that there is a vertex \(u_i\) \((1 \leq i \leq a)\) such that \(u_i u_{i,s}\) is red and \(u_i u_{i,t}\) is blue for some \(s, t\) with \(1 \leq s, t \leq a_i\). Then the coloring obtained from \(c\) by changing the color of \(uu_i\) to blue is a \(Y_1\)-coloring with fewer red edges than \(c\). This is impossible since \(c\) is a minimum \(Y_1\)-coloring. Thus for all \(i\) with \(1 \leq i \leq a\), if \(u_i\) is not an end-vertex, then either all edges \(u_i u_{i,j}\) are red for \(1 \leq i \leq a_i\) or all edges \(u_i u_{i,j}\) are blue for \(1 \leq i \leq a_i\).

First, suppose that there are two vertices \(u_i\) and \(u_j\) \((1 \leq i \neq j \leq a)\) such that all edges \(u_i u_{i,p}\) and \(u_j u_{j,q}\) are red for \(1 \leq p \leq a_i\) and \(1 \leq q \leq a_j\), say \(i = 1\) and \(j = 2\). Then the coloring obtained by changing the colors of \(uu_1\) and \(uu_2\) to blue is a \(Y_1\)-coloring with fewer red edges, which is impossible. Hence, there is at most one vertex \(u_i\) \((1 \leq i \leq a)\) such that all edges \(u_i u_{i,p}\) are red for \(1 \leq p \leq a_i\). Thus, there is at least one vertex \(u_i\)
(1 ≤ i ≠ j ≤ a) such that all edges $u_iu_{i,p}$ are blue for 1 ≤ p ≤ a_i.

We claim, in fact, that there are two vertices $u_i$ and $u_j$ (1 ≤ i ≠ j ≤ a) such that all edges $u_iu_{i,p}$ and all edges $u_ju_{j,q}$ are blue for 1 ≤ p ≤ a_i and 1 ≤ q ≤ a_j. This is certainly the case if $u$ is adjacent to no end-vertex. Thus, we may assume that $u$ is adjacent to exactly one end-vertex, say $u_a$ is an end-vertex. Thus $u_1$ and $u_2$ are not end-vertices and $uu_1$ and $uu_2$ are red. If there is $p$ with (1 ≤ p ≤ a_1) such that $u_1u_{1,p}$ is red, then we can change the colors of $uu_1$ and $uu_a$ to blue and the resulting coloring is a $Y_1$-coloring with fewer red edges. Since this is impossible, all edges $u_1u_{1,p}$ are blue for all $p$ with 1 ≤ p ≤ a_1. Similarly, all edges $u_2u_{2,q}$ are blue for all $q$ with 1 ≤ q ≤ a_2. Therefore, as claimed, all edges $u_1u_{1,p}$ and $u_2u_{2,q}$ are blue for 1 ≤ p ≤ a_1 and 1 ≤ q ≤ a_2.

With an argument similar to the one used in Case 1, we obtain a red-blue caterpillar $C$ rooted at $u$ such that the spine of $C$ is $(x_1, x_2, \ldots, x_d)$ (where $x_1 = u$, $x_2 = u_1$ and $x_3 = u_{1,1}$ and so on) and $d ≥ 3$ is odd. For each $i$ with 1 ≤ i ≤ d, let $X_i$ be the set of end-vertices of $C$ that are adjacent to $x_i$. Thus $X_i$ contains no end-vertex of $T$ for 1 ≤ i ≤ d − 1 and $X_d$ contains at least one end-vertex of $T$, say $x \in X_d$ is an end-vertex of $T$. Since $x_{d,2}$ is red, we can change the color of $x_{d,2}$ to blue and the resulting coloring is also a $Y_1$-coloring with fewer red edges, a contradiction. ■

It is useful to recall the following result on $\chi'_{Y_1}(G)$ and $\chi'_{Y_2}(G)$ for a connected graph $G$ of order at least 4 having no vertex of degree 2.

**Theorem 6.4.2** If $G$ is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) ≤ \alpha''(G)$ and $\chi'_{Y_2}(G) = \alpha''_2(G)$.

