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Measures of Travers Ability in Graphs

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MEASURES OF TRAVERSABILITY IN GRAPHS

by

Futaba Okamoto

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics
Dr. Ping Zhang, Advisor

Western Michigan University
Kalamazoo, Michigan
June 2007
ACKNOWLEDGEMENTS

First and foremost I would like to thank my dissertation advisor, Professor Ping Zhang. You have treated me as if I were your own daughter and I am very honored that I now am one of your academic children. Your patient guidance and belief in my abilities encouraged me to learn and accomplish more than I could imagine in my area of research.

I am also grateful to the members of my dissertation committee: Professors Gary Chartrand, Allen Schwenk, Arthur White, and my outside reader, Professor Garry Johns. Your suggestions and many hours taken from your busy days to review my dissertation are greatly appreciated. I thank Professors Schwenk and White in addition for being instructors of several courses that I have taken. Being in your classes strengthened my writing skills tremendously, and more importantly, I became more and more interested in graph theory. I would also like to extend my gratitude to Professor Chartrand, who always welcomed my visits to his office. Professors Zhang and Chartrand were my inspiration when it came to the career decision – a professor of mathematics.

I would also like to thank everyone in the Department of Mathematics at Western Michigan University. I always feel at home surrounded by pleasant people in the department. It is another family of mine. To my professors, thank you for your dedication to the profession of mathematics and mathematics education. From each of you, I learned everything from mathematics to teaching techniques to new jokes. To the staff, thank you for all the help to make my life easier on a great number of occasions. My life as a student was definitely made more comfortable because you were here. To my classmates, I won’t forget the time we spent together doing homework, studying for exams, or just having non-mathematical good times.

Special thanks go to my family: my parents and sister in Japan. I especially want to thank my mom for her understanding and support during these years. I cannot thank you enough. Finally, to my significant other, Shingo, I am so lucky that you are with me.

Futaba Okamoto
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Chapter 1

Introduction

A cycle in a graph $G$ that contains every vertex of $G$ is called a Hamiltonian cycle of $G$. Thus a Hamiltonian cycle of $G$ is a spanning cycle of $G$. A Hamiltonian graph is a graph that contains a Hamiltonian cycle. A circuit $C$ in a graph (or a multigraph) $G$ is called an Eulerian circuit if $C$ contains every edge of $G$. Since no edge is repeated in a circuit, every edge appears exactly once in an Eulerian circuit. A connected graph (or multigraph) that contains an Eulerian circuit is called an Eulerian graph (or Eulerian multigraph).

As described in [6], while certainly not every connected graph of order at least 3 contains a Hamiltonian cycle, every connected graph does contain a closed spanning walk (in which all vertices are encountered, possibly more than once). Indeed, if every edge of a connected graph $G$ is replaced by two parallel edges, then the resulting multigraph $M$ is Eulerian. Each Eulerian circuit $C$ in $M$ then gives rise to a closed spanning walk in $G$ in which each edge of $G$ appears twice (see Figure 1.1). That is, if $G$ is a connected graph of size $m$, there is always a closed spanning walk of length $2m$ in $G$.

![Figure 1.1: A multigraph $M$ obtained from a graph $G$](Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.)

Goodman and Hedetniemi introduced in [9] the concept of a Hamiltonian walk in a connected graph $G$, defined as a closed spanning walk of minimum length.
in $G$. They denoted the length of a Hamiltonian walk in a connected graph $G$ by $h(G)$. Therefore, for a connected graph $G$ of order $n \geq 3$, it follows that

$$h(G) \geq n$$

and

$$h(G) = n$$

if and only if $G$ is Hamiltonian.

During the 10-year period 1973-1983, this concept received considerable attention. For example, Hamiltonian walks were also studied by Asano, Nishizeki, and Watanabe [1, 2], Bermond [3], Nebesky [12], and Vacek [15].

In 2004 Hamiltonian walks were studied from a different point of view. In [6] an alternative way to define the length $h(G)$ of a Hamiltonian walk in $G$ was presented. We have seen that a Hamiltonian graph $G$ contains a spanning cycle $C : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n$. Thus Hamiltonian graphs of order $n \geq 3$ are those graphs for which there is a cyclic ordering $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of vertices of $G$ such that

$$\sum_{i=1}^{n} d(v_i, v_{i+1}) = n,$$

where $d(v_i, v_{i+1})$ is the distance between $v_i$ and $v_{i+1}$ (that is, the length of a shortest path between $v_i$ and $v_{i+1}$ in $G$) for $1 \leq i \leq n$. For a connected graph $G$ of order $n \geq 3$ and a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of vertices of $G$, the number $d(s)$ is defined in [6] as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

Therefore, for each cyclic ordering $s$ of vertices of $G$,
Furthermore, \( d(s) = n \) if and only if \( C : v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) is a Hamiltonian cycle in \( G \). The Hamiltonian number \( h^*(G) \) of \( G \) is defined in \([6]\) by

\[
h^*(G) = \min \{ d(s) \},
\]

where the minimum is taken over all cyclic orderings \( s \) of vertices of \( G \). Therefore, \( h^*(G) = n \) if \( G \) is Hamiltonian; otherwise, \( h^*(G) \geq n + 1 \).

To illustrate this concept, consider the graph \( G = K_{2,3} \) of Figure 1.2. For the cyclic orderings

\[
s_1 : v_1, v_2, v_3, v_4, v_5, v_1 \quad \text{and} \quad s_2 : v_1, v_3, v_2, v_4, v_5, v_1
\]

of vertices of \( G \), we see that \( d(s_1) = 8 \) and \( d(s_2) = 6 \). Since \( G \) is a non-Hamiltonian graph of order 5 and \( d(s_2) = 6 \), it follows that \( h^*(G) = 6 \).

![Figure 1.2: A graph \( G \) with \( h(G) = h^*(G) = 6 \)](image-url)

It was shown in \([6]\) that the Hamiltonian number of a connected graph \( G \) is, in fact, the length of a Hamiltonian walk in \( G \). We state this result below.

**Theorem 1.1** For every connected graph \( G \), \( h^*(G) = h(G) \).

As a consequence of Theorem 1.1, the Hamiltonian number of a graph \( G \) is also denoted by \( h(G) \), which is then the length of a Hamiltonian walk in \( G \). In this work, we study Hamiltonian numbers of graphs and related concepts. We refer to the book \([7]\) for graph theory notation and terminology not described here.
Chapter 2

The Hamiltonian Number of a Graph

We begin this chapter by surveying known results in Sections 2.1 and 2.2 on the Hamiltonian and upper Hamiltonian numbers of a graph. In Section 2.3, we establish a new result, namely a characterization of those connected graphs of order 3 or more whose Hamiltonian number and upper Hamiltonian number differ by 1.

2.1 The Hamiltonian Number of a Graph

In this section, we present several known results on the Hamiltonian number of a graph. The following two theorems appeared in [5, 6, 9].

**Theorem 2.1** For every connected graph $G$ of order $n \geq 2$,

$$n \leq h(G) \leq 2n - 2.$$  

Moreover, $h(G) = 2n - 2$ if and only if $G$ is a tree.

**Theorem 2.2** For every pair $n, k$ of integers with $3 \leq n \leq k \leq 2n - 2$, there exists a connected graph $G$ of order $n$ having $h(G) = k$.

A characterization of connected graphs of order $n \geq 3$ with Hamiltonian number $2n - 3$ was established in [14]. In order to present this characterization, we need some additional definitions. A connected graph with exactly one cycle is called a *unicyclic graph*. Let $\mathcal{U}_T$ be the set of all unicyclic graphs containing exactly one
triangle. Therefore, if \( G \in \mathcal{U}_T \), then \( G \) is connected and contains exactly one cycle, namely a triangle.

**Theorem 2.3** Let \( G \) be a connected graph of order \( n \geq 3 \). Then
\[
h(G) = 2n - 3 \quad \text{if and only if} \quad G \in \mathcal{U}_T.
\]

All connected graphs of order \( n \geq 5 \) whose Hamiltonian number is \( 2n-4 \) were also determined in [14]. Among the results obtained by Goodman and Hedetniemi are the following (see [8, 9]).

**Theorem 2.4** Let \( G \) be a connected graph having blocks \( B_1, B_2, \ldots, B_k \). Then the union of the edges in a Hamiltonian walk for each of the blocks \( B_i \) forms a Hamiltonian walk for \( G \) and, conversely, the edges in a Hamiltonian walk of \( G \) that belong to \( B_i \) form a Hamiltonian walk in \( B_i \).

The following is an immediate consequence of Theorem 2.4 (see [6]).

**Corollary 2.5** If \( G \) is a connected graph having blocks \( B_1, B_2, \ldots, B_k \), then
\[
h(G) = \sum_{i=1}^{k} h(B_i).
\]

In particular, every bridge of \( G \) appears twice in every Hamiltonian walk of \( G \).

In [8] an upper bound for the Hamiltonian number of a graph was established in terms of its order and other graphical parameters. For a connected graph \( G \), the diameter of \( G \) is the largest distance between two vertices of \( G \) and is denoted by \( \text{diam}(G) \). A vertex-cut in a connected graph \( G \) is a set \( U \) of vertices of \( G \) such that \( G-U \) is disconnected. A vertex-cut of minimum cardinality in \( G \) is called a minimum vertex-cut. For a graph \( G \) that is not complete, the vertex-connectivity \( \kappa(G) \) of \( G \) is defined as the cardinality of a minimum vertex-cut of \( G \); while \( \kappa(K_n) = n-1 \). For a positive integer \( k \), a graph \( G \) is said to be \( k \)-connected if \( \kappa(G) \geq k \). Thus a graph \( G \) is 1-connected if and only if \( G \) is nontrivial and connected; while \( G \) is 2-connected if and only if \( G \) is a connected graph of order 3 or more containing no cut-vertices.


general, a graph $G$ is $k$-connected if and only if the removal of fewer than $k$ vertices does not result in a disconnected or trivial graph.

Theorem 2.4 implies that the topic of Hamiltonian walks can be restricted to 2-connected graphs. In [8] an upper bound for the Hamiltonian number of a $k$-connected graph was established in terms of the order, diameter, and connectivity of the graph.

**Theorem 2.6** If $G$ is a $k$-connected graph of order $n \geq 2$ having diameter $d$, then

$$h(G) \leq 2n - \left\lfloor \frac{k}{2} \right\rfloor (2d - 2) - 2.$$ 

The Hamiltonian numbers of some well-known classes of graphs have been studied by Goodman and Hedetniemi in [8, 9], where the following result appears.

**Theorem 2.7** Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete $k$-partite graph of order

$$n = n_1 + n_2 + \cdots + n_k,$$

where $n_1 \leq n_2 \leq \cdots \leq n_k$. Then

1. $h(G) = n$ if and only if $n_1 + n_2 + \cdots + n_{k-1} \geq n_k$. In this case, the graph $G$ is Hamiltonian;

2. $h(G) = 2n_k$ if $n_1 + n_2 + \cdots + n_{k-1} < n_k$.

A well-known sufficient condition for a graph $G$ to be Hamiltonian is due to Ore [13].

**Theorem 2.8 (Ore)** Let $G$ be a graph of order $n \geq 3$. Then $G$ is Hamiltonian if $\deg u + \deg v \geq n$ for every pair $u, v$ of nonadjacent vertices of $G$.

This theorem can be restated in terms of the parameter $h(G)$.

**Theorem 2.9** Let $G$ be a graph of order $n \geq 3$. Then $h(G) = n$ if $\deg u + \deg v \geq n$ for every pair $u, v$ of nonadjacent vertices of $G$. 6
Bermond [3] obtained the following generalization of Theorem 2.9.

**Theorem 2.10** Let $G$ be a connected graph $G$ of order $n \geq 3$ and let $k$ be an integer with $0 \leq k \leq n - 2$. If $\deg u + \deg v \geq n - k$ for every pair $u, v$ of nonadjacent vertices of $G$, then

$$h(G) \leq n + k.$$  

The *clique number* $\omega(G)$ of a graph $G$ is the maximum order among the complete subgraphs of $G$. Thus if $G$ is a connected graph of order $n \geq 2$, then $\omega(G) = 2$ if and only if $G$ is triangle-free; while $\omega(G) = n$ if and only if $G$ is complete. In particular, $\omega(T) = 2$ for every nontrivial tree $T$. In [14] an upper bound for the Hamiltonian number of a connected graph was established in terms of its order and clique number.

**Theorem 2.11** If $G$ is a nontrivial connected graph of order $n$ having clique number $\omega$, then

$$h(G) \leq 2n - \omega.$$  

Furthermore, for each integer $\omega$ with $2 \leq \omega \leq n$, there exists a connected graph of order $n$ having clique number $\omega$ and Hamiltonian number $2n - \omega$.

### 2.2 The Upper Hamiltonian Number of a Graph

For a connected graph $G$, the *upper Hamiltonian number* $h^+(G)$ of $G$ is defined in [6] as

$$h^+(G) = \max \{d(s)\},$$

where the maximum is taken over all cyclic orderings $s$ of vertices of $G$. For example, consider the graph $G = K_{2,3}$ of Figure 2.1. For the cyclic orderings $s_1 : v_1, v_2, v_3, v_4, v_5, v_1$ and $s_2 : v_1, v_3, v_2, v_4, v_5, v_1$ of vertices of $G$, we have seen that $d(s_1) = 8$ and $d(s_2) = 6$. Furthermore, since $G$ is a non-Hamiltonian graph of order 5, it follows that $h(G) = 6$. In fact, it is not difficult to see that *every* cyclic ordering $s$ of vertices of $G$ has either $d(s) = 6$ or $d(s) = 8$. Thus $h^+(G) = 8$.

Obviously, for every nontrivial connected graph $G$ of order $n$,

$$n \leq h(G) \leq h^+(G) \leq n \cdot \text{diam}(G).$$  

(2.1)
For each integer $n \geq 3$, it was shown in [6] that there are only two connected graphs $G$ of order $n$, namely the complete graph $K_n$ and the star $K_{1,n-1}$, for which $h(G) = h^+(G)$. This result is stated below.

**Theorem 2.12** Let $G$ be a connected graph of order $n \geq 3$. Then

$$h(G) = h^+(G) \text{ if and only if } G \in \{K_n, K_{1,n-1}\}.$$  

As another example, consider the Petersen graph $P$ in Figure 2.2. Since $P$ is a non-Hamiltonian graph of order 10, it follows that $h(P) \geq 11$. On the other hand, $\text{diam}(P) = 2$ and so $h^+(P) \leq 2 \cdot 10 = 20$ by (4.1). Therefore,

$$11 \leq h(P) \leq h^+(P) \leq 20.$$  

In fact, $h(P) = 11$ and $h^+(P) = 20$. Moreover, consider the sequences $s_i$ ($1 \leq i \leq 10$) as follows:

$$s_1 : u_1, u_2, u_3, u_4, u_5, v_2, v_3, v_5, v_1, u_1$$
$$s_2 : u_1, u_2, u_3, u_4, u_5, v_2, v_3, v_4, v_1, u_1$$
$$s_3 : u_1, u_2, u_3, u_5, u_4, v_2, v_3, v_5, v_1, u_1$$
Since \( d(s_i) = 10 + i \) for \( 1 \leq i \leq 10 \), it follows that for each integer \( k \) with \( 11 \leq k \leq 20 \), there exists a cyclic ordering \( s \) of \( V(P) \) such that \( d(s) = k \).

For a graph \( G \), the set of all possible values of \( d(s) \), where \( s \) is a cyclic ordering of vertices of \( G \), is called the Hamiltonian spectrum of \( G \). Thus the Hamiltonian spectrum of the graph \( G \) of Figure 2.1 is \( \{6, 8\} \), while the Hamiltonian spectrum of the Petersen graph \( P \) is \( \{11, 12, \ldots, 20\} \). The Hamiltonian spectrum of a cycle \( C_n \) of order \( n \) was determined in [10].

The upper Hamiltonian number of a graph was introduced and studied in [6] and studied further in [5, 10]. The upper Hamiltonian numbers of some well-known classes of graphs were studied in [5, 6]. In particular, the following result for trees was established in [6].

**Theorem 2.13** Let \( T \) be a tree of order \( n \geq 3 \). Then

\[
2n - 2 \leq h^+(T) \leq \left\lfloor \frac{n^2}{2} \right\rfloor.
\]

Moreover,

1. \( h^+(T) = 2n - 2 \) if and only if \( T = K_{1,n-1} \);
2. \( h^+(T) = \left\lfloor \frac{n^2}{2} \right\rfloor \) if and only if \( T = P_n \).

A sharp lower bound for the upper Hamiltonian number of a graph \( G \) was established in [10] in terms of the order and diameter of \( G \).
Theorem 2.14 If $G$ is a connected graph of order $n \geq 3$ and diameter $d$, then

$$h^+(G) \geq n + \left\lceil \frac{d^2}{2} \right\rceil - 1.$$ 

Furthermore, for every pair $n, d$ of integers with $1 \leq d \leq n - 1$, there is a connected graph $G$ of order $n$ and diameter $d$ such that $h^+(G) = n + \left\lceil \frac{d^2}{2} \right\rceil - 1$.

2.3 A Characterization

We have seen that $h(G) \leq h^+(G)$ for every connected graph $G$. Furthermore, Theorem 2.12 states that for each integer $n \geq 3$, the complete graph $K_n$ and the star $K_{1,n-1}$ are only two connected graphs $G$ of order $n$ for which $h(G) = h^+(G)$. In this section, we establish a characterization of those connected graphs $G$ of order $n \geq 3$ whose hamiltonian number and upper Hamiltonian number differ by 1, that is, $h^+(G) - h(G) = 1$. In order to do this, we need some additional definitions.

Recall that for a connected graph $G$, the diameter $\text{diam}(G)$ of $G$ is the largest distance between two vertices of $G$. A tree $T$ of order 4 or more is called a double star if $\text{diam}(T) = 3$. Necessarily then, a double star $T$ contains exactly two vertices that are not end-vertices, which are called the central vertices of $T$. If $T$ is a double star with central vertices $u$ and $v$ such that $\deg u = a$ and $\deg v = b$, then we write $T = S_{a,b}$. For each integer $n \geq 4$, let $G_n$ be the graph of order $n$ obtained by adding a pendant edge to the complete graph $K_{n-1}$.

Also, it will be useful to recall Ore’s theorem (Theorem 2.8): For a graph $G$ of order $n \geq 3$, if $\deg u + \deg v \geq n$ for each pair $u, v$ of nonadjacent vertices of $G$, then $G$ is Hamiltonian.

The complete graph $K_3$ and the path $P_3 = K_{1,2}$ are the only connected graphs of order 3. By Theorem 2.12, $h^+(G) = h(G)$ for $G = K_3$ and $G = K_{1,2}$. Thus, we consider connected graphs of order 4 or more.

Theorem 2.15 Let $G$ be a connected graph of order $n \geq 4$. Then

$$h^+(G) - h(G) = 1 \text{ if and only if } G \in \{K_n - e, G_n, K_{1,n-1} + e, K_{1,1,n-2}\}.$$
Proof. First, we show that if \( G \in \{ K_{1,n-1} + e, K_n - e, K_{1,1,n-2}, G_n \} \), where \( n \geq 4 \), then \( h^+(G) - h(G) = 1 \), beginning with \( G = K_n - e \). For \( G = K_n - e \), let \( u \) and \( v \) be the only two nonadjacent vertices of \( G \) and \( s \) an arbitrary cyclic ordering of vertices of \( G \). If \( u \) and \( v \) are consecutive vertices in \( s \), then \( d(s) = n + 1 \); while if \( u \) and \( v \) are not consecutive in \( s \), then \( d(s) = n \). Thus

\[
 h^+(K_n - e) = n + 1 \quad \text{and} \quad h(K_n - e) = n.
\]

Hence if \( G = K_n - e \), then \( h^+(G) - h(G) = 1 \).

For \( G = G_n \), let \( v \) be the end-vertex of \( G \) that is adjacent to \( u \) in the subgraph \( K_{n-1} \) of \( G \) and \( s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) an arbitrary cyclic ordering of vertices of \( G \). We may assume, without loss of generality, that \( v_1 = v \). If \( v_2 = u \) or \( v_n = u \), then \( d(s) = n + 1 \); while if \( v_2 \neq u \) and \( v_n \neq u \), then \( d(s) = n + 2 \). Hence

\[
 h^+(G_n) = n + 2 \quad \text{and} \quad h(G_n) = n + 1.
\]

Thus \( h^+(G) - h(G) = 1 \) for \( G = G_n \).

Suppose that \( V(G) = \{ u, v, w, v_1, v_2, \ldots, v_{n-3} \} \), where \( \deg u = \deg w = 2 \), \( \deg v = n - 1 \), and \( \deg v_i = 1 \) for \( 1 \leq i \leq n - 3 \). Let \( s \) be an arbitrary cyclic ordering of vertices of \( G \). If \( u \) and \( w \) are consecutive vertices in \( s \), then \( d(s) \) has exactly three terms equal to 1 and so \( d(s) = 3 + 2(n - 3) = 2n - 3 \); while if \( u \) and \( w \) are not consecutive vertices in \( s \), then \( d(s) \) has exactly two terms equal to 1 and so \( d(s) = 2 + 2(n - 2) = 2n - 2 \). Thus

\[
 h^+(K_{1,n-1} + e) = 2n - 2 \quad \text{and} \quad h(K_{1,n-1} + e) = 2n - 3.
\]

Hence \( h^+(G) - h(G) = 1 \) for \( G = K_{1,n-1} + e \).

For \( G = K_{1,1,n-2} \), suppose that \( V_1 = \{ u \}, V_2 = \{ v \}, \) and \( V_3 = V(G) - \{ u, v \} \) are the partite sets of \( G \), where \( |V_3| \geq 2 \). Let \( s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \) an arbitrary cyclic ordering of vertices of \( G \). If \( u \) and \( v \) are consecutive vertices of \( s \), say \( u = v_1 \) and \( v = v_2 \), then \( s : u, v, v_3, v_4, \ldots, v_n, u \) and so \( d(s) = 3 + 2(n - 3) = 2n - 3 \). If \( u \) and \( v \) are not consecutive vertices of \( s \), say \( u = v_1 \) and \( v = v_3 \), then \( s : u, v_2, v, v_4, \ldots, v_n, u \). Since \( n \geq 4 \), it follows that \( d(s) = 4 + 2(n - 4) = 2n - 4 \). Thus either \( d(s) = 2n - 3 \) or \( d(s) = 2n - 4 \). This implies that

\[
 h^+(K_{1,1,n-2}) = 2n - 3 \quad \text{and} \quad h(K_{1,1,n-2}) = 2n - 4.
\]

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Hence \( h^+(G) - h(G) = 1 \) for \( G = K_{1,1,n-2} \).

For the converse, let \( G \) be a connected graph of order \( n \geq 4 \) such that \( h^+(G) - h(G) = 1 \). First, we claim that diam\( (G) \) = 2. Assume, to the contrary, that diam\( (G) \) \# 2. If diam\( (G) \) = 1, then \( G = K_n \). However, \( h^+(K_n) = h(K_n) = n \) by Theorem 2.12. Hence diam\( (G) \) \# 1. If diam\( (G) \) \# 3, then \( G \) contains two vertices \( u \) and \( v \) such that \( d(u, v) = 3 \). Let \( u, x, y, v \) be a \( u - v \) path in \( G \) and \( v_1, v_2, \ldots, v_{n-4} \) the remaining vertices of \( G \). Also, let \( v_0 = v \) and

\[
\sum_{i=0}^{n-5} d(v_i, v_{i+1}) = a.
\]

For the cyclic orderings

\[
s_1 : \ u, x, y, v, v_1, v_2, \ldots, v_{n-4}, u
\]

\[
s_2 : \ u, y, x, v, v_1, v_2, \ldots, v_{n-4}, u,
\]

it follows that

\[
d(s_1) = a + 3 + d(u, v_{n-4}) \quad \text{and} \quad d(s_2) = a + 5 + d(u, v_{n-4}).
\]

Since \( h(G) \leq d(s_1) \) and \( h^+(G) \geq d(s_2) \), it follows that

\[
h^+(G) - h(G) \geq 2,
\]

a contradiction. Thus, as claimed, diam\( (G) \) = 2. Hence for every cyclic ordering \( s \) of vertices of \( G \), every term in \( d(s) \) is either 1 or 2.

We now consider two cases, depending on whether \( G \) is Hamiltonian.

**Case I.** \( G \) is Hamiltonian. Then \( h(G) = n \). Since \( G \neq K_n \), the graph \( G \) contains at least one pair of nonadjacent vertices. Assume, to the contrary, that \( G \) contains more than one such pair of nonadjacent vertices, say \( u, v \) and \( x, y \). If the vertices \( u, v, x, y \) are distinct, then every cyclic ordering \( s \) beginning with \( u, v, x, y \) has \( d(s) \geq n + 2 \), which is a contradiction. If \( \{u, v\} \cap \{x, y\} \neq \emptyset \), say \( v = x \), then every cyclic ordering \( s \) beginning with \( u, v, y \) has \( d(s) \geq n + 2 \), a contradiction. Hence \( G \) contains exactly one pair of nonadjacent vertices and so \( G = K_n - e \).
Case II. G is not Hamiltonian. Then $h(G) = n + k$ for some integer $k \geq 1$. First note that if $s$ is a cyclic ordering of vertices of $G$ for which $d(s) = h(G) = n + k$, then $d(s)$ contains exactly $n - k$ terms equal to 1 and exactly $k$ terms equal to 2. Hence $G$ contains $k$ pairwise vertex-disjoint paths $Q_1, Q_2, \ldots, Q_k$ such that $\{V(Q_1), V(Q_2), \ldots, V(Q_k)\}$ is a partition of $V(G)$. Moreover, $G$ cannot contain fewer than $k$ vertex-disjoint paths with this property. We consider three cases, according to whether $k = 1$, $k = 2$, or $k \geq 3$.

Case 1. $k = 1$. Then $h(G) = n + 1$ and $h^+(G) = n + 2$. Notice that $G$ contains a Hamiltonian path. If $s_1$ and $s_2$ are cyclic orderings of vertices of $G$ such that $d(s_1) = n + 1$ and $d(s_2) = n + 2$, then $d(s_1)$ contains exactly one term equal to 2 while $d(s_2)$ contains exactly two terms equal to 2. Hence there are at least two edges of $K_n$ that do not belong to $G$. Let $X$ be the set of edges of $K_n$ that do not belong to $G$, that is, $G = K_n - X$ and $H = \langle X \rangle$ the edge-induced subgraph in the complement $\overline{G}$ of $G$.

First, we claim that $H$ cannot contain any of $P_4$, $K_2 \cup P_3$, and $3K_2$ as a subgraph. For example, if $H$ contains $P_4$ as a subgraph, then every cyclic ordering $s$ of vertices of $G$ beginning with $v_1, v_2, v_3, v_4$ has $d(s) \geq n + 3$, which is impossible. Similarly, if $H$ contains $K_2 \cup P_3$ or $3K_2$ as a subgraph, then there is a cyclic ordering $s$ of vertices of $G$ such that $d(s) \geq n + 3$. Therefore, $H$ cannot contain any of $P_4$, $K_2 \cup P_3$, and $3K_2$ as a subgraph, as claimed.

Next, we claim that $H$ is connected. Assume, to the contrary, that $H$ is disconnected and let $H_1, H_2, \ldots, H_\ell$ be the components of $H$, where $\ell \geq 2$. Since (i) each $H_i$ ($1 \leq i \leq \ell$) contains at least one edge and (ii) $H$ cannot contain any of $P_4$, $K_2 \cup P_3$, and $3K_2$ as a subgraph, it follows that $\ell = 2$ and $H_1 = H_2 = K_2$, that is, $H = K_2 \cup K_2$. Thus $G = K_n - e_1 - e_2$, where $e_1$ and $e_2$ are nonadjacent edges of $\overline{G}$. Since $n \geq 4$, it then follows by Ore's theorem that $G$ is Hamiltonian, which is a contradiction. Therefore, $H$ is connected, as claimed.

Since $H$ cannot contain $P_4$ as a subgraph, $H = K_{1,i}$, where $2 \leq i \leq n - 2$, or $H = K_3$. We consider these subcases.

Subcase 1.1. $H = K_{1,i}$, where $2 \leq i \leq n - 2$. Let $v$ be the central vertex of $H$. If $\deg_H v = i \leq n - 3$, then $\deg_G v \geq 2$. Since $G - v = K_{n-1}$, it follows that $G$ is Hamiltonian, which is a contradiction. Thus $i = n - 2$ and so $G$ is the
graph obtained by adding a pendant edge to the complete graph \( K_{n-1} \). Therefore, \( G = G_n \) in this case.

Subcase 1.2. \( H = K_3 \). If \( n = 4 \), then \( G = K_{1,3} \). Thus \( h(G) = h^+(G) \) by Theorem 2.12, a contradiction. If \( n = 5 \), then \( G = K_{1,1,3} = K_{1,1,n-2} \). If \( n \geq 6 \), then \( G \) is Hamiltonian by Ore’s theorem, which is a contradiction. Therefore, \( G = K_{1,1,3} \) in this case.

Case 2. \( k = 2 \). Since \( h(G) = n + 2 \), the graph \( G \) contains two pairwise vertex-disjoint paths \( Q_1 \) and \( Q_2 \) such that \( \{V(Q_1), V(Q_2)\} \) is a partition of \( V(G) \). However, \( G \) does not contain Hamiltonian path. Suppose that \( Q_i \) is an \( x_i - y_i \) path for \( i = 1,2 \). Furthermore, let \( x_1, \ldots, y_i \) denote the \( x_i - y_i \) path \( Q_i \) for \( i = 1,2 \). Observe that \( x_i x_j, x_i y_j, y_i y_j \notin E(G) \) for all \( i \) and \( j \) with \( 1 \leq i, j \leq 2 \) and \( i \neq j \), for otherwise \( G \) has a Hamiltonian path. Then the cyclic ordering

\[
s : x_1, \ldots, y_1, x_2, \ldots, y_2, x_1
\]

of vertices of \( G \) has the property that \( d(s) = h(G) = n + 2 \). We show, in this case, that \( G = K_{1,1,4} \) or \( G = K_{1,4} + e \).

First, we claim that each of \( x_2 \) and \( y_2 \) is adjacent to every vertex in \( V(Q_1) - \{x_1, y_1\} \) (if \( V(Q_1) - \{x_1, y_1\} \neq \emptyset \)) and each of \( x_1 \) and \( y_1 \) is adjacent to every vertex in \( V(Q_2) - \{x_2, y_2\} \) (if \( V(Q_2) - \{x_2, y_2\} \neq \emptyset \)). It suffices to show that \( y_2 \) is adjacent to every vertex in \( V(Q_1) - \{x_1, y_1\} \) if \( V(Q_1) - \{x_1, y_1\} \neq \emptyset \). If \( y_2 \) is not adjacent to a vertex \( z \in V(Q_1) - \{x_1, y_1\} \), then let \( s \) be a cyclic ordering of vertices of \( G \) beginning with \( x_1, x_2, y_1, y_2, z \). Then \( d(s) \) contains at least 4 terms equal to 2. Thus

\[
h^+(G) \geq d(s) \geq (n - 4) + 2 \cdot 4 = n + 4 = h(G) + 2,
\]

which is a contradiction. Thus \( y_2 \) is adjacent to every vertex in \( V(Q_1) - \{x_1, y_1\} \), as claimed. Similarly, \( x_2 \) is adjacent to every vertex in \( V(Q_1) - \{x_1, y_1\} \) and each of \( x_1 \) and \( y_1 \) is adjacent to every vertex in \( V(Q_2) - \{x_2, y_2\} \) if \( V(Q_2) - \{x_2, y_2\} \neq \emptyset \).

Next, we claim that

\[
|V(Q_i) - \{x_i, y_i\}| \leq 1 \text{ for } i = 1, 2.
\]

If \( |V(Q_i) - \{x_i, y_i\}| \geq 2 \), then let \( Q_i : x_1, z, w, \ldots, y_1 \). Since each of \( x_2 \) and \( y_2 \) is adjacent to every vertex in \( V(Q_1) - \{x_1, y_1\} \), it follows that \( G \) contains a Hamiltonian path.
which is a contradiction. Thus $|V(Q_1) - \{x_1, y_1\}| \leq 1$, as claimed. Similarly, $|V(Q_2) - \{x_2, y_2\}| \leq 1$ as well.

Since $G$ is connected and $x_ix_j, x_iy_j, y_ix_j \notin E(G)$ for all $i$ and $j$ with $1 \leq i, j \leq 2$ and $i \neq j$, it follows that either

$$|V(Q_1) - \{x_1, y_1\}| = 1 \text{ or } |V(Q_2) - \{x_2, y_2\}| = 1.$$ 

We consider these two cases.

**Subcase 2.1.** $|V(Q_i) - \{x_i, y_i\}| = 1$ for $i = 1, 2$. Let $Q_1 : x_1, u_1, y_1$ and $Q_2 : x_2, u_2, y_2$. In this case, the order of $G$ is 6 and $h(G) = 8$. If $u_1u_2 \notin E(G)$, then the cyclic ordering

$$s : x_1, x_2, y_1, y_2, u_1, u_2, x_1$$

of vertices of $G$ has $d(s) = 10$ and so $h^+(G) \geq 10 = h(G) + 2$, a contradiction. Thus $u_1u_2 \in E(G)$. Hence $G$ contains the graph of Figure 2.3(a) as a subgraph, where each dashed line indicates that there is no edge between the two vertices. Observe that $x_1y_1, x_2y_2 \notin E(G)$, for otherwise, $G$ contains a Hamiltonian path. (For example, if $x_1y_1 \in E(G)$, then $x_2, u_1, y_2, x_1, y_1$ is a Hamiltonian path in $G$.) Therefore, $G = K_{1,1,4}$ whose partite sets are $\{u_1\}$, $\{u_2\}$, and $\{x_1, x_2, y_1, y_2\}$, as shown in Figure 2.3(b).

\[\text{(a)}\]

\[\text{(b)}\]

**Figure 2.3: Graphs in Subcase 2.1**

**Subcase 2.2.** Either $|V(Q_1) - \{x_1, y_1\}| = 1$ or $|V(Q_2) - \{x_2, y_2\}| = 1$, but not both, say $|V(Q_1) - \{x_1, y_1\}| = 1$ and $|V(Q_2) - \{x_2, y_2\}| = 0$. In this case, the order of $G$ is 5 and $h(G) = 7$. Let $Q_1 : x_1, u_1, y_1$ and $Q_2 : x_2, y_2$, where $x_2$ and...
$y_2$ are adjacent to $u_1$. Observe that $x_1y_1 \notin E(G)$, for otherwise, $G$ contains the Hamiltonian path $x_2, y_2, u_1, x_1, y_1$, a contradiction. Therefore, $G = K_{1,4} + e$, as shown in Figure 2.4.

$$G: \begin{array}{c}
x_1 \\
\ \ | \\
\ \ x_2 \\
\ | \\
\ \ | \\
y_1 \end{array}$$

Figure 2.4: The graph $G = K_{1,4} + e$ in Subcase 2.2

**Case 3.** $k \geq 3$. Thus $G$ contains $k$ pairwise vertex-disjoint paths $Q_1, Q_2, \ldots, Q_k$ such that $\{V(Q_1), V(Q_2), \ldots, V(Q_k)\}$ is a partition of $V(G)$. However, $G$ does not contain fewer than $k$ pairwise vertex-disjoint paths with this property. Suppose that $Q_i$ is an $x_i - y_i$ path for $1 \leq i \leq k$. Furthermore, let $x_i, \ldots, y_i$ denote the $x_i - y_i$ path $Q_i$ for $1 \leq i \leq k$. Recall that $x_ix_j, x_iy_j, y_iy_j \notin E(G)$ for all $i$ and $j$ with $1 \leq i, j \leq k$ and $i \neq j$, for otherwise $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$. Then the cyclic ordering

$$s : x_1, \ldots, y_1, x_2, \ldots, y_2, \ldots, y_{k-1}, x_k, \ldots, y_k, x_{k+1} = x_1$$

of vertices of $G$ has the property that $d(s) = h(G) = n + k$. Moreover, $d(s)$ contains exactly $k$ terms, namely $d(y_i, x_{i+1})$ for $1 \leq i \leq k$, that equal 2 with all other terms equal to 1.

We claim that exactly one of the paths $Q_i$ ($1 \leq i \leq k$) has order 2 or more. First, observe that if $|V(Q_i)| = 1$ for all $i$ with $1 \leq i \leq k$, then $G = \overline{K_k}$, which contradicts the fact that $G$ is connected. Thus at least one of the paths $Q_i$ ($1 \leq i \leq k$) has order 2 or more. Next, assume, to the contrary, that there are two such paths, say $Q_1$ and $Q_2$. Let $s$ be a cyclic ordering of vertices of $G$ beginning with $x_1, x_2, y_1, y_2$ and containing the pairs $y_i, x_{i+1}$ ($2 \leq i \leq k$) as consecutive terms (note that $x_{k+1} = x_1$). Then $d(s)$ contains at least $3 + (k - 1) = k + 2$ terms equal to 2. Thus

$$h^+(G) \geq d(s) \geq 2(k + 2) + |n - (k + 2)| = n + k + 2 = h(G) + 2,$$

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which is a contradiction. Thus, as claimed, exactly one of the paths $Q_i$ ($1 \leq i \leq k$) has order 2 or more, say $Q_1$. Thus $x_i = y_i$ for $2 \leq i \leq k$. Since $G$ is connected and none of $x_i x_1, x_i y_1$ are edges of $G$ for each $i$ with $2 \leq i \leq k$, each of the vertices $x_i$ ($2 \leq i \leq k$) must be adjacent to at least one interior vertex of $Q_1$. Therefore, the path $Q_1$ has order 3 or more. We consider four subcases.

Subcase 3.1. $Q_1$ has order 3, say $Q_1$ is the path $x_1, v, y_1$. Then the order of $G$ is $n = 3 + (k - 1) = k + 2$. Since $G$ is connected and $x_i$ ($2 \leq i \leq k$) is adjacent to neither $x_1$ nor $y_1$, it follows that $vx_i \in E(G)$ for $2 \leq i \leq k$. Thus $x_1 y_1 \notin E(G)$ for otherwise, $x_1, y_1, v, x_2$ is a path in $G$ and so $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$. Therefore, $G = K_{1, n-1}$ whose central vertex is $v$. However,

$$h^+(K_{1, n-1}) = h(K_{1, n-1}) = 2n - 2$$

by Theorem 2.12, which is a contradiction. Thus, this case cannot occur.

Subcase 3.2. $Q_1$ has order 4, say $Q_1$ is the path $x_1, u, v, y_1$. Then the order of $G$ is $n = 4 + (k - 1) = k + 3$. By an argument similar to the one used in Subcase 3.1, each of the vertices $x_i$ ($2 \leq i \leq k$) must be adjacent to at least one interior vertex of $Q_1$. Thus $x_1 y_1 \notin E(G)$, for otherwise $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. If there is a vertex $x_i$ ($2 \leq i \leq k$) that is adjacent to both $u$ and $v$, then $x_1, u, x_i, v, y_1$ is a path in $G$ and so $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. Thus each vertex $x_i$ ($2 \leq i \leq k$) is adjacent to exactly one of $u$ and $v$. If $x_1 v \notin E(G)$ and $u y_1 \notin E(G)$, then $G$ is a double star whose central vertices are $u$ and $v$. However then, $\text{diam}(G) = 3$, which is a contradiction. Thus, we may assume that $x_1 v \in E(G)$ or $u y_1 \in E(G)$, say the former. Observe then that $x_i u \notin E(G)$ for all $i$ with $2 \leq i \leq k$, for otherwise $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$. Since $G$ is connected and $x_i u \notin E(G)$ for $2 \leq i \leq k$, it follows that $x_i v \in E(G)$ for $2 \leq i \leq k$. This implies that $u y_1 \notin E(G)$. Therefore, in this case, $G = K_{1, n-1} + e$, where $\deg u = n - 1$, $\deg x_1 = \deg u = 2$, and all other vertices are end-vertices of $G$.

Subcase 3.3. $Q_1$ has order 5, say $Q_1$ is the path $x_1, u, v, w, y_1$. Then the order of $G$ is $n = 5 + (k - 1) = k + 4$. As we saw in Subcases 3.1 and 3.2, each
of the vertices $x_i$ ($2 \leq i \leq k$) must be adjacent to an interior vertex of $Q_1$ and $x_1y_1 \notin E(G)$. Furthermore, there is no vertex $x_i$ ($2 \leq i \leq k$) that is adjacent to both $u$ and $v$ or to both $v$ and $w$, for if some vertex $x_i$ ($2 \leq i \leq k$) is adjacent to both $u$ and $v$ or to both $v$ and $w$, say the former, then $x_1, u, x_i, v, w, y_1$ is a path in $G$ and so $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. Thus each vertex $x_i$ ($2 \leq i \leq k$) is not adjacent to at least one vertex in $\{u, v, w\}$. We now assume that $x_2$ is not adjacent to a vertex $z \in \{u, v, w\}$.

First, we claim that each vertex $x_i$ ($3 \leq i \leq k$) is adjacent to each of the remaining two vertices in $\{u, v, w\} - \{z\}$. Assume, to the contrary, that there is a vertex $x_i$ ($3 \leq i \leq k$) that is not adjacent to one of the remaining two vertices in $\{u, v, w\} - \{z\}$, say $x_3$ is not adjacent to $z' \in \{u, v, w\} - \{z\}$. Let $\{z''\} = \{u, v, w\} - \{z, z'\}$. Thus $\{u, v, w\} = \{z, z', z''\}$, where $x_2$ is not adjacent to $z$ and $x_3$ is not adjacent to $z'$. Recall that $x_1y_1 \notin E(G)$. Consider the cyclic ordering

$$s : z, x_2, y_1, x_1, x_4, \ldots, x_k, x_3, z', z'', z$$

of vertices of $G$. Observe that every term in $d(s)$ is 2, except possibly $d(z', z'')$ and $d(z'', z)$; that is, $d(s)$ contains at most two terms equal to 1. Since $n = k + 4$, it follows that

$$h^+(G) \geq d(s) \geq 2 + 2(n - 2) = 2n - 2 = n + k + 2 = h(G) + 2,$$

which is a contradiction. Thus every vertex $x_i$ ($3 \leq i \leq k$) is adjacent to each of the remaining two vertices in $\{u, v, w\} - \{z\}$, as claimed.

Necessarily then, $z = v$ and $x_iv \notin E(G)$ for all $i$ with $2 \leq i \leq k$. Furthermore, observe that $uw \in E(G)$, for otherwise, consider the cyclic ordering

$$s : u, x_2, y_1, x_1, x_3, \ldots, x_k, u, w, v$$

of vertices of $G$. Exactly two terms in $d(s)$ are equal to 1 (namely, $d(x_k, u)$ and $d(w, v)$) and so

$$h^+(G) \geq d(s) \geq 2 + 2(n - 2) = 2n - 2 = n + k + 2 = h(G) + 2,$$

which is a contradiction.

Therefore, we have the following:
(i) $z = v$ (that is, $x_2$ is not adjacent to $v$ in $G$) and so $x_2$ is adjacent to at least one of $u$ and $w$, say $ux_2 \in E(G)$;

(ii) Every vertex $x_i$ ($3 \leq i \leq k$) is adjacent to both $u$ and $w$ and so no vertex $x_i$ ($3 \leq i \leq k$) is adjacent to $v$;

(iii) $uw \in E(G)$.

By (i)-(iii), the graph $G$ contains the graph of Figure 2.5 as a subgraph.

![Diagram of Figure 2.5: A subgraph of $G$ in Subcase 3.3](image)

Since $\text{diam}(G) = 2$ and $x_1y_1 \notin E(G)$, it follows that either

(i*) $x_1v \in E(G)$ and $vy_1 \in E(G)$; or

(ii*) $x_1w \in E(G)$ or $y_1u \in E(G)$.

If $x_1v \in E(G)$, then $x_3, u, x_1, v, w, y_1$ is a path in $G$ and so $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. Thus $x_1v \notin E(G)$. Similarly, $y_1v \notin E(G)$. We now consider two subcases, according to whether $x_2w \notin E(G)$ or $x_2w \in E(G)$.

**Subcase 3.3.1.** $x_2w \notin E(G)$. Since $d(x_2, y_1) = 2$ and $u$ is the only vertex that is adjacent to $x_2$, it follows that $uy_1 \in E(G)$. Recall that $x_1v \notin E(G)$ and $y_1v \notin E(G)$. Thus the graph $G$ contains the graph of Figure 2.6 as a subgraph, where each dashed line indicates that there is no edge between the two vertices.

We now consider the cyclic ordering

$$s : x_2, x_1, x_3, \ldots, x_k, y_1, v, u, w, x_2$$
of vertices of $G$. Since there are exactly two terms in $d(s)$ equal to 1 (namely $d(v, u)$ and $d(u, w)$), it follows that

$$h^+(G) \geq d(s) = 2 + 2(n - 2) = 2n - 2 = n + k + 2 = h(G) + 2,$$

which is a contradiction.

**Subcase 3.3.2.** $x_2w \in E(G)$. Then $G$ contains a subgraph as shown in Figure 2.7.

We have seen that $x_1v \notin E(G)$ and $vy_1 \notin E(G)$. This implies that either $x_1w \in E(G)$ or $y_1u \in E(G)$. We may assume, without loss of generality, that $x_1w \in E(G)$. Hence the graph $G$ contains the graph of Figure 2.8 as a subgraph, where each dashed line indicates that there is no edge between the two vertices. Also, as we have mentioned, $x_iy_1 \notin E(G)$ for $1 \leq i \leq k$ and $x_ix_j \notin E(G)$ for
1 ≤ i, j ≤ k and i ≠ j (which are not indicated in Figure 2.8).

**Figure 2.8:** A subgraph of $G$ in Subcase 3.3.2

There is only one other possible edge that $G$ may have, namely $uy_1$. We claim that $uy_1 \in E(G)$. Assume, to the contrary, that $uy_1 \notin E(G)$. Then consider the cyclic ordering

$$s : u, y_1, v, x_1, x_2, \ldots, x_k, w, u$$

of vertices of $G$. There are exactly two terms in $d(s)$ equal to 1 (namely $d(x_k, w)$ and $d(w, u)$). Thus

$$h^+(G) ≥ d(s) = 2 + 2(n - 2) = 2n - 2 = n + k + 2 = h(G) + 2,$$

which is a contradiction. Hence $uy_1 \in E(G)$, as claimed. Therefore, in this case, $G = K_{1,1,n-2}$ is the only graph with the desired properties, which is shown in Figure 2.9.

**Figure 2.9:** The graph $G = K_{1,1,n-2}$ in Subcase 3.3

Subcase 3.4. $Q_1$ has order 6 or more. Suppose that
where $a \geq 4$. By an argument similar to the one used in Subcase 3.3, we may assume that $x_2$ is not adjacent to an interior vertex $v$ of $Q_1$, say $v = v_p$ for some integer $p$ with $1 \leq p \leq a$. Next, we claim that every vertex $x_i$ $(3 \leq i \leq k)$ is adjacent to each vertex of $V(Q_1) - \{x_1, y_1, v_p\}$.

Assume, to the contrary, that some vertex $x_i$ $(3 \leq i \leq k)$ is not adjacent to a vertex of $V(Q_1) - \{x_1, y_1, v\}$, say $x_3$ is not adjacent to the vertex $v_q$ in $V(Q_1) - \{x_1, y_1, v_p\}$, where $q \neq p$. For every cyclic ordering $s$ of vertices of $G$ beginning with

\[v_p, x_2, x_1, y_1, x_4, \ldots, x_k, x_3, v_q,\]

there are at least $k + 2$ terms in $d(s)$ equal to 2 and so

\[
h^+(G) \geq d(s) \geq 2(k + 2) + [n - (k + 2)]
= n + k + 2 = h(G) + 2,
\]

which is a contradiction. Therefore, as claimed, every vertex $x_i$ $(3 \leq i \leq k)$ is adjacent to each vertex of $V(Q_1) - \{x_1, y_1, v_p\}$. Since $|V(Q_1) - \{x_1, y_1, v\}| \geq 3$, it follows that every vertex $x_i$ $(3 \leq i \leq k)$ is adjacent to at least two consecutive interior vertices $v_i$ and $v_{i+1}$ $(1 \leq i \leq a - 1)$ of $Q_1$. We may assume that $x_3$ is adjacent to two consecutive interior vertices $v_i$ and $v_{i+1}$ of $Q_1$ for some integer $i$ with $1 \leq i \leq a - 1$. However then,

\[x_1, v_1, v_2, \ldots, v_i, x_3, v_{i+1}, \ldots, v_a, y_1\]

is a path in $G$ and so $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction, implying that this case cannot occur.

Therefore, the graphs in $\{K_n - e, G_n, K_{1,n-1} + e, K_{1,1,n-2}\}$ are the only connected graphs of order $n \geq 4$ for which $h^+(G) - h(G) = 1$. \[\blacksquare\]
Chapter 3

The Traceable Number of a Graph

3.1 Introduction

A graph has been called traceable if it contains a Hamiltonian path. Therefore, every Hamiltonian graph is traceable. The converse is not true of course. For example, the path $P_n$ of order $n$ clearly contains a Hamiltonian path, but $P_n$ contains no cycles at all. For a connected graph $G$ of order $n \geq 2$ and an ordering (also called a linear ordering) $s : v_1, v_2, \ldots, v_n$ of vertices of $G$, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1}).$$

The traceable number $t(G)$ of $G$ is defined by

$$t(G) = \min \{d(s)\},$$

where the minimum is taken over all linear orderings $s$ of vertices of $G$. Thus if $G$ is a connected graph of order $n \geq 2$, then $t(G) \geq n - 1$. Furthermore, $t(G) = n - 1$ if and only if $G$ is traceable. For example, since the graph $G$ of Figure 3.1 is traceable and has order 5, it follows that $t(G) = 4$.

As with Hamiltonian numbers of graphs, we now see that there is an alternative way to define the traceable number of a connected graph. Denote the length of a walk $W$ in a graph by $L(W)$. 

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Proposition 3.1  Let $G$ be a nontrivial connected graph. Then $t(G)$ is the minimum length of a spanning walk in $G$.

Proof.  Suppose that the minimum length of a spanning walk in a graph $G$ is $\ell$. Furthermore, let $s : v_1, v_2, \ldots, v_n$ be a linear sequence of vertices of $G$ such that $d(s) = t(G)$. For each integer $i$ with $1 \leq i \leq n - 1$, let $Q_i$ be a $v_i - v_{i+1}$ path of length $d(v_i, v_{i+1})$ in $G$. Let $W'$ be the $v_1 - v_n$ spanning walk of $G$ obtained by proceeding along the paths $Q_1, Q_2, \ldots, Q_{n-1}$ in the given order. Thus the length of $W'$ is $L(W') = d(s) = t(G)$. Since $\ell \leq L(W')$, it follows that $\ell \leq t(G)$.

Next, let $W : x_1, x_2, \ldots, x_{\ell+1}$ be a spanning walk in $G$ of length $\ell$. Then $\ell + 1 \geq n$. Define $u_1 = x_1$ and $u_2 = x_2$. For $3 \leq i \leq n$, define $u_i$ to be $x_{j_i}$, where $j_i$ is the smallest positive integer such that $x_{j_i} \notin \{u_1, u_2, \ldots, u_{i-1}\}$. Then $s : u_1, u_2, \ldots, u_n$ is an ordering of vertices of $G$. For each integer $i$ with $1 \leq i \leq n-1$, let $W_i$ be the $u_i - u_{i+1}$ subwalk of $W$ determined by the terms $u_i$ and $u_{i+1}$ in $s$. Thus $d(u_i, u_{i+1}) \leq L(W_i)$. Since

$$t(G) \leq d(s) = \sum_{i=1}^{n-1} d(u_i, u_{i+1}) \leq \sum_{i=1}^{n-1} L(W_i) = L(W) = \ell,$$

it follows that $t(G) \leq \ell$, giving the desired result. \hfill \box

3.2 Bounds for the Traceable Number of a Graph

In this section, we establish bounds for the traceable number of a graph in terms of its order and other graphical parameters. We have seen that for every connected graph $G$ of order $n \geq 2$, the Hamiltonian number $h(G) \leq 2n - 2$. As expected, there is a smaller upper bound for the traceable number of $G$. 

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Theorem 3.2 If $G$ is a nontrivial connected graph of order $n$ such that the length of a longest path in $G$ is $\ell$, then

$$t(G) \leq 2n - 2 - \ell.$$  

Proof. To show that $t(G) \leq 2n - 2 - \ell$, we proceed by induction on $n$. Since it is straightforward to see that $t(G) = 2n - 2 - \ell$ for every connected graph $G$ of order $n$ with $2 \leq n \leq 4$, the inequality holds for every connected graph of order $n$ with $2 \leq n \leq 4$. For an integer $n \geq 5$, assume, for every connected graph $H$ of order $n - 1$ such that the length of a longest path in $H$ is $\ell'$, that $t(H) \leq 2n - 4 - \ell'$. Let $G$ be a connected graph of order $n$ such that the length of a longest path in $G$ is $\ell$. We show that $t(G) \leq 2n - 2 - \ell$. If $G$ contains a Hamiltonian path, then $\ell = n - 1$ and $t(G) = n - 1$; so $t(G) = 2n - 2 - \ell$. Hence we may assume that $G$ does not contain a Hamiltonian path. Let $P$ be a path of length $\ell < n - 1$ in $G$. Among the vertices of $G$ not on $P$, let $w$ be a vertex of $G$ such that the length of a path from $w$ to a vertex on $P$ is maximum. Thus $G - w$ has order $n - 1$, is connected, and the length of a longest path in $G - w$ is $\ell$. By the induction hypothesis, $t(G - w) \leq 2n - 4 - \ell$.

Let

$$s : v_1, v_2, \ldots, v_{n-1}$$

be a linear sequence of vertices of $G - w$ for which $d(s) = t(G - w)$. Suppose that $w$ is adjacent to $v_i$ ($1 \leq i \leq n - 1$). If $i = n - 1$, then let

$$s' : v_1, v_2, \ldots, v_{n-1}, w.$$  

Thus

$$t(G) \leq d(s') = d(s) + d(v_{n-1}, w) = d(s) + 1$$

$$= t(G - w) + 1 \leq (2n - 4 - \ell) + 1 < 2n - 2 - \ell.$$  

If $1 \leq i \leq n - 2$, then insert $w$ immediately after $v_i$ in $s$, producing the sequence

$$s'' : v_1, v_2, \ldots, v_i, w, v_{i+1}, \ldots, v_{n-1}.$$  

Thus

$$d(s'') = d(s) - d(v_i, v_{i+1}) + d(v_i, w) + d(w, v_{i+1})$$

$$\leq d(s) - d(v_i, v_{i+1}) + d(v_i, w) + d(w, v_i) + d(v_i, v_{i+1})$$

$$= t(G - w) + 2 \leq (2n - 4 - \ell) + 2 = 2n - 2 - \ell.$$  

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Since \( t(G) \leq d(s^n) \), it follows that \( t(G) \leq 2n - 2 - \ell \).

A graph is a \textit{linear forest} if each of its components is a path. The following result gives a lower bound for the traceable number of a connected graph in terms of its order and the maximum size of a spanning linear forest.

**Proposition 3.3** If \( G \) is a nontrivial connected graph of order \( n \) such that the maximum size of a spanning linear forest in \( G \) is \( p \), then

\[
t(G) > 2n - 2 - p.
\]

**Proof.** Let \( s : v_1, v_2, \ldots, v_n \) be an arbitrary sequence of vertices of \( G \). Since the maximum size of a spanning linear forest in \( G \) is \( p \), at most \( p \) of the \( n - 1 \) numbers \( d(v_i, v_{i+1}) \) \((1 \leq i \leq n - 1)\) are 1 and the remaining \( n - 1 - p \) numbers are at least 2. Thus

\[
d(s) \geq p + 2(n - 1 - p) = 2n - 2 - p.
\]

Therefore, \( t(G) \geq 2n - 2 - p \).

The following corollary is an immediate consequence of Theorem 3.2 and Proposition 3.3.

**Corollary 3.4** Let \( G \) be a nontrivial connected graph of order \( n \) such that \( \ell \) is the length of a longest path in \( G \) and \( p \) is the maximum size of a spanning linear forest in \( G \). Then

\[
2n - 2 - p \leq t(G) \leq 2n - 2 - \ell.
\]

The graph \( G \) of Figure 3.2 has order \( n = 11 \). The length of a longest path in \( G \) is \( \ell = 6 \) and the maximum size of a spanning linear forest in \( G \) is \( p = 8 \). By Corollary 3.4, \( 12 \leq t(G) \leq 14 \). Actually, \( t(G) = 13 \) and \( s : v_1, v_2, \ldots, v_{11} \) is a linear ordering of vertices of \( G \) such that \( d(s) = 13 \). This example brings up a natural problem, which we state below.
Problem 3.5  For given positive integers \( n, \ell, p, \) and \( k \) for which \( n \geq 2 \) and \( \ell \leq k \leq p \leq n - 1 \), does there exist a connected graph \( G \) of order \( n \) such that a longest path in \( G \) has length \( \ell \), the maximum size of a spanning linear forest in \( G \) is \( p \), and \( t(G) = 2n - 2 - k \)?

In order to investigate Problem 3.5, we need more information about the traceable number of a graph and so we postpone the study of this problem to Section 3.6. On the other hand, for a connected graph of diameter 2, we have the following.

Proposition 3.6  If \( G \) is a nontrivial connected graph of order \( n \) and diameter 2 such that the maximum size of a spanning linear forest in \( G \) is \( p \), then

\[
t(G) = 2n - 2 - p.
\]

Proof.  Since the maximum size of a spanning linear forest in \( G \) is \( p \), there exists a sequence \( s : v_1, v_2, \ldots, v_n \) of vertices of \( G \) such that \( p \) of the \( n - 1 \) distances \( d(v_i, v_{i+1}) \) \((1 \leq i \leq n - 1)\) are 1 and the remaining \( n - 1 - p \) numbers are 2. Thus

\[
d(s) = p + 2(n - 1 - p) = 2n - 2 - p.
\]

Hence \( t(G) \leq 2n - 2 - p \). Since \( t(G) \geq 2n - 2 - p \) by Proposition 3.3, it follows that \( t(G) = 2n - 2 - p \).

