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NEW ESTIMATORS OF A CIRCULAR MEDIAN

by

Sauwanit Ratanaruamkarn

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Statistics
Dr. Magdalena Niewiadomska-Bugaj, Advisor

Western Michigan University
Kalamazoo, Michigan
April 2006

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NEW ESTIMATORS OF A CIRCULAR MEDIAN

Sauwanit Ratanaruamkarn, Ph.D.

Western Michigan University, 2006

The specific properties of probability distributions on a circle require different definitions of several statistical concepts. For example, a median that can always be found for linear data not always exists on the circle. Several estimators of a circular median were proposed, lately by Otieno (2002), Otieno and Anderson - Cook (2003). Their work is, however, focused on the "preferred direction" which coincides with the median, mean and mode in the case of symmetric, unimodal distributions. This dissertation is focused on the estimators of a population median in a wider range of population distributions on a circle including distributions with a low concentration around the center, or skewed distributions. Three new median estimators were proposed (MIR, CQM, and CSQM) and their properties were studied. A simulation study was performed to study their properties in small samples from unimodal symmetric, and skewed distributions without and with contamination.

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Sauwanit Ratanaruamkarn

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Chapter 1

Basic Concepts of Circular Data

1.1 Introduction

Data in the form of angles or two-dimensional orientations can be found almost everywhere. They are common in biology, geography, geology, geophysics, psychology, medicine, meteorology and oceanography, and in many other areas. Typical examples include directions of birds or animals departing from points of release, directional movement of animals in response to stimuli, circadian and other biorhythms, orientations of planes fracture and linear geographical features, wind and ocean current directions, times of day of accident occurrences, patients' arrival times in an emergency ward of a hospital and so on (Fisher, 1993).

Two-dimensional directions can be represented as angles measured with respect to some suitably chosen "zero direction"(starting point) and a direction of rotation taken as positive (clockwise or anti-clockwise). Since direction has no magnitude, these can be conveniently represented as points on the circumference of a unit circle centered at the origin or as unit vectors connecting the origin to these points. Because of the circular representation, such observations are also called *circular data*. Similarly, directions in three dimensions may be represented by two angles (akin to the representation of points on the earth's surface by their longitude and latitude), as unit vectors in three dimensions, or as points on the surface of a unit sphere. Because of this, directional data in three dimensions are also referred to as *spherical data* (Jammalamadaka and SenGupta, 2001).

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1.2 Examples of Circular Data

Here we introduce some of the circular data sets that can be found in the literature and are often used in examples.

- A roulette wheel was allowed to revolve and its stopping positions were measured in angles with a fixed direction. The stopping positions in 9 trials were 43°, 45°, 52°, 61°, 75°, 88°, 88°, 279°, 357°(Mardia, 1972). Its representation in Figure 1.1 shows that the wheel seems to have a *preferred* direction.

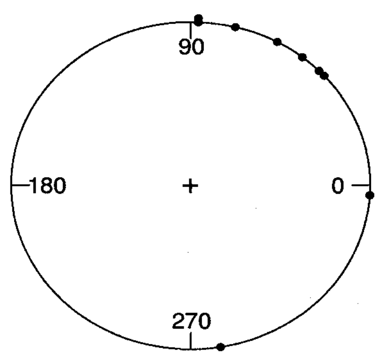

Figure 1.1: Circular Plot of the Roulette Data.

Using the standard procedure (1.51) for the circular mean, we obtain $\bar{\theta} = 51.05$ degrees, $\bar{R} = 0.711$, the median direction is 52 degrees. The median of these points when considered on the line is 75 degrees.

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- Orientations of turtles after laying eggs. The data represent the directions taken by the sea turtles after laying their eggs (Jammalamadaka and SenGupta, 2001). The direction "north-east" means that turtles are going back to the sea after laying eggs.

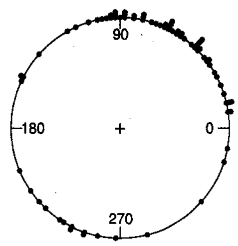

Figure 1.2: Circular Plot of the Turtles Data.

- Directions chosen by 100 ants in response to an evenly illuminated black target. The circular plot in Figure 1.3 suggests that ants most likely tend to run toward 180° which corresponds to the direction in which a black target has been placed. (N.I. Fisher, 1993)

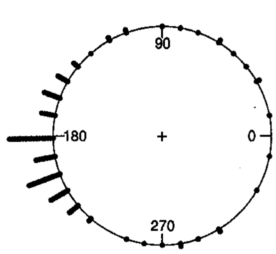

Figure 1.3: Circular Plot for Movement of Ants.

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- Time of traffic accidents in a city recorded during several days. The circular plot in Figure 1.4 suggests that data seem to be clustered around the early evening hours (6:00 PM or equivalently 270°). This is a potentially useful information for drivers, city police etc. In Table 1.1, the time points are converted into angles measured in degrees in steps of 1° or in minutes in steps of 4 (Batschelet, 1981).

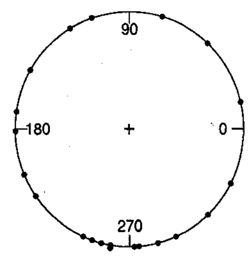

Figure 1.4: Circular Plot of the Traffic Accidents.

Table 1.1: Major Traffic Accidents.

time	degrees	time	degrees	time	degrees
00:56*h*	14	03:08	47	04:52	73
07:16	109	08:08	122	10:00	150
11:24	171	12:08	182	13:28	202
14:16	214	16:20	245	16:44	251
17:04	256	17:20	260	17:24	261
18:08	272	18:16	274	18:56	284
19:32	293	20:52	313	22:08	332

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1.3 Directional Measurement and Distance

Two dimensional directions are referred to as *circular data or angular data*, while three-dimensional directions are referred to as *spherical data*. For circular data, direction measurements are angles in the range [a; a + 2π) usually in [0, 2π) or [-π, π) intervals. The choice of the interval is arbitrary since there is no maximum or minimum on the circle. The choice of zero (usually either "north" or "east" on the graph) and the direction of rotation (counter-clockwise or clockwise) are also arbitrary. Circular data are usually measured either in degrees or in radians. We can convert degrees to radians or vice versa by multiplying by $\pi/180$ (or $180/\pi$, respectively). In this dissertation, all angles will be measured in radians, unless stated otherwise.

Circular (Angular) Distance

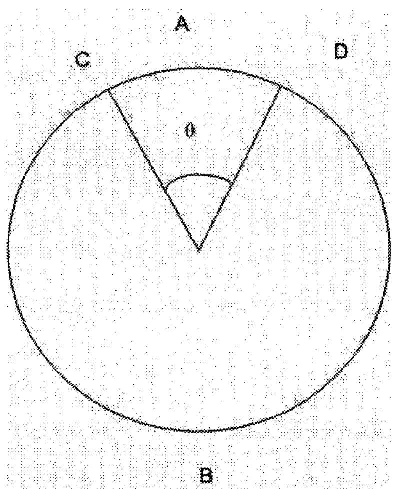

Figure 1.5: Circular Distance Between C and D

The circular distance between C and D is defined to be the length of the arc CAD. The shortest of the two lengths : CAD and CBD (Figure 1.5). For any two angles θ and γ, the angular distance can be obtained as

$$\begin{aligned} \text{dist}(\theta, \gamma) &= \min(|\theta - \gamma|, 2\pi - |\theta - \gamma|) && (1.1) \\ &= \pi - |\pi - |\theta - \gamma||. && (1.2) \end{aligned}$$

where $0 \leq \text{dist}(\theta, \gamma) \leq \pi$.

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Example 1.1

Suppose $\theta = 95°$ and $\gamma = 75°$ then $\theta - \gamma = 20°$ since

$$\begin{aligned}
\text{dist}(\theta, \gamma) &= \min(|\theta - \gamma|, 360° - |\theta - \gamma|) = \min(|20°|, 360° - |20°|) \\
&= \min(20°, 340°) = 20°, \text{ or} \\
\text{dist}(\theta, \gamma) &= 180° - |180° - |\theta - \gamma|| = 180° - |180° - |20°|| \\
&= 180° - 160° = 20°.
\end{aligned}$$

1.4 Probability Distribution on a Circle

The cumulative distribution function (cdf) F for a random angle θ is defined as:

$$\text{F}(x) = \Pr(0 \le \theta \le x), \qquad 0 \le x < 2\pi, \tag{1.3}$$

and

$$\lim_{x \to 2\pi} \text{F}(x) - \text{F}(0) = 1 \tag{1.4}$$

where $\text{F}(0) = 0$

For $\beta \le \gamma < \beta + 2\pi$,

$$\Pr(\beta < \theta \le \gamma) = \text{F}(\gamma) - \text{F}(\beta) = \int_\beta^\gamma dF(x), \tag{1.5}$$

and

$$\int_\beta^\gamma f(\theta) d\theta = \text{F}(\gamma) - \text{F}(\beta), \quad -\infty < \beta \le \gamma < \infty. \tag{1.6}$$

The probability density function, $f(\theta)$, of a circular random variable Θ must satisfy:

1. $f(\theta) \ge 0$,
2. $\int_0^{2\pi} f(\theta)\, d\theta = 1$, and
3. $f(\theta + 2\kappa\pi) = f(\theta)$ for any integer κ

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1.5 The Characteristic Function

Definition The *characteristic function* of a random angle θ is the doubly-infinite sequence of complex number $\phi_k : k = 0, \ \pm 1, \ldots$ given by

$$\phi_k = E[e^{ik\theta}] = \int_0^{2\pi} e^{ik\theta} dF(\theta), \quad k = 0, \ \pm 1, \ \pm 2, \ldots. \tag{1.7}$$

where

$$\phi_0 = 1, \qquad \overline{\phi}_{-k} = \phi_{-k}, \qquad |\phi_k| \leq 1, \tag{1.8}$$

where $\overline{\phi}_{-k}$ denotes the complex conjugate of ϕ_k. and

$$\phi_k = \alpha_k + i\beta_k, \tag{1.9}$$

where

$$\alpha_k = \mathrm{E}[\cos k\theta] = \int_0^{2\pi} \cos k\theta dF(\theta) \tag{1.10}$$

and

$$\beta_k = \mathrm{E}[\sin k\theta] = \int_0^{2\pi} \sin k\theta dF(\theta). \tag{1.11}$$

where

$$\alpha_{-k} = \alpha_k, \qquad \beta_{-k} = -\beta_k, \qquad |\alpha_k| \leq 1, \qquad |\beta_k| \leq 1. \tag{1.12}$$

1.6 Trigonometric Moments

The *trigonometric moments*

$$\alpha_k = \mathrm{E}[\cos k\theta], \qquad \beta_k = \mathrm{E}[\sin k\theta] \tag{1.13}$$

have already been defined in (1.10) - (1.11).

Note that the sequence $(\alpha_k, \beta_k) : k = 0, \ \pm 1, \ldots$ of trigonometric moments of a random angle θ is equivalent to the characteristic function of θ.

For $k \geq 0$,

$$\phi_k = \rho_k e^{i\mu_k}, \qquad \rho_k \geq 0 \tag{1.14}$$

In case $k = 1$, we obtain

$$\rho_1 = \rho, \qquad \mu_1 = \mu, \tag{1.15}$$

i.e.,

$$\phi_1 = \rho e^{i\mu}. \tag{1.16}$$

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1.7 Population Median

Mardia (1972) defined a population median direction ξ_0 as any solution of

$$\int_{\xi_0}^{\xi_0+\pi} f(\theta)d\theta = \int_{\xi_0+\pi}^{\xi_0+2\pi} f(\theta)d\theta = \frac{1}{2} \tag{1.17}$$

and with the population density $f(\theta)$ satisfying

$$f(\xi_0) > f(\xi_0 + \pi). \tag{1.18}$$

For a symmetric distribution, the median direction will be in the axis of symmetry. Condition (1.18) implies

$$f(\xi_0) - f(\xi_0 + \pi) > 0. \tag{1.19}$$

Mardia and Jupp (2000) also defined the population median as a direction ϕ which minimizes

$$\mathrm{E}[\pi - |\pi - |\theta - \phi|||], \qquad 0 \leq \theta < 2\pi. \tag{1.20}$$

where θ is a random variable with a cdf F_θ.
Note that (1.20) is the population analogue of (1.54). A median direction ϕ satisfies

$$\Pr(\theta \in [\phi, \phi + \pi)) \geq \frac{1}{2}, \qquad \Pr(\theta \in (\phi - \pi, \phi]) \geq \frac{1}{2} \tag{1.21}$$

1.8 Most Distributions on the Circle

Two most common distributions on the circle are: uniform distribution, von Mises distribution. The von Mises distribution is symmetric and unimodal.

(i) Uniform Distribution

The probability density function is defined as:

$$f(\theta) = \frac{1}{2\pi}, \qquad 0 \leq \theta < 2\pi. \tag{1.22}$$

Cumulative distribution function is defined by

$$F(\theta) = \frac{\theta}{2\pi}, \qquad 0 \leq \theta < 2\pi. \tag{1.23}$$

Probability density function is rotationally invariant but cumulative distribution function is not rotational invariant.

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(ii) von Mises Distribution

The probability density function is defined as:

$$f(\theta) = [2\pi I_0(\kappa)]^{-1}\exp[\kappa\cos(\theta-\mu)], \qquad 0 \le \theta < 2\pi. \tag{1.24}$$

where $0 \le \mu < 2\pi$, $0 \le \kappa < \infty$ are mean and concentration parameters, and

$$I_0(\kappa) = (2\pi)^{-1}\int_0^{2\pi} \exp[\kappa\cos(\theta-\mu)]\,d\theta \tag{1.25}$$

is the modified Bessel function of the first kind and of order zero. $I_0(\kappa)$ has power series expansion

$$I_0(\kappa) = \sum_{r=0}^{\infty}\frac{1}{(r!)^2}\left(\frac{\kappa}{2}\right)^{2r} \tag{1.26}$$

The distribution is unimodal and is symmetric about $\theta = \mu$ (mean=median= mode). The larger value of κ, the greater is the concentration around the mean. However, κ is not a scale parameter. When $\kappa \ge 2$, the von Mises distribution vM (μ, κ) can be approximated by the wrapped normal distribution WN (μ, ρ), which is a symmetric unimodal distribution obtained by wrapping a normal distribution $N(\mu, \sigma^2)$ around the circle (Otieno, 2002) where

$$\begin{aligned} \sigma^2 &= -2\log\rho, \qquad (\text{or } \rho = \exp(-\sigma^2/2)) \\ \rho = A(\kappa) &= \frac{I_1(\kappa)}{I_0(\kappa)} \approx 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} - \dots \end{aligned} \tag{1.27}$$

(see Jammalamadaka and SenGupta, 2001) For large value of κ, tails are light and distribution is clustered around the mean. Small values of κ correspond to distributions with heavier tails. For $\kappa = 0$, vM$(\mu, 0)$ is a uniform distribution.

Many useful and interesting circular models may be generated from known probability distributions on the real line by a general methods below:

1. By *wrapping* a linear distribution around the unit circle;
2. By transforming a bivariate linear random vector (X, Y) into polar coordinates (R, θ) and integrate over R for a given θ, one obtains the so-called *offset distribution* -as of a marginal distribution. For example if

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$f(x,y)$ denotes the joint bivariate distribution on the plane, the circular offset distribution, $g(\theta)$ can be obtained by

$$g(\theta) = \int_0^{\infty} f(r\cos\theta, r\sin\theta) r dr.$$

3. One may start with a distribution on the real line $\mathbb{R}$, and apply a *stereographic projection* that identifies points x on $\mathbb{R}$ with those on the circumference of the circle, say θ. This correspondence is one-to-one except for the fact that the mass if any, at both $+\infty$ and $-\infty$, are identified with π.

(iii) Wrapped Exponential Distribution

Given a distribution on the line, we can wrap it around the circumference of a unit circle. That is, if X is a random variable on the line (not necessarily symmetric), the corresponding random variable X_w of the wrapped distribution is given by

$$X_w = X \pmod{2\pi}. \tag{1.28}$$

If X has cdf F then the cdf F_w of X_w is given by

$$F_w(\theta) = \sum_{k=-\infty}^{\infty} \{F(\theta + 2\pi\kappa) - F(2\pi\kappa)\}, \qquad 0 \leq \theta < 2\pi \tag{1.29}$$

In particular, if X has a probability density function f then the corresponding probability density function f_w of X_w is

$$f_w(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2\pi\kappa), \qquad 0 \leq \theta < 2\pi \tag{1.30}$$

This is a many-to-one mapping when $X = \theta, \theta \pm 2\pi, \theta \pm 4\pi, \ldots$. So that the left hand side of (1.30) represents the circular distribution while the right hand side is the density of the real-valued random variable (Jammalamadaka and Kozubowski, 2003).

When we apply (1.30) to the exponential distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $x > 0$, we obtain a wrapped exponential distribution, denoted by $WE(\lambda)$ with $\lambda > 0$. The probability density function (p.d.f.), and cumulative distribution function (c.d.f.) of the wrapped exponential distribution $WE(\lambda)$, $\lambda \in R$ are as follows:

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$$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \qquad 0 \le \theta < 2\pi \tag{1.31}$$

and the cumulative distribution function (c.d.f) is

$$F(\theta) = \frac{1 - e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \qquad 0 \le \theta < 2\pi \tag{1.32}$$

The p.d.f. should be extended in a periodic fashion for the values of θ outside of the interval $[0, 2\pi)$, i.e., $f(\theta) = f(\theta + 2\pi\kappa)$, for any integer k.

The case $\lambda < 0$ results from wrapping the "negative" exponential distribution with parameter $|\lambda| > 0$, whose p.d.f. is $f(x) = |\lambda| e^{|\lambda| x}, x < 0$. Moreover, we have the relation

$$\Theta \sim WE(\lambda) \text{ if and only if } 2\pi - \Theta \sim WE(-\lambda). \tag{1.33}$$

Jammalamadaka and Kozubowski, (2003) noticed that the restriction of the linear exponential random variable X to the interval $[0, 2\pi)$ (that is $X|X < 2\pi$) has the same distribution as the wrapped random variable X_w given by (1.28).

The population mean direction for wrapped exponential distribution is defined as:

$$\overline{\alpha} = \begin{cases} \arctan \frac{1}{\lambda}, & \lambda > 0 \\ 2\pi + \arctan \frac{1}{\lambda}, & \lambda < 0. \end{cases} \tag{1.34}$$

The population median direction for wrapped exponential distribution is defined as:

$$\overline{\theta} = \frac{1}{\lambda} \ln \frac{2}{1 + e^{-\lambda\pi}} + \begin{cases} 0, & \lambda > 0 \\ \pi, & \lambda < 0. \end{cases} \tag{1.35}$$

When $\lambda = 0$ wrapped exponential distribution is uniform distribution with no direction (see Jammalamadaka and Kozubowski, 2004a).

(iv) Wrapped Laplace Distribution

The classical Laplace distribution and its skew generalizations are competing with normal and other distributions in stochastic modeling, particularly in financial applications. Their circular analogue can find interesting applications in directional data, when they resemble the characteristic shape of the Laplace distribution (with its possible skewness and sharp peak at the mode). Such data frequently result from orientation experiments in biology.

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The Laplace distribution on the line is a symmetric distribution on R with p.d.f.

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad x \in R,\ \lambda > 0. \tag{1.36}$$

The Laplace distribution (1.36) can be generalized and written as a mixture of a positive and a negative exponential distributions

$$f(x) = pf_1(x) + (1-p)f_2(x), \qquad x \in R, \tag{1.37}$$

with p $= 1/2$, where

$$f_1(x) = \lambda_1 e^{-\lambda_1 x}(x > 0), \qquad f_2(x) = \lambda_2 e^{\lambda_2 x}(x < 0) \tag{1.38}$$

More generally, one can consider mixtures of the form (1.38) with any $p \in (0,1)$ and two distinct values λ_1 and λ_2 ($\lambda_i > 0, i = 1, 2$) for the parameters of the positive exponential and the negative exponential distributions, leading to asymmetric Laplace laws. The wrapped distribution corresponding to (1.38) takes the form

$$f_w(\theta) = pf_{w1}(\theta) + (1-p)f_{w2}(\theta), \qquad 0 \le \theta < 2\pi \tag{1.39}$$

where f_{w1} and f_{w2} are the densities of the wrapped exponential laws $WE(\lambda_1)$ and $WE(-\lambda_2)$, corresponding to f_1 and f_2. Thus, the p.d.f. (1.38) takes an explicit form

$$f(\theta) = p\frac{\lambda_1 e^{-\lambda_1 \theta}}{1 - e^{-2\pi\lambda_1}} + (1-p)\frac{\lambda_2 e^{\lambda_2 \theta}}{e^{2\pi\lambda_2} - 1}, \qquad 0 \le \theta < 2\pi \tag{1.40}$$

Jammalamadaka and Kozubowski (2003) introduced the specific class of asymmetric Laplace laws where in the representation (1.37) parameters p, λ_1, λ_2 are such that

$$p = \frac{1}{\kappa^2 + 1}, \qquad \lambda_1 = \lambda\kappa, \qquad \lambda_2 = \lambda/\kappa. \tag{1.41}$$

for some κ, $\lambda > 0$. The asymmetric Laplace density (1.38) with the above parameters takes the form

$$f(x) = \lambda(1/k + k)^{-1} \begin{cases} e^{-\lambda\kappa|x|} & : \ \text{for } x \ge 0, \\ e^{-(\lambda/\kappa)|x|} & : \ \text{for } x < 0, \end{cases} \tag{1.42}$$

For $\kappa = 1$ we obtain the classical (symmetric) Laplace density. When we apply (1.30) to the above distribution, we obtain a *wrapped Laplace distribution*, denoted by $WL(\lambda, \kappa)$. The probability density function (p.d.f.), and cumulative distribution function (c.d.f.) of the wrapped Laplace distribution are:

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$$f(\theta) = \frac{\lambda\kappa}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{e^{(\lambda/\kappa)\theta}}{e^{2\pi\lambda/\kappa}-1}\right), \qquad 0 \le \theta < 2\pi \tag{1.43}$$

and

$$F(\theta) = \frac{1}{1+\kappa^2}\frac{1-e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{\kappa^2}{1+\kappa^2}\frac{e^{(\lambda/\kappa)\theta}-1}{1-e^{2\pi\lambda/\kappa}}, \qquad 0 \le \theta < 2\pi \tag{1.44}$$

The p.d.f. is a periodic function for the values of θ outside of the interval $[0, 2\pi)$ as we can see from (1.30). Observed that the p.d.f. in (1.37) integrates to 1 on $[0, 2\pi)$ for any values of λ, including $\lambda < 0$, and we have

$$WL(-\lambda, \kappa) = WL(\lambda, 1/\kappa). \tag{1.45}$$

It is also easy to see that

$$\Theta \sim WL(\lambda, \kappa) \text{ if and only if } 2\pi - \Theta \sim WE(\lambda, 1/\kappa). \tag{1.46}$$

The population mean direction for wrapped Laplace distribution is defined as:

$$\overline{\alpha} = \begin{cases} \arctan\frac{1}{\lambda\kappa} - \arctan\frac{\kappa}{\lambda}, & \kappa \le 1 \\ 2\pi + \arctan\frac{1}{\lambda\kappa} - \arctan\frac{\kappa}{\lambda}, & \kappa > 1. \end{cases} \tag{1.47}$$

The population median direction for wrapped Laplace distribution is defined as:

$$\xi_0 = \begin{cases} \xi^*, & \lambda > 0, 0 < \kappa < 1 \\ \xi^* + \pi, & \lambda > 0, \kappa > 1 \\ 0, & \kappa = 1, \end{cases} \tag{1.48}$$

with $\xi \in [0, \pi]$ such that

$$\frac{1}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\xi}}{1+e^{-\lambda\kappa\pi}} + \kappa^2\frac{e^{\lambda\xi/\kappa}}{1+e^{\lambda\pi/\kappa}}\right) = \frac{1}{2} \tag{1.49}$$

where λ and κ must satisfy (1.41).

Wrapped exponential distribution and wrapped Laplace distribution will be used in the simulation study of the performance of estimators of location parameter on the circle. The R codes for transforming linear data to circular data are provided in Appendix I.

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1.9 Descriptive Statistics on a Circle

The directional position can be easily determined by two coordinates. The rectangular coordinate system has O and two (or three) perpendicular axes X, Y (and Z) through O. Any point P on the plane can be represented as (X, Y) in terms of its rectangular coordinates or as (r, α) in terms of its polar coordinates where r is the distance to the origin and α is its direction. For the point of origin, O, $r = 0$ so no direction is indicated i.e., corresponding angle α is undefined (Jammalamadaka and SenGupta, 2001).

We can convert polar coordinates into rectangular coordinates or vice versa by using trigonometric functions *sine* and *cosine*. A point with polar coordinates (r, α) has rectangular coordinates (x, y) where $x = r\cos\alpha$, $y = r\sin\alpha$, $r = \sqrt{x^2 + y^2}$, and

$$\alpha = \arctan(y/x) \tag{1.50}$$

If the point P lies on the circumference of the unit circle, the conversion between polar and rectangular coordinates is simply

$$(1, \alpha) \Leftrightarrow (x = \cos\alpha, y = \sin\alpha).$$

1.9.1 Measures of Location

Mean Direction

Sample mean is defined on the line as an average but analogous definition cannot be applied to circular data as explained in the following example.

Example 1.2

Assume that a sample of two directions is given by the angles $\theta_1 = 15°$, $\theta_2 = 345°$. However, the average direction (arithmetic mean) which is 180°, points the wrong way.

When we replace 345° by the equivalent angle −15°, we get arithmetic mean = 0° (see Figure 1.6) which is the midpoint between 15° and 345°. However, with more than two directions, we need other procedures to handle the angles.

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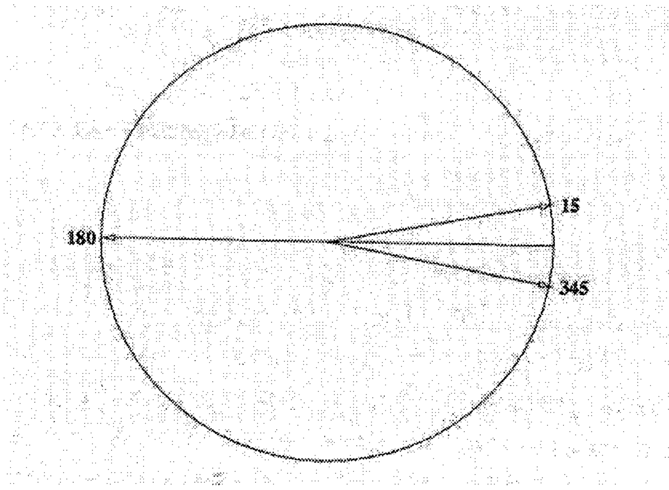

Figure 1.6: Arithmetic Mean Points the Wrong Way

Let $\alpha_1, \ldots, \alpha_n$ be a set of circular observations (angles) with corresponding rectangular coordinates $(\cos\alpha_i, \sin\alpha_i)$, $i = 1, \ldots, n$.

We obtain the resultant vector $\mathbf{R}$ of these n unit vectors by summing them component-wise,

$$\mathbf{R} = (C,\ S) = \left(\sum_{i=1}^{n} \cos\alpha_i, \sum_{i=1}^{n} \sin\alpha_i\right)$$

The length R of the resultant vector $\mathbf{R}$ is

$$R = \|\mathbf{R}\| = \sqrt{C^2 + S^2}$$

The direction of the resultant vector $\mathbf{R}$, which is called the circular mean direction, is denoted by $\overline{\alpha}_0$, and is obtained as the "quadrant-specific" inverse of the tangent,

$$\overline{\alpha}_0 = \arctan^*(S/C),$$

where

$$\overline{\alpha}_0 = \arctan^*(S/C) = \begin{cases} \arctan(S/C) & : \ C > 0, S \geq 0, \\ \pi/2 & : \ C = 0, S > 0, \\ -\pi/2 & : \ C = 0, S < 0, \\ \arctan(S/C) + \pi & : \ C < 0, \\ \arctan(S/C) + 2\pi & : \ C > 0, S < 0, \\ \text{undefined} & : \ C = 0, S = 0. \end{cases} \tag{1.51}$$

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Such definition of a quadrant-specific inverse of the tangent is necessitated by the fact that $\tan(\theta) = \tan(\theta + \pi)$, so that there are two inverses for any given θ within the $[0, 2\pi)$ range. Since *arctan* is usually defined so as to take values in $(-\frac{\pi}{2}, +\frac{\pi}{2})$, formula (1.51) provides the correct unique inverse in $[0, 2\pi)$ (Jammalamadaka and SenGupta, 2001, page 13), since it takes into account the signs of C and S. The following proposition shows that such an $\overline{\alpha}_0$ reflects the center for the data set and does not depend on the choice of origin or the sense of rotation.

Proposition 1.1 (Jammalamadaka and SenGupta, 2001) $\overline{\alpha}_0$ *is rotationally invariant, i.e., if the data is shifted by a certain amount, the value of* $\overline{\alpha}_0$ *also changes by the same amount.*

Proof: Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ have mean direction $\overline{\alpha}_0$. We will show that $(\alpha_1 + c, \alpha_2 + c, \ldots, \alpha_n + c)$ have mean direction $\overline{\alpha}_0 + c$. Suppose that $\mathbf{R}'$ is the vector of the new set of observations (after the shift). Then

$$\mathbf{R}' = \left(\sum_{i=1}^{n} \cos(\alpha_i + c), \sum_{i=1}^{n} (\sin \alpha_i + c) \right) = (C', S'),$$

where,

$$\begin{aligned} C' &= \sum_{i=1}^{n} \cos(\alpha_i + c) = \sum_{i=1}^{n} (\cos \alpha_i \cos c - \sin \alpha_i \sin c) \\ &= C \cos c - S \sin c \\ &= R \cos \overline{\alpha}_0 \cos c - R \sin \overline{\alpha}_0 \sin c \\ &= R \cos(\overline{\alpha}_0 + c). \end{aligned}$$

Similarly, $S' = R \sin(\overline{\alpha}_0 + c)$. Now,

$$R' = \|\mathbf{R}'\| = \sqrt{C'^2 + S'^2} = R = \sqrt{C^2 + S^2}.$$

Hence,

$$\frac{C'}{R'} = \cos(\overline{\alpha}') = \cos(\overline{\alpha}_0 + c), \ \frac{S'}{R'} = \sin(\overline{\alpha}') = \sin(\overline{\alpha}_0 + c),$$

proving the result.