**Theorem 6.4.3** If $T$ is a tree of order at least 4 having no vertex of degree 2, then

$$\chi'_{Y_1}(T) = \alpha''(T).$$

**Proof.** By Theorem 6.4.2, it remains to show that $\chi'_{Y_1}(T) ≥ \alpha''(T)$. Assume, to the contrary, that $\chi'_{Y_1}(T) = k ≤ \alpha''(T) - 1$. Let $c$ be a minimum $Y_1$-coloring of $T$ such that the edge independence number of the red subgraph $G_{c,r}$ is maximum. By Lemma 6.4.1, every non-end-vertex of $T$ is incident with at least two blue edges. We claim that $E_{c,r}$ is an independent set of edges of $T$. For otherwise, suppose that $uv$ and $vw$ are adjacent edges in $E_{c,r}$. By Lemma 6.4.1, $v$ is incident with two blue edges. If $u$ is an end-vertex of $T$, then the coloring obtained from $c$ by changing the color of $uv$ to blue is an $Y_1$-coloring with fewer red edges than $c$. This is impossible since $c$ is a minimum $Y_1$-coloring. Thus $u$ is not an end-vertex. Similarly, $w$ is not an end-vertex of
Thus, we may assume that \( N(u) = \{v, u_1, u_2, \ldots, u_\alpha\} \), \( N(v) = \{u, w, v_1, v_2, \ldots, v_\beta\} \) and \( N(w) = \{v, w_1, w_2, \ldots, w_\gamma\} \), where \( \alpha, \beta, \gamma \geq 2 \). By Lemma 6.4.1, we may assume that \( uu_1, uu_2, vv_1, vv_2, ww_1, ww_2 \) are blue. If there exists \( i \) with \( 1 \leq i \leq \alpha \) such that \( uu_i \) is red, then the coloring obtained from \( c \) by changing the color of \( uv \) to blue is an \( Y_1 \)-coloring with fewer red edges than \( c \), which is impossible. Hence all edges \( uu_i \) (\( 1 \leq i \leq \alpha \)) are blue. Similarly, all edges \( ww_i \) (\( 1 \leq i \leq \gamma \)) are blue. If there is \( i \) (\( 1 \leq i \leq \alpha \)) such that \( u_i \) is not incident with any red edge, then the coloring \( c' \) obtained from \( c \) by interchanging the colors \( uu_i \) and \( uv \) is a minimum \( Y_1 \)-coloring with a larger number of independent edges in \( E_{c',r} \), which contradicts the defining property of \( c \). Thus each vertex \( u_i \) (\( 1 \leq i \leq \alpha \)) is incident with at least one red edge. Similarly, each vertex \( w_i \) (\( 1 \leq i \leq \beta \)) is incident with at least one red edge. By Lemma 6.4.1, each \( u_i \) (\( 1 \leq i \leq \alpha \)) is incident with at least two blue edges. Thus, the coloring obtained from \( c \) by changing the color of \( uv \) to blue is a \( Y_1 \)-coloring with fewer red edges than \( c \), which is impossible. Thus, as claimed, \( E_{c,r} \) is an independent set of edges of \( T \). Since \( |E_{c,r}| = k \leq \alpha''(T) - 1 \), it follows that \( E_{c,r} \) is not a maximal independent set of edges of \( T \). Thus there is a blue edge \( e \notin E_{c,r} \) such that \( E_{c,r} \cup \{e\} \) is an independent set of edges of \( T \). However then, the blue edge \( e \) is not incident with any red edge and so \( c \) is not an \( Y_1 \)-coloring, which is a contradiction.

