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NILPOTENT ORBITS ON INFINITESIMAL  
SYMMETRIC SPACES

by  
Joseph A. Fox

A Dissertation  
Submitted to the  
Faculty of The Graduate College  
in partial fulfillment of  
requirements for the  
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Department of Mathematics

Western Michigan University  
Kalamazoo, Michigan  
April 2006

NILPOTENT ORBITS ON INFINITESIMAL  
SYMMETRIC SPACES

Joseph A. Fox, Ph.D.

Western Michigan University, 2006

Let  $G$  be a reductive linear algebraic group defined over an algebraically closed field  $k$  whose characteristic is good for  $G$ . Let  $\theta$  be an involution defined on  $G$ , and let  $K$  be the subgroup of  $G$  consisting of elements fixed by  $\theta$ . The differential of  $\theta$ , also denoted  $\theta$ , is an involution of the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ , and it decomposes  $\mathfrak{g}$  into  $+1$ - and  $-1$ -eigenspaces,  $\mathfrak{k}$  and  $\mathfrak{p}$ , respectively. The space  $\mathfrak{p}$  identifies with the tangent space at the identity of the symmetric space  $G/K$ . In this dissertation, we are interested in the adjoint action of  $K$  on  $\mathfrak{p}$ , or more specifically, on the nullcone  $\mathcal{N}(\mathfrak{p})$ , which consists of the nilpotent elements of  $\mathfrak{p}$ . The main result is a new classification of the  $K$ -orbits on  $\mathcal{N}(\mathfrak{p})$ .



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Joseph A. Fox



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## INTRODUCTION

The main result of this dissertation can be motivated by the following classical result in matrix theory. Let  $O_n(k)$  be the group of orthogonal  $n \times n$  matrices defined over an algebraically closed field  $k$ . This group acts on the set of nilpotent symmetric  $n \times n$  matrices by conjugation. Using the fact (proven in [16] when  $k = \mathbb{C}$  and in [9] for fields of good characteristic) that every nilpotent matrix is similar to a symmetric nilpotent matrix and that any two similar symmetric matrices are orthogonally similar, we have that the set of  $O_n(k)$ -orbits in the set of nilpotent symmetric matrices is in one-to-one correspondence with the set of similarity classes of  $n \times n$  nilpotent matrices. Since the latter set is parameterized by partitions of  $n$  (corresponding to the possible nilpotent Jordan forms), we have that the former set is as well.

To generalize this result, let  $G$  be a linear algebraic group defined over an algebraically closed field  $k$ , and let  $\theta$  be an involution on  $G$ . Let  $K$  be the subgroup of  $\theta$ -fixed points in  $G$ . The differential of  $\theta$ , also denoted  $\theta$ , is an involution on the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ . Let  $\mathfrak{p}$  be the  $-1$ -eigenspace of  $\theta$  in  $\mathfrak{g}$ , and let  $\mathcal{N}(\mathfrak{p})$  be the variety of nilpotent elements in  $\mathfrak{p}$ . For  $x, y \in \mathfrak{p}$ , it is not necessarily true that  $[x, y] \in \mathfrak{p}$ , so  $\mathfrak{p}$  is not a Lie subalgebra of  $\mathfrak{g}$ . However, for  $x, y, z \in \mathfrak{p}$ , we have  $[x, [y, z]] \in \mathfrak{p}$ , giving  $\mathfrak{p}$  the structure of an algebraic object known as a Lie triple system. The group  $K$  acts on  $\mathfrak{p}$  and on  $\mathcal{N}(\mathfrak{p})$  via the adjoint action of  $G$  restricted to  $K$ , and these actions are central to this dissertation. We can recover the situation in the previous paragraph by letting  $G = GL_n(k)$  and defining  $\theta(g) = (g^{-1})^\top$ . Then  $\mathfrak{g} = \mathfrak{gl}_n(k)$ , and the differential of  $\theta$  is given by  $\theta(x) = -x^\top$ . Subsequently,  $K = O_n(k)$  and  $\mathfrak{p}$  is the space of symmetric  $n \times n$  matrices.

When  $k = \mathbb{C}$ , the  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  have been classified in various ways (see [26] and [25]). In [20], Kawanaka classified the orbit set  $\mathcal{N}(\mathfrak{p})/K$  using a method similar

to weighted Dynkin diagrams, which holds when the characteristic of  $k$  is good for  $G$ . In [9],  $\mathcal{N}(\mathfrak{p})/K$  was classified in the special case when  $G$  is a classical matrix group defined over a field of good characteristic (see § for a definition of good characteristic). The methods used were largely linear algebraic and combinatorial. This dissertation continues the work started in [9] by giving a classification of  $\mathcal{N}(\mathfrak{p})/K$  that includes the case where  $G$  is a group of exceptional type. This new classification employs a method similar to the one given in [25], which uses certain types of Lie subalgebras of  $\mathfrak{g}$ .

Nilpotent  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  in the setting where  $k = \mathbb{C}$  are important in the representation theory of real Lie groups. One example which illustrates their importance is the Kostant-Sekiguchi correspondence. This says there is a one-to-one correspondence between  $\mathcal{N}(\mathfrak{p})/K$  and  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})/G_{\mathbb{R}}$ , where  $G_{\mathbb{R}}$  is the real adjoint group of the real form  $\mathfrak{g}_{\mathbb{R}}$  of  $\mathfrak{g}$  associated to the involution  $\theta$ . For example, let  $G = GL_{2m}(\mathbb{C})$ , and define  $\theta$  by  $\theta(g) = J^{-1}(g^{-1})^{\top}J$ , where  $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ ,  $I_m$  being the  $m \times m$  identity matrix. Then  $G_{\mathbb{R}} = SL_m(\mathbb{H})$ , where  $\mathbb{H}$  is the quaternions,  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_m(\mathbb{H})$ ,  $K = Sp_{2m}(\mathbb{C})$ , and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ C & A^{\top} \end{pmatrix} \mid A, B, C \in \mathfrak{gl}_m(\mathbb{C}), B^{\top} = -B, C^{\top} = -C \right\},$$

otherwise known as the set of skew-Hamiltonian matrices. Thus, the Kostant-Sekiguchi bijection says that the nilpotent  $Sp_{2m}(\mathbb{C})$ -orbits consisting of skew-Hamiltonian matrices correspond to nilpotent  $SL_m(\mathbb{H})$ -orbits in  $\mathcal{N}(\mathfrak{sl}_m(\mathbb{H}))$ .

When  $\text{char}(k) = p$  is good for  $G$ , the study of the orbit set  $\mathcal{N}(\mathfrak{p})/K$  is motivated by topics in the cohomology theory of Lie algebras. In this case, the Lie algebra  $\mathfrak{g}$  and the Lie triple system  $\mathfrak{p}$  also have an additional “restricted” structure defined by the  $p$ th power map  $x \mapsto x^{[p]}$  (see [18] for more details), and we can define the restricted nullcone  $\mathcal{N}_1(\mathfrak{g})$  (resp.,  $\mathcal{N}_1(\mathfrak{p})$ ) to be the set of all elements  $x$  of  $\mathcal{N}(\mathfrak{g})$  (resp.,  $\mathcal{N}(\mathfrak{p})$ ) such that  $x^{[p]} = 0$ . It can be shown that  $\mathcal{N}_1(\mathfrak{g})$  (resp.,  $\mathcal{N}_1(\mathfrak{p})$ ) is a closed subvariety of  $\mathcal{N}(\mathfrak{g})$  (resp.,  $\mathcal{N}(\mathfrak{p})$ ). Let  $\mathcal{U}(\mathfrak{g})$  be the universal enveloping algebra of  $\mathfrak{g}$ , and set  $u(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/(x^p - x^{[p]})$ , a finite-dimensional cocommutative Hopf algebra. Denote by

$R$  the cohomology ring  $H^{2\bullet}(u(\mathfrak{g}), k)$ , a finitely generated  $k$ -algebra (see [10] and [11]). If  $M$  is a  $u(\mathfrak{g})$ -module, then  $R$  acts on  $\text{Ext}_{u(\mathfrak{g})}^{\bullet}(M, M)$ . Let  $J_M = \text{Ann}_R \text{Ext}_{u(\mathfrak{g})}^{\bullet}(M, M)$  be the annihilator of this action, and define  $V_{\mathfrak{g}}(M) = \text{Spec}(R/J_M)$ , a variety known as the *support variety* of  $M$ . It was proven in [38] that  $V_{\mathfrak{g}}(k) = \mathcal{N}_1(\mathfrak{g})$ , and  $V_{\mathfrak{g}}(M)$  can be regarded as a subset of  $\mathcal{N}_1(\mathfrak{g})$ . Thus, support varieties, and  $\mathcal{N}_1(\mathfrak{g})$  in particular, give a geometric interpretation of cohomology. An open question is whether there is an analogous cohomological interpretation of  $\mathcal{N}_1(\mathfrak{p})$ , perhaps in terms of the cohomology of restricted Lie triple systems as defined in [14] and [15]. This is one motivation behind our present study of  $\mathcal{N}(\mathfrak{p})/K$  when  $\text{char}(k)$  has good characteristic.

In the “Background” section, we provide some background from the theory of algebraic groups that will be needed later, including, among other things, topics such as root systems, Lie algebras, and certain important subgroups of algebraic groups. A more detailed exposition of this material can be found in any standard text on linear algebraic groups, such as [4] or [35].

An algebraic group  $G$  acts on the variety of nilpotent elements  $\mathcal{N}(\mathfrak{g})$  of its Lie algebra  $\mathfrak{g}$  via the adjoint action, and this action partitions  $\mathcal{N}(\mathfrak{g})$  into orbits consisting of nilpotent elements, i.e., nilpotent orbits. Nilpotent orbits in Lie algebras are the subject of the “Nilpotent  $G$ -Orbits on  $\mathfrak{g}$ ” section. The variety  $\mathcal{N}(\mathfrak{g})$  has many applications in representation theory. In addition to its connection with support varieties as described above, the nullcone also plays a crucial role in the representation theory (over  $\mathbb{C}$ ) of Weyl groups, as stated by the Springer correspondence (see [33]). There are also applications of the nullcone to primitive ideals (by definition, a *primitive ideal* of a Lie algebra  $\mathfrak{g}$  is the annihilator of an irreducible  $\mathfrak{g}$ -module), which are surveyed in Chapter 10 of [8]. In this section, we discuss some of the basic theory of nilpotent orbits and describe several ways in which nilpotent orbits can be classified.

The “Nilpotent  $K$ -Orbits on  $\mathfrak{p}$ ” section sets the stage for the main result. Let  $G$  be an algebraic group equipped with an involution  $\theta$ . Let  $K$ ,  $\mathfrak{p}$ , and  $\mathcal{N}(\mathfrak{p})$  be as

above. The action of  $K$  on  $\mathfrak{p}$  and on  $\mathcal{N}(\mathfrak{p})$  fits into the theory of symmetric spaces as follows. Define a morphism  $\tau : G \rightarrow G$  by  $\tau(g) = g\theta(g)g^{-1}$ . Then  $\tau(G) = P$  is a closed subvariety of  $G$ , stable under conjugation by  $K$ . This map induces an isomorphism of  $K$ -varieties between  $P$  and the symmetric space  $G/K$ . Furthermore  $\mathfrak{p}$  identifies with the tangent space at the identity of  $P$  (hence the term *infinitesimal symmetric space*) and has the structure of a Lie triple system. Kostant and Rallis gave an extensive study of the action of  $K$  on  $\mathfrak{p}$  in [22] when  $k = \mathbb{C}$ . Richardson dealt with a global analogue of Kostant and Rallis' work in [31], giving results related to the action of  $K$  on  $G/K$ . Recently, Levy has extended many of the results of [22] to fields of good characteristics in [23]. In this section, we summarize various classification schemes for nilpotent  $K$ -orbits in  $\mathfrak{p}$ , including those found in [20], [25], and [9].

The ‘‘Classification of  $\mathcal{N}(\mathfrak{p})/K$ ’’ section lays out the main result of this dissertation, which is a new classification of nilpotent  $K$ -orbits in  $\mathfrak{p}$  which holds whenever the characteristic of the field  $k$  is good. This classification is similar in some ways to the one given by No el in [25], but it avoids the characteristic 0 machinery used there, such as the Jacobson-Morozov Theorem and normal triples. Instead, the approach given here depends heavily on the notion of an associated cocharacter, as developed by Pommerening and explained in [19,  4]. We first reduce to a certain type of orbit called a featured orbit. A classification of these orbits leads immediately to a classification of all nilpotent  $K$ -orbits in  $\mathfrak{p}$ . Given a featured orbit, we assign to it one of Pommerening's associated cocharacters and then use that cocharacter to produce an object called a featured pair, which consists of a parabolic subalgebra and a certain subspace of that subalgebra. We then show that this assignment induces a one-to-one map from the set of featured nilpotent orbits to the set of  $K$ -conjugacy classes of featured pairs. We do this by constructing an inverse map using an idea similar to Richardson's method of induction of nilpotent orbits (see [30]). This gives a classification of nilpotent  $K$ -orbits on  $\mathfrak{p}$  similar in spirit to the one given by Bala-Carter and Pommerening for nilpotent  $G$ -

orbits on  $\mathfrak{g}$ . This dissertation thus continues the work begun in [9] to give an elementary classification of nilpotent  $K$ -orbits in good characteristics.

To conclude, the section “A Result on Polynomials Defined on  $\mathfrak{p}$ ” uses a result of Levy from [23] to give a description of the algebra of polynomial functions defined on  $\mathfrak{p}$  which has a potential application in determining characters of  $\mathcal{N}(\mathfrak{p})$ . The final section then discusses directions for further research and applications.

## BACKGROUND

This section contains some basic facts about algebraic groups, Lie algebras, and root systems which will be needed in what follows. Throughout this entire work, we will assume that  $k$  is an algebraically closed field.

An *affine algebraic variety* is a subset of  $k^n$  defined by

$$\{(x_1, x_2, \dots, x_n) \mid f(x_1, x_2, \dots, x_n) = 0 \text{ for all } f \in I\},$$

where  $I$  is some ideal of the polynomial ring  $k[T_1, T_2, \dots, T_n]$ . (In this work, affine varieties are all that are needed; we will not need to consider the generalizations of affine varieties known as *schemes*.) From now on, the term *variety* will refer to an affine algebraic variety. Given a variety  $\mathcal{V}$ ,  $k[\mathcal{V}]$  denotes the ring of polynomial functions defined on  $\mathcal{V}$ , which is isomorphic to  $k[T_1, T_2, \dots, T_n]/I$ . For  $f \in k[\mathcal{V}]$ , let  $X_f = \{x \in \mathcal{V} \mid f(x) \neq 0\}$ . Then the sets  $\{X_f \mid f \in k[\mathcal{V}]\}$  form a basis for a topology on  $\mathcal{V}$  called the *Zariski topology*. In what follows, any topological concepts such as closed and irreducible refer to the Zariski topology. If  $\mathcal{V}$  and  $\mathcal{W}$  are varieties, then a function  $\phi : \mathcal{V} \rightarrow \mathcal{W}$  is a *morphism* of varieties if  $\phi$  is continuous with respect to the Zariski topology.

An *algebraic group* is a variety  $G$  equipped with a multiplication morphism  $\mu : G \times G \rightarrow G$  and an inversion morphism  $\iota : G \rightarrow G$  which make  $G$  into a group. (More precisely, this is an *affine* algebraic group, which we consider here instead of the more general group schemes.) If  $\text{id}$  is the identity element of  $G$ , then  $G^\circ$  denotes the irreducible component of  $G$  containing  $\text{id}$ . (We note that for an algebraic group  $G$ , a subset of  $G$  is irreducible if and only if it is connected. See [35, Prop. 2.2.1].) Addition in the field  $k$  gives  $k$  itself the structure of an algebraic group, and we will denote  $k$  by  $\mathbb{G}_a$  when considering it thus. Likewise, the multiplicative group  $k^*$  is an algebraic group which we will denote by  $\mathbb{G}_m$ . Another important example is the general linear group,



$GL_n(k)$ . Since our underlying varieties are always affine, our algebraic groups can be embedded as closed subsets of some  $GL_n(k)$ ; see [35, Theorem 2.3.7]. For this reason, the algebraic groups we are considering are called *linear* algebraic groups. A morphism of algebraic groups is a morphism of varieties which is also a homomorphism of groups.

Let  $G$  be a linear algebraic group. An element  $x \in G$  is defined to be *semisimple* (resp., *unipotent*) if for any isomorphism  $\phi$  from  $G$  onto a closed subgroup of some  $GL_n(k)$ , we have that  $\phi(x)$  is a semisimple (resp., unipotent) matrix. Given any element  $x \in G$ , there exist unique elements  $x_s, x_u \in G$  such that  $x_s$  is semisimple,  $x_u$  is unipotent, and  $x = x_s x_u = x_u x_s$ . This is known as the *Jordan decomposition* in  $G$ . Let  $R_u(G)$  be the maximal closed, connected, subgroup of  $G$  which consists of unipotent elements. This is called the *unipotent radical* of  $G$ . If  $R_u(G) = \{\text{id}\}$ , then  $G$  is said to be *reductive*. The general linear group, for example, is reductive.

There are certain subgroups of  $G$  which will play important roles. A *torus* of  $G$  is a subgroup of  $G$  which is isomorphic to a direct product  $\mathbb{G}_m \times \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ . Given any two maximal tori  $T$  and  $T'$  of  $G$ , there is an element  $g \in G$  such that  $gTg^{-1} = T'$ . Also, any semisimple element of  $G$  lies in some maximal torus. A *Borel subgroup* of  $G$  is a maximal connected, closed, solvable subgroup of  $G$ . Any two Borel subgroups of  $G$  are  $G$ -conjugate, and every element of  $G$  lies in some Borel subgroup.

Let  $T$  be a maximal torus of  $G$ . We let  $X^*(T)$  denote the set of all algebraic group morphisms from  $T$  to  $\mathbb{G}_m$ , and we let  $X_*(T)$  denote the set of algebraic group morphisms from  $\mathbb{G}_m$  to  $T$ . Elements of  $X^*(T)$  are called *characters*, and elements of  $X_*(T)$  are called *cocharacters*. Both  $X^*(T)$  and  $X_*(T)$  have the structure of abelian groups. There is a pairing  $X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$ ,  $(\alpha, \lambda) \mapsto \langle \alpha, \lambda \rangle$ , defined by  $\alpha(\lambda(t)) = t^{\langle \alpha, \lambda \rangle}$  for all  $t \in \mathbb{G}_m$ . This pairing has the property that  $\alpha \mapsto \langle \alpha, \bullet \rangle$  defines an isomorphism  $X^*(T) \rightarrow \text{Hom}(X_*(T), \mathbb{Z})$  of abelian groups. Similarly, we get an isomorphism  $X_*(T) \rightarrow \text{Hom}(X^*(T), \mathbb{Z})$ . We identify  $X^*(T)$  (resp.,  $X_*(T)$ ) with a subgroup of the Euclidean space  $E = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$  (resp.,  $E^* = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ ). The pairing  $\langle, \rangle$  of  $X^*(T)$  and  $X_*(T)$

extends to one of  $E$  and  $E^*$ .