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Median Direction

The sample median on the circle is defined as follows:

Suppose we are given a set of sample points on the unit circle. Any point P such that:

1. Half of the sample points are on each side of the diameter PQ through P,

2. The majority of the sample points are nearer to P than to Q,

is called a sample median on the circle. The vector $\overrightarrow{OP}$ is called a median direction of the sample (K.V. Mardia, 1972). Like in a linear case, for a sample of an odd size the median is an actual observation while for a sample of an even size the median is the midpoint (circular mean) of two consecutive observations. Such definition of a median does not guarantee uniqueness and therefore Mardia (1972) proposed that in such case the median additionally has to *minimize circular mean absolute deviation* defined as:

$$\mathrm{CMAD}(\theta_0) = \pi - \frac{1}{n}\sum_{i=1}^{n} |\pi - |\theta_i - \theta_0||,$$

($\theta_1, \ldots, \theta_n$ are data points, and θ_0 is a candidate median satisfying 1. and 2. above). We will refer to the median direction with the minimum circular mean absolute deviation as a "*Mardia Median*" (MM).

Circular median direction can still not be unique but such cases are very rare, especially under unimodal distribution. Also, in small samples, MM can be undefined as explained in following examples.

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Example 1.3

Figure 1.7 shows a data set (10, 87, 180, 269, and 278 degrees) for which none of the points satisfies conditions 1 and 2 so the median is undefined.

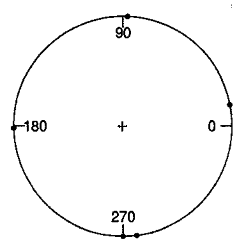

Figure 1.7: A Data Set with Non-existing Median on the Circle

Example 1.4

The frog data consist of 104, 110, 117, 121, 127, 130, 136, 144, 152, 178, 184, 192, 200, and 316 degrees. Figure 1.8 shows that 133 and 140 are two Mardia medians with minimum CMAD of 0.6508.

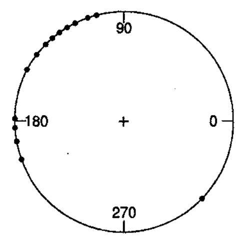

Figure 1.8: A Data Set with More than One Mardia Median on the Circle

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Quartiles

The definition of quartiles that applies to the data on the line can not be used on the circle because of no beginning and ending. Instead, quartiles can be determined in relation to the median direction. Since the direction of rotation on the circle can be determined in two ways, one can label quartiles as left and right instead of first and third subsequently. We propose that each quartile is found as a median among the data points on each semi-circle obtained by cutting the circle according to the median direction (P). It has to be noticed that within each semi-circle, median is obtained analogously as on the line. We will use symbols Q_L and Q_R corresponding to the left quartile and right quartile when the median coincides with the "north" direction.

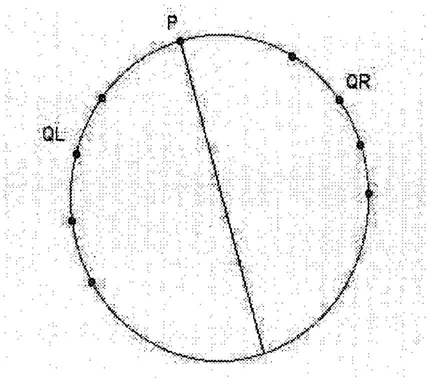

Figure 1.9: Quartiles on the Circle

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1.9.2 Measures of Variation

Circular Variance

The length R of the resultant vector $\mathbf{R}$ is a useful measure of variation or concentration of the data. When all the angles (unit vectors) point in the same direction indicating large (perfect) concentration, R will be as large as n. Conversely, if the data are evenly spread over the circle (indicating no concentration), (see Figure 1.10), R can be as small as zero. Therefore $0 \leq R \leq n$. The *mean resultant length* $\overline{R}$, associated with the mean direction $\overline{\alpha}$, is defined as

$$\overline{R} = \frac{R}{n}, \qquad 0 \leq \overline{R} \leq 1. \tag{1.52}$$

However, $\overline{R} = 0$ does *not* only imply uniform dispersion around the circle.

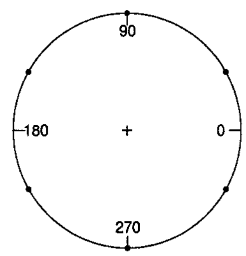

Figure 1.10: A Data Set with Mean Resultant Length Equal Zero.

In Example 1.5 and related Figure 1.11, we obtained $\overline{R} = 0.01066$, but this data set is bimodal with 2 directions (north and south). So $\overline{R} = 0$ does not guarantee uniform spread around the circle.

The *sample circular variance* is defined as

$$CV = 1 - \overline{R}, \tag{1.53}$$

So that the smaller the value of the circular variance, the more concentrated is the distribution. Unlike an ordinary linear variance, circular variance is bounded, $0 \leq CV \leq 1$.

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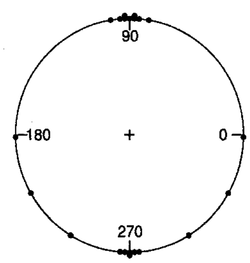

Figure 1.11: A Data Set with Mean Resultant Length Almost Zero. Data Points are not Uniformly Spread Over the Circle.

In the light of the above remark CV = 1 does not necessarily imply a maximally dispersed distribution (see Example 1.5) below.

Example 1.5

This example shows a data set that is not maximally disperse (evenly spread) but $CV \approx 1$. The data in Figure 1.11 consist of 81, 84, 87, 87, 90, 93, 93, 96, 99, 180, 210, 240, 264, 267, 270, 271, 273, 276, 300, 330, and 359 degrees. Using the routine *circ.summary* and *circ.disp* in the CircStats package in R, we obtained CV = 0.9893.

Circular Mean Absolute Deviation (CMAD)

The circular mean absolute deviation from any θ_0 is defined as:

$$\text{CMAD}(\theta_0) = \pi - \frac{1}{n}\sum_{i=1}^{n} |\pi - |\theta_i - \theta_0||. \tag{1.54}$$

where $\theta_1, \ldots, \theta_n$ are data points.

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Circular Median Absolute Deviation (CMEAD)

Similarly to CMAD, we define the circular median absolute deviation from θ_0, as:

$$\text{CMEAD}(\theta_0) = \text{median}(\pi - |\pi - |\theta_i - \theta_0||). \tag{1.55}$$

where $\theta_1, \ldots, \theta_n$ are data points.

1.9.3 Outliers

Outliers on the line are observations with extreme values relatively small or relatively large when compared with the rest of the data. Inconsistent with the rest of the sample, they can be contaminated observations, recording errors, and so on. The definition of outliers on the circle is quite different. They need not to be relatively large or relatively small, but could also be in the "*central*" part of the data. For example consider the following data set (in degrees): 23, 45, 52, 61, 75, 88, 357.

Figure 1.12: A Data Set with an Outlier on the Line.

From Figure 1.12, 357° would be an extreme value or an outlier on the line.

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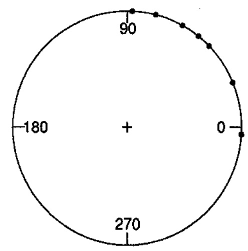

Figure 1.13: A Data Set Without Outliers on the Circle.

The same value (357°) should not be considered an outlier on the circle (see Figure 1.13). Extreme value on the line might not be recognize on the circle as an extreme.

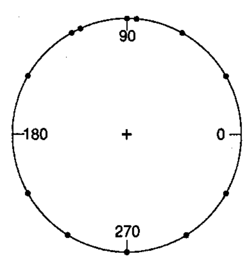

Figure 1.14: Outliers from Uniform Distribution.

Moreover, outliers on the circle can be in the central part of the data if it seems inconsistent with the distribution. (see Figure 1.14 where the data are obtained from a uniform distribution).

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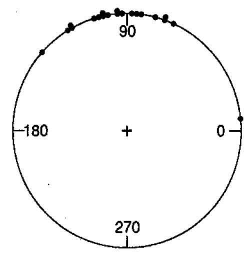

Figure 1.15: Outliers from von Mises Distribution.

Figure 1.15 shows outliers from von Mises distribution which is far apart from main part of data.

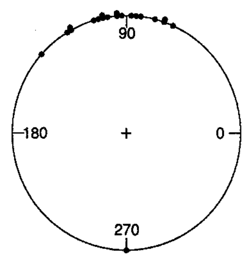

Figure 1.16: An Outlier Causes Most Damage.

An outlier that causes most damage is the one that is pointing 180° away from the trend of the bulk of the data (Wessel, 2000), see Figure 1.16.

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Among the tools that can be used to detect outliers on the circle, the most common one is so-called P-P plot of the data for the von Mises distribution, a simple graphical way to detect outliers by first finding the best fit von Mises distribution, $\hat{F}(\theta; \hat{\mu}, \hat{\kappa})$ to the data, and then plot

$$(j/(n+1), \hat{F}(\theta_{(j)}; \hat{\mu}, \hat{\kappa})), \qquad j = 1, \ldots, n.$$

A P-P plot can be obtained by using the routine *pp.plot*, in the CircStats package in R. Similar to Q-Q plot for detection of outliers on the line, i.e., if a few points on the circular P-P plot seem to be quite away from the diagonal, one may suspect these to be possible outliers (see Figure 1.17). More details for outliers and related problems can be found in Jammalamadaka and SenGupta (2001).

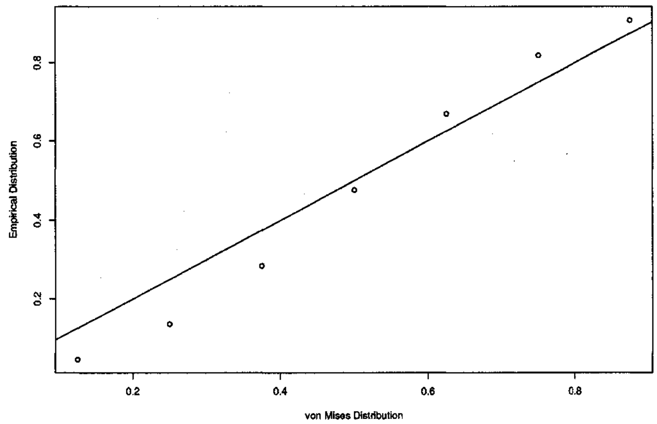

Figure 1.17: P-P Plots of an Example Dataset.

More details about how to detect outliers on the circle can be found in Collett (1980).

1.10 Evaluating Estimators on the Circle

In this section we propose the criteria of evaluating estimators.

Suppose we have an estimator, say θ^* of θ. The performance of this estimator depends on how "close"is θ^* from θ.

The error $\pi - |\pi - |\theta^* - \theta||$ is a circular difference between θ^* and θ. So that the smaller the error, the higher is the performance of this estimator.

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To build a general theory, we may assume that the situation can be adequately represented by specifying the circular loss function $\mathcal{CL}(\theta^*, \theta)$, where

$$\mathcal{CL}(\theta^*, \theta) = \pi - |\pi - |\theta^* - \theta||. \tag{1.56}$$

Definition 1 *The estimator T_1 is $\mathcal{CL}$ – dominating estimator T_2, or is $\mathcal{CL}$ – better than T_2, if for all $\theta \in \Theta$ we have*

$$\mathcal{CL}(T_1, \theta) \leq \mathcal{CL}(T_2, \theta),$$

where the inequality is strict for at least one value of θ.

Moreover, an estimator T will be called $\mathcal{CL}$ – *inadmissible* if there exists an estimator T' which is $\mathcal{CL}$-better than T. Otherwise, T will be called $\mathcal{CL}$ – *admissible*.

Circular Mean Absolute Error

Definition 2 *The Mean Circular Absolute Error(CAE) of an estimator θ^* of θ is its expected circular distance from the true value θ, obtained as*

$$\text{CAE}(\theta^*) = E(\pi - |\pi - |\theta^* - \theta||, \tag{1.57}$$

estimated as

$$\hat{\text{CAE}}(\theta^*) = \frac{1}{n}\sum_{i=1}^{n}(\pi - |\pi - |\theta_i - \theta||), \tag{1.58}$$

where $\theta_1, \ldots, \theta_n$ are estimates of θ.

Circular Median Absolute Error(CMEAE)

Definition 3 *The Median Absolute Error(MEAE) of an estimator is the median of its absolute error.i.e.,*

$$\text{MEAE}(\theta^*) = \text{circ.median}(\pi - |\pi - |\hat{\theta^*} - \theta||). \tag{1.59}$$

where θ is the parameter of interest.

We will also obtain a circular variance (CV) -see (1.53), mean and median absolute deviation (CMAD and CMEAD -see (1.54) and (1.55).

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Chapter 2

Literature Review

The most important books on circular data are: "*Statistics of Directional Data*" by K.V. Mardia (1972), "*Statistical Analysis of Circular Data*" N.I. Fisher (1993) with a lot of raw data, "*Directional Statistics*" by K.V. Mardia and P.E. Jupp (2000). "*Topics in Circulars Statistic*" by S.R. Jamalamadaka and A. SenGupta (2001) shows recent research in circular statistics and also includes CircStats, a package of computational subroutines in Splus. More applied book, with many practical examples is "*Circular Statistics in Biology*" by E. Batschelet (1981). The theory of three-dimensional directional (or spherical) data, is presented in "*Statistics on Spheres*" by G.S. Watson (1983).

Ducharme and Milasevic (1987) investigated asymptotic properties of the circular median, they derived asymptotic distributions for the circular mean, median and for the estimator μ_n^* of the mean proposed by Watson (1983), ($\mu_n^* = \arctan(\nu_1/\nu_2)$, where (ν_1, ν_2) is an eigenvector associated with the greatest eigenvalue of $M_n = n^{-1}\Sigma x_i x_i{}'$, and $x_i{}' = (\cos\theta_i, \sin\theta_i)$). Simulation study was performed under von Mises distribution with concentration parameter $\kappa = 0.0, 0.1, 0.5, 1.0, 2.0, 3.0, 5.0, 10.0$. They found that for small and moderate values of κ ($\kappa = 0.1, 0.5, 1.0, 2.0$) circular median would be preferred over μ_n^*. To study robustness, they performed simulation study under contaminated von Mises distribution (1-ϵ) vM (μ, κ)+ϵ vM $(\mu, \lambda\kappa)$ where $\lambda > 0$ and $\epsilon \in$ [0,1] . They found that in the presence of outliers the circular median is usually more efficient than both: the circular mean direction and the Watson's estimator of the mean (μ_n^*).

Lenth (1981) adapted technique for robust M estimator for directional data by using function $\psi(t) = (d/dt)\rho(t)$ where ρ is some function chosen to provide an estimator with the desired robustness properties. This technique

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was illustrated with results of small simulation study (250 random samples of size 20) under von Mises distributions with mean $\mu = 0, \kappa = 4$ and $\kappa = 16$. In simulations they used heavy tailed distribution (1-ϵ) vM (0, κ) $+\epsilon$ U$(-\pi, \pi)$ where von Mises distribution is contaminated by the uniform distribution on a circle. In such case M estimator for directional data seems to be more robust than the mean direction, but it is less efficient than mean direction under light-tailed distributions vM(0, κ).

Wehrly and Shine (1981) used the influence curve introduced by Hample (1974) as a heuristic tool in evaluating the robustness of some estimators of location for directional data. They introduced the circular population mean in functional form as

$$\mathrm{T}(F) = \tan^{-1}\left(\frac{\int_0^{2\pi} \sin\theta \, \mathrm{d}F(\theta)}{\int_0^{2\pi} \cos\theta \, \mathrm{d}F(\theta)}\right)$$

Straightforward calculations yield the influence curve for the circular mean

$$\mathrm{IC}(\theta) = \frac{\sin(\theta - \mu)}{\rho}.$$

For any given value of ρ, the influence curve and its first derivative are bounded by $\pm\rho^{-1}$. Thus, the circular mean has a bounded sensitivity to fixed amounts of contamination and to local shifts. They also introduced the influence curve for the circular median. This influence curve has jumps at the mode and anti mode because we are estimating on the circle. Thus, the circular median is sensitive to rounding or grouping of data. Since the influence curve is bounded, the circular median has finite gross error sensitivity.

Collett (1980) investigated problems connected with the presence of surprising values in samples of univariate circular data. He examined how their presence might be detected and described four possible tests of discordance L, C, D and M for assessing the possible discordance of a single angular outlier. L and C require an underlying von Mises distribution while the remaining are based on intuitive considerations and could be used for other models. The investigation was conducted using Monte Carlo methods for two thousand samples of size n = 5, 10, 15; $\kappa = 1, 2, 5, 10$ and λ (same as ϵ, the percentage of contamination), is 0.0, 0.1, 1.0. To evaluate performance of the four tests of discordance, he chooses n-1 observations in the sample of n from vM(0, κ) while one comes from vM($\lambda\pi$, κ), i.e., a "mean-shift" alternative. He suggested that as sample size increased, differences in the

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performance of the four tests become less noticeable.

Otieno (2002) proposed three versions of the Hodges-Lehmann estimator on the circle, which are analogues to the HL estimators on the line, namely

- HL1: the circular pseudo-median obtained by using circular means of all pairs of observations (i.e., $\binom{n}{2}$ distinct pairs)
- HL2: the circular pseudo-median obtained by using circular means of all pairs of observations including individual observations (i.e., $\binom{n}{2}+n$), and
- HL3: the circular pseudo-median of all possible pairwise circular means (i.e.,n^2 of Walsh averages).

Overall, the three measures are virtually identical. The circular Hodges-Lehmann estimator is defined as:

$$\hat{\theta}^c_{HL} = \text{circular median}(\bar{\theta}^c_{1,1}, \bar{\theta}^c_{1,2}, \ldots, \bar{\theta}^c_{n,n-1}, \bar{\theta}^c_{n,n}), \tag{2.1}$$

where $\bar{\theta}^c_{i,j}$ is the pairwise circular mean of observations θ_i and θ_j and can be defined as

$$\bar{\theta}^c_{i,j} = \arctan\left(\frac{\sin\theta_i + \sin\theta_j}{\cos\theta_i + \cos\theta_j}\right), \; i \leq j \leq n, \tag{2.2}$$

$\bar{\theta}^c_{i,j}$ is approximately distributed as $\text{vM}(\theta^c_{HL}, \frac{3n\kappa}{\pi})$. (see Otieno (2002))

Theoretical results show that the HLs are asymptotically more efficient than the circular median (MM) and their asymptotic efficiency relative to the circular mean is quite comparable. HLs are less robust when compared to circular median (MM) and these three estimators require extensive computations.

Otieno and Anderson-Cook (2003) introduced circular mean of all candidate medians to obtain a unique median estimator (pseudo-median) for circular data under symmetric unimodal distribution where mean = median = mode. This estimator is called *New Median*. The new median is computationally easier and faster to work with than the median estimator proposed by Mardia (Mardia Median), one of the existing alternatives, which requires comparison of circular mean deviation for each hypothetical median, that is data value satisfying conditions 1. and 2. on page 17.

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Cabrera, Maguluri, and Singh (1994) observed that the sample median (on the line) for the even sample size (n = 2m) is more efficient than the sample median with the odd sample size (n = 2m+1). To eliminate this property, they proposed a slight modification in the definition of the sample median for the odd sample size the so-called "$1/\sqrt{2}$-median" as the linear combination of the three most central order statistics (M_n = a$X_{(m+1)}$ $+\frac{1}{2}(1-a)\{X_{(m)}+X_{(m+2)}\}$), where $a=\sqrt{2}$. This "$1/\sqrt{2}$-median" alters the definition of the classical median very little and hence it retains the intuitive appeal of the classical median as a "middle point" of the data set.

Hodges and Lehmann (1967) found that asymptotic efficiency of the median ($\tilde{X}$) to the mean ($\bar{X}$) for normal distribution, which is known as $2/\pi \approx 0.637$, is approached through consistently higher values for even n than for odd n. They proposed the following adjustment in term of order $1/n$:

$$e(n) = \begin{cases} \frac{2}{\pi} + \frac{1}{n}\left[\frac{4}{\pi} - 1\right] & \text{for n = odd} \\ \frac{2}{\pi} + \frac{1}{n}\left[\frac{6}{\pi} - 1\right] & \text{for n = even.} \end{cases} \tag{2.3}$$

The result is valid for other distributions under mild regularity conditions. In fact, to terms of order $1/n$, the efficiency of $\tilde{X}$ to $\bar{X}$ is given by

$$e(n) = \begin{cases} 4f^2\sigma^2\left[1 + \frac{g+8}{4n}\right] & \text{for n = odd} \\ 4f^2\sigma^2\left[1 + \frac{g+12}{n}\right] & \text{for n = even.} \end{cases} \tag{2.4}$$

where

$$g = f''(0)/f^3(0), \qquad f = f(0), \qquad \sigma^2 = \mathrm{Var}(X_i) \tag{2.5}$$

with $f(x)$ denoting the density of X_i at x.

Formula (2.3) is the adjustment for accuracy of the value $2/\pi$ for odd and even sample while (2.4) is the efficiency for any symmetric (not necessarily normal) distribution. The variance is given, up to term of order $1/n^2$, by

$$\mathrm{Var}(\tilde{X}) = \frac{1}{8mf^2}\left[1 - \frac{g+12}{8mf^2}\right]. \tag{2.6}$$

Comparison of (2.6) and (2.4) shows that it is always better to base the median on an even number of observations. A sample of an odd size larger by 1

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provides a median, which is just as accurate. This conclusion is independent of the shape of the symmetric distribution of the $X's$. As a consequence Hodges and Lehmann proposed a *Quasi Medians* as an average of two order statistics of the same "order" when counted from both ends:

$$M_r = \begin{cases} (Y_{m+1-r} + Y_{m+1+r})/2 & \text{if } n = 2m+1 \\ (Y_{m-r} + Y_{m+1+r})/2 & \text{if } n = 2m, \end{cases} \tag{2.7}$$

where $Y_1 < Y_2 < ... < Y_n$ are the ordered observations, r is fixed, and $2r < n$. It turns out that up to terms of order $1/n^2$

$$\text{Var}(M_r) = \begin{cases} \frac{1}{4f^2n} - \frac{1}{16f^2n^2}(g + 8r + 8), & \text{if } n = odd \\ \frac{1}{4f^2n} - \frac{1}{16f^2n^2}(g + 8r + 12), & \text{if } n = even \end{cases} \tag{2.8}$$

(r is assumed fixed, n is large, and f and g have the same meaning as before). Putting successively $n = 2m+1, 2m, 2m-1$ and using the fact that $1/(4m+2) \approx (1 - 1/2m)/4m$, etc., we obtain,

$$\text{Var}(M_r) = \begin{cases} \frac{1}{8mf^2} - \frac{1}{64m^2f^2}(g + 8r + 12) & n = 2m, 2m+1 \\ \frac{1}{8mf^2} - \frac{1}{64m^2f^2}(g + 8(r-1) + 12) & n = 2m-1. \end{cases} \tag{2.9}$$

Formula (2.9) above, based on successively fewer observation, all have the same accuracy.

Jammalamadaka and Kozubowski (2004) discuss circular distributions obtained by wrapping the classical exponential and Laplace distributions on the real line around the circle. They present explicit forms of the densities, distribution functions, as well as their trigonometric moments and related parameters. They also discuss main properties of these laws. Both distributions are very useful as asymmetric models for directional data and will be used in a simulation study.

J.J. Fernández-Durán (2004) proposed a new family of distributions for circular random variables. It is based on nonnegative trigonometric sums and can be used to model data sets which present skewness and/or multi modality. In this family of distributions, the trigonometric moments are easily expressed in term of the parameters of the distribution. The proposed family is applied to two data sets, one related to the directions taken by ants

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and the other to the directions taken by turtles, to compare their fit to the proposed distribution versus to other possible distributions on the circle.

Otieno and Anderson-Cook (2005) evaluated the robustness of the three common choices for summarizing the preferred direction (the sample circular mean, sample circular median and a circular analog of the Hodges-Lehmann estimator on the line) by using their influence functions. The result shows that HL2 is less robust to outliers than the median, but more robust than the circular mean. The HL2 is an alternative robust estimator comparable to the mean in the case when the von Mises distribution is in doubt.

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Chapter 3

Proposed Estimators of the Population Median on the Circle

Basic concepts and definitions on the circle were introduced in Chapter 1 and 2. Many of them are different than those, on the line. For example, median on the circle is not only defined in a specific way, but it is not necessarily unique. Different definitions of a sample median as an estimator of a population median were proposed over the years in an effort to make it unique. The classical definition requires median on the circle, being either an actual observation or a midpoint (circular mean) of two consecutive observations, to split data evenly. Mardia (1972) proposed that potential median minimizes circular mean deviation of the data points. This definition, however, still does not guarantee uniqueness. Otieno and Anderson-Cook (2003) proposed a New Median (NM) estimator defined as a circular mean of all candidate medians in view of conditions 1. and 2. (page 17). Also, Otieno (2002) proposed three versions of the Hodges-Lehmann estimator on the circle which are analogues to the HL estimators on the line. In this chapter, we introduce three new estimators of a population median:

1. a mid-interquartile range (MIR) estimator of a median–a circular mean of the left and right quartile (a pseudo-median on the circle).

2. a central quasi median (CQM) estimator–a circular mean of the two observations closest to the median.

3. a central smoothed quasi median (CSQM) estimator–a circular mean of the median and two closest observations (one on each side).

Performance of these estimators will be studied theoretically (section 3.4) and by Monte Carlo simulations (Chapter 4 and Appendix A-F).

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3.1 Mid-Interquartile Range (MIR)

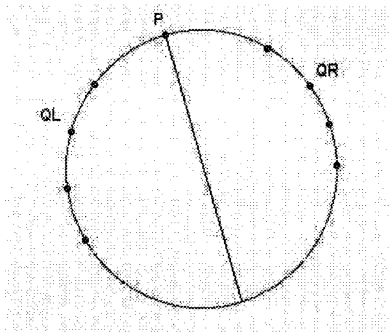

Figure 3.1: Mid-Interquartile Range (MIR) on the Circle.

Motivation behind MIR

In the population with a symmetric distribution on the real line, the average of two order statistics $(X_\alpha + X_{1-\alpha})/2$ is an L-estimator of a location parameter. The optimal choice of α is 0.27 if the distribution is normal, and such estimator has efficiency above 80% when compared with the sample mean $\overline{X}_n$. It is also robust against outliers (see Staudte and Sheather, 1990).

We want to explore a similar idea on the circle. As a new estimator of a circular median we propose a Mid Interquartile Range (MIR) a circular mean of two quartiles (called on the circle the left and right quartiles see page 19). The performance of this estimator is expected to be good especially in the case of symmetric distribution (for example von Mises) with small concentration (large κ). However, it will be also studied for von Mises distribution with a large concentration (small κ), and in the case of skewed distributions.

$$\hat{\theta}_{MIR} = circ.mean(\theta_{Q_L}, \theta_{Q_R}) \tag{3.1}$$

$$= \arctan\left(\frac{\sin\theta_{Q_L} + \sin\theta_{Q_R}}{\cos\theta_{Q_L} + \cos\theta_{Q_R}}\right) \tag{3.2}$$

The following algorithm explains how the MIR estimate is obtained.(See Appendix I for R codes). Suppose $\theta_1, \theta_2, \ldots, \theta_n$ is a random sample from a unimodal distribution on a circle.

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Step 1: Consider all values satisfying the definition of a median according to Mardia (page. 9). For even size samples, the potential values are the mid-points of two adjacent observations, and for odd size samples, the potential values are the observations themselves.

Step 2: Identify all angles satisfying the definition of median obtained in step 1. Then select the angle with the lowest CMAD. This is the so-called "Mardia Median". If there is more than one Mardia median, we obtain a circular mean all Mardia medians to get the unique estimator. This step is analogue to "New Median" proposed in Otieno and Anserson-Cook (2003).

Step 3: Find a sample median for each half of the circle (analogous to finding a sample median in the data on the line), these will be the left and the right quartile (or vice versa).

Step 4: The MIR is defined as a circular mean of these two quartiles.

3.2 Central Quasi Median (CQM) and Central Smoothed Quasi Median (CSQM)

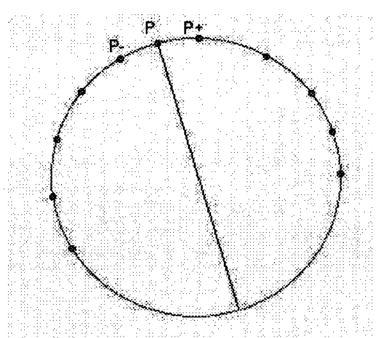

Figure 3.2: Quasi Median on the Circle.

Motivation behind CQM and CSQM

During investigating the performance of MIR and comparing it to that of the Mardia median and other existing estimators of the population median on a circle, we found that the variability of Media median is higher for odd size as measured by CV, CMAD, and CMEAD when compared to MIR. This motivated us to propose the circular mean of two most central order statistics,

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i.e., the circular mean of two order statistics closest to the MM as an analogue to the so-called *Quasi Median* on the line which was first proposed in Hodges and Lehmann (1967). Moreover, we introduced another estimator by smoothing the three most central order statistics, including Mardia median which is analogue to the linear combination of the three most central order statistics on the line proposed in Cabrera, Maguluri, and Singh (1994).

step 1: Let θ_P be the Mardia median and let θ_{P-} and θ_{P+} be the observations closest to Mardia median on the left and right.

step 2A: The circ.mean$(\theta_{P-}, \theta_{P+})$ is the *central quasi median* (CQM) on the circle, or

$$\hat{\theta}_{CQM} = circ.mean(\theta_{P-}, \theta_{P+}) \tag{3.3}$$

$$= \arctan\left(\frac{\sin\theta_{P-} + \sin\theta_{P+}}{\cos\theta_{P-} + \cos\theta_{P+}}\right) \tag{3.4}$$

step 2B: The circ.mean$(\theta_{P-}, \theta_P, \theta_{P+})$ is the *central smoothed quasi median* (CSQM) on the circle, or

$$\hat{\theta}_{CSQM} = circ.mean(\theta_{P-}, \theta_P, \theta_{P+}) \tag{3.5}$$

$$= \arctan\left(\frac{\sin\theta_{P-} + \sin\theta_P + \sin\theta_{P+}}{\cos\theta_{P-} + \cos\theta_P + \cos\theta_{P+}}\right) \tag{3.6}$$

More details of the algorithm can be found in Appendix I.

where $\theta_1 < \theta_2 < \ldots < \theta_{Q_L} < \ldots < \theta_P- < \theta_P < \theta_P+ < \ldots < \theta_{Q_R} < \ldots < \theta_n$ are the ordered observations with respect to the Mardia median as a center on the circle.

3.3 Normal Approximation to von Mises

Proposition 3.1 (Jammalamadaka and SenGupta, 2001) *Suppose* $\alpha \sim vM(\gamma, \kappa)$. *If* $\beta = \sqrt{\kappa}(\alpha - \gamma)$. *Then as* $\kappa \to \infty$, $\beta \xrightarrow{d} N(0,1)$.