If \( G \) is a nontrivial connected graph of order \( n \) and diameter 3 or more such that the maximum size of a spanning linear forest in \( G \) is \( p \), then either \( t(G) = 2n - 2 - p \) or \( t(G) > 2n - 2 - p \). Each of the graphs \( G_1 \) and \( G_2 \) of Figure 3.3 has order \( n = 10, \) diameter 3, and the maximum size of a spanning linear forest of each graph is \( p = 7 \). Such a spanning linear forest \( F_i \) of \( G_i \) \((i = 1, 2)\) is also shown in Figure 3.3.
By Proposition 3.3, \( t(G_i) \geq 2n - 2 - p = 11 \) for \( i = 1, 2 \). While \( t(G_1) = 11 \), it turns out that \( t(G_2) = 12 \). In the sequence \( s_1 : v_1, v_2, \ldots, v_{10} \) of vertices of \( G_1 \), exactly \( p = 7 \) of the 9 distances \( d(v_i, v_{i+1}) \) \((1 \leq i \leq 9) \) are 1 and the other distances are 2. On the other hand, there is no sequence of vertices of \( G_2 \) with this property and so \( t(G_2) \geq 12 \). Because \( d(s_2) = 12 \) for the sequence \( s_2 : u_1, u_2, \ldots, u_{10} \), it follows that \( t(G_2) = 12 \).

The following lemma establishes expected upper and lower bounds for \( h(G) - t(G) \) for a nontrivial connected graph \( G \).

**Lemma 3.7** \emph{For every nontrivial connected graph} \( G \),

\[ 1 \leq h(G) - t(G) \leq \text{diam}(G). \]

**Proof.** The lower bound is immediate. To verify the upper bound, let \( s : v_1, v_2, \ldots, v_n \) be an ordering of vertices of \( G \) such that \( d(s) = t(G) \) and let \( s_c : v_1, v_2, \ldots, v_n, v_1 \) be the cyclic ordering of vertices of \( G \) obtained from \( s \). Then

\[ h(G) \leq d(s_c) = d(s) + d(v_n, v_1) \leq t(G) + \text{diam}(G). \]

Therefore, \( h(G) - t(G) \leq \text{diam}(G) \). \( \blacksquare \)

We now determine all connected graphs \( G \) for which \( h(G) - t(G) = 1 \).
Proposition 3.8 For a nontrivial connected graph $G$, 

$$h(G) - t(G) = 1 \text{ if and only if } G \text{ is Hamiltonian.}$$

Proof. Observe first that if $G$ is a Hamiltonian graph of order $n$, then $h(G) = n$ and $t(G) = n - 1$; so $h(G) - t(G) = 1$. For the converse, assume that $G$ is a connected graph such that $h(G) - t(G) = 1$. Let 

$$s_c : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$$

be a cyclic ordering of vertices of $G$ with $d(s_c) = h(G)$. We show that $d_G(v_i, v_{i+1}) = 1$ for $1 \leq i \leq n$, which implies that $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle of $G$. Consider the linear ordering 

$$s_l : v_1, v_2, \ldots, v_n$$

of vertices of $G$ obtained from $s_c$. Since 

$$d(s_l) = d(s_c) - d(v_1, v_n) = h(G) - d(v_1, v_n),$$

it follows that $t(G) \leq d(s_l) = h(G) - d(v_1, v_n)$ and so $1 \leq d(v_1, v_n) \leq h(G) - t(G) = 1$. Thus $d(v_1, v_n) = 1$. Consequently, $d(v_{i-1}, v_i) = 1$ for $2 \leq i \leq n$ as well. Therefore, $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle of $G$ and so $G$ is Hamiltonian. 

3.3 Traceable Numbers of Trees

If $G$ is a connected graph and $H$ is a connected spanning subgraph of $G$, then $d_G(u, v) \leq d_H(u, v)$ for all $u, v \in V(G) = V(H)$. Thus for every linear ordering $s : v_1, v_2, \ldots, v_n$ of vertices of $G$ (or $H$), 

$$d_G(s) = \sum_{i=1}^{n-1} d_G(v_i, v_{i+1}) \leq \sum_{i=1}^{n-1} d_H(v_i, v_{i+1}) = d_H(s)$$

and so $t(G) \leq t(H)$. We state this useful observation below.

Observation 3.9 If $G$ is a connected graph and $H$ is a connected spanning subgraph of $G$, then $t(G) \leq t(H)$. In particular, if $G$ is a connected graph and $T$ is a spanning tree of $G$, then $t(G) \leq t(T)$. 

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Observation 3.9 suggests the usefulness of knowing the traceable numbers of trees. Since a tree $T$ is traceable if and only if $T$ is a path, it follows for a tree $T$ of order $n$ that $t(T) = n - 1$ if and only if $T = P_n$ and so $t(T) \geq n$ if $T \neq P_n$.

Since the length of a longest path in $T$ is the diameter of $T$, we have the following consequence of Corollary 3.4.

**Corollary 3.10** If $T$ is a nontrivial tree of order $n$ such that the maximum size of a spanning linear forest in $T$ is $p$, then

$$2n - 2 - p \leq t(T) \leq 2n - 2 - \text{diam}(T).$$

A caterpillar is a tree $T$ the removal of whose end-vertices is a path. The trees $T_1$ and $T_2$ of Figure 3.4 are caterpillars of the same order $n = 10$. While the maximum size of a spanning linear forest of $T_1$ is $\text{diam}(T_1)$, the maximum size of a spanning linear forest of $T_2$ is $\text{diam}(T_2) + 1$. In Figure 3.4, $F_i$ is a spanning linear forest of maximum size in $T_i$ for $i = 1, 2$.

![Figure 3.4: Spanning linear forests of maximum size in caterpillars](image)

Since the maximum size of a spanning linear forest of $T_1$ is $\text{diam}(T_1)$, it follows by Corollary 3.10 that

$$t(T_1) = 2n - 2 - \text{diam}(T_1).$$

In fact,

$$s_1 : u_1, u_2, u_3, u_7, u_8, u_4, u_5, u_6, u_9, u_{10}$$

is a linear ordering of vertices of $T_1$ for which $d(s_1) = t(T_1)$. For the caterpillar $T_2$, however, the maximum size $p$ of a spanning linear forest is $\text{diam}(T_2) + 1$. Consequently, by Corollary 3.10 either
$t(T_2) = 2n - 2 - \text{diam}(T_2)$ or $t(T_2) = 2n - 3 - \text{diam}(T_2)$.

The linear ordering

$s_2 : v_1, v_2, v_7, v_8, v_9, v_4, v_5, v_6, v_{10}$

of vertices of $T_2$ has the property that

$$d(s_2) = 2n - 2 - \text{diam}(T_2).$$

A total of $p = 6$ terms of the $n - 1 = 9$ terms in the sum $d(s_2)$ are 1. All of the remaining terms in $d(s_2)$ are 2, except for one which is 3. If fewer than $p$ terms in the sum $d(s')$ for a linear ordering $s'$ of vertices of $T_2$ are 1, then $d(s') \geq 2n - 2 - \text{diam}(T_2)$. Hence if there is a linear ordering $s$ of vertices of $T_2$ for which $d(s) = 2n - 3 - \text{diam}(T_2)$, then there must be $p$ terms in $d(s)$ equal to 1. We may assume that both $v_1, v_2, v_8$ (or $v_8, v_2, v_1$) and $v_9, v_3, v_4$ (or $v_4, v_3, v_9$) are subsequences of $s$. Assume, without loss of generality, that the vertices $v_1, v_2, v_8$ occur in $s$ before $v_9, v_3, v_4$. Then the first vertex in $s$ that follows the last vertex of $v_1, v_2, v_8$ or the last vertex of $v_1, v_2, v_8, v_7$ is a vertex whose distance is at least 3 from that vertex. Hence $d(s) \geq 2n - 2 - \text{diam}(T_2)$ and so $t(T_2) = 2n - 2 - \text{diam}(T_2)$. Proceeding in a similar manner for every caterpillar gives us the following result.

**Corollary 3.11** If $T$ is a caterpillar of order $n$, then

$$t(T) = 2n - 2 - \text{diam}(T).$$

We now show that the formula presented in Corollary 3.11 for the traceable number of a caterpillar holds in fact for all trees.

**Theorem 3.12** If $T$ is a nontrivial tree of order $n$, then

$$t(T) = 2n - 2 - \text{diam}(T).$$

**Proof.** Since $h(T) = 2n - 2$ for every tree $T$ of order $n$, it follows by Lemma 3.7 that $t(T) \geq 2n - 2 - \text{diam}(T)$. Furthermore, since the length of a longest path in $T$ is $\text{diam}(T)$, it follows by Theorem 3.2 that $t(T) \leq 2n - 2 - \text{diam}(T)$, giving the desired result.

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If \( T \) is a tree of order \( n \geq 3 \), then \( 2 \leq \text{diam}(T) \leq n - 1 \). Therefore by Theorem 3.12, if \( T \) is a tree of order \( n \geq 3 \), then

\[
n - 1 \leq t(T) \leq 2n - 4. \tag{3.1}
\]

We saw that

\[
t(T) = n - 1 \quad \text{if and only if} \quad T = P_n.
\]

Furthermore, only stars have diameter 2. So

\[
t(T) = 2n - 4 \quad \text{if and only if} \quad T = K_{1,n-1}
\]

by Theorem 3.12. More generally, we have the following realization result.

**Proposition 3.13** For each pair \( k,n \) of integers with \( 3 \leq n - 1 \leq k \leq 2n - 4 \), there exists a tree \( T \) of order \( n \) with \( t(T) = k \).

**Proof.** Let \( P : v_1, v_2, \ldots, v_{2n-1-k} \) be a path of length \( 2n - 2 - k \). A tree \( T \) is constructed by adding \( k + 1 - n \) new vertices \( w_1, w_2, \ldots, w_{k+1-n} \) and joining each of these vertices to \( v_2 \). Since \( \text{diam}(T) = 2n - 2 - k \), it follows by Theorem 3.12 that \( t(T) = 2n - 2 - (2n - 2 - k) = k \).

With the aid of Theorem 3.12, it is straightforward to determine those non-trivial trees \( T \) of order \( n \) such that \( t(T) = n \).

**Proposition 3.14** Let \( T \) be a tree of order \( n \geq 4 \). Then \( t(T) = n \) if and only if \( T \) is a caterpillar with maximum degree \( \Delta(T) = 3 \) and having exactly one vertex of degree 3.

**Proof.** By Theorem 3.12, \( t(T) = n \) if and only if \( 2n - 2 - \text{diam}(T) = n \) and so \( \text{diam}(T) = n - 2 \). Hence \( T \) contains a path \( P : v_1, v_2, \ldots, v_{n-1} \) of length \( n - 2 \) and a vertex \( w \) not on \( P \) that is adjacent to some vertex \( v_i \) with \( 2 \leq i \leq n - 2 \).

By (3.1) and Observation 3.9, if \( G \) is a connected graph of order \( n \geq 3 \), then

\[
n - 1 \leq t(G) \leq 2n - 4. \tag{3.2}
\]

We now determine all those connected graphs \( G \) of order \( n \) such that \( t(G) = 2n - 4 \) or \( t(G) = 2n - 5 \).
Proposition 3.15  Let $G$ be a connected graph of order $n \geq 3$. Then

$$t(G) = 2n - 4 \text{ if and only if } G = K_3 \text{ or } G = K_{1,n-1}.$$  

Proof.  Let $G$ be a connected graph of order $n \geq 3$ such that $t(G) = 2n - 4$. If $G$ contains a path of length 3 or more, then it follows by Theorem 3.2 that $t(G) \leq 2n - 5$. Hence the length of a longest path in $G$ is 2. This implies that $\Delta(G) = n - 1$ and so $G = K_3$ or $G = K_{1,n-1}$. Furthermore, note that $t(K_3) = 2 = 2n - 4$ and $t(K_{1,n-1}) = 2n - 4$.  

A tree $T$ is a double star if $T$ contains exactly two vertices that are not end-vertices. Necessarily these vertices are adjacent in $T$. For two integers $a, b \geq 2$, let $S_{a,b}$ denote the double star whose two vertices that are not end-vertices have degrees $a$ and $b$.

Proposition 3.16  Let $G$ be a connected graph of order $n \geq 4$. Then $t(G) = 2n - 5$ if and only if

1. $n = 4$ and $G \neq K_{1,3}$; or
2. $n \geq 5$ and $G = K_{1,n-1} + e$ or $G = S_{a,b}$ for some integers $a$ and $b$ with $2 \leq a, b \leq n - 2$ and $a + b = n$.

Proof.  Let $G$ be a connected graph of order $n \geq 4$ such that $t(G) = 2n - 5$. From Theorem 3.2 and Proposition 3.15, it follows that the length of a longest path in $G$ is 3. This implies that (i) $n = 4$ and $G \neq K_{1,3}$; or (ii) $n \geq 5$, $\Delta(G) = n - 1$, and $G = K_{1,n-1} + e$; or (iii) $n \geq 5$, $\Delta(G) \leq n - 2$ and $G$ is a double star. The converse is straightforward.

We have seen that if $G$ is a connected graph and $H$ is a connected spanning subgraph of $G$, then

$$t(G) \leq t(H).$$

In particular, if $T$ is a spanning tree of $G$, then

$$t(G) \leq t(T).$$
This brings up the following problem:

**Problem 3.17** Suppose that $G$ is a connected graph and $T$ is a spanning tree of $G$ such that $t(G) = a$ and $t(T) = b$. If $a < b$ and $k$ is an integer for which $a < k < b$, does there exist a spanning subgraph $H$ of $G$ such that $t(H) = k$?

If $G$ is a tree, then the only connected spanning subgraph of $G$ is $G$ itself, while if $G = C_n$ ($n \geq 3$), then each connected spanning subgraph of $G$ is $C_n$ itself or a path $P_n$ and $t(C_n) = t(P_n) = n - 1$. Hence we consider those connected graphs that are neither trees nor cycles. Since it is not easy in general to determine the traceable number of an arbitrary connected graph, we consider a well-known class of graphs, namely the class of unicyclic graphs. Recall that a unicyclic graph is a connected graph that contains exactly one cycle. Thus each proper connected spanning subgraph of a unicyclic graph $G$ is a spanning tree of $G$, which is obtained from $G$ by deleting an edge from the cycle of $G$. Furthermore, if we remove all the edges of the cycle in a unicyclic graph, we obtain a forest.

For each positive integer $i$, let $\mathcal{F}_i$ be the family of unicyclic graphs $G$ such that $G \in \mathcal{F}_i$ if and only if removing all the edges on the cycle in $G$ results in a linear forest with exactly $i$ nontrivial components. For example, the graphs $G_i$ and $G'_i$ (shown in Figure 3.5) belong to $\mathcal{F}_i$ for $i = 1, 2$. Observe that, for each positive integer $i$, the maximum degree $\Delta(G)$ of $G \in \mathcal{F}_i$ is either 3 or 4 and $G$ contains exactly $i$ vertices of degree 3 or 4.

For $j = 3, 4$, let the set $\mathcal{F}_{1,j}$ consist of all graphs $G$ in $\mathcal{F}_1$ with $\Delta(G) = j$. Thus $\mathcal{F}_1 = \mathcal{F}_{1,3} \cup \mathcal{F}_{1,4}$. For example, for the graphs $G$ and $G'$ in Figure 3.6, we have
$G \in \mathcal{F}_{1,3}$ and $G' \in \mathcal{F}_{1,4}$. The three non-isomorphic spanning trees of $G$ and $G'$ are also shown in Figure 3.6. Observe that $t(G) = 7$ and $t(T_i) = t(G) + i$ for $0 \leq i \leq 2$. The graph $G'$ has order 13 and $t(G') = 13$. Furthermore, $t(T'_i) = t(G') + 3$ for $0 \leq i \leq 2$. Thus there is no connected spanning subgraph $H$ of $G'$ for which $t(H) = 14$ or $t(H) = 15$. Therefore, for the graph $G'$ and each spanning tree $T'_i$ ($0 \leq i \leq 2$) of $G'$, there is no connected spanning subgraph $H$ of $G'$ for which $t(G) < t(H) < t(T'_i)$.

![Figure 3.6: Graphs in \( \mathcal{F}_1 \) and their spanning trees](image)

We now consider the traceable number of a graph $G \in \mathcal{F}_1$ and the traceable numbers of its connected spanning subgraphs in general. First, we make an observation.

**Observation 3.18** Let $G$ be a graph of order $n$ belonging to $\mathcal{F}_1$. If $G \in \mathcal{F}_{1,3}$, then $n \geq 4$ and $t(G) = n - 1$; while if $G \in \mathcal{F}_{1,4}$, then $n \geq 5$ and $t(G) = n$.

We first consider the traceable numbers of the connected spanning subgraphs of graphs in $\mathcal{F}_{1,3}$.

**Proposition 3.19** Let $G$ be a unicyclic graph of order $n = r + s \geq 4$ obtained from a cycle $C$ of order $r$ ($r \geq 3$) and a path $P$ of length $s$ ($s \geq 1$) by identifying one of the $r$ vertices of $C$ and one of the end-vertices of $P$. If $H$ is a proper connected...
spanning subgraph of $G$, then $H$ is a spanning tree and

$$n - 1 \leq t(H) \leq n - 1 + \min \left\{ \left\lfloor \frac{r - 1}{2} \right\rfloor, s \right\}.$$  

**Proof.** Let $H$ be a proper connected spanning subgraph of $G$. Then $H = G - e$ for some edge $e$ belonging to $C$. Hence $H$ is a tree obtained by taking the path $P : u_0, u_1, u_2, \ldots, u_s$ of length $s$ and a path $Q : v_0, v_1, v_2, \ldots, v_{r-1}$ of length $r - 1$ and identifying $u_q$ and $v_i$ for some $i$, where $0 \leq i \leq r - 1$. By symmetry, we may further assume that $0 \leq i \leq \left\lfloor \frac{r - 1}{2} \right\rfloor$. For each $i$ with $0 \leq i \leq r - 1$, we denote the tree obtained in this manner by $T_i$. Note that the length of a longest path in $T_i$ (the diameter of $T_i$) is

$$\text{diam}(T_i) = \ell = (r - 1 - i) + \max\{s, i\}.$$  

Hence

$$t(H) = t(T_i) = 2n - 2 - (r - 1 - i + \max\{s, i\}) = n - 1 + (s + i) - \max\{s, i\} = n - 1 + \min\{s, i\},$$  

implying that

$$n - 1 \leq t(H) \leq n - 1 + \min \left\{ \left\lfloor \frac{r - 1}{2} \right\rfloor, s \right\},$$  

as desired.  

**Theorem 3.20** Let $G$ be a unicyclic graph of order $n = r + s \geq 4$ obtained from a cycle $C$ of order $r$ ($r \geq 3$) and a path $P$ of length $s$ ($s \geq 1$) by identifying one of the $r$ vertices of $C$ with one of the end-vertices of $P$. For a positive integer $k$, there exists a connected spanning subgraph $H$ of $G$ with $t(H) = k$ if and only if

$$n - 1 \leq k \leq n - 1 + \min \left\{ \left\lfloor \frac{r - 1}{2} \right\rfloor, s \right\}.$$  

**Proof.** If $H$ is a connected spanning subgraph of $G$ for which $t(H) = k$, then by Proposition 3.19,

$$n - 1 \leq k \leq n - 1 + \min \left\{ \left\lfloor \frac{r - 1}{2} \right\rfloor, s \right\}.$$  

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For the converse, let \( k = n - 1 + j \), where
\[
0 \leq j \leq \min \left\{ \left\lfloor \frac{r - 1}{2} \right\rfloor, s \right\}.
\]

Construct the spanning tree \( T_j \) of \( G \) in the manner described in the proof of Proposition 3.19. Since \( \min\{s, j\} = j \), it follows that \( t(T_j) = n - 1 + j = k \). 

Next we consider the traceable numbers of the connected spanning subgraphs of a graph in \( \mathcal{F}_{1,4} \).

**Proposition 3.21** Let \( G \) be a unicyclic graph of order \( n = r + s_1 + s_2 \geq 5 \) obtained from a cycle \( C \) of order \( r \) \((r \geq 3)\) and two paths \( P \) and \( P' \) of lengths \( s_1 \) and \( s_2 \), respectively \((s_1 \geq s_2 \geq 1)\), by identifying one of the \( r \) vertices of \( C \) with one of the end-vertices of each of \( P \) and \( P' \). If \( H \) is a proper connected spanning subgraph of \( G \), then \( H \) is a spanning tree and
\[
n - 1 + s_2 \leq t(H) \leq n - 1 + s_2 + \min \left\{ \left\lfloor \frac{r - 1}{2} \right\rfloor, s \right\}.
\]

**Proof.** Let \( H \) be a proper connected spanning subgraph of \( G \). Then \( H = G - e \) for some edge \( e \) belonging to \( C \). Hence \( H \) is a tree obtained by taking the path
\[
P : u_0, u_1, u_2, \ldots, u_{s_1}
\]
of length \( s_1 \), the path
\[
P' : w_0, w_1, w_2, \ldots, w_{s_2}
\]
of length \( s_2 \), and a path
\[
Q : v_0, v_1, v_2, \ldots, v_{r-1}
\]
of length \( r - 1 \) and identifying \( u_{s_1}, w_{s_2}, \) and \( v_i \) for some \( i \), where \( 0 \leq i \leq r - 1 \). By symmetry, we may further assume that \( 0 \leq i \leq \left\lfloor \frac{r-1}{2} \right\rfloor \). We denote the tree obtained in this manner by \( T_i \). Note that the length of a longest path in \( T_i \) is
\[
diam(T_i) = \ell = (r - 1 - i) + \max\{s_1, i\}.
\]
Hence
\[
t(H) = t(T_i) = 2n - 2 - (r - 1 - i + \max\{s_1, i\}) \\
= n - 1 + s_2 + (s_1 - i) - \max\{s_1, i\} = n - 1 + s_2 + \min\{s_1, i\},
\]
implying that
\[
n - 1 + s_2 \leq t(H) \leq n - 1 + s_2 + \min\left\{\left\lfloor \frac{r-1}{2} \right\rfloor, s_1 \right\},
\]
which establishes the result.

**Theorem 3.22**  Let \(G\) be a unicyclic graph of order \(n = r + s_1 + s_2 \geq 5\) obtained from a cycle \(C\) of order \(r\) \((r \geq 3)\) and two paths \(P\) and \(P'\) of lengths \(s_1\) and \(s_2\), respectively \((s_1 \geq s_2 \geq 1)\), by identifying one of the \(r\) vertices of \(C\) with one of the end-vertices of each of \(P\) and \(P'\). For a positive integer \(k\), there exists a connected spanning subgraph \(H\) of \(G\) with \(t(H) = k\) if and only if
\[
n - 1 + s_2 \leq k \leq n - 1 + s_2 + \min\left\{\left\lfloor \frac{r-1}{2} \right\rfloor, s_1 \right\}.\]

**Proof.** If \(H\) is a connected spanning subgraph of \(G\) for which \(t(H) = k\), then by Proposition 3.21,
\[
n - 1 + s_2 + \leq k \leq n - 1 + s_2 + \min\left\{\left\lfloor \frac{r-1}{2} \right\rfloor, s_1 \right\}.\]

For the converse, let \(k = n - 1 + s_2 + j\), where
\[
0 \leq j \leq \min\left\{\left\lfloor \frac{r-1}{2} \right\rfloor, s_1 \right\}.
\]
Construct the spanning tree \(T_j\) of \(G\) in the manner described in the proof of Proposition 3.21. Since \(\min\{s_1, j\} = j\), it follows that \(t(T_j) = n - 1 + s_2 + j = k\).

As a consequence of Propositions 3.20 and 3.22, we have the following:

**Remarks:** Let \(G\) be a graph of order \(n\).
(1) If \( G \in \mathcal{F}_{1,3} \) is a graph of order \( n \geq 4 \) and \( T \) is a spanning tree of \( G \), then \( t(G) = n - 1 \) and for each integer \( k \) with \( t(G) \leq k \leq t(T) \), there exists a connected spanning subgraph \( H \) (which is also a spanning tree of \( G \)) for which \( t(H) = k \).

(2) If \( G \in \mathcal{F}_{1,4} \) is a graph of order \( n \geq 5 \), then \( t(G) = n \). If \( T \) is a spanning tree of \( G \) with the maximum diameter among all spanning trees of \( G \) and \( t(T) \geq t(G) + 2 \), then there is no connected spanning subgraph \( H \) of \( G \) for which \( t(G) < t(H) < t(T) \).

Now we consider a certain type of graphs in \( \mathcal{F}_2 \). If \( G \in \mathcal{F}_2 \), then \( 3 \leq \Delta(G) \leq 4 \) and \( G \) contains exactly two vertices whose degrees are 3 or 4. Consider the unicyclic graphs \( G \) and \( G' \) in Figure 3.7, where the three non-isomorphic spanning trees for each graph are also shown. Observe that \( t(G) = 5 \) and \( t(T_i) = t(G) + i \) for \( 0 \leq i \leq 2 \); while \( t(G') = t(T'_0) = 7 \) and \( t(T'_1) = t(T'_2) = 9 \). Thus there is no connected spanning subgraph \( H \) of \( G' \) for which \( t(H) = 8 \). Therefore, for the graph \( G' \) and each spanning tree \( T'_i \) \((i = 1, 2)\) of \( G' \), there is no connected spanning subgraph \( H \) of \( G' \) for which \( t(G') < t(H) < t(T'_i) \).

![Figure 3.7: Graphs in \( \mathcal{F}_2 \) and their spanning trees](image)

**Proposition 3.23** Let \( G \) be a unicyclic graph of order \( n = r + 2s \geq 5 \), where \( r \geq 3 \) and \( s \geq 1 \), obtained from a path

\[
P : \ x_0, x_1, x_2, \ldots, x_{n-1} = x_{r+2s-1}
\]

of length \( n - 1 = r + 2s - 1 \) by joining \( x_s \) and \( x_{r+s-1} \). If \( H \) is a proper connected spanning subgraph of \( G \), then \( H \) is a spanning tree of \( G \) and
(1) $t(H) = n - 1$; or

(2) $n - 1 + \min\{r - 2, s\} \leq t(H) \leq n - 1 + \min\{r - 2, \lfloor \frac{r}{2} \rfloor + s - 1, 2s\}$.

**Proof.** First, observe that $G$ is traceable and so $t(G) = n - 1$. Let $H$ be a proper connected spanning subgraph of $G$. If $H = G - x_s x_{r + s - 1}$, then $H = P_n$ and therefore $t(H) = n - 1$. Thus suppose that $H = G - e$ for some cycle edge $e \in E(G)$ and $e \neq x_s x_{r + s - 1}$. Observe that $H$ is a spanning tree of $G$ obtained from the two disjoint paths $Q_1 : u_0, u_1, u_2, \ldots, u_s$ and $Q_2 : w_0, w_1, w_2, \ldots, w_s$ of the same length $s$ ($s \geq 1$) and a path $Q : v_0, v_1, v_2, \ldots, v_{r - 1}$ of length $r - 1$ and identifying (i) $u_s$ and $v_i$ and (ii) $w_s$ and $v_{i+1}$, for some $i$ with $0 \leq i \leq r - 2$. By symmetry, we may further assume that

$$0 \leq i \leq \left\lfloor \frac{r - 2}{2} \right\rfloor = \left\lfloor \frac{r}{2} \right\rfloor - 1.$$

Notice that $i \leq r - 2 - i$. We denote the tree obtained in this manner by $T_i$. Then the length of a longest path in $T_i$ is

$$\ell = \max\{s, i\} + \max\{s, r - 2 - i\} + 1.$$

If $i \geq s$, then

$$\ell = i + (r - 2 - i) + 1 = r - 1 \quad \text{and} \quad 2s \leq s + i \leq r - 2.$$

Thus

$$t(T_i) = 2n - 2 - \ell = 2(r + 2s) - 2 - (r - 1) = 4s + 1 = n - 1 + 2s.$$

If $i < s \leq r - 2 - i$, then

$$\ell = s + (r - 2 - i) + 1 = r + s - 1 - i.$$

Also, $s + i \leq r - 2$, $s + i < 2s$, and

$$t(T_i) = 2n - 2 - \ell = 2(r + 2s) - 2 - (r + s - 1 - i) = r + 3s - 1 + i = n - 1 + (s + i).$$

Finally, if $r - 2 - i < s$, then
\[
\ell = 2s + 1 \quad \text{and} \quad r - 2 < s + i < 2s.
\]

Thus
\[
t(T_i) = 2n - 2 - \ell = 2(r + 2s) - 2 - (2s + 1) = 2r + 2s - 3 = n - 1 + (r - 2).
\]

Therefore, if \( H = G - e \) with \( e \neq x_sx_{r+s-1} \), then
\[
t(H) = t(T_i) = n - 1 + \min\{r - 2, s + i, 2s\},
\]
where \( i \) is an integer with \( 0 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor - 1 \). Thus if \( H = G - e \) with \( e \neq x_sx_{r+s-1} \), then
\[
n - 1 + \min\{r - 2, s\} \leq t(H) \leq n - 1 + \min \left\{r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s\right\},
\]
as claimed. \[\square\]

**Proposition 3.24** Let \( G \) be a unicyclic graph of order \( n = r + 2s \geq 5 \), where \( r \geq 3 \) and \( s \geq 1 \), obtained from a path
\[
P : x_0, x_1, x_2, \ldots, x_{n-1} = x_{r+2s-1}
\]
of length \( n - 1 = r + 2s - 1 \) by joining \( x_s \) and \( x_{r+s-1} \). For each positive integer \( k \), there exists a connected spanning subgraph \( H \) of \( G \) with \( t(H) = k \) if and only if

1. \( k = n - 1 \); or
2. \( n - 1 + \min\{r - 2, s\} \leq k \leq n - 1 + \min \left\{r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s\right\} \).

**Proof.** If \( H \) is a connected spanning subgraph of \( G \) for which \( t(H) = k \), then by Proposition 3.23, we have \( k = n - 1 \) or
\[
n - 1 + \min\{r - 2, s\} \leq k \leq n - 1 + \min \left\{r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s\right\}.
\]
For the converse, let \( k \) be an integer such that \( k = n - 1 \) or
\[
n - 1 + \min\{r - 2, s\} \leq k \leq n - 1 + \min \left\{r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s\right\}.
\]
If \( k = n - 1 \), then let \( H = G - x_sx_{r+s-1} \). Otherwise, \( k = n - 1 + j \) for some \( j \), where
\[
\min\{r - 2, s\} \leq j \leq \min \left\{r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s\right\}.
\]
Note that since \( r \geq 3 \) and \( s \geq 1 \), it follows that \( \min \{ r - 2, s \} \geq 1 \). We consider three cases.

Case 1. \( r = 3 \). Then \( n = 2s + 3 \) and

\[
\min \{ r - 2, s \} = \min \left\{ r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s \right\} = 1.
\]

Hence \( k = n - 1 + 1 = n \). Let \( C : v_1, v_2, v_3, v_1 \) be the cycle in \( G \) such that \( \deg(v_1) = \deg(v_2) = 3 \) and \( \deg(v_3) = 2 \). Then the length of a longest path in \( H = G - v_1v_3 \) is \( \ell = 2s + 1 = n - 2 \) and so

\[
t(H) = 2n - 2 - \ell = 2n - 2 - (n - 2) = n = k.
\]

Case 2. \( r \geq 4 \) and \( s = 1 \). Then

\[
\min \{ r - 1, s \} = 1 \quad \text{and} \quad \min \left\{ r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s \right\} = 2,
\]

hence \( k \in \{ n, n + 1 \} \). For the spanning trees \( T_0 \) and \( T_1 \) of \( G \) constructed in the manner described in the proof of Proposition 3.23, we have

\[
t(T_i) = n - 1 + \min \{ r - 2, s + i, 2s \} = \begin{cases} 
   n & \text{if } i = 0 \\
   n + 1 & \text{if } i = 1.
\end{cases}
\]

Case 3. \( r \geq 4 \) and \( s \geq 2 \). Then \( \min \{ r - 2, s \} \geq 2 \). Let \( k = n - 1 + \min \{ r - 2, s \} + i \), where \( i \) is an integer with

\[
0 \leq i \leq \min \left\{ r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s \right\} - \min \{ r - 2, s \}.
\]

If \( r - 2 \geq s \), then \( k = n - 1 + (s + i) \) and

\[
s + i = \min \{ r - 2, s \} + i \leq \min \{ r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s \}.
\]

For the spanning tree \( T_i \) constructed in the manner described in the proof of Proposition 3.23, we have

\[
t(T_i) = n - 1 + \min \{ r - 2, s + i, 2s \} = n - 1 + (s + i) = k.
\]

If \( r - 2 < s \), then \( k \geq n - 1 + (r - 2) \). Since

\[
n - 1 + (r - 2) \leq k \leq n - 1 + \min \left\{ r - 2, \left\lfloor \frac{r}{2} \right\rfloor + s - 1, 2s \right\} \leq n - 1 + (r - 2),
\]

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it follows that $k = n - 1 + (r - 2)$. For each $i$ with $0 \leq i \leq \left\lceil \frac{r}{2} \right\rceil - 1$, observe that

$$t(T_i) = n - 1 + \min\{r - 2, s + i, 2s\} = n - 1 + (r - 2) = k,$$

providing the desired result. \hfill \blacksquare

**Proposition 3.24** implies the following.

**Remarks:** Let $r \geq 4$ and $s \geq 2$ be integers. If $G$ is a unicyclic graph of order $n = r + 2s$ obtained from a path

$$P : x_0, x_1, x_2, \ldots, x_{n-1} = x_{r+2s-1}$$

by joining $x_s$ and $x_{r+s-1}$, then $t(G) = n - 1$ and for every spanning tree $T$ of $G$ with

$$t(T) \geq n - 1 + \min\{r - 2, s\} \geq n + 1 = t(G) + 2,$$

there exists no connected spanning subgraph $H$ of $G$ for which

$$t(G) < t(H) < t(T).$$

### 3.4 Traceable Numbers of Vertices

Let $G$ be a connected graph of order $n$. For a vertex $v$ of $G$, the **traceable number** $t(v)$ of $v$ is defined by

$$t(v) = \min\{d(s)\},$$

where the minimum is taken over all linear orderings $s$ of vertices of $G$ whose first term is $v$. Thus

$$t(v) \geq n - 1$$

for every vertex $v$ of $G$. Furthermore, $t(v) = n - 1$ if and only if $G$ contains a Hamiltonian path with initial vertex $v$. Observe that

$$t(G) = \min\{t(v) : v \in V(G)\}.$$

Using an argument similar to that used in the proof of Proposition 3.1, we have the following.
Proposition 3.25 Let $G$ be a nontrivial connected graph and $v \in V(G)$. Then $t(v)$ is the minimum length of a spanning walk in $G$ whose initial vertex is $v$.

We present a result concerning the traceable number of adjacent vertices in a connected graph.

Proposition 3.26 Let $G$ be a connected graph and $u, v$ a pair of adjacent vertices of $G$. Then

$$|t(u) - t(v)| \leq 1.$$ 

Proof. Let

$$s : v = v_1, v_2, \ldots, v_n$$

be a linear ordering of vertices of $G$ such that $d(s) = t(v)$. Thus $u = v_i$ for some integer $i$ with $2 \leq i \leq n$. We consider two cases.

Case 1. $u = v_i$, where $2 \leq i \leq n - 1$. Let

$$s' : u = v_i, v_{i-1}, \ldots, v_2, v_1 = v, v_{i+1}, v_{i+2}, \ldots, v_n.$$ 

Then

$$t(u) \leq d(s') = d(s) - d(u, v_{i+1}) + d(v, u_{i+1})$$
$$\leq d(s) - d(u, v_{i+1}) + d(v, u) + d(u, v_{i+1})$$
$$= d(s) + 1 = t(v) + 1.$$ 

Thus $t(u) - t(v) \leq 1$.

Case 2. $u = v_n$. Consider the sequence

$$s'' : u = v_n, v_{n-1}, \ldots, v_2, v_1 = v.$$ 

Then $t(u) \leq d(s'') = d(s) = t(v)$ and so $t(u) - t(v) \leq 0$.

In either case, $t(u) - t(v) \leq 1$. Applying a similar argument to that given above, we have $t(v) - t(u) \leq 1$ as well and so $|t(u) - t(v)| \leq 1$.

For a connected graph $G$, the maximum vertex traceable number of $G$ is defined as
\[ t^*(G) = \max\{t(v) : v \in V(G)\}. \]

Obviously, \( t(G) \leq t^*(G) \) for every connected graph \( G \). For example, consider the graph \( G \) in Figure 3.8. Each vertex of \( G \) is labeled with its traceable number. Thus \( t^*(G) = \max\{4, 5\} = 5 \).

\[ G: \begin{array}{cccc}
4 & 5 & 5 & 4 \\
\end{array} \]

Figure 3.8: A graph \( G \) with \( t^*(G) = 5 \)

The maximum vertex traceable number \( t^*(G) \) of a connected graph \( G \) will be investigated further in Chapter 5. The following is a consequence of Proposition 3.26.

**Corollary 3.27** Let \( G \) be a connected graph and \( k \) an integer such that \( t(G) < k \leq t^*(G) \). Then there exists a vertex \( w \) of \( G \) such that \( t(w) = k \).

**Proof.** The statement is obvious if \( k = t(G) \) or \( k = t^*(G) \). Hence we may assume that \( t(G) < k < t^*(G) \). Let \( u \) and \( v \) be vertices such that \( t(u) = t(G) \) and \( t(v) = t^*(G) \). Since \( G \) is connected, \( G \) contains a \( u - v \) path

\[ P: u = u_1, u_2, \ldots, u_s = v. \]

By Proposition 3.26, \( |t(u_i) - t(u_{i+1})| \leq 1 \) for all \( i \) with \( 1 \leq i \leq s - 1 \). Let \( j \) be the largest integer with \( 1 \leq j < s \) such that \( t(u_j) \leq k \). Then \( t(u_j) = k \); for otherwise, \( t(u_j) \leq k - 1 \) and so \( t(u_{j+1}) \leq 1 + (k - 1) = k \), producing a contradiction.

For a vertex \( v \) in a connected graph \( G \), recall that the eccentricity \( e(v) \) of \( v \) is the largest distance between \( v \) and a vertex of \( G \).

**Theorem 3.28** If \( T \) is a nontrivial tree of order \( n \) and \( v \) is a vertex of \( T \), then

\[ t(v) = 2n - 2 - e(v). \]

**Proof.** First, we show that \( t(v) \geq 2n - 2 - e(v) \). Let

\[ s: v = u_1, u_2, \ldots, u_n \]

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be a linear ordering of vertices of $T$ such that $d(s) = t(v)$ and let

$$s' : v = v_1, v_2, \ldots, v_n, v_1$$

be the cyclic ordering of vertices of $T$ obtained by adding $v_1 = v$ at the end of $s$. Then

$$2n - 2 = h(T) \leq d(s') = d(s) + d(v_n, v_1) \leq t(v) + e(v)$$

and so $t(v) \geq 2n - 2 - e(v)$.

Next, we show that $t(v) \leq 2n - 2 - e(v)$ for each vertex $v$ in a nontrivial tree of order $n$. We proceed by induction on $n$. This is certainly true for a tree of order 2. Assume, for every tree $T'$ of order $n - 1$, where $n - 1 \geq 2$, and every vertex $u$ of $T'$, that $t(u) \leq 2n - 4 - e(u)$. We show that if $T$ is a nontrivial tree of order $n$ and $v$ is a vertex of $T$, then

$$t(v) \leq 2n - 2 - e(v).$$

This is certainly the case if $T$ is the path $P_n$ and $v$ is an end-vertex of $P_n$. Hence we may assume that this is not the case. Let $P$ be a longest path in $T$ with initial vertex $v$, say $P$ is a $v - w$ path. Then $d(v, w) = e(v)$. There exists an end-vertex $x$ of $T$ such that $x$ does not lie on $P$. Let $y$ be the vertex of $T$ that is adjacent to $x$. Thus $T - x$ is a tree of order $n - 1$ such that $v \in V(T - x)$ and $e_{T-x}(v) = e_T(v)$. By the induction hypothesis,

$$t_{T-x}(v) \leq 2n - 4 - e_{T-x}(v) = 2n - 4 - e_T(v).$$

Let $s_1 : v = u_1, u_2, \ldots, u_{n-1}$ be a linear ordering of vertices of $T - x$ such that $d(s_1) = t_{T-x}(v)$. Then $y = u_i$ for some $i$ with $2 \leq i \leq n - 1$. Let $z$ be the vertex of $T - x$ that immediately follows or immediately precedes $y$ in $s_1$, say $z$ immediately follows $y$ in $s_1$. Thus $z = u_{i+1}$. Let $s$ be the linear ordering of vertices of $T$ obtained by inserting $x$ between $y$ and $z$. Then

$$d(s) = d(s_1) - d(y, z) + d(y, x) + d(x, z)$$

$$\leq d(s_1) - d(y, z) + d(y, x) + [d(x, y) + d(y, z)]$$

$$= d(s_1) + 2 = t_{T-x}(v) + 2$$

$$\leq 2n - 4 - e_T(v) + 2 = 2n - 2 - e_T(v).$$
Therefore, $t_T(v) \leq d(s) \leq 2n - 2 - e_T(v)$ and so $t(v) = 2n - 2 - e(v)$.

Note that since $|e(u) - e(v)| \leq 1$ for every two adjacent vertices $u$ and $v$, Theorem 3.28 provides an alternative proof of Proposition 3.26.

We have seen that if $T$ is a tree of order $n$, then the Hamiltonian number of $T$ is

$$h(T) = 2n - 2. \tag{3.3}$$

By Theorem 3.28,

$$t(v) = h(T) - e(v)$$

for every tree $T$ and every vertex $v$ of $T$. Since

$$t(T) = \min \{t(v) : v \in V(G)\},$$

it follows that

$$t(T) = h(T) - \max \{e(v) : v \in V(T)\} = 2n - 2 - \text{diam}(T),$$

which provides us with an alternative proof of Theorem 3.12.

Theorem 3.28 is not true in general for connected graphs that are not trees. Consider the graphs $G$ and $H$ in Figure 3.9. Each vertex of $G$ and $H$ is labeled with its traceable number. The Hamiltonian number of $G$ is $h(G) = 7$. Since $e(u) = e(y) = 3$ and $e(v) = e(w) = e(x) = 2$, it follows that $t(z) = h(G) - e(z)$ for every vertex $z$ of $G$. On the other hand, the Hamiltonian number of $H$ is $h(H) = 6$. While $t(z) = h(H) - e(z)$ for $z \in \{w, x\}$, this is not true otherwise.

![Figure 3.9: Graphs G and H](image)

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3.5 Realization Results on Two Parameters

We have seen in Lemma 3.7 that for every nontrivial connected graph $G$,

$$1 \leq h(G) - t(G) \leq \text{diam}(G).$$

Furthermore, by Proposition 3.8, Hamiltonian graphs are the only connected graphs $G$ for which $h(G) - t(G) = 1$. By (3.3) and Theorem 3.12, if $T$ is a tree, then $h(T) - t(T) = \text{diam}(T)$. However, trees are not the only connected graphs with this property. For example, if $G = K_{n_1, n_2, \ldots, n_k}$ is a complete $k$-partite graph, where $k \geq 2$, $n_1 \leq n_2 \leq \cdots \leq n_k$, and $n_1 + n_2 + \cdots + n_{k-1} < n_k$, then $h(G) - t(G) = (2n_k) - (2n_k - 2) = 2 = \text{diam}(G)$. Next, we show that for each pair $k, d$ of integers with $1 \leq k \leq d$, there exists a connected graph $G$ with $\text{diam}(G) = d$ such that $h(G) - t(G) = k$. In order to do this, we first recall a useful fact about the Hamiltonian number of a graph and the Hamiltonian numbers of the blocks of the graph; that is, if $G$ is a connected graph having blocks $B_1, B_2, \ldots, B_\ell$, then

$$h(G) = \sum_{i=1}^{\ell} h(B_i). \quad (3.4)$$

**Proposition 3.29** For each pair $k, d$ of integers with $1 \leq k \leq d$, there exists a connected graph $G$ with diameter $d$ such that $h(G) - t(G) = k$.

**Proof.** If $k = d$, let $G$ be a tree with $\text{diam}(G) = d$. It then follows by (3.3) and Theorem 3.12 that

$$h(G) - t(G) = (2n - 2) - (2n - 2 - d) = d.$$ 

Thus, we may assume that $k < d$. For $k = 1$, the cycle $C_{2d}$ of order $2d$ has the desired property. For $k \geq 2$, let $G$ be the graph obtained from a cycle

$$C_{2(d-k+1)} : u_1, u_2, \ldots, u_{2(d-k+1)}, u_1$$

and a path

$$P_{k-1} : v_1, v_2, \ldots, v_{k-1}$$
by joining \( u_1 \) and \( v_{k-1} \). Then the order of \( G \) is \( n = 2d - k + 1 \) and its diameter is \( \text{diam}(G) = d \). By (3.4),

\[
h(G) = h(C_{2(d-k+1)}) + (k-1)h(P_2) = 2(d - k + 1) + 2(k-1) = 2d.
\]

Since \( G \) is traceable, \( t(G) = n - 1 = 2d - k \). Therefore, \( h(G) - t(G) = k \).

Since \( h(G) \leq t(G) + \text{diam}(G) \) for every nontrivial connected graph \( G \) and, trivially, \( t(G) \geq \text{diam}(G) \), it follows that

\[
t(G) < h(G) < 2t(G).
\]

Thus if \( G \) is a connected graph with \( t(G) = a \) and \( h(G) = b \), then \( a < b \leq 2a \). Next, we show that every pair \( a, b \) of positive integers with \( a < b \leq 2a \) is realizable as the traceable number and Hamiltonian number, respectively, of some connected graph.

**Proposition 3.30** For every pair \( a, b \) of positive integers with \( a < b \leq 2a \), there is a connected graph \( G \) with \( t(G) = a \) and \( h(G) = b \).

**Proof.** If \( b = 2a \), then \( G = P_{2a+1} \) has the desired properties. Hence we may assume that \( a < b < 2a \). Let \( k = b - a \). Thus \( 1 < k < a \). Let \( G \) be the graph obtained from a path \( P : u_1, u_2, \ldots, u_a, u_{a+1} \) by joining \( u_{a+1} \) and \( u_k \). By (3.4),

\[
h(G) = h(C_{a-k+2}) + (k-1)h(P_2) = (a - k + 2) + 2(k-1) = b.
\]

Since \( G \) is traceable, \( t(G) = (a + 1) - 1 = a \).

We have seen that if \( G \) is a connected graph of order \( n \geq 2 \), then \( n \leq h(G) \leq 2n - 2 \). Thus by Lemma 3.7 and (3.2), if \( G \) is a connected graph of order \( n \geq 3 \) with \( t(G) = a \) and \( h(G) = b \), then

\[
1 \leq n - 1 \leq a < b \leq 2n - 2.
\]

Next we determine all triples \( (a, b, n) \) of positive integers satisfying (3.5) that can be realized as the traceable number, Hamiltonian number, and order, respectively, of some connected graph.

**Theorem 3.31** For each triple \( (a, b, n) \) of positive integers with \( 1 \leq n - 1 \leq a < b \leq 2n - 2 \) and \( n \geq 3 \), there is a connected graph \( G \) of order \( n \) such that \( t(G) = a \) and \( h(G) = b \) if and only if (1) \( b = a + 1 = n \) or (2) \( b \geq a + 2 \).
Proof. Let $G$ be a connected graph of order $n$ with $t(G) = a$ and $h(G) = b$. If $b = a + 1$, then $h(G) - t(G) = 1$. By Proposition 3.8, $G$ is Hamiltonian. Thus $t(G) = n - 1$ and $h(G) = n$ and so $b = a + 1 = n$. If $b \neq a + 1$, then $b \geq a + 2$ by Lemma 3.7.

For the converse, let $(a, b, n)$ be a triple of positive integers with $1 \leq n - 1 \leq a < b \leq 2n - 2$ such that $b = a + 1 = n$ or $b \geq a + 2$. If $b = a + 1 = n$, then every Hamiltonian graph of order $n$ has the desired property. Thus, we may assume that $b \geq a + 2$. Observe that $b - a - 1 \geq 1$ and $2n - b \geq 2$. We consider two cases.

Case 1. $a = n - 1$. Let $G_1$ be the graph obtained from a path $P_{b-a-1}$: $u_1, u_2, \ldots, u_{b-a-1}$ of order $b - a - 1$ and a complete graph $K_{2n-b}$ with $V(K_{2n-b}) = \{v_1, v_2, \ldots, v_{2n-b}\}$ by joining $u_{b-a-1}$ to $v_1$. Then the order of $G_1$ is $(b - a - 1) + (2n - b) = n$. By (3.4),

$$h(G_1) = (b - a - 1)h(P_2) + h(K_{2n-b}) = 2(b - a - 1) + (2n - b) = b.$$ 

Since $G_1$ is traceable, $t(G_1) = n - 1 = a$.

Case 2. $a \geq n$. Let $G_2$ be the graph obtained from the graph $G_1$ in Case 1 by adding $a - n + 1$ new vertices $w_1, w_2, \ldots, w_{a-n+1}$ and joining each $w_i$ to $v_1$ for $1 \leq i \leq a - n + 1$. Then the order of $G_2$ is $(b - a - 1) + (2n - b) + (a - n + 1) = n$ and $diam(G_2) = b - a$. By (3.4),

$$h(G_2) = (b - a - 1)h(P_2) + h(K_{2n-b}) + (a - n + 1)h(P_2) = 2(b - a - 1) + (2n - b) + 2(a - n + 1) = b.$$ 

It remains to show that $t(G_2) = a$. By Lemma 3.7,

$$t(G_2) \geq h(G_2) - diam(G_2) = b - (b - a) = a.$$ 

Since the length of a longest path in $G_2$ is $\ell = (b - a - 1) + (2n - b - 1) = 2n - a - 2$, it follows by Theorem 3.2 that $t(G_2) \leq 2n - 2 - \ell = a$. Thus $t(G_2) = a$. $\blacksquare$

3.6 Graphs with Prescribed Parameters

We have seen in Corollary 3.4 that if $G$ is a nontrivial connected graph of order $n$ such that $\ell$ is the length of a longest path in $G$ and $p$ is the maximum size of a
spanning linear forest in $G$, then

$$2n - 2 - \ell \leq t(G) \leq 2n - 2 - \ell.$$ 

Furthermore, if $T$ is a nontrivial tree of order $n$, then the length of a longest path in $T$ is the diameter of $T$. Hence by Theorem 3.12, if $T$ is a nontrivial tree of order $n$, then

$$t(T) = 2n - 2 - \text{diam}(T).$$

We now investigate Problem 3.5, that is, we investigate those 4-tuples $(n, \ell, p, k)$ of positive integers with

$$n \geq 2 \text{ and } \ell \leq k \leq p \leq n - 1$$

for which there exists a connected graph $G$ of order $n$ whose longest path has length $\ell$, the maximum size of a spanning linear forest in $G$ is $p$, and $t(G) = 2n - 2 - k$.

For positive integers $n$, $\ell$, $p$, and $k$, where $n \geq 2$ and $\ell \leq k \leq p \leq n - 1$, let $G(n, \ell, p, k)$ be the family of graphs such that $G \in G(n, \ell, p, k)$ if and only if $G$ is a connected graph of order $n$ such that

1. the length of a longest path in $G$ is $\ell$;
2. the maximum size of a spanning linear forest in $G$ is $p$; and
3. $t(G) = 2n - 2 - k$.

Since $\ell = 1$ if and only if $G = K_2$, we may assume that $\ell \geq 2$. We begin with the case where $\ell = p$. In this case then $\ell = k = p$.

**Proposition 3.32** Let $n$ and $\ell$ be two integers such that $n \geq 3$ and $2 \leq \ell \leq n - 1$. Then there exists a graph $G \in G(n, \ell, \ell, \ell)$.

**Proof.** Let $G$ be the graph obtained from a path $P : v_0, v_1, \ldots, v_\ell$ by adding $n - \ell - 1$ new vertices $u_1, u_2, \ldots, u_{n-\ell-1}$ and joining each $u_i$ ($1 \leq i \leq n - \ell - 1$) to the vertex $v_1$. Since $G$ is a tree with diameter $\ell$, the length of a longest path in $G$ is $\ell$. The spanning linear forest consisting of the path $P$ and the $(n - \ell - 1)$ copies of $K_1$ has the maximum size $\ell$. Furthermore, $t(G) = 2n - 2 - \ell$ by Theorem 3.12. 

Therefore from now on, we will only consider the case where $\ell < p$. Thus $\ell \leq k \leq p$. We first make two observations.
Observation 3.33 Let $G$ be a connected graph of order $n \geq 3$ such that the length of a longest path in $G$ is $\ell$. Then $\ell = 2$ if and only if $G \in \{K_3, K_{1,n-1}\}$.

Observation 3.34 Let $G$ be a connected graph of order $n \geq 3$ such that the length of a longest path in $G$ is $\ell$ and the maximum size of a spanning linear forest in $G$ is $p$. Then $\ell = 2$ if and only if $p = 2$.

By Observation 3.34, $\mathcal{G}(n, 2, p, k) = \emptyset$ if $p \geq 3$. Furthermore, $\mathcal{G}(n, 2, 2, k) \neq \emptyset$ if and only if $n \geq 3$ and $k = 2$. In fact, more can be said.

Proposition 3.35 Let $G$ be a graph of order $n \geq 5$ such that the length of a longest path in $G$ is $\ell$ and the maximum size of a spanning linear forest in $G$ is $p$. If $\ell < p$, then $\ell \geq 3$ and $p \leq n-2$.

Proof. Let $F$ be a spanning linear forest of size $p$ in $G$. Since $p > \ell$, it follows that $F$ contains at least two nontrivial components. Let $G_1, G_2, \ldots, G_s$, with $s \geq 2$, be $s$ components of $F$ that are nontrivial and suppose that each component $G_i$ contains $n_i$ vertices. Since $n \geq \sum_{i=1}^{s} n_i$ and each $G_i$ is a path, it follows that

\[
p = \sum_{i=1}^{s} (n_i - 1) = \sum_{i=1}^{s} n_i - s \leq n - s \leq n - 2,
\]

as desired. \[\square\]

By Observation 3.34 and Proposition 3.35, if $G$ is a connected graph of order $n$ for which the length of a longest path in $G$ is $\ell$ and the maximum size of a spanning linear forest in $G$ is $p > \ell$, then

\[3 \leq \ell < p \leq n - 2.
\]

Furthermore, $\ell \leq k \leq p$. We first consider the 4-tuples $(n, 3, p, k)$.

Proposition 3.36 Suppose that $G$ is a connected graph of order $n$ such that the length of a longest path in $G$ is 3 and the maximum size of a spanning linear forest in $G$ is $p$. Then
(1) $p = 3$ or $p = 4$; and

(2) $t(G) = 2n - 5$ (that is, $k = 3$).

**Proof.** By Observation 3.34, it follows that $p \geq 3$. We consider two cases.

**Case 1.** $G$ is a tree. Then $\ell = \text{diam}(G)$, implying that $G$ is a double star. Hence $p \leq 4$. Let $P : x_1, v_1, v_2, x_2$ be a path of length 3 in $G$. If at least one of $\deg v_1$ and $\deg v_2$ is 2, then $p = 3$. Otherwise, there are two vertices $y_1$ and $y_2$ such that $y_i \neq x_i$ and $y_i v_i \in E(G)$ for $i = 1, 2$. Then every spanning linear forest containing the two paths $Q_1$ and $Q_2$, where $Q_i : x_i, u_i, y_i$ for $i = 1, 2$, has size 4. Hence $p = 3$ or $p = 4$. Moreover, since $G$ is a tree, it follows by Theorem 3.12 that $t(G) = 2n - 2 - \text{diam}(G) = 2n - 5$.

**Case 2.** $G$ is not a tree. Since $\ell = 3$, it follows that $G$ contains no cycles of length 5 or greater. Now we consider two subcases.

**Subcase 2.1.** $G$ contains a 4-cycle. Then the order of $G$ must be 4, for otherwise a path of length at least 4 is produced. Hence $G \in \{C_4, K_4 - e, K_4\}$ and so $p = 3$. Since $G$ is traceable, $t(G) = n - 1 = 4 - 1 = 2 \cdot 4 - 5 = 2n - 5$.

