Detectable Coloring of Graphs

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DETECTABLE COLORING OF GRAPHS

by

Henry E. Escuadro

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Submitted to the
Faculty of The Graduate College
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Degree of Doctor of Philosophy
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Henry E. Escuadro

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A basic problem in graph theory is to distinguish the vertices of a connected graph from one another in some manner. In this study, we investigate the problem of coloring the edges of a graph in a manner that distinguishes the vertices of the graph. The method we use combines many of the features of previously introduced methods.

Let $G$ be a connected graph of order $n \geq 3$ and let $c : E(G) \rightarrow \{1, 2, \ldots, k\}$ be a coloring of the edges of $G$ (where adjacent edges may be colored the same). For each vertex $v$ of $G$, the color code of $v$ with respect to $c$ is the $k$-tuple $c(v) = (a_1, a_2, \ldots, a_k)$, where $a_i$ is the number of edges incident with $v$ that are colored $i$ ($1 \leq i \leq k$). The coloring $c$ is detectable if distinct vertices have distinct color codes. The detection number $\text{det}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has a detectable $k$-coloring.

The detection number of stars, double stars, cycles, paths, complete graphs, and complete bipartite graphs are determined. It is also shown that a pair $k, n$ of positive integers is realizable as the detection number and the order of some nontrivial connected graph if and only if $k = n = 3$ or $2 \leq k \leq n - 1$.

Extremal problems on detectable colorings of graphs are investigated in this study. If $G$ is a connected graph of order $n$ and size $m$, then the number of edges that must be deleted from $G$ to obtain a spanning tree of $G$ is $m - n + 1$. The number $m - n + 1$ is called the cycle rank of $G$. For integers $\psi$ and $n$, where $\psi \geq 0$ and $n \geq \left\lceil \frac{3 + \sqrt{1 + 8\psi}}{2} \right\rceil$, let $D_\psi(n)$ denote the maximum detection number among all connected graphs of order $n$ with cycle rank $\psi$ and let $d_\psi(n)$ denote the minimum detection number among all connected graphs of order $n$ with cycle rank $\psi$. Hence, if $G_{\psi,n}$ denotes the set of all connected graphs of order $n$ with cycle rank $\psi$, then

$$
D_\psi(n) = \max \{\text{det}(G) : G \in G_{\psi,n}\}
$$

$$
d_\psi(n) = \min \{\text{det}(G) : G \in G_{\psi,n}\}.
$$
Formulas for $D_\psi(n)$ and $d_\psi(n)$ for $\psi = 0, 1, 2$ (where $n \geq 3$ when $\psi = 0, 1$ and $n \geq 4$ when $\psi = 2$) are developed. It is also shown that if $k \geq 2$, then there exists a connected graph $G$ of order $n$ with cycle rank $\psi$ having detection number $k$ if and only if $d_\psi(n) \leq k \leq D_\psi(n)$ for $\psi = 0, 1, 2$.

Detectable colorings of graphs are studied from another point of view. For a connected graph $G$ of order $n \geq 3$ and an ordered factorization $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ of $G$ into $k$ spanning subgraphs $G_i$ ($1 \leq i \leq k$), the color code of a vertex $v$ of $G$ with respect to $\mathcal{F}$ is the ordered $k$-tuple $c(v) = (a_1, a_2, \ldots, a_k)$ where $a_i = \deg_{G_i} v$. If distinct vertices have distinct color codes, then the factorization $\mathcal{F}$ is called a detectable factorization of $G$; while the detection number $\det(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has a detectable factorization into $k$ factors.

Since it is most interesting and most challenging to find minimum detectable colorings for graphs having many vertices of the same degree, regular graphs are considered. In particular, the detection numbers of connected cubic graphs of order at most 10 are determined. It is shown that there is a unique graph $F$ for which the Petersen graph has a detectable $F$-factorization into three factors. Furthermore, if $G$ is a connected cubic graph of order $\binom{k+2}{3}$ with $\det(G) = k$, then $k \equiv 2 \pmod{4}$ or $k \equiv 3 \pmod{4}$. The problem of determining the largest order of a connected cubic graph with prescribed detection number is also investigated. Moreover, irregular and isomorphic factorizations are considered.
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Chapter 1

Introduction

1.1 Motivation and Background

A problem in graph theory that has received increased attention during the past 35 years concerns studying methods of distinguishing the vertices of a connected graph from one another.

One of the earlier methods was suggested by Sumner [19] and by Entringer and Gassman [10]. They studied graphs $G$ for which the equality of the open neighborhoods of every two vertices of $G$ implies that the vertices are the same. In this case, the vertices of $G$ are uniquely determined by their open neighborhoods.

Another idea is to look at the automorphism group Aut($G$) of a connected graph $G$. Of course, if Aut($G$) is the identity group, then all the vertices of $G$ are distinguishable. However, if the automorphism group of $G$ is not the identity group, then this group has at least one nontrivial orbit and the vertices that belong to such an orbit are indistinguishable. This means that the vertices of a nontrivial graph cannot be distinguished in all cases by its automorphism group alone. Erwin and Harary [11] introduced the idea of selecting a subset $S$ of the vertex set of a graph $G$ such that the subgroup of Aut($G$) that fixes every vertex of $S$ is the identity group. Another method, introduced by Albertson and Collins [3] and by Harary [13], involves coloring the vertices of $G$ in such a way that the subgroup of color-preserving automorphisms of Aut($G$) is the identity group, thereby distinguishing the vertices of $G$ from one another.
Harary and Melter [14] and Slater [17] introduced the idea of selecting an ordered set

\[ W = \{w_1, w_2, \ldots, w_k\} \]

of vertices in a connected graph \( G \) and assigning to each vertex \( v \) the ordered \( k \)-tuple

\[ c_W(v) = (a_1, a_2, \ldots, a_k), \]

called the distance code of \( v \) (with respect to \( W \)), where \( a_i = d(v, w_i), 1 \leq i \leq k \). If distinct vertices of \( G \) have distinct distance codes, then the vertices of \( G \) are distinguishable. That is, the vertices of \( G \) are distinguished by their distances to some ordered set of vertices in \( G \).

Harary and Plantholt [15] considered assigning colors to the edges of a graph \( G \) in such a way that for every two vertices of \( G \), one of the vertices is incident with an edge assigned a color that the other vertex is not. They referred to the minimum number of colors needed to accomplish this as the point-distinguishing chromatic index of \( G \).

A well-known result from graph theory concerning the degrees of the vertices of a graph is the following.

**Theorem 1.1**  Every nontrivial graph contains at least two vertices having the same degree.

By Theorem 1.1, it follows that the vertices of a nontrivial graph cannot be distinguished by their degrees alone. However, it has been observed that weights can be assigned to the edges of every connected graph of order 3 or more in such a way that the degrees of the vertices (defined as the sum of the weights of its incident edges) of the resulting weighted graph are distinct.

For a connected graph \( F \) of order 2 or more and a vertex \( v \) of a graph \( G \), Chartrand, Holbert, Oellermann and Swart [6] defined the \( F \)-degree of \( v \) in \( G \) as the number of copies of \( F \) containing \( v \). It has been conjectured that if \( F \) has order 3 or more, then there exists some graph \( G \) whose vertices can be distinguished by their \( F \)-degrees.
In this study, we present another manner in which differences among the vertices of a connected graph $G$ can be detected and which combines many of the features of the methods given above.

1.2 Basic Definitions and Notation

We refer to the book [8] for graph theory notation and terminology not described in this dissertation. Let $G$ be a connected graph of order $n \geq 3$ and let
\[ c : E(G) \to \{1, 2, \ldots, k\}, \]
be a coloring of the edges of $G$ for some positive integer $k$ (where adjacent edges may be colored the same). The color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered $k$-tuple
\[ \text{code}_c(v) = (a_1, a_2, \ldots, a_k) \]
where $a_i$ denotes the number of edges incident with $v$ that are colored $i$ for $1 \leq i \leq k$. If the coloring $c$ is clear, we use $\text{code}(v)$ to denote the color code of a vertex $v$. For convenience, we sometimes write $a_1a_2\ldots a_k$ instead of $(a_1, a_2, \ldots, a_k)$. It follows that
\[ \sum_{i=1}^{k} a_i = \deg_G v. \]

If distinct vertices have distinct color codes, then the coloring $c$ is called detectable. This means that if $c$ is a detectable coloring of a graph $G$, then for every pair of vertices of $G$, there exists a color such that the number of incident edges with that color is different for these two vertices.

For example, consider the graph $G$ shown in Figure 1.1(a). A coloring $c$ of the edges of $G$ is shown in Figure 1.1(b). For this 3-coloring, the color codes of its vertices are
\[ \text{code}_c(u) = 110, \quad \text{code}_c(v) = 021, \quad \text{code}_c(w) = 210, \]
\[ \text{code}_c(x) = 201, \quad \text{code}_c(y) = 101, \quad \text{code}_c(z) = 001. \]
Since the vertices of $G$ have distinct color codes, $c$ is a detectable coloring.

$$G: \quad \begin{array}{ccc}
  & u & \\
  v & & w \\
  & x & \\
  & y & \\
  & z & \\
\end{array} \quad \begin{array}{ccc}
  & u & \\
  v & 2 & w \\
  & 2 & \\
  & x & 1 \\
  & 1 & \\
  3 & y & \\
  & z & \\
\end{array}$$

Figure 1.1: A detectable 3-coloring of a graph

It is sometimes useful to look at this type of edge coloring from another point of view. For a connected graph $G$ of order $n \geq 3$ and a factorization

$$\mathcal{F} = \{G_1, G_2, \cdots, G_k\}$$

of $G$ into $k$ subgraphs $G_i$, the color code of a vertex $v$ of $G$ (with respect to $\mathcal{F}$) is the ordered $k$-tuple

$$\text{code}_\mathcal{F}(v) = (a_1, a_2, \cdots, a_k)$$

where $a_i = \deg_{G_i} v$. Thus

$$\sum_{i=1}^{k} \deg_{G_i} v = \deg_G v$$

for every vertex $v$ in $G$. If distinct vertices have distinct color codes, then the factorization $\mathcal{F}$ is called detectable. If $\mathcal{F}$ consists of $k$ factors, we say that $\mathcal{F}$ is a $k$-tuple factorization of $G$. This means that if $\mathcal{F} = \{G_1, G_2, \cdots, G_k\}$ is a detectable $k$-tuple factorization of a graph $G$, then for every pair of vertices $v$ and $w$ of $G$, there exists a factor $G_i$ in $\mathcal{F}$ such that $\deg_{G_i} v \neq \deg_{G_i} w$. To illustrate this concept, consider the graph $G$ of Figure 1.1 (a). The graph $G$ is redrawn in Figure 1.2 together with a detectable 3-tuple factorization of $G$ that results from the detectable 3-coloring of $G$ shown in Figure 1.1(b).
Figure 1.2: A detectable 3-tuple factorization of the graph $G$ of Figure 1.1(a)

Observe that every detectable $k$-coloring of a graph $G$ induces a detectable $k$-tuple factorization of $G$ into factors $G_i$ ($1 \leq i \leq k$), where the edges of $G_i$ are the edges of $G$ colored $i$ for $1 \leq i \leq k$. Conversely, each detectable $k$-tuple factorization of $G$ into factors $G_1, G_2, \ldots, G_k$ gives rise to a detectable $k$-coloring of $G$ where the edges of $G_i$ are colored $i$ in $G$ for $1 \leq i \leq k$. It thus follows that a coloring of the edges of a graph $G$ is detectable if and only if the corresponding $k$-tuple factorization of $G$ is detectable.

The minimum integer $k$ for which $G$ has a detectable $k$-coloring (or a $k$-tuple factorization) is called the detection number of $G$ and is denoted by $\text{det}(G)$. A detectable coloring of $G$ using $\text{det}(G)$ colors is called a minimum detectable coloring of $G$. By Theorem 1.1, if all edges of a nontrivial connected graph $G$ were colored by only one color, then at least two vertices of $G$ would have the same code. Thus, the following observation is an immediate consequence of Theorem 1.1.

**Observation 1.2** The detection number of every nontrivial connected graph is at least 2.

A detectable 2-coloring $\mathcal{c}'$ and the corresponding 2-tuple factorization for the graph in Figure 1.1(a) is given in Figure 1.3. For this coloring,

$$\text{code}_{\mathcal{c}'}(u) = 20, \text{code}_{\mathcal{c}'}(v) = 30, \text{code}_{\mathcal{c}'}(w) = 21,$$
$$\text{code}_{\mathcal{c}'}(x) = 12, \text{code}_{\mathcal{c}'}(y) = 02, \text{code}_{\mathcal{c}'}(z) = 01.$$

By Observation 1.2, the detection number of every nontrivial connected graph is at least 2. From this, it follows that for the graph $G$ of Figure 1.3, $\text{det}(G) = 2$ and $\mathcal{c}'$ is a minimum detectable coloring of $G$. 
Note that since every coloring that assigns distinct colors to the edges of a connected graph of order \( n \geq 3 \) is detectable, the detection number of a connected graph is always defined. However, this not the case for \( n = 2 \). Hence, we restrict our study to connected graphs of order \( n \geq 3 \). Therefore, if \( G \) is a connected graph of order \( n \geq 3 \) and size \( m \), then

\[
2 \leq \det(G) \leq m
\]  
(1.1)

### 1.3 Some Known Results

It was shown by Burris [4] that if \( G \) is a connected regular graph of order \( n \geq 3 \), then \( \det(G) \geq 3 \). We state this result and give a proof that is different from the one given in [4].

**Theorem 1.3**  
If \( G \) is a regular connected graph of order \( n \geq 3 \), then \( \det(G) \geq 3 \).

**Proof.** Assume, to the contrary, that there exists an \( r \)-regular connected graph \( G \) of order \( n \geq 3 \) such that \( \det(G) = 2 \). Hence there exists a detectable 2-tuple factorization \( \mathcal{F} = \{F_1, F_2\} \) of \( G \). By Theorem 1.1, \( F_1 \) contains two vertices \( u \) and \( v \) having the same degree in \( F_1 \), say \( \deg_{F_1} u = \deg_{F_1} v = s \), where \( 0 \leq s \leq r \). Let \( c \) be the edge coloring of \( G \) corresponding to \( \mathcal{F} \). Then \( \text{code}(u) = \text{code}(v) = (s, r - s) \), a contradiction. ■

Theorem 1.3 implies that if \( n \geq 3 \), then \( \det(K_n) \geq 3 \). It was shown in [1] that \( \det(K_n) = 3 \) for every integer \( n \geq 3 \). In order to present our proof of this result, we first
state a well-known result in graph theory.

**Lemma 1.4** For every integer $n \geq 3$, there is a unique (up to isomorphism) connected graph of order $n$ containing exactly two vertices of the same degree.

**Theorem 1.5** The detection number of $K_n$ is 3 for every integer $n \geq 3$.

**Proof.** Let $F$ be the unique connected graph of order $n$ containing exactly two vertices with equal degree. Without loss of generality, we may assume that

$$\deg_F v_i = \begin{cases} 
  i & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
  i - 1 & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n.
\end{cases}$$

Thus $v_{\left\lfloor \frac{n}{2} \right\rfloor}$ and $v_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ are the only two vertices of $F$ having the same degree. Define a 3-tuple factorization $\mathcal{F} = \{F_1, F_2, F_3\}$ of $K_n$, where $F_1 = F - v_{\left\lfloor \frac{n}{2} \right\rfloor}v_n$, $F_2 = \overline{F}$, and $F_3 \cong K_2 \cup (n - 2)K_1$ with $E(F_3) = \{v_{\left\lfloor \frac{n}{2} \right\rfloor}v_n\}$. Figure 1.4 illustrates such a 3-tuple factorization for $K_{10}$.

![Figure 1.4: The detectable 3-tuple factorization of $K_{10}$ described in the proof of Theorem 1.5](image)

Since $\deg_{F_2} v_i = (n - 1) - \deg_F v_i$ for $1 \leq i \leq n$ and $v_{\left\lfloor \frac{n}{2} \right\rfloor}$ and $v_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ are the only two vertices having the same degree in $F$, it follows that $v_{\left\lfloor \frac{n}{2} \right\rfloor}$ and $v_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ are the only two vertices having the same degree in $F_2$. Since $\deg_{F_3} v_{\left\lfloor \frac{n}{2} \right\rfloor} = 1$ and $\deg_{F_3} v_{\left\lfloor \frac{n}{2} \right\rfloor + 1} = 0$, it follows that $\mathcal{F}$ is a detectable 3-tuple factorization of $K_n$ and so $\det(K_n) \leq 3$. It then follows by Theorem 1.3 that $\det(K_n) = 3$. ■
The following is an immediate consequence of the proof of Theorem 1.5.

**Corollary 1.6**  Let $e$ be an edge of the complete graph $K_n$, where $n \geq 3$. Then the detection number of $K_n - e$ is 2.

**Proof.** Observe that in the detectable 3-tuple factorization of $K_n$ in the proof of Theorem 1.5, the color codes of every pair of vertices differ in at least one of the first two entries. In particular,

$$
\text{code} (v_n) = (n - 2, 0, 1),
$$

$$
\text{code} (v_{n-1}) = (n - 2, 1, 0),
$$

$$
\text{code} (v_{\lfloor \frac{n}{2} \rfloor}) = (\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil - 1, 1), \text{ and }
$$

$$
\text{code} (v_{\lfloor \frac{n}{2} \rfloor - 1}) = (\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil, 0).
$$

Without loss of generality, we may assume that $e = v_{\lfloor \frac{n}{2} \rfloor} v_n$, that is, $e$ is the edge in $F_3$ (or $e$ is the only edge colored 3). Removing $e$ from $K_n$ gives us a detectable 2-tuple factorization of $K_n - e$. It follows by (1.1) that $\det(K_n - e) = 2$.

A result that is more general than Corollary 1.6 was given in [4].

**Theorem 1.7**  Let $n \geq 3$. If $E' \subseteq E(K_n)$ such that $1 \leq |E'| \leq \frac{n}{2}$, then

$$
\det(K_n - E') = 2.
$$

The following upper bound for the detection number of a connected regular graph in terms of its order and its degree of regularity was given in [4].
Theorem 1.8  If $G$ is a connected $r$-regular graph of order $n \geq 3$, then
\[
\det(G) \leq (5e(r + 1)!)^n r,
\]
where $e$ is the natural base.

In [5], the following upper bound for the detection number of a tree in terms of the number of its end-vertices and the number of vertices of degree 2 was determined.

Theorem 1.9  For any tree $T$ with at least three vertices,
\[
\det(T) \leq \max\{n_1, 4.62\sqrt{n_2}, 8\} + 1,
\]
where $n_i$ denotes the number of vertices of $T$ of degree $i$ for $i = 1, 2$. The absolute constant 4.62 given above may be replaced by a constant $k_T$ which depends on $T$, where $2\sqrt{2} < k_T < 4.62$. 
Chapter 2

Preliminary Results

2.1 Results From Graph Theory

In this section, we state some well-known results from graph theory that will be useful in our study. The first two results deal with factorizations of the complete graphs into Hamiltonian cycles and 1-factors.

**Theorem 2.1** For each positive integer $\ell$, the complete graph $K_{2\ell}$ can be factored into $\ell - 1$ Hamiltonian cycles and a 1-factor.

**Theorem 2.2** For each positive integer $\ell$, the complete graph $K_{2\ell+1}$ is Hamiltonian factorable.

A concept that plays a major role in our study is that of the cycle rank of a connected graph. Let $G$ be a connected graph of order $n$ and size $m$. The number of edges that must be deleted from $G$ to obtain a spanning tree of $G$ is $m - n + 1$. The number $m - n + 1$ is called the cycle rank of $G$. Thus the cycle rank of a tree is 0 and the cycle rank of a unicyclic graph (a connected graph with exactly one cycle) is 1.

The next result provides a formula that expresses the number of end-vertices in a connected graph $G$ having cycle rank $\psi$ in terms of $\psi$ and the numbers of vertices of degree 3 or more in $G$. 
Proposition 2.3  Let $G$ be a nontrivial connected graph having maximum degree $\Delta$ and cycle rank $\psi$. If $n_i$ is the number of vertices of degree $i$ in $G$, where $1 \leq i \leq \Delta$, then

$$n_1 = (2 - 2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_{\Delta}. \quad (2.1)$$

Proof. Suppose that $G$ has order $n$ and size $m$. Then $m = (n - 1) + \psi$.

$$n = \sum_{i=1}^{\Delta} n_i, \quad \text{and} \quad 2m = \sum_{i=1}^{\Delta} in_i.$$ 

Therefore, $2m = 2(n - 1 + \psi) = 2n - 2 + 2\psi$ and so

$$\sum_{i=1}^{\Delta} in_i = 2\sum_{i=1}^{\Delta} n_i - 2 + 2\psi. \quad (2.2)$$

Solving for $n_1$ in (2.2), we obtain (2.1). ■

In particular, if $G$ is a tree having $n_i$ vertices of degree $i$ for $1 \leq i \leq \Delta$, then

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_{\Delta}. \quad (2.3)$$

Furthermore, if $G$ is a unicyclic graph having $n_i$ vertices of degree $i$ for $1 \leq i \leq \Delta$, then

$$n_1 = n_3 + 2n_4 + 3n_5 + \cdots (\Delta - 2)n_{\Delta}. \quad (2.4)$$

2.2 Some Observations on Detection Numbers

This section deals with results concerning the detection number of a graph, which will be useful in proving many of our main results. We also state some results from combinatorics that are important in our study.

Verifying that a given coloring of the edges of a graph is detectable may be tedious and long. However, the following observation tells us that if we are to determine whether
a given coloring of the edges of a graph is detectable, we only need to be concerned with sets of vertices of the same degree in the graph.

**Observation 2.4**  
Let $c$ be a coloring of the edges of a connected graph $G$ of order at least 3. If $u$ and $v$ are two vertices of $G$ with $\text{deg}_G(u) \neq \text{deg}_G(v)$, then $\text{code}(u) \neq \text{code}(v)$.

The next observation will be useful to us.

**Observation 2.5**  
If $G$ is a connected graph of order at least 3 with $k$ end-vertices, then $\det(G) \geq k$.

If $T$ is a tree of order at least 3, then $T$ has at least $\Delta(T)$ end-vertices, where $\Delta(T)$ is the maximum degree of $T$. The following is a consequence of Observation 2.5.

**Observation 2.6**  
If $T$ is a tree of order at least 3, then $\det(T) \geq \Delta(T)$.

The following theorem in combinatorics concerns the number of combinations with repetition (see [9]).

**Theorem 2.7**  
Let $A$ be a set containing $k$ different kinds of elements, where there are at least $r$ elements of each kind. The number of different selections of $r$ elements from $A$ is $\binom{r+k-1}{r}$.

This gives us the following result on graphs.

**Theorem 2.8**  
Let $c$ be a $k$-coloring of the edges of a connected graph $G$. The number of possible color codes for the vertices of degree $r$ in $G$ is $\binom{r+k-1}{r}$.

As a consequence of Theorem 2.8, we obtain an upper bound for the number of vertices of degree $r$ in a graph having a detectable $k$-coloring.

**Corollary 2.9**  
If $c$ is a detectable $k$-coloring of a connected graph $G$ of order at least 3, then $G$ contains at most $\binom{r+k-1}{r}$ vertices of degree $r$. 

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It follows from Corollary 2.9 that if \( c \) is a detectable 2-coloring of the edges of a connected graph \( G \), then \( G \) has at most \( r + 1 \) vertices of degree \( r \); while if \( c \) is a detectable 3-coloring of the edges of \( G \), then \( G \) contains at most \( \frac{r^2 + 3r + 2}{2} \) vertices of degree \( r \). Suppose, for example, that a detectable 3-coloring \( c \) of the edges of a connected graph \( G \) is given. By Corollary 2.9, \( G \) can have at most \( \binom{3+3-1}{3} = 10 \) vertices of degree 3 and the possible color codes for these vertices are

300, 030, 003, 210, 201, 120, 102, 021, 012, and 111.

An example of a cubic graph of order 10 having a detectable 3-coloring is given in Figure 2.1. On the other hand, if a connected graph \( G \) has at least 11 vertices of degree 3, then no detectable 3-coloring of \( G \) exists.

![Figure 2.1: Detetable 3-coloring of a cubic graph of order 10](image_url)

We consider, for the moment, a type of edge coloring that has been studied extensively. An assignment of colors to the edges of a nonempty graph \( G \) so that adjacent edges are colored differently is called a proper edge coloring of \( G \). The minimum number of colors for which a graph \( G \) has a proper edge coloring is called the edge chromatic number of \( G \) and is denoted by \( \chi_1(G) \). It is well-known that if \( H \) is a subgraph of \( G \), then \( \chi_1(H) \leq \chi_1(G) \). This however is not true, in general, for the detection number of a graph. Figure 2.2 shows minimum detectable colorings of connected graphs of small orders. Observe that \( G_1 \subseteq G_3 \subseteq G_5 \), \( \det(G_1) = 2 < 3 = \det(G_3) \), and \( \det(G_3) = 3 > 2 = \det(G_6) \).

Furthermore, if \( F, G, \) and \( H \) are graphs with \( F \leq G \leq H \), then the fact that \( \det(F) = \det(H) \) does not imply that \( \det(F) = \det(G) = \det(H) \) as the graphs \( G_1, G_2, \) and \( G_6 \) in Figure 2.2 show. As another example, observe that \( G_3 \subseteq G_6 \subseteq G_8 \), \( \det(G_3) = \)
Figure 2.2: Minimum detectable colorings of connected graphs of small orders

\( \det(G_8) = 3 \), while \( \det(G_6) = 2 \). The following theorem, however, provides a relationship between \( \det(G) \) and \( \det(H) \). For a graph \( F \), let \( m(F) \) denote the size of \( F \).

**Proposition 2.10**  Let \( G \) be a connected graph of order at least 3. If \( H \) is a connected subgraph of \( G \) of order at least 3, then

\[
\det(G) - \det(H) \leq m(G) - m(H).
\]

**Proof.** Color the \( m(H) \) edges of \( H \) using \( k = \det(H) \) colors and color the remaining \( m(G) - m(H) \) edges of \( G \) using the colors \( k+1, k+2, \ldots, k+(m(G)-m(H)) \). This gives us a detectable \( (m(G) - m(H) + k) \)-coloring of \( G \) and so \( \det(G) \leq m(G) - m(H) + \det(H) \).  

**2.3 Some Preliminary Results on Detection Numbers**

We are now prepared to present some preliminary results on the detection number of graphs. In particular, we determine the detection number of complete bipartite graphs, beginning with regular complete bipartite graphs. We also determine those pairs \( k \) and \( n \) of positive integers that are realizable as the detection number and the order of some connected graph of order at least 3.

**Theorem 2.11**  The detection number of \( K_{r,r} \) is 3 for every integer \( r \geq 2 \).

**Proof.** Figure 2.2 shows that the detection number of \( K_{2,2} = C_4 \) is 3. Thus, we may assume that \( r \geq 3 \). Let the partite sets of \( K_{r,r} \) be
Let $F$ be the spanning subgraph of $K_{r,r}$ such that

$$E(F) = \{u_iw_j : i + j \geq r + 1\}$$

and let $X = \{u_iw_r : 1 \leq i \leq r\}$. Define a 3-tuple factorization $\mathcal{F} = \{F_1, F_2, F_3\}$, where $F_1 = F - X$, $F_2$ is the complement of $F$ in $K_{r,r}$, and $F_3$ is the factor of $K_{r,r}$ with $E(F_3) = X$.

Observe that

1. $\deg_{F_2} u_i = \deg_{F_2} w_i = r - i$ for $1 \leq i \leq r$,
2. $\deg_{F_1} u_r = r - 1$ and $\deg_{F_1} w_r = 0$, and
3. $\deg_{F_3} u_i = 1$ and $\deg_{F_3} w_i = 0$ for $1 \leq i \leq r - 1$.

It follows that $\mathcal{F}$ is a detectable 3-factorization of $K_{r,r}$ and so $\det(K_{r,r}) \leq 3$. It then follows by Theorem 1.3 that $\det(K_{r,r}) = 3$.

We now consider nonregular complete bipartite graphs. By Observation 2.5, we have that $\det(K_{1,n-1}) \geq n - 1$ for $n \geq 3$. On the other hand, coloring the $n - 1$ edges of $K_{1,n-1}$ using $n - 1$ different colors gives a detectable coloring for $K_{1,n-1}$ where $n \geq 3$. This yields the following.

**Proposition 2.12** For each integer $n \geq 3$, $\det(K_{1,n-1}) = n - 1$.

Next, we determine the detection number of nonregular complete bipartite graphs that are not stars, beginning with the graphs $K_{s,s+1}$, where $s \geq 2$.

**Theorem 2.13** For every integer $s \geq 2$, $\det(K_{s,s+1}) = 2$.

**Proof.** Let the partite sets of $K_{s,s+1}$ be

$$U = \{u_1, u_2, \ldots, u_s\} \text{ and } W = \{w_1, w_2, \ldots, w_{s+1}\}.$$
Let $F_1$ be the spanning subgraph of $K_{s,s+1}$ such that

$$E(F_1) = \{u_iw_j : i + j \geq s + 2\}$$

and let $F_2$ be the complement of $F_1$ in $K_{s,s+1}$. Since the 2-factorization $\{F_1, F_2\}$ is detectable, $\det(K_{s,s+1}) = 2$. 

We now determine the detection numbers of the remaining complete bipartite graphs.

**Theorem 2.14** For integers $s$ and $t$ with $2 \leq s < t - 1$, the detection number of the complete bipartite graph $K_{s,t}$ is $\det(K_{s,t}) = k$, where $k$ is the unique integer for which

$$\left(\frac{s + k - 2}{s}\right) < t \leq \left(\frac{s + k - 1}{s}\right).$$

**Proof.** Because $K_{s,t}$ has $t$ vertices of degree $s$ and $t > \binom{s+k-1}{s}$, it follows from Corollary 2.9 that $\det(K_{s,t}) \geq k$. To verify that $\det(K_{s,t}) \leq k$, it suffices to establish the existence of a detectable $k$-coloring of $K_{s,t}$.

Suppose first that $t = \binom{s+k-1}{s}$. Let

$$U = \{u_1, u_2, \ldots, u_s\} \text{ and } W = \{w_1, w_2, \ldots, w_t\}$$

be the partite sets of $K_{s,t}$. Since $t = \binom{s+k-1}{s}$, distinct color codes $(a_{i1}, a_{i2}, \ldots, a_{ik})$, $1 \leq i \leq t$, can be assigned to the vertices $w_i$, where $a_{ij} \geq 0$ for each integer $j \in \{1, 2, \ldots, k\}$ and $\sum_{j=1}^k a_{ij} = s$.