The condition in Theorem 6.4.3 that \( T \) has no vertex of degree 2 is necessary. For example, let \( k \) be an arbitrary positive integer and let \( T \) be the tree obtained from \( P_{3k+1} = (v_1, v_2, \ldots, v_{3k+1}) \) by adding the pendant edge \( vv_2 \) at the vertex \( v_2 \). Then \( \chi'_{Y_1}(T) = 3k - 1 \) and \( \alpha''(T) = k \) and so \( \chi'_{Y_1}(T) - \alpha''(T) = 2k - 1 \), which can be arbitrarily large. In fact, this is also true for trees without non-claw edges. To see this, we first construct the tree \( T_0 \) from the subdivision graph \( S(K_{1,3}) \) of \( K_{1,3} \) by adding two pendant edges at each end-vertex of \( S(K_{1,3}) \). Suppose that the central vertex of \( K_{1,3} \) is \( t \) and \( t \) is adjacent to three vertices \( u, v, w \) of degree 2. Furthermore, suppose that \( u \) is adjacent to \( x \), \( v \) is adjacent to \( y \) and \( w \) is adjacent to \( z \). Thus each of \( x, y, z \) is adjacent to two end-vertices in \( T \). Observe that \( \alpha''(T_0) = 3 \) and \( \{ux, vy, wz\} \) is a maximal matching in \( T_0 \). Since \( T_0 \) has four edge-disjoint copies of \( K_{1,3}, \) any two of which have only an end-vertex in common, a \( Y_1 \)-coloring of \( T_0 \) must assign red to at least one edge in each of these four copies of \( K_{1,3} \) and so \( \chi'_{Y_1}(T_0) \geq 4 \). On the other hand, the red-blue coloring \( c \) with \( E_{c,r} = \{tu, ux, vy, wz\} \) is a \( Y_1 \)-coloring and so \( \chi'_{Y_1}(T_0) = 4 \). Let \( T_1, T_2, \ldots, T_k \) be \( k \) copies of \( T_0 \). For each \( i \) with \( 1 \leq i \leq k \), let \( v_i \) be a vertex of degree 2 in \( T_i \) that corresponds to \( v \) in \( T_0 \). The tree \( T \) is then constructed from \( T_1, T_2, \ldots, T_k \) by adding the edges \( v_i v_{i+1} \) for \( 1 \leq i \leq k - 1 \) (see Figure 6.13 for \( k = 3 \)). Although \( T \) contains vertices of degree 2, it contains no non-claw edges. It can be shown that \( \chi'_{Y_1}(T) = 4k \)
and \( \alpha''(T) = 3k \). Therefore, \( \chi'_{Y_1}(T) - \alpha''(T) = k \), which can be arbitrarily large.

Theorems 6.3.1 and 6.4.3 provide us an upper bound for \( \chi'_{Y_2}(T) \) of a tree \( T \) in terms of \( \alpha''(T) \).

**Corollary 6.4.4** If \( T \) is a tree of order at least 4 having no vertex of degree 2, then

\[
\chi'_{Y_2}(T) \leq 3\alpha''(T).
\]

The upper bound in Corollary 6.4.4 is sharp. In fact, for each positive odd integer \( k \), there is a tree \( T_k \) having no vertex of degree 2 such that \( \chi'_{Y_2}(T_k) = 3k \) and \( \chi'_{Y_1}(T_k) = k \), as we show next. For each \( i \) with \( 1 \leq i \leq k \), let \( S_i = S_{3,4} \) be the double star with central vertices \( u_i \) and \( v_i \), where \( u_i \) is adjacent to the two end-vertices and \( v_i \) is adjacent to the three end-vertices one of which is \( w_i \). If \( k = 1 \), let \( T_1 = S_1 \); while if \( k \geq 3 \), let \( k = 2\ell + 1 \) where \( \ell \geq 1 \) and we construct \( T_k \) in the following two steps: (1) For each \( i \) with \( 1 \leq i \leq \ell \), identifying \( w_i \) in \( S_i \) with \( w_{i+1} \) in \( S_{i+1} \) and labeling the identified vertex by \( x_i \), resulting in a tree \( T' \) in which each \( x_i \) has degree 2 for \( 1 \leq i \leq \ell \); (2) For each \( i \) with \( 1 \leq i \leq \ell \), identify \( w_{\ell+1+i} \) in \( S_{\ell+1+i} \) with the vertex \( x_i \) in \( T' \) constructed in (1), producing the tree \( T_k \). The tree \( T_7 \) containing seven copies of \( S_{3,4} \) is shown in Figure 6.14. Then \( T_k \) has the desired properties for each \( k \geq 3 \).

![Figure 6.13: A tree T with \( \chi'_{Y_1}(T) - \alpha''(T) = 3 \)](image)

![Figure 6.14: The tree \( T_7 \) with \( \chi'_{Y_2}(T_7) = 21 = 3\chi'_{Y_1}(T_7) \)](image)
If $G$ is a connected graph of order at least 4 having no vertex of degree 2 and $c$ is a minimum $Y_1$-coloring of $G$, then the structure of the red subgraph induced by $c$ can be determined. A graph $H$ is a galaxy if each component of $H$ is a star of order 2 or more.

**Theorem 6.4.5** Let $G$ be a connected graph having no vertices of degree 2. If $c$ is a minimum $Y_1$-coloring of $G$, then the red subgraph induced by $c$ is a galaxy in $G$.

