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OPTIMUM FAILURE TRUNCATED TESTING
STRATEGIES

by

Erik Kostandyan

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Industrial and Manufacturing Engineering
Advisors: Azim Houshyar, Ph.D. and Leonard Lamberson, Ph.D.

Western Michigan University
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May 2010

OPTIMUM FAILURE TRUNCATED TESTING STRATEGIES

Erik Kostandyan, Ph.D.

Western Michigan University, 2010

Accumulated fatigue damage on mechanical components due to random stress loads eventually causes failure. Therefore products with lower failure rates are more desirable. Testing mechanical components for their intended purpose under predetermined working conditions is a common practice used by industries to prevent failures. Fatigue tests are categorized as Time Truncated or Failure Truncated, known in the literature as Type I and Type II tests, respectively. In failure truncated tests, the mechanical components are tested until the desired number of results is obtained. The parameters of a typical failure truncated test include the capacity of the test facility, the actual number of components placed on the test, the termination of the test once a predetermined number of test results has been collected and the duration of the test. Also, important is the cost for test time and components as well as the desired confidence in the results. The investigation of varying Type II testing strategies to determine optimal test methods is the essence of this research. Also, in this research a new failure truncated test is investigated. This research considers two different Type II test strategies. The strategies are termed: the Modified Sudden Death Test (MSDT) and the Classified Sudden Death Test (CSDT). In this study, the time and cost domains for MSDT and CSDT are investigated. The theoretical research in test completion time for the MSDT and CSDT is done to establish the most advantageous testing strategy from both a time and cost perspective.

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LIST OF NOTATIONS

A	Number of available testing facilities
N	Number of components in the test
R	Predetermined number of failures for the test
K	Number of Groups
N	Number of components in the Group
R	Designated number of failures for the Group
$F(t)$	Cumulative distribution function of the population
$f(t)$	Probability density function of the population
$W(t)$	Cumulative distribution function of the Weibull distributed population
$w(t)$	Probability density function of the Weibull distributed population
$G_{r:n}(t)$	Cumulative distribution function of r^{th} out of n ordered random variable
$g_{r:n}(t)$	Probability density function of r^{th} out of n ordered random variable
G	Number of Grouping Combinations
$t_{r:n}$	The r^{th} failure-time out of n components
$t_{r:n,k}$	The r^{th} failure-time out of n components in the k^{th} Group
T_s	Run time for the Serial Failure Tests
T_p	Run time for the Parallel Failure Tests
T_{ac}	Total accumulated time for a Group
$T_{\text{test_ac}}$	Total accumulated time for the Test
C_{TTC}	Total Testing Time Cost

List of Notations—Continued

C_C	Total Components Cost
C_S	Supervision or Technician Cost
C_O	Total Operational Cost
c_1	Unit testing time cost
c_2	Unit component cost
c_3	Technician cost per unit testing time
B_p	(1-p) level reliability
$Q_{r,n}(p_c)$	Cumulative distribution function of $p_{r,n}=P(t \leq t_{r,n})$ random variable
$q_{r,n}(p_c)$	Probability density function of $p_{r,n}=P(t \leq t_{r,n})$ random variable
$t_{r,n,k}^{\max}$	Maximum of the r^{th} out of n components from the k^{th} Group random variable
$M_{r,n,k}(t)$	Cumulative distribution function of $t_{r,n,k}^{\max}$ random variable
$m_{r,n,k}(t)$	Probability density function of $t_{r,n,k}^{\max}$ random variable
$O_{r,n,k}(t)$	The ratio of the $m_{r,n,k}(t)$ to $M_{r,n,k}(t)$
$s(r,n,k)$	Sum of the r^{th} out of n components from the k^{th} Group random variable
$b(r,n,k)$	Average of the r^{th} out of n components from the k^{th} Group random variable
$f_{b(r,n,k)}(t)$	Probability density function of $b(r,n,k)$ random variable
$F_{b(r,n,k)}(t)$	Cumulative distribution function of $b(r,n,k)$ random variable
$H_{(r,n,k)}(p_c)$	Cumulative distribution function of $p_{(r,n,k)}=P(t \leq b(r,n,k))$ random variable
$h_{(r,n,k)}(p_c)$	Probability density function of $p_{(r,n,k)}=P(t \leq b(r,n,k))$ random variable

CHAPTER 1

INTRODUCTION

1.1 Introduction and Problem Statement

Reliability is defined as the probability that the product will be able to perform its intended function in a specified working environment for a specified time. To evaluate the product reliability, companies execute reliability tests. In reliability tests, a random sample of the product is tested under assigned conditions. The test conditions are adjusted as closely as possible to the product working environment. Based on the results of the reliability test, statistical inference is used to estimate the product's reliability. Generally, two major testing procedures exist, termed "Time Truncated" and "Failure Truncated" tests.

In "Time Truncated" tests, the products are tested to a predetermined time limit. Whereas in "Failure Truncated" tests, the test is terminated once the predetermined test results are collected. In the literature, "Time Truncated" tests are known as Type I tests and "Failure Truncated" tests, as Type II tests.

Time and cost are inseparable parts of a reliability test. The cost and the time might be categorized as follows:

Cost

Components cost

Unit testing time cost

Total testing time cost

Supervision cost

Test equipment cost

Operational cost

Etc.

Time

Run time or time to finish the test

Total accumulated time or sum of times for all results

Total failing time

Total unfailing time

Etc.

Companies use specialized testing laboratories to perform the tests, where the capacity of the test stations is limited. This is a major restriction for reliability tests. Different testing strategies will vary in cost and time, so an optimum strategy for the reliability test would be desirable, to obtain the least expensive and fastest results.

The parameters of a Type II test include the capacity of the test facility, the actual number of components placed on test, the termination of the test once a predetermined number of test results have been collected and the duration of the test. Also, important is the cost for test time and components as well as the desired confidence in the results.

The investigation of varying test strategies to determine optimal test methods is the essence of this research. The efficiency of the test from a cost and time stand point will be considered in this research.

This research will consider different Type II test strategies. The strategies are termed: the Modified Sudden Death Test and the Classified Sudden Death Test.

Definition: A Modified Sudden Death Test (MSDT) is a failure (Type II) test, where the components are divided into Groups, and each Group is tested without replacement until a predetermined number of failures occur. Once there is the predetermined number of failures in a Group, the test is terminated for that Group and the next Group can be tested. The predetermined number of failures in each group is the same.

Definition: A Classified Sudden Death Test (CSDT) is defined herein as a failure (Type II) test where the components are randomly divided into Groups and all of the Groups are tested simultaneously until the predetermined number of failures is collected from each Group. The predetermined number of failures for each Group is the same.

1.2 The Research Objectives

The goals of the research are:

1. To develop a simulation study in the time domain for the MSDT and CSDT to compare the total test duration.
2. To do theoretical research in time duration for the MSDT and CSDT to analytically determine the total test duration.
3. To compare the simulation study results as a validation for the theoretical results.
4. To develop cost models as a basis for comparing the MSDT and CSDT.
5. To determine optimum test strategies from a cost perspective for the MSDT and CSDT.
6. From the cost models, establish optimum test strategies, considering number of components for the test and number of groups, as well as number of test results in each group under the budget limitation and with the desired confidence level.

For this research the following assumptions will be made:

- The components under consideration are mechanical and non-repairable.
- The component life will be modeled by a two-parameter Weibull distribution, with a shape parameter greater than one.
- For the sake of the comparison of MSDT and CSDT, the available testing facilities utilization and the number of test results collected by either test strategies will be the same.
- The predetermined number of failures for each Group is the same for either test strategy.

CHAPTER 2

LITERATURE REVIEW

2.1 Failure of Mechanical Components

In real life, mechanical components fail due to unpredictable stress loads, where each load affects the material's molecular-microscopic strength by an amount proportional to the loads' stress level and duration. Each load contributes to the history of the molecules' bond damage or fatigue. As time passes, the history of progressive damage produces crack propagation at the highest stress concentrated point, which eventually causes failure. It is important to notice that mechanical components retain a fatigue history.

The failed mechanical components sometimes may cause million-dollar projects to crash and shake the companies' shares in the financial market. E.g., "Suzlon Energy Ltd.", the largest wind-turbine maker, faced a 39% drop in share value in 2008. The cause was a breakage of one of the three 140 feet long wind-turbines, which was installed by "Suzlon" in Illinois.¹

Freudental and Gumbel (Freudenthal & Gumbel, 1953) discussed the fatigue phenomenon and showed that fatigue life has a Weibull distribution. The disruption produced by random stress cycles of the same amplitude S is inversely proportional to the number of stress cycles. Also, they showed that the probability density function $p(N)_s$ of the number of load cycles with constant stress amplitude S has a Weibull distribution.

2.2 Weibull Distribution

In 1951 the Swedish engineer Waloddi Weibull (Weibull, 1951) published an article entitled "A Statistical Distribution Function of Wide Applicability", which was a fundamental manuscript defining the essence of the Weibull distribution. In this article

¹ "The Wall Street Journal", October 25-26, 2008, pp-B4A.

W. Weibull suggested that the model of the system, consisted of n subsystems, and that the system will fail when any one of the n subsystems fails to function. It was assumed that all subsystems fail independently. If one denotes the probability that the subsystem will fail by P and the probability that the system will fail by P_n , then the probability that the system will function would be the probability of the event that all n th subsystems are functioning. So, the probability that the system will function would be

Equation 1
$$(1 - P_n) = (1 - P)^n$$

Let the random variable X have a cumulative probability function $F(X)$, which can be written in the form

Equation 2
$$F(x) = 1 - e^{-\varphi(x)}$$

where $\varphi(x)$ is a function of x and $F(x) = P(X \leq x)$ or this is the probability of the event that the random variable X will be less than or equal to the threshold x .

If the cumulative probability function of each subsystem to fail has a form

Equation 3
$$F(x) = 1 - e^{-\varphi(x)}$$

then the cumulative probability function for the system to fail would be

Equation 4
$$P_n = F(x) = 1 - e^{-n\varphi(x)}$$

The function $\varphi(x)$ must be positive and nondecreasing, and does not necessarily have to equal zero. Also, the function $\varphi(x)$ has a relationship to the $F(x)$ in the following way

Equation 5
$$\varphi(x) = Ln \frac{1}{1 - F(x)}, \quad 0 \leq F(x) \leq 1$$

and

Equation 6
$$n\varphi(x) = nLn \frac{1}{1-F(x)}, 0 \leq F(x) \leq 1$$

Thus, the simplest function, satisfying those conditions, has the following equation

Equation 7
$$\frac{(x-x_u)^m}{x_0}$$

where x_u , x_0 and m are constants and could be described as location, scale and shape parameters, respectively.

So, the cumulative probability function for the system to fail would be

Equation 8
$$F(x) = 1 - e^{-\frac{(x-x_u)^m}{x_0}}$$

This is the Weibull distribution function as given in Equation 8. The Weibull distribution is useful for many engineering component-reliability analyses.

The cumulative probability function for the system failure, based on Weibull's article, is

Equation 9
$$F(x) = 1 - e^{-\frac{(x-x_u)^m}{x_0}}$$

where x_u -location parameter, x_0 - scale parameter and m-shape parameter.

One may write $F(x)$ in the form of

Equation 10
$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\beta}$$

where the location parameter is zero and the scale and shape parameters are θ and β , respectively. (See Appendix A for Weibull distribution function).

2.3 Distribution of the r^{th} Ordered Statistic Out of n

“Definition: Order Statistics is the technical name given to the items in a collection of data when they have been arranged in numerical order from the smallest value to the largest value.”

When dealing with random samples, those samples come in a random fashion. To answer particular questions like “What is a likelihood that in the random sample of size n , the r^{th} in magnitude random variable would be less than or equal to any defined threshold?”, it is necessary to know what would be the cumulative probability or probability distribution functions of the r^{th} out of n ordered random variable. In life test a group of products are all tested at the same time and the failure time data is always ordered.

If the random variable “ t ” has the cumulative distribution function (cdf) $F(t)$ and probability density functions (pdf) $f(t)$, then the r^{th} out of n ordered random variable $t_{r:n}$ has the cumulative distribution function $G_{r:n}(t)$ and probability density functions $g_{r:n}(t)$ (See Appendix B) which may be represented by:

$$\text{Equation 11} \quad G_{r:n}(t) = P(t_{r:n} \leq t) = \sum_{i=r}^n \binom{n}{i} F(t)^i (1 - F(t))^{n-i}$$

$$\text{Equation 12} \quad g_{r:n}(t) = n \binom{n-1}{r-1} F(t)^{r-1} (1 - F(t))^{n-r} f(t)$$

2.5 Background for Sudden Death Test Strategy

Johnson (Johnson, 1964a; Johnson, 1964b) discussed the procedure of running simultaneously more specimens than one intends to fail. Such a test is called an incomplete test. He showed that the testing time required to fail r specimens out of n would be significantly less than to fail r out of r .

In fact, for a Weibull distributed failure time with the slope equal to 1, the median time to fail 10 specimens out of 20 is about 23.86% of the median time required to fail 10 out of 10. If the slope is equal to 2, then it is about 48.85%, and if the slope is 0.5 then it is about 5.69%.

This means that running more specimens than one intends to fail, reduces testing time, but logically increases the cost of the test by the cost of the unfailed specimens, as well as the need for more testing capacity.

Johnson described the testing technique and called it a “Sudden Death” test. This is a test where specimens to be tested are grouped into sets of two or more and run simultaneously until the first failure occurs in each set. Once there is a first failure in a set, the test is terminated for that set. The first failure in each set is the first Order Statistic of the set. Looking upon each set as an assembly where the assembly fails if any one of the k specimens in it fails with Weibull failure distribution $W(\theta, \beta)$, then the assembly has a cumulative distribution function of the form:

Equation 13

$$F_{assembly} = 1 - e^{-\left(\frac{x}{\theta_1}\right)^\beta}$$

$$\text{where } \theta_1 = \frac{\theta}{k^{\frac{1}{\beta}}}$$

Thus, the specimen's characteristic life is equal to the Sudden Death characteristic life times $k^{1/\beta}$.

If the probability that the assembly fails by the time x is F_{assembly} then

Equation 14
$$x = \theta_1 \left(-\ln(1 - F_{\text{assembly}}) \right)^{\frac{1}{\beta}}$$

So, the median time to fail 1 out of k is

Equation 15
$$x = \frac{\theta}{k^{\frac{1}{\beta}}} (\ln(2))^{\frac{1}{\beta}}$$

For example, if the assembly consist of 8 specimens, $k=8$, then x in Equation 15 is an estimate of $B_{8,3}$ of the population. ($B_{8,3}$ is equivalent to 91.7% reliability of the specimens).

And the median time required for the r series Sudden Death failures is

Equation 16
$$\varphi r x = \varphi r \frac{\theta}{k^{\frac{1}{\beta}}} (\ln(2))^{\frac{1}{\beta}}$$

where φ converts the sum of medians into the median of the sum and is an empirical function, suggested by Johnson to be

Equation 17
$$\varphi = \frac{(r-1)! \Gamma(1+1/\beta) (r-1 + \ln 2)^{1/\beta}}{(\ln 2)^{1/\beta} \Gamma(r+1/\beta)}$$

The median time required to fail n out of n specimens is

Equation 18

$$x_1 = \theta \left(-\ln(1 - \lambda_n(n)) \right)^{\frac{1}{\beta}}$$

where the proportion of the population below the n^{th} out of n ordered random variable is $\lambda_n(n)$.

So the proportion of the median time of r series Sudden Death failures to the n out of n median time becomes

Equation 19

$$\frac{\varphi r x}{x_1} = \frac{\varphi r}{k^{\frac{1}{\beta}}} \left(\frac{\ln(2)}{-\ln(1 - \lambda_n(n))} \right)^{\frac{1}{\beta}}$$

Johnson gave an example of having 80 specimens for the test and compared the median time to fail 80 out of 80 with the median time required for 10 Sudden Death sets of 8 specimens each. He pointed out that

“For a Weibull slope=1, the ratio $\frac{\varphi r x}{x_1}$ reduces to 25.54% of the time required to fail 80 out of 80, assuming the sets of eight are tested serially”².

Sudden Death testing is useful when the early deaths are inferior to the later ones because it is a point estimate of the 1st failure time out of k rank (e.g., in bearing applications one is interested in early failures rather than in later ones).

2.6 Background for Modified Sudden Death Test Strategy and Related Works

In 1998 Pascual and Meeker (Pascual & Meeker, 1998) compared sample sizes and the corresponding cumulative testing time estimators for a technique, which they termed Modified Sudden Death Test (MSDT). The comparison was to the traditional experiment

² Johnson, Leonard, The Statistical Treatment of Fatigue Experiments, 1964, 115, Elsevier Publishing Company, New York, pg.105.

test, assuming the life is Weibull distributed. They assumed that the traditional experiment test is a sequential test to a predetermined time length.

The Sudden Death Test is a special case of the Modified Sudden death test when the number of failures in each set is equal to one. In the MSDT the number of failures in each set may be less than or equal to the number of components in the set. If r is the number of failures in the set and n is the number of components in the set, then $r \leq n$ holds.

Pascual and Meeker assumed that in the MSDT the total testing time is the sum of the times for each set.

Let Y_{gr} be the g^{th} -set testing time with r number of failures in it, then the total testing time of the MSDT might be expressed as:

Equation 20
$$L = \sum_{i=1}^g Y_{ir}$$

In order to estimate the mean and variance of the Y_{ir} , the moment generating function of the Weibull order statistic was used. After which, the standardized version of the testing time L was given as L' and cumulants of L' were calculated. The standardized testing time L' was used to compute the q quantiles of L' by the Cornish-Fisher expansion approximation. Subsequently, these q quantiles of L' were used to compute q quantiles of L .

The maximum likelihood estimators \hat{Y}_q for the q quantiles of the Y_{ir} and the asymptotic variance of the maximum likelihood estimators of the $\text{Log}(\hat{Y}_q)$ were computed. They found that the MSDT plans required a shorter time to estimate small quantiles than do traditional plans.

Vlcek, Hendricks and Zaretsky (Vlcek, 2004) in 2003 did a simulation study for virtual bearing life. The virtual bearings under examination were 50-mm bore deep-groove ball bearings. A total of 30,960 bearings were assessed in 33 sudden death test strategies comprising of 36, 72 and 144 bearing groups. Based on the Lunderberg-Palmgren work (Lundberg G., Palmgren A., 1949), it was assumed that bearing life was a two-parameter Weibull distributed, with the slope parameter equal to 1.11. Results from past studies have shown that most bearings with rolling elements have a slope parameter between one and two.

The simulation study results were compared to the calculated theoretical results, based on the Lunderberg-Palmgren equations and Zaretsky's rule. The comparison was done between B_{10} and slope parameter of simulated and calculated results, respectively. Based on the authors' previous work, the maximum and minimum variation equations for the B_{10} life and slope parameter were introduced. Also, simulated results were compared to these maximum and minimum variation lines.

To achieve the predetermined results, the sudden death test requires less testing time than the sequential test (failure of the entire population by multiple testers). Nevertheless, in comparison to the variation lines, the trustworthiness of the results from the sudden death test were not as precisely descriptive for the entire population as those from the sequential test. However, the sudden death test, with some assurance, will provide the predetermined number of failures and/or test results. It was concluded that the variations in slope and shape parameters were functions of failed bearings, rather than number of bearings tested. Also, the authors stated that approximately 40% of testing time could be saved for achieving a predetermined number of test results, in comparison to the sudden death testing, provided that each subgroup is tested to failure or B_{50} (whichever comes first) and the test is terminated when the predetermined number of test results is collected.

Jun, Balamurali and Lee (Jun & Balamurali, 2006) considered single and double sampling plans for lot acceptance, in which sudden death was the test strategy. They

assumed that the failure time was Weibull distributed and the shape parameter was known. They proposed a lot acceptability criteria under sudden death testing with fixed test positions and selectable number of groups. If one allocates n components to g groups of r , so that $n=gr$ holds, and by letting Y_i be the time to the first failure from the i^{th} group ($i=1, \dots, g$), under the assumption that the failure time is Weibull distributed with shape parameter β and scale parameter λ then they proved that:

Equation 21

$$2\lambda^\beta rv \sim \chi_{2g}$$

$$\text{where } v = \sum_{i=1}^g (Y_i)$$

For the single sampling plan, the lot acceptability criterion was suggested to be $v \geq kL^\beta$, where $v = \sum_{i=1}^g Y_i^\beta$ and Y_i is the first failure time in the i^{th} group, L is the lower specification limit and the constant k is calculated based on either consumer or producer risk levels.

The number of groups, which one may select, could be determined independently from the group size and the shape parameter.

Arizono and Kawamura (Arizono & Kawamura, 2008) discussed a reliability test for the Weibull distribution with variation shape parameter based on sudden death lifetime data, where the Weibull distribution had the following form:

Equation 22

$$f(t) = \begin{cases} \frac{\beta}{\theta} t^{\beta-1} e^{-\frac{t^\beta}{\theta}}, & \text{when } t > 0 \\ 0, & \text{when } t \leq 0 \end{cases}$$

They developed the sudden death reliability test, assuming that under given first and second type errors, the shape parameters of the acceptable and rejectable distributions did not match and belong to different intervals. They assumed that there were N testing

facilities, and each of them might run n specimens and all N testing facilities could run simultaneously. From each testing facility the first failure was collected and based on these N first failures the acceptance or rejection criterion was developed. The average, based on these N first failures, was computed and compared to the acceptance limit.

The economical plan for the proposed reliability test was considered, in which the expected testing time was assumed to be equal to the largest value out of the first N failures.

Motyka (Motyka, 2007) did a simulation study to compare a sudden death testing time with one from a traditional time terminated life test. Two sets of a hundred random samples from the same Weibull distribution were generated. One of them was assigned for the sudden death test, another one was assigned for the time terminated life test. The number of groups and sample sizes under the investigation for the sudden death test were the following: $\{25, 4\}$, $\{20, 5\}$, $\{14, 7\}$, $\{10, 10\}$, $\{7, 14\}$ and $\{5, 20\}$. For each set of group and sample size, the random sample designated for the sudden death test was used. The lowest number from each subgroup was the time duration of the test for that particular subgroup. The maximum time from these lowest times in each subgroup represented the sudden death test duration, for the assigned set of groups and sample size. Based on these lowest times in each group, the Weibull parameters were estimated by a probability plot. The sudden death test duration was used as a cut off for the time terminated life test. Based on this cut off point, all the lowest random variables from the second generated random sample were selected. The Weibull parameters were estimated from the selected random variables. One of the findings of this study was that the shape parameter estimator has better properties than the scale parameter estimator, regardless of the test method.

CHAPTER 3

PILOT STUDY AND METHODS

3.1 MSDT and CSDT Testing Plans Simulation in Time and Cost Domains

3.1.1 Number of Grouping Combinations

The number of arrangements from the “N” elements, taken “n” at a time, without repetitions is known as the combination of “N chosen n”. So, the combination of “N chosen n” is the number of sets that can be made up from the “N” elements, such that in each set there are exactly “n” elements, and no two sets are the same.

The number of sets, where each set has exactly “k” subsets made up from the N elements, such that in each subset exactly “n” elements, where no two subsets are the same, is defined herein as the “Number of Grouping Combinations”.

The Number of Grouping Combinations is computed by:

Equation 23
$$G = \frac{N!}{(n!)^k}$$

For example, from the 6 elements ({1, 2, 3, 4, 5, 6}), it is possible to arrange 20 not-repeating grouping combinations, such that each grouping combination is made up from 2 subgroups where exactly 3 elements in each subgroup (see Table 1).

Table 1: All possible grouping combinations from six elements in two groups.

# of Combinations	Subgroup 1			Subgroup 2		
1	1	2	3	4	5	6
2	1	2	4	3	5	6
3	1	2	5	3	4	6
4	1	2	6	3	4	5
5	1	3	4	2	5	6
6	1	3	5	2	4	6
7	1	3	6	2	4	5
8	1	4	5	2	3	6
9	1	4	6	2	3	5
10	1	5	6	2	3	4
11	2	3	4	1	5	6
12	2	3	5	1	4	6
13	2	3	6	1	4	5
14	2	4	5	1	3	6
15	2	4	6	1	3	5
16	2	5	6	1	3	4
17	3	4	5	1	2	6
18	3	4	6	1	2	5
19	3	5	6	1	2	4
20	4	5	6	1	2	3

3.1.2 Run Time for the Failure Test

The run time or time to finish the test is the completion time for the reliability test. For the failure truncated tests the run time depends on the mode of the test.

3.1.2.1 Run time for the serial failure tests

If k Groups of n components are tested serially until the r^{th} failure in each Group, then the run time for the test will be the sum of the r^{th} failure-times in each Group.

Let $t_{r:n,k}$ be the r^{th} failure-time out of n components in the k^{th} Group, then the time to finish the test serially is computed by the following:

Equation 24

$$T_s = \sum_{i=1}^k t_{r:n,i}$$

3.1.2.2 Run time for the parallel failure tests

If k Groups of n components are tested in parallel until the r^{th} failure in each Group, then the time to finish the test will be the maximum time out of the r^{th} failure-times of each Group.

Let $t_{r,n,k}$ would be the r^{th} failure-time out of n components in the k^{th} Group, then the time to finish the parallel test is computed by the following:

Equation 25
$$T_p = \max\{t_{r,n,1} \dots t_{r,n,i} \dots t_{r,n,k}\}$$

3.1.3 Total Accumulated Time for the Failure Test

The total accumulated time or Total time on test is the total observed testing times for both the failed and survived components in the test.

If n components are tested simultaneously until the r^{th} failure, then the accumulated time for the test is a summation of the all “r” failures plus “n-r” survived components testing times.

Let $t_{r,n}$ would be the r^{th} failure time for a Group of n components, then the accumulated time for the Group is computed by the following:

Equation 26
$$T_{ac} = \sum_{i=1}^r t_{i,n} + (n-r) * t_{r,n}$$

If k Groups of n components are tested serially or in parallel until the r^{th} failure in each Group, then the total accumulated time for the test is the sum of the accumulated times in each Group.

Let $t_{r,n,k}$ would be the r^{th} failure time out of n components in the k^{th} Group, then the total accumulated time for the test (parallel or serially) is computed by the following:

Equation 27

$$T_{\text{test_ac}} = \sum_{j=1}^k T_{ac}^j = \sum_{j=1}^k \left(\sum_{i=1}^r t_{i,n,j} + (n-r) * t_{r,n,j} \right)$$

3.1.4 Run Time Cost or Supervision/Technician Cost

Run time cost or Supervision/ Technician Cost associated with how long the test was run and depend on the test mode.

If c_3 is the technician cost per unit testing time and T_s or T_p is the run time for the test in serial and parallel modes, then Supervision/Technician Cost is computed by

Equation 28

$$C_s = c_3 * T_s$$

or

Equation 29

$$C_s = c_3 * T_p$$

3.1.5 Total Accumulated Time Cost

Total accumulated time cost or Total testing time cost is the cost related to the facilities or equipment usage during the reliability test. The facilities and equipment costs might also be expressed proportional to the unit testing time.

Total accumulated time cost for the serially or parallel tests is equal to the total accumulated time of the test multiplied by the unit testing time cost.

If c_1 is unit testing time cost and T_{test_ac} is total accumulated time for the test, then the total testing time cost is computed by the following:

Equation 30
$$C_{TTC} = c_1 * T_{test_ac}$$

3.1.6 Total Components Cost

Total Components Cost, associates with the tested components, is the cost of all components used in the reliability test.

If during the reliability test, a total of N components were used, disregarding the number of results collected, and the cost for the unit component is c_2 , then the total cost for the components is calculated by the following:

Equation 31
$$C_C = c_2 * N$$

3.1.7 Total Operational Cost

If C_s is the supervision cost (run time cost), C_{TTC} is total testing time cost (total accumulated time cost), C_c is total components cost and for the reliability test, then total operational cost is the sum of these costs.

Equation 32
$$C_{OP} = C_s + C_{TTC} + C_C$$

If k Groups of n components are tested serially until the r^{th} failure in each Group, then total operational cost could be computed by:

Equation 33

$$C_{OP} = c_1 \left(\sum_{j=1}^k \left(\sum_{i=1}^r t_{i,n,j} + (n-r) * t_{r,n,j} \right) \right) + c_2 * n * k + c_3 * \sum_{i=1}^k t_{r,n,i}$$

If k Groups of n components are tested parallel until the r^{th} failure in each Group, then total operational cost could be computed by:

Equation 34

$$C_{OP} = c_1 \left(\sum_{j=1}^k \left(\sum_{i=1}^r t_{i,n,j} + (n-r) * t_{r,n,j} \right) \right) + c_2 * n * k + c_3 * \max\{t_{r,n,1} \dots t_{r,n,i} \dots t_{r,n,k}\}$$

3.1.8 Computational Formulas Discussion

The time to finish the test T_s or T_p could be computed by Equation 24 or Equation 25. Also, this depends on the number of components in the test, the predetermined number of results for the test and the testing mode (either a serially or parallel test).

The time to finish the test in serial mode, is a random variable, which is the summation of the ordered random variables. Time to finish the test in parallel mode, is another random variable, which is the maximum value of these ordered random variables.

In the cost domain, the operational cost Equation 32 is constituted from three major components: total testing time cost Equation 30, total components cost Equation 31 and run time cost Equation 28 or Equation 29.

Total components cost could be computed by Equation 31 and it primarily depends on the unit cost of a component and the total number of components tested during the test.

The total component cost is a part of the variable cost or operational cost, but it is not affected by any random variability pattern.

Total accumulated time is a random variable and total testing time cost depends on the total accumulated time, because total accumulated time comes from the ordered random variables.

Supervision cost primarily depends on the run time, which is also constituted from random variables.

In this research, the investigation to the “supervision cost” and “total accumulated time cost” as random variables would be done. Also, by the desired confidence, the easiest and fastest way to estimate a B_p life for the time to finish the test will be investigated.

3.1.9 Simulation Study for MSDT and CSDT in Time and Cost Domains

The goal of this simulation study is to compare MSDT and CSDT in the time and cost domains.

In the time domain the emphasis is given to the “Time to Finish the Test” category by using Equation 24 and Equation 25. In cost domain the emphasis is given to the “Total Testing Time Cost” category by using the Equation 30.

Let us assume that the component under the investigation is a coupling³, which has a Weibull failure distribution with scale parameter equal to 2.5 and shape parameter equal to 75,000.

The parameters of the tests are the following:

- Number of available testing facilities is sixteen ($A=16$)
- Number of components in the test is sixteen ($N=16$)

³ Source: <http://www.barringer1.com/wdbase.htm>

- Predetermined number of failures for the test is twelve ($R=12$)
- Number of Groups is four ($k=4$)
- Number of components in the Group for MSDT is sixteen ($n=16$)
- Number of components in the Group for CSDT is four ($n=4$)
- Designated number of failures for any Group in MSDT and CSDT is three ($r=3$)
- Unit testing time cost is one dollar ($c_1 = \$1$)

Single Simulation Run Steps Description for of MSDT.

Step 1: Draw k random samples of size 16.

Step 2: Put each random sample in ascending order.

Step 3: For each random sample, choose the r^{th} element in magnitude as a time to finish the test for the Group.

Step 4: For each random sample, compute Accumulated Time by using Equation 26.

Step 5: Sum all k results in Step 3 as a Total Time to Finish the Test.

Step 6: Sum all k results in Step 4 as a total accumulated time.

Single Simulation Run Steps Description for CSDT.

Step 1: Draw k random samples of size 4.

Step 2: Put each random sample in ascending order.

Step 3: For each random sample, choose the r^{th} element in magnitude as a time to finish the test for the Group.

Step 4: For each random sample, compute Accumulated Time by using Equation 26.

Step 5: Choose maximum of all k results in Step 3 as a Total Time to Finish the Test.

Step 6: Sum all k results in Step 4 as a total accumulated time.

Results of the Simulation Study for 20,000 Runs.

Figure 1 shows “Time to Finish the Test” histograms for MSDT and CSDT for 20,000 runs.

Figure 1: "Run time" histograms for MSDT and CSDT.

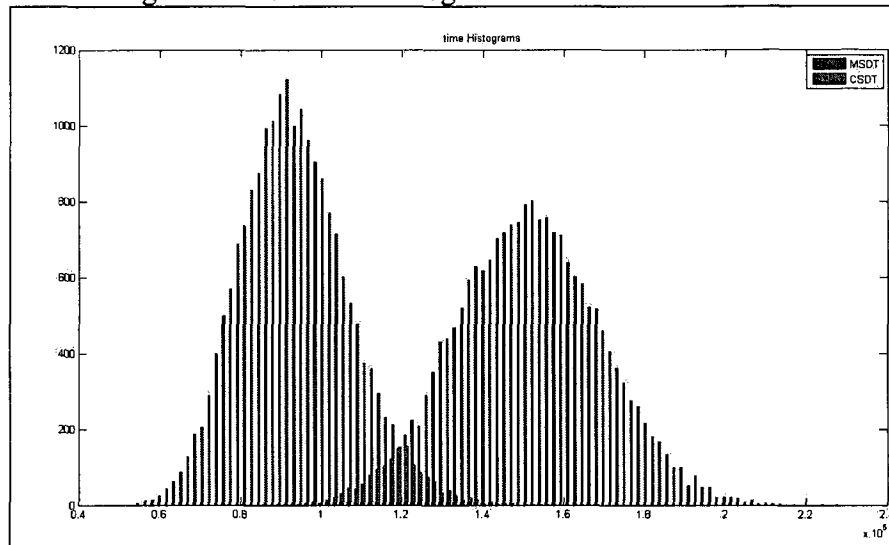


Figure 2 shows "Total accumulated time cost" histograms for MSDT and CSDT for 20,000 runs.

Figure 2: "Total accumulated time cost" histograms for MSDT and CSDT.

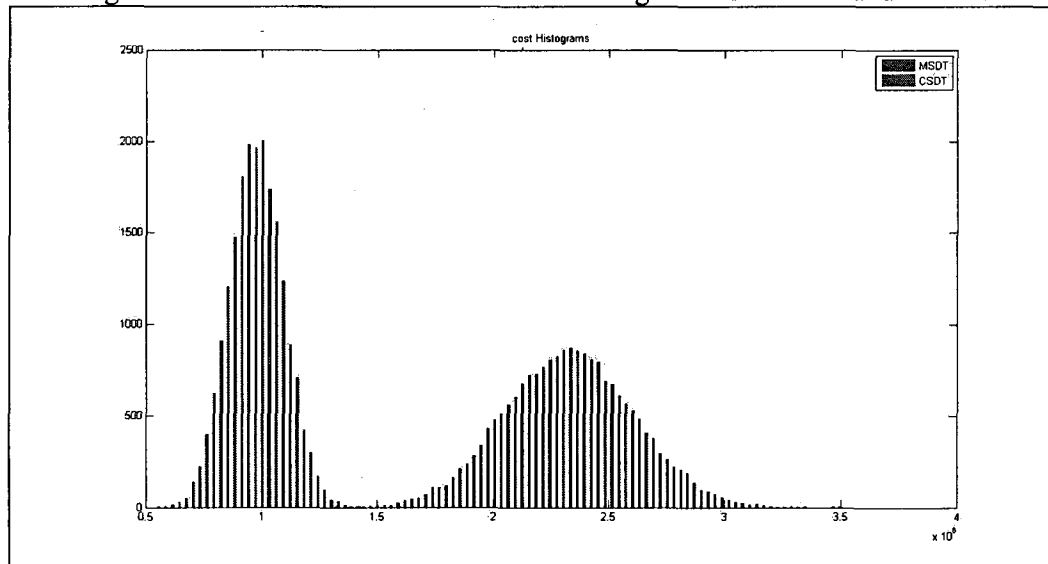


Figure 3: Empirical cumulative distribution functions for MSDT and CSDT.

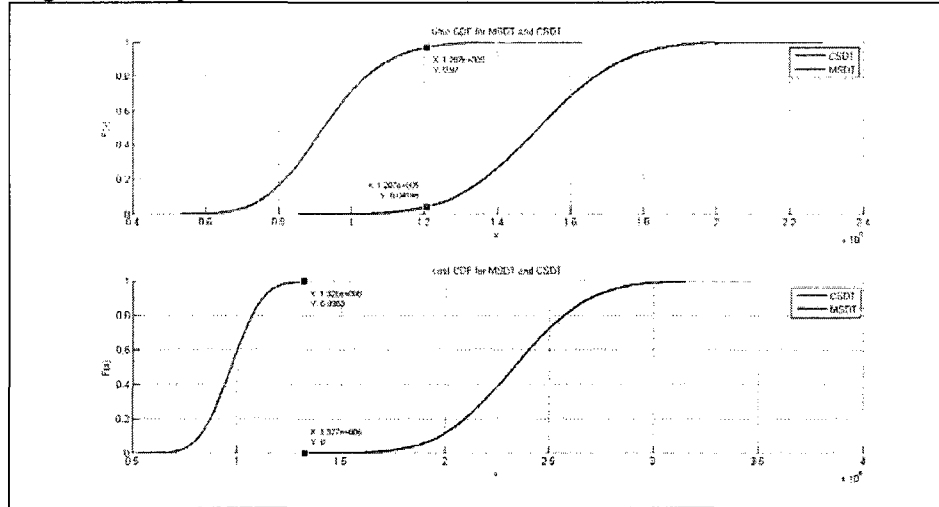


Figure 3 shows the Empirical cumulative distribution functions for MSDT and CSDT Cost and Time Domains, if Time is in ascending order.

Figure 4: “Time to finish the test” and “Total testing time cost” values by ascending order.

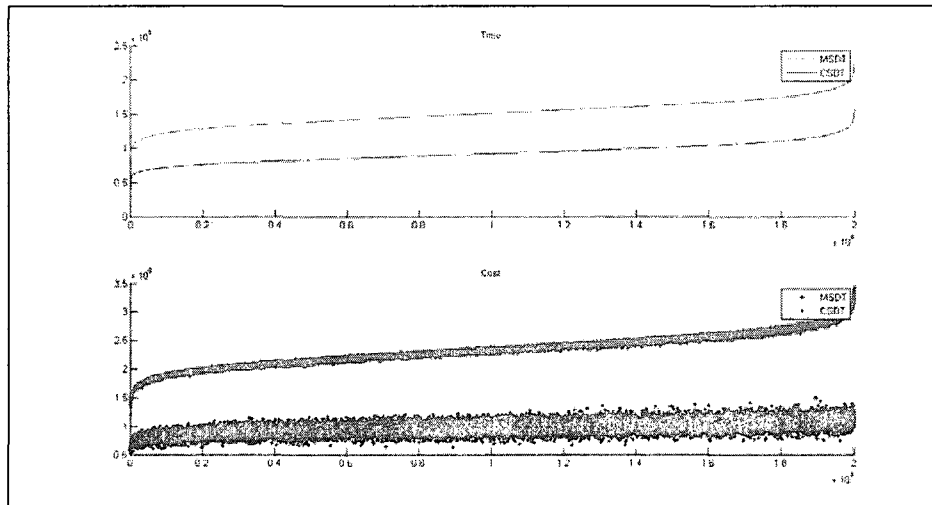


Figure 4 shows “Time to finish the test” and “Total testing time cost” plots by ascending order.

3.1.10 Simulation Study Discussion

Based on the Figure 1 and Figure 2, the cost of the test and time to finish the test for CSDT are superior to MSDT.

From Figure 3 with 97% of assurance it can be claimed that the time to finish the test for CSDT is smaller than the MSDT. For the first type error of 3% , corresponding second type error is about 4.2%.

From Figure 3, with 99.83% of assurance it could be claimed that the cost to finish the test for CSDT is smaller than for the MSDT. For the first type error of 0.17% , corresponding second type error is about 0%.