For $1 \leq i \leq t$, we color the edges incident with $w_i$ in the order $u_1w_i, u_2w_i, \ldots, u_kw_i$. The first $a_{i1}$ of these edges are colored 1, the next $a_{i2}$ of these edges are colored 2, and so on, until we arrive at the last $a_{ik}$ of these edges, which are colored $k$ (see Figure 2.3). In general, for an integer $\ell$ with $1 \leq \ell \leq s$, the edge $u_\ell w_i$ is assigned the color $p \in \{1, 2, \ldots, k\}$, where $p$ is the smallest positive integer for which

$$\ell \leq \sum_{j=1}^{p} a_{ij}. $$
If we remove those vertices $w_i$ from $K_{s,t}$ for which $a_{i1} \neq 0$, then the resulting graph is $K_{s,t'}$, where

$$t' = \binom{s + k - 2}{s}.$$

Let $(b_{11}', b_{21}', \ldots, b_{sk}')$, $1 \leq i \leq s$, be the resulting color code of $u_i$ in $K_{s,t'}$. From the manner in which the edges of $K_{s,t}$ have been colored, it follows that

$$b_{1k}' < b_{2k}' < \cdots < b_{sk}'.$$

Now suppose that $\binom{s+k-2}{s} < t \leq \binom{s+k-1}{s}$. We add back $t - \binom{s+k-2}{s}$ of those vertices $w_i$ for which $a_{i1} \neq 0$ to $K_{s,t'}$ to produce the graph $K_{s,t}$. Let $(b_{11}, b_{21}, \ldots, b_{sk})$, $1 \leq i \leq s$, be the color code of $u_i$ in $K_{s,t}$. Since each vertex $w_i$ for which $a_{i1} \neq 0$ is joined to a vertex $u_j$ by an edge colored $k$ only if each edge $w_iu_r$, $j < r \leq s$, is also colored $k$, it follows that

$$b_{1k} < b_{2k} < \cdots < b_{sk}.$$

Hence the vertices of $K_{s,t}$ have distinct color codes and so this $k$-coloring of $K_{s,t}$ is detectable. Thus $\det(K_{s,t}) \leq k$, giving the desired result that $\det(K_{s,t}) = k$. 

We summarize what we have obtained for the detection number of complete bipartite graphs as follows.
Corollary 2.15  
For integers $s$ and $t$ with $1 \leq s \leq t$ and $s + t \geq 3$,  
\[
\det(K_{s,t}) = \begin{cases} 
3 & \text{if } s = t \geq 2 \\
t & \text{if } 1 = s < t \\
2 & \text{if } t = s + 1 \\
k & \text{if } 2 \leq s < t - 1 \text{ and } k \text{ is the unique integer} \\
& \text{for which } \binom{s+k-2}{s} < t \leq \binom{s+k-1}{s} 
\end{cases}
\]

We have seen by (1.1) that if $G$ is a connected graph of order $n \geq 3$ and size $m$, then $2 \leq \det(G) \leq m$. A better upper bound is presented in the next result. We have already noted that $\det(K_3) = 3$. It therefore follows that for $n = 3$, there is a connected graph $G$ of order $n$ with $\det(G) = n$. On the other hand, there is no connected graph of order $n \geq 4$ having detection number greater than $n - 1$.

Proposition 2.16  
If $G$ is a connected graph of order $n \geq 4$, then  
\[
\det(G) \leq n - 1.
\]

Proof. We proceed by induction on $n$. The result is true for $n = 4$ as the graphs of order 4 in Figure 2.2 show. Assume that $\det(G) \leq n - 1$ for every connected graph of order $n$, where $n \geq 4$. Let $H$ be a connected graph of order $n+1$. Let $v$ be a vertex of $H$ that is not a cut-vertex. Then $H - v$ is a connected graph of order $n$. By the induction hypothesis, $\det(H - v) \leq n - 1$. Hence there exists a detectable $(n - 1)$-coloring of $E(H - v)$. If we extend this coloring of $E(H - v)$ to a coloring of $E(H)$ by coloring every edge incident to $v$ with the color $n$, then we obtain a detectable $n$-coloring of $H$ and so $\det(H) \leq n$.

Observe that the upper bound in Proposition 2.16 is sharp since $\det(K_{1,n-1}) = n - 1$. In fact, more can be said. The next result shows that every pair $k, n$ of integers with $2 \leq k \leq n - 1$ and $n \geq 4$ is realizable as the detection number and order, respectively, of some connected graph.

Theorem 2.17  
For every pair $k, n$ of integers with $2 \leq k \leq n - 1$ and $n \geq 4$, there exists a connected graph of order $n$ having detection number $k$.  

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Proof. First, we consider the case where \( k = 2 \). Let \( G \) be the unique connected graph of order \( n \) having exactly two vertices \( u \) and \( v \) of the same degree. Let \( e \) be an edge of \( G \) that is incident with \( u \) but not \( v \). Assign the edge \( e \) the color 1 and assign the remaining edges of \( G \) the color 2. Since this coloring is a detectable 2-coloring, \( \det(G) = 2 \). Since \( \det(K_n) = 3 \) by Theorem 1.5, the theorem holds when \( k = 3 \) and every integer \( n \geq 4 \).

Because \( \det(K_{1,n-1}) = n - 1 \), the theorem follows when \( k = n - 1 \). Let \( H \) be the graph of order \( n \) obtained by subdividing a single edge of \( K_{1,n-2} \). Since \( \det(H) = n - 2 \), the theorem holds for every integer \( n \geq 4 \) when \( k = n - 2 \).

Hence we may assume that \( 4 \leq k \leq n - 3 \) and so \( n \geq 7 \). Let \( G \) be the graph obtained from \( K_{n-k} \) by adding \( k \) new vertices \( v_1, v_2, \ldots, v_k \) and joining each vertex \( v_i \) (\( 1 \leq i \leq k \)) to a fixed vertex \( v \) in \( K_{n-k} \). Since \( G \) contains \( k \) end-vertices, \( \det(G) \geq k \) by Observation 2.5. Because \( n - k \geq 3 \), it follows that \( \det(K_{n-k}) = 3 \). Let there be given a detectable 3-coloring of \( K_{n-k} \) using the three colors 1, 2, and 3. If we assign the edge \( vv_i \) the color \( i \) for \( 1 \leq i \leq k \), a detectable \( k \)-coloring results and so \( \det(G) \leq k \). Hence \( \det(G) = k \).

We have seen by (1.1) that if \( G \) is a nontrivial connected graph, then \( \det(G) \geq 2 \). Therefore, the following is a consequence of Proposition 2.16 and Theorem 2.17.

**Corollary 2.18** A pair \( k, n \) of positive integers is realizable as the detection number and the order of some nontrivial connected graph if and only if \( k = n = 3 \) or \( 2 \leq k \leq n - 1 \).

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. As we will see next, the addition of an edge to a connected graph \( G \) can result in a graph whose detection number is the same as that of \( G \) or differs from that of \( G \) by 1 or 2. The same can be said if an edge (that is not a bridge) of a graph is removed.

**Theorem 2.19** Let \( G \) be a noncomplete connected graph of order at least 3. If \( e \) is an edge of \( G \), then

\[
\det(G + e) \leq \det(G) + 1.
\]
If $f$ is an edge of $G$ that is not a bridge, then

$$\det(G - f) \leq \det(G) + 2.$$  

**Proof.** Let $\det(G) = k$. If $c$ is a detectable $k$-coloring of $G$, then we can extend $c$ to a detectable $(k + 1)$-coloring of $G + e$ by assigning the color $k + 1$ to $e$. Thus

$$\det(G + e) \leq k + 1 = \det(G) + 1.$$  

Suppose now that $H = G - f$ where $f = xy$ is an edge of $G$ that is not a bridge. Let $\det(H) = k$. Then $G$ is a connected graph that is not a tree. Let $c$ be a detectable $k$-coloring of $G$. If the coloring $c$ restricted to $H$ is detectable, then

$$\det(H) = \det(G - f) \leq k < k + 2 = \det(G) + 2.$$  

Hence we may assume that the $k$-coloring of $H$ (obtained by restricting the coloring $c$ to $H$) is not detectable. Consequently, there are vertices in $H$ having the same color code. Since the coloring $c$ of $G$ is detectable, $x$ and $y$ have distinct color codes in $G$. This implies that $x$ and $y$ also have distinct color codes in $H$. Observe that $x$ and $y$ are the only vertices whose color codes in $G$ are different from their color codes in $H$. This means that any set of vertices having the same color code in $H$ must contain one of $x$ and $y$ and one other vertex. Hence there is either exactly one pair or exactly two pairs of vertices having the same color code in $H$. We consider these two possibilities.

**Case 1: Exactly one pair of vertices have the same color code in $H$.**

Without loss of generality, we may assume that $x$ and $w$ are the only two vertices of $H$ having the same color code in $H$. Since $\deg_H x = \deg_H w$, necessarily $x$ is adjacent to a vertex different from $w$; for otherwise, $x$ is adjacent only to $w$ and $\deg_H x = \deg_H w = 1$, which implies that $H$ is disconnected. Hence there is an edge $e$ that is incident to $x$ but not to $w$. Changing the color of $e$ to $k + 1$ gives us a detectable $(k + 1)$-coloring of $H$. Hence,

$$\det(G - f) = \det(H) \leq k + 1 < k + 2 = \det(G) + 2.$$  

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Case 2: Exactly two pairs of vertices have the same color codes in $H$.

Suppose that $x$ and $y$ have the same color codes in $H$ as $w$ and $z$, respectively. If $w$ and $z$ are adjacent, then we change the color of the edge $wz$ to color $k + 1$. This gives us a detectable $(k + 1)$-coloring of $H$. Thus,

$$
\det(G - f) = \det(H) \leq k + 1 < k + 2 = \det(G) + 2.
$$

We may therefore assume that $w$ and $z$ are not adjacent. By an argument similar to the one in Case 1, we can find distinct edges $e_1$ and $e_2$ such that $e_1$ is incident to $x$ but not to $w$ and $e_2$ is incident to $y$ but not to $z$. Changing the colors of $e_1$ and $e_2$ to $k + 1$ and $k + 2$, respectively, gives us a detectable $(k + 2)$-coloring of $H$. Hence,

$$
\det(G - f) = \det(H) \leq k + 2 = \det(G) + 2.
$$

as desired.

**Corollary 2.20** Let $G$ be a noncomplete connected graph of order at least 3. If $e$ is an edge of $G$, then

$$
\det(G) - 2 \leq \det(G + e).
$$

If $f$ is an edge of $G$ that is not a bridge, then

$$
\det(G) - 1 \leq \det(G - f).
$$

**Proof.** Let $e$ be an edge of $G$ and let $H = G + e$. Then $e$ is an edge of $H$ that is not a bridge. Moreover, $G = H - e$. Thus, by Theorem 2.19, we have

$$
\det(G) = \det(H - e) \leq \det(H) + 2 = \det(G + e) + 2.
$$

This implies that

$$
\det(G) - 2 \leq \det(G + e).
$$
Now, let \( f \) be an edge of \( G \) that is not a bridge and let \( H = G - f \). Then \( H \) is a noncomplete connected graph and \( G = H + f \). Hence, by Theorem 2.19, we have

\[
\det(G) = \det(H + f) \leq \det(H) + 1 = \det(G - f) + 1.
\]

It follows that

\[
\det(G) - 1 \leq \det(G - f).
\]

The next result summarizes Theorem 2.19 and Corollary 2.20.

**Corollary 2.21** Let \( G \) be a noncomplete connected graph of order at least 3. If \( e \) is an edge of \( G \), then

\[
\det(G) - 2 \leq \det(G + e) \leq \det(G) + 1.
\]

If \( f \) is an edge of \( G \) that is not a bridge, then

\[
\det(G) - 1 \leq \det(G - f) \leq \det(G) + 2.
\]

Figure 2.4 shows that each bound in Corollary 2.21 is attainable.

We now illustrate Corollary 2.21 for stars and double stars. By Proposition 2.12, \( \det(K_{1, n-1}) = n - 1 \) for \( n \geq 3 \). A tree \( T \) is a **double star** if \( T \) contains exactly two vertices that are not end-vertices; necessarily, these two vertices are adjacent. By Observation 2.5, if \( T \) is a double star of order \( n \geq 4 \), then \( \det(T) \geq n - 2 \). On the other hand, assume that \( x \) and \( y \) are the two vertices that are not end-vertices in \( T \) such that \( x \) is adjacent to end-vertices \( x_1, x_2, \ldots, x_a \) and \( y \) is adjacent to end-vertices \( y_1, y_2, \ldots, y_b \). Then \( a + b = n - 2 \).

The coloring \( c \) defined by \( c(xx_i) = i \) for \( 1 \leq i \leq a \), \( c(yy_j) = a + j \) for \( 1 < j < b \), and \( c(xy) = 1 \) is a detectable \((n - 2)\)-coloring of \( T \) and so \( \det(T) \leq n - 2 \). Therefore, if \( G \) is a double star of order \( n \geq 4 \), then \( \det(G) = n - 2 \). We now determine \( \det(G + e) \) where \( G \) is a star or a double star.
Figure 2.4: Graphs showing that each bound in Corollary 2.21 can be attained

**Proposition 2.22** Let $e \in E(K_{1,n-1})$ where $n \geq 3$. Then

$$\det(K_{1,n-1} + e) = \begin{cases} 
3 & \text{if } n = 3 \\
2 & \text{if } n = 4 \\
n - 3 & \text{if } n \geq 5.
\end{cases}$$

**Proof.** For $n = 3$, $K_{1,2} + e = K_3$ and $\det(K_3) = 3$; for $n = 4$, $K_{1,3} + e = K_1 + (K_2 \cup K_1)$ and $\det(K_1 + (K_2 \cup K_1)) = 2$. Thus, we may assume that $n \geq 5$. Since $K_{1,n-1} + e$ has $n - 3$ end-vertices, $\det(K_{1,n-1} + e) \geq n - 3$ by Observation 2.5. On the other hand, if we assign the edge $e$ the color 1, assign the two edges adjacent to $e$ the colors 1 and 2, and color the remaining $n - 3$ edges $1, 2, \ldots, n - 3$, we obtain a detectable $(n - 3)$-coloring of $K_{1,n-1} + e$. Thus, $\det(K_{1,n-1} + e) \leq n - 3$. Consequently, $\det(K_{1,n-1} + e) = n - 3$.
Proposition 2.23  Let $G$ be a double-star with $n - 2$ end-vertices where $n \geq 4$. Suppose that $e \in E(G)$. Then

$$
\det(G + e) = \begin{cases} 
2 \text{ or } 3 & \text{if } n = 4 \\
2 & \text{if } n = 5 \\
n - 3 \text{ or } n - 4 & \text{if } n \geq 6.
\end{cases}
$$

Proof. Observe that the result holds for $n = 4, 5, 6$ as shown in Figure 2.5.

We now assume that $G$ is a double star of order $n \geq 7$. Let $e = uv$ and $f = xy$, where $x$ and $y$ are the two vertices that are not end-vertices of $G$. Without loss of generality, we
may assume that \( \deg_G u = 1 \) and that \( u \) is adjacent to \( x \). We consider two possibilities.

**Case 1: \( v \neq y \)**

This means that \( \deg_G v = 1 \). There are two subcases, according to whether \( v \) is adjacent to \( x \) or \( v \) is adjacent to \( y \).

**Subcase 1.1: \( v \) is adjacent to \( x \).**

In this case, \( \left\{ u, v, x \right\} \) = \( K_3 \). Assign the color 1 to the edges \( ux \) and \( xy \), assign the color 2 to the edges \( uv \) and \( vx \), and assign the remaining \( n - 4 \) edges the colors 1, 2, \ldots, \( n - 4 \) starting with the edges incident with \( x \). This gives us a detectable \((n - 4)\)-coloring of \( G + e \). Since \( G + e \) has \( n - 4 \) end-vertices, by Observation 2.5, we also have \( \det(G + e) \geq n - 4 \). It follows that \( \det(G + e) = n - 4 \).

**Subcase 1.2: \( v \) is adjacent to \( y \).**

In this case, \( \left\{ u, v, x, y \right\} \) = \( C_4 \). Assign the color 1 to the edges \( ux \) and \( uv \), assign the color 2 to the edges \( xv \) and \( vy \), and assign the remaining \( n - 4 \) edges the colors 1, 2, \ldots, \( n - 4 \) starting with the edges incident with \( x \). This gives us a detectable \((n - 4)\)-coloring of \( G + e \). Since \( G + e \) has \( n - 4 \) end-vertices, by Observation 2.5, we also have \( \det(G + e) \geq n - 4 \). It follows that \( \det(G + e) = n - 4 \).

**Case 2: \( v = y \).**

Assign the edges \( ux, uy = uv \) and \( xy \) the color 1 and assign the remaining \( n - 3 \) edges the colors 1, 2, \ldots, \( n - 3 \). This gives us a detectable \((n - 3)\)-coloring of \( G + e \). Since \( G + e \) has \( n - 3 \) end-vertices, by Observation 2.5, we also have \( \det(G + e) \geq n - 3 \). It follows that \( \det(G + e) = n - 3 \).

This completes the proof.
Chapter 3

Cycles and Paths

3.1 Introduction

It was observed in Chapter 2 that vertices of different degrees in a connected graph $G$ have distinct color codes for every coloring of the edges of $G$. Hence, it is most challenging and most interesting to find minimum detectable colorings for graphs having many vertices of the same degree. In particular, this suggests considering regular graphs. Since the only connected 1-regular graph is $P_2$ and the detection number of $P_2$ does not exist, we study the connected 2-regular graphs, namely cycles $C_n$ for $n \geq 3$. In Section 3.2, we consider cycles and provide a formula for the detection number of a cycle in terms of its order. We do the same for paths $P_n$ for $n \geq 3$ in Section 3.3. We also exhibit minimum detectable colorings for graphs in these classes.

3.2 Detection Numbers of Cycles

Let us first investigate minimum detectable colorings of cycles of small orders. According to Corollary 2.9, if $\det(C_n) = k$, then $n \leq \binom{k + 1}{2}$. Hence if $\binom{k}{2} < n \leq \binom{k + 1}{2}$ and $C_n$ has a detectable $k$-coloring, then $\det(C_n) = k$. A detectable 3-coloring for each cycle $C_n$, where $3 \leq n \leq 6$, is shown in Figure 3.1. Thus $\det(C_n) = 3$ for $3 \leq n \leq 6$.

From our earlier observation, in order for $\det(C_n) = 3$, we must have $n \leq \binom{4}{2} = 6$. Consequently, $\det(C_7) \geq 4$. In fact, for $6 = \binom{4}{2} < n \leq \binom{5}{2} = 10$, $\det(C_n) \geq 4$. Since there exists a detectable 4-coloring of $C_8$ for $n = 7.8$ shown in Figure 3.2, it follows that
Figure 3.1: Minimum detectable colorings of some cycles of small orders

\[ \det(C_7) = \det(C_8) = 4. \]

Figure 3.2: Minimum detectable colorings of \( C_7 \) and \( C_8 \)

We now turn to the cycles \( C_9 \) and \( C_{10} \). From our observation above, \( \det(C_n) \geq 4 \) for \( n = 9, 10 \). Since there exists a detectable 5-coloring of \( C_n \) for \( n = 9, 10 \) shown in Figure 3.3, it follows that \( \det(C_n) \leq 5 \) for \( n = 9, 10 \). Therefore, either \( \det(C_n) = 4 \) or \( \det(C_n) = 5 \) for \( n = 9, 10 \). We show that \( \det(C_9) = \det(C_{10}) = 5. \)

Figure 3.3: Minimum detectable colorings of \( C_9 \) and \( C_{10} \)

Let us consider \( \det(C_{10}) \) first. Assume, to the contrary, that \( \det(C_{10}) = 4 \). Since there are at most ten distinct color codes of length 4 for vertices of degree 2, each of the following ordered 4-tuples must be the color code of exactly one vertex of \( C_{10} \):

\[
2000, 0200, 0020, 1100, 1010, 1001, 0110, 0101, 0011.
\]

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Observe that for every integer \( i, 1 \leq i \leq 4, \) the entries in the \( i \)th coordinate of the codes used comprise the degree sequence of the \( i \)th factor in the corresponding 4-tuple factorization of \( C_{10} \). Hence, if \( G_1 \) is the factor of \( C_{10} \) in which each edge is colored 1, then \( G_1 \) contains exactly three vertices of odd degree. This is impossible and so, \( \det(C_{10}) = 5 \) as claimed. Similarly, if \( \det(C_9) = 4 \), then there must be a detectable 4-coloring of \( C_9 \) that results in nine of the ten color codes listed above. Regardless of which nine codes result, at least two of the resulting factors \( G_1, G_2, G_3, G_4 \) have an odd number of vertices of odd degree, which is impossible. Since there exists a detectable 5-coloring of \( C_9 \), it follows that \( \det(C_9) = 5 \).

We now determine the detection numbers of all cycles. In order to do this, we first present two results. The first is a consequence of Theorem 1.3 and Corollary 2.9 and the second has a straightforward proof, which we omit.

**Corollary 3.1** Let \( k \geq 3 \) be an integer. If \( n > \binom{k}{2} \), then \( \det(C_n) \geq k \).

**Lemma 3.2** For each integer \( n \geq 3 \), there exists a unique positive integer \( \ell \) such that

\[
2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2.
\]

Furthermore, \( \ell = \left\lceil \sqrt{n/2} \right\rceil \).

We are now prepared to present a formula for the detection number of a cycle in terms of its order.

**Theorem 3.3** Let \( n \geq 3 \) be an integer and let \( \ell = \left\lceil \sqrt{n/2} \right\rceil \). Then

\[
\det(C_n) = \begin{cases} 
2\ell & \text{if } 2\ell^2 - \ell + 1 \leq n \leq 2\ell^2 \\
2\ell - 1 & \text{if } 2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - \ell
\end{cases}
\]

**Proof.** Since \( \ell = \left\lceil \sqrt{n/2} \right\rceil \), it follows that \( 2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 \) by Lemma 3.2. We consider two cases, according to whether

\[
2\ell^2 - \ell + 1 \leq n \leq 2\ell^2 \text{ or } 2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - \ell.
\]
Case 1: \(2\ell^2 - \ell + 1 \leq n \leq 2\ell^2\).

Since

\[ n \geq 2\ell^2 - \ell + 1 > 2\ell^2 - \ell = \left(\frac{2\ell}{2}\right). \]

it follows by Corollary 3.1 that \(\det(C_n) \geq 2\ell\). We now show that \(\det(C_n) \leq 2\ell\) by considering two subcases, according to whether \(n = 2\ell^2\) or \(n < 2\ell^2\).

Subcase 1.1: \(n = 2\ell^2\).

Let \(V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}\). We now describe a method to color the edges of \(C_{2\ell}\) with the elements of \(V(K_{2\ell})\). By Theorem 2.1, there exist \(\ell - 1\) pairwise edge-disjoint Hamiltonian cycles

\[ H_1, H_2, \ldots, H_{\ell-1} \]

of \(K_{2\ell}\). For each integer \(i\) with \(1 \leq i \leq \ell - 1\), suppose that

\[ H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell}, 1, \]

where \(a_{i,j} (1 \leq j \leq 2\ell)\) is the \(j\)th vertex of \(H_i\). We may assume, without loss of generality, that

\[ H_1 : 1, 2, \ldots, 2\ell, 1. \]

Therefore, \(a_{1,j} = j\) for \(1 \leq j \leq 2\ell\). Suppose that the edges of \(C_{2\ell}\) are encountered in the order

\[ e_1, e_2, \ldots, e_{2\ell^2}, e_{2\ell^2+1} = e_1, \]

as we proceed about the cycle in some direction. For each integer \(k\) with \(1 \leq k \leq 2\ell^2\), either \(1 \leq k \leq 4\ell\) or \(k = i(2\ell) + j\) for some integers \(i\) and \(j\) with \(2 \leq i \leq \ell - 1\) and \(1 \leq j \leq 2\ell\). We now define a coloring \(c : E(C_{2\ell}) \rightarrow V(K_{2\ell})\) of the edges of \(C_{2\ell}\) by

\[
c(e_k) = \begin{cases} 
  a_{1,\lceil \frac{k}{2} \rceil} & \text{if } 1 \leq k \leq 4\ell \\
  a_{i,j} & \text{if } k = i(2\ell) + j, \text{ where } \ 2 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq 2\ell.
\end{cases}
\]
In other words, we assign the color \([k/2]\) to the edge \(e_k\) for \(1 \leq k \leq 4\ell\). Assign the colors \(a_{2,j}\) to the next \(2\ell\) edges \(e_{2(2\ell)+j}\) \((1 \leq j \leq 2\ell)\) of \(C_{2\ell}\), assign the colors \(a_{3,j}\) to the next \(2\ell\) edges \(e_{3(2\ell)+j}\) \((1 \leq j \leq 2\ell)\) and so on. The last \(2\ell\) edges \(e_{(\ell-1)(2\ell)+j}\) \((1 \leq j \leq 2\ell)\) are then colored \(a_{\ell-1,j}\). Since every vertex of \(C_{2\ell}\) is incident with two edges having a unique pair of colors, \(c\) is a detectable \(2\ell\)-coloring of \(C_{2\ell}\) and so \(\det(C_{2\ell}) \leq 2\ell\).

We now illustrate how the edges of \(C_{2\ell}\) can be colored for \(\ell = 3\). By Theorem 2.1, \(K_{2\ell} = K_6\) has a factorization into two Hamiltonian cycles \(H_1\) and \(H_2\) and a 1-factor \(F\). Such a factorization is shown in Figure 3.4 along with the resulting detectable coloring of \(C_{2\ell} = C_{18}\).

![Figure 3.4: A detectable coloring of \(C_{18} = C_{2\ell}\) for \(\ell = 3\)](image)

Subcase 1.2 : \(n = 2\ell^2 - p\), where \(1 \leq p \leq \ell - 1\).

For each integer \(q\) with \(1 \leq q \leq p\), let \(v_q\) be the vertex incident with \(e_{2q-1}\) and \(e_{2q}\)
on \( C_{2\ell^2} \). Suppressing the vertex \( v_q \) (\( 1 \leq q \leq p \)) so that \( e_{2q-1} \) and \( e_{2q} \) become the single edge \( f_q \), we obtain a cycle \( C_{2\ell^2-p} \) of order \( 2\ell^2-p \). Let \( c \) be the detectable \( 2\ell \)-coloring of \( C_{2\ell^2} \) defined in Subcase 1.1. Define an edge coloring \( c^* : E(C_{2\ell^2-p}) \rightarrow V(K_{2\ell}) \) of \( C_{2\ell^2-p} \) by

\[
c^*(e) = \begin{cases} 
   c(e_{2q-1}) & \text{if } e = f_q \text{ for some } q \text{ with } 1 \leq q \leq p \\
   c(e) & \text{otherwise.}
\end{cases}
\]

The codes of the vertices of \( C_{2\ell^2-p} \) are all those of \( C_{2\ell^2} \) except the \( p \) \( 2\ell \)-tuples for which 2 occurs in the \( q \)th coordinate for \( 1 \leq q \leq p \). This is a detectable \( 2\ell \)-coloring of \( C_{2\ell^2-p} \) and so \( \det(C_{2\ell^2-p}) \leq 2\ell \). Figure 3.5 illustrates such a detectable coloring of \( C_{2\ell^2-p} = C_{16} \) for \( \ell = 3 \) and \( p = 2 \).

\[\text{Figure 3.5: A detectable coloring of } C_{16} = C_{2\ell^2-p} \text{ for } \ell = 3 \text{ and } p = 2\]

\[\text{Case 2: } 2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - \ell.\]

First we show that \( \det(C_n) \geq 2\ell - 1 \). If \( 2\ell^2 - 3\ell + 1 < n \leq 2\ell^2 \), then

\[n > \left(\frac{2\ell - 1}{2}\right) = 2\ell^2 - 3\ell + 1.\]
It follows, by Corollary 3.1, that \( \det(C_n) \geq 2\ell - 1 \). Hence we may restrict our attention to those integers \( n \) for which
\[
2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - 3\ell + 1.
\]
Since
\[
\binom{2\ell - 2}{2} = 2\ell^2 - 5\ell + 3 < 2(\ell - 1)^2 < n,
\]
it follows by Corollary 3.1 that \( \det(C_n) \geq 2\ell - 2 \). However, \( \det(C_n) \neq 2\ell - 2 \) when \( 2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - 3\ell + 1 \) as we now show.

Assume, to the contrary, that there exists a detectable \((2\ell - 2)\)-coloring of \( C_n \) for some integer \( n \) with
\[
2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - 3\ell + 1.
\]
Since the code of every vertex of \( C_n \) is a \((2\ell - 2)\)-tuple such that either (a) one coordinate is 2 and the remaining \( 2\ell - 3 \) coordinates are 0 or (b) two coordinates are 1 and the remaining \( 2\ell - 4 \) coordinates are 0, it follows that the total number of such codes is
\[
\binom{2\ell - 2}{1} + \binom{2\ell - 2}{2} = 2\ell^2 - 3\ell + 1.
\]
In particular, if \( n = 2\ell^2 - 3\ell + 1 \), then each such \((2\ell - 2)\)-tuple is the code of some vertex of \( C_n \). Furthermore, for each integer \( j \) with \( 1 \leq j \leq 2\ell - 2 \), there are \( 2\ell - 3 \) vertices of \( C_n \) whose codes have 1 in the \( j \)th coordinate.

For each \( j \in \{1, 2, \ldots, 2\ell - 2\} \), let \( F_j \) denote the factor of \( C_n \) consisting of all those edges colored \( j \). Therefore, \( F_j \) contains one vertex of degree 2, \( 2\ell - 3 \) vertices of degree 1, and the remaining vertices are isolated. However then, \( F_j \) has an odd number of odd vertices, which is impossible. Therefore, \( \det(C_n) \neq 2\ell - 2 \), as claimed, and so \( \det(C_n) \geq 2\ell - 1 \) when \( n = 2\ell^2 - 3\ell + 1 \).

Hence we may assume that \( n = (2\ell^2 - 3\ell + 1) - p \) for some integer \( p \) with \( 1 \leq p \leq \ell - 2 \). Consequently, \( p \) of the \( 2\ell^2 - 3\ell + 1 \) possible \((2\ell - 2)\)-tuples are not codes of the vertices.
of $C_n$. Therefore, among all of the $(2\ell - 2)$-tuples having 1 as two coordinates and 0 as the remaining $2\ell - 4$ coordinates, at most $\ell - 2$ are not used. Hence there are at least 
$(2\ell - 2) - 2(\ell - 2) = 2$ coordinates, say $s$ and $t$ (where $1 \leq s, t \leq 2\ell - 2$), for which the factors $F_s$ and $F_t$ contain exactly $2\ell - 3$ vertices of odd degree, again a contradiction. Therefore, for every integer $n$ with
$$2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - 3\ell + 1,$$
we have $\det(C_n) \neq 2\ell - 2$ and so $\det(C_n) \geq 2\ell - 1$.

It remains to show that $\det(C_n) \leq 2\ell - 1$, which is accomplished by finding a detectable $(2\ell - 1)$-coloring of $C_n$. We consider two subcases.