If  $G$  is an algebraic group, then its underlying variety is smooth ([35, Theorem 4.3.7]). Thus, we know that tangent spaces  $T_x(G)$  exist at all points of  $x \in G$ . In particular, we have  $T_{\text{id}}(G)$  is a vector space of dimension equal to that of  $G$ , which one can identify with the Lie algebra of left-invariant derivations of the  $k$ -algebra  $k[G]$ . This gives  $T_{\text{id}}(G)$  the structure of a Lie algebra. We will denote  $T_{\text{id}}(G)$  by either  $\text{Lie}(G)$  or by  $\mathfrak{g}$ . (In general, we will denote algebraic groups by capital Roman letters and their Lie algebras by the corresponding lower-case fraktur.) If  $\phi : G_1 \rightarrow G_2$  is a morphism of algebraic groups, then its differential  $d\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a homomorphism of Lie algebras. If  $B$  is a Borel subgroup of  $G$ , then  $\mathfrak{b} = \text{Lie}(B)$  is called a *Borel subalgebra* of  $\mathfrak{g}$ .

A Lie algebra  $\mathfrak{g}$  is said to be *semisimple* if it can be decomposed as a direct sum of simple ideals of  $\mathfrak{g}$ . Let  $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$  denote the *center* of  $\mathfrak{g}$ . A Lie algebra  $\mathfrak{g}$  is *reductive* if  $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ , with  $[\mathfrak{g}, \mathfrak{g}]$  a semisimple ideal of  $\mathfrak{g}$ . Every semisimple Lie algebra is reductive. If  $G$  is a semisimple (resp., reductive) algebraic group, then  $\text{Lie}(G)$  is a semisimple (resp., reductive) Lie algebra.

An element  $x \in \mathfrak{g}$  is said to be *semisimple* (resp., *nilpotent*) if for any representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_n(k)$  we have that  $\rho(x)$  is a semisimple (resp., nilpotent) matrix. There is also a Jordan decomposition for Lie algebras which states that for any  $x \in \mathfrak{g}$ , there exist unique semisimple and nilpotent elements  $x_s$  and  $x_n$ , respectively, such that  $x = x_s + x_n$  and  $[x_s, x_n] = 0$ .

For a reductive Lie algebra  $\mathfrak{g}$  (the only type we will be considering), a *Cartan subalgebra*  $\mathfrak{h}$  of  $\mathfrak{g}$  is a maximal abelian subalgebra consisting of semisimple elements. If  $T$  is a maximal torus of  $G$ , then  $\text{Lie}(T)$  is a Cartan subalgebra of  $\mathfrak{g}$ .

If  $g \in G$ , then  $\text{Int}(g)$  is the inner automorphism of  $G$  defined by  $\text{Int}(g)(x) = gxg^{-1}$  for  $x \in G$ . The differential of  $\text{Int}(g)$  gives an automorphism of  $\mathfrak{g}$  denoted by  $\text{Ad}(g)$ . This defines the *adjoint representation* of  $G$  on  $\mathfrak{g}$ . By differentiating  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ , we also get the adjoint representation of  $\mathfrak{g}$  on  $\mathfrak{g}$ , denoted by  $\text{ad}$  and defined by

$\text{ad}(x)(y) = [x, y]$  for  $x, y \in \mathfrak{g}$ . For  $g \in G$  and  $x \in \mathfrak{g}$ , we will often abbreviate  $\text{Ad}(g)(x)$  by  $g \cdot x$ .

Let  $G$  be a reductive group with  $B$  a Borel subgroup of  $G$  and  $T$  a maximal torus of  $G$  contained in  $B$ . Let  $\mathfrak{h} = \text{Lie}(T)$ , a Cartan subalgebra of  $\mathfrak{g}$ . Since  $T$  is abelian and consists of semisimple elements, then  $\text{Ad}(T)$  is a family of commuting semisimple automorphisms of  $\mathfrak{g}$ , which means that we can decompose  $\mathfrak{g}$  as

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha,$$

where for each  $\alpha \in X^*(T)$ ,

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \text{Ad}(t)x = \alpha(t)x \text{ for all } t \in T\},$$

or equivalently

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = d\alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

We define the *root system* of  $G$  relative to  $T$  (resp., the root system of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ ) to be the set of all nonzero  $\alpha$  (resp., nonzero  $d\alpha$ ) in the above decomposition such that  $\mathfrak{g}_\alpha \neq 0$ , and we denote this set by  $\Phi = \Phi(G, T)$  (resp.,  $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ ). It is a finite subset of  $X^*(T)$  (resp.,  $\mathfrak{h}^*$ ) and a root system in  $E$  in the axiomatic sense. Given a root system  $\Phi$ , there exists a subset  $\Delta \subset \Phi$  such that every root in  $\Phi$  can be written as a  $\mathbb{Z}$ -linear combination of roots in  $\Delta$  in such a way that all the coefficients are nonnegative or all are nonpositive. The roots of  $\Delta$  are called the *simple roots* of  $\Phi$ . The *rank* of  $\mathfrak{g}$  is defined to be the cardinality of  $\Delta$ . This equals the dimension of  $\mathfrak{h}$  (since  $\Delta$  is a basis for  $\mathfrak{h}^*$ ). Those roots which can be written as  $\mathbb{Z}$ -linear combinations of  $\Delta$  with all nonnegative coefficients are called *positive roots*, the set of which is denoted by  $\Phi^+$ . The positive roots in  $\Phi$  are the roots of  $B$  relative to  $T$ . Groups with the property that  $\mathbb{Z}\Phi = X^*(T)$  are of *adjoint type*. Such groups will be very important for us later.

Let  $N_G(T)$  be the normalizer of  $T$  in  $G$ . Then  $T = (N_G(T))^\circ$  (see [35, Cors. 3.2.9 and 7.6.4]), which implies  $W := N_G(T)/T$  is a finite group which acts on  $T$  and

hence on  $X^*(T)$ . Furthermore,  $W$  permutes  $\Phi(G, T)$ . The group  $W$  is called the *Weyl group* of  $\Phi(G, T)$ .

There exists a bijective map,  $\alpha \mapsto \alpha^\vee$ , from  $\Phi = \Phi(G, T)$  onto a subset  $\Phi^\vee = \Phi^\vee(G, T)$  of  $X_*(T)$ .  $\Phi^\vee$  is called the *dual root system* to  $\Phi$ , and its elements are called *coroots*. It is a root system in the vector space  $E^*$ .

So given a reductive group  $G$  and maximal torus  $T \subset G$ , we get a *root datum*

$$R(G, T) = (X^*(T), X_*(T), \Phi(G, T), \Phi^\vee(G, T))$$

associated to  $G$  and  $T$ . Root data can also be defined axiomatically, independent of algebraic groups. There is also a notion of an isomorphism of root data. The following theorem, whose proof can be found in [35], shows that connected reductive groups are classified by their root data. It was originally proven by Chevalley.

**Theorem 1.**

- (a) *Two connected reductive groups are isomorphic if and only if their root data are isomorphic.*
- (b) *If  $R$  is a root datum, then there exists a connected reductive group whose root datum is  $R$ .*

A root system is said to be *irreducible* if it cannot be partitioned into two proper subsets such that each root in one subset is orthogonal (relative to the standard inner product in  $E$ ) to each root in the other. If  $\Phi$  is a root system then it can be written uniquely as a disjoint union of irreducible root systems

$$\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_t.$$

If  $G$  is the connected reductive group corresponding to  $\Phi$ , then we have

$$G = G_1 \times G_2 \times \cdots \times G_t,$$

where  $G_i$  is the group corresponding to  $\Phi_i$  for all  $i$ . Each  $G_i$  has the property that its derived subgroup is *quasi-simple*, meaning it has no proper connected, closed, normal subgroups. Thus, when studying connected reductive groups, it is often enough to consider groups whose root systems are irreducible. These have been classified using Dynkin diagrams and are listed below along with one of the corresponding groups in some cases.

- Type  $A_n$ ,  $G = SL_{n+1}(k)$
- Type  $B_n$ ,  $G = SO_{2n+1}(k)$
- Type  $C_n$ ,  $G = Sp_{2n}(k)$
- Type  $D_n$ ,  $G = SO_{2n}(k)$
- Type  $E_6$
- Type  $E_7$
- Type  $E_8$
- Type  $F_4$
- Type  $G_2$

Types  $A_n$ ,  $B_n$ ,  $C_n$ , and  $D_n$  are called the *classical* types. The others are called the *exceptional* types. Their corresponding groups are often realized as automorphism groups of certain algebraic structures.

Let  $(\bar{w})_{w \in W}$  be a set of coset representatives in  $N_G(T)$  of the elements in  $W$ . Then  $G$  is the disjoint union of the double cosets  $B\bar{w}B$ ,  $w \in W$ . This is known as the *Bruhat decomposition* of  $G$ . A *parabolic subgroup* of  $G$  is one which contains a Borel subgroup of  $G$ . There is a one-to-one correspondence between  $G$ -conjugacy classes of parabolic subgroups and subsets of  $\Delta$  defined in terms of the Bruhat decomposition

as follows. Let  $I$  be a subset of  $\Delta$ , and let  $W_I$  be the Weyl group of the root system generated by  $I$ . Then  $Q_I = \cup_{w \in W_I} B\bar{w}B$  is a parabolic subgroup of  $G$  containing  $B$ . Every parabolic subgroup of  $G$  containing  $B$  has this form for some  $I \subset \Delta$ . By the conjugacy of Borel subgroups, every parabolic subgroup of  $G$  is conjugate to a parabolic of this form. Parabolic subgroups can also be described in terms of cocharacters of  $G$  as follows. Given  $\lambda \in X_*(G)$ ,

$$Q(\lambda) = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$$

is a parabolic subgroup of  $G$ , and every parabolic has this form for some cocharacter  $\lambda$ . This link between parabolic subgroups and cocharacters will be of great importance.

Let  $Q$  be a parabolic subgroup of  $G$ , and let  $U$  be its unipotent radical. A *Levi subgroup* of  $Q$  is a closed subgroup  $L$  such that the product map  $L \times U \rightarrow Q$  is bijective. The decomposition  $Q = LU$  is called a *Levi decomposition* of  $Q$ . Notice that Levi subgroups are necessarily reductive. For a subset  $I \subset \Delta$ , let  $S_I = (\cap_{\alpha \in I} \ker \alpha)^\circ$ . Then  $L_I = C_G(S_I)$  is a Levi subgroup of the parabolic subgroup  $Q_I$ . Every Levi subgroup of  $G$  is conjugate to a Levi subgroup of this form.

The Lie algebras of parabolic subgroups (resp., Levi subgroups) are known as *parabolic subalgebras* (resp., *Levi subalgebras*). The Levi decomposition of a parabolic subgroup  $Q$  induces one in  $\mathfrak{q} = \text{Lie}(Q)$ :  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ , where  $\mathfrak{l}$  is a Levi subalgebra, and  $\mathfrak{u}$  is the nilpotent radical of  $\mathfrak{g}$ . Again, we can define these subalgebras in terms of simple roots and cocharacters. Let  $\Delta$  be the set of simple roots corresponding to a Borel subgroup  $B \subset G$ , and let  $\mathfrak{h} = \text{Lie}(T)$ , for  $T$  a maximal torus in  $B$ , be a Cartan subalgebra of  $\mathfrak{g}$ . For  $I \subset \Delta$ , let  $\mathfrak{q}_I$  be the subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{h}$  and all root spaces  $\mathfrak{g}_\alpha$  such that  $\alpha \in \Delta$  or  $-\alpha \in I$ . Then  $\mathfrak{q}_I$  is a parabolic subalgebra of  $\mathfrak{g}$  with Levi decomposition  $\mathfrak{q}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ , where

$$\mathfrak{l}_I = \mathfrak{h} \oplus \sum_{\alpha \in \langle I \rangle} \mathfrak{g}_\alpha$$

and

$$\mathfrak{u}_I = \sum_{\alpha \in \Phi^+ \setminus \langle I \rangle^+} \mathfrak{g}_\alpha.$$

Every parabolic subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{b}$  has this form for some  $I$ , and every parabolic in  $\mathfrak{g}$  is  $G$ -conjugate to a parabolic of this form.

If  $Q(\lambda)$  is a parabolic subgroup defined by a cocharacter  $\lambda$ , then  $\mathfrak{q}(\lambda) = \text{Lie}(Q(\lambda))$  can be described as follows:

$$\mathfrak{q}(\lambda) = \bigoplus_{i \geq 0} \mathfrak{g}_i,$$

where  $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \text{Ad}(\lambda(t))x = t^i x \text{ for all } t \in k^*\}$ . Then  $\mathfrak{q}(\lambda) = \mathfrak{l}(\lambda) \oplus \mathfrak{u}(\lambda)$  is a Levi decomposition of  $\mathfrak{g}$ , where  $\mathfrak{l}(\lambda) = \mathfrak{g}_0$  and  $\mathfrak{u}(\lambda) = \bigoplus_{i > 0} \mathfrak{g}_i$ .

In what follows, unless noted otherwise, we assume that  $G$  is (at least) a reductive linear algebraic group defined over an algebraically closed field  $k$  of characteristic  $p > 2$ .

## NILPOTENT $G$ -ORBITS ON $\mathfrak{g}$

In this section,  $G$  is a reductive linear algebraic group and  $\mathfrak{g}$  is its Lie algebra. We will consider the adjoint action of  $G$  on  $\mathfrak{g}$ . The set of nilpotent elements in  $\mathfrak{g}$  is called the *nullcone* of  $\mathfrak{g}$ , and is denoted by  $\mathcal{N}(\mathfrak{g})$ . If  $x \in \mathfrak{g}$  is nilpotent, then so is  $\text{Ad}(g)x$  for all  $g \in G$ . Thus,  $G$  acts on  $\mathcal{N}(\mathfrak{g})$ . If  $x \in \mathcal{N}(\mathfrak{g})$ , then the orbit  $\mathcal{O}_x := \text{Ad}(G)x$  is called a *nilpotent orbit*. In this section, we give some standard results on nilpotent  $G$ -orbits in  $\mathfrak{g}$ , including several classifications of the orbit set  $\mathcal{N}(\mathfrak{g})/G$ .

For  $x \in \mathfrak{g}$ , let  $C_G(x)$  (resp.,  $\mathfrak{g}^x$ ) denote the centralizer of  $x$  in  $G$  (resp.,  $\mathfrak{g}$ ) with respect to the adjoint representation. We have  $\text{Lie } C_G(x) \subset \mathfrak{g}^x$ , but when  $\text{char}(k) \neq 0$ , it is not true in general that  $\text{Lie } C_G(x) = \mathfrak{g}^x$ . For example, let  $G = SL_n(k)$ , and suppose that  $\text{char}(k) > 0$  and  $\text{char}(k)$  divides  $n$ . Let  $x \in \mathfrak{sl}_n(k)$  be the  $n \times n$  nilpotent matrix with one Jordan block. Then  $\dim C_G(x) = n - 1$ , while  $\dim \mathfrak{g}^x = n$ .

However, it is true that  $\text{Lie } C_G(x) = \mathfrak{g}^x$  when  $x$  is a semisimple element of  $\mathfrak{g}$ . It follows that if  $\phi : G \rightarrow G$  is an automorphism defined by  $\phi(g) = sgs^{-1}$ , where  $s$  is a semisimple element in a linear algebraic group containing  $G$ , then  $\text{Lie}(G^\phi) = \mathfrak{g}^\phi$ , where  $G^\phi$  (resp.,  $\mathfrak{g}^\phi$ ) is the set of fixed points of  $\phi$  in  $G$  (resp.,  $\mathfrak{g}$ ). An automorphism of this form is called a *semisimple automorphism*. A fact that will be of great importance to us is the following well-known result.

**Proposition 2.** *Let  $\phi$  be an automorphism of  $G$ . If  $p = \text{char}(k) > 0$  and the order of  $\phi$  is relatively prime to  $p$ , then  $\phi$  is a semisimple automorphism.*

*Proof.* Let  $\langle \phi \rangle$  be the subgroup of  $\text{Aut}(G)$  generated by  $\phi$ , and let  $S = G \rtimes \langle \phi \rangle$ . Then  $\phi(g) = \phi g \phi^{-1}$ , where the multiplication on the right side of the equation takes place in  $S$ . The element  $\phi \in S$  has Jordan decomposition  $\phi = \phi_s \phi_u$  where  $\phi_s$  is a semisimple element of  $S$  and  $\phi_u$  is a unipotent element of  $S$ . Since  $\phi_u$  is unipotent,  $(\phi_u - \text{id})^{p^t} = 0$



for some integer  $t$ . Thus,  $\phi_u^{p^t} = \text{id}$ . Suppose that  $\phi_u \neq \text{id}$ . Then  $p$  divides the order  $|\phi_u|$  of  $\phi_u$ , and thus  $p$  divides  $|\phi|$ , which contradicts the hypothesis stated above. Thus,  $\phi_u = \text{id}$ , and hence  $\phi$  is semisimple. This means that  $\phi$  acts on  $G$  by conjugation by a semisimple element. Therefore,  $\phi$  is a semisimple automorphism of  $G$ .  $\square$

In what follows, we will need to assume that  $\text{Lie } C_G(x) = \mathfrak{g}^x$  for all  $x \in \mathfrak{g}$ , but to do so we will have to assume extra hypotheses on  $G$ . Let  $\Phi$  be the root system for  $G$ . A prime  $p$  is said to be *good* for  $G$  (or equivalently, for  $\Phi$ ) if  $p$  is larger than any coefficient of any root  $\alpha \in \Phi$  when  $\alpha$  is expressed as a  $\mathbb{Z}$ -linear combination of simple roots. Otherwise  $p$  is said to be *bad* for  $G$ . The good primes for the irreducible root systems (equivalently, for the connected reductive groups with quasi-simple derived subgroups) are:

- Type A: all primes are good
- Type B, C, D:  $p \geq 3$
- Type E<sub>6</sub>, E<sub>7</sub>, F<sub>4</sub>, G<sub>2</sub>:  $p \geq 5$
- Type E<sub>8</sub>:  $p \geq 7$

The characteristic of  $k$  is said to be *good* for  $G$  if  $\text{char}(k) = 0$  or if  $\text{char}(k)$  is a good prime for  $G$ . The following are the *standard hypotheses* on  $G$ :

- (SH1) The characteristic of  $k$  is good for  $G$ .
- (SH2) The derived subgroup  $\mathcal{D}G$  of  $G$  is simply connected.
- (SH3) There exists a nondegenerate  $G$ -invariant bilinear form  $(,)$  defined on  $\mathfrak{g}$ .

