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Understanding Similarity: Bridging Geometric and Numeric Contexts for Proportional Reasoning

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UNDERSTANDING SIMILARITY: BRIDGING GEOMETRIC AND NUMERIC CONTEXTS FOR PROPORTIONAL REASONING

by

Dana Christine Cox

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics
Dr. Steve Ziebarth and Dr. Jane-Jane Lo, Advisors

Western Michigan University
Kalamazoo, Michigan
August 2008
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2008
ACKNOWLEDGMENTS

A colleague once remarked that if you want good collaboration, you should assign a group-worthy task. Although the doctoral dissertation is ostensibly a solitary task, anyone who has completed one must know that it deserves the "group worthy" moniker. I certainly could not have completed the work without a supportive and intelligent group of people about whom I care deeply. I would like to thank these people for being around and participating in my life and this work.

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I dedicate this dissertation to parents who continue to offer love and protection to their child even as they take new interest in the person she has become as an adult. I
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Dana Christine Cox
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CHAPTER I

INTRODUCTION

While attempts to standardize learning expectations in mathematics are not a new phenomenon, the production of published standards has received a tremendous amount of recent attention on local, state, and national platforms. Organizations and communities struggle to come to consensus on what mathematics students should learn and when they should learn it, often sending very mixed messages about the relative value of topics and the speed at which they can and should be mastered (Reys, 2006).

This confusion is not only present at the state level, but also at the national level where various organizations vie for the right to determine national standards for mathematics instruction. Although certainly seminal and groundbreaking, the National Council of Teachers of Mathematics (NCTM, 2000) *Principles and Standards for School Mathematics* has drawn criticism in the age of national legislation such as *No Child Left Behind*, for lacking grade level-specificity and for being difficult to interpret. More recently, other groups (i.e., Achieve, 2008; College Board, 2006) have submitted documents outlining course-specific and grade-specific learning objectives.

Amongst all of the confusion and debate, there emerges some consensus about the topic of proportionality and the development of proportional reasoning. The recommendations of many national organizations regarding the importance of this topic in K-12 education are remarkably unified. The National Council of Teachers of Mathematics refers to proportionality as “an important integrative thread that connects
many of the mathematics topics studied in grades 6-8" (NCTM, 2000, p. 217). The College Board (2006), like NCTM, positions proportional reasoning as a middle school topic with far-reaching connections to more advanced mathematics. It cites proportional reasoning as a “critical foundation for algebra and the rest of the high school mathematics curriculum” (College Board, 2006, p. xiv).

Individual researchers in mathematics education have also agreed on the importance of developing strong proportional reasoning in the middle grades. Lesh, Post, and Behr (1988) invoke the mental imagery of a “linchpin,” describing proportional reasoning as a pivotal concept, simultaneously the capstone of children’s elementary school arithmetic and the cornerstone of all that is to follow. Reaching as far back as Piaget, proportional reasoning has been characterized as a difficult middle school topic, closely connected to the study of many whole and rational number concepts (Vergnaud, 1988). Lamon (2007) elucidates the difficulty as well as the importance:

Fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites. (p. 629)

It is also a research site that she feels is rich enough to support many lifetimes of work.
Scope and Significance

Similarity is only one context available for the study of proportion. However, as the only geometric context for proportion, it is an important one. Similarity provides a way for students to connect spatial and numerical reasoning, and provides the basis for advanced mathematical topics such as projective geometry, slope and trigonometric ratios. The applications of similarity include surveying, as well as map and model making. Students at even young ages have extensive experiences from daily life including playing with miniature toys, building scale models, enlarging or shrinking images (Lehrer, Strom, & Confrey, 2002), resizing typographic fonts (Cox, Lo, & Mingus, 2007), and viewing posters or illustrations of everyday objects (Van den Brink & Streefland, 1979).

These rich experiences translate into a subtext of intuition and imagery that supports students as they engage with similarity tasks. It has been documented that children as young as 8 years old can employ visual perception and preproportional strategies to solve similarity problems (Lehrer et al., 2002; Swoboda & Tocki, 2002; Van den Brink & Streefland, 1979). For example, van den Brink and Streefland (1979) observed children making estimates of the “real life size” of various objects based on their memory of these objects in relation to others in drawings or photographs. In a study of sixth-grade students who had not received any instruction in ratio or proportion, Lamon (1993) also noted that students predominantly used more primitive visual or additive strategies on problems that were of what she calls Stretchers and Shrinkers, a category that includes similarity tasks, even if they were able to reason correctly with proportions in other contexts. While this research corroborates the
findings regarding the use of visual perception strategies, the presence of additive strategies and lack of proportional reasoning also indicates that transcending this imagery and making the leap to abstract proportional reasoning in this context is very difficult.

Other research studies have shown repeatedly that similarity is one of the most difficult of contexts for proportional reasoning (cf. Kaput & West, 1994). Research by the Rational Number Project has shown that like students, even pre-service and in-service teachers struggle to make sense of similarity (Post, Cramer, Harel, Kieren, & Lesh, 1998). Hart (1988) found that close to 40% of the 15-year olds in her study still focused on additive strategies when solving for missing side lengths. Even after instruction, students struggle to remember and utilize procedures and symbolism for numeric strategies that seem to replace visual perceptions and intuition (Karplus, Pulos, & Stage, 1983) rather than extend or support them. Studies have been done to characterize the nature of this student difficulty (Chazan, 1987), but there exists a gap between documenting the useful visual perceptions of younger children and the quantitative inadequacies of the older ones.

In strictly numerical contexts for proportional reasoning, this gap has been narrowed and work has been done to identify intermediate qualitative strategies such as building up, norming, or unitizing (Lamon, 2007). These numeric strategies give us some power to hypothesize about intermediate conceptions of similarity; however, they are not entirely translatable to a geometric context. Given the position of similarity at the crossroads of geometric and proportional reasoning, it is likely that the conceptions students have are influenced by their development in both areas. It is also likely that the
progression of students from visual to *proportional thinker* (Cramer & Post, 1993b) can be better interpreted by the combination of more specific theories in both fields, in particular, Piaget’s (Inhelder & Piaget, 1958) and van Hiele’s (van Hiele, 1986) theories about the development of proportional and geometric thinking.

Lamon (1993) laments that much of the research on student difficulties with proportion in general has only focused on cataloging the variety of proportion problems and students’ (in)ability to solve them. In her earlier study and reiterated in her recent review of the literature, Lamon argues for research that identifies the ideas that students have that contribute to proportional reasoning, and investigates the contexts and models that “offer more explanatory power” (p. 42) to students in their work. Similarity, being multiplicative and continuous in nature, may be such a context that provides an entry point for students moving beyond additive reasoning. The relationship of students’ geometric thinking to their ability to reason proportionally about similarity problems will be investigated in this study, guided by two research questions.

**Research Questions**

The problem that stands out most, from the background presented above, is the gap between what we know about students’ resources prior to instruction and what we know about their difficulties after instruction. Although the research literature suggests that students struggle to develop abilities to reason proportionally and to make sense of similarity, the fact remains that some students actually do develop these abilities. We still do not know how these students advance from using visual and additive reasoning strategies to using multiplicative proportional reasoning on similarity tasks. Before we
can study the transition, however, we must admit that we have only incomplete theories about what intermediate student strategies would look like on these tasks. In order to help fill this gap in the literature, this study will document more carefully student strategies and conceptions bridging visual perception and proportional reasoning. Two research questions guide this inquiry:

1. What strategies do students use to differentiate similar figures from non-similar figures? What types of geometric and numeric reasoning are indicated by these strategies?

2. What strategies do students use to construct similar figures? What types of geometric and numeric reasoning are indicated by these strategies?

Design Limitations

This study is not a longitudinal study and, as such, cannot track individual learning trajectories over time. It would be very interesting indeed to study similarity in the course of a teaching experiment, as Lehrer, Strom, and Confrey (2002) chose to do when they modeled the development of more primitive conceptions of the concept of similarity. However, before a study of such magnitude could be done, it would be useful to explore further the complexity of the connection between spatial and proportional reasoning as it occurs for middle school students. This is the limited focus of this study. Furthermore, this study is a narrow look at the thinking and behavior of students in the earlier stages of development with respect to geometric and proportional thinking. As such, it does not take into consideration more advanced conceptions of similarity that may be gained as students return to the topic in high school.
CHAPTER II

LITERATURE REVIEW

Orienting the Reader

Understanding similarity relies both on one’s visual perceptions of shape and ability to negotiate second order relationships. Thus, the concept of similarity is located at the intersection of geometric and spatial thinking and proportional reasoning. With a genesis in Piagetian interviews (Piaget, 1966; Piaget, Inhelder, & Szeminska, 1966), similarity tasks have been used with versatility to explore what a student understands about ratio and proportion, but also how students perceive the properties of shape and how to measure and compare them. With versatility comes complexity. In order to interpret what students understand of the concept of similarity, it is unlikely that a theory purely numeric or geometric will suffice; it is necessary to understand how that student is incorporating geometric, spatial, and numeric ways of reasoning.

This review of the literature is organized into three sections. The first section is a review of the literature on the development of proportional reasoning in numeric contexts. Becoming a proportional thinker is widely viewed as a developmental journey, taken over the course of many years (Cramer & Post, 1993b; Inhelder & Piaget, 1958; Lamon, 2007). In this literature there are theories about how students become proportional thinkers, and research results on the preproportional strategies that are key to developing proportional reasoning numerically. Similarity tasks require
students to reason proportionally about dimensional growth, but differ from other numerical tasks in significant qualitative ways. This qualitative divergence may prevent students from using previously developed numerical strategies in a contextual manner to make sense of similarity.

The second section is a review the literature on the development of conceptions of geometry and space, including the theory of Piaget. Piaget included specific work with similarity tasks while developing his theory, while others have not. However, the validity of Piaget’s work has been contested as researchers struggle to replicate his results. Pierre van Hiele’s (1986) theory is largely accepted as being the most accurate theory describing the development of geometric thinking (Battista, 2007). Originally investigated to describe how students come to reason using 2-d shapes, van Hiele described five discrete levels of reasoning ability. The discrete nature of these levels has since been challenged, and researchers have sought to describe sublevels of thinking. Researchers have also sought to extend the levels to describe reasoning in other geometric domains such as 3-d shape. This section highlights these efforts and describes how the van Hiele levels, as well as proposed sublevels, might be interpreted in the context of similarity to explain some of the nature of student difficulty in the domain.

The third and final section is a review and discussion of the literature specific to similarity. It documents the existence of rich conceptual imagery in students’ minds and conjectures how this could be leveraged to transition to a formal understanding of similarity. The section includes description in greater detail of what is already known about students’ informal visual perceptions of similar figures. The implication of this work is that it is not a simple matter to interpret student behavior on similarity tasks, but
also that the successful numerical solving of a similarity task does not necessarily imply that proportional reasoning has taken place.

Proportional Reasoning

Over 20 years ago, Tourniaire and Pulos (1985) described proportional reasoning as both useful and difficult to master, a description that is as relevant now as it was then. Lamon (2007) is even more emphatic when describing the importance and complexity of this topic and others related to rational numbers. Even after decades of work, the topic remains complex for students and researchers alike, and the domain is still an active site for productive work in the field of mathematics education research.

To address the question of how students develop from intuition to quantitative proportional reasoning on similarity tasks, it is first necessary to describe what it means to reason proportionally in a broad sense, and to describe theories of how students come to reason this way. Then, it is possible to locate similarity within this broader domain. At the conclusion of this section is a discussion of how proportional strategies can and cannot be interpreted in the context of similarity, and how an over reliance on simple convex figures in the literature may cloud this interpretation.

Definition of Terms

A proportion is the comparison of two equal ratios in the form $a/b = c/d$. These relationships are ubiquitous in real life and can be used by proportional thinkers (Cramer & Post, 1993b) to solve a myriad of different problems in a variety of different forms. In order to reason proportionally, students must be aware of the relationship
between two relationships rather than just recognizing the relationship between two objects or quantities (Inhelder & Piaget, 1958). To illustrate, consider the problem of scaling a rectangle. The student must note the relationship of the length to the width in the given rectangle as well as the scaled rectangle. The key awareness for the student is that these relationships need to be the same for the scaling to be done correctly. Furthermore, students must be able to discern when a multiplicative relationship exists and the task warrants proportional reasoning, and when it does not (Cramer & Post, 1993b).

Lamon (2007) makes a point to clarify the distinction between proportional reasoning and proportionality. Proportional reasoning develops alongside an understanding of rational number and is a way of working within the broader scheme of proportionality. However, individuals learn to reason about proportions long before they understand the full range of proportionality. She defines proportional reasoning specifically as “supplying reasons in support of claims made about the structural relationships among four quantities (say a, b, c, d), in a context simultaneously involving covariance of quantities and invariance of ratios or products” (pp. 637-638). These structural relationships are what constitutes proportionality, which has direct, inverse, square, and cubic variations.

Similarity is a context for direct variation. To be similar, figures need to have exactly the same shape, but could be different in size. Formally, two figures are similar if their corresponding angles are of equal measure and all pairs of corresponding sides are in proportion. As a consequence, if one constructs the ratios between pairs of corresponding sides, all such ratios will be equivalent. Also, the ratio of any two lengths
within a given figure will be equivalent to the corresponding ratio on all other similar figures.

The term proportional thinker has been used throughout this section defined in the broader sense by Cramer and Post (1993b) as a set of expected student behaviors. It would be useful to establish such a set of behaviors that would identify geometric proportional thinkers, or a proportional thinker who can operate within the geometric context of similarity. Such a description is provided in Table 1. Each behavior of a proportional thinker has a parallel behavior described in similarity-specific terms and including behaviors discussed in the context of geometry.

Becoming a Proportional Thinker

Piaget hypothesized that adults think differently and are capable of far more sophisticated reasoning than children (Piaget, 1964). As they age, children progress through four levels of development: sensorimotor, preoperational, concrete operational, and formal operational, aided by maturity, experience, social transmission, and equilibration. The ability to reason with proportions is the hallmark of the most sophisticated level of thinking, the formal operations stage. Even though Piaget found that this ability develops sometime during early adolescence, it does not mean that on one miraculous birthday a child wakes up with new skills. Instead, children develop these skills over long periods of time and exhibit behaviors and thinking patterns that are emergent long before we can be considered proportional thinkers.

En route to becoming proportional thinkers, children engage with proportionality tasks in increasingly sophisticated ways. Inhelder and Piaget (1985)
Table 1

*Behaviors of a Geometric Proportional Thinker*

<table>
<thead>
<tr>
<th>Proportional Thinker</th>
<th>Geometric Proportional Thinker</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knowing the...</td>
<td>Knowing the properties of...</td>
</tr>
<tr>
<td>Being able to...</td>
<td>Being able to recognize...</td>
</tr>
<tr>
<td>Understanding...</td>
<td>Understanding the principles...</td>
</tr>
<tr>
<td>Realizing that...</td>
<td>Realizing that both...</td>
</tr>
<tr>
<td>Knowing how to...</td>
<td>Knowing how to scale...</td>
</tr>
<tr>
<td>Being unaffected by</td>
<td>Being unaffected by the...</td>
</tr>
</tbody>
</table>

identified three levels of sophistication: *additive, preproportional*, and *proportional*. In the additive stage, a child is only partially aware of proportionality, able to recognize one relationship at a time. The child may compare shapes along one dimension only, and in quantifying a size comparison, calculate the difference rather than a ratio. As the child enters the concrete operational stage, he or she is able to instinctively deal with
ratio, using additive strategies to build up patterns. Finally, a child is able to abstract the
concept of ratio in the formal operations stage, and is able to represent the second-order
relationship symbolically.

Beginning with Piaget’s (Inhelder & Piaget, 1958) identified levels of
proportional thinking (additive, preproportional, proportional), researchers have
continued to map out the developmental stages through which a child progresses into a
mature notion of proportional relationships. It is generally agreed that a student’s
abilities in mathematics and with reasoning can be measured according to his or her
performance compared to general age group responses. “From the Piagetian
perspective, proportional reasoning is a late achievement because it requires second-
order reasoning or considering relations among relations” (Sophian & Wood, 1997).

Since Piaget, some researchers have questioned a late position of proportional
reasoning in instruction relative to other multiplicative concepts (Vergnaud, 1988),
including multiplication and division with whole and rational number, ratio, rate,
dimensional analysis, and vector spaces (Lamon, 2007). Vergnaud’s (1988)
*multiplicative concept field* is a fusion of these concepts and ways of representing them
with student understanding—procedural as well as conceptual.

Steffe “devotes considerable attention to articulating how a child might be
viewing a task as a continuation of earlier actions” (as cited in Harel & Confrey, 1994,
p. xiv). Taken from this perspective, it may be possible that proportional reasoning
develops alongside and even as a result of the development of other multiplicative
concepts. Lo and Watanabe (1997) found evidence to support this in their work with
Bruce, a student in the fifth grade. Although initial confusion about proportion indicated
that Bruce possessed a limited understanding of division and multiplication, further work found that he was capable of solving very complex problems using sophisticated counting and unitizing strategies.

Lamon (1999) hypothesizes that students develop into proportional thinkers gradually and that it requires a composite of knowledge in the six areas of relative thinking, partitioning, unitizing, ratio sense, rational numbers, and quantities and change. It is her position that by allowing students to experience all of these areas, we contribute to a student's power to think proportionally.

**Student Strategies**

Piaget documented great variation in the strategies of students solving proportion tasks. On one end of the spectrum are primitive (Tourniaire and Pulos, 1985) or nonconstructive (Lamon, 1993) strategies. These early strategies may cause students to ignore available and relevant information in problems or mistake the multiplicative relationship for an additive one. The latter strategy permeates the literature on students' solving of similarity problems (Lamon, 2007), and Chazan (1987) found that activity such as constructing similar triangles by extending sides and relying on the parallelism of the third side is particularly prone to additive reasoning. Nonetheless, early strategies are not reliable in solving proportion problems, but as students develop, their strategies get more sophisticated until the students are able to reason proportionally. From additive strategies, students seem to move on to other preproportional strategies.

A significant contribution of the review by Tourniaire and Pulos (1985) and the accompanying study by Tourniaire (1986) was to give credence to the hypothesis that
the strategies selected by students on a variety of tasks could give us insight into how students developed into proportional thinkers and to "define developmental sequences" (p. 199). The authors posited that what remained to be articulated was a model, which integrated a developmental sequence (Piaget, 1964) as well as attention to the relative difficulty of tasks. These authors also speculated that the study of "elementary proportional strategies" might give insight into how students develop strategies and how instruction might be improved.

In their review of the literature on proportional reasoning, Behr, Harel, Post, and Lesh (1992) added what was known about the intuitive qualitative reasoning strategies of children. Similar to what Tourniaire and Pulos (1985) referred to as elementary, intuitive qualitative strategies were seen as precursors to genuine proportional reasoning (Inhelder & Piaget, 1958). These strategies gave students methods of answering contextual problems involving proportion without explicitly recognizing the invariance of the relationships or referring to the proportion in quantitative terms. Instead, students relied heavily on experiences with the inherent relationship of quantities in the task as well as contextual cues. Lamon (1993) referred to these strategies by Piaget’s original term, preproportional (Inhelder & Piaget, 1958). Her original outline of a developmental sequence is summarized in Table 2.

Other studies have looked at particular strategies in many contexts (e.g., Lamon, 1993; Lo & Watanabe, 1997) which can be leveraged to successfully solve some types of proportion problems. In particular, strategies such as partitioning, building-up, norming, and unitizing have been established as successful in the domain of rational
number and proportional reasoning (Lamon, 2007). These strategies fit within Lamon’s preproportional strategies category and can be described further.

Table 2

*Lamon’s Strategies for Solving Ratio and Proportion Problems (1993, p. 46)*

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nonconstructive Strategies</strong></td>
<td></td>
</tr>
<tr>
<td>Avoiding</td>
<td>No serious interaction with the problem</td>
</tr>
<tr>
<td>Visual or Additive</td>
<td>Trial and error or Responses without reasons or Purely visual judgments (“It looks like…”) or Incorrect additive approaches</td>
</tr>
<tr>
<td>Pattern Building</td>
<td>Use of oral or written patterns without understanding numerical relationships</td>
</tr>
<tr>
<td><strong>Constructive strategies</strong></td>
<td></td>
</tr>
<tr>
<td>Preproportional reasoning</td>
<td>Intuitive, sense-making activities (pictures, charts modeling, manipulating) and Use of some relative thinking</td>
</tr>
<tr>
<td>Qualitative proportional reasoning</td>
<td>Use of ratio as a unit and Use of relative thinking and Understanding of some numerical relationships</td>
</tr>
<tr>
<td>Quantitative proportional reasoning</td>
<td>Use of algebraic symbols to represent proportions with full understanding of functional and scalar relationships</td>
</tr>
</tbody>
</table>

*Partitioning* is the strategy of “fair sharing” and has a firm foothold in the literature on rational number (e.g., Lamon, 2007; Pothier & Sawada, 1983). Relative to proportional reasoning, partitioning is seen as laying groundwork for determining the equivalence of ratio. *Building-up* strategies are defined as extending a ratio relationship by addition. This strategy relies on pattern recognition and replication (Lamon, 2007).
Norming involves taking one of the ratios and using it to reinterpret the other ratio. In order to use such a strategy, the student counts two different types of objects simultaneously using different units, for example, counting off eight people every time two cars are counted off to imply the ratio of 8 people per 2 cars. Lastly, unitizing is the act of chunking or reorganizing a given quantity into manageable chunks (Lamon, 1993). Lamon (2007) sees the construction of complex units as an advancement over the construction of single units for the purposes of comparing ratios.

What separates these strategies from the other more primitive (non-constructive) strategies and defines a student as "preproportional" is that they are all successful strategies that can be used to solve many problems correctly. An additive strategy would illustrate a non-preproportional strategy because it does not give correct solutions. On the other end of the spectrum, what sets these strategies apart from proportional reasoning is that students lack the awareness of the relationship of the relationships, in other words, the proportion. Lamon (1993) describes this as "without understanding scalar and functional relationships" (p. 45).

There is a firm distinction between preproportional reasoning and proportional reasoning. Lamon (1993) defines proportional reasoning as occurring when "a student could demonstrate understanding of the equivalence of appropriate scalar ratios and the invariance of the function ratio between two measure spaces, whether or not the student could represent these relationships symbolically" (p. 45). As a final distinction, if a student was able to represent the relationship symbolically in a proportion, that student was said to be using quantitative proportional reasoning, as opposed to qualitative proportional reasoning.
Definition of Problem Types

The domain of ratio and proportion tasks is the space defined by four particular problem types. Traditionally, Comparison and Missing-Value Problems have been used by researchers and curriculum developers to test students' ability to reason proportionally, while Qualitative Prediction and Qualitative Comparison tasks have been identified only in more recent literature (Cramer & Post, 1993a; Lamon, 2007; Lesh et al., 1988; Tourniaire & Pulos, 1985).

Missing value problems are those where students are expected to identify a fourth, "missing value" when presented with three other values from the proportion. The literature is rife with examples of these tasks, some of which include finding missing side lengths on similar rectangles or identifying how many pellets of food a longer fish would eat. Comparison problems involve the comparison of two scenarios to determine an order relationship. Instead of one of four values, the relationship of the ratios is held in reserve. A well-known context for comparison problems are juice recipes where the student identifies an ordering based on the strength of the taste (Noelting, 1980). Another comparison problem would be to analyze two differently sized tables with varying amounts of pizza on them to determine where one might sit to get the largest serving (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998a).

Qualitative prediction and comparison problems are similar in nature to the aforementioned types, however they do not include numerical values. Cramer and Post (1993a) describe these tasks as targeting the conceptual understanding that one must have to truly reason proportionally. Without this awareness, students are prone to following rote procedures or solving problems algorithmically without understanding.
By Lamon's (2007) estimate, only 10% of adults are able to reason proportionally. This number is generally overestimated, particularly when success on numerical comparison and missing value problems is taken as indicative of the presence of reasoning.

*Locating Similarity*

Tourniaire and Pulos (1985) list four main categories of contexts for tasks found in the literature: physical tasks, rate problems, mixture problems, and probability tasks. Provided examples of physical tasks include the Piagetian balance beams and projection of shadows (Inhelder & Piaget, 1958). Rate problems include, but are not limited to, the fish and food task (Inhelder and Piaget, 1958) and Mr. Tall and Mr. Short task. Mixture problems are those such as comparing orange juice recipes taste (Noelting, 1980). Note the absence of examples of probability and similarity tasks. Even though it was defined earlier as a problem type, the omission of probability tasks at this stage may suggest that probability tasks were not used extensively by researchers, particularly in the studies reviewed here. The absence of similarity tasks could reflect that these tasks were inducted into the working set used by proportional reasoning theorists after 1985. Prior, they had been included only in the literature of geometric and spatial thinking (Piaget, 1966; Piaget et al., 1966).