**Subcase 2.1**: $n = 2\ell^2 - \ell$.

Let $V(K_{2\ell - 1}) = \{1, 2, \ldots, 2\ell - 2, 2\ell - 1\}$. We now describe a method of coloring the edges of $C_{2\ell^2 - \ell}$ with the elements of $V(K_{2\ell - 1})$. By Theorem 2.2, there exist $\ell - 1$ pairwise edge-disjoint Hamiltonian cycles

$$H_1, H_2, \ldots, H_{\ell - 1}$$

of $K_{2\ell - 1}$. For each integer $i$ with $1 \leq i \leq \ell - 1$, suppose that

$$H_i : 1 = a_{i,1} \cdot a_{i,2} \cdot \ldots \cdot a_{i,2\ell - 1} \cdot 1,$$

where $a_{i,j}$ $(1 \leq j \leq 2\ell)$ is the $j$th vertex of $H_i$. We may assume, without loss of generality, that

$$H_1 : 1, 2, \ldots, 2\ell - 1, 1.$$

Therefore, $a_{1,j} = j$ for $1 \leq j \leq 2\ell - 1$. Suppose that the edges of $C_{2\ell^2 - \ell}$ are encountered in the order

$$e_1, e_2, \ldots, e_{2\ell^2 - \ell}, e_{2\ell^2 - \ell + 1} = e_1,$$

as we proceed about the cycle in some direction. We now define a coloring.
of the edges of $C_{2\ell^2-\ell}$ by

$$c(e_k) = \begin{cases} 
  a_{1,\lceil \frac{k}{2} \rceil} = \lfloor k/2 \rfloor & \text{if } 1 \leq k \leq 4\ell - 2 \\
  a_{i,j} & \text{if } k = i(2\ell - 1) + j, \text{ where} \\
  & 2 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq 2\ell - 1.
\end{cases}$$

In other words, we assign the first $4\ell - 2$ edges $e_k$ ($1 \leq k \leq 4\ell - 2$) of $C_{2\ell^2-\ell}$ the color $\lfloor k/2 \rfloor$, assign the next $2\ell - 1$ edges $e_{2(2\ell - 1) + j}$ ($1 \leq j \leq 2\ell - 1$) of $C_{2\ell^2-\ell}$ the color $a_{2,j}$, assign the next $2\ell - 1$ edges $e_{3(2\ell - 1) + j}$ ($1 \leq j \leq 2\ell - 1$) the color $a_{3,j}$ and so on. The last $2\ell - 1$ edges $e_{(\ell-1)(2\ell - 1) + j}$ ($1 \leq j \leq 2\ell - 1$) are then colored $a_{\ell-1,j}$. Since every vertex of $C_{2\ell^2-\ell}$ is incident with two edges having a unique pair of colors, $c$ is a detectable $(2\ell - 1)$-coloring of $C_{2\ell^2-\ell}$ and so $\det(C_{2\ell^2-\ell}) \leq 2\ell - 1$. Hence $\det(C_{2\ell^2-\ell}) = 2\ell - 1$.

Figure 3.6 illustrates how the edges of $C_{2\ell^2-\ell}$ can be colored when $\ell = 4$. By Theorem 2.2, $K_{2\ell-1} = K_7$ has a Hamiltonian factorization into three Hamiltonian cycles $H_1$, $H_2$, and $H_3$. This factorization gives rise to a detectable $(2\ell - 1)$-coloring of $C_{2\ell^2-\ell}$.

Subcase 2.2 : $2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - \ell - 1$.

Let

$$n = (2\ell^2 - \ell) - p, \text{ where } 1 \leq p \leq 3\ell - 3.$$ 

We consider two subcases, according to whether

$$1 \leq p \leq 2\ell - 1 \text{ or } 2\ell \leq p \leq 3\ell - 3.$$ 

Subcase 2.2.1 : $1 \leq p \leq 2\ell - 1$.

For each integer $q$ with $1 \leq q \leq p$, let $v_q$ be the vertex incident with $e_{2q - 1}$ and $e_{2q}$ on $C_{2\ell^2-\ell}$. Suppressing the vertex $v_q$ ($1 \leq q \leq p$) so that $e_{2q - 1}$ and $e_{2q}$ become a single edge $f_q$, we obtain a cycle $C_{2\ell^2-\ell-p}$ of order $2\ell^2 - \ell - p$. Let $c$ be the detectable $(2\ell - 1)$-coloring of $C_{2\ell^2-\ell}$ defined in Subcase 2.1. Define an edge coloring

$$c : E(C_{2\ell^2-\ell}) \to V(K_{2\ell-1})$$
Figure 3.6: A detectable coloring of $C_{28} = C_{2\ell^2 - \ell}$ for $\ell = 1$

$$c^* : E(C_{2\ell^2 - \ell - p}) \rightarrow V(K_{2\ell - 1})$$

of $C_{2\ell^2 - \ell - p}$ by

$$c^*(e) = \begin{cases} 
  c(e_{2q-1}) & \text{if } e = f_q \text{ for some } q \text{ with } 1 \leq q \leq p \\
  c(c) & \text{otherwise.}
\end{cases}$$

The codes of the vertices of $C_{2\ell^2 - \ell - p}$ are all those of $C_{2\ell^2 - \ell}$ except those $(2\ell-1)$-tuples for which 2 occurs in the $q$th coordinate for $1 \leq q \leq p$ (there are $p$ such $(2\ell-1)$-tuples). Since this is a detectable $(2\ell - 1)$-coloring of $C_{2\ell^2 - \ell - p}$, it follows that $\det(C_{2\ell^2 - \ell - p}) \leq 2\ell - 1$.

Figure 3.7 illustrates such a detectable $(2\ell - 1)$-coloring of $C_{2\ell^2 - \ell - p}$ for $\ell = 4$ and $p = 3$. 

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Figure 3.7: A detectable coloring of $C_{25} = C_{2\ell^2 - \ell - p}$ for $\ell = 1$ and $p = 3$

Subcase 2.2.2: $2\ell \leq p \leq 3\ell - 3$.

Let $p = (2\ell - 1) + h$. Then $1 \leq h \leq \ell - 2 < 2\ell - 1$. Recall that the edges of $C_{2\ell^2 - \ell}$ are encountered in the order

$$e_1, e_2, \ldots, e_{2\ell^2 - \ell}, e_{2\ell^2 - \ell + 1} = e_1,$$

as we proceed around the cycle in some direction. Let $v_i$ denote the vertex of $C_{2\ell^2 - \ell}$ incident with $e_i$ and $e_{i+1}$ for $1 \leq i \leq 6\ell - 3$. First, we construct a cycle $C_{2\ell^2 - \ell - (2\ell - 1)}$ from $C_{2\ell^2 - \ell}$ by

1. deleting the vertices $v_{4\ell - 1}, v_{4\ell}, \ldots, v_{6\ell - 1}$ and therefore the $2\ell - 1$ edges $e_{4\ell - 1}, e_{4\ell}, \ldots, e_{6\ell - 3}$ (which correspond to the Hamiltonian cycle $H_2$) are deleted as well, and

2. identifying the vertices $v_{4\ell - 2}$ and $v_{6\ell - 3}$.

Next, we suppress the vertex $v_{2j - 1}$ for $1 \leq j \leq h$, where the two edges $e_{2j - 1}$ and $e_{2j}$ become the single edge $f_j$. This produces a cycle $C_{2\ell^2 - \ell - (2\ell - 1) + h} = C_n$. Let $e$ be the detectable $(2\ell - 1)$-coloring of $C_{2\ell^2 - \ell}$ defined in Subcase 2.1. Define an edge coloring
$c' : E(C_n) \rightarrow V(K_{2^\ell - 1})$ by

$$c'(e) = \begin{cases} 
  c(\ell_{2j-1}) & \text{if } e = f_j \text{ for } 1 \leq j \leq h \\
  c(e) & \text{otherwise.}
\end{cases}$$

Figure 3.7 illustrates such a detectable $(2^\ell - 1)$-coloring of $C'_{2^\ell \cdot \ell \cdot (2^\ell - 1 \cdot h)}$ for $\ell = 1$ and $h = 1$.

![Figure 3.7: A detectable coloring of $C'_{2^\ell \cdot \ell \cdot (2^\ell - 1 \cdot h)}$ for $\ell = 1$ and $h = 1$.](image)

The codes of the vertices of $C_n$ are all those of $C'_{2^\ell \cdot \ell}$ except for

1. those $(2\ell - 1)$-tuples for which 2 occurs in the $j$th coordinate for $1 \leq j \leq h$ (there are $h$ such $(2\ell - 1)$-tuples) and

2. those $(2\ell - 1)$-tuples that are produced from the Hamiltonian cycle $H_2$; that is, the codes of the vertices $v_{4\ell - 1}, v_{6\ell}, \ldots, v_{6\ell - 3}$ in the cycle $C'_{2^\ell \cdot \ell}$ (there are $2\ell - 1$ such $(2\ell - 1)$-tuples).

Since $c'$ is a detectable $(2^\ell - 1)$-coloring of $C_n$, it follows that $\det(C_n) \leq 2^\ell - 1$. 

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3.3 Detection Numbers of Paths

In this section, we determine the detection numbers of all paths. In order to do this, we first present two results. The first is a consequence of Corollary 2.9 and the second has a straightforward proof, which we omit.

Corollary 3.4 Let \( k \geq 2 \) be an integer. If \( n > \begin{pmatrix} k \\ 2 \end{pmatrix} + 2 \), then \( \det(P_n) > k \).

Lemma 3.5 For each integer \( n \geq 3 \), there exists a unique positive integer \( \ell \) such that

\[
\left( \frac{2\ell}{2} \right) + 2 = 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3\ell + 2 = \left( \frac{2\ell + 2}{2} \right) + 1.
\]

Furthermore, \( \ell = \left\lfloor \frac{-3 + \sqrt{8n - 7}}{4} \right\rfloor \).

Theorem 3.6 Let \( n \geq 3 \) and let \( \ell = \left\lfloor \frac{-3 + \sqrt{8n - 7}}{4} \right\rfloor \). Then

\[
\det(P_n) = \begin{cases} 
2\ell & \text{if } 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3 \\
2\ell + 1 & \text{if } 2\ell^2 + 4 \leq n \leq 2\ell^2 + 3\ell + 2.
\end{cases}
\]

Proof. Since \( \ell = \left\lfloor \frac{-3 + \sqrt{8n - 7}}{4} \right\rfloor \), it follows that \( 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3\ell + 2 \) by Lemma 3.5. Figure 3.9 shows detectable 2-colorings for \( P_n \) where \( 3 \leq n \leq 5 \). It follows that \( \det(P_n) = 2 \) for \( 3 \leq n \leq 5 \) and so the result holds for \( 3 \leq n \leq 5 \). Hence, we may restrict our attention to \( n \geq 6 \). We consider two cases, according to whether

\[
2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3 \quad \text{or} \quad 2\ell^2 + 4 \leq n \leq 2\ell^2 + 3\ell + 2.
\]

Figure 3.9: Minimum detectable colorings of some paths of small orders
Case 1: $2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3$.

By Corollary 3.4, if $n \geq \binom{2\ell}{2} + 3 = 2\ell^2 - \ell + 3$, then $\det(P_n) \geq 2\ell$. We now show that if $n = 2\ell^2 - \ell + 2$, then $\det(P_n) \geq 2\ell$. Since

$$n > \binom{2\ell-1}{2} + 2 = 2\ell^2 - 3\ell + 3$$

for every $\ell \geq 1$, it follows that $\det(P_n) \geq 2\ell - 1$. Suppose that there exists a detectable $(2\ell - 1)$-coloring $c$ of $P_n$ where $n = 2\ell^2 - \ell + 2$. Since the maximum number of vertices in a path with detection number $2\ell - 1$ is

$$\binom{(2\ell - 1) + 1}{2} + 2 = \binom{2\ell}{2} + 2 = 2\ell^2 - \ell + 2,$$

it follows that all possible color codes for the vertices of degree 2 are used in the coloring $c$. Observe that among the possible color codes for vertices of degree 2, there is a total of $2\ell - 2$ codes starting with 1. Indeed, among the codes containing exactly two 1's, there is a total of $2\ell - 2$ codes having 1 in the $j$th position for every $j = 1, 2, \ldots, 2\ell - 1$. Since the code of each end-vertex of $P_n = P_{2\ell^2-\ell+2}$ contains exactly one 1, it follows that in the corresponding detectable $(2\ell - 1)$-tuple factorization of $P_n = P_{2\ell^2-\ell+2}$, two of the factors have $2\ell - 1$ vertices of degree 1, which is not possible. Hence, $\det(P_n) = \det(P_{2\ell^2-\ell+2}) \neq 2\ell - 1$. Consequently, $\det(P_n) = \det(P_{2\ell^2-\ell+2}) \geq 2\ell$. This shows that

$$\det(P_n) \geq 2\ell \text{ if } 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3.$$ 

We now show that $\det(P_n) \leq 2\ell$ if $2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3$ by considering two subcases, depending on whether

$$n = 2\ell^2 + 3 \text{ or } 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 2.$$

Subcase 1.1 : $n = 2\ell^2 + 3$.

Let $V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}$. We now describe a method to assign a detectable coloring of the edges of $P_{2\ell^2+3}$ with the elements of $V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}$. By Theorem 2.1, there exists a factorization of $K_{2\ell}$ into $\ell - 1$ Hamiltonian cycles
and a 1-factor $F$. For each integer $i$ with $1 \leq i \leq \ell - 1$, suppose that

$$H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell}, 1,$$

where $a_{i,j}$ ($1 \leq j \leq 2\ell$) is the $j$th vertex of $H_i$. We may assume, without loss of generality, that

$$H_1 : 1, 2, \ldots, 2\ell, 1.$$

Therefore, $a_{1,j} = j$ for $1 \leq j \leq 2\ell$. Also, let $b_1$ be the neighbor of 1 in the 1-factor $F$ of $K_{2\ell}$. Note that $b_1 \neq a_{i,2}$ and $b_1 \neq a_{i,2\ell}$ for every $i$ with $1 \leq i \leq \ell - 1$. Suppose that the edges of $P_{2\ell^2 + 3}$ are encountered in the order

$$e_1, e_2, \ldots, e_{2\ell^2 + 2}$$

as we proceed along the path. For each integer $k$ with $1 \leq k \leq 2\ell^2$, either $1 \leq k \leq 4\ell$ or $k = i(2\ell) + j$ for some integers $i$ and $j$ with $2 \leq i \leq \ell - 1$ and $1 \leq j \leq 2\ell$. We now define a coloring $c : E(P_{2\ell^2 + 3}) \rightarrow V(K_{2\ell})$ of the edges of $P_{2\ell^2 + 3}$ by

$$c(e_k) = \begin{cases} 
    a_{1, \left\lfloor \frac{k}{2} \right\rfloor} = [k/2] & \text{if } 1 \leq k \leq 4\ell \\
    a_{i,j} & \text{if } k = i(2\ell) + j, 2 \leq i \leq \ell - 1, 1 \leq j \leq 2\ell \\
    1 & \text{if } k = 2\ell^2 + 1 \\
    b_1 & \text{if } k = 2\ell^2 + 2.
\end{cases}$$

In other words, we assign the color $[k/2]$ to the edge $e_k$ for $1 \leq k \leq 4\ell$. Color the next $2\ell$ edges $e_{2(2\ell) + j}$ ($1 \leq j \leq 2\ell$) of $P_{2\ell^2 + 3}$ by $a_{2,j}$, color the next $2\ell$ edges $e_{3(2\ell) + j}$ ($1 \leq j \leq 2\ell$) by $a_{3,j}$, and so on. We continue this process until we have used all of the Hamiltonian cycles $H_1, H_2, \ldots, H_{\ell - 1}$. We have now assigned colors to the first $2\ell^2$ edges of $P_{2\ell^2 + 3}$. We assign the colors 1 and $b_1$ (in that order) to the last two edges. Since every vertex of degree 2 of $P_{2\ell^2 + 3}$ is incident with two edges having a unique pair of colors and the edges incident with the end-vertices are colored 1 and $b_1$ ($\neq 1$), $c$ is a detectable $2\ell$-coloring of $P_{2\ell^2 + 3}$ and so $\text{det}(P_{2\ell^2 + 3}) \leq 2\ell$. Figure 3.10 illustrates a detectable $2\ell$-coloring for $P_{2\ell^2 + 3} = P_{21}$ for $\ell = 3$. 

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Subcase 1.2: $n = 2\ell^2 + 3 - p$, for some integer $p$ with $1 \leq p \leq \ell + 1$.

For each integer $q$ with $1 \leq q \leq p$, let $v_q$ be the vertex incident with $e_{2q-1}$ and $e_{2q}$ on $P_{2\ell^2+3}$. Suppressing the vertex $v_q (1 \leq q \leq p)$ so that $e_{2q-1}$ and $e_{2q}$ become the single edge $f_q$, we obtain a path $P_{2\ell^2+3-p}$ of order $2\ell^2 + 3 - p$. Let $c$ be the detectable $2\ell$-coloring of $P_{2\ell^2+3}$ defined in Subcase 1.1. Define an edge coloring $c^*: E(P_{2\ell^2+3-p}) \to V(K_{2\ell})$ of $P_{2\ell^2+3-p}$ by

$$c^*(e) = \begin{cases} c(e_{2q-1}) & \text{if } e = f_q \text{ for some } q \text{ with } 1 \leq q \leq p \\ c(e) & \text{otherwise.} \end{cases}$$

The codes of the vertices of $P_{2\ell^2+3-p}$ are all those of $P_{2\ell^2+3}$ except the $p$ $2\ell$-tuples for which 2 occurs in the $q$th coordinate for $1 \leq q \leq p$. This is a detectable $2\ell$-coloring of $P_{2\ell^2+3-p}$ and so $\det(P_{2\ell^2+3-p}) \leq 2\ell$. Figure 3.11 illustrates a detectable $2\ell$-coloring of $P_{2\ell^2+3-p} = P_{17}$ for $\ell = 3$ and $p = 4 = \ell + 1$.

Figure 3.11: The detectable coloring of $P_{17}$ in Subcase 1.2
Case 2: $2\ell^2 + 4 \leq n \leq 2\ell^2 + 3\ell + 2$.

By Corollary 3.4, if $n > \binom{2\ell+1}{2} + 2 = 2\ell^2 + \ell + 2$, then $\det(P_n) \geq 2\ell + 1$. Thus, if $n \geq 2\ell^2 + \ell + 3$, then $\det(P_n) \geq 2\ell + 1$. Now, let $n = 2\ell^2 + \ell + 2$. Since $n > \binom{2\ell}{2} = 2\ell^2 - \ell + 2$, it follows that $\det(P_n) = \det(P_{2\ell^2 + \ell + 2}) \geq 2\ell$. Suppose now that there exists a detectable $2\ell$-coloring $c$ of $P_n = P_{2\ell^2 + \ell + 2}$. Because the largest possible number of vertices in a path with detection number $2\ell$ is $\left(\binom{2\ell+1}{2}\right) + 2 = 2\ell^2 + \ell + 2$, all possible color codes for the vertices of degree 2 are used in the coloring $c$. Observe that among the codes containing exactly two 1's, there is a total of $2\ell - 1$ codes having 1 in the $j$th position for every $j = 1, 2, \ldots, 2\ell$. Also, the code of each end-vertex of $P_n = P_{2\ell^2 + \ell + 2}$ contains exactly one 1. This implies that in the corresponding detectable $2\ell$-tuple factorization of $P_n = P_{2\ell^2 + \ell + 2}$, all but two of the factors have an odd number of vertices of degree 1, which is not possible. Hence, $\det(P_n) = \det(P_{2\ell^2 + \ell + 2}) \geq 2\ell + 1$. Suppose now that $1 \leq p \leq \ell - 2$. Then

$$2\ell^2 + \ell + 2 - p \geq 2\ell^2 + \ell + 2 - (\ell - 2) = 2\ell^2 + 4.$$

But $2\ell^2 + 4 > 2\ell^2 - \ell + 2$. It follows that $\det(P_{2\ell^2 + \ell + 2 - p}) \geq 2\ell$ for every $p = 1, 2, \ldots, \ell - 2$. If $c$ is a detectable $2\ell$-coloring of $P_{2\ell^2 + \ell + 2 - p}$, then in the corresponding $2\ell$-tuple factorization of $P_{2\ell^2 + \ell + 2 - p}$, there would be at least $2\ell - (2 + 2(\ell - 2)) = 2$ factors having an odd number of vertices of degree 1 which is not possible. It follows then that

$$\det(P_n) \geq 2\ell + 1 \text{ if } n \geq 2\ell^2 + 4.$$

Since $2\ell^2 + 3\ell + 2 > 2\ell^2 + 4$ for all $\ell \geq 1$, we have

$$\det(P_n) \geq 2\ell + 1 \text{ if } 2\ell^2 + 4 \leq n \leq 2\ell^2 + 3\ell + 2.$$

It remains to show that $\det(P_n) \leq 2\ell + 1$ whenever $2\ell^2 + 1 \leq n \leq 2\ell^2 + 3\ell + 2$. This is accomplished by finding a detectable $(2\ell + 1)$-coloring of $P_n$. We consider two subcases.

**Subcase 2.1**: $n = 2\ell^2 + 3\ell + 2$.

Let $V(K_{2\ell+1}) = \{1, 2, \ldots, 2\ell, 2\ell + 1\}$. We now describe a method of assigning a detectable coloring of $P_{2\ell^2 + 3\ell + 2}$ with the elements of $V(K_{2\ell+1})$. By Theorem 2.2, $K_{2\ell+1}$ can be factored into $\ell$ Hamiltonian cycles.
For each integer \( i \) with \( 1 \leq i \leq \ell \), suppose that

\[ H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell + 1}, 1. \]

where \( a_{i,j} \) (\( 1 \leq j \leq 2\ell + 1 \)) is the \( j \)th vertex of \( H_i \). We may assume, without loss of generality, that

\[ H_1 : 1, 2, \ldots, 2\ell + 1, 1. \]

Therefore, \( a_{1,j} = j \) for \( 1 \leq j \leq 2\ell + 1 \). Suppose that the edges of \( P_{2\ell^2 + 3\ell + 2} \) are encountered in the order

\[ e_1, e_2, \ldots, e_{2\ell^2 + 3\ell + 1} \]

as we proceed along the path. We now define a coloring \( c : E(P_{2\ell^2 + 3\ell + 2}) \to V(K_{2\ell + 1}) \) of the edges of \( P_{2\ell^2 + 3\ell + 2} \) by

\[
c(e_k) = \begin{cases} a_{1,\lceil k/2 \rceil} & \text{if } 1 \leq k \leq 4\ell + 2 \\ a_{i,j} & \text{if } k = i(2\ell + 1) + j, \ 2 \leq i \leq \ell, \ 1 \leq j \leq 2\ell + 1. \end{cases}
\]

That is, we assign the first \( 4\ell + 2 \) edges \( e_k \) (\( 1 \leq k \leq 4\ell + 2 \)) of \( P_{2\ell^2 + 3\ell + 2} \) the color \( \lceil k/2 \rceil \), assign the next \( 2\ell + 1 \) edges \( e_{2(2\ell + 1) + j} \) (\( 1 \leq j \leq 2\ell + 1 \)) of \( P_{2\ell^2 + 3\ell + 2} \) the color \( a_{2,j} \), assign the next \( 2\ell + 1 \) edges \( e_{3(2\ell + 1) + j} \) (\( 1 \leq j \leq 2\ell + 1 \)) the color \( a_{3,j} \) and so on. The last \( 2\ell + 1 \) edges \( e_{(2\ell + 1) + j} \) (\( 1 \leq j \leq 2\ell + 1 \)) are then colored \( a_{\ell,j} \). Since every vertex of degree 2 of \( P_{2\ell^2 + 3\ell + 2} \) is incident with two edges having a unique pair of colors and the pendant edges are assigned the colors 1 and \( a_{\ell,2\ell + 1} \neq 1 \), it follows that \( c \) is a detectable \((2\ell + 1)\)-coloring of \( P_{2\ell^2 + 3\ell + 2} \) and so \( \text{det}(P_{2\ell^2 + 3\ell + 2}) \leq 2\ell + 1 \). Hence, \( \text{det}(P_{2\ell^2 + 3\ell + 2}) = 2\ell + 1 \). Figure 3.12 illustrates the detectable \((2\ell + 1)\)-coloring of \( P_{2\ell^2 + 3\ell + 2} = P_{2\ell^2} \) for \( \ell = 3 \).

Subcase 2.2 : \( 2\ell^2 + 4 \leq n \leq 2\ell^2 + 3\ell + 1 \).

Let \( n = (2\ell^2 + 3\ell + 2) - p \), where \( 1 \leq p \leq 3\ell - 2 \). We consider two subcases, according to whether
Subcase 2.2.1: $1 \leq p \leq 2\ell + 1$.

For each integer $q$ with $1 \leq q \leq p$, let $v_q$ be the vertex incident with $e_{2q-1}$ and $e_{2q}$ on $P_{2\ell^2+3\ell+2}$. Suppressing the vertex $v_q$ ($1 \leq q \leq p$) so that $e_{2q-1}$ and $e_{2q}$ become the single edge $f_q$, we obtain a path $P_{2\ell^2+3\ell+2-p}$ of order $2\ell^2 + 3\ell + 2 - p$. Let $c$ be the detectable $(2\ell+1)$-coloring of $P_{2\ell^2+3\ell+2}$ defined in Subcase 2.1. Define an edge coloring $c^*: E(P_{2\ell^2+3\ell+2-p}) \rightarrow V(K_{2\ell+1})$ of $P_{2\ell^2+3\ell+2-p}$ by

$$c^*(e) = \begin{cases} 
  c(e_{2q-1}) & \text{if } e = f_q \text{ for some } q \text{ with } 1 \leq q \leq p \\
  c(e) & \text{otherwise}.
\end{cases}$$

The codes of the vertices of $P_{2\ell^2+3\ell+2-p}$ are all those of $P_{2\ell^2+3\ell+2}$ except the $(2\ell+1)$-tuples for which 2 occurs in the $q$th coordinate for $1 \leq q \leq p$. Since this is a detectable $(2\ell+1)$-coloring of $P_{2\ell^2+3\ell+2-p}$, it follows that

$$\det(P_{2\ell^2+3\ell+2-p}) \leq 2\ell + 1.$$ 

Figure 3.13 illustrates the detectable $(2\ell+1)$-coloring of $P_{2\ell^2+3\ell+2-P} = P_{25}$ when $\ell = 3$ and $p = 4$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig3_12}
\caption{The detectable coloring of $P_{29}$ in Subcase 2.1}
\end{figure}

$$1 \leq p \leq 2\ell + 1 \text{ or } 2\ell + 2 \leq p \leq 3\ell - 2.$$
Figure 3.13: The detectable coloring of $P_{25}$ in Subcase 2.2.1

Subcase 2.2.2: $2\ell + 2 \leq p \leq 3\ell - 2$.

Note that this subcase can only occur when $\ell \geq 4$. Let $p = (2\ell + 1) + h$ where $1 \leq h \leq \ell - 3$. Observe that $\ell - 3 < 2\ell + 1$ for all positive integers $\ell$. Recall that the edges of $P_{2\ell^2+3\ell+2}$ are encountered in the order

\[ e_1, e_2, \ldots, e_{2\ell^2+3\ell}, e_{2\ell^2+3\ell+1} \]

as we proceed along the path. Let $v_i$ denote the vertex of $P_{2\ell^2+3\ell+2}$ incident with $e_i$ and $e_{i+1}$ for $1 \leq i \leq 6\ell + 3$. First, we construct a path $P_{(2\ell^2+3\ell+2)-(2\ell+1)}$ from $P_{2\ell^2+3\ell+2}$ by

1. deleting the vertices $v_4\ell+3, v_4\ell+4, \ldots, v_6\ell+2$ and therefore, deleting the $2\ell + 1$ edges $e_4\ell+3, e_4\ell+4, \ldots, e_6\ell+3$ (which correspond to the Hamiltonian cycle $H_2$), and

2. identifying the vertices $v_4\ell+2$ and $v_6\ell+3$.

This produces a path $P_{(2\ell^2+3\ell+2)-(2\ell+1)}$ of order $(2\ell^2 + 3\ell + 2) - (2\ell + 1)$. Next, we suppress the vertex $v_{2j-1}$ for $1 \leq j \leq h$, where the two edges $e_{2j-1}$ and $e_{2j}$ become the single edge $f_j$. This produces a path $P_{(2\ell^2+3\ell+2)-(2\ell+1)+h} = P_n$. Let $c$ be the detectable $(2\ell + 1)$-coloring of $P_{2\ell^2+3\ell+2}$ defined in Subcase 2.1. Define an edge coloring $c': E(C_n) \to V(K_{2\ell+1})$ by

\[
c'(e) = \begin{cases} 
  c(e_{2j-1}) & \text{if } e = f_j \text{ for } 1 \leq j \leq h \\
  c(e) & \text{otherwise.}
\end{cases}
\]

Figure 3.14 illustrates a detectable $(2\ell+1)$-coloring of $P_{2\ell^2+3\ell+2} = P_{36}$ where $\ell = 4$ and a detectable $(2\ell + 1)$-coloring of $P_{(2\ell^2+3\ell+2)-(2\ell+1)+h} = P_{36}$ (for $\ell = 4$ and $h = 1$) obtained from the coloring of $P_{46}$.
The codes of the vertices of $P_n$ are all those of $P_{2\ell^2+3\ell+2}$ except

(1) the $(2\ell+1)$-tuples for which 2 occurs in the $j$th coordinate for $1 \leq j \leq h$ (there are $h$ such $(2\ell+1)$-tuples) and

(2) the $(2\ell+1)$-tuples that are produced from the Hamiltonian cycle $H_2$; that is, the codes of the vertices $v_{1\ell+3}$, $v_{1\ell+4}$, ..., $v_{6\ell+3}$ in the path $P_{2\ell^2+3\ell+2}$ (there are $2\ell+1$ such $(2\ell+1)$-tuples).

Since $c'$ is a detectable $(2\ell+1)$-coloring of $P_n$, it follows that $\text{det}(P_n) \leq 2\ell + 1$.

### 3.4 Some Concluding Remarks

Using the formula for the detection number of a cycle given in Theorem 3.3 and the formula for the detection number of a path given in Theorem 3.6, we obtain the table shown in Figure 3.15, which gives the orders of the cycles and paths having detection number $k$ for $2 \leq k \leq 10$. 

---

Figure 3.14: Detectable colorings of $P_{16}$ and $P_{36}$ in Subcase 2.2.2 in the proof of Theorem 3.6
The table in Figure 3.15 suggests the existence of a relationship between the detection numbers of $C_n$ and $P_n$. The following results show that this is the case.