The standard hypotheses imply that  $\text{Lie } C_G(x) = \mathfrak{g}^x$  for all  $x \in \mathfrak{g}$ . See [19, Prop. 5.9] for a proof. Thus we assume in what follows that  $G$  satisfies the standard

hypotheses. Since we are assuming that  $\text{Lie } C_G(x) = \mathfrak{g}^x$  for all  $x \in \mathfrak{g}$ , the tangent space  $T_x(\mathcal{O}_x)$  to the orbit  $\mathcal{O}_x$  at  $x$  is equal to  $[\mathfrak{g}, x]$  (see [19, Section 2.2]).

There is a bijection from the set of nilpotent  $G$ -orbits in  $\mathfrak{g}$  to the set of nilpotent  $G/Z(G)$ -orbits in  $\text{Lie}(G/Z(G))$ . For a proof of this, see [19, Prop. 2.7a]. Since  $G/Z(G)$  is semisimple of adjoint type, we may thus assume that  $G$  also has these properties when studying nilpotent orbits. Since such groups are direct products of simple groups, we may further assume that  $G$  is simple if necessary.

It is known that there are always only finitely many  $G$ -orbits in  $\mathcal{N}(\mathfrak{g})$ , no matter what  $\text{char}(k)$  is. We now give a summary of some of the classification schemes of  $\mathcal{N}(\mathfrak{g})/G$  in various situations.

**Partition Classification** Assume that  $G$  is a group of classical type, meaning it is a closed subgroup of  $GL_n(k)$  for some  $n$ , and that  $\text{char}(k) \neq 2$ . Every matrix in a nilpotent  $G$ -orbit is similar to one common matrix in lower triangular Jordan form with zeros on the diagonal. Thus, each nilpotent orbit determines a partition of  $n$ . (The parts are the sizes of the Jordan blocks.) In this way, the nilpotent  $G$ -orbits can be classified in terms of partitions of  $n$ . The results are summarized below.

- **Type  $A_n$**  Nilpotent orbits in  $\mathfrak{sl}_{n+1}(k)$  are in one-to-one correspondence with the set of partitions of  $n + 1$ .
- **Type  $B_n$**  Nilpotent orbits in  $\mathfrak{so}_{2n+1}(k)$  are in one-to-one correspondence with the set of partitions of  $2n + 1$  in which even parts occur with even multiplicity.
- **Type  $C_n$**  Nilpotent orbits in  $\mathfrak{sp}_{2n}(k)$  are in one-to-one correspondence with the set of partitions of  $2n$  in which odd parts occur with even multiplicity.
- **Type  $D_n$**  Nilpotent orbits in  $\mathfrak{so}_{2n}(k)$  are in one-to-one correspondence with the set of partitions of  $2n$  in which even parts occur with even multiplicity, except that “very even” partitions (those with only even parts, each having even multiplicity) correspond to two orbits.

**Dynkin-Kostant Classification** Assume  $G$  is an arbitrary reductive group and that  $\text{char}(k) = 0$  or  $\text{char}(k) \gg 0$ . The Jacobson-Morozov Theorem says that, given a nilpotent element  $x \in \mathfrak{g}$ , there exists a semisimple element  $h \in \mathfrak{g}$  and a nilpotent element  $y \in \mathfrak{g}$  such that  $(h, x, y)$  is an  $\mathfrak{sl}_2$ -triple; i.e.,  $[h, x] = 2x$ ,  $[h, y] = -2y$ , and  $[x, y] = h$ . Here,  $h$  is called the *neutral* element,  $x$  the *nilpositive* element, and  $y$  the *nilnegative* element. The Jacobson-Morozov Theorem thus gives a map from  $\mathcal{N}(\mathfrak{g})/G$  to the set of  $G$ -conjugacy classes of  $\mathfrak{sl}_2$ -triples in  $\mathfrak{g}$ , which actually turns out to be a bijection. Furthermore, a theorem of Mal'cev says that two  $\mathfrak{sl}_2$ -triples are  $G$ -conjugate if and only if their neutral elements are  $G$ -conjugate. Thus, nilpotent orbits are completely determined by the  $G$ -conjugacy class of the neutral element of the corresponding  $\mathfrak{sl}_2$ -triple.

Knowing this, we can associate to each orbit  $\mathcal{O}$  a *weighted Dynkin diagram*  $\Delta_{\mathcal{O}}$  as follows. Let  $x$  be an element of  $\mathcal{O}$ , and let  $(h, x, y)$  be its  $\mathfrak{sl}_2$ -triple. Label each vertex  $v$  of the Dynkin diagram of  $G$  with the value  $\alpha(h)$ , where  $\alpha$  is the simple root corresponding to  $v$ . It was proved by Dynkin that  $\alpha(h) \in \{0, 1, 2\}$ .

It turns out that a weighted Dynkin diagram is a complete invariant of its orbit; i.e.,  $\mathcal{O}_1 = \mathcal{O}_2$  if and only if  $\Delta_{\mathcal{O}_1} = \Delta_{\mathcal{O}_2}$ . This shows that there are at most  $(\text{rank } \mathfrak{g})^3$  nilpotent  $G$ -orbits in  $\mathfrak{g}$ . Historically, this result is important because it shows there are only finitely many nilpotent  $G$ -orbits in  $\mathfrak{g}$ . However it is not easy to determine when a given weighted Dynkin diagram actually corresponds to an orbit. A more precise classification of  $\mathcal{N}(\mathfrak{g})/G$  was given by Bala and Carter, and is discussed next.

**Bala-Carter Classification** Let  $G$  be an arbitrary reductive algebraic group, and let  $\text{char}(k) = 0$  or  $\text{char}(k) \gg 0$ . The partition method discussed above is no longer valid because we are allowing  $G$  to possibly be a group of exceptional type. Instead, nilpotent  $G$ -orbits are classified by certain Levi subalgebras of  $\mathfrak{g}$  and certain parabolic subalgebras of those Levi subalgebras. Since this is related to the approach we take for the main result in the ‘‘Classification of  $\mathcal{N}(\mathfrak{p})/K$ ’’ section, we will go into some detail

here.

An element  $x \in \mathcal{N}(\mathfrak{g})$  is said to be *distinguished* in  $\mathfrak{g}$  if the only Levi subalgebra containing  $x$  is  $\mathfrak{g}$  itself. If  $x$  is distinguished, then so is  $\text{Ad}(g)x$  for any  $g \in G$ . A nilpotent orbit is said to be distinguished if any (and hence, all) of its elements are distinguished. Obviously, if  $\mathfrak{l}$  is a minimal Levi subalgebra of  $\mathfrak{g}$  containing  $x$ , then  $x$  is distinguished in  $\mathfrak{l}$ . Furthermore, we can always find a minimal Levi subalgebra  $\mathfrak{l}$  containing  $x$ ; just take  $\mathfrak{l} = \mathfrak{g}^{\mathfrak{t}}$  where  $\mathfrak{t}$  is a maximal toral subalgebra in  $\mathfrak{g}^x$ . Therefore, if we replace  $\mathfrak{g}$  by a minimal Levi subalgebra containing  $x$ , we may assume that  $x$  is distinguished. The Bala-Carter classification first parameterizes the distinguished nilpotent orbits, which is done in terms of certain parabolic subalgebras of  $\mathfrak{g}$ .

We first need a way to associate a nilpotent orbit to a parabolic subalgebra. Let  $\mathfrak{q}$  be a parabolic subalgebra with nilradical  $\mathfrak{u}$ . Then we may write

$$\mathfrak{u} = \bigcup_{\mathcal{O} \in \mathcal{N}(\mathfrak{g})/G} (\mathcal{O} \cap \mathfrak{u}).$$

Because there are only finitely many  $G$ -orbits in  $\mathcal{N}(\mathfrak{g})$ , this is a finite disjoint union. However,  $\mathfrak{u}$  is irreducible, and hence,  $\mathcal{O} \cap \mathfrak{u}$  is dense in  $\mathfrak{u}$  for some orbit  $\mathcal{O}$ . If  $\mathcal{O}'$  is another orbit which meets  $\mathfrak{u}$ , then  $\mathcal{O}' \cap \mathfrak{u} \subset \mathfrak{u} \setminus (\mathcal{O} \cap \mathfrak{u})$ , which means  $\mathcal{O}' \cap \mathfrak{u}$  is not dense in  $\mathfrak{u}$ . Thus,  $\mathcal{O}$  is the unique orbit which meets  $\mathfrak{u}$  in a dense set, and  $\mathcal{O}$  is called the *Richardson orbit* associated to  $\mathfrak{q}$ . Elements  $x \in \mathcal{O} \cap \mathfrak{u}$  are called *Richardson elements* for  $\mathfrak{q}$ .

Now let  $\mathfrak{q}$  be a parabolic subalgebra determined by a subset of simple roots  $I \subset \Delta$ . Define a function  $f : \Delta \rightarrow \mathbb{Z}$  by

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \in I \\ 2 & \text{if } \alpha \in \Delta \setminus I \end{cases},$$

and extend linearly to  $\mathbb{Z}\Phi$ . This defines a grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i),$$

where

$$\mathfrak{g}(i) = \begin{cases} \bigoplus_{f(\alpha)=i} \mathfrak{g}_\alpha & \text{if } i \neq 0 \\ \mathfrak{h} \oplus \bigoplus_{f(\alpha)=0} \mathfrak{g}_\alpha & \text{if } i = 0 \end{cases}.$$

We define  $\mathfrak{q}$  to be a *distinguished parabolic subalgebra* if  $\dim \mathfrak{g}(0) = \dim \mathfrak{g}(2)$ . This is equivalent to saying  $\dim \mathfrak{l} = \dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ , where  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  is a Levi decomposition of  $\mathfrak{q}$ .

The main result in the Bala-Carter Classification is that the map which associates a parabolic subalgebra to its Richardson orbit induces a bijection from the set of  $G$ -conjugacy classes of distinguished parabolics in  $\mathfrak{g}$  to the set of distinguished nilpotent orbits in  $\mathfrak{g}$ . Proofs of the following statements can be found in [7, Chapter 5].

**Proposition 3.** *Let  $\mathfrak{q}$  be a distinguished parabolic subalgebra of  $\mathfrak{g}$ , and let  $x$  be Richardson for  $\mathfrak{q}$ . Then  $x$  is a distinguished element of  $\mathfrak{g}$ .*

This says that the Richardson orbit associated to a distinguished parabolic is a distinguished nilpotent orbit, and we thus get a well defined map  $\psi$  from the set of  $G$ -conjugacy classes of distinguished parabolic subalgebras of  $\mathfrak{g}$  to the set of distinguished nilpotent  $G$ -orbits in  $\mathfrak{g}$ . It remains to show that  $\psi$  is a bijection.

**Proposition 4.** *Suppose  $x \in \mathfrak{g}$  is Richardson for the distinguished parabolic subalgebras  $\mathfrak{q}$  and  $\mathfrak{q}'$ . Then  $\mathfrak{q} = \mathfrak{q}'$ .*

This proves that  $\psi$  is injective.

**Proposition 5.** *Let  $x$  be a distinguished nilpotent element in  $\mathfrak{g}$ . Then there exists a distinguished parabolic subalgebra  $\mathfrak{q}$  such that  $x$  is Richardson for  $\mathfrak{q}$ .*

The idea behind Proposition 5 is as follows. The Jacobson-Morozov Theorem says that given a nilpotent element  $x$ , there exists a semisimple element  $h$  and a nilpotent element  $y$  in  $\mathfrak{g}$  such that  $(h, x, y)$  is an  $\mathfrak{sl}_2$ -triple. We then get the following decomposition of  $\mathfrak{g}$ :

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

where  $\mathfrak{g}_i = \{w \in \mathfrak{g} \mid [h, w] = iw\}$ . We can then take  $\mathfrak{q}$  to be

$$\mathfrak{q} = \bigoplus_{i \geq 0} \mathfrak{g}_i.$$

It turns out that if  $x$  is distinguished, then so is  $\mathfrak{q}$ , and  $x$  is Richardson for  $\mathfrak{q}$ . This proves that  $\psi$  is surjective and hence bijective.

Now let  $x$  be an arbitrary nilpotent element, not necessarily distinguished in  $\mathfrak{g}$ . Recall that  $x$  is distinguished in any minimal Levi subalgebra containing it.

**Lemma 6.** *Any two minimal Levi subalgebras of  $\mathfrak{g}$  which contain an element  $x \in \mathcal{N}(\mathfrak{g})$  are  $C_G(x)$ -conjugate.*

Using this fact, we may state the Bala-Carter Classification:

**Theorem 7.** *There is a one-to-one correspondence between the set of  $G$ -conjugacy classes of pairs  $(\mathfrak{l}, \mathfrak{q}_{\mathfrak{l}})$ , where  $\mathfrak{l}$  is a Levi subalgebra in  $\mathfrak{g}$  and  $\mathfrak{q}_{\mathfrak{l}}$  is a distinguished parabolic subalgebra of  $\mathfrak{l}$ , and the set of nilpotent  $G$ -orbits in  $\mathfrak{g}$ . A pair  $(\mathfrak{l}, \mathfrak{q}_{\mathfrak{l}})$  corresponds to the nilpotent orbit containing the Richardson orbit in  $\mathfrak{l}$  associated to  $\mathfrak{q}_{\mathfrak{l}}$ .*

**Pommerening's Classification** In [27] and [28], Pommerening extended the Bala-Carter classification to groups defined over fields of good characteristic. He found that the classification remained the same as the one given by Bala and Carter when  $k = \mathbb{C}$ . The main problem was not being able to use the Jacobson-Morozov Theorem, which only holds when  $\text{char}(k) \gg 0$ . Given  $e \in \mathcal{N}(\mathfrak{g})$ , his solution was to use a cocharacter  $\lambda_e$  associated with  $e$ . Associated cocharacters provide a partial substitute for  $\mathfrak{sl}_2$ -triples used prominently by Bala and Carter, and they will be used heavily in the classification we give in our classification section. We will precisely define and discuss some properties of associated cocharacters there.

Pommerening's proof was computational in nature and relied on case-checking by root system type. In [29], Premet gave a fairly short, conceptual proof of Pommerening's theorem using the theory of *optimal cocharacters*, as introduced by Kempf and Rousseau.

The relationship between the optimal cocharacters used by Premet and the associated cocharacters used by Pommerening was stated precisely by McNinch in [24, Theorem 21]. He showed that if a cocharacter  $\lambda$  is associated to  $e \in \mathcal{N}(\mathfrak{g})$ , then  $\lambda$  is optimal for  $e$ . Conversely, if  $\lambda$  is optimal for  $e$ , then either  $\lambda$  or  $2\lambda$  is associated to  $e$ .

## NILPOTENT $K$ -ORBITS ON $\mathfrak{p}$

In this section, we look at nilpotent orbits related to the study of symmetric spaces. Let  $M$  be a Riemannian manifold with a geodesic symmetry  $s_p$  at each point  $p$ . Then  $M$  is a *symmetric space* if each  $p \in M$  is an isolated fixed point of  $s_p$ . Symmetric spaces arise when studying Lie groups equipped with involutions (Lie group automorphisms of order two) as follows. Let  $\theta$  be an involution on a Lie group  $G$ , and let  $K = G^\theta$  be the closed subgroup of  $G$  consisting of the fixed points of  $\theta$ . In other words,  $K = \{g \in G \mid \theta(g) = g\}$ . Then  $G/K$  has the structure of a symmetric space (see [12, Theorem 3.3]). If  $G$  is an algebraic group instead of a Lie group and  $K$  is the fixed-point subgroup of an involution  $\theta$  (which is now an automorphism of algebraic groups), we will still refer to  $G/K$  as a symmetric space, even though it no longer has the structure of a Riemannian manifold. The differential of  $\theta$  is an involution on  $\mathfrak{g} = \text{Lie}(G)$ , which we will still denote by  $\theta$ . We thus get a decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\mathfrak{k}$  (resp.,  $\mathfrak{p}$ ) is the  $+1$ - (resp.,  $-1$ -) eigenspace of  $\theta$  in  $\mathfrak{g}$ . Since  $\theta$  is a semisimple automorphism,  $\text{Lie}(K) = \mathfrak{k}$ . Also,  $\mathfrak{p}$  identifies with the tangent space at the identity of  $G/K$ . For this reason, we call  $\mathfrak{p}$  an *infinitesimal symmetric space*.

In this section, we assume that  $G$  is a reductive linear algebraic group and  $\text{char}(k) \neq 2$ , at least. We show that  $K$  acts on  $\mathfrak{p}$  and  $\mathcal{N}(\mathfrak{p})$  and then give several known classifications of the orbit set  $\mathcal{N}(\mathfrak{p})/K$ . This section sets the stage for §, which gives a new classification of  $\mathcal{N}(\mathfrak{p})/K$ .

**Example 8.** Let  $G = SL_n(k)$ , and define an involution  $\theta : G \rightarrow G$  by  $\theta(g) = (g^{-1})^\top$ . Then  $K = SO_n(k)$ , and the involution induced on  $\mathfrak{g} = \mathfrak{sl}_n(k)$  is given by  $\theta(x) = -x^\top$ . Thus  $\mathfrak{k} = \mathfrak{so}_n(k)$ , and  $\mathfrak{p}$  is the subspace of all symmetric matrices of trace zero. The adjoint action of  $K$  on  $\mathcal{N}(\mathfrak{p})$  is just conjugation by matrices in  $K$ . In the special case



that  $n = 2$ , we have that

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathcal{N}(\mathfrak{p}) \iff a^2 + b^2 = 0.$$

Thus,  $\mathcal{N}(\mathfrak{p})$  is the variety  $\mathcal{V}(x^2 + y^2)$ , which is reducible, being the union of the lines  $y = \pm ix$ , where  $i = \sqrt{-1}$  by definition. Furthermore, it can be shown that there are exactly three  $K$ -conjugacy classes in  $\mathcal{N}(\mathfrak{p})$  with representatives  $0$ ,  $\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$ , and  $\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}$ .

**Example 9.** Let  $G$  be any linear algebraic group with  $\mathfrak{g} = \text{Lie}(G)$ . Let  $H = G \times G$ , and define  $\theta : H \rightarrow H$  by  $\theta(h_1, h_2) = (h_2, h_1)$ . Then  $K = H^\theta \cong G$  and  $\mathfrak{p} \cong \mathfrak{g}$ , and the  $K$ -orbits in  $\mathfrak{p}$  (resp.,  $\mathcal{N}(\mathfrak{p})$ ) identify with the  $G$ -orbits in  $\mathfrak{g}$  (resp.,  $\mathcal{N}(\mathfrak{g})$ ). Thus, the study of  $K$ -orbits in  $\mathfrak{p}$  and  $\mathcal{N}(\mathfrak{p})$  includes that of  $G$ -orbits in  $\mathfrak{g}$  and  $\mathcal{N}(\mathfrak{g})$ .