**Subcase 2.2.** $G$ does not contain a 4-cycle. Then every cycle contained in $G$ is a 3-cycle. First we show that $G$ contains exactly one 3-cycle. Assume, to the contrary, that there are two distinct 3-cycles $C$ and $C'$ in $G$. If $|V(C) \cap V(C')| = 2$, then $G$ contains a 4-cycle; while if $|V(C) \cap V(C')| = 1$, then $G$ contains a path of length at least 4. In either case, a contradiction is produced. Hence $C$ and $C'$ are disjoint. However, since $G$ is connected, there is a path from a vertex in $C$ to a vertex in $C'$, producing a path of length at least 5, a contradiction. Hence $G$ contains a unique 3-cycle $C : u, v, w, u$. Since $\ell = 3$, at least one of the three vertices $u, v,$ and $w$ has degree at least 3, say $\deg(u) \geq 3$. Let $U = N(u) - \{v, w\}$. Note that $U \neq \emptyset$ and that every vertex in $U$ must be an end-vertex to avoid a path of length greater than 3. Moreover, $\deg(v) = \deg(w) = 2$, for otherwise a path of length at least 4 is produced. Hence $G$ of order $n$ described in this subcase is obtained from $C$ by joining $n - 3$ end-vertices $x_1, x_2, \ldots, x_{n-3}$ to $v$ and therefore $p = 3$. Moreover, $t(G) \geq 2n - 2 - p = 2n - 5$ by Corollary 3.4. Let $s$ be the linear sequence of vertices of $G$ given by

$$s : x_1, u, v, w, x_2, x_3, \ldots, x_{n-3}$$
and observe that

\[2n - 5 \leq t(G) \leq d(s) = 1 + 3 + 2(n - 5) + 1 = 2n - 5.\]

Hence \(t(G) = 2n - 5.\)

Proposition 3.36 implies that \(G(n, 3, p, k) = \emptyset\) for \(p \geq 5\) and \(k \geq 4.\) Furthermore, \(G(n, 3, p, k) \neq \emptyset\) if and only if \((p, k) \in \{(3, 3), (3, 4)\}.\) So only the case where \(4 \leq \ell < p\) remains. Since \(5 \leq p \leq n - 2,\) it follows that \(n \geq 7\) and \(\ell \leq n - 3.\) We begin with those \(4\)-tuples \((n, \ell, p, p),\) where \(p = \ell + b\) for some positive integer \(b.\)

**Proposition 3.37** Let \(n\) and \(\ell\) be two integers such that \(\ell \geq 4\) and \(n \geq \ell + 3.\) Then there exists a graph \(G \in G(n, \ell, \ell + 1, \ell + 1).\)

**Proof.** Let \(n = (\ell + 3) + a,\) where \(a \geq 0.\) We first construct a graph \(H.\) Let \(C : v_1, v_2, v_3, v_1\) be a 3-cycle. Consider two paths

\[Q_1 : u_0, u_1, u_2, \ldots, u_{\lceil \frac{\ell}{2} \rceil}\] and \(Q_2 : w_0, w_1, w_2, \ldots, w_{\lfloor \frac{\ell}{2} \rfloor} \]

of lengths \(\lceil \frac{\ell}{2} \rceil\) and \(\lfloor \frac{\ell}{2} \rfloor\), respectively. The graph \(H\) is obtained from \(C, Q_1,\) and \(Q_2\) by identifying the vertices \(v_1, w_0,\) and \(w_0.\) Then the graph \(G\) is obtained from \(H\) by joining \(a\) new vertices \(x_1, x_2, \ldots, x_a\) to \(v_1.\) Observe that the order of \(G\) is \(n = \ell + 3 + a\) and the path \(P\) given by

\[P : u_{\lceil \frac{\ell}{2} \rceil}^1, u_{\lceil \frac{\ell}{2} \rceil}^1-1, \ldots, u_1, u_0 = w_0, w_1, w_2, \ldots, w_{\lfloor \frac{\ell}{2} \rfloor}^1 \]

is a longest path in \(G\) of length \(\ell.\) Furthermore, the spanning linear forest \(F\) consisting of \(P,\) the edge \(v_2v_3,\) and the \(a\) isolated vertices has the maximum size \(\ell + 1.\)

It remains to show that \(t(G) = 2n - 2 - (\ell + 1).\) By Corollary 3.4, it follows that \(t(G) \geq 2n - 2 - (\ell + 1).\) Consider the linear sequence \(s\) given by

\[s : u_{\lceil \frac{\ell}{2} \rceil}^1, u_{\lceil \frac{\ell}{2} \rceil}^1-1, \ldots, u_1, u_0, v_2, v_3, x_1, x_2, \ldots, x_a, w_1, w_2, \ldots, w_{\lfloor \frac{\ell}{2} \rfloor}^1. \]

Observe that

\[t(G) \leq d(s) = \lceil \frac{\ell}{2} \rceil + 3 + 2a + \lfloor \frac{\ell}{2} \rfloor = \ell + 3 + 2a\]

\[= 2(\ell + 3 + a) - 2 - (\ell + 1)\]

\[= 2n - 2 - (\ell + 1).\]
Hence \( t(G) = 2n - 2 - (\ell + 1) \).

Proposition 3.37 can be extended. In order to do this, we first prove the following proposition.

**Proposition 3.38** Let \( b \) be a positive integer. If \( G \) is a connected graph for which

1. the length of a longest path in \( G \) is \( \ell \geq 4 \);
2. the order of \( G \) is \( n \geq \ell + b + 2 \); and
3. \( t(G) = n \) (that is, \( k = n - 2 \)),

then \( \ell \geq 2b + 2 \).

**Proof.** Since \( t(G) = n \), there exists a spanning walk \( W \) given by

\[
W : v_1, v_2, v_3, \ldots, v_n, v_{n+1}
\]

of length \( n \). Since the order of \( G \) is \( n \), it follows that \( W \) contains exactly two vertices \( v_{i_1} \) and \( v_{i_2} \), where \( 1 \leq i_1 < i_1 + 2 \leq i_2 \leq n + 1 \), for which \( v_{i_1} = v_{i_2} \). Then \( G \) contains three paths \( Q_1, Q_2, \) and \( Q_3 \), given by

\[
Q_1 : v_1, v_2, \ldots, v_{i_1} = v_{i_2}, v_{i_2+1}, v_{i_2+2}, \ldots, v_{n+1} \\
Q_2 : v_1, v_2, \ldots, v_{i_1}, \ldots, v_{i_2-1} \\
Q_3 : v_{i_1+1}, v_{i_1+2}, \ldots, v_{i_2}, \ldots, v_{n+1}.
\]

Observe that \( Q_1, Q_2, \) and \( Q_3 \) have lengths \( n + i_1 - 1, i_2 - 2, \) and \( n - i_1, \) respectively. Since every path in \( G \) has length at most \( \ell \), we have

\[
n + i_1 - 1 \leq \ell, \quad i_2 - 2 \leq \ell, \quad \text{and} \quad n - i_1 \leq \ell. \quad (3.6)
\]

From the three inequalities in (3.6), we obtain

\[
i_1 + (n - \ell) \leq i_2 \leq \ell + 2 \quad \text{and} \quad n - \ell \leq i_1
\]

and so

\[
2(n - \ell) \leq \ell + 2. \quad (3.7)
\]
From the inequality (3.7) with $n \geq \ell + b + 2$, we obtain

$$\ell + 2 \geq 2n - 2\ell \geq 2(\ell + b + 2) - 2\ell = 2b + 4,$$

implying that $\ell \geq 2b + 2$, as claimed.

By Proposition 3.38, for each integer $b \geq 2$, if $4 \leq \ell < 2b + 2$ and $n \geq \ell + b + 2$, then $G(n, \ell, p, n - 2) = \emptyset$ for all $p > \ell$.

**Proposition 3.39** Let $b$ be a positive integer. If $n$ and $\ell$ are two integers such that $n \geq \ell + b + 2$ and $\ell \geq 2b + 2$, then there exists a graph $G \in G(n, \ell, \ell + b, \ell + b)$.

**Proof.** Let $n = (\ell + b + 2) + a$, where $a \geq 0$. We first construct a graph $H$. Let

$$C : v_1, v_2, \ldots, v_b, v_1$$

be a cycle of length $b + 2$. Consider two paths

$$Q_1 : u_0, u_1, u_2, \ldots, u_{\left\lceil \frac{\ell}{2} \right\rceil} \quad \text{and} \quad Q_2 : w_0, w_1, w_2, \ldots, w_{\left\lceil \frac{\ell}{2} \right\rceil}$$

of lengths $\left\lceil \frac{\ell}{2} \right\rceil$ and $\left\lceil \frac{\ell}{2} \right\rceil$, respectively. The graph $H$ is obtained from $C$, $Q_1$, and $Q_2$ by identifying the vertices $v_1$, $u_0$, and $w_0$. Then the graph $G$ is obtained from $H$ by joining a new vertices $x_1, x_2, \ldots, x_a$ to $v_1$. Since $\ell \geq 2b + 2$, the path $P$ given by

$$P : u_{\left\lceil \frac{\ell}{2} \right\rceil}, u_{\left\lceil \frac{\ell}{2} \right\rceil - 1}, \ldots, u_1, u_0 = w_0, w_1, w_2, \ldots, w_{\left\lceil \frac{\ell}{2} \right\rceil}$$

is a longest path in $G$ of length $\ell$. Furthermore, the spanning linear forest

$$F = P \cup (C - v_1) \cup aK_1$$

has the maximum size $\ell + b$. It remains to show that $t(G) = 2n - 2 - (\ell + b)$. By Corollary 3.4, it follows that $t(G) \geq 2n - 2 - (\ell + b)$. Consider the linear sequence $s$ of given by

$$s : u_{\left\lceil \frac{\ell}{2} \right\rceil}, u_{\left\lceil \frac{\ell}{2} \right\rceil - 1}, \ldots, u_1, v_1, v_2, v_3, \ldots, v_{b+2}, x_1, x_2, \ldots, x_a, w_1, w_2, \ldots, w_{\left\lceil \frac{\ell}{2} \right\rceil}.$$  

Observe that

$$t(G) \leq d(s) = \left\lceil \frac{\ell}{2} \right\rceil + (b + 2) + 2a + \left\lfloor \frac{\ell}{2} \right\rfloor = \ell + b + 2 + 2a$$

$$= 2(\ell + b + 2 + a) - 2 - (\ell + b)$$

$$= 2n - 2 - (\ell + b).$$

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Hence $t(G) = 2n - 2 - (\ell + b)$.

By Proposition 3.38, if $\ell$ and $b$ are integers with $\ell \geq 4$ and $b \geq 1$ and $G$ is a connected graph of order $n = \ell + b + 2$ such that

(1) the length of a longest path in $G$ is $\ell$; and

(2) $t(G) = 2n - 2 - (\ell + b)$,

then $\ell \geq 2b + 2$. However, if $n \geq \ell + b + 3$, then there exists a graph $G$ of order $n$ which satisfies (1) and (2) and $\ell \leq 2b + 1$. For example, let $b = 3$ and consider $G_1$ and $G_2$ in Figure 3.10. The length of a longest path in the graph $G_1$ is $\ell_1 = 9 \geq 2b + 2$, the order of $G_1$ is $n_1 = 14 = \ell_1 + b + 2$, and $t(G_1) = 14 = 2n_1 - 2 - (\ell_1 + b)$. On the other hand, the length of a longest path in the graph $G_2$ is $\ell_2 = 7 < 2b + 2$, the order of $G_2$ is $n_2 = 13 = \ell_2 + b + 3$, and $t(G_2) = 14 = 2n_2 - 2 - (\ell_2 + b)$.

\[ G_1: \quad \begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \quad \circ \\
\end{array} \]

\[ G_2: \quad \begin{array}{c}
\circ \quad \circ \\
\end{array} \]

Figure 3.10: Graphs $G_1$ and $G_2$

Proposition 3.40 Let $b \geq 2$ be an integer. If $G$ is a connected graph for which

(1) the length of a longest path in $G$ is $\ell \geq 4$;

(2) the order of $G$ is $n \geq \ell + b + 3$; and

(3) $t(G) = n + 1$ (that is, $k = n - 3$),

then $\ell \geq 2 \left\lceil \frac{b}{2} \right\rceil + 2$.

Proof. Since $t(G) = n + 1$, there exists a spanning walk $W$ given by

$W : v_1, v_2, v_3, \ldots, v_{n+1}, v_{n+2}$
of length $n + 1$. Since the order of $G$ is $n$, it follows that $W$ contains either

(i) exactly four vertices $v_{i_1}, v_{i_2}, v_{i_3},$ and $v_{i_4}$, where

$$1 \leq i_1 < i_1 + 2 \leq i_2 < i_2 + 2 \leq i_3 < i_3 + 2 \leq i_4 \leq n + 2,$$

for which $v_{i_1} = v_{i_2}, v_{i_3} = v_{i_4},$ and $v_{i_2} \neq v_{i_3};$ or

(ii) exactly three vertices $v_{i_1}, v_{i_2},$ and $v_{i_3}$, where

$$1 \leq i_1 < i_1 + 2 \leq i_2 < i_2 + 2 \leq i_3 \leq n + 2,$$

for which $v_{i_1} = v_{i_2} = v_{i_3}$.

We consider two cases.

**Case 1.** $W$ satisfies the condition (i). Then $G$ contains two paths $Q_1$ and $Q_2$ given by

$$Q_1 : v_1, v_2, \ldots, v_{i_1} = v_{i_2}, v_{i_2+1}, v_{i_2+2}, \ldots, v_{i_3} = v_{i_4}, v_{i_4+1}, v_{i_4+2}, \ldots, v_{n+2}$$

$$Q_2 : v_{i_1+1}, v_{i_1+2}, \ldots, v_{i_2}, v_{i_2+1}, \ldots, v_{i_3}, v_{i_3+1}, \ldots, v_{i_4-1}.$$

Observe that $Q_1$ and $Q_2$ have lengths

$$(i_1 - 1) + (i_3 - i_2) + (n + 2 - i_4) = n + i_1 - i_2 + i_3 - i_4 + 1 \text{ and}$$

$$(i_4 - 1) - (i_1 + 1) = -i_1 + i_4 - 2,$$

respectively. Since every path in $G$ has length at most $\ell$,

$$n + i_1 - i_2 + i_3 - i_4 + 1 \leq \ell \text{ and } -i_1 + i_4 - 2 \leq \ell.$$  \hfill (3.8)

From the inequalities in (3.8) and the fact that $n \geq \ell + b + 3$, we obtain

$$(\ell + b + 3) - i_2 + i_3 - 1 \leq n - i_2 + i_3 - 1 \leq 2\ell$$

and so

$$b + 2 + (i_3 - i_2) \leq \ell.$$
Since \( i_3 - i_2 \geq 2 \), it follows that

\[
\ell \geq b + 4 > 2 \left\lceil \frac{b}{2} \right\rceil + 2.
\]

**Case 2.** \( W \) satisfies the condition (ii). Then \( G \) contains six paths \( Q_j \) (1 \( \leq j \leq 6 \)) given by

\[
\begin{align*}
Q_1 : & \quad v_1, v_2, \ldots, v_{i_1} = v_{i_3}, v_{i_3+1}, v_{i_3+2}, \ldots, v_{n+2} \\
Q_2 : & \quad v_1, v_2, \ldots, v_{i_1}, v_{i_1+1}, v_{i_1+2}, \ldots, v_{i_2} \\
Q_3 : & \quad v_1, v_2, \ldots, v_{i_1} = v_{i_2}, v_{i_2+1}, v_{i_2+2}, \ldots, v_{i_3} \\
Q_4 : & \quad v_{i_1+1}, v_{i_1+2}, \ldots, v_{i_2} = v_{i_3}, v_{i_3+1}, \ldots, v_{n+2} \\
Q_5 : & \quad v_{i_2+1}, v_{i_2+2}, \ldots, v_{i_3}, v_{i_3+1}, \ldots, v_{n+2} \\
Q_6 : & \quad v_{i_1+1}, v_{i_1+2}, \ldots, v_{i_2}, v_{i_2+1}, \ldots, v_{i_3-1}.
\end{align*}
\]

Denote the length of the path \( Q_j \) by \( L_j \) for 1 \( \leq j \leq 6 \). Then \( L_j \leq \ell \) for every \( j \) with \( 1 \leq j \leq 6 \) and so

\[
\begin{align*}
L_1 &= (i_1 - 1) + (n + 2 - i_3) = n + i_1 - i_3 + 1 \leq \ell \quad (3.9) \\
L_2 &= i_2 - 2 \leq \ell \quad (3.10) \\
L_3 &= (i_1 - 1) + (i_3 - i_2 - 1) = i_1 - i_2 + i_3 - 2 \leq \ell \quad (3.11) \\
L_4 &= (i_2 - i_1 - 1) + (n + 2 - i_3) = n - i_1 + i_2 - i_3 + 1 \leq \ell \quad (3.12) \\
L_5 &= n - i_2 + 1 \leq \ell \quad (3.13) \\
L_6 &= -i_1 + i_3 - 2 \leq \ell. \quad (3.14)
\end{align*}
\]

From the inequalities (3.10) and (3.13) and the fact that \( n \geq \ell + b + 3 \), we obtain

\[
b + 2 = (\ell + b + 3 - \ell - 1) \leq (n - \ell + 1) - 2 \leq i_2 - 2 \leq \ell. \quad (3.15)
\]

Hence if \( b \) is even, then \( \ell \geq b + 2 = 2 \left\lceil \frac{b}{2} \right\rceil + 2 \). So suppose that \( b = 2c + 1 \) for some integer \( c \geq 1 \). We show that \( \ell \geq b + 3 = 2 \left\lceil \frac{b}{2} \right\rceil + 2 \). Assume, to the contrary, that \( \ell = b + 2 = 2c + 3 \). Then by (3.10) and (3.13),

\[
2c + 5 = b + 4 = (\ell + b + 3) - \ell + 1 \leq n - \ell + 1 \leq i_2 \leq \ell + 2 = 2c + 5,
\]

implying that \( n - \ell + 1 = \ell + 2 = i_2 = 2c + 5 \). Then by (3.11) and (3.12),

\[
2(2c + 5) = (n - \ell + 1) + i_2 \leq i_1 + i_3 \leq (\ell + 2) + i_2 = 2(2c + 5).
\]
Hence

\[ i_3 + i_3 = 4c + 10. \]  
\[ (3.16) \]

By (3.9) and (3.14),

\[ 2c + 5 = n - \ell + 1 \leq i_3 - i_1 \leq \ell + 2 = 2c + 5 \]

and so

\[ i_3 - i_1 = 2c + 5. \]  
\[ (3.17) \]

However, by (3.16) and (3.17), we obtain

\[ 2i_3 = 6c + 15, \]

which is a contradiction. Hence \( \ell \geq 2 \left\lfloor \frac{b}{2} \right\rfloor + 2 \), as claimed. \( \blacksquare \)

**Proposition 3.41** Let \( b \geq 2 \) be an integer. If \( n \) and \( \ell \) are two integers such that \( \ell \geq 2 \left\lfloor \frac{b}{2} \right\rfloor + 2 \) and \( n \geq \ell + b + 3 \), then there exists a graph \( G \in \mathcal{G}(n, \ell, \ell + b, \ell + b) \).

**Proof.** Let \( n = (\ell + b + 3) + a \), where \( a \geq 0 \). We first construct a graph \( H \).

Consider three paths

\[ Q : u_0, v_1, \ldots, v_\ell \]
\[ Q_1 : u_0, u_1, u_2, \ldots, u_{\left\lfloor \frac{b}{2} \right\rfloor} \]
\[ Q_2 : u_0, w_1, w_2, \ldots, w_{\left\lfloor \frac{b}{2} \right\rfloor} \]

of lengths \( \ell, \left\lfloor \frac{b}{2} \right\rfloor \), and \( \left\lfloor \frac{b}{2} \right\rfloor \), respectively. The graph \( H \) is obtained from \( Q, Q_1, \) and \( Q_2 \) by joining the vertices \( u_0, u_{\left\lfloor \frac{b}{2} \right\rfloor}, u_0, \) and \( u_{\left\lfloor \frac{b}{2} \right\rfloor} \) to the vertex \( v_{\left\lfloor \frac{b}{2} \right\rfloor} \) on \( Q \). Then the graph \( G \) is obtained from \( H \) by joining a new vertices \( x_1, x_2, \ldots, x_\alpha \) to \( v_{\left\lfloor \frac{b}{2} \right\rfloor} \). Since \( \ell \geq 2 \left\lfloor \frac{b}{2} \right\rfloor + 2 \), the path \( Q \) is a longest path in \( G \). Furthermore, the spanning linear forest

\[ F = Q \cup Q_1 \cup Q_2 \cup aK_1 \]

has the maximum size \( \ell + b \). It remains to show that \( t(G) = 2n - 2 - (\ell + b) \). By Corollary 3.4, it follows that \( t(G) \geq 2n - 2 - (\ell + b) \). Consider the linear sequence

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$s$ given by

$$s : v_0, v_1, \ldots, v_{\left\lfloor \frac{\ell}{2} \right\rfloor}, u_0, u_1, \ldots, u_{\left\lfloor \frac{\ell}{2} \right\rfloor}, w_0, w_1, \ldots, w_{\left\lfloor \frac{\ell}{2} \right\rfloor}, x_1, x_2, \ldots, x_a, v_{\left\lfloor \frac{\ell}{2} \right\rfloor+1}, v_{\left\lfloor \frac{\ell}{2} \right\rfloor+2}, \ldots, v_{\ell}.$$ 

Observe that

$$t(G) \leq d(s) = \left\lceil \frac{\ell}{2} \right\rceil + \left( \left\lceil \frac{b}{2} \right\rceil + 2 \right) + \left( \left\lceil \frac{b}{2} \right\rceil + 2 \right) + 2a + \left\lceil \frac{\ell}{2} \right\rceil$$

$$= \ell + b + 4 + 2a = 2(\ell + b + 3 + a) - 2 - (\ell + b)$$

$$= 2n - 2 - (\ell + b).$$

Hence $t(G) = 2n - 2 - (\ell + b)$.

Propositions 3.39 and 3.41 suggest that if $n$ is sufficiently large, then for positive integers $b$ and $\ell \geq 4$, there exists a connected graph $G \in \mathcal{G}(n, \ell, \ell + b, \ell + b)$. We state this in the following proposition.

**Proposition 3.42** Let $b$ be a positive integer. If $\ell \geq 4$ and $n \geq \ell + 2b + 1$, then there exists a connected graph $G \in \mathcal{G}(n, \ell, \ell + b, \ell + b)$.

**Proof.** Let $Q : v_0, v_1, v_2, \ldots, v_{\ell}$ be a path of length $\ell$ and let $Q_j : u_j, w_j$ be a copy of $K_2$ for each integer $j$ with $1 \leq j \leq b$. Then construct a graph $G$ by (i) joining each vertex of $Q_j$ to $v_2$ for each $j$ with $1 \leq j \leq b$ and then (ii) joining a new vertices $x_1, x_2, \ldots, x_a$ to $v_2$. Observe that the order of $G$ is

$$n = (\ell + 1) + 2b + a \geq \ell + 2b + 1$$

and the path $Q$ is a longest path in $G$. Moreover, the spanning linear forest

$$F = Q \cup Q_1 \cup Q_2 \cup \cdots \cup Q_b \cup aK_1$$

has the maximum size of $\ell + b$. To show $t(G) = 2n - 2 - (\ell + b)$, first observe that by Corollary 3.4,

$$t(G) \geq 2n - 2 - (\ell + b).$$

Consider the linear sequence $s$ given by
Observe that
\[
  t(G) \leq d(s) = 2 + 3b + 2a + (\ell - 2) = \ell + 3b + 2a \\
  = 2(\ell + 2b + a + 1) - 2 - (\ell + b) \\
  = 2n - 2 - (\ell + b).
\]
Hence \( t(G) = 2n - 2 - (\ell + b) \), as claimed. \( \blacksquare \)

We now consider those 4-tuples \( (n, \ell, p, b) \), where \( p = \ell + b \) for some positive integer \( b \).

**Proposition 3.43** Suppose that \( G \) is a connected graph of order \( n \geq 7 \) such that
\[ 4 \leq \ell < p = n - 2, \]
where \( \ell \) is the length of a longest path in \( G \) and \( p \) is the maximum size of a spanning linear forest of \( G \). Then
\[ \left\lfloor \frac{n}{2} \right\rfloor \leq \ell \leq n - 3. \]

**Proof.** Since \( \ell < p = n - 2 \), it follows that \( \ell \leq n - 3 \). By Proposition 3.35, every spanning linear forest of \( G \) contains at least two nontrivial components. Suppose that \( F \) is a spanning linear forest of \( G \) containing \( s \) nontrivial components. Then
\[ n - 2 = p \leq n - s \leq n - 2 \]
and so \( s = 2 \). Assume, to the contrary, that \( \ell < \left\lfloor \frac{n}{2} \right\rfloor \). We first show that \( F \) must contain at least one component of size \( \ell \). Suppose this is not the case. Then each of the two nontrivial components of \( F \) has at most \( \ell - 1 \) edges. Hence
\[
  n - 2 = p \leq 2(\ell - 1) \leq 2 \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right) \\
  = \begin{cases} 
    n - 4 & \text{if } n \text{ is even} \\
    n - 3 & \text{if } n \text{ is odd},
  \end{cases}
\]
which is a contradiction. Therefore, \( F \) contains at least one component isomorphic to a path of length \( \ell \). Now let \( \ell' \) be the size of the other nontrivial component of \( F \). Then
\[
  \left\lfloor \frac{n}{2} \right\rfloor - 1 \geq \ell \geq \ell' = p - \ell \geq (n - 2) - \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \\
  \geq \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]
(3.18)
Thus, either $\ell' = \ell$ or $\ell' = \ell - 1$. We now show that $\ell' = \ell$ if and only if $n$ is even. If $n$ is even, then $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{\ell}{2} \right\rfloor$ and so $\ell' = \ell = \frac{n}{2} - 1$ by the inequality in (5.2). Next suppose that $\ell = \ell'$ and assume, to the contrary, that $n$ is odd. Then $n - 2 = p = 2\ell$, which is a contradiction since $n - 2$ is odd. Consequently, if $n$ is odd, then $\ell = \frac{n-1}{2}$ and $\ell' = \ell - 1 = \frac{n-3}{2}$. Let $Q_1$ and $Q_2$ be the two nontrivial paths of $F$, whose lengths are $\ell$ and $\ell'$, respectively, and let $r$ be the number of trivial components in $F$. Observe that

$$n = (\ell + 1) + (\ell' + 1) + r = (p + 2) + r = n + r.$$ 

This implies that $r = 0$ and so $F = Q_1 \cup Q_2$. However, since $G$ is connected, there is a vertex of $Q_1$ that is adjacent in $G$ to a vertex of $Q_2$, creating a path of length at least $\ell + 1$ in $G$, a contradiction. Hence $\ell \geq \left\lfloor \frac{n}{2} \right\rfloor$, as claimed. \hfill \blacksquare

\textbf{Proposition 3.44} Let $n$ and $\ell$ be two integers such that $n \geq 7$ and

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \ell \leq n - 3.$$ 

Then there exists a graph $G \in G(n, \ell, n - 2, \ell)$.

\textbf{Proof.} We consider four cases, according to the residue classes to which $n$ belongs modulo 4.

\textit{Case 1.} $n \equiv 0 \pmod{4}$. Then $n = 4a$ for some integer $a \geq 2$. Hence $\ell = 2a + b$, where $b$ is an integer with $0 \leq b \leq 2a - 3$.

\textit{Subcase 1.1.} $b = 2r$, where $0 \leq r \leq a - 2$. Consider the two paths

$$Q_1 : u_0, u_1, \ldots, u_a, v_1, \ldots, v_{2a-1} \quad \text{and} \quad Q_2 : v_0, v_1, \ldots, v_{a+r-1}, \ldots, v_{2a-2}.$$ 

The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r-1}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Since $G$ is not a path, the size of every spanning linear forest in $G$ is at most $n - 2$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

\textit{Subcase 1.2.} $b = 2r + 1$, where $0 \leq r \leq a - 2$. Consider the two paths

$$Q_1 : u_0, u_1, \ldots, u_a, v_1, \ldots, v_{2a-1} \quad \text{and} \quad Q_2 : v_0, v_1, \ldots, v_{a+r}, \ldots, v_{2a-1}.$$ 

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The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

Case 2. $n \equiv 1 \pmod{4}$. Then $n = 4a + 1$ for some integer $a \geq 2$. Hence $\ell = 2a + b$, where $b$ is an integer with $1 \leq b \leq 2a - 2$.

Subcase 2.1. $b = 2r$, where $1 \leq r \leq a - 1$. Consider the two paths

$Q_1 : u_0, u_1, \ldots, u_{a+r}, \ldots, u_{2a}$ and $Q_2 : v_0, v_1, \ldots, v_{a+r-1}, \ldots, v_{2a-1}$.

The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r-1}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

Subcase 2.2. $b = 2r + 1$, where $0 \leq r \leq a - 2$. Consider the two paths

$Q_1 : u_0, u_1, \ldots, u_{a+r}, \ldots, u_{2a}$ and $Q_2 : v_0, v_1, \ldots, v_{a+r}, \ldots, v_{2a-1}$.

The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r-1}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

Case 3. $n \equiv 2 \pmod{4}$. Then $n = 4a + 2$ for some integer $a \geq 2$. Hence $\ell = 2a + b$, where $b$ is an integer with $1 \leq b \leq 2a - 1$.

Subcase 3.1. $b = 2r$, where $1 \leq r \leq a - 1$. Consider the two paths

$Q_1 : u_0, u_1, \ldots, u_{a+r}, \ldots, u_{2a+1}$ and $Q_2 : v_0, v_1, \ldots, v_{a+r-1}, \ldots, v_{2a-1}$.

The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r-1}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

Subcase 3.2. $b = 2r + 1$, where $0 \leq r \leq a - 1$. Consider the two paths

$Q_1 : u_0, u_1, \ldots, u_{a+r}, \ldots, u_{2a}$ and $Q_2 : v_0, v_1, \ldots, v_{a+r}, \ldots, v_{2a}$.

The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$. 

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Case 4. $n \equiv 3 \pmod{4}$. Then $n = 4a + 3$ for some integer $a \geq 1$. Hence $\ell = 2a + b$, where $b$ is an integer with $2 \leq b \leq 2a$.

Subcase 4.1. $b = 2r$, where $1 \leq r \leq a$. Consider the two paths

\[ Q_1 : u_0, u_1, \ldots, u_{a+r}, \ldots, u_{2a+1} \quad \text{and} \quad Q_2 : v_0, v_1, \ldots, v_{a+r-1}, \ldots, v_{2a}. \]

The graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r-1}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$.

Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

Subcase 4.2. $b = 2r + 1$, where $1 \leq r \leq a - 1$. Let

\[ Q_1 : u_0, u_1, \ldots, u_{a+r}, \ldots, u_{2a+1} \quad \text{and} \quad Q_2 : v_0, v_1, \ldots, v_{a+r}, \ldots, v_{2a}. \]

Then the graph $G$ is obtained from $Q_1$ and $Q_2$ by adding the edge $u_{a+r}v_{a+r}$. Then $G$ is a tree of diameter $\ell$. Thus the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2$ is a spanning linear forest in $G$ of size $n - 2$.

Proposition 3.44 can be generalized as follows.

**Theorem 3.45** Let $n$, $\ell$, and $p$ be three integers such that $p \geq 5$, $n \geq p + 2$, and

\[ \left\lceil \frac{p}{2} \right\rceil + 1 \leq \ell \leq p - 1. \]

Then there exists a graph $G \in \mathcal{G}(n, \ell, p, \ell)$.

**Proof.** If $n = p + 2$, then $n \geq 7$, $p - 1 = n - 3$, and

\[ \left\lceil \frac{p}{2} \right\rceil + 1 = \left\lceil \frac{n - 2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil \]

and so the result follows from Proposition 3.44. Therefore we may assume that $n = p + 2 + a$, where $a \geq 1$. Let

\[ n' = n - a = p + 2. \]

Then $n' \geq 7$ and

\[ \left\lceil \frac{n'}{2} \right\rceil \leq \ell \leq n' - 3. \]
By Proposition 3.44, there exists a graph $G' \in \mathcal{G}(n', \ell, p, \ell)$. Let $G'$ be such a graph of order $n'$ constructed from two paths $Q_1$ and $Q_2$, as described in the proof of Proposition 3.44. Furthermore, let $v$ be the vertex in $Q_2$ for which $\deg_{G'} v = 3$. Then the graph $G$ of order $n$ is constructed from $G'$ by joining a new vertices to the vertex $v$. Since $G$ is a tree of diameter $\ell$, it follows that the length of a longest path in $G$ is $\ell$ and $t(G) = 2n - 2 - \ell$. Furthermore, $Q_1 \cup Q_2 \cup aK_1$ is a spanning linear forest in $G$ with the maximum size $n' - 2 = p$. Therefore, $G$ possesses the desired property.

A 4-tuple $(n, \ell, p, k)$ of positive integers, where $n \geq 2$ and $\ell \leq k \leq p \leq n - 1$, is said to be realizable if $\mathcal{G}(n, \ell, p, k) \neq \emptyset$. We now summarize what we have obtained thus far.

1. A 4-tuple $(2, \ell, p, k)$ is realizable if and only if $(2, \ell, p, k) = (2, 1, 1, 1)$.

2. Each 4-tuple $(n, \ell, \ell, \ell)$, where $n \geq 3$ and $2 \leq \ell \leq n - 1$, is realizable.

3. A 4-tuple $(n, 2, p, k)$ is realizable if and only if $(n, 2, p, k) = (n, 2, 2, 2)$ for $n \geq 3$.

4. Let $n \geq 4$. A 4-tuple $(n, 3, p, k)$ is realizable if and only if $(n, 3, p, k) = (n, 3, 3, 3)$ or $(n, 3, p, k) = (n, 3, 4, 3)$.

5. Let $n \geq 7$ and $\ell \geq 4$. A 4-tuple $(n, \ell, \ell + 1, \ell + 1)$ is realizable if and only if $n \geq \ell + 3$.

6. Let $b$ be a positive integer. If $4 \leq \ell < 2b+2$ and $n \geq \ell + b + 2$, then $(n, \ell, p, n-2)$ is not realizable for all $p > \ell$.

7. Let $b$ be a positive integer. If $\ell \geq 2b+2$ and $n \geq \ell + b + 2$, then $(n, \ell, \ell + b, \ell + b)$ is realizable.

8. Let $b \geq 2$ be an integer. If $4 \leq \ell < 2\lfloor b/2 \rfloor + 2$ and $n \geq \ell + b + 3$, then $(n, \ell, p, n-3)$ is not realizable for all $p > \ell$.

9. Let $b \geq 2$ be an integer. If $\ell \geq 2\lfloor b/2 \rfloor + 2$ and $n \geq \ell + b + 3$, then $(n, \ell, \ell + b, \ell + b)$ is realizable.

10. Let $b$ be a positive integer, $\ell \geq 4$, and $n \geq \ell + 2b + 1$. Each 4-tuple $(n, \ell, \ell + b, \ell + b)$ is realizable.
11. Let $n \geq 7$ and $\ell \geq 4$. A 4-tuple $(n, \ell, n - 2, \ell)$ is realizable if and only if $[n/2] \leq \ell \leq n - 3$.

12. If $p \geq 5$, $n \geq p + 2$, and $[p/2] + 1 \leq \ell \leq p - 1$, then $(n, \ell, p, \ell)$ is realizable.

We have also studied realizable 4-tuples $(n, \ell, p, k)$ of positive integers, where $n \geq 2$ and

$$4 \leq \ell < k < p \leq n - 1.$$ 

In fact, there are infinitely many such realizable 4-tuples and infinitely many such 4-tuples that are not realizable. Some of these realizable or non-realizable 4-tuples are listed below.

1. Each 4-tuple $(n, \ell, \ell + 2, \ell + 1)$ is realizable if and only if $n \geq \ell + 4$.

2. A 4-tuple $(n, \ell, \ell + 3, \ell + 1)$ is realizable if and only if (1) $\ell = 4$ and $n \geq 10$ or (2) $\ell \geq 5$ and $n \geq \ell + 5$.

3. A 4-tuple $(n, \ell, \ell + 3, \ell + 2)$ is realizable if and only if (1) $\ell = 4$ and $n \geq 10$ or (2) $\ell \geq 5$ and $n \geq \ell + 5$.

4. A 4-tuple $(n, \ell, \ell + 4, \ell + 1)$ is realizable if and only if (1) $\ell = 4$ and $n \geq 11$, or (2) $\ell = 5$ and $n \geq 12$, or (3) $\ell \geq 6$ and $n \geq \ell + 6$.

5. A 4-tuple $(n, \ell, \ell + 4, \ell + 2)$ is realizable if and only if (1) $\ell = 4$ and $n \geq 11$, or (2) $\ell = 5$ and $n \geq 12$, or (3) $\ell \geq 6$ and $n \geq \ell + 6$.

6. A 4-tuple $(n, \ell, \ell + 4, \ell + 3)$ is realizable if and only if (1) $\ell = 4$ and $n \geq 12$, or (2) $\ell = 5$ and $n \geq 12$, or (3) $\ell \geq 6$ and $n \geq \ell + 6$.

As an illustration, we prove the following.

**Theorem 3.46** For $\ell \geq 4$, a 4-tuple $(n, \ell, \ell + 4, \ell + 3)$ is realizable if and only if $n$ and $\ell$ satisfy one of the following:

1. $\ell = 4$ and $n \geq 12$;
2. $\ell = 5$ and $n \geq 12$;
(3) $\ell \geq 6$ and $n \geq \ell + 6$.

**Proof.** We first show that each 4-tuple $(n, \ell, \ell + 4, \ell + 3)$ satisfying one of (1)-(3) is realizable, that is, we construct a graph $G_{\ell, n}$ such that $G_{\ell, n} \in \mathcal{G}(n, \ell, \ell + 4, \ell + 3)$. We consider three cases.

**Case 1.** $\ell = 4$ and $n \geq 12$. Let $P_5 : u_1, u_2, u_3, u_4, u_5$ be a path of order 5. For each $i$ with $1 \leq i \leq 3$, let $Q_i : x_i, y_i$ be a copy of $K_2$. Then the graph $G_{4,12}$ is obtained from $P_5$ and $Q_i$ ($1 \leq i \leq 3$) by (i) joining each vertex of $Q_i$ ($1 \leq i \leq 3$) to the vertex $u_3$ and (ii) joining a new vertex $v$ to $u_2$. For $n = 12 + a$, where $a \geq 1$, let $G_{4, n}$ be the graph obtained from $G_{4,12}$ by joining a new vertices $w_1, w_2, \ldots, w_a$ to $u_3$. Then $P_5$ is a longest path in $G_{4, n}$ and so the length of a longest path in $G_{4, n}$ is $\ell = 4$. The spanning linear forest $F$ consisting of the path $u_1, u_2, v$, the path $x_1, y_1, x_3, y_3$, the path $u_4, u_5$, and the isolated vertices $w_1, w_2, \ldots, w_a$ has the maximum size $p = 8 = \ell + 4$. Furthermore, consider the linear ordering $s_1$ of vertices of $G_{4, n}$ given by

$$s_1 : u_1, u_2, v, u_3, x_1, y_1, x_2, y_2, x_3, y_3, w_1, w_2, \ldots, w_a, u_4, u_5.$$ 

Observe that

$$t(G_{4, n}) = d(s_1) = 2a + 15 = 2(12 + a) - 2 - 7 = 2n - 2 - (\ell + 3).$$

**Case 2.** $\ell = 5$ and $n \geq 12$. Let $P_6 : u_1, u_2, u_3, u_4, u_5, u_6$ be a path of order 6. For each $i$ with $1 \leq i \leq 3$, let $Q_i : x_i, y_i$ be a copy of $K_2$. Then the graph $G_{5,12}$ is obtained from $P_6$ and $Q_i$ ($1 \leq i \leq 3$) by (i) joining each vertex of $Q_i$ ($i = 1, 2$) to the vertex $u_3$ and (ii) joining each vertex of $Q_3$ to $u_4$. For $n = 12 + a$, where $a \geq 1$, let $G_{5, n}$ be the graph obtained from $G_{5,12}$ by joining a new vertices $w_1, w_2, \ldots, w_a$ to $u_3$. Then $P_6$ is a longest path in $G_{5, n}$ and so the length of a longest path in $G$ is $\ell = 5$. The spanning linear forest $F$ consisting of the path $u_1, u_2$, the path $x_1, y_1, u_3, x_2, y_2$, the path $x_3, y_3, u_4, u_5, u_6$, and the isolated vertices $w_1, w_2, \ldots, w_a$ has the maximum size $p = 9 = \ell + 4$. Furthermore, consider the linear ordering $s_2$ of vertices of $G_{5, n}$ given by

$$s_2 : u_1, u_2, u_3, x_1, y_1, x_2, y_2, w_1, w_2, \ldots, w_a, u_4, x_3, y_3, u_5, u_6.$$
Observe that

\[ t(G_{5,n}) = d(s_2) = 2a + 14 = 2(12 + a) - 2 - 8 = 2n - 2 - (\ell + 3). \]

**Case 3.** \( \ell \geq 6 \) and \( n \geq \ell + 6 \). Let \( P_7 : u_1, u_2, u_3, u_4, u_5, u_6, u_7 \) be a path of order 7, \( C : v_1, v_2, \ldots, v_{\ell-2}, v_1 \) a cycle of order \( \ell - 2 \), and \( Q : x, y \) be a copy of \( K_2 \). Then the graph \( G_{\ell,\ell+6} \) is obtained from \( P_7, C, \) and \( Q \) by (i) identifying \( v_1 \) with \( u_4 \) and (ii) joining each of \( x \) and \( y \) to \( u_5 \). For \( n = \ell + 6 + a \), where \( a \geq 1 \), let \( G_{\ell,n} \) be the graph obtained from \( G_{\ell,\ell+6} \) by joining a new vertices \( w_1, w_2, \ldots, w_a \) to \( u_4 \). Then the path \( P : u_1, u_2, u_3, u_4, v_2, \ldots, v_{\ell-2} \) is a longest path in \( G_{\ell,n} \), whose length is \( \ell \). The spanning linear forest \( F \) consisting of the path \( P \), the path \( x, y, u_5, u_6, u_7 \), and the isolated vertices \( w_1, w_2, \ldots, w_a \) has the maximum size \( p = \ell + 4 \). Furthermore, consider the linear ordering \( s_3 \) of vertices of \( G_{\ell,n} \) given by

\[ s_3 : u_1, u_2, u_3, u_4, v_2, v_3, \ldots, v_{\ell-3}, v_{\ell-2}, w_1, w_2, \ldots, w_a, u_5, x, y, u_6, u_7. \]

Observe that

\[ t(G_{\ell,n}) = d(s_3) = 2a + \ell + 7 = 2(\ell + 6 + a) - 2 - (\ell + 3) = 2n - 2 - (\ell + 3). \]

For the converse, let \((n, \ell, \ell + 4, \ell + 3)\) be a realizable 4-tuple. We show that \((n, \ell, \ell + 4, \ell + 3)\) satisfies one of (1)–(3). We consider three cases.

**Case 1.** \( \ell = 4 \). Then \( p = 8 \) and \( k = 7 \). Thus \((n, \ell, \ell + 4, \ell + 3) = (n, 4, 8, 7)\). Since \( \ell < p \), it follows by Proposition 3.35 that \( n \geq p + 2 = 10 \). If \( n = 10 \), then by Proposition 3.43,

\[ 5 = \left\lceil \frac{n}{2} \right\rceil \leq \ell = 4, \]

a contradiction. Hence \( n \geq 11 \). Next we show that \( n \neq 11 \) by showing that \( G(11, 4, 8, 7) = \emptyset \). Assume, there exists a graph \( G \in G(11, 4, 8, 7) \). Thus \( t(G) = 2 \cdot 11 - 2 - 7 = 13 \). This implies that there is a spanning walk

\[ W : v_1, v_2, \ldots, v_{14} \]

of length 13. Hence \( W \) satisfies one of the following:

(A) \( W \) contains exactly six vertices \( v_{i_1}, v_{i_2}, \ldots, v_{i_6} \), where

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1 \leq i_1 < i_1 + 2 \leq i_2 < i_2 + 2 \leq \cdots < i_6 \leq 14

and \(v_1 = v_{i_2}, v_3 = v_{i_4}, v_5 = v_{i_6}, v_2 \neq v_{i_3}, \) and \(v_4 \neq v_{i_5} \).

(B) \(W\) contains exactly five vertices \(v_{i_1}, v_{i_2}, \ldots, v_{i_5}\), where
\[
1 \leq i_1 < i_1 + 2 \leq i_2 < i_2 + 2 \leq \cdots < i_5 \leq 14
\]
for which either

(B1) \(v_{i_1} = v_{i_2} = v_{i_3}, v_{i_4} = v_{i_5}\), and \(v_{i_3} \neq v_{i_4}\); or

(B2) \(v_{i_1} = v_{i_2}, v_{i_3} = v_{i_4}, v_5 \neq v_{i_3},\)

(C) \(W\) contains exactly four vertices \(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\), where
\[
1 \leq i_1 < i_1 + 2 \leq i_2 < i_2 + 2 \leq i_3 < i_3 + 2 \leq i_4 \leq 14
\]
and \(v_1 = v_{i_2} = v_{i_3} = v_{i_4}\).

Observe that both (A) and (B) imply that \(G\) contains a path of length 5 or more, which is impossible. Thus \(W\) satisfies (C). Since \(G\) has order 11 and contains no path of length 5, it follows that \(G\) is the graph shown in Figure 3.11. However then, every spanning linear forest of \(G\) has size at most 7, which is a contradiction.

\[
\begin{align*}
G: & \quad \circ \quad \circ \quad \circ \\
& \quad \circ \quad \circ \quad \circ
\end{align*}
\]

Figure 3.11: The graph \(G\) in Case 1

Therefore, \(n \geq 12\) and \((n, \ell, \ell + 4, \ell + 3)\) satisfies the condition (1).

Case 2. \(\ell = 5\). Then \(p = 9\) and \(k = 8\). Since \(\ell < p\), it follows by Proposition 3.35 that \(n \geq p + 2 = 11\). If \(n = 11\), then by Proposition 3.43,
\[
6 = \left\lfloor \frac{n}{2} \right\rfloor \leq \ell = 5,
\]
a contradiction. Hence \(n \geq 12\) and \((n, \ell, \ell + 4, \ell + 3)\) satisfies the condition (2).

Case 3. \(\ell \geq 6\). By Proposition 3.35,
\[
n \geq p + 2 \geq \ell + 6.
\]
Therefore, \(n \geq \ell + 6\) and \((n, \ell, \ell + 4, \ell + 3)\) satisfies the condition (3).
Chapter 4

The Upper Traceable Number of a Graph

4.1 Introduction

For a connected graph $G$, recall that the upper Hamiltonian number $h^+(G)$ is defined by

$$h^+(G) = \max \{d(s)\},$$

where the maximum is taken over all cyclic orderings $s$ of vertices of $G$. As expected, for a connected graph $G$, we define the upper traceable number $t^+(G)$ as

$$t^+(G) = \max \{d(s)\},$$

where the maximum is taken over all linear orderings $s$ of vertices of $G$. Consequently,

$$t(G) \leq t^+(G)$$

for every connected graph $G$.

To illustrate this concept, consider the graph $G$ of Figure 4.1. Since $G$ is traceable and the order of $G$ is 5, it follows that $t(G) = 4$. On the other hand, the diameter of $G$ is 2 and so the distance between every two vertices of $G$ is at most 2. Thus $t^+(G) \leq 8$. Since the linear ordering $s : y, v, x, u, w$ has $d(s) = 8$, it follows that $t^+(G) = 8$.

If $s : v_1, v_2, \ldots, v_n$ is a linear ordering of vertices of a connected graph, then for each vertex $v_i$, both $d(v_{i-1}, v_i) \leq e(v_i)$ ($2 \leq i \leq n$) and $d(v_i, v_{i+1}) \leq e(v_i)$.
Figure 4.1: A graph $G$ with $t(G) = 4$ and $t^+(G) = 8$

$(1 \leq i \leq n - 1)$. Thus, if $G$ is a connected graph of order $n \geq 2$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$t^+(G) \leq \sum_{i=1}^{n-1} e(v_i).$$

Since the eccentricity of a vertex in $G$ is at most the diameter of $G$, we have the following observation, which provides an upper bound for the upper traceable number of a graph in terms of its order and diameter.

**Observation 4.1** If $G$ is a nontrivial connected graph of order $n$, then

$$t^+(G) \leq (n - 1) \text{diam}(G).$$

The upper bound for the upper traceable number of a graph described in Observation 4.1 is sharp, as we will see soon. Next, we study the relationship between the upper traceable number and upper Hamiltonian number of a graph.

**Proposition 4.2** For every connected graph $G$ of order $n \geq 3$,

$$1 \leq h^+(G) - t^+(G) \leq \text{rad}(G).$$

**Proof.** We first show that $h^+(G) - t^+(G) \leq \text{rad}(G)$. Let $v$ be a central vertex of $G$. Then

$$1 \leq d(v, u) \leq \text{rad}(G)$$

for every vertex $u \in V(G) - \{v\}$. Let

$$s_c : v = v_1, v_2, \ldots, v_n, v_{n+1} = v$$
be a cyclic ordering of vertices of $G$ with $d(s_c) = h^+(G)$. Then

$$s_\ell : v = v_1, v_2, \ldots, v_n$$

is a linear ordering of vertices of $G$ and

$$t^+(G) \geq d(s_\ell) = d(s_c) - d(v, v_n) \geq h^+(G) - \text{rad}(G).$$

To show $h^+(G) - t^+(G) \geq 1$, let

$$s'_\ell : w_1, w_2, \ldots, w_n$$

be a linear ordering of vertices of $G$ with $d(s'_\ell) = t^+(G)$. Then

$$s'_\ell : w_1, w_2, \ldots, w_n, w_{n+1} = w_1$$

is a cyclic ordering of vertices of $G$ and

$$h^+(G) \geq d(s'_\ell) = d(s'_\ell) + d(w_1, w_n) \geq t^+(G) + 1.$$

Thus $1 \leq h^+(G) - t^+(G) \leq \text{rad}(G)$.

**Corollary 4.3** Let $G$ be a connected graph $G$ of order $n \geq 3$. If $\text{rad}(G) = 1$, then $h^+(G) - t^+(G) = 1$.

By Corollary 4.3, $G = K_n, K_n - e, K_{1,n-1}$ where $n \geq 3$ are examples of the graphs whose upper Hamiltonian number and upper traceable number differ exactly by 1. On the other hand, the converse of Corollary 4.3 is not true, as we will later show that $h^+(T) - t^+(T) = 1$ for every nontrivial tree $T$. Also, we will see later that if $G = C_n$, where $n \geq 3$ is an odd integer, then $h^+(G) - t^+(G) = \text{rad}(G)$.

### 4.2 Preliminary Results

For each integer $n \geq 3$, it was shown in [6] that $K_n$ and $K_{1,n-1}$ are the only connected graphs $G$ of order $n$ for which $h(G) = h^+(G)$. In fact, there is only one nontrivial connected graph $G$ of order $n$ for which $t(G) = t^+(G)$. Observe that

$$t(K_n) = t^+(K_n) = n - 1$$
for \( n \geq 2 \). On the other hand, if \( G \neq K_n \) is a connected graph of order \( n \geq 3 \), then \( G \) contains two nonadjacent vertices \( x \) and \( y \) such that \( d(x, y) = 2 \). Let \( x, z, y \) be an \( x - y \) path in \( G \) and let

\[
s : x, z, y, w_1, w_2, \ldots, w_{n-3} \text{ and } s' : z, x, y, w_1, w_2, \ldots, w_{n-3}
\]

be two linear orderings of vertices of \( G \). Then \( d(s') = d(s) + 1 \) and so \( t(G) \neq t^+(G) \).

**Observation 4.4** Let \( G \) be a nontrivial connected graph of order \( n \). Then

\[
t(G) = t^+(G) \text{ if and only if } G = K_n.
\]

We now establish the upper traceable numbers of some well-known classes of graphs, beginning with complete multipartite graphs and hypercubes.

**Proposition 4.5** If \( G = K_{n_1, n_2, \ldots, n_k} \), where \( n = n_1 + n_2 + \cdots + n_k \) and \( k \geq 2 \), then

\[
t^+(G) = 2n - k - 1.
\]

**Proof.** For each integer \( i \) with \( 1 \leq i \leq k \), let \( V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\} \) be a partite set of \( G \). Then

\[
s_0 : v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}, v_{2,1}, v_{2,2}, \ldots, v_{2,n_2}, \ldots, v_{k,1}, v_{k,2}, \ldots, v_{k,n_k}
\]

is a linear ordering of vertices of \( G \). Since

\[
d(s_0) = (k - 1) + \sum_{i=1}^{k} 2(n_i - 1) = 2n - k - 1,
\]

it follows that \( t^+(G) \geq 2n - k - 1 \). On the other hand, let \( s : x_1, x_2, \ldots, x_n \) be an arbitrary linear ordering of vertices of \( G \). Since \( \text{diam}(G) = 2 \), it follows that \( d(x_j, x_{j+1}) = 1 \) or \( d(x_j, x_{j+1}) = 2 \) for \( 1 \leq j \leq n - 1 \). Furthermore, there are at most

\[
\sum_{i=1}^{k} (n_i - 1) = n - k
\]
pairs $x_j, x_{j+1}$ ($1 \leq j \leq n - 1$) of vertices for which $d(x_j, x_{j+1}) = 2$. Thus

$$d(s) \leq 2(n - k) + [(n - 1) - (n - k)] = 2n - k - 1$$

and so $t^+(G) \leq 2n - k - 1$. Therefore, $t^+(G) = 2n - k - 1$. ■

**Proposition 4.6** For each integer $n \geq 2$,

$$t^+(Q_n) = 2^{n-1}(2n - 1) - n + 1.$$  

**Proof.** First, we show that $t^+(Q_n) \leq 2^{n-1}(2n - 1) - n + 1$. Let $s$ be an arbitrary linear ordering of vertices of $Q_n$ with $d(s) = t^+(Q_n)$. Since $\text{diam}(Q_n) = n$ and for each vertex $v$ in $Q_n$, there is exactly one vertex in $Q_n$ whose distance from $v$ is $n$, it follows that there are at most $2^{n-1}$ terms in $d(s)$ equal to $n$. Consequently, each of the remaining $2^{n-1} - 1$ terms in $d(s)$ is at most $n - 1$. Thus

$$d(s) \leq 2^{n-1}n + (2^{n-1} - 1)(n - 1) = 2^{n-1}(2n - 1) - n + 1$$

and so $t^+(Q_n) \leq 2^{n-1}(2n - 1) - n + 1$.

Next we show that $t^+(Q_n) \geq 2^{n-1}(2n - 1) - n + 1$. Since the result is true for $Q_2$, we may assume that $n \geq 3$. Let $G = Q_n$. Then $G$ consists of two disjoint copies $G_1$ and $G_2$ of $Q_{n-1}$, where corresponding vertices of $G_1$ and $G_2$ are adjacent. For each vertex $v$ of $G$, there is a unique vertex $\overline{v}$ of $G$ such that $d(v, \overline{v}) = n = \text{diam}(Q_n)$. Necessarily, exactly one of $v$ and $\overline{v}$ belongs to $G_1$ for each vertex $v$ of $G$. It is well-known that $Q_n$ is Hamiltonian for $n \geq 2$ and so $Q_n$ is traceable. Let

$$P : v_1, v_2, \ldots, v_{2n-1}$$

be a Hamiltonian path in $G_1$. Now define a linear ordering $s$ of $V(G)$ by

$$s : v_1, \overline{v_1}, v_2, \overline{v_2}, \ldots, v_{2n-1}, \overline{v_{2n-1}}.$$  

Since $d(v_i, \overline{v_i}) = n$ and $d(v_i, v_{i+1}) = 1$ for $1 \leq i \leq 2^{n-1} - 1$, it follows by the triangle inequality that

$$n = d(v_i, \overline{v_i}) \leq d(v_i, v_{i+1}) + d(v_{i+1}, \overline{v_i}) = 1 + d(v_{i+1}, \overline{v_i}).$$

Thus $d(v_{i+1}, \overline{v_i}) \geq n - 1$, which implies that $d(v_{i+1}, \overline{v_i}) = n - 1$. Hence

$$t^+(Q_n) \geq d(s) = 2^{n-1}n + (2^{n-1} - 1)(n - 1) = 2^{n-1}(2n - 1) - n + 1,$$
as desired.

Next we determine the upper traceable numbers of cycles.

**Proposition 4.7** For each integer $n \geq 3$,

$$t^+(C_n) = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor.$$

**Proof.** Let $C_n : v_1, v_2, \ldots, v_n, v_1$ and let $d = \text{diam}(C_n) = \lfloor n/2 \rfloor$. We consider two cases, according to whether $n$ is odd or $n$ is even.

**Case 1.** $n$ is odd. Then $n = 2k + 1$ for some positive integer $k$ and so $d = k = (n - 1)/2$. By Observation 4.1, it follows that $t^+(C_n) \leq (n-1)d$. Let

$$s_0 : v_1, v_{k+1}, v_{2k+1}, v_{3k+1}, \ldots, v_{(2k)k+1}$$

be a linear ordering of vertices of $C_n$, where each subscript is expressed modulo $2k + 1$ as one of the integers $1, 2, \ldots, 2k + 1$. Since

$$d(s_0) = (2k)k = (n-1)d,$$

it follows that $t^+(C_n) \geq (n-1)d$. Thus

$$t^+(C_n) = (n-1)d = \frac{(n-1)^2}{2} = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor$$

if $n$ is odd.

**Case 2.** $n$ is even. Then $n = 2k$ for some integer $k \geq 2$ and so $d = k = n/2$. Let $s$ be a linear ordering of vertices of $C_n$ with $d(s) = t^+(C_n)$. Since $\text{diam}(C_n) = k$ and for each $v \in V(C_n)$ there is exactly one vertex in $C_n$ whose distance from $v$ is $k$, it follows that at most $k$ terms in $d(s)$ equal $k$. Consequently, at least $k - 1$ terms in $d(s)$ are $k - 1$ or less. Thus

$$d(s) \leq k^2 + (k-1)^2 = 2k^2 - 2k + 1 = \frac{(n-1)^2 + 1}{2}$$

and so $t^+(C_n) \leq \frac{(n-1)^2+1}{2}$. On the other hand, let

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Let $G$ be a connected graph of order $n \geq 3$. Then

$$t^+(G) - t(G) = 1 \text{ if and only if } G = K_n - e \text{ or } G = K_{1,n-1}.$$ 

**Proof.** First observe that for $n \geq 3$, $t^+(K_n - e) = n$ and $t(K_n - e) = n - 1$; while $t^+(K_{1,n-1}) = 2n - 3$ and $t(K_{1,n-1}) = 2n - 4$. Hence, if $G = K_n - e$ or $G = K_{1,n-1}$, then $t^+(G) - t(G) = 1$. It remains therefore to verify the converse.