**Proposition 3.7** Let $n \geq 3$. Then

$$\det(P_n) \leq \det(C_n) \leq \det(P_n) + 1.$$ 

**Proof.** The right inequality follows from Theorem 2.19. Suppose now that $\det(C_n) = k$ and let $c$ be the detectable $k$-coloring of $C_n$ described in Theorem 3.3. Observe that there is an edge $e$ of $C_n$ such that $c(e) = 1$ and $c(f_1) \neq c(f_2)$ where $f_1$ and $f_2$ are the edges adjacent to $e$ in $C_n$. Deleting $e$ from $C_n$ gives us a detectable $k$-coloring of $P_n$, and this proves the left inequality. \[ \Box \]

**Corollary 3.8** Let $n \geq 3$. Then

$$\det(C_n) - 1 \leq \det(P_n) \leq \det(C_n).$$

In order to present a conjecture for the detection number of $P_n$ ($n \geq 3$) in terms of its order $n$ and the number $\left\lceil \sqrt{n/2} \right\rceil$, we now illustrate how a minimum detectable coloring of a path can be obtained from a minimum detectable coloring of a cycle. Consider $C_6$ together with the minimum detectable coloring described in Theorem 3.3. We can
obtain $P_7$ from $C_6$ as follows: (a) delete $u$, (b) add vertices $x$ and $y$, (c) join $x$ to $v$, and (d) join $y$ to $w$. If we color the edges $wy$ and $vx$ by 1 and 3, respectively, then we obtain the minimum detectable coloring for $P_7$ described by Theorem 3.6 (see Figure 3.16).

![Figure 3.16: Obtaining a minimum detectable coloring of $P_7$ from a minimum detectable coloring of $C_6$](image)

On the other hand, given $C_8$ together with the minimum detectable coloring described by Theorem 3.3, we can obtain $P_{11}$ as follows: (a) delete $u$, (b) add vertices $a, b, c$, and $d$, (c) join $a$ to $v$ and $b$, (d) join $b$ to $c$, and (e) join $d$ to $w$. If we assign the edges $ab$ and $dw$ by color 1, the edge $bc$ by color 3, and the edge $av$ by color 4, we obtain the minimum detectable coloring of $P_{11}$ described by Theorem 3.6 (see Figure 3.17).

![Figure 3.17: Obtaining a minimum detectable coloring of $P_{11}$ from a minimum detectable coloring of $C_8$](image)

Recall that by Theorem 3.3, if $n \geq 3$ and $\ell = \left\lfloor \sqrt{\frac{n}{2}} \right\rfloor$, then

$$\det(C_n) = \begin{cases} 2\ell^2 - \ell + 1 & \text{if } 2\ell^2 - \ell + 1 \leq n \leq 2\ell^2 \\ 2\ell - 1 & \text{if } 2(\ell - 1)^2 + 1 \leq n \leq 2\ell^2 - \ell. \end{cases}$$

The table in Figure 3.15, the proofs of Theorem 3.3 and Theorem 3.6, and the discussion
above suggest the following conjecture.

**Conjecture 3.9**  Let $n \geq 3$ be an integer and let $\ell = \left\lfloor \sqrt{n/2} \right\rfloor$. Then

$$\det(P_n) = \begin{cases} 2\ell & \text{if } 2\ell^2 - \ell + 2 \leq n \leq 2\ell^2 + 3 \\ 2\ell - 1 & \text{if } 2(\ell - 1)^2 + 4 \leq n \leq 2\ell^2 - \ell + 1. \end{cases}$$
Chapter 4

Extremal Problems

4.1 Introduction

We now study several extremal problems concerning detection numbers of connected graphs. More precisely, we investigate how large and how small the detection number of a connected graph of a fixed order and a fixed size can be. Specifically, we consider trees, unicyclic graphs, and connected graphs having cycle rank 2. First, we need some additional definitions and notation.

Recall that if $G$ is a connected graph of order $n$ and size $m$, then the number of edges that must be deleted from $G$ to obtain a spanning tree of $G$ is $m - n + 1$. The number $m - n + 1$ is called the cycle rank of $G$. Observe that if $G$ is a connected graph of order $n$ and size $m$ whose cycle rank is $\psi$, then

$$m = (n - 1) + \psi \leq \binom{n}{2}.$$

This implies that

$$n \geq \left\lceil \frac{3 + \sqrt{1 + 8\psi}}{2} \right\rceil.$$

For integers $\psi$ and $n$, where $\psi \geq 0$ and $n \geq \left\lceil \frac{3 + \sqrt{1 + 8\psi}}{2} \right\rceil$, let $D_\psi(n)$ denote the maximum detection number among all connected graphs of order $n$ with cycle rank $\psi$. 

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and let \( d_\psi(n) \) denote the minimum detection number among all connected graphs of order \( n \) with cycle rank \( \psi \). Hence, if \( G_{\psi,n} \) denotes the set of all connected graphs of order \( n \) with cycle rank \( \psi \), then

\[
D_\psi(n) = \max \{ \det(G) : G \in G_{\psi,n} \}
\]

\[
d_\psi(n) = \min \{ \det(G) : G \in G_{\psi,n} \}.
\]

It follows that \( D_0(n) \) is the maximum detection number among all trees of order \( n \), \( D_1(n) \) is the maximum detection number among all unicyclic graphs of order \( n \), and \( D_2(n) \) is the maximum detection number among all connected graphs of order \( n \) whose cycle rank is 2. Furthermore, \( d_0(n) \), \( d_1(n) \) and \( d_2(n) \) are the minimum detection numbers among all trees, unicyclic graphs, and connected graphs with cycle rank 2 of order \( n \), respectively.

In this chapter, we develop formulas for \( D_\psi(n) \) and \( d_\psi(n) \) for \( \psi = 0, 1, 2 \) (where \( n \geq 3 \) when \( \psi = 0, 1 \) and \( n \geq 4 \) when \( \psi = 2 \)). We also show that if \( k \geq 2 \), then there exists a connected graph \( G \) of order \( n \) with cycle rank \( \psi \) having detection number \( k \) if and only if \( d_\psi(n) \leq k \leq D_\psi(n) \) for \( \psi = 0, 1, 2 \).

## 4.2 Trees

In this section, we establish formulas for the maximum and minimum detection numbers among all trees of order \( n \geq 3 \). Proposition 2.12 states that \( \det(K_{1,n-1}) = n - 1 \) for each integer \( n \geq 3 \). This means that \( D_0(n) \geq n - 1 \) for \( n \geq 3 \). On the other hand, Corollary 2.18 implies that \( D_0(n) \leq n - 1 \) for every integer \( n \geq 3 \). We thus have the following observation.

**Observation 4.1**  
For each integer \( n \geq 3 \), \( D_0(n) = n - 1 \).

We now determine a formula for \( d_0(n) \). According to Corollary 2.9, every tree of order \( n \geq 3 \) having detection number \( k \) contains at most \( k \) end-vertices and at most \( \frac{k(k+1)}{2} \) vertices of degree 2. By (2.3) and the argument above,

\[
n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta = n_1 - 2 \leq k - 2.
\]
Since
\[ n_3 + n_4 + \cdots + n_\Delta \leq n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta, \]
it follows that
\[ n \leq k + \frac{k(k + 1)}{2} + (k - 2) = \frac{k^2 + 5k - 4}{2}. \]

Suppose now that \( T \) is a tree of order \( n = \frac{k^2 + 5k - 4}{2} \) with \( \det(T) = k \). Then
\[ \frac{k^2 + 5k - 4}{2} = n_1 + n_2 + \cdots + n_\Delta \]
\[ \leq n_1 + n_2 + n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta \]
\[ \leq \frac{k^2 + 5k - 4}{2}. \]

This implies that \( n_i = 0 \) for \( i = 4, 5, \ldots, \Delta \). Moreover,
\[ n_1 = k, n_2 = \frac{k(k + 1)}{2} \text{ and } n_3 = k - 2. \]

In other words, \( T \) must contain exactly \( k \) end-vertices, exactly \( \frac{k(k + 1)}{2} \) vertices of degree 2, and exactly \( k - 2 \) vertices of degree 3.

First, we determine \( d_0(n) \) for \( n = \frac{k^2 + 5k - 4}{2} \) where \( k \geq 2 \) is an integer.

**Theorem 4.2** Let \( k \geq 2 \) be an integer. If \( n = \frac{k^2 + 5k - 4}{2} \), then \( d_0(n) = k \).

**Proof.** Let \( n = \frac{k^2 + 5k - 4}{2} \). First, we show that \( d_0(n) \geq k \). Assume, to the contrary, that there exists a tree \( T \) of order \( \frac{k^2 + 5k - 4}{2} \) such that \( \det(T) \leq k - 1 \). By Corollary 2.9, \( T \) has at most \( k - 1 \) end-vertices and at most \( \frac{k(k - 1)}{2} \) vertices of degree 2. Therefore, \( T \) contains at least
\[ \frac{k^2 + 5k - 4}{2} - (k - 1) - \frac{k(k - 1)}{2} = 2k - 1 \]
vertices of degree 3 or more. It then follows by (2.3) that \( T \) contains at least \( 2k + 1 \) end-vertices, which is impossible. Next, we show that \( d_0(n) \leq k \). To do so, we construct a tree \( T_k \) of order \( n = \frac{k^2+5k-4}{2} \) having detection number \( k \) such that \( T_k \) contains exactly \( k \) end-vertices, exactly \( \frac{k^2+k}{2} \) vertices of degree 2, and exactly \( k - 2 \) vertices of degree 3. We consider two cases, according to whether \( k \) is odd or \( k \) is even.

**Case 1: \( k \) is odd.**

Then \( k = 2\ell + 1 \) for some positive integer \( \ell \). For each integer \( i \) with \( 0 \leq i \leq k - 1 \), let

\[
Q_i : u_{i,1}, u_{i,2}, \ldots, u_{i,\ell+2}
\]

be a copy of the path \( P_{\ell+2} \) of order \( \ell + 2 \) and let

\[
Q : v_1, v_2, \ldots, v_{k-2}
\]

be a path of order \( k - 2 \). The tree \( T_k \) is obtained from the path \( Q \) and the paths \( Q_i \) (\( 0 \leq i \leq k - 1 \)) by adding the edges \( u_{0,1}v_1, u_{k-1,1}v_{k-2} \) and \( u_{i,1}v_i \) (\( 1 \leq i \leq k - 2 \)). Then the order of \( T_k \) is \( \frac{k^2+5k-4}{2} \). Since \( T_k \) has \( k \) end-vertices, \( \text{det}(T_k) \geq k \). It remains to show that \( \text{det}(T_k) \leq k \).

For each pair \( r, s \) of integers with \( 0 \leq r \leq k - 1 \) and \( 1 \leq s \leq \ell + 1 \), let \( e_{r,s} = u_{r,s}u_{r,s+1} \) be the edge joining the vertices \( u_{r,s} \) and \( u_{r,s+1} \); for each integer \( t \) with \( 1 \leq t \leq k - 2 \), let \( e_{t,0} = u_{t,1}v_t \) be the edge joining the vertices \( u_{t,1} \) and \( v_t \); and let \( e_{0,0} = u_{0,1}v_1 \) and \( e_{k-1,0} = u_{k-1,1}v_{k-2} \). Such a labeling is shown for \( T_5 \) (so \( k = 5 \) and hence \( \ell = 2 \)) in Figure 4.1.

Consider the complete graph \( K_{2\ell+1} \) with \( V(K_{2\ell+1}) = \{1, 2, \ldots, 2\ell + 1\} \). By Theorem 2.2, \( K_{2\ell+1} \) can be factored into \( \ell \) Hamiltonian cycles, say

\[
H_1, H_2, \ldots, H_\ell.
\]

For each integer \( i \) with \( 1 \leq i \leq \ell \), suppose that

\[
H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell+1}, 1
\]

where \( a_{i,j} \) (\( 1 \leq j \leq 2\ell + 1 \)) is the \( j \)-th vertex of \( H_i \). We may assume, without loss of generality, that
Observe that for each integer $i$ with $1 \leq i \leq \ell$, the factor $H_i$ induces a permutation $\sigma_i$ of $V(K_{2\ell+1}) = \{1, 2, \ldots, 2\ell+1\}$, where

$$\sigma_i = (a_{i,1} \ a_{i,2} \ \ldots \ a_{i,2\ell+1}).$$

In particular, $\sigma_1 = (1 \ 2 \ \ldots \ 2\ell+1)$ and, in general, $\sigma_i(a_{i,j}) = a_{i,j+1}$ if $1 \leq j \leq 2\ell$ and $\sigma_i(a_{i,2\ell+1}) = a_{i,1}$ for each integer $i$ with $1 \leq i \leq \ell$.

We now define a coloring $c : E(T_k) \rightarrow V(K_{2\ell+1})$ of the edges of $T_k$. For each pair $r, s$ of integers with $0 \leq r \leq k - 1$ and $0 \leq s \leq \ell + 1$, let

$$c(e_{r,s}) = \begin{cases} 
  k & \text{if } r = 0 \text{ and } s = 0, 1 \\
  r & \text{if } 1 \leq r \leq k - 1 \text{ and } s = 0, 1 \\
  \sigma_{s-1}(c(e_{r,s-1})) & \text{if } 0 \leq r \leq k - 1 \text{ and } 2 \leq s \leq \ell + 1 
\end{cases}$$

and let $c(u_iu_{i+1}) = 1$ for $1 \leq i \leq k - 3$. For example, if $k = 5$, then $K_5$ can be factored into the two Hamiltonian cycles

$$H_1 : 1, 2, 3, 4, 5, 1 \text{ and } H_2 : 1, 3, 5, 2, 4, 1,$$

where their corresponding permutations are

$$\sigma_1 = (1 \ 2 \ 3 \ 4 \ 5) \text{ and } \sigma_2 = (1 \ 3 \ 5 \ 2 \ 4),$$

respectively. The coloring $c$ is illustrated in Figure 4.2 for the tree $T_5$. 

![Figure 4.1: A labeling of $T_5$ in Case 1](image)
Figure 4.2: A detectable 5-coloring of $T_5$ in Case 1

Observe that:

(1) If $0 \leq r \neq t \leq k - 1$ and $0 \leq s \leq \ell + 1$, then $c(e_r,s) \neq c(e_t,s)$.

(2) If $e$ and $f$ are two different edges of $T_k$ each of which is incident to an end-vertex, then $c(e) \neq c(f)$. It follows that the end-vertices of $T_k$ have distinct color codes.

(3) The color codes of the vertices of degree 3 of $T_k$, namely, $v_1, v_2, \ldots, v_{k-2}$, are also distinct since $c(v_1) = (2, 0, \ldots, 0, 1), c(v_{k-2}) = (1, 0, \ldots, 0, 1, 1, 0)$ and for each integer $i$ with $2 \leq i \leq k - 3$, $c(v_i) = (2, 0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 occurs in the $i$th position.

(4) Every vertex of degree 2 of $T_k$ is incident with two edges having a unique pair of colors.

Therefore, $c$ is a detectable $k$-coloring of $T_k$ and so $\text{det}(T_k) \leq k$. It follows that $\text{det}(T_k) = k$.

Case 2: $k$ is even.

Then $k = 2\ell$ for some positive integer $\ell$. Let $T_2$ be the path $P_3 : v_1 v_2 v_3 v_4 v_5$. The edge coloring $c$ defined by $c(v_1 v_2) = c(v_2 v_3) = 1$ and $c(v_3 v_4) = c(v_4 v_5) = 2$ is a detectable 2-coloring of $T_2$. Thus we may assume that $k \geq 4$.

For each integer $i$ with $0 \leq i \leq \frac{k}{2} - 1$, let

$$Q_i : u_{i,1}, u_{i,2}, \ldots, u_{i,\ell+2}$$

be a copy of the path $P_{\ell+2}$ of order $\ell + 2$ and for each integer $i$ with $\frac{k}{2} \leq i \leq k - 1$, let
be a copy of the path $P_{\ell+1}$ of order $\ell + 1$. Also, let

$$Q : v_1, v_2, \ldots, v_{k-2}$$

be a path of order $k - 2$. The tree $T_k$ is obtained from the path $Q$ and the paths $Q_i$, $(0 \leq i \leq k - 1)$ by adding the edges $u_{0,1}v_1$, $u_{k-1,1}v_{k-2}$, and $u_{i,1}v_i$ ($1 \leq i \leq k - 2$). Then the order of $T_k$ is $\frac{k^2 + 5k - 4}{2}$. Since $T_k$ has $k$ end-vertices, $\det(T_k) \geq k$. It remains to show that $\det(T_k) \leq k$.

For each pair $r, s$ of integers with $0 \leq r \leq k - 1$ and $1 \leq s \leq \ell$, let $e_{r,s} = u_{r,s} u_{r,s+1}$ be the edge joining the vertices $u_{r,s}$ and $u_{r,s+1}$; for each integer $t$ with $1 \leq t \leq k - 2$, let $e_{t,0} = u_{t,1}v_t$ be the edge joining the vertices $u_{t,1}$ and $v_t$; let $e_{0,0} = u_{0,1}v_1$ and $e_{k-1,0} = u_{k-1,1}v_{k-2}$; and for each integer $t$ with $0 \leq t \leq \frac{k}{2} - 1$, let $e_{t,\ell+1} = u_{t,\ell+1} u_{t,\ell+2}$ be the edge joining the vertices $u_{t,\ell+1}$ and $u_{t,\ell+2}$. Such a labeling is shown for $T_8$ (so $k = 8$ and hence $\ell = 4$) in Figure 4.3.

Consider the complete graph $K_{2\ell}$ with $V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}$. By Theorem 2.1, $K_{2\ell}$ can be factored into $\ell - 1$ Hamiltonian cycles, say

$$H_1, H_2, \ldots, H_{\ell-1},$$

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and a 1-factor $F$. For each integer $i$ with $1 \leq i \leq \ell - 1$, suppose that

$$H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell}, 1,$$

where $a_{i,j}$ ($1 \leq j \leq 2\ell$) is the $j$th vertex of $H_i$. We may assume, without loss of generality, that

$$H_1 : 1, 2, \ldots, 2\ell - 1, 2\ell, 1.$$

Observe that for each integer $i$ with $1 \leq i \leq \ell - 1$, the factor $H_i$ induces a permutation $\sigma_i$ of $V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}$, where

$$\sigma_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell}).$$

In particular, $\sigma_1 = (1 \ 2 \ \ldots \ 2\ell)$ and in general $\sigma_i(a_{i,j}) = a_{i,j+1}$ if $1 \leq j \leq 2\ell - 1$ and $\sigma_i(a_{i,2\ell}) = a_{i,1}$ for each integer $i$ with $1 \leq i \leq \ell - 1$. On the other hand, the 1-factor $F$ induces the permutation $\alpha$ of $V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}$ given by

$$\alpha = (x_1 \ y_1)(x_2 \ y_2) \cdots (x_{\frac{k}{2}} \ y_{\frac{k}{2}}),$$

where $x_i$ and $y_i$ are neighbors in $F$ for $i = 1, 2, \ldots, \frac{k}{2}$.

We now define a coloring $c : E(T_k) \to V(K_{2\ell})$ of the edges of $T_k$. For integers $r$ and $s$ where $0 \leq r \leq k - 1$ and $0 \leq s \leq \ell + 1$, let

$$c(e_{r,s}) = \begin{cases} 
    y_{\frac{k}{2}} & \text{if } r = 0, k - 1 \text{ and } s = 0 \\
    y_r & \text{if } 1 \leq r \leq \frac{k}{2} - 1 \text{ and } s = 0 \\
    y_{r-\frac{k}{2}+1} & \text{if } \frac{k}{2} \leq r \leq k - 2 \text{ and } s = 0 \\
    x_{\frac{k}{2}} & \text{if } r = 0 \text{ and } s = 1 \\
    x_r & \text{if } 1 \leq r \leq \frac{k}{2} - 1 \text{ and } s = 1 \\
    \sigma_{s-2}(c(e_{r,s-1})) & \text{if } 0 \leq r \leq \frac{k}{2} - 1 \text{ and } 2 \leq s \leq \ell + 1 \\
    \sigma_{s-1}(c(e_{r,s-1})) & \text{if } \frac{k}{2} \leq r \leq k - 1 \text{ and } 1 \leq s \leq \ell.
\end{cases}$$

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where $\sigma_0$ is the identity permutation and

$$c(v_i v_{i+1}) = \begin{cases} x_i & \text{if } 1 \leq i \leq \frac{k}{2} - 1 \\ y_{i - \frac{k}{2} + 1} & \text{if } \frac{k}{2} \leq i \leq k - 3. \end{cases}$$

For example, if $k = 8$, then $K_8$ can be factored into the three Hamiltonian cycles

$$H_1 : 1, 2, 3, 4, 5, 6, 7, 8, 1, \quad H_2 : 1, 3, 5, 2, 7, 4, 8, 6, 1, \quad \text{and } H_3 : 1, 5, 7, 3, 8, 2, 6, 4, 1.$$  

where their corresponding permutations are

$$\sigma_1 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8), \quad \sigma_2 = (1 \ 3 \ 5 \ 2 \ 7 \ 4 \ 8 \ 6), \quad \text{and } \sigma_3 = (1 \ 5 \ 7 \ 3 \ 8 \ 2 \ 6 \ 4),$$

respectively, and a 1-factor $F$ with $E(F) = \{17, 24, 36, 58\}$. The permutation $\alpha$ corresponding to the 1-factor $F$ is then

$$\alpha = (1 \ 7)(2 \ 4)(3 \ 6)(5 \ 8).$$

The coloring $c$ is illustrated in Figure 4.4 for the tree $T_8$.

![Figure 4.4: A detectable 8-coloring of $T_8$ in Case 2](image)

Observe that:

(1) If $A_i = \{c(e_{0,i}), c(e_{1,i}), \ldots, c(e_{\frac{k}{2},i}), c(e_{\frac{k}{2}+1,i}), \ldots, c(e_{k,i})\}$ for $i = 1, 2, \ldots, k + 1$, then $A_i = V(K_{2\ell}) = \{1, 2, \ldots, 2\ell\}$. In particular,
\[ A_1 = \{ x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1} \}. \]

(2) The set \( A_{\ell+1} \) corresponds to the set of colors used for the edges incident to the end-vertices of \( T_k \). It follows that the color codes of the end-vertices are distinct.

(3) The color codes of the vertices of degree 3 of \( T_k \), namely, \( v_1, v_2, \ldots, v_{k-2} \), are also distinct since

(a) \( c(v_1) \) consists of exactly three 1's and \((2\ell - 3)\) 0's with two of these three 1's occurring in the \( y_{1\text{th}} \) and \( y_{2\text{th}} \) positions;

(b) \( c(v_1) \) consists of exactly three 1's and \((2\ell - 3)\) 0's with \( v_1 \) being the only vertex having 1 in both the \( x_{i-1\text{th}} \) and \( x_i \text{th} \) positions for \( 2 \leq i \leq \frac{k}{2} - 1 \);

(c) \( c(v_i) \) consists of one 1, one 2 and \((2\ell - 3)\) 0's with \( v_i \) being the only vertex having 2 in the \( y_{i-\frac{k}{2} + 1\text{th}} \) position for \( \frac{k}{2} \leq i \leq k - 3 \); and

(d) \( c(v_{k-2}) \) consists of exactly three 1's and \((2\ell - 3)\) 0's with two of these three 1's occurring in the \( y_{k-1\text{th}} \) and \( y_{k\text{th}} \) positions.

(4) Every vertex of degree 2 in \( T_k \) is incident with two edges having a unique pair of colors.

Therefore, \( c \) is a detectable \( k \)-coloring of \( T_k \) and so \( \operatorname{det}(T_k) \leq k \). Hence \( \operatorname{det}(T_k) = k \).  

If \( T \) is a tree of order \( n \) with \( \operatorname{det}(T) = 2 \), then \( n \leq 5 \) by Corollary 2.9 and (2.3). By Theorem 3.6, \( \operatorname{det}(P_n) = 2 \) for \( 3 \leq n \leq 5 \). Thus \( d_0(n) = 2 \) if \( 3 \leq n \leq 5 \). With the aid of Theorem 4.2, we now establish the following.

**Theorem 4.3** Let \( k \geq 3 \). If

\[
\frac{k^2 + 3k - 6}{2} \leq n \leq \frac{(k - 1)^2 + 5(k - 1) - 4}{2} + 1 \leq n \leq \frac{k^2 + 5k - 4}{2},
\]

then \( d_0(n) = k \).

**Proof.** Let \( n \) be an integer such that \( \frac{k^2 + 3k - 6}{2} \leq n \leq \frac{k^2 + 5k - 4}{2} \). First, we show that \( d_0(n) \geq k \). Assume, to the contrary, that there exists a tree \( T \) of order \( n \geq \frac{k^2 + 3k - 6}{2} \) such
that $\det(T) \leq k - 1$. By Corollary 2.9, $T$ has at most $k - 1$ end-vertices and at most $\frac{k^2 - k}{2}$ vertices of degree 2. Therefore, $T$ contains at least

$$n - (k - 1) - \frac{k^2 - k}{2} \geq \frac{k^2 + 3k - 6}{2} - (k - 1) - \frac{k^2 - k}{2} = k - 2$$

vertices of degree 3 or more. It then follows by (2.3) that $T$ contains at least $k$ end-vertices, which is impossible. Therefore, $d_0(n) \geq k$.

We next show that $d_0(n) \leq k$. Theorem 4.2 shows that this is true when $n = \frac{k^2 + 5k - 4}{2}$. Assume therefore that

$$n = \frac{k^2 + 5k - 4}{2} - p,$$

where $1 \leq p \leq k + 1$. We consider two cases, according to whether $k$ is odd or even.

**Case 1: $k$ is odd.**

Then $k = 2\ell + 1$ for some positive integer $\ell$. There are two subcases, according to whether $1 \leq p \leq k$ or $p = k + 1$.

**Subcase 1.1: $1 \leq p \leq k$.**

Construct the tree $T$ of order $\frac{k^2 + 5k - 4}{2} - p$ from the tree $T_k$ described in Theorem 4.2 by suppressing the vertices $u_{i,1}$ so that the edges $e_{i,0}$ and $e_{i,1}$ become the single edge $f_i$ where $0 \leq i \leq p - 1$. Define an edge coloring $c^* : E(T) \to V(K_{2\ell + 1})$ of $T$ by

$$c^*(e) = \begin{cases} c(e_{i,0}) & \text{if } e = f_i \text{ for some } i \text{ with } 0 \leq i \leq p - 1 \\ c(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of $T$ are all those of $T_k$ except those $(2\ell + 1)$-tuples for which 2 occurs in the $k$th coordinate and those for which 2 occurs in the $i$th coordinate for $1 \leq i \leq p - 1$. Thus $c^*$ is a detectable $k$-coloring of $T$. Since $T$ has $k$ end-vertices, it follows that $\det(T) = k$. Consequently, $d_0(n) = k$. 


Subcase 1.2: \( p = k + 1 \) and so \( n = \frac{k^2 + 3k - 6}{2} \).

Consider the tree \( T \) together with the edge coloring described in Subcase 1.1 when \( p = k \). We delete the edge \( v_1v_2 \), identify the vertices \( v_1 \) and \( v_2 \), and call this new vertex \( v \). This gives us a tree \( T' \) of order \( n = \frac{k^2 + 3k - 6}{2} \). Observe that the color codes of the vertices of \( T' \) are those of \( T \) except for those of \( v_1 \) and \( v_2 \) and that \( v \) is the only vertex of \( T' \) with degree greater than 3. Hence, we have a detectable \( k \)-coloring of \( T' \). Since \( T' \) has \( k \)-end-vertices, it follows that \( \det(T') = k \). Consequently, \( d_0(n) = k \).

Examples are shown in Figure 4.5 for \( k = 5, n = 20 \) (where \( 1 \leq p = 3 \leq k \)) and \( k = 5, n = 17 \) (where \( p = 6 = k + 1 \)).

![Figure 4.5: Detectable colorings in Case 1](image)

Case 2: \( k \) is even.

Then \( k = 2\ell \) for some positive integer \( \ell \geq 2 \). There are three subcases, according to whether

\[
1 \leq p \leq \frac{k}{2} - 1 \leq p \leq k, \text{ or } p = k + 1.
\]

Subcase 2.1: \( 1 \leq p \leq \frac{k}{2} \).

Construct the tree \( T \) of order \( \frac{k^2 + 5k - 4}{2} - p \) from the tree \( T_k \) described in Theorem 4.2 by suppressing the vertices \( u_{i,2} \) so that the edges \( e_{i,1} \) and \( e_{i,2} \) become the single edge \( f_i \).
where $0 \leq i \leq p - 1$. Define an edge coloring $c^* : E(T) \to V(K_{2\ell + 1})$ of $T$ by

$$c^*(e) = \begin{cases} c(e_{i,1}) & \text{if } e = f_i \text{ for some } i \text{ with } 0 \leq i \leq p - 1 \\ c(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of $T$ are all those of $T_k$ except the $(2\ell)$-tuples for which 2 occurs in the $x_{\frac{i}{2}}$-th coordinate and those for which 2 occurs in the $x_i$-th position for $1 \leq i \leq p - 1$. Thus $c^*$ is a detectable $k$-coloring of $T$. Since $T$ has $k$-end-vertices, it follows that $\text{det}(T) = k$. Consequently, $d_0(n) = k$.

Subcase 2.2: $\frac{k}{2} + 1 \leq p \leq k$.

Construct the tree $T'$ of order $\frac{k^2 + 3k - 6}{2} - p$ from the tree $T$ in Subcase 2.1 when $p = k$ by suppressing the vertices $u_{i,1}$ so that the edges $e_{i,0}$ and $e_{i,1}$ become the single edge $f_i$ where $\frac{k}{2} \leq i \leq p - 1$. Define an edge coloring $c' : E(T') \to V(K_{2\ell + 1})$ of $T'$ by

$$c'(e) = \begin{cases} c^*(e_{i,0}) & \text{if } e = f_i \text{ for some } i \text{ with } \frac{k}{2} \leq i \leq p - 1 \\ c^*(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of $T'$ are all those of $T$ except the $(2\ell)$-tuples for which 2 occurs in the $y_{i - \frac{k}{2} + 1}$-th coordinate for $\frac{k}{2} \leq i \leq p - 1$. Thus $c'$ is a detectable $k$-coloring of $T'$. Since $T'$ has $k$-end-vertices, it follows that $\text{det}(T') = k$. Consequently, $d_0(n) = k$.

Subcase 2.3: $p = k + 1$.

In this case, $n = \frac{k^2 + 3k - 6}{2}$. Consider the tree $T'$ together with the edge coloring described in Subcase 2.2 when $p = k$. We delete the edge $v_1v_2$, identify the vertices $v_1$ and $v_2$, and call this new vertex $v$. This gives us a tree $T''$ of order $n = \frac{k^2 + 3k - 6}{2}$. Observe that the color codes of the vertices of $T''$ are those of $T'$ except for the color codes of $v_1$ and $v_2$, and that $v$ is the only vertex of $T''$ with degree greater than 3. Hence, we have a detectable $k$-coloring of $T''$. Since $T''$ has $k$-end-vertices, it follows that $\text{det}(T'') = k$. Consequently, $d_0(n) = k$.