Let  $\theta$  be an involution on  $G$ . Since  $\theta$  is an automorphism, we have the following elementary result, where we go against the convention stated above and denote the induced involution on  $\mathfrak{g}$  by  $d\theta$  instead of  $\theta$  for the sake of clarity.

**Lemma 10.** *For all  $x \in G$ ,  $d\theta \circ \text{Ad}(x) = \text{Ad}(\theta(x)) \circ d\theta$ .*

*Proof.* Choose  $y \in G$ . Then

$$\begin{aligned} (\theta \circ \text{Int}(x))(y) &= \theta(xyx^{-1}) \\ &= \theta(x)\theta(y)\theta(x^{-1}) \\ &= (\text{Int}(\theta(x)) \circ \theta)(y). \end{aligned}$$

Thus  $\theta \circ \text{Int}(x) = \text{Int}(\theta(x)) \circ \theta$ . Taking the differential of both sides, we get  $d\theta \circ \text{Ad}(x) = \text{Ad}(\theta(x)) \circ d\theta$ .  $\square$

We will need a few more basic facts about involutions and groups with involutions. Since involutions are semisimple automorphisms, [37, Theorem 7.5] gives that every algebraic group  $G$  contains a  $\theta$ -stable Borel subgroup  $B$  which in turn contains a  $\theta$ -stable torus  $T$ . Then the subgroup  $\langle \theta \rangle$  of  $\text{Aut}(G)$  acts on  $X^*(T)$  with an action

defined by  $\theta\alpha = \alpha \circ \theta$  for  $\alpha \in X^*(T)$ , and on  $X_*(T)$  with an action defined by  $\theta\lambda = \theta \circ \lambda$  for  $\lambda \in X_*(T)$ . The following lemma will be useful in §.

**Lemma 11.** *The pairing  $\langle, \rangle$  between  $X^*(T)$  and  $X_*(T)$  is  $\theta$ -equivariant.*

*Proof.* We have for all  $t \in k^*$ ,  $\alpha \in X^*(T)$ , and  $\tau \in X_*(T)$ ,

$$t^{\langle \theta\alpha, \theta\tau \rangle} = \theta\alpha(\theta\tau(t)) = \alpha(\theta(\theta(\tau(t)))) = \alpha(\tau(t)) = t^{\langle \alpha, \tau \rangle}.$$

Thus,  $\langle \theta\alpha, \theta\tau \rangle = \langle \alpha, \tau \rangle$ . □

If  $\Phi$  is the root system of  $G$  defined by  $T$  and  $\Phi^+$  is the set of positive roots of  $\Phi$  defined by  $B$ , then these are  $\langle \theta \rangle$ -stable subsets of  $X^*(T)$ . To see this, let  $x$  be a nonzero element of  $\mathfrak{g}_\alpha$ , which exists because  $\alpha$  is a root. This of course implies  $\theta(x) \neq 0$ . Using Lemma 10, for any  $t \in k$ ,

$$\begin{aligned} \text{Ad}(t)(\theta(x)) &= \text{Ad}(\theta(\theta(t)))(\theta(x)) \\ &= \theta(\text{Ad}(\theta(t))x) \\ &= \theta(\alpha(\theta(t))x) \\ &= \theta(\theta\alpha(t)x) \\ &= \theta\alpha(t)\theta(x). \end{aligned}$$

This shows that  $\theta(x)$  is a nonzero element of  $\mathfrak{g}_{\theta\alpha}$ , so that  $\theta\alpha$  is indeed a root.

Now, if  $\alpha$  is a positive root, then  $\alpha$  is a root of  $B$  relative to  $T$ . Thus, there is a nonzero element  $x \in \text{Lie}(B)$  such that for all  $t \in k$ ,  $\text{Ad}(t)(x) = \alpha(t)x$ . We showed above that this implies that  $\text{Ad}(t)(\theta(x)) = \theta\alpha(t)\theta(x)$ , i.e., that  $\theta(x)$  is a nonzero element of  $\mathfrak{g}_{\theta\alpha}$ . But  $B$ , and hence  $\text{Lie}(B)$ , is  $\theta$ -stable, which means that  $\theta(x) \in \text{Lie}(B)$ , so  $\theta\alpha$  is also a root of  $B$  relative to  $T$ . In other words,  $\Phi^+$  is  $\theta$ -stable. The following lemma follows easily from the definition of the action of  $\theta$  on  $\Phi$ .

**Lemma 12.** *Let  $\Phi = \Phi(G, T)$  for a  $\theta$ -stable torus  $T$ . Then for all  $\alpha \in \Phi$ ,  $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\theta\alpha}$ .*

By Lemma 10, if  $g \in K$  and  $x \in \mathfrak{p}$ ,

$$\begin{aligned}\theta(\operatorname{Ad} g(x)) &= \theta(\operatorname{Ad} \theta g(\theta(-x))) \\ &= \theta(\theta(\operatorname{Ad} g(-x))) \\ &= -\operatorname{Ad} g(x),\end{aligned}$$

i.e.,  $\operatorname{Ad} g(x) \in \mathfrak{p}$ . This shows that the adjoint action of  $G$  on  $\mathfrak{g}$  restricts to an action of  $K$  on  $\mathfrak{p}$ . Furthermore,  $K$  acts on the set of nilpotent elements  $\mathcal{N}(\mathfrak{g}) \cap \mathfrak{p}$ , denoted  $\mathcal{N}(\mathfrak{p})$ . The action of  $K$  on  $\mathcal{N}(\mathfrak{p})$  and the orbits that arise from it will be the focus of the rest of this dissertation. The remaining part of this section summarizes some known results on this topic. The next section gives a new classification of the orbit set  $\mathcal{N}(\mathfrak{p})/K$ .

When  $k = \mathbb{C}$ , involutions of  $G$  are closely related to real forms of  $\mathfrak{g}$ . A *real form*  $\mathfrak{g}_{\mathbb{R}}$  of  $\mathfrak{g}$  is a real Lie subalgebra of  $\mathfrak{g}$  whose complexification  $\mathfrak{g}_{\mathbb{R}} + i\mathfrak{g}_{\mathbb{R}}$  is  $\mathfrak{g}$ . A real Lie algebra  $\mathfrak{g}_{\mathbb{R}}$  is *compact* if  $\operatorname{Int}(\mathfrak{g}_{\mathbb{R}})$  is a compact subgroup of  $\operatorname{Aut}(\mathfrak{g}_{\mathbb{R}})$ . Any real simple Lie algebra  $\mathfrak{g}_{\mathbb{R}}$  admits a *Cartan decomposition*  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ , where  $\mathfrak{k}_{\mathbb{R}}$  is a maximal compact subalgebra of  $\mathfrak{g}_{\mathbb{R}}$ , and  $\mathfrak{p}_{\mathbb{R}}$  is a  $\mathfrak{k}_{\mathbb{R}}$ -module such that  $[\mathfrak{p}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] \subset \mathfrak{k}_{\mathbb{R}}$ . From this, we can define an involution  $\theta$  on  $\mathfrak{g}_{\mathbb{R}}$ , and hence on its complexification  $\mathfrak{g}$ , by letting  $\theta$  be the identity on  $\mathfrak{k}_{\mathbb{R}}$  and the negative of the identity on  $\mathfrak{p}_{\mathbb{R}}$ . Exponentiating gives an involution on  $G$ , also denoted  $\theta$ . It turns out that this induces a bijection between the noncompact real forms of  $\mathfrak{g}$  and the conjugacy classes of involutions of  $G$  in  $\operatorname{Aut}(\mathfrak{g})$ . This bijection allowed Cartan to classify the Riemannian symmetric spaces using his classification of the real simple Lie algebras.

For example, let  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ . Then  $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R})$  is a real form of  $\mathfrak{g}$  with Cartan decomposition  $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}_n(\mathbb{R}) \oplus \mathfrak{p}_{\mathbb{R}}$ , where  $\mathfrak{p}_{\mathbb{R}}$  is the subspace of symmetric matrices of trace 0. The associated involution on  $G$  is the inverse transpose  $\theta(g) = (g^{-1})^{\top}$ .

Furthermore, it turns out that there is a bijection between the  $G_{\mathbb{R}}$ -orbits in  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$  and the  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ , where  $K = G^{\theta}$  and  $\mathfrak{p}$  is the  $-1$ -eigenspace of  $\theta$  in  $\mathfrak{g}$  for the involution  $\theta$  corresponding to  $\mathfrak{g}_{\mathbb{R}}$ . This is known as the *Kostant-Sekiguchi bijection*,

and it gives motivation to study  $\mathcal{N}(\mathfrak{p})$  and its  $K$ -orbits. The seminal work in this area is [22] by Kostant and Rallis, who assume throughout that  $\text{char}(k) = 0$ . Many of their results were adapted to the case where  $\text{char}(k)$  is good for  $G$  by Levy in [23].

In [31], Richardson formulated a global analogue of [22] by studying the action of  $K = G^\theta$  on a symmetric space  $P$ , defined as follows. Define  $\tau : G \rightarrow G$  by  $\tau(g) = g\theta(g)^{-1}$ , and let  $P = \tau(G)$ , a closed subvariety of  $G$ . Then  $P$  is stable under conjugation by  $K$ , and  $P$  and  $G/K$  are isomorphic as affine  $K$ -varieties. The tangent space at the identity to  $P$  is  $\mathfrak{p}$ . Many of the results in Richardson's paper are independent of  $\text{char}(k)$ , as long as  $\text{char}(k) \neq 2$ . In particular, he proves the following proposition (which assumes  $G$  is reductive and  $\text{char}(k) \neq 2$ ), whose corollary will be important for us in the next section.

**Proposition 13.** [31, Prop. 7.4] *Let  $\mathcal{U}(P)$  denote the set of unipotent elements of  $P$ . Then there are only finitely many  $K^\circ$ -orbits in  $\mathcal{U}(P)$ .*

This of course implies that there are only finitely many  $K$ -orbits in  $\mathcal{U}(P)$ . We will also need the following result, found in [2]. Assume that  $G$  and  $k$  satisfy the same hypotheses as in the previous proposition.

**Proposition 14.** [2, Prop. 10.1] *There exists a  $K$ -morphism  $P \rightarrow \mathfrak{p}$  which restricts to a  $K$ -isomorphism  $\mathcal{U}(P) \rightarrow \mathcal{N}(\mathfrak{p})$ .*

**Corollary 15.** *There are only finitely many  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ .*

*Proof.* This follows immediately from Propositions 13 and 14. □

The rest of this section will be devoted to summarizing a few existing classifications of  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ .

**Kawanaka's Classification** We start with one given by Kawanaka in [20] which is similar in spirit to the weighted Dynkin diagram classification described in the "Nilpotent  $G$ -Orbits on  $\mathfrak{g}$ " section. Kawanaka's classification is valid when  $\text{char}(k)$  is

good for  $G$ . We use Kawanaka's notation, and we refer the reader to [20] or [23, Section 5.2] for more details.

Let  $H = H(\Phi, \Delta)$  be the group of all homomorphisms from  $\mathbb{Z}\Phi \rightarrow \mathbb{Z}$ . Any element  $h \in H$  is completely determined by the values  $h(\alpha)$  for  $\alpha \in \Delta$ . Thus, we may identify  $h$  with the weighted Dynkin diagram for  $\Delta$  whose nodes are labeled with the values  $h(\alpha)$  for  $\alpha \in \Delta$ . For  $h \in H$ , let  $\mathfrak{g}(i, h) = \bigoplus_{h(\alpha)=i} \mathfrak{g}_\alpha$  when  $i \neq 0$ , and  $\mathfrak{g}(0, h) = \mathfrak{h} \oplus \bigoplus_{h(\alpha)=0} \mathfrak{g}_\alpha$ . Let  $\mathfrak{k}(i, h) = \{x \in \mathfrak{g}(i, h) \mid \theta(x) = x\}$  and  $\mathfrak{p}(i, h) = \{x \in \mathfrak{g}(i, h) \mid \theta(x) = -x\}$ . Kawanaka showed that a certain subset of  $H$ , which we will describe next, parameterizes  $\mathcal{N}(\mathfrak{p})/K$ .

Let  $H^+ = \{h \in H \mid h(\alpha) \geq 0 \forall \alpha \in \Delta\}$ . The Weyl group  $W$  acts on  $H$ , and for any  $h \in H$ , there exists  $w \in W$  and a unique  $h_+ \in H^+$  such that  $w(h) = h_+$ .

When  $k = \mathbb{C}$ , we saw that  $\mathcal{N}(\mathfrak{g})/G$  is parameterized by a set of weighted Dynkin diagrams, which Kawanaka denotes  $H_n$ . Thanks to Pommerening's classification,  $H_n$  also parameterizes  $\mathcal{N}(\mathfrak{g})/G$  when  $\text{char}(k)$  is good.

Suppose  $h \in H$  is  $\theta$ -stable. Define a subalgebra  $\bar{\mathfrak{g}}_h$  of  $\mathfrak{g}$  to be the direct sum of the graded components  $\bar{\mathfrak{g}}_h(i)$ ,  $i \in \mathbb{Z}$ , where

$$\bar{\mathfrak{g}}_h(i) = \begin{cases} \mathfrak{k}(i, h), & i \equiv 0 \pmod{4}, \\ \mathfrak{p}(i, h), & i \equiv 2 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\bar{G}_h$  be the connected subgroup of  $G$  whose Lie algebra is  $\bar{\mathfrak{g}}_h$ . Then  $T(0) = (T \cap K)^\circ$  is a maximal torus of  $\bar{G}_h$ . We let  $\Phi_h = \Phi(\bar{G}_h, T(0))$ , and we let  $\Delta_h$  denote a basis of simple roots in  $\Phi_h$ .

Since  $h \in H$ , and  $G$  is of adjoint type, there exists  $\lambda : k^* \rightarrow T \cap \mathcal{D}G$  such that  $\langle \alpha, \lambda \rangle = h(\alpha)$  for all  $\alpha \in \Delta$ . Define  $h$  to be *slim* with respect to  $\theta$  if  $\lambda(k^*) \subset \mathcal{D}(\bar{G}_h)$ .

Now define  $H(\Phi, \Delta, \theta)'_n$  to be the set of all  $h \in H$  such that

- $h_+ \in H_n$ ,

- $h$  is  $\theta$ -stable,
- $h$  is slim with respect to  $\theta$ ,
- $\bar{h}_+ \in H(\Phi_h, \Delta_h)_n$ .

Kawanaka's theorem (see [20, Theorem 3.1.5]) states that  $H(\Phi, \Delta, \theta)'_n$  parameterizes  $\mathcal{N}(\mathfrak{p})/K$ . The correspondence is stated as follows. Let  $\bar{G}_h(0)$  be the connected subgroup of  $\bar{G}_h$  whose Lie algebra is  $\bar{\mathfrak{g}}_h(0)$ . Then  $\bar{\mathfrak{g}}_h(2)$  is  $\bar{G}_h(0)$ -stable. Let  $N_h$  be an element of the open  $\bar{G}_h(0)$ -orbit in  $\bar{\mathfrak{g}}_h(2)$ . Kawanaka's bijection is obtained by associating each element  $h \in H(\Phi, \Delta, \theta)'_n$  to the orbit  $K \cdot N_h$ .

**Noël's Classification** In [25], Noël gave a classification of  $\mathcal{N}(\mathfrak{p})/K$  similar to the one Bala and Carter gave for  $\mathcal{N}(\mathfrak{g})/G$ . Throughout, Noël assumes  $k = \mathbb{C}$ . By the Kostant-Sekiguchi bijection, this then gives a classification of  $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})/G_{\mathbb{R}}$ . As usual  $\theta$  is an involution on  $\mathfrak{g}$ . Also, let  $\sigma$  denote the complex conjugation of  $\mathfrak{g}$  with respect to  $\mathfrak{g}_{\mathbb{R}}$ .

Noël defines an element  $e \in \mathcal{N}(\mathfrak{p})$  to be *noticed* if the only  $(\theta, \sigma)$ -stable Levi subalgebra of  $\mathfrak{g}$  containing  $e$  is  $\mathfrak{g}$  itself. Noticed elements are analogous to the distinguished elements used by Bala and Carter. (Every distinguished element is noticed, but not vice versa.) Since every nilpotent element  $e$  is noticed in the minimal  $(\theta, \sigma)$ -stable Levi subalgebra  $\mathfrak{r}$  containing it, then replacing  $\mathfrak{g}$  by  $\mathfrak{r}$ , we may assume that all nilpotent elements are noticed.

While Bala and Carter classified the distinguished orbits in terms of distinguished parabolic subalgebras, Noël classifies noticed orbits in terms of pairs  $(\mathfrak{q}, \mathfrak{w})$ , where  $\mathfrak{q}$  is a parabolic subalgebra of  $\mathfrak{g}$ , and  $\mathfrak{w}$  is a subset of the nilradical  $\mathfrak{u}$  of  $\mathfrak{q}$  which satisfies certain properties. Some of these properties are stated in terms of a bilinear form  $\kappa'$  on  $\mathfrak{g}$ , which we now define.

Let  $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ , where  $i = \sqrt{-1}$ . This is a compact real form of  $\mathfrak{g}$ , and  $\tau = \theta \circ \sigma$  is conjugation of  $\mathfrak{g}$  with respect to  $\mathfrak{g}_u$ . Now define the form  $\kappa'$  on  $\mathfrak{g}$  by  $\kappa'(x, y) = -\kappa(x, \tau(y))$  for all  $x, y \in \mathfrak{g}$ , where  $\kappa$  is the Killing form on  $\mathfrak{g}$ .

Now let  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  be a  $\theta$ -stable parabolic subalgebra of  $\mathfrak{g}$ , and let  $\mathfrak{m}$  be the orthogonal complement of  $[\mathfrak{u} \cap \mathfrak{k}, [\mathfrak{u} \cap \mathfrak{k}, \mathfrak{u} \cap \mathfrak{p}]]$  relative to  $\kappa'$  inside  $\mathfrak{u} \cap \mathfrak{p}$ . Suppose  $L$  is a Levi subgroup of  $G$  whose Lie algebra is  $\mathfrak{l}$ . Let  $\mathfrak{w}$  be an  $(L \cap K)$ -submodule of  $\mathfrak{m}$ , and  $\hat{\mathfrak{w}} = \mathfrak{w} \oplus [\mathfrak{l} \cap \mathfrak{p}, \mathfrak{w}]$ . Noël now defines  $\mathcal{L}$  to be the set of all pairs  $(\mathfrak{q}, \mathfrak{w})$  as defined above satisfying the following properties:

- $\mathfrak{w}$  has a dense  $(L \cap K)$ -orbit:  $(L \cap K) \cdot \hat{e}$ ,
- $\dim(\mathfrak{l} \cap \mathfrak{k}) = \dim \mathfrak{w}$ ,
- $L \cdot \hat{e}$  is dense in  $\hat{\mathfrak{w}}$ ,
- $\hat{\mathfrak{w}} \perp [\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$ ,
- $[\mathfrak{u}, \hat{\mathfrak{w}}] \perp \hat{\mathfrak{w}}$ ,
- $[\mathfrak{u} \cap \mathfrak{k}, \mathfrak{u} \cap \mathfrak{p}] \subset [\mathfrak{q} \cap \mathfrak{k}, \mathfrak{w}]$ .