Let $G$ be a connected graph of order $n \geq 3$ such that $t^+(G) - t(G) = 1$. We claim that $\text{diam}(G) = 2$. Assume, to the contrary, that $\text{diam}(G) \neq 2$. If $\text{diam}(G) = 1$, then $G = K_n$. However,

$$t^+(K_n) - t(K_n) = n - 1.$$
If \( \text{diam}(G) \geq 3 \), then \( G \) contains two vertices \( u \) and \( v \) such that \( d(u,v) = 3 \). Let \( u, x, y, v \) be a \( u - v \) path in \( G \), and let \( v_1, v_2, \ldots, v_{n-4} \) be the remaining vertices of \( G \). Also, let \( v_0 = v \) and let

\[
\sum_{i=0}^{n-5} d(v_i, v_{i+1}) = a.
\]

For the linear orderings

\[
s_1 : u, x, y, v, v_1, v_2, \ldots, v_{n-4} \quad \text{and} \quad s_2 : u, y, x, v, v_1, v_2, \ldots, v_{n-4},
\]

\( d(s_1) = a + 3 \) and \( d(s_2) = a + 5 \). Since \( t(G) \leq d(s_1) \) and \( t^+(G) \geq d(s_2) \), it follows that \( t^+(G) - t(G) \geq 2 \), a contradiction. Thus, as claimed, \( \text{diam}(G) = 2 \).

We now consider two cases, depending on whether \( G \) is traceable.

**Case 1.** \( G \) is traceable. Then \( t(G) = n - 1 \). Since \( G \neq K_n \), the graph \( G \) contains at least one pair of nonadjacent vertices. Suppose that \( G \) contains two pairs \( u, v \) and \( x, y \) of nonadjacent vertices. If \( \{u, v\} \cap \{x, y\} = \emptyset \), then every linear ordering \( s' \) beginning with \( u, v, x, y \) has \( d(s') \geq n + 1 \), which is a contradiction. If \( \{u, v\} \cap \{x, y\} \neq \emptyset \), say \( v = x \), then every linear ordering \( s'' \) beginning with \( u, v, y \) has \( d(s'') \geq n + 1 \), a contradiction. Hence \( G \) contains exactly one pair of nonadjacent vertices and so \( G = K_n - e \).

**Case 2.** \( G \) is not traceable. Then \( t(G) = n + k - 2 \) for some integer \( k \geq 2 \). Thus \( G \) contains \( k \) pairwise vertex-disjoint paths \( Q_1, Q_2, \ldots, Q_k \) such that \( \{V(Q_1), V(Q_2), \ldots, V(Q_k)\} \) is a partition of \( V(G) \). However, \( G \) does not contain fewer than \( k \) pairwise vertex-disjoint paths with these properties. Suppose that \( Q_i \) is an \( x_i - y_i \) path for \( 1 \leq i \leq k \). Furthermore, let \( x_i, \ldots, y_i \) denote the \( x_i - y_i \) path \( Q_i \) for \( 1 \leq i \leq k \). Then the linear ordering

\[
s : x_1, \ldots, y_1, x_2, \ldots, y_2, \ldots, y_k, x_k, \ldots, y_k
\]

of vertices of \( G \) has the property that \( d(s) = t(G) = n + k - 2 \). Furthermore, \( d(s) \) contains exactly \( k - 1 \) terms, namely \( d(y_i, x_{i+1}) \) for \( 1 \leq i \leq k - 1 \), that equal \( 2 \) with all other terms equal to \( 1 \).

Observe that \( x_i x_j, x_i y_j, y_i y_j \notin E(G) \) for all \( i \) and \( j \) with \( 1 \leq i, j \leq k \) and \( i \neq j \), for otherwise \( G \) contains fewer than \( k \) vertex-disjoint paths whose vertex sets form a partition of \( V(G) \).
Next we claim that at most one of the paths $Q_i$ ($1 \leq i \leq k$) has order 2 or more, for suppose that there are two such paths, say $Q_1$ and $Q_2$. Let $s_0$ be a linear ordering of vertices of $G$ beginning with $x_1, x_2, y_1, y_2$ and containing the pairs $y_i, x_{i+1}$ ($2 \leq i \leq k - 1$) as consecutive terms. Then $d(s_0)$ contains at least $3 + (k - 2) = k + 1$ terms equal to 2. Thus

$$d(s_0) \geq 2(k + 1) + [(n - 1) - (k + 1)] = n + k,$$

which is a contradiction. Thus, as claimed, at most one of the paths $Q_i$ ($1 \leq i \leq k$) has order 2 or more, say $Q_1$. Since $G$ is connected and none of $x_ix_j, x_iy_j, y_iy_j$ are edges of $G$ for $i$ and $j$ with $1 \leq i, j \leq k$ and $i \neq j$, the path $Q_1$ has order 3 or more. If $Q_1$ has order 3, say $Q_1$ is the path $x_1, v, y_1$, then $vx_i \in E(G)$ for $2 \leq i \leq k$ and $x_1y_1 \notin E(G)$ and so $G = K_{1,n-1}$.

Suppose then that $Q_1$ has order 4 or more. Each of the vertices $x_i$ ($2 \leq i \leq k$) must be adjacent to an interior vertex of $Q_1$. Thus $x_1y_1 \notin E(G)$, for otherwise, $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. Indeed, we claim that each vertex $x_i$ ($2 \leq i \leq k$) must be adjacent to every interior vertex of $Q_1$; for assume, to the contrary, that some vertex $x_i$, say $x_2$, is not adjacent to the interior vertex $v$ of $Q_1$. Let $s^*$ be a linear ordering of vertices of $G$ beginning with $v, x_2, y_1, x_1, x_3, x_4, \ldots, x_k$. Then $d(s^*)$ contains at least $k + 1$ terms equal to 2. Thus

$$d(s^*) \geq 2(k + 1) + [(n - 1) - (k + 1)] = n + k,$$

which is a contradiction. Since $x_2$ is adjacent to all interior vertices of $Q_1$, there is a path in $G$ with vertex set $V(Q_1) \cup \{x_2\}$. However then, $G$ contains fewer than $k$ vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction.

4.3 Upper Traceable Numbers of Trees

In this section, we establish a formula for the traceable number of a tree. As an illustration, we first determine the upper traceable number of a path $P_n$ for $n \geq 2$. It was shown in [6] that

$$h^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor$$

(4.1)
for \( n \geq 2 \).

**Proposition 4.9** For each integer \( n \geq 2 \),

\[
    t^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1.
\]

**Proof.** Since \( h^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor \), it follows by Proposition 4.2 that \( t^+(P_n) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \).

To verify that \( t^+(P_n) \geq \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \), it suffices to show that there exists a linear ordering \( s \) of vertices of \( P_n \) for which \( d(s) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \). Let \( P_n : u_1, u_2, \ldots, u_n \) and we consider two cases, according to whether \( n \) is odd or \( n \) is even.

**Case 1.** \( n \) is odd. Then \( n = 2k + 1 \) for some positive integer \( k \). Let

\[
 s_0 : u_{k+1}, u_1, u_{2k+1}, u_2, u_{2k}, u_3, u_{2k-1}, \ldots, u_k, u_{k+2}
\]

be a linear ordering of vertices of \( P_n \). Since

\[
 d(s_0) = k + (2k) + (2k - 1) + (2k - 2) + \cdots + 2 \\
 = k + (1 + 2 + 3 + \cdots + 2k) - 1 = k + k(2k + 1) - 1 \\
 = k(2k + 2) - 1 = \frac{n^2 - 1}{2} - 1 = \left\lfloor \frac{n^2}{2} \right\rfloor - 1,
\]

it follows that \( t^+(P_n) \geq \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \). Thus \( t^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \) if \( n \) is odd.

**Case 2.** \( n \) is even. Then \( n = 2k \) for some integer \( k \geq 2 \). Let

\[
 s_1 : u_{k+1}, u_1, u_{2k}, u_2, u_{2k-1}, u_3, u_{2k-2}, \ldots, u_{k-1}, u_{k+2}, u_k
\]

be a linear ordering of vertices of \( P_n \). Since

\[
 d(s_1) = k + (2k - 1) + (2k - 2) + \cdots + 2 \\
 = k + [1 + 2 + 3 + \cdots + (2k - 1)] - 1 = k + k(2k - 1) - 1 \\
 = 2k^2 - 1 = \frac{n^2}{2} - 1 = \left\lfloor \frac{n^2}{2} \right\rfloor - 1,
\]

it follows that \( t^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \) if \( n \) is even.
it follows that \( t^+(P_n) \geq \left\lceil \frac{n^2}{2} \right\rceil - 1 \). Thus \( t^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \) if \( n \) is even.

We now consider trees in general. For each edge \( e \) of a tree \( T \), the component number \( cn(e) \) of \( e \) is defined in [6] as the minimum order of a component of \( T - e \). For example, the edge \( e_5 \) of the tree \( T \) of Figure 4.2(a) has component number 3 since the order of the smaller component of \( T - e_5 \) is 3. Each edge of this tree is labeled with its component number in Figure 4.2(b).

![Figure 4.2: Component numbers of edges](image)

An upper bound for the upper Hamiltonian number of a tree was established in [6] in terms of the component numbers of its edges, which we state as follows.

**Theorem 4.10** If \( T \) is a nontrivial tree, then

\[
h^+(T) \leq 2 \sum_{e \in E(T)} cn(e).
\]

For the tree \( T \) of Figure 4.2,

\[
\sum_{i=1}^{8} cn(e_i) = 1 + 1 + 3 + 1 + 4 + 1 + 2 + 1 = 14.
\]

Thus \( h^+(T) \leq 28 \) by Theorem 4.10. With the aid of Theorem 4.10 and Proposition 4.2, we are able to establish a formula for the upper traceable number of a tree.

**Theorem 4.11** If \( T \) is a nontrivial tree, then

\[
t^+(T) = 2 \sum_{e \in E(T)} cn(e) - 1.
\]

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Proof. By Theorem 4.10 and Proposition 4.2,

\[ t^+(T) \leq h^+(T) - 1 \leq 2 \sum_{e \in E(T)} \text{cn}(e) - 1. \]

Thus it remains to show that \( t^+(T) \geq 2 \sum_{e \in E(T)} \text{cn}(e) - 1 \). Since the theorem holds if \( T \) has order 2, we may assume that \( T \) has order 3 or more. Suppose that \( T_1 = T \) has order \( n \geq 3 \). Let \( v_2 \) be an end-vertex of \( T \). Furthermore, let \( Q_2 \) be a maximal path in \( T \) whose initial edge \( e_1 \) is incident with \( v_2 \) and such that each successive edge in \( Q_2 \) is chosen so that it has the maximum component number (among all edges available). Suppose that \( Q_2 \) is a \( v_2 - v_3 \) path. Necessarily, \( v_3 \) is an end-vertex of \( T \). Let \( T_2 = T - \{v_2\} \) and let \( Q_3 \) be a maximal path in \( T_2 \) whose initial edge \( e_2 \) is incident with \( v_3 \) and such that each successive edge in \( Q_3 \) is chosen so that it has the maximum component number in \( T_2 \) (among all edges available). We continue this process until we arrive at the \( v_{n-1} - v_n \) path \( Q_{n-1} \). The final vertex of \( T \) is denoted by \( v_1 \), which is necessarily adjacent to \( v_n \). Let \( e_{n-1} = v_n v_1 \). This procedure is illustrated in Figure 4.3 for the tree \( T \) of Figure 4.2, where each \( v_{i+1} - v_{i+2} \) path \( Q_{i+1} \) for \( 1 \leq i \leq n - 2 \) is indicated in bold.

![Figure 4.3: A step in the proof of Theorem 4.11](image-url)
For $2 \leq i \leq n - 2$, the edge $e_i$ is the initial edge of the $v_{i+1} - v_{i+2}$ path $Q_{i+1}$ in the tree $T_i = T - \{v_2, v_3, \ldots, v_i\}$. Furthermore, let $Q_1$ be the $v_1 - v_2$ path in $T = T_1$. Consider the linear ordering

$s : v_1, v_2, \ldots, v_n$

of vertices of $T$. We show that

$$d(s) = 2 \sum_{e \in E(T)} cn(e) - 1. \quad (4.2)$$

To verify (4.2), we show that for every integer $i$ with $1 \leq i \leq n - 2$, the edge $e_i$ is traversed $2cn(e_i)$ times by the paths $Q_1, Q_2, \ldots, Q_{n-1}$, while $e_{n-1}$ is traversed $2cn(e_{n-1}) - 1$ times by the paths $Q_1, Q_2, \ldots, Q_{n-1}$. It is certainly the case when an edge is a pendant edge, so suppose that $e$ is an edge of $T$ that is not a pendant edge.

For each tree $T_j$ containing $e$, let $T_{j,1}$ and $T_{j,2}$ be the components of $T_j - e$ such that $|V(T_{j,1})| \leq |V(T_{j,2})| + 1$. We claim that if the initial vertex $v_{j+1}$ of the path $Q_{j+1}$ belongs to $T_{j,1}$, then the terminal vertex $v_{j+2}$ belongs to $T_{j,2}$, that is, the edge $e$ is traversed by $Q_{j+1}$. Let $c_j = cnT_j(e)$ and $e = xy$ such that $x$ belongs to $T_j$.

If $|V(T_{j,1})| \leq |V(T_{j,2})|$, then note first that every edge in $T_{j,1}$ has component number at most $c_j - 1$. Assume, to the contrary, that the terminal vertex $v_{j+2}$ of the path $Q_{j+1}$ belongs to $T_{j,1}$. Let $Q_A : v_{j+1} = u_1, u_2, \ldots, u_k = x$ and $Q_B : v_{j+2} = w_1, w_2, \ldots, w_{\ell} = x$ be the $v_{j+1} - x$ path and $v_{j+2} - x$ path, respectively. Obviously, both $Q_A$ and $Q_B$ are entirely contained in $T_{j,1}$. Furthermore,

$$Q_{j+1} : v_{j+1} = u_1, u_2, \ldots, u_{k'} = w_{\ell'}, w_{\ell'-1}, \ldots, w_1 = v_{j+2}$$

for some integers $k'$ and $\ell'$ with $2 \leq k' \leq k$ and $2 \leq \ell' \leq \ell$. This implies that

$$cnT_j(u_{k'}u_{k'+1}) \leq cnT_j(w_{\ell'}w_{\ell'-1}).$$

On the other hand, however, observe that

$$cnT_j(u_{k'}u_{k'+1}) \geq cnT_j(u_{k'-1}u_{k'}) + cnT_j(w_{\ell'}w_{\ell'-1}) > cnT_j(w_{\ell'}w_{\ell'-1}),$$

a contradiction.
If \(|V(T_{ji,1})| = |V(T_{ji,2})| + 1\), then at most one edge in \(T_{ji,1}\) has component number \(c_j\) and each of the remaining edges in \(T_{ji,1}\) has component number at most \(c_j - 1\). Then by a similar argument given for the case where \(|V(T_{ji,1})| \leq |V(T_{ji,2})|\), if \(v_{j+1}\) belongs to \(T_{ji,1}\), then \(v_{j+2}\) must belong to \(T_{ji,2}\).

Now let \(T'\) and \(T''\) be the components of \(T - e\), where the order of \(T'\) is \(c = cn(e)\). Suppose that \(V(T') = \{v_{n_1}, v_{n_2}, \ldots, v_{n_c}\}\), where \(n_1 \leq n_2 \leq \cdots \leq n_c\). Furthermore, let \(e = xy\) such that \(x\) belongs to \(T'\). Necessarily then, \(x = v_{n_c}\). In each tree \(T_j\) containing \(e\), let \(T'_j\) and \(T''_j\) be the components of \(T_j - e\) containing \(x\) and \(y\), respectively. Then by the claim given above, we have the following:

(1) \(|V(T'_j)| \leq |V(T''_j)|\).

(2) \(v_1\) belongs to \(T''\).

(3) No two vertices of \(T'\) are consecutive in \(s\).

If \(x \neq v_n\), then \(e \neq e_{n-1}\). Since \(v_{n_1+1}, v_{n_2+1}, \ldots, v_{n_c+1}\) belong to \(T''\), it follows that \(e\) is traversed \(2c\) times by the paths \(Q_1, Q_2, \ldots, Q_{n-1}\). On the other hand, if \(x = v_n\), then \(e = e_{n-1}\). Since \(v_{n_1+1}, v_{n_2+1}, \ldots, v_{n_c+1+1}\) belong to \(T''\), it follows that \(e\) is traversed \(2c - 1\) times by the paths \(Q_1, Q_2, \ldots, Q_{n-1}\). Thus, as claimed, \(d(s) = 2 \sum_{e \in E(T)} cn(e) - 1\). Therefore,

\[t^+(T) \geq d(s) = 2 \sum_{e \in E(T)} cn(e) - 1,\]

providing the desired result.

Since \(h^+(T) \geq t^+(T) + 1\) for every nontrivial tree \(T\) by Proposition 4.2, the following corollary is a consequence of Theorems 4.10 and 4.11.

**Corollary 4.12** If \(T\) is a nontrivial tree, then

\[h^+(T) = 2 \sum_{e \in E(T)} cn(e).\]

We now illustrate Theorem 4.11 and Corollary 4.12. For the tree \(T\) of Figure 4.2, we have seen that \(\sum_{i=1}^{8} cn(e_i) = 14\). Thus by Theorem 4.11 and Corollary 4.12, \(t^+(T) = 28 - 1 = 27\) and \(h^+(T) = 28\). On the other hand, using
the technique described in the proof of Theorem 4.11, we obtain a linear ordering $s : v_1, v_2, \ldots, v_9$ of vertices of $T$ with $d(s) = t^+(T) = 27$. Observe that for the cyclic ordering $s_c : v_1, v_2, \ldots, v_9, v_1$ of vertices of $T$, $d(s_c) = h^+(T) = 28$.

Recall that upper and lower bounds for the upper Hamiltonian number of a tree were established in [6] in terms of its order, that is, if $T$ is a tree of order $n \geq 3$, then

$$2n - 2 \leq h^+(T) \leq \left\lceil \frac{n^2}{2} \right\rceil.$$ 

Moreover,

1. $h^+(T) = 2n - 2$ if and only if $T = K_{1,n-1}$;
2. $h^+(T) = \left\lceil \frac{n^2}{2} \right\rceil$ if and only if $T = P_n$.

Therefore, the following corollary is a consequence of Proposition 4.2, Theorem 4.11, and Corollary 4.12.

**Corollary 4.13** Let $T$ be a tree of order $n \geq 3$. Then

$$2n - 3 \leq t^+(T) \leq \left\lceil \frac{n^2}{2} \right\rceil - 1.$$ 

Furthermore,

1. $t^+(T) = 2n - 3$ if and only if $T = K_{1,n-1}$;
2. $t^+(T) = \left\lceil \frac{n^2}{2} \right\rceil - 1$ if and only if $T = P_n$.

If $G$ is a connected graph and $H$ is a connected spanning subgraph of $G$, then $d_G(u,v) \leq d_H(u,v)$ for all $u, v \in V(G) = V(H)$ and so $d_G(s) \leq d_H(s)$ for every linear ordering $s$ of vertices of $G$ (or $H$). Hence $t^+(G) \leq t^+(H)$. We state this observation as follows.

**Observation 4.14** Suppose that $G$ is a nontrivial connected graph and $H$ is a connected spanning subgraph of $G$. Then $t^+(G) \leq t^+(H)$. In particular, if $T$ is a spanning tree of $G$, then $t^+(G) \leq t^+(T)$. 

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We now present upper and lower bounds for $t^+(G)$ for a graph $G$ in terms of its order.

**Corollary 4.15** If $G$ is a connected graph of order $n \geq 3$, then

$$n - 1 \leq t^+(G) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1.$$  \hfill (4.3)

Furthermore,

1. $t^+(G) = n - 1$ if and only if $G = K_n$;
2. $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1$ if and only if $G = P_n$.

**Proof.** The inequalities in (4.3) and (1) follow by Observations 4.4 and 4.14 and Corollary 4.13. Thus, it remains only to verify (2). If $G = P_n$, then $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1$ by Corollary 4.13. For the converse, let $G$ be a connected graph of order $n \geq 3$ such that $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1$. If $G$ is a tree, then by Corollary 4.13, it follows that $G = P_n$. Now suppose that $G$ is not a tree and let $T$ be a spanning tree of $G$. By Observation 4.14,

$$t^+(G) \leq t^+(T) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1.$$  

Thus $t^+(T) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1$, implying that $T = P_n$. That is, every spanning tree of $G$ is isomorphic to $P_n$, implying that $G = C_n$. Then by Proposition 4.7,

$$\left\lfloor \frac{n^2}{2} \right\rfloor - 1 = t^+(G) = t^+(C_n) = \left\lfloor \frac{(n - 1)^2}{2} \right\rfloor .$$

However, this equality holds only for $n = 2$, a contradiction. Hence $G = P_n$ is the only connected graph of order $n \geq 3$ for which $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1$. \hfill ■

### 4.4 Realization Results

By Corollary 4.13, if $T$ is a tree of order $n \geq 3$, then

$$2n - 3 \leq t^+(T) \leq \left\lfloor n^2/2 \right\rfloor - 1.$$
Furthermore, by Theorem 4.11, if $T$ is a nontrivial tree, then

$$t^+(T) = 2 \sum_{e \in E(T)} cn(e) - 1,$$

where $cn(e)$ is the component number of the edge $e$ in $T$. For a nontrivial tree $T$, let

$$cn(T) = \sum_{e \in E(T)} cn(e).$$

Then $t^+(T) = 2cn(T) - 1$, that is, the upper traceable number of a nontrivial tree $T$ is always odd. Note that for $n = 3$, we have $2n - 3 = \left\lfloor \frac{n^2}{2} \right\rfloor - 1 = 3$ and $t^+(P_3) = 3$.

For $n = 4$, we have $2n - 3 = 5$ and $\left\lfloor \frac{n^2}{2} \right\rfloor - 1 = 7$, and note that each tree $T_{4,k}$ of order 4 with $k \in \{5, 7\}$ shown in Figure 4.4 has $t^+(T_{4,k}) = k$. Similarly for $n = 5$, we have $2n - 3 = 7$ and $\left\lfloor \frac{n^2}{2} \right\rfloor - 1 = 11$, and note that each tree $T_{5,k}$ of order 5 with $k \in \{7, 9, 11\}$ shown in Figure 4.4 has $t^+(T_{5,k}) = k$.

![Figure 4.4: Trees $T_{n,k}$ for $n = 4, 5$](image)

Next, we show that every pair $k, n$ of integers for which $n \geq 3$, $k$ is odd, and $2n - 3 \leq k \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1$ can be realized as the upper traceable number and order, respectively, of a tree. Recall that a double star with central vertices $u$ and $v$ such that $\deg u = a$ and $\deg v = b$ is denoted by $T = S_{a,b}$.

**Theorem 4.16** Let $n \geq 3$ be an integer and $k$ an odd integer such that

$$2n - 3 \leq k \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1.$$

Then there exists a tree $T$ of order $n$ for which $t^+(T) = k$. 

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Proof. Let $k = 2\ell - 1$, where $2 \leq n - 1 \leq \ell \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. We first consider the case where $n$ is odd. We have seen that the theorem holds for $n = 3$ and $n = 5$, so we may assume that $n = 2a + 1$, where $a \geq 3$. Hence $k = 2\ell - 1$, where $2a \leq \ell \leq a^2 + a$.

We now construct a caterpillar $T$ of order $n$ for which $cn(T) = \ell$, whose construction depends on the value of $\ell$.

**Case 1.** $\ell = 2a$. Let $T = K_{1,n-1}$ and observe that every edge of $K_{1,n-1}$ is a pendant edge. Hence $cn(K_{1,n-1}) = n - 1 = 2a$.

**Case 2.** $2a + 1 \leq \ell \leq 3a - 1$. Let $\ell = 2a + 1 + i$, where $0 \leq i \leq a - 2$. Let $T$ be a double star $S_{2+i,n-2-i}$ and $e$ the unique edge of $S_{2+i,n-2-i}$ that is not a pendant edge. Since $2 + i < n - 2 - i$, it follows that $cn(e) = 2 + i$. Hence

$$cn(S_{2+i,n-2-i}) = 2 + i + (n - 2) = n + i = 2a + 1 + i$$

for $0 \leq i \leq a - 2$.

**Case 3.** $3a \leq \ell \leq \frac{1}{2}(a^2 + a) + 2a - 1$. First consider the function

$$f(x) = -\frac{1}{2}x^2 + \frac{a+1}{2}x + 2a - 1$$

defined on the set $[1, a]$ (of real numbers). Observe that $f$ is continuous and strictly increasing on $[1, a]$. Let $b \in [1, a] \cap \mathbb{Z}$. Note that $f(b) \in \mathbb{Z}$, and

$$f(1) + 1 = 3a \quad \text{and} \quad f(a) = \frac{a^2 + a}{2} + 2a - 1.$$

Let

$$P : x_b, x_{b-1}, \ldots, x_1, x_0 = y_0, y_1, y_2$$

be a path of length $b + 2$. We first construct the caterpillar $T^*_b$ of order $n$ from $P$ by adding $2a - b - 2$ new end-vertices, $a - b$ of which are joined to $x_{b-1}$ and $a - 2$ of which are joined to $y_1$. If $b = 1$, then $T^*_1 = S_{a,a+1}$ and so

$$cn(T^*_1) = a + (n - 2) = 3a - 1 = f(1).$$

If $2 \leq b \leq a$, then let

$$N(x_{b-1}) = \{x_{b-2}, x_b, u_1, u_2, \ldots, u_{a-b}\}$$

$$N(y_1) = \{y_0, y_2, v_1, v_2, \ldots, v_{a-2}\}.$$
Observe that \( cn(x_0x_1) = cn(y_0y_1) = a \) and for \( 1 \leq i \leq b - 2 \),
\[
    cn(x_ix_{i+1}) = a - i.
\]
The remaining \( n - 1 - b \) edges are pendant edges; so
\[
    cn(T_b^*) = a + a + (a - 1) + (a - 2) + \ldots + (a - b + 2) + (a - 1 - b)
    = a + \frac{a(a + 1)}{2} - \frac{(a - b + 1)(a - b + 2)}{2} + 2a - b
    = a + \frac{(a - 1)(2a - b + 2)}{2} + 2a - b
    = f(b).
\]
Now, since \( f \) is strictly increasing on \([1, a]\) and \( f(1) + 1 \leq \ell \leq f(a) \), it follows that there exists a unique integer \( b \in [2, a] \) such that
\[
    f(b - 1) + 1 \leq \ell \leq f(b).
\]
Since \( f(b - 1) + 1 = f(b) - (a - b) \), it follows that \( \ell = f(b) - j \) for some \( j \) with \( 0 \leq j \leq a - b \). We construct \( T_{b,j} \) of order \( n \) from \( T_b^* \) by (i) first deleting the \( j \) vertices \( u_1, u_2, \ldots, u_j \) and then (ii) adding \( j \) new end-vertices \( w_1, w_2, \ldots, w_j \) and joining each of them to \( x_{b-2} \). Observe that
\[
    cn_{T_{b,j}}(e) = cn_{T_b^*}(e)
\]
for each edge \( e \in E(T_b^*) \cap E(T_{b,j}) - \{x_{b-2}x_{b-1}\} \) and
\[
    cn_{T_{b,j}}(x_{b-2}x_{b-1}) = a - b + 2 - j = cn_{T_b^*}(x_{b-2}x_{b-1}) - j.
\]
Since the \( j \) new edges are pendant edges,
\[
    cn(T_{b,j}) = cn(T_b^*) - j = f(b) - j = \ell.
\]

Case 4. \( \frac{1}{2}(a^2 + a) + 2a \leq \ell \leq a^2 + a \). First consider the function
\[
    g(x) = -\frac{1}{2} x^2 + 2a + 1 + \frac{a^2 + a}{2} x + \frac{a^2 + a}{2}
\]
defined on the set \([2, a]\) (of real numbers). Observe that \( g \) is continuous and strictly increasing on \([2, a]\). Let \( c \in [2, a] \cap \mathbb{Z} \). Note that \( g(c) \in \mathbb{Z} \) and
\[
    g(2) + 1 = \frac{a^2 + a}{2} + 2a \quad \text{and} \quad g(a) = a^2 + a.
\]
Let \( Q : x_a, x_{a-1}, \ldots, x_1, x_0 = y_0, y_1, \ldots, y_c \) be a path of length \( a + c \). We first construct the caterpillar \( T'_c \) of order \( n \) from \( Q \) by joining \( a - c \) new end-vertices \( u_1, u_2, \ldots, u_{a-c} \) to \( y_{c-1} \).

If \( c = 2 \), then \( T'_2 = T'_a \) and so

\[
\text{cn}(T'_2) = f(a) = \frac{a^2 + a}{2} + 2a - 1 = g(2).
\]

If \( 3 \leq c \leq a \), then observe that

\[
\text{cn}(x_i; x_{i+1}) = a - i \quad \text{for } 0 \leq i \leq a - 1
\]

\[
\text{cn}(y_i; y_{i+1}) = a - i \quad \text{for } 0 \leq i \leq c - 2,
\]

and the remaining \( n - 1 - (a + c - 1) \) edges are pendant edges. Hence

\[
\text{cn}(T'_c) = 2(1 + 2 + \cdots + a) - (1 + 2 + \cdots + a - c + 1) + (n - a - c)
\]

\[
= a(a + 1) - \frac{(a - c + 1)(a - c + 2)}{2} + a - c + 1
\]

\[
= g(c).
\]

Now, since \( g \) is strictly increasing on \([2, a]\) and \( g(2) + 1 \leq \ell \leq g(a) \), it follows that there exists a unique integer \( c \in [3, a] \) such that

\[
g(c - 1) + 1 \leq \ell \leq g(c).
\]

Since \( g(c - 1) + 1 = g(c) - (a - c) \), it follows that \( \ell = g(c) - j \) for some \( j \) with \( 0 \leq j \leq a - c \). We construct \( T'_{c,j} \) of order \( n \) from \( T'_c \) by (i) first deleting the \( j \) vertices \( u_1, u_2, \ldots, u_j \) and then (ii) adding \( j \) new end-vertices \( w_1, w_2, \ldots, w_j \) and joining each of them to \( y_{c-2} \). Observe that

\[
\text{cn}_{T'_{c,j}}(e) = \text{cn}_{T'_c}(e)
\]

for each edge \( e \in E(T'_c) \cap E(T'_{c,j}) \setminus \{y_{c-2}y_{c-1}\} \) and

\[
\text{cn}_{T'_{c,j}}(y_{c-2}y_{c-1}) = a - c + 2 - j = \text{cn}_{T'_c}(y_{c-2}y_{c-1}) - j.
\]

Since the \( j \) new edges are pendant edges,

\[
\text{cn}(T'_{c,j}) = \text{cn}(T'_c) - j = g(c) - j = \ell.
\]
A similar argument applies for the case where \( n \) is even. Hence for each odd integer \( k = 2\ell - 1 \) with \( 2n - 3 \leq k \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \), there exists a tree \( T \) for which \( cn(T) = \ell \) and so

\[
t^+(T) = 2\ell - 1 = k,
\]

providing the desired result.

We now illustrate the proof of Theorem 4.16 for \( n = 11 \) (so \( a = 5 \)). Since \( 19 \leq k = 2\ell - 1 \leq 59 \), it follows that \( 10 \leq \ell \leq 30 \). In this case,

\[
f(x) = -\frac{1}{2}x^2 + \frac{11}{2}x + 9 \quad \text{and} \quad g(x) = -\frac{1}{2}x^2 + \frac{11}{2}x + 15.
\]

<table>
<thead>
<tr>
<th>( b )</th>
<th>( f(b) )</th>
<th>( c )</th>
<th>( g(c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>2</td>
<td>24</td>
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<tr>
<td>2</td>
<td>18</td>
<td>3</td>
<td>27</td>
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<tr>
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<td>21</td>
<td>4</td>
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<tr>
<td>4</td>
<td>23</td>
<td>5</td>
<td>30</td>
</tr>
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</table>

There are four cases.

**Case 1.** \( \ell = 2a = 10 \). Let \( T = K_{1,10} \).

**Case 2.** \( 2a + 1 \leq \ell \leq 3a - 1 \), that is, \( 11 \leq \ell \leq 14 \). For \( \ell = 11 + i \), where \( 0 \leq i \leq 3 \), let \( T = S_{2+i,11-2-i} = S_{2+i,9-i} \). Thus, in this case, \( T \in \{ S_{2,9}, S_{3,8}, S_{4,7}, S_{5,6} \} \).

**Case 3.** \( 3a \leq \ell \leq \frac{3}{2}(a^2 + a) + 2a - 1 \), that is, \( 15 \leq \ell \leq 24 \). In this case, \( b \in \{ 2, 3, 4, 5 \} \).

If \( b = 2 \), then \( f(2) - 3 \leq \ell \leq f(2) \) and so the possible values of \( \ell \) are

\[
\ell = 15 = f(2) - 3 \quad \text{and so} \quad T = T_{2,3};
\ell = 16 = f(2) - 2 \quad \text{and so} \quad T = T_{2,2};
\ell = 17 = f(2) - 1 \quad \text{and so} \quad T = T_{2,1};
\ell = 18 = f(2) \quad \text{and so} \quad T = T^*_{2,0}.
\]

If \( b = 3 \), then \( f(3) - 2 \leq \ell \leq f(3) \) and so the possible values of \( \ell \) are

\[
\ell = 19 = f(3) - 2 \quad \text{and so} \quad T = T_{3,2};
\ell = 20 = f(3) - 1 \quad \text{and so} \quad T = T_{3,1};
\ell = 21 = f(3) \quad \text{and so} \quad T = T^*_{3,0}.
\]
If $b = 4$, then $f(4) - 1 \leq \ell \leq f(4)$ and so the possible values of $\ell$ are

\[ \ell = 22 = f(4) - 1 \quad \text{and so} \quad T = T_{4,1}; \]
\[ \ell = 23 = f(4) \quad \text{and so} \quad T = T_{4}^* = T_{4,0}. \]

If $b = 5$, then $\ell = 24 = f(5)$ and $T = T_{5}^* = T_{5,0}$.

Case 4. $\frac{1}{2}(a^2 + a) + 2a \leq \ell \leq a^2 + a$, that is, $25 \leq \ell \leq 30$. In this case, $c \in \{3, 4, 5\}$.

If $c = 3$, then $g(3) - 2 \leq \ell \leq g(3)$ and so the possible values of $\ell$ are

\[ \ell = 25 = g(3) - 2 \quad \text{and so} \quad T = T_{3,2}; \]
\[ \ell = 26 = g(3) - 1 \quad \text{and so} \quad T = T_{3,1}; \]
\[ \ell = 27 = g(3) \quad \text{and so} \quad T = T_{3}^* = T_{3,0}. \]

If $c = 4$, then $g(4) - 1 \leq \ell \leq g(4)$ and so the possible values of $\ell$ are

\[ \ell = 28 = g(4) - 1 \quad \text{and so} \quad T = T_{4,1}; \]
\[ \ell = 29 = g(4) \quad \text{and so} \quad T = T_{4}^* = T_{4,0}. \]

If $c = 5$, then $\ell = 30 = g(5)$ and $T = T_{5}^* = T_{5,0} = P_{11}$.

We have seen that if $T$ is a tree of order $n \geq 3$, then

\[ 2n - 2 \leq h^+(T) \leq \left\lfloor \frac{n^2}{2} \right\rfloor. \]

Furthermore, by Corollary 4.12, if $T$ is a nontrivial tree, then

\[ h^+(T) = 2 \sum_{e \in E(T)} cn(e) = t^+(T) + 1. \]

Thus the upper Hamiltonian number of a nontrivial tree is always even. Therefore, the following corollary is a consequence of Theorem 4.16.

**Corollary 4.17** Let $n \geq 3$ be an integer and $k$ an even integer such that

\[ 2n - 2 \leq k \leq \left\lfloor \frac{n^2}{2} \right\rfloor. \]

Then there exists a tree $T$ of order $n$ for which $h^+(T) = k$. 

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We now consider connected graphs in general. By Corollary 4.15, if $G$ is a connected graph of order $n$ with $t^{+}(G) = k$, then $n - 1 \leq k \leq \lfloor n^2/2 \rfloor - 1$. This gives rise to the following question:

**Problem 4.18** Which pairs $k, n$ of integers with $n \geq 3$ and $n - 1 \leq k \leq \lfloor n^2/2 \rfloor - 1$ are realizable as the upper traceable number and order, respectively, of some connected graph?

For each integer $n \geq 3$, let $G_n$ be the set of connected graphs of order $n$ and let

$$S_n = \left\{ k \in \mathbb{Z} : n - 1 \leq k \leq \lfloor n^2/2 \rfloor - 1 \right\}.$$  

Thus $t^{+}(G) \in S_n$ for every $G \in G_n$. Furthermore, let

$$S_n^+ = \left\{ t^{+}(G) : G \in G_n \right\}.$$ 

Therefore, $S_n^+ \subseteq S_n$ for each $n \geq 3$.

If $n = 3$, then $G_3 = \{K_3, P_3\}$. Since $t^{+}(K_3) = 2$ and $t^{+}(P_3) = 3$, it follows that $S_3^+ = S_3$. For $n = 4$, since

$$t^{+}(K_4) = 3, \quad t^{+}(K_4 - e) = 4, \quad t^{+}(K_{1,3}) = 5, \quad t^{+}(P_4) = 7,$$

it follows that $\{3, 4, 5, 7\} \subseteq S_4^+$. For $n = 5$, since

$$t^{+}(K_5) = 4, \quad t^{+}(K_5 - e) = 5, \quad t^{+}(K_5 - e - f) = 6, \quad t^{+}(K_{1,4}) = 7,$$

$$t^{+}(C_5) = 8, \quad t^{+}(S_{2,3}) = 9, \quad t^{+}(P_5) = 11,$$

it follows that $\{4, 5, 6, 7, 8, 9, 11\} \subseteq S_5^+$. In fact, it can be shown that

$$S_4^+ = \{3, 4, 5, 7\} = S_4 - \{6\}$$

$$S_5^+ = \{4, 5, 6, 7, 8, 9, 11\} = S_5 - \{10\}.$$ 

That is, if $n = 4, 5$, then $t^{+}(G) \neq \left\lfloor \frac{n^2}{2} \right\rfloor - 2$ for each $G \in G_n$. This is true in general, as we show next.

**Proposition 4.19** For every connected graph $G$ of order $n \geq 4$,

$$t^{+}(G) \neq \left\lfloor \frac{n^2}{2} \right\rfloor - 2.$$
Proof. Assume, to the contrary, that there exists a connected graph $G$ of order $n \geq 4$ for which $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 2$. Since $\left\lfloor \frac{n^2}{2} \right\rfloor - 2$ is even, $G$ is not a tree. Let $T$ be a spanning tree of $G$. By Observation 4.14,

$$t^+(G) \leq t^+(T) \leq \left\lceil \frac{n^2}{2} \right\rceil - 1.$$ 

Thus $t^+(T) = \left\lceil \frac{n^2}{2} \right\rceil - 1$ and so $T = P_n$. That is, every spanning tree of $G$ is isomorphic to $P_n$. Since $G$ is not a tree, this implies that $G = C_n$. By Proposition 4.7,

$$\left\lfloor \frac{n^2}{2} \right\rfloor - 2 = t^+(G) = t^+(C_n) = \left\lceil \frac{(n-1)^2}{2} \right\rceil.$$ 

However, this equality holds only for $n = 3$. \hfill \blacksquare

For each integer $n \geq 4$,

$$S^+_n \subseteq S_n - \left\{ \left\lfloor \frac{n^2}{2} \right\rfloor - 2 \right\}.$$ 

by Proposition 4.19. We will show in Chapter 6 that

$$S^+_n \neq S_n - \left\{ \left\lfloor \frac{n^2}{2} \right\rfloor - 2 \right\}$$

for $n \geq 6$. 

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Chapter 5

The Maximum Traceable Number of a Graph

5.1 Introduction

For a nontrivial connected graph $G$ and a vertex $v$ of $G$, recall that the traceable number $t(v)$ of $v$ is defined by

$$t(v) = \min\{d(s)\},$$

where the minimum is taken over all linear sequences $s$ of vertices of $G$ whose initial vertex is $v$. We have seen that the traceable number $t(G)$ of $G$ is

$$t(G) = \min\{t(v) : v \in V(G)\}$$

and the maximum vertex traceable number (or simply maximum traceable number) of $G$ is defined as

$$t^*(G) = \max\{t(v) : v \in V(G)\}.$$ 

Obviously, if $G$ is a nontrivial connected graph of order $n$, then

$$n - 1 \leq t^*(G) \leq (n - 1) \text{diam}(G). \quad (5.1)$$

By the definitions of $t(G)$, $t^*(G)$, and $t^+(G)$, we have the following observation.
**Observation 5.1**  For every nontrivial connected graph $G$,

$$t(G) \leq t^*(G) \leq t^+(G).$$  \hfill (5.2)

For example, consider the graph $G$ shown below in Figure 5.1. Each vertex of $G$ is labeled by its vertex traceable number. Thus $t(G) = 4$ and $t^*(G) = 5$. Furthermore, $t^+(G) = 9$. Observe that the diameter of $G$ is 3 and there is only one pair of vertices of distance 3, namely $d(u,y) = 3$. Thus $t^+(G) \leq 9$. On the other hand, for the linear sequence $s : x, w, u, y, v$ of vertices of $G$, $d(s) = 9$ and so $t^+(G) = 9$.

![Graph G with $t(G) = 4$, $t^*(G) = 5$, and $t^+(G) = 9$](image.png)

**Figure 5.1:** A graph $G$ with $t(G) = 4$, $t^*(G) = 5$, and $t^+(G) = 9$

Let $G$ be a nontrivial connected graph of order $n$. By Observation 4.4, $t(G) = t^+(G)$ if and only if $G = K_n$. Furthermore, by Theorem 7.10, $t^+(G) = t(G) = 1$ if and only if $G = K_n - e$ or $G = K_{1,n-1}$. We first determine the maximum traceable numbers of these graphs.

**Proposition 5.2**  For each integer $n \geq 3$,

$$t^*(K_n) = n - 1$$

$$t^*(K_n - e) = \begin{cases} 
n & \text{if } n = 3 \\
n - 1 & \text{if } n \geq 4 \end{cases}$$

$$t^*(K_{1,n-1}) = 2n - 3.$$  

**Proof.**  Since $t(v) = n - 1$ for each vertex $v \in V(K_n)$, it follows that $t^*(K_n) = n - 1$. Let $G = K_n - e$. If $n = 3$, then $G = K_3 - e = P_3$. Let $G = P_3 : v_1, v_2, v_3$ and observe
that \( t(v_1) = t(v_3) = 2 \) and \( t(v_2) = 3 \). Hence \( t^*(K_3 - e) = 3 = n \). If \( n \geq 4 \), then \( t(v) = n - 1 \) for every \( v \in V(K_n - e) \), implying that \( t^*(K_n - e) = n - 1 \). Next let \( G = K_{1,n-1} \). If \( v \) is the central vertex in \( G \), then \( t(v) = 2(n-2)+1 = 2n-3 \); while if \( v \) is an end-vertex of \( G \), then \( t(v) = 2(n-3)+2 = 2n-4 \). Thus \( t^*(K_{1,n-1}) = 2n-3 \). 

The following is a consequence of Observation 4.4, Theorem 7.10, and Proposition 5.2.

**Corollary 5.3** Let \( G \) be a connected graph of order \( n \geq 3 \). Then

1. \( t(G) = t^*(G) = t^+(G) \) if and only if \( G = K_n \);
2. \( t(G) = t^*(G) = t^+(G) - 1 \) if and only if \( G = K_n - e \) where \( n \geq 4 \);
3. \( t(G) + 1 = t^*(G) = t^+(G) \) if and only if \( G = K_{1,n-1} \).

We have seen in Corollary 3.27 that if \( G \) is a connected graph and \( k \) is an integer with \( t(G) \leq k \leq t^*(G) \), then there exists a vertex \( w \) of \( G \) such that \( t(w) = k \). Thus if we define the *traceable vertex spectrum* of a connected graph \( G \) as

\[
\text{spec}(G) = \{ t(v) : v \in V(G) \}
\]

and let \([a, b]\) denote the set of integers inclusively between the integers \( a \) and \( b \), then

\[
\text{spec}(G) = [t(G), t^*(G)].
\]

In particular, if \( t(G) = t^*(G) \), then \( \text{spec}(G) \) is a singleton and so \( t(v) \) is a constant for all \( v \in V(G) \). For this reason, a connected graph \( G \) is defined to be *traceably singular* if \( t(G) = t^*(G) \). We first present a class of traceably singular graphs.

A graph \( G \) is *vertex-transitive* if for every two vertices \( u, v \in V(G) \), there exists an automorphism \( \phi : V(G) \to V(G) \) such that \( \phi(u) = v \). For example, the Petersen graph \( P \), complete graphs \( K_n \), and cycles \( C_n \) are vertex-transitive. Suppose that \( G \) is a nontrivial connected vertex-transitive graph and let \( v \in V(G) \) such that \( t(v) = t(G) \). Then \( G \) contains a spanning walk

\[
W : v = v_0, v_1, v_2, \ldots, v_{t(G)}
\]

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of length \( t(G) \) whose initial vertex is \( v \). Let \( u \in V(G) \) be a vertex in \( G \) distinct from \( v \), and let \( \phi : V(G) \to V(G) \) be an automorphism such that \( u = \phi(v) \). Then \( G \) contains a spanning walk

\[
W' : u = \phi(v) = \phi(v_0), \phi(v_1), \phi(v_2), \ldots, \phi(v_{t(G)})
\]

of length \( t(G) \) whose initial vertex is \( u \). That is,

\[
t(v) = t(G) \quad \text{for all } v \in V(G).
\]

Therefore, we have the following observation.

**Observation 5.4** Every connected vertex-transitive graph is traceably singular.

By Observation 5.4, if \( G \) is the Petersen graph, a complete graph, a complete bipartite graph, or a cycle, then \( G \) is traceably singular. On the other hand, the converse of Observation 5.4 is not true. For example, the graph \( K_n - e \) \((n \geq 4)\) is not vertex-transitive, yet \( t(K_n - e) = t^*(K_n - e) \) by Corollary 5.3 and so \( K_n - e \) is traceably singular.

Next, we present a necessary condition for connected graphs to be traceably singular. The *connectivity* \( \kappa(G) \) of a graph \( G \) is the minimum value of \(|U|\) among all subsets \( U \) of \( V(G) \) such that \( G - U \) is either disconnected or trivial. For a positive integer \( k \), a graph \( G \) is \( k \)-connected if \( \kappa(G) \geq k \).

**Proposition 5.5** Every connected traceably singular graph of order 3 or more is 2-connected.

**Proof.** Let \( G \) be a connected graph of order \( n \geq 3 \). If \( n = 3, 4 \), then \( G \) is traceably singular if and only if \( G \) is Hamiltonian. Thus the result holds for \( n = 3, 4 \). We now assume that \( G \) is a connected graph of order \( n \geq 5 \) that is not 2-connected. Then \( G \) contains cut-vertices. Let \( v \) be a cut-vertex of \( G \). We show that \( t(G) < t^*(G) \). Let

\[
s : v = v_1, v_2, \ldots, v_n
\]

be a linear ordering of vertices of \( G \) whose initial vertex is \( v \) and such that \( t(v) = d(s) \). Since \( v \) is a cut-vertex, \( G - v \) contains at least two components. Let \( H \) be the component of \( G - v \) that contains \( v_2 \) and

\[
j = \min\{i : v_i \notin V(H) \cup \{v\}\}.
\]

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Since $v_j \neq v$, it follows that $3 \leq j \leq n$. Moreover, since $v_{j-1} \in V(H)$ and $v_j \notin V(H)$, it follows that every $v_{j-1} - v_j$ geodesic contains $v$. Thus

$$d(v_{j-1}, v_j) = d(v_{j-1}, v) + d(v, v_j). \quad (5.3)$$

Now consider the linear ordering

$$s' : v_2, \ldots, v_{j-1}, v, v_j, \ldots, v_n$$

of vertices of $G$ whose initial vertex is $v_2$. If $j = 3$, then by (5.3)

$$t(v_2) \leq d(s') = d(v_2, v) + d(v, v_3) + \sum_{i=3}^{n-1} d(v_i, v_{i+1})$$

$$= d(v_2, v_3) + \sum_{i=3}^{n-1} d(v_i, v_{i+1}) = \sum_{i=2}^{n-1} d(v_i, v_{i+1}) = t(v) - d(v, v_2)$$

$$< t(v).$$

If $4 \leq j \leq n - 1$, then by (5.3)

$$t(v_2) \leq d(s') = \left[ \sum_{i=2}^{j-2} d(v_i, v_{i+1}) \right] + d(v_{j-1}, v) + d(v, v_j) + \left[ \sum_{i=j}^{n-1} d(v_i, v_{i+1}) \right]$$

$$= \left[ \sum_{i=2}^{j-2} d(v_i, v_{i+1}) \right] + d(v_{j-1}, v_j) + \left[ \sum_{i=j}^{n-1} d(v_i, v_{i+1}) \right]$$

$$= \sum_{i=2}^{n-1} d(v_i, v_{i+1}) = t(v) - d(v, v_2) < t(v).$$

Finally if $j = n$, then by (5.3)

$$t(v_2) \leq d(s') = \left[ \sum_{i=2}^{n-2} d(v_i, v_{i+1}) \right] + d(v_{n-1}, v) + d(v, v_n)$$

$$= \left[ \sum_{i=2}^{n-2} d(v_i, v_{i+1}) \right] + d(v_{n-1}, v_n) = \sum_{i=2}^{n-1} d(v_i, v_{i+1}) = t(v) - d(v, v_2)$$

$$< t(v).$$

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In each case we obtain
\[ t(G) \leq t(v_2) < t(v) \leq t^*(G), \]
providing the desired result.

The converse of Proposition 5.5 is false. For example, the graph \( K_{2,3} \) is 2-connected, but \( t(K_{2,3}) = 4 \) while \( t^*(K_{2,3}) = 5 \). Thus \( K_{2,3} \) is not traceably singular.

On the other hand, let \( P - e \) be the graph obtained from the Petersen graph \( P \) by removing an edge \( e \). Then \( P - e \) is 2-connected. Since \( t(P - e) = t^*(P - e) = 9 \), it follows that \( P - e \) is traceably singular. Thus there are 2-connected graphs that are traceably singular and 2-connected graphs that are not traceably singular.

### 5.2 Bounds for the Maximum Traceable Number

We have seen that if \( G \) is a connected graph and \( H \) is a connected spanning subgraph of \( G \), then \( d_G(u,v) \leq d_H(u,v) \) for all \( u, v \in V(G) = V(H) \). Thus if \( v \in V(G) \), then for every linear ordering \( s : v = v_1, v_2, \ldots, v_n \) of vertices of \( G \) (or \( H \)) whose initial vertex is \( v \), we have \( d_G(s) \leq d_H(s) \) and so \( t_G(v) \leq t_H(v) \). Hence
\[
\begin{align*}
    t^*(G) &= \max\{t_G(v) : v \in V(G)\} \\
          &\leq \max\{t_H(v) : v \in V(H) = V(G)\} \\
          &= t^*(H).
\end{align*}
\]

We state this observation as follows.

**Observation 5.6** Suppose that \( G \) is a nontrivial connected graph and \( H \) is a connected spanning subgraph of \( G \), and let \( v \in V(G) = V(H) \). Then \( t_G(v) \leq t_H(v) \), and therefore \( t^*(G) \leq t^*(H) \). In particular, if \( G \) is a connected graph and \( T \) is a spanning tree of \( G \), then \( t^*(G) \leq t^*(T) \).

Therefore, it is useful to know the maximum traceable number of nontrivial trees. We have seen that if \( T \) is a nontrivial tree of order \( n \), then
\[
t(T) = 2n - 2 - \text{diam}(T). \tag{5.4}
\]
Recall that we presented two proofs to establish the formula (5.4). The second proof employed the fact that if \( T \) is a nontrivial tree of order \( n \) and \( v \in V(T) \), then
\[
t(v) = 2n - 2 - e(v). \tag{5.5}
\]
Again, with the aid of this fact, we now present a formula for the maximum traceable number of a tree in terms of its order and radius.

**Proposition 5.7** If $T$ is a nontrivial tree of order $n$, then

$$t^*(T) = 2n - 2 - \text{rad}(T).$$

**Proof.** Since $t(v) = 2n - 2 - e(v)$ for $v \in V(T)$ by (5.5), it follows that

$$t^*(T) = \max\{2n - 2 - e(v) : v \in V(T)\} = 2n - 2 - \min\{e(v) : v \in V(T)\} = 2n - 2 - \text{rad}(T),$$

which gives us the desired result. ■

**Corollary 5.8** If $T$ is a tree of order $n$ and $v$ is a vertex of $T$, then $t(v) = t^*(T)$ if and only if $e(v) = \text{rad}(T)$.

It is well known that if $T$ is a nontrivial tree of order $n$, then

$$1 \leq \text{rad}(T) \leq \left\lfloor \frac{n - 1}{2} \right\rfloor.$$

This fact together with Proposition 5.7 provides us with upper and lower bounds for $t^*(T)$, which we state in the following corollary.

**Corollary 5.9** Let $T$ be a tree of order $n \geq 3$. Then

$$2n - 2 - \left\lfloor \frac{n - 1}{2} \right\rfloor \leq t^*(T) \leq 2n - 3.$$

Since $\text{rad}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$ and $\text{rad}(K_{1,n-1}) = 1$ for $n \geq 3$, it follows by Proposition 5.7 that

$$t^*(P_n) = 2n - 2 - \left\lfloor \frac{n - 1}{2} \right\rfloor \quad \text{and} \quad t^*(K_{1,n-1}) = 2n - 3.$$
Thus both upper and lower bounds in Corollary 5.9 are sharp. Furthermore, for each pair \( r, n \) of integers such that \( n \geq 3 \) and

\[
1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor,
\]

there exists a tree \( T \) of order \( n \) with \( \text{rad}(T) = r \) and so

\[
t^*(T) = 2n - 2 - r.
\]

This observation yields the following realization result.

**Proposition 5.10** For each pair \( n, k \) of integers with \( n \geq 3 \) and

\[
2n - 2 - \left\lfloor \frac{n-1}{2} \right\rfloor \leq k \leq 2n - 3,
\]

there exists a tree \( T \) of order \( n \) with \( t^*(T) = k \).

The following is an immediate consequence of Observation 5.6, Corollary 5.9, and the inequality in (5.1).

**Corollary 5.11** If \( G \) is a connected graph of order \( n \geq 3 \), then

\[
n - 1 \leq t^*(G) \leq 2n - 3.
\]

In fact, both upper and lower bounds in (5.6) are sharp. In order to show this, we present an additional definition. A graph \( G \) is defined in [4] to be **vertex-traceable** if every vertex of \( G \) is the initial vertex of a Hamiltonian path of \( G \). Thus a graph \( G \) of order \( n \) is vertex-traceable if and only if the traceable number of every vertex of \( G \) is \( n - 1 \). In this case, \( t(G) = t^*(G) = n - 1 \) and so \( G \) is traceably singular. We state this observation as follows.

**Observation 5.12** Every vertex-traceable graph is traceably singular.

We know of no traceably singular graph that is not vertex-traceable.
Theorem 5.13  Let $G$ be a connected graph of order $n \geq 3$. Then

(1) $t^*(G) = n - 1$ if and only if $G$ is vertex traceable;

(2) $t^*(G) = 2n - 3$ if and only if $G = K_{1,n-1}$.

Proof.  We have seen that (1) is true and so it remains to verify (2). If $G = K_{1,n-1}$, then it follows by Proposition 5.7 that $t^*(K_{1,n-1}) = 2n - 3$. For the converse, let $G$ be a graph of order $n \geq 3$ for which $t^*(G) = 2n - 3$. If $G$ is a tree, then by Proposition 5.7,

$$t^*(G) = 2n - 2 - \text{rad}(G).$$

Thus $\text{rad}(G) = 1$, implying that $G = K_{1,n-1}$. Now suppose that $G$ is not a tree and let $T$ be a spanning tree of $G$. By Observation 5.6 and Proposition 5.7,

$$2n - 3 = t^*(G) < t^*(T) < 2n - 3$$

and so $t^*(T) = 2n - 3$. Then by the above argument, $T = K_{1,n-1}$. That is, $T = K_{1,n-1}$ is the unique spanning tree of $G$. If $n = 3$, then $G = K_3$ is the only connected graph that is not a tree and

$$t^*(K_3) = 2 < 2 \cdot 3 - 3,$$

a contradiction. Hence $n \geq 4$. Let $V(G) = \{v, u_1, u_2, \ldots, u_{n-1}\}$, where $\text{deg}_T(v) = n - 1$. Since $G$ is not a tree, $G$ contains a cycle. Hence there are two vertices $u_i$ and $u_j$ ($1 \leq i < j \leq n - 1$) such that $u_iu_j \in E(G)$. Without loss of generality, suppose that $u_1u_2 \in E(G)$. Then $G$ contains the spanning tree $T'$ such that

$$E(T') = [E(T) \cup \{u_1u_2\}] - \{vu_1\}.$$

Since $T'$ is a double star, this is a contradiction. Hence $G = K_{1,n-1}$ is the only connected graph of order $n \geq 3$ such that $t^*(G) = 2n - 3$.  

By Corollary 5.11, if $G$ is a connected graph of order $n$ with $t^*(G) = k$, then $n - 1 \leq k \leq 2n - 3$. Next, we show that every pair $k, n$ of integers with $n \geq 3$ and $n - 1 \leq k \leq 2n - 3$ is realizable as the maximum traceable number and order, respectively, of some connected graph.

Theorem 5.14  For each pair $k, n$ of integers with $n \geq 3$ and $n - 1 \leq k \leq 2n - 3$, there exists a connected graph $G$ of order $n$ such that $t^*(G) = k$. 

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Proof. Let \( n \geq 3 \) and \( k \) be integers such that \( n - 1 \leq k \leq 2n - 3 \). Then
\[
k = 2n - 2 - r
\]
for some integer \( r \), where \( 1 \leq r \leq n - 1 \). By Proposition 5.10, if \( 1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), then there exists a tree \( T \) of order \( n \) such that \( t^*(T) = 2n - 2 - r = k \). Also, if \( r = n - 1 \), then by Observation 5.4,
\[
t^*(C_n) = n - 1 = 2n - 2 - (n - 1) = 2n - 2 - r = k
\]
since \( C_n \) is a traceable vertex-transitive graph. Hence we may assume that \( \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \leq r \leq n - 2 \). We consider two cases.