**Proof.** Let $c$ be a minimum $Y_1$-coloring of $G$ and let $G_{c,r}$ be the red subgraph induced by $c$. Assume, to the contrary, that $G_{c,r}$ is not a galaxy. We then have the following two cases.

**Case 1.** $G_{c,r}$ contains a path $P$ of length 3, say $P = (u,v,x,y)$. If $vx$ is adjacent to a blue edge in $G$, then the red-blue coloring obtained from $c$ by re-coloring $vx$ blue is a $Y_1$-coloring with fewer red edges. This is a contradiction. Thus, we may assume that all edges adjacent to $vx$ are red. Let $N(x) = \{v\} \cup \{x_1 = y, x_2, \ldots, x_p\}$, where then $p \geq 2$ and $xx_1, xx_2, \ldots, xx_p$ are all red edges in $G$. Hence $x$ is incident with at least three red edges.

If there is some $x_i$ ($1 \leq i \leq p$) such that $x_i$ is an end-vertex of $G$, say $x_i = x_1$, then the red-blue coloring obtained from $c$ by re-coloring $vx$ and $xx_1$ blue is a $Y_1$-coloring with fewer red edges, a contradiction. Thus $\deg_G x_i \geq 3$ for each $i$ with $1 \leq i \leq p$. If there is some $x_i$ ($1 \leq i \leq p$) such that $\deg_{G_{c,r}} x_i \geq 2$, say $x_i = x_1$. Then the red-blue coloring obtained from $c$ by re-coloring $vx$ and $xx_1$ blue is a $Y_1$-coloring with fewer red edges. This is a contradiction. Thus $\deg_{G_{c,r}} x_i = 1$ for $1 \leq i \leq p$. For each integer $i$ with $1 \leq i \leq p$, let $N(x_i) = \{x\} \cup \{x_{i,1}, x_{i,2}, \ldots, x_{i,q_i}\}$, where then $q_i \geq 2$ and $x_ix_{i,1}, x_ix_{i,2}, \ldots, x_ix_{i,q_i}$ are all blue for each $i$ with $1 \leq i \leq p$.

Consider the vertex $x_1$. First, suppose that there is some $j$ with $1 \leq j \leq q_1$ such that all edges incident with $x_{1,j}$ are blue, say $x_{1,j} = x_{1,1}$. Then the red-blue coloring $c^*$ obtained from $c$ by (1) interchanging the colors of $xx_1$ and $x_1x_{1,1}$ and (2) re-coloring $vx$ to blue is a $Y_1$-coloring with fewer red edges, which is a contradiction. Next, suppose that every vertex $x_{1,j}$ is incident with at least one red edge for all $j$ with $1 \leq j \leq q_1$. If there is $j_0$ with $1 \leq j_0 \leq q_1$ such that $x_{1,j_0}$ is incident with exactly one blue edge (namely $x_1x_{1,j_0}$), then the red-blue coloring obtained by (1) interchanging the colors of $xx_1$ and $x_1x_{1,j_0}$ and (2) re-coloring $vx$ to blue is a $Y_1$-coloring with fewer red edges, which is a contradiction. Thus each vertex $x_{1,j}$ ($1 \leq j \leq q_1$) is incident with at least two blue edges.

Then the red-blue coloring obtained by re-coloring $vx$ and $xx_1$ to blue is a $Y_1$-coloring with fewer red edges, which is a contradiction.
Case 2. $G_{c,r}$ contains a 3-cycle $C$, say $C = (u, v, w, u)$. Then each of $u, v, w$ has degree at least 3. If one of $u, v, w$ is incident with a blue edge, say, $u$ is incident with a blue edge, then the red-blue coloring obtained from $c$ by re-coloring $uv$ blue is a $Y_1$-coloring with fewer red edges. This is a contradiction. Thus each of $u, v, w$ is only incident with red edges and so each of $u, v, w$ is incident with at least three red edges. However then, the red-blue coloring obtained from $c$ by re-coloring $uv, uw, vw$ to blue is a $Y_1$-coloring with fewer red edges, a contradiction.

By Cases 1 and 2, it follows that $G_{c,r}$ contains no path of length 3 and no 3-cycle, which implies that each component of $G_{c,r}$ is a star. Therefore, $G_{c,r}$ is a galaxy.