Lamon (1993) reorganized these contexts into four more global categories that encompass tasks used in the literature and provide an inclusive structure for new tasks. Lamon calls these semantic types. They include Well-Chunked Measures, Part-Part Whole, Associated Sets, and Stretcher and Shrinkers. These types are summarized in Table 3.
### Table 3

_Lamon's (1993, pp. 42-43) Framework of Semantic Problem Types_

<table>
<thead>
<tr>
<th>Semantic Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type 1: Well-Chunked Measures</strong></td>
<td>This semantic type involves the comparison of two extensive measures, resulting in an intensive measure (or rate). The term well-chunked (Kaput, 1985) refers to the fact that a third quantity, which is the relationship between the two quantities forming the rate, is itself a well-known entity.</td>
</tr>
<tr>
<td><strong>Type 2: Part-Part Whole</strong></td>
<td>In a part-part whole context, the extensive measure (cardinality) of a single subset of a whole is given in terms of the cardinalities of two or more sub-subsets of which it is composed.</td>
</tr>
<tr>
<td><strong>Type 3: Associated Sets</strong></td>
<td>Sets may have no commonly known connection or an ill-defined connection (e.g., people and pizzas) until some explicit statement in the problem indicates that rate pairs should be formed.</td>
</tr>
<tr>
<td><strong>Type 4: Stretchers and Shrinkers</strong></td>
<td>When a one-to-one continuous ratio-preserving mapping exists between two quantities representing a specific characteristic of an element, namely, a measure of distance such as height, length, width, circumference, and so on, or between two quantities representing two such characteristics of an element (e.g., length and width), the situation involves scaling up.</td>
</tr>
</tbody>
</table>

The Well-Chunked Measure refers not to the four quantities in a proportion, but in the rate-defining relationship. In these problems, a rate is easily identified from the given quantities and it is clear how the three or four quantities given are to be compared. An example of a Well-Chunked Measure would be miles per hour. If the intended rate or relationship between the quantities is obfuscated, then the task becomes an associated set problem. The relationship between hot dogs and adults at a picnic, for
example, is not a common rate and is not readily identifiable. It must be provided for the student in the statement of the problem.

The quantities given in a part-part-whole problem are the mutually exclusive subsets of a bipartite set. If combined, the quantities yield the whole, which can then be compared to the parts. Tasks can involve any bipartite set including boys and girls in a population. Discreteness is what separates these problems from those of the Stretcher and Shrinker type. Stretcher and Shrinker problems incorporate a continuous mapping of one quantity onto another and include all proportional tasks that fall under the category of linear functions (Karplus, Pulos, & Stage, 1983; Lamon, 1993). This is the type that also includes similarity tasks. Involved in the stretching and shrinking of shapes is a continuous all-over growth.

**Student Difficulties with Proportion and Similarity**

In Lamon’s (1993) study of all semantic types, students were the least successful with Stretchers and Shrinkers, which showed the highest number of avoidance and incorrect approaches. There are many factors described in the proportional reasoning literature which may indicate why similarity tasks in particular are more difficult and do not appear to students as places where more proportional reasoning can be used. First, continuous quantities are more difficult for students to manage than discrete (Rupley, 1981). Second, while some components of shape can be represented visually, the functional act of stretching a quantity is difficult to represent physically. The lack of physical representation increases the difficulty of tasks (Lamon, 1993). Third, because of the nature of scaling, similarity tasks are particularly prone to inconsistent visual
cues, another source of contextual difficulty that can be added to problems (Behr, Lesh, Post, & Silver, 1983; Cramer, Post, & Behr, 1989).

However, these contextual factors are not enough to completely explain the nature of student difficulties with similarity. Lamon (1993) noted that even students who had exhibited the ability to reason proportionally on other tasks did not do so on Stretchers and Shrinkers. This would indicate that the tasks themselves were not seen by students as places where proportional reasoning was required or useful. In order to understand why these tasks did not solicit the same types of responses from students, we look at the main difference, that being the geometric rather than numerical context.

Consider a student who has learned one of the four preproportional strategies used above. Each (building up, norming, unitizing, and partitioning) is heavily dependent on a numerical context where quantities and objects can be grouped and ungrouped at the student’s will. In fact, the grouping of the quantity is synonymous with the grouping of the objects. The act of grouping and ungrouping may not translate well to a geometric context. Consider the complex task of scaling a paperclip to twice the original size. The result cannot be understood as a grouping of two of the original set. In a geometric context, the manipulation of quantities in a problem is not the same as manipulating the objects in the problem, as is the case with quarters or candies, for example. Thus, the nature of the geometric context of similarity is significant. The following section reviews what is known about the development of geometric thought and the implications of this development on the conceptions students have of similarity.
Geometric and Spatial Reasoning

Definition of Terms

In this section, two terms will be used that are somewhat intertwined, geometric reasoning and spatial reasoning. Used here are Battista’s (2007) definitions of these terms, which represent their use in the field. Geometric Reasoning can be defined as the act of “inventing or using formal conceptual systems to investigate shape and space” (p. 843). Spatial Reasoning is “the ability to ‘see,’ inspect, and reflect on spatial objects, images, relationships, and transformations” (p. 843). Battista (2007) goes on to say that “spatial reasoning provides not only the ‘input’ for formal geometric reasoning, but critical cognitive tools for formal geometric analysis” (pp. 843-844). In this sense, individuals reason spatially when they observe and operate on shapes and images in order to describe, define, and prove statements about the images geometrically.

Becoming a Geometric Proportional Thinker

Piaget. Piaget’s theory of how children come to reason geometrically and spatially must be interpreted in the structure of the four stages: sensorimotor, preoperational, concrete operational, and, finally, formal operational. At the sensorimotor stage, students are aware of shape only topologically and are not able to perceive shape from projective or Euclidean perspectives. It is only during later stages of development that children are able to perceive measurements and quantitative relationships of shapes and their components (Piaget, 1966; Piaget et al., 1966). This would imply that young children using visual perception would look at the overall
topology of the shapes to determine similarity, for example, the number of sides and angles rather than their measure. This theory might explain why children have difficulty seeing ratio as a sorting criteria for classifying rectangles (Vollrath, 1977).

This particular component of Piaget’s work has come under scrutiny. Geeslin and Shar (1979) theorize that students instead compare shapes on the basis of how much one shape would have to be stretched or distorted to transform it into the other. Even after one has been transformed using isometries or similarity transformations, it still may differ from the original. In this case, the authors suggest that the student then considers a measure of how much the figures still differ, or how much they are distorted. In a study of young elementary students, they found evidence to support this theory—that students showed ability to perceive both topological and Euclidean similarities in shape and define similar figures to be those which are least distorted. In their study, students were given an image along with two variants. One variant was topologically equivalent to the original and the other equivalent in a Euclidean sense. Neither was congruent to the original, and for each, the amount of distortion was calculated based on a model of examining how much of the figure they would need to change in order to achieve congruency. The amount of distortion was varied, sometimes favoring the topological variant and sometimes the Euclidean. The model was successful in determining the likelihood that a child would pick a given variant. Their study helps to resolve some contradictory findings that emerged in testing the original Piagetian conclusion (Fuson, 1977).

In work specifically with similarity tasks, Piaget (1966) found that students were not able to scale images at all until they had reached the preoperational stage, and even
then, children could only make sense of scaling in one dimension and doubling or tripling was seen as making lengths longer by indiscriminant amounts. At the concrete operational stage, children can scale objects according to one dimension, but lack the ability to coordinate two-dimensional growth such as that required to double the size of a rectangle. Finally, at the formal operational stage, students are able to perceive proportional and multi-dimensional growth and are capable of the insight required to connect this growth to the concept of a ratio.

Although Piaget (1966) observed students making the conceptual leap from visual perception to ratio during interviews, other researchers claim that this leap should be attributed to the instructional trajectory and logical path of the tasks used by Piaget rather than a cognitive development of students. In an unpublished study of late adolescents, Spyrou and Kospentaris (n.d.) found that students do not connect similarity with the constant ratio of the sides, regardless of formal instruction on the topic. They asked students to do two tasks; first, given an original image, find a reduced version of the image from four choices with dimensions indicated. Second, find a pair of similar rectangles from a provided group with no original indicated. They found that students do successfully differentiate similar figures, but do so using visual perception rather than quantitative proportional thinking. This was the case for both adolescents who received high school training in similarity and proportion and those who did not.

van Hiele. Like Piaget's theory, the van Hiele theory of geometric development is based on discrete and hierarchical levels of reasoning. While it has not been explicated in terms of how students conceptualize similarity, the van Hiele theory is largely seen as the most accurate and useful model available (Battista, 2007) for
geometric reasoning. Hoffer's (1981) modifications of the original model are summarized in Table 4 and will be used for the purposes of this study.

Table 4

van Hiele Model of Development in Geometry (Hoffer, 1981)

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0</td>
<td>The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.</td>
</tr>
<tr>
<td>(Visualization)</td>
<td></td>
</tr>
<tr>
<td>Level 1</td>
<td>The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.</td>
</tr>
<tr>
<td>(Analysis)</td>
<td></td>
</tr>
<tr>
<td>Level 2</td>
<td>The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.</td>
</tr>
<tr>
<td>(Abstraction)</td>
<td></td>
</tr>
<tr>
<td>Level 3</td>
<td>The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.</td>
</tr>
<tr>
<td>(Deduction)</td>
<td></td>
</tr>
<tr>
<td>Level 4</td>
<td>The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.</td>
</tr>
<tr>
<td>(Rigor)</td>
<td></td>
</tr>
</tbody>
</table>

The difficulty researchers have with diagnosing individuals using this framework suggests that the complexity of this development has not been fully modeled or understood (Battista, 2007). Most importantly, as Lamon (1993) documented different levels of reasoning for students engaging with different contexts of proportion, Clements and Battista (2001) hypothesize that students can exhibit multiple levels of
reasoning simultaneously on different subtopics of geometry. The important implication of this hypothesis for this study is that it assumes that the van Hiele levels are relevant beyond the domain of the study of two-dimensional shapes and can be applied to other domains of geometry such as similarity. In fact, studies have been done to extend the descriptions of the van Hiele levels to student conceptions of three-dimensional shapes (Gutierrez, Jaime, & Fortuny, 1991), transformations (Lewellen, 1992) and even outside the domain of geometry to algebra and the language of functions (Isoda, 1996).

"Pre-Abstraction" Strategies: Using Visual Perception

Children as young as third grade have intuitions about what it means for two figures to be the "same shape" (Lehrer et al., 2002; Swoboda & Tocki, 2002; Van den Brink & Streefland, 1979). These intuitions are based on visual perception, which Swoboda and Tocki (2002) describe as a "natural" occurrence. Van den Brink and Streefland (1979) observed children making estimates of the "real life size" of various objects based on their memory of these objects in relation to others in drawings or photographs. One boy, having seen a picture of a ship propeller juxtaposed with a man, expressed that he believes that a real propeller would not fit in a living room since the drawing of the propeller was bigger than the drawing of the man.

This student, 7 years and 4 months old, and other young students in the same study, demonstrate that visual perception and unconscious conceptions of invariance play a key role in understanding geometric ratio and proportion. This ability to visually perceive the relational size of objects and extrapolate the size of such objects in a new context is largely unexplained. Freudenthal, quoted by his colleagues, says, "I go even
as far as saying that congruencies and similarities are built into the part of our central nervous system that processes our visual perceptions.” The mechanism by which it is built, and the method by which we perceive is to him “an enigma” (Van den Brink & Streefland, 1979, p. 408).

Lehrer et al. (2002) argue that despite the mathematical importance of similarity, it is a topic conspicuously absent from curriculum prior to the middle grades. Using the methodology of the design experiment (Cobb & Steffe, 1983), this group looked at supporting and documenting the emergence of early conceptions of similarity. Rather than expecting a mature understanding of similarity to develop, they focused instead on developing introductory conceptions, based on student visual perception and unconscious conceptions that could serve as a foundation for later study.

Since they and other researchers have documented that we can perceive proportion and make it operational even at a young age, it would seem that the human brain, like Freudenthal suspected, is programmed to make sense of geometric proportion and similarity. It would also seem that our perceptions and intuitions would smooth the conceptual development, and that similarity, as a context for proportional reasoning, would certainly be no harder than any other. Evidence has been presented to suggest the contrary, that it is the hardest context for making sense of ratio and proportion (Hart, 1984; Kaput & West, 1994; Lamon, 1993). Even students who have demonstrated a sophistication of multiplicative reasoning in other contexts still use additive strategies in the context of similarity (Chazan, 1987; Lamon, 1993). Additionally, proportionality seems to be confounded by the concept of dimension, and students are perplexed by differences in the magnitude of linear growth versus area or
volume growth (Chazan, 1987). And so, even young students have intuitions about what it means for two things to be the “same shape” (Swoboda & Tocki, 2002), but young and old students utilize additive strategies to construct shapes that do not preserve these intuitions about proportion (Chazan, 1987; Cox, Lo, & Mingus, 2007).

Similarity at the Crossroads

Why is it so difficult for students to transcend visual strategies into preproportional or proportional thinking in the context of similarity? What is it about these tasks that prevents students from advancing into the quantitative realm, even after they have done so in other contexts? Swoboda and Tocki (1985) comment:

It is not easy to cross the path between a visual perception and a mathematical description of the numerous relationships between lengths of the segments. Visual perception is spontaneous, natural. The mathematization process needs a conscious act of abstraction, and the ability of paying attention to the isolated parts of the figures. (Section 3.3)

The answers to these questions lie in bridging the previous work characterizing similarity tasks with an analysis of where students access them visually, and what barriers exist preventing more sophisticated solution strategies from being used. Furthermore, it would be important to identify how visual perception is related to other preproportional and proportional strategies. Understanding the nature of visual perception and the boundaries of qualitative and quantitative proportional reasoning will provide important information about how intermediate tasks might be designed to
support conceptual growth. Lehrer et al. (2002) have already hypothesized that identifying equivalence classes of shape may be one of these intermediate tasks.

Some initial work has been done to characterize the visual perceptions of similar figures. In a study with seventh-grade students, Cox, Lo, and Mingus (2007) investigated the impact of a variety of features on students’ ability to perceive geometric proportion. Two components of research gave different perspectives on this question. First, students were asked to visually determine if pairs of figures were “dilations” of one another. This provided a diverse set of data from which to begin to identify differences in visual perception, and to identify, in a broad sense, features that supported students in identifying the presence of proportionality, and features that seemed to hide it. Second, individual students were asked in one-on-one interviews to describe the features of shapes that they found helpful and to explain how they made decisions about proportionality.

From the perspective of the van Hiele levels, the transition from visual strategies to more analytical strategies may primarily occur as a child transitions from level 0 to level 1. Reliance on visual perception is a hallmark of early geometric thinking, representative of van Hiele level 0. At this level, students perceive the gestalt of a shape, but not individual properties. In terms of the work by Cox et al. (2007), students who identify shapes as being non-similar because one appears “smooshed” or distorted might characterize a level 0 response. In fact, distortions may be identified in terms of a dominant property such as an angle or a length. Students who comment that a figure is “too fat,” to be similar to another, or that one rectangle is “longer” than another are informally identifying that the ratios of width to length are not equal. Swoboda and
Tocki (2002) suggest that students do use distortion transformations as early tests for similarity. Students then use the absence of distortion as a definition by which to identify similar figures as those that “look the same” as Geeslin and Shar (1979) conjectured. The perception of distortion at a visual level still exists holistically, observable in the overall image rather than in isolated parts.

In comparison to the description of distorted shapes above, students who identify angles as “pointier” or noses that are “longer” may be more sophisticated describers of distortion. These students provide comparison of individual components of figures and look at informal quantitative relationships between those parts in order to describe the distortions they perceive. While the original van Hiele levels provide a gross characterization of these students as level 1, it can be argued that they are just beginning to exhibit this type of reasoning, and have not achieved the full sophistication that level 1 thinking requires. A fully sophisticated response would require students to formally quantify relationships between the two images in a multiplicative way and students would need to verbalize an expected quantified relationship between given similar figures. The need to subtend the coarse levels that van Hiele identified is not a new need. Other researchers have struggled to pinpoint specific levels for individuals and have identified sublevels (Battista, 2007). Table 5 summarizes identified sublevels for the first three levels (visualization, analysis, abstraction).

The literature on proportional reasoning says that as students begin to advance beyond basic intuitions they reach for ways to quantify the relationships they perceive. In the context of similarity, students often look to additive relationships to describe proportional size changes (Chazan, 1987; Hart, 1988; Lamon, 1993, 2007). Even though
Table 5

*Battista's (2007) van Hiele Sublevels*

<table>
<thead>
<tr>
<th>Level</th>
<th>Sublevels (Battista, 2007)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0 (Visualization)</td>
<td>0.1 Pre-recognition: students are unable to identify many common shapes (p. 851)</td>
</tr>
<tr>
<td></td>
<td>0.2 Recognition: students correctly identify many common shapes (p. 851)</td>
</tr>
<tr>
<td>Level 1 (Analysis)</td>
<td>1.1 Visual-informal componential reasoning: students describe parts and properties of shapes informally and without precision (p. 851)</td>
</tr>
<tr>
<td></td>
<td>1.2 Informal and insufficient-formal componential reasoning</td>
</tr>
<tr>
<td></td>
<td>1.3 Sufficient formal property-based reasoning: students explicitly and exclusively use formal geometric concepts and language to describe and conceptualize shapes in a way that attends to a sufficient set of properties to specify the shapes.</td>
</tr>
<tr>
<td>Level 2 (Abstraction)</td>
<td>2.1 Empirical relations: students use empirical evidence to conclude that if a shape has one property, it has another. (p. 852)</td>
</tr>
<tr>
<td></td>
<td>2.2 Componential analysis: students deduce that when one property occurs, another must occur in a part by part analysis</td>
</tr>
<tr>
<td></td>
<td>2.3 Logical inference: students make logical inferences about properties; they mentally operate on property statements, not images.</td>
</tr>
<tr>
<td></td>
<td>2.4 Hierarchical shape classification based on logical inference: students use logical inference to reorganize their classification of shapes into a logical hierarchy.</td>
</tr>
</tbody>
</table>

this is erroneous, it places students at a midrange abstract (level 2) reasoning, where they have advanced beyond informal comparisons such as “longer,” “fatter,” “pointier”
to describe the relationship of corresponding parts of figures, but have not yet identified the multiplicative relationship necessary for similarity.

Even if a student could abstract the definition of similar figures and apply it formally to classify and construct figures, in order to have achieved a deductive (level 3) response, some reasoning about necessary and sufficient conditions must be exhibited. A student would need, for example, to be able to relate that not all correspondences in two triangles must be checked in order to determine if they are similar—that a set of sufficient conditions is available and reasonable. Furthermore, elaboration of similarity as an equivalence relation complete with acknowledgment of transitivity would be necessary. Transitivity is not intuitive for beginning students (Cox et al., 2007).

Certainly, it is not expected that student reasoning in this study would be advanced enough to warrant being labeled as rigorous (level 4). This would require abstraction of not only the concept of similarity, but metric spaces in general, material that is generally reserved for study at the post-secondary level.

Rethinking Geometric Proportional Problem Types

In an analysis of tasks used in various middle school curricula, evidence is presented that would suggest that two main types of similarity tasks exist (Lo, Cox, & Mingus, 2006). This analysis of two NSF-funded curricula, including MathScape (Education Development Center [EDC], 1998), and the Connected Mathematics Project (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2007), and one Japanese text (Shin, 2000) identified differentiating and constructing as major types of activities that are used by
curricula to develop the notions of similarity. In a differentiating activity, students are either asked to determine if a given pair of figures are of the same shape, or to identify those figures with the same shape among a set of given figures. The assumed basis for this determination is either the intuitive notion of same shape, or the properties of similarity. In a *constructing* activity, curricula may provide students with specific tools (e.g., grid paper, ruler, and protractor) and/or step-by-step instructions (e.g., rubber band stretcher) when carrying out this type of activity. The activity may include a specific scale factor or leave it more open-ended while providing other information such as measures of side lengths or angles. In the latter case, students’ choices are typically bounded by the limitations of the materials or space afforded to them.

The monikers of “comparison” and “missing value” are problematic in describing these tasks in that differentiating activity requires comparison, but is not as tightly controlled as a four-quantity comparison problem. A constructing activity includes finding missing quantities, but again, is not limited to finding one missing value and also requires recontextualization once the values have been found. The student must translate the discovered value and transform it into the length of a side, correctly oriented. Further description of the mathematical activity embedded within differentiating and constructing activity is provided next.

*Differentiation*

A differentiating activity requires students to compare and classify figures as similar or non-similar. To do so requires a comparison of ratios in order to determine the existence of an equality relationship. These tasks may require the comparison of
ratios. However, the goal is not to establish an ordering relationship; in the context of similarity, only equal and non-equal make sense. Furthermore, while in the case of rectangles it is sufficient to compare two ratios, this is not the case in more complex figures. To determine that irregular hexagons are similar, for example, the comparer must manage a comparison of six ratios. Tasks where students compare triangles or rectangles might limit the extent to which conceptual power can be observed.

It is important to note that even if the ratios of the sides have been successfully managed and compared, differentiating similar figures also relies on the comparison of corresponding angles. In the case of irregular hexagons, six angle pairs must be shown to be equal in measure. In the case of rectangle tasks, where angles are defined as equal, the definition of the shape may mask even more variance in student conceptual power. By ignoring more complex shapes, we miss opportunities to note where visual perception provides support in solving the problem, but obfuscates the true nature of the concept.

The majority of experiences students have had with shape involve more complexity than rectangles. With shapes such as hexagons or paperclips, comparing shapes is more complex than the comparison problems described in the literature of proportional reasoning (Lamon, 2007; Tourniaire & Pulos, 1985). Furthermore, deciding if two figures are similar is a different kind of task than picking two similar figures from an expanded set. In this case, multiple comparisons must be made, organized, and sorted in order to locate the intended pair and using ratios of lengths within a shape can provide a significant advantage. If students have not yet learned to use these reliably, it may explain why they fall back to more visual strategies.
Construction

Construction activities, like differentiating activities, have some of the structure of a missing value task, but are slightly more complex in nature. Students can be asked to make two essential constructions, types that are determined by the information provided in the problem.

In the first type of construction activities, an example of which is in Figure 1, the model of a missing value task is more directly followed. In order to complete the task, a student must first recognize the equality of the ratios and then scale up the remaining side. Finding the scale factor is prerequisite to finding the remaining missing value, the length of the image. As figures get more complex, students must find more missing measurements. The actual number of missing values for the student is masked, and the student can fall back on visual strategies to determine the overall shape, rather than proportional reasoning.

**Here I have a rectangle that is 3 units tall and 5 units long. Draw a rectangle that is similar to this rectangle that is 6 units tall.**

![Rectangle](image)

*Figure 1. Construction Activity*

In the second case, students are provided with a preimage and a magnification factor, in which case the missing values are the measurements of the resulting image. In this case, the task does not entirely fit the notion of a missing value task. The task
makes explicit the invariance of the scale factor, thereby preventing the student from challenging or considering the second order relationship, which prevents proportional reasoning from occurring. However, there are different ways of representing the scale factor, some of which may present more explanatory information to students. Representing the scale factor as a percentage or by including the stipulation that a shape is “three times” bigger may push students to use a multiplicative structure rather than referring to the scale factor as a functional magnitude which leaves open for students whether an additive or multiplicative model is expected or useful. These tasks mask the actual number of missing values for the student, who has no way to verify if all required values have been found. To illustrate, in the case of the flower in Figure 2, a student must find the overall dimensions of the center as well as the circle circumscribed about the figure, must negotiate the growing petals and attend to the graduated width of each, and must be sure to include the exact number of petals as in the original.

**Conclusion**

At the heart of this study is the question of how students understand similarity, and the transition of students from visual perception of distortion to being able to construct and utilize the formal concept of similarity. This transition is not fully described by any known theory or literature, existing as a moment of insight for Piaget, and hidden in the complexity of Abstraction (level 1) for van Hiele. Much can be learned by studying how this transition occurs, and what intermediate stages emerge as helpful.
The clinical interview, using a stratified purposeful sample chosen using the revised Similarity Perception Test, has been chosen as the primary method of data collection. Tasks developed for these interviews included both differentiating and constructing activities and included more complex figures. Although this will be discussed in more detail in the following chapter, it is important to note that as a theory about this identified transition emerges, only by studying the actual reality of student thinking can the limits of the theory be tested.
CHAPTER III

METHODOLOGY

This study sought to uncover how students make sense of similarity and geometrical proportion. A population of students in a Midwestern, urban school district was identified to target racial, economical, and academic diversity. An assessment, the revised Similarity Perception Test (rSPT) was administered to the entire population for sampling purposes. From this population of 91 seventh-grade students, a stratified purposeful sample \( (n = 21) \) was selected for task-based interviews. While the rSPT returned data useful in sample selection, the corpus of data used to answer the questions raised by this study was videotaped interview data and accompanying student work.

The task-based interview exemplified by the work of Piaget has been an often-used method in exploring student reasoning for the purpose of modeling student conceptions and learning. It has been used in the context of proportional reasoning (i.e., Lamon, 1993), geometric reasoning (i.e., Burger & Shaughnessy, 1986), and, specifically, similarity (Chazan, 1987; Piaget, 1966). A researcher may use an interview format to “uncover the meaning structures that participants use to organize their experiences and make sense of their worlds” (Hatch, 2002, p. 91). One limitation of choosing to rely on interviews is that it is not feasible to include a large number of participants and requires sampling a given population. It was important to sample a broad range of students; thus, a stratified purposeful sample (Hatch, 2002) was used.

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This method of sampling intentionally included individuals who exhibited varying abilities, perceptions, and strategies.

In order to find such a sample, some preliminary assessment of the entire population was conducted. The chosen assessment, the revised Similarity Perception Test (rSPT), provided information about students' visual perception of shape, correspondence, and size transformation and divided students into subgroups according to their responses, and representatives from select large and medium subgroups were chosen for interviews. Additional students who showed unique response patterns were added to the sample.