Proof: Recall the vM density

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$$f(\alpha) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\alpha-\gamma)}, \qquad 0 \le \alpha < 2\pi. \tag{3.7}$$

Let $\beta = \sqrt{\kappa}(\alpha-\gamma)$. Then, for large κ and hence small $\frac{\beta}{\sqrt{\kappa}}$,

$$\cos(\alpha-\gamma) = \cos(\frac{\beta}{\sqrt{\kappa}}) \tag{3.8}$$

$$\simeq 1 - \frac{\beta^2}{\sqrt{2\kappa}}. \tag{3.9}$$

From the Taylor series expansion, $\cos\theta \simeq 1 - \frac{\theta^2}{2}$. Using the change-of-variable formula and the fact that for large κ, $I_0(\kappa) \simeq \exp(\kappa)/\sqrt{2\pi\kappa}$, we have

$$\begin{aligned} g(\beta) &= \frac{\exp(\kappa\cos(\frac{\beta}{\sqrt{\kappa}}))}{2\pi I_0(\kappa)} \frac{1}{\sqrt{\kappa}} \\ &\simeq \frac{\exp(\kappa\cos(\frac{\beta}{\sqrt{\kappa}}))}{2\pi \frac{\exp(\kappa)}{\sqrt{2\pi\kappa}}} \frac{1}{\sqrt{\kappa}} \\ &\simeq \frac{\exp(\kappa(1-\frac{\beta^2}{2\kappa}))}{e^{\kappa}\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\beta^2}{2}\right). \end{aligned}$$

Therefore for large κ, a von Mises distribution can be approximated by a normal distribution. Specifically, if $\alpha \sim \text{vM}(\mu,\kappa)$, then

$$\beta = \sqrt{\kappa}(\alpha-\mu) = \sqrt{\kappa}\theta \xrightarrow{d} N(0,1), \tag{3.10}$$

or for large κ

$$\alpha \sim \text{vM}(\mu,\kappa) \sim N(\mu, \frac{1}{\kappa}) \tag{3.11}$$

This allows to use normal approximation to von Mises distribution for suitably large κ.

If $\alpha_1, \ldots, \alpha_n$ are observations from $\text{vM}(\mu,\kappa)$, then

$$\theta_i = (\alpha-\mu) \sim N(0, \frac{1}{\kappa}). \tag{3.12}$$

This approximation is quite accurate for $\kappa > 10$ (Mardia and Jupp, 2000, page 41). We will use the normal approximations to assess asymptotic variances for CQM, MIR and CSQM respectively.

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3.4 Asymptotic Variance for CSQM, CQM and MIR

The following derivative of $f(\theta)$ and g are necessary for computing asymptotic variance in subsequent subsections.

As mentioned in Section (3.3) for large κ von Mises distribution vM(μ, κ) can be approximated by $N(\mu, \frac{1}{\kappa})$.

For $\theta \sim N(\mu, \frac{1}{\kappa})$

$$f(\theta) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{\kappa}}} e^{-\frac{1}{2}\frac{(\theta-\mu)^2}{\frac{1}{\kappa}}} = \frac{\sqrt{\kappa}}{\sqrt{2\pi}} e^{-\frac{\kappa(\theta-\mu)^2}{2}}$$

$$f'(\theta) = \frac{\sqrt{\kappa}}{\sqrt{2\pi}} e^{-\frac{\kappa(\theta-\mu)^2}{2}} d\frac{-\kappa(\theta-\mu)^2}{2} = \frac{\sqrt{\kappa}}{\sqrt{2\pi}} e^{-\frac{\kappa(\theta-\mu)^2}{2}} (-\kappa)(\theta-\mu)$$

$$\begin{aligned} f'(\theta) &= \frac{-\kappa^{\frac{3}{2}}}{\sqrt{2\pi}} (\theta-\mu) e^{-\frac{\kappa(\theta-\mu)^2}{2}} \\ f''(\theta) &= \frac{-\kappa^{\frac{3}{2}}}{\sqrt{2\pi}} [e^{-\frac{\kappa(\theta-\mu)^2}{2}} + (\theta-\mu) e^{-\frac{\kappa(\theta-\mu)^2}{2}} (-\kappa)(\theta-\mu)] \\ &= \frac{-\kappa^{\frac{3}{2}}}{\sqrt{2\pi}} [e^{-\frac{\kappa(\theta-\mu)^2}{2}} - \kappa(\theta-\mu)^2 e^{-\frac{\kappa(\theta-\mu)^2}{2}}] \\ &= \frac{-\kappa^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{\kappa(\theta-\mu)^2}{2}} [1 - \kappa(\theta-\mu)^2]. \end{aligned}$$

Therefore

$$\begin{aligned} f''(0) &= \frac{-\kappa^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{\kappa\mu^2}{2}} [1-\kappa\mu^2], \\ f(0) &= \frac{\sqrt{\kappa}}{\sqrt{2\pi}} e^{-\frac{\kappa\mu^2}{2}}, \\ f^2(0) &= \frac{\kappa}{2\pi} e^{-\kappa\mu^2}, \\ f^3(0) &= \frac{\kappa^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} e^{\frac{-3\kappa\mu^2}{2}}. \end{aligned} \tag{3.13}$$

And

$$g = \frac{f''(0)}{f^3(0)} = \frac{\frac{-\kappa^{\frac{3}{2}}}{\sqrt{2\pi}} e^{-\frac{\kappa\mu^2}{2}} [1-\kappa\mu^2]}{\frac{\kappa^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} e^{\frac{-3\kappa\mu^2}{2}}} = -2\pi e^{\kappa\mu^2} [1-\kappa\mu^2] \tag{3.14}$$

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3.4.1 Asymptotic Variance for CSQM

Using asymptotic variance under normal approximation when $\kappa \rightarrow \infty$, and Theorem 2. from Cabrera, Maguluri and Sindh (1994) (see Appendix J):

$$\text{MSE}(\hat{\theta}_{CSQM}) = \frac{1}{4f^2(0)n} - \frac{3}{4f^2(0)n^2} + 0(1/n^2) \tag{3.15}$$

for n even and odd, and $f^2(0)$ from (3.14) we found that

$$\text{MSE}(\hat{\theta}_{CSQM}) = \frac{\pi}{2n\kappa e^{-\kappa\mu^2}} - \frac{3\pi}{2n^2\kappa e^{-\kappa\mu^2}} \tag{3.16}$$

Let $\mu = 0$

$$\text{MSE}(\hat{\theta}_{CSQM}) = \frac{\pi}{2n\kappa}\left(1 - \frac{3}{n}\right). \tag{3.17}$$

3.4.2 Asymptotic Variance for CQM

For CQM, using equation (1.6) from Hodges and Lehmann (1967) (see Appendix J), we have

$$\text{MSE}(\hat{\theta}_{CQM}) = \begin{cases} \frac{1}{4f^2(0)n} - \frac{1}{16f^2(0)n^2}(g + 8r + 8), & \text{if n = odd} \\ \frac{1}{4f^2(0)n} - \frac{1}{16f^2(0)n^2}(g + 8r + 12), & \text{if n = even} \end{cases} \tag{3.18}$$

Here r is assumed fixed, n large, g from (3.14) and $f(0)$ from (3.13), we get

$$\text{MSE}(\hat{\theta}_{CQM}) = \begin{cases} \frac{\pi}{2n\kappa e^{-\kappa\mu^2}} - \frac{\pi}{8n^2\kappa e^{-\kappa\mu^2}}(g + 8r + 8), & \text{if n = odd} \\ \frac{\pi}{2n\kappa e^{-\kappa\mu^2}} - \frac{\pi}{8n^2\kappa e^{-\kappa\mu^2}}(g + 8r + 12), & \text{if n = even} \end{cases} \tag{3.19}$$

Let $\mu = 0$

$$\text{MSE}(\hat{\theta}_{CQM}) = \begin{cases} \frac{\pi}{2n\kappa} - \frac{\pi}{8n^2\kappa}(-2\pi + 8r + 8), & \text{if n = odd} \\ \frac{\pi}{2n\kappa} - \frac{\pi}{8n^2\kappa}(-2\pi + 8r + 12), & \text{if n = even} \end{cases} \tag{3.20}$$

or

$$\text{MSE}(\hat{\theta}_{CQM}) = \begin{cases} \frac{\pi}{2n\kappa}\left(1 - \frac{1}{4n}(-2\pi + 8r + 8)\right), & \text{if n = odd} \\ \frac{\pi}{2n\kappa}\left(1 - \frac{1}{4n}(-2\pi + 8r + 12)\right), & \text{if n = even} \end{cases} \tag{3.21}$$

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3.4.3 Asymptotic Variance for MIR

MIR defined as a circular mean of two quartiles, Q_L and Q_R, is a quasi-median in cases: $n = 4r + 1$ (odd) and $n = 4r + 2$ (even).

Using (3.21) for $r = (n-1)/4$ or for $r = (n-2)/4$, we get

$$\text{MSE}(\hat{\theta}_{MIR}) = \begin{cases} \frac{\pi}{2n\kappa}\left(1 - \frac{1}{4n}(-2\pi + 8(\frac{n-1}{4}) + 8)\right), & \text{if n = odd} \\ \frac{\pi}{2n\kappa}\left(1 - \frac{1}{4n}(-2\pi + 8(\frac{n-2}{4}) + 12)\right), & \text{if n = even} \end{cases} \quad (3.22)$$

or

$$\text{MSE}(\hat{\theta}_{MIR}) = \begin{cases} \frac{\pi}{2n\kappa}\left(1 - \frac{1}{2n}(-\pi + (n-1) + 4)\right), & \text{if n = odd} \\ \frac{\pi}{2n\kappa}\left(1 - \frac{1}{2n}(-\pi + (n-2) + 6)\right), & \text{if n = even} \end{cases} \quad (3.23)$$

or

$$\text{MSE}(\hat{\theta}_{MIR}) = \begin{cases} \frac{\pi}{2n\kappa}\left(1 + \frac{\pi}{2n} - \frac{1}{2} - \frac{3}{2n}\right), & \text{if n = odd} \\ \frac{\pi}{2n\kappa}\left(1 + \frac{\pi}{2n} - \frac{1}{2} - \frac{2}{n}\right), & \text{if n = even} \end{cases} \quad (3.24)$$

3.5 Asymptotic Relative Efficiency

Formula for ARE of circular mean, circular median (MM) and circular Hodges-Lehmann (HL2) estimators were provided in Otieno (2002).

Definition 1 *The asymptotic relative efficiency (ARE) of T_1 in relation to T_2 is defined by this ratio*

$$ARE_\theta(T_1, T_2) = \frac{MSE_\theta(T_2)}{MSE_\theta(T_1)}. \quad (3.25)$$

Clearly, T_1 is preferred over T_2 if $\text{ARE}_\theta(T_1, T_2) \geq 1$ with strictly inequality for at least one θ under von Mises distribution.

$$\begin{aligned} ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{mean}) &= \frac{\text{MSE}(\hat{\theta}_{mean})}{\text{MSE}(\tilde{\theta}_{CSQM})} \\ &= \frac{\frac{1}{n\rho\kappa}}{\frac{\pi}{2n\kappa}\left(1 - \frac{3}{n}\right)} \\ &= \frac{2}{\rho\pi}, \end{aligned} \quad (3.26)$$

as $n \to \infty$ and $\kappa \to \infty$ (so $\rho = e^{-\frac{1}{2\kappa}} \to 1$), we obtain $ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{mean}) = \frac{2}{\pi}$ or mean is more efficient estimator than CSQM under von Mises distribution.

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$$
\begin{aligned}
ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{MM}) &= \frac{\text{MSE}(\hat{\theta}_{MM})}{\text{MSE}(\tilde{\theta}_{CSQM})} \\
&= \frac{\frac{\pi}{2n\kappa(1-e^{-2\kappa})^2}}{\frac{\pi}{2n\kappa}\left(1-\frac{3}{n}\right)} \\
&= \frac{1}{(1-e^{-2\kappa})^2} \qquad (3.27)
\end{aligned}
$$

as $n \longrightarrow \infty$ and $\kappa \longrightarrow \infty$, we obtain $ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{MM}) = 1.0$ or CSQM has the same efficiency as MM under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{HL2}) &= \frac{\text{MSE}(\hat{\theta}_{HL2})}{\text{MSE}(\tilde{\theta}_{CSQM})} \\
&= \frac{\frac{\pi}{3n\kappa}}{\frac{\pi}{2n\kappa}\left(1-\frac{3}{n}\right)} \\
&= \frac{2}{3} \\
&= 0.667 \qquad (3.28)
\end{aligned}
$$

as $n \longrightarrow \infty$, we obtain $ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{HL2}) = 0.667$. Therefore HL2 is more efficient than CSQM under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{mean}) &= \frac{\text{MSE}(\hat{\theta}_{mean})}{\text{MSE}(\tilde{\theta}_{CQM})} \\
&= \begin{cases} \frac{\frac{1}{n\rho\kappa}}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+8)\right)}, & \text{if n} = \text{odd} \\ \frac{\frac{1}{n\rho\kappa}}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+12)\right)}, & \text{if n} = \text{even} \end{cases} \\
&= \frac{2}{\rho\pi}, \qquad (3.29)
\end{aligned}
$$

as $n \longrightarrow \infty$ and $\kappa \longrightarrow \infty$ (so $\rho = e^{-\frac{1}{2\kappa}} \longrightarrow 1$), we obtain $ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{mean}) = \frac{2}{\pi}$ or mean is more efficient than CQM under von Mises distribution.

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$$
\begin{aligned}
ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{MM}) &= \frac{\mathrm{MSE}(\hat{\theta}_{MM})}{\mathrm{MSE}(\tilde{\theta}_{CQM})} \\
&= \begin{cases} \frac{\frac{\pi}{2n\kappa(1-e^{-2\kappa})^2}}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+8)\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{2n\kappa(1-e^{-2\kappa})^2}}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+12)\right)}, & \text{if n = even} \end{cases} \\
&= \frac{1}{(1-e^{-2\kappa})^2} \qquad (3.30)
\end{aligned}
$$

as $n \to \infty$ and $\kappa \to \infty$, we obtain $ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{MM}) = 1.0$ or CQM is as efficient as MM under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{HL2}) &= \frac{\mathrm{MSE}(\hat{\theta}_{HL2})}{\mathrm{MSE}(\tilde{\theta}_{CQM})} \\
&= \begin{cases} \frac{\frac{\pi}{3n\kappa}}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+8)\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{3n\kappa}}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+12)\right)}, & \text{if n = even} \end{cases} \\
&= \frac{2}{3} \\
&= 0.667 \qquad (3.31)
\end{aligned}
$$

as $n \to \infty$ and an even as well as odd sample size n, we obtain $ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{HL2}) = 0.667$. Therefore HL2 is more efficient than CQM under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{MIR}, \hat{\theta}_{mean}) &= \frac{\mathrm{MSE}(\hat{\theta}_{mean})}{\mathrm{MSE}(\tilde{\theta}_{MIR})} \\
&= \begin{cases} \frac{\frac{1}{n\rho\kappa}}{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{3}{2n}\right)}, & \text{if n = odd} \\ \frac{\frac{1}{n\rho\kappa}}{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{2}{n}\right)}, & \text{if n = even} \end{cases} \\
&= \frac{4}{\rho\pi}, \qquad (3.32)
\end{aligned}
$$

as $n \to \infty$ and $\kappa \to \infty$ (so $\rho = e^{-\frac{1}{2\kappa}} \to 1$), we obtain $ARE(\tilde{\theta}_{MIR}, \hat{\theta}_{mean}) = \frac{4}{\pi}$. Therefore MIR is more efficient than mean under von Mises distribution.

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$$
\begin{aligned}
ARE(\tilde{\theta}_{MIR}, \hat{\theta}_{MM}) &= \frac{\text{MSE}(\hat{\theta}_{MM})}{\text{MSE}(\tilde{\theta}_{MIR})} \\
&= \begin{cases} \frac{\frac{\pi}{2n\kappa(1-e^{-2\kappa})^2}}{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{3}{2n}\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{2n\kappa(1-e^{-2\kappa})^2}}{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{2}{n}\right)}, & \text{if n = even} \end{cases} \\
&= \frac{2}{(1-e^{-2\kappa})^2} \qquad (3.33)
\end{aligned}
$$

As $\kappa \to \infty$ (so $\rho = e^{-\frac{1}{2\kappa}} \to 1$), we obtain $ARE(\tilde{\theta}_{MIR}, \hat{\theta}_{MM}) = 2.0$. Therefore MIR is two times more efficient than MM under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{MIR}, \hat{\theta}_{HL2}) &= \frac{\text{MSE}(\hat{\theta}_{HL2})}{\text{MSE}(\tilde{\theta}_{MIR})} \\
&= \begin{cases} \frac{\frac{\pi}{3n\kappa}}{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{3}{2n}\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{3n\kappa}}{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{2}{n}\right)}, & \text{if n = even} \end{cases} \\
&= \frac{4}{3} \\
&= 1.333 \qquad (3.34)
\end{aligned}
$$

as $n \to \infty$ and an even as well as odd sample size n, we obtain $ARE(\tilde{\theta}_{MIR}, \hat{\theta}_{HL2}) = 1.333$. Therefore MIR is more efficient than HL2 under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{CSQM}) &= \frac{\text{MSE}(\hat{\theta}_{CSQM})}{\text{MSE}(\tilde{\theta}_{CQM})} \\
&= \begin{cases} \frac{\frac{\pi}{2n\kappa}\left(1-\frac{3}{n}\right)}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+8)\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{2n\kappa}\left(1-\frac{3}{n}\right)}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+12)\right)}, & \text{if n = even} \end{cases} \\
&\approx 1.0 \qquad (3.35)
\end{aligned}
$$

as $n \to \infty$ and an even as well as odd sample size n, we obtain $ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{CSQM}) = 1.0$. CSQM is as efficient as CQM under von Mises distribution.

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$$
\begin{aligned}
ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{MIR}) &= \frac{\text{MSE}(\hat{\theta}_{MIR})}{\text{MSE}(\tilde{\theta}_{CQM})} \\
&= \begin{cases} \frac{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{3}{2n}\right)}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+8)\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{2}{n}\right)}{\frac{\pi}{2n\kappa}\left(1-\frac{1}{4n}(-2\pi+8r+12)\right)}, & \text{if n = even} \end{cases} \\
&\approx \frac{1}{2} \qquad (3.36)
\end{aligned}
$$

as $n \to \infty$ and an even as well as odd sample size n, we obtain $ARE(\tilde{\theta}_{CQM}, \hat{\theta}_{MIR}) = \frac{1}{2}$. Therefore MIR is two times more efficient than CQM under von Mises distribution.

$$
\begin{aligned}
ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{MIR}) &= \frac{\text{MSE}(\hat{\theta}_{MIR})}{\text{MSE}(\tilde{\theta}_{CSQM})} \\
&= \begin{cases} \frac{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{3}{2n}\right)}{\frac{\pi}{2n\kappa}\left(1-\frac{3}{n}\right)}, & \text{if n = odd} \\ \frac{\frac{\pi}{2n\kappa}\left(1+\frac{\pi}{2n}-\frac{1}{2}-\frac{2}{n}\right)}{\frac{\pi}{2n\kappa}\left(1-\frac{3}{n}\right)}, & \text{if n = even} \end{cases} \\
&\approx \frac{1}{2} \qquad (3.37)
\end{aligned}
$$

as $n \to \infty$ and an even as well as odd sample size n. We obtain $ARE(\tilde{\theta}_{CSQM}, \hat{\theta}_{MIR}) = \frac{1}{2}$. Therefore MIR is twice as efficient as CSQM under von Mises distribution.

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3.5.1 Summary for ARE

Tables 3.1 and 3.2 below contain ratios of variances (variance of the estimator on the top and variance of the estimator from the left most column).

Table 3.1: ARE for 6 Estimators

Estimator	Mean	MM	HL2	CSQM	CQM
MM	$\frac{2(1-e^{-2\kappa})^2}{\pi e^{-\frac{1}{2\kappa}}}$				
HL2	$\frac{3}{\pi e^{-\frac{1}{2\kappa}}}$	$\frac{3}{2(1-e^{-2\kappa})^2}$			
CSQM	$\frac{2}{\pi e^{-\frac{1}{2\kappa}}}$	$\frac{1}{(1-e^{-2\kappa})^2}$	$\frac{2}{3}$		
CQM	$\frac{2}{\pi e^{-\frac{1}{2\kappa}}}$	$\frac{1}{(1-e^{-2\kappa})^2}$	$\frac{2}{3}$	1.0	
MIR	$\frac{4}{\pi e^{-\frac{1}{2\kappa}}}$	$\frac{2}{(1-e^{-2\kappa})^2}$	$\frac{4}{3}$	2.0	2.0

As $\kappa \to \infty$ (so $\rho = e^{-\frac{1}{2\kappa}} \to 1$). The Table 3.2 below contains limiting values of ARE

Table 3.2: ARE for 6 Estimators when $\kappa \to \infty$

Estimator	Mean	MM	HL2	CSQM	CQM
MM	$\frac{2}{\pi}$				
HL2	$\frac{3}{\pi}$	$\frac{3}{2}$			
CSQM	$\frac{2}{\pi}$	1.0	$\frac{2}{3}$		
CQM	$\frac{2}{\pi}$	1.0	$\frac{2}{3}$	1.0	
MIR	$\frac{4}{\pi}$	2.0	$\frac{4}{3}$	2.0	2.0

Estimator MIR has highest asymptotic efficiency when compared with existing estimators such as circular mean, MM and HL2. Also, MIR is asymptotically twice as efficient as estimators CSQM and CQM, although these two estimators perform better in the case of a small sample (see Section. 4.3). Estimator CQM and its smoothed version CSQM have equal asymptotic efficiency.

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Chapter 4

Monte Carlo Simulations

4.1 Monte Carlo Simulation

Monte Carlo simulation is a stochastic technique used to solve mathematical problems that are difficult or impossible to solve theoretically.

In Monte Carlo simulation, the random generation is repeated many times to produce a random sample from a distribution of interest. Each time such sample is randomly selected, it forms one possible scenario and solution to the problem. Together, these scenarios give a range of possible solutions, some of which more probable and some less probable.

When repeated for many times (10,000 or more), the average solution will give an approximate answer to the problem. If necessary the accuracy of the answer can always be improved by simulating more samples. In fact, the accuracy of a Mone Carlo simulation is proportional to the square root of the number of samples used.

Monte Carlo simulations offer researchers an approach alternative to the theoretical investation. With powerful computers more widely available than ever, this computing-intensive approach is becoming more popular with quantitative researchers.

Monte Carlo simulations are used in this dissertation to compare sampling distributions of different estimators of population median on a circle for the small and moderate sample sizes.

For example, the required quantity x might be the mathematical expectation $E\xi$ of a certain random variable. The Monte Carlo method for determin-

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ing the approximate value of the quantity x consists of an N-fold sampling of the value of the variable ξ in a series of independent tests: $\xi_1, \xi_2, \ldots, \xi_N$, and the computation of their mean value:

$$\bar{\xi} = \frac{\xi_1 + \xi_2 + \ldots + \xi_N}{N}.$$

According to the law of large numbers,

$$\bar{\xi} \approx E\xi \; = \; x,$$

with a probability which is as close as required to unity for sufficiently large N. Accordingly, the quantity $\bar{\xi}$, which has been determined by observation of the random process, is approximately equal to the require quantity x.

Monte Carlo methods differ from other methods of numerical analysis in yielding an *estimate* rather than an *approximation*. The "numerical error" in a Monte Carlo estimate is due to a pseudo variance associated with a pseudo random variable; but the numerical error in standard numerical analysis is associated with approximations, including discretization, truncation, and roundoff. If the simulated data are used just to estimate one or more parameters, rather than to study the probability model more generally, we generally use the term *Monte Carlo* to refer to the method. Whenever simulated data are used in the broader problem of studying the complete process and building models, the method is often called *simulations*. This distinction between a simulation method and a Monte Carlo method is by no means universally employed; and we will sometimes use the terms "simulation" and "Monte Carlo" synonymously (Gentle, 2002).

4.2 Overview

The distributions included in the simulation study are: von Mises distribution, von Mises distribution contaminated on spread and location, and skewed distributions. The simulation study designed to compare performance of nine estimators of a circular median: circular mean, new median (NM), Mardia median (MM), MIR, central quasi median(CQM), central smoothed quasi median (CSQM), and three version of Hodges-Lehmann estimator on the circle: HL1, HL2 and HL3. Circular mean was included only when it was relevant as an estimator of a median, when the population distribution was symmetric (von Mises distribution).

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Note: HL1 requires $\binom{n}{2}$ distinct pairs, HL2 requires $\binom{n}{2}$ distinct pairs plus observations themselves $[\binom{n}{2} + n]$ and HL3 requires the total number n^2 of Walsh averages.

To determine the relative performance of all estimators on the circle, data were simulated from 4 types of distribution:

1. von Mises distribution: vM(μ, κ).
   where $\mu = 0$, $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0$, and $n = 7, 8, 19, 20$. vM(0, 1.0) and vM(0, 2.0) for small sample size $n = 7, 8, 9, 10, 11$ and moderate sample size $n = 17, 18, 19, 20, 21$.

2. von Mises distribution contaminated on spread:
   $(1-\epsilon)$vM$(0, \kappa) + \epsilon$U$(-\pi, \pi)$.
   where $\mu = 0, \epsilon = 0.1$
   and $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0$.
   $n = 7, 8, 19, 20$.

3. von Mises distribution contaminated on location:
   $(1-\epsilon)$vM$(0, \kappa) + \epsilon$vM(μ, κ),
   where $\mu = \frac{\pi}{4}$, $\epsilon = 0.1$
   and $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0$.
   $n = 7, 8, 19, 20$.

4. Skewed distribution: wrapped exponential distribution and wrapped Laplace distribution.
   where $\lambda = 1.0, 2.0$ for wrapped exponential distribution and $\lambda_1 = 2.0, 0.8$, $\lambda_2 = 0.5, 1.25$, and $p = 0.2, 0.61$ respectively for wrapped Laplace distribution.

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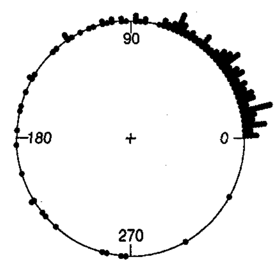

Figure 4.1: Circular Plot of Wrapped Exponential Distribution with $\lambda = 1$ and $n = 180$.

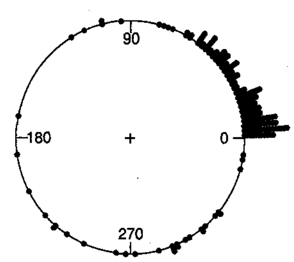

Figure 4.2: Circular Plot of Wrapped Laplace Distribution with $\lambda_1 = 2$, $\lambda_2 = 0.5$, p = 1/5 and $n = 180$.

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4.3 Results

4.3.1 Symmetric Distribution

Small Sample Size

Effect of small sample size (n) for vM(0, 2), n = 7, 8, 9, 10, 11 (data from Table C.3) is shown below:

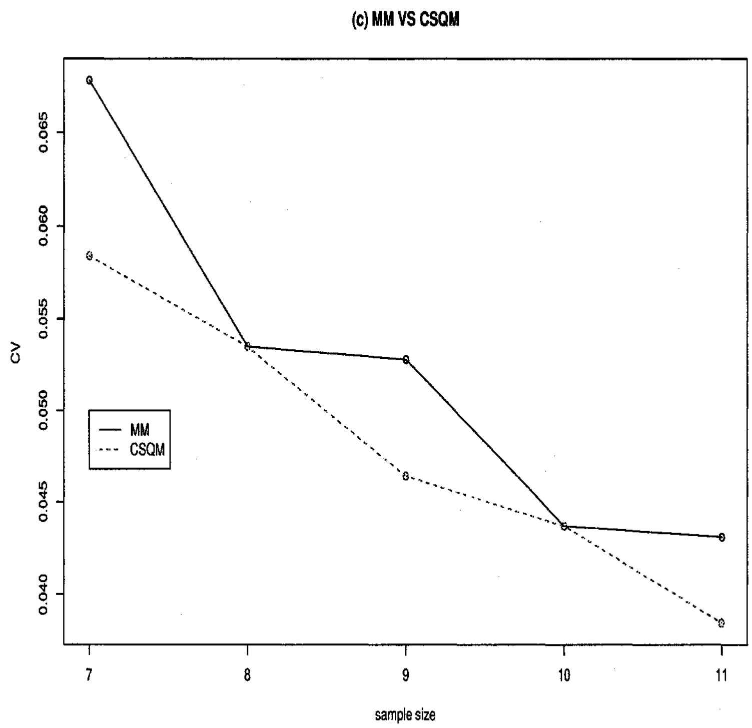

Figure 4.3: Compare CV of MM and CSQM VS Small Sample Size for vM(0, 2), Based on 10,000 Random Samples.

Figure 4.3 illustrates the trend of CV (circular variance) for Mardia median (MM) VS CSQM. We can see that when we add one more observation from 8 to 9 or 10 to 11, MM's CV does not decrease as n increase, but CSQM's CV decrease as n increase. Therefore CSQM dominate MM at this point.

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4.3.2 Distribution Contaminated on Spread

Effect of κ

Effect of κ for contaminated on spread: $(1-\epsilon)\text{VM}(0,2)+\epsilon\text{U}(-\pi,\pi)$, where $\epsilon=0.1$, sample size $=7$, $\kappa=0.1,0.2,0.3,0.4,0.5$ based on 10,000 random samples (data from Table D.1) is shown below:

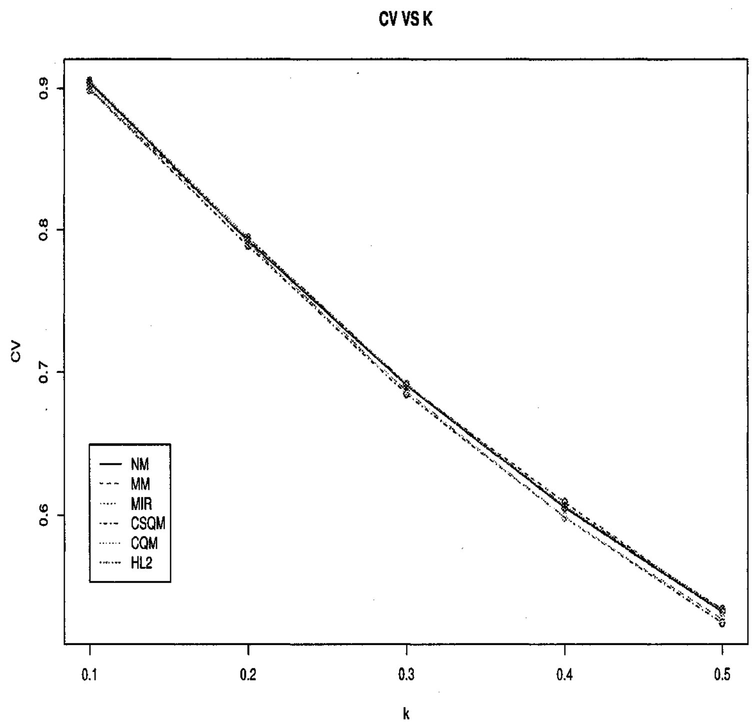

Figure 4.4: Contaminated on Spread: Compare CV of NM, MM, CQM, CSQM and MIR VS κ.