6.5 Minimal $Y$-Colorings

For a given color frame $F$, an $F$-coloring $c$ of a graph $G$ is a minimal $F$-coloring of $G$ if no proper subset of $E_{c,r}$ is the set of red edges of an $F$-coloring of $G$. Thus a minimal $F$-coloring has the property that if any red edge of $G$ is re-colored blue, then the resulting red-blue coloring of $G$ is not an $F$-coloring of $G$. For example, a minimal $Y_1$-coloring of a tree with 7 red edges is shown in Figure 6.15, where each red edge is drawn in a bold line. The maximum number of red edges in a minimal $F$-coloring of $G$ is the upper $F$-chromatic index $\chi''_F(G)$ of $G$. Since every minimum $F$-coloring of a graph $G$ is minimal, $\chi'_F(G) \leq \chi''_F(G)$. For the tree $T$ of Figure 6.15, $\chi''_{Y_1}(T) = 7$ and it follows by Theorem 6.4.3 that $\chi'_{Y_1}(T) = \alpha''(T) = 6$. The concepts of minimal $F$-colorings and upper $F$-chromatic indexes were introduced and studied in [39].

![Figure 6.15: A minimal $Y_1$-coloring of a tree $T$](image)

By Theorem 6.4.5, the red subgraph induced by a minimum $Y_1$-coloring in a connected graph having no vertices of degree 2 is a galaxy; while this may not be the case for a minimal $Y_1$-coloring. For example, the red subgraph induced by the minimal $Y_1$-coloring shown in Figure 6.15 is a double star.

By Theorem 6.4.2, if $G$ is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_1}(G) \leq \alpha''(G)$. We now show that $\chi''_{Y_1}(G) \geq \alpha'(G)$ for such a graph $G$. 
Theorem 6.5.1 \textit{If} $G$ \textit{is} a connected graph having no vertices of degree 2, \textit{then}

$$
\chi_{Y_1}''(G) \geq \alpha'(G).
$$

\textbf{Proof.} Let $M$ be a maximum matching. Then $|M| = \alpha'(G)$. Consider a red-blue coloring $c$ of $G$ that assigns the color red to every edge in $M$ and the color blue to all other edges of $G$. Let $e = uv$ be a blue edge in $G$. Since $M$ is a maximum matching, $M \cup \{e\}$ is not a matching and so either $u$ or $v$ is incident with at least one red edge in $M$, say $v$ is incident with a red edge $vw$. Since $G$ has no vertices of degree 2, it follows that $\deg v \geq 3$ and so $v$ is incident two or more blue edges. Thus $c$ is a $Y_1$-coloring. Next, we show that $c$ is minimal. Let $c'$ be a red-blue coloring obtained from $c$ by changing the color of an edge $f \in M$ to blue. However then, the blue edge $f$ in $c'$ is not adjacent to any red edge in $M - \{f\}$ and so $c'$ is not a $Y_1$-coloring. Therefore, $c$ is minimal $Y_1$-coloring with $\alpha'(G)$ red edges and so $\chi_{Y_1}''(G) \geq |M| = \alpha'(G)$.

The lower bound in Theorem 6.5.1 is sharp. In order to show this, we determine the upper $Y_1$-chromatic index of the corona of an $n$-cycle. It is shown in [14] that $\chi_{Y_1}'(G) = \lfloor n/2 \rfloor$ if $G$ is the corona of an $n$-cycle where $n \geq 3$.

Proposition 6.5.2 \textit{If} $G$ \textit{is} the corona of an $n$-cycle where $n \geq 3$, \textit{then} $\chi_{Y_1}''(G) = \alpha'(G)$.