In this chapter, I will provide more detail about how this sample was chosen, how the interviews were conducted, and how the collected data were interpreted. First, however, it is necessary to make explicit the underlying theoretical perspective and framework. The framework describes a basic model for interpreting van Hiele levels in the context of similarity, and a characterization of geometric proportional thinkers modeled after Cramer and Post's (1993) characterization of a proportional thinker. This framework guided the development of interview tasks as well as the revision of the original SPT for use in this study. The remainder of the chapter will describe the population and setting and provide detail about the collection and analysis of data.
Methodological Considerations

Theoretical Perspective

It is important to delineate what assumptions about knowledge are being made and how these assumptions impact the design of the study and analysis of the data. There are three assumptions that must be addressed:

1. Knowledge is individually constructed.
2. Students do not operate at the limits of their understanding, but in a Zone of Proximal Development.
3. Students have a rich store of concept imagery that impacts their understanding of formal definitions.

Knowledge is individually constructed. This study assumes a constructivist perspective as a theory of learning for the inquiry into student conceptions and the modeling process. This has two implications for the study at hand. First, there is the direct implication that without observations of students themselves, no theory can stand apart from the limitations of the mathematical understanding and biases of the researcher (Cobb & Steffe, 1983). The previous chapter outlined a model representing a theory about what it means to do similarity tasks and understand similarity. This model takes into consideration individual prior knowledge and experience, qualities of provided and supposed tasks, and the relation of this mathematical topic to proportional and geometric reasoning. This theory alone, however, is not enough. Theories help to make sense of the actions students take and the words they use to communicate, but without the direct interaction with students, the theory is but a shell and cannot fully
explain what it means to understand or what it is like to learn a concept. By observing students interacting with a task and engaging in the process of learning, the researcher engages in formative assessment, revising the theory if tensions arise between expected and observed behaviors. By observing students interacting with the ideas behind the theory, we open the theory up to the unexpected (Cobb & Steffe, 1983). Thus, the method of clinical interview (Cobb & Steffe, 1983) was chosen as the primary method of data collection. A clinical interview enables the researcher to study the strategies used by children and to trace from intuition into the process of abstraction how a student learns a concept (Cobb & Steffe, 1983).

The second implication is in registering the significance of the data that are collected. It is possible to have as a goal the empirical vetting of a theory, marking instances where the predictive power is great and where it is not. However, another goal, responds to Vergnaud’s (1987) challenge to “understand better the processes by which students learn, construct or discover mathematics and to help teachers, curriculum and test devisers, and other actors in mathematics education to make better decisions” (as quoted in Confrey & Kazak, 2006, p. 311).

Although a longitudinal study would, perhaps be best suited to answering the question of how students might come to learn similarity, this study is focused on identifying and describing stages of that learning trajectory. This description will provide road signs for later work. This is not an attempt to locate specific exemplars of what we would expect student behaviors to look like, but to give students the full range of tasks and investigate what behaviors emerge.
Students do not operate at the limits of their understanding, but in a Zone of Proximal Development. The implication of this assumption that is of particular importance in this study is that we cannot ascertain empirically the limits of what a student has learned or understands. The clinical interview, however, is one method by which we can observe a range of behaviors, which determine the zone of proximal development (ZPD) (Vygotsky, 1978). This theory describes the difference between what we are capable of in our own heads, solving problems without interactions with others, and what we are capable of solving when we work collaboratively with others. When we are alone, we work at our actual developmental level. For all students in the study, the actual developmental level was assessed using the SPT, where each student worked individually. During interviews, the researcher had the opportunity to offer probes and scaffolding to assist the student in solving tasks. At this time, the researcher observed the potential level of development of the student. Vygotsky (1978) claims that the difference between what we are capable of in these two levels is our ZPD.

Students have a rich store of concept imagery that affects understanding of formal definitions. Tall and Vinner (1981) draw a distinction between formal concept definitions and the concept imagery that surrounds them. A concept definition is a form of words used to specify a concept while concept images include all mental images and associated properties and processes, which may or may not be conscious to the individual. Researchers have found evidence to support that our everyday experiences impact our understanding of mathematics in helpful ways as well as unhelpful ways (Mack, 1990, 1995). These daily experiences exist within our minds either consciously
or unconsciously as concept imagery, which colors the context of the concepts we are introduced to in more formal circumstances.

While it is natural to expect great variation in the concept imagery held by individuals, there is also great variation found in concept definitions, which have the connotation of being rigid or inflexible. Instead, it is very possible to define a concept in a myriad of ways depending on the audience or intended use of the concept. In terms of similarity, it is not uncommon for a textbook to include a definition of similar figures as those which are the same shape, but different sizes (Lo et al., 2006). This definition is used as an informal marker to enable students to investigate what is meant by same shape. As students are asked to classify shapes on the basis of similarity classes, it becomes necessary to include more specific criteria, leading to a more sophisticated definition. For example, "two figures are similar if and only if their corresponding angles are equal and the ratios of all pairs of corresponding sides are equal" (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998b, p. 186).

Concept imagery surrounds these definitions and enables students to put them to use. Concept images are developed from experiences both immediately related to the statement or exploration of definitions during instruction and in daily life that could be quite far removed from the classroom. These images vary with respect to richness and cohesion. This imagery is both conscious and unconscious, and may be contradictory in nature. We may not be aware of this contradiction if the images are not within the same context or called upon simultaneously. Furthermore, these images vary with respect to richness and connectedness. Some images are used to support a myriad of different concepts, while others are quite limited in their explanatory power.
The representation of experiences and intuitions as concept images helps to formalize what is included in the individual backgrounds of students. It also helps explain the impact prior experiences and intuitions have on a student’s conceptions of similarity. It will be important to ascertain not only what a student thinks as they solve a task, but why they might think this way and why other students might not. Providing an opportunity to explore these inconsistencies was key to the design of the interview protocol used in this study. It was also important to engage students in thinking about where they have seen similar figures before and what they may have come to know unconsciously about “same shape.”

Theoretical Framework

Proportional reasoning has been described as the space defined by two major types of tasks, comparison tasks and missing value tasks (Lamon, 1993, 2007; Lesh et al., 1988; Tourniaire & Pulos, 1985). In a conceptual analysis of middle grades textbooks conducted by Lo, Cox, and Mingus (2006), this space was redefined in the context of similarity to include differentiating and constructing. These two types of activity are related to the tasks of comparing and finding missing values, but describe more authentically the types of activity in which students are expected to engage relative to using proportion to solve similarity tasks. Constructing and differentiating activity are at the center of the developed assessment and interview tasks. Furthermore, a geometric lens was required in analyzing student strategies on these tasks. It was insufficient to analyze these data only for instances of proportional reasoning. Students
also used geometric and spatial reasoning to solve the tasks and it was imperative that this be accounted for.

Lamon (1993) outlines a conceptual progression for the development of proportional reasoning that stems from visual and intuitive solutions and grows through successful preproportional strategies up into mature proportional reasoning. Table 2 on page 16 summarizes how she parsed out beginning and advanced conceptions absent of the context in which they are used and served as a preliminary lens for interpreting the data collected from clinical interviews. This progression was useful in describing and organizing the strategies that are used by students during interviews, and in relating those strategies to other pattern building or preproportional reasoning strategies identified in the literature on proportion.

During an interview, the researcher can probe for further clarification or verbalization, a luxury reserved for the interview environment. In order to gain insight into the subtle differences in student thinking and reasoning, the rSPT was modified to stratify the population and select an appropriate interview sample. It was first revised to include a greater variety of items that would show subtle differences in visual perceptions and in the way students might quantify factors of proportional growth and the numerical relationship of similar figures. In order to modify the original items, a theory to predict what these subtle differences might be was needed. The van Hiele levels of geometric thought and subsequent subdivision (Battista, 2007) provided a geometric lens through which student responses can be interpreted. The following framework, shown in Table 6, reiterates conjectured descriptions of identified sublevels for van Hiele levels 0-2, and recontexts them for similarity.
Table 6

Battista’s (2007) van Hiele Sublevels with Similarity Context Added

<table>
<thead>
<tr>
<th>Level</th>
<th>General Sublevels (Battista, 2007)</th>
<th>Similarity Sublevels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0 (Visualization)</td>
<td><em>0.1 Pre-recognition:</em> Students are unable to identify many common shapes (p. 851).</td>
<td><em>0.1 Pre-recognition:</em> Students are unable to visually discern distorted shapes from those that are similar.</td>
</tr>
<tr>
<td></td>
<td><em>0.2 Recognition:</em> Students correctly identify many common shapes (p. 851).</td>
<td><em>0.2 Recognition:</em> Students are able to visually discern distortion and comment that shapes look “fatter” or “smooshed.” Descriptions are limited to overall shapes rather than specific parts.</td>
</tr>
<tr>
<td>Level 1 (Analysis)</td>
<td><strong>1.1 Visual-informal componential reasoning:</strong> Students describe parts and properties of shapes informally and without precision (p. 851).</td>
<td><strong>1.1 Visual-informal componential reasoning:</strong> Students select portions of shapes for informal (i.e., pointier angles) description of distortion.</td>
</tr>
<tr>
<td></td>
<td><strong>1.2 Informal and insufficient-formal componential reasoning</strong></td>
<td><strong>1.2 Informal and insufficient-formal componential reasoning:</strong> Students describe shapes as in level 1.1, but are able to describe specific quantitative relationships. The term <em>insufficient</em> may indicate additive or incorrect quantification or it may indicate that ratio comparisons are incomplete.</td>
</tr>
<tr>
<td></td>
<td><strong>1.3 Sufficient formal property-based reasoning:</strong> Students explicitly and exclusively use formal geometric concepts and language to describe and conceptualize shapes in a way that attends to a sufficient set of properties to specify the shapes.</td>
<td><strong>1.3 Sufficient formal property-based reasoning:</strong> Students are able to discriminate between and construct similar and non-similar shapes; this requires the identification and pairwise comparison of correspondences.</td>
</tr>
</tbody>
</table>
These conjectured sublevels are based upon student responses gathered during a pilot study for this dissertation, and were vetted during the course of this project. They have been incorporated into the design and scoring of the rSPT, which stratified students on the basis of their verbal perceptions and understandings of similarity. More detail about this instrument and the interview protocol and how these frameworks were used to analyze collected data can be found in subsequent sections.

Participants and Setting

In this study, the population is defined as middle school students. A sample of middle school students was selected to show racial and socio-economic diversity, and includes students with a variety of prior achievement levels in mathematics as measured by state assessments. The inclusion of diversity in the sample for study was not intended to highlight differences between groups of students, but rather to ensure that a broader extent of prior student experience and knowledge is included in the results. As an illustration of potential differences, students from urban areas may have significantly different experiences related to geometric proportionality, such as reading bus maps, that influence conceptions of scale or correspondence. Alternatively, lower socio-economic status may indicate a more limited access to technology, photographic or otherwise, and a different repertoire of imagery that others take for granted.

Secondly, in the sense that the majority of student data related to research on proportion and similarity are more than a decade old (Lamon, 2007), it is likely that technological advances have increased and altered potential imagery for students. Even if no changes were made in the make-up of the population studied, there is great
potential that students would reason differently in this study than in previous studies. The availability of publishing software, photo enlargement machines, and multidimensional video gaming systems may alter potential student imagery and visual acuity, and provide a more fertile ground for developing quantitative strategies.

**Setting**

All data in this study relevant to race, socio-economic status, and aggregate prior achievement are taken from schooldatadirect.org, an online service of the Council of Chief State School Officers (CCSSO) State Education Data Center (CCSSO, 2007). The Urban Meadows School District is an urban, Midwestern school district. It has a student population of 11,684, with approximately 1,800 attending one of three middle schools: Heathside, Fieldstone, and Prairiewood. Fieldstone was selected as the site for data collection particularly because of its diverse student body, but also because of the available technology, which will be described later.

The racial make-up of the district is pictured in Figure 3. White and black students together make up 87.7% of the district’s population, which also includes Hispanic, Asian/Pacific Islander, and American Indian/Alaskan individuals. In addition to racial diversity, students from different socio-economic backgrounds attend the district schools and, according to 2006 data, 61% of the students are economically disadvantaged. This has been determined according to enrollment in free and reduced lunch programs. The racial make-up of the Fieldstone Middle School reflects that of the district; however, Fieldstone has a higher percentage of economically disadvantaged students (70.3%).
The *School Data Direct* website provides the percentages of middle school students who are deemed proficient in mathematics and reading performance on state tests. The percentages of seventh-grade students at all three middle schools who are deemed proficient in reading and math are compared in Figure 4. Note that at Fieldstone, 27% of students are proficient in mathematics, lower than the other two schools in the district. Data for the 2006-2007 school year was unavailable at the time of publication; however, fewer than half of Fieldstone’s seventh-grade students were proficient in math for the 2006 school year, a condition that has been typical for many years.

*Figure 3. Racial Make-Up of Urban Meadows School District*
Because of the long-term failure to raise math and reading scores, Fieldstone Middle School was recently restructured in accordance with the No Child Left Behind Act. It was converted into a district-wide magnet school that emphasized topics in mathematics and science and became a regional leader in providing access to instructional technology. All classrooms were recently outfitted with state of the art digital projection systems and smart board technology as well as individual Classroom Response Systems. The systems arm each student with a remote control device that allows teachers to integrate immediate student response into their formative assessment repertoire. Mathematics classrooms are also equipped with a classroom set of TI-84+ graphing calculators. TI-Navigators are available for mathematics teachers to check out,
which carry with them the capability of networking student calculators and coordinating numeric data and display.

Fieldstone Middle School, like the other middle schools in the district, offers a two-track program. The majority of students opt to take a grade-level general mathematics course. To be eligible for an honors course, students must achieve better than average scores on district assessments as well as show a history of high achievement in previous courses. The top 20% of students are chosen. The honors course uses an accelerated version of the general mathematics curriculum, using the same textbook materials. By the time students enter ninth grade, general mathematics students should be ready to take a first course in algebra. Honors students will satisfy the algebra course requirement in middle school and enter a first course in geometry in ninth grade.

Urban Meadows has a policy of mainstreaming students who qualify for special education services, a policy that is enacted at all three middle schools. This population of students attends general math courses with their peers. There is a self-contained classroom reserved for extremely impaired students; however, this has limited enrollment. Students are placed in the least restrictive environment possible.

Participants

The participants in this study were students who attended seventh grade at Fieldstone Middle School and were enrolled in either the general mathematics course or the honors mathematics course. Out of convenience, a teacher was selected who was assigned to teach both courses. All participants were taught by this teacher and all
students who are taught by this teacher were included in the population. This better ensured that students of all subcategories were included. Students who were returning to the district began studying from the *Connected Mathematics Project (CMP)* (Lappan et al., 2007) in sixth grade; however, the heavy migration of students in and out of the district makes tracking the location and nature of prior instruction and curriculum very complex to monitor or to control as a research variable. Additionally, migration within the district from building to building exacerbates the complexity as not all sixth-grade classes are taught at the middle schools, and not all sixth-grade teachers use the CMP curriculum.

**Data Collection**

Initially, 91 students were administered the *revised Similarity Perception Test (rSPT)*, from which a stratified purposeful sample of 21 students was selected for follow-up interviews. The initial administration occurred at the beginning of an instructional unit on similarity, which concluded before interviews were conducted. In total, 21 student interviews were conducted over the course of 6 weeks during ordinary class periods. Each interview lasted between 1-3 class periods over the span of 3 days.

During the span of this study, students received instruction on similarity. In order to ascertain the nature of this instruction and its potential impact on collected data, five observations of the classroom were made. During these observations, the researcher noted the language and terms used in the classroom related to similarity, potential sources of concept imagery, and the types of activity in which the students engaged. These data shaped the language used during student interviews and helped the
researcher interpret student responses. Prior to each observation and directly following, the classroom teacher was informally interviewed. The foci of these interviews were the observed lesson, lessons planned prior to the next scheduled observation, the teacher's impressions of classroom generated language, and student achievement.

The classroom observations were also a means by which to familiarize the researcher to the students and develop rapport. Interactions in the classroom between students and researcher also served to broadly familiarize the students with the roles that would be assumed during the clinical interview. During periods of time when students were working individually at their seats, the observer visited with students working on similarity problems. Quietly, the researcher probed students to think aloud as they were solving problems, listening for the terms used to describe their thinking and then encouraging students to provide needed clarity. This time was used to establish the role of the researcher as questioner and listener, and the role of the student as explainer and solver. Being able to assume these roles was helpful during clinical interviews.

Assessment and Examination of Students Using the rSPT

In order that a broad spectrum of student strategies be observed in one-on-one interviews, a pre-interview assessment was given to all participants. The instrument used, the rSPT, was piloted by Cox, Lo and Mingus (2007) with seventh graders at Fieldstone during the 2005-2006 school year. The Similarity Perception Test (SPT) was initially developed to mark differences in visual perceptions of geometric proportion, which represented only a limited portion of the spectrum of interest in this study. A
description of the pilot instrument and the results of the study using that instrument are
given in Appendix C. Described here is the revised Similarity Perception Test along
with a rationale for changes made to the instrument.

*Description of the rSPT.* While the pilot version SPT was able to pick up
variations in the visual perceptions of students and identified some early conceptions
about similarity and geometric proportion, it did not return data about progression
beyond visualization. As this study was concerned with identifying strategies used
during the entire progression from visualization to proportional thinking, revisions were
needed to be able to coarsely differentiate students by their ability to visualize
geometric proportional growth, as well as their likely understanding of scaling and the
relationships between similar figures.

For the current study, items were rearranged and new items were generated to
expand the SPT, which is included as Appendix A. The process of reviewing the
organization of existing items and generating new items was undertaken by the author
of this dissertation, assisted by a member of the mathematics education faculty. Items
were then reviewed by three additional mathematics education faculty prior to
administration.

The *revised* SPT (rSPT) is organized into seven sections labeled 0-6. Two of
these sections are entirely comprised of new items; the remaining five sections include
reorganized piloted items. These sections are described in Table 7. Section 0 was
intended only to provide instructions to children, give opportunities to practice
procedures, and calibrate the definition of “same shape, different size” with children.
Vollrath (1977) and Syprou and Kospentaris (n.d.) have shown that the colloquial use of
Table 7

*Overview of the Revised Similarity Perception Test*

<table>
<thead>
<tr>
<th>Section (Number of Included Questions)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 0: Introduction (2)</td>
<td>Description of Purpose, instructions, and calibration of terms <em>same shape, different size</em>.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Perceptions of &quot;Same Shape&quot;</td>
<td></td>
</tr>
<tr>
<td>Section 1: General Shape Identification (5)</td>
<td>Identifies visual perceptions of distortion in the general shape.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Perceptions of &quot;Different Size&quot;</td>
<td></td>
</tr>
<tr>
<td>Section 2: Coordination of Vertical and Horizontal Growth (7)</td>
<td>Identifies visual perceptions of distortion in terms of simultaneous horizontal and vertical growth.</td>
</tr>
<tr>
<td>Section 3: Coordination of Interior and Exterior Growth (7)</td>
<td>Identifies visual perceptions of zooming, understanding of distortion as changing the perspective of the image.</td>
</tr>
<tr>
<td>Section 4: Continuity of Growth Factor (4)</td>
<td>Identifies visual perceptions of continuous all-over growth.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Quantifying perceptions</td>
<td></td>
</tr>
<tr>
<td>Section 5: Quantifying Growth (4)</td>
<td>Identifies strategies of students finding a scale factor quantitatively when presented with consistent and inconsistent visual cues.</td>
</tr>
<tr>
<td>Section 6: Quantifying Measurement (3)</td>
<td>Identifies whether students can use an implicit or explicit scale factor to indirectly measure images.</td>
</tr>
</tbody>
</table>
the terms *similar or same shape* does not carry with it the full geometrical meaning of
the concept of similarity. It was necessary to standardize what is meant by the phrase
*same shape, different size* as well as define *distortion* before continuing to give the SPT.
Although data were collected from responses in this section, they were not used in
subsequent analysis of student perception.

The yes/no format for items from the original SPT was retained for sections 0-5. In each item, students are given a pair of figures and asked if they are different sizes of
the same shape. In each section, similar pairs have been mixed with non-similar pairs.
Further descriptions and illustrations of these items are organized by section.

Section 1 was designed to assess student perceptions of “same shape.” Items test
topological perceptions as well as Euclidean perceptions. The non-similar pairings are
such because of inconsistent features or non-proportionality. An example of an item
from this section would be the pair of As from the original SPT, an item that has been
retained and placed in this section.

Sections 2-4 were designed to assess student perceptions of “different size.”
Each of the three sections includes items that are designed to illustrate different features
of dilation. In section 2, students must coordinate multiplicative vertical and horizontal
size changes. The items in this section were constructed by varying vertical and
horizontal scale factors. In section 3, students must coordinate interior and exterior
growth. This was shown to be an area of tension for students in the pilot test. The
“zoom outs” were retained and put into this section along with one item where an
additive change was made on the exterior while the interior was dilated. Section 4
deepens the ideas from section 3 by requiring students to coordinate the scale factors of
all components of a shape. The U-shape is featured in this section because it is easily partitioned into components that are not related by interior/exterior or vertical/horizontal. Pairs of non-similar U-shapes, such as that featured in Figure 5, were constructed by scaling the legs of the U by a scale factor of 2, while scaling the bottom middle section by a scale factor of 1.5.

Figure 5. Revised SPT Item: Component-wise Dilation

Sections 5 and 6 were designed to assess students’ quantitative strategies. These items are presented in a multiple-choice format where answer choices are selected deliberately to attend to different perceptions of growth in section 5 and measurement in section 6. In section 5, four new items were included, three of which were constructed to provide consistent visual cues. In all items, the dimensions and overall shape of the figures used were carefully selected so that both additive and multiplicative strategies can be indicated. Shapes that are used in section 5 include rectangles, an M-shape, and a square. The squares in the fourth item are designed to incorporate inconsistent visual cues. The labeled measurements indicate a scale factor of 3, while the squares were
constructed using a scale factor of 2. In the first three items, the choices available to students represent horizontal additive, vertical additive, multiplicative linear and area strategies. The option "none of these" is also available for students. On the fourth item featuring inconsistent visual cues, choices include those indicating additive, visual linear, visual area, indicated linear, and indicated area strategies.

In section 6, there are three items featuring rectangles and parallelograms. Students are asked to identify missing linear measurement values given three different scenarios of growth: (1) indicated growth factor ("tripled"), (2) non-indicated integral, and (3) non-integral growth factors ("blown up").

Administration of the rSPT. The rSPT was administered to 95 students in five separate classes. Four of these students were not able to complete the test and were subsequently removed from the study. The administration took approximately 30 minutes and was conducted using a wireless Classroom Response System. Students responded to each item using numbered devices. Students were given as much time as they needed to respond to each item before viewing the next. Ratings were assigned anonymously to students based on the number of their device. A master list, which matched students' names to devices, was kept. This list was used only to match students who consented to participate in a follow-up interview with assigned rSPT ratings.

Student Interviews

Student interviews are the main source of data used to answer the two research questions:
1. What strategies do students use to differentiate similar figures from non-similar figures? What types of geometric and numeric reasoning are indicated by these strategies?

2. What strategies do students use to construct similar figures? What types of geometric and numeric reasoning are indicated by these strategies?

Procedure. Six tasks, included as Appendix B, were administered individually to students in videotaped clinical interviews. The students were told that they would be asked questions about their responses on the SPT and about size changes. Colored markers, plain and grid paper, compasses, and straight edges were made available to students as they worked on provided tasks. After each task was completed, additional probes were used to help the student clarify the given response or to provide additional information. Sample probes are listed in the protocol and include the example, “How did you measure this?”

The interviews were conducted privately in a separate room during the school day. Only the interviewer and the student were present. Each interview lasted approximately 60 minutes; however, some students were asked to return for additional time if the interview ran long. Other students finished the protocol in less time. Videos were transferred to DVD format for storage and to Quicktime format for transcription and analysis.

Interviews were transcribed using Transana. Transcription consisted of typing verbatim relevant dialogue between researcher and participant while watching the video to document actions in written word format. While transcribing, the researcher made clips of each task. These clips were initially made to document compelling exchanges or
verbalized reasoning. Initial informal codes were assigned as a method of creating memos and organizing what was compelling about these clips. As more data were transcribed and more clips created, the memo codes became more nuanced—partially to enable the researcher to differentiate one clip from another, but also to reflect the increasing number of notable themes that were emerging. Later, the printed transcripts were used along with the video to do a more formal coding of the data that incorporated many of the themes noted during the transcription phase.

*Interview Protocol.* The Interview Protocol, included in Appendix B, has four parts: (A) deepening responses to the rSPT (task 1), (B) constructing similar figures (tasks 2 and 3), (C) differentiating similar figures (tasks 4 and 5), and (D) identifying numerical relationships (task 6). Throughout the interview, the researcher was cognizant of student imagery and inquired when students referenced classroom and informal experiences or used nonstandard descriptive language. Furthermore, if it was unclear from a student's verbalized reasoning what imagery they were drawing from, the researcher inquired generally if they had seen shapes that had been enlarged before. If students were unsure, specific probes regarding photography, clip art, computers, magnifying glasses, or observed classroom activities were given.

In part A, students were asked to complete one task: revisit the items on the rSPT. With the researcher, they responded a second time to selected slides at their own pace, describing his or her decision on each response. The original responses were not made available to students as they completed the task, and all new responses were recorded. Students were asked to elaborate on how they decided if the figures were the "same shape, different size." Students were encouraged to point, draw, or otherwise
describe aspects of the shape that were of interest or assistance to them. Students who identified correspondences as justification for their decision, or who quantified lengths or angle measures, were prompted to identify a scale factor for those shapes they identified as similar.

In part B, students were asked to complete up to four construction tasks involving three different shapes: the rectangle with embedded square, the L-shape, and the heart. While most of the tasks were assigned as written, some modifications were made. For example, Tom showed a strong preference for multiplicative models for growth and extremely sophisticated perceptions of proportion. Modifications were made during his interview to add tasks that were more challenging. These tasks involved the same shapes but different scale factors. Other minor modifications for the remaining 20 participants are indicated and described below.