Case 1. \( n \) is odd. Let \( n = 2a + 1 \), where \( a \) is a positive integer. Observe that
\[
a + 1 = \left\lceil \frac{n-1}{2} \right\rceil + 1 \leq r \leq n - 2 = 2a - 1.
\]
Hence \( a \geq 2 \). Let \( b \) be an integer such that
\[
r = 2a - b.
\]
Then \( 1 \leq b \leq a - 1 \). Note that \( n - 2b \geq (2a + 1) - 2(a - 1) = 3 \). For each integer \( i \) with \( 1 \leq i \leq b \), let \( Q_i : x_i, y_i \) be a copy of \( K_2 \). We construct a graph \( G \) of order \( n \) from \( Q_1, Q_2, \ldots, Q_b \) and a cycle \( C : v, v_1, v_2, \ldots, v_{n-2b-1}, v_{n-2b} = v \) of order \( n - 2b \) by joining each vertex of \( Q_i \) to \( v \) (\( 1 \leq i \leq b \)). Observe that for all \( i \) (\( 1 \leq i \leq b \)) and for all \( j \) (\( 1 \leq j \leq n - 2b - 1 \)),
\[
t(x_i) = t(y_i) = 2 + 3(b - 1) + (n - 2b - 1) = n + b - 2;
\]
\[
t(v_j) = (n - 2b - j) + 3b + (j - 1) = n + b - 1;
\]
\[
t(v) = 3b + (n - 2b - 1) = n + b - 1.
\]
Hence
\[
t^*(G) = n + b - 1 = (2a + 1) + b - 1 = 2(2a + 1) - 2 - (2a - b) = 2n - 2 - r = k.
\]

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Case 2. $n$ is even. Let $n = 2a$, where $a$ is an integer with $a \geq 2$. Observe that

$$a + 1 = \left\lfloor \frac{n - 1}{2} \right\rfloor + 1 \leq n - 2 = 2a - 2.$$ 

Hence $a \geq 3$. Let $b$ be an integer such that

$$r = 2a - 1 - b.$$ 

Then $1 \leq b \leq a - 2$. Note that $n - 2b \geq 2a - 2(a - 2) = 4 > 3$. For each integer $i$ with $1 \leq i \leq b$, let $Q_i : x_i, y_i$ be a copy of $K_2$. We construct a graph $G$ of order $n$ from $Q_1, Q_2, \ldots, Q_b$ and a cycle $C : v, v_1, v_2, \ldots, v_{n-2b-1}, v_{n-2b} = v$ of order $n - 2b$ by joining each vertex of $Q_i$ to $v$. Observe that for all $i$ ($1 \leq i \leq b$) and for all $j$ ($1 \leq j \leq n - 2b - 1$),

$$t(x_i) = t(y_i) = 2 + 3(b - 1) + (n - 2b - 1) = n + b - 2;$$

$$t(v_j) = (n - 2b - j) + 3b + (j - 1) = n + b - 1;$$

$$t(v) = 3b + (n - 2b - 1) = n + b - 1.$$ 

Hence

$$t^*(G) = n + b - 1 = 2a + b - 1$$

$$= 2(2a) - 2 - (2a - 1 - b)$$

$$= 2n - 2 - r = k.$$ 

Therefore, for each integer $k$ with $n - 1 \leq k \leq 2n - 3$, there exists a connected graph $G$ of order $n$ for which $t^*(G) = k$. □

We have seen in (5.1) that if $G$ is a nontrivial connected graph of order $n$, then

$$n - 1 \leq t^*(G) \leq (n - 1) \text{diam}(G)$$ 

and that $t^*(G) = n - 1$ if and only if $G$ is vertex traceable by Theorem 5.13. With the aid of Corollary 5.11, we show next that the complete graph $K_n$ is the only connected graph $G$ of order $n$ for which $t^*(G) = (n - 1) \text{diam}(G)$.

**Proposition 5.15** Let $G$ be a connected graph of order $n \geq 1$. Then

$$t^*(G) = (n - 1) \text{diam}(G) \text{ if and only if } G = K_n.$$
Proof. Suppose that \( G = K_n \). Observe that \( t^*(K_1) = 0 \) and \( t^*(K_2) = 1 \). Also by Proposition 5.2, we know \( t^*(K_n) = n - 1 \) for \( n \geq 3 \). Hence \( t^*(K_n) = n - 1 \) for every \( n \geq 1 \). Since \( \text{diam}(K_1) = 0 \) and \( \text{diam}(K_n) = 1 \) for \( n \geq 2 \), it follows that for every \( n \geq 1 \),

\[
(n - 1) \text{diam}(K_n) = \begin{cases} 
(1 - 1) \cdot 0 & \text{if } n = 1 \\
(n - 1) \cdot 1 & \text{if } n \geq 2
\end{cases}
\]

\[= n - 1 = t^*(K_n).\]

For the converse, let \( G \) be a connected graph of order \( n \geq 1 \) for which

\[t^*(G) = (n - 1) \text{diam}(G). \tag{5.7}\]

Note that \( K_1 \) and \( K_2 \) are the only connected graphs of order 1 and 2, respectively, and both graphs satisfy the equation (5.7). Hence we may assume that \( n \geq 3 \). Then \( \text{diam}(G) \geq 1 \). By Corollary 5.11,

\[(n - 1) \text{diam}(G) \leq 2n - 3 = 2(n - 1) - 1\]

and therefore

\[1 \leq (n - 1) [2 - \text{diam}(G)].\]

Since \( n - 1 \geq 2 \), this implies that \( 2 - \text{diam}(G) > 0 \). Hence \( \text{diam}(G) = 1 \). Since \( K_n \) is the only graph of order \( n \) having diameter \( 1 \), we have the desired result.

The results obtained in this section leave us the following question.

**Problem 5.16** If \( G \) is a traceably singular graph of order \( n \), is it the case that \( \text{spec}(G) = \{n - 1\} \)? Equivalently, is every traceably singular graph vertex-traceable?

### 5.3 Two Special Classes of Unicyclic Graphs

In general, it is difficult to determine the maximum traceable number of an arbitrary graph. To illustrate this, we consider the maximum traceable number of a unicyclic graph. A unicyclic graph can be constructed by adding an edge to a tree of order at least 3. Although we have established formulas for the maximum traceable numbers of trees and cycles, there is no general formula for the maximum traceable number of a unicyclic graph. By Proposition 5.7 and Theorem 5.13,
\[ t^*(P_n) = 2n - 2 - \left\lceil \frac{n-1}{2} \right\rceil \quad \text{and} \quad t^*(C_n) = n - 1 \]

for all \( n \geq 3 \). Next, we determine the maximum traceable number of two special classes of unicyclic graphs.

Let \( n \) and \( k \) be integers such that \( k \geq 3 \) and \( n \geq k + 1 \), and let \( G_{n,k} \) be the unicyclic graph of order \( n \) obtained from a cycle \( C \) of order \( k \) and a path \( P \) of order \( n - k + 1 \) by identifying one of the \( k \) vertices on \( C \) and one of the two end-vertices of \( P \). Note that the smaller the value of \( k \) (the closer the value of \( k \) is to 3), the closer \( G_{n,k} \) is to being \( P_n \), and the larger the value of \( k \) (the closer the value of \( k \) is to \( n - 1 \)), the closer \( G_{n,k} \) is to being \( C_n \). We next consider the maximum traceable number of \( G_{n,k} \).

**Observation 5.17** Let \( n \) and \( k \) be integers such that \( k \geq 3 \) and \( n \geq k + 1 \). Then

\[
\text{diam}(G_{n,k}) = n - \left\lceil \frac{k}{2} \right\rceil; \\
\text{rad}(G_{n,k}) = \begin{cases} 
\left\lceil \frac{n-1}{2} \right\rceil & \text{if } n \geq \left\lceil \frac{3k-2}{2} \right\rceil \\
\left\lfloor \frac{k}{2} \right\rfloor & \text{if } k + 1 \leq n \leq \left\lceil \frac{3k-2}{2} \right\rceil.
\end{cases}
\]

For the graphs \( G_{15,6} \), \( G_{15,10} \), and \( G_{15,13} \) shown in Figure 5.2,

\[
\text{diam}(G_{15,6}) = 12, \quad \text{rad}(G_{15,6}) = 6; \\
\text{diam}(G_{15,10}) = 10, \quad \text{rad}(G_{15,10}) = 4; \\
\text{diam}(G_{15,13}) = 8, \quad \text{rad}(G_{15,13}) = 6.
\]

In each of the graphs \( G_{15,6} \), \( G_{15,10} \), and \( G_{15,13} \), the end-vertex is a peripheral vertex and each solid vertex is a central vertex of the graph.

**Proposition 5.18** Let \( n \) and \( k \) be integers such that \( k \geq 3 \) and \( n \geq k + 1 \). Then

\[
t^*(G_{n,k}) = \begin{cases} 
\left\lceil \frac{3n-k-1}{2} \right\rceil & \text{if } n \geq 2k - 3 \\
\left\lfloor \frac{2n+k-4}{2} \right\rfloor & \text{if } \left\lceil \frac{3k-2}{2} \right\rceil \leq n \leq 2k - 3 \\
2n - k - 1 & \text{if } k + 1 \leq n \leq \left\lceil \frac{3k-2}{2} \right\rceil.
\end{cases}
\]
Proof. We consider two cases, according to whether $k$ is odd or $k$ is even.

Case 1. $k$ is even. Let $k = 2b + 2$, where $b \geq 1$. We consider two subcases, according to whether $n$ is odd or $n$ is even.

Subcase 1.1. $n$ is odd. Let $n = 2a + 1$, where $a \geq b + 1 \geq 2$. Then the length of $P$ is $(2a + 1) - (2b + 2) = 2a - 2b - 1$. Let

$$P : v_0, v_1, \ldots, v_{2a-2b-1} = u$$
$$C : u, w_0, w_1, \ldots, w_{2b}, u.$$ 

We next determine the values of $t(v_i) (0 \leq i \leq 2a - 2b - 1)$ and $t(w_j) (0 \leq j \leq 2b)$. By symmetry, we may assume that $0 \leq j \leq b$. Observe that $t(v_0) = t(w_0) = n - 1 = 2a$. For $1 \leq i \leq 2a - 2b - 1$, we see that either of the two linear orderings

$$s_i : v_i, v_{i+1}, \ldots, v_{2a-2b-1}, w_0, w_1, \ldots, w_{2b}$$
$$s'_i : v_i, v_{i+1}, \ldots, v_{2a-2b-1}, w_0, w_1, \ldots, w_{2b}, v_{i-1}, \ldots, v_0$$
gives us $t(v_i)$. Hence

$$t(v_i) = \min\{n-1+i, 2(2a-2b-1) + (2b+2) - i\}$$

$$= \min\{2a+i, 4a-2b-i\}$$

and so

$$\max_{0 \leq i \leq 2n-2k-1} t(v_i) = t(v_{a-b}) = 3a-4 = 2a + (a-b).$$

On the other hand, for $1 \leq j \leq b$, either of the two linear orderings

$$s^j_1: w_j, w_{j+1}, \ldots, w_{2b}, u, w_0, w_1, \ldots, w_{j-1}, v_{2a-2b-2}, v_{2a-2b-3}, \ldots, v_1, v_0$$

$$s^j_2: w_j, w_{j+1}, \ldots, w_{2b}, u = v_{2a-2b-1}, v_{2a-2b-2}, \ldots, v_1, v_0, w_0, w_1, \ldots, w_{j-1}$$

gives us $t(w_j)$. Hence

$$t(w_j) = (2b+2) - (j+1) + 2 \min\{j, 2a-2b-1\} + \max\{j, 2a-2b-1\}$$

$$= 2b - j + 1 + (j + 2a - 2b - 1) + \min\{j, 2a-2b-1\}$$

$$= 2a + \min\{j, 2a-2b-1\}$$

and so

$$\max_{0 \leq j \leq b} t(w_j) = t(w_b) = 2a + \min\{b, 2a-2b-1\}.$$ 

Therefore,

$$t^*(G_{a,k}) = \max\{2a + (a-b), 2a + \min\{b, 2a-2b-1\}\}$$

$$= 2a + \max\{a-b, \min\{b, 2a-2b-1\}\}.$$ 

Observe that

$$\left\lfloor \frac{3n-k-1}{2} \right\rfloor = \frac{3(2a+1) - (2b+2) - 1}{2} = 3a - b;$$

$$\left\lfloor \frac{2n+k-4}{2} \right\rfloor = \frac{2(2a+1) + (2b+2) - 4}{2} = 2a + b;$$

$$2n - k - 1 = 2(2a+1) - (2b+2) - 1 = 4a - 2b - 1.$$
If $n \geq 2k - 3 = 4b + 1$, then

$$t^*(G_{n,k}) = 2a + \max\{a - b, \min\{b, 2a - 2b - 1\}\}$$

$$= 2a + (a - b) = 3a - b.$$  

If $3b + 2 = \left\lfloor \frac{3k - 2}{2} \right\rfloor \leq n \leq 2k - 3 = 4b + 1$, then

$$t^*(G_{n,k}) = 2a + \max\{a - b, \min\{b, 2a - 2b - 1\}\} = 2a + b.$$  

If $2b + 3 = k + 1 \leq n \leq \left\lfloor \frac{3k - 2}{2} \right\rfloor = 3b + 2$, then

$$t^*(G_{n,k}) = 2a + \max\{a - b, \min\{b, 2a - 2b - 1\}\}$$

$$= 2a + (2a - 2b - 1) = 4a - 2b - 1.$$  

Subcase 1.2. $n$ is even. Let $n = 2a$, where $a \geq b + 2 \geq 3$. Then the length of $P$ is $2a - (2b + 2) = 2a - 2b - 2$. Let

$$P : v_0, v_1, \ldots, v_{2a - 2b - 2} = u$$

$$C : u, w_0, w_1, \ldots, w_{2b}, u.$$  

We determine the values of $t(v_i)$ ($0 \leq i \leq 2a - 2b - 2$) and $t(w_j)$ ($0 \leq j \leq 2b$). By symmetry, we may assume that $0 \leq j \leq b$. Observe that $t(v_0) = t(w_0) = n - 1 = 2a - 1$. For $1 \leq i \leq 2a - 2b - 2$, we see that either of the two linear orderings

$$s_i : v_i, v_{i-1}, \ldots, v_1, v_0, v_{i+1}, v_{i+2}, \ldots, v_{2a-2b-2}, w_0, w_1, \ldots, w_{2b}$$

$$s'_i : v_i, v_{i+1}, \ldots, v_{2a-2b-2}, w_0, w_1, \ldots, w_{2b}, v_{i-1}, \ldots, v_1, v_0$$

gives us $t(v_i)$. Hence

$$t(v_i) = \min\{n - 1 + i, 2(2a - 2b - 2) + (2b + 2) - i\}$$

$$= \min\{2a - 1 + i, 4a - 2b - 2 - i\}$$

and so

$$\max_{0 \leq i \leq 2a - 2b - 2} t(v_i) = t(v_{a-b-1}) = t(v_{a-b}) = 3a - b - 2$$

$$= 2a - 1 + (a - b - 1).$$
On the other hand, for $1 \leq j \leq b$, either of the two linear orderings

\[ s'_j : w_j, w_{j+1}, \ldots, w_{2b}, u, w_0, w_1, \ldots, w_{j-1}, v_{2a-2b-3}, v_{2a-2b-4}, \ldots, v_1, v_0 \]
\[ s''_j : w_j, w_{j+1}, \ldots, w_{2b}, u = v_{2a-2b-2}, v_{2a-2b-3}, \ldots, v_1, v_0, w_0, w_1, \ldots, w_{j-1} \]

gives us $t(w_j)$. Hence

\[
t(w_j) = (2b + 2) - (j + 1) + 2 \min\{j, 2a - 2b - 2\} + \max\{j, 2a - 2b - 2\} = 2b - j + 1 + (j + 2a - 2b - 2) + \min\{j, 2a - 2b - 2\} = 2a - 1 + \min\{j, 2a - 2b - 2\}
\]

and so

\[
\max_{0 \leq j \leq b} t(w_j) = t(w_b) = 2a + \min\{b, 2a - 2b - 2\}.
\]

Therefore,

\[
t^*(G_{n,k}) = \max\{2a - 1 + (a - b - 1), 2a - 1 + \min\{b, 2a - 2b - 2\}\} = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\}.
\]

Observe that

\[
\begin{align*}
\left\lfloor \frac{3n - k - 1}{2} \right\rfloor &= \left\lfloor \frac{3(2a) - (2b + 2) - 1}{2} \right\rfloor = 3a - b - 2; \\
\left\lfloor \frac{2n + k - 4}{2} \right\rfloor &= \left\lfloor \frac{2(2a) + (2b + 2) - 4}{2} \right\rfloor = 2a + b - 1;
\end{align*}
\]

\[2n - k - 1 = 2(2a) - (2b + 2) - 1 = 4a - 2b - 3.\]

If $n \geq 2k - 3 = 4b + 1$, then

\[
t^*(G_{n,k}) = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\} = 2a - 1 + (a - b - 1) = 3a - b - 2.
\]

If $3b + 2 = \left\lfloor \frac{3k - 2}{2} \right\rfloor \leq n \leq 2k - 3 = 4b + 1$, then

\[
t^*(G_{n,k}) = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\} = 2a + b - 1.
\]

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If $2b + 3 = k + 1 \leq n \leq \left\lfloor \frac{3b - 3}{2} \right\rfloor = 3b + 2$, then
\[
t^*(G_{n,k}) = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\}
\]
\[
= 2a - 1 + (2a - 2b - 2) = 4a - 2b - 3.
\]

Case 2. $k$ is odd. Let $k = 2b + 3$, where $b \geq 0$. Again, we consider two subcases, according to whether $n$ is odd or $n$ is even.

Subcase 2.1. $n$ is odd. Let $n = 2a + 1$, where $a \geq b + 2 \geq 2$. Then the length of $P$ is $(2a + 1) - (2b + 3) = 2a - 2b - 2$. Let
\[
P: v_0, v_1, \ldots, v_{2a - 2b - 2} = u
\]
\[
C: u, w_0, w_1, \ldots, w_{2b + 1}, u.
\]
We determine the values of $t(v_i)$ $(0 \leq i \leq 2a - 2b - 2)$ and $t(w_j)$ $(0 \leq j \leq 2b + 1)$. By symmetry, we may assume that $0 \leq j \leq b$. Observe that $t(v_0) = t(w_0) = n - 1 = 2a$. For $1 \leq i \leq 2a - 2b - 2$, we see that either of the two linear orderings
\[
s_i: v_1, v_i-1, \ldots, v_1, v_0, v_{i+1}, v_{i+2}, \ldots, v_{2a-2b-2}, w_0, w_1, \ldots, w_{2b+1}
\]
\[
s'_i: v_1, v_{i+1}, \ldots, v_{2a-2b-2}, w_0, w_1, \ldots, w_{2b+1}, v_{i-1}, \ldots, v_1, v_0
\]
gives us $t(v_i)$. Hence
\[
t(v_i) = \min\{n - 1 + i, 2(2a - 2b - 2) + (2b + 3) - i\}
\]
\[
= \min\{2a + i, 4a - 2b - 1 - i\}
\]
and so
\[
\max_{0 \leq i \leq 2a - 2b - 2} t(v_i) = t(v_{a-b-1}) = t(v_a-b) = 3a - b - 1
\]
\[
= 2a + (a - b - 1).
\]
On the other hand, for $1 \leq j \leq b$, either of the two linear orderings
\[
s'_j: w_j, w_{j+1}, \ldots, w_{2b+1}, u, w_0, w_1, \ldots, w_{j-1}, v_{2a-2b-3}, v_{2a-2b-4}, \ldots, v_1, v_0
\]
\[
s''_j: w_j, w_{j+1}, \ldots, w_{2b+1}, u = v_{2a-2b-2}, v_{2a-2b-3}, \ldots, v_1, v_0, w_0, w_1, \ldots, w_{j-1}
\]
gives us $t(w_j)$. Hence

$$t(w_j) = (2b + 3) - (j + 1) + 2 \min\{j, 2a - 2b - 2\} + \max\{j, 2a - 2b - 2\}$$

$$= 2b + j + 2 + (j + 2a - 2b - 2) + \min\{j, 2a - 2b - 2\}$$

$$= 2a + \min\{j, 2a - 2b - 2\}$$

and so

$$\max_{0 \leq j \leq b} t(w_j) = t(w_b) = 2a + \min\{b, 2a - 2b - 2\}.$$ 

Therefore,

$$t^*(G_{n,k}) = \max\{2a + (a - b - 1), 2a + \min\{b, 2a - 2b - 2\}\}$$

$$= 2a + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\}.$$ 

Observe that

$$\left\lfloor \frac{3n - k - 1}{2} \right\rfloor = \left\lfloor \frac{3(2a + 1) - (2b + 3) - 1}{2} \right\rfloor = 3a - b - 1;$$

$$\left\lfloor \frac{2n + k - 4}{2} \right\rfloor = \left\lfloor \frac{2(2a + 1) + (2b + 3) - 4}{2} \right\rfloor = 2a + b;$$

$$2n - k - 1 = 2(2a + 1) - (2b + 3) - 1 = 4a - 2b - 2.$$ 

If $n \geq 2k - 3 = 4b + 3$, then

$$t^*(G_{n,k}) = 2a + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\}$$

$$= 2a + (a - b - 1) = 3a - b - 1.$$ 

If $3b + 3 = \lfloor \frac{3k - 2}{2} \rfloor \leq n \leq 2k - 3 = 4b + 3$, then

$$t^*(G_{n,k}) = 2a + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\} = 2a + b.$$ 

If $2b + 4 = k + 1 \leq n \leq \lfloor \frac{3k - 2}{2} \rfloor = 3b + 3$, then

$$t^*(G_{n,k}) = 2a + \max\{a - b - 1, \min\{b, 2a - 2b - 2\}\}$$

$$= 2a + (2a - 2b - 2) = 4a - 2b - 2.$$
**Subcase 2.2. n is even.** Let \( n = 2a \), where \( a \geq b + 2 \geq 2 \). Then the length of \( P \) is \( 2a - (2b + 3) = 2a - 2b - 3 \). Let

\[
P : v_0, v_1, \ldots, v_{2a-2b-3} = u
\]

\[
C : u, w_0, w_1, \ldots, w_{2b+1}, u.
\]

We determine the values of \( t(v_i) \) (\( 0 \leq i \leq 2a - 2b - 3 \)) and \( t(w_j) \) (\( 0 \leq j \leq 2b + 1 \)). By symmetry, we may assume that \( 0 \leq j \leq b \). Observe that \( t(v_0) = t(w_0) = n - 1 = 2a - 1 \). For \( 1 \leq i \leq 2a - 2b - 3 \), we see that either of the two linear orderings

\[
s_i : v_i, v_{i-1}, \ldots, v_1, v_0, v_{i+1}, v_{i+2}, \ldots, v_{2a-2b-3}, w_0, w_1, \ldots, w_{2b+1}
\]

\[
s_i' : v_i, v_{i+1}, \ldots, v_{2a-2b-3}, w_0, w_1, \ldots, w_{2b+1}, v_{i-1}, \ldots, v_1, v_0
\]

gives us \( t(v_i) \). Hence

\[
t(v_i) = \min\{n - 1 + i, 2(2a - 2b - 3) + (2b + 3) - i\}
\]

\[
= \min\{2a - 1 + i, 4a - 2b - 3 - i\}
\]

and so

\[
\max_{0 \leq i \leq 2a-2b-3} t(v_i) = t(v_{a+b-1}) = 3a - b - 2
\]

\[
= 2a - 1 + (a - b - 1).
\]

On the other hand, for \( 1 \leq j \leq b \), either of the two linear orderings

\[
s_j : w_j, w_{j+1}, \ldots, w_{2b+1}, u, w_0, w_1, \ldots, w_{j-1}, v_{2a-2b-4}, v_{2a-2b-5}, \ldots, v_1, v_0
\]

\[
s_j' : w_j, w_{j+1}, \ldots, w_{2b+1}, u = v_{2a-2b-3}, v_{2a-2b-4}, \ldots, v_1, v_0, w_0, w_1, \ldots, w_{j-1}
\]

gives us \( t(w_j) \). Hence

\[
t(w_j) = (2b + 3) - (j + 1) + 2 \min\{j, 2a - 2b - 3\} + \max\{j, 2a - 2b - 3\}
\]

\[
= 2b - j + 2 + (j + 2a - 2b - 3) + \min\{j, 2a - 2b - 3\}
\]

\[
= 2a - 1 + \min\{j, 2a - 2b - 3\}
\]

and so

\[
\max_{0 \leq j \leq b} t(w_j) = t(w_b) = 2a + \min\{b, 2a - 2b - 3\}.
\]
Therefore, 
\[ t^*(G_{n,k}) = \max\{2a - 1 + (a - b - 1), 2a - 1 + \min\{b, 2a - 2b - 3\}\} \]
\[ = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 3\}\}. \]

Observe that
\[ \frac{3n - k - 1}{2} = \left\lfloor \frac{3(2a) - (2b + 3) - 1}{2} \right\rfloor = 3a - b - 2; \]
\[ \frac{2n + k - 4}{2} = \left\lfloor \frac{2(2a) + (2b + 3) - 4}{2} \right\rfloor = 2a + b - 1; \]
\[ 2n - k - 1 = 2(2a) - (2b + 3) - 1 = 4a - 2b - 4. \]

If \( n \geq 2k - 3 = 4b + 3 \), then
\[ t^*(G_{n,k}) = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 3\}\} \]
\[ = 2a - 1 + (a - b - 1) = 3a - b - 2. \]

If \( 3b + 3 = \left\lfloor \frac{3k - 2}{2} \right\rfloor \leq n \leq 2k - 3 = 4b + 3 \), then
\[ t^*(G_{n,k}) = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 3\}\} \]
\[ = 2a + b - 1. \]

If \( 2b + 4 = k + 1 \leq n \leq \left\lfloor \frac{3k - 2}{2} \right\rfloor = 3b + 3 \), then
\[ t^*(G_{n,k}) = 2a - 1 + \max\{a - b - 1, \min\{b, 2a - 2b - 3\}\} \]
\[ = 2a - 1 + (2a - 2b - 3) = 4a - 2b - 4. \]

Hence in each case, we obtain the desired result. 

By Corollary 5.8, if \( T \) is a tree of order \( n \geq 3 \) and \( v \in V(T) \), then \( t(v) = t^*(T) \) if and only if \( v \) is a central vertex in \( T \). For a graph \( G \), let
\[ V^*(G) = \{ v \in V(G) : t(v) = t^*(G) \}. \]

Then \( V^*(G) \) is the set of central vertices of \( G \) if \( G \) is a tree. This is not true in general, however, if \( G \) is not a tree. For example, consider the three graphs \( G_{15,6} \), \( G_{15,10} \), and \( G_{15,13} \) of order 15 shown in Figure 5.3.
We have seen that the solid vertices in each of these graphs are central vertices of the graph and so

\[ \text{rad}(G_{15,6}) = 6, \quad \text{rad}(G_{15,10}) = 5, \quad \text{and} \quad \text{rad}(G_{15,13}) = 6. \]

On the other hand, the traceable number of the vertex labeled by \( v^* \) in each of these graphs is the maximum traceable number of the graph. Thus

\begin{align*}
     t^*(G_{15,6}) &= \left\lfloor \frac{3 \cdot 15 - 6 - 1}{2} \right\rfloor = 19; \\
     t^*(G_{15,10}) &= \left\lfloor \frac{2 \cdot 15 + 10 - 4}{2} \right\rfloor = 18; \\
     t^*(G_{15,13}) &= 2 \cdot 15 - 13 - 1 = 16.
\end{align*}

Observe that for \( n = 15 \) and each \( k \in \{6, 10, 13\} \),

\[ t^*(G_{n,k}) \neq 2n - 2 - \text{rad}(G_{n,k}) \]

and so \( V^*(G_{n,k}) \) is not equal to the set of central vertices. In fact, these two sets are disjoint for \( n = 15 \) and each \( k \in \{6, 10, 13\} \).
We now consider another class of unicyclic graphs. Let $k$, $\ell_1$, and $\ell_2$ be integers such that $k \geq 3$ and $1 \leq \ell_2 \leq \ell_1$. Let $G_{k,\ell_1,\ell_2}$ be the unicyclic graph of order $n = k + \ell_1 + \ell_2 \geq 5$ obtained from a cycle $C$ of order $k$ and two paths $Q_1$ and $Q_2$ of length $\ell_1$ and $\ell_2$, respectively, by identifying one of the $k$ vertices on $C$ and one of the two end-vertices of $Q_i$ for $i = 1, 2$. We determine the maximum traceable number of $G_{k,\ell_1,\ell_2}$.

**Proposition 5.19** Let $n$, $k$, $\ell_1$, and $\ell_2$ be positive integers such that $k \geq 3$, $\ell_2 \leq \ell_1$, and $n = k + \ell_1 + \ell_2$. Then

$$t'(G_{k,\ell_1,\ell_2}) = \begin{cases} n + \ell_1 + \ell_2 - 1 & \text{if } k \geq 2\ell_1 + 2 \\ n + \left\lceil \frac{\ell_1 + \ell_2}{2} \right\rceil & \text{if } k \leq 2\ell_1 + 1 \text{ and } \ell_1 - \ell_2 \geq 2 \left\lfloor \frac{k}{2} \right\rfloor - 4 \\ n + \ell_2 + \left\lfloor \frac{k}{2} \right\rfloor - 2 & \text{if } k \leq 2\ell_1 + 1 \text{ and } \ell_1 - \ell_2 \leq 2 \left\lfloor \frac{k}{2} \right\rfloor - 5. \end{cases}$$

**Proof.** Let $C : x, w_0, w_1, \ldots, w_{k-2}, w_{k-1} = x$ be a cycle of length $k$ and let

\begin{align*}
Q_1 &: u_0, u_1, \ldots, u_{\ell_1-1}, u_{\ell_1} = x \\
Q_2 &: v_0, v_1, \ldots, v_{\ell_2-1}, v_{\ell_2} = x
\end{align*}

be two paths of lengths $\ell_1$ and $\ell_2$, respectively. We first determine the values of $t(u_i)$ for $0 \leq i \leq \ell_2$. Observe that the linear ordering

$$s_i : u_0, u_1, \ldots, v_0, v_1, \ldots, u_{\ell_2}, w_0, w_1, \ldots, w_{k-2}, u_{\ell_1-1}, u_{\ell_1-2}, \ldots, u_0$$

gives us $t(u_i)$ for each $i$. Hence $t(u_i) = n + i$ and so

$$\max_{0 \leq i \leq \ell_2} t(u_i) = t(u_{\ell_2}) = n + \ell_2.$$

Next we determine the values of $t(u_i)$ for $0 \leq i \leq \ell_1$. Observe that either of the two linear orderings

\begin{align*}
s_i' &: u_0, u_1, \ldots, u_{\ell_1-1}, u_0, u_1, u_2, \ldots, u_{\ell_1}, w_0, w_1, \ldots, w_{k-2}, u_{\ell_2-1}, u_{\ell_2-2}, \ldots, v_0 \\
s_i'' &: u_0, u_{\ell_1+1}, \ldots, u_0, u_{\ell_1}, w_0, w_1, \ldots, w_{k-2}, v_0, v_1, \ldots, v_{\ell_2-1}, u_{\ell_1-1}, u_{\ell_1-2}, \ldots, u_0
\end{align*}

gives us $t(u_i)$ for each $i$. Hence

$$t(u_i) = \min \{n + i, (\ell_1 - i) + k + 2\ell_2 + \ell_1\} = \min \{n + i, n + \ell_1 + \ell_2 - i\}$$

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and so
\[
\max_{0 \leq i \leq \ell_1} t(u_i) = t\left(u_{\left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor}\right) = t\left(u_{\left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor}\right) = n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor.
\]

Note that \(n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor \geq n + \ell_2\). Finally we determine the values of \(t(w_i)\) for \(0 \leq i \leq k - 2\). By symmetry, we may assume that \(0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor - 1\). Observe that either of the two linear orderings

1. \(s^*: w_i, w_{i+1}, \ldots, w_{k-1} = x, w_0, w_1, \ldots, w_{i-1}, v_0, v_1, \ldots, v_{\ell_2-1}, u_{\ell_1-1}, u_{\ell_1-2}, \ldots, u_0\)

2. \(s^{**}_i: w_i, w_{i+1}, \ldots, w_{k-1} = x, u_0, w_1, \ldots, u_{\ell_1-1}, v_0, v_1, \ldots, v_{\ell_2-1}, w_0, w_1, \ldots, w_{i-1}\)

gives us \(t(w_i)\) for each \(i\). Hence

\[
t(w_i) = \min \{(k - 1) + i + 2\ell_2 + \ell_1, (k - 1) + 2\ell_1 + 2\ell_2\}
\]
\[
= \min \{n + \ell_2 - 1 + i, n + \ell_1 + \ell_2 - 1\}
\]

and so

\[
\max_{0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor - 1} t(w_i) = t\left(w_{\left\lfloor \frac{k}{2} \right\rfloor - 1}\right) = \begin{cases} 
\ n + \ell_2 + \left\lfloor \frac{k}{2} \right\rfloor - 2 & \text{if } k \leq 2\ell_1 + 1 \\
\ n + \ell_1 + \ell_2 - 1 & \text{if } k \geq 2\ell_1 + 2.
\end{cases}
\]

Therefore if \(k \geq 2\ell_1 + 2\), then \(k \geq 4\) and

\[
t^*(G_{k,\ell_1,\ell_2}) = \max \left\{ n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor, n + \ell_1 + \ell_2 - 1 \right\}
\]
\[
= n + \ell_1 + \ell_2 - 1.
\]

If \(k \leq 2\ell_1 + 1\), then

\[
t^*(G_{k,\ell_1,\ell_2}) = \max \left\{ n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor, n + \ell_2 + \left\lfloor \frac{k}{2} \right\rfloor - 2 \right\}.
\]

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Thus for $3 \leq k \leq 5$, we obtain

$$t^*(G_{k,\ell_1,\ell_2}) = n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor.$$ 

In general, since

$$n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor = n + \left\lfloor \frac{2\ell_2 + (\ell_1 - \ell_2)}{2} \right\rfloor = n + \ell_2 + \left\lfloor \frac{\ell_1 - \ell_2}{2} \right\rfloor,$$

it follows that for $k \leq 2\ell_1 + 1$,

$$t^*(G_{k,\ell_1,\ell_2}) = \begin{cases} 
  n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor & \text{if } \ell_1 - \ell_2 \geq 2 \left\lfloor \frac{k}{2} \right\rfloor - 4 \\
  n + \ell_2 + \left\lfloor \frac{k}{2} \right\rfloor - 2 & \text{if } \ell_1 - \ell_2 \leq 2 \left\lfloor \frac{k}{2} \right\rfloor - 5.
\end{cases}$$

This gives us the desired result. 

\[\blacksquare\]
Chapter 6

Comparing Traceable Parameters

6.1 Introduction

We have seen that for every nontrivial connected graph $G$,

$$t(G) \leq t^*(G) \leq t^+(G).$$

Thus if $G$ is a nontrivial connected graph with $t(G) = a$, $t^*(G) = b$, and $t^+(G) = c$, then $a \leq b \leq c$. For this reason, we define a triple $(a, b, c)$ of positive integers $a, b, c$ to be realizable if $a \leq b \leq c$ and there exists a nontrivial connected graph $G$ such that $t(G) = a$, $t^*(G) = b$, and $t^+(G) = c$. There is a natural question related to these three parameters.

**Problem 6.1** For which triples $(a, b, c)$ of positive integers with $a \leq b \leq c$, does there exist a connected graph $G$ for which

$$t(G) = a, \ t^*(G) = b, \text{ and } t^+(G) = c?$$

For a connected graph $G$, let

$$f(G) = (t(G), t^*(G), t^+(G)).$$

Thus our goal is to identify those triples $(a, b, c)$ of positive integers with $a \leq b \leq c$ for which there exists a connected graph $G$ such that $f(G) = (a, b, c)$. 

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As an illustration, we consider triples \((a, b, c)\) for which \(1 \leq a \leq 4\) as follows:

- \(a = 1\) : \(G = K_2\) and \(f(K_2) = (1, 1, 1)\).
- \(a = 2\) : \(G \in \{K_3, P_3\}\). Note that \(f(K_3) = (2, 2, 2)\) and \(f(P_3) = (2, 3, 3)\).
- \(a = 3\) : Observe that every graph \(G\) with \(t(G) = 3\) must be a traceable graph of order 4, namely \(K_4, K_4 - e, c_4, (K_2 \cup K_1) + K_1, \) and \(P_4\), and
  \[
  f(K_4) = (3, 3, 3), \quad f(K_4 - e) = (3, 3, 4), \quad f(c_4) = (3, 3, 5), \\
  f((K_2 \cup K_1) + K_1) = (3, 4, 5), \quad f(P_4) = (3, 4, 7).
  \]

Hence

\[
\{ f(G) : t(G) = 3 \} = \{ (3, 3, 3), (3, 3, 4), (3, 3, 5), (3, 4, 5), (3, 4, 7) \}.
\]

- \(a = 4\) : Then \(n = 4\) or \(n = 5\). If \(n = 4\), then \(G = K_{1,3}\) and \(f(K_{1,3}) = (4, 5, 5)\).
  For \(n = 5\), all traceable graphs of order 5 are shown in Figure 6.1, where each graph \(G\) is labeled with \(f(G)\).

\[
\begin{align*}
K_5 & \quad K_5 - e & \quad K_{2,3} & \quad C_5 \\
(4, 4, 4) & \quad (4, 4, 5) & \quad (4, 5, 7) & \quad (4, 4, 8) \\
(4, 5, 6) & \quad (4, 5, 7) & \quad (4, 5, 8) & \\
(4, 5, 9) & \quad (4, 6, 11)
\end{align*}
\]

Figure 6.1: Graphs \(G\) of order 5 with \(t(G) = 4\)

Hence

\[
\{ f(G) : t(G) = 4 \} = \{ (4, 4, 4), (4, 4, 5), (4, 4, 6), (4, 4, 7), (4, 4, 8), (4, 5, 5), \\
(4, 5, 6), (4, 5, 7), (4, 5, 8), (4, 5, 9), (4, 6, 11) \}.
\]
Notice that in all of these examples,

\[
\left\lfloor \frac{t^+(G) + t(G)}{2} \right\rfloor \geq t^*(G).
\]

The following is a consequence of Corollary 5.3.

**Observation 6.2** Each triple \((a, b, c)\) is realizable if \(a = b = c\), \(a = b = c - 1\), or \(b = c = a + 1\).

Next, we consider the realizability of \((a, a + 1, a + 2)\) for some positive integer \(a\).

- Let \(G\) be a connected graph of order \(n \geq 4\) obtained from \(K_{n-1}\) by joining a new end-vertex \(w\) to one of the \(n-1\) vertices of \(K_{n-1}\). Let \(V(G) = \{u, w, v_1, v_2, \ldots, v_{n-2}\}\) such that \(\text{deg} \ u = n - 1\), \(\text{deg} \ w = 1\), and \(\text{deg} \ v_i = n - 2\) for each \(i\) with \(1 \leq i \leq n - 2\). Observe that \(t(v) = n - 1\) for \(v \in V(G) - \{u\}\) and \(t(u) = n\). Thus \(t(G) = n - 1\) and \(t^*(G) = n\). Hence we only need to verify that \(t^+(G) = n + 1\). Let \(x\) and \(y\) be two distinct vertices and observe that

\[
d(x, y) = \begin{cases} 
2 & \text{if } w \in \{x, y\} \text{ and } u \notin \{x, y\} \\
1 & \text{otherwise.}
\end{cases}
\]

Thus for every linear ordering \(s\) of vertices of \(G\),

\[
d(s) \leq (n - 3) + 2 \cdot 2 = n + 1
\]

and so \(t^+(G) \leq n + 1\). Consider the linear ordering \(s'\) of vertices of \(G\) given by

\[
s' : v_1, w, v_2, v_3, \ldots, v_{n-2}, u
\]

and observe that \(d(s') = n + 1\). Therefore \(t^+(G) = n + 1\) and

\[
f(G) = (t(G), t^*(G), t^+(G)) = (n - 1, n, n + 1).
\]

- Let \(G = K_{1,1,n-2}\) be the complete 3-partite graph of order \(n \geq 5\) with partite sets \(U_1 = \{u_1\}\), \(U_2 = \{u_2\}\), and \(W = \{w_1, w_2, \ldots, w_{n-2}\}\). First observe that
\[ t(u_i) = 2(n - 3) + 1 = 2n - 5 \text{ for } i = 1, 2 \text{ and } t(w_i) = 1 + 2(n - 4) + 1 = 2n - 6 \]

for \( 1 \leq i \leq n - 2 \). Hence \( t(G) = 2n - 6 \) and \( t^*(G) = 2n - 5 \). By Proposition 4.5, \( t^+(G) = 2n - 3 - 1 = 2n - 4 \) and so

\[ f(G) = (t(G), t^*(G), t^+(G)) = (2n - 6, 2n - 5, 2n - 4). \]

- Let \( n \geq 4 \) and \( G = K_{1,n-1} + e \). Let \( V(G) = \{u, v_1, v_2, w_1, w_2, \ldots, w_{n-3}\} \), where \( \deg u = n - 1 \), \( \deg v_i = 2 \) for \( i = 1, 2 \), and \( \deg w_i = 1 \) for \( 1 \leq i \leq n - 3 \). Observe that \( t(u) = 2n - 4 \) while \( t(x) = 2n - 5 \) for \( x \in V(G) - \{u\} \), so \( t(G) = 2n - 5 \) and \( t^*(G) = 2n - 4 \). Also, for an arbitrary linear ordering \( s \) of vertices of \( G \), there is at least one term equal to 1 in \( d(s) \) and so

\[ d(s) \leq 1 + 2(n - 2) = 2n - 3, \]

implying that \( t^+(G) \leq 2n - 3 \). Now consider the linear ordering \( s' \) of vertices of \( G \) given by

\[ s': v_1, w_1, w_2, \ldots, w_{n-3}, v_2, u \]

and observe that \( d(s') = 2n - 3 \). Thus \( t^+(G) = 2n - 3 \) and

\[ f(G) = (t(G), t^*(G), t^+(G)) = (2n - 5, 2n - 4, 2n - 3). \]

These examples yield the following.

**Proposition 6.3** For each integer \( a \geq 3 \), the triple \((a, a + 1, a + 2)\) is realizable.

In order to shed some light on Problem 6.1, we first study pairs \( a, b \) of positive integers such that either (1) there exists a connected graph \( G \) for which \( t(G) = a \) and \( t^*(G) = b \); or (2) there exists a connected graph \( G \) for which \( t(G) = a \) and \( t^+(G) = b \). We will then study the related traceable ratio problems in Sections 6.3 and 6.5.

### 6.2 Traceable and Maximum Traceable Numbers

In this section, we investigate the following question.

**Problem 6.4** For which pairs \( a, b \) of positive integers with \( a \leq b \), does there exist a connected graph \( G \) for which \( t(G) = a \) and \( t^*(G) = b \)?
First, we recall the following two facts about the traceable number and maximum traceable number of a graph, which appear in Chapter 3.

**Theorem 6.5** If $G$ is a connected graph of order $n \geq 3$, then

$$n - 1 \leq t(G) \leq 2n - 4.$$  

Furthermore, $t(G) = 2n - 4$ if and only if $G \in \{K_3, K_{1,n-1}\}$.

**Theorem 6.6** If $G$ is a connected graph of order $n \geq 2$, then

$$n - 1 \leq t^*(G) \leq 2n - 3.$$  

Furthermore, $t^*(G) = 2n - 3$ if and only if $G = K_{1,n-1}$.

For a connected graph $G$, let

$$g(G) = (t(G), t^*(G)).$$

Observe that for $1 \leq a \leq 4$, we have the following:

- $a = 1$: $\{g(G) : t(G) = 1\} = \{(1,1)\}$;
- $a = 2$: $\{g(G) : t(G) = 2\} = \{(2,2), (2,3)\}$;
- $a = 3$: $\{g(G) : t(G) = 3\} = \{(3,3), (3,4)\}$;
- $a = 4$: $\{g(G) : t(G) = 4\} = \{(4,4), (4,5), (4,6)\}$.

Let $a \geq 5$ be an integer. If $G$ is a connected graph of order $n$ for which $t(G) = a$, then by Theorem 6.5,

$$\left\lceil \frac{a + 4}{2} \right\rceil \leq n \leq a + 1. \quad (6.1)$$

Moreover, if $n = a + 1$, then $G$ is traceable. To continue our discussion, the following proposition will be useful.

**Proposition 6.7** Let $n$ and $k$ be integers such that $k \geq 3$ and $n \geq k + 1$, and let $G_{n,k}$ be the graph of order $n$ obtained from a complete graph $K_k$ of order $k$ and a
path $P$ of order $n-k+1$ by identifying one of the $k$ vertices of $K_k$ and one of the two end-vertices of $P$. Then

$$t^*(G_{n,k}) = \left\lfloor \frac{3n-k-1}{2} \right\rfloor.$$ 

**Proof.** Let $V(K_k) = \{u = w_1, w_2, \ldots, w_k\}$ and

$$P : v_0, v_1, \ldots, v_{n-k+1} = u.$$ 

First observe that

$$t(v_0) = t(u) = n - 1$$

for $2 \leq i \leq k$. Observe that for each $i$, either of the two linear orderings

$$s_i : v_i, v_{i-1}, \ldots, v_1, v_0, v_{i+1}, v_{i+2}, \ldots, v_{2\ell+1} = w_1, w_2, w_3, \ldots, w_k$$

$$s'_i : v_i, v_{i+1}, \ldots, v_{2\ell+1} = w_1, w_2, w_3, \ldots, w_k, v_{i-1}, \ldots, v_1, v_0$$

gives us $t(v_i)$. Hence

$$t(v_i) = \min\{n - 1 + i, 2(n - k) + k - i\} = \min\{n - 1 + i, 2n - k - i\}.$$ 

We consider two cases, according to whether $n-k$ is even or odd.

**Case 1.** $n-k$ is even. Let $n-k = 2\ell$ for some integer $\ell \geq 1$. We determine the values of $t(v_i)$ for $1 \leq i \leq 2\ell + 1$. Observe that

$$\max_{1 \leq i \leq 2\ell+1} t(v_i) = t(v_{\ell}) = t(v_{\ell+1}) = n - 1 + \ell = n - 1 + \frac{n-k}{2} = \frac{3n - k - 2}{2}.$$ 

**Case 2.** $n-k$ is odd. Let $n-k = 2\ell - 1$ for some integer $\ell \geq 1$. We determine the values of $t(v_i)$ for $1 \leq i \leq 2\ell$. Observe that

$$\max_{1 \leq i \leq 2\ell} t(v_i) = t(v_\ell) = n - 1 + \ell = n - 1 + \frac{n-k+1}{2} = \frac{3n - k - 1}{2}.$$
Therefore in each case we obtain the desired result.

Recall that

\[ t^*(P_n) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor. \]

**Proposition 6.8** Let \( a, b \) be a pair of integers with \( 1 \leq a \leq b \). Then there exists a traceable graph \( G \) of order \( n = a + 1 \) with \( g(G) = (a, b) \) if and only if \( a \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor \).

**Proof.** The statement is true for \( 1 \leq a \leq 4 \), so we assume that \( a \geq 5 \). Suppose first that \( G \) is a traceable graph of order \( n = a + 1 \) whose maximum traceable number is \( b \). Then \( t(G) = n - 1 = a \). Since \( G \) is traceable, \( G \) contains \( P_n \) as a spanning tree. Hence

\[ b = t^*(G) \leq t^*(P_n) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor = \left\lfloor \frac{3a}{2} \right\rfloor. \]

For the converse, let \( b = a + i \), where \( i \) is an integer with \( 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \). If \( i = 0 \), then let \( G = C_n \). If \( i = \left\lfloor \frac{n}{2} \right\rfloor \), then let \( G = P_n \). Hence assume that \( 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). If \( a \) is even, let \( a = 2\ell \) for some integer \( \ell \geq 3 \). Then \( 1 \leq i \leq \ell - 1 \). Also note that \( 4 \leq a + 2 - 2i = n - (2i - 1) \leq n - 1 \). If \( a \) is odd, let \( a = 2\ell + 1 \) for some integer \( \ell \geq 2 \). Then \( 1 \leq i \leq \ell - 1 \). Note that \( 5 \leq a + 2 - 2i \leq n - 1 \). Let \( G_{n,a+2-2i} \) be the traceable graph of order \( n = a + 1 \) constructed in the way described in Proposition 6.7. Then

\[ t^*(G_{n,a+2-2i}) = \left\lfloor \frac{3(a+1) - (a+2-2i) - 1}{2} \right\rfloor = a + i = b. \]

Hence \( g(G_{a+1,3a-2b+2}) = (a, b) \), providing the desired result.

By the inequality in (6.1), if \( G \) is a connected graph of order \( n \) with \( t(G) = 5 \), then \( n = 5 \) or \( n = 6 \). We have seen that by Proposition 6.8,

\[ \{g(G) : G \ is \ a \ traceable \ graph \ of \ order \ 6\} = \{(5, 5), (5, 6), (5, 7)\}. \]

For \( n = 5 \), observe that \( t(G) = 5 \) if and only if \( G \in \{ S_{2,3}, K_{1,4} + e \} \) and

\[ t^*(S_{2,3}) = t^*(K_{1,4} + e) = 6. \]

Hence for \( a = 5 \), we have

\[ \{g(G) : t(G) = 5\} = \{(5, 5), (5, 6), (5, 7)\}. \]
For \( a = 6 \), we first note that by the inequality in (6.1), if \( G \) is a connected graph of order \( n \) with \( t(G) = 6 \), then \( 5 \leq n \leq 7 \). By Theorem 6.5, we see that \( G = K_{1,4} \) is the only graph of order 5 with \( t(G) = 6 \), for which \( g(K_{1,4}) = (6,7) \). Also, Proposition 6.8 implies that

\[
\{ g(G) : G \text{ is a traceable graph of order } 7 \} = \{(6,6), (6,7), (6,8), (6,9)\}.
\]

For a connected graph \( G \) of order 6 with \( t(G) = 6 \), we see by Theorem 6.6 that \( t^*(G) \leq 2 \cdot 6 - 3 = 9 \). Hence we conclude that for \( a = 6 \), we have

\[
\{ g(G) : t(G) = 6 \} = \{(6,6), (6,7), (6,8), (6,9)\}.
\]

Before we present another result, we first make the following observation, which is a consequence of Theorem 6.6.

**Observation 6.9** Let \( G \) be a connected graph of order \( n \). If \( G \) is nontraceable and \( G \neq K_{1,n-1} \), then

\[
n \leq t^*(G) \leq 2n - 4.
\]

**Proposition 6.10** Let \( a \) be an integer with \( 1 \leq a \leq 8 \). Then

\[
\{ g(G) : t(G) = a \} = \{ (a,b) : a \leq b \leq \left\lceil \frac{3a}{2} \right\rceil \}.
\]

**Proof.** The result immediately follows for \( a = 1, 2 \). Hence suppose that \( 3 \leq a \leq 8 \). Let \( G \) be a connected graph of order \( n \) having \( t(G) = a \). If \( G \) is traceable, then \( n = a + 1 \). By Proposition 6.8, it follows that

\[
\{ g(G) : t(G) = a \} \supseteq \{ g(G) : G \text{ is a traceable graph of order } n = a + 1 \}
\]

\[
= \left\{ (a,b) : a \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor \right\}.
\]

Next suppose that \( G \) is nontraceable. If \( G \) is the star \( K_{1,n-1} \), then \( t(K_{1,n-1}) = 2n - 4 = a \), implying that

\[
t^*(K_{1,n-1}) = 2n - 3 = a + 1 \leq \left\lfloor \frac{3a}{2} \right\rfloor.
\]
On the other hand, if $G$ is nontraceable and $G \neq K_{1,n-1}$, then $n \leq a$ and so

$$t^*(G) \leq 2n - 4 \leq 2a - 4 \leq \left\lfloor \frac{3a}{2} \right\rfloor$$

by Observation 6.9. Hence $a \leq t^*(G) \leq \left\lfloor \frac{3a}{2} \right\rfloor$ for every connected graph $G$ with $t(G) = a$. This means that

$$\{g(G) : t(G) = a\} \subseteq \{(a, b) : a \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor\},$$

completing the proof. \hfill \square

Let us next consider the case where $a = 9$. If $G$ is a connected graph of order $n$ with $t(G) = 9$, we know again by the inequality in (6.1) that $7 \leq n \leq 10$. Also by Proposition 6.8, there exists a traceable graph $G$ of order 10 with $t^*(G) = b$ for each integer $b$ with $9 \leq b \leq 13$. Now, if $G$ is a connected graph of order 9 for which $t(G) = 9$, then $G$ is not a star by Theorem 6.5. Then by Observation 6.9, it follows that $t^*(G) \leq 2 \cdot 9 - 4 = 14$. Our question is:

**Is there a connected graph $G$ of order 9 such that $g(G) = (9, 14)$?**

The answer to this question turns out to be no. In fact, every connected graph $G$ of order 9 for which $t(G) = 9$ satisfies $t^*(G) \leq 12$. The following proposition shows why.

For three positive integers $k, \ell_1$, and $\ell_2$ such that $k \geq 3$ and $\ell_1 \geq \ell_2$, let $G_{k,\ell_1,\ell_2}$ be the unicyclic graph of order $n = k + \ell_1 + \ell_2 \geq 5$ obtained from a cycle $C$ of order $k$ and two paths $Q_1$ and $Q_2$ of lengths $\ell_1$ and $\ell_2$, respectively, by identifying one of the $k$ vertices of $C$ and one of the two end-vertices of $Q_i$ for $i = 1, 2$. The maximum traceable number of $G_{k,\ell_1,\ell_2}$ was determined in Proposition 5.20.

**Proposition 6.11** Let $G$ be a connected graph for which $t(G) = 9$. Then

$$\{g(G)\} = \{(9, b) : 9 \leq b \leq 13\}.$$

**Proof.** If the order of $G$ is $n$, then $7 \leq n \leq 10$ by (6.1). For each integer $b$ with $9 \leq b \leq 13$, there exists a traceable graph $G$ (of order $n = 10$) for which $t^*(G) = b$
by Proposition 6.8. If \( G \) is nontraceable, then observe that \( G \) is not a star since otherwise we would have \( 9 = t(G) = 2n - 4 \), which is impossible. Hence for \( n = 7, 8 \), we have \( 9 \leq t^*(G) \leq 2 \cdot 8 - 4 = 12 \) by Observation 6.9. Thus we only consider those graphs of order \( n = 9 \) with the given condition. Assume, to the contrary, that there exists a graph \( G \) of order 9 with \( g(G) = (9, b) \), where \( b \geq 13 \). Let \( T \) be a spanning tree of \( G \) having the largest diameter. Then by Proposition 5.7,

\[
13 \leq t^*(G) \leq t^*(T) = 2 \cdot 9 - 2 - \text{rad}(T) = 16 - \text{rad}(T).
\]

Hence diam\((T) \leq 6\). Let \( W : v_1, v_2, \ldots, v_{10} \) be a spanning walk in \( G \) of length 9. Then \( v_i = v_j \) for some \( i \) and \( j \), where \( 2 \leq i < i + 2 \leq j \leq 9 \). Let \( H \) be the spanning subgraph of \( G \) whose edge set \( E(H) \) consists of those nine edges that appear in \( W \). Since \( G \) (and \( H \)) contains no spanning tree of diameter greater than 6, it follows that \( H \) must be one of the three graphs \( H_1 = G_{5,2,2} \), \( H_2 = G_{4,3,2} \), and \( H_3 = G_{3,3,3} \) shown in Figure 6.2.

![Figure 6.2: Graphs \( H_1, H_2, \text{ and } H_3 \)](image)

However, observe that \( t^*(H_1) = t^*(H_2) = 11 \) and \( t^*(H_3) = 12 \) by Proposition 5.20. This implies that

\[
13 \leq t^*(G) \leq t^*(H) \leq 12,
\]

a contradiction. Hence we conclude that \( 9 \leq t^*(G) \leq 13 \) for every connected graph \( G \) with \( t(G) = 9 \).

Similarly, we can exactly identify the set \( \{g(G) : t(G) = 10\} \).

**Proposition 6.12** Let \( G \) be a connected graph for which \( t(G) = 10 \). Then

\[
\{g(G)\} = \{(10, b) : 10 \leq b \leq 15\}.
\]

**Proof.** If the order of \( G \) is \( n \), then \( 7 \leq n \leq 11 \) by (6.1). For each integer \( b \) with \( 10 \leq b \leq 15 \), there exists a traceable graph \( G \) (of order \( n = 11 \)) for which \( t^*(G) = b \)
by Proposition 6.8. Let us now consider those graphs satisfying the given condition that are nontraceable. If $G$ is a star, then $n = 7$ since $t(G) = 2n - 4 = 10$, implying that $t^*(G) = 2 \cdot 7 - 3 = 11$. Hence for $n = 8, 9$, we know that $G$ is not a star and therefore $t^*(G) \leq 2 \cdot 9 - 4 = 14$. Thus we only consider those graphs of order $n = 10$. Assume, to the contrary, that there exists a graph $G$ of order 10 with $g(G) = (10, b)$ where $b \geq 15$. Let $T$ be a spanning tree of $G$ having the largest diameter. Then by Proposition 5.7,

$$15 \leq t^*(G) \leq t^*(T) = 2 \cdot 10 - 2 - \text{rad}(T) = 18 - \text{rad}(T).$$

Hence diam$(T) \leq 6$. Let $W : v_1, v_2, \ldots, v_{11}$ be a spanning walk in $G$ of length 10. Then $v_i = v_j$ for some $i$ and $j$, where $2 \leq i < i + 2 \leq j \leq 10$. Let $H$ be the spanning subgraph of $G$ whose edge set $E(H)$ consists of those ten edges that appear in $W$. Since $G$ (and $H$) contains no spanning tree of diameter greater than 6, it follows that $(i, j) = (4, 8)$, that is, $H = G_{4,3,3}$. However, observe that $t^*(H) = 13$ by Proposition 5.20, implying that

$$15 \leq t^*(G) \leq t^*(H) \leq 13,$$

a contradiction. Hence we conclude that $10 \leq t^*(G) \leq 15$ for every connected graph $G$ with $t(G) = 10$.