Based on the Figure 4, there is a positive correlation between the time to finish the test and the cost of the test for CSDT and MSDT. Generally, based on the simulation results the proposed CSDT is superior to the MSDT.

3.2 Derivation and Development of the Computational Formulas and Methods

3.2.1 An Approach to Calculate the r^{th} Out of n Ordered Random Variable Quantiles

3.2.1.1 The rank distribution

If one wants to know “What is the percentage of the population below the r^{th} out of n ordered random variable”, one may answer this question by using the rank distribution.

Given that the probability distribution and density functions of the r^{th} ordered statistic out of n are known and may be represented by Equation 11 and Equation 12:

$$G_{r:n}(t) = P(t_{r:n} \leq t) = \sum_{i=r}^n \binom{n}{i} F(t)^i (1 - F(t))^{n-i}$$

$$g_{r:n}(t) = n \binom{n-1}{r-1} F(t)^{r-1} (1-F(t))^{n-r} f(t)$$

Let p_c be the percentage of the population below some time t_c , that is $p_c = P(t \leq t_c)$.

If $t_c = t_{r:n}$ then $p_{r:n} = P(t \leq t_{r:n})$ would be the percentage of population below the time of the r^{th} out of n ordered statistic (Lamberson & Kapur, 1977, 297-303).

Note that, if $p_{r:n} = P(t \leq t_{r:n})$ then $t_{r:n} = F^{-1}(p_{r:n})$.

Also,

$$\text{Equation 35} \quad G_{r:n}(t_c) = P(t_{r:n} \leq t_c) = P(F^{-1}(p_{r:n}) \leq t_c) = P(p_{r:n} \leq p_c) = Q_{r:n}(p_c)$$

From Equation 11 and Equation 12 we know that

$$G_{r:n}(t_c) = \int_{-\infty}^{t_c} g_{r:n}(t) dt = \int_{-\infty}^{t_c} n \binom{n-1}{r-1} F(t)^{r-1} (1-F(t))^{n-r} f(t) dt$$

Let $p = P(T \leq t) = F(t)$, then $dp = f(t) dt$, also when $t = -\infty$; $p = 0$ and $t = t_c$; $p = p_c$.

So,

$$\text{Equation 36} \quad Q_{r:n}(p_c) = P(p_{r:n} \leq p_c) = \int_0^{p_c} n \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} dp$$

And

$$\text{Equation 37} \quad q_{r:n}(p_c) = n \binom{n-1}{r-1} p_c^{r-1} (1-p_c)^{n-r} = \frac{n!}{(r-1)!(n-r)!} p_c^{r-1} (1-p_c)^{n-r}$$

where $Q_{r:n}(p_c)$ and $q_{r:n}(p_c)$ are the CDF and PDF of the random variable $p_{r:n}$, and p_c is defined within $0 \leq p_c \leq 1$.

One could recognize that $q_{r,n}(p_c)$ is the well known Beta distribution:

$$\text{Equation 38} \quad f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

where $0 \leq x \leq 1$

$$\text{Equation 39} \quad F(x) = \int_0^x \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1} du$$

$$\text{Equation 40} \quad E(x) = \frac{\alpha}{\alpha + \beta}$$

$$\text{Equation 41} \quad Var(x) = \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$$

Let

$r = \alpha$ and $n - r + 1 = \beta$ and using the fact that $\Gamma(n) = (n-1)!$

Then

$$\text{Equation 42} \quad q_{r,n}(p_c) = \frac{n!}{(r-1)!(n-r)!} p_c^{r-1} (1-p_c)^{n-r} = \frac{\Gamma(r+n-r+1)}{\Gamma(r)\Gamma(n-r+1)} p_c^{r-1} (1-p_c)^{n-r}$$

where $0 \leq p_c \leq 1$

$$\text{Equation 43} \quad Q_{r,n}(p_c) = \int_0^{p_c} \frac{\Gamma(r+n-r+1)}{\Gamma(r)\Gamma(n-r+1)} u^{r-1} (1-u)^{n-r} du$$

$$\text{Equation 44} \quad E(p_c) = \frac{r}{r+n-r+1} = \frac{r}{n+1}$$

Equation 45
$$Var(p_c) = \frac{r(n-r+1)}{(r+n-r+1)^2(r+n-r+1+1)} = \frac{r(n-r+1)}{(n+1)^2(n+2)}$$

3.2.1.2 Technique of computing the ordered statistic time for a desired quantile

Recall from Equation 11 that the probability distribution function of the r^{th} order statistic out of n has the following form:

$$G_{r,n}(t_c) = P(t_{r:n} \leq t_c) = \sum_{i=r}^n \binom{n}{i} F(t_c)^i (1-F(t_c))^{n-i}.$$

If one would like to know the $(1-\alpha)$ level quantile of the time t_c , then the inverse transformation of Equation 1 can be applied:

Equation 46
$$t_c = G_{r,n}^{-1}(1-\alpha)$$

This calculation is difficult to do manually and is more suitable to a computer algorithm.

Recall that Equation 42 is a Beta distribution and p_c can be computed for a $(1-\alpha)$ quantile level if we apply an inverse transformation of Equation 43:

Equation 47
$$p_c = Q_{r,n}^{-1}(1-\alpha)$$

but $p_c = P(t \leq t_c)$, therefore the inverse of it will be

Equation 48
$$t_c = F^{-1}(p_c)$$

and because of the relationship in Equation 36, this is the $(1-\alpha)$ level quantile of the time t_c .

3.2.1.3 The results verification for the suggested approach

In this Section the following steps will be accomplished:

- By computer algorithms the time for the desired level quantile by Equation 11 will be calculated.
- Next, the same time for the same desired quantile by Equation 47 and Equation 48 will be found.
- Then the results from both techniques will be compared.

Let's assume that the r.v. t has a Weibull parent distribution with the scale parameter 100 and shape parameter 1.5. Our goal is to find the 95% level quantile of the 5th out of 10 ordered random variable.

By Equation 11 and a computer algorithm, the empirical cumulative distribution function (CDF) was calculated for the values of abscissa from 0 to 200 with 0.5 increments. Then, based on the closest values to 0.95 the linear interpolated value for " t_c " was found, which is 112.44 (see Figure 5).

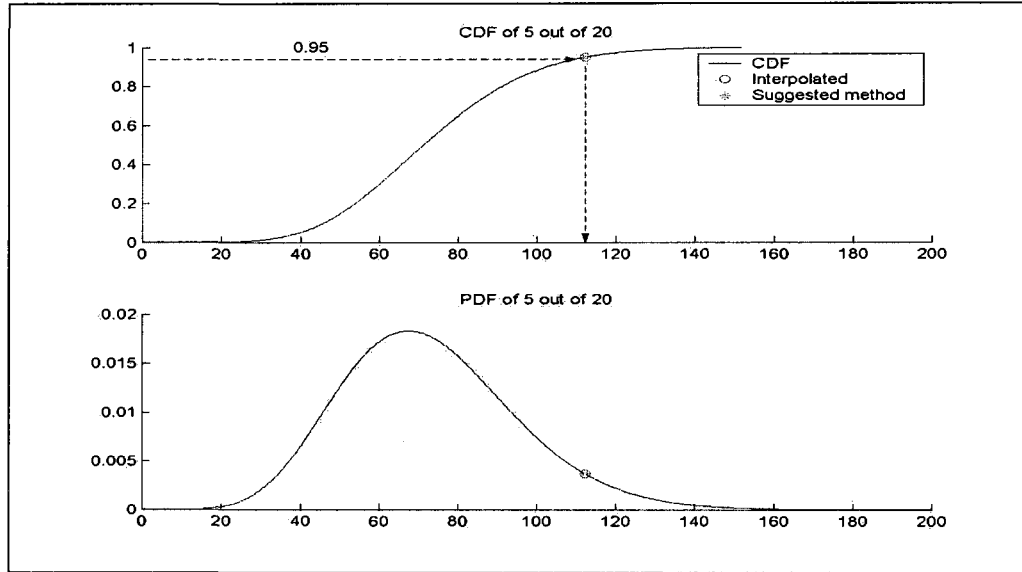
By Equation 47 and Equation 48 the value of " t_c " was calculated at 0.95 level quantile, which is 112.43.

Microsoft Office Excel 2003⁴ has a spreadsheet function for the Beta distribution, this Excel function can be applied for our example to find the value for the p_c as " $=\text{BETAINV}(0.95,5,6)$ ", which yields $p_c = 0.6964$. Since, the parent distribution is Weibull (1.5; 100), the inverse of it at the point 0.6964 will be $t_c = 100(-\ln(1 - 0.6964))^{\frac{1}{1.5}}$, which yields $t_c = 112.43$.

⁴ Trademark, Microsoft Corporation, Redmond, WA, 98052

The slight difference in values of “ t_c ” was noticed, between the suggested method and interpolated one. As the increment of abscissa is decreasing, the interpolated value gets closer to the suggested method value.

Figure 5: 95% quantile level value for the 5th out of 20 ordered random variable, assuming the parent distribution is Weibull with shape parameter 1.5 and scale parameter 100.



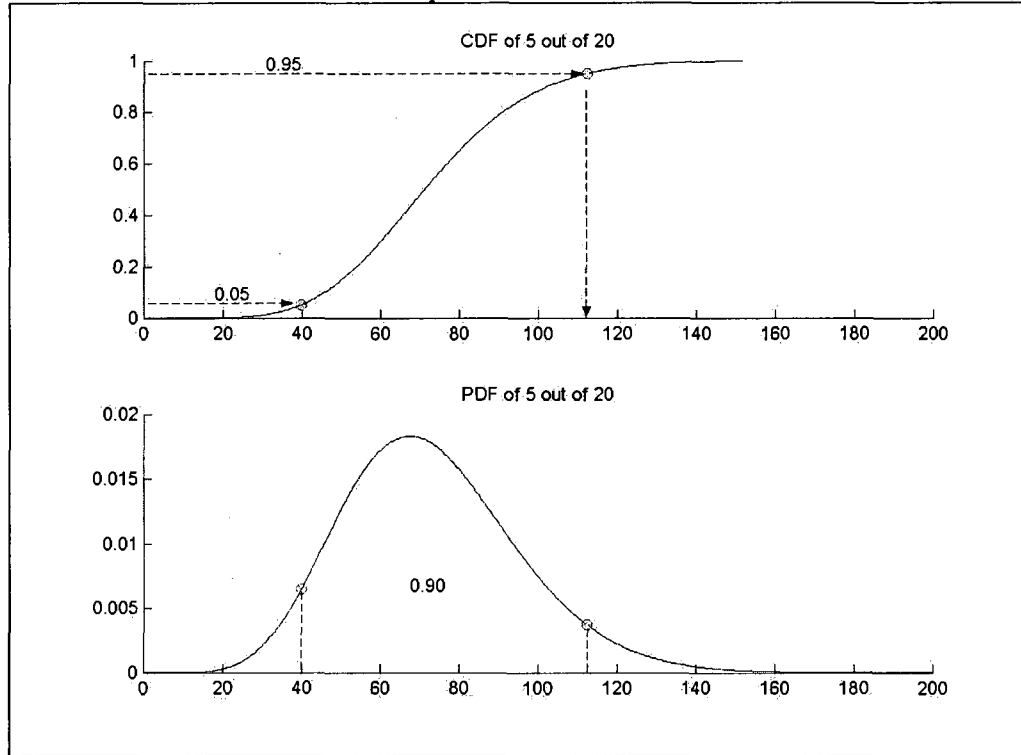
In Figure 5 the values of interpolated and suggested methods are depicted.

3.2.1.4 For the suggested quantile levels, the interval limits for the r^{th} out of n ordered random variable

Let's assume that the r.v. t has a Weibull parent distribution with the scale parameter 100 and shape parameter 1.5. Our goal is to find the interval limits for the 5th out of 10 ordered random variable, where it lies in the interval with probability 0.90. So we have to find $P(L \leq T_{5:10} \leq U) = 0.90$. Which means L and U are 0.05 and 0.95 level quantiles.

Using the suggested method the limits were found to be $L = 39.85$ and $U = 112.44$ (see Figure 6).

Figure 6: 5% and 95% quantile level values for the 5th out of 20 ordered random variable, assuming the parent distribution is Weibull with shape parameter 1.5 and scale parameter 100.



Please note that the Upper and Lower limits do not depend on 5th out of 10 ordered random variable value. This means that 5th out of 10 ordered random variable value, from Weibull parent distribution with shape parameter 1.5 and scale parameter 100, will be within the interval limits by 0.90 probability.

3.2.1.5 Illustrative example

Assume the failure distribution of a roller bearing is a Weibull distribution with shape parameter 1.7 and scale parameter 100,000. There is a mechanism, consisting of 20 such bearings and it is functioning, if at least any 5 bearings are working. Quote the limits within which the mechanism will fail, with 90% of probability (proportionally tailed)?

The mechanism will function until the 15th failure out of 20, where parent distribution is Weibull with shape parameter 1.7 and scale parameter 100,000. So we have to find the 5% and 95% quantiles of the 15th out of 20 ordered random variable.

For 5% quantile: by Equation 11, the $p_c=0.54442$, where Beta distribution parameters are $\alpha=15$ and $\beta=6$. Using Equation 12 for Weibull distribution, $t_c=86,804$.

For 95% quantile: by Equation 11, the $p_c=0.86045$, where $\alpha=15$ and $\beta=6$. Using Equation 12 for Weibull distribution, $t_c=148,979$.

So, the mechanism will fail with 90% probability between 86,804 and 148,979.

3.2.1.6 Verification of the proposed approach, using computer algorithm assuming the parent distribution is Weibull, Normal or Uniform

In this Section a computer algorithm will be used to verify the proposed method for several distribution functions, commonly used in reliability testing. In this numerical analysis, the 10% and 90% quantile times for the $r=13$ out of $n=20$ ordered random variable for each of the following parent distribution will be computed.

- Weibull, with the shape parameter 5 and scale parameter 10,000,
- Normal with mean 10,000 and standard deviation 500, or
- Uniform within the interval [8,500; 11,500].

Using Equation 1 the empirical CDF will be constructed for the following fixed points in time:

- For the Weibull parent distribution, the abscissa values vary from 0 to 15,000 with increments of 1.

- For the Normal parent distribution, the abscissa values vary from 8,000 to 12,000 with increments of 1.
- For the Uniform parent distribution, the abscissa values vary from 8,500 to 11,500 with increments of 1.

Then, a linear approximation will be applied to find the 10% and 90% quantile times. The interpolated results will be compared with the proposed method, using Equation 47 and Equation 48. Figure 7, Figure 8 and Figure 9 depict the interpolated and proposed method values for the 10% and 90% quantile times of the 13th out of 20 ordered random variable, from Weibull, Normal and Uniform parent distributions, respectively. It was noticed that as the increment of abscissa is decreasing, the interpolated value gets closer to the proposed method.

Figure 7: 10% and 90% quantile level values for the 13th out of 20 ordered random variable, assuming the parent distribution is Weibull with shape parameter 5 and scale parameter 10,000.

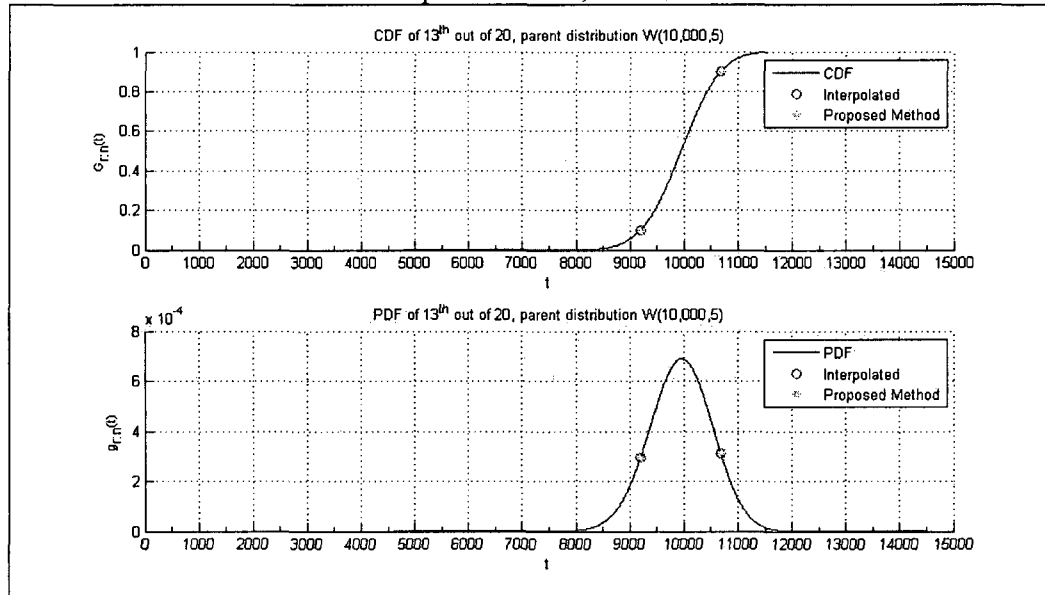


Figure 8: 10% and 90% quantile level values for the 13th out of 20 ordered random variable, assuming the parent distribution is Normal with mean 10,000 and standard deviation 500.

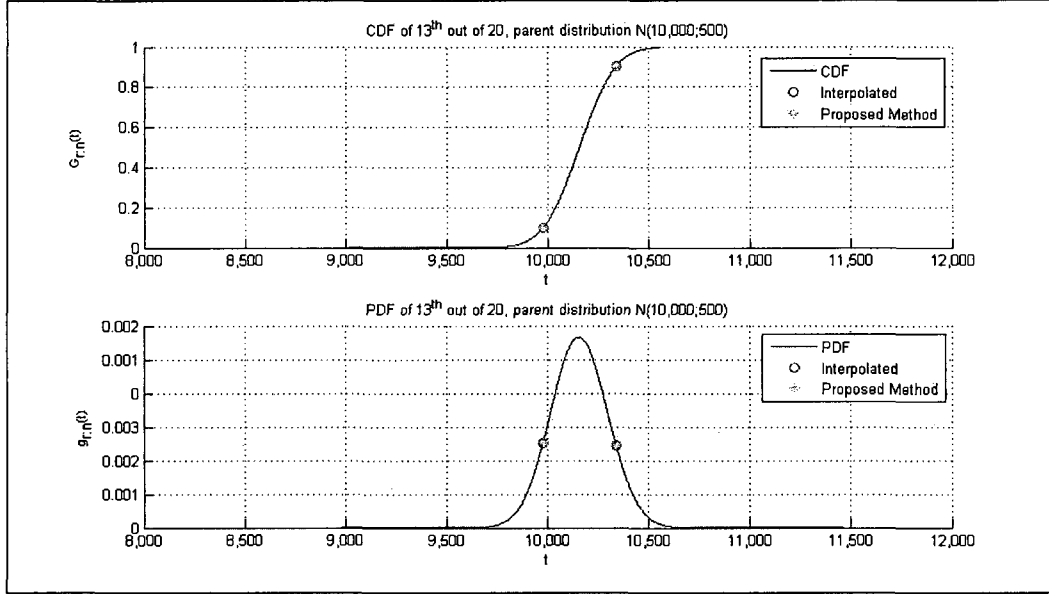
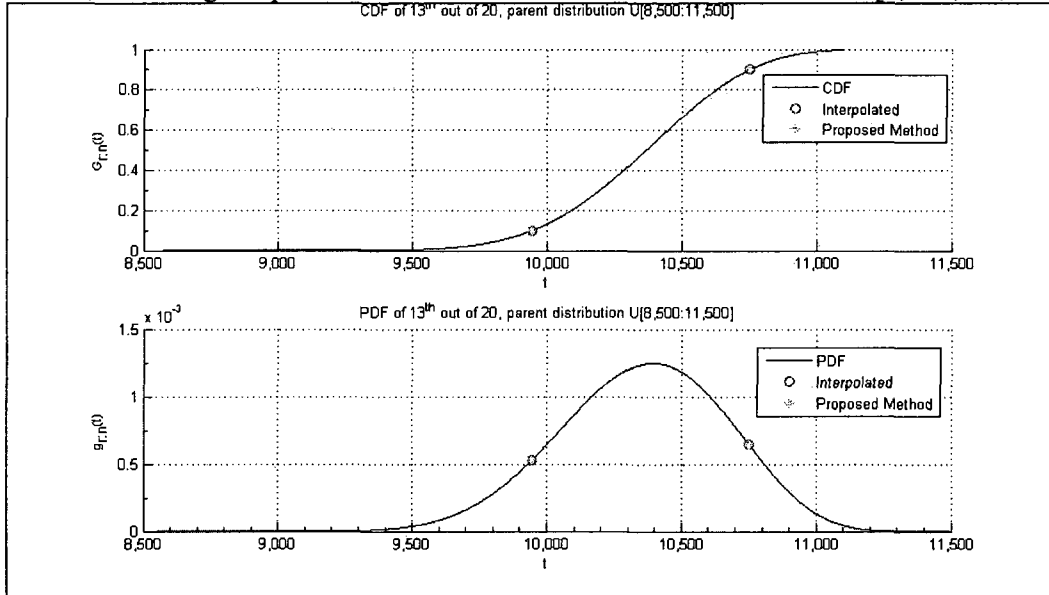


Figure 9: 10% and 90% quantile level values for the 13th out of 20 ordered random variable, assuming the parent distribution is Uniform within the interval [8,500; 11,500].



As we see in Figures 7,8 and 9, the proposed method is accurate and theoretically proved.

3.2.1.7 The r^{th} out of n ordered random variable behavior under a Weibull parent distribution, with shape parameter greater than unity

In this Section, we will investigate the distributional behavior of the ordered random variable for a Weibull parent distribution, where the shape parameter is greater than unity. We will use the previously developed method for the quantile time computation.

The following assumptions will be made:

- Number of components on the test is 20 ($n=20$).
- Number of failures varies from 1 until n . ($r=1:n$).
- The shape parameter is increasing by 0.2 increments, starting from 1 until 6.
- The scale parameter is 100.
- The 0.10 and 0.90 level quantiles limits are of interest.

Figure 10: 10% quantile times depending on the shape and r^{th} order random variable.

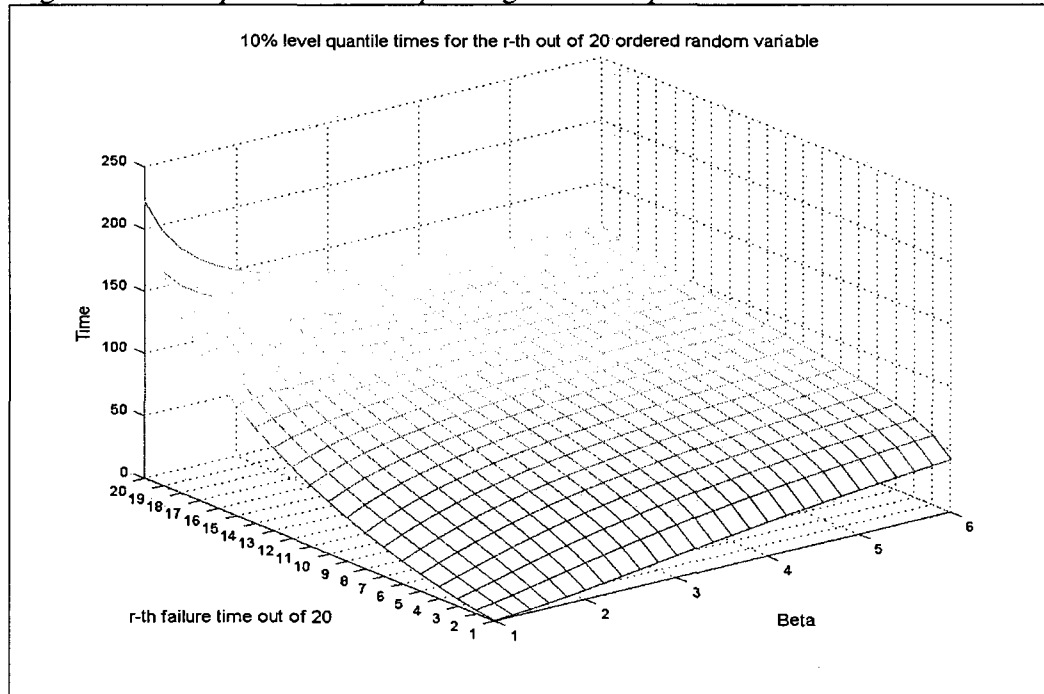


Figure 10 shows the 10% quantile limits depending on r^{th} ordered random variable and the shape parameter beta, respectively.

For the fixed beta, an increase in the r^{th} ordered random variable out of 20 results in an increase of the 10% level quantile times. But for a fixed r^{th} ordered random variable out of 20, an increase in beta results in an increase of the 10% quantile times until the 16th ordered random variable out of 20.

Figure 11: 90% quantile times depending on the shape and r^{th} order random variable.

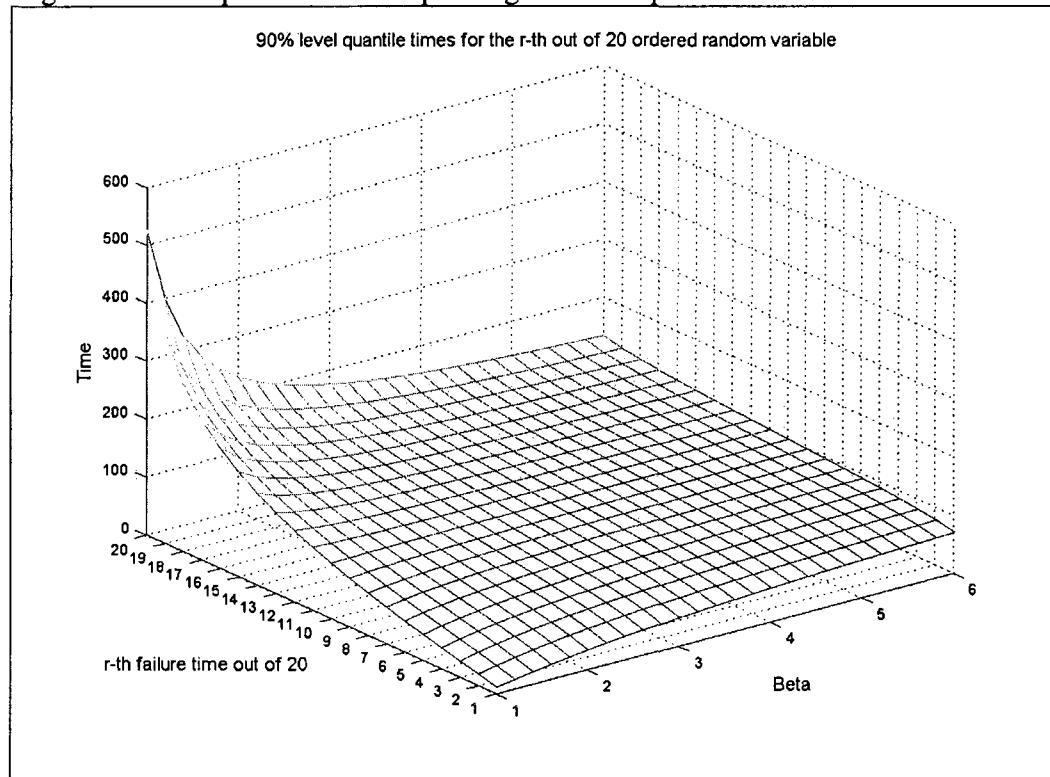


Figure 11 shows the 90% quantile limits depending on r^{th} ordered random variable and shape parameter beta, respectively.

For a fixed beta, an increase in the r^{th} ordered random variable out of 20 results in an increase of the 90% level quantile times. But for a fixed r^{th} ordered random variable out

of 20, an increase in beta results in an increase of the 90% quantile times until the 11th ordered random variable out of 20.

Figure 12: 63.2121% quantile times depending on the shape and r^{th} order random variable.

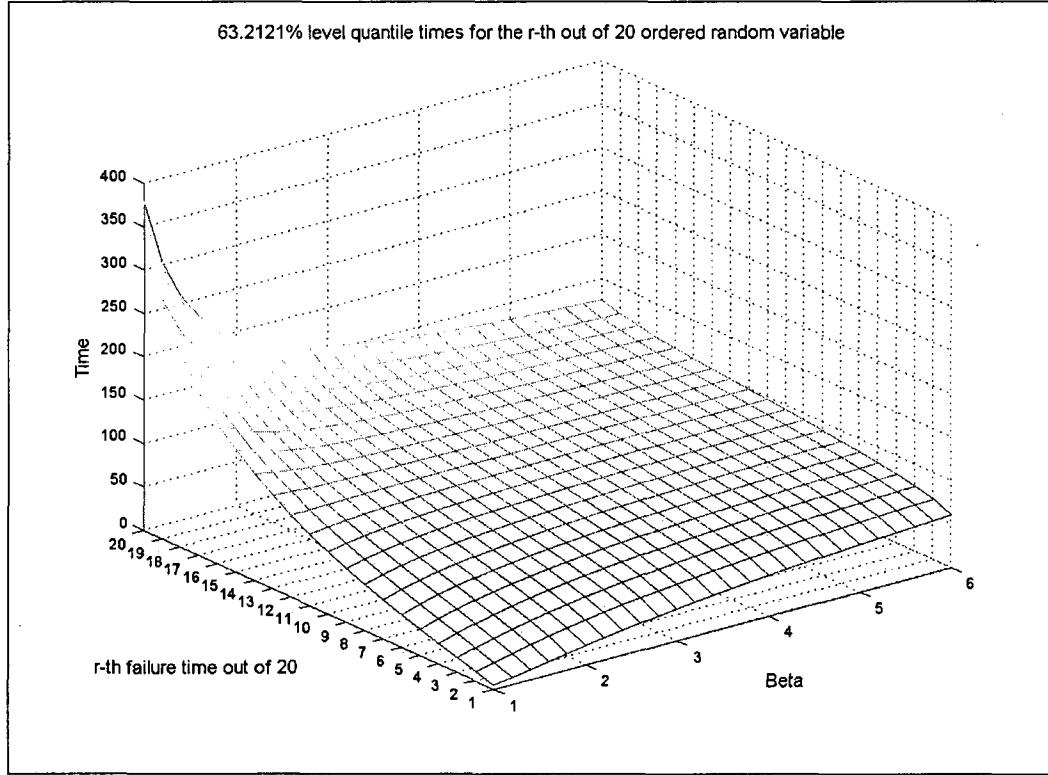


Figure 12 shows the 63.21% quantile limits depending on r^{th} ordered random variable and shape parameter beta, respectively.

For a fixed beta, an increase in the r^{th} ordered random variable out of 20 results in an increase of the 63.21% level quantile times. But for a fixed r^{th} ordered random variable out of 20, an increase in beta results in an increase of the 63.21% quantile times until the 13th ordered random variable out of 20.

Figure 13: 50% quantile times depending on the shape and r^{th} order random variable.

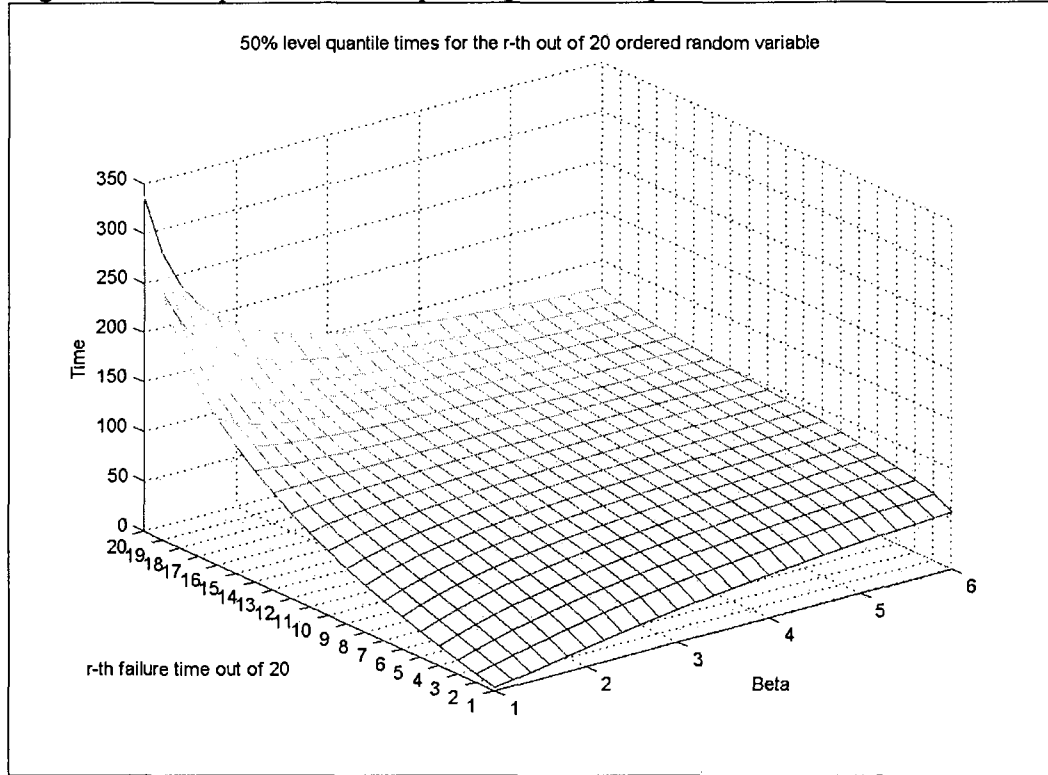


Figure 13 shows the 50% quantile limits depending on r^{th} ordered random variable and shape parameter beta, respectively.

For a fixed beta, an increase in the r^{th} ordered random variable out of 20 results in an increase of the 50% level quantile times. But for a fixed r^{th} ordered random variable out of 20, an increase in beta results in an increase of the 50% quantile times until the 14th ordered random variable out of 20.

Figure 14: 10% and 90% quantiles depending on the shape and r^{th} order random variable.

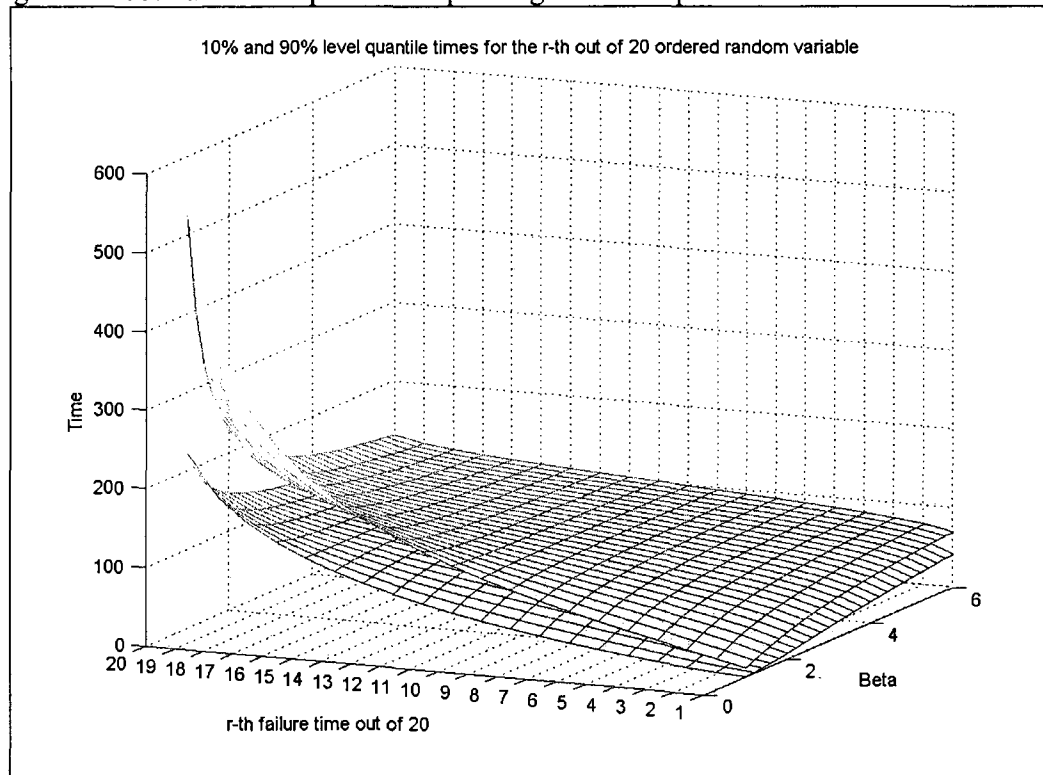


Figure 14 depicts both the 10% and 90% levels quantile times for the r^{th} out of 20 ordered random variable in one plot.

The difference in these surfaces is the 80% range for the r^{th} out of 20 ordered random variable, from the 10% quantile to 90% quantile times.

As the range lowers, the variability for the r^{th} out 20 ordered random variable lowers.

Figure 15: Quantile range depending on the shape and r^{th} order random variable.

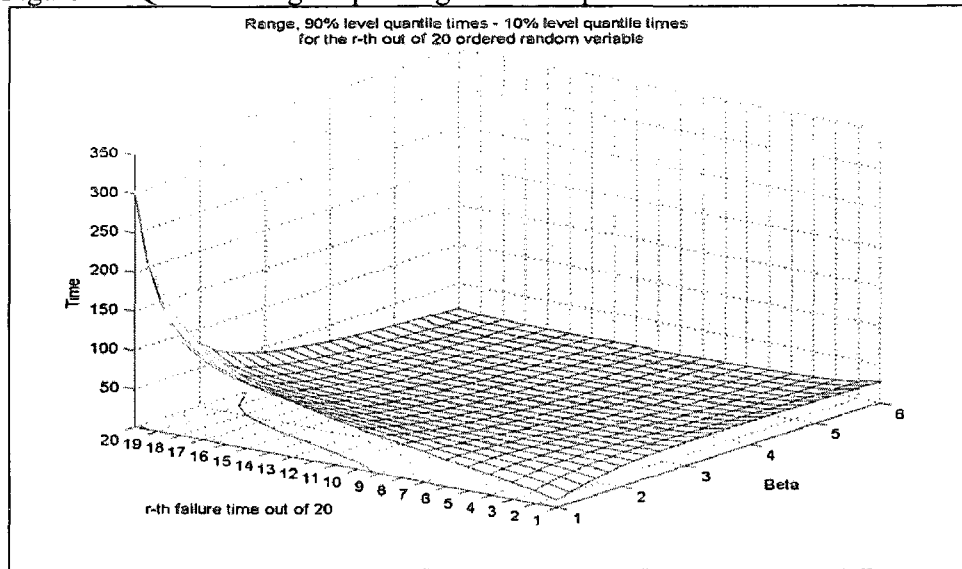


Figure 15 illustrates the 80% range surface, from 10% quantile to 90% quantile times, depending on the r^{th} out of 20 ordered random variable and beta values.

Figure 16: Contour plot of quantile ranges.