Examples are shown in Figure 4.6 for (a) $k = 8, n = 47$ (where $1 \leq p = 3 \leq \frac{k}{2}$), (b) $k = 8, n = 44$ (where $\frac{k}{2} + 1 \leq p = 6 \leq k$), and (c) $k = 8, n = 41$ (where $p = 9 = k + 1$).
This completes the proof of Theorem 4.3.

Figure 4.6: Detectable colorings in Case 2 of Theorem 4.3

By solving for the smallest positive integer $n \geq 3$ for which $n \leq \frac{k^2 + 5k - 1}{2}$, we obtain the following result.
Proposition 4.4  For each integer \( n \geq 3 \),

\[
d_0(n) = \left\lfloor \frac{-5 + \sqrt{8n + 41}}{2} \right\rfloor.
\]

By Proposition 4.4, \( d_0(n) \approx \sqrt{2n} \). We are now able to determine all pairs \( k, n \) of integers for which there exists a tree \( T \) of order \( n \) having detection number \( k \).

Theorem 4.5  Let \( k \geq 2 \) and \( n \geq 3 \) be integers. Then there exists a tree \( T \) of order \( n \) such that \( \det(T) = k \) if and only if

\[
d_0(n) = \left\lfloor \frac{-5 + \sqrt{8n + 41}}{2} \right\rfloor \leq k \leq n - 1 = D_0(n).
\] (4.1)

Proof.  We have seen that if \( T \) is a tree of order \( n \) such that \( \det(T) = k \), then (4.1) holds. It remains to verify the converse. For each integer \( i \) with \( i = 0, 1, \ldots, n - d_0(n) - 1 \), we construct a tree \( G_i \) such that \( G_i \) has order \( n \) and \( \det(G_i) = d_0(n) + i \).

Let \( G_0 \) be the tree of order \( n \) described in the proof of Theorem 4.3 together with the detectable \( d_0(n) \)-coloring, which we denote by \( c_0 \) here. Then \( \det(G_0) = d_0(n) \) and \( G_0 \) has exactly \( d_0(n) \) end-vertices. Let \( w_0 \) be a vertex of maximum degree in \( G_0 \) (there may be more than one choice for \( w_0 \)). Since \( \det(G_0) \neq n - 1 \) (as \( G_0 \neq K_{1,n-1} \)), it follows that there exists a vertex \( x_0 \) in \( G_0 \) adjacent to \( w_0 \) such that \( \deg_{G_0} x_0 \neq 1 \). Construct \( G_1 \) by deleting the edge \( w_0x_0 \), identifying the vertices \( w_0 \) and \( x_0 \), naming the new vertex \( w_1 \), introducing a new vertex \( y_1 \), and joining \( y_1 \) to \( w_1 \). We note that \( G_1 \) has order \( n \) and has \( d_0(n) + 1 \) end-vertices. Thus, \( \det(G_1) \geq d_0(n) + 1 \).

We now show that \( \det(G_1) \leq d_0(n) + 1 \) by giving a detectable \( (d_0(n) + 1) \)-coloring of \( G_1 \). Define \( c_1 : E(G_1) \to \{1, 2, \ldots, d_0(n) + 1\} \) by

\[
c_1(e) = \begin{cases} c_0(e) & \text{if } e \in E(G_0) \\ d_0(n) + 1 & \text{if } e = w_1y_1. \end{cases}
\]

Then \( c_1 \) is detectable \( (d_0(n) + 1) \)-coloring of \( G_1 \). This implies that \( \det(G_1) = d_0(n) + 1 \).
In general, we construct $G_{i+1}$ from $G_i$ and obtain the edge coloring $c_{i+1}$ from $c_i$, $0 \leq i \leq n - d_0(n) - 2$, as follows:

1. Let $w_i$ be a vertex of maximum degree in $G_i$.
2. Let $x_i$ be a vertex in $G_i$ that is adjacent to $w_i$ such that $\deg_{G_i} x_i \neq 1$.
3. Construct $G_{i+1}$ by deleting the edge $w_i x_i$, identifying the vertices $w_i$ and $x_i$, naming the new vertex $w_{i+1}$, introducing a new vertex $y_{i+1}$, and joining $y_{i+1}$ to $w_{i+1}$.
4. Define $c_{i+1} : E(G_{i+1}) \to \{1, 2, \ldots, d_0(n) + i + 1\}$ by

$$c_{i+1}(e) = \begin{cases} c_i(e) & \text{if } e \in E(G_i) \\ d_0(n) + i + 1 & \text{if } e = w_{i+1} y_{i+1}. \end{cases}$$

Observe that for $i = 0, 1, \ldots, n - d_0(n) - 1$:

1. $G_i$ is a tree of order $n$ with $d_0(n) + i$ end-vertices;
2. $c_i$ is a detectable $(d_0(n) + i)$-coloring of $G_i$;
3. (1) and (2) imply that $\det(G_i) = d_0(n) + i$.

An example showing the steps in the construction of the trees $G_i$ is given in Figure 4.7 for $n = 7$. This completes the proof.

### 4.3 Unicyclic Graphs

A connected graph with exactly one cycle is called a unicyclic graph. A graph $G$ of order $n$ and size $m$ is unicyclic if and only if $G$ is connected and $m = n$. In this section, we study some extremal problems concerning detection numbers of unicyclic graphs, in particular, the problems of determining how large and how small the detection number of a unicyclic graph of a fixed order can be, that is, we determine $D_1(n)$ and $d_1(n)$ for $n \geq 3$. Recall (see (2.4)) that if $n_i$ is the number of vertices of degree $i$ in a unicyclic graph $G$ with maximum degree $\Delta$, then
Figure 4.7: The graphs $G_i$ for $n = 7$ in Theorem 4.5

$$n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta.$$  

Figure 4.8 shows all the unicyclic graphs of order $n$ for $3 \leq n \leq 5$ together with a minimum detectable coloring for each. Hence $D_1(3) = d_1(3) = 3$, and $D_1(n) = 3$ and $d_1(n) = 2$ for $n = 4, 5$. Thus, we may assume that $n \geq 6$. We first determine $D_1(n)$ for $n \geq 6$. In order to do this, we present the following result which is an immediate consequence of Proposition 2.10.

Corollary 4.6  Let $G$ be a connected graph of order $n \geq 3$ and size $m \geq n$. If $g$ is the girth of $G$, then

$$\det(G) \leq m - g + \det(C_g).$$
Proposition 4.7  
For \( n \geq 6 \), \( D_1(n) = n - 3 \).

Proof.  By Proposition 2.23, \( \det(K_{1,n-1} + e) = n - 3 \) for \( n \geq 6 \) and so \( D_1(n) \geq n - 3 \) for \( n \geq 6 \). It remains to show that \( D_1(n) \leq n - 3 \) for \( n \geq 6 \). Let \( G \) be a unicyclic graph of order \( n \geq 6 \) and let \( g \) be the girth of \( G \). If \( 3 \leq g \leq 5 \), then \( G \) contains a subgraph \( F \) such that \( F \in \{ F_1, F_2, F_3, F_4, F_5 \} \), where \( F_i \ (1 \leq i \leq 4) \) is shown in Figure 4.8 and \( F_5 \) is the graph shown in Figure 4.9.

![Figure 4.9: The graph \( F_5 \) in the proof of Proposition 4.7](image)

Observe that \( \det(F) = m(F) - 3 \) for each \( F \in \{ F_1, F_2, F_3, F_4, F_5 \} \). It then follows by Proposition 2.10 that

\[
\det(G) \leq m(G) + \det(F) - m(F) = n + (m(F) - 3) - m(F) = n - 3 \quad \text{for} \quad 3 \leq g \leq 5.
\]

If \( g \geq 6 \), then \( \det(C_g) \leq g - 3 \) by Theorem 3.3. It then follows by Corollary 4.6 that

\[
\det(G) \leq n - g + \det(C_g) \leq n - g + (g - 3) = n - 3.
\]

Thus, \( D_1(n) \leq n - 3 \) for all \( n \geq 6 \).

Next, we determine the minimum detection number among all unicyclic graphs of order \( n \). According to Corollary 2.9, every unicyclic graph of order \( n \geq 3 \) having detection number \( k \) contains at most \( k \) end-vertices and at most \( \frac{k(k+1)}{2} \) vertices of degree 2. It then follows by (2.4) that

\[
n \leq k + \frac{k(k+1)}{2} + k = \frac{k^2 + 5k}{2}.
\]

Furthermore, if \( G \) is a unicyclic graph of order \( n = \frac{k^2 + 5k}{2} \) with \( \det(G) = k \), then \( G \) must contain exactly \( k \) end-vertices, exactly \( \frac{k(k+1)}{2} \) vertices of degree 2, and exactly \( k \) vertices of degree 3. We first determine \( d_1(n) \) for values of \( n = \frac{k^2 + 5k}{2} \), where \( k \geq 2 \) is an integer.
Theorem 4.8 \textit{Let} \( k \geq 2 \) \textit{be an integer. If} \( n = \frac{k^2 + 5k}{2} \), \textit{then} \( d_1(n) = k \).}

\textbf{Proof.} First, \textit{we show that if} \( n = \frac{k^2 + 5k}{2} \), \textit{then} \( d_1(n) \geq k \). Assume, \textit{to the contrary, that} \textit{there exists a unicyclic graph} \( G \) \textit{of order} \( \frac{k^2 + 5k}{2} \) \textit{such that} \( \det(G) \leq k - 1 \). \textit{By Corollary 2.9}, \( G \) \textit{has at most} \( k - 1 \) \textit{end-vertices and at most} \( \frac{k(k-1)}{2} \) \textit{vertices of degree 2}. Therefore, \( G \) \textit{contains at least}

\[
\frac{k^2 + 5k}{2} - (k - 1) - \frac{k(k-1)}{2} = 2k + 1
\]

\textit{vertices of degree 3 or more. It then follows by (2.4) that} \( G \) \textit{contains at least} \( 2k + 1 \) \textit{end-vertices, which is impossible. Thus,} \( d_1(n) \geq k \).

\text{To show that} \( d_1(n) \leq k \), \text{we construct a unicyclic graph} \( G_k \) \text{of order} \( n = \frac{k^2 + 5k}{2} \) \text{having detection number} \( k \) \text{such that} \( G_k \) \text{has exactly} \( k \) \text{end-vertices, exactly} \( \frac{k^2 + k}{2} \) \text{vertices of degree 2, and exactly} \( k \) \text{vertices of degree 3}. \text{We consider two cases, according as to whether} \( k \) \text{is odd or even.}

\textit{Case 1:} \( k \) \text{is odd.}

\text{Then} \( k = 2\ell - 1 \) \text{for some integer} \( \ell \geq 2 \). \text{We now construct} \( G_k \). \text{Let}

\[
C_{2\ell^2 - \ell} : v_1, v_2, \ldots, v_{2\ell^2 - \ell}, v_1
\]

\text{be a cycle of length} \( 2\ell^2 - \ell \) \text{and for} \( 1 \leq i \leq k \), \text{let} \( Q_i \) \text{be a copy of} \( K_2 \) \text{with} \( V(Q_i) = \{u_{i,1}, u_{i,2}\} \). \text{Then the graph} \( G_k \) \text{is obtained from} \( C_{2\ell^2 - \ell} \) \text{and the} \( k \) \text{graphs} \( Q_i \) \text{by adding the edges} \( v_{2i}u_{i,1} \) \text{for} \( 1 \leq i \leq k \). \text{Observe that} \( G_k \) \text{is a unicyclic graph of order}

\[
n = (2\ell^2 - \ell) + 2(2\ell - 1) = \frac{k^2 + 5k}{2}.
\]

\text{We now define a} \( k \)-\text{coloring} \( c \) \text{of the edges of} \( G_k \). \text{First, we color the} \( 2\ell^2 - \ell \) \text{edges of} \( C_{2\ell^2 - \ell} \) \text{with the elements of} \( V(K_{2\ell - 1}) = \{1, 2, \ldots, 2\ell - 2, 2\ell - 1\} \) \text{as follows. Let} \( H_1, H_2, \ldots, H_{\ell - 1} \) \text{be} \( \ell - 1 \) \text{pairwise edge-disjoint Hamiltonian cycles of} \( K_{2\ell - 1} \). \text{For each integer} \( i \) \text{with} \( 1 \leq i \leq \ell - 1 \), \text{suppose that} \( H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell - 1}, 1 \), \text{where} \( a_{i,j} \) \text{is the} \( j \)\text{th vertex of} \( H_i \) \text{and we assume that} \( H_1 : 1, 2, \ldots, 2\ell - 1, 1 \). \text{Therefore,} \( a_{1,j} = j \) \text{for} \( 1 \leq j \leq 2\ell - 1 \). \text{Suppose that the edges of} \( C_{2\ell^2 - \ell} \) \text{are encountered in the order}

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as we proceed about the cycle in some direction. We then define

\[
c(e_k) = \begin{cases} 
    a_1, \left[\frac{k}{2}\right] & \text{if } 1 \leq k \leq 4\ell - 2 \\
    a_i, j & \text{if } k = i(2\ell - 1) + j, 2 \leq i \leq \ell - 1, \text{ and } 1 \leq j \leq 2\ell - 1.
\end{cases}
\]

It was shown in the proof of Theorem 3.3 that this coloring of the cycle $C_{2\ell^2 - \ell}$ is detectable. Furthermore, let

\[c(v_{2i, u_{i,1}}) = c(u_{i,1}u_{i,2}) = i\]

for $1 \leq i \leq k$. Thus $c$ uses $k$ colors. It remains to show that $c$ is detectable. Note that the color codes of the vertices of $G_k$ consist of all possible color codes for vertices of degrees 1 and 2 together with all the $k$-tuples whose only nonzero entry is 3 occurring in the $i$th coordinate for $1 \leq i \leq k$. Since each of the color codes described above occurs exactly once, $c$ is a detectable $k$-coloring for $G_k$. Therefore, $\det(G_k) \leq k$ and consequently $\det(G_k) = k$.

**Case 2 : $k$ is even.**

Then $k = 2\ell$ for some positive integer $\ell$. If $k = 2$, then $n = 7$. Since the unicyclic graph $G_2$ of order 7 in Figure 4.10 has detection number 2 (as shown in that figure), the result holds for $k = 2$. Thus we may assume that $k \geq 4$ and so $\ell \geq 2$.

![Figure 4.10: A detectable 2-coloring of $G_2$ in Case 2](image)

Let $C_{2\ell^2} : v_1, v_2, \ldots, v_{2\ell^2}, v_1$ be a cycle of length of $2\ell^2$. For $1 \leq i \leq \ell$, let $Q_i$ be a copy of $K_2$ with $V(Q_i) = \{u_{i,1}, u_{i,2}\}$ and for $\ell + 1 \leq i \leq 2\ell$, let $Q_i : u_{i,1}, u_{i,2}, u_{i,3}$ be a copy of a path of length 2. Then the graph $G_k$ is obtained from $C_{2\ell^2}$ and the paths $Q_i$ $(1 \leq i \leq k)$ by adding the edges $v_{2i, u_{i,1}}$ ($1 \leq i \leq \ell$). Observe that $G_k$ is a unicyclic graph of order

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We now define a $k$-coloring $c$ of the edges of $G_k$. First, we color the $2\ell^2$ edges of the cycle $C_{2\ell^2}$ with the elements of $V(K_{2\ell^2}) = \{1, 2, \ldots, 2\ell\}$ as follows. Let $H_1, H_2, \ldots, H_{\ell-1}$ be $\ell - 1$ pairwise edge-disjoint Hamiltonian cycles of $K_{2\ell}$ and let $F$ be the 1-factor of $K_{2\ell}$ with $E(F) = \{x_iy_i : 1 \leq i \leq \ell\}$, where $x_\ell = 2\ell = k$. For each integer $i$ with $1 \leq i \leq \ell - 1$, suppose that $H_i : 1 = a_{i,1}, a_{i,2}, \ldots, a_{i,2\ell}$, where $a_{i,j}$ ($1 \leq j \leq 2\ell$) is the $j$th vertex of $H_i$ and $H_1 : 1, 2, \ldots, 2\ell, 1$. Therefore, $a_{1,j} = j$ for $1 \leq j \leq 2\ell$. Suppose that the edges of $C_{2\ell^2}$ are encountered in the order

$$e_1, e_2, \ldots, e_{2\ell^2}, e_{2\ell^2+1} = e_1,$$

as we proceed about the cycle in some direction. For each integer $k$ with $1 \leq k \leq 2\ell^2$, either $1 \leq k \leq 4\ell$ or $k = i(2\ell) + j$ for some integers $i$ and $j$ with $2 \leq i \leq \ell - 1$ and $1 \leq j \leq 2\ell$. We now define

$$c(e_k) = \begin{cases} a_{1, \lceil k/2 \rceil} = \lceil k/2 \rceil & \text{if } 1 \leq k \leq 4\ell \\ a_{i,j} & \text{if } k = i(2\ell) + j, 2 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq 2\ell. \end{cases}$$

This coloring of $C_{2\ell^2}$ is detectable, which was shown in the proof of Theorem 3.3. Furthermore, for $1 \leq i \leq \ell$, let

$$c(v_{2i}u_{i,1}) = c(u_{i,1}u_{i,2}) = x_i$$

and for $\ell + 1 \leq i \leq 2\ell$, let

$$c(v_{2i}u_{i,1}) = x_{i-\ell} \text{ and } c(u_{i,1}u_{i,2}) = c(u_{i,2}u_{i,3}) = y_{i-\ell}.$$

Thus $c$ uses $k$ colors. It remains to show that $c$ is detectable. Note that in the coloring $c$ all possible color codes for the vertices of degree 1 or 2 are used exactly once. The vertices of degree 3, namely $v_{2i}$ ($i = 1, 2, \ldots, \ell$), also have distinct color codes since $v_{2i}$ is the only vertex whose code has an entry that is at least 2 in the $i$th position. Therefore, $\det(G_k) \leq k$ and consequently, $\det(G_k) = k$. 

With the aid of Theorem 4.8, we are now able to establish the following.
Theorem 4.9  For each integer $k \geq 2$, if $n$ is an integer such that

$$\frac{(k-1)^2 + 5(k-1)}{2} + 1 \leq n \leq \frac{k^2 + 5k}{2},$$

then $d_1(n) = k$.

Proof. First, we show that if

$$\frac{k^2 + 3k - 2}{2} = \frac{(k-1)^2 + 5(k-1)}{2} + 1 \leq n \leq \frac{k^2 + 5k}{2},$$

then $d_1(n) \geq k$. Assume, to the contrary, that there exists a unicyclic graph $G$ of order $n \geq \frac{k^2 + 3k - 2}{2}$ such that $\det(G) \leq k - 1$. By Corollary 2.9, $G$ has at most $k - 1$ end-vertices and at most $\frac{k^2 - k}{2}$ vertices of degree 2. Therefore, $G$ contains at least

$$n - (k - 1) - \frac{k^2 - k}{2} \geq \frac{k^2 + 3k - 2}{2} - (k - 1) - \frac{k^2 - k}{2} = k$$

vertices of degree 3 or more. It then follows by (2.4) that $G$ has at least $k$ end-vertices, which is impossible.

We next show that $d_1(n) \leq k$ if $\frac{k^2 + 3k - 2}{2} \leq n \leq \frac{k^2 + 5k}{2}$. Theorem 4.8 shows that this is true when $n = \frac{k^2 + 5k}{2}$. Assume therefore that $n = \frac{k^2 + 5k}{2} - p$ where $1 \leq p \leq k + 1$. We consider two cases, according to whether $k$ is odd or $k$ is even.

Case 1: $k$ is odd.

Then $k = 2\ell - 1$ for some integer $\ell \geq 2$. There are two subcases, depending on whether $1 \leq p \leq k$ or $p = k + 1$.

Subcase 1.1: $1 \leq p \leq k$.

Construct the unicyclic graph $G$ of order $\frac{k^2 + 5k}{2} - p$ from the unicyclic graph $G_k$ described in Theorem 4.8 by suppressing the vertices $u_i,1$ so that the edges $v_{2i,1}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,1}$, where $1 \leq i \leq p$. Define a $k$-coloring $c^*$ of $G$ by

$$c^*(e) = \begin{cases} c(v_{2i}u_{i,1}) & \text{if } e = v_{2i}u_{i,1}, \\ c(e) & \text{otherwise.} \end{cases}$$
The color codes for the vertices of $G$ are all those of $G_k$ except the $(2\ell - 1)$-tuples for which $2$ occurs in the $i$th coordinate (and $0$ occurs everywhere else) for $1 \leq i \leq p$. Thus $c^*$ is a detectable $k$-coloring of $G$. Since $G$ has $k$ end-vertices, it follows that $\det(G) = k$. Consequently, $d_1(n) \leq k$.

Subcase 1.2: $p = k + 1$ and so $n = \frac{k^2 + 3k - 2}{2}$.

Consider the unicyclic graph $G$ of order $\frac{k^2 + 3k - 2}{2} + 1 = \frac{k^2 + 3k}{2}$ together with the edge coloring described in Subcase 1.1 (that is, when $p = k$ in Subcase 1.1). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices $v_{2k}$ and $v_{2k-1}$, and label this new vertex by $v$. This gives us a unicyclic graph $G'$ of order $n = \frac{k^2 + 3k - 2}{2}$. Observe that the color codes of the vertices of $G'$ are those of $G$ except for the color codes of $v_{2k}$ and $v_{2k-1}$, and that $v$ is the only vertex of $G'$ of degree 3 whose color code has 2 as the $k$th coordinate. Hence, we have a detectable $k$-coloring of $G'$. Since $G'$ has $k$ end-vertices, it follows that $\det(G') \geq k$ and consequently $\det(G') = k$. Therefore, $d_1(n) \leq k$.

Case 2: $k$ is even.

Then $k = 2\ell$ for some positive integer $\ell$. The result holds for $k = 2$ (that is, $\ell = 1$) as the graphs in Figure 4.11 show. For $k \geq 4$ (and so $\ell \geq 2$), we consider three subcases, according to whether

\[1 \leq p \leq \frac{k}{2}, \quad \frac{k}{2} + 1 \leq p \leq k, \text{ or } p = k + 1.\]

Figure 4.11: Detectable colorings when $k = 2$ in Case 2

Subcase 2.1: $1 \leq p \leq \frac{k}{2}$.

Construct the unicyclic graph $G$ of order $\frac{k^2 + 5k}{2} - p$ from the unicyclic graph $G_4$ described in Theorem 4.8 by suppressing the vertices $u_{6,1}$ so that the edges $v_1u_{6,1}$ and
$u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$ where $1 \leq i \leq p$. Define a $k$-coloring $c^*$ of $G$ by

$$c^*(e) = \begin{cases} c(v_{2i}u_{i,1}) & \text{if } e = v_{2i}u_{i,2} \text{ for some } i \text{ with } 1 \leq i \leq p \\ c(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of $G$ are all those of $G_k$ except the $(2\ell)$-tuples for which 2 occurs in the $x_i$th coordinate (and 0 occurs everywhere else) for $1 \leq i \leq p$. Thus $c^*$ is a detectable $k$-coloring of $G$. Since $G$ has $k$ end-vertices, it follows that $\det(G) \leq k$. Consequently, $d_1(n) \leq k$.

Subcase 2.2 : $\frac{k}{2} + 1 \leq p \leq k$.

Construct the unicyclic graph $G'$ of order $\frac{k^2+5k}{2} - p$ from the unicyclic graph $G$ of order $\frac{k^2+5k}{2} - \frac{k}{2} = \frac{k^2+4k}{2}$ described in Subcase 2.1 (that is, when $p = \frac{k}{2}$ in Subcase 2.1) by suppressing the vertices $u_{i,2}$ so that the edges $u_{i,1}u_{i,2}$ and $u_{i,2}u_{i,3}$ become the single edge $u_{i,1}u_{i,3}$ where $\frac{k}{2} + 1 \leq i \leq p$. Define a $k$-coloring $c'$ of $G'$ by

$$c'(e) = \begin{cases} c^*(u_{i,2}u_{i,3}) & \text{if } e = u_{i,1}u_{i,3} \text{ for some } i \text{ with } \frac{k}{2} + 1 \leq i \leq p \\ c^*(e) & \text{otherwise.} \end{cases}$$

The color codes for the vertices of $G'$ are all those of $G$ except for the color codes of $v_{2k}$ and $v_{2k-1}$, and that $v$ is the only vertex of $G''$ of degree 3 whose color code has 2 as the $k$th coordinate. Hence, we have a detectable $k$-coloring of $G''$. Since $G''$ has $k$ end-vertices, it follows that $\det(G'') = k$. Consequently, $d_1(n) \leq k$.

Subcase 2.3 : $p = k + 1$.

In this case, $n = \frac{k^2+3k-2}{2}$. Consider the unicyclic graph $G'$ of order $\frac{k^2+3k-2}{2} + 1 = \frac{k^2+3k}{2}$ together with the edge coloring described in Subcase 2.2 (that is, when $p = k$ in Subcase 2.2). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices $v_{2k}$ and $v_{2k-1}$, and label this new vertex by $v$. This gives us a unicyclic graph $G''$ of order $n = \frac{k^2+3k-2}{2}$. Observe that the color codes of the vertices of $G''$ are those of $G'$ except for the color codes of $v_{2k}$ and $v_{2k-1}$, and that $v$ is the only vertex of $G''$ of degree 3 whose color code has 2 as the $k$th coordinate. Hence, we have a detectable $k$-coloring of $G''$. Since $G''$ has $k$ end-vertices, it follows that $\det(G'') = k$. Consequently, $d_1(n) \leq k$. ■
Solving for the smallest integer \( k \) for which \( n \leq \frac{k^2+5k}{2} \), we obtain the following.

**Theorem 4.10** For each integer \( n \geq 4 \),

\[
d_1(n) = \left\lceil \frac{-5 + \sqrt{8n + 25}}{2} \right\rceil.
\]

By Theorem 4.10, \( d_1(n) \approx \sqrt{2n} \). We now determine all pairs \( k, n \) of integers for which there exists a unicyclic graph of order \( n \) having detection number \( k \).

**Theorem 4.11** Let \( k \geq 2 \) and \( n \geq 3 \) be integers. There exists a unicyclic graph \( G \) of order \( n \) such that \( \det(G) = k \) if and only if \( d_1(n) \leq k \leq D_1(n) \).

**Proof.** By definition, if \( G \) is a unicyclic graph of order \( n \) such that \( \det(G) = k \), then \( d_1(n) \leq k \leq D_1(n) \). It remains to verify the converse. The result holds for \( 3 \leq n \leq 5 \) as the graphs in Figure 4.8 show. Furthermore, the graphs in Figure 4.12 show that the result holds for \( n = 6, 7 \) as well.

![Figure 4.12: Minimum detectable colorings for graphs of order \( n = 6, 7 \)](image)

We now assume that \( n \geq 8 \) and so \( k \geq 3 \). In this case, we show that if

\[
d_1(n) = \left\lceil \frac{-5 + \sqrt{8n + 25}}{2} \right\rceil \leq k \leq n - 3 = D_1(n),
\]

then there is a unicyclic graph \( G \) of order \( n \) such that \( \det(G) = k \). For \( i = 0, 1, \ldots, n - d_1(n) - 3 \), we construct a unicyclic graph \( H_i \) such that \( H_i \) has order \( n \) and
Let \( H \) be the unicyclic graph of order \( n \) described in the proof of Theorem 4.9 and \( c \) the \( d_1(n) \)-coloring described in the proof of Theorem 4.9 as well. We first construct a unicyclic graph \( H_0 \) from \( H \) as follows.

1. If vertex \( u_{1,1} \in V(H) \), then delete the vertex \( u_{1,2} \); while if \( u_{1,1} \not\in V(H) \), then delete the vertex \( v_3 \) and join the vertices \( v_2 \) and \( v_4 \).

2. Delete the edge \( v_1v_2 \), add the vertex \( v \), and join \( v \) to \( v_1 \) and \( v_2 \).

Then \( H_0 \) has exactly \( d_1(n) \) end-vertices and so \( \det(H_0) \geq d_1(n) \). Define the coloring \( c_0 : E(H_0) \to \{1, 2, \ldots, d_1(n)\} \) by

\[
c_0(e) = \begin{cases} 
    c(e) & \text{if } e \in E(H) \\
    1 & \text{if } e \not\in E(H).
\end{cases}
\]

Then \( c_0 \) is a detectable \( d_1(n) \)-coloring of \( H_0 \). Thus \( \det(H_0) \leq d_1(n) \) and so \( \det(H_0) = d_1(n) \).

Observe that if \( \ell = \left\lceil \frac{d_1(n)}{2} \right\rceil \), then the girth of \( H_0 \) is \( 2\ell^2 - \ell \), \( 2\ell^2 \), \( 2\ell^2 - \ell + 1 \), or \( 2\ell^2 + 1 \), depending on (a) the parity of \( d_1(n) \) and (b) whether the vertex \( u_{1,1} \) belongs to \( H \). In each case, we denote the girth of \( H_0 \) by \( g(\ell) \). Then \( g(\ell) > 3 \) and so \( H_0 \neq K_{1,n-1} + e \).

Note that the vertices \( v_1, v, \) and \( v_2 \), in this order, are consecutive vertices in the cycle of \( H_0 \). For the purpose of notation, we relabel the vertex \( v_2 \) as \( w_0 \). Since \( H_0 \neq K_{1,n-1} + e \) (as \( g(\ell) > 3 \)), it follows that there exists a vertex \( x_0 \) in \( H_0 \) adjacent to \( w_0 \) such that \( \deg_{H_0} x_0 \neq 1 \) and \( x_0 \not\in \{v, v_1\} \).

We now construct a unicyclic graph \( H_1 \) from \( H_0 \) by deleting the edge \( w_0x_0 \), identifying the vertices \( w_0 \) and \( x_0 \), labeling the new vertex by \( w_1 \), introducing a new vertex \( y_1 \), and joining \( y_1 \) to \( w_1 \). We note that \( H_1 \) has order \( n \) and has \( d_1(n) + 1 \) end-vertices. Thus, \( \det(H_1) \geq d_1(n) + 1 \). To show that \( \det(H_1) \leq d_1(n) + 1 \), we provide a detectable \((d_1(n) + 1)\)-coloring of \( H_1 \). Define \( c_1 : E(H_1) \to \{1, 2, \ldots, d_1(n) + 1\} \) by

\[
c_1(e) = \begin{cases} 
    c_0(e) & \text{if } e \in E(H_0) \\
    d_1(n) + 1 & \text{if } e = w_1y_1.
\end{cases}
\]
Then $c_1$ is detectable $(d_1(n) + 1)$-coloring of $H_1$. This implies that $\det(H_1) = d_1(n) + 1$.

In general, we construct $H_{i+1}$ from $H_i$ and obtain the edge coloring $c_{i+1}$ from $c_i$, where $0 \leq i \leq n - d_1(n) - 4$, as follows.

1. Let $x_i$ be a vertex in $H_i$ that is adjacent to $w_i$ such that $\deg_{H_i} x_i \neq 1$ and $x_i \notin \{v, v_1\}$.