To see an example of such a pair, let  $e \in \mathcal{N}(\mathfrak{p})$  be noticed, and let  $(x, e, f)$  be an  $\mathfrak{sl}_2$ -triple containing  $e$ . (Remember,  $k = \mathbb{C}$ , so this triple exists by the Jacobson-Morozov Theorem.) We may actually choose  $x \in \mathfrak{k}$  and  $f \in \mathfrak{p}$  (see for example [8, Theorem 9.4.2]). Such triples are called *normal triples*. Let  $\mathfrak{q}_x$  be the Jacobson-Morozov parabolic subalgebra associated to  $(x, e, f)$ . Then  $\mathfrak{q}_x$  is  $\theta$ -stable since  $x \in \mathfrak{k}$ . Let  $\mathfrak{w} = \mathfrak{g}(2) \cap \mathfrak{p}$ , where  $\mathfrak{g}(2) = \{z \in \mathfrak{g} \mid [x, z] = 2z\}$ . Then  $\hat{\mathfrak{w}} = \mathfrak{g}(2)$ , and one can show that the pair  $(\mathfrak{q}_x, \mathfrak{g}(2) \cap \mathfrak{p})$  satisfies the six conditions above. This gives a map from the set of noticed  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  to the set of  $K$ -conjugacy classes in  $\mathcal{L}$ , which Noël proves to be a bijection (see [25, Theorem 3.2.3]).

If  $e$  is not noticed in  $\mathfrak{g}$ , then it is noticed in  $\mathfrak{r}$ , where  $\mathfrak{r}$  is a minimal  $(\theta, \sigma)$ -stable Levi subalgebra containing  $e$ . Now, we associate the orbit  $K \cdot e$  to the triple  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}}, \mathfrak{w}_{\mathfrak{r}})$ , where  $(\mathfrak{q}_{\mathfrak{r}}, \mathfrak{w}_{\mathfrak{r}})$  is a pair defined above inside of  $\mathfrak{r}$ . This induces a bijection from the set of all nilpotent  $K$ -orbits in  $\mathfrak{p}$  to the set of  $K$ -conjugacy classes of triples  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}}, \mathfrak{w}_{\mathfrak{r}})$

consisting of  $(\theta, \sigma)$ -stable Levi subalgebras  $\mathfrak{r}$  of  $\mathfrak{g}$  and pairs  $(\mathfrak{q}_{\mathfrak{r}}, \mathfrak{w}_{\mathfrak{r}})$  of  $\mathcal{L}$  inside of  $\mathfrak{r}$  (see [25, Theorem 3.2.4]).

**Partition Classification** When  $G$  is of classical type, there exists a classification of the nilpotent  $K$ -orbits in  $\mathfrak{p}$  in terms of partitions of  $n$ . Ohta gave such a classification when  $k = \mathbb{C}$  in [26]. In several critical spots, Ohta used techniques which are in general unhelpful or unavailable when  $\text{char}(k) > 0$ , such as the Jacobson-Morozov Theorem and the representation theory of  $\mathfrak{sl}_2(k)$ . Fox, Hodge, and Parshall in [9] have recently given a partition classification of  $\mathcal{N}(\mathfrak{p})/K$  in the classical cases which uses primarily linear algebra rather than Lie theory, avoiding Ohta's characteristic 0 methods. This gives a classification which only requires that  $\text{char}(k) \neq 2$ , which we now summarize.

The classification in [9] relies on a result of Springer. In [34], he classified the involutions on  $G$  for each Cartan type up to conjugation by an inner automorphism of  $G$ . There are seven involution classes when  $G$  is of classical type, which we list below. The  $m \times m$  identity matrix is denoted by  $I_m$ . In some cases there is an infinite family of non-conjugate involutions indexed by a pair of positive integers  $(a, b)$ .

1. Type AI:  $G = GL_n(k)$ ,  $\theta(g) = (g^{-1})^{\top}$
2. Type AII:  $G = GL_n(k)$ ,  $n$  even,  $\theta(g) = J(g^{-1})^{\top}J^{-1}$ , where  $J^2 = -I_n$
3. Type AIII $_{a,b}$ :  $G = GL_n(k)$ ,  $a + b = n$ ,  $a \leq b$ ,  $\theta(g) = JgJ^{-1}$ , where  $J = \text{diag}(I_a, -I_b)$
4. Type BDI $_{a,b}$ :  $G = SO_n(k)$ ,  $a + b = n$ ,  $a \leq b$ ,  $\theta(g) = JgJ^{-1}$ , where  $J = \text{diag}(I_a, -I_b)$
5. Type CI:  $G = Sp_n(k)$ ,  $n$  even,  $\theta(g) = JgJ^{-1}$ , where  $J^2 = -I_n$
6. Type CII $_{a,b}$ :  $G = Sp_n(k)$ ,  $n = 2m$  even,  $0 < a \leq b < m$ ,  $a + b = m$ ,  $\theta(g) = JgJ^{-1}$ , where  $J = \text{diag}(I_a, -I_b)$



7. Type DIII:  $G = SO_n(k)$ ,  $n$  even,  $\theta(g) = JgJ^{-1}$ , where  $J^2 = -I_n$

The  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  are then classified in [9] case-by-case for each classical involution type using linear algebra methods which apply to any algebraically closed field whose characteristic is not 2. The classification in [9] does not require one to already know the classification of  $\mathcal{N}(\mathfrak{g})/G$  (except for the case  $G = GL_n(k)$  or  $G = SL_n(k)$ ). We now give a summary of the results for each involution class. In what follows, the set of  $m \times n$  matrices with entries in  $k$  is denoted by  $M_{m,n}(k)$ .

**Type AI:** In this case,  $K = O_n(k)$  and  $\mathfrak{p}$  is the space of symmetric matrices. Let  $\lambda$  be a partition of  $n$ . Then it is shown that the  $G$ -orbit  $\mathcal{O}_\lambda$  in  $\mathcal{N}(\mathfrak{g})$  corresponding to  $\lambda$  intersects  $\mathfrak{p}$  in a single  $K$ -orbit (see [9, Theorem 6.1.1]). Thus,  $\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of all partitions of  $n$ . When  $G = GL_n(k)$  is replaced by  $G = SL_n(k)$ , then  $K$  becomes  $SO_n(k)$ , and each  $G$ -orbit still intersects  $\mathfrak{p}$  in a single  $K$ -orbit except in the case when  $\lambda$  has all even parts. In this case,  $\mathcal{O}_\lambda \cap \mathfrak{p}$  splits into two  $K$ -orbits (see [9, Prop. 6.1.6]).

**Type AII:** Here,  $K \cong Sp_n(k)$ , and  $\mathfrak{p}$  is the set of skew-Hamiltonian matrices, i.e. ,

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ C & A^\top \end{pmatrix} \mid A, B, C \in M_{n,n}(k), B^\top = -B, C^\top = -C \right\}.$$

Again let  $\lambda$  be a partition of  $n$ . Then the  $G$ -orbit  $\mathcal{O}_\lambda$  intersects  $\mathfrak{p}$  in a nonempty set if and only if all of the parts of  $\lambda$  occur an even number of times. Thus,  $\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of all partitions of  $m$ , where  $m = n/2$  (see [9, Theorem 6.2.1]). The classification does not change when  $GL_n(k)$  is replaced by  $SL_n(k)$ .

The  $K$ -orbits for Types AI and AII were classified by considering the Jordan canonical forms of their representatives. This gave a correspondence between orbits and partitions of  $n$ , or equivalently *Young diagrams*, which are defined as follows. Let  $\lambda = (\lambda_1, \lambda_2, \dots)$ . The Young diagram of shape  $\lambda$  is the collection of left-justified rows

of boxes of lengths, counting from top to bottom,  $\lambda_1, \lambda_2, \dots$ . For example, the Young diagram corresponding to the partition  $(5, 2, 1)$  is shown in Figure 1 below.

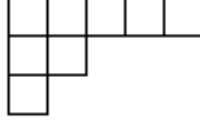


Figure 1. Young diagram for the partition  $(5, 2, 1)$  of 8.

The  $K$ -orbits for the remaining types will be classified using more specific data, namely *signed Young diagrams*. A signed Young diagram is a Young diagram whose boxes are filled in with the signs  $+$  and  $-$  in such a way that the signs alternate across every row. There is an equivalence relation  $\sim$  on the set of all signed Young diagrams of shape  $\lambda$ , where  $\lambda$  is a partition of  $n$ , which says that for signed Young diagrams  $Y_1$  and  $Y_2$ ,  $Y_1 \sim Y_2$  if and only if  $Y_1$  can be obtained from  $Y_2$  by some permutation of its rows.

**Type AIII<sub>a,b</sub>:** Fix positive integers  $a \leq b$  such that  $a + b = n$ , and define the matrix  $J = J_{a,b} = \text{diag}(I_a, -I_b)$ . Then conjugation by  $J$  defines an involution of Type AIII. We have  $K = GL_a(k) \times GL_b(k)$  and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A \in M_{a,b}(k), B \in M_{b,a}(k) \right\}.$$

Let  $V_+$  be the  $+1$ -eigenspace of  $J$  in  $k^n$ , and let  $V_-$  be the  $-1$ -eigenspace. Now, let  $X \in \mathcal{N}(\mathfrak{p})$  have Jordan canonical form corresponding to the partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  of  $n$ . Then we may choose vectors  $v_1, v_2, \dots, v_l \in k^n$  such that

$$v_1, Xv_1, \dots, X^{\lambda_1-1}v_1, v_2, Xv_2, \dots, X^{\lambda_2-1}v_2, \dots, v_l, Xv_l, \dots, X^{\lambda_l-1}v_l$$

is a basis for  $k^n$ . This is the basis relative to which  $X$  is in Jordan canonical form. Furthermore, we may choose each  $v_i$ ,  $1 \leq i \leq l$ , to be in either  $V_+$  or  $V_-$ . Notice that  $X(V_\pm) \subset V_\mp$ . Form the Young diagram  $Y_\lambda$  of shape  $\lambda$ . The boxes correspond to the basis vectors above as follows: the  $i$ th box in the  $j$ th row corresponds to the basis vector  $X^{i-1}v_j$ . Now fill in each box with a  $+$  or a  $-$  depending on whether the corresponding basis vector is in  $V_+$  or  $V_-$ , respectively. In this way, we create a signed Young diagram associated to the  $K$ -orbit  $K \cdot X$ . It will contain  $a$   $+$ 's and  $b$   $-$ 's, so we say that its *signature* is  $(a, b)$ . For example, let

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the Jordan canonical form of  $X$  relative to the ordered basis  $(e_3, e_2, e_4, e_1, e_5)$ , where  $e_i$  is the  $i$ th standard basis vector in  $k^5$ , is

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the partition  $(4, 1)$  of 5. Further, for  $J = \text{diag}(1, 1, -1, -1, -1)$ ,  $e_3$ ,  $e_4$ , and  $e_5$  are in  $V_-$ , and  $e_1$  and  $e_2$  are in  $V_+$ . Thus, the signed Young diagram associated to the orbit  $K \cdot X$  is the one shown in Figure 2 below.

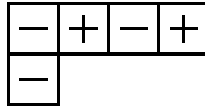


Figure 2. Signed Young diagram of signature  $(2,3)$ .

It turns out that when  $\theta$  is of Type AIII $_{a,b}$ ,  $\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of equivalence classes of signed Young diagrams of signature  $(a, b)$

(see [9, Theorem 6.3.2]). The classification is unchanged when  $GL_n(k)$  is replaced by  $SL_n(k)$ .

**The Other Classical Types** For the remaining cases, one can associate each  $K$ -orbit in  $\mathcal{N}(\mathfrak{p})$  to a signed Young diagram similar to the manner described in the Type AIII case. The results of the remaining classifications are summarized below (see [9, Theorem 7.3]):

- In Type BDI $_{a,b}$ ,  $K = S(O_a(k) \times O_b(k))$ , and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A \\ -A^\top & 0 \end{pmatrix} \mid A \in M_{a,b}(k) \right\}.$$

$\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of equivalence classes of signed Young diagrams of signature  $(a,b)$  that have the property that rows of even length occur in pairs of the same length in which one row begins with a + and one with a -.

- In Type CI (with  $n = 2m$ ),  $K \cong GL_m(k)$  and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A = A^\top \in M_{m,m}(k), B = B^\top \in M_{m,m}(k) \right\}.$$

$\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of equivalence classes of signed Young diagrams of signature  $(m,m)$  that have the property that rows of odd length occur in pairs in which one row begins with a + and the other with a -.

- In Type CII $_{a,b}$ ,  $K \cong Sp_a(k) \times Sp_b(k)$  and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A, B, C, D \in \mathfrak{p}^{AIII_{a,b}}, A^\top = -D, B^\top = B, C^\top = C \right\},$$

where  $\mathfrak{p}^{AIII_{a,b}}$  is the set of matrices in  $\mathfrak{p}$  when  $\theta$  is of Type AIII $_{a,b}$ .  $\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of equivalence classes of signed Young diagrams of signature  $(2a, 2b)$  that have the property that all rows occur in pairs

of equal length such that (i) for each pair of odd length rows, both rows begin with the same sign, and (ii) for each pair of even length rows, one row begins with a + and the other with a -.

- In Type DIII (with  $n = 2m$ ),  $K \cong GL_m(k)$  and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A, B \in M_{m,m}(k), A^\top = -A, B^\top = -B \right\}.$$

$\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with the set of equivalence classes of signed Young diagrams of signature  $(m, m)$  that have the property that all rows occur in pairs of the same length such that (i) for each pair of even length rows, both rows begin with the same sign, and (ii) for each pair of odd length rows, one row begins with a + and the other with a -.

The main result of this dissertation is a classification of  $\mathcal{N}(\mathfrak{p})/K$  similar to the one which Bala-Carter and Pommerening gave for  $\mathcal{N}(\mathfrak{g})/G$  in terms of Levi and parabolic subalgebras, which we give in the next section. Noël did this when  $\text{char}(k) = 0$ , but the one given in the next section is valid whenever  $\text{char}(k)$  is good.

## A CLASSIFICATION OF $\mathcal{N}(\mathfrak{p})/K$

In this section, we give a new classification of the nilpotent  $K$ -orbits in  $\mathfrak{p}$ . This is significantly different from Kawanaka's classification in that it uses parabolic and Levi subalgebras instead of weighted Dynkin diagrams. It is also different from Noël's classification since we are not assuming that  $\text{char}(k) = 0$ , which was an essential condition in [25] needed to utilize the theory of normal triples. Instead, we make use of certain cocharacters associated to nilpotent elements to give a classification that only assumes  $\text{char}(k)$  is good. Since this also includes the case when  $\text{char}(k) = 0$ , we get a simpler version of the classification given in [25].

Let  $k$  be an algebraically closed field, and let  $G$  be a connected reductive group defined over  $k$ . In this section, we will assume the standard hypotheses on  $G$ :

- (SH1) The characteristic of  $k$  is good for  $G$ .
- (SH2) The derived subgroup  $\mathcal{D}G$  is simply connected.
- (SH3) There exists a  $G$ -invariant nondegenerate bilinear form on  $\mathfrak{g}$ .

Recall that under these hypotheses, we have that  $\text{Lie } C_G(x) = \mathfrak{g}^x$  for all  $x \in \mathfrak{g}$ . The following fact was stated in the nilpotent  $G$ -orbits section:

**Proposition 16.** *[19, Prop. 2.7a] There is a bijection from the set of nilpotent  $G$ -orbits in  $\mathfrak{g}$  to the set of nilpotent  $G/Z(G)$ -orbits in  $\text{Lie}(G/Z(G))$ .*

This proposition says that when classifying nilpotent orbits, we can reduce to the case where  $G$  is semisimple of adjoint type. We will make these assumptions from now on, unless stated otherwise.

Let  $\theta$  be an involution on  $G$ . We will also denote the induced involution on  $\mathfrak{g} = \text{Lie}(G)$  by  $\theta$ . Let  $K = G^\theta$ . Also, let  $\mathfrak{k}$  be the  $+1$ -eigenspace of  $\theta$  in  $\mathfrak{g}$ , and let  $\mathfrak{p}$

be the  $-1$ -eigenspace. As stated in §,  $G$  has a  $\theta$ -stable maximal torus  $T$  and a  $\theta$ -stable Borel subgroup  $B$  containing  $T$ . Let  $\Phi$  be the  $\theta$ -stable root system of  $G$  corresponding to  $T$ , and let  $\Delta$  be the  $\theta$ -stable basis of simple roots of  $\Phi$  corresponding to  $B$ . Let  $\mathfrak{h} = \text{Lie}(T)$ , a  $\theta$ -stable Cartan subalgebra of  $\mathfrak{g}$ .

A cocharacter  $\lambda : k^* \rightarrow G$  is said to be *associated* with  $e \in \mathcal{N}(\mathfrak{g})$  if  $\text{Ad}(\lambda(t))e = t^2e$  for all  $t \in k^*$ , and if  $\text{Im } \lambda$  is contained in the derived subgroup of a Levi subgroup  $L$  of  $G$  such that  $\text{Lie}(L)$  is a minimal Levi subalgebra containing  $e$ .

Given a cocharacter  $\lambda \in X_*(G)$ , we get a grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\lambda, i)$$

of  $\mathfrak{g}$ , where  $\mathfrak{g}(\lambda, 0) = \mathfrak{h} \oplus \bigoplus_{\langle \alpha, \lambda \rangle = 0} \mathfrak{g}_\alpha$  and  $\mathfrak{g}(\lambda, i) = \bigoplus_{\langle \alpha, \lambda \rangle = i} \mathfrak{g}_\alpha$  for  $i \neq 0$ . Define

$$\mathfrak{q} = \mathfrak{q}(\lambda) = \bigoplus_{i \geq 0} \mathfrak{g}(\lambda, i).$$

This is a parabolic subalgebra of  $\mathfrak{g}$ . If  $\mathfrak{l} = \mathfrak{l}(\lambda) = \mathfrak{g}(\lambda, 0)$  and  $\mathfrak{u} = \mathfrak{u}(\lambda) = \bigoplus_{i > 0} \mathfrak{g}(\lambda, i)$ , then  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  is a Levi decomposition of  $\mathfrak{q}$ . One can show that  $L = C_G(\lambda)$  is a Levi subgroup of  $G$  such that  $\text{Lie}(L) = \mathfrak{l}$ , where  $C_G(\lambda) = \{g \in G \mid g\lambda(t)g^{-1} = \lambda(t) \text{ for all } t \in k^*\}$ .