All of the remaining 20 students were given Task 2: *double the size of a rectangle with an imbedded square*. Based on the response to this task as well as task 1, students were assigned task 3A, 3B, or 3C or a combination thereof. The differences in these three tasks were intended to provide challenge, but also to provide different avenues for cognitive tension. Task 3A was similar to Task 2, but students were asked to draw a second enlargement of the square and the rectangle that was *somewhere in between the original and the one they just drew*. Task 3A was given to 13 students. Task 3B asked students to scale an L-shape and was given to 10 participants. Five of these students were given the task without modification to evoke cognitive challenge with additive thinking. The remaining 5 were given the task of shrinking the L shape by a scale factor of 1/2 or 1/3 as a means of adding challenge and also so that strategies
used to shrink could be compared to those used to enlarge shapes. Task 3D, given to 9 students, was used only when time permitted. Students were given a symmetric heart and asked to construct a bigger version of the same shape. In certain cases, the student was told specifically to double the size. In others, the student was given freedom to use any scale factor.

In part C, a maximum of four differentiation tasks were completed by the student. For 5 students, no differentiation tasks were given due to time constraints. Preference was given to the drawing tasks in the event that the interview time was limited. On all tasks, students were given a cluster of shapes and asked to identify those that are different sizes of the same shape. Task 4A featured a cluster of four non-square rectangles and was given to 13 students. Task 5A featured a cluster of five double arrow shapes with some non-right angles and was given to 6 students. In each cluster, there were at least two similar shapes. The remainder of the shapes were constructed using either an additive strategy or by blowing up the figure component-wise using different methods for each component.

It was intended that students who had already demonstrated that they were capable of using multiplicative strategies, indicating a more sophisticated conception of similarity, would be given alternatives to tasks 4A and 5A. Task 4B featured a cluster of double arrow shapes and was given to 7 students, and task 5B featured a cluster of heart shapes and was given to 10 students. However, the rectangles and the hearts were eventually prioritized. In early interviews, the rectangle tasks showed high potential to help students verbalize methods they used to compare shapes numerically. More will be shared about this potential in subsequent chapters. The arrow tasks were used
interchangeably with participants. Both tasks provoked similar tension with integral
versus non-integral scale factors as well as angle comparisons, which the rectangles did
not generate. It is important to note that after the first interview (Jules), the arrows tasks
were revised so that the shapes were larger and more visible on camera. Finally, the
hearts task was given to students to probe more deeply into how strategies differed
when curved lines were used. This obfuscated the correspondences and made numerical
comparisons more difficult for students.

Lastly, in part D, students completed task 6. They were given an assortment of
pairings of figures from the rSPT on paper with limited measurements indicated.
Students were asked to identify other measurements as they could. The pairings were
selected from a pool based on student performance on prior tasks. The pool of six
possible pairings is summarized in Table 8.

Table 8

Description of Pairings on Task 6 from the Interview Protocol

<table>
<thead>
<tr>
<th>Pairing</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>6A</td>
<td>Parallelograms related by a scale factor of 5/3.</td>
</tr>
<tr>
<td>6B</td>
<td>U shapes related by a scale factor of 2.</td>
</tr>
<tr>
<td>6C</td>
<td>(Not given).</td>
</tr>
<tr>
<td>6D</td>
<td>Parallelograms with interior parallelogram with one coincident angle, related by a scale factor of 2 with non-integral side lengths.</td>
</tr>
<tr>
<td>6E</td>
<td>Identical to task 6D but parallelograms arranged concentrically with no portions touching.</td>
</tr>
<tr>
<td>6F</td>
<td>“Valentines”: heart shapes with circumscribed rectangles related by a scale factor of 3/4.</td>
</tr>
</tbody>
</table>
Each student received up to four pairings depending on their prior performance. Three students did not complete task 6 due to time constraints. Of the remaining 18, 16 participants were given task A. Elijah had functioned at a visual/additive level prior to task 6, and was not given this task in lieu of being given pairing 6B. The researcher thought it best to assign to him a task with a familiar integral scale factor (2). Kyle had shown multiplicative reasoning on prior tasks, and the researcher opted to forgo this task in lieu of task 6F. This task required students to use a scale factor less than 1 to solve for a missing side length.

Students were given additional pairings as time permits: 7 were given pairing B, 5 were given pairing D, 3 were given pairing E, and 5 were given pairing F. This arrangement provided suitable levels of challenge for all students, but allowed students to have some pairings in common. Students were given additional prompts to continue to justify and deepen their responses as well as any non-responses. They were also asked, when appropriate, “How much bigger is the larger shape?”

Data Analysis

Scoring the Revised Similarity Perception Test

The rSPT returns three ratings for each student. Students were rated high, medium, or low according to their performance on visual perception items. Students were placed into one of five categories (multiplicative, additive, area, mixed, or none) according to their performance on quantifying growth items. Students were given a score of 0-3 according to their performance on quantifying measurement items. All three ratings were assigned objectively according to response patterns, not by the
induction of the researcher. In the following section, the results of the rSPT and how they were used to select a sample of 21 students to be interviewed are described.

Assigning Visual Perception (VP) ratings. Students were given 23 visual perception items organized into four sections: General Shape (GS), Coordinating Vertical/Horizontal Growth (VH), Coordinating Internal/External Growth (IE), and Continuity of Growth Factor (CG). Individual section ratings were assigned based on the percentage of items answered correctly. Low indicated 50% of items or fewer correct, Medium indicated between 51% and 100% correct, and High indicated all items correct. Students were given an overall rating based on the simple majority of their four section ratings. For 76 students, a simple majority was present. For 15 students, however, no majority was present and they had section ratings split evenly between two levels. Fourteen students were evenly divided between mediums and highs, and one student was evenly divided between lows and mediums. As students who scored High, Medium, Low, Low received a rating of Low, it can be justified that the one student who scored Medium, Medium, Low, Low should also receive a rating of Low. Furthermore, the number of correct items for the Medium-Low was less than or equal to the number answered correctly by all of the identified Low students. The rating of the 14 Medium-High students was not as clear-cut.

In order to determine whether a rating of Medium-High was significantly different from a rating of Medium or High, the total number of visual perception items answered correctly was used and an ANOVA was used to determine significant differences of the mean. According to Levine’s Test of Homogeneity of Variance (sig. = .162), equal variance can be assumed between the groups established as Low,
Medium, Medium/High, and High. An ANOVA returned a significant value of .000, indicating that there is significant difference in the performance of these groups. A Post-Hoc Comparison, summarized in Table 9, would suggest that the rating groups of Medium/High and High are not significantly different. The mean difference is significant at the 0.05 level. As a result, all 14 ratings of Medium/High were revised to High. The revised VP ratings assigned are summarized in Table 10.

Table 9

_Bonferroni Comparisons of Visual Perception Ratings_

<table>
<thead>
<tr>
<th>(I) RATING</th>
<th>(J) RATING</th>
<th>Mean Difference (I-J)</th>
<th>Standard Error</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>Medium</td>
<td>-3.6532</td>
<td>.95494</td>
<td>.001</td>
</tr>
<tr>
<td></td>
<td>Medium/High</td>
<td>-7.1071</td>
<td>1.04947</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>High</td>
<td>-7.7045</td>
<td>1.08081</td>
<td>.000</td>
</tr>
<tr>
<td>Medium</td>
<td>Medium/High</td>
<td>-3.4539</td>
<td>.54774</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>High</td>
<td>-4.0513</td>
<td>.60562</td>
<td>.000</td>
</tr>
<tr>
<td>Medium/High</td>
<td>High</td>
<td>-.5974</td>
<td>.74583</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 10

_Summary of Visual Perception Ratings_

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>4</td>
<td>12.7500</td>
<td>2.06155</td>
<td>1.03078</td>
</tr>
<tr>
<td>Medium</td>
<td>62</td>
<td>16.4032</td>
<td>1.95406</td>
<td>.24817</td>
</tr>
<tr>
<td>High</td>
<td>25</td>
<td>20.1200</td>
<td>1.50886</td>
<td>.30177</td>
</tr>
<tr>
<td>Total</td>
<td>91</td>
<td>17.2637</td>
<td>2.64926</td>
<td>.27772</td>
</tr>
</tbody>
</table>
Assigning Quantifying Growth (QG) ratings. According to their responses, students were given a categorical rating of multiplicative, additive, area, or none on each of four quantifying growth items. Items were designed in a multiple-choice format so that answers indicated a likely strategy for quantifying growth. A rating of “none” does not indicate that there was no strategy, but rather that the strategy used was not multiplicative, additive, or an area model. A composite rating based on simple majority was then assigned. These ratings are summarized in Table 11.

Table 11

Quantifying Growth Ratings

<table>
<thead>
<tr>
<th></th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive</td>
<td>8</td>
<td>8.8</td>
</tr>
<tr>
<td>Area</td>
<td>11</td>
<td>12.1</td>
</tr>
<tr>
<td>Mixed</td>
<td>25</td>
<td>27.5</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>34</td>
<td>37.4</td>
</tr>
<tr>
<td>None</td>
<td>13</td>
<td>14.3</td>
</tr>
<tr>
<td>Total</td>
<td>91</td>
<td>100.0</td>
</tr>
</tbody>
</table>

For the 25 students (27.4%) that did not exhibit a preferred strategy, the rating of “mixed” was assigned. Mixed ratings could indicate that the student guessed on these items, or it could indicate that students quantified the growth in the items in different ways. The items were designed to vary in complexity of shape and in the explicitness of numerical quantities and it is possible that for some students, the items solicited varying strategies.
Assigning Quantifying Measurement (QM) Ratings. The three quantifying measurement items were designed to build in difficulty and to have correct answers. All students completed all items and earned ratings depending on their correct answers to items 1-3. It is unlikely that a student would get the third question correct but not the second or first. Thus, a student's rating for this item corresponded to the most advanced question that was answered correctly. To illustrate, if a student got the first question wrong, he or she was given a rating of 0 and no other items were scored. If a student got the first two questions correct, but missed the third, he or she received a rating of 2. All questions correct received a score of 3. A student answering only the first and third question correct was flagged for possible inclusion in the interview set, but was given a score of 1 indicating that question 2 was incorrectly answered. Ratings given are summarized in Table 12.

Table 12

Quantifying Measurement Ratings

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>41</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>91</td>
</tr>
</tbody>
</table>

Selecting a stratified purposeful sample. On the basis of their performance on the rSPT, students were anonymously prioritized for inclusion in a stratified purposeful
sample. Priority was given to both students whose ratings were commonly found and those that were extremely rare. For comparison, the most common combination of ratings given to students was (Medium, Mixed, 0), but only one student was given the combination (Low, Multiplicative, 1). As students consented to be interviewed, they were accepted on the basis of this prioritization. Only in the cases where multiple students with identical ratings consented to participate did ethnicity or gender influence the inclusion of students. In these cases, the goal was to represent the diversity of the population and to be as inclusive as possible.

The sample of 21 students, described in Table 13, was a diverse group in terms of demographics and achievement. However, because the group was sampled primarily on rSPT performance, the sample does not represent the general population in terms of demographics or prior student achievement (as measured by their assigned math class).

Analysis of the Interview Protocol

The interview data were qualitatively analyzed to answer the two research questions in the study and to refine the hypothesized characterizations of van Hiele sublevels. As the data were analyzed, attempts were made to refine the characterization of van Hiele levels 0-2, but not levels 3 and 4. A refinement of Battista’s (2007) sublevels in the context of similarity is an implication of this study (see Chapter V). The specific data analyzed and the method of analysis is described next.

Part 1: Expanded SPT. For each student, all verbal explanations of each response on the SPT were partially transcribed and their final answers noted. Responses given during the interview were compared to the original responses given during the
Table 13

Description of Sample

<table>
<thead>
<tr>
<th>Name</th>
<th>Gender</th>
<th>Ethnicity</th>
<th>Math Class</th>
<th>Visual Perception</th>
<th>Quantifying Growth</th>
<th>Quantifying Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alecia</td>
<td>Female</td>
<td>African American</td>
<td>General</td>
<td>2</td>
<td>Mixed</td>
<td>0</td>
</tr>
<tr>
<td>Andre</td>
<td>Male</td>
<td>African American</td>
<td>Honors</td>
<td>3</td>
<td>Mixed</td>
<td>3</td>
</tr>
<tr>
<td>Anna</td>
<td>Female</td>
<td>Caucasian</td>
<td>Honors</td>
<td>2</td>
<td>Mixed</td>
<td>2</td>
</tr>
<tr>
<td>Chris</td>
<td>Male</td>
<td>Caucasian</td>
<td>General</td>
<td>1</td>
<td>Multiplicative</td>
<td>1</td>
</tr>
<tr>
<td>David</td>
<td>Male</td>
<td>Caucasian</td>
<td>Honors</td>
<td>2</td>
<td>Multiplicative</td>
<td>3</td>
</tr>
<tr>
<td>Elaine</td>
<td>Female</td>
<td>Caucasian</td>
<td>Honors</td>
<td>3</td>
<td>Area</td>
<td>2</td>
</tr>
<tr>
<td>Eli</td>
<td>Male</td>
<td>Caucasian</td>
<td>Honors</td>
<td>3</td>
<td>Multiplicative</td>
<td>2</td>
</tr>
<tr>
<td>Elijah</td>
<td>Male</td>
<td>African American</td>
<td>Honors</td>
<td>1</td>
<td>Multiplicative</td>
<td>1</td>
</tr>
<tr>
<td>Ian</td>
<td>Male</td>
<td>Caucasian</td>
<td>Honors</td>
<td>3</td>
<td>Multiplicative</td>
<td>3</td>
</tr>
<tr>
<td>Jeff</td>
<td>Male</td>
<td>Caucasian</td>
<td>General</td>
<td>2</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>Jorge</td>
<td>Male</td>
<td>Hispanic</td>
<td>General</td>
<td>2</td>
<td>Additive</td>
<td>2</td>
</tr>
<tr>
<td>Jules</td>
<td>Male</td>
<td>African American</td>
<td>Honors</td>
<td>2</td>
<td>Additive</td>
<td>3</td>
</tr>
<tr>
<td>Kyle</td>
<td>Male</td>
<td>Caucasian</td>
<td>General</td>
<td>3</td>
<td>Multiplicative</td>
<td>2</td>
</tr>
<tr>
<td>Marquon</td>
<td>Male</td>
<td>African American</td>
<td>General</td>
<td>2</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>Matt</td>
<td>Male</td>
<td>Caucasian</td>
<td>General</td>
<td>2</td>
<td>Multiplicative</td>
<td>2</td>
</tr>
<tr>
<td>Naomi</td>
<td>Female</td>
<td>African American</td>
<td>Honors</td>
<td>2</td>
<td>Multiplicative</td>
<td>1</td>
</tr>
<tr>
<td>Pedro</td>
<td>Male</td>
<td>Hispanic</td>
<td>Honors</td>
<td>2</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>Ryann</td>
<td>Female</td>
<td>African American</td>
<td>General</td>
<td>2</td>
<td>Mixed</td>
<td>2</td>
</tr>
<tr>
<td>Shanice</td>
<td>Female</td>
<td>African American</td>
<td>Honors</td>
<td>2</td>
<td>Mixed</td>
<td>0</td>
</tr>
<tr>
<td>Tate</td>
<td>Male</td>
<td>Caucasian</td>
<td>General</td>
<td>3</td>
<td>Area</td>
<td>0</td>
</tr>
<tr>
<td>Tom</td>
<td>Male</td>
<td>Caucasian</td>
<td>Honors</td>
<td>3</td>
<td>Multiplicative</td>
<td>2</td>
</tr>
</tbody>
</table>
classroom administration of the rSPT. This comparison was conducted using an Excel spreadsheet. Each student’s reasoning on each item was also recorded and analyzed for themes, first by student and second by item. Of particular interest were the characteristics of shape that capture student attention or were useful in helping justify a response. This is directly related to the first research question.

Tasks 2 and 3: Construction tasks. Student constructions, drawings and measurements were all digitally scanned. These constructions were analyzed related to the second research question. A description of student strategies used on each assigned task was written. These descriptions were labeled, categorized, and compared to other strategies used by the student on other tasks, noting similarities and differences in particular uses. In order to validate these general descriptions, they were compared to original student responses and revised when necessary. Finally, these general descriptions of construction strategies were compared to the strategies found in the literature on proportional reasoning. This is directly related to the second research question.

Tasks 4 and 5: Differentiation tasks and quantitative application. Student responses were partially transcribed and summarized in a similar fashion described for the construction tasks above. Data were not analyzed relative to either research question, but were collected for future analysis.

To conclude, data collected from an administration of the rSPT and in student interviews were analyzed to answer the two research questions outlined in Chapter I. The broad essence of each research question and the data that were analyzed to answer it is provided in Table 14. Non-analyzed data, such as responses to parts C and D from
the interview protocol, influenced analysis in a more broad fashion. Student responses to all of the interview tasks contributed to the researcher’s overall impressions of a student’s understanding and strategy and, as with classroom observations and interactions, were influential as background information.

Table 14

*Summary of Data Analysis*

<table>
<thead>
<tr>
<th>Research Question</th>
<th>rSPT Data</th>
<th>Student Interview Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Differentiation Strategies</td>
<td>Visual Perception Strategies: Sections 1-4</td>
<td>Part A</td>
</tr>
<tr>
<td>2. Construction Strategies</td>
<td></td>
<td>Part B, Tasks 2 and 3</td>
</tr>
</tbody>
</table>
CHAPTER IV

RESULTS

This chapter is intended to report findings relative to answering the two research questions outlined in Chapter I. The analysis is based on data collected from an administration of the rSPT and follow-up interviews with 21 seventh-grade students. The core purpose of the analysis was intended to illuminate how students solve the two main types of similarity task, differentiation and construction, each treated separately in the two sections of this chapter. It was the intent of this study to better describe student strategies and conceptions that bridge visual perception and proportional reasoning.

Differentiation Strategies

The data most suited to ascertaining the strategy by which a student determines if two figures are similar were think-aloud responses to selected rSPT items shown at the beginning of the student interviews. As time did not permit the review of every item on the rSPT with every student, students were given individually selected subsets of the items. These subsets were chosen carefully to include not only items the student had answered correctly, but also those that challenged the student. Less challenging items provided opportunities for students to verbalize in a reflective manner how they perceived the figures and ultimately made their decisions. More challenging items, including those a student did not get correct, provided extended opportunities to capture the decision making process in situ. The subsets were adjusted to include items featuring
different classes of figure types, both similar and non-similar. The goal was to capture the most diverse reasoning possible from each student.

Three distinct classes of figures featured on the rSPT items were included in the subsets given to each student. These distinctive classes are the U-shape, simple convex, and complex figures, which are illustrated with examples in Figure 6. The U-shape is an eight-sided concave polygon with all right angles. All of the simple convex figures are rectangles (as the one shown) or parallelograms. The complex figure class includes figures that were created by setting a second image (stars, cartoon girls, or parallelograms similar to the exterior) inside of a parallelogram. The variation in figure type included in the rSPT items meant that different arrays of characteristics such as side lengths within the figure were available to students on each item. The items also provided four different sources for distortion between the figures. These sources of distortion are described in more detail relative to the design of the instruments in Chapter III.

Figure 6. U-shape, Simple Convex, and Complex Figures Used on rSPT Tasks

Whether a student determines that two figures are similar or not depends on how the student has perceived the figures to be compared and the relationship between them. Thus, student responses could be analyzed according to the characteristics (or properties)
of figures students noticed and referred to while deciding and justifying whether the two figures presented were similar. Using an iterative process of viewing, recording, summarizing and reviewing student responses, a framework for six main characteristics and several sub-characteristics emerged. This section begins with an introduction of this framework. Examples of student responses are utilized to illustrate how these main or sub-characteristics were used alone or combined to help students decide if two figures are similar or not similar.

It was initially intuitive to include such characteristics as "angle" or "side length" in such a framework. In the traditional analytic sense, these are the characteristics by which similar figures are defined, but other characteristics such as the ways students described the overall appearance of the figures, variety in the lengths students referred to, or types of relationships between the two figures emerged during the analysis of student responses. Four main characteristic types, defined in Table 15, encompassed the overall variety that was documented: Appearance, Angle, Length, and Relationship.

The characteristics that students note are highly intertwined with the strategies that students are using. The narrative description of student responses given next will highlight some of the variation within each of the five characteristic types as well as the strategies they indicate. Each characteristic type is first described in more detail, followed by a discussion of the related strategies and their application.
Table 15

*A Framework for Analyzing the Characteristics of Shape That Students Use While Differentiating Similar From Non-Similar Figures*

<table>
<thead>
<tr>
<th>Characteristic Type</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonspecific</td>
<td>Response does not include any references to specific characteristics of the shape. If a student response is given this code, no other codes are given.</td>
</tr>
<tr>
<td>Appearance</td>
<td>Reference to the overall appearance of the shape including color, blur, shape type, general size or style.</td>
</tr>
<tr>
<td>Length</td>
<td>Indication that an individual has measured in a qualitative, quantitative, or relative way the distance between any two points within a figure. Comparisons of this type establish a relationship between one pair of lengths.</td>
</tr>
<tr>
<td>Angle</td>
<td>Indicates that an individual has measured in a qualitative, quantitative, or relative way any angle of a figure.</td>
</tr>
<tr>
<td>Relationship</td>
<td>Indication of an external or internal relationship between two characteristics of the figures in the pair.</td>
</tr>
</tbody>
</table>

*Non-Specific Responses*

In 40 cases, or 11% of total responses, a student’s response was not specific enough to ascertain what characteristics of the shape the student perceived, or how the student arrived at a conclusion about similarity:

Elijah: “They are the same. I forgot what I was going to say.”

Interviewer: “How would you turn this one into that one?”

Elijah: “Stretch it.” (Elijah, 2.5, Similar)
In all of these cases, the interviewer attempted to solicit more information about the characteristics and the strategy in use, but as with Elijah’s response to Item 2.5 above, this was not always provided by the student.

Individuals answered between 0-30% of assigned items using non-specific strategies, and used them more often in reference to similar figures than non-similar. Approximately 25% of responses to mathematically similar pairings were non-specific as compared to 5% of responses to mathematically non-similar pairings. When the responses are reanalyzed according to the students’ perceptions of the figures as similar or non-similar, the results are just as dramatic. When students perceived the figures to be similar, 24% of the responses were non-specific as compared to 4% of responses to perceived non-similar figures.

Appearance-type Strategies

In reference to the first item of the rSPT, Jeff responded, “The A on the left is more like dull and smaller. It's not...doesn't have like...not as exciting or not...very cool as the other one,” (Jeff, 1.1, Non-similar). Jeff’s response indicated two things. First, he had made a decision about the non-similarity of the figures in the item. Second, his response indicated a line of reasoning to support this decision; Jeff shared evidence to support his conclusion. This line of reasoning was based on the general appearance of the figures. Overall, 36% of responses included reasoning based on the general appearance of the figures.

In the initial examination of these responses, three types of responses regarding the general appearance of shapes were made: (1) descriptions (like Jeff's above) of
cosmetic features of figures such as color or blur, (2) comparisons of shape type such as “rectangle,” and (3) the relative positions of subshapes within a figure. These holistic descriptions of the shape did not reference specific components of the figures, but did help students describe their visual perceptions of figures and were indicative of different levels of sophistication even within the broader category of Appearance-type strategies. In many cases, students used general appearance characteristics to describe initial judgments of figures and were subsequently supported with more reasoning involving other characteristic types. In other cases, students were content with their initial visual judgments. These are the cases described here; these strategies involve only visual perceptions of the figures as a whole.

The Cosmetic Strategy: Two figures are similar/non-similar if they are visually alike/different. A geometric proportional thinker is aware that two parallelograms can be mathematically similar even if one is red and the other blue. However, the cosmetic appearances such as color and transparency (light/dark) of figures were mentioned in responses both as evidence for similarity and non-similarity. Some students seemed to expect the shapes to be exact in even non-geometric ways. This phenomenon has been reported in other studies where the everyday connotation of the term similar interferes with mathematical understanding. A common fix to introducing the term similar at the onset is to first introduce the informal definition of “same shape, different size.” This, too, seemed to interfere. For example, in Ryann’s case, “same shape” meant that the shapes were the same in a duplication sense—that both came from the same original shape. While differentiating, she invoked concept imagery related to experience with enlarging shapes using a computer. In her experience, shapes became blurry when you
enlarged them. "Yes. Hold on. Yeah, but this one would look more blurry than that one. When you blow it up, it will be blurrier but it will be the same thing," (Ryann, 3.1, Non-Similar). Although Ryann concedes that the two shapes look alike, they are not "the same shape" because the larger one would appear blurry if they were.

**The Shape Type Strategy:** Two figures are similar/non-similar if they are different sizes of the same/different shape type. A second characteristic related to the overall appearance of the shape was shape type. Students can use shape type as evidence for similarity, over-generalizing similarity to shape classes such as "rectangles." Item 1.4 featured two rectangles that were non-similar, though students reasoned that they were similar "since they are both rectangles."