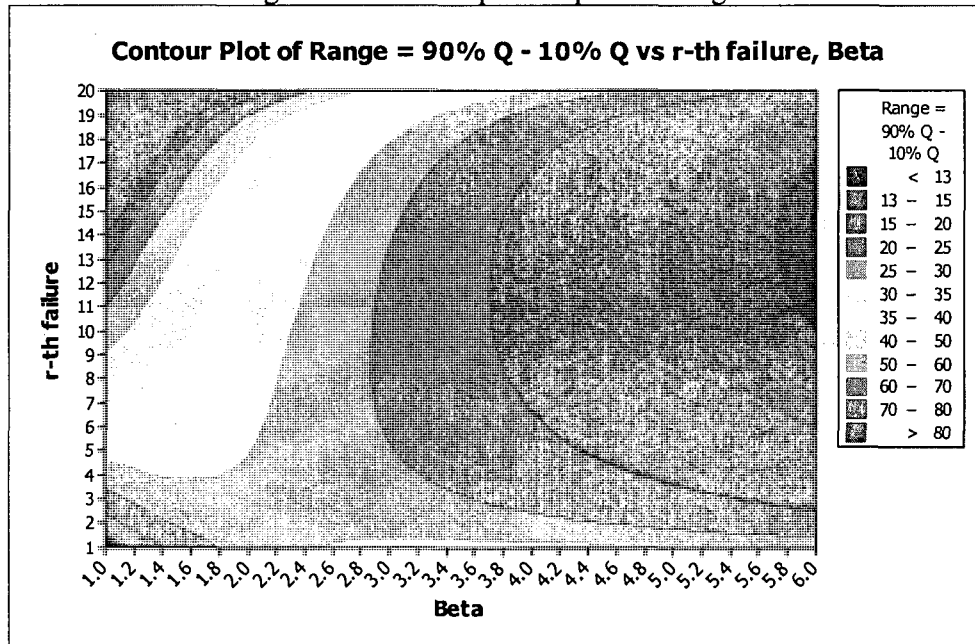


Figure 16 is the contour plot for the 80% ranges, from 10% quantile to 90% quantile times. The lowest ranges were observed for the following cases:

- Case 1. Beta between 1 and 1.1 and r is 1.
- Case 2. Beta between 5.8 to 6 and r between 10 and 17.

Another way to interpret this: these are the cases where the variability of the r^{th} ordered random variable out of 20 is the lowest, in the possible ranges.

3.2.1.8 Conclusion

The proposed technique uses the beta distribution to find the proportion of the population that is less than the r^{th} out of n ordered random variable, for any desired quantile. Then, by using the inverse of the parent distribution at the designated proportion, the time could be found. It is shown that this time corresponds to the r^{th} out of n ordered random variable quantile.

The proposed technique is easy, straight forward and theoretically sound. Using the proposed technique, the time for the r^{th} out of n ordered random variable for the desired level quantile could be found. It is shown that the developed method is accurate and could be applied to any continuous type parent distribution.

Moreover, in reliability testing the r^{th} out of n ordered random variable time corresponds to the Type II testing time (Failure truncated testing time). So, by the proposed method the test completion time by the desired probability could be computed, assuming the parent distribution and its parameters are known.

If the parent distribution is Weibull, then an increase in the r^{th} ordered random variable out of n for a fixed shape parameter greater than unity, will increase the time of the r^{th} out of n ordered random variable.

If the parent distribution is Weibull, then an increase in shape parameter, starting from unity, for the r^{th} out of n ordered random variable, will not always affect an increase in the time of the r^{th} out of n ordered random variable.

The behavior of the r^{th} out of 20 ordered random variable time, with the increase of the shape parameter starting from unity and the Weibull parent distribution where scale parameter is 100, is summarized in Table 2.

Table 2: r^{th} out of 20 ordered random variable behavior.

	Quantile Levels			
	10%	50%	63%	90%
Until r^{th} failure Increases	15	13	12	10
After r^{th} failure Decreases	16	14	13	11

3.2.2 Maximum Ordered Random Variable from the Sample Size k

3.2.2.1 Derivation of the CDF and PDF for the maximum ordered random variable from the sample size k

Time to finish the test for the CSDT approach to the life testing is a random variable defined as a maximum of all k times, which are r^{th} out of n ordered random variables. Based on CSDT definition, the test will be terminated when the r^{th} failure occurs in each of k groups, comprehended from n components.

In this Section we will develop CDF and PDF for the time to finish the test under CSDT approach.

Given that the probability distribution and density functions of the r^{th} ordered statistic out of n are known and may be represented by Equation 11 and Equation 12:

$$G_{r,n}(t) = P(t_{r,n} \leq t) = \sum_{i=r}^n \binom{n}{i} F(t)^i (1 - F(t))^{n-i}$$

$$g_{r:n}(t) = r \binom{n}{r} F(t)^{r-1} (1 - F(t))^{n-r+1-1} f(t)$$

Let the sample of size k of the r^{th} ordered statistic out of n be: $t_{r:n,1}; t_{r:n,2}; \dots; t_{r:n,k}$.

The random variable $t_{r:n,k}^{\max}$ will be formed from the k^{th} sample in the following way:

$$\text{Equation 49} \quad t_{r:n,k}^{\max} = \max\{t_{r:n,1}; t_{r:n,2}; \dots; t_{r:n,k}\}$$

This random variable describes the termination time of a CSDT with n components in k groups.

Also, let us define the CDF and PDF for the random variable $t_{r:n,k}^{\max}$ as $M_{r:n,k}(t)$ and $m_{r:n,k}(t)$, respectively.

By definition from Equation 49, it follows that:

$$\begin{aligned} \text{Equation 50} \quad M_{r:n,k}(t) &= P(t_{r:n,k}^{\max} \leq t) = P(t_{r:n,1} \leq t \cap t_{r:n,2} \leq t \cap \dots \cap t_{r:n,k} \leq t) = \\ &= P(t_{r:n,1} \leq t) P(t_{r:n,2} \leq t) \dots P(t_{r:n,k} \leq t) = (P(t_{r:n} \leq t))^k = (G_{r:n}(t))^k \end{aligned}$$

$$\begin{aligned} \text{Equation 51} \quad m_{r:n,k}(t) &= \frac{dM_{r:n,k}(t)}{dt} = \frac{d(G_{r:n}(t))^k}{dt} = k * g_{r:n}(t) * (G_{r:n}(t))^{k-1} = \\ &= k * m_{r:n,1}(t) * M_{r:n,k-1}(t) \end{aligned}$$

Please note that:

$$\text{Equation 52} \quad M_{r:n,1}(t) = G_{r:n}(t)$$

And

Equation 53
$$m_{r,n,1}(t) = m_{r,n}(t)$$

Based on Equation 50 and Equation 51, $M_{r,n,k}(t) = (G_{r,n}(t))^k$ and

$$m_{r,n,k}(t) = k * g_{r,n}(t) * (G_{r,n}(t))^{k-1}.$$

From Equation 50 and Equation 52 follows that:

Equation 54
$$M_{r,n,k}(t) = (G_{r,n}(t))^{k-1} * G_{r,n}(t) = M_{r,n,k-1}(t) * M_{r,n,1}(t)$$

From Equation 54, it is noticed that CFD for $t_{r,n,k}^{\max}$ is a recursive function.

Substituting $M_{r,n,k-1}(t)$ from Equation 54 into the Equation 51 we will get:

Equation 55
$$m_{r,n,k}(t) = k * g_{r,n}(t) * \frac{M_{r,n,k}(t)}{M_{r,n,1}(t)}$$

Let the function $O_{r,n,k}(t)$ be defined as:

Equation 56
$$O_{r,n,k}(t) = \frac{m_{r,n,k}(t)}{M_{r,n,k}(t)}$$

From Equation 55 and Equation 56 follows that:

Equation 57
$$k = \frac{O_{r,n,k}(t)}{O_{r,n,1}(t)}$$

Please note that:

Equation 58
$$O_{r,n,1}(t) = \frac{m_{r,n,1}(t)}{M_{r,n,1}(t)} = \frac{g_{r,n}(t)}{G_{r,n}(t)}$$

So function $O_{r,n,k}(t)$ is proportional to the number of groups (k) and $m_{r,n,k}(t)$ could be expressed as:

Equation 59
$$m_{r,n,k}(t) = k * O_{r,n,1}(t) * (G_{r,n}(t))^k$$

The tables for the different values of r and n for the function $O_{r,n,1}(t)$ might be provided. So the PDF $m_{r,n,k}(t)$ might be evaluated for any number of groups (k).

Time to finish the test under CSDT approach as a random variable, the PDF $m_{r,n,k}(t)$ and CDF $M_{r,n,k}(t)$ are derived. These might be computed by Equation 50 and Equation 59.

3.2.2.2 Quantile limits for the maximum ordered random variable from the sample size k

Let p be the percentage of the population below some time t_c , such that $p_c = P(T \leq t_c)$.

If $t_c = t_{r,n}$ then $p_{r,n} = P(t \leq t_{r,n})$ would be percentage of population below the time of the r^{th} out of n ordered statistic and $t_{r,n} = F^{-1}(p_{r,n})$.

If $t_c = t_{r,n,k}^{\max}$ then $p_{r,n,k} = P(t \leq t_{r,n,k}^{\max})$ would be the percentage of population below the maximum time of the r^{th} out of n ordered statistic from sample size k and $t_{r,n,k}^{\max} = F^{-1}(p_{r,n,k})$.

From Equation 50,

$$\text{Equation 60} \quad M_{r,n,k}(t_c) = P(t_{r,n,k}^{\max} \leq t_c) = P(F^{-1}(p_{r,n,k}) \leq t_c) = P(p_{r,n,k} \leq p_c)$$

Also

Equation 61

$$M_{r,n,k}(t_c) = P(t_{r,n,k}^{\max} \leq t_c) = (P(t_{r,n,k} \leq t_c))^k = (P(F^{-1}(p_{r,n,k}) \leq t_c))^k = (P(p_{r,n,k} \leq p_c))^k$$

From Equation 60 and Equation 61 follows:

$$\text{Equation 62} \quad M_{r,n,k}(t_c) = P(t_{r,n,k}^{\max} \leq t_c) = P(p_{r,n,k} \leq p_c) = (P(p_{r,n,k} \leq p_c))^k = (Q_{r,n}(p_c))^k$$

In Section 3.2.1 it was shown that the random variable $p_{r,n}$ has a Beta distribution function $Q_{r,n}(p_c)$ with $\alpha = r$ and $\beta = n - r + 1$ parameters, where the CDF is defined as:

$$\text{Equation 63} \quad Q_{r,n}(p_c) = \int_0^{p_c} \frac{\Gamma(r+n-r+1)}{\Gamma(r)\Gamma(n-r+1)} u^{r-1} (1-u)^{n-r} du$$

If one would like to know the $(1-\alpha)$ level quantile of the time $t_{r,n,k}^{\max}$, then for fixed points of time t , the $M_{r,n,k}(t)$ must be computed using the relationship in Equation 50. After which, using the interpolation technique the $(1-\alpha)$ level quantile could be determined.

This calculation is very hard to do manually and it is more suitable to use a computer algorithm to do the computation.

Herein, a novel approach is proposed to find the $(1-\alpha)$ level quantile of the time $t_{r,n,k}^{\max}$ by using the rank distribution.

Recall that Equation 63 is a Beta distribution with parameters $\alpha = r$ and $\beta = n - r + 1$ and p_c can be computed for $(1 - \alpha)^{\frac{1}{k}}$ quantile level if we apply an inverse transformation of Equation 63:

$$\text{Equation 64} \quad p_c = Q_{r,n}^{-1}\left((1 - \alpha)^{\frac{1}{k}}\right)$$

but $p_c = P(t \leq t_c)$, therefore the inverse of it will be

$$\text{Equation 65} \quad t_c = F^{-1}(p_c)$$

and because of the relationship in Equation 14, this is the $(1 - \alpha)$ level quantile of the time $t_{r,n,k}^{\max}$.

3.2.2.3 Verification of the proposed approach, using computer algorithm assuming the parent distribution is Weibull, Normal or Uniform

Verification of the proposed method for the several distribution functions (Weibull, Normal and Uniform) by using a computer algorithm was done. In this numerical analysis, the 10% and 90% quantile times for the $r=13$ out of $n=20$ ordered random variable from the sample size of $k=30$ for each of the following parent distribution will be computed.

- Weibull, with the shape parameter 5 and scale parameter 10,000,
- Normal with mean 10,000 and standard deviation 500, or
- Uniform within the interval [8,500; 11,500].

Using Equation 50 the empirical CDF will be constructed for the following fixed points in time:

- For the Weibull parent distribution, the abscissa values vary from 0 to 15,000 with increments of 1.
- For the Normal parent distribution, the abscissa values vary from 8,000 to 12,000 with increments of 1.
- For the Uniform parent distribution, the abscissa values vary from 8,500 to 11,500 with increments of 1.

Then, a linear approximation will be applied to find the 10% and 90% quantile times. The interpolated results will be compared with the proposed method, using Equation 64 and Equation 65.

Figure 17, Figure 18 and Figure 19 depict the interpolated and proposed method values for the 10% and 90% quantile times of the maximum 13th out of 20 ordered random variable from the sample size 30, from Weibull, Normal and Uniform parent distributions, respectively. It was noticed that as the increment of abscissa is decreasing, the interpolated value gets closer to the proposed method.

Figure 17: 10% and 90% quantile level values for the 13th out of 20 ordered random variable from the sample size 30, assuming the parent distribution is Weibull with shape parameter 5 and scale parameter 10,000.

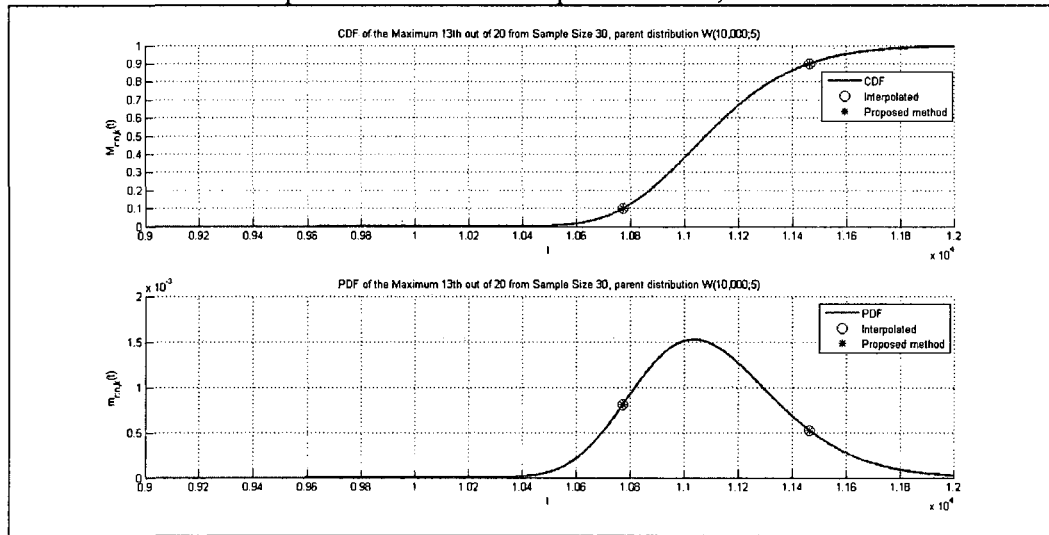


Figure 18: 10% and 90% quantile level values for the 13th out of 20 ordered random variable from the sample size 30, assuming the parent distribution is Normal with mean 10,000 and standard deviation 500.

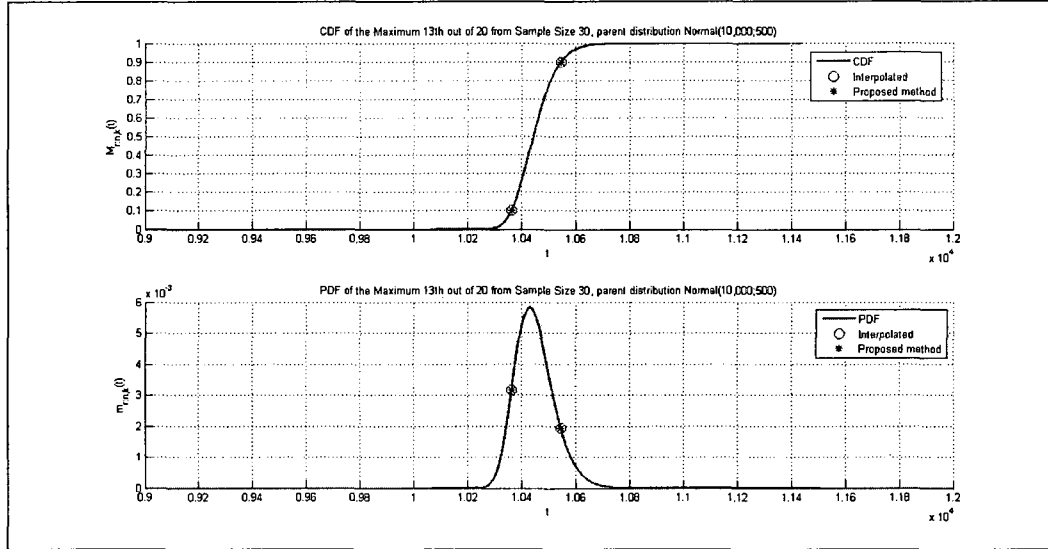
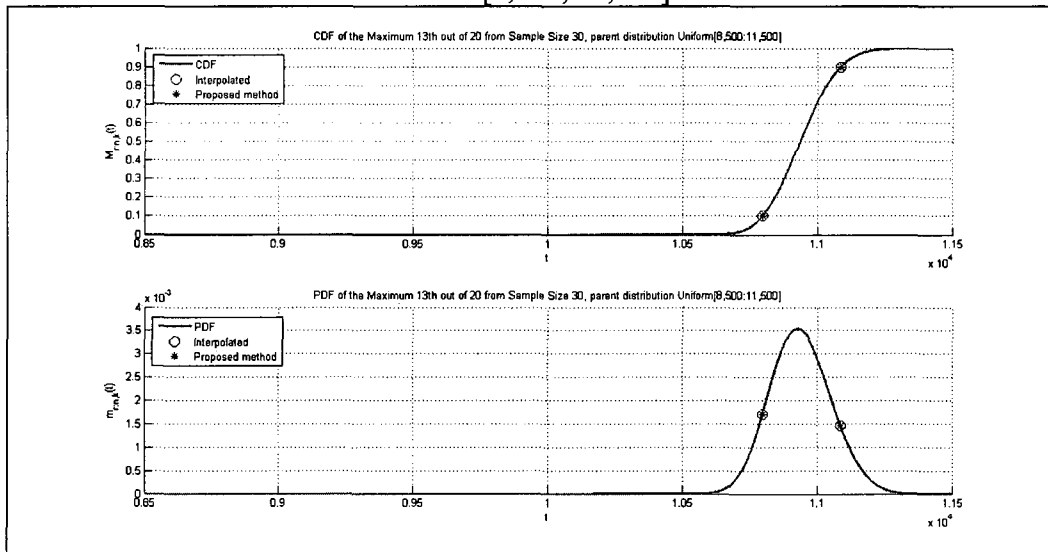


Figure 19: 10% and 90% quantile level values for the 13th out of 20 ordered random variable from the sample size 30, assuming the parent distribution is Uniform within the interval [8,500; 11,500].



3.2.2.4 Conclusion

The pdf and cdf for the Maximum Ordered Random Variable from the Sample Size k were derived. It was noticed that cdf is a recursive function.

Also, a new approach to calculate quantile for the Maximum Ordered Random Variable from the Sample Size k was proposed. The proposed method is accurate and theoretically proofed. The proposed method is fast, easy and straightforward in calculation.

3.2.3 k^{th} Sum of the Order Statistics

3.2.3.1 Background: the sums of the ordered random variables

Definition: Let X and Y be two independent continuous random variables with density functions $f(x)$ and $g(y)$, respectively. Assume that both $f(x)$ and $g(y)$ are defined for all real numbers. Then the convolution $f * g$ of f and g is the function given by:

Equation 66

$$(f * g)(z) = \int_{-\infty}^{+\infty} f(z-y)g(y)dy = \int_{-\infty}^{+\infty} g(z-x)f(x)dx$$

If the CDF and PDF of the Weibull distribution are defined as:

Equation 67

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\beta},$$
$$f(t) = \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta}$$

where $t, \beta, \theta > 0$

then the PDF of the r^{th} out of n ordered random variable, with the Weibull parent distribution, will be:

Equation 68

$$w_{r,n}(t) = n \binom{n-1}{r-1} \left(1 - e^{-\left(\frac{t}{\theta}\right)^\beta} \right)^{r-1} \left(e^{-\left(\frac{t}{\theta}\right)^\beta} \right)^{n-r} \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta}$$

The convolution function for two independent r^{th} out of n order random variables, with the same parent Weibull distribution is defined as:

Equation 69

$$Z_{r,n,2}(z) = \left(n \binom{n-1}{r-1} \frac{\beta}{\theta^\beta} \right)^2 * \int_0^z \left(1 - e^{-\left(\frac{z-t}{\theta}\right)^\beta} \right)^{r-1} \left(1 - e^{-\left(\frac{t}{\theta}\right)^\beta} \right)^{r-1} \left(e^{-\left(\frac{z-t}{\theta}\right)^\beta} \right)^{n-r} \left(e^{-\left(\frac{t}{\theta}\right)^\beta} \right)^{n-r} ((z-t)t)^{\beta-1} e^{-\left(\frac{z-t}{\theta}\right)^\beta} e^{-\left(\frac{t}{\theta}\right)^\beta} dt$$

where z is a random variable defined as sum of the two r^{th} out of n order random variables, with the same parent Weibull distribution.

An extended research of the literature did not reveal the explicit form for the PDF of the sum of the more than one ordered random variables with the same parent Weibull distribution. As we see from Equation 69 the integration is complicated and becomes more complicated when the sum is defined for more random variables.

As we mentioned in the literature review, the sums of the random variables also could be approximated by the Cornish Fisher expansion. This is a complicated and hard approach, where computer software has to be used. To extend this approach for the ordered random variable in order to find an approximate quantile value, the mean and variances of the ordered random variable are required (It should be mentioned that mean and variances of the ordered random variable with the Weibull parent distribution might be computed by the Equation). Then it might be assumed that the k^{th} sum of the ordered random variable distribution is approximately normal and by using the standardization technique to find the quantile values.

3.2.3.2 Simulation study of the convolution of the ordered random variables from the Weibull distribution and the comparison with the $1-\alpha$ level quantile limit sums

Let's assume k groups of n components are tested in series to the r^{th} failure in a group. So, once the r^{th} failure from the first group is obtained the second group will be tested. If the same logic will be extended for the k groups, then the time to finish the test will be the k^{th} sum of the r^{th} out of n ordered random variables.

In the previous Section we develop the technique, to find the $1-\alpha$ level quantile limit for the r^{th} out of n ordered random variable, from any continuous type PDF parent distribution.

For the parent distribution, if the parameters of the parent distribution, the number of components in the test (n) and the number of the results from each group (r) are the same, then for the same $1-\alpha$ level quantile the limits are going to be the same for the all k groups. Adding these limits together or just multiplying the $1-\alpha$ level quantile limit by the number of groups (k) and comparing the result with the $1-\alpha$ level quantile limit from the empirical distribution is the goal of this chapter.

Let's assume that the two groups of twenty components are tested in a series mode until the fifth failure, from a Weibull parent distribution with shape parameter equal to 1.5 and scale parameter equal to 100. So,

- Number of Groups is 2 ($k=2$).
- Number of components in a Group is 20 ($n=20$).
- Number of failures per Group is 5 ($r=5$).
- The shape parameter is 1.5.
- The scale parameter is 100.
- 0.10 and 0.90 level quantile limits are of interest.
- Number of runs, 20,000.

Figure 20: Empirical CDF of 2 sums of 5th out 20.

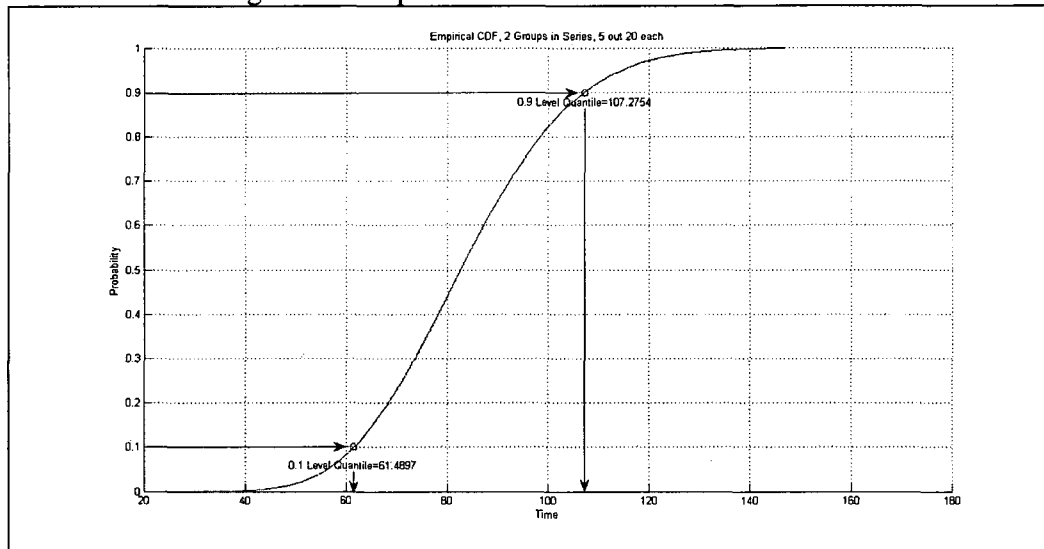


Figure 21: Empirical CDF of 2 sums of 5th out 20 and 10% and 90% quantiles multiplied by the number of groups (k).

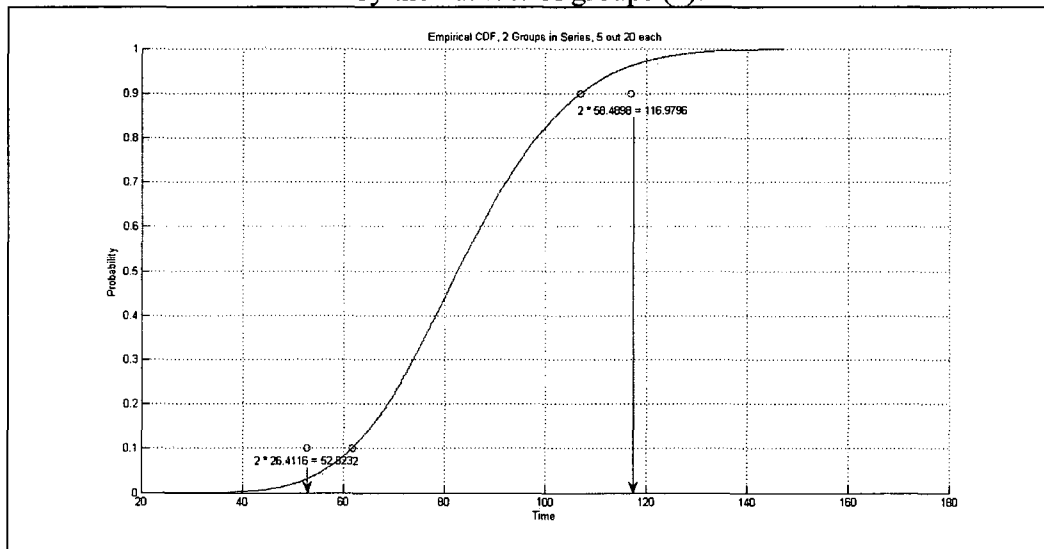


Figure 20 shows the empirical CDF of sums of the two, 5 out of 20 ordered random variables and interpolated 0.1 and 0.9 quantile levels. The 0.1 quantile level limit is approximately 61.48 and the 0.9 quantile level limit is approximately 107.27.

In the previous chapter we showed how to calculate the $1-\alpha$ level quantile limit by using the rank distribution. For 5 out of 20 ordered random variable, with the assumed Weibull distribution parameters, the 0.1 quantile limit is 26.41 and 0.9 quantile level limit is 58.48. If we multiply the quantile limits by the number of groups ($k=2$) we will get 52.82 for the 0.1 quantile and 116.97 for the 0.9 quantile (see Figure 21).

Let us assume that the interpolated values for 0.1 and 0.9 quantiles (from Figure 20) are the true quantile values for the 2 groups in series mode test, terminated at the 5th out of 20 failures. So by 80% probability the test will be terminated within the interval (61.48 and 107.27).

If we find the 5th out of 20 failure time at 0.1 and 0.9 quantiles and multiply them by 2, we will get the wider interval than the true one. In this case we got [52.82:116.97].

So, it seems that the interval [52.82:116.97] includes in it the true [61.48:107.27] interval and one might be able to use [52.82:116.97] interval as an approximate 80% probability interval. Moreover, the approximate interval always contains the true interval. This claim is proved in the next Section.

3.2.3.3 Claim

If X and Y are iid and have a continuous PDF $f(x)$ and random variable S is constructed such as $S=X+Y$ and has $s(S)$ continuous PDF, then for the random variable S the following is always true:

If $P(L \leq X \leq U) = P(L \leq Y \leq U) = 1-\alpha$, where L and U are $\alpha/2$ quantile limits from $f(x)$, and $P(L_true \leq S \leq U_true) = 1-\alpha$ and L_true and U_true are $\alpha/2$ quantile limits from $s(S)$, then:

$$P(2L \leq S \leq 2U) > P(L_true \leq S \leq U_true).$$

3.2.3.4 Proof

Let there exist r.v. Z that has zero and one, as the mean and standard deviation, also has a continuous PDF $z(Z)$.

Let X and Y have mean μ_x and standard deviation σ_x ,
then

$$E(S)=2\mu_x \text{ and } \text{Var}(S)=2\sigma_x^2.$$

Given that:

$$P(L \leq X \leq U) = 1 - \alpha, \text{ it follows that } P\left(\frac{L - \mu_x}{\sigma_x} \leq Z \leq \frac{U - \mu_x}{\sigma_x}\right) = 1 - \alpha$$

From another hand we have that:

$$P(2L \leq S \leq 2U) = P\left(\frac{2}{\sqrt{2}}\left(\frac{L - \mu_x}{\sigma_x}\right) \leq Z \leq \frac{2}{\sqrt{2}}\left(\frac{U - \mu_x}{\sigma_x}\right)\right) > 1 - \alpha \quad (1)$$

and it is given that:

$$P(L_{\text{true}} \leq S \leq U_{\text{true}}) = 1 - \alpha \quad (2)$$

So from (1) and (2) follows that:

$$P\left(\frac{2}{\sqrt{2}}\left(\frac{L - \mu_x}{\sigma_x}\right) \leq Z \leq \frac{2}{\sqrt{2}}\left(\frac{U - \mu_x}{\sigma_x}\right)\right) > P(L_{\text{true}} \leq S \leq U_{\text{true}})$$

The claim could be extended for the random variable S for k sums.

Figure 22: Empirical CDF of 2 sums of 5th out 20 and quantiles multiplied by the number of groups (k).

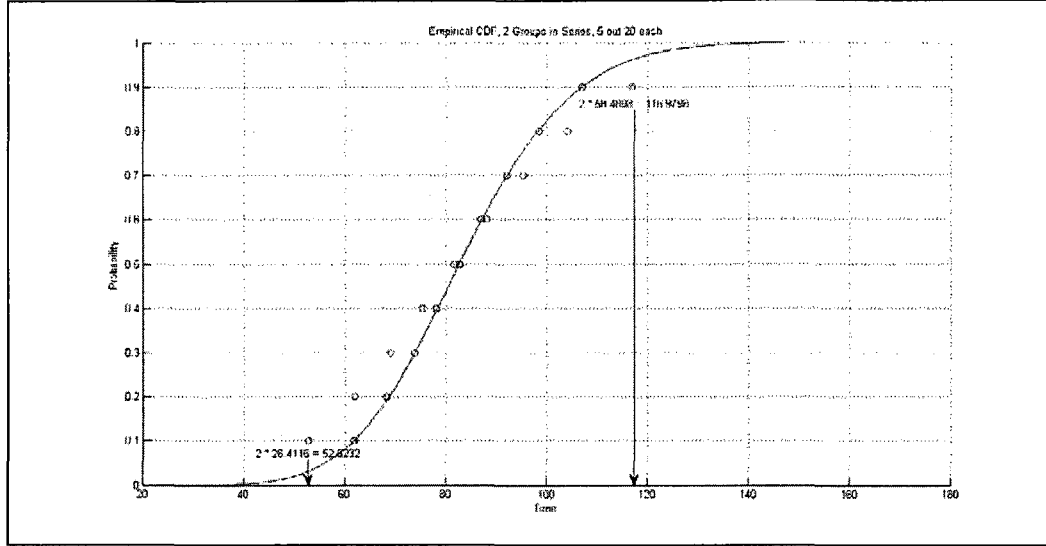


Figure 22 shows the empirical CDF of the time to complete the test in 2 groups under series mode with 20 components each, terminated at the 5th failure for each group. We see that the approximated ranges always contain the true ranges for the different levels.

This is a one way to approximate the desired quantile ranges under series mode. In next Section we will develop another approximation method for the quantiles in series mode test. That will be used for the different configuration tests comparison.

3.2.3.5 Approximate quantile limits for the k^{th} sum of the ordered random variables

Given in Equation 11 and Equation 12 that the probability distribution and density functions of the r^{th} ordered statistic out of n are known and may be represented by:

$$G_{r,n}(t) = P(t_{r,n} \leq t) = \sum_{i=r}^n \binom{n}{i} F(t)^i (1 - F(t))^{n-i}$$

$$g_{r,n}(t) = r \binom{n}{r} F(t)^{r-1} (1-F(t))^{n-r+1} f(t)$$

Let the sample of size k of the r^{th} ordered statistic out of n be: $t_{r,n,1}; t_{r,n,2}; \dots; t_{r,n,k}$.

The random variable $s(r, n, k)$ will be formed from the sample in the following way:

$$s(r, n, k) = t_{r,n,1} + t_{r,n,2} + \dots + t_{r,n,k} = \sum_{i=1}^k t_{r,n,i}.$$

Let's assume that there exists some random variable $b(r, n, k)$ that could define $s(r, n, k)$ in the following way:

$$\text{Equation 70} \quad s(r, n, k) \approx k * b(r, n, k)$$

And has $F_{b(r,n,k)}(t)$ and $f_{b(r,n,k)}(t)$ as an CDF and PDF, where the PDF has the following form:

$$\text{Equation 71} \quad f_{b(r,n,k)}(t) = \frac{(k(n+1)-1)!}{(kr-1)!(k(n-r+1)-1)!} F(t)^{kr-1} (1-F(t))^{k(n-r+1)-1} f(t)$$

where, $t \geq 0$

or

$$\text{Equation 72} \quad f_{b(r,n,k)}(t) = kr \binom{k(n+1)-1}{kr} F(t)^{kr-1} (1-F(t))^{k(n-r+1)-1} f(t)$$

where, $t \geq 0$

Let p be the percentage of the population below some time t_c , such that $p_c = P(T \leq t_c)$. If $t_c = b(r, n, k)$ then $p_{(r, n, k)} = P(T \leq b(r, n, k))$ would be the percentage of population below the $b(r, n, k)$ and $b(r, n, k) = F^{-1}(p_{(r, n, k)})$.

Equation 73

$$F_{b(r, n, k)}(t_c) = P(b(r, n, k) \leq t_c) = P(F^{-1}(p_{(r, n, k)}) \leq t_c) = P(p_{(r, n, k)} \leq p_c) = H_{(r, n, k)}(p_c)$$

From Equation 71 we know that

$$F_{b(r, n, k)}(t_c) = \int_{-\infty}^{t_c} f_{b(r, n, k)}(t) dt = \int_{-\infty}^{t_c} \frac{(k(n+1)-1)!}{(kr-1)!(k(n-r+1)-1)!} F(t)^{kr-1} (1-F(t))^{k(n-r+1)-1} f(t) dt$$

Let

$$p = P(T \leq t) = F(t) \text{ then } dp = f(t) dt$$

$$\text{when } t = -\infty; p = 0 \text{ and } t = t_c; p = p_c$$

So

Equation 74

$$F_{b(r, n, k)}(p_c) = P(p_{(r, n, k)} \leq p_c) = \int_0^{p_c} \frac{(k(n+1)-1)!}{(kr-1)!(k(n-r+1)-1)!} p^{kr-1} (1-p)^{k(n-r+1)-1} dp$$

Equation 75

$$\begin{aligned} h_{(r, n, k)}(p_c) &= \frac{(k(n+1)-1)!}{(kr-1)!(k(n-r+1)-1)!} p_c^{kr-1} (1-p_c)^{k(n-r+1)-1} = \\ &= kr \binom{k(n+1)-1}{kr} p_c^{r-1} (1-p_c)^{n-r} \end{aligned}$$

$$\text{where } 0 \leq p_c \leq 1$$

One can recognize that $h_{(r, n, k)}(p_c)$ as a Beta distribution of the form:

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \text{ where } 0 \leq x \leq 1,$$

$$F(x) = \int_0^x \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1} du$$

$$E(x) = \frac{\alpha}{\alpha + \beta} \quad \text{Var}(x) = \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$$

Let $kr = \alpha$ and $k(n-r+1) = \beta$ and using the fact that $\Gamma(n) = (n-1)!$

Then

$$\begin{aligned} \text{Equation 76} \quad h_{(r,n,k)}(p_c) &= \frac{(k(n+1)-1)!}{(kr-1)!(k(n-r+1)-1)!} p^{kr-1} (1-p)^{k(n-r+1)-1} = \\ &= \frac{\Gamma(k(n+1))}{\Gamma(kr)\Gamma(k(n-r+1))} p^{kr-1} (1-p)^{k(n-r+1)-1} \end{aligned}$$

where, $0 \leq p_c \leq 1$

$$\text{Equation 77} \quad H_{(r,n,k)}(p_c) = \int_0^{p_c} \frac{\Gamma(k(n+1))}{\Gamma(kr)\Gamma(k(n-r+1))} u^{kr-1} (1-u)^{k(n-r+1)-1} du$$

$$\text{Equation 78} \quad E(p_c) = \frac{kr}{k(n+1)} = \frac{r}{n+1}$$

$$\begin{aligned} \text{Equation 79} \quad \text{Var}(p_c) &= \frac{kr * k(n-r+1)}{(kr + k(n-r+1))^2 (kr + k(n-r+1) + 1)} = \\ &= \frac{r(n-r+1)}{(n+1)^2 (k(n+1) + 1)} \end{aligned}$$

From Equation 76 the “ p_c ” could be computed for $(1-\alpha)$ quantile level if we apply an inverse transformation of the Equation 76 for the desired r , n and k , where $\alpha = kr$ and $\beta = k(n-r+1)$.

So,

$$\text{Equation 80} \quad p_{(r,n,k)} = H_{(r,n,k)}^{-1}(1-\alpha)$$

and as far as $p_{(r,n,k)} = P(T \leq b_{(r,n,k)})$, then taking the inverse of it will be

$$\text{Equation 81} \quad b(r, n, k) = F^{-1}(p_{(r,n,k)})$$

and using Equation 70, the $s(r, n, k)$ approximately would be $k * b(r, n, k)$, which is claimed herein to be the approximate $(1-\alpha)$ level quantile for the random variable $s(r, n, k)$.

In summary: to find an approximate $(1-\alpha)$ level quantile for the k^{th} sum of the r^{th} out of n ordered random variable, first the $p_{(r,n,k)}$ should be computed from Equation 80, next the $b(r, n, k)$ by Equation 81 and $k * b(r, n, k)$ would be an approximate $(1-\alpha)$ level quantile.

The logic for defining the Equation 70 and Equation 71 as they are described is the following.