2. Construct $H_{i+1}$ by deleting the edge $w_i x_i$, identifying the vertices $w_i$ and $x_i$, labeling the new vertex by $w_{i+1}$, introducing a new vertex $y_{i+1}$, and joining $y_{i+1}$ to $w_{i+1}$.

3. Define $c_{i+1} : E(H_{i+1}) \rightarrow \{1, 2, \ldots, d_1(n) + i + 1\}$ by

   $$c_{i+1}(e) = \begin{cases} c_i(e) & \text{if } e \in E(H_i) \\ d_1(n) + i + 1 & \text{if } e = w_{i+1} y_{i+1}. \end{cases}$$

Observe that for $i = 0, 1, \ldots, n - d_1(n) - 3$:

1. $H_i$ is a unicyclic graph of order $n$ with $d_1(n) + i$ end-vertices;

2. $c_i$ is a detectable $(d_1(n) + i)$-coloring of $H_i$;

3. parts (1) and (2) imply that $\det(H_i) = d_1(n) + i$.

Figure 4.13 illustrates how to construct the unicyclic graphs $H_i$ $(0 \leq i \leq 6)$ for $n = 12$. In this case, $d_1(12) = 3$, $D_1(12) = 9$, and $\det(H_i) = d_1(12) + i = 3 + i$ for $0 \leq i \leq 6$.

4.4 Connected Graphs With Cycle Rank 2

Let us now consider connected graphs with cycle rank 2. We have seen that if $G$ is a connected graph of order $n$ having cycle rank 2, then $n \geq 4$. Observe that $G_{2,4} = \{K_4 - e\}$. Since $\det(K_4 - e) = 2$, it follows that $D_2(4) = d_2(4) = 2$. All connected graphs of order 5 or 6 with cycle rank 2 are shown in Figure 4.14 together with a minimum detectable coloring for each. Therefore, $D_2(5) = D_2(6) = 3$ and $d_2(5) = d_2(6) = 2$. 

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We now determine $D_2(n)$ for $n \geq 7$. Unlike the cases for trees and unicyclic graphs, it is more challenging to establish a formula for $D_2(n)$. Observe that the graph $G \in \mathcal{G}_{2,n}$ shown in Figure 4.15 has order $n$ and detection number $n - 4$ for $n \geq 7$. Thus $D_2(n) \geq n - 4$ for $n \geq 7$. We show, in fact, that $D_2(n) = n - 4$ for $n \geq 7$. In order to verify this, we first establish several preliminary results.

We begin by considering some special classes of graphs in $\mathcal{G}_{2,n}$. Observe that each of the four graphs in Figure 4.16 has cycle rank 2. Note that the subscript of each of these graphs equals the number of pendant edges that it contains. A minimum detectable coloring for each such graph is also shown in Figure 4.16. We now describe four classes of connected graphs with cycle rank 2 constructed from the four graphs shown in Figure 4.16.
(1) Let $B$ be the set of all graphs that are subdivisions of the graph $B_0$ in Figure 4.16. That is, $B$ consists of all connected graphs obtained from two cycles $C$ and $C'$ by identifying a vertex in $C$ and a vertex in $C'$. In particular, $B_0 \in B$. For each integer $n \geq 5$, let $B_n$ be the set of graphs of order $n$ in $B$. So $B_5 = \{B_0\}$. If $B \in B_n$, then $B$ has one vertex of degree 4 and $n - 1$ vertices of degree 2.

(2) Let $\mathcal{H}$ be the set of all subdivisions of the graph $H_1$ in Figure 4.16. That is, $\mathcal{H}$ consists of all connected graphs obtained from a graph $B \in B$ by adding a new vertex, joining this vertex to the unique vertex of degree 4 in $B$, and then possibly subdividing the newly added pendant edge. For each integer $n \geq 6$, let $\mathcal{H}_n$ be the set of graphs of order $n$ in $\mathcal{H}$. Thus if $H \in \mathcal{H}_n$, then $H$ has one end-vertex, one vertex of degree 5, and $n - 2$ vertices of degree 2.
(3) Let $\mathcal{F}$ be the set of all subdivisions of the graph $F_1$ in Figure 4.16. That is, $\mathcal{F}$ consists of all connected graphs obtained from a graph $B \in \mathcal{B}$ by adding a new vertex, joining this vertex to a vertex of degree 2 in $B$, and then possibly subdividing the newly added pendant edge. For each integer $n \geq 6$, let $\mathcal{F}_n$ be the set of graphs of order $n$ in $\mathcal{F}$. Thus if $F \in \mathcal{F}_n$, then $F$ has one end-vertex, one vertex of degree 3, one vertex of degree 4, and $n - 3$ vertices of degree 2.

(4) Let $\mathcal{J}$ be the set of all subdivisions of the graph $J_2$ in Figure 4.16. That is, $\mathcal{J}$ consists of all connected graphs obtained from a graph $B \in \mathcal{B}$ by adding two vertices, joining each of the two new vertices to the unique vertex of degree 4 in $B$, and then possibly subdividing one or both newly added pendant edges. For each integer $n \geq 7$, let $\mathcal{J}_n$ be the set of graphs of order $n$ in $\mathcal{J}$. Thus if $J \in \mathcal{J}_n$, then $J$ has two end-vertices, one vertex of degree 6, and $n - 3$ vertices of degree 2.

In order to establish upper bounds for the detection numbers of graphs belonging to any of the four classes (1)-(4) in terms of their orders, we first present two useful results, which are consequences of Proposition 2.10 and Theorem 3.3, respectively. For a graph $G$, we denote the order of $G$ by $n(G)$ and the size of $G$ by $m(G)$.

**Lemma 4.12** Let $G$ be a connected graph with cycle rank 2 and let $H$ be a connected subgraph of $G$.
(1) If $H$ has cycle rank 2, then $\det(G) \leq \det(H) + n(G) - n(H)$.

(2) If $H$ is unicyclic, then $\det(G) \leq \det(H) + n(G) - n(H) + 1$.

**Proof.** We first verify (1). Since $G$ and $H$ both have cycle rank 2, it follows that $m(G) = n(G) + 1$ and $m(H) = n(H) + 1$. It then follows by Proposition 2.10 that

$$\det(G) \leq \det(H) + m(G) - m(H) = \det(H) + n(G) - n(H).$$

The proof of (2) is similar to that of (1) except $m(H) = n(H)$. $lacksquare$

**Lemma 4.13** Let $n \geq 3$ be an integer. Then $\det(C_n) = 3$ for $3 \leq n \leq 5$ and $\det(C_n) \leq n - 3$ for $n \geq 6$.

**Lemma 4.14** If $B \in B_n$ where $n \geq 5$, then

$$\det(B) \leq \begin{cases} n - 2 & \text{if } n = 5 \\ n - 3 & \text{if } n = 6 \\ n - 4 & \text{if } n \geq 7. \end{cases}$$

**Proof.** Let $v$ be the vertex of degree 4 in $B$. Suppose that $N(v) = \{w, x, y, z\}$, where $w$ and $y$ belong to one component of $B - v$ and $x$ and $z$ belong to the other component of $B - v$. Let $C$ be the cycle of order $n - 1$ obtained from $B - v$ by joining $w$ to $x$ and joining $y$ to $z$. Suppose that $\det(C) = k$ and let $c$ be a detectable $k$-coloring of $C$. Define a $k$-coloring $c' : E(B) \rightarrow \{1, 2, \ldots, k\}$ of $B$ by

$$c'(e) = \begin{cases} c(wx) & \text{if } e = vw \text{ or } e = vx \\ c(yz) & \text{if } e = vy \text{ or } e = vz \\ c(e) & \text{otherwise.} \end{cases}$$

The color codes of the vertices of $B$ are all those of $C$ together with the color code of $v$. Since $v$ is the only vertex of degree 4 in $B$, it follows that $c'$ is a detectable $k$-coloring of $B$ and so $\det(B) \leq k = \det(C)$. Since $\det(C_4) = \det(C_5) = 3$ and $\det(C) \leq (n - 1) - 3 = n - 4$ if the order $n - 1$ of $C$ is at least 6 (or $n \geq 7$) by Lemma 4.13, we have the desired result. $lacksquare$
Lemma 4.15  If \( H \in \mathcal{H}_n \), where \( n \geq 6 \), then

\[
\det(H) \leq \begin{cases} 
  n - 3 & \text{if } n = 6 \\
  n - 4 & \text{if } n \geq 7.
\end{cases}
\]

**Proof.** Let \( B \in \mathcal{B} \) be a subgraph of \( H \), let \( v \) be the unique vertex of degree 4 in \( B \), and let \( w \) be the vertex in \( V(H) - V(B) \) that is adjacent to \( v \). Furthermore, let \( H' \) be the subgraph of \( H \) obtained from \( B \) by adding the pendant edge \( vw \) to \( B \). Thus \( n(H') = n(B) + 1 \). Suppose that \( \det(B) = k \) and let \( c \) be a detectable \( k \)-coloring of \( B \). The coloring \( c' : E(H') \to \{1, 2, \ldots, k\} \) of \( H' \) defined by

\[
c'(e) = \begin{cases} 
  1 & \text{if } e = vw \\
  c(e) & \text{if } e \neq vw
\end{cases}
\]

is a detectable \( k \)-coloring of \( H' \). Thus, \( \det(H') \leq k = \det(B) \). By Lemma 4.14,

\[
\det(B) \leq \begin{cases} 
  n(B) - 2 & \text{if } n(B) = 5 \\
  n(B) - 3 & \text{if } n(B) = 6 \\
  n(B) - 4 & \text{if } n(B) \geq 7.
\end{cases}
\]

Since \( \det(H') \leq \det(B) \) and \( n(B) = n(H') - 1 \), it follows that

\[
\det(H') \leq \begin{cases} 
  n(H') - 3 & \text{if } n(H') = 6 \\
  n(H') - 4 & \text{if } n(H') = 7 \\
  n(H') - 5 & \text{if } n(H') \geq 8.
\end{cases}
\]

Thus \( \det(H') \leq n(H') - 3 \) if \( n(H') = 6 \) and \( \det(H') \leq n(H') - 4 \) if \( n(H') \geq 7 \). If \( n = 6 \), then \( H = H' \), which is isomorphic to the graph \( H_1 \) of Figure 4.16; so \( \det(H) = \det(H') = 3 = n - 3 \). Thus we may assume that \( n \geq 7 \). If \( n(H') \geq 7 \), then by Lemma 4.12,

\[
\det(H) \leq \det(H') + \det(H) - n(H') \leq n(H') - 4 + n(H) - n(H') = n - 4.
\]

If \( n(H') = 6 \), then \( H \) contains a subgraph \( H'' \) that is isomorphic the graph of Figure 4.17. Since \( H'' \) has a (minimum) detectable 3-coloring, as shown in Figure 4.17, it follows that \( \det(H''') = 3 = n(H'') - 4 \). Thus \( \det(H) \leq n - 4 \) by Lemma 4.12.
Lemma 4.16  If $F \in \mathcal{F}_n$ where $n \geq 6$, then $\det(F) \leq n - 4$.

Proof. Let $B \in \mathcal{B}$ be a subgraph of $F$. If the order of $B$ is 5, then $F$ contains a subgraph $F'$ that is isomorphic to the graph $F_1$ in Figure 4.16. Since $\det(F_1) = 2 = n(F_1) - 4$, it follows by Lemma 4.12 that $\det(F) \leq n - 4$. If the order of $B$ is 6, then $F$ contains a subgraph $F'$ that is isomorphic to one of the graphs of order 7 in Figure 4.18. In each case, $\det(F') = 3 = n(F') - 4$ and so $\det(F) \leq n - 4$ by Lemma 4.12. If the order of $B$ is 7 or more, then $\det(B) \leq n(B) - 4$ by Lemma 4.14. It then follows by Lemma 4.12 that $\det(F) \leq n - 4$. ■

Figure 4.18: Subgraphs of $F$ in the proof of Lemma 4.16

Lemma 4.17  If $J \in \mathcal{J}_n$ where $n \geq 7$, then $\det(J) \leq n - 4$.

Proof. Let $B \in \mathcal{B}$ be a subgraph in $J$. If the order of $B$ is 5, then $J$ contains a subgraph isomorphic to the graph $J_2$ in Figure 4.16. Since $\det(J_2) = 3 = n(J_2) - 4$, it follows by Lemma 4.12 that $\det(J) \leq n - 4$. If the order of $B$ is 6 or more, then $J$ contains a subgraph $H \in \mathcal{H}$ of order 7 or more. Since $\det(H) \leq n(H) - 4$ by Lemma 4.15, it follows that $\det(J) \leq n - 4$ by Lemma 4.12. ■

The following lemma will be useful to us.

Lemma 4.18  Let $G \in \mathcal{G}_{2,n}$ where $n \geq 7$ and let $g$ be the girth of $G$. If $g \geq 6$, then

\[ \det(G) \leq \begin{cases} 
 n - 3 & \text{if } g = 6 \\
 n - 4 & \text{if } g \geq 7.
\end{cases} \]
Proof. Let $C_g : v_1, v_2, \ldots, v_g, v_1$ be a cycle of order $g$ in $G$. Since $G \neq C_g$, there is a vertex $w$ of $G$ such that $w \notin V(C_g)$ and $w$ is adjacent to a vertex on $C_g$, say $w$ is adjacent to $v_g$. Let $F$ be the graph obtained from $C_g$ by adding the edge $v_gw$. Then $F$ is a unicyclic subgraph of order $g + 1$ and size $g + 1$ in $G$. Let $C_{g-1} : v_1, v_2, \ldots, v_{g-1}, v_1$ be a cycle of order $g - 1$. Suppose that $\det(C_{g-1}) = k$ and let $c$ be a detectable $k$-coloring of $C_{g-1}$. The $k$-coloring $c' : E(F) \rightarrow \{1, 2, \ldots, k\}$ of $F$ defined by

$$c'(e) = \begin{cases} c(v_{g-1}v_1) & \text{if } e = v_gw, v_gv_1, v_gv_{g-1} \\ c(e) & \text{otherwise} \end{cases}$$

is detectable. Thus, $\det(F) \leq \det(C_{g-1})$. If $g = 6$, then $\det(C_{g-1}) = \det(C_5) = 3$. Thus $\det(F) \leq 3$ and so

$$\det(G) \leq \det(F) + m(G) - m(F) \leq 3 + (n + 1) - 7 = n - 3$$

by Proposition 2.10. If $g \geq 7$ (and so $g - 1 \geq 6$), then $\det(C_{g-1}) \leq (g - 1) - 3 = g - 4$ by Lemma 4.13. Thus $\det(F) \leq g - 4$ and so

$$\det(G) \leq \det(F) + m(G) - m(F) \leq (g - 4) + (n + 1) - (g + 1) = n - 4$$

by Proposition 2.10.

We are now prepared to establish the following.

Theorem 4.19. If $G \in G_{2,n}$ where $n \geq 7$, then $\det(G) \leq n - 4$.

Proof. Since $G$ contains at least two cycles, $G$ has a subgraph $F$ that is isomorphic to one of three types of graphs in Figure 4.19:

![Figure 4.19: Three possible types of subgraphs](image)

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(1) a graph obtained from two cycles $C$ and $C'$ by identifying a vertex in $C$ and a vertex in $C'$, that is, a graph in $B$ as shown in Figure 4.19(a);

(2) a graph obtained from two disjoint cycles $C$ and $C'$ and a path $P$ of length 1 or more by identifying an end-vertex $u$ of $P$ with a vertex of $C$ and identifying the other end-vertex $v$ of $P$ with a vertex of $C'$ as shown in Figure 4.19(b);

(3) a subdivision of $K_4 - e$, that is, a graph consisting of three internally disjoint $u - v$ paths $P_i$ $(1 \leq i \leq 3)$, as shown in Figure 4.19(c), where at least two paths $P_i$ $(1 \leq i \leq 3)$ have length 2 or more.

We now consider these three cases.

**Case 1: $F$ is isomorphic to some graph $B \in B$.**

If the order of $F$ is 7 or more, then $\det(F) \leq n(F) - 4$ by Lemma 4.14. It then follows by Lemma 4.12 that $\det(G) \leq n - 4$. If the order of $F$ is 6, then $G$ contains a subgraph $F'$ of order 7 or more such that $F' \in \mathcal{H}$ or $F' \in \mathcal{F}$. Thus $\det(F') \leq n(F') - 4$ by Lemma 4.15 or by Lemma 4.16. Hence $\det(G) \leq n - 4$ by Lemma 4.12.

We now assume that the order of $F$ is 5. If $G$ contains a subgraph $F'$ of order 7 or more that belongs to one of the classes $\mathcal{H}$, $\mathcal{F}$, or $\mathcal{J}$, then $\det(F') \leq n(F') - 4$ by Lemmas 4.15, 4.16, or 4.17 and so $\det(G) \leq n - 4$ by Lemma 4.12. Otherwise, $G$ contains a subgraph $F'$ that is isomorphic to one of the graphs of order 7 in Figure 4.20, where a minimum detectable 2-coloring is given for each graph. In each case, $\det(F') = 2 = n(F') - 5$. Thus $\det(G) \leq n - 5 < n - 4$ by Lemma 4.12.

![Figure 4.20: Subgraphs of $G$ in Case 1](image)

**Case 2: $F$ is isomorphic to a graph belonging to the class of graphs described in (2).**

If the girth $g$ of $G$ is 7 or more, then $\det(G) \leq n - 4$ by Lemma 4.18. On the other hand, if $3 \leq g \leq 6$, then $G$ contains a subgraph $F$ that is isomorphic to one of the graphs
shown in Figure 4.21, where a minimum detectable coloring is given for each such graph. If $F$ is isomorphic to one of the unicyclic graphs in Figure 4.21, then $\det(F) = n(F) - 5$; while if $F$ is isomorphic to one of the graphs with cycle rank 2 in Figure 4.21, then $\det(F) \leq n(F) - 4$. In either case, $\det(G) \leq n - 4$ by Lemma 4.12.

Case 3: $F$ is isomorphic to a graph belonging to the class of graphs described in (3).

If the girth $g$ of $G$ is 7 or more, then $\det(G) \leq n - 4$ by Lemma 4.18. If the girth $g$ of $G$ is 5 or 6, then $G$ contains a subgraph $F$ that is isomorphic to one of the graphs shown in Figure 4.22, where a minimum detectable 3-coloring is given for each such graph. If $F$ is isomorphic to one of the two unicyclic graphs in Figure 4.22, then $\det(F) = n(F) - 5$; while if $F$ is isomorphic to the graph with cycle rank 2 in Figure 4.22, then $\det(F) = n(F) - 4$. In either case, $\det(G) \leq n - 4$ by Lemma 4.12.

If the girth $g$ of $G$ is 4, then $G$ contains a subgraph $F$ that is isomorphic to one of the graphs shown in Figure 4.23, where a minimum detectable 2-coloring is given for each such graph. If $F$ is isomorphic to one of the unicyclic graphs in Figure 4.23, then
\( \det(F) = n(F) - 5 \); while if \( F \) is isomorphic to one of the graphs with cycle rank 2 in Figure 4.23, then \( \det(F) = n(F) - 4 \). In either case, \( \det(G) \leq n - 4 \) by Lemma 4.12.

Figure 4.23: Possible subgraphs of girth 4

If the girth \( g \) of \( G \) is 3, then \( G \) contains a subgraph \( F \) that is isomorphic to one of the graphs shown in Figure 4.24, where a minimum detectable coloring is given for each such graph. If \( F \) is isomorphic to the unicyclic graph in Figure 4.24, then \( \det(F) = n(F) - 5 \); while if \( F \) is isomorphic to one of the graphs with cycle rank 2 in Figure 4.24, then \( \det(F) \leq n(F) - 4 \). In either case, \( \det(G) \leq n - 4 \) by Lemma 4.12.

Figure 4.24: Possible subgraphs of girth 3

We have seen that \( D_2(4) = 2 \) and \( D_2(5) = D_2(6) = 3 \). Furthermore, there is a graph \( G \in \mathcal{G}_{2,n} \) \( (n \geq 7) \) having detection number \( n - 4 \). These observations together with Theorem 4.19 yield the following.

**Corollary 4.20**  Let \( n \geq 4 \). Then

\[
D_2(n) = \begin{cases} 
2 & \text{if } n = 4 \\
3 & \text{if } n = 5, 6 \\
n - 4 & \text{if } n \geq 7.
\end{cases}
\]
We now determine the minimum detection number among all connected graphs of order $n$ having cycle rank 2, that is, we find a formula for $d_2(n)$ for $n \geq 4$. In order to do this, we first recall a useful result that is a consequence Proposition 2.3.

**Corollary 4.21** Let $G$ be a nontrivial connected graph having maximum degree $\Delta$ and cycle rank 2. If $n_i$ is the number of vertices of degree $i$ in $G$, where $1 \leq i \leq \Delta$, then

$$n_1 + 2 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta.$$  

Furthermore, we make an observation. According to Corollary 2.9, if $G \in G_{2,n}$ and $\det(G) = k$, then $G$ has at most $k$ end-vertices and at most $\frac{k^2 + k}{2}$ vertices of degree 2. It then follows by Corollary 4.21 that $G$ has at most $k + 2$ vertices of degree 3 or more. Consequently,

$$n \leq k + \frac{k^2 + k}{2} + (k + 2) = \frac{k^2 + 5k + 4}{2}.$$  

We are now prepared to present the following.

**Theorem 4.22** Let $n \geq 6$ be an integer. If $k \geq 2$ is the unique integer such that

$$\frac{(k - 1)^2 + 5(k - 1) + 4}{2} + 1 \leq n \leq \frac{k^2 + 5k + 4}{2}$$

then $d_2(n) = k$.

**Proof.** First, we show that if

$$n \geq \frac{(k - 1)^2 + 5(k - 1) + 4}{2} + 1 = \frac{k^2 + 3k + 2}{2},$$

then $d_2(n) \geq k$. Assume, to the contrary, that there exists a connected graph $G$ of order $n \geq \frac{k^2 + 3k + 2}{2}$ having cycle rank 2 such that $\det(G) \leq k - 1$. By Corollary 2.9, $G$ has at most $k - 1$ end-vertices and at most $\frac{k(k - 1)}{2}$ vertices of degree 2. Since $G$ has at most $k - 1$ end-vertices, it follows from Corollary 4.21 that $G$ has at most $k + 1$ vertices of degree 3 or more. Thus
\[ n \leq (k - 1) + \frac{k^2 - k}{2} + (k + 1) = \frac{k^2 + 3k}{2}, \]

which is a contradiction. Thus, as claimed, \( d_2(n) \geq k \).

Next, we show for each integer \( n \) with

\[ \frac{k^2 + 3k + 2}{2} \leq n < \frac{k^2 + 5k + 4}{2} \]

that \( d_2(n) \leq k \) by constructing a graph \( G \in G_{2,n} \) of order \( n \) having detection number \( k \). For \( 6 \leq n \leq 9 \), it follows that \( k = 2 \). Figure 4.25 shows four graphs of order \( n \) for \( n = 6, 7, 8, 9 \), respectively, having cycle rank 2 and detection number 2. Thus the theorem holds for \( 6 \leq n \leq 9 \).

![Figure 4.25: Graphs in \( G_{2,n} \) with detection number 2 for \( 6 \leq n \leq 9 \)](image)

We may now assume that \( n \geq 10 \). Hence there exists a unique integer \( k \geq 3 \) such that

\[ \frac{(k - 1)^2 + 5(k - 1) + 4}{2} + 1 \leq n \leq \frac{k^2 + 5k + 4}{2}. \]

Therefore,

\[ \frac{(k - 1)^2 + 5(k - 1)}{2} + 1 \leq n - 2 \leq \frac{k^2 + 5k}{2}. \]

It then follows by the proof of Theorem 4.11 that there is a unicyclic graph of order \( n - 2 \) with detection number \( k \). With the aid of the proof of Theorem 4.11, we construct a graph of order \( n \), cycle rank 2, and detection number \( k \) in the following three steps.

1. We construct a unicyclic graph \( H_k \) of order \( \frac{k^2 + 5k}{2} \) having detection number \( k \).
(2) From $H_k$ we construct a unicyclic graph $H$ of order $n - 2$ having detection number $k$.

(3) From $H$ we construct a graph $G$ of order $n$ having cycle rank 2 and detection number $k$.

We now consider two cases, according to whether $k$ is odd or $k$ is even.

**Case 1 : $k$ is odd.**

Then $k = 2\ell - 1$ for some integer $\ell \geq 2$.

We now construct a unicyclic graph $H_k$ of order $\frac{k^2 + 5k}{2}$ having detection number $k$. Let $C_{2\ell^2 - \ell} : v_1, v_2, \ldots, v_{2\ell^2 - \ell}, v_1$ be a cycle of length $2\ell^2 - \ell$ and, for $1 \leq i \leq k$, let $Q_i$ be a copy of $K_2$ with $V(Q_i) = \{u_{i,1}, u_{i,2}\}$. Then the graph $H_k$ is obtained from $C_{2\ell^2 - \ell}$ and the graphs $Q_i (1 \leq i \leq k)$ by adding the edges $v_{2i}u_{i,1} (1 \leq i \leq k)$. Observe that $H_k$ is a unicyclic graph of order

$$\left(2\ell^2 - \ell\right) + 2(2\ell - 1) = \frac{k^2 + 5k}{2}.$$ 

By the proof of Theorem 4.11, $H_k$ has detection number $k$. We now construct a unicyclic graph $H$ of order $n - 2$ having detection number $k$. Let

$$n - 2 = \frac{k^2 + 5k}{2} - p,$$

where $0 \leq p \leq k + 1$.

If $p = 0$, then $H = H_k$ has the desired properties. We next consider two subcases, according to whether $1 \leq p \leq k$ or $p = k + 1$.

**Subcase 1.1 : $1 \leq p \leq k$.**

Let $H$ be the unicyclic graph of order $\frac{k^2 + 5k}{2} - p$ obtained from $H_k$ by suppressing the vertices $u_{i,1}$ so that the edges $v_{2i}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$ where $1 \leq i \leq p$.

**Subcase 1.2 : $p = k + 1$.**

Thus $n - 2 = \frac{k^2 + 3k - 2}{2}$. In this case, consider the unicyclic graph of order $\frac{k^2 + 3k - 2}{2} + 1 = \frac{k^2 + 3k}{2}$ described in Subcase 1.1 (that is, when $p = k$ in Subcase 1.1). We delete the edge
\[ v_{2k}v_{2k-1}, \text{ identify the vertices } v_{2k} \text{ and } v_{2k-1}, \text{ and label this new vertex by } v. \text{ This gives us a unicyclic graph } H \text{ of order } \frac{k^2 + 3k - 2}{2}. \]

Therefore, we have constructed a unicyclic graph \( H \) of order \( n - 2 \) and size \( n - 2 \) having exactly \( k \) end-vertices. Furthermore, it was shown in the proof of Theorem 4.11 that \( \det(H) = k \).

Next, let \( G \) be the graph obtained from \( H \) by (a) adding the vertices \( u_{1,3} \) and \( u_{2,3} \) and (b) joining \( u_{1,3} \) to \( u_{1,2} \), joining \( u_{2,3} \) to \( u_{2,2} \), and joining \( u_{1,2} \) to \( u_{2,2} \). Then \( G \) is a connected graph of order \( n \) and size \( n + 1 \), that is, \( G \in G_{2,n} \). By the proof of Theorem 4.11, there exists a detectable \( k \)-coloring \( c \) of \( H \) such that \( c(v_{2u_{1,2}}) = 1 \), \( c(v_{4u_{2,2}}) = 2 \), and \( (200\ldots01) \) and \( (0200\ldots01) \) are not the color codes of any vertices of degree 3 in \( H \).

Thus, we define a \( k \)-coloring \( c^* : E(G) \to \{1, 2, \ldots, k\} \) of \( G \) by

\[
c^*(e) = \begin{cases} 
1 & \text{if } e = u_{1,2}u_{1,3} \\
2 & \text{if } e = u_{2,2}u_{2,3} \\
k & \text{if } e = u_{1,2}u_{2,2} \\
c(e) & \text{otherwise.}
\end{cases}
\]

Observe that the color codes of the vertices of \( H \) are those of \( G \) except that

1. the color code of \( u_{1,2} \) in \( H \) is now the color code of \( u_{1,3} \) in \( G \),
2. the color code of \( u_{2,2} \) in \( H \) is now the color code of \( u_{2,3} \) in \( G \), and
3. the color codes of \( u_{1,2} \) and \( u_{2,2} \) in \( G \) are \( (200\ldots01) \) and \( (0200\ldots01) \), respectively.

Hence \( c^* \) is a detectable \( k \)-coloring for \( G \) and so \( \det(G) \leq k \). The graph \( G \) is shown in Figure 4.26 for \( n = 21 \) and \( k = 5 \). In this case, the graph \( H_5 \) has order 25, the graph \( H \) has order \( n - 2 = 19 \), and \( H \) is constructed from \( H_5 \) in Subcase 1.2 for \( p = k + 1 = 6 \). The graph \( G \) is then obtained from \( H \) by adding the two vertices \( u_{1,3} \) and \( u_{2,3} \) and the three edges \( u_{1,2}u_{1,3}, u_{2,2}u_{2,3}, \) and \( u_{1,3}u_{2,3} \).

Case 2: \( k \) is even.

Then \( k = 2\ell \) for some integer \( \ell \geq 2 \).
Figure 4.26: A graph $G \in \mathcal{G}_{2,21}$ with a detectable 5-coloring in Case 1

As with Case 1, we first construct a graph $H_k$ of order $\frac{k^2 + 5k}{2}$ having detection number $k$. Let $C_{2\ell^2} : v_1, v_2, \ldots, v_{2\ell^2}, v_1$ be a cycle of length of $2\ell^2$. For $1 \leq i \leq \ell$, let $Q_i$ be a copy of $K_2$ with $V(Q_i) = \{u_{i,1}, u_{i,2}\}$ and for $\ell + 1 \leq i \leq 2\ell$, let $Q_i = u_{i,1}, u_{i,2}, u_{i,3}$ be a path of length 2. Then the graph $H_k$ is obtained from $C_{2\ell^2}$ and the graphs $Q_i$ ($1 \leq i \leq k$) by adding the edges $v_{2i}u_{i,1}$ $(1 \leq i \leq k)$. The graph $H_k$ is a unicyclic graph of order

$$2\ell^2 + 2\ell + 3\ell = \frac{k^2 + 5k}{2}$$

and detection number $k$. We next construct a unicyclic graph $H$ of order $n - 2$ having detection number $k$. Let

$$n - 2 = \frac{k^2 + 5k}{2} - p,$$

where $0 \leq p \leq k + 1$. If $p = 0$, then $H = H_k$ has the desired properties. We now consider three subcases, according to whether

$$1 \leq p \leq \ell, \ell + 1 \leq p \leq k, \text{ or } p = k + 1.$$

Subcase 2.1: $1 \leq p \leq \ell$.