More commonly, however, shape type was used as evidence for non-similarity. Shanice struggled to find an abstract description of shape type, but used it correctly to identify two non-similar rectangles. She described the differences that she saw using a concrete analogy to two rectangular objects: "That [rectangle] is just bigger, wider, and it's like a big giant square. The first [rectangle] is like a bookmark; the other one is like a bigger piece of wood like you'd use to make a desk," (Shanice, 1.4, Non-similar). On items that featured the U-shape in particular, students were able to imagine the valley, legs or bottom join of the U as a rectangular subshape of the whole. Evidence against similarity was found in these subshapes when one looked "more rectangular" or "more square" than another. U-shapes were not the only figure type to be subdivided. The spaces between interior and exterior images in complex shapes were also recognized and used.
The Relative Positioning Strategy: Two figures are similar/non-similar if the relative positions of the sub-shapes do not/do change. Regarding complex shapes, students used the relative positioning of the interior and exterior shapes to decide if shapes were similar. Item 1.5 was specifically designed with this sort of characteristic in mind—the girl in the center of the parallelogram was deliberately moved off-center in the larger figure. Students were able to identify this shift and use it as evidence of non-similarity. Surprisingly, it was used on other items that were not designed in this way, such as Item 3.6, illustrated in Figure 7. The larger figure was designed as a “zoom out” of the smaller (i.e., the exterior was enlarged, but not the interior) with the interior anchored at the bottom of the shape. In some cases, rather than noticing the zoom quality, students remarked that the girl had been moved from the center of the parallelogram toward the bottom of the shape. This strategy is related to the constant gap strategy described in the section on Length-type strategies.

#3.6

Figure 7. Item 3.6 from the rSPT
The Angle-type Strategy

Students in this study indicated that they believed two figures are similar/non-similar if corresponding angles are of equal/different measures. Because the rSPT was initially designed to look at how students made use of visual and spatial reasoning relative to proportion, the items were oriented more toward exposing reasoning and strategy relative to other characteristics rather than the property that all corresponding angles had the same measure. Angle was not a differentiating factor for most of the figure pairs on the rSPT, although it was a factor on Item 1.2, which features non-similar parallelograms. Most students successfully determined that these two figures were non-similar using the non-equality of corresponding angles as evidence. Anna responded, “I could tell the difference between these two because it's got more of an angle. Like, it's angled more over. This one is up...er...bent over more. I can't really explain it,” (Anna, 1.2, Non-similar). In the language that students used and methods of angle comparison, there were subtle differences in the ways angles were invoked. Anna talked about the overhang of the slanted side, another student talked about the “slantiness” of the side length, and still another talked about the perceived non-parallelism between corresponding side lengths. All three of these examples were references to angle; however, they invoked very different images and measure. Angles were also used, albeit infrequently, as evidence of similarity. Although it was not used often in cases where the figures were not similar, parallelism was very common as a reason two shapes were similar.
Length-type Strategies

Three distinct length measures were noticed and compared by students while differentiating figures: primary, secondary, and space lengths. One of each of the three measures are illustrated in the case of the U-shape in Figure 8. Primary and secondary lengths are both measurements of drawn lines within the figure, generally edges. A primary length is a length that defines the height or width of the entire figure. In the case of a parallelogram, all four edges were defined as primary because they frame the figure and determine both horizontal width and a “slant height,” which most students referred to as “height.” All other lengths including remaining edges or drawn lines within the figure are secondary. A space length measures the width of a gap in the figure not represented by a drawn line. As with the Appearance-type characteristics that students noticed, students compared and used these lengths in a myriad of ways. Two particular Length-type strategies were documented: Corresponding Length Comparison and The Constant Gap Strategy.

Figure 8. Illustration of Length Subtypes
Corresponding Length Comparison: Two figures are similar/non-similar if one length is longer or the same as another. This strategy indicated that a student made a decision about similarity based on the comparison of two (and only two) lengths. Arguments that students used that indicate this strategy sound like “these figures are similar since they are the same height,” or, “these figures are similar, one is just taller than the other.” Two students using this strategy may have different reactions to the prompt “How much taller?” Elijah reacted to the prompt by measuring the difference in the heights with his thumb and index finger demonstrating “this much taller.” Other students used a multiplicative comparison, indicating that one figure was twice as tall as the other. Chris was inconsistent in his method of comparison, sometimes using a ratio model and other times difference model of growth. This reasoning pattern will be discussed at further length in the section on construction strategies.

The Constant Gap Strategy: Two figures are similar/non-similar if there exists/does not exist consistency of a space length within two figures. For some students, there is a conceptual difference between a space length and one that is represented by a drawn line. The difference lies in the fact that a drawn line can be perceived as an object that has properties such as length. A space length is not as easily perceived as an object, although it can be represented as the unmarked distance between two points. In this sense, space lengths can be classified by students differently from drawn lengths, and may not be expected to scale in the same way. The constant gap strategy was based on the expectation that while lengths may get longer (or even scale multiplicatively), gaps remained constant. Elijah’s solution to Item 3.2, shown in Figure 9, illustrates this strategy and the expectation.
On Item 3.2, the complex figures to be differentiated are similar; the star has been scaled by the same factor as the parallelogram. Elijah responded to this item by deciding incorrectly that the two figures were non-similar.

Elijah: Because how the stars are placed. There is more space in that one than that.

Interviewer: Where do you see the space? [Elijah points to the left of the larger star.] You see more space here [Interviewer indicates the space to the left of the star.] How much more space?

Elijah: Like half. Half here (small) compared to that (large).

Interviewer: You see the stars being moved.

Elijah: Yeah. (Elijah, 3.2, Non-Similar)

Elijah had stumbled blindly upon evidence of a very sophisticated property of similar figures, that being that all lengths—not just those marked by drawn edges or lines—are scaled by the same factor. He showed relative thinking in that he identified that the gap to the left of the star had been scaled by a factor of 1/2. This was particularly
remarkable in light of the overall absence of ratio and relative thinking in Elijah’s reasoning about differentiation and construction throughout the interview and on the initial administration of the rSPT. Instead of using this ratio as evidence that scaling had occurred, it was evidence to Elijah that someone had moved the star—the term similar meant (to Elijah) that the gap should remain constant in width. Here, although all of the other primary and secondary lengths had been made larger in some way, Elijah expected that the gap would remain constant.

Relationship-type Strategies

Three other types of strategies were identified in this study that relied on a student’s perception of relationships within the pair of figures. Two of these strategies were numerical in nature, while the third relied on a qualitative assessment of the relationships. Two different structures for these relationships were of interest: external and internal. Pedro responded to Item 4.4 by describing an external relationship. “I figured this right here [indicates the horizontal distance between inner and outer parallelograms], if you enlarge [the figure], it’d be thicker than this side,” (Pedro, 4.4, Non-similar). In contrast, Andre responded to Item 4.4 by describing an internal relationship between the heights of the interior and exterior parallelograms within one figure. “And in this one [smaller], the inside is about half the height [of the outside], but in this one it looks a little bit taller by half. If you compare this space [distance between the top of interior and the top of exterior] to that shape, it will be too big. It looks about like the inner shape got too tall,” (Andre, 4.4, Non-similar).
The Constant Difference Strategy: Two figures are similar/non-similar if there is/isn't a constant difference relationship between pairs of corresponding lengths. The Constant Difference strategy is related to the additive strategy identified in other studies (cf., Lamon, 1993) relative to proportional reasoning. A numerical application of this strategy requires a student to read or measure two corresponding length measurements and then calculate the subtracted difference between them. The student can then check to see if this is the same difference as found between other pairs of corresponding length measurements. In this study, Pedro measured lengths by marking the endpoints of a length on a cardboard strip that he held up to the screen. He compared a short length to a longer corresponding length by first marking the start and end of the shorter length on the cardboard. Then, using the same start point, Pedro marked the end point of the longer length on the same cardboard strip. This process is illustrated in Figure 10. Pedro was then able to construct another length—the gap between the end of the shorter length and the end of the corresponding longer length. When considering two similar parallelograms, Pedro was able to find two such gaps that were not equal in length, thus concluding incorrectly that the two shapes were not similar.

Interviewer: What would that look like if it was a yes?

Pedro: They would be the same amount added to both sides (Pedro, 2.3).

The Constant Ratio Strategy: Two figures are similar/non-similar if there exists/does not exist constant ratio between corresponding lengths. Tom was one student who consistently used extrinsic ratios of primary and secondary side lengths to determine similarity. On most items, he chose to compare between two and four pairs of lengths. Tom used this strategy on almost all items including Item 1.2, described above. Tom
coarsely estimated a constant ratio between the corresponding slant heights and widths of the simple convex parallelograms, but failed to notice that the angles had changed. (Later in the interview, Tom remembered that his teacher had taught them a rule about angles, which he "should have been using all along.")

Tom’s method was also unreliable on items featuring U-shapes—where angle was not a differentiating factor. Responding to another item, Item 2.4, Tom made a statement that unintentionally explained why his method was unreliable:

Interviewer:  You coordinated three things there: the width, then the width of this leg thing, then the height. Is that because this is a different kind of shape? How come three things?

Tom:  On the other ones you've chose, you have...There's really not much you can...You can go like that [compares the overall height gesturally] which I kinda did. You, but, you can't really measure...The more complicated the shape, the
different you...the more different ways you could change it to look similar but it could be different (Tom, 2.4, Similar).

On the U-shape items like Item 2.6 shown in Figure 11, it was possible to compare multiple pairs of corresponding lengths, correctly conclude that there was a constant ratio in all three pairings and still conclude falsely that the shapes were similar. It was difficult for students to know when they had made sufficient comparisons to conclude similarity. Even students, like Tom, who were cognizant of the complexity of the figure, lacked a systematic way of knowing when enough is enough.

#2.6

Figure 11. Item 2.6 from the rSPT

The Qualitative Relationship Strategy: Two figures are similar/non-similar if there is/is not a constant qualitative relationship in the components of two figures. The relationship a student used, whether intrinsic or extrinsic, did not have to be a numerical relationship. In the case of items featuring complex figures students recognized
qualitative relationships between different components of the figures. The language used
to describe these relationships was reminiscent of the language used in the length
comparison strategy. Relative terms like “bigger” and “smaller” and “the same as” were
used to describe relationships qualitatively without indicating a specific numerical
structure such as difference or ratio. Three different approaches to using a Qualitative
Relationship strategy are transcribed below. Elijah used an external relationship between
the interiors and exteriors of the figures; Chris used an internal relationship between the
heights of the girls and the heights of the parallelograms; and Matt used both types to
describe why the figures in Item 3.6 are not similar. The following exchanges illustrate
these different approaches:

Elijah: The person. They are the same size. Even though the parallelograms get
bigger, the person stays the same. (Elijah, 3.6, non-similar).

Chris: Different. The picture would have to be all the way up there [Chris
gestures toward the top of the parallelogram] just like that one. (Chris,
3.6, non-similar)

Matt: The girl is the same size as the other one. If the whole shape changed, if
it went from here to here, then the girl would get smaller. [The girl in the
larger figure] only comes up half way and [the girl in the smaller figure]
comes up all the way. (Matt, 3.6, Non-similar).

Transformative Strategies

Students using Transformative strategies decided if two shapes were similar/non-
similar based on whether one could/could not be transformed into the other. Two
Transformative strategies were observed: The *Dilation Strategy* and The *Tiling Strategy*. Both strategies were used by students who perceived a dynamic relationship between the two figures in the item and imagined one as transformed from or into the other either by dilation or by tiling.

*Dilation Strategy*: Two figures are similar if one can be transformed into the other. David used a dilating action radiating from the lower corner to describe the relationship between two parallelograms. Figure 12 shows David's depiction. As he drew, David said, "*I tried to fit that in that and make it bigger. And see... kinda picture it,*" (David, 1.3, Similar).

![Figure 12. David Illustrates an Imagined Dilation of a Small into a Large Parallelogram](image)

Above, David uses qualifying language such as "like" or "kinda" (kind of) with a tentative tone, as if he is searching for a way to describe what he imagines and is trying words out. Students used common phrases like "fit it in" or "filling up," and prior experiences became tools by which to identify specific perceptions. It was also common for students to make sketches of one shape inside of the other, as David did above with
the parallelograms. To highlight the use of many of these tools in a slightly more difficult context than the parallelograms above, the following transcript was taken from David's description of why he thought two U-shapes were similar. David interpreted the larger and smaller shapes as related by a dilation action.

David: I think I said these are the same. Like, these are parallel. Just the same shape as it. Kinda like just tried to fit [the small leg] in [the large leg].

Interviewer: How does this one fit in there?

David: It kinda doesn't.

Interviewer: What do you imagine?

David: I kinda just imagined like...[David draws a sketch.] I try to picture it as it getting bigger and fitting.

Interviewer: Filling it up?

David: Yeah! This, like, this corner go into this corner. (David, 2.4, Similar)

Although the language used is not formal language typically associated with dilation, it is not lacking in sophistication. In Figure 13, dots are visible. The dots were used by David to describe the destinations within the larger U-shape of specific points within the smaller U-shape. This illustrated a very sophisticated conception of correspondence and continuous all-directional growth, even if this conception did not translate into a strategy that was easily verbalized or uniformly applied.
Figure 13. David Illustrates an Imagined Dilation of a Small into a Large U-shape

Tiling Strategy: Two figures are similar if one can be transformed into the other. A tiling action was also a popular way to transform one shape into another. Ian used the following argument to describe how he knows that two parallelograms are similar. “Two of those equal the same height. Yes. Same height...means two of these make the height of larger figure. Two more make up the area. You could fit three more pieces in there. Scale factor of 2,” (Ian, 3.2, Similar). Ian’s perception of the external relationship was of the large figure as a frame or puzzle and the smaller figure as the pieces that fit inside. Ian first checked to make sure that if he stacked two of the smaller parallelograms the resulting figure would have the same height as the larger parallelogram. Having established that, he was able to imagine four parallelograms (the original plus “three more pieces”) fitting inside of the larger in a $2 \times 2$ array. He then concluded that the scale factor between the small and the large parallelograms was 2.

This strategy was not always easy for students to apply. The following examples indicate sources of cognitive tension related to this strategy. Elaine tiled parallelograms much like Ian did. However, when the parallelogram was a part of a complex figure with
a star embedded inside, Elaine expressed some doubt about whether this was a valid strategy:

It'd fit four small ones in there...if the star wasn't there, pretty much four of them would. If the star was there, you couldn't do that because the star would be, in between, er, like...behind all the lines. It would have to show the star, pretty much. The star would be behind all the lines. The lines would get in the way.

There would be two. The star would be crossed out. (Elaine, 3.3, Similar)

The difficulty for Elaine wasn't in the tessellating but in the fact that tessellating did not create a larger version of the interior of the shape—only the edges. Anna also had difficulty using this strategy when featured pairs of figures that were related by non-whole scale factors. After seeing that she couldn't tile the larger figure with the smaller figure, Anna decided that the two figures must not be similar. However, doubt was present in her response and it seemed that intuitively Anna sensed that the parallelograms were similar even if this strategy told her that they were not.

Discussion

The analysis of the types of characteristics students referenced in their responses was a method by which to gain insight into the strategies students were using, but also to identify ways that students perceived the figures in the items visually and made sense of their relationships. By considering the individual components that students perceived and compared, it was possible to parse out even subtle differences in the strategies students were applying and the level of reasoning they indicated. For instance, by documenting that it was a space length that Elijah was paying attention to rather than a drawn edge, it
was possible to discern subtle differences between the Relative Position strategy and the Constant Gap strategy. Along those lines, it was also possible to differentiate when students were making a judgment based on the overall appearance of the figures, and when they had noticed a qualitative relationship between various components within those figures. By paying attention to the particulars of the comparisons that students made, it was possible to characterize eleven specific differentiation strategies spanning the first two van Hiele levels of geometric reasoning: visualization and analysis.

In Table 16, each of the twelve identified strategies (non-specific is included) are listed in the order that they appeared in this section, along with a characterization of the level of reasoning each indicated in the responses of students. The levels used in this characterization are the first two van Hiele levels, Visualization and Analysis. In some cases, a strategy is marked as having indicated both levels. This does not mean that both levels are exhibited in every response. Rather, it means that different responses could be characterized as one or the other even as they used the same strategy. A one-to-one correspondence between strategy and reasoning seemed unlikely from this data and was not sought here. However, with more data, it may be possible to further differentiate analytic from visual reasoning using strategy.

Four strategies were used in exclusively visual ways: Non-specific, Cosmetic, Relative Position, and Dilation. In non-specific responses, students gave a global judgment about the similarity of the figures based on non-verbalized and probably unconscious concept imagery. In their lack of specificity, these responses indicated a purely visual strategy not likely linked to any single characteristic of the figures. Responses using the Cosmetic and Relative Position strategies indicated comparisons of
Table 16

*Summary of Differentiation Strategies and Indicated Levels of Geometric Reasoning in This Study*

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Level 0: Visualization</th>
<th>Level 1: Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-Specific</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Appearance-Type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cosmetic</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Shape Type</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Relative Position</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Angle-Type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angle</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Length-Type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length Comparison</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Constant Gap</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Relationship-Type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant Difference</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Constant Ratio</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Qualitative Relationship</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Dilation</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Tiling</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

only the general appearances of the figures. Although it is certainly possible to imagine a student using dilation analytically to reason that two figures are similar, it was not

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1 Not observed in this study.
observed in this study. Thus, Dilation is included here as an exclusively visual strategy. Although it is classified as level 0 thinking, this strategy is markedly different from the others classified in this way. Visualized dilation is more sophisticated. In order to use this strategy, students must likely have at least beginning conceptions of correspondence and geometric proportion.

Three strategies were used in exclusively analytic ways: Constant Gap, Constant Difference, and Constant Ratio. Students noticed and numerically compared not only pairs of lengths, but also relationships between them. Students were conscious of multiple corresponding length pairs and used abstract mathematical tools such as differences or ratios to describe them in relation to one another.

Other strategies, such as Shape Type, Angle, Length Comparison, Qualitative Relationship, and Tiling were hybrid strategies that incorporated visualization with analytical reasoning. For example, these strategies may have included comparisons of components in qualitative ways. Visually, figures were “more rectangular” than others or relatively tall, short, wide, or skinny. Analytically, students identified and compared components; primary edges were longer or shorter, specific angles were the same or different, and identified relationships were constant or non-constant. As the reasoning became more analytical, students depended on quantifiable characteristics such as angles and length measurements and students indicated more recognition and coordination of multiple characteristics in their arguments. Although these responses were dominated by more quantifiable characteristics, this did not prevent students from making use of visual perceptions of the overall appearance of the figures—particularly the presence of
distortion. These hybrid strategies also indicated an awareness of correspondence that other exclusively visual strategies did not.

With few exceptions, all of these strategies were sufficient to prove two shapes were non-similar; however, no single strategy was sufficient to prove that two figures were similar. When figures are non-similar, flexible combinations of strategies reflect awareness, conscious or unconscious, of the particular array of characteristics available in a pair of figures and intuitions, conscious or unconscious, about the presence of distortion/absence of proportion. In Matt’s response to Item 3.6 on page 90, he used both external and internal relationships to illustrate why the figures were not similar. Although one strategy would have been sufficient, Matt’s solution can be considered an indication of sophisticated thinking because he was able to fluidly group and ungroup components and compared the resulting composite units qualitatively. In a formal mathematical sense, when the figures were similar, combinations of strategies were required to argue a formal proof. In this study, there were no responses rigorous enough to be considered a proof, formal or informal, that the two figures in an item were similar.

In search of a way to glimpse the overall strategy selections of the sample and of individuals, the twelve differentiation strategies were organized above according to characteristic type: Non-Specific, Appearance, Angle, Length, and Relationship. Relationships include external relationships such as, but not limited to extrinsic ratios, and internal relationships such as, but not limited to, intrinsic ratio. All Transformation-type strategies were interpreted as referencing relationships and are included within this category. In Figure 14, an Expanded Profile for Ryann’s responses to rSPT items is
shown according to the characteristic types of the strategies she used on each item. Where Ryann used multiple types of strategies, the vertical bar is divided to indicate this.

**Figure 14.** Expanded Profile of Ryann's Strategies by Characteristic Type

There is considerable variance in Ryann's profile, as there is in others, and there is evidence that she does not consistently rely on the same type of strategy from item to item. In order to create Ryann's expanded profile, each of her responses was coded according to the characteristic-type of the strategies she used. Characteristic-type was chosen as the level of analysis so that generalities and themes could be noted across the limited number of responses. Ryann combined strategies in her responses. Thus, responses were given multiple codes if multiple strategies were used. As an example, if we submerge, briefly, into the level of specific strategies that Ryann used, we find that on Item 1.5 Ryann combined a Relative Position strategy and the Angle strategy. Thus, her response was coded as both Appearance-type and Angle-type.
All student responses were coded in this manner, allowing for some exploration of themes in strategy choices by item. Two factors seemed to influence the type of strategies that students chose to use: (1) whether the figures in the item were similar or non-similar, and (2) the type of distortion in non-similar figures. Figure 15 compares the types of strategies used on items featuring similar and non-similar figures and shows students were more likely to respond to similar items with non-specific strategies. Of the responses to similar figures, 25% were non-specific compared to 5% of responses to non-similar figures. When responses were specific, 51% of responses to non-similar figures indicted Relationship-type strategies compared to only 34% of responses to similar figures.

![Figure 15. Comparison of Strategy Types Used on Similar and Non-Similar Items](chart.png)
In order to look at how this sample of students may have selected strategies differently based on the distortion type in an item, the items were sorted according to the nature of the distortion between them. Each of the four sections of the rSPT was designed to feature a different variety of distortion. Section 1 featured distortions in shape type. Of the four non-similar pairings, one featured letters of different font types, two featured simple convex figures related additively, and one featured a complex figure whose interior image had been repositioned off-center. Section 2 featured distortions arising from the non-coordination of vertical and horizontal scale factors. Section 3 featured distortions arising from the non-coordination of interior and exterior images. Section 4 featured distortions arising from the non-coordination of the growth rates of certain subshapes within the shape. We can see how students perceive each type of distortion by comparing the strategies that they used on each individual section. Only items that featured distortions (non-similar figures) were included in this analysis. Table 17 lists by section the percentages of responses that used strategies of each type.

Table 17

Percent of Strategy Types Used on Non-Similar Figures by Section

<table>
<thead>
<tr>
<th></th>
<th>Non-Specific</th>
<th>Appearance</th>
<th>Length</th>
<th>Angle</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1 (n = 67)</td>
<td>1.5%</td>
<td>59.7%</td>
<td>13.4%</td>
<td>25.4%</td>
<td>34.3%</td>
</tr>
<tr>
<td>Section 2 (n = 50)</td>
<td>10.0%</td>
<td>34.0%</td>
<td>44.0%</td>
<td>4.0%</td>
<td>26.0%</td>
</tr>
<tr>
<td>Section 3 (n = 64)</td>
<td>6.3%</td>
<td>29.7%</td>
<td>4.7%</td>
<td>12.5%</td>
<td>75.0%</td>
</tr>
<tr>
<td>Section 4 (n = 58)</td>
<td>5.2%</td>
<td>20.7%</td>
<td>24.1%</td>
<td>3.4%</td>
<td>65.5%</td>
</tr>
</tbody>
</table>
On items that feature non-similar figures, student strategy reflected the type of distortion in the figures. For each section, the predominant strategy choices were those that described the type of distortion in the figures. The distortions in Section 1 could be discerned and described naturally by using the overall appearance of the figure, and approximately 60% of the responses used Appearance-type strategies. The pairs in Section 2 can also naturally be described as one figure being “too tall” or “too wide,” and 44% of student responses used Length-type strategies. Another way of describing uncoordinated horizontal and vertical growth is by describing one figure as “smooshed,” thus Appearance-type strategies were also common on this section. The distortions in Section 3 were most naturally detected by noticing that the exterior and interior images were scaled by different amounts. Student strategies were reflective of this distortion; 75% of responses were of a Relationship-type (i.e., the Qualitative Relationship strategy).

Lastly, Relationship-type strategies were used in 64% of student responses to items in Section 4, almost three times as many as the next most common type for this section.

Construction Strategies

In an interview setting, each of the 21 students were assigned between two and six construction tasks, based on the figures shown in Figure 16, and asked to think aloud as they worked. These tasks were taken from the pool of seven possible tasks described in Table 18. In a few cases, slight adaptations were made to the tasks to provide additional challenge or scaffolding for students. Students had access to paper (both grid and plain) and colored markers. If a student expressed that they had made an error, they were instructed to pick up a new color, make the correction, and then continue working on the
same drawing. Student constructions became amalgams of multiple attempts at solving the specified tasks. Efforts were taken to document not only the finished product, but also the layered attempts.

![Figure 16. Figures Depicted in Construction Tasks](image)

Students were given different subsets of the construction tasks based on time available for the interview and previous performance. Decisions about which tasks to assign were made according to student performance and time available. With the exception of Tom, all students started with the Double Rectangle and Embedded Square (Double) tasks. If the student scaled additively on either of these tasks, they were given the L-shape (Double) task next. If the student scaled multiplicatively, they were instead given the Medium Rectangle and Embedded Square (Medium) tasks. If time permitted, the student was given additional tasks such as the Heart \((k)\) or L-Shape Reduced \((\bar{k})\).