Let  $e$  be an element in  $\mathcal{N}(\mathfrak{g})$ , and let  $N(e) = \{g \in G \mid \text{Ad}(g)e \in ke\}$ , a closed subgroup of  $G$ . Any cocharacter of  $G$  associated with  $e$  is in  $X_*(N(e))$ . By [37] again,  $N(e)$  has a maximal torus which is  $\theta$ -stable.

**Lemma 17.** *Let  $e \in \mathcal{N}(\mathfrak{p})$ , and let  $S$  be a  $\theta$ -stable maximal torus of  $N(e)$ . There is a unique cocharacter  $\lambda$  in  $X_*(S \cap K)$  associated with  $e$ .*

*Proof.* McNinch [24, Lemma 25] proved that every nilpotent element in  $\mathfrak{g}$  has a unique cocharacter in  $X_*(S)$  associated with it. Let  $\lambda$  be such a cocharacter associated with  $e$ . Since  $S$  is  $\theta$ -stable,  $\theta \circ \lambda \in X_*(S)$ . Since  $p > 2$ ,  $\theta$  is a semisimple automorphism by Proposition 2, which means that  $\theta$  is conjugation by a semisimple element  $s$  in a

linear algebraic group  $\overline{G}$  containing  $G$ . Then  $\text{Ad}(\theta \circ \lambda)(t)e = \text{Ad}(s\lambda(t)s^{-1})e = t^2e$  for all  $t \in k^*$ . Also,  $(\theta \circ \lambda)(k^*) \subset \mathcal{D}(sLs^{-1})$  and  $\text{Lie}(sLs^{-1}) = \text{Ad } s\text{Lie}(L)$  is a minimal Levi subalgebra containing  $e$ . Thus  $\theta \circ \lambda$  is associated with  $e$ . The uniqueness of  $\lambda$  now implies that  $\theta \circ \lambda = \lambda$ , which means  $\lambda$  is also a cocharacter of  $K$ .  $\square$

Given  $e \in \mathcal{N}(\mathfrak{p})$ , Lemma 17 gives an associated cocharacter  $\lambda \in X_*(S \cap K)$ . Then there exists an element  $x \in K$  such that  $x\lambda x^{-1} \in X_*(T \cap K)$ , and  $x\lambda x^{-1}$  is associated to  $x \cdot e$ . Replacing  $e$  by  $x \cdot e$  does not affect the orbit  $K \cdot e$ , so we may assume that  $\lambda \in X_*(T \cap K)$ . We will denote the unique cocharacter in  $X_*(T \cap K)$  associated to  $e \in \mathcal{N}(\mathfrak{p})$  by  $\lambda_e$ .

**Lemma 18.** *Let  $e \in \mathcal{N}(\mathfrak{p})$ , and let  $\lambda = \lambda_e$ . Let  $L = C_G(\lambda)$ , and let  $\mathfrak{l} = \text{Lie } L$ .*

- (a)  $\text{Lie}(L \cap K) = \mathfrak{l} \cap \mathfrak{k}$ .
- (b)  $(C_{L \cap K}(e))^\circ$  is reductive.
- (c)  $\text{Lie}(C_{L \cap K}(e)) = (\mathfrak{l} \cap \mathfrak{k})^e$ .
- (d)  $\text{Lie}(C_K(e)) = \mathfrak{k}^e$ .

*Proof.* (a) Let  $g \in L$ . Then for all  $t \in k^*$ ,  $\theta(g)\lambda(t)\theta(g)^{-1} = \theta(g\lambda(t)g^{-1})$  since  $\lambda(t) \in K$ . But this equals  $\theta(\lambda(t))$  since  $g \in L$ , and this in turn is just  $\lambda(t)$ . Thus  $\theta(g) \in L$ , and we have that  $\theta$  restricts to a semisimple automorphism of  $L$ . Thus  $\text{Lie}(L \cap K) = \text{Lie}(L^\theta) = \text{Lie}(L)^\theta = \mathfrak{l}^\theta = \mathfrak{l} \cap \mathfrak{k}$ .

(b) Since  $e \in \mathcal{N}(\mathfrak{g})$ , the centralizer  $C_L(e)$  is reductive (see [19]). It thus follows from [39] that  $(C_L(e)^\theta)^\circ = (C_{L \cap K}(e))^\circ$  is also reductive.

(c) By [19, Prop. 5.10],  $\text{Lie}(C_L(e)) = \mathfrak{l}^e$ . Using this fact, and the fact that  $\theta$  restricts to a semisimple automorphism of  $C_L(e)$ , an argument similar to the one in part (a) gives the desired result.

(d) Since  $\theta$  is semisimple,  $\text{Lie } C_K(e) = \text{Lie}(C_G(e))^\theta = (\mathfrak{g}^e)^\theta = \mathfrak{k}^e$ .  $\square$



Suppose  $\mathfrak{g}$  is semisimple. If  $\mathfrak{t}$  is a Levi subalgebra of  $\mathfrak{g}$ , then  $\mathfrak{t} = \mathfrak{g}^{\mathfrak{s}}$  for some subset  $\mathfrak{s}$  of  $\mathfrak{g}$  consisting of semisimple elements. This follows from the fact stated in the background section that all Levi subgroups of  $G$  are conjugate to a subgroup of the form  $C_G(S)$  for some torus  $S$ . We call a Levi subalgebra *special* if  $\mathfrak{s} \subset \mathfrak{k}$ . We call an element  $e \in \mathcal{N}(\mathfrak{p})$  *featured* in  $\mathfrak{g}$  if the only special Levi subalgebra of  $\mathfrak{g}$  containing  $e$  is  $\mathfrak{g}$  itself. Featured elements are analogous to Noël's noticed elements and Bala and Carter's distinguished elements. Every distinguished element is featured, but the converse need not be true. The relationship between featured and noticed elements is less clear and is still being investigated.

Since any nilpotent element is featured in a minimal special Levi subalgebra containing it, we can obtain the general classification of nilpotent orbits by first classifying the featured orbits, i.e., the orbits containing featured elements. We have the following characterizations of featured elements.

**Proposition 19.** *Let  $e$  be an element in  $\mathcal{N}(\mathfrak{p})$ . Let  $\lambda = \lambda_e$ , as defined after Lemma 17. The following are equivalent:*

- (a)  $e$  is featured in  $\mathfrak{g}$ .
- (b)  $\mathfrak{k}^e$  contains no nonzero semisimple elements.
- (c)  $\dim(\mathfrak{g}(\lambda, 0) \cap \mathfrak{k}) = \dim(\mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$ .

*Proof.* ((a)  $\Leftrightarrow$  (b)) By definition, if  $e$  is featured then all semisimple elements of  $\mathfrak{k}^e$  are contained in the center of  $\mathfrak{g}$ . But  $\mathfrak{g}$  is semisimple, so its center is  $\{0\}$ . Conversely, suppose  $\mathfrak{k}^e$  contains no nonzero semisimple elements, and let  $\mathfrak{t} = \mathfrak{g}^{\mathfrak{s}}$  be a special Levi subalgebra containing  $e$ . Then  $\mathfrak{s} \subset \mathfrak{k}^e$ , so  $\mathfrak{s} = \{0\}$ . Thus,  $\mathfrak{t} = \mathfrak{g}$ , and hence  $e$  is featured.

((b)  $\Leftrightarrow$  (c)) Let  $\mathfrak{l} = \mathfrak{g}(\lambda, 0)$ . Since  $e \in \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$ , the map  $\text{ad } e : \mathfrak{l} \cap \mathfrak{k} \rightarrow \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$  is onto, which means  $\dim(\mathfrak{l} \cap \mathfrak{k}) = \dim(\mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$  if and only if this map is one-to-one. The kernel of this map is  $(\mathfrak{l} \cap \mathfrak{k})^e$ . But  $(\mathfrak{l} \cap \mathfrak{k})^e$  is the Lie algebra of the reductive group

$(C_{L \cap K}(e))^\circ$  (Lemma 18(b) and (c)), so it will be nonzero if and only if it contains nonzero semisimple elements. Thus, we have so far that  $\dim(\mathfrak{l} \cap \mathfrak{k}) = \dim(\mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$  if and only if  $(\mathfrak{l} \cap \mathfrak{k})^e$  contains no nonzero semisimple elements. Now since  $\lambda_e$  is associated to  $e$ , by [19, Prop. 5.9]  $\mathfrak{g}^e = \mathfrak{q}^e$ , which implies  $\mathfrak{k}^e = (\mathfrak{l} \oplus \mathfrak{u})^e \cap \mathfrak{k}$ . Thus  $(\mathfrak{l} \cap \mathfrak{k})^e$  contains no nonzero semisimple elements if and only if  $\mathfrak{k}^e$  contains no nonzero semisimple elements.  $\square$

Let  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  be a parabolic subalgebra of  $\mathfrak{g}$ , where  $\mathfrak{l}$  is a Levi subalgebra and  $\mathfrak{u}$  is the nilpotent radical of  $\mathfrak{q}$ . Let  $Q$ ,  $L$ , and  $U$  be the connected subgroups of  $G$  having Lie algebras  $\mathfrak{q}$ ,  $\mathfrak{l}$ , and  $\mathfrak{u}$ , respectively. Suppose  $[\mathfrak{u}, \mathfrak{u}]$  has an  $L$ -stable complement  $V_{\mathfrak{u}}$  in  $\mathfrak{u}$ . (This is automatically true in characteristic 0 because  $L$  is a reductive group.) This implies that  $[\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$  has an  $L$ -stable complement  $V_{[\mathfrak{u}, \mathfrak{u}]}$  in  $[\mathfrak{u}, \mathfrak{u}]$ . Let  $\mathfrak{m}$  be the closure of an  $(L \cap K)^\circ$ -orbit in  $V_{\mathfrak{u}} \cap \mathfrak{p}$  or in  $V_{[\mathfrak{u}, \mathfrak{u}]} \cap \mathfrak{p}$ . Then  $\mathfrak{m}$  is an irreducible subset of  $\mathfrak{p}$  since  $(L \cap K)^\circ$  is irreducible. Since there are only finitely many  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ , by the argument used in the nilpotent  $G$ -orbits section there must be a unique  $K$ -orbit  $\mathcal{O}_{(\mathfrak{q}, \mathfrak{m})}$  such that  $\mathcal{O}_{(\mathfrak{q}, \mathfrak{m})} \cap \mathfrak{m}$  is open and dense in  $\mathfrak{m}$ . Notice that  $\mathcal{O}_{(\text{Ad}h \cdot \mathfrak{q}, \text{Ad}h \cdot \mathfrak{m})} = \mathcal{O}_{(\mathfrak{q}, \mathfrak{m})}$  for all  $h \in K$ . This shows that the definition of  $\mathcal{O}_{(\mathfrak{q}, \mathfrak{m})}$  is independent of the  $K$ -conjugacy class of the pair  $(\mathfrak{q}, \mathfrak{m})$ . We say that the pair  $(\mathfrak{q}, \mathfrak{m})$  as defined above is a *featured pair* of  $\mathfrak{g}$  if the following conditions are satisfied:

- $\mathfrak{q}$  is  $K$ -conjugate to a standard parabolic subalgebra
- $\mathfrak{q}$  is  $\theta$ -stable
- $[\mathfrak{u}, \mathfrak{u}]$  has an  $L$ -stable complement in  $\mathfrak{u}$
- $\dim \mathfrak{l} \cap \mathfrak{k} = \dim \mathfrak{m}$

Furthermore if  $e \in \mathcal{O}_{(\mathfrak{q}, \mathfrak{m})} \cap \mathfrak{m}$ , we call  $e$  a *Richardson element* associated with  $(\mathfrak{q}, \mathfrak{m})$ .

**Example 20.** Let  $e$  be a featured element in  $\mathcal{N}(\mathfrak{p})$ , and let  $\lambda = \lambda_e$ . Let  $\mathfrak{q} = \mathfrak{q}(\lambda) = \bigoplus_{i \geq 0} \mathfrak{g}(\lambda, i)$ , and let  $\mathfrak{m} = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$ . Then  $e \in \mathfrak{m}$  since  $\lambda$  is an associated cocharacter for  $e$ . We may take  $L = C_G(\lambda)$ . ( $L$  is  $\theta$ -stable since  $\lambda(k^*) \subset K$ .)

Now,  $\mathfrak{q}$  is the standard parabolic subalgebra of  $\mathfrak{g}$  defined by the subset  $\{\alpha \in \Delta \mid \langle \alpha, \lambda \rangle = 0\}$ , hence  $\mathfrak{q}$  is trivially  $K$ -conjugate to a standard parabolic. Also,  $\mathfrak{q}$  is  $\theta$ -stable because  $\lambda \in X_*(K)$ . We say  $e$  is *even* if  $\mathfrak{g}(\lambda, 1) = 0$  and *odd* if  $\mathfrak{g}(\lambda, 1) \neq 0$ . If  $e$  is even, then  $\mathfrak{u} = \mathfrak{g}(\lambda, 2) \oplus [\mathfrak{u}, \mathfrak{u}]$ , and if  $e$  is odd, then  $\mathfrak{u} = \mathfrak{g}(\lambda, 1) \oplus [\mathfrak{u}, \mathfrak{u}]$ . Either way, this shows  $[\mathfrak{u}, \mathfrak{u}]$  has an  $L$ -stable complement in  $\mathfrak{u}$  since  $\mathfrak{g}(\lambda, 1)$  and  $\mathfrak{g}(\lambda, 2)$  are both  $L$ -stable. Thus, the third condition is satisfied. If  $e$  is even, take  $V_{\mathfrak{u}} = \mathfrak{g}(\lambda, 2)$ , and if  $e$  is odd, take  $V_{[\mathfrak{u}, \mathfrak{u}]} = \mathfrak{g}(\lambda, 2)$ . We claim that either way  $\mathfrak{m}$  is the closure of an  $(L \cap K)^\circ$ -orbit in  $V_{\mathfrak{u}} \cap \mathfrak{p}$  or in  $V_{[\mathfrak{u}, \mathfrak{u}]} \cap \mathfrak{p}$ , which is shown as follows. By Proposition 19,  $[\mathfrak{l} \cap \mathfrak{k}, e] = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$ , which implies that  $(L \cap K)^\circ \cdot e$  is dense in  $\mathfrak{g}(\lambda, 2) \cap \mathfrak{p} = \mathfrak{m}$ . Finally, since  $e$  is featured,

$$\dim \mathfrak{l} \cap \mathfrak{k} = \dim \mathfrak{g}(\lambda, 0) \cap \mathfrak{k} = \dim \mathfrak{g}(\lambda, 2) \cap \mathfrak{p} = \dim \mathfrak{m}.$$

Thus,  $(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$  is a featured pair, and  $\mathcal{O}_{\mathfrak{q}, \mathfrak{m}} = (L \cap K)^\circ \cdot e$ . Since  $e \in \mathcal{O}_{\mathfrak{q}, \mathfrak{m}} \cap \mathfrak{m}$ ,  $e$  is a Richardson element for the featured pair  $(\mathfrak{q}, \mathfrak{m})$ .

**Lemma 21.** *Let  $\mathfrak{q}$  be a parabolic subalgebra defined by  $\Delta_0 \subset \Delta$  with nilradical  $\mathfrak{u}$ . A root  $\alpha \in \Phi$  such that  $\mathfrak{g}_\alpha$  is in  $\mathfrak{u}$  but not in  $[\mathfrak{u}, \mathfrak{u}]$  is the sum of one simple root in  $\Delta \setminus \Delta_0$  plus various simple roots in  $\Delta_0$ .*

*Proof.* See Proposition 8.2.7 in [8]. □

**Lemma 22.** *Let  $\mathfrak{q}$  be a parabolic subalgebra defined by  $\Delta_0 \subset \Delta$  with nilradical  $\mathfrak{u}$ . A root  $\alpha \in \Phi$  such that  $\mathfrak{g}_\alpha$  is in  $[\mathfrak{u}, \mathfrak{u}]$  but not in  $[\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$  is the sum of two simple roots in  $\Delta \setminus \Delta_0$  plus various simple roots in  $\Delta_0$ .*

*Proof.* Any root of  $\mathfrak{u}$  is the sum of at least one simple root in  $\Delta \setminus \Delta_0$  and various simple roots in  $\Delta_0$ . Thus, any root of  $[\mathfrak{u}, \mathfrak{u}]$  is the sum of at least two simple roots in  $\Delta \setminus \Delta_0$  and various simple roots in  $\Delta_0$ . Suppose  $\alpha$  is a root of  $[\mathfrak{u}, \mathfrak{u}]$  which is not a root of  $[\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$ , and suppose further that  $\alpha$  is the sum of more than two roots in  $\Delta \setminus \Delta_0$  and various simple roots in  $\Delta_0$ . Let  $\beta_1, \beta_2$ , and  $\beta_3$  be simple roots in  $\Delta \setminus \Delta_0$  which are summands of  $\alpha$ . Then  $\mathfrak{g}_\alpha \subset [\mathfrak{g}_{\beta_1}, [\mathfrak{g}_{\beta_2}, \mathfrak{g}_{\beta_3}]] \subset [\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]]$ , which is a contradiction. □

We say a featured pair  $(\mathfrak{q}, \mathfrak{m})$  is *even* if  $\mathfrak{m} \subset V_{\mathfrak{u}}$  and *odd* if  $\mathfrak{m} \subset V_{[\mathfrak{u}, \mathfrak{u}]}$ . Let  $(\mathfrak{q}, \mathfrak{m})$  be a featured pair. We assume that  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$  is a standard parabolic corresponding to a subset  $\Delta_0$  of  $\Delta$ . If  $(\mathfrak{q}, \mathfrak{m})$  is odd (resp., even) let  $\Phi_1$  (resp.,  $R_2$ ) be the subset of  $\Phi$  consisting of the roots described in Lemma 22 (resp., Lemma 21), and let  $\Delta_1 = \{\alpha \in \Delta \setminus \Delta_0 \mid \alpha \text{ is a summand of a root in } \Phi_1\}$  (resp.,  $\Pi_2 = \{\alpha \in \Delta \setminus \Delta_0 \mid \alpha \text{ is a summand of a root in } R_2\}$ ). Then by Lemma 22 (resp., Lemma 21),  $V_{[\mathfrak{u}, \mathfrak{u}]} = \bigoplus_{\alpha \in \Phi_1} \mathfrak{g}_\alpha$  (resp.,  $V_{\mathfrak{u}} = \bigoplus_{\alpha \in R_2} \mathfrak{g}_\alpha$ ).

If  $(\mathfrak{q}, \mathfrak{m})$  is an odd featured pair, define a function  $f : \Delta \rightarrow \mathbb{Z}$  by

$$f(\alpha) = \begin{cases} 0, & \alpha \in \Delta_0 \\ 1, & \alpha \in \Delta_1 \\ 2, & \text{otherwise} \end{cases}$$

and extend  $f$  linearly to  $\mathbb{Z}\Phi$ .