Let $H$ be the unicyclic graph of order $\frac{k^2 + 5k}{2} - p$ obtained from $H_k$ by suppressing the vertices $u_{i,1}$ for $1 \leq i \leq p$ so that the edges $v_{2i}u_{i,1}$ and $u_{i,1}u_{i,2}$ become the single edge $v_{2i}u_{i,2}$.
Subcase 2.2: $\ell + 1 \leq p \leq k$.

Let $H$ be the unicyclic graph of order $\frac{k^2 + 5k}{2} - p$ obtained from the unicyclic graph of order $\frac{k^2 + 5k}{2} - \frac{k}{2} = \frac{k^2 + 4k}{2}$ described in Subcase 2.1 (that is, when $p = \ell$ in Subcase 2.1) by suppressing the vertices $u_{i,2}$ so that the edges $u_{i,1}u_{i,2}$ and $u_{i,2}u_{i,3}$ become the single edge $u_{i,1}u_{i,3}$ where $\ell + 1 \leq i \leq p$.

Subcase 2.3: $p = k + 1$.

Hence $n - 2 = \frac{k^2 + 3k - 2}{2}$. Consider the unicyclic graph of order $\frac{k^2 + 3k - 2}{2} + 1 = \frac{k^2 + 3k}{2}$ described in Subcase 2.2 (that is, when $p = k$ in Subcase 2.2). We delete the edge $v_{2k}v_{2k-1}$, identify the vertices $v_{2k}$ and $v_{2k-1}$, and label this new vertex by $v$. This gives us a unicyclic graph $H$ of order $\frac{k^2 + 3k - 2}{2}$.

As in Case 1, the desired graph $G \in \mathcal{G}_{2,n}$ is then obtained from $H$ by (a) adding the vertices $u_{1,3}$ and $u_{2,3}$ and (b) joining $u_{1,3}$ to $u_{1,2}$, joining $u_{2,3}$ to $u_{2,2}$ and joining $u_{1,2}$ to $u_{2,2}$. An argument similar to the one in Case 1 shows that $\det(G) \leq k$. Figure 4.27 shows such a graph for $n = 33$ and $k = 6$. In this case, the graph $H_6$ has order 33, the graph $H$ has order $n - 2 = 31$, and $H$ is constructed from $H_6$ in Subcase 2.1 for $p = 2$. The graph $G$ then is obtained from $H$ by adding the two vertices $u_{1,3}$ and $u_{2,3}$ and the three edges $u_{1,2}u_{1,3}$, $u_{2,2}u_{2,3}$, and $u_{1,3}u_{2,3}$.

![Figure 4.27: A graph $G \in \mathcal{G}_{2,33}$ with a detectable 6-coloring in Case 2](image)
We have seen that \( d_2(4) = d_2(5) = 2 \). These observations together with Theorem 4.22 yield the following.

**Corollary 4.23** For each integer \( n \geq 4 \),

\[
d_2(n) = \begin{cases} 
2 & \text{if } n = 4, 5 \\
\frac{-5 + \sqrt{8n + 9}}{2} & \text{otherwise.}
\end{cases}
\]

By Corollary 4.23, \( d_2(n) \approx \sqrt{2n} \). We state the following result, which describes all pairs \( k, n \) of integers for which there exists a connected graph in \( G_{2,n} \) having detection number \( k \). Since the proof is long, heavily case-oriented, and similar in nature to the proof of Theorem 4.11, we do not present it.

**Theorem 4.24** Let \( k \geq 2 \) and \( n \geq 4 \) be integers. There exists a connected graph \( G \) of order \( n \) having cycle rank 2 and \( \text{det}(G) = k \) if and only if \( d_2(n) \leq k \leq D_2(n) \).

Since \( d_2(14) = 3 \) and \( D_2(14) = 10 \), there is, according to Theorem 4.24, a connected graph \( G_k \) of order 14 having cycle rank 2 and detection number \( k \) for each integer \( k \) with \( 3 \leq k \leq 10 \). This is illustrated in Figure 4.28.

### 4.5 Some Concluding Remarks

Recall that for integers \( \psi \) and \( n \), with \( \psi \geq 0 \) and \( n \geq \left\lceil \frac{3 + \sqrt{1 + 8\psi}}{2} \right\rceil \), \( D_{\psi}(n) \) denotes the maximum detection number among all connected graphs of order \( n \) having cycle rank \( \psi \) and \( d_{\psi}(n) \) denotes the minimum detection number among all connected graphs of order \( n \) having cycle rank \( \psi \). By Observation 4.1, Propositions 4.4 and 4.7, Theorem 4.10, and Corollaries 4.20 and 4.23, we have the following:

- For \( \psi = 0 \), \( D_0(n) = n - 1 \) for \( n \geq 3 \) and \( d_0(n) = \begin{cases} 
\frac{-5 + \sqrt{8n + 41}}{2} & \text{for } n \geq 3; \\
\frac{-5 + \sqrt{8n + 25}}{2} & \text{for } n \geq 4; \\
\frac{-5 + \sqrt{8n + 9}}{2} & \text{for } n \geq 6.
\end{cases} \)

- For \( \psi = 1 \), \( D_1(n) = n - 3 \) for \( n \geq 6 \) and \( d_1(n) = \begin{cases} 
\frac{-5 + \sqrt{8n + 41}}{2} & \text{for } n \geq 3; \\
\frac{-5 + \sqrt{8n + 25}}{2} & \text{for } n \geq 4; \\
\frac{-5 + \sqrt{8n + 9}}{2} & \text{for } n \geq 6.
\end{cases} \)

- For \( \psi = 2 \), \( D_2(n) = n - 4 \) for \( n \geq 7 \) and \( d_2(n) = \begin{cases} 
\frac{-5 + \sqrt{8n + 41}}{2} & \text{for } n \geq 3; \\
\frac{-5 + \sqrt{8n + 25}}{2} & \text{for } n \geq 4; \\
\frac{-5 + \sqrt{8n + 9}}{2} & \text{for } n \geq 6.
\end{cases} \)
For integers $\psi, t, n$ with $t \geq 3$, $n \geq t + 3$, and $\left(\frac{t-2}{2}\right) + 1 \leq \psi \leq \left(\frac{t-1}{2}\right)$, it is possible to construct a connected graph $G$ of order $n$ having cycle rank $\psi$ such that $\det(G) \geq n - t$.

For example, let $T$ be a tree of order $n$ with exactly $n - t$ end-vertices and let $V(T) = U_1 \cup U_2$, where $U_1$ is the set of end-vertices of $T$ and $U_2 = V(T) - U_1$. Thus $|U_2| = t$, the subgraph $(U_2)$ induced by $U_2$ is connected, and $|E((U_2))| = t - 1$. Since

$$\psi \leq \left(\frac{t-1}{2}\right) = \left(\frac{t}{2}\right) - (t-1),$$

we can construct a connected graph $G \in G_{\psi,n}$ from $T$ by adding $\psi$ edges to the vertices of $U_2$ in $T$. Since $G$ contains exactly $n - t$ end-vertices, $\det(G) \geq n - t$. These observations yield the following.

**Proposition 4.25**  
*For integers $\psi \geq 1$, $t \geq 3$, and $n \geq t + 3$,*

$$D_\psi(n) \geq n - t \quad \text{if} \quad \left(\frac{t-2}{2}\right) + 1 \leq \psi \leq \left(\frac{t-1}{2}\right).$$
Let $G$ be a nontrivial connected graph having maximum degree $\Delta$ and cycle rank $\psi$. Recall that if $n_i$ is the number of vertices of degree $i$ in $G$ where $1 \leq i \leq \Delta$, then

$$n_1 = (2 - 2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_{\Delta}$$

Furthermore, if $\det(G) = k$, then $G$ has at most $k$ end-vertices and at most $\frac{k^2+k}{2}$ vertices of degree 2. Since

$$n_1 = (2 - 2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_{\Delta}$$
$$\geq (2 - 2\psi) + n_3 + n_4 + n_5 + \cdots + n_{\Delta},$$

it follows that

$$n_3 + n_4 + n_5 + \cdots + n_{\Delta} \leq n_1 - (2 - 2\psi) \leq k + 2\psi - 2$$

and so

$$n \leq k + \frac{k^2 + k}{2} + (k + 2\psi - 2) = \frac{k^2 + 5k + 4\psi - 4}{2}.$$

Hence the largest possible order of a connected graph having cycle rank $\psi$ and detection number $k$ is $\frac{k^2 + 5k + 4\psi - 4}{2}$. That is, if $G$ is a connected graph of order $n$ with

$$\frac{(k - 1)^2 + 5(k - 1) + 4\psi - 4}{2} + 1 \leq n \leq \frac{k^2 + 5k + 4\psi - 4}{2}.$$

and having cycle rank $\psi$, then $\det(G) \geq k$, implying that $d_\psi(n) \geq k$. As a consequence of these observations, we have the following.

**Proposition 4.26** For integers $\psi \geq 1$ and $n \geq 2 + 2\psi$,

$$d_\psi(n) \geq \left\lceil \frac{-5 + \sqrt{8n + (16 - 16\psi)}}{2} \right\rceil.$$

Note that for Proposition 4.26 to be nontrivial, we assume that
\[
\left\lfloor \frac{-5 + \sqrt{8n + (41 - 16\psi)}}{2} \right\rfloor \geq 2
\]

and so \( n \geq 2 + 2\psi \) as stated in the result.

We conclude this section with the following conjectures.

**Conjecture 4.27** For integers \( \psi \geq 1, t \geq 3, \) and \( n \geq t + 3, \)

\[
D_\psi(n) = n - t \quad \text{if} \quad \binom{t - 2}{2} + 1 \leq \psi \leq \binom{t - 1}{2}.
\]

**Conjecture 4.28** For integers \( \psi \geq 1 \) and \( n \geq 2 + 2\psi, \)

\[
d_\psi(n) = \left\lfloor \frac{-5 + \sqrt{8n + (41 - 16\psi)}}{2} \right\rfloor.
\]
Chapter 5

Detectable Factorizations of Regular Graphs

5.1 Introduction

It was noted in Chapter 3 that it is most interesting and most challenging to find minimum detectable colorings for graphs having many vertices of the same degree. This suggests investigating detectable colorings of regular graphs. The two results stated below are consequences of Corollary 2.9.

**Theorem 5.1** If $G$ is a connected $r$-regular graph of order $n$ having detection number $k$, then

$$n \leq \binom{r + k - 1}{r}.$$

The contrapositive of Theorem 5.1 gives the following.

**Theorem 5.2** Let $G$ be a connected $r$-regular graph of order $n$. If $n > \binom{r + k - 1}{r}$ for some positive integer $k$, then $\det(G) > k$.

In this chapter, we consider the problem of finding minimum detectable colorings of regular graphs by finding minimum detectable factorizations of these graphs. Recall that for a connected graph $G$ of order $n \geq 3$ and an ordered factorization
\[ \mathcal{F} = \{G_1, G_2, \cdots, G_k\} \]

of \( G \) into \( k \) factors \( G_i \), the color code of a vertex \( v \) of \( G \) (with respect to \( \mathcal{F} \)) is the ordered \( k \)-tuple

\[ \text{code}_\mathcal{F}(v) = (a_1, a_2, \cdots, a_k) \]

where \( a_i = \deg_{G_i} v \) (\( i \leq i \leq k \)). If distinct vertices have distinct color codes, then the factorization \( \mathcal{F} \) is called detectable. We say that an ordered factorization of a graph \( G \) into \( k \) factors is a \( k \)-tuple factorization of \( G \). The minimum integer \( k \) for which \( G \) has a detectable \( k \)-tuple factorization is called the detection number of \( G \), denoted by \( \text{det}(G) \), and such a detectable \( k \)-tuple factorization is a minimum detectable factorization of \( G \).

In Section 2, we consider detectable factorizations of cubic graphs of small orders. The detection numbers of all connected cubic graphs of order 10 are determined in Section 3. Section 4 presents more general results. We also consider irregular and isomorphic factorizations of regular graphs.

### 5.2 Detectable Factorizations of Cubic Graphs

Of course, the order of every cubic graph is at least 4. The cubic graph of smallest order is \( K_4 \) and \( \text{det}(K_4) = 3 \) by Theorem 1.5. There are two cubic graphs of order 6, namely \( K_{3,3} \) and \( K_3 \times K_2 \). We have seen that \( \text{det}(K_{3,3}) = 3 \) by Theorem 2.14. A detectable 3-coloring of \( K_3 \times K_2 \) is shown in Figure 5.1. Since the detection number of every connected regular graph of order 3 or more is at least 3, it follows that \( \text{det}(K_3 \times K_2) = 3 \).

![Figure 5.1: A minimum detectable 3-coloring of \( K_3 \times K_2 \)](image)

Not only is the detection number of \( K_3 \times K_2 = C_3 \times K_2 \) equal to 3, so too are the
detection numbers of $C_4 \times K_2 = Q_3$ and $C_5 \times K_2$ equal to 3. Detectable 3-colorings of these two cubic graphs are shown in Figure 5.2.

![Figure 5.2: Detectable 3-colorings of $Q_3$ and $C_5 \times K_2$](image)

An interesting feature of the 3-coloring of the edges of $C_5 \times K_2$ shown in Figure 5.2 is that each factor whose edges are colored the same is isomorphic to the forest $J$ shown in Figure 5.3. For graphs $F$ and $G$, an $F$-factorization of $G$ is a factorization $\mathcal{F}$ of $G$ in which every factor in $\mathcal{F}$ is isomorphic to $F$. A factorization $\mathcal{F}$ of $G$ in which each factor in $\mathcal{F}$ is isomorphic to a fixed graph is also called an isomorphic factorization of $G$. Thus the detectable 3-coloring of $C_5 \times K_2$ in Figure 5.2 gives rise to an isomorphic factorization of $C_5 \times K_2$ into the forest $J$ of Figure 5.3.

![Figure 5.3: A factor in a detectable 3-tuple factorization of $C_5 \times K_2$](image)

The following observation follows from Theorem 5.1 and the fact that the detection number of every connected regular graph of order 3 or more is at least 3.

**Observation 5.3**  
*If the detection number of a cubic graph $G$ is 3, then the order of $G$ is at most 10.*

In the case of cubic graphs, we have the following result.
Theorem 5.4  If $G$ is a connected cubic graph of order 10 with $\det(G) = 3$ and $\mathcal{F}$ is a detectable 3-tuple factorization of $G$, then every factor in $\mathcal{F}$ has degree sequence

$$3, 2, 2, 1, 1, 0, 0, 0, 0.$$ 

Furthermore, every factor in $\mathcal{F}$ is isomorphic to one of the three graphs shown in Figure 5.4.

![Image](image)

Figure 5.4: The possible factors in a detectable 3-tuple factorization of a connected cubic graph of order 10

Proof. Let $G$ be a connected cubic graph of order 10 and let $F$ be a factor in a detectable 3-tuple factorization $\mathcal{F}$ of $G$. We first show that $F$ has degree sequence

$$3, 2, 2, 1, 1, 0, 0, 0, 0.$$ 

Since there are at most \(\binom{3+3^{-1}}{3} = \binom{5}{3} = 10\) color codes for the vertices of $G$, every color code is used in the factorization $\mathcal{F}$. These ten color codes are

$$300, 030, 003, 210, 201, 120, 102, 012, 021, 111$$

where the $i$th coordinates of these codes form the degree sequence for the $i$th factor $F_i$ in $\mathcal{F}$ for $i = 1, 2, 3$. Observe that all of these sequences are the same and that each sequence is

$$3, 2, 2, 1, 1, 0, 0, 0, 0.$$ 

We now show that $F$ is isomorphic to one of the graphs shown in Figure 5.4. We need only consider the six vertices of $F$ whose degrees are at least 1. Let these vertices...
be \( u, v, w, x, y \) and \( z \). Without loss of generality, we may assume that \( \deg_F u = 3 \) and that the neighbors of \( u \) in \( F \) are \( v, w \) and \( x \). Since there are exactly two vertices of degree 2 in \( F \), at least one of \( v, w \) and \( x \) has degree 1 in \( F \), say \( v \). Also, not all of \( v, w \) and \( x \) have degree 1 in \( F \) since otherwise \( y \) and \( z \) have degree 2, which would be impossible. This implies that at least one of \( w \) and \( x \) has degree 2 in \( F \), say \( w \) (see Figure 5.5).

![Figure 5.5: A subgraph of \( G \) in the proof of Theorem 5.4](image)

We now consider two possible cases.

**Case 1: \( x \) has degree 2 in \( F \).**

If \( w \) and \( x \) are adjacent in \( F \), then \( y \) and \( z \) are adjacent in \( F \), which is therefore isomorphic to \( H_1 \). On the other hand, if \( w \) and \( x \) are not adjacent in \( F \), then each of \( w \) and \( x \) is adjacent to exactly one of \( y \) and \( z \). In this case, \( F \) is isomorphic to \( H_3 \).

**Case 2: \( x \) has degree 1 in \( F \).**

Thus \( w \) is adjacent to exactly one of \( y \) and \( z \) in \( F \), say to \( y \). This implies that \( y \) and \( z \) are adjacent in \( F \), which is therefore isomorphic to \( H_2 \).

Among all cubic graphs of order 10, the Petersen graph is undoubtedly the best known. We now investigate possible detectable 3-tuple factorizations of the Petersen graph \( P \). Figure 5.6 shows two detectable 3-colorings of \( P \). The first coloring results in a factorization of \( P \) (into \( G_1, G_2 \) and \( G_3 \)) that is not an isomorphic factorization, while the second coloring results in an \( H_2 \)-factorization, where \( H_2 \) is shown in Figure 5.4. In fact, \( H_2 \) is the only graph for which the Petersen graph has a detectable isomorphic factorization into three factors. In order to show this, we first present two lemmas.
Figure 5.6: Two minimum detectable 3-colorings of the Petersen graph and the resulting factorizations

**Lemma 5.5**  There exist exactly two $F$-factorizations of the Petersen graph $P$, where $F$ is the forest in Figure 5.7.

**Proof.** Label the vertices of $P$ as shown in Figure 5.7. Let $\mathcal{F}$ be an $F$-factorization of the Petersen graph $P$, where $F$ is the forest of Figure 5.7. Since $P$ is vertex-transitive, we may assume that $r$ is the vertex of degree 3 in the first factor $F_1$ of $\mathcal{F}$. We consider two cases.

**Case 1:** The vertices $r, u, v, w, t$ and $s$ are the nonisolated vertices of $F_1$.

Since $x$ and $y$ are adjacent, not both $x$ and $y$ can be vertices of degree 3 in factors in $\mathcal{F}$. Therefore, at least one of $q$ and $z$ has degree 3 in a factor of $\mathcal{F}$. Assume, without loss of generality, that $z$ has degree 3 in the factor $F_2$ of $\mathcal{F}$.

We claim that $y$ must have degree 1 in $F_2$ for assume, to the contrary, that $y$ has degree 2 in $F_2$. Then $F_2$ contains either $xy$ or $uy$. If $xy \in E(F_2)$, then $F_3$ contains a component isomorphic to $K_2$ with the edge $uy$ and so $F_3 \neq F$, producing a contradiction. Thus $F_2$ contains $uy$. Necessarily then, $x$ is the vertex of degree 3 in $F_3$, which implies that $qw \in E(F_2)$. However then, $F_3$ contains the path $x, q, t, s$ and so $F_3 \neq F$, again a contradiction. Thus, as claimed, $y$ has degree 1 in $F_2$. 

\[ G_1 = G_3 = H_3 \]
\[ G_2 = H_2 \]
Since $y$ has degree 1 in $F_2$, the vertices $w$ and $s$ have degree 2 in $F_2$. However then, the factor $F_3$ is isomorphic to $F$, resulting in the $F$-factorization shown in Figure 5.8(a).

**Case 2**: The vertices $r, u, v, w, y$ and $s$ are the nonisolated vertices of $F_1$.

Observe that if either $x$ or $t$ is a vertex of degree 3 in a factor in $\mathcal{F}$, then $q$ has degree 2 in that factor and $q$ cannot be the vertex of degree 3 in a factor of $\mathcal{F}$. Furthermore, this says that not both $x$ and $t$ can be the vertex of degree 3 in a factor in $\mathcal{F}$. However, if $q$ is a vertex of degree 3 in a factor in $\mathcal{F}$, then $x$ and $t$ have degree 1 or 2 in that factor, implying that neither $x$ nor $t$ is a vertex of degree 3 in a factor in $\mathcal{F}$. Consequently, $z$
must be a vertex of degree 3 in a factor, say \( F_2 \), in \( F \). Since \( \deg_{F_3} v = 1 \) and \( \deg_{F_3} y \leq 1 \), it follows that \( x \) cannot be the vertex of degree 3 in \( F_3 \). Furthermore, since \( \deg_{F_3} u = 1 \) and \( \deg_{F_3} s \leq 1 \), it follows that \( t \) cannot be the vertex of degree 3 in \( F_3 \). This implies that \( q \) is the vertex of degree 3 in \( F_3 \). Thus the \( F \)-factorization \( F \) is uniquely determined (see Figure 5.8(b)).

Since the vertex \( w \) is adjacent to the three vertices of degree 3 in the \( F \)-factorization shown in Figure 5.8(b) and there is no such vertex for the \( F \)-factorization shown in Figure 5.8(a), these two factorizations are distinct. ■

We are now prepared to show that the Petersen graph \( P \) has a unique detectable isomorphic factorization into three factors.

**Theorem 5.6** The only graph \( F \) for which the Petersen graph has a detectable \( F \)-factorization into three factors is when \( F \) is isomorphic to the graph \( H_2 \) of Figure 5.4.

**Proof.** By Theorem 5.4, the only graphs \( F \) for which the Petersen graph \( P \) could have a detectable \( F \)-factorization into three factors are the three graphs of Figure 5.4. Since \( P \) is triangle-free, \( P \) cannot have an \( H_1 \)-factorization. We have seen in Figure 5.6 that there is a detectable \( H_2 \)-factorization of \( P \). By Lemma 5.5, there are exactly two distinct \( H_3 \)-factorizations \( F_1 \) and \( F_2 \) of the Petersen graph, where \( F_1 \) is the \( F \)-factorization described in Figure 5.8(a) and \( F_2 \) is the \( F \)-factorization described in in Figure 5.8(b). In \( F_1 \), \( u \) and \( v \) have the same color codes; while in \( F_2 \), \( y \) and \( s \) have the same color codes. Therefore, neither \( F_1 \) nor \( F_2 \) is detectable and so the graph \( H_2 \) of Figure 5.4 is the only graph \( F \) for which \( P \) has a detectable \( F \)-factorization into three factors. ■

The graphs \( G \) and \( H \) in Figure 5.9 are also cubic graphs of order 10. The detection numbers of both \( G \) and \( H \) are also 3, as is shown in Figure 5.9. The resulting detectable 3-tuple factorization of \( G \) is an \( H_3 \)-factorization, where \( H_3 \) is shown in Figure 5.4; while the resulting detectable 3-tuple factorization of \( H \) is an \( H_1 \)-factorization, where \( H_1 \) is also shown in Figure 5.4.

Another cubic graph \( G \) of order 10 is shown in Figure 5.10. That \( \det(G) = 3 \) is shown by the factorization \( F' \) of \( G \), where \( F' = \{ F_1, F_2, F_3 \} \). The next two propositions tell us about the detectable 3-tuple factorizations of this graph \( G \).
Figure 5.9: Detectable 3-colorings of two cubic graphs of order 10

**Proposition 5.7**  There is no detectable isomorphic factorization of the connected cubic graph G of Figure 5.10 into three factors.

**Proof.** Suppose that there were such a detectable F-factorization \( F \) of \( G \). If \( F \) is a forest, then either \( F = H_2 \) or \( F = H_3 \) (for the graphs \( H_2 \) and \( H_3 \) of Figure 5.4). Then the bridge \( e \) of \( G \) belongs to one factor. Since the nontrivial component \( T \) of \( F \) has order 6, \( T \) is not a subgraph of \( G - e \). If \( F = H_1 \) of Figure 5.4, then the bridge \( e \) of \( G \) must be the component of some factor in \( F \) that is isomorphic to \( K_2 \). However, if \( G_1 \) denotes the component of order 4 and size 4 in \( F \), then one of the components of \( G - e \) must contain two edge-disjoint copies of \( G_1 \). However, each such component has size 7 and so this is impossible. \( \blacksquare \)

**Proposition 5.8**  Every detectable 3-tuple factorization of the graph \( G \) of Figure 5.10 has two isomorphic factors.
Proof. Suppose, to the contrary that there is a detectable factorization $\mathcal{F}$ of $G$ such that no two factors in $\mathcal{F}$ are isomorphic. Then by Theorem 5.4, $\mathcal{F} = \{H_1, H_2, H_3\}$ where $H_1, H_2$ and $H_3$ are the graphs shown in Figure 5.4. Consider the edge $e$ (see Figure 5.11). Note that $G - e$ must contain a copy of $H_2$ or $H_3$. Since each nontrivial component of $H_2$ and $H_3$ has six vertices, it follows that $G - e$ has at least one component containing six vertices. However, $G - e$ consists of two components each of which has five vertices. This produces a contradiction.

By Propositions 5.7 and 5.8, exactly two factors in every detectable 3-tuple factorization of the graph $G$ in Figure 5.10 are isomorphic. This is not the case for the connected cubic graph $H$ of Figure 5.12, however. The detectable 3-coloring of the the graph $H$ shown in Figure 5.12 results in three factors, no two of which are isomorphic. Necessarily, these three factors are the three graphs in Figure 5.4. Such a factorization is
called an irregular factorization of a graph. Therefore, the graph $H$ of Figure 5.12 has a detectable irregular 3-tuple factorization. Since the Petersen graph is triangle-free, the Petersen graph has no irregular factorization. Similarly, the graph $G$ of Figure 5.10 fails to have an irregular factorization by Proposition 5.8.

Figure 5.12: A cubic graph of order 10 with an irregular detectable 3-tuple factorization

We have now seen that some cubic graphs of order 10 do not have a detectable isomorphic 3-tuple factorization and some cubic graphs of order 10 do not have a detectable irregular 3-tuple factorization. For example, the Petersen graph has neither a detectable 3-tuple $H_3$-factorization nor a detectable irregular 3-tuple factorization; while the graph $G$ of Figure 5.10 has neither a detectable isomorphic 3-tuple factorization nor a detectable irregular 3-tuple factorization. On the other hand, the graph in Figure 5.13 has a detectable 3-tuple factorization containing $a$ factors isomorphic to $H_1$, $b$ factors isomorphic to $H_2$, and $c$ factors isomorphic to $H_3$ for all possible triples $(a, b, c)$ of nonnegative integers $a$, $b$ and $c$ for which $a + b + c = 3$. That is, $(a, b, c)$ is one of the following:

$$
(3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 0, 1), (2, 1, 0),
(1, 2, 0), (0, 2, 1), (0, 1, 2), (1, 0, 2), (1, 1, 1).
$$

These factorizations are shown in Figure 5.13.

5.3 Detection Numbers of Cubic Graphs of Small Orders

In this section, we show that every connected cubic graph of order at most 10 has detection number 3. We have seen that every cubic graph of order 4 or 6 has detection
number 3. Thus we consider connected cubic graphs of order 8 or 10. Figure 5.14 shows all connected cubic graphs of order 8 together with a detectable 3-tuple factorization for each (see [16]). This shows that every connected cubic graph of order 8 has detection number 3.

Let us now consider connected cubic graphs of order 10. Every cubic graph of order 10 we have considered, thus far, has detection number 3. We show that every connected
cubic graph of order 10 has detection number 3. The following lemma will help us classify this family of graphs.

Lemma 5.9. Let $r \geq 3$ be an integer. If $G$ is an $r$-regular graph with girth $g \geq 5$, then the order of $G$ is at least $2g$.

Proof. Suppose that $C_g : v_1, v_2, \ldots, v_g, v_1$ is a cycle in $G$. Observe that $\text{diam}(C_g) = \lceil \frac{g}{2} \rceil$. Since $r \geq 3$, there exists vertices $u_1, u_2, \ldots, u_g$, not necessarily distinct, such that $u_iv_i \in E(G)$ and $u_i \notin \{v_1, v_2, \ldots, v_g\}$ for every integer $i$ with $1 \leq i \leq g$. We claim that $u_i \neq u_j$ for $1 \leq i < j \leq g$. Assume, to the contrary, that $u_i = u_j$ for some $i$ and $j$ with $1 \leq i < j \leq g$. Let $P$ be a shortest $v_i - v_j$ path in $C_g$. Then the cycle obtained from $P$ by adding $u_i = u_j$ and the edges $u_iv_i$ and $u_jv_j$ has length

$$2 + d_{C_g}(v_i, v_j) \leq 2 + \left\lceil \frac{g}{2} \right\rceil \leq 2 + \frac{g}{2} = \frac{4 + g}{2} < \frac{2g}{2} = g,$$

which contradicts the fact that $G$ has girth $g$. Thus, as claimed, $u_i \neq u_j$ for all integers $i$ and $j$ with $1 \leq i < j \leq g$.

The next result is an immediate consequence of Lemma 5.9

Corollary 5.10. If $G$ is a cubic graph of order 10, then $G$ has girth at most 5.
We now investigate connected cubic graphs of order 10 by classifying them according to their girth.

**Proposition 5.11** The only connected cubic graph of order 10 having girth 5 is the Petersen graph.

**Proof.** Let $G$ be a connected cubic graph of order 10 and let $C : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of length 5 in $G$. For $1 \leq i \leq 5$, let $u_i$ be the unique neighbor of $v_i$ in $G$ such that $u_i \not\in \{v_1, v_2, v_3, v_4, v_5\}$. The graph shown in Figure 5.15 is therefore a subgraph of $G$.

![Figure 5.15: A subgraph of the graph $G$ of Proposition 5.11](image)

Since $G$ has girth 5 and $\deg u_1 = 3$, it follows that the neighbors of $u_1$ in $G$ are $v_1, u_3$ and $u_4$. In a similar manner, we can determine the neighbors of $u_i$ in $G$ for $i = 2, 3, 4, 5$. It follows that $G$ is the Petersen graph (see Figure 5.16).

![Figure 5.16: The graph $G$ of Proposition 5.11](image)
Since the Petersen graph has detection number 3, the next result follows immediately from Proposition 5.11.

**Corollary 5.12**   If $G$ is a connected cubic graph of order 10 having girth 5, then $\det(G) = 3$.

We now investigate the connected cubic graphs of order 10 having girth 4.

**Proposition 5.13**   Let $G$ be a connected cubic graph of order 10 having girth 4. Then $G$ is one of the following: a Möbius ladder, $K_2 \times C_5$, $G_1, G_2$ or $G_3$, where $G_1, G_2,$ and $G_3$ are shown in Figure 5.17.

![Graphs G1, G2, and G3](image)

**Figure 5.17**: The graphs $G_1, G_2,$ and $G_3$ of Proposition 5.13

**Proof.** Let $C : v_1, v_2, v_3, v_4, v_1$ be a cycle of length 4 in $G$ and for $1 \leq i \leq 4$, let $u_i$ be the unique neighbor of $v_i$ in $G$ such that $u_i \not\in \{v_1, v_2, v_3, v_4\}$ for every integer $i$ ($1 \leq i \leq 4$). There are three possibilities according to whether the vertices $u_i$ are distinct.
Case 1: \(u_1, u_2, u_3\) and \(u_4\) are distinct.