The random variable $s(r, n, k)$ was defined as:

$$s(r, n, k) = t_{r,n,1} + t_{r,n,2} + \dots + t_{r,n,k} = \sum_{i=1}^k t_{r,n,i}.$$

Also, it was assumed that there exists some random variable $b(r,n,k)$ that could define $s(r,n,k)$ in the following way:

$$s(r,n,k) \approx k * b(r,n,k)$$

If $p_{r,n}$ will be defined as $p_{r,n} = P(t \leq t_{r,n})$ (see Section 3.2.1.1), then for some r.v. S we will have the following:

$$P(S \leq s(r,n,k)) = p_{r,n,1} + p_{r,n,2} + \dots + p_{r,n,k}$$

So,

$$E(P(S \leq s(r,n,k))) = k * E(p_{r,n}) = \frac{k * r}{n+1}, \quad (\text{see Equation 44}).$$

It follows that we have to formulate $f_{b(r,n,k)}(t)$ so that $k * E(P(T \leq b(r,n,k)))$ would be equal to the $E(P(S \leq s(r,n,k)))$.

The only possible way to formulate $f_{b(r,n,k)}(t)$ so that

$$E(P(S \leq s(r,n,k))) = k * E(P(T \leq b(r,n,k))) \text{ is the formulation in Equation 71.}$$

Indeed:

$$k * E(P(T \leq b(r,n,k))) = k * E(p_{(r,n,k)}) = k * \frac{r}{n+1} \quad (\text{see$$

Equation 78)

3.2.3.6 Verification of the proposed approach, using computer simulation algorithm assuming the parent distribution is Weibull, Normal or Uniform

In this Section a computer algorithm will be used to verify the proposed method for several distribution functions, commonly used in reliability testing. In this numerical analysis, 10% and 90% quantile times for the k^{th} sum of the ordered random variable for each of the following parent distribution will be computed.

- Uniform within the interval [8,500; 11,500],
- Normal with mean 10,000 and standard deviation 500,
- Weibull, with the shape parameter 5 and scale parameter 10,000.

The following assumption will be made:

- Number of Groups is 5 ($k=5$).
- Number of components in a Group is 20 ($n=20$).
- Number of failures per Group varies from 1 until n ($r=1:n$).

Five r^{th} out of 20 order random variables will be generated and added together, the procedure will be repeated 1000 times. Then, based on these 1000 points, the empirical cumulative function will be constructed and a linear approximation will be applied to find the 10% and 90% quantile times.

The interpolated results will be compared with the proposed method, using Equation 80, Equation 81 and Equation 70.

Figure 23, Figure 24 and Figure 25 show the approximation method values (red asterix) compared with the interpolated values (blue circles).

Figure 23: CDF for the $k=5$, r -th out of 20 ordered random variables, assuming parent distribution is Uniform.

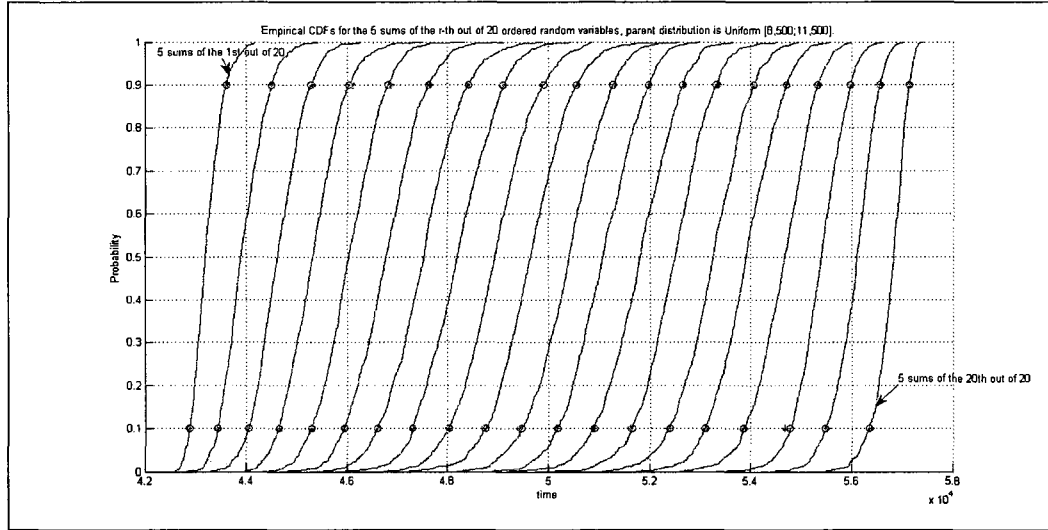


Figure 24: CDF for the $k=5$, r -th out of 20 ordered random variables, assuming parent distribution is Normal.

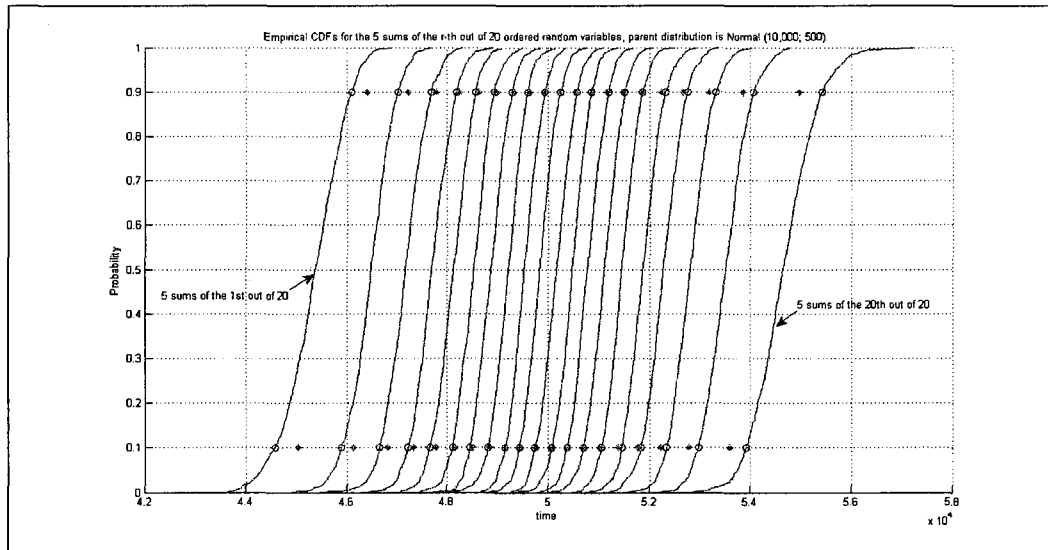
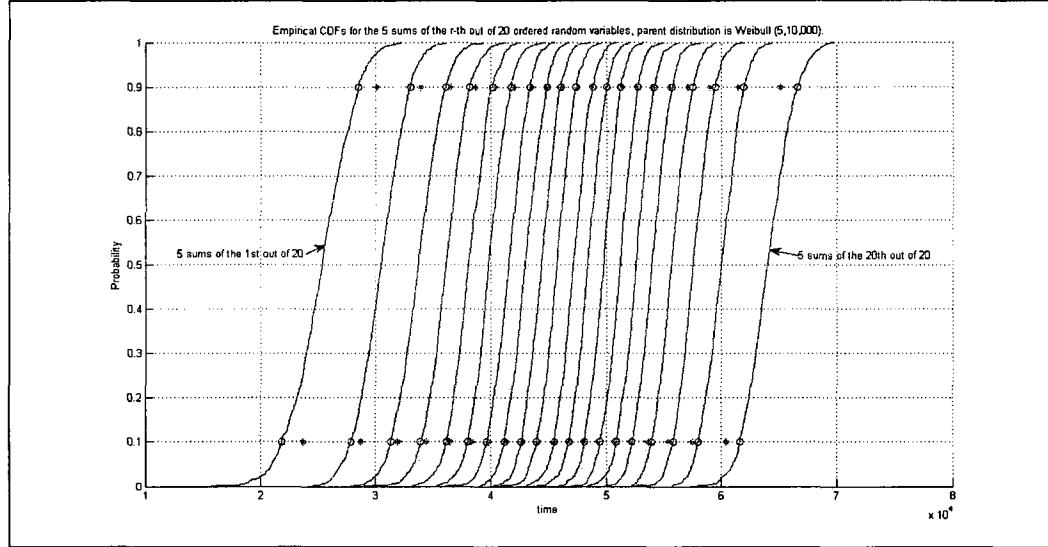


Figure 25: CDF for the $k=5$, r -th out of 20 ordered random variables, assuming parent distribution is Weibull.



It was noticed that when the parent distribution is Uniform, the proposed method is very precise for any number k^{th} sum of the r^{th} out of n ordered random variable. For the Normal and Weibull parent distributions, some “distribution recollection adjustment factor” could be used. Besides the Uniform parent distribution, this adjustment factor should be a function of the parent distribution, in a way when:

$$E(p_c) = \frac{r}{n+1}$$

- is close to 0.5 then the adjustment factor should be 1,
- is less than 0.5 then the adjustment factor should be less than 1,
- is greater than 0.5 then the adjustment factor should be greater than 1.

3.2.3.7 Comparison of the proposed method quantile times with the approximate quantile times by the Cornish-Fisher expansion

In 1960 Fisher and Cornish developed the asymptotic expansions, which allow expressing desired quantile points of the distributions in terms of the known cumulants. Later, in 1998, Pascual and Meeker used this approach for their study⁵.

Particularly, they computed and published 5%, 50% and 95% approximate quantile times for a series test that consisted of 10 ($k=10$) groups with 5 ($n=5$) components in each group and varying the number of failures per test from 1 to 5 ($r=1:5$). They assumed that failure has a Weibull distribution with scale 19.59 and shape 2.35.

I have computed, by the presented method, the quantile limits for the mentioned levels for the series tests with the same parameters as Pascual and Meeker did. The results are summarized in Table 3.

Table 3: Comparison of the approximate quantile times by the proposed method and the Cornish-Fisher expansion.

r	n	k	Approximate Quantile Times based on the Cornish-Fisher Expansion by Escobar and Meeker*.			Calculated Quantile Times by the Proposed Method		
			5%	50%	95%	5%	50%	95%
1	5	10	67	87	109	73.46	93.96	115.54
2	5	10	110	131	153	112.35	133.02	154.49
3	5	10	147	169	192	146.09	167.61	189.94
4	5	10	186	210	236	181.02	204.34	228.69
5	5	10	239	269	301	224.81	252.42	282.02

*Source: Pascual, F., & Meeker, W. (1998). The modified sudden death test: Planning life tests with a limited number of test positions. *Journal of Testing and Evaluation*, 26(5), 434.

⁵ Pascual, F., & Meeker, W. (1998). The modified sudden death test: Planning life tests with a limited number of test positions. *Journal of Testing and Evaluation*, 26(5), 434.

As we see in Table 3, the suggested approximation method deviates more on extreme ordered statistics, while for the rest ordered statistics the values are close enough.

3.2.3.8 The distributional behavior of the k^{th} sum of the ordered random variable from Weibull parent distribution, where the shape parameter is greater than unity

Using the technique described in Section 3.2.3.5, we will use a computer software package to investigate the behavior of the k^{th} sum of the ordered random variable, assuming a Weibull parent distribution with shape parameter greater than unity.

The following assumptions will be made:

- Number of Groups is varies from 1 until 5 ($k=1:5$).
- Number of components in a Group is 20 ($n=20$).
- Number of failures per Group varies from 1 until n ($r=1:n$).
- The shape parameter is increasing by 0.2 increments, starting from 1 until 6.
- The scale parameter is 100.
- 0.10 and 0.90 level quantile limits are under the interest.
- Number of runs, 10,000 for each combination.

Figure 26: For $k=1:5$ number of groups, 10% and 90% quantiles depending on the shape and r^{th} ordered random variable.

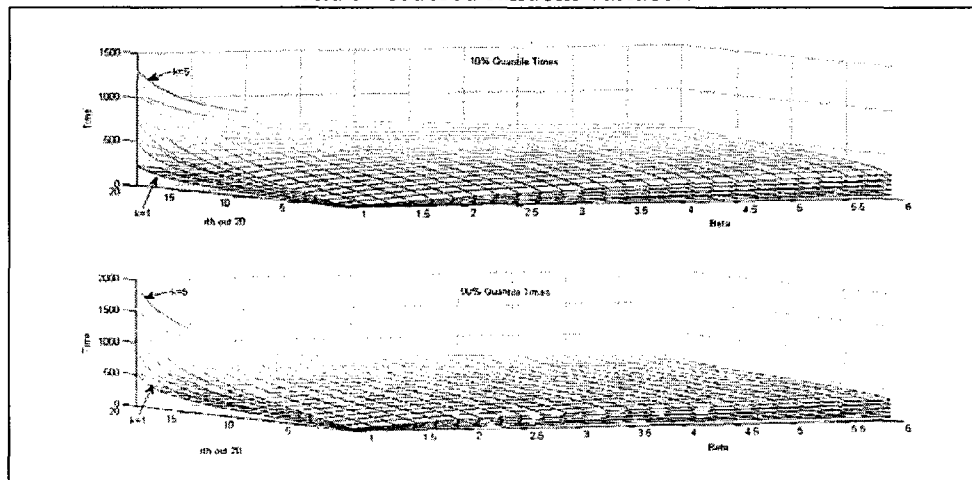


Figure 26 shows the 10% and 90% quantile times for the defined number of groups ($k=1:5$) depending on the r^{th} out 20 and beta values.

Figure 27: For r^{th} ordered random variable, 10% quantile depending on the shape and $k=1:5$ number of groups.

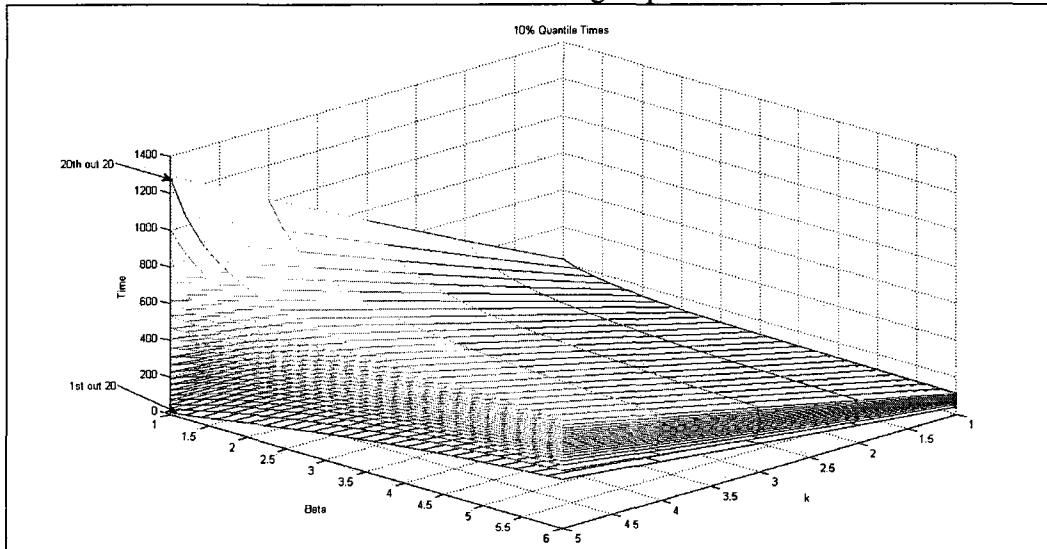


Figure 28: For r^{th} ordered random variable, 10% quantile depending on the shape and $k=1:5$ number of groups.

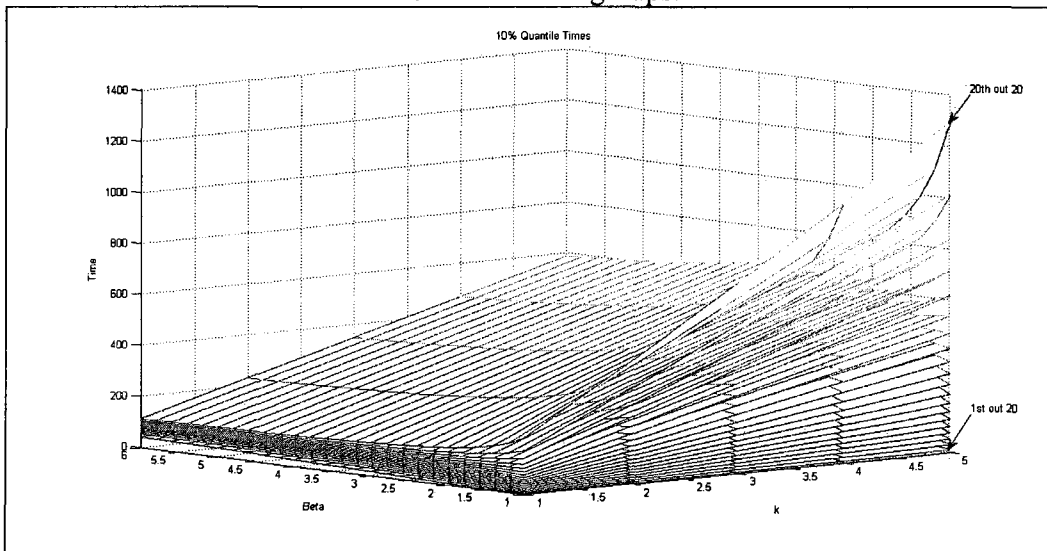


Figure 27 and Figure 28 depict the 10% quantile times for the different r^{th} out 20 values, depending on beta and number of groups. It is not always true to assume that as beta increases the time to finish the test will increase too. As we see it depends on the r^{th} value.

Figure 29: For r^{th} ordered random variable, 90% quantile depending on the shape and $k=1:5$ number of groups.

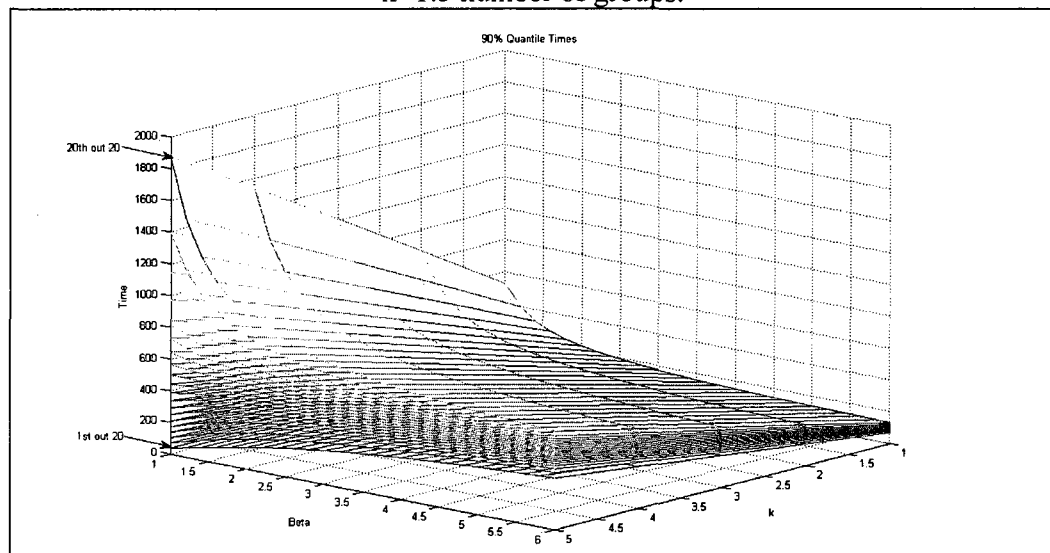


Figure 30: For r^{th} ordered random variable, 90% quantile depending on the shape and $k=1:5$ number of groups.

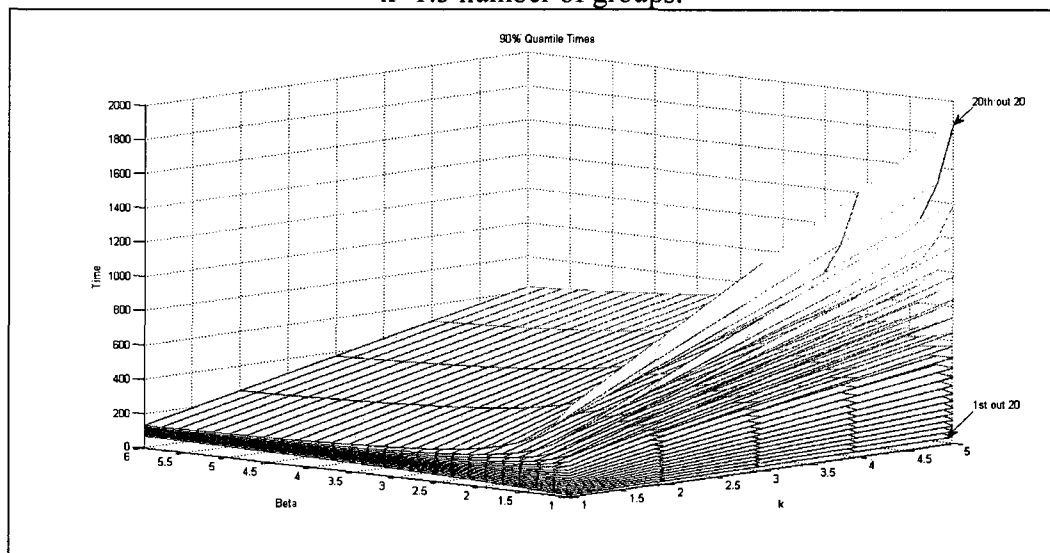


Figure 29 and Figure 30 depict the 90% quantile times for the different r^{th} out 20 values, depending on beta and number of groups. The beta effect was observed at 90% Quantile times also.

Figure 31: For the shape parameters from 1 to 6, 10% quantile depending on $k=1:5$ number of groups and r^{th} ordered random variable.

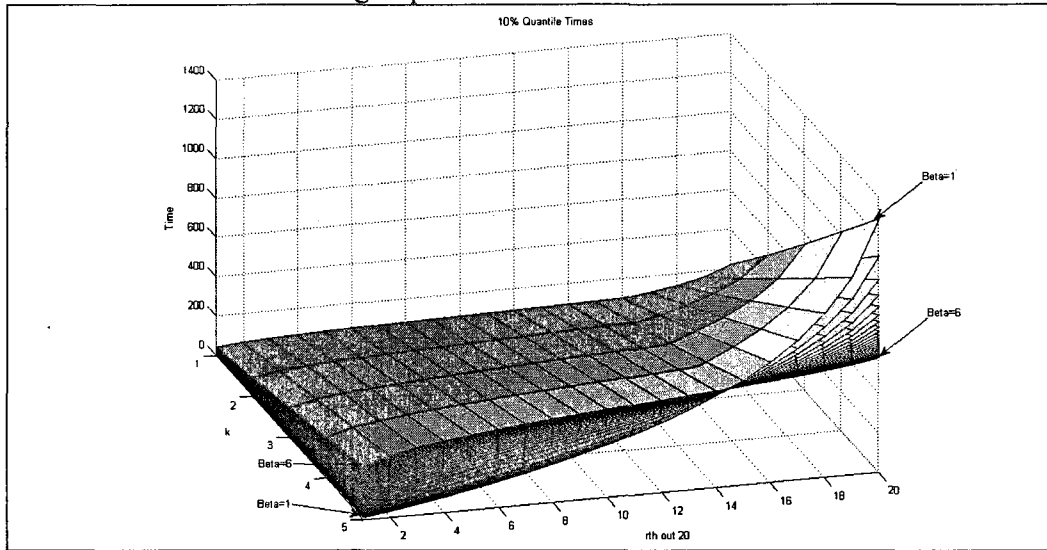


Figure 32: For the shape parameters 1 and 6, 10% quantile depending on $k=1:5$ number of groups and r^{th} ordered random variable.

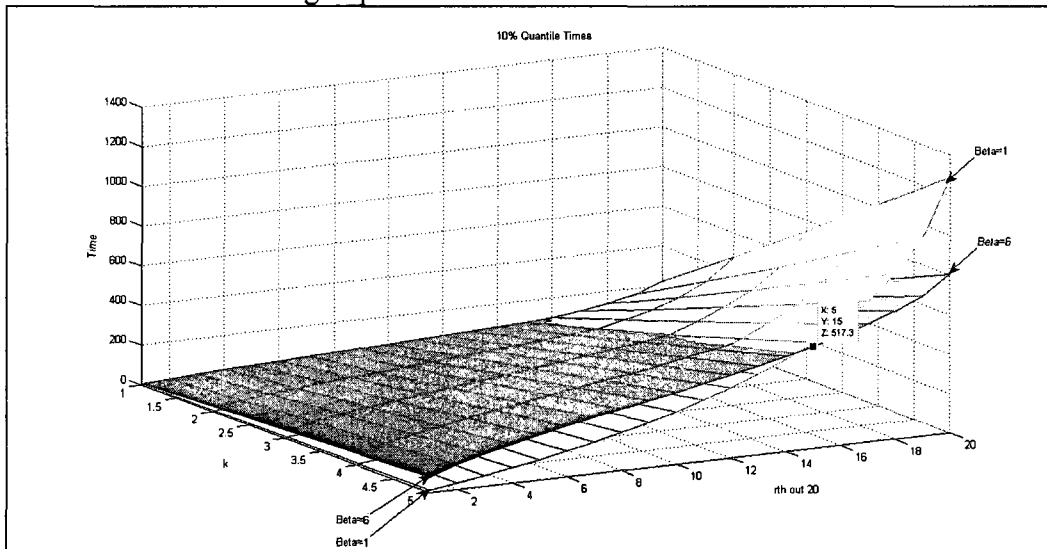


Figure 31 and Figure 32 show the 10% quantile times for the different beta values, depending on r^{th} out 20 and number of groups. There are cases where small beta will have a negative affect on the duration of the test. The truncated point was observed for the $k=5$ and $r=15$ and as k decreases, r decreases too.

Figure 33: For the shape parameters from 1 to 6, 90% quantile depending on $k=1:5$ number of groups and r^{th} ordered random variable.

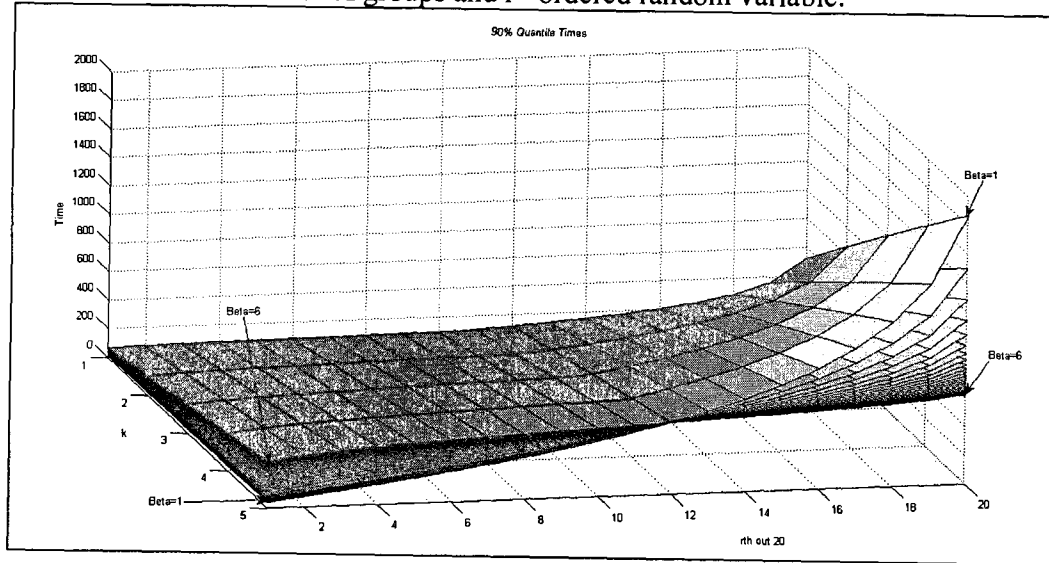


Figure 34: For the shape parameters 1 and 6, 90% quantile depending on $k=1:5$ number of groups and r^{th} ordered random variable.

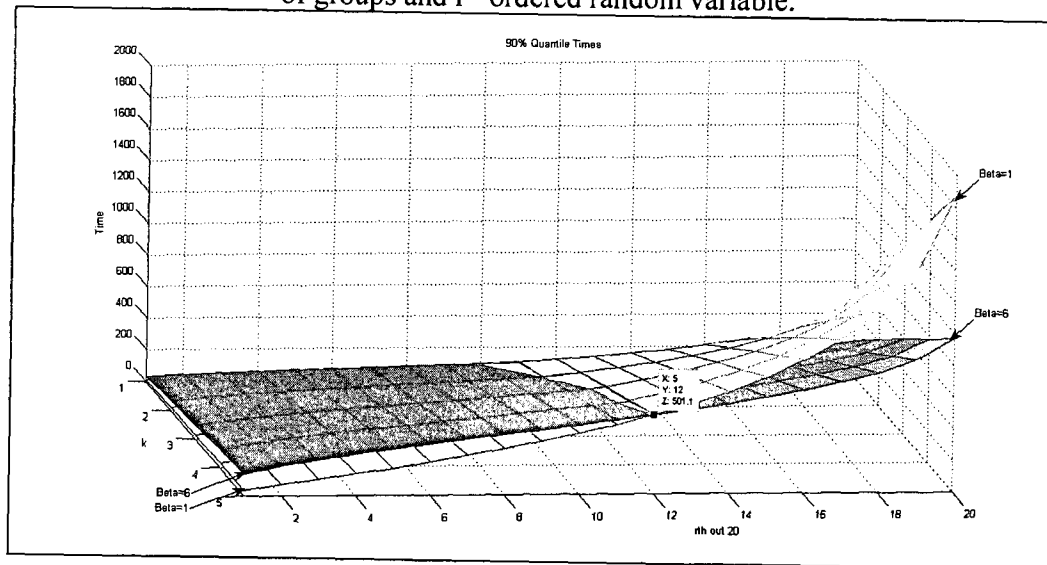


Figure 33 and Figure 34 show the 90% quantile times for the different beta values, depending on r^{th} out 20 and number of groups. The truncated point was observed for the $k=5$ and $r=12$ and as k decreases r decreases too.

3.2.3.9 Verification of the beta effect by the simulation study

In this Section, we will verify that the beta effect exists.

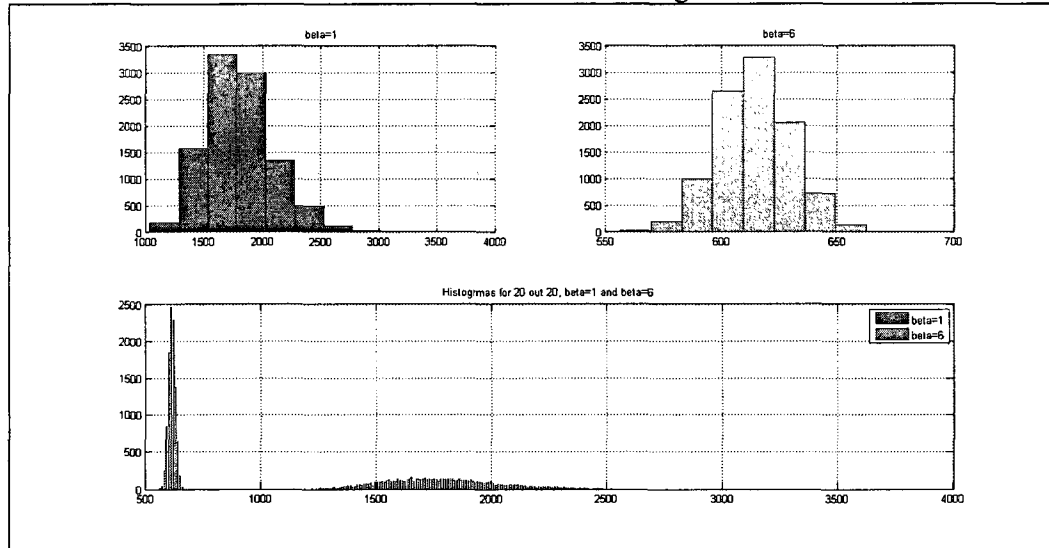
Let us assume the following:

- Number of Groups is 5 ($k=5$).
- Number of components in a Group is 20 ($n=20$).
- Number of failures per Group is 20 ($r=20$).
- The shape parameter is 1 and 6.
- The scale parameter is 100.
- Number of runs, 10,000 for each combination.

From the Weibull distribution with the shape=1 and scale=100, the 5 random samples of size 20 are taken and the maximum values from these 5 samples were added together. The procedure was repeated 10,000 times. Next, the same procedure was repeated for the shape=6.

The results are depicted in Figure 35.

Figure 35: With two different parent distribution shape parameters, the 20th out 20 ordered random variable histograms.



3.2.3.10 Conclusion

I am proposing the approximation method of finding the k^{th} Sum of the r^{th} out of n ordered random variable quantile limits for any continuous parent distribution. The proposed method is distribution free, meaning that without knowing the PDF of the k^{th} Sum as a random variable, the quantile of the k^{th} Sum can be found. It is a computationally easy method and does not involve any complicated formulas.

The proposed method is close enough to the simulated values; maximum deviations are observed for the extreme ordered statistics.

For the Weibull parent distribution, the shape parameter has an affect on the series test times. It is not always true to assume that higher shape parameter would result in longer testing time. It depends on the number of groups, number of the components in each group and the number of results per group.

3.2.4 Total Accumulated Time

3.2.4.1 Introduction

Total accumulated time is described in Section 3.1.3 by Equation 26. This is the total run time for all devices on the test.

$$T_{ac} = \sum_{i=1}^r t_{i:n} + (n-r) * t_{r:n}$$

Let a random variable w_i be defined as:

$$w_1 = n * t_1$$

$$w_i = (n-i+1) * (t_i - t_{i-1}), \quad i = 2, \dots, r$$

Then

Equation 82

$$T_{ac} = \sum_{i=1}^r w_i$$

If the parent distribution is an exponential distribution with rate parameter $1/\theta$, then $2T_{ac} / \theta$ has a chi-square distribution with $2r$ degrees of freedom^{6,7,8}.

So,

Equation 83

$$E(T_{ac}) = r * \theta$$

⁶ Lawless, J. (1982). Statistical models and methods for lifetime data. New York: Wiley, pg.-102.

⁷ Meeker, W., & Escobar, L. (1998). Statistical methods for reliability data. New York: Wiley, pg-167.

⁸ Lamberson, L., & Kapur, K. (1977). *Reliability in engineering design*. New York: Wiley, pg.-285.

The total accumulated time for the test (parallel or serially) is given in Equation 27. Regardless of the testing strategy, the expected total accumulated time of the test will be

Equation 84
$$E(T_{test_ac}) = k * E(T_{ac}) = k * r * \theta$$

Note that $k * r = R$ and this is the total results collected from the test.

3.2.4.2 Another way to represent the total accumulated time

Let assume that a random variable t has an exponential distribution with rate parameter $1/\theta$.

Let $t_{1:n}, t_{2:n}, \dots, t_{r:n}$ be an ordered statistics from this distribution and total accumulated time T will be defined as:

Equation 85
$$T_{ac} = n * t_{r:n} - \sum_{i=2}^r (i-1) * (t_r - t_{r-1})$$

A random variable $y_i = t_{i:n} - t_{i-1:n}$ has an exponential distribution⁹ with the rate parameter $(n-i+1)/\theta$.

So, T_{ac} in Equation 85 might be expressed as:

Equation 86
$$T_{ac} = n * \sum_{i=1}^r y_i - \sum_{i=2}^r (i-1) * y_i$$

Noting that $E(y_i) = \theta / (n-i+1)$, it follows that $E(T_{ac})$ in Equation 86 would be:

⁹ Lamberson, L., & Kapur, K. (1977). *Reliability in engineering design*. New York: Wiley, pg-285.

$$\begin{aligned}
E(T_{ac}) &= n * \sum_{i=1}^r \frac{\theta}{n-i+1} - \sum_{i=2}^r (i-1) * \frac{\theta}{n-i+1} = \\
&= \theta \left(1 + \frac{n}{n-1} + \frac{n}{n-2} + \dots + \frac{n}{n-r+1} - \left(\frac{1}{n-1} + \frac{2}{n-2} + \dots + \frac{r-1}{n-r+1} \right) \right) = \\
&= \theta(1+r-1) = \theta r
\end{aligned}$$

If k groups of n components are tested until the r^{th} failure in series or parallel, then the expected total accumulated time for the test will be:

Equation 87
$$E(T_{ac_test}) = k * r * \theta = R * \theta$$

3.2.4.3 Total accumulated time for the test, assuming Weibull parent distribution

Total accumulated time is described in Section 3.1.3 by Equation 26.

$$T_{ac} = \sum_{i=1}^r t_{i:n} + (n-r) * t_{r:n}$$

Techniques described in Sections 3.2.4.1 and Sections 3.2.4.2 are applicable if a parent distribution is an exponential distribution.

In this research the simulation technique will be used to evaluate the total accumulated time for the MSDT and CSDT with respect to the increase in a shape parameter.

CHAPTER 4

RESULTS

4.1 Results Comparison between MSDT and CSDT in Time Domain

4.1.1 Run Time

4.1.1.1 Introduction

In this chapter we will do a pilot study to compare MSDT and CSDT in the time domain. It is assumed that the components under investigation are non-repairable and have a Weibull failure distribution with the shape parameter greater than unity.

For a fixed shape parameter of the parent distribution, the underlying hypothesis is as follows:

Run time for the CSDT plan with a specific k_1 , r_1 , and n_1 is equal to the run time for the MSDT plan with a specific k_2 , r_2 , and n_2 .

$$H_0: \text{CSDT}(k_1, r_1, n_1) = \text{MSDT}(k_2, r_2, n_2)$$

$$H_a: \text{CSDT}(k_1, r_1, n_1) < \text{MSDT}(k_2, r_2, n_2)$$

$$H_a: \text{CSDT}(k_1, r_1, n_1) \neq \text{MSDT}(k_2, r_2, n_2)$$

$$H_a: \text{CSDT}(k_1, r_1, n_1) > \text{MSDT}(k_2, r_2, n_2)$$

The run times as random variables for MSDT and CSDT were given in Equation 24 and Equation 25. To find the 10% and 90% quantiles of these random variables, the techniques described in Section 3.2 will be used.

4.1.1.2 Logic of the comparison

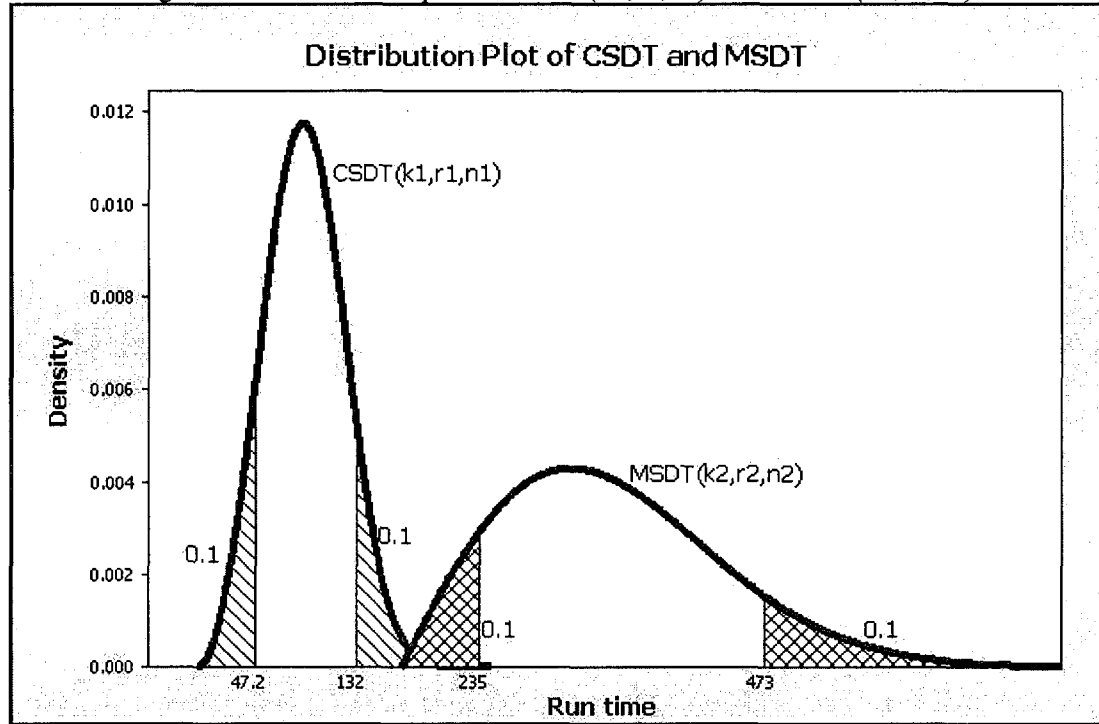
Assume for the arbitrary k_1, r_1, n_1 for CSDT and k_2, r_2, n_2 for the MSDT, the 10% and 90% quantile values for the run time are given in Table 4.