If  $(\mathfrak{q}, \mathfrak{m})$  is an even featured pair, define  $f : \Delta \rightarrow \mathbb{Z}$  by

$$f(\alpha) = \begin{cases} 0, & \alpha \in \Delta_0 \\ 2, & \alpha \in \Pi_2 \\ 4, & \text{otherwise} \end{cases}$$

and extend  $f$  linearly to  $\mathbb{Z}\Phi$ . Since  $G$  is of adjoint type,  $\mathbb{Z}\Phi = X^*(T)$ . Thus,  $f$  has the property that  $f(X^*(T)) \subset \mathbb{Z}$ . The proof of [19, Lemma 5.2] shows that this is equivalent to the existence of a cocharacter  $\tau \in X_*(T)$  such that  $f(\alpha) = \langle \alpha, \tau \rangle$  for all  $\alpha \in \Phi \cup \{0\}$ .

Since  $\mathfrak{q}$  is  $\theta$ -stable, we may assume without loss of generality that  $\mathfrak{l} = \mathfrak{g}(\tau, 0)$  is also  $\theta$ -stable. Since  $\mathfrak{h}$  and  $\Delta$  are  $\theta$ -stable, we can easily show that  $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\theta\alpha}$  for all  $\alpha \in \Delta$ . Using this fact, one can prove that  $\Delta_0$  is  $\theta$ -stable. Using the fact that  $\Phi^+$  is  $\theta$ -stable, we also get that  $\Delta_1$ ,  $\Pi_2$ ,  $\Delta_2 := \Delta \setminus (\Delta_0 \cup \Delta_1)$ , and  $\Pi_4 := \Delta \setminus (\Delta_0 \cup \Pi_2)$  are  $\theta$ -stable.

This, together with the fact that the pairing  $\langle, \rangle$  is  $\theta$ -equivariant (see Lemma 11), gives

$$\langle \alpha, \tau \rangle = f(\alpha) = f(\theta\alpha) = \langle \theta\alpha, \tau \rangle = \langle \alpha, \theta\tau \rangle$$

for all  $\alpha \in \Phi$ . This means that  $\theta\tau = \tau$ , and hence,  $\tau \in X_*(K)$ .

Let  $\mathfrak{g}(\tau, i) = \bigoplus_{\langle \alpha, \tau \rangle = i} \mathfrak{g}_\alpha$  when  $i \neq 0$ , and let  $\mathfrak{g}(\tau, 0) = \mathfrak{h} \oplus \bigoplus_{\langle \alpha, \tau \rangle = 0} \mathfrak{g}_\alpha$ . Then  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\tau, i)$ , and  $\mathfrak{q} = \bigoplus_{i \geq 0} \mathfrak{g}(\tau, i)$ . We also have that  $V_{[\mathfrak{u}, \mathfrak{u}]} \subset \mathfrak{g}(\tau, 2)$  when  $(\mathfrak{q}, \mathfrak{m})$  is odd, and  $V_{\mathfrak{u}} \subset \mathfrak{g}(\tau, 2)$  when  $(\mathfrak{q}, \mathfrak{m})$  is even. In either case, we have  $\mathfrak{m} \subset \mathfrak{g}(\tau, 2)$ . Since  $e \in \mathfrak{m}$  we have  $\tau(t) \cdot e = t^2 e$  for all  $t$ . Also, since  $G$  is of adjoint type, then  $\lambda$  is a  $\mathbb{Z}$ -linear combination of co-roots, which means  $\lambda(k^*) \subset \mathcal{D}G$  by [35, Proposition 8.1.8(iii)].

Now choose  $h_1 \in \mathfrak{h}$  such that  $\alpha(h_1) = \langle \alpha, \tau \rangle$  for all  $\alpha \in \Delta$ . Because  $\Delta_0, \Delta_1, \Delta_2, \Pi_2$ , and  $\Pi_4$  are all  $\theta$ -stable, we actually have  $h_1 \in \mathfrak{h} \cap \mathfrak{k}$ . Notice that  $[h_1, e] = \alpha(h_1)e = \langle \alpha, \tau \rangle e = 2e$  since  $e \in \mathfrak{m} \subset \mathfrak{g}(\tau, 2)$ . Also, if  $L = C_G(h_1)$ , then  $L$  is a  $\theta$ -stable Levi subgroup such that  $\text{Lie } L = \mathfrak{l}$ .

**Proposition 23.** *Let  $(\mathfrak{q}, \mathfrak{m})$  be a featured pair, and let  $e$  be a Richardson element associated with  $(\mathfrak{q}, \mathfrak{m})$ . With  $L$  and  $\mathfrak{l}$  as in the previous paragraph, we have*

(a)  $\mathfrak{k}^e \subset \mathfrak{q} \cap \mathfrak{k}$ , and

(b)  $\text{Lie } C_{L \cap K}(e) = (\mathfrak{l} \cap \mathfrak{k})^e$ .

*Proof.* (a) Let  $D$  be the unique nilpotent  $G$ -orbit which meets  $\mathfrak{u}$  in a dense set. Since  $\mathcal{O} = \mathcal{O}_{(\mathfrak{q}, \mathfrak{m})}$  is the unique nilpotent  $K$ -orbit meeting  $\mathfrak{m}$  in a dense set,  $\mathcal{O} \subset D$ . Thus,  $e \in D \cap \mathfrak{u}$ ; i.e.,  $e$  is a Richardson element for  $Q$ . Then by a theorem of Richardson (see [7, Cor. 5.2.4], which only requires that  $\text{char}(k)$  be good),  $C_G(e)^\circ \subset Q$ . Now, it is easy to show that  $C_G(e)^\circ$  is  $\theta$ -stable, and  $Q$  is  $\theta$ -stable by definition. Thus, taking the Lie algebras of the  $\theta$ -fixed point subgroups of both sides, we get  $\text{Lie}(C_G(e)^\circ)^\theta \subset \text{Lie}(Q^\theta)$ , which simplifies to  $\mathfrak{k}^e \subset \mathfrak{q} \cap \mathfrak{k}$ .

(b) Since  $h_1$  is semisimple and normalizes  $C_G(e)$ , we get by [4, §9.1],  $\text{Lie } C_L(e) = C_{C_G(e)}(h_1) = (\mathfrak{g}^e)^{h_1} = \mathfrak{l}^e$ . Since  $\theta$  restricts to a semisimple automorphism of  $C_L(e)$ , we have

$$\text{Lie } C_{L \cap K}(e) = \text{Lie}(C_L(e)^\theta) = (\text{Lie } C_L(e))^\theta = (\mathfrak{l} \cap \mathfrak{k})^e.$$

□

**Proposition 24.** *If  $(\mathfrak{q}, \mathfrak{m})$  is a featured pair, then  $\mathcal{O}_{(\mathfrak{q}, \mathfrak{m})}$  is a featured orbit.*

*Proof.* Let  $e$  be a Richardson element associated with  $(\mathfrak{q}, \mathfrak{m})$ . It suffices to show that  $e$  is featured. Since  $\text{Lie } C_{L \cap K}(e) = (\mathfrak{l} \cap \mathfrak{k})^e$ ,  $T_e((L \cap K)^\circ \cdot e) = [\mathfrak{l} \cap \mathfrak{k}, e]$  (see [19, §2.2]). Then  $\dim[\mathfrak{l} \cap \mathfrak{k}, e] = \dim(L \cap K)^\circ \cdot e = \dim \mathfrak{m} = \dim \mathfrak{l} \cap \mathfrak{k}$ . This implies  $(\mathfrak{l} \cap \mathfrak{k})^e = \{0\}$ . Since  $\mathfrak{k}^e = (\mathfrak{q} \cap \mathfrak{k})^e$  by Proposition 23,  $\mathfrak{k}^e = (\mathfrak{u} \cap \mathfrak{k})^e$ , which means  $\mathfrak{k}^e$  contains no nonzero semisimple elements. Thus  $e$  is featured.  $\square$

This proposition gives a well-defined map  $\phi$  from the set of  $K$ -conjugacy classes of featured pairs to the set of featured  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$ . It remains to show that  $\phi$  is a bijection. The following proposition, which was proved in Example 20, shows that  $\phi$  is onto.

**Proposition 25.** *Let  $e \in \mathcal{N}(\mathfrak{p})$  be featured, and let  $\lambda = \lambda_e$ . Then*

(a)  $(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$  is a featured pair, and

(b)  $e \in \mathcal{O} := \mathcal{O}_{(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})}$ .

We now show that  $\phi$  is one-to-one. Recall the grading  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\tau, i)$  introduced above.

**Proposition 26.** *Let  $e \in \mathcal{N}(\mathfrak{p})$  be a Richardson element associated to a featured pair  $(\mathfrak{q}, \mathfrak{m})$ , and let  $\lambda = \lambda_e$ . Then  $(\mathfrak{q}, \mathfrak{m})$  is  $K$ -conjugate to  $(\mathfrak{q}(\lambda), \mathfrak{g}(\lambda, 2) \cap \mathfrak{p})$ .*

*Proof.* By definition, we may assume that  $\mathfrak{q}$  is a standard parabolic subalgebra defined by  $\Delta_0 \subset \Delta$ . Then  $\mathfrak{q} = \bigoplus_{i \geq 0} \mathfrak{g}(\tau, i)$ . We will show that  $\mathfrak{g}(\tau, i) = \mathfrak{g}(\lambda, i)$  for all  $i \in \mathbb{Z}$ . This will imply that  $\mathfrak{q} = \mathfrak{q}(\lambda)$ .

Since  $\lambda \in X_*(K)$ , we can show that the subsets  $\{\alpha \in \Delta \mid \langle \alpha, \lambda \rangle = i\}$  (for  $i \in \mathbb{Z}$ ) of  $\Delta$  are  $\theta$ -stable. This allows us to choose  $h_2 \in \mathfrak{h} \cap \mathfrak{k}$  such that  $\alpha(h_2) = \langle \alpha, \lambda \rangle$  for all  $\alpha \in \Delta$ . Since  $e \in \mathfrak{g}(\lambda, 2)$ ,  $[h_2, e] = 2e$ . Thus  $[h_1 - h_2, e] = 0$ , which implies  $h_1 = h_2$  since  $e$  is featured and  $h_1 - h_2$  is a semisimple element of  $\mathfrak{k}^e$ . Therefore,  $f(\alpha) = \langle \alpha, \lambda \rangle$  for all  $\alpha \in \Phi$ , and hence,  $\mathfrak{g}(\tau, i) = \mathfrak{g}(\lambda, i)$  for all  $i \in \mathbb{Z}$ . As noted above, this means  $\mathfrak{q} = \mathfrak{q}(\lambda)$ .

It remains to show that  $\mathfrak{m} = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$ . We know that  $\mathfrak{m} \subset \mathfrak{g}(\tau, 2) \cap \mathfrak{p}$ . Since  $(\mathfrak{q}, \mathfrak{m})$  is a featured pair,

$$\begin{aligned}
\dim \mathfrak{m} &= \dim \mathfrak{l} \cap \mathfrak{k} \\
&= \dim \mathfrak{g}(\tau, 0) \cap \mathfrak{k} \\
&= \dim \mathfrak{g}(\lambda, 0) \cap \mathfrak{k} \\
&= \dim \mathfrak{g}(\lambda, 2) \cap \mathfrak{p} \\
&= \dim \mathfrak{g}(\tau, 2) \cap \mathfrak{p}.
\end{aligned}$$

Then  $\mathfrak{m} = \mathfrak{g}(\tau, 2) \cap \mathfrak{p} = \mathfrak{g}(\lambda, 2) \cap \mathfrak{p}$  since  $\mathfrak{g}(\tau, 2) \cap \mathfrak{p}$  is irreducible and  $\mathfrak{m}$  is closed.  $\square$

We have proved the following:

**Theorem 27.** *There is a one-to-one correspondence between featured  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  and  $K$ -conjugacy classes of featured pairs. The  $K$ -orbit corresponding to a featured pair  $(\mathfrak{q}, \mathfrak{m})$  is the unique one which intersects  $\mathfrak{m}$  in a dense subset of  $\mathfrak{m}$ . Even (resp., odd) featured pairs correspond to orbits consisting of even (resp., odd) elements.*

Before stating the general classification, we need a couple of lemmas.

**Lemma 28.** *All minimal special Levi subalgebras of  $\mathfrak{g}$  containing a fixed element  $e \in \mathcal{N}(\mathfrak{p})$  are  $C_K(e)$ -conjugate.*

*Proof.* By definition, a special Levi subalgebra of  $\mathfrak{g}$  containing  $e$  has the form  $\mathfrak{g}^{\mathfrak{s}}$  for some subset  $\mathfrak{s} \subset \mathfrak{k}^e$  consisting of semisimple elements. Thus, a special Levi subalgebra containing  $e$  is minimal when  $\mathfrak{s}$  is a maximal toral subalgebra of  $\mathfrak{k}^e$ . The result now follows from Lemma 18(d) and from the fact that all such subalgebras are  $C_K(e)$ -conjugate.  $\square$

**Lemma 29.** *Let  $\mathfrak{v}$  be a minimal special Levi subalgebra of  $\mathfrak{g}$  containing a nilpotent element  $e$ . Then  $e \in \mathfrak{v}' := [\mathfrak{v}, \mathfrak{v}]$ , and  $e$  is a featured element of  $\mathfrak{v}'$ .*

*Proof.* Since all elements of  $\mathfrak{z}(\mathfrak{r})$  are semisimple, we have  $e \in \mathfrak{r}'$ . The fact that  $e$  is featured in  $\mathfrak{r}'$  follows from Proposition 19.  $\square$

We can now give the classification of  $K$ -orbits on  $\mathcal{N}(\mathfrak{p})$ . The cocharacter  $\lambda_e$  constructed in Lemma 17 has the property that  $\lambda_e(k^*) \subset \mathcal{D}L$ , where  $L = C_G(M)$  for a maximal torus  $M$  in  $C_G(e)$  (see [24]). Let  $\overline{M} = M \cap K$ , a maximal torus in  $C_K(e)$ , and let  $R = C_G(\overline{M})$ , a Levi subgroup of  $G$ . Since  $\text{Lie}(C_K(e)) = \mathfrak{k}^e$  and  $\text{Lie } R = \mathfrak{g}^{\text{Lie } \overline{M}}$ , then  $e$  is featured in  $\text{Lie } R$  by definition, and  $\lambda_e(k^*) \subset \mathcal{D}R$ .

**Theorem 30.** *There is a one-to-one correspondence between  $K$ -orbits on  $\mathcal{N}(\mathfrak{p})$  and  $K$ -conjugacy classes of triples  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ , where  $\mathfrak{r}$  is a special Levi subalgebra of  $\mathfrak{g}$  and  $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$  is a featured pair of the semisimple part  $\mathfrak{r}'$  of  $\mathfrak{r}$ . The  $K$ -orbit corresponding to a given triple  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$  contains the  $(R \cap K)$ -orbit which intersects  $\mathfrak{m}$  in a dense subset of  $\mathfrak{m}$ .*

*Proof.* We again have a well-defined map  $\psi$  from  $K$ -conjugacy classes of triples  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$  to nilpotent  $K$ -orbits on  $\mathfrak{p}$  which sends the triple  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$  to the  $K$ -orbit containing the  $(R \cap K)$ -orbit which intersects  $\mathfrak{m}$  in a dense subset of  $\mathfrak{m}$ .

Let  $e \in \mathcal{N}(\mathfrak{p})$ . We have that the image of  $\lambda = \lambda_e$  is in  $\mathcal{D}R$ . This allows us to replace  $G$  by  $\mathcal{D}R$  in the preceding arguments. By Lemma 29,  $e$  is a featured element in the semisimple subalgebra  $\mathfrak{r}'$ . Thus by Proposition 25, there is a featured pair  $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$  of  $\mathfrak{r}'$  such that  $e$  is a Richardson element associated with  $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ . This shows that  $\psi$  is onto.

We say that an element  $e \in \mathcal{N}(\mathfrak{p})$  is associated with a triple  $(\mathfrak{r}, \mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$  if  $\mathfrak{r}$  is a minimal special Levi subalgebra containing  $e$  and  $e$  is a Richardson element associated with the pair  $(\mathfrak{q}_{\mathfrak{r}'}, \mathfrak{m})$ . Suppose  $e \in \mathcal{N}(\mathfrak{p})$  is a Richardson element associated with the triples  $(\mathfrak{r}_1, \mathfrak{q}_{\mathfrak{r}'_1}, \mathfrak{m}_1)$  and  $(\mathfrak{r}_2, \mathfrak{q}_{\mathfrak{r}'_2}, \mathfrak{m}_2)$ . By Lemma 28,  $\mathfrak{r}_1$  and  $\mathfrak{r}_2$  are conjugate by an element in  $C_K(e)$ , and thus by Proposition 26,  $(\mathfrak{q}_{\mathfrak{r}'_1}, \mathfrak{m}_1)$  and  $(\mathfrak{q}_{\mathfrak{r}'_2}, \mathfrak{m}_2)$  are conjugate by an element in  $\mathcal{D}(R_1) \cap K$  (or equivalently, by an element in  $\mathcal{D}(R_2) \cap K$ ). Thus, the



triples  $(\mathfrak{r}_1, \mathfrak{q}_{\mathfrak{r}_1}, \mathfrak{m}_1)$  and  $(\mathfrak{r}_2, \mathfrak{q}_{\mathfrak{r}_2}, \mathfrak{m}_2)$  are in the same  $K$ -conjugacy class. This shows that  $\psi$  is also one-to-one.  $\square$

**Example 31.** Let  $G = SL_3(k)$ , and let  $\theta$  be the involution on  $G$  defined by  $\theta(g) = (g^{-1})^\top$ . Then  $K = SO_3(k)$ ,  $\mathfrak{g} = \mathfrak{sl}_3(k)$ , and the induced involution  $\theta$  on  $\mathfrak{g}$  is defined by  $\theta(x) = -x^\top$  for all  $x \in \mathfrak{g}$ . Then  $\mathfrak{p}$  is the subspace of symmetric matrices in  $\mathfrak{g}$ . By [9], we know that the nilpotent  $K$ -orbits in  $\mathfrak{p}$  correspond to partitions of 3. Thus, there are 2 non-zero orbits, which correspond to the partitions (2, 1) and (3).