Figure 4.4 illustrates the trend of CV (circular variance) for NM, MM, CQM, MIR, and CSQM. The CV trends are identical for all existing estimators.

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4.3.3 Distribution Contaminated on Location

Effect of κ

Effect of κ for contaminated on location: $(1-\epsilon)\mathrm{VM}(0,2)+\epsilon\mathrm{VM}(\frac{\pi}{4},2)$, where $\epsilon = 0.1$, based on 10,000 random samples (data from Table E.1) is shown below:

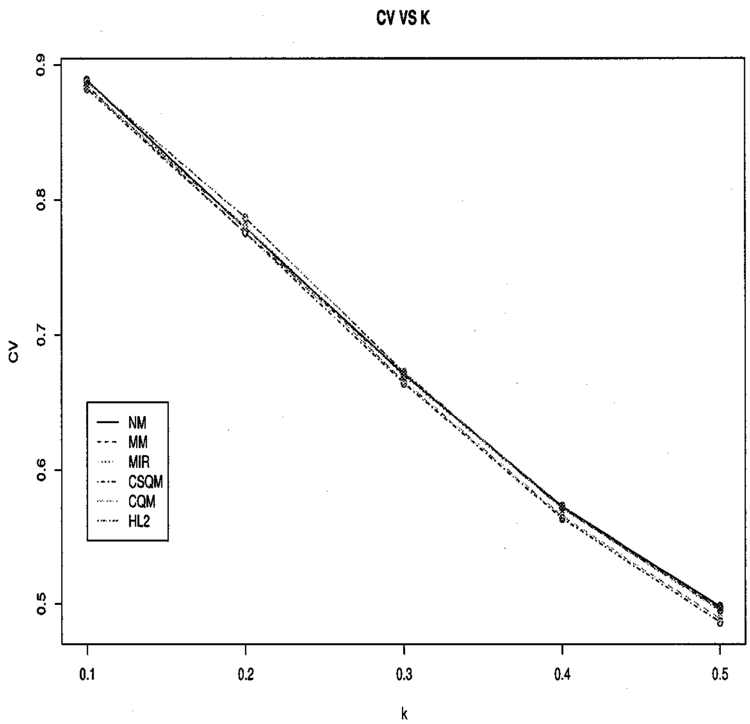

Figure 4.5: Contaminated on Location: Compare CV of NM, MM, CQM, CSQM and MIR VS κ.

Figure 4.5 illustrates the trend of CV (circular variance) for NM, MM, CQM, MIR, and CSQM. The CV trends are identical for all existing estimators.

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4.3.4 Skewed Distribution

Effect of Sample Size (n) for Wrapped Exponential Distribution

Effect of sample size (n) for wrapped exponential distribution($\lambda = 1$), mean $= \text{atan}(1/\lambda) = 0.7854$, median $= 0.6508$, based on 10,000 random samples (data from Table F.1) is shown below:

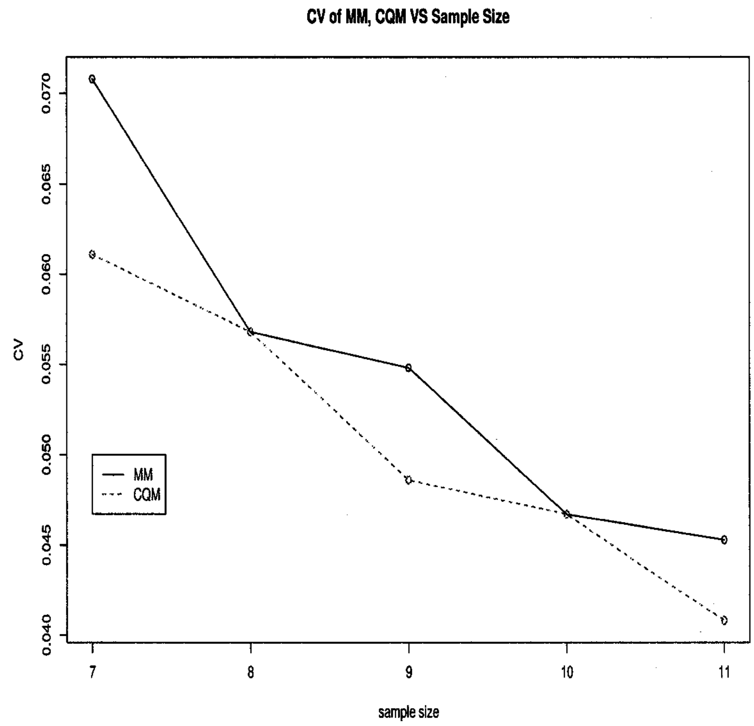

Figure 4.6: Wrapped Exponential: Compare CV of MM and CQM VS Small Sample Size.

Figure 4.6 illustrates the trend of CV (circular variance) for wrapped exponential distribution, from this picture, CQM have smaller variance for odd size sample when n increase from 8 to 9 and 10 to 11.

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Effect of Sample Size (n) for Wrapped Laplace Distribution

Effect of sample size (n) for wrapped Laplace distribution, $\lambda_1 = 2.0$, $\lambda_2 = 0.5$, p = 0.2, mean = -0.6435 (see Appendix I), median = -0.6147 (see Appendix I), based on 10,000 random samples (data from Table G.1) is shown below:

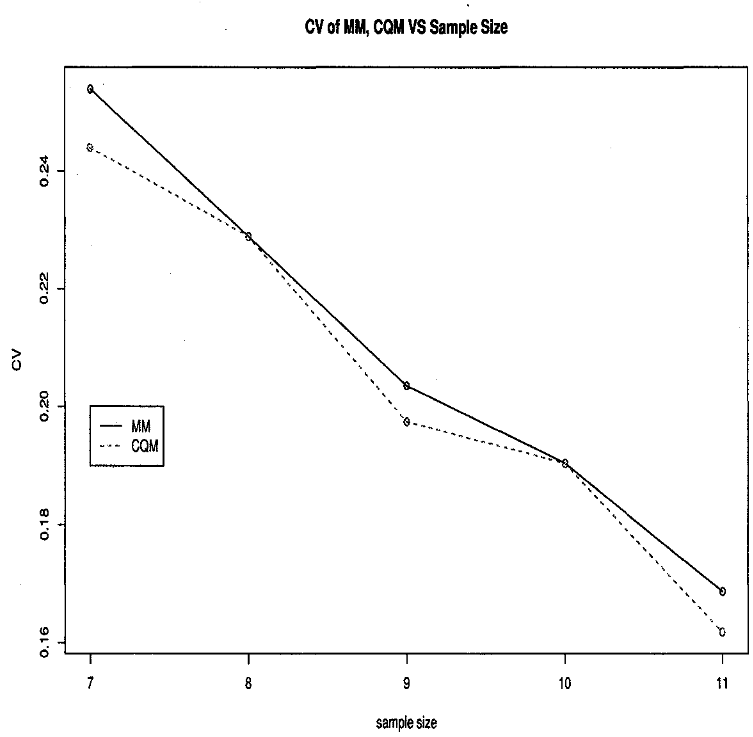

Figure 4.7: Wrapped Laplace: Compare CV of MM and CQM VS Small Sample Size.

Figure 4.7 illustrates the trend of CV (circular variance) for wrapped Laplace distribution, CQM have lower CV compare to MM.

Trends are usually the same for other measures of variability of estimators (CMAD, CMEAD).

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4.4 Ranking Performance

Ranking Performance (CV) of Estimators of a Population Median on a Circle.

We compared 8 estimators by ranking circular variance from minimum (1) to maximum (8), the results based on 10,000 repetitons from Tables B.1 - G.4.

Here we present an example of a ranking of the circular variance of all estimators in the case of small sample size ($n = 7$) and all four types of distributions.

Table 4.1: **Symmetric Distribution:** CV of Estimators NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 7$, $\kappa = 0.1$ - 1.0.

	Rank							
$\hat{\kappa}$	1	2	3	4	5	6	7	8
0.1	CSQM	CQM	MIR	MM	NM	HL2	HL1	HL3
0.2	CSQM	CQM	MIR	NM	MM	HL2	HL3	HL1
0.3	CSQM	CQM	NM	MM	MIR	HL3	HL1	HL2
0.4	CSQM	CQM	MM	MIR	HL2	NM	HL3	HL1
0.5	CSQM	CQM	HL2	HL1	MIR	HL3	NM	MM
0.6	CSQM	CQM	MIR	HL2	MM	HL3	NM	HL1
0.7	CSQM	CQM	HL2	HL3	MIR	HL1	MM	NM
0.8	CSQM	CQM	HL2	HL3	MIR	HL1	NM	MM
0.9	CSQM	CQM	HL2	MIR	HL3	NM	HL1	MM
1.0	CSQM	HL2	CQM	HL3	MIR	HL1	NM	MM

The results of the simulation study are contributed in Appendix A-F. Variability of estimators is measured by circular variance (CV), mean and median absolute deviation and by the mean absolute error.

To illustrate better the trend in the circular variance (the trend is identical for other variability measures). Their values were ranked from the smallest to the largest and displayed in the table in the respective order.

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Table 4.2: **Contamination on Spread:** CV of Estimators NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 0.1$ - 1.0.

	Rank							
$\hat{\kappa}$	1	2	3	4	5	6	7	8
0.1	CSQM	CQM	MM	MIR	NM	HL1	HL2	HL3
0.2	CSQM	CQM	NM	MIR	MM	HL2	HL1	HL3
0.3	CSQM	CQM	NM	MM	MIR	HL2	HL3	HL1
0.4	CQM	CSQM	MIR	NM	HL2	HL3	HL1	MM
0.5	CSQM	CQM	MIR	NM	MM	HL2	HL3	HL1
0.6	CSQM	CQM	NM	MIR	MM	HL2	HL3	HL1
0.7	CSQM	CQM	MIR	HL2	HL3	MM	NM	HL1
0.8	CSQM	CQM	HL2	MIR	MM	HL3	NM	HL1
0.9	CSQM	CQM	HL2	HL3	MIR	HL1	MM	NM
1.0	CSQM	HL2	CQM	HL3	HL1	MIR	NM	MM

Table 4.3: **Contamination on Location:** CV of Estimators NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 0.1 - 1.0$.

	Rank							
$\hat{\kappa}$	1	2	3	4	5	6	7	8
0.1	CSQM	CQM	MIR	MM	HL1	HL2	HL3	NM
0.2	MM	CSQM	CQM	NM	MIR	HL2	HL3	HL1
0.3	CSQM	CQM	MIR	NM	MM	HL2	HL3	HL1
0.4	CSQM	CQM	HL2	MIR	NM	MM	HL3	HL1
0.5	CSQM	CQM	MIR	MM	HL2	NM	HL3	HL1
0.6	CSQM	CQM	MIR	HL2	HL3	HL1	NM	MM
0.7	CSQM	CQM	MIR	HL2	HL3	HL1	MM	NM
0.8	CSQM	CQM	HL2	HL3	MIR	HL1	MM	NM
0.9	CSQM	CQM	HL2	MIR	HL3	HL1	MM	NM
1.0	CSQM	CQM	HL2	HL3	MIR	HL1	NM	MM

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Table 4.4: **Skewed Distribution:** CV of Estimators NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Exponential Distribution ($\lambda = 1$), Mean = atan($1/\lambda$) = 0.7854, Median = 0.6508 (See (1.35)).

	Rank							
n	1	2	3	4	5	6	7	8
7	CSQM	MIR	CQM	HL2	HL3	NM	MM	HL1
9	CSQM	CQM	HL2	HL3	HL1	MIR	NM	MM
11	MIR	CQM	CSQM	HL2	HL3	HL1	NM	MM
13	CQM	HL2	CSQM	HL3	HL1	MIR	NM	MM
15	HL2	HL1	HL3	MIR	CQM	CSQM	NM	MM

Table 4.5: **Skewed Distribution:** CV of Estimators NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Laplace Distribution, λ_1 = 2.0, λ_2 = 0.5, p = 0.2, Mean = -0.6435 (See Appendix I), Median = -0.6147 (See Appendix I).

	Rank							
n	1	2	3	4	5	6	7	8
7	CSQM	CQM	HL2	NM	MM	HL3	HL1	MIR
9	CSQM	CQM	HL2	NM	HL3	MM	HL1	MIR
11	CSQM	CQM	NM	HL2	MM	HL3	HL1	MIR
13	CSQM	CQM	HL2	NM	HL3	MM	HL1	MIR
15	HL2	CQM	CSQM	HL3	HL1	NM	MM	MIR

From Tables 4.1-4.3: one can see that CSQM has lowest circular variance compared to other estimators for small odd sample size (n = 7, 9) and small κ ($\kappa = 0.1$). Table 4.4: CSQM has lowest circular variance for odd sample size 7, 9 while MIR, CQM and HL2 have lowest circular variance for odd sample size 11, 13, 15 respectively. Table 4.5: CSQM has lowest circular variance for odd sample size 7, 9, 11 and 13 but lost efficiency to HL2 when n = 15. Overall proposed estimators: CSQM and CQM have the smallest circular variance, dominate other estimators for small odd sample size, here for $7 \leq n \leq 13$.

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4.5 Comparison of Variability

In this section we compare variability of 8 estimators of a population median, compute 95 % CL and evaluating by using 3 measure of variations, CV, CMAD, CMEAD as shown in Table (4.6 - 4.10) in one selected case -a small sample size, $n = 7$. The ranking obtained in section 4.3 was based on Table like Table 4.6 - 4.10.

Table 4.6: **Symmetric Distribution:** NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 7$, $\kappa = 0.1$, Based on 10,000 Random Samples.

Estimators	$\hat{\mu}$	95 % CL	CV	CMAD	CMEAD
NM	0.0027	(-2.933, 2.945)	0.8788	1.4156	1.3259
MM	0.0027	(-2.933, 2.945)	0.8788	1.4156	1.3259
MIR	**-0.0104**	**(-2.938, 2.954)**	**0.8770**	**1.4141**	**1.3237**
CSQM	**-0.0063**	**(-2.936, 2.936)**	**0.8762**	**1.4123**	**1.3218**
CQM	**-0.0182**	**(-2.931, 2.940)**	**0.8763**	**1.4127**	**1.3229**
HL1	-0.0016	(-2.937, 2.930)	0.8829	1.4209	1.3351
HL2	0.0060	(-2.924, 2.933)	0.8823	1.4200	1.3420
HL3	0.0034	(-2.937, 2.929)	0.8836	1.4216	1.3367

Table 4.7: **Contamination on Spread:** NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa) + \epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 0.1$, Based on 10,000 Random Samples.

Estimators	$\hat{\mu}$	95 % CL	CV	CMAD	CMEAD
NM	0.1049	(-2.951, 2.920)	0.9043	1.4486	1.3846
MM	0.0804	(-2.950, 2.940)	0.9002	1.4434	1.3743
MIR	**0.0836**	**(-2.946, 2.951)**	**0.9029**	**1.4488**	**1.3779**
CSQM	**0.0859**	**(-2.969, 2.918)**	**0.8992**	**1.4431**	**1.3627**
CQM	**0.0923**	**(-2.948, 2.914)**	**0.8998**	**1.4442**	**1.3630**
HL1	0.1631	(-2.960, 2.940)	0.9047	1.4507	1.3742
HL2	0.1361	(-2.942, 2.957)	0.9054	1.4509	1.3808
HL3	0.1438	(-2.959, 2.943)	0.9063	1.4522	1.3820

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Table 4.8: **Contamination on Location:** NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa) + \epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 0.1$, Based on 10,000 Random Samples.

Estimators	$\hat{\mu}$	95 % CL	CV	CMAD	CMEAD
NM	-0.0444	(-2.940, 2.947)	0.8885	1.4286	1.3598
MM	-0.0452	(-2.934, 2.948)	0.8846	1.4235	1.3525
MIR	**-0.0656**	**(-2.953, 2.944)**	**0.8834**	**1.4222**	**1.3422**
CSQM	**-0.0538**	**(-2.941, 2.936)**	**0.8823**	**1.4207**	**1.3395**
CQM	**-0.0588**	**(-2.935, 2.946)**	**0.8829**	**1.4209**	**1.3385**
HL1	-0.1124	(-2.947, 2.914)	0.8876	1.4267	1.3560
HL2	-0.1040	(-2.946, 2.925)	0.8877	1.4271	1.3619
HL3	-0.1144	(-2.947, 2.923)	0.8883	1.4277	1.3596

Table 4.9: **Skewed Distribution:** NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Exponential Distribution($\lambda = 2$), Mean = $\text{atan}(1/\lambda) = 0.4637$, Median = 0.3456 (see 1.35), Sample Size $n = 7$, Based on 10,000 Random Samples.

Estimators	$\hat{\mu}$	95 % CL	CV	CMAD	CMEAD	CAE
NM	0.3743	(0.100, 0.837)	0.0181	0.1495	0.1282	0.1473
MM	0.3745	(0.100, 0.837)	0.0181	0.1496	0.1283	0.1475
MIR	**0.4325**	**(0.158, 0.846)**	**0.0156**	**0.1394**	**0.1181**	**0.1484**
CSQM	**0.3877**	**(0.125, 0.811)**	**0.0154**	**0.1379**	**0.1181**	**0.1374**
CQM	**0.3944**	**(0.128, 0.812)**	**0.0158**	**0.1397**	**0.1200**	**0.1401**
HL1	0.4640	(0.167, 0.915)	0.0181	0.1500	0.1269	0.1689
HL2	0.4272	(0.146, 0.848)	0.0162	0.1424	0.1202	0.1499
HL3	0.4370	(0.152, 0.873)	0.0169	0.1453	0.1229	0.1552

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Table 4.10: **Skewed Distribution:** NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Laplace Distribution, $\lambda_1 = 0.8$, $\lambda_2 = 1.25$, p = 0.61, Mean = 0.2213 (See Appendix I), Median = 0.3955 (See Appendix I), Sample Size $n = 7$, Based on 10,000 Random Samples.

Estimators	$\hat{\mu}$	95 % CL	CV	CMAD	CMEAD	CAE
NM	0.2100	(-0.949, 1.564)	0.1571	0.4410	0.3207	0.4794
MM	0.2106	(-0.940, 1.553)	0.1579	0.4434	0.3244	0.4822
MIR	**0.2368**	**(-0.995, 1.480)**	**0.1646**	**0.4617**	**0.3559**	**0.4831**
CSQM	**0.2184**	**(-0.918, 1.457)**	**0.1505**	**0.4337**	**0.3246**	**0.4673**
CQM	**0.2224**	**(-0.952, 1.451)**	**0.1553**	**0.4433**	**0.3361**	**0.4737**
HL1	0.2388	(-0.996, 1.505)	0.1642	0.4605	0.3548	0.4820
HL2	0.2300	(-0.951, 1.472)	0.1554	0.4441	0.3362	0.4709
HL3	0.2338	(-0.988, 1.476)	0.1592	0.4509	0.3426	0.4754

4.6 Sample Size for Odd and Even

Sample median on a circle not always exists, since necessary conditions 1. and 2. (see page 17) sometimes are not satisfied. This is more likely to happen if the sample size is small, than if it is odd, and/or the variability the data is large (κ is small). Table 4.11 illustrates these trends. Numbers of times conditions 1. and 2. are satisfied is given for n = 7, 8, 19, and 20 and different values of κ.

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Table 4.11: Frequency of Cases with Satisfied Conditions 1. and 2. in 10,000 Repetitions for Odd and Even Sample Size.

odd	κ	Cond. 1 and 2 satisfied	even	κ	Cond. 1 and 2 satisfied
7	0.1	9503	8	0.1	9989
	0.2	9546		0.2	9978
	0.3	9554		0.3	9979
	0.4	9608		0.4	9991
	0.5	9642		0.5	9992
	1.0	9846		1.0	9996
	1.5	9972		1.5	9999
	2.0	9992		2.0	10000
	2.5	10000		2.5	10000
19	0.1	9831	20	0.1	9967
	0.2	9833		0.2	9962
	0.3	9854		0.3	9982
	0.4	9887		0.4	9967
	0.5	9930		0.5	9982
	1.0	9992		1.0	10000
	1.5	10000		1.5	10000
	2.0	10000		2.0	10000
	2.5	10000		2.5	10000

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Chapter 5

Conclusions and Future Research

Three proposed estimators, MIR, CQM and CSQM were investigated and compared with existing estimators of a population median on a circle. Existing estimators of population median either have a small variation (HL1, HL2, HL3) but require extensive computations, or are computationally simple (MM, NM) but have larger variability. Also, properties of existing estimators were studied neither in the case of small samples nor relatively small concentration of the distribution.

Result of a simulation study can be found in the Tables (B.1 - G.4). Wide variety of cases have been covered -different sample sizes, a whole range of concentration parameters and different distribution types (symmetric, symmetric contaminated on location or on scale, skewed). In the case of a small sample a CQM (Central Quasi Median), and especially its smoothed version CSQM (Central Smoothed Quasi Median), performs the best. CSQM has additional advantage since its variability decreases with increasing sample size regardless of the sample size being even or odd, which is not true for quasi-medians (Here MIR and CQM)

It can be easily seen that the MIR estimator has very good asymptotic relative efficiency (see Table 3.1 - 3.2).

The work that has been presented in this dissertation can be extended in few possible directions.

First, since the method of obtaining quartiles on the circle has been proposed, the performance of their bootstrap CI's could be investigated.

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Secondly, it could be studied how one can compare two or more population medians on the circle using the proposed estimators. Bootstrap technique as a tool in this types of statistical itself is an interesting research question. We are aware that condition 1. and 2. (page 17) of the median might cause additional difficulties because of likely repeated values in the bootstrap sample.

It would be interesting to investigate if the frequency of the cases when the median cannot be found (see Section 4.5) in the bootstrap samples is not higher and if yes then to propose a suitable solution to this problem.

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Appendix A

Notations

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There are some notations that will be used in our simulation:

Est = Estimator
κ = Concentration Parameter
Mean = Circ.mean
NM = New Median
MM = Mardia Median
MIR = Mid Interquartile Range
CSQM = Central Smoothed Quasi Median
CQM = Central Quasi Median
HL1 = Hodges-Lehmann 1
HL2 = Hodges-Lehmann 2
HL3 = Hodges-Lehmann 3
CV = Circular Variance
Mean Abs Dev = Circular Mean Absolute Deviation (CMAD)
Median Abs Dev = Circular Median Absolute Deviation (CMEAD)
Mean Abs Error = Circular Mean Absolute Error (CAE)
n = Frequency of cases with satisfied conditions 1. and 2. (see page. 17)

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Appendix B

vM(0, $\boldsymbol{k}$)

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Table B.1: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 7$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.012346	(-2.943, 2.926)	0.8818	1.4200	1.3296	1.4201	10000
	NM	0.015665	(-2.924, 2.949)	0.8819	1.4197	1.3365	1.4198	9503
	MM	0.002741	(-2.933, 2.945)	0.8788	1.4156	1.3259	1.4156	9503
	MIR	**-0.010448**	**(-2.938, 2.954)**	**0.8770**	**1.4141**	**1.3237**	**1.4141**	**9503**
	CSQM	**-0.006272**	**(-2.936, 2.936)**	**0.8762**	**1.4123**	**1.3218**	**1.4123**	**9503**
	CQM	**-0.018168**	**(-2.931, 2.940)**	**0.8763**	**1.4127**	**1.3229**	**1.4127**	**9503**
	HL1	-0.001587	(-2.937, 2.930)	0.8829	1.4209	1.3351	1.4209	9883
	HL2	0.006001	(-2.924, 2.933)	0.8823	1.4200	1.3420	1.4201	10000
	HL3	0.003420	(-2.937, 2.929)	0.8836	1.4216	1.3367	1.4216	9997
0.2	Mean	0.023683	(-2.878, 2.841)	0.7518	1.2547	1.1004	1.2546	10000
	NM	0.018366	(-2.835, 2.878)	0.7546	1.2573	1.1134	1.2573	9546
	MM	0.025352	(-2.838, 2.872)	0.7551	1.2586	1.1161	1.2585	9546
	MIR	**0.003446**	**(-2.867, 2.848)**	**0.7540**	**1.2563**	**1.1133**	**1.2563**	**9546**
	CSQM	**0.015365**	**(-2.869, 2.844)**	**0.7500**	**1.2518**	**1.1031**	**1.2517**	**9546**
	CQM	**0.006831**	**(-2.862, 2.843)**	**0.7512**	**1.2527**	**1.1041**	**1.2527**	**9546**
	HL1	0.011466	(-2.889, 2.864)	0.7573	1.2615	1.1005	1.2615	9887
	HL2	0.018253	(-2.883, 2.869)	0.7551	1.2590	1.1062	1.2589	10000
	HL3	0.013013	(-2.879, 2.868)	0.7552	1.2588	1.0964	1.2588	10000
0.3	Mean	0.011497	(-2.763, 2.780)	0.6723	1.1511	0.9743	1.1511	10000
	NM	0.012164	(-2.796, 2.777)	0.6717	1.1520	0.9691	1.1520	9554
	MM	0.011317	(-2.803, 2.778)	0.6722	1.1527	0.9693	1.1526	9554
	MIR	**0.012942**	**(-2.805, 2.783)**	**0.6730**	**1.1515**	**0.9763**	**1.1514**	**9554**
	CSQM	**0.009745**	**(-2.796, 2.780)**	**0.6658**	**1.1431**	**0.9668**	**1.1430**	**9554**
	CQM	**0.007198**	**(-2.804, 2.756)**	**0.6678**	**1.1447**	**0.9706**	**1.1447**	**9554**
	HL1	0.000704	(-2.774, 2.792)	0.6784	1.1589	0.9827	1.1589	9870
	HL2	0.007736	(-2.782, 2.788)	0.6785	1.1589	0.9780	1.1588	10000
	HL3	0.006082	(-2.777, 2.789)	0.6783	1.1586	0.9743	1.1586	9994
0.4	Mean	0.027662	(-2.659, 2.685)	0.5541	1.0007	0.7863	1.0009	10000
	NM	0.030891	(-2.712, 2.684)	0.5609	1.0104	0.8040	1.0107	9608
	MM	0.026003	(-2.706, 2.643)	0.5597	1.0068	0.8067	1.0069	9608
	MIR	**0.031481**	**(-2.676, 2.664)**	**0.5599**	**1.0077**	**0.8010**	**1.0079**	**9608**
	CSQM	**0.027667**	**(-2.659, 2.645)**	**0.5522**	**0.9969**	**0.7922**	**0.9970**	**9608**
	CQM	**0.029786**	**(-2.658, 2.661)**	**0.5553**	**1.0010**	**0.7971**	**1.0010**	**9608**
	HL1	0.029661	(-2.686, 2.699)	0.5629	1.0123	0.8008	1.0124	9890
	HL2	0.034608	(-2.691, 2.672)	0.5607	1.0090	0.8019	1.0091	10000
	HL3	0.035792	(-2.672, 2.691)	0.5618	1.0105	0.8021	1.0108	9997
0.5	Mean	0.016632	(-2.495, 2.570)	0.4760	0.8959	0.6871	0.8961	10000
	NM	0.014993	(-2.563, 2.554)	0.4846	0.9091	0.6991	0.9093	9642
	MM	0.022937	(-2.600, 2.507)	0.4872	0.9134	0.7081	0.9138	9642
	MIR	**0.026910**	**(-2.506, 2.533)**	**0.4827**	**0.9046**	**0.7049**	**0.9051**	**9642**
	CSQM	**0.021885**	**(-2.537, 2.515)**	**0.4766**	**0.8985**	**0.6884**	**0.8988**	**9642**
	CQM	**0.020478**	**(-2.541, 2.542)**	**0.4778**	**0.8987**	**0.6843**	**0.8990**	**9642**
	HL1	0.022758	(-2.572, 2.518)	0.4826	0.9051	0.6980	0.9053	9905
	HL2	0.019712	(-2.589, 2.514)	0.4804	0.9022	0.6921	0.9023	10000
	HL3	0.022034	(-2.582, 2.502)	0.4828	0.9051	0.6967	0.9053	9999