\textbf{Proof.} Let $G = cor(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ for some integer $n \geq 3$. Suppose that $u_iv_i$ is a pendant edge of $G$ at $v_i$ for $1 \leq i \leq n$. Since the order of $G$ is $2n$ and $\{u_1v_i : 1 \leq i \leq n\}$ is a matching in $G$, it follows that $\alpha'(G) = n$. By Theorem 6.5.1, $\chi_{Y_1}''(G) \geq \alpha'(G) = n$. It remains to show that $\chi_{Y_1}'(G) \leq n$. Let $c$ be a minimal $Y_1$-coloring of $G$ with $|E_{c,r}| = \chi_{Y_1}''(G)$ and let $G_r$ be the red subgraph induced by $c$. We claim that each vertex of $C_n$ is incident with exactly one red edge in $G_r$. First, suppose that there is $v_i \in V(C_n)$ where $1 \leq i \leq n$ such that $v_i$ is incident with no red edge of $G_r$. Then the blue edge $v_iu_i$ does not belong to any copy of $Y_1$, which is impossible. Next, suppose that there is $v_j \in V(C_n)$ where $1 \leq j \leq n$ such that $v_j$ is incident with at least two red edges of $G_r$, say $j = 1$. If the two red edges are $v_nv_1$ and $v_1v_2$, then the blue edge $v_1u_1$ does not belong to any copy of $Y_1$, a contradiction. If, on the other hand, one of these two red edges is $v_1u_1$, then the red-blue coloring obtained from $c$ by changing the color of $v_1u_1$ to blue is a $Y_1$-coloring of $G$ with fewer red edges, which is again a contradiction. Therefore, as claimed, every vertex of $C_n$ is incident with exactly one red edge in $G_r$. This implies that $E_{c,r}$ is an independent set of edges in $G$ and so $\chi_{Y_1}'(G) = |E_{c,r}| \leq \alpha'(G) = n$. Therefore, $\chi_{Y_1}''(G) = n$. \hfill \blacksquare
The value of $\chi''_{Y_1}(G) - \alpha'(G)$ can also be arbitrarily large for a connected graph $G$, as we show next. For each positive integer $k$, let $W_{6k} = C_{6k} + K_1$ be the wheel of order $6k+1$, where the vertex $v$ of $W_{6k}$ is adjacent to every vertex of $C_{6k} = (v_1, v_2, \ldots, v_{6k}, v_{6k+1} = v_1)$ in $W_{6k}$. Let $G_k$ be the graph obtained from $W_{6k}$ by adding $k$ edges $v_iv_{i+3k}$ for each integer $i$ with $i \equiv 2 \pmod{3}$ and $2 \leq i \leq 3k - 1$. The order of $G_k$ is $6k + 1$. The graph $G_2$ of order 13 is shown in Figure 6.16. Since $M = \{v_iv_{i+1} \colon i \text{ is odd and } 1 \leq i \leq 6k - 1\}$ is a maximum matching in $G_k$, it follows that $\alpha'(G_k) = 3k$. The red-blue coloring $c$ of $G_k$ defined by

$$E_{c,r} = \{v_iv_{i+1}, v_{i+1}v_{i+2} \colon i \equiv 1 \pmod{3} \text{ and } 1 \leq i \leq 6k - 2\}$$

is a minimal $Y_1$-coloring of $G_k$. (The coloring $c$ is shown in Figure 6.16 for $k = 2$.) Thus $\chi''_{Y_1}(G_k) \geq |E_{c,r}| = 4k$. Therefore, $\chi''_{Y_1}(G_k) - \alpha'(G_k) \geq k$, which can be arbitrarily large.

Figure 6.16: The graph $G_2$ and a minimal $Y_1$-coloring of $G_2$

The argument employed in the proof of Proposition 6.5.1 also shows that a maximal matching $M$ in a graph $G$ gives rise to a minimal $Y_1$-coloring $c$ of $G$ such that $E_{c,r} = M$.

**Corollary 6.5.3** If $G$ is a connected graph of order at least 4 having no vertices of degree 2 and $M$ is maximal matching, then the $Y_1$-coloring $c$ of $G$ with $E_{c,r} = M$ is a minimal $Y_1$-coloring.

The converse of Corollary 6.5.3 is not true in general; that is, there are minimal $Y_1$-colorings $c$ of a connected graph order at least 4 having no vertices of degree 2 such that $E_{c,r}$ is not even a matching, as the graph of Figure 6.16 shows.

By Theorem 6.4.2, if $G$ is a connected graph of order at least 4 having no vertex of degree 2, then $\chi'_{Y_2}(G) = \alpha''_2(G)$. By an argument similar to the proof of Theorem 6.5.1, we now show that $\chi''_{Y_2}(G) \geq \alpha'_2(G)$ for every such connected graph $G$. 
Theorem 6.5.4 If $G$ is a connected graph having no vertices of degree 2, then