During the interviews, spontaneous modifications to the overall scheme were made in two cases: Tom and Marquon. Tom had indicated on the rSPT items that he was able to use relational thinking and understood a whole-number scale factor to be multiplicative and not additive. Thus, to provide more of a challenge and opportunity for
Table 18

*Explanation of Construction Tasks*

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double Rectangle</td>
<td>Student is shown an opaque rectangle (4 units) × (6 units) superimposed on grid paper. The student is asked to draw a rectangle that is similar and double the original size.</td>
</tr>
<tr>
<td>Medium Rectangle</td>
<td>Student is shown an opaque rectangle (4 units) × (6 units) superimposed on grid paper. The student is asked to draw a similar rectangle that is larger than the original size, but smaller than double.</td>
</tr>
<tr>
<td>L-Shape (Double)</td>
<td>Student is shown an opaque L-shape superimposed on grid paper. The student is asked to construct a similar figure that is double the original size.</td>
</tr>
<tr>
<td>L-Shape Reduced (k)</td>
<td>Student is shown an opaque L-shape superimposed on grid paper. The student is asked to construct a similar figure that is $k$ times the original size.</td>
</tr>
<tr>
<td>Heart (k)</td>
<td>Student is handed a heart that had been cut from paper. With an option of using grid or plain paper, the student is asked to construct a similar figure that is $k$ times the original size.</td>
</tr>
<tr>
<td>Embedded Square (k)</td>
<td>Student is shown a unit square drawn inside the original 4 × 6 rectangle. The student is asked to draw and position the square, as it would appear inside a rectangle scaled by a factor of $k$.</td>
</tr>
<tr>
<td>Embedded Square (M)</td>
<td>Student is shown a unit square drawn inside the original 4 × 6 rectangle. The student is asked to draw and position the square, as it would appear inside the middle rectangle as described above.</td>
</tr>
</tbody>
</table>

novel problem solving, the series of tasks was modified for Tom to emphasize non-whole scale factors and complex figures. On the other hand, the Medium rectangle task was too difficult for Marquon, who responded with avoidance and impatience. To provide more
scaffolding and opportunities to document solution strategies, the series was modified to scale back the complexity of the remaining tasks.

Additional modifications were made to the scale factors in the tasks within the structure above, although the figures themselves were never modified. The L-shape task was designed to provoke cognitive tension for students using an additive strategy. In cases where time allowed for additional assignments, this task was modified to investigate strategies students used to reduce a shape since all of the other constructions were enlargements. Thus, the scale factor was changed to a value less than one.

In three cases, students were given two versions of the task with different scale factors. After reducing the size of the L-shape \((k = 1/2)\), Elaine was given the charge to construct another similar L-shape smaller than the original, yet larger than the image she had just constructed. Tom and Jorge were each given two versions of the heart task. After being asked to double the size of one heart, Tom was given a second, larger heart and asked to draw one half the size using plain paper. Jorge and Tom were asked to scale the heart twice. Before being asked to double the heart, Jorge was first given the task to draw a “bigger” version of the heart. The modified scale factors are indicated on the summary of tasks given to each student and overall performance on each task in Table 19. Bold indicates a correct construction while non-bold indicates an incorrect construction. Tasks are arranged from left to right conveying a decreasing trend in student success on the tasks. This gives some indication of the difficulty of the particular tasks as well as a global picture of the performance of individuals.

It should be noted that the figures that were to be scaled do not all lend themselves to the exact science of scaling. In order to get a broad range of strategies in
Table 19

**Summary of Performance on Construction Tasks by Student**

<table>
<thead>
<tr>
<th>Name</th>
<th>Double Rectangle</th>
<th>Medium Rect.</th>
<th>L-Shape (Double)</th>
<th>Embedded Square (k)</th>
<th>L-shape Reduced (k)</th>
<th>Emb. Square (M)</th>
<th>Heart (k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alecia</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X (2)</td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Andre</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X (2)</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Anna</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X (2)</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Chris</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X (2)</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>David</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X (2)</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Elaine</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X (1/2)</td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Eli</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X (2)</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Elijah</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X (2)</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Ian</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Jeff</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Jorge</td>
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<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Jules</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Kyle</td>
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<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Marquon</td>
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<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Matt</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
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<td>Naomi</td>
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<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Pedro</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Ryann</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Shanice</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Tate</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Tom</td>
<td>X (1/2)</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X (2)</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact Scale</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

novel situations, tasks were deliberately designed to be challenging and to push the limits of what students understood numerically and geometrically about similar figures. Two tasks in particular were extremely challenging: reducing the L-shape and scaling the
heart. The low percentage of solutions that were exactly similar was a function of the
difficulty of representing non-whole lengths and curved lines rather than the lack of
successful strategies applied. As is depicted in this section, students were resourceful, if
not completely successful, in overcoming these challenges.

Although Lamon's framework did guide the initial description of student
strategies, it was insufficient in classifying all of the strategies observed in this study
because of the nature of the tasks. Unlike missing value tasks or comparison tasks,
construction (scaling) tasks, such as those analyzed here, do not require or encourage
students to set up proportions and solve them algebraically. Also, strategies describe how
students used either given or implied scale factors to act upon the given figure. Some
modifications were made to the framework to make it applicable here. Avoidance and
Additive strategies are aligned to Lamon's original characterization. Other strategies, such
as the visual strategy identified by Lamon, needed to be adjusted to fit the context of
similarity. Seven strategy types, described in Table 20, were identified in this study:
Avoidance, Additive, Visual, Betweening, Pattern Building, Unitizing, and Functional
Scaling. Table 20 lists the observed strategies along with a description of each. A
narrative description and illustration of each strategy follows this table.

Visual Strategies

In Lamon's framework, visual strategies were characterized as guesses or answers
that did not use multiplicative reasoning; in fact, visual strategies were grouped along
with additive strategies. In this context, and in the responses collected here, a distinction
is captured between a student who relied on visual perceptions and one who applied
Table 20

*Strategies Used by Students to Construct Scaled Images of Given Figures*

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avoidance (AV)</td>
<td>No serious interaction with the problem.</td>
</tr>
<tr>
<td>Additive (AD)</td>
<td>Student determines scaled lengths by adding the scale factor to corresponding lengths in the original figure.</td>
</tr>
<tr>
<td>Visual (VI)</td>
<td>Student determines the size or placement of figures by sight or intuition rather than measurement or arithmetic calculation.</td>
</tr>
<tr>
<td>Betweening (BE)</td>
<td>Student initially determines scaled lengths additively, but uses visual judgment to improve the overall quality of the drawing by lengthening or shortening the result.</td>
</tr>
<tr>
<td>Pattern Building (PB)</td>
<td>Use of oral or written patterns without understanding the functional nature of the scale factor (tiling; median finding, angle matching).</td>
</tr>
<tr>
<td>Unitizing (UN)</td>
<td>Use of an original length as one unit. Scale factor indicates the number of units in the corresponding image length.</td>
</tr>
<tr>
<td>Functional Ratio (FR)</td>
<td>Application of scale factor as a functional ratio.</td>
</tr>
</tbody>
</table>

additive methods. The distinction is sufficient to warrant separate categories for these approaches to the task. *Visual* solutions, in the context of constructing similar figures, were not guesses or unreasoned responses, but a visual reliance on concept imagery related to proportion, conscious or unconscious. Elaine’s drawing in Figure 17 illustrated the presence of this intuition. Elaine’s image was reasonably similar to the original heart and was based on only visual measurements. As Elaine described, “*All I did was look at the heart; the design of what it was drawn. I looked at it while I was drawing, too. I was trying to make it exactly like it was.*” When she described her strategy, she claimed that the two centers of the hearts were the same point. Later, she recognized that they were in
slightly different places, but they were reasonably close—especially for a visual estimate. Rather than use this as evidence that her image was flawed, Elaine accepted that the centers could be distinct. This showed that her concept image was not entirely in line with the concept of dilation.

*Figure 17. Elaine Scales the Heart*

This concept imagery related to dilation was robust and did not occur accidentally or by chance. This was not the only instance where Elaine demonstrated her intuitions about geometric proportion and dilation, which were remarkably reliable. The following exchange occurred spontaneously prior to the construction task above in response to a prompt given by the interviewer.

**Interviewer:** Do you have experiences with making things bigger or enlarging things or shrinking things down?
Elaine: In drawing, or computers?

Interviewer: Any way.

Elaine: Sometimes when I draw I like to make things bigger.

Interviewer: Can you give me an example?

Elaine: Like...if I wanted to make a heart and it was that small, sometimes I like to make it a little bigger than what it looks like. [Elaine draws two hearts side by side, one small and the other slightly larger.]

Elaine: Sometimes, I try to draw hearts inside of it and I don't like it. [Elaine draws Figure 18B.] Sometimes I draw hearts and I don't like the way they look. Because they are so small and like that. They hardly have any room in it like that heart on the outside. If I do it, I go like this and draw a smaller one like that inside of it. [Elaine draws Figure 18A]

Interviewer: This [Figure 18A] is more pleasing to you than this [Figure 18B].

Elaine: I don't like it when it gets all smooshed up like that kind of. It looks...I don't like it.

Interviewer: So, this one can't possibly be the same shape as the outer one because it gets smooshed up.

Elaine: It's smaller on the inside and...Because...it's all smooshed and not wider like that one on the inside.

Interviewer: Are these similar? are these the same heart? Or do you think they are different?
Elaine: They are kind of different. This one [inside] looks more slanted than this one [outside] does. Other than that...looks the same.

Interviewer: If you were very careful, do you think that you could draw a similar heart to the outside edge?

Elaine: I don’t know.

Figure 18. Elaine Sketches Two Spontaneous Versions of Concentric Hearts

Elaine’s drawings illustrated intuitions she has about correspondence and the implications of this intuition on the way she visualized scaling. On Elaine’s drawings, there was a framing theme; however, the correspondence lines that Elaine drew in Figure 17 were not constant in length. While not made explicit during the interview, this might illustrate an informal understanding of proportion—the distance between corresponding points on the original and image depends multiplicatively on the distance the original is from the point of dilation. Elaine suggested that when this distance was held constant, the result was displeasing to her (Figure 18B). This intuition was not evident when a student took an additive approach and accepted the result.


Additive Strategies

When doubling rectangles, Chris took just such an additive approach and despite rigorous testing, accepted the result. In the traditional sense, an additive strategy describes a student who notes a constant difference relationship between two given quantities and scales by adding this constant to the third known value. Chris scaled the original rectangle (4 units x 6 units) into a larger rectangle (5 units x 7 units) beside the original. His construction is featured in Figure 19. Initially, Chris miscounted the vertical length. On his own, he corrected the drawing by adding another row of grid squares to the bottom of the figure.

Figure 19. Chris Scales a Rectangle Using an Additive Strategy

In order to test the strength of Chris’ convictions that an additive strategy produced similar figures, the interviewer drew another rectangle using an additive strategy, this time concentric to the original, shown in Figure A. Chris identified this
rectangle as similar to the original. The interviewer then drew a smaller rectangle and asked Chris to construct more rectangles, both larger and smaller, using this concentric additive strategy. The following transcript occurred during and after these constructions, shown in Figure 20B. Chris' confidence in the additive strategy was unshaken, despite visual evidence that it created figures that were cosmetically dissimilar to the original.

Interviewer: Is this orange one mathematically similar to the original?
Chris: Yup.
Interviewer: Use that method to make a bigger one.
Chris: You'd just go around it.

[Draws the next larger rectangle in the series.]

Interviewer: Could you do the same to make another bigger one? Keep going keep going. Do two more for me. [Chris continues to add rectangles in series.] Could you do it in the opposite direction?
Chris: Like inside?
Interviewer: Could you make a smaller version?
Chris: That would just be that.
Interviewer: Would it?
Chris: That's not what it would be. Can't make it smaller. [Pause] Unless you draw it in the square right there. Do that?

Interviewer: What would it look like?
Chris: Like this. [Draws a rectangle inside of the original.]

Interviewer: Are all these similar?
Chris: Mm hmmm.
Interviewer: Look at this small one. Describe that shape.

Chris: It's a little rectangle.

Interviewer: Is it fat, skinny, long? Use some adjectives.

Chris: Wide and it's small.

Interviewer: Is the outer one wide, too?


Figure 20. Additive Strategy Used Concentrically with Chris

The Betweening Strategy

When scaling geometric figures, however, an additive strategy can create a shape that looks distorted when compared to the original. Certainly, the shapes are not mathematically similar and this occasionally becomes visually apparent either during or after construction. Students were observed making minor alterations to lengths that were scaled additively in order to help the figure conform visually to expectations of the
image. A shape with six distinct edges may require the student to add a different amount to each edge—which yields a much different image than a traditional additive strategy would. This strategy is referred to as *Betweening*.

Another of Elaine’s constructions can be used to illustrate clearly how *Betweening* differs from both visual and additive thinking. Elaine outwardly measured as she drew on two different tasks relative to the L-shape. First, Elaine was asked to reduce the size of the L-shape to half of the original size. She drew this just below the original L-shape in Figure 21, and did so by dividing each side length by two in her head. The second task that Elaine was given was to draw a figure whose size is “in the middle of the two.” Elaine’s solution is pictured on the bottom right of Figure 21.

*Figure 21. Elaine Scales the L-shape*
Elaine initially drew a sketch that appears to scale the L-shape to 2/3 of its original size. In her justification, however, she listed the side lengths as having values in between the corresponding edges on the small and short figure. For instance, she made the top 4, which was one more than 3 and was a number between 3 and 6. The other sides also had lengths that were between the original and smallest image lengths. While describing the figure, she altered the figure by widening the left portion of the shape and thereby reducing the width of the right portion ("nose") of the shape. She provided no mathematical reasoning for this shift beyond visual intuition. Even though she had adeptly scaled the L-shape with a scale factor of 1/2, when given the task of drawing one "in between," this strategy broke down for her and she began to temper her quantitative responses with visual judgment.

Pattern-Building Strategies

Pattern-building is an umbrella term for the use of oral or written patterns without indicating an understanding of the functional nature of the scale factor. Median Finding, and Angle Matching are two particular strategies that were observed in this study that can be described in this way. Tiling is another pattern-building strategy that students can use in select cases. Not illustrated here, tiling is the process of building an enlargement of a figure by tessellating copies of the original. This method is not useful when scaling by non-whole factors including those less than 1. Because the tasks incorporating shapes that do tessellate were not presented with freely moving figures, tiling may not have been interpreted as a strategic option. The only construction task to incorporate a freely
moving figure was the Heart task and this shape is not an example of a figure that
tessellates.

Median Finding. Median Finding is a strategy that is particularly suited to
situations requiring multiple constructions. Most of the students in this study were asked
to construct multiple versions of the rectangle and this was when median finding was
observed. Students who were given the Medium Rectangle (MR) task had already
doubled the size of the same image and therefore had two “bookend” images as models.
They were instructed to draw an image smaller than the second but still larger than the
first. Jeff’s construction of the medium rectangle was a rectangle, 6 units by 9 units large,
which was similar to the original rectangle (4 units by 6 units). Jeff used the relative sizes
of the two existing figures to interpolate a median figure formed by finding median
lengths. He explains his method of finding the dimensions of the new rectangle (6 × 9)
based on the dimensions of the original (4 × 6) and the large (8 × 12):

Interviewer: How did you know to do 6?
Jeff: It was the closest I got to in between 8 and 4.
Interviewer: When you say in between 8 and 4, what does that mean to you?
Jeff: Uh...I'm not sure. Like, the numbers in between 8 and 4...4, 5...uh 5, 6, and 7...Those numbers in between 8 and the one I got to was 6
because I thought it'd be the closest from in between those.
Interviewer: You're looking for the very center in between.
Jeff nods.
Jeff: I did that with both sides. With the 12 and 6 side and did that the same with the square.

*Angle-matching.* Angle-matching is a strategy particularly suited to scaling figures with non-right angles. In this study, it was particularly suited to the heart-shaped task. When using this strategy, Jeff began his work by tracing the bottom angle in the heart, without tracing the rest of the heart. He finished scaling the figure visually around this tracing. This is simulated, as closely as possible, in Figure 22. Jeff's first attempt is the inner heart. He then modified the figure visually by enlarging it slightly.

![Figure 22. Jeff Uses an Angle-Matching Strategy to Scale the Heart](image)

*Unitizing*

The most sophisticated strategies were *Unitizing* and *Functional Scaling.*

*Unitizing*, as observed in this study, is closely related to tiling, but differs in that a student acts on lengths as units rather than entire figures. In addition, it is not limited to use with
figures that tile the plane. Tate used this method on the Heart task. He scaled the image by using the edge of the heart as a measuring unit and drew a corresponding length twice as large, illustrated in Figure 23. He then repeated this for the remaining straight edge before sketching the curves freehand.

![Figure 23. Scaling with Units Using Tate’s Method](image)

*Figure 23. Scaling with Units Using Tate’s Method*

*Functional Scaling*

*Functional Scaling* was the most common strategy. Students using this strategy multiplied select original lengths (dimensional, secondary, and space) by the indicated scale factor to determine, pre-construction, corresponding image lengths. Students knew how long the lengths would be before they even began drawing them. This was not always the case when students used pattern-building strategies or even unitizing strategies.

Eleven students can be portrayed as using a functional scaling approach in a dominant way. These students used the strategy in multiple contexts including whole and
non-whole scale factors, different figure types, and both to reduce or to enlarge shapes. This indicated that these students were more advanced in the transition from visual thinker to geometric proportional thinker. They seemed to have abstracted the functional role of the scale factor, and chose to apply it consistently across multiple problem contexts. David used the strategy successfully on all four tasks that he was given. For others, the strategy broke down at the intersection of similarity and three key concepts: measurement, rational number, and spatial thinking.

Alecia’s solution broke down as she struggled with measurement issues. She did not use the strategy successfully on the double rectangle task or the embedded square task. Her mistake was not in the application of the strategy, which she did flawlessly, but in measuring the primary, secondary, and space lengths in the picture. Once she began counting the grid squares instead of touched grid lines to measure lengths, she applied the strategy perfectly to double the L-shape.

Andre applied the strategy on three out of five tasks, but resorted to a visual strategy when embedding the square in the medium rectangle and when scaling the heart figure. This implied that Andre’s conceptual difficulty is related to spatial reasoning rather than proportional reasoning. Andre’s drawing of the embedded square is shown in Figure 24. He applied the scale factor (1.5) functionally to all edges of whole-number lengths, but did not transfer that strategy to lines or spaces that were not. All attempts at placing the square within the rectangle were done visually, and the square changed in dimension as he made three attempts at moving the location inside of the figure.

Eli also had spatial difficulties on the same task. He began the task by scaling the rectangle by multiplying each side length by 1.5 and correctly drawing the new rectangle.
He had no trouble measuring the given side lengths. However, while scaling and positioning the unit square within the rectangle, Eli struggled to quantify the space between the shapes and did not accurately scale the square. On the original figure, he described the square as being located in the "second box down and third box over," which would make doing multiplication $2 \times 1.5$ and $3 \times 1.5$ relevant (if he were continuing to scale the figure). Instead, he added 1.5 units to each length and did not change the size of the square. The placement of his unit square, drawn to match the original in size, is illustrated in the darker of the two colors in Figure 25.

It was clear that Eli also failed to scale the inner square. On his first attempt, the square was drawn so that it was the same size as the original. Following a prompt about its size, Eli rescaled the image, shown in the lighter color in Figure 25. He started by doing the calculation $1 \times 1.5$ on the calculator, which led to a key revelation: 1.5 is only one half box more than 1. He added one half of a box onto the right side and then
commented that he had to do the same thing to the height. He then asked an important question, whether he had to “do the other sides.” Eli answered his own question affirmatively, “Yeah because there are four sides and you have to do every side,” and adjusted one of the remaining sides by tacking on one half of a box. This created a three-sided plus figure. Once he had completed three sides, Eli enclosed the figure to turn it back into a square, leaving the fourth edge of the plus sign implicit.

Figure 25. Eli Finds the Location of the Square on the Medium Rectangle

Ian’s difficulties stemmed from his understanding of rational number. Drawn to the left in Figure 26, Ian tried out two different possibilities for how the unit square would look when scaled by a factor of 1.5. The first was a unit square translated away from the grid paper a half-unit horizontally and vertically. The second was a square that was 1.5 units long. He eventually chose the second image and positioned it in the correct location vertically, but not horizontally. Here he seemed to have added a half unit in both
directions, which worked out in his favor when the distance was 1 unit, but not when it was 2.

Figure 26. Ian Scales the Rectangle with Embedded Square

In both Ian’s case and Eli’s before him, the scale factor (1.5) was initially used multiplicatively. However, when scaling the unit square, the equivalence of the two expressions $1.5 \times 1$ and $1 + 0.5$ became confused. This equivalence may have implied to both students that the scale factor could be used interchangeably in an additive or
multiplicative way on this task. Although both boys indicated elsewhere in the interview that they knew scale factors to act multiplicatively, it was not uncommon for students to doubt this rule or others like it in select cases. Chris, for example, believed that the rules regarding scale factor were entirely case-specific.

Interviewer: When you said [take] half, were you adding and subtracting, or dividing by 2?

Chris: I wasn't doing anything really I was thinking times and dividing, plusing and minusing.

Interviewer: Just to kinda see what would work?

Chris: Mmm hmmm.

Interviewer: So the object is to find what relationship works. Is there one relationship that works for every problem, or is every problem different?

Chris: I think every one is different. Sometimes you plus and sometimes you multiply.

Interviewer: Is it only based on which ones work? [Chris nods.] Is that the only way you can figure out which ones are plus and which are multiply?

Chris: That's how I do it.

Discussion

Seven construction strategy types were observed in this study: Avoidance (AV), Additive, Visual, Betweening, Pattern Building (PB), Unitizing, and the Functional
Scaling (FS) strategy. Visual and Functional Scaling were the most common types of strategies. Avoidance, Betweening, and Unitizing were the least common types. Although there are some common features of this framework and the framework Lamon (1993) used to look at proportional reasoning tasks, the variety of types and the nature of some types are quite different given the geometric context. While Lamon grouped additive and visual strategies together, in a geometric context these two strategies were quite distinct. The main distinction between additive and visual strategies is the incorporation of intuitions regarding the constant of proportion, correspondence, and dilation. Furthermore, the combination of numeric and visual reasoning yielded a new type of strategy, Betweening, which incorporates numerical calculation with visual judgment and remediation.

In Table 21, strategies selected by each student on each assigned task are summarized. The conventions established in Table 19 are continued: bold indicates success, while non-bold indicates an incorrect construction. Functional Scaling was the most sophisticated strategy observed. Most students were able to apply a functional scaling strategy on the double rectangle task; however, some were not. As the figures became more complex, incorporating secondary and space lengths, and as the scale factors were changed from whole numbers to non-whole numbers to numbers less than 1, fewer students applied the functional ratio strategy and began to use visual strategies and other strategies tempered by visual judgment. When a student did not use a functional scaling strategy correctly, it was generally because the student lacked the understanding of one or more of three related concepts: measurement, rational number, or spatial thinking.
<table>
<thead>
<tr>
<th>Name</th>
<th>Double Rectangle</th>
<th>Medium Rect.</th>
<th>L-Shape (Double)</th>
<th>Embedded Square (k)</th>
<th>L-shape Reduced (k)</th>
<th>Emb. Square (M)</th>
<th>Heart (k)</th>
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<td>FS</td>
<td>FS</td>
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<td>Visual</td>
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<td>Visual</td>
<td>FS</td>
<td>Unitizing</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Total | 21               | 12            | 5                | 21                   | 6                 | 11             | 10        |
| Exact |                  |               |                  |                     |                   |                |           |
| Scale | 85.7%            | 66.7%         | 60%              | 57%                  | 33%               | 18.2%         | 0%        |

There was a distinctive nature to the profiles of student strategies elicited by each of the tasks, but not in entirely expected ways. The two L-shape tasks should have skewed results. Only students who indicated using an additive strategy on some or all of their previous responses were asked to double the L-shape. Thus, it is expected that the profile for this task would be skewed toward less sophisticated strategies. Only students who had shown proficiency on other tasks or who finished tasks more quickly were asked...
to reduce the size of the L-shape. Thus, it is expected that the profile for this task would be skewed toward more sophisticated strategies. In fact, neither skew is observed. Reducing the L-shape inspired the greatest variety of strategies with no single strategy dominating student responses. Doubling the L-shape did have the highest frequency of additive strategy use, but this is to be expected. A few students used a visual approach, but students tended to favor the functional scaling approach—even if, like Chris, they did not use a functional scaling approach on previous tasks.

In Figure 27, profiles of the strategies used on each task are compared. Of the 21 students given the double rectangle task, 18 used the scale factor as a functional ratio. As a baseline task for comparison, this task incorporated a whole scale factor, was an enlargement task, and students needed to apply the ratio functionally to only dimensional lengths in order to be successful. The prevalence of the functional scaling strategy dropped when the Double Rectangle (DR) task was compared to every other given task.

Figure 27. Relative Frequency of Strategy Use by Task
The tasks incorporating whole scale factors (DR, L-shape [Double], Embedded Square [Double]) are clustered at the top end when the profiles are ranked according to the percentage of students utilizing the functional scaling strategy. As the scale factors change to non-whole numbers, fewer students utilized this strategy in lieu of a variety of other strategies. A smaller percentage of students (42%) utilized the functional scaling strategy on the Medium Rectangle (MR) task, a task that is different from the Double Rectangle (DR) task only in scale factor. No students avoided solving the DR task, but this behavior emerged in the MR profile along with the pattern building strategy. Pattern building emerged as a strategy on the MR task even though it was not used on the DR task; it was a strategy used by students to reduce the L-shape but not to double it. In fact, on the L-shape Reduction task, a task that incorporated non-whole factors less than 1, there was much variety in student strategies. No particular strategy type seemed more prevalent than any other.

The percentage of students who used the functional scaling strategy also dropped as the figures in the task became more complex. The Heart and Embedded Square (M) tasks each required students to rely on secondary and space lengths. Only 40% of students used a functional scaling strategy to scale the heart and 9.1% of students used it to scale and embed the square.

On the other end of the proposed spectrum were students who utilized a visual strategy. The number of students who utilized a visual strategy did not seem to be impacted by a non-whole scale factor. On tasks where students used whole and non-whole factors to scale the same figure, the frequencies of visual strategies were remarkably close. However, as the lengths to be scaled became more oriented toward
secondary and space lengths and away from primary lengths, the visual strategy was used more frequently. Very few students used visual strategies to scale each of the rectangles, but 40% of students used a visual strategy on the heart task, 27% of students used a visual strategy to embed squares inside the double rectangle, and 36% to embed squares inside the medium rectangle.