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Table B.2: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 7$, $\kappa = 1.0,1.5,2.0,2.5,3.0$ Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.000784	(-1.363, 1.338)	0.1900	0.4999	0.3762	0.4999	10000
	NM	0.001573	(-1.395, 1.411)	0.2019	0.5213	0.4018	0.5213	9846
	MM	0.001450	(-1.391, 1.411)	0.2040	0.5246	0.4031	0.5246	9846
	MIR	**-0.002288**	**(-1.430, 1.423)**	**0.1987**	**0.5124**	**0.3895**	**0.5124**	**9846**
	CSQM	**0.000088**	**(-1.374, 1.358)**	**0.1930**	**0.5057**	**0.3838**	**0.5057**	**9846**
	CQM	**-0.000494**	**(-1.389, 1.392)**	**0.1956**	**0.5093**	**0.3877**	**0.5092**	**9846**
	HL1	0.002980	(-1.407, 1.391)	0.2001	0.5157	0.3945	0.5157	9964
	HL2	0.001993	(-1.364, 1.381)	0.1951	0.5084	0.3886	0.5084	10000
	HL3	0.002741	(-1.373, 1.410)	0.1977	0.5124	0.3924	0.5124	9999
1.5	Mean	-0.006266	(-0.835, 0.838)	0.0853	0.3276	0.2618	0.3276	10000
	NM	-0.005826	(-0.930, 0.909)	0.1009	0.3593	0.2913	0.3593	9972
	MM	-0.006455	(-0.937, 0.909)	0.1020	0.3619	0.2939	0.3620	9972
	MIR	**-0.006053**	**(-0.877, 0.872)**	**0.0911**	**0.3383**	**0.2694**	**0.3384**	**9972**
	CSQM	**-0.005323**	**(-0.864, 0.861)**	**0.0907**	**0.3385**	**0.2733**	**0.3386**	**9972**
	CQM	**-0.004794**	**(-0.869, 0.874)**	**0.0918**	**0.3407**	**0.2727**	**0.3407**	**9972**
	HL1	-0.005545	(-0.866, 0.862)	0.0905	0.3375	0.2692	0.3375	9989
	HL2	-0.005763	(-0.851, 0.836)	0.0878	0.3323	0.2653	0.3323	10000
	HL3	-0.005564	(-0.863, 0.850)	0.0889	0.3345	0.2665	0.3346	10000
2.0	Mean	-0.006692	(-0.658, 0.652)	0.0533	0.2610	0.2166	0.2611	10000
	NM	-0.004682	(-0.731, 0.741)	0.0675	0.2958	0.2428	0.2959	9992
	MM	-0.004099	(-0.737, 0.742)	0.0678	0.2965	0.2439	0.2965	9992
	MIR	**-0.007008**	**(-0.680, 0.679)**	**0.0567**	**0.2683**	**0.2223**	**0.2684**	**9992**
	CSQM	**-0.006518**	**(-0.689, 0.680)**	**0.0584**	**0.2735**	**0.2258**	**0.2736**	**9992**
	CQM	**-0.007975**	**(-0.691, 0.681)**	**0.0588**	**0.2737**	**0.2287**	**0.2737**	**9992**
	HL1	-0.006974	(-0.677, 0.665)	0.0565	0.2682	0.2213	0.2682	9996
	HL2	-0.006150	(-0.677, 0.667)	0.0553	0.2656	0.2191	0.2657	10000
	HL3	-0.006238	(-0.673, 0.663)	0.0558	0.2668	0.2207	0.2668	10000
2.5	Mean	-0.004758	(-0.562, 0.556)	0.0381	0.2208	0.1841	0.2208	10000
	NM	-0.004099	(-0.644, 0.627)	0.0495	0.2521	0.2126	0.2522	10000
	MM	-0.003970	(-0.644, 0.633)	0.0498	0.2528	0.2133	0.2528	10000
	MIR	**-0.005480**	**(-0.578, 0.572)**	**0.0401**	**0.2263**	**0.1870**	**0.2263**	**10000**
	CSQM	**-0.003916**	**(-0.594, 0.586)**	**0.0424**	**0.2331**	**0.1946**	**0.2332**	**10000**
	CQM	**-0.003903**	**(-0.592, 0.586)**	**0.0427**	**0.2335**	**0.1929**	**0.2335**	**10000**
	HL1	-0.004680	(-0.578, 0.563)	0.0404	0.2270	0.1880	0.2270	10000
	HL2	-0.005318	(-0.575, 0.560)	0.0394	0.2244	0.1854	0.2245	10000
	HL3	-0.005476	(-0.577, 0.556)	0.0398	0.2255	0.1868	0.2255	10000
3.0	Mean	-0.005142	(-0.493, 0.476)	0.0297	0.1955	0.1631	0.1955	10000
	NM	-0.007046	(-0.570, 0.559)	0.0403	0.2283	0.1913	0.2284	10000
	MM	-0.006907	(-0.570, 0.561)	0.0404	0.2285	0.1913	0.2286	10000
	MIR	**-0.005999**	**(-0.507, 0.486)**	**0.0312**	**0.2006**	**0.1682**	**0.2007**	**10000**
	CSQM	**-0.006903**	**(-0.526, 0.504)**	**0.0341**	**0.2107**	**0.1776**	**0.2107**	**10000**
	CQM	**-0.006922**	**(-0.524, 0.502)**	**0.0341**	**0.2107**	**0.1774**	**0.2107**	**10000**
	HL1	-0.005636	(-0.502, 0.487)	0.0313	0.2007	0.1680	0.2008	10000
	HL2	-0.006503	(-0.500, 0.482)	0.0311	0.2002	0.1677	0.2003	10000
	HL3	-0.005958	(-0.502, 0.483)	0.0312	0.2005	0.1680	0.2006	10000

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Table B.3: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size n = 7, κ = 4.0,5.0,6.0,7.0, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	-0.000128	(-0.397, 0.405)	0.0208	0.1633	0.1376	0.1633	10000
	NM	-0.000585	(-0.461, 0.473)	0.0286	0.1916	0.1613	0.1916	10000
	MM	-0.000597	(-0.461, 0.473)	0.0286	0.1917	0.1613	0.1917	10000
	MIR	**-0.000179**	**(-0.410, 0.413)**	**0.0219**	**0.1680**	**0.1412**	**0.1680**	**10000**
	CSQM	**-0.000534**	**(-0.433, 0.439)**	**0.0242**	**0.1763**	**0.1472**	**0.1763**	**10000**
	CQM	**-0.000505**	**(-0.435, 0.435)**	**0.0242**	**0.1765**	**0.1490**	**0.1765**	**10000**
	HL1	0.000495	(-0.407, 0.416)	0.0218	0.1673	0.1410	0.1673	10000
	HL2	0.000161	(-0.409, 0.414)	0.0219	0.1678	0.1419	0.1678	10000
	HL3	0.000080	(-0.410, 0.415)	0.0220	0.1680	0.1421	0.1680	10000
5.0	Mean	0.001780	(-0.347, 0.342)	0.0155	0.1412	0.1188	0.1412	10000
	NM	0.001418	(-0.418, 0.421)	0.0219	0.1674	0.1403	0.1674	10000
	MM	0.001428	(-0.418, 0.421)	0.0219	0.1674	0.1403	0.1674	10000
	MIR	**0.001934**	**(-0.354, 0.349)**	**0.0162**	**0.1445**	**0.1237**	**0.1445**	**10000**
	CSQM	**0.001721**	**(-0.377, 0.376)**	**0.0181**	**0.1526**	**0.1302**	**0.1526**	**10000**
	CQM	**0.001837**	**(-0.372, 0.372)**	**0.0180**	**0.1529**	**0.1311**	**0.1529**	**10000**
	HL1	0.002063	(-0.355, 0.348)	0.0162	0.1442	0.1221	0.1442	10000
	HL2	0.001811	(-0.355, 0.357)	0.0162	0.1445	0.1215	0.1446	10000
	HL3	0.002025	(-0.355, 0.351)	0.0162	0.1444	0.1221	0.1444	10000
6.0	Mean	-0.001253	(-0.316, 0.322)	0.0131	0.1295	0.1095	0.1295	10000
	NM	-0.002236	(-0.379, 0.379)	0.0184	0.1535	0.1301	0.1535	10000
	MM	-0.002236	(-0.379, 0.379)	0.0184	0.1535	0.1301	0.1535	10000
	MIR	**-0.001576**	**(-0.325, 0.333)**	**0.0138**	**0.1329**	**0.1108**	**0.1329**	**10000**
	CSQM	**-0.002321**	**(-0.343, 0.350)**	**0.0153**	**0.1401**	**0.1188**	**0.1401**	**10000**
	CQM	**-0.002368**	**(-0.342, 0.353)**	**0.0153**	**0.1396**	**0.1182**	**0.1396**	**10000**
	HL1	-0.001611	(-0.323, 0.332)	0.0137	0.1323	0.1117	0.1323	10000
	HL2	-0.001694	(-0.322, 0.334)	0.0138	0.1326	0.1117	0.1326	10000
	HL3	-0.001876	(-0.321, 0.333)	0.0138	0.1326	0.1118	0.1326	10000
7.0	Mean	0.000102	(-0.295, 0.298)	0.0112	0.1192	0.1002	0.1192	10000
	NM	0.000508	(-0.357, 0.354)	0.0161	0.1440	0.1216	0.1440	10000
	MM	0.000508	(-0.357, 0.354)	0.0161	0.1440	0.1216	0.1440	10000
	MIR	**0.000240**	**(-0.308, 0.303)**	**0.0119**	**0.1232**	**0.1034**	**0.1232**	**10000**
	CSQM	**0.000482**	**(-0.322, 0.322)**	**0.0133**	**0.1304**	**0.1096**	**0.1304**	**10000**
	CQM	**0.000485**	**(-0.321, 0.320)**	**0.0133**	**0.1300**	**0.1086**	**0.1300**	**10000**
	HL1	0.000916	(-0.303, 0.302)	0.0117	0.1221	0.1022	0.1221	10000
	HL2	0.000906	(-0.304, 0.305)	0.0119	0.1229	0.1030	0.1229	10000
	HL3	0.000963	(-0.303, 0.302)	0.0118	0.1227	0.1029	0.1227	10000
8.0	Mean	-0.000635	(-0.267, 0.269)	0.0094	0.1102	0.0944	0.1102	10000
	NM	-0.001870	(-0.315, 0.324)	0.0134	0.1315	0.1130	0.1315	10000
	MM	-0.001870	(-0.315, 0.324)	0.0134	0.1315	0.1130	0.1315	10000
	MIR	**-0.000318**	**(-0.277, 0.277)**	**0.0100**	**0.1136**	**0.0967**	**0.1136**	**10000**
	CSQM	**-0.000725**	**(-0.288, 0.296)**	**0.0111**	**0.1199**	**0.1016**	**0.1199**	**10000**
	CQM	**-0.000136**	**(-0.289, 0.298)**	**0.0112**	**0.1201**	**0.1020**	**0.1201**	**10000**
	HL1	-0.000589	(-0.275, 0.275)	0.0099	0.1131	0.0968	0.1131	10000
	HL2	-0.000902	(-0.274, 0.276)	0.0099	0.1136	0.0977	0.1136	10000
	HL3	-0.000772	(-0.274, 0.276)	0.0099	0.1135	0.0973	0.1135	10000

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Table B.4: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 8$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	-0.089556	(-2.949, 2.949)	0.8752	1.4117	1.3283	1.4125	10000
	NM	-0.095881	(-2.953, 2.949)	0.8841	1.4221	1.3476	1.4229	9989
	MM	-0.089424	(-2.935, 2.956)	0.8848	1.4238	1.3532	1.4240	9989
	MIR	**-0.126076**	**(-2.958, 2.956)**	**0.8865**	**1.4274**	**1.3572**	**1.4278**	**9989**
	CSQM	**-0.089424**	**(-2.935, 2.956)**	**0.8848**	**1.4238**	**1.3532**	**1.4240**	**9989**
	CQM	**-0.089424**	**(-2.935, 2.956)**	**0.8848**	**1.4238**	**1.3532**	**1.4240**	**9989**
	HL1	-0.110705	(-2.958, 2.943)	0.8788	1.4167	1.3297	1.4179	9988
	HL2	-0.112057	(-2.953, 2.949)	0.8767	1.4139	1.3224	1.4153	10000
	HL3	-0.109881	(-2.971, 2.937)	0.8786	1.4163	1.3229	1.4179	9996
0.2	Mean	-0.003597	(-2.870, 2.871)	0.7465	1.2485	1.0828	1.2485	10000
	NM	-0.009011	(-2.874, 2.866)	0.7545	1.2597	1.1061	1.2597	9978
	MM	-0.001956	(-2.879, 2.855)	0.7512	1.2538	1.0999	1.2538	9978
	MIR	**-0.019338**	**(-2.847, 2.872)**	**0.7595**	**1.2630**	**1.1077**	**1.2631**	**9978**
	CSQM	**-0.001956**	**(-2.879, 2.855)**	**0.7512**	**1.2538**	**1.0999**	**1.2538**	**9978**
	CQM	**-0.001956**	**(-2.879, 2.855)**	**0.7512**	**1.2538**	**1.0999**	**1.2538**	**9978**
	HL1	-0.003960	(-2.857, 2.876)	0.7526	1.2540	1.1017	1.2540	9989
	HL2	-0.014047	(-2.850, 2.872)	0.7496	1.2510	1.1034	1.2510	10000
	HL3	-0.012613	(-2.855, 2.877)	0.7504	1.2519	1.0999	1.2518	9998
0.3	Mean	0.000751	(-2.779, 2.795)	0.6440	1.1155	0.9144	1.1155	10000
	NM	0.013556	(-2.811, 2.821)	0.6635	1.1410	0.9365	1.1410	9979
	MM	-0.002386	(-2.806, 2.786)	0.6555	1.1308	0.9310	1.1309	9979
	MIR	**0.003509**	**(-2.774, 2.811)**	**0.6638**	**1.1400**	**0.9401**	**1.1399**	**9979**
	CSQM	**-0.002386**	**(-2.806, 2.786)**	**0.6555**	**1.1308**	**0.9310**	**1.1309**	**9979**
	CQM	**-0.002386**	**(-2.806, 2.786)**	**0.6555**	**1.1308**	**0.9310**	**1.1309**	**9979**
	HL1	0.011050	(-2.792, 2.798)	0.6495	1.1227	0.9226	1.1226	9994
	HL2	0.008738	(-2.774, 2.819)	0.6462	1.1182	0.9203	1.1182	10000
	HL3	0.011442	(-2.788, 2.804)	0.6481	1.1205	0.9212	1.1205	9999
0.4	Mean	0.001106	(-2.601, 2.625)	0.5285	0.9662	0.7587	0.9662	10000
	NM	-0.003161	(-2.676, 2.704)	0.5504	0.9944	0.7830	0.9944	9991
	MM	0.007805	(-2.675, 2.651)	0.5444	0.9867	0.7802	0.9867	9991
	MIR	**-0.006397**	**(-2.668, 2.656)**	**0.5546**	**0.9994**	**0.8000**	**0.9994**	**9991**
	CSQM	**0.007805**	**(-2.675, 2.651)**	**0.5444**	**0.9867**	**0.7802**	**0.9867**	**9991**
	CQM	**0.007805**	**(-2.675, 2.651)**	**0.5444**	**0.9867**	**0.7802**	**0.9867**	**9991**
	HL1	0.001304	(-2.631, 2.627)	0.5373	0.9769	0.7731	0.9769	9990
	HL2	0.003035	(-2.613, 2.627)	0.5322	0.9701	0.7646	0.9701	10000
	HL3	0.003292	(-2.637, 2.619)	0.5347	0.9734	0.7661	0.9734	9996
0.5	Mean	-0.010962	(-2.450, 2.477)	0.4530	0.8652	0.6602	0.8652	10000
	NM	-0.008215	(-2.539, 2.499)	0.4750	0.8948	0.6863	0.8949	9992
	MM	-0.007415	(-2.488, 2.493)	0.4691	0.8872	0.6833	0.8873	9992
	MIR	**0.000920**	**(-2.531, 2.482)**	**0.4727**	**0.8904**	**0.6783**	**0.8904**	**9992**
	CSQM	**-0.007415**	**(-2.488, 2.493)**	**0.4691**	**0.8872**	**0.6833**	**0.8873**	**9992**
	CQM	**-0.007415**	**(-2.488, 2.493)**	**0.4691**	**0.8872**	**0.6833**	**0.8873**	**9992**
	HL1	-0.008842	(-2.478, 2.489)	0.4583	0.8729	0.6713	0.8729	9989
	HL2	-0.011076	(-2.503, 2.497)	0.4564	0.8699	0.6722	0.8699	10000
	HL3	-0.009166	(-2.477, 2.516)	0.4577	0.8718	0.6707	0.8718	9999

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Table B.5: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 8$, $\kappa =$ 1.0,1.5,2.0,2.5,3.0, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.002215	(-1.231, 1.318)	0.1706	0.4699	0.3580	0.4699	10000
	NM	-0.000024	(-1.324, 1.450)	0.1892	0.4978	0.3718	0.4978	9996
	MM	0.000379	(-1.324, 1.402)	0.1870	0.4963	0.3743	0.4963	9996
	MIR	**-0.001929**	**(-1.376, 1.391)**	**0.1903**	**0.5007**	**0.3823**	**0.5007**	**9996**
	CSQM	**0.000379**	**(-1.324, 1.402)**	**0.1870**	**0.4963**	**0.3743**	**0.4963**	**9996**
	CQM	**0.000379**	**(-1.324, 1.402)**	**0.1870**	**0.4963**	**0.3743**	**0.4963**	**9996**
	HL1	0.000772	(-1.252, 1.338)	0.1768	0.4800	0.3692	0.4800	9997
	HL2	0.001428	(-1.252, 1.346)	0.1750	0.4769	0.3645	0.4769	10000
	HL3	0.000929	(-1.263, 1.328)	0.1758	0.4784	0.3686	0.4784	10000
1.5	Mean	-0.001069	(-0.808, 0.797)	0.0772	0.3108	0.2526	0.3108	10000
	NM	-0.001170	(-0.866, 0.861)	0.0907	0.3377	0.2704	0.3377	9999
	MM	-0.000527	(-0.863, 0.867)	0.0895	0.3362	0.2697	0.3362	9999
	MIR	**0.000771**	**(-0.861, 0.853)**	**0.0877**	**0.3299**	**0.2636**	**0.3299**	**9999**
	CSQM	**-0.000527**	**(-0.863, 0.867)**	**0.0895**	**0.3362**	**0.2697**	**0.3362**	**9999**
	CQM	**-0.000527**	**(-0.863, 0.867)**	**0.0895**	**0.3362**	**0.2697**	**0.3362**	**9999**
	HL1	-0.000642	(-0.823, 0.830)	0.0817	0.3194	0.2568	0.3194	10000
	HL2	-0.000551	(-0.823, 0.815)	0.0805	0.3172	0.2556	0.3172	10000
	HL3	-0.000599	(-0.823, 0.820)	0.0810	0.3183	0.2571	0.3183	10000
2.0	Mean	-0.003758	(-0.600, 0.608)	0.0460	0.2417	0.2013	0.2418	10000
	NM	-0.004816	(-0.653, 0.650)	0.0542	0.2628	0.2163	0.2628	10000
	MM	-0.004741	(-0.654, 0.653)	0.0541	0.2630	0.2165	0.2631	10000
	MIR	**-0.004151**	**(-0.645, 0.644)**	**0.0509**	**0.2543**	**0.2111**	**0.2544**	**10000**
	CSQM	**-0.004741**	**(-0.654, 0.653)**	**0.0541**	**0.2630**	**0.2165**	**0.2631**	**10000**
	CQM	**-0.004741**	**(-0.654, 0.653)**	**0.0541**	**0.2630**	**0.2165**	**0.2631**	**10000**
	HL1	-0.003660	(-0.614, 0.623)	0.0478	0.2458	0.2022	0.2459	10000
	HL2	-0.003513	(-0.607, 0.622)	0.0472	0.2445	0.2007	0.2446	10000
	HL3	-0.003557	(-0.613, 0.624)	0.0476	0.2453	0.2016	0.2454	10000
2.5	Mean	-0.003576	(-0.512, 0.497)	0.0325	0.2032	0.1694	0.2032	10000
	NM	-0.002936	(-0.567, 0.545)	0.0389	0.2229	0.1860	0.2229	10000
	MM	-0.002953	(-0.567, 0.547)	0.0392	0.2233	0.1861	0.2233	10000
	MIR	**-0.004151**	**(-0.532, 0.521)**	**0.0350**	**0.2103**	**0.1743**	**0.2104**	**10000**
	CSQM	**-0.002953**	**(-0.567, 0.547)**	**0.0392**	**0.2233**	**0.1861**	**0.2233**	**10000**
	CQM	**-0.002953**	**(-0.567, 0.547)**	**0.0392**	**0.2233**	**0.1861**	**0.2233**	**10000**
	HL1	-0.003958	(-0.532, 0.504)	0.0338	0.2067	0.1715	0.2067	10000
	HL2	-0.003656	(-0.528, 0.504)	0.0336	0.2066	0.1713	0.2066	10000
	HL3	-0.004162	(-0.532, 0.505)	0.0338	0.2070	0.1709	0.2070	10000
3.0	Mean	0.001106	(-2.601, 2.625)	0.5285	0.9662	0.7587	0.9662	10000
	NM	-0.003161	(-2.676, 2.704)	0.5504	0.9944	0.7830	0.9944	9991
	MM	0.007805	(-2.675, 2.651)	0.5444	0.9867	0.7802	0.9867	9991
	MIR	**-0.006397**	**(-2.668, 2.656)**	**0.5546**	**0.9994**	**0.8000**	**0.9994**	**9991**
	CSQM	**0.007805**	**(-2.675, 2.651)**	**0.5444**	**0.9867**	**0.7802**	**0.9867**	**9991**
	CQM	**0.007805**	**(-2.675, 2.651)**	**0.5444**	**0.9867**	**0.7802**	**0.9867**	**9991**
	HL1	0.001304	(-2.631, 2.627)	0.5373	0.9769	0.7731	0.9769	9990
	HL2	0.003035	(-2.613, 2.627)	0.5322	0.9701	0.7646	0.9701	10000
	HL3	0.003292	(-2.637, 2.619)	0.5347	0.9734	0.7661	0.9734	9996

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Table B.6: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 8$, $\kappa = 4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.000252	(-0.368, 0.370)	0.0179	0.1518	0.1280	0.1518	10000
	NM	-0.000543	(-0.418, 0.421)	0.0226	0.1706	0.1433	0.1706	10000
	MM	-0.000559	(-0.418, 0.421)	0.0226	0.1705	0.1433	0.1705	10000
	MIR	**0.000984**	**(-0.383, 0.383)**	**0.0192**	**0.1570**	**0.1318**	**0.1570**	**10000**
	CSQM	**-0.000559**	**(-0.418, 0.421)**	**0.0226**	**0.1705**	**0.1433**	**0.1705**	**10000**
	CQM	**-0.000559**	**(-0.418, 0.421)**	**0.0226**	**0.1705**	**0.1433**	**0.1705**	**10000**
	HL1	0.000950	(-0.376, 0.381)	0.0187	0.1549	0.1300	0.1549	10000
	HL2	0.000615	(-0.376, 0.381)	0.0187	0.1550	0.1315	0.1550	10000
	HL3	0.000706	(-0.377, 0.381)	0.0187	0.1552	0.1305	0.1552	10000
5.0	Mean	-0.000975	(-0.336, 0.329)	0.0139	0.1329	0.1123	0.1329	10000
	NM	-0.001676	(-0.378, 0.362)	0.0178	0.1506	0.1273	0.1506	10000
	MM	-0.001681	(-0.378, 0.362)	0.0178	0.1506	0.1273	0.1506	10000
	MIR	**-0.000676**	**(-0.348, 0.343)**	**0.0150**	**0.1379**	**0.1159**	**0.1379**	**10000**
	CSQM	**-0.001681**	**(-0.378, 0.362)**	**0.0178**	**0.1506**	**0.1273**	**0.1506**	**10000**
	CQM	**-0.001681**	**(-0.378, 0.362)**	**0.0178**	**0.1506**	**0.1273**	**0.1506**	**10000**
	HL1	-0.000868	(-0.344, 0.337)	0.0146	0.1364	0.1143	0.1364	10000
	HL2	-0.000936	(-0.343, 0.338)	0.0147	0.1367	0.1153	0.1367	10000
	HL3	-0.000870	(-0.346, 0.337)	0.0147	0.1369	0.1152	0.1369	10000
6.0	Mean	0.000802	(-0.293, 0.293)	0.0111	0.1189	0.1008	0.1189	10000
	NM	0.000898	(-0.338, 0.333)	0.0143	0.1351	0.1145	0.1351	10000
	MM	0.000898	(-0.338, 0.333)	0.0143	0.1351	0.1145	0.1351	10000
	MIR	**0.000615**	**(-0.301, 0.305)**	**0.0119**	**0.1233**	**0.1042**	**0.1233**	**10000**
	CSQM	**0.000898**	**(-0.338, 0.333)**	**0.0143**	**0.1351**	**0.1145**	**0.1351**	**10000**
	CQM	**0.000898**	**(-0.338, 0.333)**	**0.0143**	**0.1351**	**0.1145**	**0.1351**	**10000**
	HL1	0.000663	(-0.299, 0.299)	0.0117	0.1223	0.1033	0.1223	10000
	HL2	0.000649	(-0.301, 0.299)	0.0117	0.1224	0.1029	0.1224	10000
	HL3	0.000576	(-0.300, 0.299)	0.0118	0.1224	0.1037	0.1224	10000
7.0	Mean	0.001149	(-0.271, 0.278)	0.0098	0.1124	0.0950	0.1124	10000
	NM	0.000188	(-0.309, 0.316)	0.0128	0.1279	0.1090	0.1279	10000
	MM	0.000188	(-0.309, 0.316)	0.0128	0.1279	0.1090	0.1279	10000
	MIR	**0.001387**	**(-0.278, 0.288)**	**0.0105**	**0.1160**	**0.0983**	**0.1160**	**10000**
	CSQM	**0.000188**	**(-0.309, 0.316)**	**0.0128**	**0.1279**	**0.1089**	**0.1279**	**10000**
	CQM	**0.000188**	**(-0.309, 0.316)**	**0.0128**	**0.1279**	**0.1089**	**0.1279**	**10000**
	HL1	0.000611	(-0.277, 0.283)	0.0103	0.1152	0.0974	0.1152	10000
	HL2	0.000287	(-0.279, 0.284)	0.0104	0.1157	0.0986	0.1157	10000
	HL3	0.000468	(-0.278, 0.285)	0.0104	0.1158	0.0983	0.1158	10000
8.0	Mean	0.002008	(-0.259, 0.256)	0.0084	0.1041	0.0881	0.1041	10000
	NM	0.002110	(-0.299, 0.291)	0.0111	0.1196	0.1008	0.1196	10000
	MM	0.002110	(-0.299, 0.291)	0.0111	0.1196	0.1008	0.1196	10000
	MIR	**0.001933**	**(-0.262, 0.262)**	**0.0089**	**0.1072**	**0.0913**	**0.1072**	**10000**
	CSQM	**0.002110**	**(-0.299, 0.291)**	**0.0111**	**0.1196**	**0.1008**	**0.1196**	**10000**
	CQM	**0.002110**	**(-0.299, 0.291)**	**0.0111**	**0.1196**	**0.1008**	**0.1196**	**10000**
	HL1	0.001986	(-0.266, 0.261)	0.0089	0.1069	0.0907	0.1070	10000
	HL2	0.002087	(-0.268, 0.261)	0.0089	0.1073	0.0897	0.1073	10000
	HL3	0.002023	(-0.267, 0.261)	0.0090	0.1074	0.0900	0.1074	10000

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Table B.7: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 19$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.003595	(-2.931, 2.915)	0.8197	1.3407	1.2161	1.3406	10000
	NM	0.038858	(-2.913, 2.903)	0.8281	1.3512	1.2419	1.3513	9831
	MM	0.013820	(-2.932, 2.918)	0.8277	1.3508	1.2327	1.3509	9831
	MIR	**0.005979**	**(-2.916, 2.911)**	**0.8247**	**1.3465**	**1.2272**	**1.3465**	**9831**
	CSQM	**0.009823**	**(-2.931, 2.918)**	**0.8274**	**1.3504**	**1.2318**	**1.3505**	**9831**
	CQM	**0.007811**	**(-2.923, 2.923)**	**0.8277**	**1.3509**	**1.2296**	**1.3509**	**9831**
	HL1	0.003621	(-2.897, 2.912)	0.8179	1.3386	1.2071	1.3386	9992
	HL2	0.009246	(-2.901, 2.913)	0.8203	1.3408	1.2125	1.3408	9999
	HL3	0.015003	(-2.894, 2.915)	0.8197	1.3404	1.2110	1.3405	9997
0.2	Mean	0.005483	(-2.767, 2.743)	0.6269	1.0938	0.8827	1.0939	10000
	NM	0.013909	(-2.779, 2.757)	0.6405	1.1089	0.8998	1.1090	9833
	MM	0.010914	(-2.802, 2.762)	0.6374	1.1063	0.8954	1.1063	9833
	MIR	**0.007657**	**(-2.771, 2.795)**	**0.6422**	**1.1123**	**0.9058**	**1.1124**	**9833**
	CSQM	**0.009374**	**(-2.794, 2.756)**	**0.6361**	**1.1047**	**0.8886**	**1.1047**	**9833**
	CQM	**0.008641**	**(-2.803, 2.761)**	**0.6364**	**1.1052**	**0.8876**	**1.1052**	**9833**
	HL1	0.015657	(-2.756, 2.793)	0.6307	1.0991	0.8958	1.0993	9991
	HL2	0.009995	(-2.767, 2.788)	0.6299	1.0979	0.8920	1.0980	9998
	HL3	0.012718	(-2.761, 2.791)	0.6308	1.0990	0.8956	1.0992	9997
0.3	Mean	0.000469	(-2.527, 2.566)	0.4797	0.9013	0.7011	0.9013	10000
	NM	0.019442	(-2.580, 2.576)	0.4991	0.9269	0.7226	0.9269	9854
	MM	0.005570	(-2.569, 2.555)	0.4932	0.9188	0.7185	0.9187	9854
	MIR	**0.010761**	**(-2.561, 2.527)**	**0.4952**	**0.9207**	**0.7216**	**0.9207**	**9854**
	CSQM	**0.008350**	**(-2.571, 2.554)**	**0.4902**	**0.9147**	**0.7138**	**0.9147**	**9854**
	CQM	**0.009850**	**(-2.570, 2.558)**	**0.4899**	**0.9143**	**0.7130**	**0.9142**	**9854**
	HL1	0.006089	(-2.522, 2.522)	0.4852	0.9080	0.7029	0.9079	9997
	HL2	0.005142	(-2.526, 2.517)	0.4829	0.9051	0.7007	0.9051	10000
	HL3	0.005249	(-2.543, 2.522)	0.4841	0.9064	0.7000	0.9064	9997
0.4	Mean	-0.004035	(-2.107, 2.074)	0.3508	0.7305	0.5604	0.7305	10000
	NM	-0.000308	(-2.158, 2.149)	0.3679	0.7537	0.5756	0.7537	9887
	MM	-0.003961	(-2.190, 2.131)	0.3707	0.7561	0.5762	0.7561	9887
	MIR	**-0.006528**	**(-2.151, 2.168)**	**0.3723**	**0.7588**	**0.5721**	**0.7588**	**9887**
	CSQM	**-0.003005**	**(-2.142, 2.156)**	**0.3664**	**0.7500**	**0.5696**	**0.7500**	**9887**
	CQM	**-0.002530**	**(-2.141, 2.157)**	**0.3658**	**0.7492**	**0.5703**	**0.7492**	**9887**
	HL1	-0.005481	(-2.152, 2.110)	0.3567	0.7386	0.5629	0.7386	9994
	HL2	-0.003118	(-2.134, 2.103)	0.3552	0.7364	0.5597	0.7363	10000
	HL3	-0.005050	(-2.143, 2.097)	0.3557	0.7369	0.5618	0.7369	9999
0.5	Mean	-0.007799	(-1.686, 1.690)	0.2552	0.5969	0.4483	0.5969	10000
	NM	-0.007792	(-1.814, 1.737)	0.2777	0.6293	0.4764	0.6293	9930
	MM	-0.005732	(-1.787, 1.789)	0.2769	0.6281	0.4779	0.6281	9930
	MIR	**-0.006545**	**(-1.812, 1.757)**	**0.2756**	**0.6252**	**0.4716**	**0.6253**	**9930**
	CSQM	**-0.006289**	**(-1.769, 1.741)**	**0.2717**	**0.6200**	**0.4656**	**0.6201**	**9930**
	CQM	**-0.006596**	**(-1.768, 1.728)**	**0.2708**	**0.6187**	**0.4659**	**0.6187**	**9930**
	HL1	-0.007739	(-1.757, 1.647)	0.2611	0.6058	0.4545	0.6058	9999
	HL2	-0.007062	(-1.718, 1.673)	0.2593	0.6033	0.4524	0.6033	10000
	HL3	-0.007065	(-1.711, 1.690)	0.2596	0.6039	0.4539	0.6039	9999