Table 4: 10% and 90% quantile values for CSDT and MSDT.

Quantile	CSDT(k_1, r_1, n_1)	MSDT(k_2, r_2, n_2)
10%	47.2	235
90%	132	473

So, we will accept the null hypothesis if and only if there is no overlapping between these regions. See Figure 36.

Figure 36: Distribution plot of CSDT(k_1, r_1, n_1) and MSDT(k_2, r_2, n_2).



If one would like to know the second type error value for this specific example, then it might be calculated by finding the area 132 from the MSDT(k_2, r_2, n_2) distribution. (which is approximately zero).

4.1.1.3 Run time comparison of CSDT and MSDT

The following assumptions will be made:

- Number of testing facilities available is 50 ($N=50$).
- Number of results required is 20 ($R=20$).
- The shape parameter is increasing by 0.2 increments, starting from 1 until 6.
- The scale parameter is 75,000.

Based on above mentioned assumptions on the number of testing facilities and number of results required, the following testing configurations could be formed to perform CSDT or MSDT, respectively (see Table 5 and Table 6).

Table 5: CSDT possible configurations.

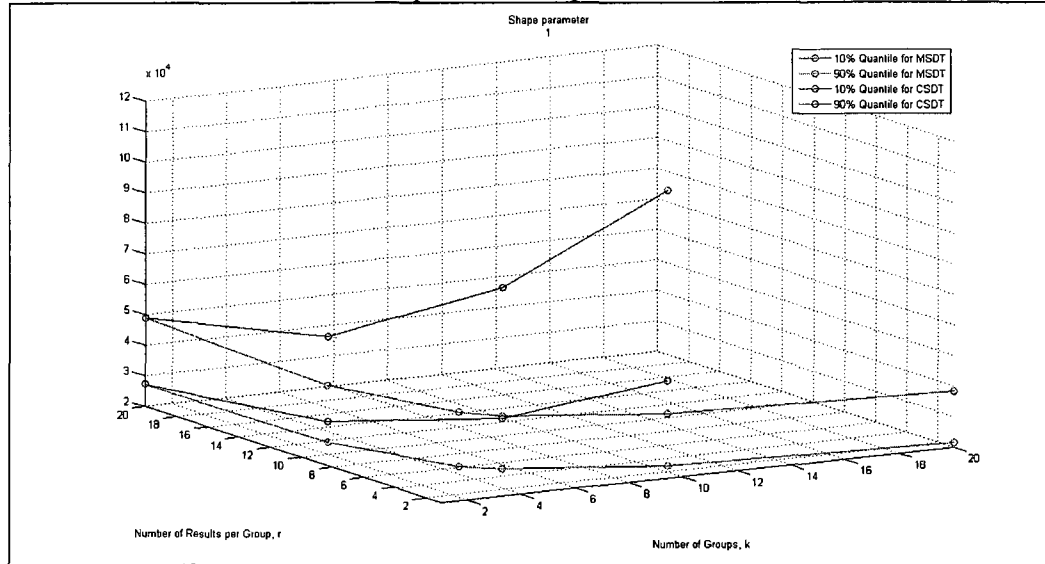
Number of groups (k)	Number of components per group (n)	Number of results per group (r)	Total results Collected from the test (R)	Total components used in the test
1	50	20	20	50
2	25	10	20	50
5	10	4	20	50
10	5	2	20	50

Table 6: MSDT possible configurations.

Number of groups (k)	Number of components per group (n)	Number of results per group (r)	Total results Collected from the test (R)	Total components used in the test
1	50	20	20	50
2	50	10	20	100
4	50	5	20	200
5	50	4	20	250
10	50	2	20	500
20	50	1	20	1000

For a shape parameter increased by 0.2 increments, starting from 1 to 6, and a scale parameter of 75,000, for each CSDT configuration with 10% and 90% quantile times, the approximate 10% and 90% quantile times for each MSDT configuration were computed. (See Figure 37-Figure 62).

Figure 37: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 1.



By increasing number of groups in a way so that the number of collected results are the same from each group and the total collected results from the tests is 20, the 10% and 90% quantiles of CSDT tends to increase when shape parameter is one (see Figure 37). The lowest 10% and 90% quantiles are observed when number of groups is one.

By increasing number of groups in a way so that the number of collected results are the same from each group and the total collected results from the tests is 20, the 10% and 90% quantiles of MSDT tends to decrease when shape parameter is one (see Figure 37). The lowest 10% and 90% quantiles are observed when number of groups is twenty and number of results collected from each group is one (Sudden Death Test). As far as this is a special case of MSDT when $r=1$, then the results from the Jun and Balamurali's work¹⁰ might be applied to calculate and compare the exact 10% and 90% quantiles (also, this work is summarized in literature review section).

¹⁰ Jun, C., & Balamurali, S. (2006). Variables sampling plans for Weibull distributed lifetimes under sudden death testing. *IEEE Transactions on Reliability*, 55(1), 53.

Table 7: For the MSDT($k=20, r=1, n=50$) test mode, the 10% and 90% quantiles exact and suggested approximation method values.

MSDT($k=20, n=50, r=1$)		
Quantiles	10%	90%
Exact Quantile Values (Jun and Balamurali)	21,788	38,854
Suggested Approximation Method Quantile Values (derived in Section 3.2.3)	21,583	38,490
Suggested Approximation Method Quantile Values deviation from the Exact Quantile Values (in percentage, about)	0.9%	0.9%

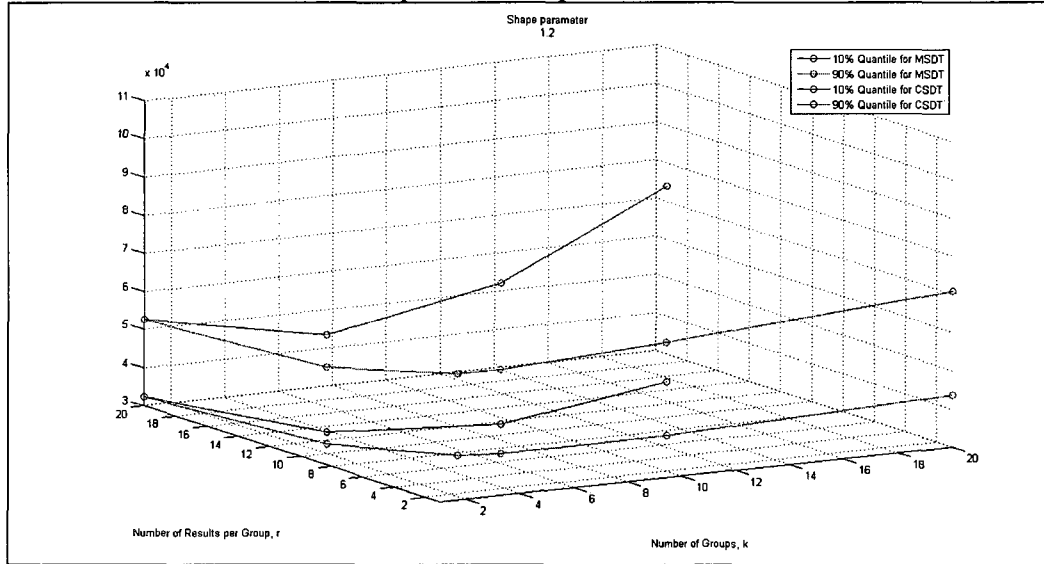
Table 8: For the CSDT($k=1, r=20, n=50$) test mode, the 10% and 90% quantiles exact values based on derived method.

CSDT($k=1, n=50, r=20$)		
Quantiles	10%	90%
Exact Quantile Values (derived in Section 3.2.2)	27,375	49,098

It was noticed that for the MSDT ($k=20, r=1, n=50$) test mode, the suggested approximation method 10% and 90% quantile values deviate from the exact 10% and 90% quantile values by no more than 1% (see Table 7).

Also, it might be concluded that MSDT($k=20, n=50, r=1$) test mode completion time and CSDT($k=1, n=50, r=20$) test mode completion time are not significantly different at 80% confidence.

Figure 38: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 1.2.

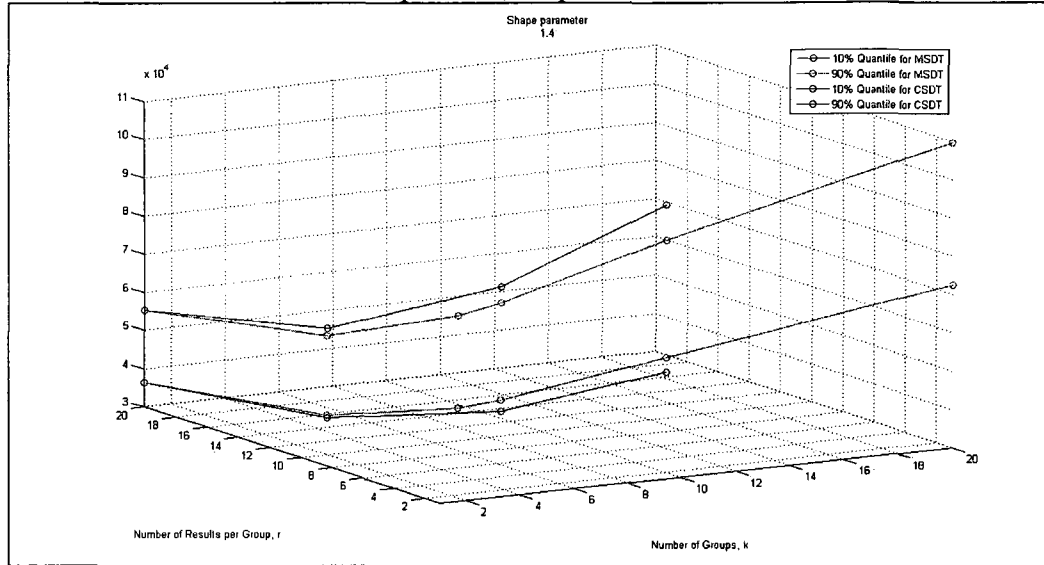


By increasing number of groups in a way so that the number of collected results are the same from each group and the total collected results from the tests is 20, the 10% and 90% quantiles of CSDT tends to increase when shape parameter is 1.2 (see Figure 38). The lowest 10% and 90% quartiles are observed when the number of groups is one.

By increasing the number of groups in a way so that the number of collected results are the same from each group and the total collected results from the tests is 20, the 10% and 90% quantiles of MSDT tends to increase when the shape parameter is 1.2 (see Figure 38). The lowest 10% and 90% quartiles are observed when number of groups is one.

For both MSDT and CSDT testing modes, when the number of groups is one it implies that the testing mode is a Classical Test (CT). So when the shape parameter is 1.2, the classical test is preferable.

Figure 39: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 1.4.



By increasing number of groups in a way so that the number of collected results are the same from each group and the total collected results from the tests is 20, the 10% and 90% quantiles of CSDT tends to increase when shape parameter is 1.4 (see Figure 39). The lowest 10% and 90% quartiles are observed when number of groups is one.

By increasing number of groups in a way so that the number of collected results are the same from each group and the total collected results from the tests is 20, the 10% and 90% quantiles of MSDT tends to increase when shape parameter is 1.4 (see Figure 39). The lowest 10% and 90% quartiles are observed when number of groups is one.

For both MSDT and CSDT testing modes, when the number of groups is one it implies that the testing mode is a Classical Test (CT). So when the shape parameter is 1.4, the classical test is preferable. The same patten was observed for the rest of the comparisons, when the shape parameter is increasing until 6 (see Figure 40 - Figure 62).

Figure 40: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 1.6.

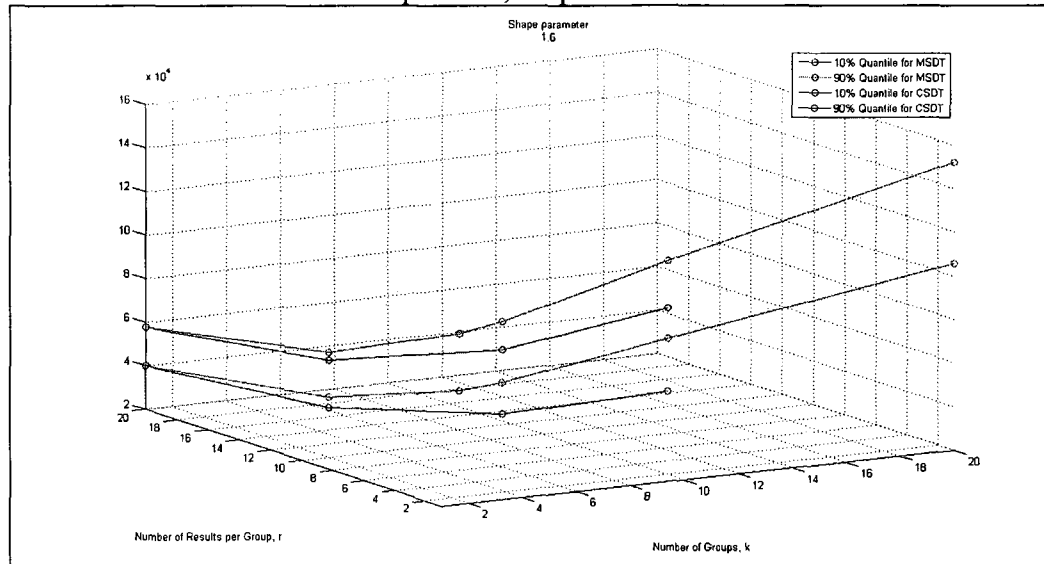


Figure 41: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 1.8.

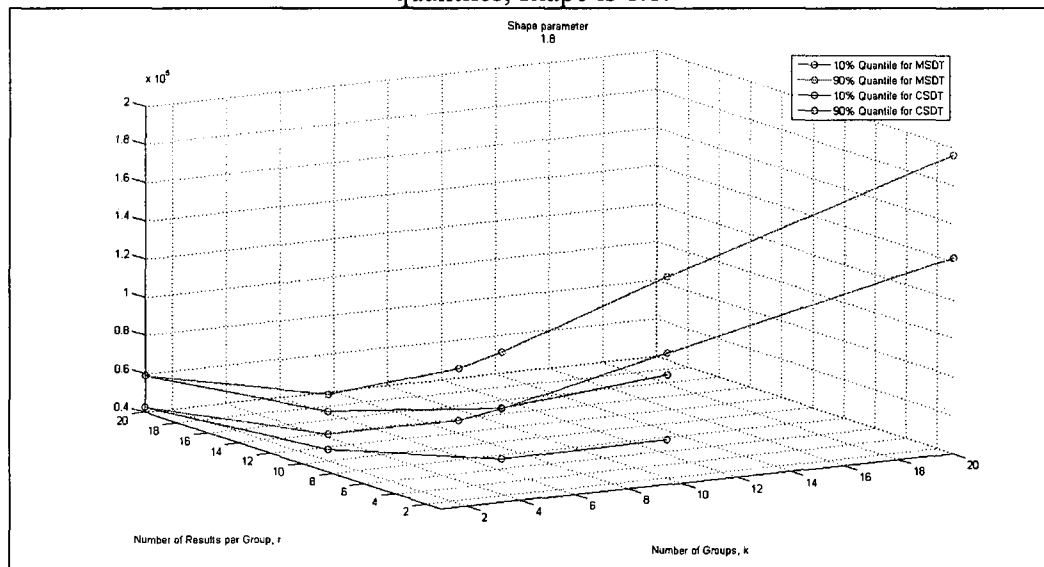


Figure 42: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 2.

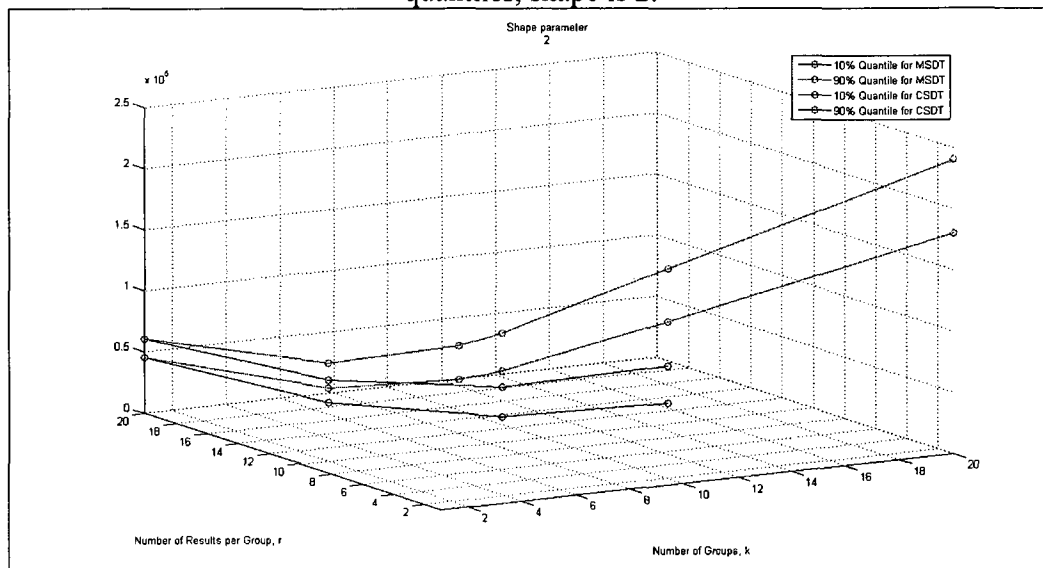


Figure 43: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 2.2.

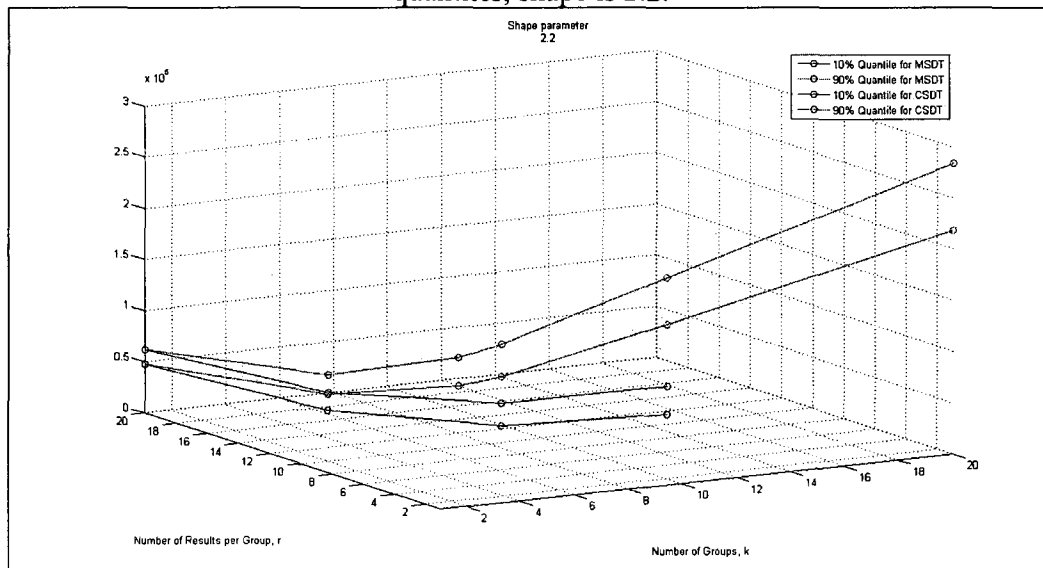


Figure 44: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 2.4.

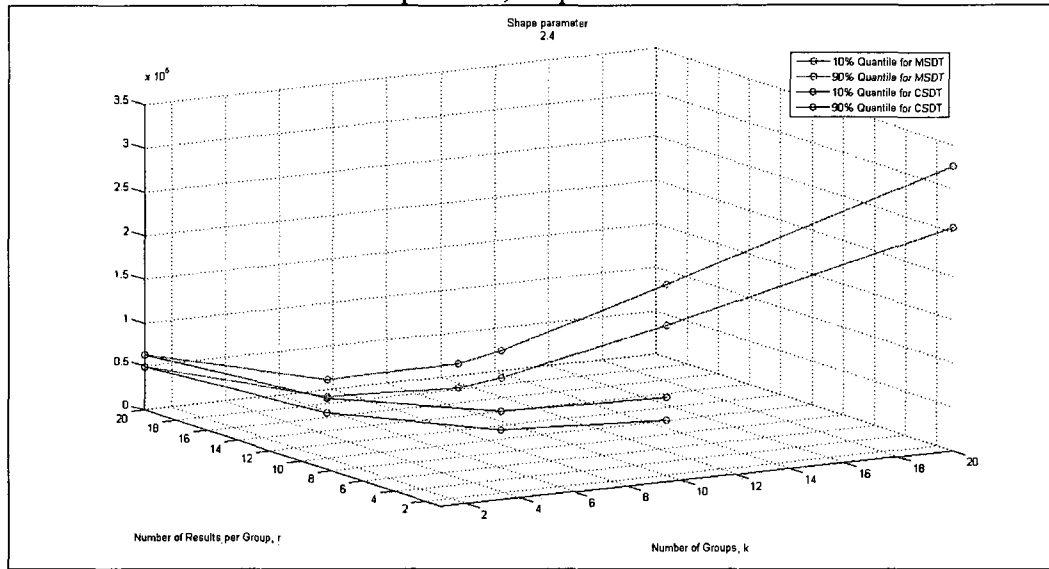


Figure 45: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 2.6.

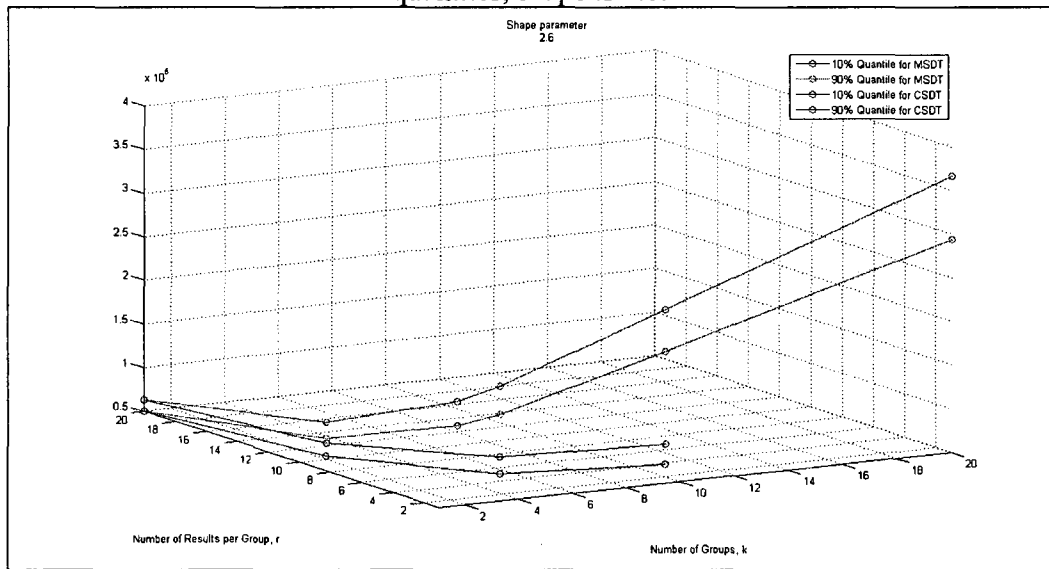


Figure 46: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 2.8.

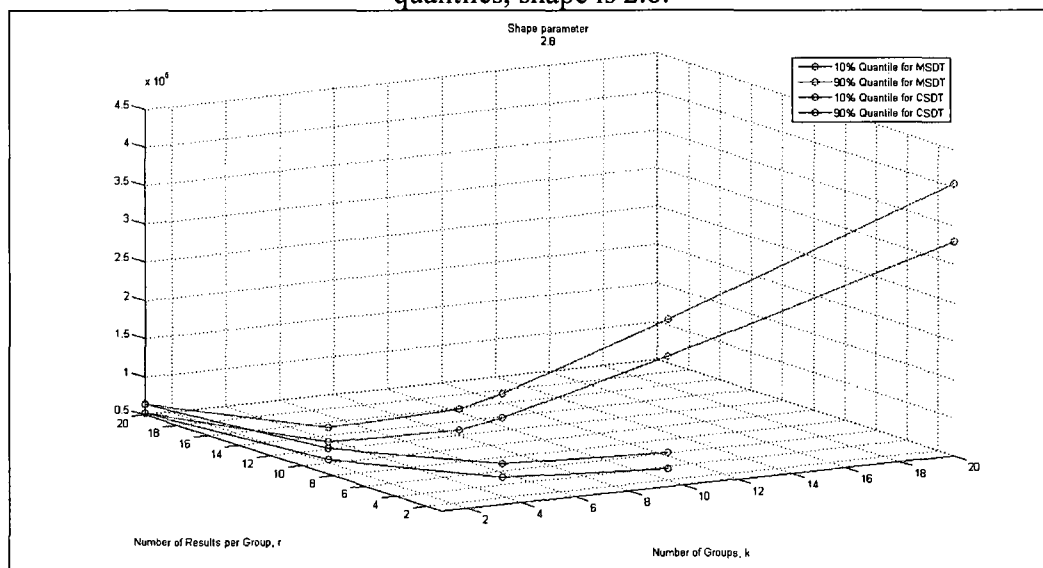


Figure 47: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 3.

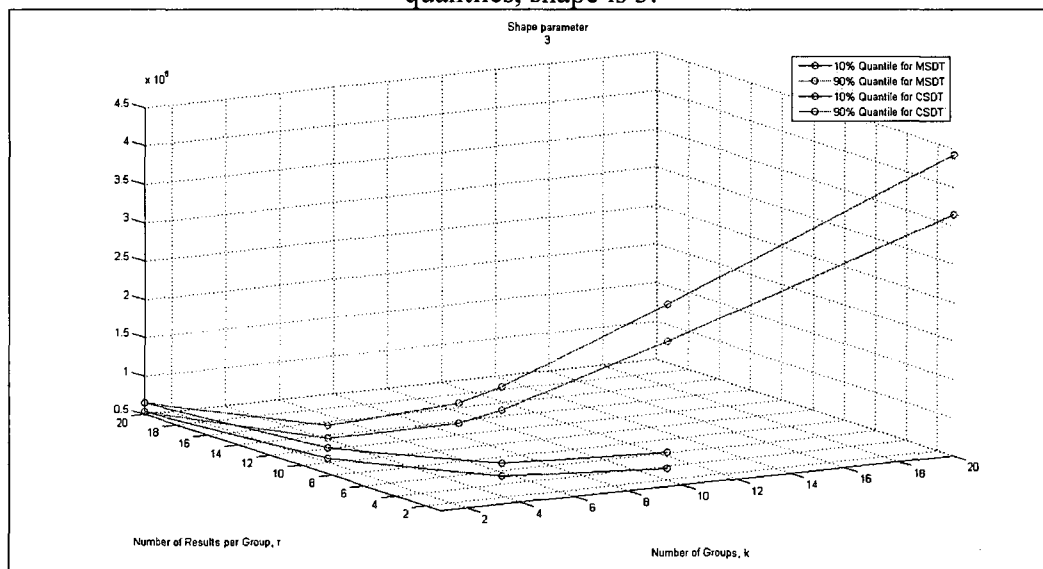


Figure 48: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 3.2.

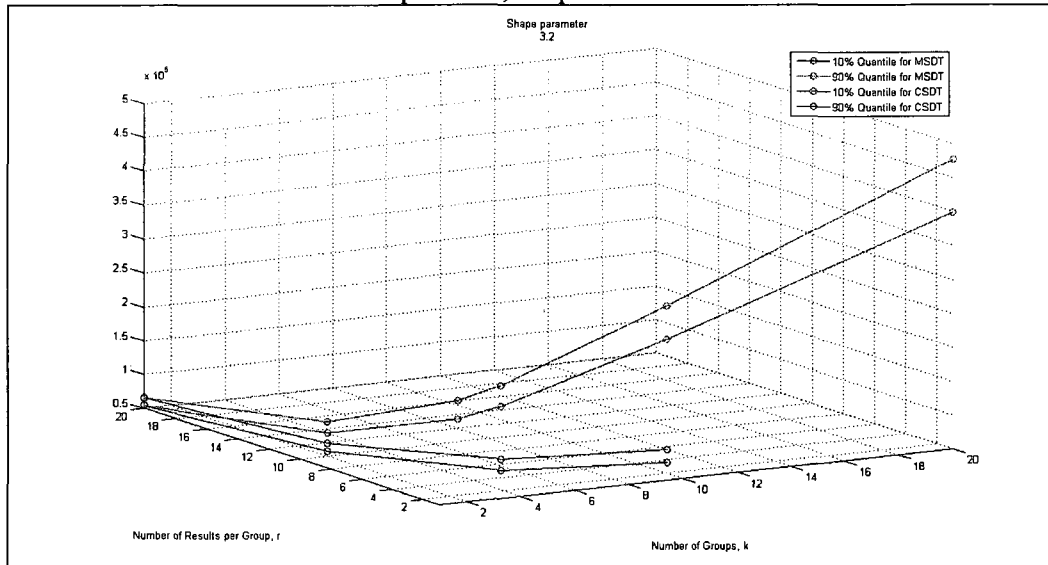


Figure 49: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 3.4.

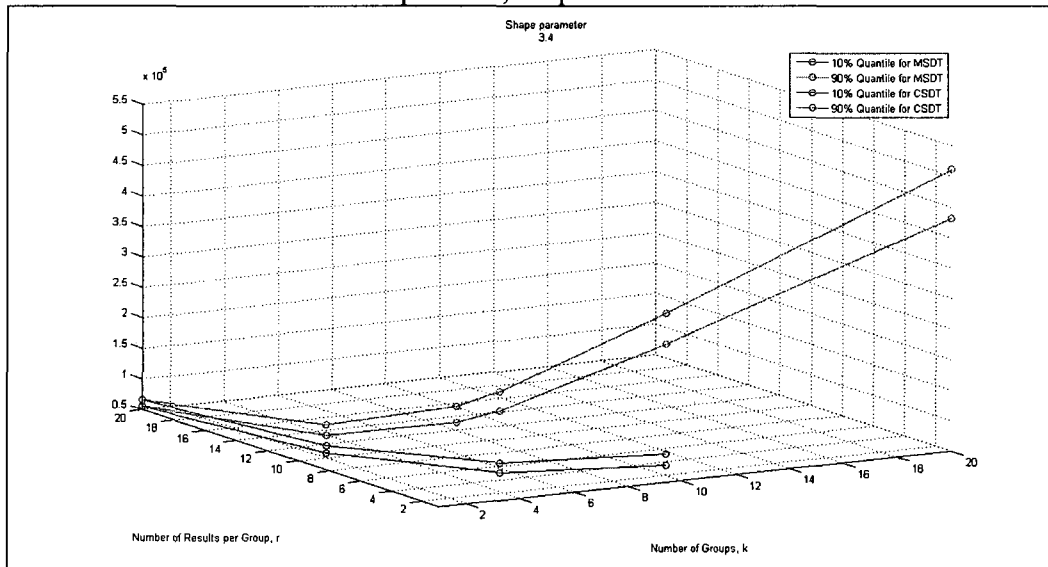


Figure 50: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 3.6.

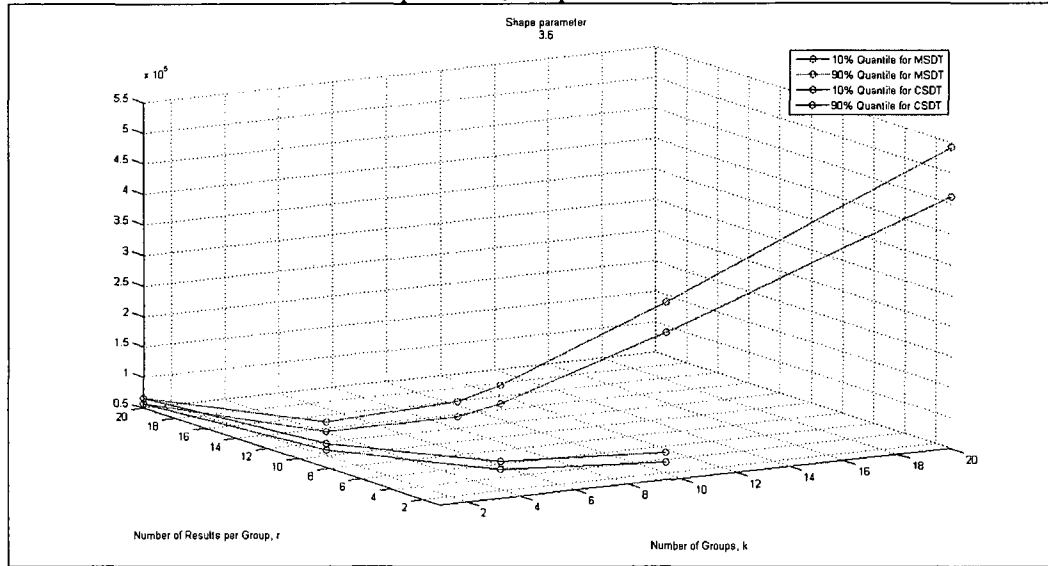


Figure 51: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 3.8.

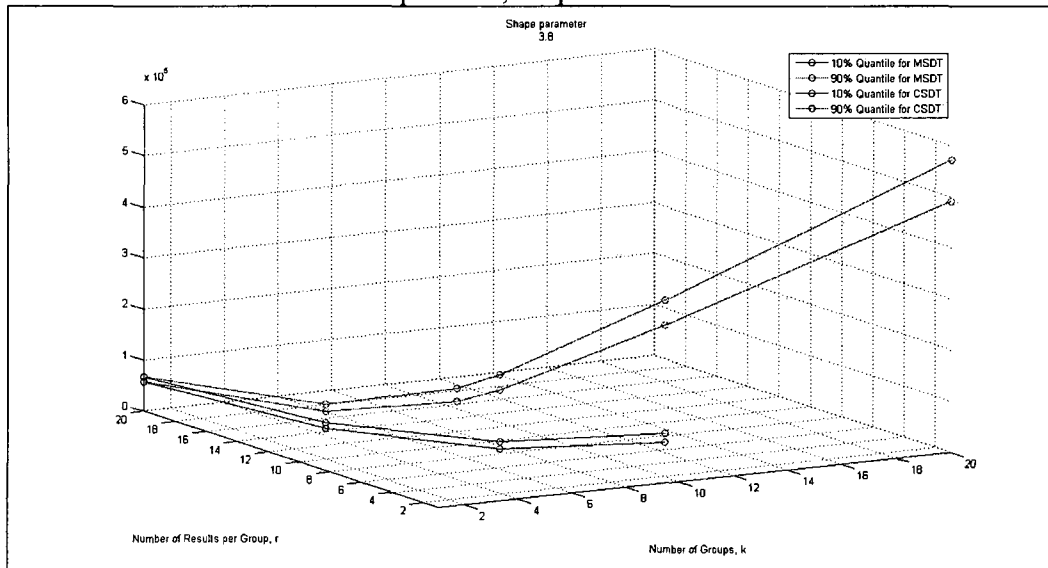


Figure 52: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 4.

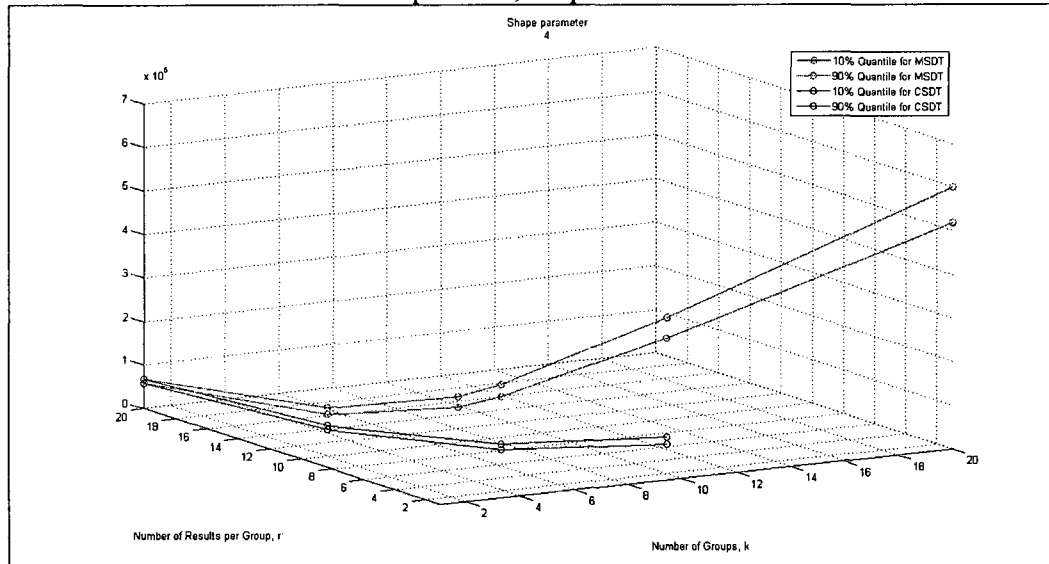


Figure 53: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 4.2.

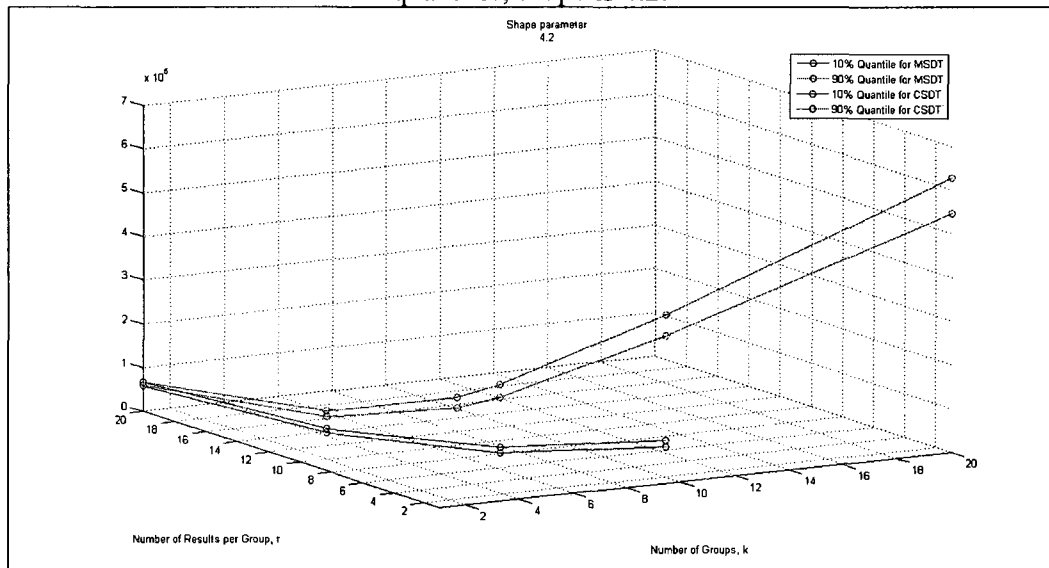


Figure 54: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 4.4.