$$\chi''(G) \geq \alpha'_2(G).$$

Proof. Let $X$ be a $\Delta_2$-set of maximum size where then $|X| = \alpha'_2(G)$. Consider a red-blue coloring $c$ of $G$ that assigns the color red to every edge in $X$ and the color blue to all other edges of $G$. Let $e = uv$ be a blue edge in $G$. Since $X$ is a maximum $\Delta_2$-set, $X \cup \{e\}$ is not a $\Delta_2$-set and so either $u$ or $v$ is incident with at least two red edge in $X$ and $e$ belongs to a copy of $Y_2$. Thus $c$ is a $Y_2$-coloring. Next, we show that $c$ is minimal. Let $X'$ be a proper subset of $X$ and let $c'$ be the red-blue coloring of $G$ such that $E_{c'} = X'$. Now let $f \in X - X'$ be a blue edge in $c'$. Since $\Delta(G[X]) = 2$, it follows that the blue edge $f$ in $c'$ is not adjacent to two red edges in $X - \{f\}$ and so $f$ does not belong to a copy of $Y_2$. Hence $c'$ is not a $Y_2$-coloring. Therefore, $c$ is minimal $Y_2$-coloring with $\alpha'_2(G)$ red edges and so $\chi''(G) \geq |X| = \alpha'_2(G).$ \hfill \blacksquare

To show that the lower bound in Theorem 6.5.4 is sharp, we determine the upper $Y_2$-chromatic index of the corona of an $n$-cycle. It is shown in [14] that $\chi''(G) = n$ if $G$ is the corona of an $n$-cycle where $n \geq 3$.

Proposition 6.5.5 If $G$ is the corona of an $n$-cycle where $n \geq 3$, then

$$\chi''(G) = \alpha'_2(G) = 2n - \lceil n/2 \rceil.$$

Proof. Let $G = cor(C_n)$ where $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ for some integer $n \geq 3$. Suppose that $u_i v_i$ is the pendant edge of $G$ at $v_i$ for $1 \leq i \leq n$. We first show that $\chi''(G) = 2n - \lceil n/2 \rceil$. First, we show that there is a minimal $Y_2$-coloring $c$ of $G$ having exactly $2n - \lceil n/2 \rceil$ red edges. For an even integer $n \geq 4$, let

$$E_{c,r} = \{v_i v_{i+1} : i \text{ is odd, } 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\} \quad (6.5)$$

and so $E_{c,b} = \{v_i v_{i+1} : i \text{ is even, } 2 \leq i \leq n\}$. For an odd integer $n \geq 3$, let

$$E_{c,r} = \{v_i v_{i+1} : i \text{ is odd, } 1 \leq i \leq n\} \cup \{u_i v_i : 2 \leq i \leq n\} \quad (6.6)$$

and so $E_{c,b} = \{v_i v_{i+1} : i \text{ is even, } 2 \leq i \leq n - 1\} \cup \{u_1 v_1\}$. (This coloring is shown in Figure 6.17 for $n = 8$ and $n = 9$.) Then $|E_{c,b}| = \lceil n/2 \rceil$ and so $\chi''(G) \geq |E_{c,r}| = 2n - \lceil n/2 \rceil$.

Next, we show that $\chi''(G) \leq 2n - \lceil n/2 \rceil$. Assume, to the contrary, that

$$\chi''(G) = t \geq 2n - \lceil n/2 \rceil + 1. \quad (6.7)$$
Let $c^*$ be a minimal $Y_2$-coloring of $G$ having exactly $t$ red edges and let $G_r$ be the red subgraph induced by $c^*$. Thus the size of $G_r$ is $t$. First, suppose that $G_r$ contains a vertex $v$ such that $\deg_{G_r}(v) = 3$, say $v = v_2$ and $v_1v_2, v_2v_3$ and $v_2v_4$ are red. Then the red-blue coloring obtained from $c^*$ by changing the color of $v_2v_4$ to blue is an $Y_2$-coloring, which is impossible. Thus $\deg_{G_r}(v) \leq 2$ for every vertex $v$ of $G_r$. Since (i) the order of $G_r$ is at most $2n$ and (ii) at most $n$ vertices in $G_r$ have degree 2 and the remaining vertices of $G_r$ are end-vertices, the size $t$ of $G_r$ is at most $1/2(2n + n) = n + n/2$. By (6.7), $2n - [n/2] + 1 \leq t \leq n + n/2$ or $n/2 + 1 \leq [n/2]$, which is impossible. Therefore, $\chi''_{Y_2}(G) = 2n - [n/2]$.

It remains to show that $\alpha'_2(G) = 2n - [n/2]$. If $n$ is even, then the subgraph induced by the set $E_{c,r}$ described in (6.5) is a $\Delta_2$-set in $G$; while is $n$ is odd, then the subgraph induced by the set $E_{c,r}$ described in (6.6) is a $\Delta_2$-set in $G$. Thus $\alpha'_2(G) \geq |E_{c,r}| = 2n - [n/2]$. Since $\alpha'_2(G) \leq \chi''_{Y_2}(G) = 2n - [n/2]$ by Theorem 6.5.4, it follows that $\chi''_{Y_2}(G) = \alpha'_2(G)$.