Proposition 6.8 shows that if $G$ is a traceable graph of order $n = a + 1$, then $t(G) = a$ and $a \leq t^*(G) \leq \left\lfloor \frac{3a}{2} \right\rfloor$. On the other hand, in Propositions 6.10, 6.11, and 6.12, we saw that if $G$ is a connected (nontraceable) graph of order $n = a$ such that $t(G) = a$, then $a \leq t^*(G) \leq \left\lfloor \frac{3a}{2} \right\rfloor - 1$ for $1 \leq a \leq 10$. In particular, $a \leq t^*(G) \leq \left\lfloor \frac{3a}{2} \right\rfloor - 1$ for $a = 9, 10$. We generalize this in the next proposition.

**Proposition 6.13** Let $a, b$ be a pair of integers with $4 \leq a \leq b$. If $G$ is a connected nontraceable graph of order $n = a$ with $g(G) = (a, b)$, then $a \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor - 1$.

**Proof.** Let $G$ be a connected graph of order $n = a$ such that $t(G) = a$. Note that $t^*(G) \geq t(G) = a$. If $G$ is a tree, then $G$ is a caterpillar with diam$(G) = a - 2$ by Proposition 3.14 and therefore

$$t^*(G) = 2a - 2 - \text{rad}(G) = 2a - 2 - \left\lfloor \frac{a - 2}{2} \right\rfloor = \left\lfloor \frac{3a}{2} \right\rfloor - 1.$$
If $G$ is not a tree, then $G$ contains a spanning subgraph $H = G_{k, \ell_1, \ell_2}$ where $k \geq 3$, $1 \leq \ell_2 \leq \ell_1$, and $a = k + \ell_1 + \ell_2$. We consider three cases.

**Case 1.** $k \geq 2\ell_1 + 2$. Observe that

$$2(\ell_1 + \ell_2) \leq (\ell_1 + \ell_2) + (2\ell_1) \leq (\ell_1 + \ell_2) + k - 2 = a - 2.$$ 

Hence by Proposition 5.20,

$$t^*(G) \leq t^*(H) = a + (\ell_1 + \ell_2) - 1 \leq a + \left[ \frac{a - 2}{2} \right] - 1$$

$$= \left[ \frac{3a}{2} \right] - 2 < \left[ \frac{3a}{2} \right] - 1.$$

**Case 2.** $k \leq 2\ell_1 + 2$ and $\ell_1 - \ell_2 \geq 2 \left[ \frac{k}{2} \right] - 4$. By Proposition 5.20,

$$t^*(G) \leq t^*(H) = a + \left[ \frac{\ell_1 + \ell_2}{2} \right] \leq a + \left[ \frac{\ell_1 + \ell_2 + k - 2}{2} \right]$$

$$= a + \left[ \frac{a - 2}{2} \right] \leq \left[ \frac{3a}{2} \right] - 1.$$

**Case 3.** $k \leq 2\ell_1 + 2$ and $\ell_1 - \ell_2 \leq 2 \left[ \frac{k}{2} \right] - 5$. By Proposition 5.20,

$$t^*(G) \leq t^*(H) = a + \ell_2 + \left[ \frac{k}{2} \right] - 2 = a + \left[ \frac{2\ell_2 + k}{2} \right] - 2$$

$$\leq a + \left[ \frac{\ell_1 + \ell_2 + k}{2} \right] - 2 = a + \left[ \frac{a}{2} \right] - 2 = \left[ \frac{3a}{2} \right] - 2$$

$$< \left[ \frac{3a}{2} \right] - 1,$$

completing the proof.

Therefore, we are able to determine all pairs $a, b$ of integers with $1 \leq a \leq 12$ and $a \leq b$ for which there is a connected graph $G$ such that $g(G) = (a, b)$.

---

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Theorem 6.14  Let $a$ be an integer with $1 \leq a \leq 12$. Then

$$\{g(G) : t(G) = a\} = \left\{ (a, b) : a \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor \right\}.$$  

Proof. The result follows for $1 \leq a \leq 10$ by Propositions 6.10, 6.11, and 6.12. So suppose that $a \in \{11, 12\}$. By Proposition 6.8,

$$\{g(G) : t(G) = a\} \supseteq \left\{ (a, b) : a \leq b \leq \left\lfloor \frac{3a}{2} \right\rfloor \right\}.$$  

Let $G$ be a connected graph of order $n$ with $t(G) = a$.

Case 1. $a = 11$. Observe that $8 \leq n \leq 12$ by (6.1) and $G$ is traceable if and only if $n = 12$. We will show that if $8 \leq n \leq 11$, then $t^*(G) \leq \left\lfloor \frac{311}{2} \right\rfloor = 16$. First observe that $G$ is not a star since otherwise $11 = t(G) = 2n - 4$, which is impossible. Hence for $8 \leq n \leq 10$, it follows that $t^*(G) \leq 2 \cdot 10 - 4 = 16$ by Observation 6.9. If $n = 11$, then $t^*(G) \leq \left\lfloor \frac{311}{2} \right\rfloor - 1 = 15$ by Proposition 6.13.

Case 2. $a = 12$. Then $8 \leq n \leq 13$ and $G$ is traceable if and only if $n = 13$. We will show that if $8 \leq n \leq 12$, then $t^*(G) \leq \left\lfloor \frac{312}{2} \right\rfloor = 18$. If $G$ is a star $K_{1,n-1}$, then $12 = t(G) = 2n - 4$, implying that $n = 8$ and $t^*(K_{1,7}) = 2 \cdot 8 - 3 = 13$. Thus for $9 \leq n \leq 11$, Observation 6.9 guarantees that $t^*(G) \leq 2 \cdot 11 - 4 = 18$. For $n = 12$, we have $t^*(G) \leq \left\lfloor \frac{312}{2} \right\rfloor - 1 = 17$ by Proposition 6.13.  

If $G$ is a connected graph of order $n$ with $t(G) = 13$, then $9 \leq n \leq 14$. By Proposition 6.8, there exists a traceable graph $G$ of order 14 with $t^*(G) = b$ for each integer $b$ with $13 \leq b \leq 19$. If $G$ is nontraceable graph, then $G \neq K_{1,n-1}$. Hence if $9 \leq n \leq 11$, then $t^*(G) \leq 2 \cdot 11 - 4 = 18$. Also if $n = 13$, then $t^*(G) \leq \left\lfloor \frac{313}{2} \right\rfloor - 1 = 18$. If $n = 12$, then $t^*(G) \leq 2 \cdot 12 - 4 = 20$. Whether there is a connected graph $G$ of order 12 with $g(G) = (13,19)$ or $g(G) = (13,20)$ is not known.
6.3 Traceable and Maximum Traceable Ratios

Let \( G \) be a connected graph of order \( n \geq 2 \). We define the *traceable ratio* \( r_t(G) \) and *maximum traceable ratio* \( r_{t^*}(G) \) of \( G \) by

\[
 r_t(G) = \frac{t(G)}{n} \quad \text{and} \quad r_{t^*}(G) = \frac{t^*(G)}{n}.
\]

Observe that \( r_t(G) \) and \( r_{t^*}(G) \) are positive rational numbers and since \( t(G) \leq t^*(G) \), it follows that \( r_t(G) \leq r_{t^*}(G) \). Also, since \( n-1 \leq t(G) \leq 2n-4 \) and \( n-1 \leq t^*(G) \leq 2n-3 \), we have

\[
\frac{1}{2} \leq 1 - \frac{1}{n} \leq r_t(G) \leq r_{t^*}(G) \leq 2 - \frac{3}{n} < 2.
\]

For a nontrivial connected graph \( G \), let

\[
g_r(G) = (r_t(G), r_{t^*}(G)).
\]

We next consider the following question.

**Problem 6.15** For which pairs \( \alpha, \beta \) of rational numbers with

\[
\frac{1}{2} \leq \alpha \leq \beta < 2,
\]

does there exist a connected graph \( G \) for which \( g_r(G) = (\alpha, \beta) \)?
then observe that \( g_r(K_n) = (1 - \frac{1}{n}, 1 - \frac{1}{n}) = (\alpha, \beta) \). Moreover, every nontrivial connected graph \( G \) with \( g_r(G) = (1 - \frac{1}{n}, 1 - \frac{1}{n}) \) must have order \( n \). We state this observation as follows.

**Observation 6.16** Let \( \alpha \) and \( \beta \) be rational numbers such that \( \frac{1}{2} \leq \alpha \leq \beta < 2 \). If \( \alpha < 1 \), then there exists a nontrivial connected graph \( G \) with \( g_r(G) = (\alpha, \beta) \) if and only if \( \alpha = 1 - \frac{1}{n} \) for some integer \( n \geq 2 \). Moreover, if \( \beta < 1 \), then there exists a nontrivial connected graph \( G \) with \( g_r(G) = (\alpha, \beta) \) if and only if \( \alpha = \beta = 1 - \frac{1}{n} \) for some integer \( n \geq 2 \). In either case, such a graph \( G \) must be a traceable graph of order \( n \). In the case where \( \alpha = \beta = 1 - \frac{1}{n} \), such a graph \( G \) must be traceable and traceably singular.

From now on we assume that \( \beta \geq 1 \). Let \( G \) be a connected graph with \( g_r(G) = (\alpha, \beta) \). The order of \( G \) must be at least 3 since there is no connected graph of order 2 with maximum traceable number 2. If \( \frac{1}{2} \leq \alpha < 1 \), then \( \alpha = 1 - \frac{1}{n} \) for some integer \( n \geq 3 \) and \( G \) is a traceable graph of order \( n \) by Observation 6.16. Since \( G \) contains a Hamiltonian path, \( n \leq t^*(G) \leq t^*(P_n) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor \). Hence \( 1 \leq \beta \leq \frac{1}{n} \left\lfloor \frac{3(n-1)}{2} \right\rfloor \). If \( \beta = 1 \), then let \( G \) be the graph of order \( n \) obtained by adding a pendant edge to \( K_{n-1} \). Then \( G \) is traceable and \( t^*(G) = n \). Hence \( g_r(G) = (1 - \frac{1}{n}, 1) \).

**Observation 6.17** The ordered pair \((1 - \frac{1}{n}, 1)\) is realizable as the value \( g_r(G) \) of some nontrivial connected graph \( G \) if and only if \( n \geq 3 \). In addition, such a graph \( G \) must be a traceable graph of order \( n \).

Next suppose that \( \frac{1}{2} \leq \alpha < 1 < \beta \). If \( G \) is a connected graph of order \( n \) with \( g_r(G) = (\alpha, \beta) \), then \( G \) is traceable and \( \alpha = 1 - \frac{1}{n} \) by Observation 6.16. Also \( n \geq 5 \) since for \( n \leq 4 \) there is no traceable graph of order \( n \) with maximum traceable number greater than \( n \). In general, if \( n = 2k + 1 \) for some positive integer \( k \geq 1 \), then observe that \( t^*(P_{2k+1}) = 3k \) and

\[
\beta \leq t^*(P_{2k+1}) = \frac{3k}{2k+1} < \frac{3}{2}. \tag{6.2}
\]

Similarly, if \( n = 2k \) for some integer \( k \geq 2 \), then \( t^*(P_{2k}) = 3k - 2 \) and

\[
\beta \leq t^*(P_{2k}) = \frac{3k - 2}{2k} < \frac{3}{2}. \tag{6.3}
\]
Hence suppose that $1 < \beta < \frac{3}{2}$. Let $a$ and $b$ be positive integers such that $\beta = 1 + \frac{a}{b}$ and $\gcd(a, b) = 1$. Then $1 \leq a < \frac{b}{2}$, that is, $b - 2a \geq 1$. Since $\gcd(a, b) = 1$, it follows that $n$ is divisible by $b$. Let $n = bc$ where $c$ is a positive integer. We first show that $c(b - 2a) \geq 3$. Since this is true for $c \geq 3$, let $c \leq 2$. Assume, to the contrary, that $c(b - 2a) \leq 2$. If $c = 1$, then $n = b \in \{2a + 1, 2a + 2\}$. If $n = b = 2a + 1$, then observe that

$$\beta = \frac{3a + 1}{2a + 1} > \frac{3a}{2a + 1},$$

contradicting the inequality in (6.2). Similarly, if $n = b = 2a + 2$, then

$$\beta = \frac{3a + 2}{2a + 2} = \frac{3(a + 1) - 1}{2(a + 1)} > \frac{3(a + 1) - 2}{2(a + 1)},$$

contradicting the inequality in (6.3). If $c = 2$, then $n = 2b \in \{4a + 1, 4a + 2\}$. Since $n$ is even, $n = 2b = 4a + 2$. However,

$$\beta = \frac{2(a + b)}{2b} = \frac{6a + 2}{4a + 2} = \frac{3(2a + 1) - 1}{2(2a + 1)} > \frac{3(2a + 1) - 2}{2(2a + 1)},$$

contradicting the inequality in (6.3). Hence $c(b - 2a) \geq 3$. We now consider two cases, namely (1) $c(b - 2a) = 3$ and (2) $c(b - 2a) \geq 4$. If $c(b - 2a) = 3$, then (i) $c = 1$ and $n = b = 2a + 3 \geq 5$; or (ii) $c = 3$ and $n = 3b = 6a + 3 \geq 9$. For both cases, let $G = P_n$ be the path of order $n$. It is routine to verify that $t^r(G) = \beta$ using the equation in (6.2). If $c(b - 2a) \geq 4$, then let $G = G_{bc,c(b-2a)-1}$ be the connected graph of order $n = bc$ obtained from a complete graph $K_{c(b-2a)-1}$ of order $c(b - 2a) - 1 \geq 3$ and a path $P$ of order $n - c(b - 2a) + 1 = 2ac + 1$ by identifying one of the $c(b - 2a) - 1$ vertices of $K_{c(b-2a)-1}$ and one of the two end-vertices of $P$. Note that $G$ is traceable and by Proposition 6.7,

$$t^*(G) = \left[\frac{3(bc) - (c(b - 2a) - 1) - 1}{2}\right] = ac + bc.$$

Hence

$$r_1^r(G) = \frac{ac + bc}{bc} = \beta.$$

In this case $r_1(G) = 1 - \frac{1}{bc}$. We can state this as follows.
Observation 6.18  Let $\beta$ be a rational number such that $1 < \beta < 2$ and let $n \geq 4$ be an integer. The ordered pair $(1 - \frac{1}{n}, \beta)$ is realizable as the value $g_r(G)$ of some nontrivial connected graph $G$ if and only if $\beta$ satisfies both of the following two conditions:

(1) $\beta = 1 + \frac{b}{a}$ for some integers $a$ and $b$ such that $1 \leq a < \frac{b}{2}$ (that is, $1 < \beta < \frac{3}{2}$);

(2) There exists a positive integer $c$ such that $n = bc$ and $c(b - 2a) \geq 3$.

In addition, such a graph $G$ must be a traceable graph of order $n$.

We summarize Observations 6.16, 6.17, and 6.18 in the following result.

Proposition 6.19  Let $\alpha$ and $\beta$ be rational numbers such that $\frac{1}{2} \leq \alpha \leq \beta < 2$. If $\alpha < 1$, then $\alpha$ is realizable as the traceable ratio of some nontrivial connected graph $G$ if and only if $\alpha = 1 - \frac{1}{n}$ for some integer $n \geq 2$. In this case $G$ must be a traceable graph of order $n$. Moreover, the ordered pair $(\alpha, \beta) = (1 - \frac{1}{n}, \beta)$ is realizable as the value $g_r(G)$ of some nontrivial connected graph $G$ if and only if the pair $\beta, n$ satisfies one of the following:

(1) $\beta = 1 - \frac{1}{n}$ and $n \geq 2$ (In this case $G$ is traceably singular.);

(2) $\beta = 1$ and $n \geq 3$;

(3) $\beta = 1 + \frac{a}{b}$ for some integers $a$ and $b$, where $1 \leq a < \frac{b}{2}$ and there exists a positive integer $c$ such that $n = bc$ ($\geq 5$) and $c(b - 2a) \geq 3$.

Next let $1 \leq \alpha \leq \beta < 2$ and suppose that $G$ is a nontrivial connected graph such that $g_r(G) = (\alpha, \beta)$. Then $G$ is not traceable since $r_t(G) \geq 1$. What remains unknown here is the following.

Problem 6.20  For a rational number $\alpha$ with $1 \leq \alpha < 2$, does there exist a connected graph $G$ such that $g_r(G) = (\alpha, \alpha)$?

Notice that Problem 6.20 is equivalent to the following, which we have asked in Section 5.2:

Does there exist a traceably singular graph that is not traceable?
Suppose next that $\alpha = 1 < \beta < 2$.

**Proposition 6.21** Let $\beta$ be a rational number such that $1 < \beta$. If there exists a connected graph $G$ for which $g_r(G) = (1, \beta)$, then $\beta < \frac{3}{2}$.

**Proof.** Assume, to the contrary, that there exists a connected graph $G$ of order $n \geq 2$ for which $g_r(G) = (1, \beta)$ with $\beta \geq \frac{3}{2}$. Since $r_t(G) = 1$, it follows that $t(G) = n$. We first show that $G$ cannot be a tree. If $G$ is a tree, then $G$ is a caterpillar of diameter $n - 2$ by Proposition 3.14. However, this implies that

$$t^*(G) = 2n - 2 - \left\lfloor \frac{n - 2}{2} \right\rfloor < \frac{3n}{2},$$

a contradiction. Hence $G$ is not a tree. Then $G$ contains a spanning walk

$$W : v_1, v_2, \ldots, v_{n+1}$$

of length $n$, where $v_i = v_j$ for some $i$ and $j$ with $2 \leq i < i + 3 \leq j \leq n$. Without loss of generality, assume that $i - 1 \geq n + 1 - j$, that is, $i + j \geq n + 2$. Let $H$ be the spanning subgraph of $G$ of size $n$ whose edge set consists of the $n$ edges of $W$. Observe that $H$ is the graph obtained from a cycle $C$ of order $k = j - i$ and two paths $Q_1$ and $Q_2$ of lengths $\ell_1 = i - 1$ and $\ell_2 = n + 1 - j$, respectively, by identifying one of the $k$ vertices on $C$ with one of the two end-vertices of $Q_i$ for $i = 1, 2$. Note that $n = k + \ell_1 + \ell_2$. Also $\ell_1 \geq \ell_2$ since $i + j \geq n + 2$. By Proposition 5.20,

$$\frac{3n}{2} \leq t^*(G) \leq t^*(H) = \begin{cases} 
  n + \ell_1 + \ell_2 - 1 & \text{if } k \geq 2\ell_1 + 2 \\
  n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor & \text{if } k \leq 2\ell_1 + 1 \text{ and } \ell_1 - \ell_2 \geq 2 \left\lfloor \frac{\ell_1}{2} \right\rfloor - 4 \\
  n + \ell_2 + \left\lfloor \frac{\ell_2}{2} \right\rfloor - 2 & \text{if } k \leq 2\ell_1 + 1 \text{ and } \ell_1 - \ell_2 \leq 2 \left\lfloor \frac{\ell_1}{2} \right\rfloor - 5.
\end{cases}$$

If $t^*(H) = n + \ell_1 + \ell_2 - 1 \geq \frac{3n}{2}$, then $\ell_1 + \ell_2 \geq \frac{n}{2} + 1$. Hence

$$k = n - (\ell_1 + \ell_2) \leq n - \left( \frac{n}{2} + 1 \right) = \frac{n}{2} - 1.$$

On the other hand,

$$k \geq 2\ell_1 + 2 \geq (\ell_1 + \ell_2) + 2 \geq \left( \frac{n}{2} + 1 \right) + 2 = \frac{n}{2} + 3,$$

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a contradiction. If $t^*(H) = n + \left\lfloor \frac{\ell_1 + \ell_2}{2} \right\rfloor \geq \frac{3n}{2}$, then $\ell_1 + \ell_2 \geq n$, implying that $k = n - (\ell_1 + \ell_2) \leq 0$, a contradiction. Finally if $t^*(H) = n + \ell_2 + \left\lfloor \frac{\ell_1}{2} \right\rfloor - 2 \geq \frac{3n}{2}$, then $\left\lfloor \frac{\ell_1}{2} \right\rfloor \geq \frac{n}{2} - \ell_2 + 2$. However, this implies that

$$k \geq n - \ell_2 + 4 \geq n - (\ell_1 + \ell_2) + 4 \geq k + 4,$$

a contradiction. Hence there is no connected graph $G$ for which $g_r(G) = (1, \beta)$ with $\beta \geq \frac{3}{2}$.

Let $n$, $\ell$, and $p$ be integers such that $\ell \geq 2$, $p \geq 1$, and $n \geq \ell + p + 2$. Let $G_{n, \ell, p}$ be the graph of order $n$ obtained from a complete graph $K_{n-\ell-p}$ and a path $P$ of order $\ell$ by (i) joining one of the $n - \ell - p$ vertices, say $x$, of $K_{n-\ell-p}$ and one of the two end-vertices of $P$ and (ii) adding $p$ new vertices and joining each of them to the vertex $x$ (so $\Delta(G) = \deg x = k + p$). The following proposition will be useful in the discussion that follows.

**Proposition 6.22** Let $n$, $\ell$, and $p$ be integers such that $\ell \geq 2$, $p \geq 1$, and $n \geq \ell + p + 2$. Then

$$g(G_{n, \ell, p}) = \left(n + p - 1, n + p - 1 + \left\lfloor \frac{\ell + 1}{2} \right\rfloor \right).$$

**Proof.** Let $P : v_0, v_1, \ldots, v_{\ell-1}$ be a path of order $\ell$ and join the vertex $v_{\ell-1}$ to one of the $n - \ell - p$ vertices of $K_{n-\ell-p}$, say $x$. Let $u_1, u_2, \ldots, u_{n-\ell-p-1}$ be the remaining vertices of $K_{n-\ell-p}$. The graph $G_{n, \ell, p}$ is then obtained by adding and joining $p$ new vertices $w_1, w_2, \ldots, w_p$ to $x$. Let $x = v_\ell$ and observe that

$$t(v) = (n - \ell - p) + \ell + 2p - 1 = n + p - 1$$

for each $v \in V(G_{n, \ell, p}) - \{v_1, v_2, \ldots, v_\ell\}$. For $v_i$ $(1 \leq i \leq \ell)$, either of the two linear orderings

\[ s_i : v_i, v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_\ell = x, u_1, u_2, \ldots, u_{n-\ell-p-1}, w_1, w_2, \ldots, w_p \]

\[ s'_i : v_i, v_{i-1}, \ldots, v_\ell = x, u_1, u_2, \ldots, u_{n-\ell-p-1}, w_1, w_2, \ldots, w_p, v_{i-1}, v_{i-2}, \ldots, v_0 \]

gives us $t(v_i)$. Hence

$$t(v_i) = \min\{n + p - 1 + i, (\ell - i) + (n - \ell - p) + 2p + \ell\}$$

$$= \begin{cases} n + p - 1 + i & \text{if } i \leq \left\lfloor \frac{\ell+1}{2} \right\rfloor \\ n + p - 1 + (\ell + 1) - i & \text{if } i \geq \left\lfloor \frac{\ell+1}{2} \right\rfloor. \end{cases}$$
Hence \( t(v_i) > n + p - 1 \), that is, \( t(G_{n,\ell,\rho}) = n + p - 1 \) and

\[
\max_{1 \leq i \leq \ell} t(v_i) = n + p - 1 + \left\lfloor \frac{\ell + 1}{2} \right\rfloor.
\]

Therefore \( t^*(G_{n,\ell,\rho}) = n + p - 1 + \left\lfloor \frac{\ell + 1}{2} \right\rfloor \).

We are now ready to show that the ordered pair \((1,\beta)\) is realizable as the value \( g_r(G) \) of some connected graph \( G \) for each rational number \( \beta \) with \( 1 < \beta < \frac{3}{2} \).

Let \( \beta = \frac{a}{b} \), where \( a \) and \( b \) are some positive integers. Since \( b < a \) and \( 2a < 3b \), observe that \( 2(3b - 2a) \geq 2 \) and \( 4(a - b) \geq 4 \). Consider the connected graph \( G_{2b,4(a-b)-1,1} \) of order \( n = 2b \) constructed in the way described above (so \( \ell = 4(a-b) - 1 \geq 2 \), \( p = 1 \), and \( n = \ell + p + 2(3b - 2a) \geq \ell + p + 2 \)). Observe that by Proposition 6.22, it follows that \( r_t(G_{2b,4(a-b)-1,1}) = 1 \) and

\[
r_t^*(G_{2b,4(a-b)-1,1,1}) = \frac{1}{2b} \left( 2b + \left\lfloor \frac{4(a-b)}{2} \right\rfloor \right) = \frac{2a}{2b} = \beta.
\]

Hence \( g_r(G_{2b,4(a-b)-1,1,1}) = (1,\beta) \). We summarize this case with \( \alpha = 1 \) as follows.

**Proposition 6.23** Let \( \beta \) be a rational number. If \( G \) is a connected graph for which \( g_r(G) = (1,\beta) \), then \( 1 \leq \beta < \frac{3}{2} \). Furthermore, if \( 1 < \beta < \frac{3}{2} \), then there is a connected graph \( G \) such that \( g_r(G) = (1,\beta) \).

So far we have studied those ordered pairs of rational numbers \((\alpha,\beta)\) where \( \frac{1}{2} \leq \alpha \leq 1 \). We have seen that if there exists a connected graph \( G \) having \( g_r(G) = (\alpha,\beta) \) with \( \frac{1}{2} \leq \alpha \leq 1 \), then \( \alpha \leq \beta \leq \frac{3}{2}\alpha \). We give a generalization of this fact in the next proposition.

**Proposition 6.24** Let \( \alpha \) and \( \beta \) be rational numbers such that \( \frac{1}{2} \leq \alpha \leq \beta < 2 \). If \( G \) is a connected graph for which \( g_r(G) = (\alpha,\beta) \), then \( \beta < \frac{3}{2}\alpha \).

**Proof.** Since the result has been verified for \( \frac{1}{2} \leq \alpha \leq 1 \), we may assume that \( 1 < \alpha \leq \beta < 2 \). Let \( G \) be a connected graph of order \( n \) for which \( g_r(G) = (\alpha,\beta) \). Let \( s : v_1, v_2, \ldots, v_n \) be a linear ordering of vertices of \( G \) for which \( d(s) = t(G) \) and \( v^* \) a vertex of \( G \) for which \( t(v^*) = t^*(G) \). Then \( v^* = v_i \) for some \( i \) with \( 1 \leq i \leq n \).
If \( v^* = v_1 \) or \( v^* = v_n \), then \( t^*(G) = t(v^*) = d(s) = t(G) \) and therefore \( \beta = \alpha < \frac{3}{2} \alpha \).

Hence we assume that \( 2 \leq i \leq n - 1 \). Without loss of generality, assume that \( d(v^*, v_1) \leq d(v^*, v_n) \). Observe that \( d(v^*, v_1) \leq \frac{t(G)}{2} \). Consider the linear ordering \( s' \) whose initial vertex is \( v^* \) given by

\[
s' : v^* = v_i, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n.
\]

Observe that

\[
t^*(G) = t(v^*) \leq d(s')
\]

\[
= d(v^*, v_1) + \sum_{j=1}^{n-1} d(v_j, v_{j+1}) + d(v_{i-1}, v_{i+1}) - d(v_{i-1}, v^*) - d(v^*, v_{i+1})
\]

\[
\leq d(v^*, v_1) + \sum_{j=1}^{n-1} d(v_j, v_{j+1}) \leq \frac{t(G)}{2} + d(s) = \frac{3}{2} t(G).
\]

Hence \( t^*(G) \leq \frac{3}{2} t(G) \). Dividing the both sides of this inequality by \( n \), we obtain \( \beta \leq \frac{3}{2} \alpha \).

The following proposition gives another fact which holds for a realizable ordered pair \((\alpha, \beta)\) of rational numbers in general.

**Proposition 6.25** Let \( \alpha \) and \( \beta \) be rational numbers such that \( \frac{1}{2} \leq \alpha \leq \beta < 2 \). If \( G \) is a connected graph for which \( gr(G) = (\alpha, \beta) \), then \( \beta < 1 + \frac{\alpha}{2} \).

**Proof.** Assume, to the contrary, that there exists a connected graph \( G \) of order \( n \) for which \( gr(G) = (\alpha, \beta) \) and \( \beta \geq 1 + \frac{\alpha}{2} \). Then \( 2(\beta - 1) \geq \alpha \) and so \( 2(t^*(G) - n) \geq t(G) \). Let \( x, y \in V(G) \) such that \( t(x) = t(G) \) and \( t(y) = t^*(G) \). Let \( s : x = v_1, v_2, \ldots, v_n \) be a linear ordering of vertices of \( G \) whose initial vertex is \( x \) and such that \( d(s) = t(G) \). Observe that \( y = v_i \) for some \( i \) with \( 2 \leq i \leq n - 2 \). Consider the linear ordering \( s' \) whose initial vertex is \( y \) given by

\[
s' : y = v_i, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n.
\]
Then
\[ t^*(G) = t(y) \leq d(s') \]
\[ = d(y, v_1) + \sum_{j=1}^{n-1} d(v_j, v_{j+1}) + d(v_{j-1}, v_{j+1}) - d(v_{i-1}, y) - d(y, v_{i+1}) \]
\[ \leq d(y, v_1) + \sum_{j=1}^{n-1} d(v_j, v_{j+1}) = d(y, v_1) + d(s) = d(y, v_1) + t(G). \]

Hence \( t^*(G) - t(G) \leq d(y, v_1) \). Now let \( T \) be a spanning tree of \( G \). Then for every pair \( u, v \) of vertices of \( G \), we have \( d_G(u, v) \leq d_T(u, v) \) and so \( t_G(w) \leq t_T(w) \) for every vertex \( w \in V(G) \). Therefore

\[ t^*(G) - t(G) \leq d_G(y, v_1) \leq d_T(y, v_1) \leq c_T(y). \]

By Theorem 3.28,
\[ t_T(y) = 2n - 2 - c_T(y) \leq 2n - 2 - (t^*(G) - t(G)) \]
and therefore
\[ t^*(G) = t_G(y) \leq t_T(y) \leq 2n - 2 - t^*(G) + t(G). \]

However, this implies that
\[ t(G) \leq 2(t^*(G) - n) \leq t(G) - 2, \]
a contradiction. Hence \( \beta < 1 + \frac{\alpha}{2} \).

Finally suppose that \( 1 < \alpha < \beta < 1 + \frac{\alpha}{2} \). Let \( \alpha = \frac{a}{b} \) and \( \beta = \frac{c}{d} \), where \( a, b, c, \) and \( d \) are positive integers. Thus \( b < a < 2b, \) \( d < c < 2d, \) \( bc - ad > 0, \) and \( ad + 2bd - 2bc > 0. \) Hence \( 2(ad + 2bd - 2bc) \geq 2, \) \( 4(bc - ad) - 1 \geq 3, \) and \( 2d(a - b) + 1 \geq 3. \) Consider the connected graph \( G = G_{2(ad + 2bd - 2bc), 4(bc - ad) - 1, 2d(a - b) + 1} \) of order \( n = 2bd \) which appears in Proposition 6.22 (so \( k = 2(ad + 2bc - 2bc) \geq 2, \) \( \ell = 4(bc - ad) - 1 \geq 2, \) and \( p = 2d(a - b) + 1 \geq 1. \) By Proposition 6.22, it follows that \( t(G) = 2bd + (2d(a - b) + 1) - 1 = 2ad \) and
\[ t^*(G) = 2ad + \left( \frac{4(bc - ad)}{2} \right) = 2bc. \]
Since \( n = 2bd \), this implies that

\[
gr_r(G) = \left( \frac{2ad}{2bd}, \frac{2bc}{2bd} \right) = (\alpha, \beta).
\]

Hence we obtain the following proposition.

**Proposition 6.26** Let \( \alpha \) and \( \beta \) be rational numbers such that \( 1 < \alpha < \beta < 1 + \frac{3}{2} < 2 \). Then the ordered pair \((\alpha, \beta)\) is realizable as the value \( gr_r(G) \) of some connected graph \( G \).

Let us summarize what we have obtained.

**Theorem 6.27** Let \( \alpha \) and \( \beta \) be rational numbers such that \( \frac{1}{2} \leq \alpha \leq \beta < 2 \). Then the ordered pair \((\alpha, \beta)\) is realizable as the value \( gr_r(G) \) of some connected graph \( G \) if the pair \( \alpha, \beta \) satisfies one of the following four conditions:

1. \( \alpha = \beta = 1 - \frac{1}{n} \) for some integer \( n \geq 2 \);
2. \( \alpha = 1 - \frac{1}{n} \) for some integer \( n \geq 3 \) and \( \beta = 1 \);
3. \( \alpha = 1 - \frac{1}{n} \) for some integer \( n \geq 4 \), \( \beta = 1 + \frac{a}{b} < \frac{3}{2} \) where \( a, b \in \mathbb{N} \), and there exists a positive integer \( c \) such that \( n = bc \) and \( c(b - 2a) \geq 3 \);
4. \( 1 \leq \alpha < \beta < 1 + \frac{a}{2} \).

Therefore, we have determined all pairs \((\alpha, \beta)\) of rational numbers for which there is a connected graph \( G \) such that \( gr_r(G) = (\alpha, \beta) \) except for those pairs \((\alpha, \alpha)\) with \( 1 \leq \alpha < 2 \).

### 6.4 Traceable and Upper Traceable Numbers

In this section, we consider the following question.

**Problem 6.28** For which pairs \( a, c \) of positive integers with \( a \leq c \), does there exist a connected graph \( G \) for which \( t(G) = a \) and \( t^+(G) = c \)?
First, we recall the following facts related to the traceable number and upper traceable number of a graph, which appear in Chapters 3 and 4.

**Theorem 6.29** If $G$ is a connected graph of order $n \geq 3$, then

$$n - 1 \leq t^+(G) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1$$

and $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1$ if and only if $G = P_n$.

**Theorem 6.30** For every nontrivial tree $T$, its upper traceable number is odd. Moreover, if $n \geq 3$ and $k$ is an odd integer such that

$$2n - 3 \leq k \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1,$$

then there exists a tree $T$ of order $n$ for which $t^+(T) = k$.

**Theorem 6.31** For every connected graph $G$ of order $n \geq 4$,

$$t^+(G) \neq \left\lfloor \frac{n^2}{2} \right\rfloor - 2.$$

By the inequality in (6.1) and Theorem 6.29, if $G$ is a nontrivial connected graph of order $n$ with $t(G) = a$, then

$$a \leq t^+(G) \leq \left\lfloor \frac{(a + 1)^2}{2} \right\rfloor - 1.$$

For a connected graph $G$, let

$$q(G) = (t(G), t^+(G)).$$

Observe that for $1 \leq a \leq 4$, we have the following:

- $a = 1$ : $\{q(G) : t(G) = 1\} = \{(1, 1)\};$
- $a = 2$ : $\{q(G) : t(G) = 2\} = \{(2, 2), (2, 3)\};$
- $a = 3$ : $\{q(G) : t(G) = 3\} = \{(3, 3), (3, 4), (3, 5), (3, 7)\};$
- $a = 4$ : $\{q(G) : t(G) = 4\} = \{(4, 4), (4, 5), (4, 6), (4, 7), (4, 8), (4, 9), (4, 11)\}.$
To continue our discussion, the following proposition will be useful.

**Proposition 6.32** Let \( n \geq 5 \) be an integer. For each integer \( i \) with \( 1 \leq i \leq n - 1 \), let \( G_i \) be the connected graph of order \( n \) such that

\[
\overline{G_i} = P_{i+1} \cup (n - 1 - i)K_1.
\]

Then \( G_i \) is traceable and \( t^+(G_i) = n - 1 + i \).

**Proof.** For each \( i \), first observe that \( G_i \) contains a Hamiltonian cycle and so \( t(G_i) = n - 1 \). Also \( \text{diam}(G_i) = 2 \). Let \( V(G_i) = \{v_1, v_2, \ldots, v_{i+1}\} \cup \{u_1, u_2, \ldots, u_{n-1-i}\} \) such that \( P : v_1, v_2, \ldots, v_{i+1} \) is the path of order \( i + 1 \) in \( G_i \). Since \( G_i \) contains exactly \( i \) pairs \( x, y \) of vertices such that \( d_{G_i}(x, y) = 2 \), it follows that

\[
d(s) \leq 2i + (n - 1 - i) = n - 1 + i
\]

for every linear ordering \( s \) of vertices of \( G_i \) and so \( t^+(G_i) \leq n - 1 + i \). On the other hand, consider the linear sequence \( s' \) of vertices of \( G_i \) given by

\[
s' : v_1, v_2, \ldots, v_{i+1}, u_1, u_2, \ldots, u_{n-1-i}
\]

and observe that \( d(s') = 2i + (n - 1 - i) = n - 1 + i \). Hence \( t^+(G_i) = n - 1 + i \), as claimed. \( \blacksquare \)

Recall from Chapter 4 that \( t^+(C_n) = \left\lceil \frac{(n-1)^2}{2} \right\rceil \) and \( t^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \). By Proposition 6.32 together with the inequality in (6.1) and Theorem 6.29, we make the following observation.

**Observation 6.33** Let \( a \geq 4 \). Then

\[
\left\{ a, a + 1, \ldots, 2a, \left\lceil \frac{a^2}{2} \right\rceil, \left\lceil \frac{(a+1)^2}{2} \right\rceil - 1 \right\}
\]

\[
\subseteq \{t^+(G) : t(G) = a\} \subseteq \{a, a + 1, \ldots, \left\lfloor \frac{(a+1)^2}{2} \right\rfloor - 1\}.
\]

Let us consider the case where \( a = 5 \). If \( G \) is a connected graph of order \( n \) with \( t(G) = 5 \), then \( n = 5, 6 \). If \( n = 5 \), then \( 5 \leq t^+(G) \leq \left\lfloor \frac{5^2}{2} \right\rfloor - 1 = 11 \) by Theorem 6.29 and \( t^+(G) \neq 10 \) by Theorem 6.31. Also, observe that for a connected graph \( G \) of order 5,
\( t(G) = 5 \) if and only if \( G \in \{K_{1,4} + e, S_{2,3}\} \)

and \( t^+(K_{1,4} + e) = 7 \) while \( t^+(S_{2,3}) = 9 \). If \( n = 6 \), then \( G \) is traceable, \( 5 \leq t^+(G) \leq \left\lceil \frac{6^2}{2} \right\rceil - 1 = 17 \) by Theorem 6.29, and \( t^+(G) \neq 16 \) by Theorem 6.31. By Observation 6.33 with these facts, we have shown so far that

\[
\{5, 6, \ldots, 10, 13, 17\} \subseteq \{t^+(G) : t(G) = 5\} \subseteq \{5, 6, \ldots, 17\} - \{16\}.
\]

Let \( C : v_1, v_2, \ldots, v_6, v_7 = v_1 \) be a 6-cycle and \( G_{11} = C + v_1v_3 \) and \( G_{12} = C + v_1v_4 \).

We next show that \( t^+(G_i) = i \) for \( i = 11, 12 \). First, let

\[
s_1 : \ v_2, v_5, v_1, v_4, v_6, v_3
\]

be a linear ordering of vertices of \( G_{11} \). Then \( t^+(G_{11}) \geq d(s_1) = 11 \). To show that \( t^+(G_{11}) \leq 11 \), observe that \( \text{diam}(G_{11}) = 3 \) and that there is exactly one pair \( u, v \) of vertices for which \( d(u, v) = 3 \) (namely \( d(v_2, v_5) = 3 \)). Hence if \( s'_1 \) is a linear ordering of vertices of \( G_{11} \) for which \( d(s'_1) = t^+(G_{11}) \), then

\[
t^+(G_{11}) = d(s'_1) \leq 1 \cdot 3 + 4 \cdot 2 = 11.
\]

For \( G_{12} \), consider the linear ordering

\[
s_2 : \ v_1, v_3, v_6, v_4, v_2, v_5
\]

and observe that \( t^+(G_{12}) \geq d(s_2) = 12 \). On the other hand, \( \text{diam}(G_{12}) = 3 \) and there are exactly two pairs \( u, v \) of vertices for which \( d(u, v) = 3 \) (namely \( d(v_2, v_5) = d(v_3, v_6) = 3 \)). Hence if \( s'_2 \) is a linear ordering of vertices of \( G_{12} \) for which \( d(s'_2) = t^+(G_{12}) \), then

\[
t^+(G_{12}) = d(s'_2) \leq 2 \cdot 3 + 3 \cdot 2 = 12.
\]

Therefore, we have shown that \( t^+(G_i) = i \) for \( i = 11, 12 \). Next let \( P_6 : v_1, v_2, \ldots, v_6 \) be a path of order 6 and \( G_{15} = P_6 + v_4v_6 \). Then \( G_{15} \) is traceable. We will show that \( t^+(G_{15}) = 15 \). By Propositions 6.29 and 6.31, it follows that \( t^+(G_{15}) \leq 15 \). On the other hand, consider the linear ordering \( s_3 \) of vertices of \( G_{15} \) given by

\[
s_3 : \ v_4, v_1, v_5, v_2, v_6, v_3
\]

and observe that \( t^+(G_{15}) \geq d(s_3) = 15 \). Therefore, we now have

\[
\{t^+(G) : t(G) = 5\} \supseteq \{5, 6, \ldots, 13, 15, 17\} = \{c \in \mathbb{N} : 5 \leq c \leq 17\} - \{14, 16\}.
\]

Our question is:

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Is there a traceable graph $G$ of order 6 such that $t^+(G) = 14$?

We will shortly show that the answer to this question is no. We first make an observation.

**Observation 6.34** If $T$ is a tree of order 6, then $T \in \{P_6, S_{2,4}, S_{3,3}, K_{1,5}, T_1, T_2\}$, where $T_1$ and $T_2$ are the two caterpillars of diameter 4 shown in Figure 6.3.

![Figure 6.3: Trees $T_1$ and $T_2$ of order 6 and diameter 4](image)

Moreover,

$q(P_6) = (5, 17), ~ q(T_1) = (6, 15), ~ q(T_2) = (6, 13),

q(S_{3,3}) = (7, 13), ~ q(S_{2,4}) = (7, 11), ~ q(K_{1,5}) = (8, 9).$

**Proposition 6.35** If $G$ is a traceable graph of order 6, then $t^+(G) \neq 14$.

**Proof.** Suppose that there exists a traceable graph $G$ of order 6 with $t^+(G) = 14$. Then $G \neq P_6$. Let $P : v_1, v_2, \ldots, v_6$ be a path of order 6 and let

\[
\begin{align*}
H_1 &= P + v_4v_6 \\
H_2 &= P + v_3v_6 \\
H_3 &= P + v_2v_6 \\
H_4 &= P + v_2v_5 \\
H_5 &= P + v_2v_4 \\
H_6 &= P + v_1v_6 = C_6
\end{align*}
\]

be the six traceable graphs of order 6 and size 6. Notice that $G$ is not a tree by Theorem 6.30. Hence $G$ contains at least one of the graphs $H_i$ ($1 \leq i \leq 6$) as a spanning subgraph. Since $t^+(H_6) = t^+(C_6) = 13 < t^+(G)$, it follows that $G$ cannot contain $H_6$ as a spanning subgraph. Also, since every spanning tree $T$ of $G$ must be isomorphic to either $P_6$ or $T_1$ by Observation 6.34, it follows that among those
five graphs \(H_i\) \((1 \leq i \leq 5)\), we see that \(H_1\) is the only graph that can be a spanning subgraph of \(G\). Moreover, \(G = H_1\), since otherwise \(G\) will contain a spanning tree that is neither \(P_6\) nor \(T_1\), a contradiction. However, the linear ordering

\[
s : v_4, v_1, v_5, v_2, v_6, v_3
\]

of vertices of \(G = H_1\) yields \(t^+(G) \geq d(s) = 15\), a contradiction. Hence there is no traceable graph \(G\) of order 6 whose upper traceable number is 14.

With the aid of Proposition 6.35, we finally conclude that

\[
\{t^+(G) : t(G) = 5\} = \{c \in \mathbb{N} : 5 \leq c \leq 17\} - \{14, 16\}.
\]

In other words,

\[
\{q(G) : t(G) = 5\} = \{(5, c) : 5 \leq c \leq 17 \text{ and } c \neq 14, 16\}.
\]

Proposition 6.35 shows that if \(G\) is a traceable graph of order 6, then \(t^+(G) \neq 14 = \left\lceil \frac{6^2}{2} \right\rceil - 4\). To generalize this fact, we first make some observations. Recall that for a nontrivial tree \(T\),

\[
t^+(T) = 2 \sum_{e \in E(T)} \text{cn}(e) - 1,
\]

where \(\text{cn}(e)\) is the component number of edge \(e \in E(T)\). We denote this sum of the component numbers of edges of \(T\) by \(\text{cn}(T)\).

Let \(G\) be a nontrivial connected graph. For a vertex \(v\) and an edge \(e = uv\) in \(G\), the distance between them is

\[
d(v, e) = \min\{d(v, u), d(v, w)\}.
\]

**Proposition 6.36** Let \(T\) be a nontrivial tree of order at least 3 with \(\text{diam}(T) = d\). Then there exists an end-vertex \(v\) of \(T\) such that

\[
1 \leq \text{cn}(T) - \text{cn}(T - v) \leq \left\lfloor \frac{d}{2} \right\rfloor.
\]

**Proof.** Assume that \(T\) is a tree of order \(n \geq 3\). Let \(M = \max\{\text{cn}(e) : e \in E(T)\}\) and choose an edge \(e^* \in E(T)\) such that \(\text{cn}(e^*) = M\). If \(M = 1\), then \(T\) and \(T - v\)
are stars for every end-vertex $v$ of $T$. Hence $d = 2$ and $\text{cn}(T) - \text{cn}(T - v) = 1$, so the result holds. Thus we assume that $M \geq 2$. Let $U = \{v \in V(T) : \deg v = 1\}$ be the set of end-vertices of $T$. Let

$$\ell = \min_{v \in U} d(v, v^*)$$

and choose a vertex $v^* \in U$ such that $d(v^*, v^*) = \ell$. Note that $1 \leq \ell \leq \left\lfloor \frac{d+1}{2} \right\rfloor$. Let $P : v^* = v_0, v_1, v_2, \ldots, v_\ell, v_{\ell+1}$ be the path of length $\ell + 1$ that has the initial vertex $v^*$ and terminal edge $e^* = v_\ell v_{\ell+1}$. Let $X = \{v_\ell v_{i+1} \in E(T - v^*) : 1 \leq i \leq \ell\}$ be the set of edges in $T - v^*$ that belong to $P$. We first show that if $e \in E(T - v^*) - X$, then $\text{cn}_T(e) - \text{cn}_{T-v^*}(e) = 0$. Let $e \in E(T - v^*) - X$ and suppose that $G_1$ and $G_2$ are the two components of $T - e$. Necessarily, exactly one of $G_1$ and $G_2$ contains the entire $P$, say $G_1$ does. If $|V(G_1)| > |V(G_2)|$, then $\text{cn}_T(e) = |V(G_2)| = \text{cn}_{T-v^*}(e)$. If $|V(G_1)| \leq |V(G_2)|$, then first observe that $\text{cn}_T(e) = |V(G_1)|$. Let $H_1$ and $H_2$ be the two components of $T - e^*$. Then exactly one of $H_1$ and $H_2$ contains the entire $G_2$ and the edge $e$, say $H_2$ does. Then $|V(H_2)| \geq |V(G_2)| + 1$ and so

$$|V(H_1)| = n - |V(H_2)| \leq n - (|V(G_2)| + 1) = |V(G_1)| - 1.$$

This implies that

$$\text{cn}_T(e^*) \leq |V(H_1)| < |V(G_1)| = \text{cn}_T(e),$$

a contradiction. Hence $\text{cn}_T(e) - \text{cn}_{T-v^*}(e) = 0$ for every edge $e \in E(T - v^*) - X$. Next we show that if $e \in X$, then $0 \leq \text{cn}_T(e) - \text{cn}_{T-v^*}(e) \leq 1$. Let $G_1$ and $G_2$ be the two components of $T - e$ such that $v^*$ belongs to $G_1$. If $|V(G_1)| > |V(G_2)|$, then $\text{cn}_T(e) = |V(G_2)| = \text{cn}_{T-v^*}(e)$. If $|V(G_1)| \leq |V(G_2)|$, then $\text{cn}_T(e) = |V(G_1)|$ and

$$\text{cn}_{T-v^*}(e) = |V(G_1)| - 1 = \text{cn}_T(e) - 1.$$

Hence $0 \leq \text{cn}_T(e) - \text{cn}_{T-v^*}(e) \leq 1$ for each edge $e \in X$. Now observe that

$$\text{cn}(T) = \text{cn}_T(v^*v_1) + \sum_{e \in X} \text{cn}_T(e) + \sum_{e \in E(T-v^*)-X} \text{cn}_T(e)$$

$$= 1 + \sum_{e \in X} \text{cn}_T(e) + \sum_{e \in E(T-v^*)-X} \text{cn}_T(e).$$
\[ cn(T-v^*) = \sum_{e \in X} cn_{T-v^*}(e) + \sum_{e \in E(T-v^*)-X} cn_{T-v^*}(e) \]
\[ = \sum_{e \in X} cn_{T-v^*}(e) + \sum_{e \in E(T-v^*)-X} cn_T(e). \]

Thus \( cn(T) - cn(T-v^*) \geq 1 \) and

\[ cn(T) - cn(T-v^*) \leq 1 + \sum_{e \in X} \left( cn_T(e) - cn_{T-v^*}(e) \right) \]
\[ \leq 1 + |X| = 1 + \ell \leq 1 + \left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{d}{2} \right\rfloor, \]

completing the proof. \( \Box \)

For \( n \geq 4 \), let \( T_n \) be the caterpillar of order \( n \) with \( \text{diam}(T_n) = n - 2 \) containing two end-vertices \( u \) and \( v \) such that \( d(u,v) = 2 \).

**Proposition 6.37** Let \( T \) be a tree of order \( n \geq 4 \). Then
\[ t^+(T) = \left\lfloor \frac{n^2}{2} \right\rfloor - 3 \text{ if and only if } T = T_n. \]

**Proof.** The result follows immediately for \( 4 \leq n \leq 6 \), so we may assume that \( n \geq 7 \). We first show that \( t^+(T_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 3 \). We consider two cases, according to the parity of \( n \).

*Case 1. \( n \) is odd.* Then \( n = 2k + 1 \) for some integer \( k \geq 3 \). Observe that \( T_n \) contains exactly three edges with component number 1 and exactly one edge with component number 2. Moreover, for each integer \( i \) with \( 3 \leq i \leq k \), there are exactly two edges with component number \( i \). Therefore
\[ cn(T_n) = 3 + 2 + 2(3 + 4 + \cdots + k) \]
\[ = 5 + (k-2)(k+3) = k^2 + k - 1 \]

and so
\[ t^+(T_n) = 2cn(T_n) - 1 = 2k^2 + 2k - 3 = \left\lfloor \frac{(2k+1)^2}{2} \right\rfloor - 3. \]
**Case 2.** $n$ is even. Then $n = 2k$ for some integer $k \geq 4$. Observe that $T_n$ contains exactly three edges with component number 1, exactly one edge with component number 2, and exactly one edge with component number $k$. Moreover, for each integer $i$ with $3 \leq i \leq k - 1$, there are exactly two edges with component number $i$. Therefore

$$
\begin{align*}
\text{cn}(T_n) &= 3 + 2 + 2(3 + 4 + \cdots + (k - 1)) + k \\
&= 5 + (k - 3)(k + 2) + k = k^2 - 1
\end{align*}
$$

and so

$$
\begin{align*}
t^+(T_n) &= 2\text{cn}(T_n) - 1 = 2k^2 - 3 = \left\lceil \frac{(2k)^2}{2} \right\rceil - 3.
\end{align*}
$$

For the converse, let $T$ be a tree of order $n \geq 7$ having $t^+(T) = \left\lfloor \frac{n^2}{4} \right\rfloor - 3$. Then $T \neq P_n$ and $\text{cn}(T) = \left\lfloor \frac{n^2}{4} \right\rfloor - 1$. We first show that $\text{diam}(T) = n - 2$. Assume, to the contrary, that $d = \text{diam}(T) \leq n - 3$. Then $T - v \neq P_{n-1}$ for each end-vertex $v$ of $T$. Hence $t^+(T - v) \leq \left\lfloor \frac{(n-1)^2}{2} \right\rfloor - 3$ and consequently $\text{cn}(T - v) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - 1$.

Let $v^*$ be an end-vertex of $T$ such that $1 \leq \text{cn}(T) - \text{cn}(T - v^*)$. Let $v^*$ be an end-vertex of $T$ such that $1 \leq \text{cn}(T) - \text{cn}(T - v^*)$. Let $v^*$ be an end-vertex of $T$ such that $1 \leq \text{cn}(T) - \text{cn}(T - v^*)$. Let $v^*$ be an end-vertex of $T$ such that $1 \leq \text{cn}(T) - \text{cn}(T - v^*)$.

**Case 1.** $n$ is odd. Then $n = 2k + 1$ for some integer $k \geq 3$ and

$$
\begin{align*}
\text{cn}(T) - \text{cn}(T - v^*) &\leq \left\lceil \frac{d}{2} \right\rceil - \left\lceil \frac{n - 3}{2} \right\rceil = k - 1.
\end{align*}
$$

On the other hand,

$$
\begin{align*}
\text{cn}(T) - \text{cn}(T - v^*) &\geq \left( \left\lfloor \frac{n^2}{4} \right\rfloor - 1 \right) - \left( \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - 1 \right) \\
&= (k^2 + k - 1) - (k^2 - 1) = k,
\end{align*}
$$

a contradiction.

**Case 2.** $n$ is even. Then $n = 2k$ for some integer $k \geq 4$ and

$$
\begin{align*}
\text{cn}(T) - \text{cn}(T - v^*) &\leq \left\lceil \frac{d}{2} \right\rceil - \left\lceil \frac{n - 3}{2} \right\rceil = k - 2.
\end{align*}
$$
On the other hand, 

\[
\begin{align*}
\text{cn}(T) - \text{cn}(T - v^*) & \geq \left( \left\lfloor \frac{n^2}{4} \right\rfloor - 1 \right) - \left( \left\lfloor \frac{(n-1)^2}{4} \right\rfloor - 1 \right) \\
& = (k^2 - 1) - (k^2 - k - 1) = k,
\end{align*}
\]

a contradiction.

Hence \( \text{diam}(T) = n - 2 \). Then \( T \) is a caterpillar with exactly three end-vertices obtained from a path \( P : v_1, v_2, \ldots, v_{n-1} \) of order \( n - 1 \) by joining a new vertex \( u \) to \( v_i \) for some \( i \) with \( 2 \leq i \leq n - 2 \). By symmetry, we may assume that \( 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \).

**Case 1.** \( n \) is odd. Then \( n = 2k + 1 \) for some integer \( k \geq 3 \). Thus \( \text{cn}(T) = k^2 + k - 1 \). Observe that \( T \) contains exactly three edges with component number 1, exactly one edge with component number \( i \), and for each integer \( j \) with \( 2 \leq j \leq k \) and \( j \neq i \), there are exactly two edges with component number \( j \). Hence

\[
\text{cn}(T) = 3 + 2(2 + 3 + \cdots + k) - i = k^2 + k + 1 - i
\]

and so \( i = 2 \), that is, \( T = T_n \).

**Case 2.** \( n \) is even. Then \( n = 2k \) for some integer \( k \geq 4 \). Thus \( \text{cn}(T) = k^2 - 1 \). Observe that \( T \) contains exactly three edges with component number 1. If \( i = k \), then for each \( j \) with \( 2 \leq j \leq k - 1 \), there are exactly two edges with component number \( j \). Hence

\[
\text{cn}(T) = 3 + 2(2 + 3 + \cdots + (k - 1)) = k^2 - k + 1,
\]

which is a contradiction since \( k^2 - 1 = k^2 - k + 1 \) only when \( k = 2 \). Hence \( 2 \leq i \leq k - 1 \).

Then \( T \) contains exactly one edge with component number \( i \), exactly one edge with component number \( k \), and for each integer \( j \) with \( 2 \leq j \leq k - 1 \) and \( j \neq i \), there are exactly two edges with component number \( j \). Hence

\[
\text{cn}(T) = 3 + 2(2 + 3 + \cdots + k) - k - i = k^2 + 1 - i
\]

and so \( i = 2 \), that is, \( T = T_n \). \( \blacksquare \)

**Corollary 6.38** Let \( T \) be a tree of order \( n \geq 5 \). Then
\[ 2n - 3 \leq t^+(T) \leq \left\lceil \frac{n^2}{2} \right\rceil - 5 \text{ if and only if } T \notin \{P_n, T_n\}. \]

We are now prepared to present a generalization of Proposition 6.35.

**Proposition 6.39** If \( G \) is a traceable graph of order \( n \geq 6 \), then \( t^+(G) \neq \left\lceil \frac{n^2}{2} \right\rceil - 4 \).