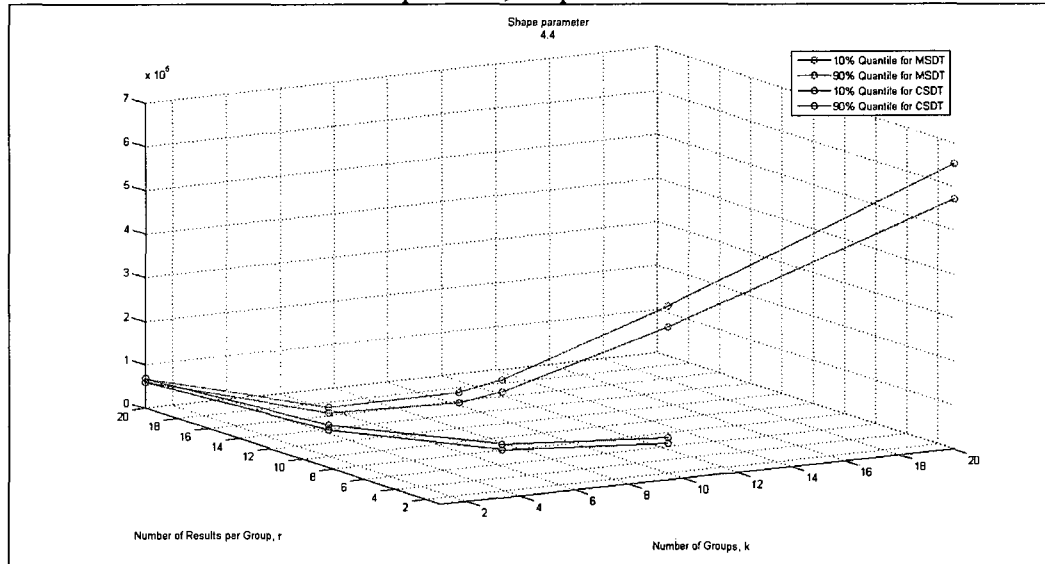


Figure 55: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 4.6.

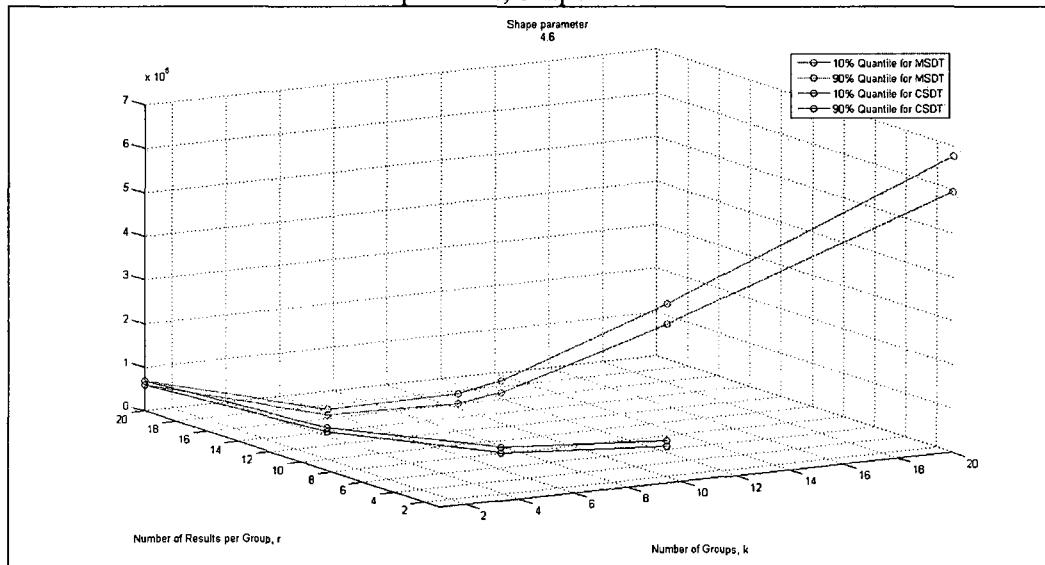


Figure 56: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 4.8.

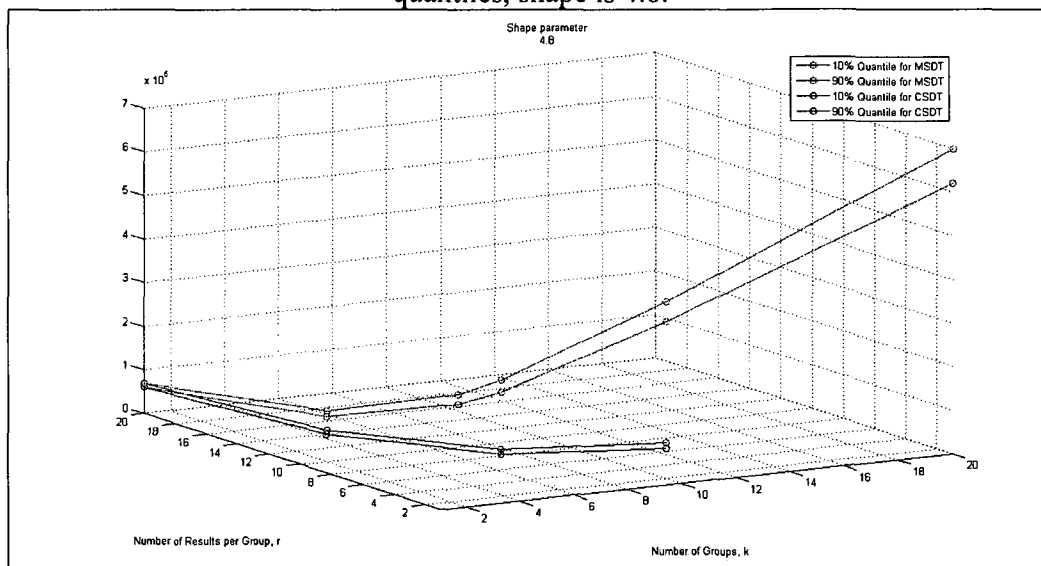


Figure 57: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 5.

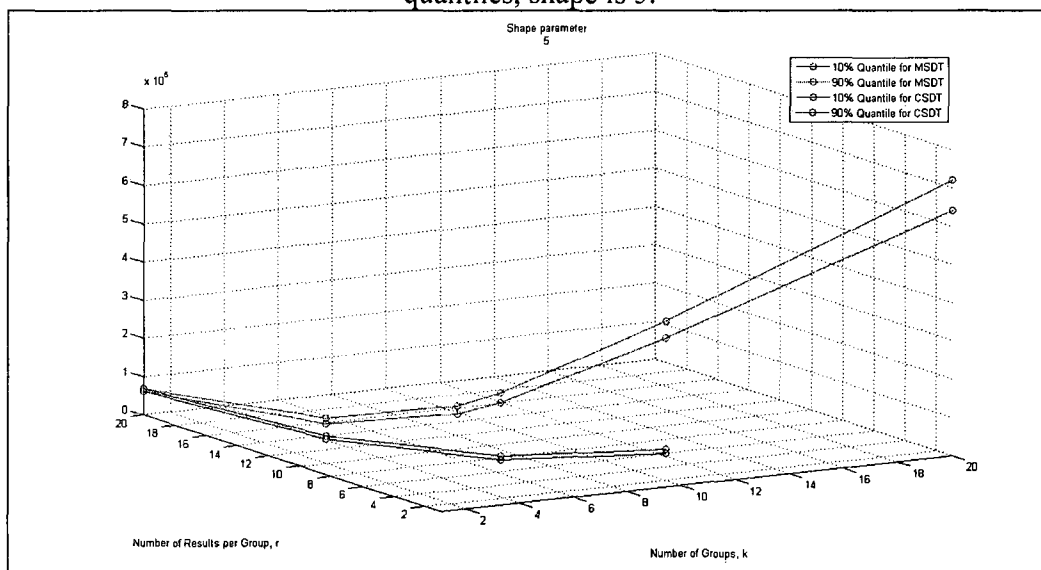


Figure 58: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 5.2.

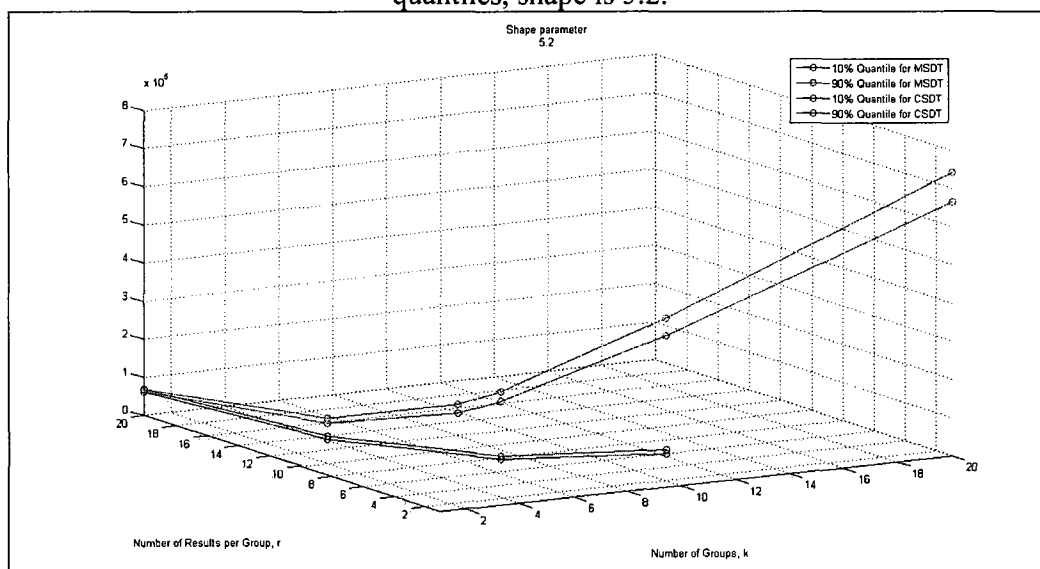


Figure 59: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 5.4.

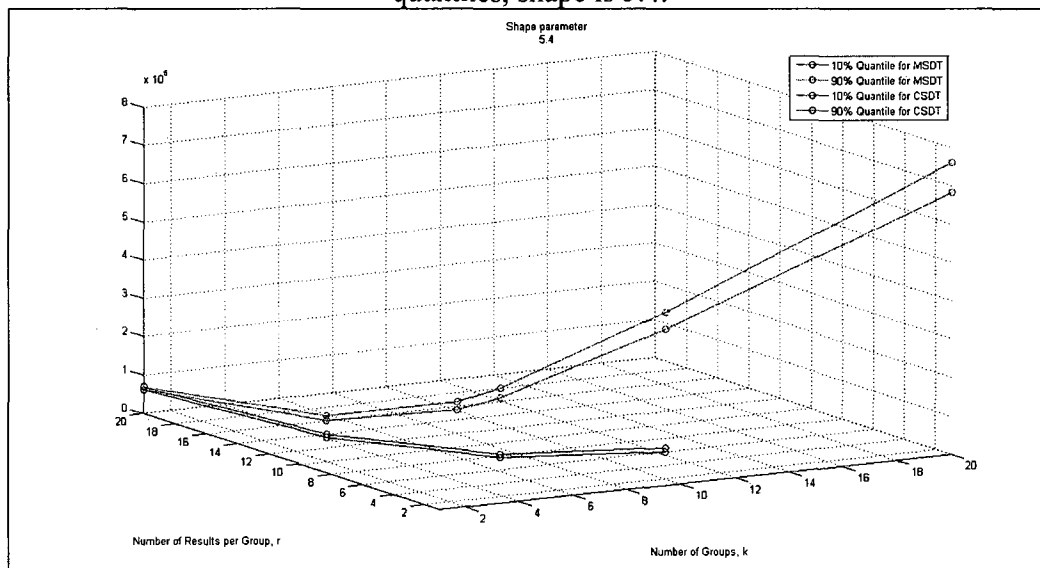


Figure 60: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 5.6.

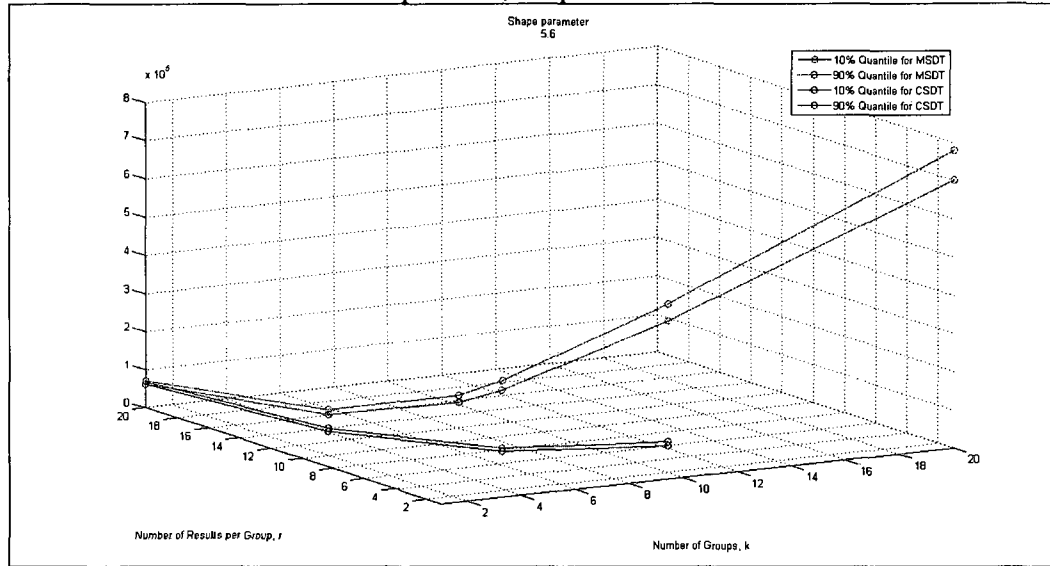


Figure 61: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 5.8.

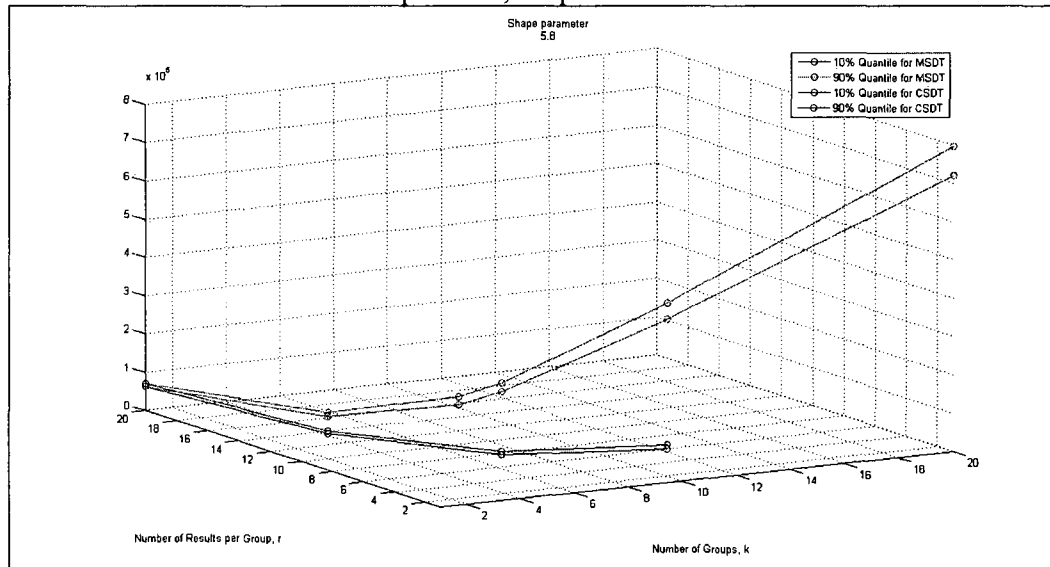
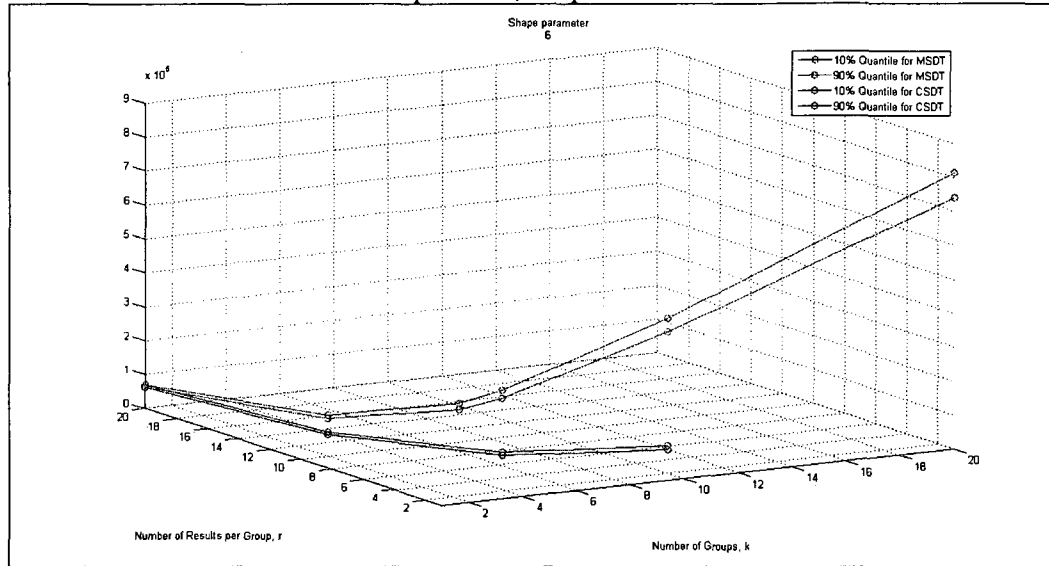


Figure 62: For the all possible configurations of MSDT and CSDT, 10% and 90% quantiles, shape is 6.



Under the assumption that:

- Number of testing facilities available is 50 ($N=50$).
- Number of results required is 20 ($R=20$).
- The scale parameter is 75,000.

When shape parameter is 2.4 or more and number of groups is different than one, by 80% of confidence might be claimed that any possible combination of parallel mode test run time $CSDT(k_1, n_1, r_1)$ is shorter than any possible combination of series mode test run time $MSDT(k_2, n_2, r_2)$, see Figure 44 - Figure 62.

4.1.1.4 Conclusion

In Section 2.1 it was mentioned that the failure of mechanical components is well explained by Weibull distribution, with a shape parameter greater than unity. This was the main argument and reason to make this assumption in the early stages of the research.

When the shape parameter is one, breaking into the groups affects on run time. That has the pattern to decrease the series tests (MSDT) and increase the parallel mode tests (CSDT) run times.

When the shape parameter is greater than one, breaking into the groups affects both run times of the tests (MSDT and CSDT) and has an increasing pattern.

The best testing strategy exists in the grouping combinations. For example, with the first type error of 20% and with shape parameter of 2.2, the testing combinations CSDT($k=5$, $r=4$, $n=10$) and CSDT($k=10$, $r=2$, $n=5$) are better than MSDT($k=4$, $r=5$, $n=50$), MSDT($k=5$, $r=4$, $n=50$), MSDT($k=10$, $r=2$, $n=50$) and MSDT($k=20$, $r=1$, $n=50$), see Figure 43.

The CSDT test mode expected run time tends to increase as well as its variance by the increase of the shape parameter starting from one. Among the all possible values of the shape parameter (starting from one to six) and the all possible CSDT mode configurations, the shortest expected run time was observed when the number of groups is equal to one. So, this is a special case of CSDT testing strategy, which is a Classical Test.

The MSDT test mode expected run time tends to increase as well as its variance by the increase of the shape parameter starting but not equal to one. Among the all possible values of the shape parameter (starting but not equal to one till six) and the all possible MSDT mode configurations, the shortest expected run time was observed when the

number of groups is equal to one. So, this is a special case of MSDT testing strategy, which is a Classical Test.

Generally, if the shape parameter is greater than unity, for either testing mode the shortest expected run time was observed for the testing strategies where the number of groups is one. So, this testing strategy becomes a Classical Test.

From the run time prospective the Sudden Death test is not reasonable if the shape parameter is greater than unity. As far as expected run time and variance of Sudden Death tend to increase.

4.1.2 Total Accumulated Time Comparison of CSDT and MSDT

The following assumptions will be made:

- Number of testing facilities available is 50 ($N=50$).
- Number of results required is 20 ($R=20$).
- The shape parameter is increasing by 0.2 increments, starting from 1 until 3.
- The shape parameter is increasing by 1 increments, starting from 4 until 6.
- The scale parameter is 75,000.
- Number of runs for each testing combination is 10,000.
- 10%, 50% and 90% quantiles are of interest.

Based on the assumptions on the number of testing facilities and the number of results required, the following testing configurations could be formed to perform CSDT or MSDT, respectively (see Table 9 and Table 10).

Table 9: CSDT possible configurations.

Number of groups (k)	Number of components per group (n)	Number of results per group (r)	Total results Collected from the test (R)	Total components used in the test
1	50	20	20	50
2	25	10	20	50
5	10	4	20	50
10	5	2	20	50

Table 10: MSDT possible configurations.

Number of groups (k)	Number of components per group (n)	Number of results per group (r)	Total results Collected from the test (R)	Total components used in the test
1	50	20	20	50
2	50	10	20	100
4	50	5	20	200
5	50	4	20	250
10	50	2	20	500
20	50	1	20	1000

Figure 63: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 1.

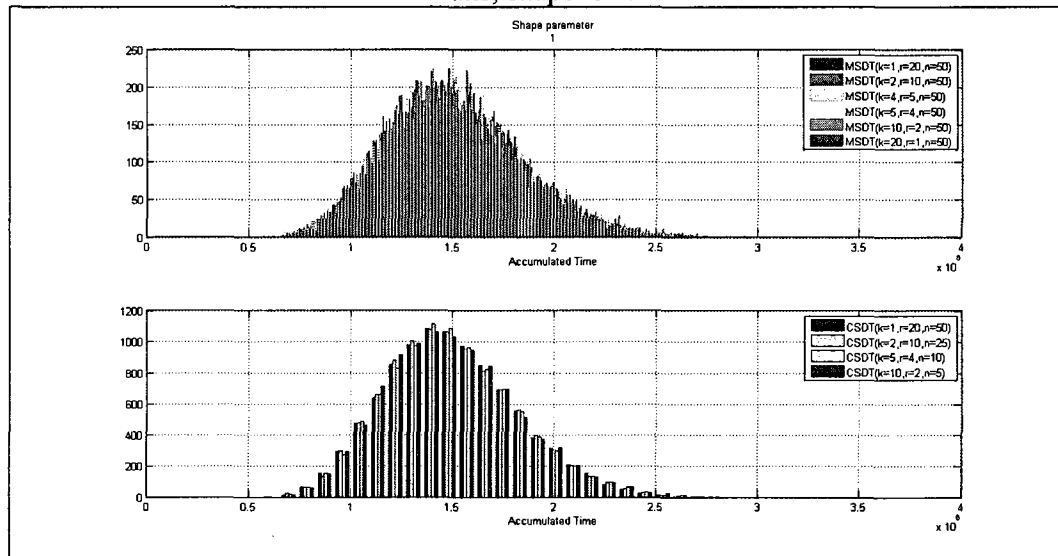
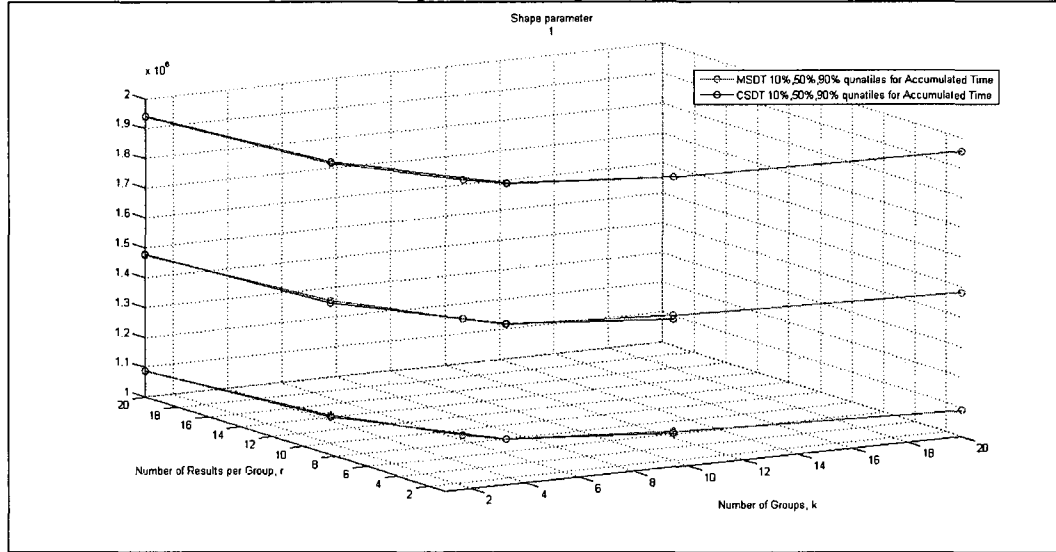


Figure 64: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 1.



As we have noticed, there is no change in total accumulated time when shape parameter is one for any MSDT and CSDT possible configurations. This result is also noticeable from Equation 87, as far as the total accumulated time for either MSDT or CSDT depends on the number of the results (R). Based on the definitions of MSDT and CSDT the number of results (R) is the same for either testing strategy, so the expected total accumulated time will not change.

Figure 65: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 1.2.

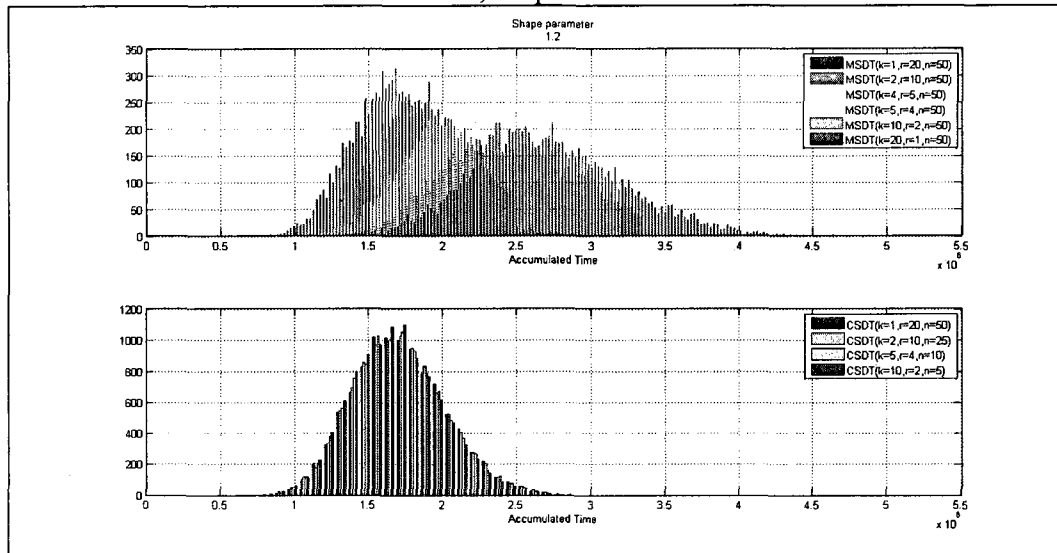
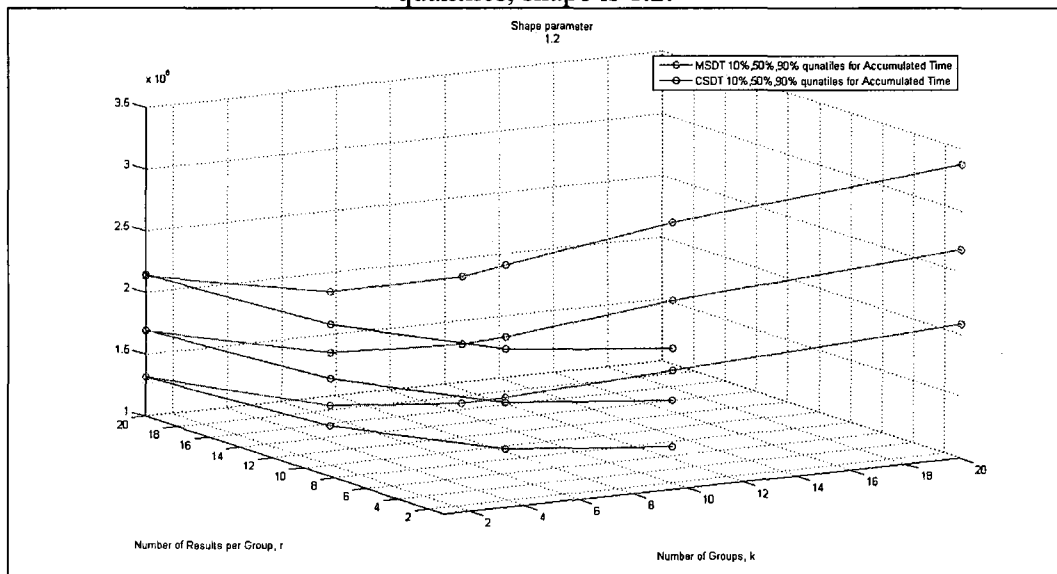


Figure 66: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 1.2.



When the shape parameter is 1.2, the increase in number of groups under MSDT strategy affects in increase of the total accumulated time. Nevertheless, dividing into the groups for the CSDT strategy does not have any significant effect on the total accumulated time.

Figure 67: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 1.4.

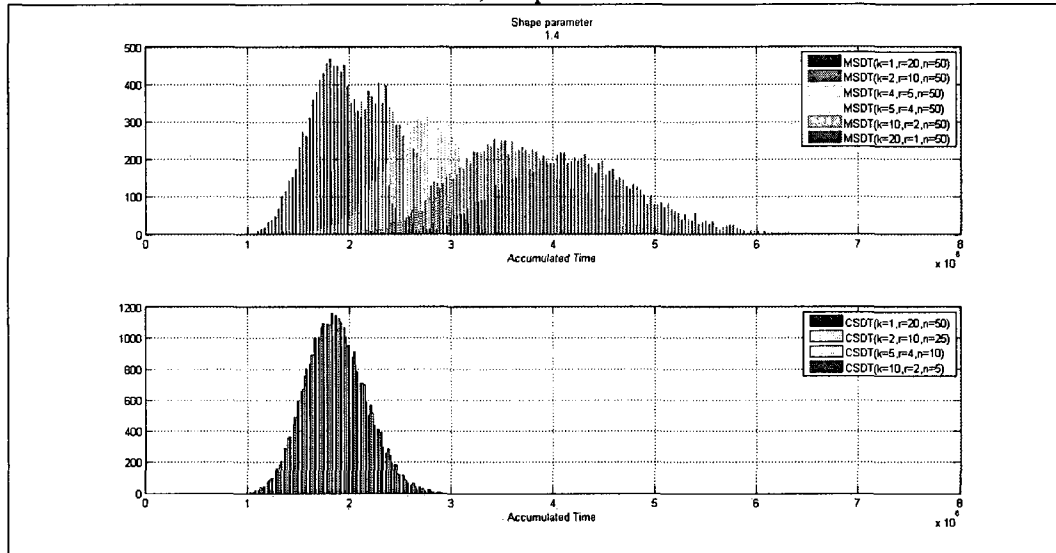
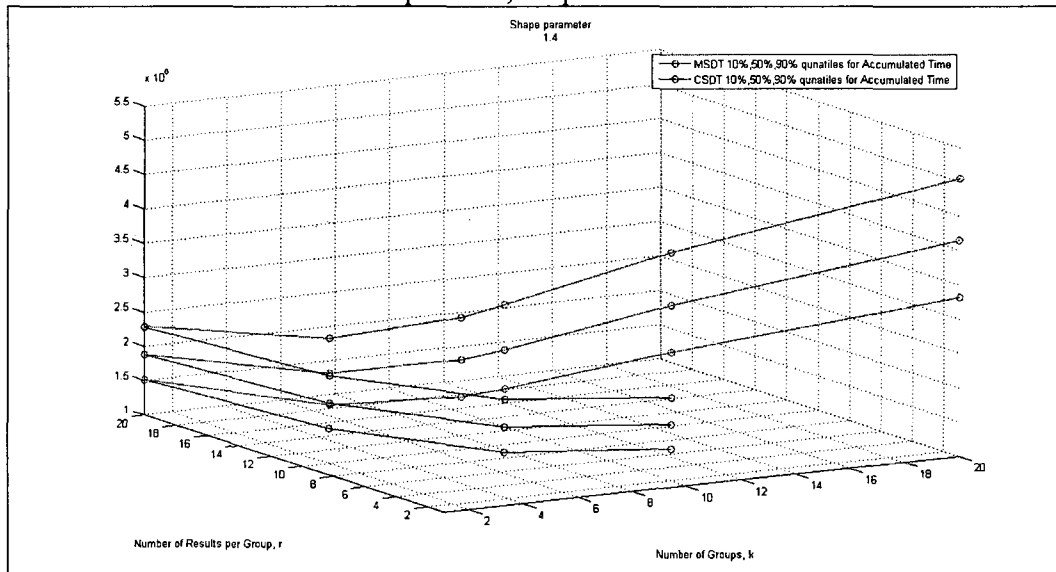


Figure 68: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 1.4.



It was noticed that the increase in shape parameter from 1.2 to 1.4 affects in increase of both the centering and variability of the total accumulated time under MSDT strategy. When shape parameter in 1.4, the total accumulated time within any possible CSDT configuration strategy doesn't change significantly. But the shape parameter increase from 1.2 to 1.4 increases the centering of the total accumulated time.

Figure 69: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 1.6.

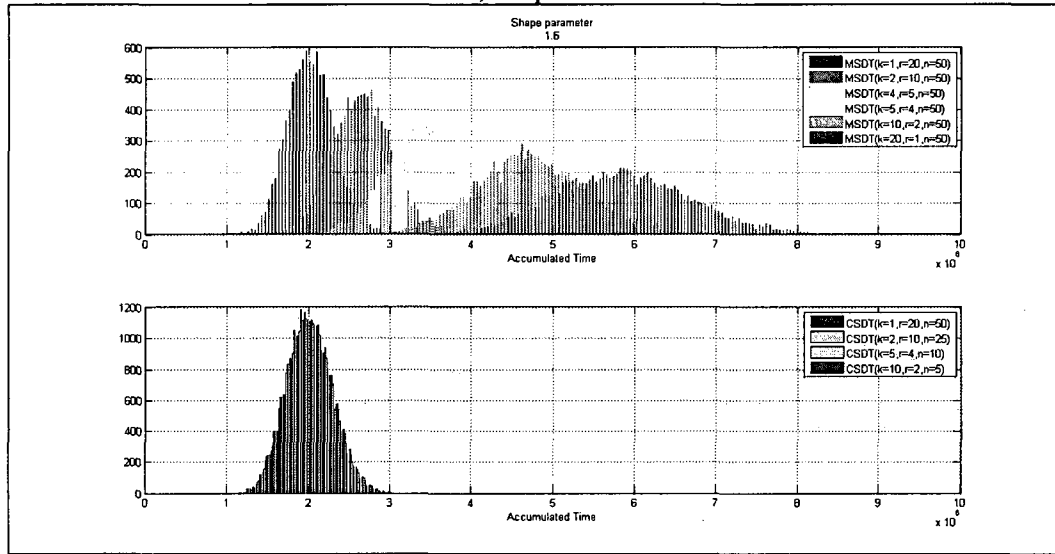


Figure 70: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 1.6.

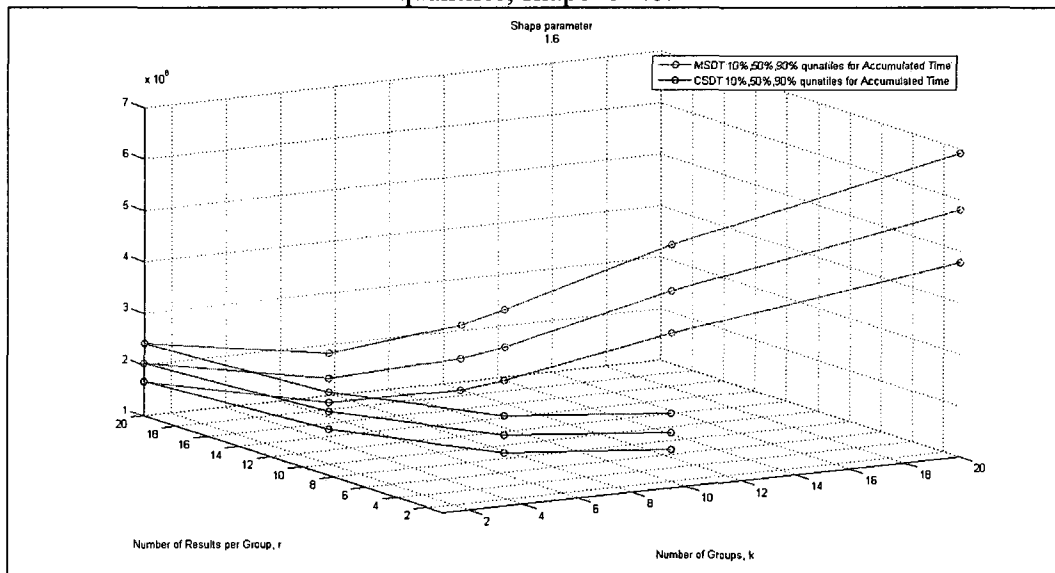


Figure 71: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 1.8.

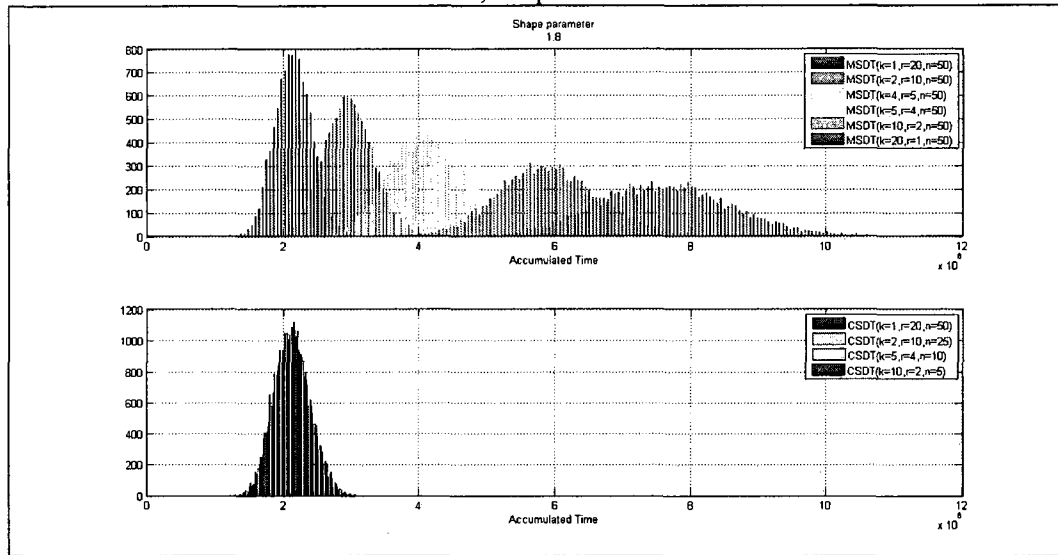


Figure 72: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 1.8.