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Table B.8: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 19$, $\kappa = 1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	-0.002890	(-0.731, 0.716)	0.0653	0.2858	0.2321	0.2858	10000
	NM	-0.000200	(-0.799, 0.793)	0.0771	0.3126	0.2551	0.3126	9992
	MM	-0.002330	(-0.816, 0.796)	0.0795	0.3177	0.2588	0.3177	9992
	MIR	**-0.002724**	**(-0.806, 0.762)**	**0.0755**	**0.3078**	**0.2500**	**0.3078**	**9992**
	CSQM	**-0.001253**	**(-0.791, 0.774)**	**0.0759**	**0.3097**	**0.2532**	**0.3097**	**9992**
	CQM	**-0.000689**	**(-0.790, 0.772)**	**0.0756**	**0.3089**	**0.2526**	**0.3089**	**9992**
	HL1	-0.001246	(-0.750, 0.726)	0.0673	0.2902	0.2367	0.2902	10000
	HL2	-0.001317	(-0.742, 0.726)	0.0669	0.2895	0.2349	0.2895	10000
	HL3	-0.001303	(-0.742, 0.726)	0.0670	0.2896	0.2356	0.2896	10000
1.5	Mean	-0.004296	(-0.490, 0.488)	0.0301	0.1964	0.1664	0.1965	10000
	NM	-0.002708	(-0.553, 0.551)	0.0382	0.2205	0.1841	0.2205	10000
	MM	-0.003156	(-0.557, 0.557)	0.0388	0.2220	0.1853	0.2221	10000
	MIR	**-0.004060**	**(-0.521, 0.522)**	**0.0341**	**0.2089**	**0.1755**	**0.2089**	**10000**
	CSQM	**-0.003174**	**(-0.537, 0.536)**	**0.0362**	**0.2144**	**0.1784**	**0.2144**	**10000**
	CQM	**-0.003180**	**(-0.535, 0.537)**	**0.0357**	**0.2136**	**0.1770**	**0.2136**	**10000**
	HL1	-0.004693	(-0.501, 0.491)	0.0307	0.1983	0.1676	0.1984	10000
	HL2	-0.004657	(-0.497, 0.491)	0.0307	0.1979	0.1669	0.1980	10000
	HL3	-0.004711	(-0.498, 0.491)	0.0307	0.1981	0.1669	0.1982	10000
2.0	Mean	0.002476	(-0.380, 0.379)	0.0185	0.1540	0.1313	0.1541	10000
	NM	0.003711	(-0.439, 0.434)	0.0246	0.1790	0.1541	0.1790	10000
	MM	0.003892	(-0.439, 0.436)	0.0248	0.1794	0.1539	0.1794	10000
	MIR	**0.001597**	**(-0.402, 0.406)**	**0.0207**	**0.1624**	**0.1371**	**0.1624**	**10000**
	CSQM	**0.003004**	**(-0.422, 0.420)**	**0.0229**	**0.1722**	**0.1458**	**0.1722**	**10000**
	CQM	**0.002548**	**(-0.420, 0.415)**	**0.0227**	**0.1714**	**0.1459**	**0.1714**	**10000**
	HL1	0.002091	(-0.387, 0.382)	0.0189	0.1554	0.1317	0.1555	10000
	HL2	0.002209	(-0.382, 0.382)	0.0189	0.1554	0.1313	0.1554	10000
	HL3	0.002125	(-0.383, 0.382)	0.0189	0.1554	0.1316	0.1554	10000
2.5	Mean	0.000671	(-0.334, 0.338)	0.0141	0.1333	0.1113	0.1333	10000
	NM	-0.000791	(-0.387, 0.389)	0.0190	0.1548	0.1281	0.1548	10000
	MM	-0.000644	(-0.387, 0.393)	0.0190	0.1550	0.1282	0.1550	10000
	MIR	**0.000965**	**(-0.347, 0.350)**	**0.0156**	**0.1403**	**0.1177**	**0.1403**	**10000**
	CSQM	**0.000155**	**(-0.375, 0.375)**	**0.0178**	**0.1498**	**0.1242**	**0.1498**	**10000**
	CQM	**0.000566**	**(-0.374, 0.376)**	**0.0177**	**0.1498**	**0.1237**	**0.1498**	**10000**
	HL1	0.000905	(-0.336, 0.340)	0.0143	0.1346	0.1124	0.1346	10000
	HL2	0.000753	(-0.339, 0.340)	0.0143	0.1347	0.1120	0.1347	10000
	HL3	0.000791	(-0.338, 0.340)	0.0143	0.1346	0.1124	0.1346	10000
3.0	Mean	0.000403	(-0.291, 0.289)	0.0109	0.1179	0.0992	0.1179	10000
	NM	-0.000465	(-0.339, 0.343)	0.0149	0.1382	0.1177	0.1382	10000
	MM	-0.000420	(-0.339, 0.344)	0.0149	0.1384	0.1178	0.1384	10000
	MIR	**0.001078**	**(-0.301, 0.309)**	**0.0122**	**0.1246**	**0.1053**	**0.1246**	**10000**
	CSQM	**-0.000112**	**(-0.327, 0.329)**	**0.0138**	**0.1328**	**0.1126**	**0.1328**	**10000**
	CQM	**0.000043**	**(-0.324, 0.328)**	**0.0137**	**0.1323**	**0.1122**	**0.1323**	**10000**
	HL1	0.000319	(-0.293, 0.292)	0.0111	0.1193	0.1002	0.1193	10000
	HL2	0.000254	(-0.295, 0.294)	0.0111	0.1193	0.1000	0.1193	10000
	HL3	0.000331	(-0.294, 0.293)	0.0111	0.1193	0.1003	0.1193	10000

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Table B.9: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size $n = 19$, $\kappa = 4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.001374	(-0.245, 0.243)	0.0077	0.0988	0.0827	0.0988	10000
	NM	0.001230	(-0.290, 0.285)	0.0108	0.1173	0.0983	0.1173	10000
	MM	0.001221	(-0.290, 0.285)	0.0108	0.1174	0.0983	0.1174	10000
	MIR	**0.001799**	**(-0.261, 0.255)**	**0.0086**	**0.1049**	**0.0888**	**0.1049**	**10000**
	CSQM	**0.001245**	**(-0.280, 0.276)**	**0.0101**	**0.1126**	**0.0946**	**0.1126**	**10000**
	CQM	**0.001258**	**(-0.279, 0.274)**	**0.0100**	**0.1122**	**0.0944**	**0.1122**	**10000**
	HL1	0.001521	(-0.249, 0.246)	0.0079	0.1001	0.0846	0.1001	10000
	HL2	0.001549	(-0.248, 0.247)	0.0079	0.1002	0.0839	0.1002	10000
	HL3	0.001541	(-0.248, 0.246)	0.0079	0.1001	0.0842	0.1001	10000
5.0	Mean	-0.000753	(-0.210, 0.215)	0.0058	0.0862	0.0725	0.0862	10000
	NM	-0.001636	(-0.252, 0.255)	0.0084	0.1035	0.0881	0.1035	10000
	MM	-0.001636	(-0.252, 0.255)	0.0084	0.1035	0.0881	0.1035	10000
	MIR	**-0.000091**	**(-0.226, 0.224)**	**0.0066**	**0.0915**	**0.0763**	**0.0915**	**10000**
	CSQM	**-0.001402**	**(-0.244, 0.243)**	**0.0077**	**0.0995**	**0.0854**	**0.0995**	**10000**
	CQM	**-0.001281**	**(-0.242, 0.243)**	**0.0076**	**0.0993**	**0.0846**	**0.0993**	**10000**
	HL1	-0.000645	(-0.209, 0.217)	0.0060	0.0875	0.0740	0.0875	10000
	HL2	-0.000791	(-0.209, 0.217)	0.0060	0.0875	0.0741	0.0876	10000
	HL3	-0.000777	(-0.210, 0.216)	0.0060	0.0875	0.0741	0.0875	10000
6.0	Mean	0.000727	(-0.191, 0.193)	0.0049	0.0794	0.0667	0.0794	10000
	NM	0.000111	(-0.234, 0.236)	0.0072	0.0960	0.0815	0.0960	10000
	MM	0.000111	(-0.234, 0.236)	0.0072	0.0960	0.0815	0.0960	10000
	MIR	**0.000943**	**(-0.206, 0.205)**	**0.0055**	**0.0844**	**0.0716**	**0.0844**	**10000**
	CSQM	**0.000239**	**(-0.226, 0.225)**	**0.0066**	**0.0920**	**0.0788**	**0.0920**	**10000**
	CQM	**0.000303**	**(-0.224, 0.225)**	**0.0066**	**0.0916**	**0.0779**	**0.0916**	**10000**
	HL1	0.000940	(-0.196, 0.196)	0.0051	0.0806	0.0683	0.0806	10000
	HL2	0.000898	(-0.198, 0.197)	0.0051	0.0809	0.0687	0.0809	10000
	HL3	0.000897	(-0.196, 0.196)	0.0051	0.0807	0.0685	0.0808	10000
7.0	Mean	-0.001202	(-0.183, 0.175)	0.0042	0.0730	0.0618	0.0730	10000
	NM	-0.000621	(-0.220, 0.219)	0.0062	0.0887	0.0746	0.0886	10000
	MM	-0.000621	(-0.220, 0.219)	0.0062	0.0887	0.0746	0.0886	10000
	MIR	**-0.001364**	**(-0.195, 0.187)**	**0.0047**	**0.0778**	**0.0650**	**0.0778**	**10000**
	CSQM	**-0.000848**	**(-0.210, 0.210)**	**0.0057**	**0.0852**	**0.0721**	**0.0852**	**10000**
	CQM	**-0.000961**	**(-0.212, 0.208)**	**0.0057**	**0.0848**	**0.0717**	**0.0848**	**10000**
	HL1	-0.001246	(-0.188, 0.182)	0.0044	0.0745	0.0623	0.0744	10000
	HL2	-0.001227	(-0.188, 0.181)	0.0044	0.0747	0.0623	0.0746	10000
	HL3	-0.001269	(-0.188, 0.182)	0.0044	0.0746	0.0624	0.0746	10000
8.0	Mean	-0.000489	(-0.161, 0.164)	0.0034	0.0661	0.0562	0.0661	10000
	NM	-0.001543	(-0.198, 0.198)	0.0052	0.0814	0.0692	0.0815	10000
	MM	-0.001543	(-0.198, 0.198)	0.0052	0.0814	0.0692	0.0815	10000
	MIR	**-0.000059**	**(-0.173, 0.177)**	**0.0039**	**0.0708**	**0.0595**	**0.0708**	**10000**
	CSQM	**-0.001146**	**(-0.191, 0.192)**	**0.0048**	**0.0781**	**0.0656**	**0.0781**	**10000**
	CQM	**-0.000948**	**(-0.191, 0.191)**	**0.0047**	**0.0777**	**0.0655**	**0.0777**	**10000**
	HL1	-0.000311	(-0.163, 0.168)	0.0036	0.0676	0.0575	0.0676	10000
	HL2	-0.000396	(-0.164, 0.169)	0.0036	0.0679	0.0577	0.0679	10000
	HL3	-0.000359	(-0.163, 0.168)	0.0036	0.0678	0.0579	0.0678	10000

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Table B.10: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size n = 20, κ = 0.1,0.2,0.3,0.4,0.5, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	-0.001932	(-2.906, 2.936)	0.8084	1.3256	1.1997	1.3256	10000
	NM	0.006163	(-2.907, 2.919)	0.8159	1.3347	1.2211	1.3348	9967
	MM	0.002381	(-2.921, 2.935)	0.8152	1.3346	1.2093	1.3346	9967
	MIR	**0.019445**	**(-2.918, 2.922)**	**0.8214**	**1.3425**	**1.2131**	**1.3425**	**9967**
	CSQM	**0.002381**	**(-2.921, 2.935)**	**0.8152**	**1.3346**	**1.2093**	**1.3346**	**9967**
	CQM	**0.002381**	**(-2.921, 2.935)**	**0.8152**	**1.3346**	**1.2093**	**1.3346**	**9967**
	HL1	0.004474	(-2.916, 2.929)	0.8073	1.3246	1.1898	1.3246	9995
	HL2	0.005137	(-2.933, 2.928)	0.8074	1.3249	1.1910	1.3249	9998
	HL3	0.004602	(-2.923, 2.933)	0.8083	1.3258	1.1903	1.3258	10000
0.2	Mean	-0.008476	(-2.799, 2.750)	0.6284	1.0963	0.9046	1.0963	10000
	NM	-0.014944	(-2.765, 2.752)	0.6386	1.1089	0.9209	1.1090	9962
	MM	-0.012645	(-2.773, 2.738)	0.6395	1.1092	0.9108	1.1092	9962
	MIR	**-0.006072**	**(-2.772, 2.810)**	**0.6558**	**1.1308**	**0.9416**	**1.1308**	**9962**
	CSQM	**-0.012645**	**(-2.773, 2.738)**	**0.6395**	**1.1092**	**0.9108**	**1.1092**	**9962**
	CQM	**-0.012645**	**(-2.773, 2.738)**	**0.6395**	**1.1092**	**0.9108**	**1.1092**	**9962**
	HL1	-0.007433	(-2.793, 2.740)	0.6309	1.0983	0.8966	1.0983	9990
	HL2	-0.007920	(-2.782, 2.754)	0.6298	1.0970	0.8990	1.0970	9999
	HL3	-0.011157	(-2.789, 2.751)	0.6305	1.0979	0.9003	1.0980	9997
0.3	Mean	-0.002750	(-2.546, 2.495)	0.4635	0.8808	0.6766	0.8808	10000
	NM	-0.001753	(-2.596, 2.540)	0.4825	0.9056	0.6969	0.9057	9982
	MM	-0.002001	(-2.538, 2.510)	0.4807	0.9027	0.6992	0.9027	9982
	MIR	**-0.012829**	**(-2.613, 2.493)**	**0.4986**	**0.9257**	**0.7193**	**0.9258**	**9982**
	CSQM	**-0.002001**	**(-2.538, 2.510)**	**0.4807**	**0.9027**	**0.6992**	**0.9027**	**9982**
	CQM	**-0.002001**	**(-2.538, 2.510)**	**0.4807**	**0.9027**	**0.6992**	**0.9027**	**9982**
	HL1	-0.004989	(-2.591, 2.498)	0.4688	0.8881	0.6815	0.8881	9996
	HL2	-0.004529	(-2.585, 2.509)	0.4677	0.8869	0.6829	0.8869	9999
	HL3	-0.004247	(-2.569, 2.510)	0.4682	0.8873	0.6822	0.8873	9999
0.4	Mean	0.001445	(-2.136, 2.167)	0.3532	0.7318	0.5514	0.7317	10000
	NM	0.008030	(-2.212, 2.162)	0.3706	0.7560	0.5777	0.7561	9967
	MM	0.000691	(-2.200, 2.172)	0.3700	0.7555	0.5728	0.7555	9967
	MIR	**0.000627**	**(-2.304, 2.293)**	**0.3892**	**0.7818**	**0.5905**	**0.7818**	**9967**
	CSQM	**0.000691**	**(-2.200, 2.172)**	**0.3700**	**0.7555**	**0.5728**	**0.7555**	**9967**
	CQM	**0.000691**	**(-2.200, 2.172)**	**0.3700**	**0.7555**	**0.5728**	**0.7555**	**9967**
	HL1	0.000093	(-2.155, 2.204)	0.3585	0.7392	0.5609	0.7392	9995
	HL2	0.000860	(-2.138, 2.193)	0.3575	0.7378	0.5587	0.7378	9999
	HL3	0.001367	(-2.157, 2.177)	0.3570	0.7371	0.5577	0.7371	9997
0.5	Mean	-0.004112	(-1.659, 1.615)	0.2447	0.5822	0.4405	0.5822	10000
	NM	-0.005106	(-1.765, 1.644)	0.2635	0.6095	0.4619	0.6095	9982
	MM	-0.001662	(-1.739, 1.673)	0.2641	0.6100	0.4637	0.6100	9982
	MIR	**-0.000820**	**(-1.822, 1.775)**	**0.2761**	**0.6258**	**0.4773**	**0.6258**	**9982**
	CSQM	**-0.001662**	**(-1.739, 1.673)**	**0.2641**	**0.6100**	**0.4637**	**0.6100**	**9982**
	CQM	**-0.001662**	**(-1.739, 1.673)**	**0.2641**	**0.6100**	**0.4637**	**0.6100**	**9982**
	HL1	-0.006369	(-1.705, 1.649)	0.2519	0.5926	0.4501	0.5926	9997
	HL2	-0.006185	(-1.683, 1.638)	0.2497	0.5895	0.4473	0.5895	9999
	HL3	-0.006209	(-1.687, 1.634)	0.2508	0.5910	0.4502	0.5910	10000

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Table B.11: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size n = 20, κ = 1.0,1.5,2.0,2.5,3.0, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.002748	(-0.685, 0.711)	0.0599	0.2745	0.2266	0.2745	10000
	NM	0.002632	(-0.741, 0.762)	0.0710	0.2989	0.2443	0.2990	10000
	MM	0.002779	(-0.756, 0.760)	0.0717	0.3014	0.2473	0.3014	10000
	MIR	**0.002535**	**(-0.747, 0.762)**	**0.0714**	**0.3008**	**0.2442**	**0.3008**	**10000**
	CSQM	**0.002779**	**(-0.756, 0.760)**	**0.0717**	**0.3014**	**0.2473**	**0.3014**	**10000**
	CQM	**0.002779**	**(-0.756, 0.760)**	**0.0717**	**0.3014**	**0.2473**	**0.3014**	**10000**
	HL1	0.003146	(-0.698, 0.715)	0.0616	0.2781	0.2272	0.2780	10000
	HL2	0.003329	(-0.689, 0.715)	0.0613	0.2774	0.2274	0.2773	10000
	HL3	0.003197	(-0.694, 0.714)	0.0614	0.2777	0.2275	0.2776	10000
1.5	Mean	0.000202	(-0.469, 0.470)	0.0277	0.1873	0.1558	0.1873	10000
	NM	-0.001260	(-0.524, 0.520)	0.0341	0.2078	0.1736	0.2078	10000
	MM	-0.001011	(-0.531, 0.527)	0.0346	0.2093	0.1750	0.2093	10000
	MIR	**-0.000695**	**(-0.506, 0.502)**	**0.0321**	**0.2017**	**0.1660**	**0.2017**	**10000**
	CSQM	**-0.001011**	**(-0.531, 0.527)**	**0.0346**	**0.2093**	**0.1750**	**0.2093**	**10000**
	CQM	**-0.001011**	**(-0.531, 0.527)**	**0.0346**	**0.2093**	**0.1750**	**0.2093**	**10000**
	HL1	0.000212	(-0.474, 0.473)	0.0282	0.1894	0.1584	0.1894	10000
	HL2	0.000034	(-0.472, 0.470)	0.0281	0.1888	0.1571	0.1888	10000
	HL3	0.000060	(-0.472, 0.473)	0.0281	0.1891	0.1575	0.1891	10000
2.0	Mean	0.000615	(-0.375, 0.376)	0.0182	0.1516	0.1268	0.1516	10000
	NM	0.001406	(-0.425, 0.427)	0.0231	0.1717	0.1445	0.1717	10000
	MM	0.001262	(-0.425, 0.427)	0.0232	0.1720	0.1451	0.1720	10000
	MIR	**0.000055**	**(-0.401, 0.398)**	**0.0204**	**0.1606**	**0.1339**	**0.1606**	**10000**
	CSQM	**0.001262**	**(-0.425, 0.427)**	**0.0232**	**0.1720**	**0.1451**	**0.1720**	**10000**
	CQM	**0.001262**	**(-0.425, 0.427)**	**0.0232**	**0.1720**	**0.1451**	**0.1720**	**10000**
	HL1	0.000359	(-0.380, 0.383)	0.0184	0.1523	0.1274	0.1523	10000
	HL2	0.000519	(-0.379, 0.381)	0.0184	0.1523	0.1285	0.1523	10000
	HL3	0.000446	(-0.379, 0.382)	0.0183	0.1523	0.1279	0.1523	10000
2.5	Mean	-0.002345	(-0.315, 0.322)	0.0130	0.1285	0.1081	0.1285	10000
	NM	-0.001663	(-0.362, 0.361)	0.0169	0.1471	0.1245	0.1471	10000
	MM	-0.001657	(-0.362, 0.362)	0.0169	0.1472	0.1245	0.1472	10000
	MIR	**-0.002937**	**(-0.332, 0.338)**	**0.0143**	**0.1347**	**0.1132**	**0.1348**	**10000**
	CSQM	**-0.001657**	**(-0.362, 0.362)**	**0.0169**	**0.1472**	**0.1245**	**0.1472**	**10000**
	CQM	**-0.001657**	**(-0.362, 0.362)**	**0.0169**	**0.1472**	**0.1245**	**0.1472**	**10000**
	HL1	-0.002480	(-0.317, 0.322)	0.0133	0.1303	0.1102	0.1303	10000
	HL2	-0.002478	(-0.318, 0.324)	0.0133	0.1303	0.1100	0.1303	10000
	HL3	-0.002471	(-0.318, 0.324)	0.0133	0.1303	0.1102	0.1303	10000
3.0	Mean	-0.002640	(-0.283, 0.278)	0.0103	0.1144	0.0950	0.1144	10000
	NM	-0.003196	(-0.328, 0.319)	0.0134	0.1307	0.1099	0.1307	10000
	MM	-0.003221	(-0.328, 0.319)	0.0134	0.1307	0.1099	0.1307	10000
	MIR	**-0.002934**	**(-0.302, 0.297)**	**0.0117**	**0.1223**	**0.1024**	**0.1223**	**10000**
	CSQM	**-0.003221**	**(-0.328, 0.319)**	**0.0134**	**0.1307**	**0.1099**	**0.1307**	**10000**
	CQM	**-0.003221**	**(-0.328, 0.319)**	**0.0134**	**0.1307**	**0.1099**	**0.1307**	**10000**
	HL1	-0.003285	(-0.286, 0.282)	0.0105	0.1155	0.0957	0.1156	10000
	HL2	-0.003282	(-0.285, 0.280)	0.0105	0.1157	0.0963	0.1157	10000
	HL3	-0.003298	(-0.286, 0.282)	0.0105	0.1155	0.0955	0.1156	10000

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Table B.12: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0, κ), Sample Size n = 20, κ = 4.0,5.0,6.0,7.0,8.0, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.000862	(-0.229, 0.235)	0.0070	0.0951	0.0808	0.0951	10000
	NM	-0.000017	(-0.275, 0.274)	0.0099	0.1123	0.0956	0.1123	10000
	MM	-0.000032	(-0.275, 0.274)	0.0099	0.1123	0.0956	0.1123	10000
	MIR	**0.001511**	**(-0.247, 0.249)**	**0.0080**	**0.1014**	**0.0869**	**0.1014**	**10000**
	CSQM	**-0.000032**	**(-0.275, 0.274)**	**0.0099**	**0.1123**	**0.0956**	**0.1123**	**10000**
	CQM	**-0.000032**	**(-0.275, 0.274)**	**0.0099**	**0.1123**	**0.0956**	**0.1123**	**10000**
	HL1	0.000883	(-0.234, 0.238)	0.0072	0.0966	0.0825	0.0966	10000
	HL2	0.000796	(-0.235, 0.235)	0.0073	0.0968	0.0823	0.0968	10000
	HL3	0.000815	(-0.235, 0.236)	0.0072	0.0967	0.0825	0.0967	10000
5.0	Mean	0.000162	(-0.205, 0.203)	0.0054	0.0833	0.0705	0.0833	10000
	NM	0.000372	(-0.243, 0.239)	0.0075	0.0979	0.0822	0.0979	10000
	MM	0.000372	(-0.243, 0.239)	0.0075	0.0979	0.0822	0.0979	10000
	MIR	**-0.000496**	**(-0.220, 0.216)**	**0.0062**	**0.0893**	**0.0756**	**0.0893**	**10000**
	CSQM	**0.000372**	**(-0.243, 0.239)**	**0.0075**	**0.0979**	**0.0822**	**0.0979**	**10000**
	CQM	**0.000372**	**(-0.243, 0.239)**	**0.0075**	**0.0979**	**0.0822**	**0.0979**	**10000**
	HL1	-0.000182	(-0.209, 0.205)	0.0056	0.0845	0.0713	0.0845	10000
	HL2	-0.000184	(-0.210, 0.205)	0.0056	0.0846	0.0713	0.0846	10000
	HL3	-0.000167	(-0.209, 0.205)	0.0056	0.0846	0.0712	0.0846	10000
6.0	Mean	-0.000226	(-0.185, 0.186)	0.0045	0.0760	0.0638	0.0760	10000
	NM	-0.001004	(-0.222, 0.224)	0.0065	0.0914	0.0784	0.0914	10000
	MM	-0.001004	(-0.222, 0.224)	0.0065	0.0914	0.0784	0.0914	10000
	MIR	**0.000123**	**(-0.199, 0.199)**	**0.0052**	**0.0809**	**0.0682**	**0.0809**	**10000**
	CSQM	**-0.001004**	**(-0.222, 0.224)**	**0.0065**	**0.0914**	**0.0784**	**0.0914**	**10000**
	CQM	**-0.001004**	**(-0.222, 0.224)**	**0.0065**	**0.0914**	**0.0784**	**0.0914**	**10000**
	HL1	-0.000110	(-0.192, 0.191)	0.0047	0.0774	0.0648	0.0774	10000
	HL2	-0.000229	(-0.191, 0.192)	0.0047	0.0777	0.0653	0.0777	10000
	HL3	-0.000186	(-0.192, 0.192)	0.0047	0.0775	0.0650	0.0775	10000
7.0	Mean	0.001285	(-0.176, 0.174)	0.0039	0.0705	0.0596	0.0705	10000
	NM	0.001542	(-0.205, 0.202)	0.0054	0.0835	0.0713	0.0835	10000
	MM	0.001542	(-0.205, 0.202)	0.0054	0.0835	0.0713	0.0835	10000
	MIR	**0.000605**	**(-0.187, 0.185)**	**0.0045**	**0.0752**	**0.0635**	**0.0752**	**10000**
	CSQM	**0.001542**	**(-0.205, 0.202)**	**0.0054**	**0.0835**	**0.0713**	**0.0835**	**10000**
	CQM	**0.001542**	**(-0.205, 0.202)**	**0.0054**	**0.0835**	**0.0713**	**0.0835**	**10000**
	HL1	0.000936	(-0.177, 0.176)	0.0041	0.0718	0.0608	0.0718	10000
	HL2	0.000926	(-0.178, 0.177)	0.0041	0.0720	0.0609	0.0720	10000
	HL3	0.000931	(-0.178, 0.177)	0.0041	0.0719	0.0606	0.0719	10000
8.0	Mean	-0.000397	(-0.165, 0.159)	0.0034	0.0654	0.0550	0.0654	10000
	NM	-0.000274	(-0.196, 0.191)	0.0049	0.0784	0.0664	0.0784	10000
	MM	-0.000274	(-0.196, 0.191)	0.0049	0.0784	0.0664	0.0784	10000
	MIR	**-0.000063**	**(-0.175, 0.173)**	**0.0039**	**0.0702**	**0.0587**	**0.0702**	**10000**
	CSQM	**-0.000274**	**(-0.196, 0.191)**	**0.0049**	**0.0784**	**0.0664**	**0.0784**	**10000**
	CQM	**-0.000274**	**(-0.196, 0.191)**	**0.0049**	**0.0784**	**0.0664**	**0.0784**	**10000**
	HL1	-0.000415	(-0.168, 0.162)	0.0035	0.0668	0.0563	0.0668	10000
	HL2	-0.000383	(-0.168, 0.163)	0.0035	0.0670	0.0565	0.0670	10000
	HL3	-0.000403	(-0.167, 0.163)	0.0035	0.0669	0.0565	0.0669	10000

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Appendix C

vM(0,1.0) and vM(0,2.0)