**Proof.** Assume, to the contrary, that there exists a traceable graph \( G \) of order \( n \) for which \( t^+(G) = \left\lceil \frac{n^2}{2} \right\rceil - 4 \). Then \( G \neq P_n \). In fact, since \( \left\lceil \frac{n^2}{2} \right\rceil - 4 \) is even, \( G \) is not a tree. Hence \( G \) contains a spanning subgraph \( H \) of size \( n \) that is traceable. Observe that \( H \neq C_n \) since otherwise

\[
\left\lceil \frac{n^2}{2} \right\rceil - 4 = t^+(G) \leq t^+(H) = \left\lceil \frac{(n-1)^2}{2} \right\rceil,
\]

which holds only for \( n \leq 5 \). Also, since every spanning tree \( T \) of \( G \) must be isomorphic to either \( P_n \) or \( T_n \) by Corollary 6.38, it follows that \( H \) must be the graph obtained from a path \( P : v_1, v_2, \ldots, v_n \) of order \( n \) by joining \( v_{n-2} \) and \( v_n \). Moreover, \( G = H \) since otherwise \( G \) contains a spanning tree that is neither \( P_n \) nor \( T_n \). If \( n = 2k + 1 \) for some integer \( k \geq 3 \), then \( t^+(G) = 2k^2 + 2k - 4 \). Consider the linear ordering

\[ s : v_{k+2}, v_1, v_{2k+1}, v_2, v_{2k}, v_3, v_{2k-1}, \ldots, v_{k-1}, v_{k+3}, v_k, v_{k+1} \]

of vertices of \( G \) and observe that

\[
t^+(G) \geq d(s) = (k + 1) + (2k - 1) + (2k - 2) + (2k - 2) + (2k - 3) + (2k - 4) + \cdots + 4 + 3 + 1 = [1 + 2 + \cdots + (2k - 1)] - 2 + (k + 1) + (2k - 2) = 2k^2 + 2k - 3,
\]

a contradiction. If \( n = 2k \) for some integer \( k \geq 3 \), then \( t^+(G) = 2k^2 - 4 \). Consider the linear ordering

\[ s' : v_{k+1}, v_1, v_{2k}, v_2, v_{2k-1}, v_3, v_{2k-2}, \ldots, v_{k-1}, v_{k+2}, v_k \]

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of vertices of $G$ and observe that

$$t^+(G) \geq d(s')$$

$$= k + (2k - 2) + (2k - 3) + (2k - 3) + (2k - 4) + \cdots + 3 + 2$$

$$= [1 + 2 + \cdots + (2k - 2)] - 1 + k + (2k - 3)$$

$$= 2k^2 - 3,$$

a contradiction. Hence there is no traceable graph $G$ of order $n$ for which $t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor - 4.$

For $n \geq 4$, let $G_{n,3}$ be the traceable graph of order $n$ containing exactly one end-vertex and a triangle obtained from $T_n$ by joining two end-vertices $u$ and $v$ with $d(u, v) = 2$. Note that $G_{n,3}$ is one of the graphs whose maximum traceable numbers were studied in Proposition 5.19.

**Corollary 6.40** Let $G$ be a connected graph of order $n \geq 6$. Then

$$q(G) = \left( n - 1, \left\lfloor \frac{n^2}{2} \right\rfloor - 3 \right) \text{ if and only if } G = G_{n,3}.$$  

Moreover,

$$f(G_{n,3}) = \left( n - 1, \left\lfloor \frac{3n}{2} \right\rfloor - 2, \left\lfloor \frac{n^2}{2} \right\rfloor - 3 \right).$$

Let us next consider the case where $a = 6$. If $G$ is a connected graph of order $n$ with $t(G) = 6$, then $5 \leq n \leq 7$ by (6.1). If $n = 5$, then $G = K_{1,4}$ and $f(G) = (6, 7, 7)$. If $n = 6$, then $G \neq P_6$ and so $6 \leq t^+(G) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 3 = 15$ by Theorems 6.29 and 6.31. If $n = 7$, then $G$ is traceable, $6 \leq t^+(G) \leq \left\lfloor \frac{7^2}{2} \right\rfloor - 1 = 23$ by Theorem 6.29, and $t^+(G) \neq 20, 22$ by Theorem 6.31 and Proposition 6.39. Hence

$$\{6, 7, \ldots, 12, 18, 23\} \subseteq \{t^+(G) : t(G) = 6\} \subseteq \{6, 7, \ldots, 23\} - \{20, 22\}.$$
Next let \( C : v_1, v_2, \ldots, v_6, v_7, v_8 = v_1 \) be a 7-cycle and consider the following five traceable graphs of order 7:

\[
G_{13} = C + v_1v_3 + v_1v_4 + v_2v_5 \\
G_{14} = C + v_1v_4 + v_2v_5 \\
G_{15} = C + v_2v_3 + v_1v_5 \\
G_{16} = C + v_1v_4 \\
G_{17} = C + v_1v_3.
\]

We show that \( t^+(G_i) = i \) for \( 13 \leq i \leq 17 \). For each \( i \) with \( 13 \leq i \leq 17 \), observe that \( \text{diam}(G_i) = 3 \) and \( G_i \) contains exactly \( i - 12 \) pairs \( u, v \) of vertices such that \( d(u, v) = 3 \). Hence for every linear sequence \( s \) of vertices of \( G_i \), at most \( i - 12 \) of the 6 terms in \( d(s) \) are 3 and consequently \( 6 - (i - 12) = 18 - i \) terms are at most 2, so

\[
t^+(G_i) \leq 3(i - 12) + 2(18 - i) = i.
\]

To show that \( t^+(G_i) \geq i \) for each \( i \), consider the following linear orderings \( s_i \) of vertices of \( G_i \):

\[
\begin{align*}
s_{13} & : v_4, v_2, v_7, v_3, v_6, v_1, v_5 \\
s_{14} & : v_2, v_4, v_7, v_3, v_6, v_1, v_5 \\
s_{15} & : v_1, v_4, v_7, v_2, v_6, v_3, v_5 \\
s_{16} & : v_1, v_5, v_2, v_6, v_3, v_7, v_4 \\
s_{17} & : v_1, v_5, v_2, v_6, v_3, v_7, v_4.
\end{align*}
\]

One can verify that \( d(s_i) = i \) and so \( t^+(G_i) \geq d(s_i) = i \) for \( 13 \leq i \leq 17 \). Finally we show that there exists a traceable graph \( G_i \) of order 7 having \( t^+(G_i) = i \) for \( i = 19, 21 \). Let \( P : v_1, v_2, \ldots, v_7 \) be a path of order 7 and let \( G_{19} = P + v_4v_7 \) and \( G_{21} = P + v_5v_7 = G_{7,3} \). By Corollary 6.40, it follows that \( t^+(G_{21}) = \left\lfloor \frac{7}{2} \right\rfloor - 3 = 21 \). For \( G_{19} \), first observe that the component number of the spanning tree \( T = G_{19} - v_6v_7 \) is 10 and so \( t^+(G_{19}) \leq t^+(T) = 2 \cdot 10 - 1 = 19 \). Consider the linear ordering \( s : v_3, v_5, v_1, v_6, v_2, v_7, v_4 \) and observe that \( t^+(G_{19}) \geq d(s) = 19 \). Therefore, we conclude that

\[
\{ t^+(G) : t(G) = 6 \} = \{ c \in \mathbb{N} : 6 \leq c \leq 23 \} - \{ 20, 22 \}.
\]

In other words,

\[
\{ q(G) : t(G) = 6 \} = \{ (6, c) : 6 \leq c \leq 23 \text{ and } c \neq 20, 22 \}.
\]

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We have seen that for $a \geq 5$,

$$\{q(G) : t(G) = a\} \subseteq \left\{(a, c) : a \leq c \leq \left\lfloor \frac{(a + 1)^2}{2} \right\rfloor - 1\right\}$$

$$- \left\{\left\lfloor \frac{(a + 1)^2}{2} \right\rfloor - 2, \left\lfloor \frac{(a + 1)^2}{2} \right\rfloor - 4\right\}.$$

In particular, we now know that these two sets are equal for $a = 5, 6$. Our next question is:

*Are those sets equal for $a \geq 7$ as well?*

### 6.5 Traceable and Upper Traceable Ratios

Let $G$ be a nontrivial connected graph of order $n$. We define the upper traceable ratio $r_{t^+}(G)$ of $G$ by

$$r_{t^+}(G) = \frac{t^+(G)}{n}.$$

Observe that $r_{t^+}(G)$ is a positive rational number and since $t(G) \leq t^+(G)$, it follows that $r_t(G) \leq r_{t^+}(G)$. Also, since $n - 1 \leq t^+(G) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1$, we have

$$\frac{1}{2} \leq 1 - \frac{1}{n} \leq r_{t^+}(G) \leq \frac{1}{n} \left(\left\lfloor \frac{n^2}{2} \right\rfloor - 1\right) < \left\lfloor \frac{n}{2} \right\rfloor.$$

Recall that for every nontrivial connected graph $G$ of order $n$,

$$t^+(G) \leq (n - 1) \text{diam}(G)$$

by Observation 4.1, which leads to another observation.

**Observation 6.41** If $G$ is a nontrivial connected graph of order $n$, then

$$r_{t^+}(G) < \text{diam}(G).$$
For a nontrivial connected graph $G$, let 

$$q_r(G) = (r_t(G), r_{t^+}(G)).$$

We next consider the following question.

**Problem 6.42** For which pairs $\alpha, \gamma$ of rational numbers with

$$\frac{1}{2} \leq \alpha \leq \gamma \text{ and } \alpha < 2,$$

does there exist a connected graph $G$ for which $q_r(G) = (\alpha, \gamma)$?

Let $\alpha$ and $\gamma$ be positive rational numbers with $\frac{1}{2} \leq \alpha \leq \gamma$ and $\alpha < 2$ and $G$ a nontrivial connected graph such that $q_r(G) = (\alpha, \gamma)$. First notice that $\alpha = \gamma$ if and only if $t(G) = t^+(G)$. In this case, we have $t(G) = t^*(G) = t^+(G)$, that is, $G$ is traceably singular.

We first suppose that $\frac{1}{2} \leq \gamma < 1$. Let $G$ be a connected graph of order $n \geq 2$ having $q_r(G) = (\alpha, \gamma)$. Then $\alpha = 1 - \frac{1}{n}$ by Theorem 6.27. Furthermore, $t^+(G) < n$. Therefore $t^+(G) = n - 1$ and so $r_{t^+}(G) = 1 - \frac{1}{n}$. Hence if there exists a nontrivial connected graph $G$ with $q_r(G) = (\alpha, \gamma)$ where $\frac{1}{2} \leq \alpha \leq \gamma < 1$, then $\alpha = \gamma = 1 - \frac{1}{n}$ for some integer $n \geq 2$. On the other hand, if $\alpha = \gamma = 1 - \frac{1}{n}$ for some integer $n \geq 2$, then observe that $q_r(K_n) = (1 - \frac{1}{n}, 1 - \frac{1}{n}) = (\alpha, \gamma)$. Moreover, every nontrivial connected graph $G$ with $q_r(G) = (1 - \frac{1}{n}, 1 - \frac{1}{n})$ must have order $n$. In fact, $G = K_n$ is the only graph such that $q_r(G) = (1 - \frac{1}{n}, 1 - \frac{1}{n})$ because $t^+(G) = n - 1$ if and only if $G = K_n$ by Observation 4.4. We state this new observation as follows.

**Observation 6.43** Let $\alpha$ and $\gamma$ be rational numbers such that $\frac{1}{2} \leq \alpha \leq \gamma$ and $\alpha < 2$. If $\alpha < 1$, then there exists a nontrivial connected graph $G$ with $q_r(G) = (\alpha, \gamma)$ if and only if $\alpha = 1 - \frac{1}{n}$ for some integer $n \geq 2$. Moreover, if $\gamma < 1$, then there exists a nontrivial connected graph $G$ with $q_r(G) = (\alpha, \gamma)$ if and only if $\alpha = \gamma = 1 - \frac{1}{n}$ for some integer $n \geq 2$. In either case, such a graph $G$ must be a traceable graph of order $n$. Furthermore, in the case where $\alpha = \gamma = 1 - \frac{1}{n}$, we have that $q_r(G) = (1 - \frac{1}{n}, 1 - \frac{1}{n})$ if and only if $G = K_n$.

From now on we assume that $\gamma \geq 1$. Let $G$ be a connected graph with $q_r(G) = (\alpha, \gamma)$. The order of $G$ must be at least 3 since there is no connected graph
of order 2 with upper traceable number 2. Since \( t(G) = t^+(G) \) if and only if \( G = K_n \), it follows that \( q_r(G) = (\alpha, \alpha) \) if and only if \( G = K_n \), and in this case \( \alpha = 1 - \frac{1}{n} \). Hence if \( \gamma \geq 1 \), then \( \alpha < \gamma \). If \( \frac{1}{2} \leq \alpha < 1 \), then \( \alpha = 1 - \frac{1}{n} \) for some integer \( n \geq 3 \) and \( G \) is a traceable graph of order \( n \) by Observation 6.43. If \( \gamma = 1 \), then consider the graph \( K_n - e \) of order \( n \). Since \( K_n - e \) is traceable and \( t^+(K_n - e) = n \) by Theorem 4.8, it follows that \( q_r(K_n - e) = (1 - \frac{1}{n}, 1) \). In fact, \( G = K_n - e \) is the only graph such that \( q_r(G) = (1 - \frac{1}{n}, 1) \) because \( q(G) = (n - 1, n) \) if and only if \( G = K_n - e \) by Theorem 4.8.

**Observation 6.44** The ordered pair \((1 - \frac{1}{n}, 1)\) is realizable as the value \( q_r(G) \) of some nontrivial connected graph \( G \) if and only if \( n \geq 3 \). In addition, \( q_r(G) = (1 - \frac{1}{n}, 1) \) if and only if \( G = K_n - e \).

Recall that for every graph \( G \) of order \( n \geq 3 \), we have \( n - 1 \leq t^+(G) \leq \left\lfloor \frac{n^2}{2} \right\rfloor - 1 \) and

- \( t^+(G) = n - 1 \) if and only if \( G = K_n \);
- \( t^+(G) = \left\lfloor \frac{n^2}{2} \right\rfloor \) if and only if \( G = P_n \).

Observe that both \( K_n \) and \( P_n \) are traceable. We have seen in Section 6.3 that if \( G \) is a traceable graph of order \( n \geq 3 \), then \( t^*(G) < \frac{3n^2}{8} \) and so \( r_t(G) < \frac{3}{8} \). However, each of the upper and lower bounds of the upper traceable numbers given above is sharp and is attained by a traceable graph. In general, therefore, the upper traceable number of a traceable graph can be as small or as large as possible within the range of the upper traceable numbers of graphs of a fixed order and so is the upper traceable ratio.

Next suppose that \( \frac{1}{2} \leq \alpha < 1 < \gamma \). If \( G \) is a connected graph with \( q_r(G) = (\alpha, \gamma) \), then the order of \( G \) must be at least 4 since for \( n \leq 3 \) there is no connected graph of order \( n \) with upper traceable number greater than \( n \). Hence \( \alpha = 1 - \frac{1}{n} \) for some integer \( n \geq 4 \) and \( G \) is a traceable graph of order \( n \) by Observation 6.43. With the aid of some earlier observations and Figure 6.1 in Section 6.1, we have the
following for $\alpha = \frac{3}{4}$ ($n = 4$) and $\alpha = \frac{4}{5}$ ($n = 5$):

$$\left\{ q_r(G) : r_t(G) = \frac{3}{4} \right\} = \left\{ \left( \frac{3}{4}, \frac{k}{4} \right) : k = 3, 4, 5, 7 \right\};$$

$$\left\{ q_r(G) : r_t(G) = \frac{4}{5} \right\} = \left\{ \left( \frac{4}{5}, \frac{k}{5} \right) : k = 4, 5, 6, 7, 8, 9, 11 \right\}.$$

Now we will show that for $n \geq 6$, the ordered pair $(1 - \frac{1}{n}, \gamma)$ is realizable for each rational number $\gamma$ with $1 < \gamma \leq \frac{3}{2}$. In order to this, we first consider the graph $G'_{n,1}$ of order $n \geq 4$ obtained by adding a new vertex $x$ to $K_{n-1}$ and joining $x$ to one of the $n-1$ vertices, say $y$, in $K_{n-1}$. Let $v_1, v_2, \ldots, v_{n-2}$ be the remaining vertices of $G'_{n,1}$. Observe that $G'_{n,1}$ is traceable, $\text{diam}(G'_{n,1}) = 2$, and every linear sequence of vertices of $G'_{n,1}$ contains at most two pairs of consecutive vertices of distance 2, so $t^+(G'_{n,1}) \leq 2 \cdot 2 + (n-3) = n + 1$. Since the linear sequence $s_1'$ given by

$$s_1' : v_1, x, v_2, v_3, \ldots, v_{n-2}, y$$

has $d(s_1') = n + 1$, it follows that $t^+(G'_{n,1}) \geq d(s_1') = n + 1$ and so $t^+(G'_{n,1}) = n + 1$.

Next consider the graph $G'_{n,2} = G'_{n,1} - v_1 v_2$ of order $n \geq 5$. Then $G'_{n,2}$ is traceable, $\text{diam}(G'_{n,2}) = 2$, and every linear sequence of vertices of $G'_{n,2}$ contains at most three pairs of consecutive vertices of distance 2, so $t^+(G'_{n,2}) \leq 3 \cdot 2 + (n-4) = n + 2$. Since the linear sequence $s_2'$ given by

$$s_2' : v_1, v_2, x, v_3, v_4, \ldots, v_{n-2}, y$$

has $d(s_2') = n + 2$, it follows that $t^+(G'_{n,2}) \geq d(s_2') = n + 2$ and so $t^+(G'_{n,2}) = n + 2$.

Finally, let $n$ and $k$ be integers such that $k \geq 3$ and $n \geq 2k$. We construct the graph $G'_{n,k}$ of order $n$ with

$$V(G'_{n,k}) = \{x, y, v_1, v_2, \ldots, v_{k-1}, u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots, w_{n-2k}\}$$

by (i) forming a complete graph $K_{n-1}$ such that $V(K_{n-1}) = V(G'_{n,k}) - \{x\}$, (ii) deleting the $k-1$ edges $v_i u_i$ ($1 \leq i \leq k-1$) from $K_{n-1}$, and then (iii) adding and joining $x$ to $y$.

**Proposition 6.45** Let $n$ and $k$ be integers such that $k \geq 3$ and $n \geq 2k$. Then

$$q(G'_{n,k}) = (n-1, n+k).$$

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Proof. Let

\[ V(G'_{n,k}) = \{x, y, v_1, v_2, \ldots, v_{k-1}, u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots, w_{n-2k}\} \]

and construct \(G'_{n,k}\) as described above. Observe that \(G'_{n,k}\) is traceable and so \( t(G'_{n,k}) = n - 1 \). To show that \( t^+(G'_{n,k}) = n + k \), observe that \( \text{diam}(G'_{n,k}) = 2 \) and every linear sequence of vertices of \(G'_{n,k}\) contains at most \( k + 1 \) pairs of consecutive vertices of distance 2. Hence \( t^+(G'_{n,k}) \leq 2(k + 1) + (n - k - 2) = n + k \). The linear ordering \( s \) of vertices of \(G'_{n,k}\) given by

\[ s : v_1, u_1, x, v_2, u_2, \ldots, v_{k-1}, u_{k-1}, w_1, w_2, \ldots, w_{n-2k}, y \]

contains exactly \( k + 1 \) pairs of consecutive vertices of distance 2. Hence

\[ t^+(G'_{n,k}) \geq d(s) = n + k, \]

providing the desired result.

By Proposition 6.45, for each pair \( n, k \) of integers with \( k \geq 3 \) and \( n \geq 2k \),

\[ q_r(G'_{n,k}) = \left(1 - \frac{1}{n}, 1 + \frac{k}{n}\right). \]

Furthermore, observe that

\[ r_{t^+}(G'_{n,k}) = 1 + \frac{k}{n} \leq 1 + \frac{k}{2k} = \frac{3}{2}. \]

Using this fact, we will show that the ordered pair \( (1 - \frac{1}{n}, \gamma) \), where \( n \geq 6 \), is realizable as the value \( q_r(G) \) of some connected graph \( G \) (of order \( n \)) for each rational number \( \gamma \) with \( 1 < \gamma \leq \frac{3}{2} \). Let \( a \) and \( b \) be positive integers such that \( \gamma = 1 + \frac{a}{b} \) and \( \gcd(a, b) = 1 \). Then \( 0 < \frac{a}{b} \leq \frac{1}{2} \) and so \( a \geq 1 \) and \( 2a \leq b \). Since \( \gcd(a, b) = 1 \), it follows that \( n = bc \) for some positive integer \( c \). If \( ac = 1 \), then \( \gamma = 1 + \frac{ac}{bc} = 1 + \frac{1}{n} \). Then \( q_r(G'_{n,1}) = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = (\alpha, \gamma) \). Similarly, if \( ac = 2 \), then \( \gamma = 1 + \frac{2c}{bc} = 1 + \frac{2}{n} \) and \( q_r(G'_{n,2}) = \left(1 - \frac{1}{n}, 1 + \frac{2}{n}\right) = (\alpha, \gamma) \). Finally if \( ac \geq 3 \), then observe that \( bc \geq 2ac \geq 6 \). Let \( G'_{bc,ac} \) be the traceable graph of order \( n = bc \) with \( k = ac \geq 3 \) constructed in the way described above. Then by Proposition 6.45, we have \( t^+(G'_{bc,ac}) = ac + bc \) and

\[ r_{t^+}(G'_{bc,ac}) = \frac{ac + bc}{bc} = \gamma. \]
Hence $q_r(G'_{bc,ac}) = (\alpha, \gamma)$.

Next we will show that for $n \geq 6$, the ordered pair $(1 - \frac{1}{n}, \gamma)$ is realizable for each rational number $\gamma$ with $\frac{3}{4} < \gamma \leq 2$. Let $K = K_{[\frac{3}{2}], [\frac{7}{2}]}$ be a complete bipartite graph whose partite sets are

$$U = \{u_1, u_2, \ldots, u_{\lceil \frac{n}{2} \rceil}\} \quad \text{and} \quad W = \{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\}.$$  

Let $G^*_{n,0} = K - \{u_1w_1, u_2w_2\}$ and $G^*_{n,1} = K - \{u_1w_1\}$.

We will show that $q(G^*_{n,k}) = (n-1, 2n-k)$ for $k = 0, 1$. It is clear that $G^*_{n,k}$ is traceable since $t(w_1) = n-1$ for both $G^*_{n,0}$ and $G^*_{n,1}$. To see why $t^+(G^*_{n,k}) = 2n-k$ for each $k$, observe that $\text{diam}(G^*_{n,k}) = 3$ and the graph $G^*_{n,k}$ contains exactly $2-k$ pairs of vertices of distance 3. Hence $t^+(G^*_{n,k}) \leq 3(2-k) + 2(n-1 - (2-k)) = 2n-k$.

For each $k$, consider the linear sequence $s_k$ of vertices of $G^*_{n,k}$ given by

$$s_0 : w_1, u_1, u_2, \ldots, u_{\lceil \frac{n}{2} \rceil}, u_2, w_2, w_3, \ldots, w_{\lfloor \frac{n}{2} \rfloor}$$

$$s_1 : w_2, w_3, \ldots, w_{\lfloor \frac{n}{2} \rfloor}, w_1, u_1, u_2, \ldots, u_{\lceil \frac{n}{2} \rceil}.$$  

Then each $d(s_k)$ contains exactly $2-k$ terms equal to 3 and exactly $n-3+k$ terms equal to 2, so

$$t^+(G^*_{n,k}) \geq d(s_k) = 3(2-k) + 2(n-3+k) = 2n-k.$$  

Therefore, $t^+(G^*_{n,k}) = 2n-k$, as claimed.

Next we construct $G^*_{n,2}$ from $K$ by deleting the vertex $w_{\lfloor \frac{n}{2} \rfloor}$ and adding and joining a new vertex $v$ to $u_1$ and $w_1$. We show that $q(G^*_{n,2}) = (n-1, 2n-2)$. Note that $t(G^*_{n,2})$ is traceable since $t(u_1) = n-1$. To verify that $t^+(G^*_{n,2}) = 2n-2$, observe that $t^+(G^*_{n,2}) \leq 2(n-1)$ since $\text{diam}(G^*_{n,2}) = 2$. On the other hand, for the linear sequence $s_2$ of vertices of $G^*_{n,2}$ given by

$$s_2 : u_1, u_2, \ldots, u_{\lceil \frac{n}{2} \rceil}, v, w_2, w_3, \ldots, w_{\lfloor \frac{n}{2} \rfloor}^{-1}, w_1,$$

every term in $d(s_2)$ is equal to 2 and so $d(s_2) = 2(n-1)$ and $t^+(G^*_{n,2}) \geq d(s_2) = 2n-2$. Hence $t^+(G^*_{n,2}) = 2n-2$, as claimed.

It was shown in Proposition 4.5 that if $G$ is a complete $i$-partite graph $K_{n_1, n_2, \ldots, n_i}$ where $n = n_1 + n_2 + \cdots + n_i$ and $i \geq 2$, then $t^+(G) = 2n-i-1$. Without loss of generality, assume that $n_1 \leq n_2 \leq \cdots \leq n_i$. Observe that if $n_i = \lceil \frac{n}{2} \rceil$, then

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$G$ is traceable. Furthermore, if $n_i = \left\lceil \frac{n}{3} \right\rceil$, then $i \leq \left\lceil \frac{n}{2} \right\rceil + 1$. For each integer $k$ with $3 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$, let $G_{n,k}^*$ be a complete $(k - 1)$-partite graph $K_{n_1,n_2,\ldots,n_k-1}$ of order $n$ where $n_{k-1} = \left\lceil \frac{n}{3} \right\rceil$ and $n_{k-2} = \left\lceil \frac{n}{3} \right\rceil + 3 - k$ with $n_1 = n_2 = \cdots = n_{k-3} = 1$ if $k \geq 4$. Then $G_{n,k}^*$ is traceable and $t^+(G_{n,k}^*) = 2n - (k - 1) - 1 = 2n - k$ by Proposition 4.5. Hence for $0 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$,

$$q_r(G_{n,k}^*) = \left(1 - \frac{1}{n}, 1 + \frac{n - k}{n}\right).$$

Now we are ready to show that the ordered pair $(1 - \frac{1}{n}, \gamma)$, where $n \geq 6$, is realizable for each rational number $\gamma$ with $\frac{3}{2} < \gamma \leq 2$. Let $a$ and $b$ be positive integers such that $\gamma = 1 + \frac{a}{b}$ and $\gcd(a, b) = 1$. Hence $a \leq b < 2a$ and $n = bc$ for some positive integer $c$. If $a = b$, then $\gamma = 2$ and $q_r(G_{n,0}^*) = (1 - \frac{1}{n}, 2)$. Next suppose that $a < b < 2a$. If $n = b$ and $a \in \{n - 1, n - 2\}$, then observe that

$$q_r(G_{n,1}^*) = \left(1 - \frac{1}{n}, 1 + \frac{n - 1}{n}\right),$$

$$q_r(G_{n,2}^*) = \left(1 - \frac{1}{n}, 1 + \frac{n - 2}{n}\right).$$

If $n = 2b$ and $a = b - 1$, then

$$r_{t^+}(G_{2b,2}^*) = 1 + \frac{2b - 2}{2b} = 1 + \frac{2a}{2b} = \gamma.$$ 

Otherwise $3 \leq bc - ac < bc - \frac{bc}{2} = \frac{n}{2}$. Consider the graph $G_{bc,bc-ac}^*$ and observe that

$$r_{t^+}(G_{bc,bc-ac}^*) = 1 + \frac{bc - (bc - ac)}{bc} = 1 + \frac{ac}{bc} = \gamma.$$ 

We summarize what we have obtained with $\alpha = 1 - \frac{1}{n}$ and $\gamma > 1$.

**Proposition 6.46** Let $\gamma$ be a rational number with $\gamma > 1$.

1. The ordered pair $(\frac{3}{4}, \gamma)$ is realizable if and only if $\gamma \in \left\{\frac{5}{4}, \frac{7}{4}\right\}$.

2. The ordered pair $(\frac{4}{5}, \gamma)$ is realizable if and only if $\gamma \in \left\{\frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, \frac{11}{5}\right\}$.

3. If $n \geq 6$ and $\gamma \leq 2$, then the ordered pair $(1 - \frac{1}{n}, \gamma)$ is realizable.
Suppose next that \( \alpha = 1 < \gamma \). We next show that the ordered pair \((1, \gamma)\) is realizable for each rational number \( \gamma \) with \( 1 < \gamma < \frac{3}{2} \). In order to do this, let \( n \) and \( k \) be integers with \( k \geq 6 \) and \( n \geq 2k - 3 \). We construct the graph \( G''_{n,k} \) of order \( n \) with

\[
V(G''_{n,k}) = \{ x_1, x_2, y, v_1, v_2, \ldots, u_{k-3}, u_1, u_2, \ldots, u_{k-3}, w_1, w_2, \ldots, w_{n-2k+3}\}
\]

by (i) forming a complete graph \( K_{n-2} \) such that \( V(K_{n-2}) = V(G_{n,k}) - \{x_1, x_2\} \), (ii) deleting the \( k \) edges \( v_i u_i \) \((1 \leq i \leq k-3)\) from \( K_{n-2} \), and then (iii) adding and joining each of \( x_1 \) and \( x_2 \) to \( y \).

**Proposition 6.47** Let \( n \) and \( k \) be integers such that \( k \geq 6 \) and \( n \geq 2k - 3 \). Then

\[
q_r(G''_{n,k}) = (1, 1 + \frac{k}{n}).
\]

**Proof.** Let

\[
V(G''_{n,k}) = \{ x_1, x_2, y, v_1, v_2, \ldots, u_{k-3}, u_1, u_2, \ldots, u_{k-3}, w_1, w_2, \ldots, w_{n-2k+3}\}
\]

and construct \( G''_{n,k} \) as described above. Observe that \( G''_{n,k} \) is not traceable and \( t(y) = n + 1 \) while \( t(v) = n \) for every vertex \( v \) distinct from \( y \). Thus \( t(G''_{n,k}) = n \). To show that \( t^+(G''_{n,k}) = n + k \), observe that \( \text{diam}(G''_{n,k}) = 2 \) and every linear sequence of vertices of \( G''_{n,k} \) contains at most \( k + 1 \) pairs of consecutive vertices of distance 2. Hence \( t^+(G''_{n,k}) \leq 2(k + 1) + (n - k - 2) = n + k \). For the linear ordering \( s \) given by

\[
s : v_1, u_1, x_1, v_2, u_2, x_2, v_3, u_3, \ldots, v_{k-3}, u_{k-3}, w_1, w_2, \ldots, w_{n-2k+3}, y,
\]

d\((s)\) contains exactly \( k + 1 \) terms that are equal to 2. Hence

\[
t^+(G''_{n,k}) \geq d(s) = n + k,
\]

providing the desired result. \[\blacksquare\]

By Proposition 6.47, for each pair \( n, k \) of integers with \( k \geq 6 \) and \( n \geq 2k - 3 \),

\[
q_r(G''_{n,k}) = \left(1, 1 + \frac{k}{n}\right).
\]

Furthermore, observe that

\[
r_{t^+}(G''_{n,k}) = 1 + \frac{k}{n} \leq 1 + \frac{k}{2k - 3},
\]

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where
\[
\frac{3}{2} < 1 + \frac{k}{2k - 3} \leq 2
\]
for \( k \geq 6. \) Using this fact, we will show that the ordered pair \((1, \gamma)\) is realizable as the value \( q_r(G) \) of some connected graph \( G \) for each rational number \( \gamma \) with 
\( 1 < \gamma \leq \frac{3}{2} \). Let \( a \) and \( b \) be positive integers such that \( \gamma = 1 + \frac{a}{b} \). Then \( 0 < \frac{a}{b} \leq \frac{1}{2} \) and so \( a \geq 1 \) and \( 2a \leq b \). Note that \( 6a \geq 6 \) and \( 6b \geq 12a > 2(6a) - 3 \). Consider the connected graph \( G_{6b,6a}^{\text{tr}} \) of order \( n = 6b \) constructed in the way described above (so \( k = 6a \geq 6 \) and \( n = 6b \geq 2k - 3 \)). Then by Proposition 6.47, it follows that
\[
r_t(G_{6b,6a}^{\text{tr}}) = 1 \text{ and } r(t)(G_{6b,6a}^{\text{tr}}) = \frac{6a + 6b}{6b} = \gamma.
\]
Hence \( g_r(G_{6b,6a}^{\text{tr}}) = (1, \gamma) \).

Next we show that the ordered pair \((1, \gamma)\) is realizable for each rational number \( \gamma \) with \( \frac{2}{3} < \gamma \leq 2 \). First observe that if \( G = K_{n_1,n_2,\ldots,n_i} \) is a complete \( i \)-partite graph with \( n = n_1 + n_2 + \cdots + n_i \) (\( i \geq 2 \)) and \( n_i \geq \left\lceil \frac{n}{3} \right\rceil + 1 \), then \( G \) is not traceable. For an even positive integer \( n = 2\ell \) where \( \ell \geq 3 \), let \( K = K_{\ell-1,\ell+1} \) be a complete bipartite graph of order \( n \) whose partite sets are
\[
U = \{u_1, u_2, \ldots, u_{\ell-1}\} \text{ and } W = \{w_1, w_2, \ldots, w_{\ell+1}\}.
\]
Then \( K \) is not traceable and in fact, \( t(K) = n \). Let \( G_{n,0}^{**} = K - \{u_1w_1, u_2w_2\} \). Observe that \( G_{n,0}^{**} \) is not traceable since \( K \) is not traceable. On the other hand, for the linear sequence
\[
s : w_1, w_2, w_3, w_4, \ldots, u_{\ell-1}, w_{\ell}, u_1, w_2, w_{\ell+1}
\]
of vertices of \( G_{n,0}^{**} \) whose initial vertex is \( w_1 \), there is exactly one term in \( d(s) \) that is equal to 2 (namely \( d(w_2, u_{\ell+1}) \)), while all other terms are equal to 1. Hence
\[
t(w_1) \leq d(s) = n \text{ and so } t(G_{n,0}^{**}) = n.
\]
Next we show that \( t(G_{n,0}^{**}) = 2n \). Observe that \( \text{diam}(G_{n,0}^{**}) = 3 \) and the graph contains exactly two pairs of vertices of distance 3. Hence \( t^+(G_{n,0}^{**}) \leq 2 \cdot 3 + 2(n - 3) = 2n \). On the other hand, for the linear sequence \( s' \) given by
\[
s' : w_1, u_1, u_3, u_4, \ldots, u_{\ell-1}, u_2, w_2, w_3, \ldots, w_{\ell+1},
\]
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d(s') contains exactly two terms that are equal to 3 and exactly \( n - 3 \) terms that are equal to 2 and so
\[
t^+(G_{n,0}^{**}) \geq d(s') = 2 \cdot 3 + 2(n - 3) = 2n.
\]
Therefore, \( t^+(G_{n,0}^{**}) = 2n \) and so \( q_r(G_{n,0}^{**}) = (1, 2) \). For an even integer \( n = 2\ell \geq 6 \) and each integer \( k \) with \( 3 \leq k \leq \ell \), let \( G_{n,k}^{**} \) be a complete \((k-1)\)-partite graph \( K_{n_1,n_2,...,n_{k-1}} \) of order \( n \), where \( n_{k-1} = \ell + 1 \) and \( n_{k-2} = \ell + 2 - k \) with \( n_1 = n_2 = \cdots = n_{k-3} = 1 \) if \( k \geq 4 \). Then \( t(G_{n,k}^{**}) = n \) and \( t^+(G_{n,k}^{**}) = 2n - (k - 1) - 1 = 2n - k \) by Proposition 4.5. Hence for an even integer \( n \geq 6 \) and \( 3 \leq k \leq \frac{n}{2} \),
\[
q_r(G_{n,k}^{**}) = \left(1, 1 + \frac{n-k}{n}\right).
\]

We are ready to show that the ordered pair \((1, \gamma)\) is realizable for each rational number \( \gamma \) with \( \frac{3}{2} < \gamma \leq 2 \). Let \( a \) and \( b \) be positive integers such that \( \gamma = 1 + \frac{a}{b} \). Hence \( a \leq b < 2a \). If \( a = b \), then \( \gamma = 2 \) and \( q_r(G_{6,0}^{**}) = (1, 2) \). If \( a < b \), then \( 6b > 6(b-a) \geq 6 \). Consider the graph \( G_{6b,6b-6a}^{**} \) of order \( n = 6b \). Observe that \( r_l(G_{6b,6b-6a}^{**}) = 1 \) and
\[
r_l^+(G_{6b,6b-6a}^{**}) = 1 + \frac{6b - (6b-6a)}{6b} = 1 + \frac{6a}{6b} = \gamma.
\]

**Proposition 6.48** If \( \gamma \) is a rational number with \( 1 < \gamma \leq 2 \), then the ordered pair \((1, \gamma)\) is realizable.

Recall, by Proposition 6.25, that if \( G \) is a nontrivial connected graph for which \( q_r(G) = (\alpha, \beta) \), then \( \alpha \leq \beta < 1 + \frac{\alpha}{2} \). Is there a similar relationship for \( \alpha \) and \( \gamma \)? Let us consider the case where \( 1 < \alpha < 1 + \frac{\alpha}{2} \leq \gamma < 2 \). Let \( n, i, \) and \( k \) be integers such that \( n = 2\ell \) for some integer \( \ell \geq 4 \), \( 2 \leq i \leq \ell - 1 \), and \( 3 \leq k \leq \ell - i + 2 \). Let \( G_{n,i,k}^{**} \) be a complete \((k-1)\)-partite graph \( K_{n_1,n_2,...,n_{k-1}} \) of order \( n \), where \( n_{k-1} = \ell + i \) and \( n_{k-2} = \ell - i - k + 3 \) with \( n_1 = n_2 = \cdots = n_{k-3} = 1 \) if \( k \geq 4 \).

**Proposition 6.49** Let \( n, i, \) and \( k \) be integers such that \( n = 2\ell \) for some integer \( \ell \geq 4 \) with \( 2 \leq i \leq \ell - 1 \) and \( 3 \leq k \leq \ell - i + 2 \). Then
\[
q(G_{n,i,k}) = (n + 2i - 2, 2n - k).
\]
Proof. Since \( t^+(G_{n,i,k}) = 2n - (k - 1) - 1 = 2n - k \) by Proposition 4.5, we only need to show that \( t(G_{n,i,k}) = n + 2i - 2 \). Let \( V_1, V_2, \ldots, V_{k-1} \) be sets of vertices such that

\[
V_{k-1} = \{u_1, u_2, \ldots, u_{\ell+i}\} \quad \text{and} \quad V_{k-2} = \{v_1, v_2, \ldots, v_{\ell-i-k+3}\}
\]

and \( V_j = \{w_{\ell-i-k+3+j}\} \) for \( 1 \leq j \leq k - 3 \) if \( k \geq 4 \). Construct \( G_{n,i,k} \) using \( V_1, V_2, \ldots, V_{k-1} \) as its partite sets. Observe that the spanning linear forest \( F \) of \( G_{n,i,k} \) consisting of the path

\[
P: \quad u_1, v_1, u_2, v_2, \ldots, u_{\ell-i-k+3}, v_{\ell-i-k+3},
\]

\[
\ldots, u_{\ell-i-k+4}, v_{\ell-i-k+4}, u_{\ell-i-k+5}, v_{\ell-i-k+5}, \ldots, u_{\ell-i}, u_{\ell-i}, u_{\ell+i+1}
\]

of length \( n - 2i \) and \( 2i - 1 \) isolated vertices \( u_{\ell-i+2}, u_{\ell-i+3}, \ldots, u_{\ell+i} \) has size \( n - 2i \), which is the maximum possible size of a spanning linear forest of \( G_{n,i,k} \). Since \( P \) is a longest path in \( G_{n,i,k} \) and its length is \( n - 2i \), it follows that \( 2n - 2 - (n - 2i) \leq t(G_{n,i,k}) \leq 2n - 2 - (n - 2i) \) by Corollary 3.4, that is, \( t(G_{n,i,k}) = n + 2i - 2 \).  

By Proposition 6.49,

\[
q_r(G_{n,i,k}) = \left(1 + \frac{2i - 2}{n}, 2 - \frac{k}{n}\right).
\]

Observe that \( 1 < r_t(G_{n,i,k}) < 2 \), \( r_t(G_{n,i,k}) < 2 \), and

\[
r_t^+(G_{n,i,k}) = 2 - \frac{k}{n} \geq \frac{4n - 2(\ell - i + 2)}{2n} = \frac{2n + (n + 2i - 2) - 2}{2n} = 1 + \frac{r_t(G_{n,i,k})}{2} - \frac{1}{n}.
\]

Using this fact, we will show that the ordered pair \((\alpha, \gamma)\) of rational numbers is realizable if \( 1 < \alpha < 2 \) and \( 1 + \frac{\alpha}{2} \leq \gamma < 2 \). Note that \( \alpha < 1 + \frac{\alpha}{2} \) since \( \alpha < 2 \). Let \( \alpha = \frac{a}{b} \) and \( \gamma = \frac{c}{d} \), where \( a, b, c, \) and \( d \) are positive integers. Since \( 1 < \frac{a}{b} < 2 \), it follows that \( b \geq 2 \) and \( b < a < 2b \). Similarly, since \( 1 + \frac{\alpha}{2} \leq \frac{a}{d} < 2 \), it follows that \( d \geq 2, d < c < 2d, \) and \( 2bd + ad \leq 2bc < 4bd \). Observe that

\[
3 \leq d(a - b) + 1 \leq d(b - 1) + 1 = bd - d + 1 \leq bd - 1
\]
and

\[4 \leq 2b(2d - c) = 2bd - (2bc - 2bd) \leq 2bd - ad \leq bd - d(a - b).\]

That is, \(3 \leq d(a - b) + 1 \leq bd - 1\) and \(4 \leq 2b(2d - c) \leq bd - d(a - b)\). Consider the connected graph \(G = G_{2bd,d(a-b)+1,2b(2d-c)}\) of order \(n = 2bd\) which appears in Proposition 6.49 (with \(3 \leq i \leq bd - 1\) and \(4 \leq k \leq bd - i + 1\)). Then \(t(G) = 2bd + 2[d(a-b)+1] - 2 = 2ad\) and \(t^+(G) = 2 \cdot 2bd - 2b(2d-c) = 2bc\). Since \(n = 2bd\), this implies that

\[q_r(G) = \left( \frac{2ad}{2bd}, \frac{2bc}{2bd} \right) = (\alpha, \gamma).\]

**Proposition 6.50** Let \(\alpha\) and \(\gamma\) be rational numbers such that \(1 < \alpha < 1 + \frac{3}{2} \leq \gamma < 2\). Then the ordered pair \((\alpha, \gamma)\) is realizable as the value \(q_r(G)\) of some connected graph \(G\).

What can we say if \(1 < \alpha < \gamma < 1 + \frac{3}{2}\)? For example, let \(\alpha = \frac{8}{5}\) and suppose that \(\frac{8}{5} < \gamma < \frac{9}{5}\). If \(\gamma = \frac{17}{10}\), then for the star \(K_{1,9}\),

\[t(K_{1,9}) = 2 \cdot 10 - 4 = 16\quad\text{and}\quad t^+(K_{1,9}) = 2 \cdot 10 - 3 = 17\]

and so

\[q_r(K_{1,9}) = \left( \frac{16}{10}, \frac{17}{15} \right) = (\alpha, \gamma).\]

If \(\gamma = \frac{35}{20}\), then consider a complete 4-partite graph \(K_{1,1,1,17}\). It is not difficult to verify that

\[t(K_{1,1,1,17}) = 32\quad\text{and}\quad t^+(K_{1,1,1,17}) = 35\]

and so

\[q_r(K_{1,1,1,17}) = \left( \frac{32}{20}, \frac{35}{20} \right) = (\alpha, \gamma).\]

**Problem 6.51** Does there exist a connected graph \(G\) for which

\[q_r(G) = \left( \frac{8}{5}, \frac{33}{20} \right)?\]
Suppose that $G$ is a connected graph of order $n$ for which

$$q_r(G) = \left(\frac{5}{2}, \frac{23}{20}\right) = \left(\frac{32}{20}, \frac{33}{20}\right).$$

Note that $n = 20\ell$ for some positive integer $\ell$ and so

$$q(G) = (32\ell, 33\ell).$$

Moreover, $n \neq 20$ since $t^+(G) - t(G) = 1$ if and only if $G \in \{K_n - e, K_{1,n-1}\}$ by Theorem 4.18. Hence

$$t^+(G) - t(G) = \ell \geq 2.$$

**Proposition 6.52** Let $G$ be a nontrivial connected graph. If $d$ is a positive integer such that

$$t^+(G) - t(G) < \frac{d(d + 1)}{2},$$

then $\text{diam}(G) \leq d$.

**Proof.** Let $n$ be the order of $G$. If $t^+(G) - t(G) = 0$, then $G = K_n$; while if $t^+(G) - t(G) = 1$, then $G \in \{K_n - e, K_{1,n-1}\}$. Since the result holds in either case, we suppose that $t^+(G) - t(G) \geq 2$ and assume, to the contrary, that $\text{diam}(G) > d + 1$. Then $G$ contains two vertices $u$ and $v$ such that $d(u, v) = d + 1$. Let $u, x_1, \ldots, x_d, v$ be a $u - v$ path in $G$ and let $y_1, y_2, \ldots, y_{n-d-2}$ be the remaining vertices of $G$. Also, let $v = y_0$ and

$$\sum_{i=0}^{n-d-3} d(v_i, v_{i+1}) = a.$$ 

Consider the linear ordering $s_1$ of vertices of $G$ given by

$$s_1: u, x_1, x_2, \ldots, x_d, v, y_1, y_2, \ldots, y_{n-d-2}$$

and observe that $t(G) \leq d(s_1) = (d + 1) + a$. We now consider two cases, according to whether $d$ is odd or even.

**Case 1.** $d$ is odd. Then $d = 2k + 1$ for some integer $k \geq 1$. Observe that for the linear ordering $s_2$ of vertices of $G$ given by

$$s_2: u, x_1, x_2, \ldots, x_d, v, y_1, y_2, \ldots, y_{n-d-2}$$

and observe that $t(G) \leq d(s_2) = (d + 1) + a$. We now consider two cases, according to whether $d$ is odd or even.
we have
\[ d(s_2) = 1 + 2 + \cdots + (2k + 2) + a = 1 + 2 + \cdots + d + (d + 1) + a \]
\[ \geq \frac{d(d + 1)}{2} + t(G). \]
Since \( t^+(G) \geq d(s_2) \), this implies that
\[ t^+(G) \geq \frac{d(d + 1)}{2} + t(G), \]
a contradiction.

**Case 2.** \( d \) is even. Then \( d = 2k \) for some integer \( k \geq 2 \). Consider the linear ordering \( s'_2 \) of vertices of \( G \) given by
\[ s'_2 : x_k, x_{k+1}, x_{k-1}, x_{k+2}, \ldots, x_1, x_{2k}, u, v, y_1, y_2, \ldots, y_{n-d-2}. \]
Observe that
\[ d(s'_2) = 1 + 2 + \cdots + (2k + 1) + a = 1 + 2 + \cdots + d + (d + 1) + a \]
\[ \geq \frac{d(d + 1)}{2} + t(G). \]
By the same argument as in Case 1, this leads to a contradiction.

**Corollary 6.53** If \( G \) is a nontrivial connected graph for which \( t^+(G) - t(G) = 1 \), then
\[ \text{diam}(G) \leq \left\lfloor \frac{1 + \sqrt{8\ell + 1}}{2} \right\rfloor. \]

**Proof.** Let \( d \) be the smallest positive integer such that \( \ell < \frac{d(d+1)}{2} \). Then
\[ \frac{d(d - 1)}{2} \leq \ell < \frac{d(d + 1)}{2}, \]
that is, \( d^2 - d - 2\ell \leq 0 \) and \( d^2 + d - 2\ell > 0 \). Since \( d^2 - d - 2\ell \leq 0 \),
\[ 1 \leq d \leq \left\lfloor \frac{1 + \sqrt{8\ell + 1}}{2} \right\rfloor. \]
On the other hand, since \( d^2 + d - 2\ell > 0 \),

\[
d \geq \left\lfloor \frac{-1 + \sqrt{8\ell + 1}}{2} \right\rfloor + 1 = \left\lfloor \frac{1 + \sqrt{8\ell + 1}}{2} \right\rfloor.
\]

Therefore, \( d = \left\lfloor \frac{1 + \sqrt{8\ell + 1}}{2} \right\rfloor \) and the result follows by Proposition 6.52.

We have seen that if \( G \) is a connected graph of order \( n \) for which

\[
q_r(G) = \left( \frac{33}{2}, \frac{33}{20} \right) = \left( \frac{33}{20}, \frac{33}{20} \right),
\]

then \( n = 20\ell \) for some integer \( \ell \geq 2 \) and \( q(G) = (32\ell, 33\ell) \). In addition, \( \text{diam}(G) \leq \left\lfloor \frac{1 + \sqrt{8\ell + 1}}{2} \right\rfloor \) by Corollary 6.53. If \( \ell = 2 \), then this implies that \( G \) is a graph of order 40 with \( q(G) = (64, 66) \) and \( \text{diam}(G) \leq 2 \). Moreover, since \( G \) is not traceable, \( \text{diam}(G) \neq 1 \). Hence \( \text{diam}(G) = 2 \). First observe that

\[
1 \leq h(G) - t(G) \leq \text{diam}(G) = 2
\]

by Lemma 3.7. Since \( G \) is not traceable and \( h(G) - t(G) = 1 \) if and only if \( G \) is Hamiltonian by Proposition 3.8, it follows that \( h(G) - t(G) = 2 \). Next we show that \( h^+(G) - t^+(G) = 1 \). Assume, to the contrary, that \( h^+(G) = t^+(G) + 2 = 68 \) and let

\[
s_c: v_1, v_2, \ldots, v_{40}, v_{41} = v_1
\]

be a cyclic ordering of vertices of \( G \) for which \( d(s_c) = h^+(G) \). Observe that \( d(s_c) \) must contain exactly 12 terms equal to 1 and exactly 28 terms equal to 2. Let \( i \) be an integer with \( 1 \leq i \leq 39 \) such that \( d(v_i, v_{i+1}) = 1 \) and consider the linear ordering \( s_\ell \) of vertices of \( G \) given by

\[
s_\ell: v_{i+1}, v_{i+2}, \ldots, v_{40}, v_1, v_2, \ldots, v_i.
\]

Then

\[
t^+(G) \geq d(s_\ell) = d(s_c) - d(v_i, v_{i+1}) = h^+(G) - 1 = t^+(G) + 1,
\]

which is a contradiction. Hence \( h^+(G) - t^+(G) = 1 \). Then

\[
h^+(G) - h(G) = (t^+(G) + 1) - (t(G) + 2) = t^+(G) - t(G) - 1 = 1.
\]
Since $G$ is not traceable, it follows by Theorem 2.15 that $G \in \{K_{1,39} + e, K_{1,1,38}\}$. If $G = K_{1,39} + e$, then the length of a longest path is 3 while the maximum size of a spanning linear forest is also 3 and so

$$64 = t(G) = 2 \cdot 40 - 2 - 3 = 75,$$

a contradiction. Thus $G = K_{1,1,38}$. By Proposition 4.5, however,

$$66 = t^+(G) = 2 \cdot 40 - 3 - 1 = 76,$$

another contradiction. Hence there is no graph $G$ of order 40 for which $q(G) = (64,66)$. This means that we must search for a connected graph $G$ of order $n = 20\ell$ with $\ell \geq 3$ such that $q(G) = (32\ell, 33\ell)$, if there is any.

We have seen that some ordered pairs $(\frac{8}{5}, \gamma)$ of rational numbers, where $\frac{8}{5} < \gamma < \frac{9}{5}$, are realizable as the value of $q_r(G)$ of some connected graph $G$ (for example, $\gamma = \frac{17}{10}$, $\frac{33}{20}$), while we haven’t been able to determine if it is also realizable when $\gamma = \frac{33}{20}$. We end this section with the following question.

**Problem 6.54** For each pair $\alpha, \gamma$ of rational numbers such that $1 < \alpha < \gamma < 1 + \frac{9}{2}$, does there exist a connected graph $G$ such that $q_r(G) = (\alpha, \gamma)$? If not, then what additional conditions on the pair $\alpha, \gamma$ determine its realizability?
Chapter 7

The Total Traceable Number of a Graph

7.1 Introduction

Recall that a graph $G$ is vertex-traceable if every vertex of $G$ is the initial vertex of a Hamiltonian path of $G$. Thus if $G$ is a vertex-traceable graph of order $n$, then the traceable number of every vertex of $G$ is $n - 1$. Certainly, every Hamiltonian graph is vertex-traceable, but the converse is not true. For example, the Petersen graph is vertex-traceable but it is not Hamiltonian. A graph $G$ is Hamiltonian-connected if $G$ has a $u - v$ Hamiltonian path for every two vertices $u$ and $v$ of $G$. Obviously, every Hamiltonian-connected graph is Hamiltonian and so every Hamiltonian-connected graph is vertex-traceable. However, the converse is not true. For example, every cycle $C_n$ of order $n \geq 4$ is vertex-traceable and Hamiltonian, but it is not Hamiltonian-connected.

For a connected graph $G$ of order $n$, the total traceable number $tt(G)$ of $G$ is defined by

$$tt(G) = \sum_{v \in V(G)} t(v),$$

where $t(v)$ is the traceable number of $v$. Since $t(v) \geq n - 1$ for every vertex $v$ of $G$, it follows that

$$tt(G) \geq n(n - 1). \quad (7.1)$$

Furthermore,
\[ tt(G) = n(n - 1) \] if and only if \( G \) is vertex-traceable.

Therefore, the total traceable number of a connected graph \( G \) of order \( n \) can be considered as a measure of how close \( G \) is to being vertex-traceable – the closer \( tt(G) \) is to \( n(n - 1) \), the closer \( G \) is to being vertex-traceable.

Consider the graphs \( H_1 \) and \( H_2 \) in Figure 7.1, where \( H_1 \) is obtained by adding a pendant edge at a vertex of a complete graph \( K_{n-1} \) and \( H_2 \cong 2K_1 + (K_{n-4} \cup 2K_1) \).

For the graph \( H_1 \), every vertex of \( H_1 \) has traceable number \( n - 1 \), except for the vertex \( v \), which has traceable number \( n \). Thus \( tt(H_1) = n(n - 1) + 1 \). Every vertex of the graph \( H_2 \) has traceable number \( n - 1 \), except for \( v_1 \) and \( v_2 \), which have traceable number \( n \). Thus \( tt(H_2) = n(n - 1) + 2 \). Note that by Proposition 3.26, if \( u \) and \( v \) are adjacent vertices in a connected graph \( G \), then \( |t(u) - t(v)| \leq 1 \). This implies that if \( G \) is a connected graph of order \( n \) with \( tt(G) = n(n - 1) + 2 \), then \( G \) must contain exactly two vertices with traceable number \( n \). In other words, it is impossible for \( G \) to contain a single vertex with traceable number \( n + 1 \).

Next consider the graphs \( G_1 \) and \( G_2 \) in Figure 7.2, where \( G_1 \) is obtained from a complete graph \( K_{n-2} \) (\( n \geq 5 \)) by adding two pendant edges and \( G_2 \) is obtained from a cycle \( C_{n-1} \) (\( n \geq 4 \)) by adding a pendant edge. The graph \( G_1 \) of order \( n \) in Figure 7.2 contains exactly two vertices with traceable number \( n - 1 \), namely \( t(u) = t(v) = n - 1 \). All other vertices of \( G_1 \) have traceable number \( n \). Thus \( tt(G_1) = n(n - 1) + (n - 2) \). The graph \( G_2 \) of order \( n \) in Figure 7.2 contains exactly three vertices with traceable number \( n - 1 \), namely \( t(u) = t(v) = t(w) = n - 1 \). All other vertices of \( G_2 \) have traceable number \( n \). Thus \( tt(G_2) = n(n - 1) + (n - 3) \).

Therefore, the graphs \( H_1 \) and \( H_2 \) in Figure 7.1 are closer to being vertex-traceable than are the graphs \( G_1 \) and \( G_2 \) in Figure 7.2.

\[ \text{Figure 7.1: Graphs } H_1 \text{ and } H_2 \]

\[ \text{Figure 7.2: Graphs } G_1 \text{ and } G_2 \]
There is no graph of order \( n \) containing exactly one vertex with traceable number \( n - 1 \), since for every nontrivial connected graph \( G \), there are at least two vertices with traceable number \( t(G) \). We know of no example of a nontrivial connected graph of order \( n \) every vertex of which has traceable number \( n \). That is, we know of no example of a non-traceable graph \( G \) of order \( n \) for which \( tt(G) = n^2 \). (Note that for \( n \geq 6 \), we do know that there exists a traceable graph \( G \) of order \( n \) for which \( tt(G) = n^2 \). Observe that the graph \( G \) shown in Figure 7.3 has order \( n \), is traceable, and \( tt(G) = n^2 \).)

On the other hand, we show that for every pair \( n, k \) of integers such that \( n \geq 3 \) and \( 2 \leq k \leq n \), there exists a connected graph \( G \) of order \( n \) containing \( k \) vertices \( v \) with \( t(v) = n - 1 \) such that

\[
\text{tt}(G) = n(n - 1) + (n - k).
\]

**Proposition 7.1** For every pair \( n, k \) of integers with \( n \geq 3 \) and \( 2 \leq k \leq n \), there exists a connected graph of order \( n \) containing \( k \) vertices with traceable number \( n - 1 \) and \( n - k \) vertices with traceable number \( n \).
Proof. Since every Hamiltonian-connected graph has the desired properties for 
k = n, we restrict our attention to those integers \( k \) for which \( 2 \leq k \leq n - 1 \). For 
\( 3 \leq n \leq 5 \), the graphs \( G_{k,n} \) of Figure 7.4 have the desired properties.

\[
\begin{array}{ll}
G_{2,3} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \\
G_{2,4} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ
\end{array} \\
G_{3,4} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
\end{array}
\]

\[
\begin{array}{ll}
G_{2,5} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array} \\
G_{3,5} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ
\end{array} \\
G_{4,5} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ
\end{array}
\end{array}
\]

Figure 7.4: Graphs \( G_{k,n} \) with \( 2 \leq k \leq n - 1 \leq 4 \)

\[
\begin{array}{ll}
G_{2,n} : & 
\begin{array}{c}
\circ \\
\circ
\end{array} \\
G_{n-2,n} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ
\end{array} \\
G_{n-1,n} : & 
\begin{array}{c}
\circ \\
\circ
\end{array}
\end{array}
\]

\[
\begin{array}{ll}
G_{n,n-1} : & 
\begin{array}{c}
\circ \\
\circ
\end{array} \\
G_{k,n} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array} \\
G_{n-k-1,n} : & 
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array}
\end{array}
\]

Figure 7.5: Graphs \( G_{k,n} \) with \( n \geq 6 \) and \( 2 \leq k \leq n - 1 \)

For \( n \geq 6 \), the graphs \( G_{k,n} \) of Figure 7.5 have the appropriate properties. ■

7.2 Bounds for the Total Traceable Number

If \( G \) is a connected graph and \( H \) is a connected spanning subgraph of \( G \), then we have 
seen that \( t_G(v) \leq t_H(v) \) for every vertex \( v \in V(G) = V(H) \). Hence \( tt(G) \leq tt(H) \). 
We state this useful observation below.