Conclusion

Students in this study used a variety of differentiation and construction strategies that were not previously classified according to existing research frameworks. This could be due to the fact that more complex figures involving different types of distortion were used in both differentiating and constructing tasks in this study. While engaging in differentiation tasks, students were given open choices about which characteristics of the figures they would attend to. Construction tasks are not designed with that kind of open freedom; depending on the nature of the figure being constructed, students are required to attend to a variety of characteristics including angles and lengths of different varieties. On differentiation tasks, the researcher has the opportunity to observe what students attend to and what they do not. By carefully attending to the complexity of the figure to be constructed, a researcher can, in a sense, require students to attend to a specific combination of characteristics and observe the response.

By increasing the complexity of the figures in both the rSPT and the interview protocol, it was possible to manipulate the characteristics that students perceived and were required to attend to. The differentiation items on the rSPT were designed by incorporating distortion and proportion in different ways, encouraging students to
become distortion detectives. By using complex and U-shaped figures along with simple convex figures, students were poised to consider the impact of scaling not only on primary lengths, but also on secondary lengths and space lengths.
CHAPTER V

CONCLUSIONS AND IMPLICATIONS

The motivation for this study was to help address the gap between what we know about students’ intuitions about similarity prior to instruction and their documented difficulties even after instruction. This would illuminate possible pathways students may take when advancing from using visual and additive reasoning strategies to using multiplicative proportional reasoning on similarity tasks. In Chapter I, similarity was established as the most difficult context for proportional reasoning when surveying the existing literature. Some students who think proportionally in numeric contexts still have difficulty recognizing similarity as a context where proportional thinking is useful. This is despite studies that show young children have instinctual awareness of geometric proportion. The second chapter situated similarity at the conceptual crossroads of proportional and geometric thinking. A review of related research was presented there including major theories about stages of development relative to proportional and geometric reasoning. Particular attention was paid to studies that explored transitional reasoning or strategies employed by students before the development of robust conceptual understanding. Chapter III described the research design and methodology as well as the instrumentation used to select a group of participants that were likely to represent diversity in reasoning and conduct clinical interviews. Data collected were analyzed and the findings presented in Chapter IV.
This final chapter will examine the questions posed in the first chapter in light of the qualitative analyses presented in Chapter IV. This chapter is organized into three sections. The first section is intended to answer each of the two stated research questions related to the exploration of differentiation and construction strategies used by students. The second section is intended to draw three final conclusions and highlight the implications of this study for the design of curriculum, instruction, and future research. Finally, the methodological limitations of this study will be discussed in light of future directions for research.

Addressing the Research Questions

The participants in the research study were seventh-grade students engaged in the study of similarity. At the beginning of this study, 91 students were administered a revised version of the Similarity Perception Test (rSPT), first piloted in a teaching experiment one year prior. The rSPT returned three ratings for each student on Visual Perception, Quantifying Growth, and Quantifying Measurement. Based on these ratings, a stratified purposeful sample of 21 students was chosen for follow-up interviews. The overall goal was to establish a sample likely to show greater variation in student conceptions and strategies.

The follow-up interviews were conducted after all instruction on similarity had been completed. A task-based interview protocol was used which incorporated two main types of similarity tasks: differentiation and construction. As a part of this protocol, students were asked to revisit a subset of the rSPT items and think aloud as they responded to the items. These responses were analyzed qualitatively to ascertain the
nature of the differentiation strategies and the characteristics students noticed and compared while deciding if two figures were similar or not. Students were also asked to complete a series of construction tasks. From which their verbal and written solutions were analyzed and compared to strategies described in the existing literature on proportional reasoning and theories about the development of geometric and proportional reasoning. This study was conducted to answer the research questions as posed in Chapter I.

Question 1: Differentiation Strategies

The first research question was: What strategies do students use to differentiate similar figures from non-similar figures? What types of geometric and numeric reasoning are indicated by these strategies? Student think-aloud responses to rSPT items during the interviews were analyzed using a constant comparative strategy. The unit of analysis was the individual response to one item on the rSPT. In addition to strategies that were non-specific in terms of the characteristics students noticed or compared, four characteristic types were outlined as indicative of particular strategies: Appearance, Angle, Length, and Relationship. Relationships between components of the figures were key in some strategies, as were transformational relationships linking the figures themselves. In addition to the group of non-specific strategies, 11 specific differentiation strategies were identified as unique. These strategies indicated geometric reasoning aligned with van Hiele levels 0 (Visualization) and 1 (Analysis). They were also indicative of multiple levels of numerical reasoning including additive thinking, preproportional reasoning, qualitative and quantitative proportional reasoning.
Student responses indicated that reasoning at the visualization level was not always easy to verbalize. Perceptions related to the cosmetic features of shape like color, blur, and tiling transformations were easily put into words or pictures, but indicated less sophisticated strategies and reasoning. On the other hand, Anna, like her cohort, struggled to describe perceptions such as angle inequality and dilation transformations. As difficult as it is to verbalize, a reliance on visualization does not always indicate a lack of sophistication. David’s description of his Dilation strategy indicated sophisticated conceptions of correspondence and geometric proportion. His notion of one shape “fitting inside” another also relies on very informal imagery about the equality of corresponding angles.

Some strategies, such as the Visual and Dilation strategies, corresponded to Visualization—reasoning at van Hiele level 0. Other strategies such as the Constant Gap, Constant Difference, and Constant Ratio corresponded entirely to Analysis—reasoning at van Hiele level 1. Other strategies such as Shape Type, Length Comparison, and Qualitative Relationship indicated that these reasoning patterns are complementary, rather than mutually exclusive. Furthermore, although only three strategies which accompany strictly analytical reasoning were numeric, students were able to qualitatively describe the relationships between components of figures in non-numeric ways and use of the Qualitative Relationship strategy showed more sophisticated conceptions of similarity than the analytic Constant Difference strategy.

Even students who are capable of analytic reasoning use visual judgment to mediate their responses to differentiation tasks. The use of this visual judgment, particularly to identify distortions between figures, supports Swoboda and Tocki’s (2002)
hypothesis that students regard distortion as a property of shapes in this context. Two findings are particular supportive: (1) there is preliminary evidence that whether two figures are similar or not impacts student strategy choices, and (2) the presence of different types of distortion has a further impact on student strategies when figures are non-similar.

First, the general presence of distortion in non-similar figures seemed to be influential in that it gave students an initial source of comparison. This led to fewer non-specific responses overall. In comparison, the number of non-specific responses when students perceived the figures to be similar was three times as high as when they perceived the figures to be non-similar. Although almost all of the strategies identified in this study are sufficient to prove two figures are not similar, none of the strategies were sufficient to prove that they were. In order to show that two figures are similar, a combination of strategies was required. However, no responses included such a combination or a rigorous proof of similarity. On the contrary, students combined multiple strategies when proving two figures were not similar. The probes used by the interviewer such as “How do you know?” were not particularly solicitous of multiple strategies and were simply intended to explore reasoning. These unsolicited combinations were not necessary, but illustrated greater facility in forming and comparing composite units on the part of the students who made them. This may also have implications for geometric proof.

Second, the particular nature of distortion seemed to be influential. The Visual Perception portion of the rSPT was designed in four sections, each section featuring a different type of distortion in the items. Section 1 featured distortions in the overall
appearance of a shape, Section 2 featured uncoordinated horizontal and vertical growth factors, Section 3 featured uncoordinated interior and exterior growth factors, and Section 4 featured uncoordinated growth factors between different subshapes within the shape. Strategies were used in different proportions on each section, indicating that some strategies were more compatible to particular distortion types. For instance, Appearance-type strategies were used most often on Section 1. Although students could have used external relationships or ratios to describe the non-coordination of vertical and horizontal growth in Section 2, students were more likely to use length-type comparisons such as “longer” or “taller” to describe this type of distortion.

**Question 2: Construction Strategies**

The second research question was: What strategies do students use to construct similar figures? What types of geometric and numeric reasoning are indicated by these strategies? In order to answer this question, student responses to construction items during the interviews were analyzed using a constant comparative strategy. The unit of analysis was the individual response to one task. From these responses, nine construction strategies were identified. Two of these strategies, avoidance and additive, were similar to strategies documented for proportional reasoning. Responses that indicated no meaningful engagement with the problem were considered indicative of an avoidance strategy. Responses that featured students scaling lengths using a constant additive approach were considered indicative of a classical additive strategy as documented by Lamon (1993) and many others.
There were other strategy types described in the literature that diverged from standard in the case of similarity. For example, a visual strategy has been identified by Lamon (1993) as a primitive approach to proportional reasoning and was aligned with the additive strategy as non-constructive. However, in the context of similarity, there was cause to differentiate this strategy more from an additive approach in the case of similarity. The main distinction between additive and visual strategies is that visual strategies in this context can indicate sophisticated conceptions of proportional growth, and can be quite constructive. While it is true that for some students, a visual strategy is more akin to a guess, this is certainly not the case for Elaine who used it to scale the heart figure. Visual strategies can incorporate a range of simple to sophisticated concept imagery regarding the constant of proportion, correspondence, and dilation. The additive strategy is more accurately depicted as primitive and non-constructive.

In some cases, it was apparent that students were making use of visual judgment to mediate numeric strategies. This combination of numeric and visual reasoning was indicative of a new type of strategy, Betweening, which is characterized by the remediation, both during and post-construction, of a numeric strategy so that it conforms to visual expectations. Betweening was used primarily, though not exclusively, to remediate a constant additive construction. Students, noticing distortion in their constructions, adjusted side lengths so that they varied by different additive amounts.

*Pattern-building* is an umbrella term for the use of oral or written patterns without indicating an understanding of the functional nature of the scale factor. In the context of differentiation, two pattern-building strategies were observed: Angle-matching and Median Finding. In the context of construction, Tiling can also be put underneath this
umbrella. All of these strategies require the use of existing angles and shapes as tools in constructing a new shape. In all three cases there are limitations about the applicability and reliability of the strategy. Angle-matching does not take into consideration the scaling of side lengths; Median Finding requires intermediate scaling. Beyond the case presented here of “finding a rectangle in the middle,” this strategy would require complex multi-stage constructions. Tiling is not applicable when the scale factor is non-whole or less than 1 or when the figure is too complex to be tiled—such as an L-shape or heart. This is not to say these strategies are not without instructional merit, simply that they are limited in their standard applicability.

Unitizing and Functional Scaling were the most sophisticated strategies observed. Unitizing, as observed in this study, is closely related to tiling, but differs in that a student acts on lengths as units rather than entire figures. In addition, it is not limited to use with figures that tile the plane. Students using the Functional Scaling strategy multiplied select original lengths (dimensional, secondary, and space) by the identified or indicated scale factor to determine corresponding image lengths before construction. Students knew how long the lengths would be before they even began drawing them. This is not always the case when students used pattern-building strategies or even unitizing strategies. Functional scaling was the most popular approach used by students to construct similar rectangles, but broke down for students when the figures incorporated secondary and space lengths or when the concept of similarity intersected with three other related concepts: measurement, rational number, and spatial thinking. At the point where the strategy broke down, students utilized less sophisticated strategies, or made modifications
to their constructions using visual judgment. Visual judgment, used in concert with other strategies, was used as a tool for mathematical reflection and evaluation.

Three Final Conclusions and Resulting Implications

In each of the next three sections, a particular conclusion will be drawn from the results summarized here. In each case, support for the conclusion will be drawn from the findings shared in Chapter IV and summarized here. Each conclusion will be placed in reference to other theories or studies in the literature that were reviewed in Chapter II. Finally, implications of each conclusion will be shared for three important stakeholder groups: curriculum designers, teachers, and researchers.

Conclusion 1

Conclusion 1: There is value in teaching students to reason visually and analytically.

Even when a strategy is distinctively visual or analytic, the sophistication indicated by these strategies was certainly not well ordered; some visual strategies denoted a far more developed conception of similarity and of proportional growth than some analytical strategies. Part of the reason that visual strategies have been characterized in the literature as less sophisticated could stem from the difficulty students have in verbalizing them. Anna’s repeated attempts at describing an angle inequality are examples of the common struggle to verbalize visual perception. Shanice’s “bookmarks versus desks” comparison, discussed on page 80, is an attempt to draw on common experiences to make the idea of “rectangularness” more concrete.
David was remarkable in his ability to explain his Transformative strategy of imagining the small figure slowly filling up the larger while maintaining its shape. David’s understanding of similarity is quite well developed and is evident in his visualization. The correspondences that he saw between dots both on the edge of the figure and in the middle indicated that he was, indeed, visualizing dilation. He concluded with a dynamic hand gesture as if he was holding a beach ball as it filled with air. This gesture could be interpreted as a method of describing continuous all-directional growth.

Elaine’s description of drawing satisfying versus dissatisfying concentric hearts is another example of how sophisticated concept imagery related to dilation impacted a student’s visual judgment. Neither Elaine nor David’s description included analytic reasoning. Yet both showed more sophistication than Pedro or Tom’s accurate, yet perhaps incomplete use of the Constant Difference or Constant Ratio strategy.

Taking a purely analytical approach to differentiation tasks can be limiting and in some cases debilitating, especially when the task is to prove that two shapes are similar. When proving two complex figures or U-shapes are similar, a student taking a purely analytical approach must compare each and every possible length including primary and secondary edges, but also space lengths. It is difficult for students to know when enough is enough. In the case of non-similar figures, a student could make multiple comparisons and still miss the one pair that deviates from the pattern. Visual judgment can greatly reduce the analytical workload by indicating to a student where to look to find likely counterevidence to similarity and help support the student in determining a sufficient argument. For example, instead of constructing all possible ratios of corresponding
lengths to test for equality, Tom could have used visual judgment to detect distortions in particular components of the U-shape before deciding which ratios to compare.

This is not to elevate all visual judgment to extreme levels of sophistication. Certainly there were examples of visual strategies (i.e., Shanice's bookmarks) that alluded to more basic conceptions. Nor is this argument intended to deny the importance of analytic reasoning. Visual judgments provide structure for mathematical description, but are not themselves numerical descriptions. In order completely mathematize and abstract the act of classifying or scaling figures, analytical reasoning is required. However, what has been observed in this study suggests that visual perception is not entirely guess-related and primitive, that the consideration of visual perception as a powerful indicator and supportive extender of conceptual understanding in this area might be warranted.

Visual strategies seem to serve an important role in developing student conceptions of similarity and geometric proportion both before and after students learn analytic strategies. Visual strategies can lead to deeper concept imagery related to continuous all-directional growth if developed before or alongside analytical strategies. Even after students have adopted analytical ways of reasoning about similarity with simple figures, visual strategies can provide students with methods of reflection on and evaluation of more analytical methods when more complex figures and sources of distortion are present. This may build a more robust understanding of continuous growth.

Implications. There are implications of this conclusion for curriculum design, instruction, and future research. During the interviews, the verbalization of visual perceptions was augmented by physical gestures and sounds that defy written expression.
This may indicate that students need oral methods of expression—with documentation—beyond written reflection. Curriculum designers may wish to expand upon the current repertoire of “explain your reasoning” phrases to encourage the documentation of more visual detail. When the prompt is vague, it is easier to write off a visual perception with a phrase like “I just see it that way.” By recognizing that visualization is a form of reasoning and making prompts more specific, more detail about what that student sees could bring powerful concept imagery into consciousness. “What differences/relationships between the shapes did you see?” is one example of a visualization-specific prompt. Curriculum designers should also be explicit in designing reflection tasks that ask students to use visualization to evaluate the sufficiency of analytical arguments.

Teachers are the key to providing opportunities for the oral expression of visualization. By being aware of common visual strategies and ways that visual judgment augments analytical reasoning, teachers can be more supportive facilitators of classroom conversations and encourage students to describe visualization with words. Providing concrete examples by which students can identify or categorize their visualizations is also important. Having experiences by which to say “it’s like throwing a coin in a fountain and watching a water circle expand” may help students make their descriptions of visual thinking more specific. In this study, students were well versed in imagery related to tiling and tessellation, but seemed to search for ways to describe dilation. Sources for dilation-type imagery in the experiences of these students included David’s inflating ball, watching a drop of colored water expand on a paper towel, and using the grab-and-drag feature for stretching shapes included in most word-processing software.
In order to diagnose a student’s difficulties with similarity and determine the level to which the student understands similar figures, it will be important to further explicate levels of reasoning from these strategies. It is not entirely clear from these data if there are logical orderings of the strategies. In this study, it is clear that students do not use the same strategy for every pair of figures. Perhaps, some strategies break down when the conditions of the figures are varied. Strategies exhibit different levels of geometric thinking based on how students verbalize the characteristics by which they are making judgments and the extent to which students are thinking numerically about those characteristics.

Conclusion 2


Conclusion 2: Distortion-detection is a skill that enables students to reflect upon and evaluate the validity and accuracy of differentiation and construction strategies.

On simple differentiation items such as those incorporated in the rSPT, students’ decisions were more evidence-based on items where they perceived distortion than when they perceived none. Of the 40 non-specific responses, three times as many were related to figures students perceived as similar as non-similar. Specific distortions were located by students in the overall Appearance-type characteristics of the figures such as the positioning of subshapes within complex figures. Students also described shapes as appearing “more square” than others or “smooshed,” references that invoked shape type and size.

Students also located distortions by invoking Relationship-type characteristics both external and internal. These relationships, pre-cursors to extensive and intensive
ratio, were invoked both qualitatively and quantitatively. Using a Qualitative Relationship strategy, students invoked relationships with comments such as that “the girl is too small in that figure.” These comments are more suggestive of visual intuitions about geometric proportion and distortion in the appearance of the shape as a whole—rather than by comparison of individual components. In the analytic sense, students noticed that different components such as subshapes, side lengths, or gaps in the figure were too long or too short in relation to the rest of the figure. These relationships had a more numeric quality about them even though they did not always directly imply a ratio. Constant Gap, Constant Difference, and Constant Ratio were three strategies that were based on different assumptions about the expected numeric relationship between various length measurements in the figures. Select students did form ratios to represent the relationships that they noticed “Two of these lengths fit inside that one and so do the heights,” but to assume that noticed relationships were ratio would belie the complexity of the numeric relationships that were formed.

Students relied on their perceptions of distortion on construction tasks as well. Many students, like Andre and Eli, made repairs to their constructions after they had completed them because they perceived qualitative differences in the appearance of the original figure and their own. Students added units to lengths or widths, moved the embedded square around inside the rectangle, and made adjustments to the curve of the heart figures. In this context, the perception of distortion enabled students to engage in reflection and evaluation activity and gave them tools by which to judge the reasonableness of their work.
Stronger conflict between numeric strategies and a student's visual expectations emerged for some students on construction tasks, particularly when the numeric strategy was additive. This conflict was marked by the perception of distortion on the part of the student. Shanice, while scaling the L-shape, made modifications to her additive strategy. This occurred at the moment she perceived that to continue with the strategy meant that she would not be able to close the figure and have it resemble the original; either the angles would change or the side lengths would not meet at a vertex. Even students who made mistakes using a functional scaling strategy made modifications after the fact based on their perceptions of distortion; the Betweening strategy is a powerful connection between visual judgment and numerical methods.

Data from this study supports the theory that distortions are identified as a dominant property of figures (Swoboda & Tocki, 2002) and that students use the presence and absence of distortion to visually decide if two figures are similar (Geeslin & Shar, 1979). The framework by which the rSPT was developed is an initial foray into the characterization of different types of visual distortion. This study adds specific strategies to existing findings, strategies that students use to identify and describe the distortions they perceive. Furthermore, students can use their detective skills as a way to reflect on their work and the validity or accuracy of their numeric strategies.

*Implications.* The implications of this conclusion are centered on making distortion detection a more explicit skill in curriculum and instruction, and continuing to identify potential sources of distortion that would help students abstract the concept of similarity and support the development of rigorous and sufficient proofs of similarity. Curriculum designers who want to build on early conceptions of distortion and visual
perception may consider inviting students to be distortion detectives—to find distortions of increasing sophistication. This would support the development of imagery related to continuous all-directional growth by giving students opportunities to develop holistic forms of ratio. It would also assist students in the formulation of analytic criteria for non-distortion to support the concept definition of similar figures.

Cox, Lo, and Mingus (2007) found evidence that curriculum designers do incorporate distortion-related tasks into units on similarity. However, these tasks are found at the very beginning of the unit and then totally abandoned as soon as the property-based definition of similarity has been introduced. Following this introduction, tasks become oriented toward showing that two figures meet the criteria for similarity rather than showing that figures are not distorted. Students still seem to compartmentalize the act of identifying or describing distortion and the act of proving two shapes are similar. Curriculum and instruction should continue to support the detection of the distortion even as formal arguments for similarity are developed. Bringing these two concepts together is an integral part of understanding the concept of similarity.

Teachers who engage their students in open discussion of differentiation should be cognizant of the power of holistic reasoning and distortion detection. Rather than replace them with traditional analytic algorithms for proving two shapes are similar, teachers can build on proportional intuitions by encouraging the recognition of external and internal relationships as a structure for comparing two shapes and slowly advancing students to think numerically about these relationships with more complex figures.

It is possible that teaching students to represent and communicate their recognitions of different types of distortion in increasingly sophisticated contexts would
support students in the identification of necessary and sufficient criteria for similarity. The rSPT items begin to expose potential levels of sophistication; however, more experimentation is necessary in terms of the degree of sophistication that can be visually discerned. The difference in perceptions about enlargement versus contractions are, as of yet, unexplored.

**Conclusion 3**

*Conclusion 3: A revision of hypothetical van Hiele sublevels is necessary to incorporate observed differences in visualization-level reasoning.*

In Chapter II, it was hypothesized that the van Hiele levels could be refined to better describe differences in the reasoning of students in the context of similarity. Battista’s (2007) proposed sublevels for geometric reasoning were used as a model for sublevels in the context of similarity. The diversity of differentiation and construction strategies observed in this study and presented in Chapter IV indicated that these sublevels could be expanded. In particular, reasoning involved in using Transformative-type strategies was not well described by the model. Although the sublevels proposed for Level 1 (Analysis) reasoning seemed adequate, more description was possible in terms of sublevels for Level 0 (Visualization). In particular, there seemed a need to describe the level of Recognition in more detail. Table 22 suggests a revised subdivision of the van Hiele levels. Three distinct levels of recognition are included: 0.2 Coarse Recognition, 0.3 Recognition by Holistic Comparison, and 0.4 Recognition by Transformation.

As in Chapter IV, differentiation and construction strategies can be interpreted as indicating a level of reasoning. Each of the strategies can be mapped onto these revised
Table 22

*Revised van Hiele Sublevels in the Context of Similarity*

<table>
<thead>
<tr>
<th>Level</th>
<th>Similarity Sublevels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0</td>
<td></td>
</tr>
<tr>
<td>(Visualization)</td>
<td>0.1 <strong>Pre-recognition:</strong> Students are unable to visually discern non-similar shapes from those that are similar.</td>
</tr>
<tr>
<td></td>
<td>0.2 <strong>Coarse Recognition:</strong> Students are able to coarsely recognize distortions in shape based on overt changes in figure type or construction.</td>
</tr>
<tr>
<td></td>
<td>0.3 <strong>Recognition by Holistic Comparison:</strong> Students are able to visually discern using general appearance comparisons. Descriptions are limited to overall shapes rather than specific parts.</td>
</tr>
<tr>
<td></td>
<td>0.4 <strong>Recognition by Transformation:</strong> Students are able to visually discern using strategies related to imagined similarity transformations such tiling and dilation. Strategies include using criteria such as “two shapes are similar if one can be transformed into the other.”</td>
</tr>
<tr>
<td>Level 1</td>
<td></td>
</tr>
<tr>
<td>(Analysis)</td>
<td>1.1 <strong>Visual-informal componential reasoning:</strong> Students select portions of shapes for informal (i.e., pointier angles) description of distortion.</td>
</tr>
<tr>
<td></td>
<td>1.2 <strong>Informal and insufficient-formal componential reasoning:</strong> Students describe shapes as in level 1.1, but are able to describe specific quantitative relationships. Insufficient may indicate additive or incorrect quantification or it may indicate that ratio comparisons are incomplete.</td>
</tr>
<tr>
<td></td>
<td>1.3 <strong>Sufficient formal property-based reasoning:</strong> Students are able to discriminate between and construct similar and non-similar shapes; this requires the identification and pair wise comparison of corresponding lengths and angles.</td>
</tr>
<tr>
<td>Level 2</td>
<td></td>
</tr>
<tr>
<td>(Abstraction)</td>
<td>The student logically orders the properties of similar figures, forms abstract definitions or theorems, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.</td>
</tr>
</tbody>
</table>
sublevels in different ways. One such mapping, which is still not one-to-one, is depicted in Table 23. This mapping represents the reasoning identified in this study rather than what could be possible. Although some strategies still span multiple levels of recognition, the descriptions more closely match the types of reasoning that these strategies indicate. In particular, the descriptions more closely match the levels of recognition evident in these strategies and more of the diversity in visualization reasoning is evident.