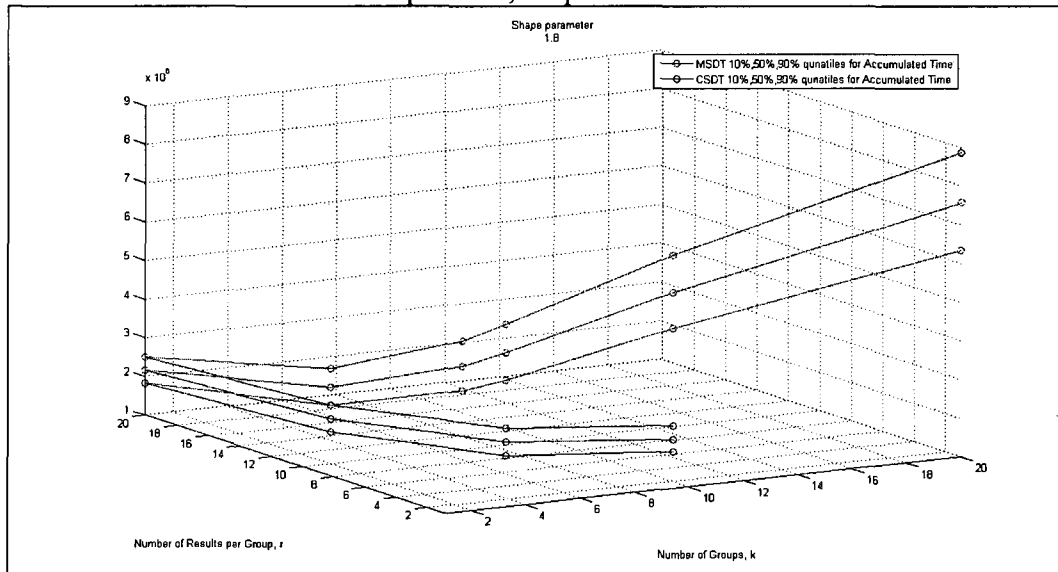


Figure 73: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 2.

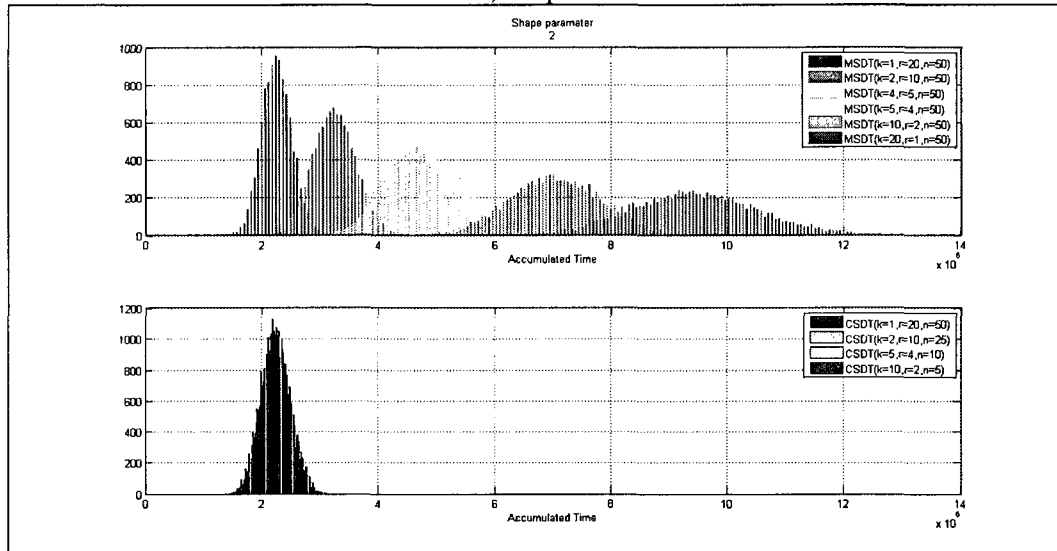


Figure 74: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 2.

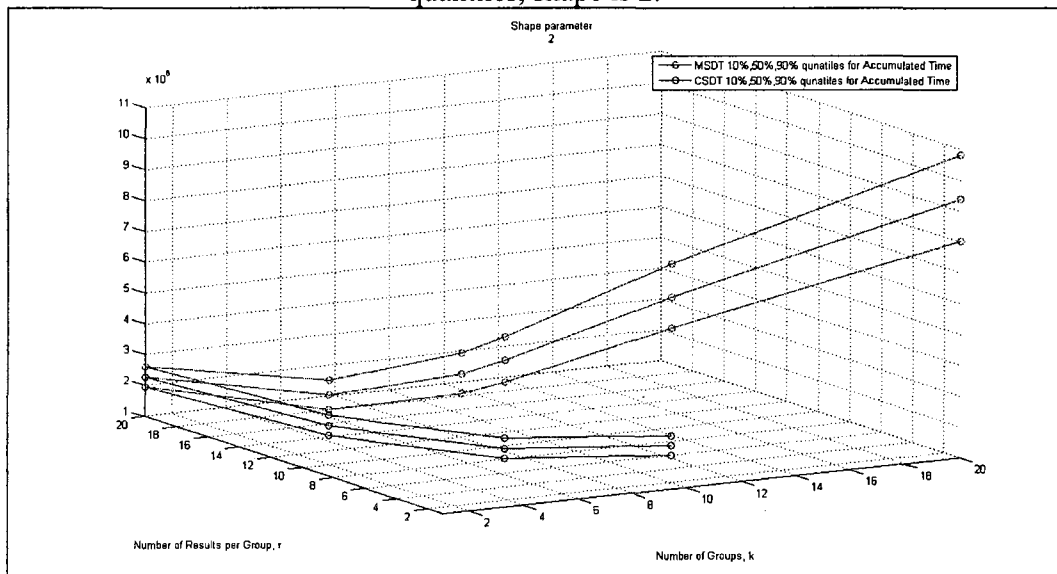


Figure 75: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 2.2.

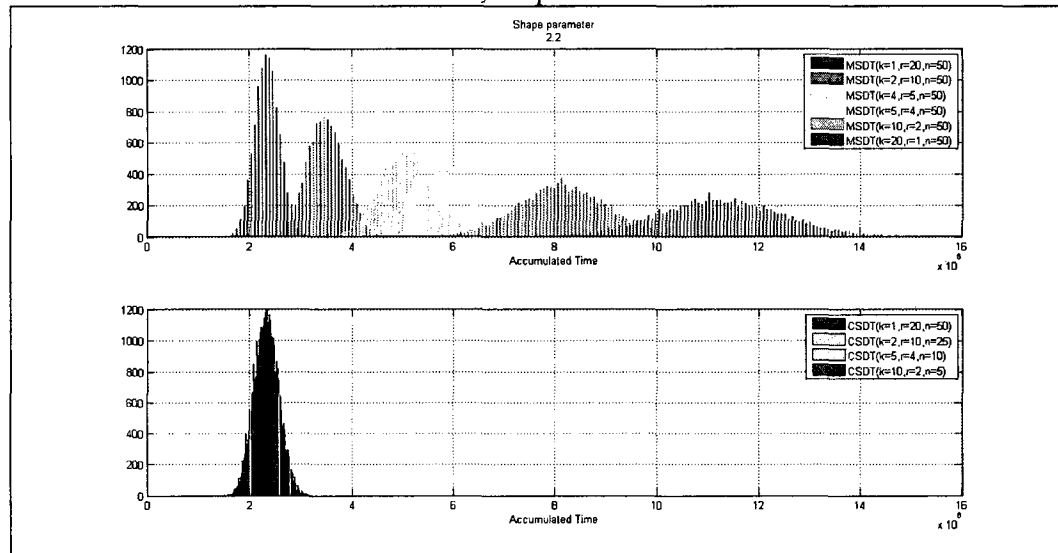


Figure 76: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 2.2.

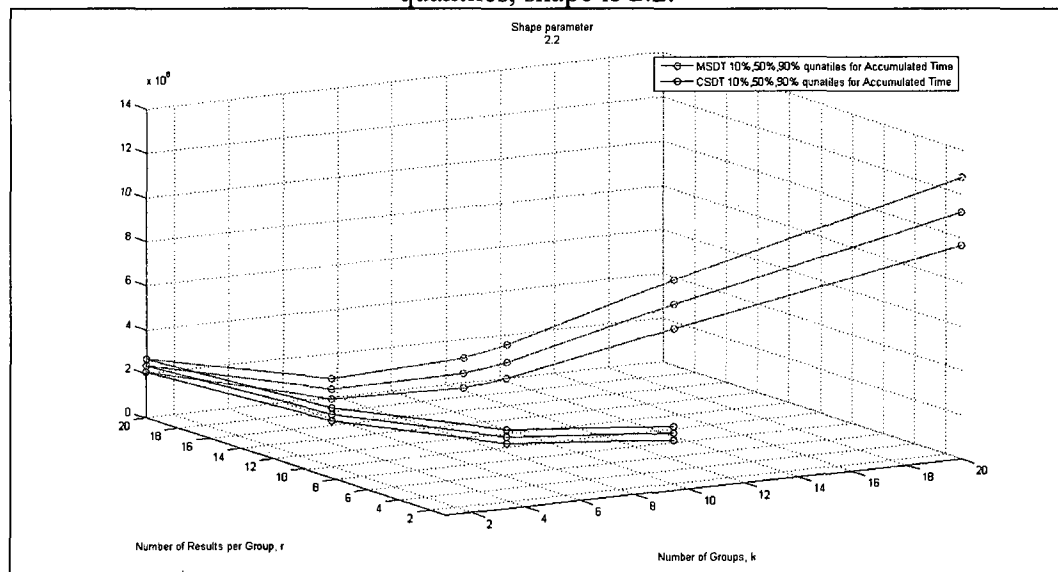


Figure 77: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 2.4.

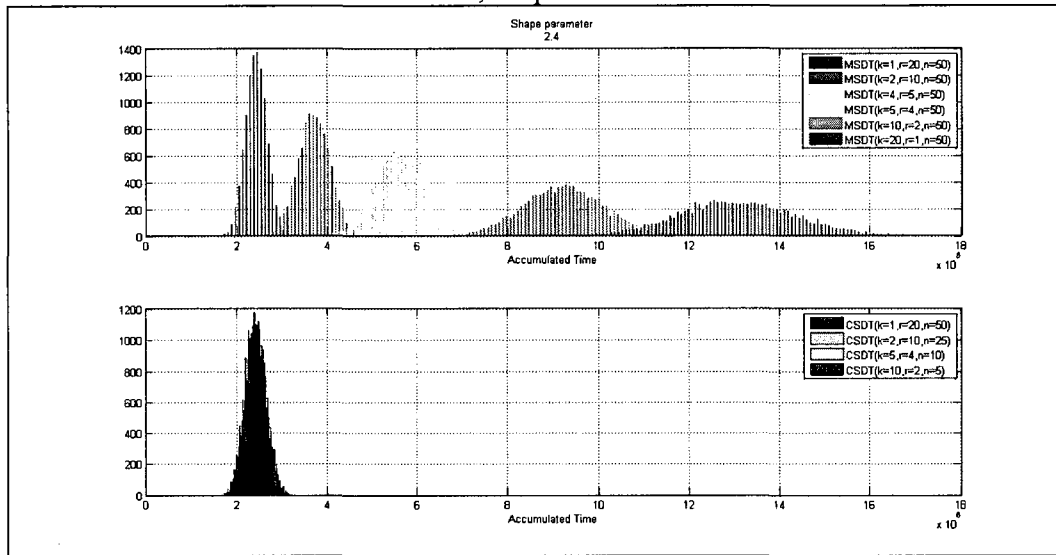


Figure 78: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 2.4.

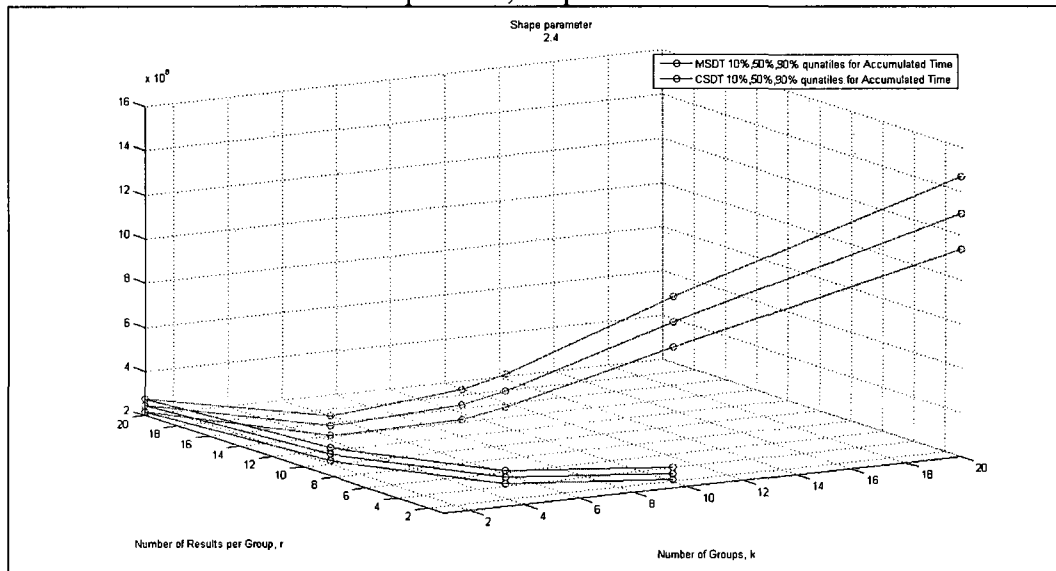


Figure 79: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 2.6.

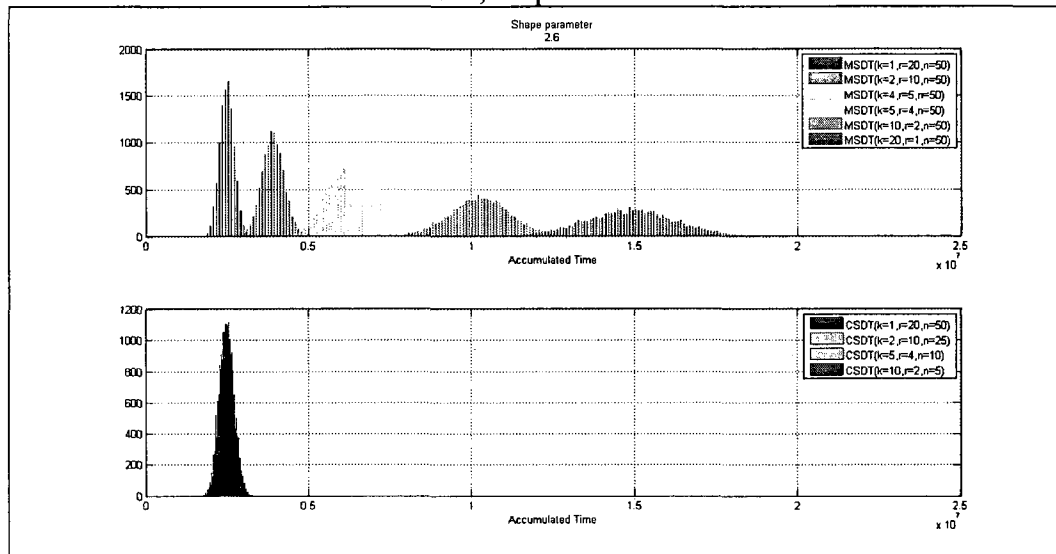


Figure 80: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 2.6.

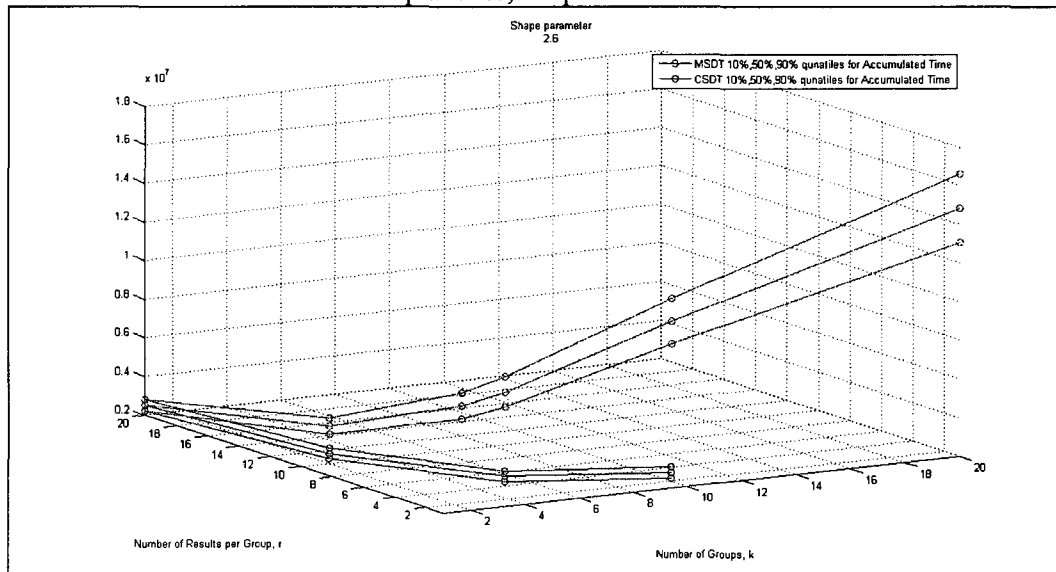


Figure 81: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 2.8.

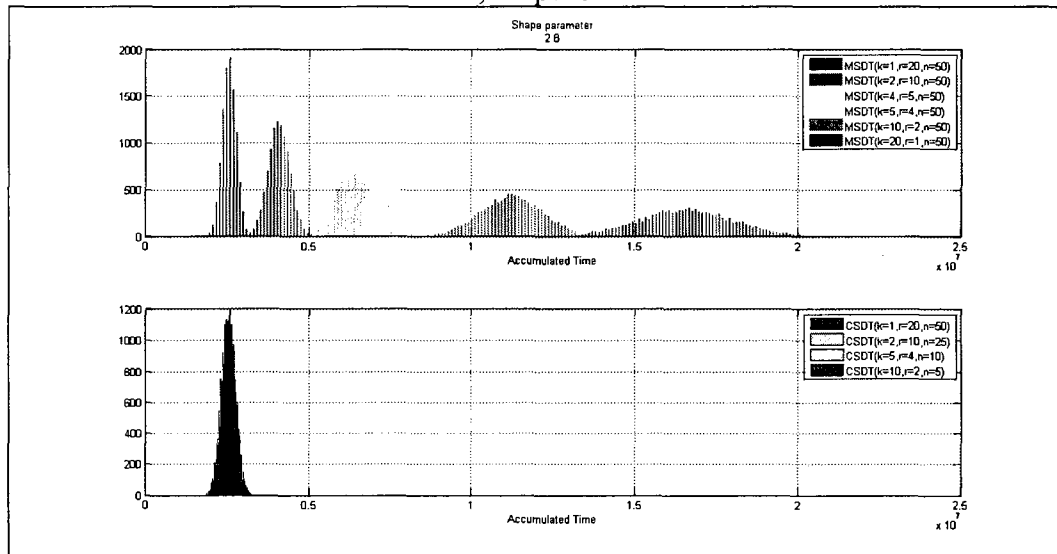


Figure 82: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 2.8.

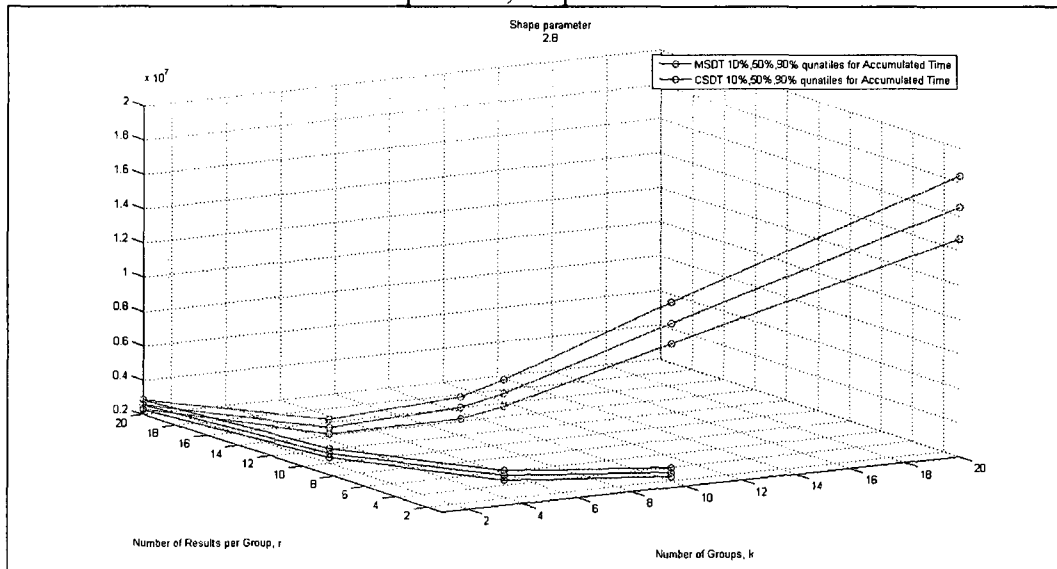


Figure 83: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 3.

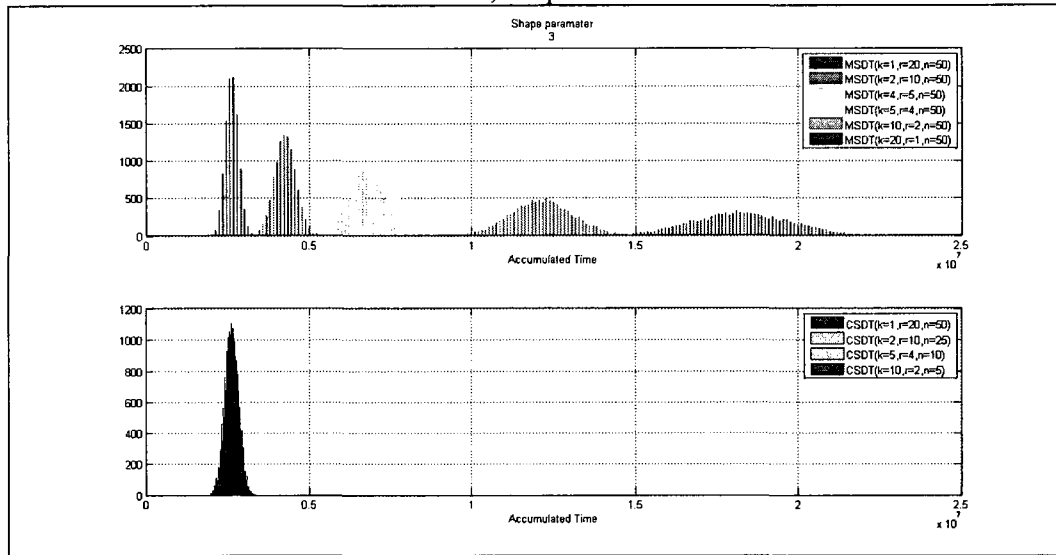


Figure 84: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 3.

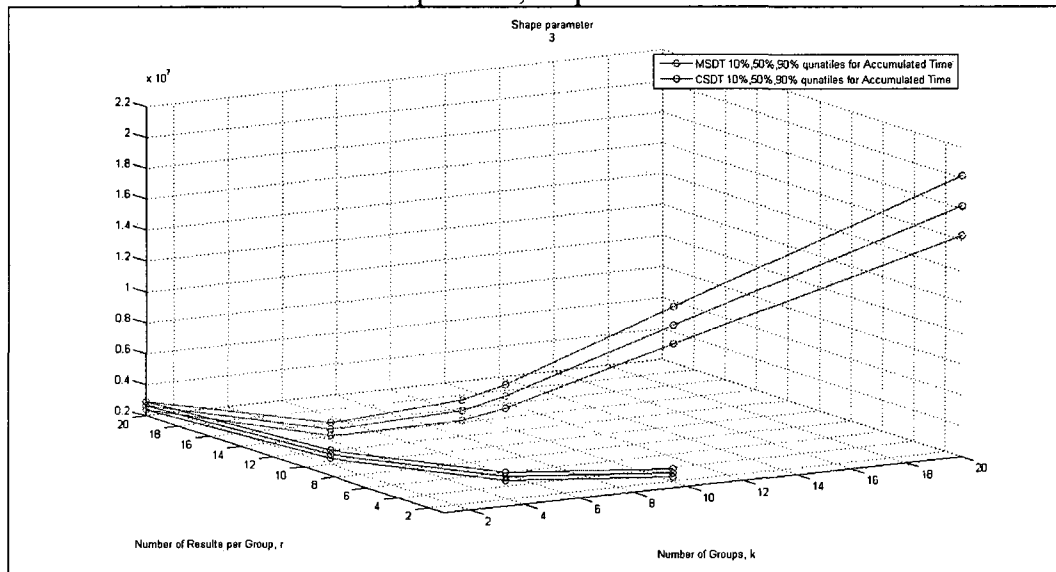


Figure 85: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 4.

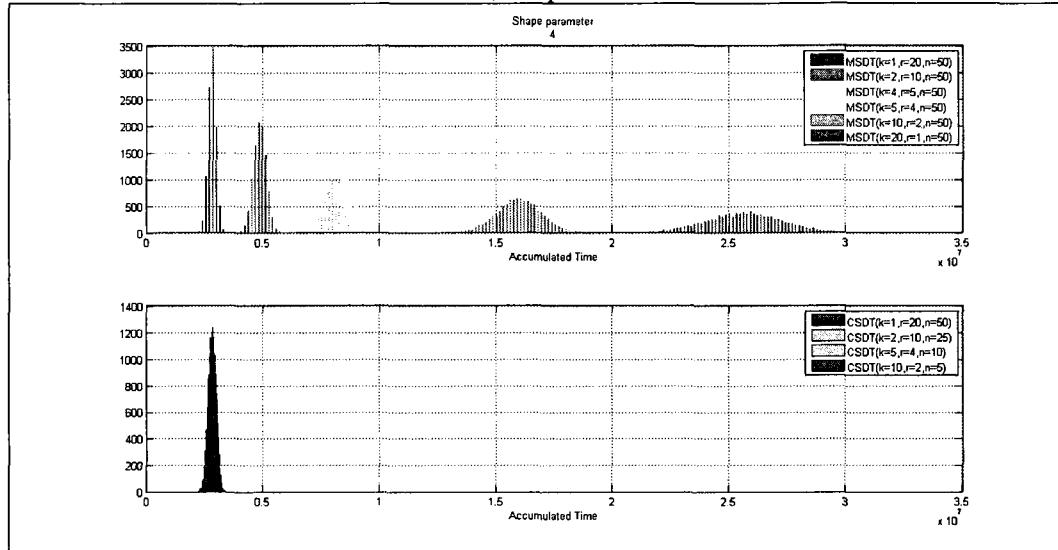


Figure 86: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 4.

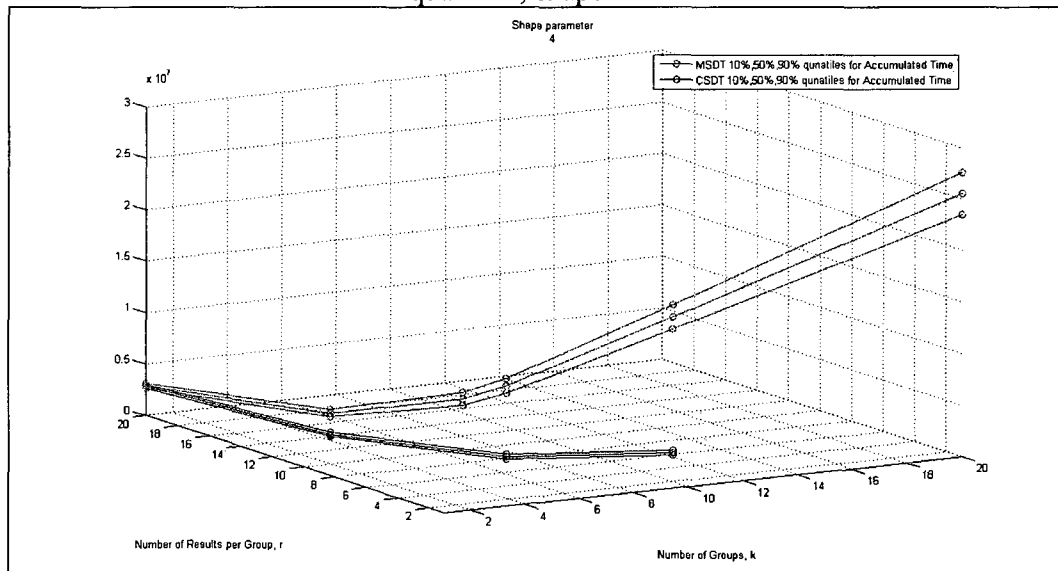


Figure 87: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 5.

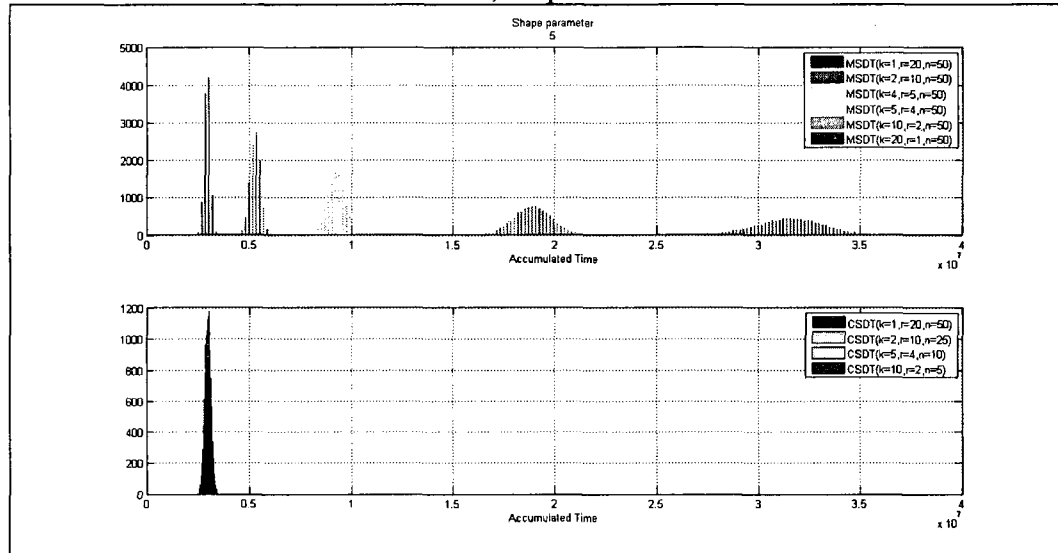


Figure 88: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 5.

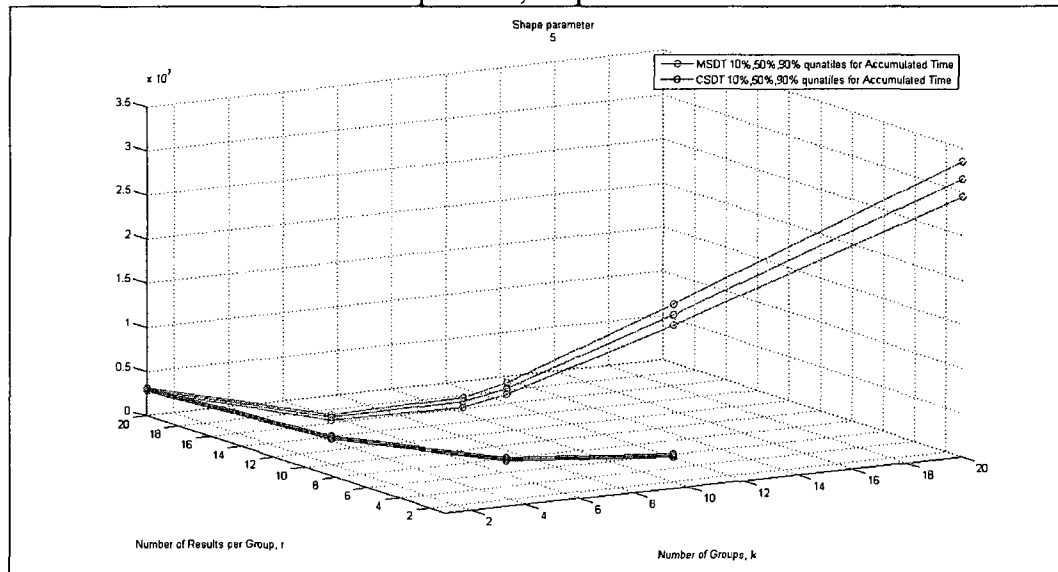


Figure 89: For the all possible configurations of MSDT and CSDT, histogram of 10,000 runs, shape is 6.

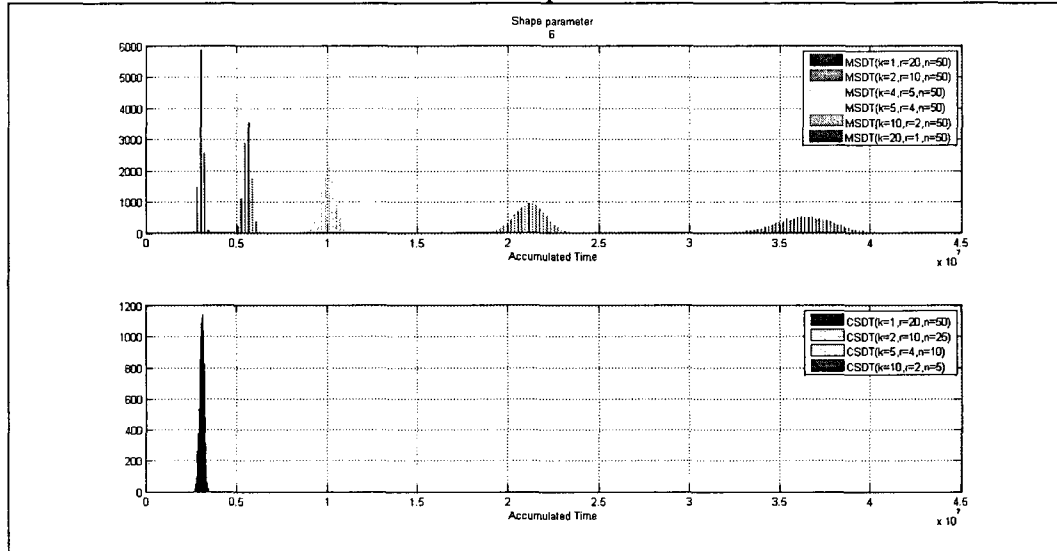


Figure 90: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles, shape is 6.

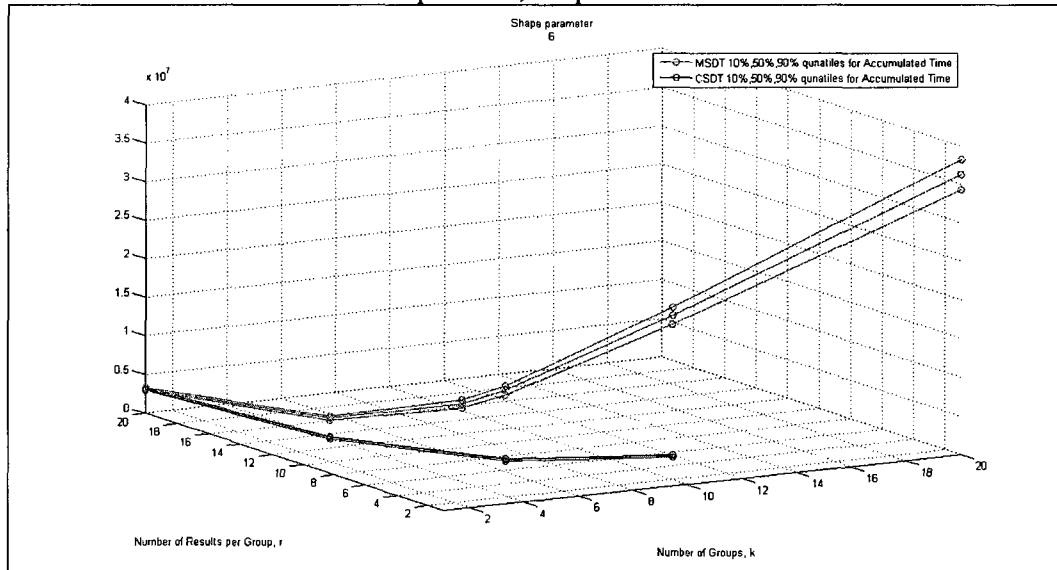
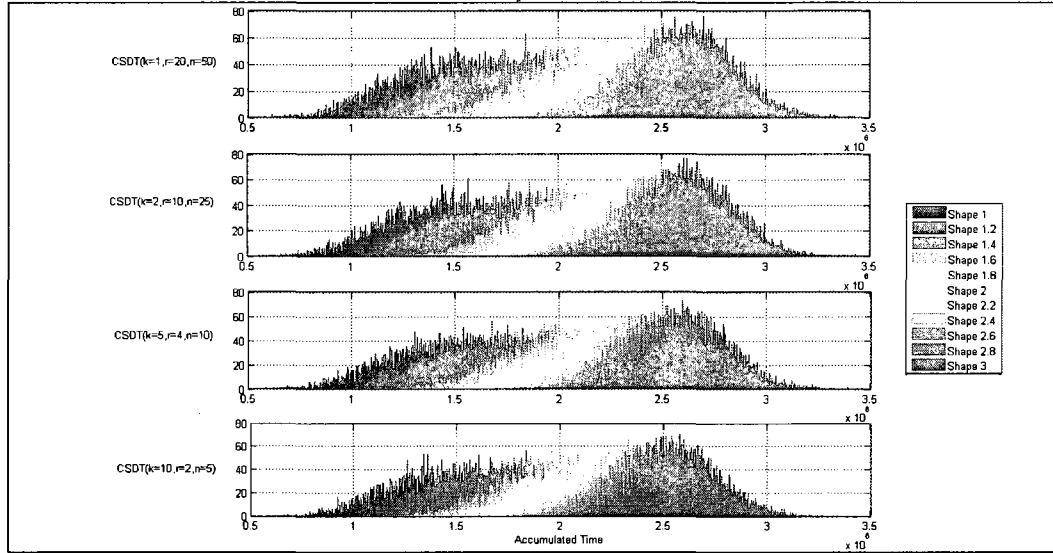


Figure 91: For the all possible configurations of CSDT, histogram of 10,000 runs, shape from 1 to 3 by 0.2 increment.



Increase in shape parameter affects in increase of the centering of the total accumulated time for the fixed CSDT strategy. Increase in number of groups affects in slightly decrease of the centering of the total accumulated time for the fixed shape parameter.

4.1.3 Conclusion

An increase in shape parameter, starting from one, affects in an increase of the variability as well as centering of the total accumulated time for the MSDT. When the number of groups is one, the total accumulated time is the lowest from the all possible configuration of the MSDT.

An increase in shape parameter increases the total accumulated time for CSDT strategy, but there is no significant change on the total accumulated time, among any CSDT combination for the fixed shape parameter. Nevertheless, it was noticed that there are a trend that total accumulated time is decreasing by the increase of the number of groups, which is more visible at high shape parameters.

When shape parameter is one, then total accumulate time for MSDT and CSDT are not different.

4.2 Results Comparison between MSDT and CSDT in Cost Domain

4.2.1 Total Components Cost

Based on the Equation 31 in Section 3.1.6, the total component cost was defined as

$$C_c = c_2 * N.$$

Where, N is the total components used during the test and c_2 is the unit cost of them.