Now, back to our classification scheme. Noël showed that up to  $K$ -conjugacy, the only  $\theta$ -stable parabolic subalgebra of  $\mathfrak{g}$  is

$$\mathfrak{q} = kH_1 \oplus kH_2 \oplus kE_1 \oplus kE_2 \oplus kE_3,$$

where

$$H_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$E_1 = \frac{1}{2} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{pmatrix}.$$

We have that  $\mathfrak{q}$  is  $K$ -conjugate to the set of upper triangular matrices in  $\mathfrak{sl}_3(k)$ , which of course is a standard parabolic. A Levi decomposition for  $\mathfrak{q}$  is  $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ , where  $\mathfrak{l} = kH_1 \oplus kH_2$  and  $\mathfrak{u} = kE_1 \oplus kE_2 \oplus kE_3$ . Then we have  $[\mathfrak{u}, \mathfrak{u}] = kE_1$ ,  $[\mathfrak{u}, [\mathfrak{u}, \mathfrak{u}]] = 0$ ,  $V_{\mathfrak{u}} \cap \mathfrak{p} = kE_2$ , and  $V_{[\mathfrak{u}, \mathfrak{u}]} \cap \mathfrak{p} = kE_1$ . Notice that  $kE_2 \oplus kE_3$  is an  $L$ -stable complement to  $[\mathfrak{u}, \mathfrak{u}]$  in  $\mathfrak{u}$ .

Let  $\mathfrak{m}_1 = kE_1$  and let  $\mathfrak{m}_2 = kE_2$ . Both of these contain dense  $L \cap K$ -orbits since  $\mathfrak{l} \cap \mathfrak{k} = kH_1$  and  $[H_1, E_1] = 2E_1$  and  $[H_1, E_2] = E_2$ . Also,

$$\dim(\mathfrak{l} \cap \mathfrak{k}) = 1 = \dim \mathfrak{m}_1 = \dim \mathfrak{m}_2.$$

Thus,  $(\mathfrak{q}, \mathfrak{m}_1)$  is an even featured pair corresponding to the orbit in  $\mathcal{N}(\mathfrak{p})/K$  containing  $E_1$ , and  $(\mathfrak{q}, \mathfrak{m}_2)$  is an odd featured pair corresponding to the orbit containing  $E_2$ . Since  $E_1$  and  $E_2$  are both featured nilpotent elements, then the nonzero nilpotent  $K$ -orbits in  $\mathcal{N}(\mathfrak{p})$  are completely classified by the triples  $(\mathfrak{g}, \mathfrak{q}, \mathfrak{m}_1)$  and  $(\mathfrak{g}, \mathfrak{q}, \mathfrak{m}_2)$ .

The classification of  $\mathcal{N}(\mathfrak{p})/K$  given here is inspired by the one given by Noël in [25]. However, that classification depends heavily on the fact that  $k = \mathbb{C}$ , whereas our classification only requires that  $k$  is algebraically closed and that  $\text{char}(k)$  is good for the group  $G$ . Thus, our classification also holds when  $\text{char}(k) = 0$ . In this case, we can drop from the definition of a featured pair the requirement that  $[\mathfrak{u}, \mathfrak{u}]$  has an  $L$ -stable complement in  $\mathfrak{u}$  since  $L$  is reductive, and hence every  $L$ -module is completely reducible when  $\text{char}(k) = 0$ . We thus get a much streamlined classification of  $\mathcal{N}(\mathfrak{p})/K$  when  $\text{char}(k) = 0$ , and hence, by the Kostant-Sekiguchi correspondence, a classification of  $\mathcal{N}(\mathfrak{g})/G$  for a real Lie group  $G$  with complexified maximal compact subgroup  $K$ .

## A RESULT ON POLYNOMIALS DEFINED ON $\mathfrak{p}$

In this section, we assume  $G$  is a reductive linear algebraic group and that  $\text{char}(k)$  does not divide the order of the Weyl group of  $G$ . This condition on  $\text{char}(k)$  is more restrictive than the condition used above, namely that  $\text{char}(k)$  is good. Let  $k[\mathfrak{g}]$  be the ring of polynomials defined on  $\mathfrak{g}$  with coefficients in the field  $k$ . Formally,  $k[\mathfrak{g}]$  is defined to be the symmetric algebra of the dual vector space  $\mathfrak{g}^*$ . The action of  $G$  on  $\mathfrak{g}$  induces an action of  $G$  on  $k[\mathfrak{g}]$  as follows. Let  $g$  be an element of  $G$ , and let  $f$  be an element of  $k[\mathfrak{g}]$ . Then for  $x \in \mathfrak{g}$ ,  $(g \cdot f)(x) := f(g^{-1}x)$ . Let  $k[\mathfrak{g}]^G$  denote the subring of  $G$ -invariant polynomials on  $\mathfrak{g}$ . The *module of covariants* of  $k[\mathfrak{g}]$  is  $k[\mathfrak{g}]$  regarded as a  $k[\mathfrak{g}]^G$ -module. For  $K$  and  $\mathfrak{p}$  defined as above, we have the analogous structures  $k[\mathfrak{p}]$  and  $k[\mathfrak{p}]^K$ . Similarly, the module of covariants of  $k[\mathfrak{p}]$  is  $k[\mathfrak{p}]$  regarded as a  $k[\mathfrak{p}]^K$ -module. The goal of this section is to prove that the module of covariants of  $k[\mathfrak{p}]$  is free. This result has potential applications in determining characters of  $\mathcal{N}(\mathfrak{p})$ , as discussed in the next section.

**Chevalley's Restriction Theorem** Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$ . A famous result of Chevalley states that when  $\text{char}(k) = 0$ , the restriction morphism  $r : k[\mathfrak{g}] \rightarrow k[\mathfrak{h}]$  induces an isomorphism  $k[\mathfrak{g}]^G \cong k[\mathfrak{h}]^W$  of  $k$ -algebras, where  $W$  is the Weyl group of  $\mathfrak{g}$  defined by  $\mathfrak{h}$  (see [6, VIII.8.3 Theorem 1]). This is a very useful result, as it reduces the study of invariants of the infinite group  $G$  to the finite group  $W$ .

There exists an analogue of Chevalley's Restriction Theorem for the situation of  $K$  acting on  $\mathfrak{p}$ . A torus  $T$  in  $G$  is  $\theta$ -split if  $\theta(g) = g^{-1}$  for all  $g \in T$ . Notice that the Lie algebra of a  $\theta$ -split torus is contained in  $\mathfrak{p}$ . Let  $\mathfrak{a}$  be a maximal toral subalgebra of  $\mathfrak{p}$ . By [23, Lemma 2.4], there is a unique maximal  $\theta$ -split torus  $A$  of  $G$  such that  $\mathfrak{a} = \text{Lie}(A)$ . Let  $W_A$  be the Weyl group of  $G$  corresponding to  $A$ , i.e.,  $W_A = N_G(A)/C_G(A)$ , which is isomorphic to  $N_K(A)/C_K(A)$  by [31, §4]. The following theorem of Levy, which holds

when  $\text{char}(k)$  is good, states an analogue of Chevalley's Restriction Theorem. Its proof can be found in [23, Theorem 4.9].

**Theorem 32.** *Let  $A$  be a maximal  $\theta$ -split torus of  $G$ , and let  $\mathfrak{a} = \text{Lie}(A)$ . Then the natural embedding  $i : \mathfrak{a} \rightarrow \mathfrak{p}$  induces an isomorphism  $k[\mathfrak{p}]^K \rightarrow k[\mathfrak{a}]^{W_A}$ .*

Another theorem of Chevalley says that for a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ ,  $k[\mathfrak{h}]$  is a free  $k[\mathfrak{h}]^W$ -module, where  $W = W(\mathfrak{g}, \mathfrak{h})$ . We have the following analogous result.

**Proposition 33.** *For  $\mathfrak{a}$ ,  $A$ , and  $W_A$  defined as above,  $k[\mathfrak{a}]$  is a free  $k[\mathfrak{a}]^{W_A}$ -module provided  $\text{char}(k)$  does not divide  $|W_A|$ .*

*Proof.* This is just a special case of Theorem 1 in [5, V.5.2], with  $V = \mathfrak{a}$  and  $G = W_A$ .  $\square$

We now need a general result, proved by Bernstein and Lunts ([3, Prop. 1.1]). Let  $W$  be a subspace of a vector space  $V$ . We then have a restriction map  $r : k[V] \rightarrow k[W]$ . Let  $I$  be a graded subalgebra of  $k[V]$ .

**Proposition 34.** *[Bernstein and Lunts] Suppose that  $I$  has the following properties:*

- (i) *The restriction morphism  $r : I \rightarrow k[W]$  is injective.*
- (ii) *The algebra  $k[W]$  is a free module over the algebra  $I' = r(I)$ .*

*Then  $k[V]$  is a free  $I$ -module.*

We can now state the main result of this section:

**Theorem 35.**  *$k[\mathfrak{p}]$  is a free  $k[\mathfrak{p}]^K$ -module.*

*Proof.* Referring to Proposition 34, let  $V = \mathfrak{p}$ ,  $W = \mathfrak{a}$ , and  $I = k[\mathfrak{p}]^K$ . Then (i) holds by Theorem 32, and (ii) holds by Proposition 33. Thus  $k[\mathfrak{p}]$  is a free  $k[\mathfrak{p}]^K$ -module.  $\square$

The action of  $K$  on  $\mathfrak{p}$  and  $k[\mathfrak{p}]$  as studied in this section has potential applications in constructing characters of  $\mathcal{N}(\mathfrak{p})$ . This is discussed further in the next section.

## APPLICATIONS AND IDEAS FOR FURTHER RESEARCH

As mentioned in the introduction, an object which could be very valuable in the representation theory of restricted Lie triple systems is the restricted nullcone,  $\mathcal{N}_1(\mathfrak{p})$ . It is hoped that the classification of nilpotent  $K$ -orbits in  $\mathfrak{p}$  for fields of good characteristics given in this dissertation will provide some framework for a more detailed study of  $\mathcal{N}_1(\mathfrak{p})$ .

For example, one would like to know when  $\mathcal{N}_1(\mathfrak{p})$  is irreducible. The classification in [9] allowed the authors to determine this in a few classical cases as follows. For two nilpotent orbits  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , set  $\mathcal{O}_1 \leq_{\text{Zar}} \mathcal{O}_2$  if and only if  $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$ , where  $\overline{\mathcal{O}_2}$  denotes the Zariski closure of  $\mathcal{O}_2$ . This defines a partial order on the set  $\mathcal{N}(\mathfrak{p})/K$ . In all of the cases in which  $G$  is a classical group,  $\mathcal{N}(\mathfrak{p})/K$  is in one-to-one correspondence with a set of equivalence classes of Young diagrams or signed Young diagrams (except that in some cases a single equivalence class will correspond to two orbits). There exists a partial ordering  $\trianglelefteq$  on the equivalence classes of these diagrams which is compatible with the ordering  $\leq_{\text{Zar}}$  on  $\mathcal{N}(\mathfrak{p})/K$ . More precisely, if  $\mathcal{O}_\lambda$  and  $\mathcal{O}_\mu$  are orbits corresponding to equivalence classes of diagrams  $[\lambda]$  and  $[\mu]$ , respectively, then  $\mathcal{O}_\lambda \leq_{\text{Zar}} \mathcal{O}_\mu$  if and only if  $[\lambda] \trianglelefteq [\mu]$ . Thus, orbits which are maximal relative to  $\leq_{\text{Zar}}$  correspond to diagrams maximal relative to  $\trianglelefteq$ , and the latter can be examined using purely combinatorial methods. This allows one to determine how many maximal orbits there are in  $\mathcal{N}(\mathfrak{p})$ , or equivalently, how many irreducible components  $\mathcal{N}(\mathfrak{p})$  has.

Now, if  $x \in \mathcal{N}_1(\mathfrak{p})$ , then  $x^{[p]} = 0$ . In the classical cases, this just means that  $x^p = 0$ , which in turn implies that all of the Jordan blocks of  $x$  have at most  $p$  rows and columns. Thus, diagrams corresponding to matrices in  $\mathcal{N}_1(\mathfrak{p})$  have rows of length less than or equal to  $p$ . Hence, irreducible components of  $\mathcal{N}_1(\mathfrak{p})$  correspond to maximal diagrams whose rows have length less than or equal to  $p$ . The following example gives an illustration of this.

**Example 36.** Let  $\theta$  be an involution of Type  $AIII_{2,3}$ , and let  $p = \text{char}(k) = 3$ . Then as proved in [9], in this case  $\mathcal{N}(\mathfrak{p})/K$  is parameterized by the set of all equivalence classes of signed Young diagrams of signature  $(2,3)$ . The Hasse diagram in Figure 3 below illustrates the partial ordering  $\leq$ , where each equivalence class of signed Young diagrams is unambiguously represented by a single diagram. Since there is a unique maximal diagram,  $\mathcal{N}(\mathfrak{p})$  is irreducible here. However, there are four maximal diagrams with rows of length at most  $p = 3$ , meaning that  $\mathcal{N}_1(\mathfrak{p})$  has four irreducible components.

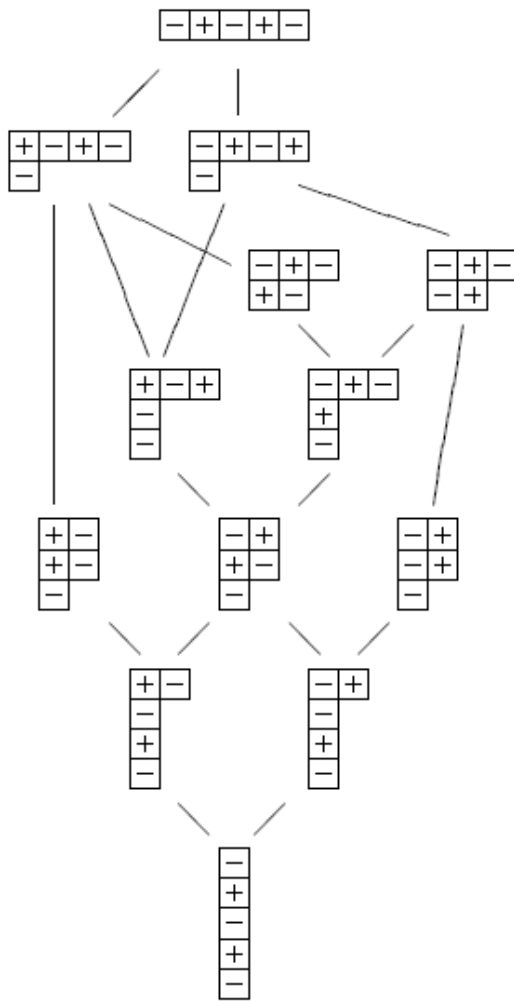


Figure 3. Hasse diagram for Type  $AIII_{2,3}$ .

Of course, when  $G$  is a group of exceptional type, this combinatorial analysis does not apply. However, an interesting question is whether the classification of  $\mathcal{N}(\mathfrak{p})/K$  given here (which includes the exceptional cases) can be used to determine of irreducibility of  $\mathcal{N}_1(\mathfrak{p})$  in general. This will be the subject of a future project.

The original purpose of this thesis was to study the action of  $K$  on  $\mathfrak{p}$  and on  $\mathcal{N}(\mathfrak{p})$  in positive characteristics, with the goal of adapting many of the results of Kostant and Rallis in [22], which were proved in the case when  $\text{char}(k) = 0$ . Our idea was to use associated cocharacters in place of the  $\mathfrak{sl}_2$  triples employed in [22]. Among other things, we wanted to prove a version of Chevalley's Restriction Theorem which would give an isomorphism  $k[\mathfrak{p}]^K \cong k[\mathfrak{a}]^{W_A}$  as described in §. However, we have recently learned that Levy has achieved many of these results in [23]. These results have been quite useful to us, particularly in proving the main result in §. Knowing that  $k[\mathfrak{p}]$  is a free  $k[\mathfrak{p}]^K$ -module (Theorem 35) leads us to believe that there may be a decomposition of  $k[\mathfrak{p}]$  similar to the one found in [22, Theorem 15] (when  $\text{char}(k) = 0$ ) which says  $k[\mathfrak{p}] \cong \mathcal{H}(\mathfrak{p}) \otimes_k k[\mathfrak{p}]^K$ , where  $\mathcal{H}(\mathfrak{p})$  is the space of *harmonic polynomials* on  $\mathfrak{p}$ . These are defined to be those polynomials  $f$  on  $\mathfrak{p}$  with the property that  $\partial_u f = 0$  for all  $u \in J_+$ , where  $J_+$  is the ideal in  $k[\mathfrak{p}]^K$  consisting of polynomials without constant terms, and  $\partial_u$  is a differential operator associated with  $u$ . Of course, differentiation in positive characteristics can be problematic, but Kostant and Rallis also showed that  $\mathcal{H}(\mathfrak{p})$  can be defined as the induced  $K$ -module  $\text{Ind}_M^K(1)$ , where  $M$  is a certain subgroup of  $K$ . An open question is to determine whether this situation extends to the case when  $\text{char}(k)$  is positive. Knowing this would lead to a decomposition of  $k[\mathfrak{p}]$  analogous to the one in [22].

There are links between the invariant theory of polynomials on  $\mathfrak{p}$  and the nullcone of  $\mathfrak{p}$ . In particular, the decomposition  $k[\mathfrak{p}] \cong \mathcal{H}(\mathfrak{p}) \otimes_k k[\mathfrak{p}]^K$ , if it exists when  $\text{char}(k) > 0$ , may have an application in determining the characters of  $\mathcal{N}(\mathfrak{p})$ . First, a little background. In [21], Kostant proved there is a decomposition  $k[\mathfrak{g}] \cong \mathcal{H}(\mathfrak{g}) \otimes_k k[\mathfrak{g}]^G$ , where  $G$

is a semisimple algebraic group over  $\mathbb{C}$ ,  $\mathfrak{g} = \text{Lie}(G)$ , and  $\mathcal{H}(\mathfrak{g})$  is the space of harmonic polynomials on  $\mathfrak{g}$ . Let  $k[\mathcal{N}(\mathfrak{g})]$  be the coordinate ring of the variety  $\mathcal{N}(\mathfrak{g})$ . Then the polynomial rings  $k[\mathcal{N}(\mathfrak{g})]$  and  $\mathcal{H}(\mathfrak{g})$  are both graded by degree, say  $k[\mathcal{N}(\mathfrak{g})] = \bigoplus_{n \geq 0} A_n$  and  $\mathcal{H}(\mathfrak{g}) = \bigoplus_{n \geq 0} H_n$ . Kostant showed that for each  $n$ ,  $A_n$  and  $H_n$  are isomorphic as  $G$ -modules. Using this information, Hesselink was able to derive a formula for the characters of  $k[\mathcal{N}(\mathfrak{g})]$  ([13, Lemma 3]), and this formula also holds when  $\text{char}(k) > 0$ , as in the proof of [1, Lemma 3.9]. An open question is whether the main result of § will lead to some appropriate decomposition  $k[\mathfrak{p}] \cong \mathcal{H}(\mathfrak{p}) \otimes_k k[\mathfrak{p}]^K$ , and whether this decomposition will lead to a character formula for  $k[\mathcal{N}(\mathfrak{p})]$ . This question and the others described in this section will be subjects of future research.



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