**Implications.** The implications of this revised theory are more aligned with proposals for future research and assessment, rather than curriculum development or instruction. While curriculum developers and teachers may take note of the sublevels of reasoning, the theory needs to be substantiated before direct implications can be drawn from it for task development or instruction. However, in substantiating this theory, there are interesting questions for possible research extensions:

1. Can “visual” construction strategies be further differentiated so that more can be inferred about the level of student reasoning?
2. Are these sublevels a useful or productive way of categorizing student reasoning? Are the sublevels distinct? How can the assessment tools used in this study be refined to capture and diagnose student reasoning on similarity tasks so that they can better be used for formative assessment?
3. Do other strategies not identified by this study exist and can the identified strategies indicated broader spectra of reasoning levels? What additional sublevels of reasoning can be identified—either visual or analytical that would make it easier to recognize unique student strategies for differentiation or construction?
Table 23

*Interpretation of Student Strategies as Indicative of Revised van Hiele Sublevels in the Context of Similarity*

<table>
<thead>
<tr>
<th>Similarity Sublevels</th>
<th>Differentiation Strategies</th>
<th>Construction Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Level 0: Visualization</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1 Pre-recognition</td>
<td>Characteristically</td>
<td>Avoidance</td>
</tr>
<tr>
<td></td>
<td>Non-Specific</td>
<td>Visual</td>
</tr>
<tr>
<td></td>
<td>Cosmetic</td>
<td></td>
</tr>
<tr>
<td>0.2 Coarse recognition</td>
<td>Shape Type</td>
<td>Visual</td>
</tr>
<tr>
<td></td>
<td>Relative Position</td>
<td></td>
</tr>
<tr>
<td>0.3 Recognition by holistic comparison</td>
<td>Shape Type</td>
<td>Visual</td>
</tr>
<tr>
<td></td>
<td>Length Comparison</td>
<td></td>
</tr>
<tr>
<td>0.4 Recognition by transformation</td>
<td>Dilation</td>
<td>Visual</td>
</tr>
<tr>
<td></td>
<td>Tiling</td>
<td>Tiling</td>
</tr>
<tr>
<td></td>
<td>Qualitative Relationship</td>
<td></td>
</tr>
<tr>
<td><strong>Level 1: Analysis</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1 Visual-informal componential reasoning</td>
<td>Shape Type</td>
<td>Visual</td>
</tr>
<tr>
<td></td>
<td>Angle</td>
<td>Betweening</td>
</tr>
<tr>
<td></td>
<td>Length Comparison</td>
<td>Angle Matching</td>
</tr>
<tr>
<td></td>
<td>Qualitative Relationship</td>
<td></td>
</tr>
<tr>
<td>1.2 Informal and Insufficient-formal componential</td>
<td>Angle</td>
<td>Betweening</td>
</tr>
<tr>
<td></td>
<td>Constant Difference</td>
<td>Additive</td>
</tr>
<tr>
<td></td>
<td>Constant Ratio</td>
<td>Median Finding</td>
</tr>
<tr>
<td></td>
<td>Constant Gap</td>
<td>Unitizing</td>
</tr>
<tr>
<td></td>
<td>Qualitative Relationship</td>
<td>Functional Scaling</td>
</tr>
<tr>
<td>1.3 Sufficient formal property-based reasoning</td>
<td>Combination:</td>
<td>Combination: Angle</td>
</tr>
<tr>
<td></td>
<td>Angle/Constant Ratio</td>
<td>Matching/Functional</td>
</tr>
<tr>
<td></td>
<td>Strategies</td>
<td>Scaling</td>
</tr>
</tbody>
</table>
4. How might future research further adjust the tools (test; interview protocol) to better understand these research questions? This study altered tools that get at answers, but also expose some methodological limitations that can be addressed with better tools.

Methodological Limitations

The methodology of this study was limited in two important ways: sample selection and interview constraints. The sample of students used in this study was small. Only 91 students were initially administered the rSPT, from which only 21 students were selected for follow-up interview. The selection of these students was deliberate in order to guarantee the widest possible variation in strategy and reasoning. While this was beneficial in terms of the breadth of description that was possible, it was limiting in the comparisons that could be made between like-responses and gave little indication of the likelihood that each strategy would be used in general.

The data collected in each interview were also intended to capture a student’s reasoning patterns broadly. In the initial research design, the intentions were to capture these data in one interview session, which in reality was not feasible. Interviews were extended in the cases where the facility, student schedule, and student willingness were favorable; however, this was not always possible. In order to mediate the time and energy constraints, decisions needed to be made about which protocol tasks to assign and which to forego. These decisions limited the degree to which comparisons could be made at the student-level.
Further research is needed to validate the work of this study and refine the instrumentation for use in practice. By expanding the time allotted for interview and by changing the method by which students are sampled, more comparisons at the level of individual students could be made. With the current data, or new data collected with a modified method, exploration in the following additional areas is possible:

1. The impact of distortion-type on the strategy choices was examined; however, the rSPT also varied items by figure type. Certain types of distortion are simply not feasible using simple complex figures; however, the overall impact of using Complex or U-shaped figures on items needs further study.

2. In response to some of the interview tasks, students experienced moments of cognitive tension where their concept images related to similarity were challenged. These moments were marked by confusion and periods of silent contemplation. Exploration of these moments would not only indicate the potential for these tasks to inspire learning, but would also show potential thresholds in understanding similarity.
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Appendix A

Revised Similarity Perception Test
Script:

Hello. My name is Dana Cox and I am a researcher from Western Michigan University. I came to talk to you a few days ago about participating in a research study. Thank you to all of you who returned your consent forms. I am eager to learn more about how you see shape and what you know about similarity.

Today we will be looking at pictures of shape. I want you to think very carefully about the question that is on the board now—you may already have some very good ideas. [Read question.]

I am going to read these instructions to you just to be very clear. [Read instructions from slide 2.]

Slide 3 and 4: Let discussion emerge but eventually settle on the statements: they have to have the same general shape, no distortion is allowed, no smooshing, fattening, stretching out of whack etc…

Slide 5: Now I want you to pick up the remote that is in front of you. Most of these questions are the same as we just answered. If the two shapes that I show you are different sizes of the same shape, I want you to press 1. If they don’t look like the same shape, I want you to press 2.

Slide 6: We’re going to test this shape just to be sure. I sense that many of you had a hard time knowing for sure what it looked like. What do you see?

Slide 8: These are the same shapes as before. Would you like to answer again?

Slide 9: I am going to summarize what we talked about. (Help read slide, answer questions.)

Slide 10: it is important for you to do all the rest of the questions on your own. Please do not talk to one another until they are all finished. There are 30 questions total. I will give you as much time as you need to answer each question. Shall we begin?
Welcome!
How can you tell if two shapes are the same even if they are different sizes?

Instructions
• Put your full name and clicker number on the paper in front of you.
• Answer every question using your clicker. Even if it's a guess.
• You may use this paper to write anything you want.
  • Reminder card
  • Questions you would like to ask me
  • Comments from you would like to express
  • Explaining why you aren't sure about an answer.

Just for Practice:
Are these figures the SAME SHAPE?

Press 1 if Yes
Press 2 if No

Just for Practice:
Are these figures the SAME SHAPE?

Let's Test
Watch again, slower.

Are these figures the SAME SHAPE?
Press 1 if Yes
Press 2 if No

We agree:
SAME SHAPE means
- Same general shape—rectangles are NOT the same as squares. There are even different kinds of rectangles.
- No Distortion is Allowed. The pictures can't look taller or wider, stretched or squished out.
- Try your best. Sometimes it can be hard to see if shapes are the same just by looking.
   I presume: I am not trying to trick you.

READY?
Remember,
Press 1 if they are the SAME SHAPE
Press 2 if you think that something looks wrong or distorted.

#1.1
A

#1.2

A
Question 5.1: The rest of the questions will be multiple choice. (check for understanding) Please pick the best choice from those that are given. If you don’t see a choice you like, select “none of these.” Does everyone understand the instructions?
**#5.4**
How many times Bigger?

**#6.1**
If I triple the size of the triangle, what will the length of the base be?

**#6.2**
This figure is the same shape, blown up.
How wide is it now?

**#6.3**
This figure is the same shape, blown up.
How wide is it now?
Appendix B

Similarity Interview Protocol
Similarity Interview Protocol
Researcher Recordings

Student Name: ____________________________

Student Code: ____________________________

Date and Time of Interview: ____________________________

Scheduled Tutor Time: ____________________________

SPT Ratings
Perceptions Segments
Perceptions Composite
Quantifying Growth
Quantifying Measurement
Interview Questioning Guide

Questions to be asked throughout the interview

1. Have you ever blown up a photograph or seen a photograph that has been blown up?
2. Where else might you have seen two versions of a shape that are different sizes?
3. Can you think of a way to enlarge or shrink a picture or drawing? Can you think of another?
4. How have you used a computer to make words or a picture larger or smaller?
5. Have you studied similarity in school?
6. Is there anything else you would like to tell me about enlarging or shrinking things?

Task 1

Instructions:
We are going to go back to the slides that we saw the last time I came to your class. What do you remember about looking at those slides?

On this laptop, you can control the slide show by clicking the mouse buttons or the arrow keys. You can go at whatever speed you would like. I want you to look at each slide and tell me if you think the two figures are different sizes of the same shape or whether you think they are distorted in some way. Then, I want you to carefully explain to me why you think this and what you are looking for. The more you can describe, the better I will understand how you are thinking. Are the instructions clear?

Prompts:
If a student’s response is yes or no without explanation:
* And how do you know that the figures are similar/non-similar?
If a student’s response is unclear:
* I don’t understand what you just explained. Can you explain it to me again? Is there another way you could say that?
If a student has changed their answer from the original:
• This is different from what you said the first time. Can you tell me how you might be looking at it differently this time? (OR)
• The first time, you answered ______. Why might you have thought that at the time?

Recording:
Record response and jot notes about the explanation given.

Task 2

Double Rectangle/ Embedded Square Task

Instructions:
Very nice work on that. I really like how you explained your answers. On the next task I need you to talk out loud while you work on the problem. I am really interested in knowing what you are thinking as you work.

For this task, you see a rectangle with a square inside on some grid paper. I would like you to draw another version of this shape that is double in size. Are the instructions clear?

Prompts:
If a student is struggling to begin:
• What am I asking you to do?
• What do you see?
• How might you draw another rectangle that is twice as big as this one?

If a student does not draw the dot, prompt the student to imagine where it might go.

If a student does not talk aloud
• How did you know to draw this rectangle?
• How did you know what size to make the rectangle?

When the student has finished
• I see that you drew a square [describe location]. How did you know to put the square there?

Recording:
Be sure that the student is drawing on the given paper and take field notes about the verbal responses given.
Medium Rectangle/Embedded Square Task

Instructions:
This is the same rectangle and same dot as before. I would like you to make another
enlargement of the shape, but not as big as the doubled version you just drew. This one
should be somewhere in between. Are the instructions clear?

Prompts:
If the student asks for clarification, restate the instructions as such:
I want you to draw a third version of this shape. I want it to be bigger than this one, but
smaller than the doubled version you just drew. It is perfectly fine for the student to look
at the doubled version while working on this task.

Use the same prompts as Task 2 to further clarify the student response.

Recording:
Be sure that the student is drawing on the given paper and take field notes about the
verbal responses given.

L-Shape Task
Give this task instead of 3A if an additive strategy is used previously.

Instructions:
The instructions for this task are the same as the last one. I would like you to draw
another version of this shape that is double in size. Are the instructions clear?

Prompts:
If a student is struggling to begin:
• What am I asking you to do?
• What do you see?
• How might you draw another shape that is twice as big as this one?

If a student again uses an additive strategy, record which lengths the student adds to and
which are drawn without measurement.
• I noticed that you added two boxes to this length [point to one as appropriate].
  Did you do that for all of the lengths?
• (If no tension) What about this length here? Can you tell me how it changed?
  [point to one of the lengths that was drawn without measure
• How did you know where this corner should be? (point to the interior vertex
  that makes the shape an L.
Continue to prompt the student to consider why you can’t add to two all six edge lengths.
Note when (if) tension occurs and why.
If a student does not talk aloud
  • How did you know to draw this shape?
  • How did you know what size to make the shape?

**Recording:**
Be sure that the student is drawing on the given paper and take field notes about the verbal responses given.

**L-Shape Task-Reduced**

**Instructions:**
The instructions for this task are the same as the last one. I would like you to draw another version of this shape that is smaller in size. Are the instructions clear?

**Prompts:**
If a student is struggling to begin:
  • What am I asking you to do?
  • What do you see?
  • How might you draw another shape that is smaller than this one?

If a student does not talk aloud
  • How did you know to draw this shape?
  • How did you know what size to make the shape?

**Recording:**
Be sure that the student is drawing on the given paper and take field notes about the verbal responses given.

**Heart Task**

**Hand the student Heart B from the heart differentiation task.**

**Instructions:**
The instructions for this task are the same as the previous tasks. I would like you to draw another version of this shape that is double in size. Are the instructions clear?

**Prompts:**
If a student is struggling to begin:
  • What am I asking you to do?
  • What do you see?
  • How is this figure different than the others?
If a student does not talk aloud
  • How did you know to draw this shape?
  • How did you know what size to make the shape?

**Recording:**

Be sure that the student is drawing on the given paper and take field notes about the verbal responses given.

**Rectangle Sorting Task**

Give the student cut-out versions of the task pages.

**Instructions:**

Very nicely drawn. The next two tasks that I am going to ask you to do are sorting tasks. Have you ever sorted items before?

Here, I would like you to look at these rectangles and tell me which you think are different sizes of the same shape. By same shape, I mean it exactly as we have been talking about it before—that the rectangles have been blown up, but not stretched out of shape or distorted. You can think of this task as sorting rectangles into groups of the same shape. As you sort them, explain to me what you are looking at and how you know that shapes belong in the same group. Are the instructions clear?

**Prompts:**

Continue to prompt for clear explanations and justification of responses as during other tasks.

When the student is done sorting:
  • You have grouped the shapes so that...[describe what you interpret as the grouping for clarity.] Do I understand this correctly?
  • Why do these shapes belong together? [indicate each group one by one]
  • Do any of the shapes belong to more than one group?
  • You mentioned that you [repeat one sorting strategy]. Is there another strategy you could use?

**Recording:**

Record the groupings by letter and by circling the group on the protocol. Jot notes or indicate on the shapes what characteristics are of interest. Particularly, note what the student points to as they talk or how the shapes are moved in relation to one another. Are they stacked? Are they measured against one another in any way? Etc...
Arrow sorting Task 1

(May be given to more advanced students if time permits)

Give the student cut-out versions of the task page.

Instructions:
The instructions for this task are the same as the other. Here, I would like you to look at these arrows and tell me which you think are different sizes of the same shape. As you sort them, explain to me what you are looking at and how you know which arrows belong in the same group. Are the instructions clear?

Prompts:
Continue to prompt for clear explanations and justification of responses as during other tasks.

When the student is done sorting:
- You have grouped the shapes so that... [describe what you interpret as the grouping for clarity.] Do I understand this correctly?
- Why do these shapes belong together? [indicate each group one by one]
- Do any of the shapes belong to more than one group?
- You mentioned that you [repeat one sorting strategy]. Is there another strategy you could use?

Recording:
Record the groupings by letter and by circling the group on the protocol. Jot notes or indicate on the shapes what characteristics are of interest. Particularly, note what the student points to as they talk or how the shapes are moved in relation to one another. Are they stacked? Are they measured against one another in any way? Etc...

Arrow Sorting Task 2

Give the student cut-out versions of the task pages.

Instructions:
Very nicely drawn. The next two tasks that I am going to ask you to do are sorting tasks. Have you ever sorted items before?

Here, I would like you to look at these pictures and tell me which you think are different sizes of the same shape. By same shape, I mean it exactly as we have been talking about it before—that the pictures have been blown up, but not stretched out of shape or distorted. You can think of this task as sorting arrows into groups of the same shape. As you sort them, explain to me what you are looking at and how you know that shapes belong in the same group. Are the instructions clear?

Prompts:
Continue to prompt for clear explanations and justification of responses as during other tasks.
When the student is done sorting:

- You have grouped the shapes so that...[describe what you interpret as the grouping for clarity.] Do I understand this correctly?
- Why do these shapes belong together? [indicate each group one by one]
- Do any of the shapes belong to more than one group?
- You mentioned that you [repeat one sorting strategy]. Is there another strategy you could use?

Recording:
Record the groupings by letter and by circling the group on the protocol. Jot notes or indicate on the shapes what characteristics are of interest. Particularly, note what the student points to as they talk or how the shapes are moved in relation to one another. Are they stacked? Are they measured against one another in any way? Etc...

Heart Sorting Task

Give the student cut-out versions of the task page.

Instructions:
The instructions for this task are the same as the other. Here, I would like you to look at these hearts and tell me which you think are different sizes of the same shape. As you sort them, explain to me what you are looking at and how you know which arrows belong in the same group. Are the instructions clear?

Prompts:
Continue to prompt for clear explanations and justification of responses as during other tasks.

When the student is done sorting:

- You have grouped the shapes so that...[describe what you interpret as the grouping for clarity.] Do I understand this correctly?
- Why do these shapes belong together? [indicate each group one by one]
- Do any of the shapes belong to more than one group?
- You mentioned that you [repeat one sorting strategy]. Is there another strategy you could use?

Recording:
Record the groupings by letter and by circling the group on the protocol. Jot notes or indicate on the shapes what characteristics are of interest. Particularly, note what the student points to as they talk or how the shapes are moved in relation to one another. Are they stacked? Are they measured against one another in any way? Etc...
Task 6

Instructions:
I have been asking you to make a lot of decisions about shapes. I have asked you if figures are the same shape but different sizes. On the previous tasks I had you sort larger groups of shapes. On this next task, you don’t have to make any decisions about the shapes. They are definitely different sizes of the same shape. You may even have ideas about how much bigger or smaller they are. If you do, I’d like to hear them.

For each pair, identify and label as many measurements as you can. Label them on the paper and explain to me how you figure them out. Are these instructions clear?

Prompts:
Continue to prompt students as before to justify the measurements they have indicated. As them to explain the comparisons they are making.
- Are there any more measurements that you can label?
- How did you know not to label [indicate an unlabeled measurement].
- I noticed that you [describe how the student measured a side length]. Why did you do this?

When students have finished with each pair, as them if they are sure there are no other measurements that can be made. Then, proceed to the next pairing. Present visual or additive students with pairings A-D, multiplicative students with pairings C-E. If time permits, students from either group may be assigned additional pairings.

Recording:
For each, record the measurements that the student labels on the protocol. Leave all of the other measurements blank. Record short notes about the justifications students provide or any new comparisons that are made.
Similarity Interview Protocol  
Researcher Recordings

Student Name: ______________________

Student Code: ______________________

Date and Time of Interview: ______________________

Scheduled Tutor Time: ______________________

SPT Ratings

Perceptions Segments
Perceptions Composite
Quantifying Growth
Quantifying Measurement
Record Task 1 responses here

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<thead>
<tr>
<th>Q</th>
<th>Response</th>
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</table>
Double Rectangle/Embedded Square Task

Part 1: Draw another rectangle and square that is twice as big as this one.
Task 3A: Enlarging the dotted rectangle

Draw another rectangle/square figure that is bigger than the original, but not quite as big as the one you just drew.
L-Shape (Double) Task

Draw an L shape that is twice as big as the original.
L-Shape (Reduce) Task

**Draw an L shape that is smaller than the original.**

![L-shape drawing](image)
Heart-Shape Task

Draw a version of this Heart that is twice as big as the original given to you. You may use the grid paper below, or plain paper on the back.
Rectangle Sorting Task

Which of the given pictures are different sizes of the same shape?

D

A

B

C
Arrow Sorting Task 1

Which of the given pictures are different sizes of the same shape?

B
C
A
D
E
Arrow Sorting Task 2

Which of the given pictures are different sizes of the same shape?
Heart Sorting Task

Which of the given pictures are different sizes of the same shape?
Task 6: Identifying Measurements.

For each pair, identify and label as many measurements as you can. Show your work on the paper and explain what you are doing out loud.

Pair A:

```
18

15
```

```
25
```
For each pair, identify and label as many measurements as you can. Show your work on the paper and explain what you are doing out loud.

Pair B:
For each pair, identify and label as many measurements as you can.
Show your work on the paper and explain what you are doing out loud.

**Pair C:**

![Diagram of pair C]

**Pair D:**

![Diagram of pair D]
For each pair, identify and label as many measurements as you can. Show your work on the paper and explain what you are doing out loud.

Pair E:
For each pair, identify and label as many measurements as you can. Show your work on the paper and explain what you are doing out loud.

Pair F:
Appendix C

Piloting the SPT
Description of the Pilot Instrument. The Similarity Perceptions Test (SPT) is an objective assessment of students’ visual perception of similarity. It was constructed to test the assumption that seventh-grade students have already developed a visual perception that enables them to classify shapes that are mathematically the “same shape” even if they are “different sizes” (Lehrer et al., 2002; Swoboda & Tocki, 2002). The test is comprised of 15 pairs of figures designed to test the envelope of visual perception. Each pair of figures is featured on a single presentation slide. Students are shown each pair separately and asked to visually discern if they can be classified as different sizes of the same shape. When all students have answered, the next pair is shown. Data can be collected using paper-and-pencil or as it was in the pilot study by using a Classroom Response System. The Classroom Response System will be used in this study as well.

The figures used in the pairings are constructed of both simple polygons and of more complex shapes. These shapes included a four-pointed star, a cartoon girl, and an interior parallelogram. Some examples of these shapes can be found in the following figure. Of the 15 pairs, eight were constructed to be similar, seven non-similar. The similar pairs were of three main types: parallelogram with no interior, parallelogram with an interior image, and a U-shape with all 90-degree angles. Most of the similar pairs had a scale factor of 2, however two of the pairs were related by a factor of 1.5 in order to identify those students who do not recognize size transformations using non-integral scale factors.
The nonsimilar pairs drew on the same basic set of figures with the addition of the letter A. The images were presented with a preimage and the transformations were of the four varieties summarized in the following table.
Description of the Relationship of Non-Similar Pairings from the SPT

<table>
<thead>
<tr>
<th>Non-Similar Varieties</th>
<th>Description</th>
<th>SPT Item Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alterations</td>
<td>Changes in the general features of a shape or their position have been made</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>Zoom-outs</td>
<td>Indicated by a proportional enlargement of the frame and unchanged interior</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>Disjoints</td>
<td>Indicated by different scaling techniques applied to exterior and interior</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>Additives</td>
<td>Pairs that are scaled using additive instead of ratio techniques.</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>
The SPT returns data on a student’s ability to perceive four things. First, “same shape” as a relationship that preserves the general features and relative position of those features. This is same shape in the topological sense. Second, “same shape” as a relationship that does not allow for distortion of the image in the Euclidean sense. This would include non-proportional size transformations. Third, perceptions of “different size” as a transformation that acts on all portions of an image in the same way. Fourth, perceptions of “different size” as a magnitude that is measurable.

Results of the Pilot Study. The data collected using the SPT showed that seventh-grade students vary in their ability to perceive same shape and different size. Students perceived same shape as a relationship that preserves the general features and relative position of those features. All readily identified two letter As that were different sizes of different fonts as non-similar on the basis that one of them had serifs and the other did not, and also identified a pairing of parallelograms with interior girls as non-similar on the basis that the interior girl had been shifted to the left in the larger figure.

Most students also perceived “same shape” as a relationship that does not allow for the distortion of an image. Where a parallelogram had been enlarged using an additive method rather than a ratio method, students identified one as looking “more squished” than the other. Some described with their hands a pulling motion they would use on two diagonal vertices to stretch the smaller one onto the larger. This motion was generally seen as destroying the shape rather than preserving it. It also seems to support findings that students consider how one shape might be altered or distorted to coincide with another when determining same shape relationships (Geeslin & Shar, 1979).
Variance in student perception was most prevalent in the perception of “different size.” Some students struggled to identify the zoom-out images. In follow-up interviews where they were asked to explain their reasoning on individual SPT items, students sometimes offered contradicting reasoning strategies for determining if zoom-outs were different sizes of the same shape. One student, Jorge, identified a zoom out of the parallelogram/star combination as similar, citing that he was happy that the star had remained the same size, even if the outside frame had gotten larger. He then correctly identified the similar images of the parallelogram/girl combination on the basis that he was happy the girl had gotten bigger just like the outside frame. Later, this was a source of some tension for Jorge as he worked to understand that “different size” must apply the same way to all components of the figure, not just some.

Lastly, the SPT was able to generate corroborating evidence that the concept of dimension interferes with the concept of scale. While some students seemed to only guess at the relationship of one shape to another, some students were able to use both linear and area models as ways of comparing the relative sizes of shapes. For example, Jorge assessed the relative size of two shapes related by a scale factor of 1.5: “you can’t fit two of them in there. Probably like 2.5 or 1.5. Somewhere in between 2 and 1.” In this sense, “fitting two of them in there” was in relation to fitting two of the heights or two of the widths inside the height or width of the larger one. A second student, Amelia, understood things differently. She described the relative size of a figure and its image under a scale factor of two: “four would cover. Add 1 more across the top.” She said this as she pointed out the location of four smaller parallelograms “inside” a larger one. In this way, she identified that two of the smaller width would fit inside the larger, but
that you would need four to cover. She also identified this as a 200% dilation.

Identifying both four times bigger and a 200% dilation did not spark tension for her.
Appendix D

Human Subjects Institutional Review Board
Letter of Approval
Date: September 6, 2007

To: Steven Ziebarth, Principal Investigator
   Jane-jane Lo, Co-Principal Investigator
   Dana Cox, Student Investigator for dissertation

From: Amy Naugle, Ph.D., Chair

Re: HSIRB Project Number: 07-08-19

This letter will serve as confirmation that your research project entitled “Understanding Similarity: Becoming a Geometric Proportional Thinker” has been approved under the expedited category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may only conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition, if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: September 6, 2008