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Table C.1: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0,1.0), n = 7,8,9,10,11, Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
7	Mean	-0.007051	(-1.387, 1.319)	0.1867	0.4964	0.3806	0.4965	10000
	NM	-0.005675	(-1.417, 1.397)	0.2025	0.5216	0.4096	0.5216	9888
	MM	-0.007317	(-1.414, 1.380)	0.2035	0.5241	0.4131	0.5242	9888
	MIR	**-0.005885**	**(-1.427, 1.349)**	**0.1956**	**0.5104**	**0.3895**	**0.5104**	**9888**
	CSQM	**-0.006719**	**(-1.370, 1.315)**	**0.1905**	**0.5040**	**0.3895**	**0.5040**	**9888**
	CQM	**-0.006537**	**(-1.393, 1.341)**	**0.1926**	**0.5066**	**0.3932**	**0.5066**	**9888**
	HL1	-0.010174	(-1.463, 1.341)	0.1972	0.5113	0.3928	0.5113	9961
	HL2	-0.009187	(-1.429, 1.341)	0.1926	0.5041	0.3865	0.5042	10000
	HL3	-0.009502	(-1.425, 1.355)	0.1949	0.5075	0.3889	0.5075	10000
8	Mean	0.002215	(-1.231, 1.318)	0.1706	0.4699	0.3580	0.4699	10000
	NM	-0.000024	(-1.324, 1.450)	0.1892	0.4978	0.3718	0.4978	9996
	MM	0.000379	(-1.324, 1.402)	0.1870	0.4963	0.3743	0.4963	9996
	MIR	**-0.001929**	**(-1.376, 1.391)**	**0.1903**	**0.5007**	**0.3823**	**0.5007**	**9996**
	CSQM	**0.000379**	**(-1.324, 1.402)**	**0.1870**	**0.4963**	**0.3743**	**0.4963**	**9996**
	CQM	**0.000379**	**(-1.324, 1.402)**	**0.1870**	**0.4963**	**0.3743**	**0.4963**	**9996**
	HL1	0.000772	(-1.252, 1.338)	0.1768	0.4800	0.3692	0.4800	9997
	HL2	0.001428	(-1.252, 1.346)	0.1750	0.4769	0.3645	0.4769	10000
	HL3	0.000929	(-1.263, 1.328)	0.1758	0.4784	0.3686	0.4784	10000
9	Mean	-0.012370	(-1.143, 1.147)	0.1450	0.4309	0.3337	0.4311	10000
	NM	-0.011004	(-1.206, 1.202)	0.1602	0.4571	0.3550	0.4571	9920
	MM	-0.013840	(-1.235, 1.222)	0.1642	0.4633	0.3594	0.4634	9920
	MIR	**-0.015126**	**(-1.253, 1.210)**	**0.1623**	**0.4588**	**0.3538**	**0.4589**	**9920**
	CSQM	**-0.015371**	**(-1.210, 1.177)**	**0.1546**	**0.4469**	**0.3470**	**0.4470**	**9920**
	CQM	**-0.016370**	**(-1.195, 1.173)**	**0.1552**	**0.4481**	**0.3481**	**0.4483**	**9920**
	HL1	-0.014044	(-1.195, 1.161)	0.1522	0.4415	0.3402	0.4416	9996
	HL2	-0.014770	(-1.175, 1.145)	0.1500	0.4387	0.3405	0.4388	10000
	HL3	-0.014676	(-1.191, 1.145)	0.1507	0.4395	0.3401	0.4396	9996
10	Mean	0.01177	(-1.063, 1.083)	0.1312	0.4088	0.3205	0.4090	10000
	NM	0.01400	(-1.119, 1.165)	0.1461	0.4335	0.3388	0.4338	9999
	MM	0.01443	(-1.123, 1.171)	0.1461	0.4340	0.3406	0.4343	9999
	MIR	**0.01371**	**(-1.168, 1.236)**	**0.1598**	**0.4517**	**0.3440**	**0.4519**	**9999**
	CSQM	**0.01443**	**(-1.123, 1.171)**	**0.1461**	**0.4340**	**0.3406**	**0.4343**	**9999**
	CQM	**0.01443**	**(-1.123, 1.171)**	**0.1461**	**0.4340**	**0.3406**	**0.4343**	**9999**
	HL1	0.01340	(-1.080, 1.133)	0.1373	0.4181	0.3220	0.4184	9994
	HL2	0.01267	(-1.054, 1.119)	0.1347	0.4143	0.3203	0.4145	10000
	HL3	0.01367	(-1.060, 1.131)	0.1353	0.4151	0.3204	0.4153	10000
11	Mean	0.001513	(-1.013, 1.023)	0.1202	0.3892	0.3069	0.3893	10000
	NM	0.001038	(-1.066, 1.084)	0.1337	0.4129	0.3234	0.4130	9951
	MM	0.002604	(-1.093, 1.098)	0.1356	0.4182	0.3323	0.4182	9951
	MIR	**0.001931**	**(-1.101, 1.070)**	**0.1317**	**0.4088**	**0.3171**	**0.4088**	**9951**
	CSQM	**0.001923**	**(-1.065, 1.058)**	**0.1288**	**0.4052**	**0.3175**	**0.4052**	**9951**
	CQM	**0.001595**	**(-1.063, 1.065)**	**0.1296**	**0.4060**	**0.3173**	**0.4060**	**9951**
	HL1	0.000534	(-1.051, 1.058)	0.1254	0.3985	0.3133	0.3985	9998
	HL2	0.000454	(-1.038, 1.037)	0.1236	0.3954	0.3096	0.3954	10000
	HL3	0.001193	(-1.048, 1.047)	0.1248	0.3973	0.3123	0.3973	10000

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Table C.2: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0,1.0), n = 17,18,19,20,21, Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
17	Mean	0.009428	(-0.743, 0.801)	0.0739	0.3042	0.2472	0.3043	10000
	NM	0.008788	(-0.807, 0.862)	0.0849	0.3263	0.2622	0.3263	9988
	MM	0.009217	(-0.818, 0.876)	0.0864	0.3296	0.2648	0.3297	9988
	MIR	**0.008364**	**(-0.827, 0.876)**	**0.0866**	**0.3290**	**0.2638**	**0.3290**	**9988**
	CSQM	**0.009306**	**(-0.808, 0.846)**	**0.0824**	**0.3215**	**0.2584**	**0.3215**	**9988**
	CQM	**0.009337**	**(-0.803, 0.838)**	**0.0823**	**0.3215**	**0.2577**	**0.3216**	**9988**
	HL1	0.009683	(-0.767, 0.809)	0.0770	0.3096	0.2511	0.3097	9999
	HL2	0.010267	(-0.765, 0.814)	0.0758	0.3075	0.2497	0.3076	10000
	HL3	0.010097	(-0.767, 0.812)	0.0763	0.3085	0.2496	0.3085	9999
18	Mean	-0.001509	(-0.738, 0.721)	0.0681	0.2918	0.2389	0.2918	10000
	NM	0.002028	(-0.790, 0.786)	0.0773	0.3131	0.2558	0.3131	9999
	MM	0.001080	(-0.814, 0.800)	0.0792	0.3170	0.2597	0.3171	9999
	MIR	**-0.001408**	**(-0.822, 0.813)**	**0.0835**	**0.3233**	**0.2592**	**0.3233**	**9999**
	CSQM	**0.001080**	**(-0.814, 0.800)**	**0.0792**	**0.3170**	**0.2597**	**0.3171**	**9999**
	CQM	**0.001080**	**(-0.814, 0.800)**	**0.0792**	**0.3170**	**0.2597**	**0.3171**	**9999**
	HL1	-0.002847	(-0.759, 0.740)	0.0707	0.2970	0.2405	0.2970	10000
	HL2	-0.002748	(-0.748, 0.726)	0.0698	0.2952	0.2392	0.2952	10000
	HL3	-0.002825	(-0.755, 0.731)	0.0702	0.2960	0.2399	0.2960	10000
19	Mean	-0.000098	(-0.724, 0.724)	0.0644	0.2836	0.2282	0.2836	10000
	NM	-0.001622	(-0.802, 0.786)	0.0765	0.3091	0.2490	0.3091	9993
	MM	-0.003099	(-0.804, 0.788)	0.0783	0.3135	0.2532	0.3135	9993
	MIR	**-0.000463**	**(-0.785, 0.781)**	**0.0752**	**0.3066**	**0.2494**	**0.3066**	**9993**
	CSQM	**-0.003059**	**(-0.789, 0.779)**	**0.0750**	**0.3065**	**0.2467**	**0.3065**	**9993**
	CQM	**-0.002985**	**(-0.785, 0.781)**	**0.0749**	**0.3063**	**0.2466**	**0.3063**	**9993**
	HL1	-0.000316	(-0.734, 0.724)	0.0664	0.2879	0.2329	0.2879	10000
	HL2	0.000033	(-0.731, 0.725)	0.0659	0.2867	0.2317	0.2867	10000
	HL3	-0.000353	(-0.737, 0.722)	0.0662	0.2873	0.2318	0.2873	10000
20	Mean	0.002748	(-0.685, 0.711)	0.0599	0.2745	0.2266	0.2745	10000
	NM	0.002632	(-0.741, 0.762)	0.0710	0.2989	0.2443	0.2990	10000
	MM	0.002779	(-0.756, 0.760)	0.0717	0.3014	0.2473	0.3014	10000
	MIR	**0.002535**	**(-0.747, 0.762)**	**0.0714**	**0.3008**	**0.2442**	**0.3008**	**10000**
	CSQM	**0.002779**	**(-0.756, 0.760)**	**0.0717**	**0.3014**	**0.2473**	**0.3014**	**10000**
	CQM	**0.002779**	**(-0.756, 0.760)**	**0.0717**	**0.3014**	**0.2473**	**0.3014**	**10000**
	HL1	0.003146	(-0.698, 0.715)	0.0616	0.2781	0.2272	0.2780	10000
	HL2	0.003329	(-0.689, 0.715)	0.0613	0.2774	0.2274	0.2773	10000
	HL3	0.003197	(-0.694, 0.714)	0.0614	0.2777	0.2275	0.2776	10000
21	Mean	-0.005026	(-0.690, 0.672)	0.0561	0.2651	0.2173	0.2650	10000
	NM	-0.003912	(-0.763, 0.734)	0.0669	0.2907	0.2379	0.2907	9997
	MM	-0.004161	(-0.773, 0.736)	0.0685	0.2947	0.2410	0.2947	9997
	MIR	**-0.006074**	**(-0.766, 0.732)**	**0.0679**	**0.2916**	**0.2367**	**0.2916**	**9997**
	CSQM	**-0.004244**	**(-0.760, 0.726)**	**0.0653**	**0.2870**	**0.2344**	**0.2870**	**9997**
	CQM	**-0.004269**	**(-0.756, 0.729)**	**0.0650**	**0.2862**	**0.2337**	**0.2862**	**9997**
	HL1	-0.005463	(-0.698, 0.682)	0.0579	0.2690	0.2192	0.2690	10000
	HL2	-0.005220	(-0.696, 0.684)	0.0575	0.2682	0.2196	0.2682	10000
	HL3	-0.005365	(-0.697, 0.682)	0.0576	0.2685	0.2190	0.2685	10000

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Table C.3: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0,2.0), n = 7,8,9,10,11, Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
7	Mean	-0.006692	(-0.658, 0.652)	0.0533	0.2610	0.2166	0.2611	10000
	NM	-0.004682	(-0.731, 0.741)	0.0675	0.2958	0.2428	0.2959	9992
	MM	-0.004099	(-0.737, 0.742)	0.0678	0.2965	0.2439	0.2965	9992
	MIR	**-0.007008**	**(-0.680, 0.679)**	**0.0567**	**0.2683**	**0.2223**	**0.2684**	**9992**
	CSQM	**-0.006518**	**(-0.689, 0.680)**	**0.0584**	**0.2735**	**0.2258**	**0.2736**	**9992**
	CQM	**-0.007975**	**(-0.691, 0.681)**	**0.0588**	**0.2737**	**0.2287**	**0.2737**	**9992**
	HL1	-0.006974	(-0.677, 0.665)	0.0565	0.2682	0.2213	0.2682	9996
	HL2	-0.006150	(-0.677, 0.667)	0.0553	0.2656	0.2191	0.2657	10000
	HL3	-0.006238	(-0.673, 0.663)	0.0558	0.2668	0.2207	0.2668	10000
8	Mean	0.004930	(-0.595, 0.603)	0.0453	0.2402	0.1990	0.2402	10000
	NM	0.004094	(-0.645, 0.652)	0.0532	0.2617	0.2176	0.2618	10000
	MM	0.003226	(-0.651, 0.654)	0.0535	0.2625	0.2180	0.2625	10000
	MIR	**0.002821**	**(-0.637, 0.636)**	**0.0499**	**0.2515**	**0.2067**	**0.2515**	**10000**
	CSQM	**0.003226**	**(-0.651, 0.654)**	**0.0535**	**0.2625**	**0.2180**	**0.2625**	**10000**
	CQM	**0.003226**	**(-0.651, 0.654)**	**0.0535**	**0.2625**	**0.2180**	**0.2625**	**10000**
	HL1	0.004086	(-0.613, 0.614)	0.0476	0.2458	0.2029	0.2458	10000
	HL2	0.003624	(-0.607, 0.610)	0.0469	0.2445	0.2029	0.2445	10000
	HL3	0.003202	(-0.615, 0.611)	0.0474	0.2455	0.2031	0.2455	10000
9	Mean	-0.002391	(-0.571, 0.562)	0.0409	0.2286	0.1915	0.2286	10000
	NM	-0.004028	(-0.641, 0.633)	0.0522	0.2594	0.2162	0.2594	10000
	MM	-0.003647	(-0.646, 0.639)	0.0528	0.2605	0.2167	0.2605	10000
	MIR	**-0.000641**	**(-0.611, 0.589)**	**0.0461**	**0.2421**	**0.2003**	**0.2421**	**10000**
	CSQM	**-0.002450**	**(-0.614, 0.591)**	**0.0464**	**0.2434**	**0.2037**	**0.2434**	**10000**
	CQM	**-0.001777**	**(-0.608, 0.586)**	**0.0463**	**0.2436**	**0.2021**	**0.2436**	**10000**
	HL1	-0.002093	(-0.592, 0.571)	0.0424	0.2328	0.1929	0.2328	10000
	HL2	-0.002247	(-0.586, 0.568)	0.0423	0.2325	0.1948	0.2325	10000
	HL3	-0.002132	(-0.584, 0.567)	0.0423	0.2326	0.1941	0.2326	10000
10	Mean	0.003456	(-0.540, 0.545)	0.0363	0.2148	0.1788	0.2148	10000
	NM	0.004153	(-0.594, 0.593)	0.0439	0.2369	0.1992	0.2369	10000
	MM	0.004304	(-0.595, 0.593)	0.0437	0.2369	0.1993	0.2369	10000
	MIR	**0.004138**	**(-0.585, 0.593)**	**0.0430**	**0.2328**	**0.1907**	**0.2328**	**10000**
	CSQM	**0.004304**	**(-0.595, 0.593)**	**0.0437**	**0.2369**	**0.1993**	**0.2369**	**10000**
	CQM	**0.004304**	**(-0.595, 0.593)**	**0.0437**	**0.2369**	**0.1993**	**0.2369**	**10000**
	HL1	0.002281	(-0.553, 0.552)	0.0380	0.2190	0.1834	0.2190	10000
	HL2	0.002544	(-0.549, 0.547)	0.0376	0.2182	0.1832	0.2182	10000
	HL3	0.002410	(-0.550, 0.551)	0.0377	0.2182	0.1823	0.2183	10000
11	Mean	0.001683	(-0.508, 0.514)	0.0331	0.2050	0.1706	0.2050	10000
	NM	0.003288	(-0.578, 0.590)	0.0429	0.2352	0.1964	0.2352	10000
	MM	0.002576	(-0.582, 0.590)	0.0431	0.2356	0.1966	0.2356	10000
	MIR	**0.000086**	**(-0.519, 0.534)**	**0.0354**	**0.2123**	**0.1758**	**0.2123**	**10000**
	CSQM	**0.001350**	**(-0.551, 0.556)**	**0.0385**	**0.2218**	**0.1863**	**0.2218**	**10000**
	CQM	**0.000696**	**(-0.549, 0.553)**	**0.0384**	**0.2211**	**0.1861**	**0.2211**	**10000**
	HL1	0.001722	(-0.513, 0.519)	0.0341	0.2084	0.1750	0.2084	10000
	HL2	0.001554	(-0.513, 0.520)	0.0339	0.2078	0.1740	0.2078	10000
	HL3	0.001702	(-0.511, 0.520)	0.0340	0.2080	0.1749	0.2080	10000

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Table C.4: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for vM(0,2.0), n = 17,18,19,20,21, Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
17	Mean	0.001699	(-0.413, 0.408)	0.0215	0.1660	0.1405	0.1661	10000
	NM	0.003352	(-0.470, 0.470)	0.0280	0.1890	0.1586	0.1890	10000
	MM	0.003084	(-0.475, 0.470)	0.0282	0.1894	0.1587	0.1895	10000
	MIR	**0.000825**	**(-0.443, 0.433)**	**0.0245**	**0.1774**	**0.1494**	**0.1774**	**10000**
	CSQM	**0.002406**	**(-0.455, 0.448)**	**0.0259**	**0.1814**	**0.1511**	**0.1814**	**10000**
	CQM	**0.002063**	**(-0.456, 0.450)**	**0.0257**	**0.1807**	**0.1509**	**0.1808**	**10000**
	HL1	0.001906	(-0.424, 0.412)	0.0220	0.1679	0.1425	0.1679	10000
	HL2	0.001974	(-0.425, 0.412)	0.0219	0.1677	0.1423	0.1677	10000
	HL3	0.002009	(-0.425, 0.411)	0.0220	0.1678	0.1424	0.1678	10000
18	Mean	-0.002229	(-0.401, 0.391)	0.0201	0.1607	0.1342	0.1607	10000
	NM	-0.002824	(-0.454, 0.446)	0.0261	0.1832	0.1545	0.1832	10000
	MM	-0.002843	(-0.455, 0.446)	0.0262	0.1837	0.1548	0.1837	10000
	MIR	**-0.002871**	**(-0.428, 0.417)**	**0.0232**	**0.1727**	**0.1454**	**0.1727**	**10000**
	CSQM	**-0.002843**	**(-0.455, 0.446)**	**0.0262**	**0.1837**	**0.1548**	**0.1837**	**10000**
	CQM	**-0.002843**	**(-0.455, 0.446)**	**0.0262**	**0.1837**	**0.1548**	**0.1837**	**10000**
	HL1	-0.002523	(-0.406, 0.396)	0.0204	0.1620	0.1357	0.1620	10000
	HL2	-0.002528	(-0.406, 0.399)	0.0205	0.1620	0.1362	0.1620	10000
	HL3	-0.002549	(-0.405, 0.396)	0.0204	0.1620	0.1360	0.1620	10000
19	Mean	0.002476	(-0.380, 0.379)	0.0185	0.1540	0.1313	0.1541	10000
	NM	0.003711	(-0.439, 0.434)	0.0246	0.1790	0.1541	0.1790	10000
	MM	0.003892	(-0.439, 0.436)	0.0248	0.1794	0.1539	0.1794	10000
	MIR	**0.001597**	**(-0.402, 0.406)**	**0.0207**	**0.1624**	**0.1371**	**0.1624**	**10000**
	CSQM	**0.003004**	**(-0.422, 0.420)**	**0.0229**	**0.1722**	**0.1458**	**0.1722**	**10000**
	CQM	**0.002548**	**(-0.420, 0.415)**	**0.0227**	**0.1714**	**0.1459**	**0.1714**	**10000**
	HL1	0.002091	(-0.387, 0.382)	0.0189	0.1554	0.1317	0.1555	10000
	HL2	0.002209	(-0.382, 0.382)	0.0189	0.1554	0.1313	0.1554	10000
	HL3	0.002125	(-0.383, 0.382)	0.0189	0.1554	0.1316	0.1554	10000
20	Mean	0.000615	(-0.375, 0.376)	0.0182	0.1516	0.1268	0.1516	10000
	NM	0.001406	(-0.425, 0.427)	0.0231	0.1717	0.1445	0.1717	10000
	MM	0.001262	(-0.425, 0.427)	0.0232	0.1720	0.1451	0.1720	10000
	MIR	**0.000055**	**(-0.401, 0.398)**	**0.0204**	**0.1606**	**0.1339**	**0.1606**	**10000**
	CSQM	**0.001262**	**(-0.425, 0.427)**	**0.0232**	**0.1720**	**0.1451**	**0.1720**	**10000**
	CQM	**0.001262**	**(-0.425, 0.427)**	**0.0232**	**0.1720**	**0.1451**	**0.1720**	**10000**
	HL1	0.000359	(-0.380, 0.383)	0.0184	0.1523	0.1274	0.1523	10000
	HL2	0.000519	(-0.379, 0.381)	0.0184	0.1523	0.1285	0.1523	10000
	HL3	0.000446	(-0.379, 0.382)	0.0183	0.1523	0.1279	0.1523	10000
21	Mean	0.003205	(-0.365, 0.372)	0.0174	0.1488	0.1260	0.1488	10000
	NM	0.004578	(-0.426, 0.426)	0.0233	0.1721	0.1433	0.1721	10000
	MM	0.004675	(-0.428, 0.428)	0.0234	0.1724	0.1430	0.1724	10000
	MIR	**0.003824**	**(-0.393, 0.397)**	**0.0202**	**0.1609**	**0.1358**	**0.1609**	**10000**
	CSQM	**0.003822**	**(-0.410, 0.416)**	**0.0218**	**0.1665**	**0.1389**	**0.1665**	**10000**
	CQM	**0.003385**	**(-0.407, 0.418)**	**0.0216**	**0.1659**	**0.1385**	**0.1659**	**10000**
	HL1	0.003277	(-0.369, 0.372)	0.0176	0.1503	0.1277	0.1503	10000
	HL2	0.003327	(-0.369, 0.373)	0.0177	0.1501	0.1270	0.1501	10000
	HL3	0.003315	(-0.369, 0.371)	0.0177	0.1501	0.1271	0.1502	10000

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Appendix D

Contamination on Spread

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Table D.1: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.099207	(-2.949, 2.933)	0.9038	1.4483	1.3786	1.4484	10000
	NM	0.104919	(-2.951, 2.920)	0.9043	1.4486	1.3846	1.4486	9506
	MM	0.080366	(-2.950, 2.940)	0.9002	1.4434	1.3743	1.4434	9506
	MIR	**0.083609**	**(-2.946, 2.951)**	**0.9029**	**1.4488**	**1.3779**	**1.4489**	**9506**
	CSQM	**0.085899**	**(-2.969, 2.918)**	**0.8992**	**1.4431**	**1.3627**	**1.4432**	**9506**
	CQM	**0.092269**	**(-2.948, 2.914)**	**0.8998**	**1.4442**	**1.3630**	**1.4442**	**9506**
	HL1	0.163073	(-2.960, 2.940)	0.9047	1.4507	1.3742	1.4521	9881
	HL2	0.136118	(-2.942, 2.957)	0.9054	1.4509	1.3808	1.4517	10000
	HL3	0.143835	(-2.959, 2.943)	0.9063	1.4522	1.3820	1.4534	9997
0.2	Mean	0.025530	(-2.864, 2.885)	0.7918	1.3043	1.1582	1.3044	10000
	NM	0.028395	(-2.883, 2.903)	0.7922	1.3060	1.1673	1.3061	9541
	MM	0.034066	(-2.867, 2.902)	0.7930	1.3060	1.1668	1.3061	9541
	MIR	**0.030900**	**(-2.850, 2.899)**	**0.7929**	**1.3071**	**1.1628**	**1.3071**	**9541**
	CSQM	**0.033912**	**(-2.896, 2.876)**	**0.7889**	**1.3016**	**1.1603**	**1.3014**	**9541**
	CQM	**0.034325**	**(-2.895, 2.882)**	**0.7904**	**1.3039**	**1.1650**	**1.3039**	**9541**
	HL1	0.035043	(-2.865, 2.921)	0.7964	1.3109	1.1811	1.3111	9874
	HL2	0.034340	(-2.859, 2.904)	0.7944	1.3087	1.1742	1.3089	10000
	HL3	0.034966	(-2.864, 2.918)	0.7967	1.3120	1.1785	1.3122	9997
0.3	Mean	0.035388	(-2.833, 2.800)	0.6890	1.1736	0.9925	1.1740	10000
	NM	0.028094	(-2.793, 2.811)	0.6910	1.1756	0.9939	1.1759	9556
	MM	0.027128	(-2.819, 2.811)	0.6911	1.1764	0.9959	1.1766	9556
	MIR	**0.044675**	**(-2.828, 2.816)**	**0.6913**	**1.1762**	**1.0066**	**1.1766**	**9556**
	CSQM	**0.032213**	**(-2.825, 2.818)**	**0.6851**	**1.1679**	**0.9951**	**1.1682**	**9556**
	CQM	**0.036906**	**(-2.818, 2.820)**	**0.6870**	**1.1703**	**0.9963**	**1.1706**	**9556**
	HL1	0.032743	(-2.823, 2.809)	0.6953	1.1819	1.0012	1.1822	9882
	HL2	0.040142	(-2.828, 2.783)	0.6917	1.1761	1.0006	1.1765	10000
	HL3	0.034274	(-2.823, 2.805)	0.6940	1.1796	1.0025	1.1799	9997
0.4	Mean	0.016947	(-2.721, 2.751)	0.6019	1.0621	0.8633	1.0622	10000
	NM	0.013125	(-2.727, 2.724)	0.6054	1.0678	0.8727	1.0678	9574
	MM	0.016931	(-2.744, 2.704)	0.6092	1.0716	0.8792	1.0716	9574
	MIR	**0.022459**	**(-2.698, 2.735)**	**0.6048**	**1.0663**	**0.8790**	**1.0665**	**9574**
	CSQM	**0.018403**	**(-2.740, 2.675)**	**0.5988**	**1.0567**	**0.8611**	**1.0568**	**9574**
	CQM	**0.019873**	**(-2.713, 2.723)**	**0.5987**	**1.0576**	**0.8595**	**1.0578**	**9574**
	HL1	0.013306	(-2.736, 2.747)	0.6084	1.0706	0.8732	1.0708	9879
	HL2	0.020790	(-2.725, 2.720)	0.6063	1.0669	0.8732	1.0672	10000
	HL3	0.021458	(-2.731, 2.748)	0.6072	1.0684	0.8743	1.0686	9995
0.5	Mean	0.006395	(-2.655, 2.643)	0.5284	0.9665	0.7611	0.9665	10000
	NM	0.011216	(-2.616, 2.617)	0.5313	0.9713	0.7662	0.9714	9625
	MM	0.005559	(-2.646, 2.604)	0.5315	0.9718	0.7687	0.9719	9625
	MIR	**-0.006124**	**(-2.642, 2.598)**	**0.5310**	**0.9692**	**0.7643**	**0.9692**	**9625**
	CSQM	**0.001853**	**(-2.645, 2.565)**	**0.5237**	**0.9606**	**0.7567**	**0.9606**	**9625**
	CQM	**-0.001028**	**(-2.606, 2.587)**	**0.5262**	**0.9627**	**0.7582**	**0.9627**	**9625**
	HL1	0.002245	(-2.621, 2.631)	0.5377	0.9764	0.7734	0.9764	9894
	HL2	0.007959	(-2.637, 2.613)	0.5330	0.9710	0.7655	0.9711	10000
	HL3	0.007302	(-2.645, 2.630)	0.5373	0.9763	0.7749	0.9763	9996

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Table D.2: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon=0.1$, Sample Size $n=7$, $\kappa=1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	-0.001037	(-1.608, 1.652)	0.2351	0.5661	0.4243	0.5661	10000
	NM	-0.003528	(-1.637, 1.670)	0.2484	0.5870	0.4472	0.5871	9829
	MM	-0.004042	(-1.660, 1.675)	0.2503	0.5900	0.4505	0.5900	9829
	MIR	**-0.001435**	**(-1.666, 1.712)**	**0.2469**	**0.5843**	**0.4410**	**0.5843**	**9829**
	CSQM	**-0.003348**	**(-1.631, 1.642)**	**0.2386**	**0.5717**	**0.4336**	**0.5717**	**9829**
	CQM	**-0.002554**	**(-1.621, 1.651)**	**0.2409**	**0.5762**	**0.4370**	**0.5762**	**9829**
	HL1	0.000856	(-1.649, 1.667)	0.2451	0.5822	0.4405	0.5822	9952
	HL2	-0.000405	(-1.624, 1.668)	0.2398	0.5733	0.4317	0.5733	10000
	HL3	0.000439	(-1.630, 1.661)	0.2421	0.5770	0.4363	0.5770	10000
1.5	Mean	0.003427	(-1.038, 1.013)	0.1218	0.3908	0.3074	0.3908	10000
	NM	0.002778	(-1.066, 1.100)	0.1371	0.4198	0.3319	0.4198	9936
	MM	0.004966	(-1.068, 1.111)	0.1377	0.4209	0.3319	0.4209	9936
	MIR	**0.005593**	**(-1.089, 1.076)**	**0.1308**	**0.4062**	**0.3176**	**0.4062**	**9936**
	CSQM	**0.005025**	**(-1.032, 1.044)**	**0.1267**	**0.4008**	**0.3161**	**0.4008**	**9936**
	CQM	**0.005339**	**(-1.062, 1.043)**	**0.1286**	**0.4043**	**0.3201**	**0.4044**	**9936**
	HL1	0.004868	(-1.094, 1.053)	0.1309	0.4055	0.3140	0.4055	9983
	HL2	0.004125	(-1.060, 1.029)	0.1265	0.3989	0.3133	0.3989	10000
	HL3	0.004083	(-1.080, 1.046)	0.1286	0.4018	0.3128	0.4018	10000
2.0	Mean	-0.003223	(-0.782, 0.798)	0.0755	0.3069	0.2490	0.3070	10000
	NM	-0.003204	(-0.836, 0.857)	0.0863	0.3299	0.2678	0.3299	9982
	MM	-0.003283	(-0.850, 0.860)	0.0878	0.3334	0.2708	0.3334	9982
	MIR	**-0.003784**	**(-0.838, 0.845)**	**0.0841**	**0.3238**	**0.2574**	**0.3238**	**9982**
	CSQM	**-0.003391**	**(-0.793, 0.803)**	**0.0792**	**0.3151**	**0.2545**	**0.3151**	**9982**
	CQM	**-0.003703**	**(-0.806, 0.824)**	**0.0806**	**0.3173**	**0.2565**	**0.3174**	**9982**
	HL1	-0.005156	(-0.818, 0.848)	0.0822	0.3195	0.2569	0.3196	9990
	HL2	-0.005108	(-0.801, 0.803)	0.0782	0.3122	0.2546	0.3122	10000
	HL3	-0.005420	(-0.806, 0.812)	0.0796	0.3148	0.2553	0.3149	10000
2.5	Mean	-0.001523	(-0.667, 0.658)	0.0550	0.2624	0.2144	0.2624	10000
	NM	-0.003115	(-0.727, 0.726)	0.0644	0.2863	0.2346	0.2863	9991
	MM	-0.003589	(-0.735, 0.744)	0.0660	0.2894	0.2364	0.2894	9991
	MIR	**-0.001655**	**(-0.725, 0.701)**	**0.0619**	**0.2766**	**0.2232**	**0.2766**	**9991**
	CSQM	**-0.003474**	**(-0.702, 0.682)**	**0.0582**	**0.2709**	**0.2208**	**0.2709**	**9991**
	CQM	**-0.003425**	**(-0.704, 0.683)**	**0.0592**	**0.2725**	**0.2237**	**0.2725**	**9991**
	HL1	-0.001038	(-0.697, 0.687)	0.0597	0.2723	0.2193	0.2723	9995
	HL2	-0.002228	(-0.691, 0.667)	0.0568	0.2663	0.2174	0.2663	10000
	HL3	-0.001874	(-0.694, 0.676)	0.0579	0.2685	0.2172	0.2685	9999
3.0	Mean	0.004543	(-0.584, 0.597)	0.0434	0.2319	0.1908	0.2319	10000
	NM	0.004750	(-0.616, 0.623)	0.0501	0.2530	0.2121	0.2530	9995
	MM	0.004614	(-0.627, 0.647)	0.0513	0.2558	0.2154	0.2558	9995
	MIR	**0.003846**	**(-0.623, 0.641)**	**0.0491**	**0.2437**	**0.1949**	**0.2438**	**9995**
	CSQM	**0.004246**	**(-0.586, 0.609)**	**0.0453**	**0.2389**	**0.1965**	**0.2389**	**9995**
	CQM	**0.003950**	**(-0.599, 0.621)**	**0.0467**	**0.2408**	**0.1973**	**0.2408**	**9995**
	HL1	0.005145	(-0.608, 0.630)	0.0478	0.2409	0.1941	0.2410	9998
	HL2	0.005486	(-0.588, 0.609)	0.0449	0.2355	0.1927	0.2355	10000
	HL3	0.005622	(-0.597, 0.619)	0.0460	0.2374	0.1927	0.2374	10000