For a fixed number of testing facilities and number of test results, the total number of components under MSDT(k_1, r_1, n_1) will not exceed, but be equal to the total number of components in CSDT(k_2, r_2, n_2) only if $k_1=1$ (See the definitions in Chapter 1, Section 1.1).

If $k_1>1$, the total number of components under MSDT will be larger than the total number of components for the CSDT (See example in Chapter 4, Section 4.1.2).

4.2.2 Run Time Cost

As defined in Section 3.1.4 by Equation 28 and Equation 29, the supervision cost depends on the testing mode. Particularly, it depends on the run time. Based on the results in Section 4.1, for a fixed number of both testing facilities and number of results, a run time depends on the shape parameter. Run time, for the CSDT and MSDT, increases as shape parameter increases. Rate of an increase for MSDT is higher than that in the CSDT and this will affect on a supervision cost, causing an increase in MSDT. The shortest expected run time was observed when the number of groups is one for either testing mode (Classical Test).

4.2.3 Total Accumulated Time Cost

In Section 3.1.5 by Equation 30 the total testing time cost is defined, which depends on the total accumulated time. Assuming unit cost is a constant, total accumulated time depends on the shape and test configuration.

For the fixed shape parameter, the lowest total accumulated time from the all possible configurations of MSDT is the same for the total accumulated time from any CSDT configuration.

4.2.4 Total Operational Cost

Total operational cost is given in Section 3.1.7 by Equation 32. For the CSDT expected Component Cost (C_C), Run time (supervision) Cost (C_S), Total accumulated time Cost (C_{TC}) are lower for those at MSDT.

4.3 Illustrative Example

Supposedly the component under the investigation is a coupling¹¹, which has a Weibull failure distribution with scale parameter equal to 2.5 and shape parameter equal to 75,000.

The following assumptions will be made:

- Number of testing facilities available is 50 ($N=50$).
- Number of results required is 20 ($R=20$).

¹¹ Source: <http://www.barringer1.com/wdbase.htm>

- Unit testing time cost is \$1 (c_1).
- Unit component cost is \$1 (c_2).
- Unit supervision/technician cost is \$1 (c_3).

Table 11: For the all CSDT possible configurations, 50% quantile run time, 50% quantile total accumulated time and total component cost.

Number of Groups (k)	Number of Components per Group (n)	Number of Results per Group (r)	Total Collected Results (R)	Total Used Components	Run time	Total accumulated time	Total Cost
					50% Quantile	50% Quantile	
1	50	20	20	50	56,634	*2,492,000	2,548,684
2	25	10	20	50	59,921	2,492,000	2,551,971
5	10	4	20	50	66,894	2,492,000	2,558,944
10	5	2	20	50	74,701	2,492,000	2,566,751

*the largest interpolated value from the all possible CSDT configurations.

Table 12: For the all MSDT possible configurations, 50% quantile run time, 50% quantile total accumulated time and total component cost.

Number of Groups (k)	Number of Components per Group (n)	Number of Results per Group (r)	Total Collected Results (R)	Total Used Components	Run time	Total accumulated time	Total Cost
					50% Quantile	50% Quantile	
1	50	20	20	50	56,634	2,492,000	2,548,684
2	50	10	20	100	81,226	3,825,100	3,906,426
4	50	5	20	200	120,250	5,792,200	5,912,650
5	50	4	20	250	136,870	6,588,900	6,726,020
10	50	2	20	500	205,670	9,717,700	9,923,870
20	50	1	20	1000	310,420	13,917,000	14,228,420

As we see from, Table 11, Table 12, Figure 92 and Figure 93 the medians of the run time and the total accumulated time are lower at CSDT than at MSDT. This does affect on the median run time cost and median total accumulated time cost. Also, the total number of components at CSDT is almost constant to the changes in the number of groups. In addition, the total number of components at MSDT is increasing by the increase of the number of groups, this will magnify the increasing cost effect at MSDT.

Figure 92: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles of run time, shape is 2.5.

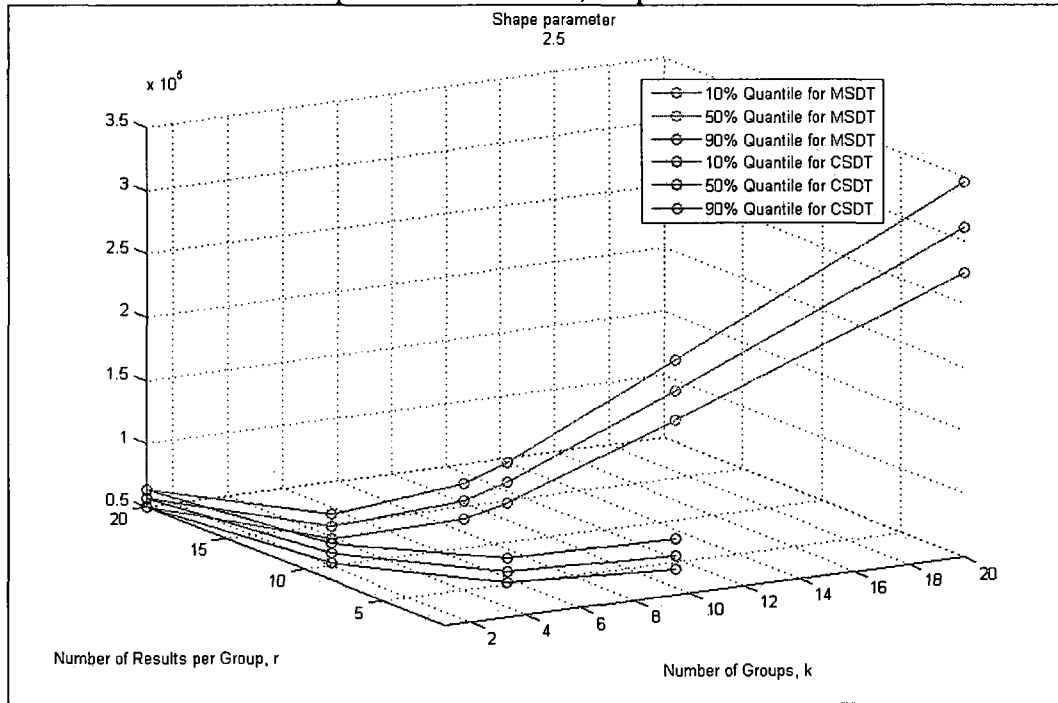
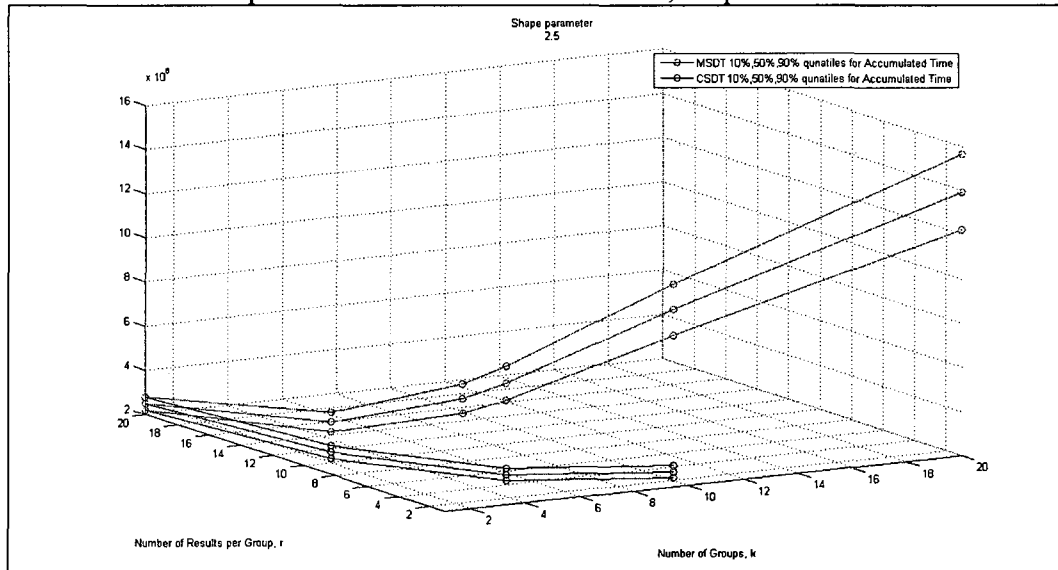


Figure 93: For the all possible configurations of MSDT and CSDT, 10%, 50% and 90% quantiles of total accumulated time, shape is 2.5.



From the both MSDT and CSDT plans, the cheapest and fastest strategy is CSDT($k=1, r=20, n=50$) or MSDT($k=1, r=20, n=50$). This is an exactly the classical failure

truncated test. Meaning that by utilizing the maximum capability of the testing facility and truncated the test at the desired number of failure is the cheapest and fastest testing configuration from the all possible configurations.

CHAPTER 5

CONCLUSION

In the Introduction Section the goals of the research were defined. We will bring the answers and conclusions to them here.

1. To develop a simulation study in the time domain for the MSDT and CSDT to compare the total test duration.

In Section 3.19 the simulation study for MSDT and CSDT in Time domain was conducted. We came to the conclusion that different testing strategies have different run time as well as total accumulated time. Also, we saw that the results primarily depend on the shape parameter.

2. To do theoretical research in time duration for the MSDT and CSDT to analytically determine the total test duration.

In Section 3.2.1.1, another approach was considered to prove that the probability of the r^{th} out of n ordered random variable be less than some threshold, has a Beta distribution with the shape parameters r and $n-r+1$.

In Section 3.2.2.1, run time for CSDT was researched. The exact PDF and CDF for the maximum ordered random variable from the sample size k were derived. It was proven that the CDF is a recursive function. With the same analogy as in Section 3.2.1.1, a new method to calculate the quantiles for the Maximum Ordered Random Variable from the sample size k was developed.

In Section 3.2.3, run time for MSDT was researched. Two approaches to approximate the quantiles were considered. In Section 3.2.3.5, a new approach to approximate quantiles

for the k^{th} sum of the r^{th} out of n ordered random variable was derived. The suggested method is fast and easy to calculate.

3. To compare the simulation study results as a validation for the theoretical results.

In Section 3.2.2.3, verification of the exact PDF and CDF values for the maximum ordered random variable from the sample size k were accomplished. The derived formulas are correct and theoretically proven.

In Section 3.2.3.6, verification to the proposed approach in Section 3.2.2.3 was conducted. If the parent distribution is Uniform, the suggested approach is accurate and exact. If the parent distribution is Weibull or Normal, then a slight difference was observed for the extreme order statistics.

4. To develop cost models as a basis for comparing the MSDT and CSDT.

In Section 3.1.4, Section 3.1.5, Section 3.1.6 and Section 3.1.7 the cost models were defined. In a study of the cost model, the run time cost, component cost and total accumulated time cost are the main cost categories.

5. To determine optimum test strategies from a cost perspective for the MSDT and CSDT.

The component cost is always less for CSDT approach. This is based on the definitions of the MSDT and CSDT.

Run time cost is proportional to the run time as a random variable. When the shape parameter is one, breaking into groups affects run time. This is causing run time to decrease in the series tests (MSDT) and increase in the parallel mode tests (CSDT). When the shape parameter is greater than one, breaking into the groups affects both run times of the tests (MSDT and CSDT) and has an increasing pattern.

The CSDT mode expected run time tends to increase as well as its variance by the increase of the shape parameter starting from one. Among the all possible values of the shape parameter and the all possible CSDT mode configurations, the shortest expected run time was observed when the number of groups is equal to one.

The MSDT test mode expected run time tends to increase as well as its variance by the increase of the shape parameter starting but not equal to one. Among all possible values of the shape parameter and all possible MSDT mode configurations, the shortest expected run time and its variance was observed when the number of groups is equal to one.

For the total accumulated time cost the simulation study was done (Section 4.1.2). The total accumulated time cost primarily depends on the total accumulated time. The total accumulated time increases by the increase of the shape parameter and the number of groups at MSDT mode. For CSDT mode, the total accumulated time is increases by the increase of the shape parameter and has a minor trend to decrease by the increase of the number of groups at the fixed shape parameter value. This minor trend is very small and doesn't have any significant effect.

6. From the cost models, establish optimum test strategies, considering number of components for the test and number of groups, as well as number of test results in each group under the budget limitation and with the desired confidence level.

If the shape parameter is greater than unity, for either testing mode the shortest expected run time was observed for the testing strategies where the number of groups is one. So, this testing strategy becomes a Classical Test.

If the shape parameter is greater than unity, for either testing mode the shortest expected total accumulated time was observed for the testing strategies where the number of groups is one. So, this testing strategy becomes a Classical Test.

The Classical Test is the cheapest among all possible testing configurations when the shape parameter is greater than unity.

CHAPTER 6

DISCUSSION AND FURTHER RESEARCH

6.1 Reliability Levels from the MSDT and CSDT Testing Strategies

In practice, there are many methods for estimating the Weibull distribution parameters from test results.

The Maximum Likelihood Estimation (MLE) is a widely used technique for estimating the Weibull distribution parameters. It is well known that MLE is asymptotically unbiased and asymptotically efficient. In 1965, Cohen (Cohen, 1965) showed shape and scale parameters MLEs from complete and Type II censored data for the Weibull distribution. (see Appendix A for derivation).

The derived MLEs are a function of the collected results and sample size. Later on, in 1974, Rockette (Rockette, Antle, & Klimko, 1974), showed that if the shape parameter is known, then the MLEs for the scale and location parameters exist and are unique. The MLE is not the only technique for estimating the parameters. For example, in 1966, Downton (Downton, 1966) derived linear estimates for the parameters of the extreme value distribution, which can easily be converted to the Weibull.

For reliability tests, when the sample sizes are small or different, the estimated parameters by MLE methods have some biases. This can be a major problem. So many methods and bias correction factors were proposed to deal with this issue. (see (Hirose, 1999)). In 2009, Cousineau (Cousineau, 2009) suggested using the weights for estimating MLEs for the Weibull parameters from complete data. Also, the author showed that by using the suggested weights, the estimated parameters are nearly unbiased.

For the MSDT and CSDT, the total number of components in the tests are different, while the test results are the same. For the reliability tests in series or parallel tests, the

likelihood function was defined and showed that the MLEs for Weibull parameters are consistent to the single group for complete and Type II censored datum results, (see Appendix). So, to estimate the parameters by MLEs from MSDT or CSDT data, the weighted MLEs can be used, based on the logic by Cousineau did. Another approach could be to use some bias correction factor method.

6.2 Sudden Death Testing

Leonard Johnson in his book showed and proved that time to finish the test is smaller if one puts on the test more components than intended to be failed. This was the main argument that more components on the test will lead to the faster results. This is true and we are not arguing this point.

But if the test in series mode, this argument is false.

Let's assume there are 50 testing facilities and 200 components available. Also 100% of the testing facility utilization is required.

Then 1st failure out of 50 will be the Sudden Death Test in a first group and so on. So the time to finish the test or run time will be the sum of all four 1st out of 50 ordered random variables, or it is MSDT($k=4, n=50, r=1$). By this testing strategy 4 results will be collected.

Now consider that if one puts 50 components at the all available testing facility and run the test until the 4th failure out of 50. So this is CSDT($k=1, n=50, r=4$) or Classical Test ($n=50, r=4$). Again, by this strategy we will collect 4 results. But in this case we will use only 50 components against 200.

If we compare these two strategies we will come to the conclusion that no doubt 1st out of 50 will have shorter time than 4th out of 50. But the sum of the all four 1st out of 50 will

be higher than 4th out of 50 (if the shape parameter is greater than 1). As we see this result in Section 4. This is the main inaccuracy and to claim that Sudden Death Test is faster regardless the shape parameter value is not justified.

In contrast, our results reveals that Classical Test or CSDT with number of groups equal to unity is the most shortest and fastest testing strategy for the failure truncated tests, if the shape parameter is greater than unity. Besides, it is much cheaper in the component and run time costs domains.

Generally, a recipe of the failure truncated test for the industrial usage is the following:

If you are testing mechanical components or you do believe that the tested product failure distribution is Weibull with the shape parameter greater than unity, then utilize testing facilities up to 100% and truncate the test on the predetermined number of failures.

If the shape parameter is exactly one, then Sudden Death Test is reasonable testing strategy. But we have to mention that not rational justification for the shape parameter will cost a money and time¹².

6.3 Total Accumulated Time

When shape parameter is one, the total accumulates time depends on the number the number of failures. When the shape parameter is grater than one, we believe that the distribution for the Total accumulated time should not only depend on the number of failures but also on the number of the components placed on the test. This is an attractive research topic for the further study.

¹² Meeker, W., & Escobar, L. (1998). Statistical methods for reliability data. New York: Wiley, pg-80.

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Appendix A

Weibull Distribution

Probability density functions of the form:

$$f(t) = \begin{cases} \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta}, & \text{when } t > 0 \\ 0, & \text{when } t \leq 0 \end{cases} \text{--Weibull_density_function}$$

It is easy to check that

$$F(t) = \begin{cases} \int_0^t \frac{\beta}{\theta^\beta} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy, & \text{if } y > 0 \\ 0, & \text{if } y \leq 0 \end{cases} \Rightarrow$$

$$\text{let } \left(\frac{y}{\theta}\right)^\beta = u, \Rightarrow \beta y^{\beta-1} dy = \theta^\beta du, \Rightarrow dy = \frac{\theta^\beta du}{\beta y^{\beta-1}}$$

when:

$$y = 0; u = 0$$

$$y = \infty; u = \infty$$

So

$$\begin{aligned} F(t) &= \int_0^t \frac{\beta}{\theta^\beta} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy = \int_0^t \frac{\beta}{\theta^\beta} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} \frac{\theta^\beta du}{\beta y^{\beta-1}} du = \int_0^t e^{-u} du = \\ &= \int_0^t e^{-\left(\frac{y}{\theta}\right)^\beta} d\left(\frac{y}{\theta}\right)^\beta = -e^{-\left(\frac{y}{\theta}\right)^\beta} \Big|_0^t = -e^{-\left(\frac{t}{\theta}\right)^\beta} - (-1) = 1 - e^{-\left(\frac{t}{\theta}\right)^\beta} \Rightarrow \end{aligned}$$

$$F(t) = \begin{cases} 1 - e^{-\left(\frac{t}{\theta}\right)^\beta}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases} \text{--Weibull_cumulativ_distribution_function}$$

$$R(t) = \begin{cases} e^{-\left(\frac{t}{\theta}\right)^\beta}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases} \text{--Weibull_Survival_Function}$$

Hazard _function

$$h(t) = \frac{f(t)}{R(t)} = \begin{cases} \frac{\frac{\beta}{\theta^\beta} t^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta}}{e^{-\left(\frac{t}{\theta}\right)^\beta}} = \frac{\beta}{\theta^\beta} t^{\beta-1}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

$$E(x^k) = \int_0^\infty y^k \frac{\beta}{\theta^\beta} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} dy$$

$$\text{let } \left(\frac{y}{\theta}\right)^\beta = u, \Rightarrow \beta y^{\beta-1} dy = \theta^\beta du, \Rightarrow dy = \frac{\theta^\beta du}{\beta y^{\beta-1}}$$

$$y = \theta u^{1/\beta}$$

when:

$$y = 0; u = 0$$

$$y = \infty; u = \infty$$

So

$$E(x^k) = \int_0^\infty y^k \frac{\beta}{\theta^\beta} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^\beta} \frac{\theta^\beta}{\beta y^{\beta-1}} du = \int_0^\infty \theta^k u^{\frac{k}{\beta}} e^{-u} du = \theta^k \int_0^\infty u^{1+\frac{k}{\beta}-1} e^{-u} du = \theta^k \Gamma\left(1 + \frac{k}{\beta}\right)$$

$$k = 1$$

$$M'(0) = E(x^1) = \theta^1 \Gamma\left(1 + \frac{1}{\beta}\right)$$

$$k = 2$$

$$M''(0) = E(x^2) = \theta^2 \Gamma\left(1 + \frac{2}{\beta}\right)$$

$$\text{Var}(x) = M''(0) - M'(0)^2 = E(x^2) - (E(x))^2 = \theta^2 \Gamma\left(1 + \frac{2}{\beta}\right) - (\theta \Gamma\left(1 + \frac{1}{\beta}\right))^2 =$$

$$= \theta^2 \Gamma\left(1 + \frac{2}{\beta}\right) - \theta^2 (\Gamma\left(1 + \frac{1}{\beta}\right))^2 = \theta^2 \left(\Gamma\left(1 + \frac{2}{\beta}\right) - (\Gamma\left(1 + \frac{1}{\beta}\right))^2\right) \Rightarrow$$

$$\text{Var}(x) = \theta^2 \left(\Gamma\left(1 + \frac{2}{\beta}\right) - (\Gamma\left(1 + \frac{1}{\beta}\right))^2\right)$$

$$E(x) = \theta \Gamma\left(1 + \frac{1}{\beta}\right)$$

or

$$E(x) = \int_0^{\infty} y \frac{\beta}{\theta^{\beta}} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^{\beta}} dy$$

$$\text{let : } \left(\frac{y}{\theta}\right)^{\beta} = u, \Rightarrow \beta y^{\beta-1} dy = \theta^{\beta} du, \Rightarrow dy = \frac{\theta^{\beta} du}{\beta y^{\beta-1}}$$

when :

$$y = 0; u = 0$$

$$y = \infty; u = \infty$$

So

$$M'(0) = E(x) = \int_0^{\infty} y \frac{\beta}{\theta^{\beta}} y^{\beta-1} e^{-\left(\frac{y}{\theta}\right)^{\beta}} \frac{\theta^{\beta}}{\beta y^{\beta-1}} du = \int_0^{\infty} \theta u^{\frac{1}{\beta}} e^{-u} du = \theta \int_0^{\infty} u^{1+\frac{1}{\beta}-1} e^{-u} du = \theta \Gamma\left(1 + \frac{1}{\beta}\right)$$

$$E(x) = \theta \Gamma\left(1 + \frac{1}{\beta}\right)$$

Appendix B

Derivation of the Cumulative Probability or Probability Distribution Functions of the r^{th} Out of n Ordered Random Variable

Let the random sample of size n be $t_1; t_2 \dots t_n$ where all t_i 's are independent, identically distributed and continuous random variables with $f(t)$ probability density and $F(t)$ cumulative distribution functions, respectively.

Consider the vector $t_{1:n}; t_{2:n} \dots t_{n:n}$ of random variables, which is composed of the t_i random variables, where $t_{i:n}$ is the i^{th} in magnitude, so that $t_{1:n} \leq t_{2:n} \leq \dots \leq t_{n:n}$. Then the $t_{1:n}; t_{2:n} \dots t_{n:n}$ random variables would have $G_{r:n}(t)$ cumulative distribution and $g_{r:n}(t)$ probability density functions, respectively.

Where

$$G_{r:n}(t) = P(t_{r:n} \leq t), \quad r \leq n.$$

If the condition is $t_{r:n} \leq t$, so “ r ” or more elements from $t_1; t_2 \dots t_n$ sample should satisfy the condition $t_i \leq t$ and since each $t_i \leq t$ has a binomial distribution with the probability of success of $F(t) = P(t_i \leq t)$ one would be able to state:

$$G_{r:n}(t) = P(t_{r:n} \leq t) = \sum_{w=r}^n \binom{n}{w} F(t)^w (1 - F(t))^{n-w}$$

To find $g_{r:n}(t)$, we have to take the derivative of $G_{r:n}(t)$ with respect to t , using the property that:

$$f(t) = \frac{dF(t)}{dt} \quad \text{and} \quad \frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}.$$

So we would get that

$$\begin{aligned}
g_{r,n}(t) &= \sum_{w=r}^n \binom{n}{w} w F(t)^{w-1} (1-F(t))^{n-w} f(t) - \sum_{w=r}^n \binom{n}{w} F(t)^w (n-w) (1-F(t))^{n-w-1} f(t) = \\
&= \sum_{w=r}^n \binom{n}{w} w F(t)^{w-1} (1-F(t))^{n-w} f(t) - \sum_{w=r}^{n-1} \binom{n}{w} F(t)^w (n-w) (1-F(t))^{n-w-1} f(t)
\end{aligned}$$

Note that for the second term “ $w=r:n-1$ ”, the variable “ w ” can be changed to “ v ”, so that “ $v=r+1:n$ ”.

$$\begin{aligned}
&= \sum_{w=r}^n \binom{n}{w} w F(t)^{w-1} (1-F(t))^{n-w} f(t) - \sum_{v=r+1}^n \binom{n}{v-1} F(t)^{v-1} (n-v+1) (1-F(t))^{n-v+1-1} f(t) = \\
&= \sum_{w=r}^n \binom{n}{w} w F(t)^{w-1} (1-F(t))^{n-w} f(t) - \sum_{v=r+1}^n \binom{n}{v-1} F(t)^{v-1} (n-v+1) (1-F(t))^{n-v} f(t) = \\
&= \sum_{w=r}^n \frac{n!}{w!(n-w)!} w F(t)^{w-1} (1-F(t))^{n-w} f(t) - \sum_{v=r+1}^n \frac{n!}{(v-1)!(n-v+1)!} F(t)^{v-1} (n-v+1) (1-F(t))^{n-v} f(t) = \\
&= \sum_{w=r}^n \frac{n!}{(w-1)!(n-w)!} F(t)^{w-1} (1-F(t))^{n-w} f(t) - \sum_{v=r+1}^n \frac{n!}{(v-1)!(n-v)!} F(t)^{v-1} (1-F(t))^{n-v} f(t) = \\
&= \frac{n!}{(r-1)!(n-r)!} F(t)^{r-1} (1-F(t))^{n-r} f(t) = n \binom{n-1}{r-1} F(t)^{r-1} (1-F(t))^{n-r} f(t)
\end{aligned}$$

So,

$$g_{r,n}(t) = n \binom{n-1}{r-1} F(t)^{r-1} (1-F(t))^{n-r} f(t)$$

To summarize the above statement, if $t_1; t_2 \dots t_n$ are each independent, identically distributed and continuous random variables with $f(t)$ probability density and $F(t)$ cumulative distribution functions respectively, then the $t_{r:n}$ ordered random variable would have $g_{r,n}(t)$ probability density and $G_{r,n}(t)$ cumulative distribution functions.¹³

$$g_{r,n}(t) = n \binom{n-1}{r-1} F(t)^{r-1} (1-F(t))^{n-r} f(t)$$

and

$$G_{r,n}(t) = P(t_{r,n} \leq t) = \sum_{w=r}^n \binom{n}{w} F(t)^w (1-F(t))^{n-w}$$

¹³ Niewiadomska-Bugaj, M., & Bartoszyński, R. (1996). *Probability and statistical inference*. New York: Wiley.

Appendix C

Derivation of the Weibull and Exponential Random Variables Relationship

If a random variable t has a Weibull distribution with a shape parameter β and scale parameter θ , then the random variable t^β has an exponential distribution with rate parameter $\theta^{-\beta}$.

Proof:

Let t has a pdf:

$$f(t) = \begin{cases} \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^\beta}, & \text{when } t > 0 \\ 0, & \text{when } t \leq 0 \end{cases}$$

Let $y = t^\beta$, then $t = y^{\frac{1}{\beta}}$ and $dt = \frac{1}{\beta} y^{\frac{1}{\beta}-1} dy$

then

$$f(y) = \begin{cases} \frac{\beta}{\theta^\beta} y^{\frac{1}{\beta}-1} e^{-\frac{y}{\theta^\beta}} \frac{1}{\beta} y^{\frac{1}{\beta}-1} = \frac{1}{\theta^\beta} e^{-\frac{y}{\theta^\beta}}, & \text{when } y > 0 \\ 0, & \text{when } y \leq 0 \end{cases}$$

Appendix D

Maximum Likelihood Estimation for the Weibull Distribution Parameters

D.1 Maximum Likelihood Estimation for the Weibull Distribution Parameters in a Group with Complete Data

Let x_1, x_2, \dots, x_n be the realization of the test of n components. Then the likelihood function will be defined as:

Equation 88
$$L(\theta, \beta / x_i) = \prod_{i=1}^n \frac{\beta}{\theta^\beta} x_i^{\beta-1} e^{-\left(\frac{x_i}{\theta}\right)^\beta} = \frac{\beta^n}{\theta^{\beta n}} \prod_{i=1}^n x_i^{\beta-1} e^{-\sum_{i=1}^n \left(\frac{x_i}{\theta}\right)^\beta}$$

Taking the natural logarithm of the likelihood function (Ln LF) we will have:

Equation 89
$$\ln(L(\theta, \beta / x_i)) = n \ln(\beta) - n \beta \ln(\theta) + \beta \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \ln(x_i) - \frac{\sum_{i=1}^n x_i^\beta}{\theta^\beta}$$

Differentiating of the Ln LF with respect to θ and equating to zero will lead to:

$$\frac{d(\ln(L(\theta, \beta / x_i)))}{d\theta} = -\frac{n\beta}{\theta} + \beta \frac{\sum_{i=1}^n x_i^\beta}{\theta^{\beta+1}}$$

$$\frac{\beta}{\theta} \left(-n + \frac{\sum_{i=1}^n x_i^\beta}{\theta^\beta} \right) = 0 \Rightarrow \theta^\beta = \frac{\sum_{i=1}^n x_i^\beta}{n} \Rightarrow$$

Equation 90
$$\hat{\theta} = \left(\frac{\sum_{i=1}^n x_i^\beta}{n} \right)^{\frac{1}{\beta}}$$

Differentiating of the Ln LF with respect to β , dividing by n, substituting the Equation 89 and equating to zero will lead to:

$$\begin{aligned}
\frac{d\left(\text{Ln}\left(L(\theta, \beta / x_i)\right)\right)}{d\beta} &= \frac{n}{\beta} - n\text{Ln}(\theta) + \sum_{i=1}^n \text{Ln}(x_i) - \sum_{i=1}^n \left(\frac{x_i^\beta}{\theta^\beta} \text{Ln}\left(\frac{x_i}{\theta}\right) \right) = \\
&= \frac{n}{\beta} - n\text{Ln}(\theta) + \sum_{i=1}^n \text{Ln}(x_i) - \sum_{i=1}^n \left(\frac{x_i^\beta}{\theta^\beta} \text{Ln}(x_i) \right) + \sum_{i=1}^n \left(\frac{x_i^\beta}{\theta^\beta} \text{Ln}(\theta) \right) = \\
&= \frac{1}{\beta} - \text{Ln}(\theta) + \frac{\sum_{i=1}^n \text{Ln}(x_i)}{n} - \frac{\sum_{i=1}^n (x_i^\beta \text{Ln}(x_i))}{n\theta^\beta} + \frac{\text{Ln}(\theta) \sum_{i=1}^n (x_i^\beta)}{n\theta^\beta} = \\
&= \frac{1}{\beta} - \text{Ln}(\theta) + \frac{\sum_{i=1}^n \text{Ln}(x_i)}{n} - \frac{\sum_{i=1}^n (x_i^\beta \text{Ln}(x_i))}{\sum_{i=1}^n (x_i^\beta)} + \frac{\text{Ln}(\theta) \sum_{i=1}^n (x_i^\beta)}{\sum_{i=1}^n (x_i^\beta)} = \\
&= \frac{1}{\beta} + \frac{\sum_{i=1}^n \text{Ln}(x_i)}{n} - \frac{\sum_{i=1}^n (x_i^\beta \text{Ln}(x_i))}{\sum_{i=1}^n (x_i^\beta)} \\
&\quad \frac{1}{\beta} + \frac{\sum_{i=1}^n \text{Ln}(x_i)}{n} - \frac{\sum_{i=1}^n (x_i^\beta \text{Ln}(x_i))}{\sum_{i=1}^n (x_i^\beta)} = 0 \Rightarrow
\end{aligned}$$

Equation 91

$$\frac{\sum_{i=1}^n \text{Ln}(x_i)}{n} = \frac{\sum_{i=1}^n (x_i^{\hat{\beta}} \text{Ln}(x_i))}{\sum_{i=1}^n (x_i^{\hat{\beta}})} - \frac{1}{\hat{\beta}}$$

So, MLEs for $\hat{\beta}$ and $\hat{\theta}$ are given in Equation 90 and Equation 91. One should use Equation 91 to find the closest value for the MLE of $\hat{\beta}$, then using Equation 90 to find the MLE of $\hat{\theta}$.

D.2 Maximum Likelihood Estimation for the Weibull Distribution Parameters in a Group with Type II Censored Data

Let x_1, x_2, \dots, x_r be the “r” realization of the test of “n” components. Rearranging the realizations in order of magnitude the $x_{1:n}, x_{2:n}, \dots, x_{r:n}$ sample will be denoted.

Then the likelihood function will be defined as:

Equation 92

$$L(\theta, \beta / x_i) = \frac{n!}{(n-r)!} \left(\prod_{i=1}^r \frac{\beta}{\theta^\beta} x_i^{\beta-1} e^{-\left(\frac{x_i}{\theta}\right)^\beta} \right) (1 - F(x_{r:n}))^{n-r}$$

Taking the natural logarithm of the likelihood function (Ln LF) we will have:

Equation 93

$$\begin{aligned} \text{Ln}(L(\theta, \beta / x_i)) &= r \text{Ln}(\beta) - r\beta \text{Ln}(\theta) + \beta \sum_{i=1}^r \text{Ln}(x_i) - \\ &- \sum_{i=1}^r \text{Ln}(x_i) - \frac{\sum_{i=1}^r x_i^\beta}{\theta^\beta} - (n-r) \frac{x_r^\beta}{\theta^\beta} + \text{const.} \end{aligned}$$

Differentiating of the Ln LF with respect to θ and equating to zero will lead to:

$$\frac{d(\text{Ln}(L(\theta, \beta / x_i)))}{d\theta} = -\frac{r\beta}{\theta} + \beta \frac{\sum_{i=1}^r x_i^\beta}{\theta^{\beta+1}} + \beta \frac{(n-r)x_r^\beta}{\theta^{\beta+1}}$$

$$\frac{\beta}{\theta} \left(-r + \frac{\sum_{i=1}^r x_i^\beta}{\theta^\beta} + \frac{(n-r)x_r^\beta}{\theta^\beta} \right) = 0 \Rightarrow$$

$$\theta^\beta = \frac{\sum_{i=1}^r x_i^\beta + (n-r)x_r^\beta}{r}$$

Equation 94

$$\hat{\theta} = \left(\frac{\sum_{i=1}^r x_i^\beta + (n-r)x_r^\beta}{r} \right)^{\frac{1}{\beta}}$$

If $\sum_{i=1}^r y_i = \sum_{i=1}^r x_i + (n-r)y_r$, then Equation 94 becomes

$$\hat{\theta} = \left(\frac{\sum_{i=1}^r x_i^\beta}{r} \right)^{\frac{1}{\beta}}$$

Equation 95

Differentiating of the Ln LF with respect to β , dividing by n, substituting the Equation 94 and equating to zero will lead to:

$$\begin{aligned} \frac{dLn(L(\theta, \beta / x_i))}{d\beta} &= \frac{r}{\beta} - rLn(\theta) + \sum_{i=1}^r Ln(x_i) - \sum_{i=1}^r \frac{x_i^\beta}{\theta^\beta} Ln\left(\frac{x_i}{\theta}\right) - (n-r) \frac{x_r^\beta}{\theta^\beta} Ln\left(\frac{x_r}{\theta}\right) = \\ &= \frac{1}{\beta} - Ln(\theta) + \frac{\sum_{i=1}^r Ln(x_i)}{r} - \frac{\sum_{i=1}^r x_i^\beta Ln(x_i)}{r\theta^\beta} + \frac{Ln(\theta) \sum_{i=1}^r x_i^\beta}{r\theta^\beta} - \\ &\quad -(n-r) \frac{x_r^\beta}{r\theta^\beta} Ln(x_r) + (n-r) \frac{x_r^\beta}{r\theta^\beta} Ln(\theta) = \\ &= \frac{1}{\beta} - Ln(\theta) + \frac{\sum_{i=1}^r Ln(x_i)}{r} - \frac{\sum_{i=1}^r x_i^\beta Ln(x_i)}{r\theta^\beta} - \\ &\quad -(n-r) \frac{x_r^\beta}{r\theta^\beta} Ln(x_r) + \frac{Ln(\theta) \left(\sum_{i=1}^r x_i^\beta + (n-r)x_r^\beta \right)}{r\theta^\beta} = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\beta} - \text{Ln}(\theta) + \frac{\sum_{i=1}^r \text{Ln}(x_i)}{r} - \frac{\sum_{i=1}^r x_i^\beta \text{Ln}(x_i)}{r\theta^\beta} - \\
&\quad - (n-r) \frac{x_r^\beta}{r\theta^\beta} \text{Ln}(x_r) + \frac{\text{Ln}(\theta) \left(\sum_{i=1}^r x_i^\beta + (n-r)x_r^\beta \right)}{\sum_{i=1}^r x_i^\beta + (n-r)x_r^\beta} = \\
&= \frac{1}{\beta} + \frac{\sum_{i=1}^r \text{Ln}(x_i)}{r} - \frac{\sum_{i=1}^r x_i^\beta \text{Ln}(x_i) + (n-r)x_r^\beta \text{Ln}(x_r)}{\sum_{i=1}^r x_i^\beta + (n-r)x_r^\beta} = 0
\end{aligned}$$

Equation 96

$$\frac{\sum_{i=1}^r \text{Ln}(x_i)}{r} = \frac{\sum_{i=1}^r x_i^{\hat{\beta}} \text{Ln}(x_i) + (n-r)x_r^{\hat{\beta}} \text{Ln}(x_r)}{\sum_{i=1}^r x_i^{\hat{\beta}} + (n-r)x_r^{\hat{\beta}}} - \frac{1}{\hat{\beta}}$$

If $\sum_{i=1}^r y_i = \sum_{i=1}^r y_i + (n-r)y_r$, then

Equation 96 becomes:

Equation 97

$$\frac{\sum_{i=1}^r \text{Ln}(x_i)}{r} = \frac{\sum_{i=1}^r x_i^{\hat{\beta}} \text{Ln}(x_i)}{\sum_{i=1}^r x_i^{\hat{\beta}}} - \frac{1}{\hat{\beta}}$$

So, MLEs for $\hat{\beta}$ and $\hat{\theta}$ are given in Equation 94 and

Equation 96. One should use

Equation 96 to find the closest value for the MLE of $\hat{\beta}$, then using Equation 94 to find the MLE of $\hat{\theta}$.

D.3 Maximum Likelihood Estimation for the Weibull Distribution Parameters in “k” Groups with Complete Data

The Likelihood function depends on the collected results, regardless of the test strategy (parallel or series).

Let $x_{1(k)}, x_{2(k)}, \dots, x_{n(k)}$ be the realization of the test of n components in the k^{th} group, where $k=1:k$ Then the likelihood function will be defined as:

$$\text{Equation 98} \quad L(\theta, \beta / x_i) = \prod_{k=1}^k \left(\prod_{i=1}^n \frac{\beta}{\theta^\beta} x_{i(k)}^{\beta-1} e^{-\left(\frac{x_{i(k)}}{\theta}\right)^\beta} \right) = \prod_{i=1}^{n \cdot k} \frac{\beta}{\theta^\beta} x_i^{\beta-1} e^{-\left(\frac{x_i}{\theta}\right)^\beta}$$

So, the rest will be the same as for Equation 89, and the MLE for $\hat{\beta}$ and $\hat{\theta}$ are given in Equation 90 and Equation 91.

With the same logic, the MLE of the $\hat{\beta}$ and $\hat{\theta}$ for the Type II censored data are given in Equation 94 and Equation 96.