Observation 7.2 If \( G \) is a connected graph and \( H \) is a connected spanning sub-
graph of \( G \), then \( tt(G) \leq tt(H) \). In particular, if \( G \) is a connected graph and \( T \) is a
spanning tree of $G$, then $tt(G) \leq tt(T)$. Therefore, $\min\{tt(T) : T$ is a spanning tree of $G\}$ is an upper bound for $tt(G)$.

Hence it is useful to know the total traceable numbers of trees. In order to do this, we first recall a useful result on the traceable number of a vertex of a tree; that is, if $T$ is a nontrivial tree of order $n$ and $v$ is a vertex of $T$, then

$$t(v) = 2n - 2 - e(v).$$

(7.2)

With the aid of (7.2), we are able to determine the total traceable numbers of some classes of trees, namely paths, stars, and double-stars.

**Example 7.3** For each integer $n \geq 2$,

$$tt(P_n) = n(n - 1) + \left\lfloor \left(\frac{n - 1}{2}\right)^2 \right\rfloor.$$

**Example 7.4** For each integer $n \geq 3$,

$$tt(K_{1,n-1}) = n(n - 1) + (n^2 - 3n + 1).$$

**Example 7.5** For each integer $n \geq 4$,

$$tt(S_{n,b}) = n(n - 1) + (n^2 - 4n + 2).$$

Recall that for a connected graph $G$, a vertex $v$ of $G$ is a central vertex in $G$ if $e(v) = \text{rad}(G)$. It is well known that for a nontrivial tree $T$, there is exactly one central vertex if $\text{diam}(T) = 2\text{rad}(T)$ and there are exactly two central vertices if $\text{diam}(T) = 2\text{rad}(T) - 1$. Moreover, for each positive integer $\ell$ such that $\text{rad}(T) < \ell \leq \text{diam}(T)$, there are at least two vertices having eccentricity $\ell$. The following result provides lower and upper bounds for the total traceable number of a nontrivial tree in terms of its order.

**Theorem 7.6** For every tree $T$ of order $n \geq 3$,

$$n(n - 1) + \left\lfloor \left(\frac{n - 1}{2}\right)^2 \right\rfloor \leq tt(T) \leq n(n - 1) + (n^2 - 3n + 1).$$

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Proof. Since diam(T) ≤ n - 1 for every tree T of order n, the largest possible radius of a tree T having odd order is (n - 1)/2, while the largest possible radius of a tree T having even order is n/2. We consider the cases when n is odd and when n is even separately.

Case 1. n is odd. Then
\[
\sum_{v \in V(T)} e(v) \leq \frac{n - 1}{2} + 2 \left[ \frac{n + 1}{2} + \frac{n + 3}{2} + \cdots + (n - 1) \right] = \frac{n - 1}{2} + (n + 1) + (n + 3) + \cdots + 2(n - 1) = \frac{n - 1}{2} + \frac{n(n - 1)}{2} + \left( \frac{n - 1}{2} \right)^2 = \frac{n^2 - 1}{2} + \left( \frac{n - 1}{2} \right)^2.
\]

It then follows by (7.2) that
\[
tt(T) = \sum_{v \in V(T)} t(v) = \sum_{v \in V(T)} (2n - 2 - e(v)) = n(2n - 2) - \sum_{v \in V(T)} e(v) \geq n(2n - 2) - \left( \frac{n^2 - 1}{2} + \left( \frac{n - 1}{2} \right)^2 \right) = n(n - 1) + \left( \frac{n - 1}{2} \right)^2.
\]

Case 2. n is even. Then
\[
\sum_{v \in V(T)} e(v) \leq 2 \left[ \frac{n}{2} + \frac{n + 2}{2} + \cdots + (n - 1) \right] = n + (n + 2) + \cdots + 2(n - 1) = \frac{n^2}{2} + \frac{n^2 - 2n}{4}
\]

It then follows by (7.2) that
\[
tt(T) = \sum_{v \in V(T)} t(v) = n(2n - 2) - \sum_{v \in V(T)} e(v) \geq n(2n - 2) - \left( \frac{n^2}{2} + \frac{n^2 - 2n}{4} \right) = n(n - 1) + \frac{n^2 - 2n}{4}.
\]

Therefore, \(tt(T) \geq n(n - 1) + \left( \frac{n - 1}{2} \right)^2\) for every tree T of order \(n \geq 3\).
If $T$ is a tree of order $n \geq 3$ and $\text{rad}(T) = 1$, then $T$ is a star and all other vertices have eccentricity 2. If $\text{rad}(T) = 2$, then at most two vertices of $T$ have eccentricity 2 with all other vertices having eccentricity 3 or 4. In each case,

$$\sum_{v \in V(T)} e(v) \geq 1 + 2(n - 1) = 2n - 1.$$ 

Consequently,

$$tt(T) = \sum_{v \in V(T)} t(v) = n(2n - 2) - \sum_{v \in V(T)} e(v) \leq n(2n - 2) - (2n - 1) = n(n - 1) + (n^2 - 3n + 1).$$

Therefore, $tt(T) \leq n(n - 1) + (n^2 - 3n + 1)$ for every tree $T$ of order $n \geq 3$. □

Hence for every tree $T$ of order $n \geq 3$, we have

$$tt(P_n) \leq tt(T) \leq tt(K_{1,n-1})$$

and by Examples 7.3 and 7.4, the lower and upper bounds are both sharp. The following result is a consequence of Theorem 7.6, which provides upper and lower bounds for a general connected graph in terms of its order.

**Corollary 7.7** For a nontrivial connected graph $G$ of order $n$,

$$n(n - 1) \leq tt(G) \leq n(n - 1) + (n^2 - 3n + 1).$$

**Proof.** We have already noted that $tt(G) \geq n(n - 1)$, so it remains only to show that $tt(G) \leq n(n - 1) + (n^2 - 3n + 1)$. By Observation 7.2, the maximum value of $tt(G)$ among all connected graphs $G$ of order $n$ occurs when $G$ is a tree. The result then follows by Theorem 7.6. □

### 7.3 Realization Results

By Corollary 7.7, if $G$ is a nontrivial connected graph of order $n$, then

$$n(n - 1) \leq tt(G) \leq n(n - 1) + (n^2 - 3n + 1).$$

In this section, we investigate the following problem.
Problem 7.8  For each pair \( n, a \) of integers such that \( n \geq 3 \) and \( n(n - 1) \leq a \leq n(n - 1) + (n^2 - 3n + 1) \), does there exist a connected graph \( G \) of order \( n \) with \( \tau(G) = a \)?

As an illustration, we consider \( n \in \{3, 4, 5\} \).

- \( n = 3 \): Then \( n(n - 1) = 6 \) and \( n(n - 1) + (n^2 - 3n + 1) = 7 \). There are only two connected graphs of order 3, namely \( K_3 \) and \( P_3 \). Observe that \( \tau(K_3) = 6 \) and \( \tau(P_3) = 7 \). Hence \( \{\tau(G): n = 3\} = \{6, 7\} \).

- \( n = 4 \): Then \( n(n - 1) = 12 \) and \( n(n - 1) + (n^2 - 3n + 1) = 17 \). Observe that \( \tau(K_4) = \tau(K_4 - e) = \tau(C_4) = 12, \ \tau(P_4) = 14, \ \tau(K_{1,3} + e) = 13, \) and \( \tau(K_{1,3}) = 17 \). Hence
  \[
  \{\tau(G): n = 4\} = \{12, 13, 14, 17\} = \{a \in \mathbb{N}: 12 \leq a \leq 17\} - \{15, 16\}.
  \]

- \( n = 5 \): Then \( n(n - 1) = 20 \) and \( n(n - 1) + (n^2 - 3n + 1) = 31 \). Observe that for every graph \( G \) of order 5 that is Hamiltonian, \( \tau(G) = 20 \). Also, each graph \( G_i \) shown in Figure 7.6 is a graph of order 5 with \( \tau(G_i) = i \).

![Graphs \( G_i \) for \( i \in \{21, 22, 23, 24, 26, 27, 31\} \)](image)

Hence
\[
\{\tau(G): n = 5\} \supseteq \{20, 21, 22, 23, 24, 26, 27, 31\} = \{a \in \mathbb{N}: 20 \leq a \leq 31\} - \{25\} \cup \{28, 29, 30\}.
\]
We will show that the two sets above are actually equal, that is, it is also true that

\[ \{tt(G) : n = 5\} \subseteq \{a \in \mathbb{N} : 20 \leq a \leq 31\} - (\{25\} \cup \{28, 29, 30\}), \]

and generalize this fact for larger integers \( n \). In order to do this, we first consider the total traceable number of a tree of order \( n \geq 5 \) whose diameter is at least 4.

**Proposition 7.9** Let \( T \) be a tree of order \( n \geq 5 \). If \( \text{diam}(T) \geq 4 \), then

\[ tt(T) \leq n(n - 1) + (n^2 - 4n - 1). \]

**Proof.** Let \( T \) be a tree of order \( n \geq 5 \) having diameter \( d \geq 4 \). We consider two cases, according to whether \( d \) is odd or even.

**Case 1.** \( d \) is odd. Thus \( d = 2k + 1 \) for some integer \( k \geq 2 \). Then \( T \) contains two central vertices with eccentricity \( \text{rad}(T) = k + 1 \). Hence the largest possible value of \( tt(T) \) occurs when for each integer \( \ell \) with \( k + 1 \leq \ell \leq 2k + 1 \), there are exactly two vertices with eccentricity \( \ell \) and the remaining \( n - (2k + 2) \) vertices have eccentricity \( k + 2 \). (Such a tree \( T \) must be a caterpillar constructed from a path \( P \) of length \( 2k + 1 \) and \( n - 2k - 2 \) copies of \( K_1 \) by joining each vertex of \( K_1 \) to exactly one of the two central vertices of \( P \).) Hence

\[
\begin{align*}
tt(T) &\leq 2(n - 2 - (2k + 1)) + 2(2n - 2 - (2k)) + \cdots \\
&+ 2(2n - 2 - (k + 1)) + (n - 2k - 2)(2n - 2 - (k + 2)) \\
&= (k + 1)(4n - 3k - 6) + [2n^2 - (5k + 8)n + (2k^2 + 10k + 8)] \\
&= 2n^2 - (k + 4)n - (k^2 - k - 2) \\
&= 2n^2 - (k + 4)n - (k + 1)(k - 2) \\
&\leq 2n^2 - (k + 4)n \\
&< 2n^2 - 5n \\
&= n(n - 1) + (n^2 - 4n).
\end{align*}
\]

**Case 2.** \( d \) is even. Thus \( d = 2k \) for some integer \( k \geq 2 \). Then \( T \) contains one central vertex with eccentricity \( \text{rad}(T) = k \). Hence the largest possible value of \( tt(T) \) occurs when for each integer \( \ell \) with \( k + 1 \leq \ell \leq 2k \), there are exactly two vertices with eccentricity \( \ell \) and the remaining \( n - (2k + 1) \) vertices have eccentricity \( k + 1 \). (In this case \( T \) must be a caterpillar constructed from a path \( P \) of length \( 2k \)
and $n - (2k + 1)$ copies of $K_1$ by joining each vertex of $K_1$ to the central vertex of $P$.) Hence

$$tt(T) \leq 2(2n - 2 - (2k)) + 2(2n - 2 - (2k - 1)) + \cdots + 2(2n - 2 - (k + 1))$$
$$+ (2n - 2 - k) + (n - 2k - 1)(2n - 2 - (k + 1))$$
$$= k(4n - 3k - 5) + (2n - 2 - k) + [2n^2 - (5k + 5)n + (2k^2 + 7k + 3)]$$
$$= 2n^2 - (k + 3)n - (k^2 - k - 1)$$
$$< 2n^2 - (k + 3)n - (k^2 - k - 2)$$
$$= 2n^2 - (k + 3)n - (k + 1)(k - 2) \leq 2n^2 - (k + 3)n$$
$$\leq 2n^2 - 5n$$
$$= n(n - 1) + (n^2 - 4n)$$

and the proof is complete. ■

Let $G$ be a connected graph of order $n \geq 5$ and $\ell$ the length of a longest path in $G$. Then $2 \leq \ell \leq n - 1$. Moreover, $\ell = 2$ if and only if $G = K_{1,n-1}$ and $\ell = 3$ if and only if $G \in \{S_{a,b}, K_{1,n-1} + e\}$. Observe that if $G = K_{1,n-1} + e$, then

$$t(v) = \begin{cases} 
2n - 4 & \text{if } v \text{ is the central vertex} \\
2n - 5 & \text{otherwise.}
\end{cases}$$

Hence

$$tt(K_{1,n-1} + e) = (2n - 4) + (n - 1)(2n - 5) = 2n^2 - 5n + 1$$
$$= n(n - 1) + (n^2 - 4n + 1).$$

If the length of a longest path in $G$ is $\ell$, then $G$ contains a spanning tree $T$ with \text{diam}(T) = \ell$. Hence we obtain the following.

\textbf{Theorem 7.10} Let $G$ be a connected graph of order $n \geq 5$. Then $tt(G)$ satisfies exactly one of the following:

(1) $n(n - 1) \leq tt(G) \leq n(n - 1) + (n^2 - 4n - 1)$;

(2) $tt(G) \in \{n(n - 1) + (n^2 - 4n + 1), n(n - 1) + (n^2 - 4n + 2)\}$;

(3) $tt(G) = n(n - 1) + (n^2 - 3n + 1)$. 

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Moreover,

\[ tt(G) = n(n - 1) + (n^2 - 4n + 1) \quad \text{if and only if} \quad G = K_{1,n-1} + e; \]
\[ tt(G) = n(n - 1) + (n^2 - 4n + 2) \quad \text{if and only if} \quad G = S_{a,b}; \]
\[ tt(G) = n(n - 1) + (n^2 - 3n + 1) \quad \text{if and only if} \quad G = K_{1,n-1}. \]

One conclusion from Theorem 7.10 is that if \( G \) is a connected graph of order \( n \geq 5 \), then

\[ tt(G) \neq n(n - 1) + (n^2 - 4n) \]

and

\[ tt(G) \notin \{ a \in \mathbb{N} : n(n - 1) + (n^2 - 4n + 3) \leq a \leq n(n - 1) + (n^2 - 3n) \}. \]

This implies that there is no connected graph \( G \) of order \( n = 5 \) for which \( tt(G) \in \{ 25 \} \cup \{ 28, 29, 30 \} \) and so we conclude that

\[ \{ tt(G) : n = 5 \} = \{ a \in \mathbb{N} : 20 \leq a \leq 31 \} - (\{ 25 \} \cup \{ 28, 29, 30 \}) . \]

Let us now consider the case where \( n \geq 6 \). For \( n = 6, 7 \), it follows from Theorem 7.10 that

\[ \{ tt(G) : n = 6 \} \subseteq \{ a \in \mathbb{N} : 30 \leq a \leq 49 \} - (\{ 42 \} \cup \{ 45, 46, 47, 48 \}) \]

and

\[ \{ tt(G) : n = 7 \} \subseteq \{ a \in \mathbb{N} : 42 \leq a \leq 71 \} - (\{ 63 \} \cup \{ 66, 67, \ldots, 70 \}) . \]

In fact, the sets are equal in both cases. This suggests a new problem that is a slight modification of Problem 7.8.

**Problem 7.11** For each pair \( n, a \) of integers with \( n \geq 3 \) and \( n(n - 1) \leq a \leq n(n - 1) + (n^2 - 4n - 1) \), does there exist a connected graph \( G \) of order \( n \) such that \( tt(G) = a \)?

By Theorem 7.10, if \( T \) is a tree of order \( n \) with \( tt(T) \leq n(n - 1) + (n^2 - 4n - 1) \), then \( \text{diam}(T) \geq 4 \) and so \( n \geq 5 \). Hence we consider a tree \( T \) of order \( n \geq 5 \) with \( 4 \leq \text{diam}(T) \leq n - 1 \). By Theorem 7.6,

\[ n(n - 1) + \left( \left( \frac{n - 1}{2} \right)^2 \right) \leq tt(T) \leq n(n - 1) + (n^2 - 4n - 1) . \]
As an illustration, we consider \( n \in \{5, 6, 7, 8\} \).

- \( n = 5 \): Then \( n(n - 1) + \left\lceil \frac{(n-1)^2}{2} \right\rceil = n(n - 1) + (n^2 - 4n - 1) = 24 \). Note that \( P_5 \) is the only tree of order 5 having diameter at least 4 and \( tt(P_5) = 24 \) by Example 7.3. Hence \( \{tt(T) : n = 5 \text{ and } \text{diam}(T) \geq 4\} = \{24\} \).

- \( n = 6 \): Then \( n(n - 1) + \left\lceil \frac{(n-1)^2}{2} \right\rceil = 36 \) and \( n(n - 1) + (n^2 - 4n - 1) = 41 \). There are exactly three trees of order 6 having diameter at least 4, listed in Figure 7.7. Observe that

\[
\{tt(T) : n = 6 \text{ and } \text{diam}(T) \geq 4\} = \{36\} \cup \{40, 41\}.
\]

- \( n = 7 \): Then \( n(n - 1) + \left\lceil \frac{(n-1)^2}{2} \right\rceil = 51 \) and \( n(n - 1) + (n^2 - 4n - 1) = 62 \). All trees of order 7 having diameter at least 4 are listed in Figure 7.8. Note that some trees have the same total traceable number. By careful inspection, we have

\[
\{tt(T) : n = 7 \text{ and } \text{diam}(T) \geq 4\} = \{51\} \cup \{55, 56\} \cup \{60, 61, 62\}.
\]
• $n = 8$: Then $n(n - 1) + \left\lfloor \frac{(n-1)^2}{2} \right\rfloor = 68$ and $n(n - 1) + (n^2 - 4n - 1) = 87$. All trees of order 8 having diameter at least 4 are listed in Figure 7.9. Again note that some trees have the same total traceable number. By careful inspection, we have

$$\{tt(T) : n = 8 \text{ and } \text{diam}(T) \geq 4\} = \{68\} \cup \{73, 74, 75\} \cup \{78, 79, 80\} \cup \{84, 85, 86, 87\}.$$  

![Figure 7.9: Trees of order 8 with diameter at least 4](image)

As we saw in the above examples, it turns out that not every integer $a$ with

$$n(n - 1) + \left\lfloor \frac{(n-1)^2}{2} \right\rfloor \leq a \leq n(n - 1) + (n^2 - 4n - 1)$$

is realizable as the total traceable number of a tree of order $n$ when $n \geq 6$. (We will show later, however, that every integer $a$ in the range above is realizable as the
total traceable number of some connected graph of order \( n \) for each \( n \geq 6 \).) We next determine the precise set of integers that are realizable as the total traceable numbers of trees of order \( n \geq 6 \).

**Proposition 7.12** Let \( T \) be a tree of order \( n \geq 5 \) and diameter \( d \geq 4 \). Then \( \text{tt}(T) = n(n - 1) + b \) for some integer \( b \) with

\[
n^2 - (d + 1)n + \left\lfloor \frac{d}{2} \right\rfloor \leq b \leq n^2 - \left\lfloor \frac{d + 5}{2} \right\rfloor n - \left\lfloor \frac{d}{2} \right\rfloor^2 + \left\lfloor \frac{d}{2} \right\rfloor + 1.
\]

**Proof.** Let \( b' \) and \( b'' \) be given by

\[
b' = n^2 - (d + 1)n + \left\lfloor \frac{d}{2} \right\rfloor \left\lceil \frac{d}{2} \right\rceil,
\]

\[
b'' = n^2 - \left\lfloor \frac{d + 5}{2} \right\rfloor n - \left\lfloor \frac{d}{2} \right\rfloor^2 + \left\lfloor \frac{d}{2} \right\rfloor + 1.
\]

We consider two cases according to whether \( d \) is even or odd.

**Case 1.** \( d \) is even. Then \( d = 2k \) for some integer \( k \geq 2 \). Thus

\[
b' = n^2 - (2k + 1)n + k^2,
\]

\[
b'' = n^2 - (k + 2)n - k^2 + k + 1.
\]

Also, \( k \leq e(v) \leq 2k \) for each vertex \( v \in V(T) \). Furthermore, \( T \) contains exactly one vertex whose eccentricity is \( k \) and for each integer \( \ell \) with \( k < \ell \leq 2k \), there are at least two vertices in \( T \) having eccentricity \( \ell \). Hence

\[
\sum_{v \in V(T)} e(v) \leq k + 2[(k + 1) + (k + 2) + \cdots + (2k)] + (n - 2k - 1)(2k)
\]

\[
= 2kn - k^2
\]

and so

\[
\text{tt}(T) = n(2n - 2) - \sum_{v \in V(T)} e(v)
\]

\[
\geq 2n^2 - 2n - 2kn + k^2
\]

\[
= n(n - 1) + n^2 - (2k + 1)n + k^2 = n(n - 1) + b'.
\]

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On the other hand,
\[
\sum_{v \in V(T)} e(v) \geq k + 2 [(k + 1) + (k + 2) + \cdots + (2k)] + (n - 2k - 1)(k + 1)
\]
\[= (k + 1)n + k^2 - k - 1\]
and so
\[
\text{tt}(T) = n(2n - 2) - \sum_{v \in V(T)} e(v)
\]
\[\leq 2n^2 - 2n - (k + 1)n - k^2 + k + 1\]
\[= n(n - 1) + n^2 - (k + 2)n - k^2 + k + 1 = n(n - 1) + b'.\]
Hence \(n(n - 1) + b' \leq \text{tt}(T) \leq n(n - 1) + b''.\)

Case 2. \(d\) is odd. Then \(d = 2k + 1\) for some integer \(k \geq 2\). Thus
\[
b' = n^2 - (2k + 2)n + k^2 + k,
\]
\[
b'' = n^2 - (k + 3)n - k^2 + k + 2.
\]
Also, \(k + 1 \leq e(v) \leq 2k\) for each vertex \(v \in V(T)\). Furthermore, for each integer \(\ell\) with \(k + 1 \leq \ell \leq 2k + 1\), there are at least two vertices in \(T\) having eccentricity \(\ell\).
Hence
\[
\sum_{v \in V(T)} e(v) \leq 2 [(k + 1) + (k + 2) + \cdots + (2k + 1)] + (n - 2k - 2)(2k + 1)
\]
\[= 2kn - k^2 - k\]
and so
\[
\text{tt}(T) = n(2n - 2) - \sum_{v \in V(T)} e(v)
\]
\[\geq 2n^2 - 2n - 2kn + k^2 + k\]
\[= n(n - 1) + n^2 - (2k + 2)n + k^2 + k = n(n - 1) + b'.\]
On the other hand,
\[
\sum_{v \in V(T)} e(v) \geq 2 [(k + 1) + (k + 2) + \cdots + (2k + 1)] + (n - 2k - 2)(k + 2)
\]
\[= (k + 2)n + k^2 - k - 2\]
and so

\[ tt(T) = n(2n - 2) - \sum_{v \in V(T)} e(v) \]

\[ \leq 2n^2 - 2n - (k + 2)n - k^2 + k + 2 \]

\[ = n(n - 1) + n^2 - (k + 3)n - k^2 + k + 2 = n(n - 1) + b''. \]

Hence \( n(n - 1) + b' \leq tt(T) \leq n(n - 1) + b'' \).

More than Proposition 7.12 can be said, which we show next.

**Proposition 7.13** Let \( n \) and \( d \) be integers such that \( n \geq 5 \) and \( 4 \leq d \leq n - 1 \). If \( a \) is an integer such that \( a = n(n - 1) + b \), where

\[ n^2 - (d + 1)n + \left\lfloor \frac{d}{2} \right\rfloor \left\lfloor \frac{d}{2} \right\rfloor \leq b \leq n^2 - \left\lfloor \frac{d + 5}{2} \right\rfloor n - \left\lfloor \frac{d}{2} \right\rfloor^2 + \left\lfloor \frac{d}{2} \right\rfloor + 1, \]

then there exists a tree \( T \) of order \( n \) and \( \text{diam}(T) = d \) for which \( tt(T) = a \).

**Proof.** For each such integer \( d \), let \( b'_{n,d} \) and \( b''_{n,d} \) be given by

\[ b'_{n,d} = n^2 - (d + 1)n + \left\lfloor \frac{d}{2} \right\rfloor \left\lfloor \frac{d}{2} \right\rfloor, \]

\[ b''_{n,d} = n^2 - \left\lfloor \frac{d + 5}{2} \right\rfloor n - \left\lfloor \frac{d}{2} \right\rfloor^2 + \left\lfloor \frac{d}{2} \right\rfloor + 1. \]

First observe that if \( d = n - 1 \), then

\[ b'_{n,n-1} = b''_{n,n-1} = \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \]

and so \( a = n(n - 1) + \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \). Taking \( T = P_n \), we obtain the desired result. Hence we assume that \( n \geq 6 \) and \( 4 \leq d \leq n - 2 \). Consider the function \( f \) defined on the set \( [0, \left\lfloor \frac{d}{2} \right\rfloor - 1] \) (of real numbers) by

\[ f(x) = (n - d - 1)x + b'_{n,d}. \]

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Observe that \( f \) is continuous and strictly increasing on \([0, \lfloor \frac{k}{2} \rfloor - 1]\). We first consider the case where \( d \) is even. Thus \( d = 2k \) for some integer \( k \geq 2 \). Then
\[
\begin{align*}
 b'_{n,d} &= n^2 - (2k + 1)n + k^2, \\
 b''_{n,d} &= n^2 - (k + 2)n - k^2 + k + 1,
\end{align*}
\]
and \( f(x) = (n - 2k - 1)x + b'_{n,d} \) is defined on the set \([0, k - 1]\). Let \( c \in [0, k - 1] \cap \mathbb{Z} \).

Note that \( f(c) \in \mathbb{Z} \) and
\[
f(0) = b'_{n,d} \quad \text{and} \quad f(k - 1) = b''_{n,d}.
\]

Let \( P : v_0, v_1, \ldots, v_{2k} \) be a path of length \( 2k \). For each integer \( c \in [0, k - 1] \), let \( T_{d,c} \) be the caterpillar of order \( n \) and diameter \( d = 2k \) obtained from \( P \) by joining \( n - 2k - 1 \) new end-vertices \( u_1, u_2, \ldots, u_{n-2k-1} \) to the vertex \( v_{c+1} \) on \( P \). Then \( k \leq e(v) \leq 2k \) for each vertex \( v \in V(T_{d,c}) \). Note that \( P \) contains exactly one vertex of eccentricity \( k \) and for each integer \( \ell \) with \( k < \ell \leq 2k \), there are exactly two vertices of eccentricity \( \ell \). Those \( n - 2k - 1 \) extra end-vertices that are added to \( P \) have eccentricity \( 2k - c \). Observe that
\[
\sum_{v \in V(P)} e(v) = k + 2 \left( (k + 1) + (k + 2) + \cdots + (2k) \right) = 3k^2 + 2k.
\]

Hence
\[
\begin{align*}
\sum_{v \in V(T_{d,c})} e(v) &= \sum_{v \in V(P)} e(v) + \sum_{i=1}^{n-2k-1} e(u_i) \\
&= 3k^2 + 2k + (n - 2k - 1)(2k - c) \\
&= (2k - c)n - k^2 + 2ck + c
\end{align*}
\]
and so
\[
\text{tt}(T_{d,c}) = n(2n - 2) - \sum_{v \in V(T_{d,c})} e(v) = n(n - 1) + (n - 2k - 1)c + n^2 - (2k + 1)n + k^2 = n(n - 1) + f(c).
\]

In particular, \( \text{tt}(T_{d,0}) = n(n - 1) + b'_{n,d} \). Hence we now assume that \( b'_{n,d} < b \leq b''_{n,d} \), that is, \( f(0) + 1 \leq b \leq f(k - 1) \). Since \( f \) is strictly increasing on \([0, k - 1]\), it follows
that there exists a unique integer \( c \in [0, k - 2] \) such that

\[
f(c) + 1 \leq b \leq f(c + 1).
\]

Furthermore, since \( f(c + 1) = f(c) + (n - 2k - 1) \), it follows that \( b = f(c) + j \) for some integer \( j \) with \( 1 \leq j \leq n - 2k - 1 \). Let \( T_{d,c,j} \) be the caterpillar of order \( n \) and diameter \( d \) obtained from \( T_d,c \) by first deleting the \( j \) vertices \( u_1, u_2, \ldots, u_j \), then by adding \( j \) new end-vertices \( w_1, w_2, \ldots, w_j \) and joining them to the vertex \( v_{c+2} \) on \( P \).

Then the \( n - 2k - 1 - j \) vertices \( u_{j+1}, u_{j+2}, \ldots, u_{n-2k-1} \) have eccentricity \( 2k - c \) while the \( j \) vertices \( w_1, w_2, \ldots, w_j \) have eccentricity \( 2k - c - 1 \). Therefore

\[
\sum_{v \in V(T_{d,c,j})} e(v) = (3k^2 + 2k) + (n - 2k - 1 - j)(2k - c) + j(2k - c - 1)
= (2k - c)n - k^2 + 2ck + c - j
\]

and so

\[
\sum_{v \in V(T_{d,c,j})} e(v) = n(2n - 2) - \sum_{v \in V(T_{d,c,j})} e(v)
= n(n - 1) + f(c) + j = n(n - 1) + b.
\]

A similar argument applies for the case where \( d \) is odd. Hence for every integer \( a \) with \( n(n - 1) + b' \leq a \leq n(n - 1) + b'' \), there exists a tree \( T \) for which \( \text{tt}(T) = a \), yielding the desired result. \(\blacksquare\)

We illustrate the proof of Proposition 7.13 for \( n = 12 \) and \( d = 7 \). Since \( 192 \leq a \leq 200 \), it follows that \( 60 \leq b \leq 68 \) and

\[
f(x) = 4x + 60 \quad \begin{array}{c|c|c|c}
\text{c} & 0 & 1 & 2 \\
\hline
\text{f(c)} & 60 & 64 & 68
\end{array}
\]

Note that \( f(0) \leq b \leq f(2) \).

If \( b = 60 = f(0) \), then \( T = T_{7,0} \) and \( \text{tt}(T_{7,0}) = 192 \).
If $61 \leq b \leq 64$, then the possible values of $b$ are

- $b = 61 = f(0) + 1$ (and $tt(T_{7,0,1}) = 193$);
- $b = 62 = f(0) + 2$ (and $tt(T_{7,0,2}) = 194$);
- $b = 63 = f(0) + 3$ (and $tt(T_{7,0,3}) = 195$);
- $b = 64 = f(0) + 4 = f(1)$ (and so $T = T_{7,0,4} = T_{7,1}$ and $tt(T_{7,1}) = 196$).

If $65 \leq b \leq 68$, then the possible values of $b$ are

- $b = 65 = f(1) + 1$ (and $tt(T_{7,1,1}) = 197$);
- $b = 66 = f(1) + 2$ (and $tt(T_{7,1,2}) = 198$);
- $b = 67 = f(1) + 3$ (and $tt(T_{7,1,3}) = 199$);
- $b = 68 = f(1) + 4 = f(2)$ (and so $T = T_{7,1,4} = T_{7,2}$ and $tt(T_{7,2}) = 200$).

Recall that by inspection, we found that

$$\{tt(T) : n = 8 \text{ and } \text{diam}(T) \geq 4\} = \{68\} \cup \{73, 74, 75\} \cup \{78, 79, 80\} \cup \{84, 85, 86, 87\}.$$

This can be verified with Propositions 7.12 and 7.13, since if $T$ is a tree of order $n = 8$ and diameter $d \geq 4$, then $4 \leq d \leq 7$, $n(n - 1) = 56$, and

<table>
<thead>
<tr>
<th>$d$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$56 + b_{6,d}'$</td>
<td>84</td>
<td>78</td>
<td>73</td>
<td>68</td>
</tr>
<tr>
<td>$56 + b_{6,d}''$</td>
<td>87</td>
<td>80</td>
<td>75</td>
<td>68</td>
</tr>
</tbody>
</table>

For each integer $n \geq 6$, define a function $g_n : \{5, \ldots, n - 1\} \to \mathbb{Z}$ by

$$g_n(d) = \max\{0, b_{n,d-1}' - b_{n,d}'' - 1\}.$$

Observe that the value of $g_n(d)$ gives us the number of integers missing between the total traceable numbers of trees of diameter $d$ and of diameter $d - 1$. For example, let $n = 8$. Since $\{tt(T) : n = 8, d = 7\} = \{68\}$ and $\{tt(T) : n = 8, d = 6\} = \{73, 74, 75\}$, there are $g_8(7) = 4$ "missing" integers, namely 69, 70, 71, 72 between the two sets. For $6 \leq n \leq 14$, see Table 7.1.
We make and verify some observations:

1. \( g_n(5) = 3 \) for \( n \geq 6 \).

Let \( n \geq 6 \). Since \( b_{n,4} = n^2 - 5n + 4 \) and \( b_{n,5}'' = n^2 - 5n \),

\[
g_n(5) = \max\{0, b_{n,4}' - b_{n,5}'' - 1\} = \max\{0, 3\} = 3.
\]

2. \( g_n(n-1) = \left\lfloor \frac{a}{2} \right\rfloor \) for \( n \geq 6 \).
Let $n \geq 6$. Observe that

$$b'_{n,n-2} = n^2 - (n-1)n + \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor = n + \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor$$

\[= \begin{cases} 
k^2 + 1 & \text{if } n = 2k \text{ for some integer } k \geq 3 \\
k^2 + k + 1 & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3;\end{cases}\]

$$b''_{n,n-1} = \left\lfloor \left( \frac{n-1}{2} \right)^2 \right\rfloor$$

\[= \begin{cases} 
k^2 - k & \text{if } n = 2k \text{ for some integer } k \geq 3 \\
k^2 & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3;\end{cases}\]

Hence

$$b'_{n,n-2} - b''_{n,n-1} - 1 = \begin{cases} 
k & \text{if } n = 2k \text{ for some integer } k \geq 3 \\
k & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3;\end{cases}$$

implying that $g_n(n-1) = \max \{0, \left\lfloor \frac{n}{2} \right\rfloor \} = \left\lfloor \frac{n}{2} \right\rfloor$.

3. For $n \geq 7$,

$$g_n(n-2) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}.\end{cases}$$

Let $n \geq 7$. Observe that

$$b'_{n,n-3} = n^2 - (n-2)n + \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = 2n + \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

\[= \begin{cases} 
k^2 + k + 2 & \text{if } n = 2k \text{ for some integer } k \geq 3 \\
k^2 + 2k + 3 & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3;\end{cases}\]

$$b''_{n,n-2} = n^2 - \left\lfloor \frac{n+3}{2} \right\rfloor n - \left\lfloor \frac{n-2}{2} \right\rfloor^2 + \left\lfloor \frac{n-2}{2} \right\rfloor + 1$$

\[= \begin{cases} 
k^2 + k - 1 & \text{if } n = 2k \text{ for some integer } k \geq 3 \\
k^2 + 2k - 1 & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3.\end{cases}\]
Hence
\[ b'_{n,n-3} - b''_{n,n-2} - 1 = \begin{cases} 2 & \text{if } n = 2k \text{ for some integer } k \geq 3 \\ 3 & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3 \end{cases} \]
implies that \( g_n(n - 2) = \max\{0, 2\} = 2 \) when \( n \) is even, while \( g_n(n - 2) = \max\{0, 3\} = 3 \) when \( n \) is odd.

4. For \( n \geq 13 \),
\[ g_n(d) = 0 \quad \text{for} \quad 6 \leq d \leq n - 3. \]

Let \( n \geq 13 \) and \( 6 \leq d \leq n - 3 \). We show that \( b'_{d-1} - b''_d - 1 \leq 0 \). If \( d \) is even, then \( d = 2k \) for some integer \( k \geq 3 \). Then \( n \geq d + 3 = 2k + 3 \) and
\[ b'_{n,d-1} - b''_{n,d} - 1 = 2k^2 - 2k - 2 - (k - 2)n \leq 2k^2 - 2k - 2 - (k - 2)(2k + 3) = 4 - k \leq 0 \]
for \( k \geq 4 \). If \( k = 3 \), then
\[ b'_{n,5} - b''_{n,6} - 1 = 2 \cdot 3^2 - 2 \cdot 3 - 2 - (3 - 2)n = 10 - n < 0 \]
since \( n \geq 13 \). If \( d \) is odd, then \( d = 2k + 1 \) for some integer \( k \geq 3 \). Then \( n \geq d + 3 = 2k + 4 \) and
\[ b'_{n,d-1} - b''_{n,d} - 1 = 2k^2 - k - 3 - (k - 2)n \leq 2k^2 - k - 3 - (k - 2)(2k + 4) = 5 - k \leq 0 \]
for \( k \geq 5 \). If \( k = 3 \), then
\[ b'_{n,6} - b''_{n,7} - 1 = 2 \cdot 3^2 - 3 - 3 - (3 - 2)n = 12 - n < 0 \]
since \( n \geq 13 \). Similarly for \( k = 4 \),
\[ b'_{n,8} - b''_{n,9} - 1 = 2 \cdot 4^2 - 4 - 3 - (4 - 2)n = 25 - 2n < 0 \]
for \( n \geq 13 \). Hence in each case, \( g_n(d) = \max\{0, b'_{n,d-1} - b''_{n,d} - 1\} = 0 \).
We next show that for \( n \geq 6 \), those "gaps" of missing three integers due to \( g_n(5) = 3 \) can be filled in; that is, for each \( i \in \{1, 2, 3\} \), the integer \( a_i = (n - 1) + b_{n-5}'' + i = n(n-1) + (n^2 - 5n) + i \) is realizable as the total traceable number of some graph of order \( n \) by constructing three connected graphs \( G_1, G_2, \) and \( G_3 \) of order \( n \) such that \( tt(G_i) = a_i \). Recall that in the proof of Theorem 7.13, we constructed the caterpillar \( T_{4,1} \) of order \( n \) and diameter \( d = 4 \) from the path \( P : v_0, v_1, v_2, v_3, v_4 \) of order 5 by adding \( n - 5 \) new vertices \( u_1, u_2, \ldots, u_{n-5} \) and joining each of them to the vertex \( v_2 \) on \( P \). (See Figure 7.10.)

![Figure 7.10: The caterpillar \( T_{4,1} \)](image)

Let

\[
G_1 = T_{4,1} + v_0v_2 + v_2v_4 \\
G_2 = T_{4,1} + v_0v_2 \\
G_3 = T_{4,1} + v_1v_3.
\]

For each \( i \in \{1, 2, 3\} \), observe that \( G_i \) contains exactly \( i \) vertices having traceable number \( 2n - 5 \) and exactly \( n - i \) vertices having traceable number \( 2n - 6 \), so

\[
\begin{align*}
\text{tt}(G_i) &= i(2n - 5) + (n - i)(2n - 6) = 2n^2 - 6n + i \\
&= n(n - 1) + n^2 - 5n + i.
\end{align*}
\]

Next we show that for \( n \geq 6 \) there exists a connected graph \( G \) of order \( n \) for which \( tt(G) = n(n - 1) + b''_{n,n-1} + 1 \).

**Proposition 7.14** Let \( n \geq 6 \). Then there exists a connected graph \( G \) of order \( n \) for which \( tt(G) = n(n - 1) + b''_{n,n-1} + 1 \).

**Proof.** We consider two cases, according to whether \( n \) is even or odd.

---

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Case 1. \( n \) is even. Then \( n = 2k \) for some integer \( k \geq 3 \). Then

\[
n(n - 1) + b''_{n,n-1} + 1 = (2k)(2k - 1) + \left( \frac{2k - 1}{2} \right)^2 + 1
\]

\[
= 5k^2 - 3k + 1.
\]

Let \( G \) be the graph of order \( n \) obtained from a path

\[P : v_0, v_1, \ldots, v_{k-3}, v_{k-2} = u_{k-2}, u_{k-3}, \ldots, u_0\]

of length \( 2k - 4 \) by adding three new vertices \( w_1, w_2, \) and \( w_3 \), where (i) \( w_1 \) and \( w_2 \) are joined to \( v_1 \) and to each other and (ii) \( w_3 \) is joined to \( v_{k-3} \) and \( v_{k-2} (= u_{k-2}) \). Then \( t(v_i) = t(u_i) = 2k + i \) for each \( i \) with \( 0 \leq i \leq k - 2 \), \( t(w_1) = t(w_2) = 2k \), and \( t(w_3) = 3k - 3 \). Hence

\[
\text{tt}(G) = 2 \left[ (2k) + (2k + 1) + \cdots + (3k - 3) \right] + (3k - 2) + 2(2k) + (3k - 3)
\]

\[
= 5k^2 - 3k + 1.
\]

Case 2. \( n \) is odd. Then \( n = 2k + 1 \) for some integer \( k \geq 3 \). Thus

\[
n(n - 1) + b''_{n,n-1} + 1 = (2k + 1)(2k) + \left( \frac{2k}{2} \right)^2 + 1
\]

\[
= 5k^2 + 2k + 1.
\]

Let \( G \) be the graph of order \( n \) obtained from a path

\[P : v_0, v_1, \ldots, v_{k-3}, v_{k-2}, u_{k-2}, u_{k-3}, \ldots, u_0\]

of length \( 2k - 3 \) by adding three new vertices \( w_1, w_2, \) and \( w_3 \), where (i) \( w_1 \) and \( w_2 \) are joined to \( v_1 \) and to each other and (ii) \( w_3 \) is joined to \( v_{k-3} \) and \( u_{k-2} \). Then \( t(v_i) = t(u_i) = 2k + i \) for each \( i \) with \( 0 \leq i \leq k - 2 \), \( t(w_1) = t(w_2) = 2k + 1 \), and \( t(w_3) = 3k - 1 \). Hence

\[
\text{tt}(G) = 2 \left[ (2k + 1) + (2k + 2) + \cdots + (3k - 1) \right] + 2(2k + 1) + (3k - 1)
\]

\[
= 5k^2 + 2k + 1,
\]

completing the proof. \( \blacksquare \)
We next show that for $n \geq 7$, if $a$ is an integer with
\[ n(n - 1) + b'_{n,n-1} + 2 \leq a \leq n(n - 1) + b'_{n,n-2} - 1, \]
then there exists a connected graph $G$ of order $n$ such that $tt(G) = a$.

**Proposition 7.15** If $n$ and $a$ are integers such that $n \geq 7$ and
\[ n(n - 1) + b'_{n,n-1} + 2 \leq a \leq n(n - 1) + b'_{n,n-2} - 1, \]
then there exists a connected graph $G$ of order $n$ with $tt(G) = a$.

**Proof.** We first consider the case where $n$ is even. If $n = 2k$ for some integer $k \geq 4$, then
\[ 5k^2 - 3k + 2 \leq a \leq 5k^2 - 2k. \]
For each integer $j$ with $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$, we construct the graph $G_j$ of order $n$ from a path
\[ P : v_0, v_1, \ldots, v_{k-2}, u_{k-2}, u_{k-3}, \ldots, u_0 \]
of order $n - 2$ by adding two new vertices $w_1$ and $w_2$, where $w_1$ and $w_2$ are joined to $v_j$ and to each other. Then $t(v_i) = t(u_i) = 2k + i$ for each $i$ with $0 \leq i \leq k - 2$ and $t(w_1) = t(w_2) = 2k - 1 + j$, so
\[ tt(G_j) = 2 \left\lfloor \frac{2k}{2} \right\rfloor + 2k + (2k + 1) + \cdots + (3k - 2) + 2(2k - 1 + j) = 5k^2 - 3k + 2j. \]

Also for each integer $j$ with $1 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor$, we construct the graph $H_j$ of order $n$ from $P$ and two new vertices $w_1$ and $w_2$ by joining (i) $w_1$ to $v_j$ and (ii) $w_2$ to $v_j$ and $v_{j+1}$. Then $t(v_i) = t(u_i) = 2k + i$ for each $i$ with $0 \leq i \leq k - 2$, $t(w_1) = 2k - 1 + j$, and $t(w_2) = 2k + j$. Hence
\[ tt(H_j) = tt(G_j) + 1 = 5k^2 - 3k + 1 + 2j. \]
Since $5k^2 - 3k + 2 \leq a \leq 5k^2 - 2k$, it follows that $a = 5k^2 - 3k + \ell$ for some integer $\ell$ with $2 \leq \ell \leq k$. If $\ell$ is even, then $a = 5k^2 - 3k + 2j$ for some $j$ with $1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$ and $tt(G_j) = a$. Similarly if $\ell$ is odd, then $a = 5k^2 - 3k + (2j + 1)$ for some $j$ with $1 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ and $tt(H_j) = a$. A similar argument applies to the case where $n$ is odd. \qed
Next we show that for $n \geq 8$, those "gaps" due to the fact that $g_n(n - 2) \in \{2, 3\}$ can be filled in. Observe that

$$n(n - 1) + b_{n,n-3}^r - 1 \begin{cases} 5k^2 - k + 1 & \text{if } n = 2k \text{ for some integer } k \geq 3 \\ 5k^2 + 4k + 2 & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3 \\
\end{cases};$$

$$n(n - 1) + b_{n,n-2}^r + 1 \begin{cases} 5k^2 - k & \text{if } n = 2k \text{ for some integer } k \geq 3 \\ 5k^2 + 4k & \text{if } n = 2k + 1 \text{ for some integer } k \geq 3. \\
\end{cases}$$

**Proposition 7.16** Let $n \geq 8$.

1. If $n = 2k$ for some integer $k \geq 4$, then for each integer $j \in \{0, 1\}$, there exists a connected graph $G_j$ of order $n$ for which

$$tt(G_j) = 5k^2 - k + j.$$

2. If $n = 2k + 1$ for some integer $k \geq 4$, then for each integer $j \in \{0, 1, 2\}$, there exists a connected graph $H_j$ of order $n$ for which

$$tt(H_j) = 5k^2 + 4k + j.$$

**Proof.** We first consider the case where $n = 2k$ for some integer $k \geq 4$. For each $j = 0, 1$, construct the graph $G_j$ from a path

$$P : v_0, v_1, \ldots, v_{k-2} = u_{k-2}, u_{k-3}, \ldots, u_0$$

of order $n - 3$ by adding three new vertices $w_1, w_2, w_3$, where (i) $w_1$ and $w_2$ are joined to $v_1$ and to each other and (ii) $w_3$ is joined to $v_{k-3}$. Observe that $t(v_i) = t(u_i) = 2k + 1 + i$ for each $i$ with $0 \leq i \leq k - 2$ (note that $v_{k-2} = u_{k-2}$), $t(w_1) = t(w_2) = 2k + 1$, and $t(w_3) = 3k - 3 + j$. Hence

$$tt(G_j) = 2[(2k + 1) + (2k + 2) + \cdots + (3k - 2)]$$

$$+(3k - 1) + 2(2k + 1) + (3k - 3 + j)$$

$$= 5k^2 - k + j.$$ 

Now we assume that $n = 2k + 1$ for some integer $k \geq 4$. For each $j = 0, 1$, construct the graph $H_j$ from a path

$$P : v_0, v_1, \ldots, v_{k-2}, u_{k-2}, u_{k-3}, \ldots, u_0$$
of order \( n - 3 \) by adding three new vertices \( w_1, w_2, \) and \( w_3 \), where (i) \( w_1 \) and \( w_2 \) are joined to \( v_1 \) and to each other and (ii) \( w_3 \) is joined to \( v_{k-3+j} \). Then \( t(v_i) = t(u_i) = 2k + 2 + i \) for each \( i \) with \( 0 \leq i \leq k-2 \), \( t(w_1) = t(w_2) = 2k + 2 \), and \( t(w_3) = 3k - 2 + j \) and so

\[
\text{tt}(H_j) = 2 \left[ (2k + 2) + (2k + 3) + \cdots + (3k) \right] + 2(2k + 2) + (3k - 2 + j)
\]

\[= 5k^2 + 4k + j.\]

For \( j = 2 \), construct \( H_2 \) from \( P \) by adding three new vertices \( w_1, w_2, \) and \( w_3 \), where (i) \( w_1 \) and \( w_2 \) are joined to \( v_2 \) and to each other and (ii) \( w_3 \) is joined to \( v_{k-3} \). Then \( t(v_i) = t(u_i) = 2k + 2 + i \) for each \( i \) with \( 0 \leq i \leq k-2 \), \( t(w_1) = t(w_2) = 2k + 3 \), and \( t(w_3) = 3k - 2 \) and so

\[
\text{tt}(H_2) = \text{tt}(H_0) + 2 = 5k^2 + 4k + 2,
\]

providing the desired result.

Combining Propositions 7.14, 7.15, and 7.16 together with some earlier observations, we then have the following.

**Theorem 7.17**  For a pair \( n, a \) of integers such that \( n \geq 13 \) and

\[
n(n-1) + \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \leq a \leq n(n-1) + (n^2 - 4n - 1),
\]

there exists a connected graph \( G \) of order \( n \) with \( \text{tt}(G) = a \).

Let \( G_{n,a} \) be a connected graph of order \( n \) with \( \text{tt}(G_{n,a}) = a \). According to Table 7.1, we still need to determine the existence of the following graphs:

\[G_{9,104}, G_{10,127}, G_{10,128}, G_{11,162}, G_{12,183}\]

It turns out that all graphs listed above do exist, shown in Figure 7.11. (Each vertex is labeled with its traceable number.) Note that these graphs \( G_{n,a} \) are not necessarily unique for each pair \( n, a \).
Theorem 7.18  For each pair \( n, a \) of integers such that \( n \geq 5 \) and

\[
n(n - 1) + \left\lfloor \left( \frac{n - 1}{2} \right)^2 \right\rfloor \leq a \leq n(n - 1) + (n^2 - 4n - 1),
\]

there exists a connected graph \( G \) of order \( n \) with \( tt(G) = a \).

Hence a pair \( n, a \) of integers with \( n \geq 5 \) and

\[
\text{tt}(P_n) \leq a \leq \text{tt}(K_{1,n})
\]
is realizable as the order and total traceable number of some connected graph if and only if \( n \) and \( a \) satisfy one of the following:

1. \( a \leq n(n - 1) + (n^2 - 4n + 2); \)
2. \( a \in \{n(n - 1) + (n^2 - 4n + 1), n(n - 1) + (n^2 - 4n + 2)\}; \)
3. \( a = \text{tt}(K_{1,n-1}). \)

Recall that \( \text{tt}(P_n) = n(n - 1) + \lfloor (\frac{n-1}{2})^2 \rfloor \). We next study the case where \( n(n - 1) \leq a < \text{tt}(P_n) \). By Theorem 7.6, if \( G \) is a connected graph of order \( n \geq 3 \) for which \( \text{tt}(G) = a \), then \( G \) is not a tree. By Proposition 7.1, each pair \( n, a \) of integers with \( n \geq 3 \) and \( n(n - 1) \leq a \leq n(n - 1) + (n - 2) \) is realizable as the order and total traceable number of some connected graph. In addition, Table 7.2 shows an example of a graph \( F \) of order \( n \) for which \( n(n - 1) + (n - 1) \leq \text{tt}(F) \leq n(n - 1) + (n + 10). \)
The graphs $F_i$ ($1 \leq i \leq 19$) are shown in Figure 7.12. Hence each pair $n,a$ of positive integers with \( n(n - 1) \leq a \leq n(n - 1) + (n + 10) \) is realizable as the order and total traceable number of some connected graph (for sufficiently large $n$). Similarly, Table 7.3 shows an example of a graph $H$ of order $n$ for which $n(n - 1) + (2n - 4) \leq tt(H) \leq n(n - 1) + (2n + 2)$. The graphs $H_i$ ($1 \leq i \leq 8$) are shown in Figure 7.13.

We next consider the case where

\[ n(n - 1) + \left\lfloor \frac{(n - 2)^2}{2} \right\rfloor \leq a \leq tt(P_n) - 1. \]

**Proposition 7.19** If $n$ and $a$ are integers such that $n \geq 5$ and $a = n(n - 1) + b$, where\[ n - 2 < b < n - 1, \]
then there exists a connected graph $G$ of order $n$ for which $tt(G) = a$.

**Proof.** Let $b = \left\lfloor \frac{(n - 2)^2}{2} \right\rfloor + i$, where $0 \leq i \leq \left\lfloor \frac{n - 3}{2} \right\rfloor$. Let $G_i$ be the graph obtained from a path $P : v_0, v_1, \ldots, v_{n-2}$ of order $n - 1$ by joining a new vertex $u$ to the two vertices $v_i$ and $v_{i+1}$ on $P$. Without loss of generality, we may assume that $0 \leq i \leq \left\lfloor \frac{n - 3}{2} \right\rfloor$. Then $t(u) = d(s)$ where $s : u, v_0, v_1, \ldots, v_{n-2}$ and so $t(u) = n + i - 1$. We next determine the values of $t(v_j)$ ($0 \leq j \leq n - 2$). Observe that

\[ t(v_j) = \min\{j, n - 2 - j\} + n - 1 \]

\[ = \begin{cases} n - 1 + j & \text{if } j \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\ 2n - 3 - j & \text{if } j \geq \left\lfloor \frac{n-2}{2} \right\rfloor. \end{cases} \]

Hence

\[ tt(G_i) = (n + i - 1) + \sum_{j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (n - 1 + j) + \sum_{j=\left\lfloor \frac{n-2}{2} \right\rfloor+1}^{n-2} (2n - 3 - j). \]
We consider two cases, according to whether \( n \) is odd or even.

**Case 1.** \( n \) is odd. Then \( n = 2k + 1 \) for some integer \( k \geq 2 \). Thus \( \left\lfloor \frac{n-3}{2} \right\rfloor = k - 1 \) and \( 0 \leq i \leq k - 1 \). Observe that

\[
\ttt(G_i) = (2k + i) + \sum_{j=0}^{k-1} (2k + j) + \sum_{j=k}^{2k-1} (4k - 1 - j)
\]

\[
= 5k^2 + k + i = (4k^2 + 2k) + (k^2 - k) + i
\]

\[
= n(n - 1) + \left\lfloor \left( \frac{n - 2}{2} \right)^2 \right\rfloor + i = n(n - 1) + b = a.
\]

**Case 2.** \( n \) is even. Then \( n = 2k \) for some integer \( k \geq 3 \). Thus \( \left\lfloor \frac{n-2}{2} \right\rfloor = k - 1 \) and \( 0 \leq i \leq \left\lfloor \frac{n-3}{2} \right\rfloor = k - 2 \). Observe that

\[
\ttt(G_i) = (2k + i - 1) + \sum_{j=0}^{k-1} (2k - 1 + j) + \sum_{j=k}^{2k-2} (4k - 3 - j)
\]

\[
= 5k^2 - 4k + 1 + i = (4k^2 - 2k) + (k^2 - 2k + 1) + i
\]

\[
= n(n - 1) + \left\lfloor \left( \frac{n - 2}{2} \right)^2 \right\rfloor + i = n(n - 1) + b = a.
\]

Hence for each \( a \) with

\[
n(n - 1) + \left\lfloor \left( \frac{n - 2}{2} \right)^2 \right\rfloor \leq a \leq n(n - 1) + \left\lfloor \left( \frac{n - 1}{2} \right)^2 \right\rfloor - 1,
\]

there exists a connected graph \( G \) of order \( n \) for which \( \ttt(G) = a \).
\[
\begin{array}{|c|c|c|}
\hline
\text{\(tt(G)\)} & n & \text{Example} \\
\hline
n(n-1) + (n-1) & n \geq 5 & F_1 \\
n(n-1) + n = n^2 & n \geq 6 & F_2 \\
n(n-1) + (n+1) & n \geq 5 & F_3 \\
n(n-1) + (n+2) & n \geq 5 & F_4 \\
n(n-1) + (n+3) & n \geq 6 & F_5 \\
n(n-1) + (n+4) & n \geq 6 & F_6 \\
n(n-1) + (n+5) & n = 6 & F_7 \\
n(n-1) + (n+6) & n \geq 7 & F_8 \\
n(n-1) + (n+7) & n = 6 & K_{1,5} + e \\
 & n = 7 & F_{10} \\
 & n \geq 8 & F_{11} \\
n(n-1) + (n+8) & n = 6 & F_{12} \\
 & n = 7 & F_{13} \\
 & n \geq 8 & F_{14} \\
n(n-1) + (n+9) & n = 7 & F_{15} \\
 & n \geq 8 & F_{16} \\
n(n-1) + (n+10) & n = 7 & F_{17} \\
 & n = 8 & F_{18} \\
 & n \geq 9 & F_{19} \\
\hline
\end{array}
\]
Table 7.2: \(tt(G)\) in terms of \(n\) and graphs \(F_i\)

\[
\begin{array}{|c|c|c|}
\hline
\text{\(tt(G)\)} & n & \text{Example} \\
\hline
n(n-1) + (2n-4) & n = 6 & H_1 \\
 & n \geq 7 & H_2 \\
n(n-1) + (2n-3) & n = 4 & P_4 \\
 & n \geq 5 & H_3 \\
n(n-1) + (2n-2) & n \geq 6 & H_4 \\
n(n-1) + (2n-1) & n \geq 6 & H_5 \\
n(n-1) + (2n) & n \geq 7 & H_6 \\
n(n-1) + (2n+1) & n \geq 6 & H_7 \\
n(n-1) + (2n+2) & n \geq 6 & H_8 \\
\hline
\end{array}
\]
Table 7.3: \(tt(G)\) in terms of \(n\) and graphs \(H_i\)
Figure 7.12: Graphs $F_i$ ($1 \leq i \leq 19$)

Figure 7.13: Graphs $H_i$ ($1 \leq i \leq 8$)

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