The value of $\chi''_{Y_2}(G) - \alpha'_2(G)$ can also be arbitrarily large for a connected graph $G$, as we show next. For each integer $k \geq 3$, let $H$ and $H'$ be two copy of $K_{2,k}$, where

$$V(H) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_k\} \text{ and } V(H') = \{u'_1, u'_2\} \cup \{v'_1, v'_2, \ldots, v'_k\}.$$ 

Let $U = \{u_1, u_2\}$, $V = \{v_1, v_2, \ldots, v_k\}$, $U' = \{u'_1, u'_2\}$ and $V' = \{v'_1, v'_2, \ldots, v'_k\}$. The graph $H_k$ is obtained from $H$ and $H'$ by adding $k$ new vertices in $W = \{w_1, w_2, \ldots, w_k\}$ and joining each $w_i$ ($1 \leq i \leq k$) to every vertex in $V \cup V'$. The order of $H_k$ is $3k + 4$. (The graph $H_3$ is shown in Figure 6.18.) Define a red-blue coloring $c$ with $E_{c,r} = E(H) \cup E(H')$. The coloring $c$ is shown in Figure 6.18 for $k = 3$. Since $c$ is a minimal $Y_2$-coloring of $H_k$, 

![Figure 6.17: Illustrate the coloring $c$ for $\text{cor}(C_8)$ and $\text{cor}(C_9)$](image)
it follows that \(\chi''_Y(H_k) \geq |E_{c,r}| = 4k\). Next, let \(X_k = \{v_iw_i, w_iv'_i : 1 \leq i \leq k\}\) and let

\[
Y_k = \begin{cases} 
X_k \cup \{u_1v_1, u_1v_2, u_2v_3, u'_1v'_1, u'_1v'_2, u'_2v'_3\} & \text{if } k = 3 \\
X_k \cup \{u_1v_1, u_1v_2, u_2v_3, u_1v'_1, u'_1v'_2, u'_2v'_3, u'_2v'_4\} & \text{if } k \geq 4.
\end{cases}
\]

Since \(Y_k\) is a maximal \(\Delta_2\)-set of \(H_k\), it follows that \(\alpha''_2(H_k) \leq |Y_k| = 2(k + \min\{k, 4\})\). Thus \(\chi''_Y(H_k) - \alpha''_2(H_k) \geq 4k - 2(k + \min\{k, 4\}) = 2(k - \min\{k, 4\})\), which can be arbitrarily large.

![Figure 6.18: The graph \(H_3\) and a minimal \(Y_2\)-coloring of \(H_3\)](image)

### 6.6 Closing Statements

It was shown in [40] that if \(G\) is a graph and \(k\) is an integer with \(\alpha''(G) \leq k \leq \alpha'(G)\), then \(G\) contains a maximal matching with \(k\) edges. It can be shown that if \(Y \in \{Y_1, Y_2\}\) is a color frame of a claw and \(G\) is the corona of an \(n\)-cycle where \(n \geq 3\), then for each integer \(k\) with \(\chi'_Y(G) \leq k \leq \chi''_Y(G)\), there is a minimal \(Y\)-coloring of \(G\) using exactly \(k\) red edges. It gives rise to the following question.

**Problem 6.6.1** Let \(Y \in \{Y_1, Y_2\}\) be a color frame of a claw. If \(G\) is a connected graph of order at least 4 and \(k\) is an integer with \(\chi'_Y(G) \leq k \leq \chi''_Y(G)\), is there a minimal \(Y\)-coloring of \(G\) using exactly \(k\) red edges?

For a connected graph \(G\) and a color frame \(F\), if \(\chi'_F(G) = a\) and \(\chi''_F(G) = b\), then \(a \leq b\) by the definitions of the \(F\)-chromatic index and upper \(F\)-chromatic index of \(G\). Thus we conclude this chapter with another question.

**Problem 6.6.2** Let \(Y \in \{Y_1, Y_2\}\) be a color frame of a claw. For which pairs \(a, b\) of positive integers with \(a \leq b\), does there exist a connected graph \(G\) such that \(\chi'_Y(G) = a\) and \(\chi''_Y(G) = b\)?
Bibliography