We consider an arbitrary vertex in \(\mathcal{U} = \{u_1, u_2, u_3, u_4\}\), say \(u_1\), and the vertices in \(\mathcal{U}\) to which \(u_1\) may be adjacent. Observe that \(u_1\) is adjacent to at most two vertices in \(\mathcal{U}\). There are five possibilities. We first consider the two subcases in which \(u_1\) is adjacent to exactly two vertices in \(\mathcal{U}\).

Subcase 1.1: \(u_1u_2, u_1u_4 \in E(G)\) and so \(u_1u_3 \notin E(G)\).

Then the graph shown in Figure 5.18 is a subgraph of \(G\). Observe that \(x\) is adjacent to at most one of \(u_2\) and \(u_4\); otherwise, \(\deg x \leq 2\). A similar statement can be said about \(y\). Hence \(x\) and \(y\) are required to be adjacent to \(u_3\) and to each other. However, this produces a triangle in \(G\), contradicting the fact that \(G\) has girth 4.

Subcase 1.2: \(u_1u_2, u_1u_3 \in E(G)\) and so \(u_1u_4 \notin E(G)\).

Then the graph shown in Figure 5.19 is a subgraph of \(G\). Observe that since \(x\) and \(y\) cannot both be adjacent to \(u_2\) (and to \(u_3\)), it follows that \(x\) and \(y\) are adjacent. Also, each of \(x\) and \(y\) is adjacent to at exactly two of \(u_2, u_3,\) and \(u_4\). This implies that \(x\) and \(y\) have a common neighbor, producing a triangle in \(G\). Since \(G\) has girth 4, we obtain a contradiction.

We now consider the two subcases in which \(u_1\) is adjacent to exactly one vertex in \(\mathcal{U}\).
Subcase 1.3: $u_1 u_2 \in E(G)$ and so $u_1 u_3, u_1 u_4 \not\in E(G)$.

Without loss of generality, we may assume that $u_1 x \in E(G)$. Since $G$ has girth 4, it follows that $u_2 x \not\in E(G)$. Furthermore, we may assume that $u_2 u_3, u_2 u_4 \not\in E(G)$ (see Subcases 1.1 and 1.2). It follows that $u_2 y \in E(G)$. This means that the graph shown in Figure 5.20 is a subgraph of $G$.

We consider three subcases of Subcase 1.3 according to which vertices are neighbors of $x$ in $G$.

Subcase 1.3.1: $u_3 x, u_4 x \in E(G)$.

It follows that $u_3 y, u_4 y \in E(G)$ and so $G$ is the graph shown in Figure 5.21. In this case, $G = G_1$. 
Subcase 1.3.2: $u_3x, xy \in E(G)$.

Since $\deg u_4 = 3$, it follows that $u_4$ is adjacent to $u_3$ and $y$ and so $G$ is the graph shown in Figure 5.22. In this case, $G$ is a Möbius ladder.

Subcase 1.3.3: $u_4x, xy \in E(G)$.

Since $\deg u_3 = 3$, it follows that $u_3$ is adjacent to $u_4$ and $y$, and so $G$ is the graph shown in Figure 5.23. In this case, $G = K_2 \times C_5$.

Subcase 1.4: $u_1u_3 \in E(G)$ and so $u_1u_2, u_1u_4 \notin E(G)$.

Without loss of generality, we may assume that $u_1x \in E(G)$. We may also assume that $u_2u_3, u_3u_4 \notin E(G)$ (see Subcase 1.2). Since $G$ has girth 4, it follows that $u_3x \notin E(G)$. This implies that $u_3y \in E(G)$. Thus, the graph shown in Figure 5.24 is a subgraph of $G$. 
We consider two possibilities according to whether $x$ and $y$ are adjacent in $G$.

**Subcase 1.4.1:** $xy \in E(G)$.

Then $x$ is adjacent to one of $u_2$ and $u_4$ while $y$ is adjacent to the other. Without loss of generality, we assume that $u_2x, u_4y \in E(G)$. Since $\deg u_2 = \deg u_4 = 3$, it follows that $u_2$ and $u_4$ are adjacent and so $G$ is the graph shown in Figure 5.25. In this case, $G = G_1$.

**Subcase 1.4.2:** $xy \notin E(G)$.

Then each of $x$ and $y$ is adjacent to both $u_2$ and $u_4$, and $G$ is the graph shown in Figure 5.26. In this case, $G = G_2$.

Finally, we consider the subcase where $u_1$ is adjacent to no vertex in $U$.

**Subcase 1.5:** $u_1u_2, u_1u_3, u_1u_4 \notin E(G)$.

It follows that every vertex in $U$ is adjacent to both $x$ and $y$. This implies that $x$
and $y$ each has degree at least 4 which contradicts the fact that $G$ is a cubic graph.

We now consider the cases where some of the vertices in $U$ are the same. There are two possibilities. We first consider the case in which exactly two vertices in $U$ are the same.

Case 2: $u_1 = u_3$.

In this case, the graph shown in Figure 5.27 is a subgraph of $G$.

We consider two possibilities according to whether $u_1$ is adjacent to one of the leaves of the graph of Figure 5.27.

Subcase 2.1: $u_1u_2 \in E(G)$.

Then the graph of Figure 5.28 is a subgraph of $G$. We break this further into two
subcases, depending on whether $u_2u_4 \in E(G)$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{subgraph_case2.png}
\caption{A subgraph of $G$ in Subcase 2.1.1 of the proof for Proposition 5.13}
\end{figure}

Subcase 2.1.1 : $u_2u_4 \in E(G)$.

Since $\deg x = 3$, it follows that $x$ is adjacent to $u_4$, $y$, and $z$ (see Figure 5.29). Note that $y$ and $z$ are not adjacent since $G$ has girth 4. This implies that $\deg y = \deg z = 1$ which contradicts the fact that $G$ is a cubic graph.

Subcase 2.1.2 : $u_2u_4 \notin E(G)$.

Without loss of generality, we may assume that $u_2$ is adjacent to $x$. Since $\deg y = 3$ it follows that $y$ is adjacent to $u_4$, $x$, and $z$ (see Figure 5.30). Observe that $z$ is adjacent to neither $u_4$ nor $x$ since $G$ has girth 4. This implies that $\deg z = 1$, which contradicts the fact that $G$ is a cubic graph.
Without loss of generality, we may assume that \( u_1 \) is adjacent to \( x \). Then the graph shown in Figure 5.31 is a subgraph of \( G \). We break this further into two subcases according to whether \( y \) and \( z \) are adjacent.

Subcase 2.2.1: \( yz \in E(G) \).

In this case, the graph shown in Figure 5.32 is a subgraph of \( G \).

Observe that if \( x \) is adjacent to \( u_2 \) and \( u_4 \), then each of \( y \) and \( z \) is adjacent to both \( u_2 \) and \( u_4 \) since \( y \) and \( z \) have degree 3. But this implies that \( \deg u_2 = \deg u_4 = 4 \). Hence, \( x \) is adjacent to at most one of \( u_2 \) and \( u_4 \). Without loss of generality, we assume that \( x \) is adjacent to \( u_2 \) and \( y \). Since \( \deg z = 3 \), it follows that \( z \) is adjacent to \( u_2 \) and \( u_4 \). Hence,
the graph in Figure 5.33 is a subgraph of $G$. Note that since $G$ has girth 4, $y$ and $u_4$ are not adjacent. This implies that $\deg u_4 = \deg y = 2$. This contradicts the fact that $G$ is a cubic graph.

**Subcase 2.2.2 :** $yz \notin E(G)$.

Since $\deg y = \deg z = 3$, it follows that each of $y$ and $z$ is adjacent to $u_2, u_4$ and $x$. This determines $G$ which is shown in Figure 5.34. In this case, $G = G_3$.

We now consider the last case.

**Case 3 :** $u_1 = u_2$ and $u_2 = u_4$.

In this case, the graph shown in Figure 5.35 is a subgraph of $G$.

Since $G$ is connected, it follows that $u_1 u_2 \notin E(G)$. Observe that if $u_1$ and $u_2$ are both adjacent to $x$, say, then two of the three vertices $v, w$ and $y$ cannot have degree 3,
which is impossible. Thus, we may assume that $u_1$ is adjacent to $x$ and $u_2$ is adjacent to $y$. Since $\deg v = 3$, it follows that $v$ is adjacent to $x, y$ and $w$ (see Figure 5.36). This implies that $\deg w = 1$ since $G$ has girth 4 and we get a contradiction as $G$ is a cubic graph.

This completes the proof. 

Recall that the Möbius ladder of order 10 and $K_2 \times C_5$ both have detection number 3. Figure 5.37 shows detectable 3-colorings for the graphs $G_1, G_2$, and $G_3$ of Figure 5.17. This together, with Proposition 5.13, imply the following result.

**Corollary 5.14** If $G$ is a connected cubic graph of order 10 having girth 4, then $\det(G) = 3$.

We now consider connected cubic graphs of order 10 having girth 3. Figure 5.38
shows a list of connected cubic graphs of order 10 having girth 3 (see [16]). The following proposition shows that this list contains all connected cubic graphs of order 10 having girth 3.

**Proposition 5.15** Let \( G \) be a connected cubic graph of order 10 having girth 3. Then \( G \) is one of the graphs shown in Figure 5.38.

**Proof.** Let \( C : v_1, v_2, v_3, v_1 \) be a cycle of length 3 in \( G \) and for \( 1 \leq i \leq 3 \), let \( u_i \) be the unique neighbor of \( v_i \) in \( G \) such that \( u_i \notin \{v_1, v_2, v_3\} \) for every integer \( i \) \((1 \leq i \leq 3)\). There are two possibilities according to whether the vertices \( u_i \) are distinct. We first consider the case when the vertices \( u_i \) are distinct.

*Case 1: \( u_1, u_2, u_3 \) are distinct.*

We consider an arbitrary vertex in \( U = \{u_1, u_2, u_3\} \), say \( u_1 \), and the vertices in \( U \) to which \( u_1 \) may be adjacent. There are three possibilities according to the number of vertices in \( U \) to which \( u_1 \) is adjacent.

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Since $G$ is connected, it follows that $u_2u_3 \notin E(G)$ and the graph in Figure 5.39 is a subgraph of $G$.

Suppose that $u_2$ and $u_3$ are adjacent to a common vertex in $\{u, v, w, x\}$, say $u$. Then $u$ is adjacent to exactly one of $v, w$ and $x$, say $v$, which implies that the degree of $w$ and $x$ are at most 2, a contradiction. Hence $u_2$ and $u_3$ have distinct neighbors in $\{u, v, w, x\}$. Without loss of generality, we assume that $u_2v, u_3u \in E(G)$. This means that $w$ is adjacent to each of $u, v$ and $x$. Similarly, $x$ is adjacent to each of $u, v$ and $w$. In this case, $G = H_1$ as shown in Figure 5.40.
Subcase 1.2 : $u_1u_2 \in E(G)$ and $u_1u_3, u_2u_3 \notin E(G)$.

Then the graph in Figure 5.41 is a subgraph of $G$.

We further divide this subcase into two subcases depending on whether $u_1$ and $u_2$ have a common neighbor in \{u, v, w, x\}.

Subcase 1.2.1 : $u_1$ and $u_2$ have a common neighbor in \{u, v, w, x\}

Without loss of generality, we assume that $u_1u, u_2u \in E(G)$. If $u_3u \in E(G)$, then not all of $v, w$ and $x$ can have degree 3, a contradiction. Hence, $u_3u \notin E(G)$. Without loss of generality, we assume that $u$ is adjacent to $v$. This implies that $w$ is adjacent to each of $v, x$ and $u_3$; while $x$ is adjacent to each of $w, v$ and $u_3$. In this case, $G = H_2$ as shown in Figure 5.42.
Subcase 1.2.2: \( u_1 \) and \( u_2 \) have distinct neighbors in \( \{ u, v, w, x \} \).

Without loss of generality, we may assume that \( u_1 \) is adjacent to \( u \) while \( u_2 \) is adjacent to \( v \). If \( u_3 \) is adjacent to both \( u \) and \( v \), then at least one of \( w \) and \( x \) does not have degree 3, a contradiction. Hence, \( u_3 \) is adjacent to at most one of \( u \) and \( v \). There are two possibilities.

Subcase 1.2.2.1: \( u_3 \) is adjacent to exactly one of \( u \) and \( v \).

By symmetry, we may assume that \( u_3 \) is adjacent to \( u \) and \( w \). This implies that \( x \) is adjacent to each of \( u, v \) and \( w \), and so \( v \) and \( w \) are adjacent. In this case, \( G = H_3 \) as shown in Figure 5.43.

Subcase 1.2.2.2: \( u_3 \) is adjacent to neither \( u \) nor \( v \).

Then \( u_3 \) is adjacent to \( w \) and \( x \). If \( u \) and \( v \) are adjacent, then \( u \) is adjacent to one of \( w \) and \( x \) while \( v \) is adjacent to the other, which further implies that \( w \) and \( x \) are adjacent. In this case, \( G = H_4 \) as shown Figure 5.44 (a). On the other hand, if \( u \) and \( v \)
are not adjacent, then each of \( u \) and \( v \) is adjacent to both \( w \) and \( x \). In this case, \( G = H_5 \) as shown in Figure 5.44 (b).

We may now assume that no vertex in \( U \) is adjacent to the other vertices of \( U \).

**Subcase 1.3**: No two vertices of \( U \) are adjacent.

Since \( \deg u_1 = \deg u_2 = \deg u_3 = 3 \), it follows that two of \( u_1, u_2 \) and \( u_3 \) have at least one common neighbor (in \( \{ u, v, w, x \} \)). Without loss of generality, we assume that \( u_1 \)
Subcase 1.3.1: $u_1$ and $u_2$ have exactly one neighbor in common.

Assume that $u_1u, u_2u \in E(G)$. Furthermore, we may assume that $u_1v, u_2w \in E(G)$. Then the graph in Figure 5.45 is a subgraph of $G$.

We further divide this subcase into three subcases depending on what other vertex is adjacent to $u$. 

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Subcase 1.3.1.1 : uu₃ ∈ E(G).

It follows that x is adjacent to each of v, w and u₃. Since \( \deg v = \deg w = 3 \), it follows that v and w are adjacent. In this case, \( G = H₆ \) as shown in Figure 5.46.

![Figure 5.46: The graph G in Subcase 1.3.1.1 of the proof of Proposition 5.15](image)

Subcase 1.3.1.2 : vw ∈ E(G).

It follow that x is adjacent to each of v, w and u₃. Since \( \deg w = \deg u₃ = 3 \), it follows that w and u₃ are adjacent. In this case, \( G = H₇ \) as shown in Figure 5.47.

![Figure 5.47: The graph G in Subcase 1.3.1.2 of the proof of Proposition 5.15](image)

Subcase 1.3.1.3 : ux ∈ E(G).

If x is adjacent to u₃, then x is adjacent to exactly one of v and w, say to w. Since \( \deg v = 3 \), it follows that v is adjacent to both w and u₃. In this case, \( G = H₈ \) as shown in Figure 5.48 (a). On the other hand, if x is not adjacent to u₃, then x is adjacent to both v and w. Since \( \deg u₃ = 3 \), it follows that u₃ is also adjacent to both v and w. Hence, \( G = H₉ \) as shown in Figure 5.48 (b).
Subcase 1.3.2: \( u_1 \) and \( u_2 \) have two neighbors in common.

Assume that \( u_1 \) and \( u_2 \) each is adjacent to both \( u \) and \( v \). Since \( G \) is connected, it follows that \( u_3 \) is adjacent to at most one of \( u \) and \( v \). If \( u_3 \) is adjacent to at least one of \( u \) and \( v \), say \( u \), then \( w \) and \( x \) must each be adjacent to both \( v \) and \( u_3 \) since \( \deg w = \deg x = 3 \). This implies that \( u_3 \) and \( v \) each has degree at least 4 which contradicts the fact that \( G \) is cubic. Hence, \( u_3 \) is adjacent to neither \( u \) nor \( v \). This means that \( uw_3, xu_3 \in E(G) \). Observe that \( wx \in E(G) \); otherwise, each of \( w \) and \( x \) must be adjacent to both \( v \) and \( w \) implying that \( v \) and \( w \) each has degree at least 4 which is not possible. Without loss of generality, we assume that \( ux, vw \in E(G) \). In this case, \( G = H_{10} \) as shown in Figure 5.49.

We now consider the case when two of the vertices \( u_i \) are identical.

Case 2: Two of the vertices \( u_1, u_2 \) and \( u_3 \) are the same.

Without loss of generality, we assume that \( u_1 = u_2 \) and \( u_3 \neq u_1, u_2 \). In this case, the graph shown in Figure 5.50 is a subgraph of \( G \).
We consider two possibilities according to whether $u_1$ and $u_3$ are adjacent.

**Subcase 2.1 :** $u_1u_3 \in E(G)$.

Since $\deg u_3 = 3$, we may assume, without loss of generality, that $u_3$ is adjacent to $u$. Furthermore, we may assume that $u$ is adjacent to $v$ and $w$. This implies that $x$ is adjacent to $v, w$ and $y$, while $y$ is adjacent to $x, v$ and $w$. In this case, $G = H_{11}$ as shown in Figure 5.51.
Subcase 2.2 : $u_1 u_3 \notin E(G)$.

Without loss of generality, we assume that $u_1 u \in E(G)$. It follows that the graph shown in Figure 5.52 is a subgraph of $G$.

![Figure 5.52: A subgraph of $G$ in Subcase 2.2 of the proof of Proposition 5.15](image)

We further divide this subcase into two subcases depending on whether $u$ and $u_3$ are adjacent.

Subcase 2.2.1 : $uu_3 \in E(G)$.

Observe that if $u$ and $u_3$ have a common neighbor, say $v$, then not all of $w, x$ and $y$ can have degree 3, which is impossible. Hence, $u$ and $u_3$ have distinct neighbors. Without loss of generality, we assume that $u_3$ is adjacent to $v$ and $u$ is adjacent to $w$. It follows that $x$ is adjacent to $v, w$ and $y$, while $y$ is adjacent to $x, v$ and $w$. In this case, $G = H_{12}$ as shown in Figure 5.53.

![Figure 5.53: The graph $G$ in Subcase 2.2.1 of the proof of Proposition 5.15](image)
Subcase 2.2.2 : \( uu_3 \notin E(G) \).

Observe that \( u \) and \( u_3 \) have at most one neighbor in common, for otherwise not all vertices \( v, w, x \) and \( y \) can have degree 3. We consider these two possibilities.

Subcase 2.2.2.1 : \( u \) and \( u_3 \) have one neighbor in common.

Without loss of generality, we assume that \( w \) is adjacent to both \( u \) and \( u_3 \). Furthermore, we assume that \( u_3 \) is adjacent to \( v \) while \( u \) is adjacent to \( x \). Thus, it follows that \( y \) is adjacent to \( v, w \) and \( x \), while \( v \) is adjacent to \( x \). In this case, \( G = H_1 \) as shown in Figure 5.54.

\[ \begin{align*}
  \text{Figure 5.54: The graph } G \text{ in Subcase 2.2.2.1 of the proof of Proposition 5.15}
\end{align*} \]

Subcase 2.2.2.2 : \( u \) and \( u_3 \) have no neighbors in common.

Without loss of generality, assume that \( u_3 \) is adjacent to both \( v \) and \( w \), and \( u \) is adjacent to both \( x \) and \( y \). Furthermore, we may assume that \( w \) and \( x \) are adjacent. If \( v \) and \( w \) are adjacent, then \( y \) is adjacent to both \( v \) and \( x \). In this case \( G = H_2 \) as shown in Figure 5.55 (a). On the other hand, if \( v \) and \( w \) are not adjacent, then \( v \) is adjacent to both \( x \) and \( y \). Thus \( w \) and \( y \) are adjacent and so \( G = H_{13} \) as shown in Figure 5.55 (b).

This completes the proof.

Figure 5.56 shows detectable 3-colorings for the graphs given in Figure 5.38. This, together with Proposition 5.15, imply the following result.

**Corollary 5.16**  If \( G \) is a connected cubic graph of order 10 having girth 3, then \( \det(G) = 3 \).
Figure 5.55: The graphs described in Subcase 2.2.2.2 of the proof of Proposition 5.15

The next result combines Corollaries 5.12, 5.14 and 5.16.

**Theorem 5.17** Every connected cubic graph of order 10 has detection number 3.

We have thus shown the following.

**Theorem 5.18** Every connected cubic graph whose order is at most 10 has detection number 3.

### 5.4 Some General Results

We have seen that if $G$ is a connected graph of order $n$ with $\text{det}(G) = k$, then $G$ contains at most $\binom{r+k-1}{r}$ vertices of degree $r$. Therefore, in the case of cubic graphs, we have the following observation.
Observation 5.19  If $G$ is a connected cubic graph of order $n$ with $\det(G) = k$, then

$$n \leq \binom{k+2}{3}.$$ 

For an integer $k \geq 3$, there need not exist a connected cubic graph having detection number $k$ and order $\binom{k+2}{3}$, however.
Theorem 5.20 \textit{If } G \textit{ is a connected cubic graph of order } \binom{k+2}{3} \textit{ with } \det(G) = k, \textit{ then}

\[ k \equiv 2 \pmod{4} \text{ or } k \equiv 3 \pmod{4}. \]

\textbf{Proof.} Assume, to the contrary, that there exists a connected cubic graph of order \( n = \binom{k+2}{3} \) with \( \det(G) = k \) such that \( k \equiv 1 \pmod{4} \) or \( k \equiv 0 \pmod{4} \). We consider these two cases.

\textit{Case 1: } \( k \equiv 1 \pmod{4} \).

Then \( k = 4q + 1 \) for some positive integer \( q \). Observe that the order of \( G \) is

\[ n = \binom{4q + 3}{3} = \frac{(4q + 3)(4q + 2)(4q + 1)}{6} = \frac{(4q + 3)(2q + 1)(4q + 1)}{3}, \]

which is odd. This is impossible.

\textit{Case 2: } \( k \equiv 0 \pmod{4} \).

Then \( k = 4q \) for some integer \( q \). This means that the order of \( G \) is

\[ n = \binom{4q + 2}{3} = \frac{(4q + 2)(4q + 1)(4q)}{6} \]

and the size of \( G \) is

\[ m = \frac{3}{2} \binom{4q + 2}{3} = \frac{(2q + 1)(4q + 1)(4q)}{2}. \]

Since \( \det(G) = k \), it follows that \( G \) has a detectable \( k \)-tuple factorization \( \mathcal{F} = \{ F_1, F_2, \ldots, F_k \} \), in which the size of each factor \( F_i \) \( (1 \leq i \leq k) \) is therefore,

\[ \frac{m}{k} = \frac{(2q + 1)(4q + 1)}{2}. \]
which is not an integer, producing a contradiction.

We have seen that the largest possible order of a connected cubic graph with detection number 3 is 10. By Observation 5.19, if $G$ is a connected cubic graph of order $n$ with detection number 4, then $n \leq 20$. By Theorem 5.20, however, there is no a connected cubic graph of order 20 having detection number 4.

**Theorem 5.21**  
*The largest order of a connected cubic graph with detection number 4 is 18.*

**Proof.** It suffices to give an example of a connected cubic graph of order 18 with detection number 4. Let $G = C_9 \times K_2$. Since $18 > 10$, it follows that $\text{det}(G) \geq 4$ by Theorem 5.2. Therefore, we need only show that there is a detectable 4-coloring of $G$. One such coloring is given in Figure 5.57.

![Figure 5.57: A detectable 4-coloring of $C_9 \times K_2$](image)

We now turn to the problem of finding the largest order of a connected cubic graph with detection number 5. By Observation 5.19 and Theorem 5.20, the largest order cannot exceed 34.

**Theorem 5.22**  
*The largest order of a connected cubic graph with detection number 5 is 32.*
Proof. In a detectable 5-coloring of a connected cubic graph of order \( n \), exactly \( n \) of the following 35 color codes must occur:

\[
\begin{array}{cccccc}
30000 & 21000 & 10020 & 11100 \\
03000 & 20100 & 01020 & 11010 \\
00300 & 20010 & 00120 & 11001 \\
00030 & 20001 & 00021 & 10101 \\
00003 & 12000 & 10002 & 10110 \\
02100 & 01002 & 10011 \\
02010 & 00102 & 01110 \\
02001 & 00012 & 01101 \\
10200 & 01011 \\
01200 & 00111 \\
00210 \\
00201
\end{array}
\]

In this list, the codes containing 2 as a coordinate are separated by a horizontal bar according to where 2 occurs. At most 34 of these can be used. Assume, to the contrary, that exactly 34 of these are used in a detectable 5-coloring of a connected cubic graph \( G \). Hence there is one color code that is not used. Since every color code contains at least two 0's, we may assume, without loss of generality, that the color code that is not used has 0 in its first coordinate. In the resulting detectable factorization \( \mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\} \) of \( G \), the degree sequence of \( F_1 \) is

\[ s : 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \]

followed by 19 0's. However, this says that \( F_1 \) contains an odd number of odd vertices, which is impossible. Consequently, the maximum order of a connected cubic graph of with detection number 5 is at most 32.

It remains to show that there exists a connected cubic graph of order 32 with detection number 5. Let \( G = C_{16} \times K_2 \). Since \( 32 > 18 \), it follows that \( \det(G) \geq 5 \) by Theorem 5.2. Therefore, we need only show that there is a detectable 5-coloring of \( G \). One such coloring is shown in Figure 5.58. Therefore, \( \det(G) = 5 \).

By Theorem 5.17, every connected cubic graph of order 10 has detection number 3. Let us now consider other regular graphs having detection number 3. The following
result is a consequence of Corollary 2.9

**Proposition 5.23**  Let $G$ be a connected $r$-regular graph of order $n \geq 3$ such that $\det(G) = 3$. Then

$$n \leq \binom{r+2}{r} = \binom{r+2}{2} = \frac{(r+2)(r+1)}{2}.$$

![Figure 5.58: A detectable 5-coloring of $C_{16} \times K_2$](image)

It follows from Theorem 5.17 that there exist cubic graphs of order $\binom{3+3-1}{2} = 10$ whose detection number is 3. Figure 5.59 shows a connected 4-regular graph of order $\binom{4+3-1}{2} = 15$ whose detection number is 3. The next result shows that for some positive integers $r$, there exists no connected $r$-regular graph having detection number 3 and order $\binom{r+2}{2}$.

**Proposition 5.24**  Let $G$ be an $r$-regular graph of order $n \geq 3$ such that $\det(G) = 3$. If $r \equiv 1 \pmod{4}$, then

$$n \leq \binom{r+2}{2} - 1$$
Proof. Let \( r = 4\ell + 1 \) where \( \ell \) is a nonnegative integer. Thus \( r \) is an odd integer. Moreover,

\[
\binom{r + 2}{2} = \frac{(4\ell + 3)(4\ell + 2)}{2} = (4\ell + 3)(2\ell + 1),
\]

which is an odd integer. Since every graph has an even number of odd vertices, it follows that

\[
n \neq \binom{r + 2}{2}.
\]

Consequently,

\[
n \leq \left( \binom{r + 2}{2} - 1 \right).
\]

Figure 5.59: A 4-regular graph of order 15 whose detection number is 3
Chapter 6

Topics for Further Study

6.1 Detectable Colorings of Disconnected Graphs

In this study, we restricted ourselves to the study of the detectable colorings of connected graphs. We plan to investigate disconnected graphs and study the following.

**Problem 6.1**  Let $G$ be a graph whose components are $G_1, G_2, \ldots, G_k$. Is there a relationship between $\det(G)$ and the numbers $\det(G_1), \det(G_2), \ldots, \det(G_k)$?

Since components of a disconnected graph may be isomorphic to $K_2$, we plan to look at the following question.

**Problem 6.2**  How should the detection number of $K_2$ be defined?

Furthermore, we wish to study the relationship between the detection number of a graph $G$ and its complement $\overline{G}$.

**Problem 6.3**  Is there a relationship between $\det(G)$ and $\det(\overline{G})$?

**Problem 6.4**  Can bounds be established for $\det(G) + \det(\overline{G})$ and $\det(G) \cdot \det(\overline{G})$ for graphs $G$ of order $n$?
6.2 Detectable Sequences

Observe that if \( F \) is an \( F \)-factorization of \( G \) for some factor \( F \), then every factor in \( F \) has the same degree sequence. The converse is not true, however. This suggests the following concept. If \( s \) is a degree sequence of a factor \( F \) of \( G \), an \( s \)-factorization is a factorization \( F \) of \( G \) in which every factor has degree sequence \( s \).

For example, the Petersen graph \( P \) has a detectable \( s \)-factorization for

\[
s : 3, 2, 2, 1, 1, 1, 0, 0, 0, 0.
\]

On the other hand, if \( G \) has a detectable \( s \)-factorization for some degree sequence \( s \), then \( G \) may not have a detectable \( F \)-factorization for any factor of \( G \) having degree sequence \( s \) as is the case for the Petersen graph. We plan to study the following:

**Problem 6.5** For which degree sequences \( s \) does a graph \( G \) have a detectable \( s \)-factorization?

**Problem 6.6** If \( s \) is a degree sequence for which a graph \( G \) has a detectable \( s \)-factorization, is there a factor \( F \) of \( G \) such that \( F \) has degree sequence \( s \) and \( G \) has a detectable \( F \)-factorization?

6.3 Detectable Colorings of Regular Graphs

In this study, we were able to determine the detection numbers of cubic graphs of small orders. Some general results concerning cubic graphs were also presented. We plan to investigate the detection number of \( r \)-regular graphs, where \( r \geq 4 \). In particular, we are interested in the following questions.

**Problem 6.7** What is the detection number of a connected 4-regular graph?

**Problem 6.8** What are the possible factors in a minimum detectable coloring of a 4-regular graph?
6.4 Proper Detectable Colorings of Graphs

Let $G$ be a connected graph of order $n \geq 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of $G$ for some positive integer $k$ (where adjacent edges may be colored the same). The color code of a vertex $v$ of $G$ has been defined as the ordered $k$-tuple

$$\text{code}(v) = (a_1, a_2, \ldots, a_k) \quad (\text{or simply}, \text{code}(v) = a_1a_2\cdots a_k),$$

where $a_i$ is the number of edges incident with $v$ that are colored $i$ for $1 \leq i \leq k$. The coloring $c$ is called properly detectable if adjacent vertices have distinct color codes; that is, for every two adjacent vertices of $G$, there exists a color such that the number of incident edges with that color is different for these two vertices. The proper detection number $pd(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has a properly detectable $k$-coloring.

We plan to study the following problems.

**Problem 6.9** What is the proper detection number of graphs belonging to some well-known classes?

**Problem 6.10** What are some properties of proper detection numbers?
Bibliography