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Table D.3: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, sample size $n = 7$, $\kappa = 4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	-0.002526	(-0.512, 0.486)	0.0319	0.1976	0.1607	0.1976	10000
	NM	-0.002319	(-0.551, 0.532)	0.0373	0.2160	0.1785	0.2160	9999
	MM	-0.002369	(-0.554, 0.537)	0.0382	0.2180	0.1800	0.2180	9999
	MIR	**-0.002543**	**(-0.557, 0.524)**	**0.0361**	**0.2064**	**0.1636**	**0.2064**	**9999**
	CSQM	**-0.002639**	**(-0.510, 0.501)**	**0.0331**	**0.2021**	**0.1651**	**0.2021**	**9999**
	CQM	**-0.002838**	**(-0.515, 0.507)**	**0.0337**	**0.2032**	**0.1654**	**0.2032**	**9999**
	HL1	-0.003036	(-0.546, 0.515)	0.0352	0.2053	0.1643	0.2053	9999
	HL2	-0.003004	(-0.513, 0.494)	0.0324	0.1993	0.1628	0.1993	10000
	HL3	-0.003034	(-0.523, 0.507)	0.0333	0.2011	0.1631	0.2011	10000
5.0	Mean	0.000029	(-0.452, 0.456)	0.0261	0.1765	0.1398	0.1765	10000
	NM	-0.001118	(-0.472, 0.473)	0.0281	0.1850	0.1509	0.1850	10000
	MM	-0.001251	(-0.478, 0.478)	0.0287	0.1867	0.1520	0.1867	10000
	MIR	**0.000143**	**(-0.490, 0.496)**	**0.0301**	**0.1850**	**0.1440**	**0.1850**	**10000**
	CSQM	**-0.000714**	**(-0.450, 0.448)**	**0.0255**	**0.1753**	**0.1415**	**0.1753**	**10000**
	CQM	**-0.000378**	**(-0.459, 0.453)**	**0.0264**	**0.1779**	**0.1437**	**0.1779**	**10000**
	HL1	0.000305	(-0.485, 0.473)	0.0296	0.1840	0.1447	0.1840	9997
	HL2	0.000018	(-0.456, 0.448)	0.0261	0.1755	0.1403	0.1755	10000
	HL3	0.000321	(-0.461, 0.459)	0.0271	0.1785	0.1424	0.1785	10000
6.0	Mean	-0.001312	(-0.417, 0.429)	0.0224	0.1640	0.1331	0.1640	10000
	NM	-0.001748	(-0.429, 0.430)	0.0238	0.1729	0.1439	0.1730	9999
	MM	-0.001836	(-0.432, 0.434)	0.0242	0.1741	0.1448	0.1741	9999
	MIR	**-0.001589**	**(-0.442, 0.448)**	**0.0256**	**0.1696**	**0.1321**	**0.1696**	**9999**
	CSQM	**-0.001510**	**(-0.400, 0.403)**	**0.0213**	**0.1616**	**0.1328**	**0.1616**	**9999**
	CQM	**-0.001368**	**(-0.400, 0.402)**	**0.0220**	**0.1627**	**0.1330**	**0.1627**	**9999**
	HL1	-0.001234	(-0.438, 0.448)	0.0260	0.1706	0.1335	0.1707	9999
	HL2	-0.001361	(-0.402, 0.415)	0.0220	0.1618	0.1311	0.1618	10000
	HL3	-0.001271	(-0.407, 0.430)	0.0228	0.1638	0.1313	0.1639	10000
7.0	Mean	0.001181	(-0.388, 0.398)	0.0200	0.1544	0.1248	0.1545	10000
	NM	0.002453	(-0.403, 0.387)	0.0202	0.1578	0.1317	0.1578	10000
	MM	0.002005	(-0.405, 0.390)	0.0205	0.1587	0.1323	0.1587	10000
	MIR	**0.000617**	**(-0.409, 0.410)**	**0.0226**	**0.1587**	**0.1237**	**0.1587**	**10000**
	CSQM	**0.001785**	**(-0.373, 0.370)**	**0.0183**	**0.1492**	**0.1237**	**0.1493**	**10000**
	CQM	**0.001667**	**(-0.380, 0.373)**	**0.0192**	**0.1513**	**0.1235**	**0.1513**	**10000**
	HL1	0.000525	(-0.417, 0.413)	0.0236	0.1607	0.1248	0.1607	10000
	HL2	0.001563	(-0.379, 0.383)	0.0194	0.1508	0.1207	0.1508	10000
	HL3	0.001196	(-0.389, 0.393)	0.0204	0.1535	0.1227	0.1535	10000
8.0	Mean	0.001912	(-0.373, 0.377)	0.0182	0.1465	0.1170	0.1465	10000
	NM	0.000420	(-0.365, 0.372)	0.0177	0.1479	0.1234	0.1479	9998
	MM	0.000470	(-0.365, 0.376)	0.0179	0.1488	0.1237	0.1488	9998
	MIR	**0.001254**	**(-0.392, 0.389)**	**0.0204**	**0.1490**	**0.1149**	**0.1489**	**9998**
	CSQM	**0.000948**	**(-0.343, 0.350)**	**0.0159**	**0.1387**	**0.1140**	**0.1387**	**9998**
	CQM	**0.001187**	**(-0.346, 0.354)**	**0.0166**	**0.1402**	**0.1143**	**0.1402**	**9998**
	HL1	0.001982	(-0.387, 0.394)	0.0210	0.1499	0.1159	0.1498	9998
	HL2	0.001540	(-0.353, 0.357)	0.0171	0.1408	0.1132	0.1408	10000
	HL3	0.001691	(-0.366, 0.368)	0.0179	0.1432	0.1143	0.1432	10000

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Table D.4: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 8$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.022509	(-2.915, 2.929)	0.8763	1.4122	1.3302	1.4122	10000
	NM	0.017543	(-2.932, 2.914)	0.8775	1.4141	1.3297	1.4141	9980
	MM	0.009154	(-2.930, 2.919)	0.8778	1.4134	1.3441	1.4134	9980
	MIR	**0.005998**	**(-2.919, 2.925)**	**0.8857**	**1.4242**	**1.3549**	**1.4242**	**9980**
	CSQM	**0.009154**	**(-2.930, 2.919)**	**0.8778**	**1.4134**	**1.3441**	**1.4134**	**9980**
	CQM	**0.009154**	**(-2.930, 2.919)**	**0.8778**	**1.4134**	**1.3441**	**1.4134**	**9980**
	HL1	-0.002218	(-2.938, 2.923)	0.8811	1.4187	1.3363	1.4187	9993
	HL2	0.019763	(-2.941, 2.901)	0.8792	1.4162	1.3345	1.4161	10000
	HL3	0.012472	(-2.934, 2.925)	0.8789	1.4161	1.3365	1.4161	9997
0.2	Mean	-0.035826	(-2.883, 2.884)	0.7704	1.2797	1.1243	1.2800	10000
	NM	-0.029137	(-2.910, 2.903)	0.7817	1.2951	1.1442	1.2955	9986
	MM	-0.034866	(-2.902, 2.884)	0.7758	1.2881	1.1395	1.2885	9986
	MIR	**-0.047226**	**(-2.879, 2.891)**	**0.7795**	**1.2905**	**1.1443**	**1.2908**	**9986**
	CSQM	**-0.034866**	**(-2.902, 2.884)**	**0.7758**	**1.2881**	**1.1395**	**1.2885**	**9986**
	CQM	**-0.034866**	**(-2.902, 2.884)**	**0.7758**	**1.2881**	**1.1395**	**1.2885**	**9986**
	HL1	-0.037350	(-2.874, 2.873)	0.7753	1.2857	1.1266	1.2860	9987
	HL2	-0.035104	(-2.872, 2.883)	0.7729	1.2829	1.1266	1.2831	9999
	HL3	-0.039373	(-2.878, 2.867)	0.7739	1.2839	1.1268	1.2842	9996
0.3	Mean	-0.019310	(-2.787, 2.825)	0.6778	1.1602	0.9855	1.1601	10000
	NM	-0.035932	(-2.828, 2.800)	0.6968	1.1843	1.0088	1.1846	9985
	MM	-0.011049	(-2.854, 2.791)	0.6860	1.1705	0.9882	1.1706	9985
	MIR	**-0.015730**	**(-2.822, 2.814)**	**0.6935**	**1.1810**	**1.0121**	**1.1810**	**9985**
	CSQM	**-0.011049**	**(-2.854, 2.791)**	**0.6860**	**1.1705**	**0.9882**	**1.1706**	**9985**
	CQM	**-0.011049**	**(-2.854, 2.791)**	**0.6860**	**1.1705**	**0.9882**	**1.1706**	**9985**
	HL1	-0.018274	(-2.803, 2.779)	0.6799	1.1630	0.9878	1.1630	9990
	HL2	-0.015205	(-2.811, 2.789)	0.6770	1.1638	0.9861	1.1638	9999
	HL3	-0.017328	(-2.813, 2.790)	0.6803	1.1640	0.9851	1.1640	9998
0.4	Mean	-0.021625	(-2.653, 2.728)	0.5717	1.0214	0.8115	1.0213	10000
	NM	-0.015847	(-2.706, 2.740)	0.5856	1.0403	0.8405	1.0404	9983
	MM	-0.015810	(-2.739, 2.718)	0.5822	1.0345	0.8343	1.0345	9983
	MIR	**-0.003980**	**(-2.722, 2.725)**	**0.5881**	**1.0433**	**0.8410**	**1.0433**	**9983**
	CSQM	**-0.015810**	**(-2.739, 2.718)**	**0.5822**	**1.0345**	**0.8343**	**1.0345**	**9983**
	CQM	**-0.015810**	**(-2.739, 2.718)**	**0.5822**	**1.0345**	**0.8343**	**1.0345**	**9983**
	HL1	-0.019418	(-2.701, 2.721)	0.5799	1.0328	0.8247	1.0329	9990
	HL2	-0.022430	(-2.681, 2.744)	0.5756	1.0271	0.8196	1.0271	9998
	HL3	-0.022154	(-2.686, 2.729)	0.5769	1.0290	0.8217	1.0290	9995
0.5	Mean	0.011168	(-2.544, 2.522)	0.4904	0.9154	0.7128	0.9155	10000
	NM	0.012448	(-2.592, 2.538)	0.5114	0.9426	0.7406	0.9427	9987
	MM	0.011455	(-2.569, 2.549)	0.5080	0.9391	0.7382	0.9391	9987
	MIR	**0.010463**	**(-2.644, 2.526)**	**0.5165**	**0.9497**	**0.7440**	**0.9497**	**9987**
	CSQM	**0.011455**	**(-2.569, 2.549)**	**0.5080**	**0.9391**	**0.7382**	**0.9391**	**9987**
	CQM	**0.011455**	**(-2.569, 2.549)**	**0.5080**	**0.9391**	**0.7382**	**0.9391**	**9987**
	HL1	0.000496	(-2.588, 2.523)	0.5039	0.9331	0.7350	0.9331	9993
	HL2	0.004388	(-2.570, 2.526)	0.4964	0.9232	0.7280	0.9232	10000
	HL3	0.002317	(-2.569, 2.529)	0.4985	0.9263	0.7340	0.9263	9996

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Table D.5: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon=0.1$, Sample Size $n=8$, $\kappa=1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	-0.013662	(-1.476, 1.518)	0.2118	0.5316	0.3995	0.5317	10000
	NM	-0.016614	(-1.623, 1.595)	0.2297	0.5575	0.4143	0.5576	9996
	MM	-0.014796	(-1.508, 1.568)	0.2245	0.5509	0.4146	0.5511	9996
	MIR	**-0.016850**	**(-1.575, 1.610)**	**0.2336**	**0.5639**	**0.4265**	**0.5640**	**9996**
	CSQM	**-0.014796**	**(-1.508, 1.568)**	**0.2245**	**0.5509**	**0.4146**	**0.5511**	**9996**
	CQM	**-0.014796**	**(-1.508, 1.568)**	**0.2245**	**0.5509**	**0.4146**	**0.5511**	**9996**
	HL1	-0.013708	(-1.538, 1.560)	0.2203	0.5450	0.4120	0.5451	10000
	HL2	-0.012619	(-1.483, 1.564)	0.2178	0.5403	0.4088	0.5404	10000
	HL3	-0.013066	(-1.519, 1.566)	0.2191	0.5427	0.4109	0.5427	10000
1.5	Mean	-0.003051	(-0.918, 0.939)	0.1021	0.3558	0.2802	0.3558	10000
	NM	0.001364	(-0.966, 0.989)	0.1139	0.3791	0.2993	0.3791	9998
	MM	0.000524	(-0.974, 0.977)	0.1136	0.3801	0.3021	0.3801	9998
	MIR	**-0.003193**	**(-1.029, 1.022)**	**0.1192**	**0.3858**	**0.3020**	**0.3858**	**9998**
	CSQM	**0.000524**	**(-0.974, 0.977)**	**0.1136**	**0.3801**	**0.3021**	**0.3801**	**9998**
	CQM	**0.000524**	**(-0.974, 0.977)**	**0.1136**	**0.3801**	**0.3021**	**0.3801**	**9998**
	HL1	-0.002830	(-0.954, 0.968)	0.1080	0.3651	0.2869	0.3651	10000
	HL2	-0.002984	(-0.927, 0.953)	0.1059	0.3621	0.2851	0.3621	10000
	HL3	-0.002984	(-0.938, 0.965)	0.1071	0.3638	0.2844	0.3638	10000
2.0	Mean	0.003840	(-0.727, 0.741)	0.0651	0.2842	0.2289	0.2842	10000
	NM	0.002233	(-0.772, 0.766)	0.0728	0.3009	0.2421	0.3009	10000
	MM	0.002709	(-0.773, 0.773)	0.0726	0.3017	0.2448	0.3017	10000
	MIR	**0.005178**	**(-0.794, 0.829)**	**0.0779**	**0.3102**	**0.2469**	**0.3102**	**10000**
	CSQM	**0.002709**	**(-0.773, 0.773)**	**0.0726**	**0.3017**	**0.2448**	**0.3017**	**10000**
	CQM	**0.002709**	**(-0.773, 0.773)**	**0.0726**	**0.3017**	**0.2448**	**0.3017**	**10000**
	HL1	0.003829	(-0.737, 0.759)	0.0691	0.2917	0.2315	0.2917	10000
	HL2	0.003762	(-0.733, 0.758)	0.0671	0.2884	0.2309	0.2884	10000
	HL3	0.003812	(-0.736, 0.759)	0.0682	0.2900	0.2307	0.2900	10000
2.5	Mean	-0.003099	(-0.596, 0.620)	0.0451	0.2372	0.1930	0.2372	10000
	NM	-0.003727	(-0.620, 0.643)	0.0498	0.2510	0.2071	0.2510	10000
	MM	-0.003829	(-0.626, 0.652)	0.0505	0.2529	0.2088	0.2529	10000
	MIR	**-0.003341**	**(-0.668, 0.679)**	**0.0551**	**0.2588**	**0.2043**	**0.2589**	**10000**
	CSQM	**-0.003829**	**(-0.626, 0.652)**	**0.0505**	**0.2529**	**0.2088**	**0.2529**	**10000**
	CQM	**-0.003829**	**(-0.626, 0.652)**	**0.0505**	**0.2529**	**0.2088**	**0.2529**	**10000**
	HL1	-0.003183	(-0.605, 0.643)	0.0482	0.2441	0.1965	0.2441	10000
	HL2	-0.003494	(-0.603, 0.629)	0.0470	0.2417	0.1954	0.2417	10000
	HL3	-0.003351	(-0.607, 0.637)	0.0476	0.2430	0.1953	0.2431	10000
3.0	Mean	-0.000587	(-0.544, 0.548)	0.0373	0.2161	0.1770	0.2161	10000
	NM	0.000741	(-0.575, 0.575)	0.0414	0.2280	0.1879	0.2280	10000
	MM	0.000259	(-0.584, 0.583)	0.0423	0.2300	0.1890	0.2300	10000
	MIR	**-0.000160**	**(-0.612, 0.620)**	**0.0458**	**0.2357**	**0.1896**	**0.2357**	**10000**
	CSQM	**0.000259**	**(-0.584, 0.583)**	**0.0423**	**0.2300**	**0.1890**	**0.2300**	**10000**
	CQM	**0.000259**	**(-0.584, 0.583)**	**0.0423**	**0.2300**	**0.1890**	**0.2300**	**10000**
	HL1	0.000188	(-0.559, 0.569)	0.0397	0.2220	0.1814	0.2220	10000
	HL2	0.000436	(-0.554, 0.560)	0.0388	0.2196	0.1803	0.2196	10000
	HL3	0.000309	(-0.559, 0.562)	0.0392	0.2206	0.1811	0.2206	10000

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Table D.6: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 8$, $\kappa = 4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.004103	(-0.457, 0.475)	0.0274	0.1845	0.1518	0.1845	10000
	NM	0.005814	(-0.485, 0.499)	0.0301	0.1944	0.1614	0.1944	10000
	MM	0.005234	(-0.490, 0.501)	0.0305	0.1955	0.1618	0.1955	10000
	MIR	**0.004832**	**(-0.506, 0.526)**	**0.0334**	**0.1989**	**0.1615**	**0.1989**	**10000**
	CSQM	**0.005234**	**(-0.490, 0.501)**	**0.0305**	**0.1955**	**0.1618**	**0.1955**	**10000**
	CQM	**0.005234**	**(-0.490, 0.501)**	**0.0305**	**0.1955**	**0.1618**	**0.1955**	**10000**
	HL1	0.004618	(-0.463, 0.482)	0.0287	0.1880	0.1546	0.1880	10000
	HL2	0.005109	(-0.459, 0.478)	0.0280	0.1863	0.1544	0.1863	10000
	HL3	0.004947	(-0.460, 0.482)	0.0283	0.1869	0.1544	0.1869	10000
5.0	Mean	-0.003979	(-0.433, 0.405)	0.0224	0.1662	0.1364	0.1662	10000
	NM	-0.002770	(-0.442, 0.423)	0.0235	0.1717	0.1424	0.1718	10000
	MM	-0.002297	(-0.443, 0.425)	0.0238	0.1727	0.1430	0.1727	10000
	MIR	**-0.003769**	**(-0.494, 0.466)**	**0.0280**	**0.1804**	**0.1421**	**0.1804**	**10000**
	CSQM	**-0.002297**	**(-0.443, 0.425)**	**0.0238**	**0.1727**	**0.1430**	**0.1727**	**10000**
	CQM	**-0.002297**	**(-0.443, 0.425)**	**0.0238**	**0.1727**	**0.1430**	**0.1727**	**10000**
	HL1	-0.003491	(-0.433, 0.411)	0.0231	0.1674	0.1376	0.1675	10000
	HL2	-0.003155	(-0.428, 0.403)	0.0223	0.1652	0.1357	0.1652	10000
	HL3	-0.003183	(-0.428, 0.406)	0.0226	0.1661	0.1364	0.1661	10000
6.0	Mean	0.002815	(-0.391, 0.398)	0.0198	0.1536	0.1246	0.1536	10000
	NM	0.003621	(-0.392, 0.390)	0.0200	0.1563	0.1292	0.1563	10000
	MM	0.003750	(-0.395, 0.392)	0.0203	0.1573	0.1299	0.1574	10000
	MIR	**0.001432**	**(-0.450, 0.437)**	**0.0249**	**0.1661**	**0.1285**	**0.1661**	**10000**
	CSQM	**0.003750**	**(-0.395, 0.392)**	**0.0203**	**0.1573**	**0.1299**	**0.1574**	**10000**
	CQM	**0.003750**	**(-0.395, 0.392)**	**0.0203**	**0.1573**	**0.1299**	**0.1574**	**10000**
	HL1	0.002891	(-0.394, 0.391)	0.0202	0.1535	0.1237	0.1535	10000
	HL2	0.003198	(-0.390, 0.385)	0.0194	0.1514	0.1222	0.1514	10000
	HL3	0.002984	(-0.390, 0.386)	0.0197	0.1521	0.1227	0.1521	10000
7.0	Mean	0.003000	(-0.368, 0.378)	0.0174	0.1442	0.1163	0.1443	10000
	NM	0.003313	(-0.366, 0.365)	0.0170	0.1443	0.1177	0.1443	10000
	MM	0.003003	(-0.368, 0.366)	0.0172	0.1450	0.1180	0.1450	10000
	MIR	**0.002711**	**(-0.418, 0.424)**	**0.0228**	**0.1559**	**0.1186**	**0.1560**	**10000**
	CSQM	**0.003003**	**(-0.368, 0.366)**	**0.0172**	**0.1450**	**0.1180**	**0.1450**	**10000**
	CQM	**0.003003**	**(-0.368, 0.366)**	**0.0172**	**0.1450**	**0.1180**	**0.1450**	**10000**
	HL1	0.002968	(-0.363, 0.367)	0.0175	0.1423	0.1142	0.1423	10000
	HL2	0.003099	(-0.358, 0.357)	0.0167	0.1403	0.1134	0.1403	10000
	HL3	0.003076	(-0.356, 0.360)	0.0170	0.1410	0.1141	0.1410	10000
8.0	Mean	0.001438	(-0.340, 0.354)	0.0150	0.1347	0.1096	0.1347	10000
	NM	0.001255	(-0.328, 0.331)	0.0139	0.1314	0.1099	0.1314	10000
	MM	0.001226	(-0.333, 0.332)	0.0140	0.1318	0.1101	0.1318	10000
	MIR	**0.001859**	**(-0.378, 0.394)**	**0.0195**	**0.1444**	**0.1099**	**0.1445**	**10000**
	CSQM	**0.001226**	**(-0.333, 0.332)**	**0.0140**	**0.1318**	**0.1101**	**0.1318**	**10000**
	CQM	**0.001226**	**(-0.333, 0.332)**	**0.0140**	**0.1318**	**0.1101**	**0.1318**	**10000**
	HL1	0.001399	(-0.329, 0.335)	0.0147	0.1310	0.1058	0.1310	10000
	HL2	0.001459	(-0.326, 0.332)	0.0140	0.1290	0.1045	0.1290	10000
	HL3	0.001446	(-0.327, 0.332)	0.0143	0.1298	0.1050	0.1298	10000

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Table D.7: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 19$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.123485	(-2.916, 2.922)	0.8364	1.3608	1.2567	1.3620	10000
	NM	0.097037	(-2.923, 2.940)	0.8377	1.3643	1.2530	1.3652	9785
	MM	0.099174	(-2.905, 2.920)	0.8375	1.3632	1.2510	1.3634	9785
	MIR	**0.110430**	**(-2.921, 2.913)**	**0.8448**	**1.3711**	**1.2573**	**1.3723**	**9785**
	CSQM	**0.096496**	**(-2.914, 2.898)**	**0.8362**	**1.3615**	**1.2516**	**1.3618**	**9785**
	CQM	**0.095077**	**(-2.912, 2.890)**	**0.8360**	**1.3613**	**1.2515**	**1.3617**	**9785**
	HL1	0.121721	(-2.945, 2.930)	0.8410	1.3666	1.2707	1.3680	9989
	HL2	0.117740	(-2.931, 2.930)	0.8397	1.3649	1.2678	1.3661	9996
	HL3	0.117986	(-2.945, 2.937)	0.8396	1.3650	1.2676	1.3663	9991
0.2	Mean	0.011385	(-2.769, 2.797)	0.6704	1.1485	0.9695	1.1485	10000
	NM	0.005936	(-2.811, 2.827)	0.6823	1.1642	0.9865	1.1642	9829
	MM	0.011683	(-2.813, 2.781)	0.6810	1.1619	0.9956	1.1618	9829
	MIR	**0.022190**	**(-2.817, 2.805)**	**0.6821**	**1.1642**	**0.9866**	**1.1641**	**9829**
	CSQM	**0.014853**	**(-2.782, 2.794)**	**0.6786**	**1.1583**	**0.9895**	**1.1583**	**9829**
	CQM	**0.016618**	**(-2.783, 2.779)**	**0.6783**	**1.1578**	**0.9889**	**1.1578**	**9829**
	HL1	0.007686	(-2.782, 2.805)	0.6733	1.1524	0.9722	1.1524	9995
	HL2	0.007913	(-2.782, 2.796)	0.6735	1.1523	0.9704	1.1523	10000
	HL3	0.006215	(-2.791, 2.795)	0.6744	1.1538	0.9727	1.1538	9999
0.3	Mean	-0.002685	(-2.581, 2.610)	0.5250	0.9604	0.7570	0.9604	10000
	NM	0.002395	(-2.604, 2.640)	0.5397	0.9788	0.7757	0.9788	9843
	MM	0.001637	(-2.621, 2.608)	0.5416	0.9827	0.7863	0.9827	9843
	MIR	**0.005819**	**(-2.631, 2.638)**	**0.5428**	**0.9846**	**0.7837**	**0.9846**	**9843**
	CSQM	**0.002975**	**(-2.623, 2.614)**	**0.5379**	**0.9774**	**0.7787**	**0.9774**	**9843**
	CQM	**0.003722**	**(-2.613, 2.628)**	**0.5371**	**0.9763**	**0.7764**	**0.9763**	**9843**
	HL1	-0.002466	(-2.613, 2.611)	0.5275	0.9634	0.7568	0.9634	9990
	HL2	-0.000815	(-2.583, 2.616)	0.5263	0.9621	0.7570	0.9621	9999
	HL3	-0.001670	(-2.582, 2.618)	0.5262	0.9615	0.7573	0.9614	9996
0.4	Mean	0.002282	(-2.314, 2.353)	0.3950	0.7883	0.5914	0.7883	10000
	NM	0.003650	(-2.347, 2.487)	0.4153	0.8172	0.6168	0.8172	9853
	MM	0.009320	(-2.379, 2.394)	0.4169	0.8184	0.6195	0.8185	9853
	MIR	**0.012016**	**(-2.406, 2.427)**	**0.4222**	**0.8245**	**0.6203**	**0.8246**	**9853**
	CSQM	**0.010067**	**(-2.378, 2.384)**	**0.4130**	**0.8126**	**0.6124**	**0.8127**	**9853**
	CQM	**0.010506**	**(-2.380, 2.373)**	**0.4124**	**0.8116**	**0.6143**	**0.8117**	**9853**
	HL1	0.001419	(-2.358, 2.337)	0.4010	0.7975	0.5994	0.7975	9992
	HL2	0.002116	(-2.340, 2.353)	0.3991	0.7945	0.6013	0.7946	9998
	HL3	0.002231	(-2.373, 2.333)	0.4004	0.7963	0.5974	0.7964	9997
0.5	Mean	0.009378	(-1.956, 1.931)	0.3062	0.6672	0.5036	0.6673	10000
	NM	0.011163	(-2.025, 1.960)	0.3236	0.6934	0.5251	0.6934	9892
	MM	0.013361	(-2.022, 1.973)	0.3232	0.6925	0.5241	0.6926	9892
	MIR	**0.012008**	**(-2.046, 1.998)**	**0.3264**	**0.6958**	**0.5247**	**0.6958**	**9892**
	CSQM	**0.012359**	**(-1.989, 1.974)**	**0.3191**	**0.6863**	**0.5181**	**0.6864**	**9892**
	CQM	**0.011922**	**(-1.984, 1.970)**	**0.3186**	**0.6856**	**0.5155**	**0.6857**	**9892**
	HL1	0.009531	(-1.993, 2.001)	0.3130	0.6765	0.5082	0.6765	9998
	HL2	0.009812	(-1.981, 1.998)	0.3111	0.6739	0.5051	0.6739	10000
	HL3	0.009404	(-2.000, 1.981)	0.3118	0.6747	0.5063	0.6748	9997

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Table D.8: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 19$, $\kappa = 1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.002158	(-0.858, 0.847)	0.0880	0.3314	0.2666	0.3314	10000
	NM	0.004766	(-0.910, 0.904)	0.1013	0.3572	0.2867	0.3572	9987
	MM	0.004004	(-0.922, 0.918)	0.1027	0.3610	0.2897	0.3610	9987
	MIR	**0.000146**	**(-0.929, 0.918)**	**0.1022**	**0.3594**	**0.2873**	**0.3594**	**9987**
	CSQM	**0.003497**	**(-0.906, 0.906)**	**0.0987**	**0.3526**	**0.2849**	**0.3526**	**9987**
	CQM	**0.003231**	**(-0.912, 0.908)**	**0.0983**	**0.3514**	**0.2838**	**0.3515**	**9987**
	HL1	0.001591	(-0.882, 0.864)	0.0908	0.3371	0.2715	0.3371	10000
	HL2	0.001764	(-0.876, 0.864)	0.0902	0.3358	0.2692	0.3358	10000
	HL3	0.001644	(-0.876, 0.865)	0.0906	0.3364	0.2704	0.3364	10000
1.5	Mean	0.007816	(-0.545, 0.576)	0.0388	0.2214	0.1826	0.2214	10000
	NM	0.007185	(-0.618, 0.621)	0.0468	0.2436	0.2016	0.2436	9998
	MM	0.007563	(-0.628, 0.629)	0.0483	0.2473	0.2048	0.2474	9998
	MIR	**0.008985**	**(-0.597, 0.635)**	**0.0463**	**0.2409**	**0.1981**	**0.2410**	**9998**
	CSQM	**0.006872**	**(-0.594, 0.616)**	**0.0453**	**0.2390**	**0.1960**	**0.2390**	**9998**
	CQM	**0.006529**	**(-0.589, 0.612)**	**0.0450**	**0.2382**	**0.1969**	**0.2383**	**9998**
	HL1	0.008394	(-0.555, 0.585)	0.0399	0.2245	0.1853	0.2245	10000
	HL2	0.008192	(-0.552, 0.584)	0.0397	0.2239	0.1835	0.2239	10000
	HL3	0.008190	(-0.553, 0.583)	0.0398	0.2242	0.1839	0.2242	10000
2.0	Mean	-0.000046	(-0.455, 0.434)	0.0252	0.1791	0.1500	0.1791	10000
	NM	0.001659	(-0.507, 0.489)	0.0311	0.1991	0.1642	0.1991	10000
	MM	0.001765	(-0.508, 0.493)	0.0318	0.2013	0.1665	0.2013	10000
	MIR	**-0.000292**	**(-0.487, 0.480)**	**0.0298**	**0.1934**	**0.1600**	**0.1934**	**10000**
	CSQM	**0.001135**	**(-0.491, 0.481)**	**0.0298**	**0.1948**	**0.1618**	**0.1948**	**10000**
	CQM	**0.000817**	**(-0.486, 0.480)**	**0.0297**	**0.1945**	**0.1611**	**0.1945**	**10000**
	HL1	0.000535	(-0.455, 0.446)	0.0257	0.1805	0.1501	0.1805	10000
	HL2	0.000621	(-0.457, 0.441)	0.0256	0.1800	0.1501	0.1801	10000
	HL3	0.000581	(-0.456, 0.445)	0.0256	0.1802	0.1500	0.1802	10000
2.5	Mean	0.000922	(-0.376, 0.380)	0.0183	0.1524	0.1271	0.1525	10000
	NM	-0.000250	(-0.428, 0.431)	0.0231	0.1708	0.1427	0.1708	10000
	MM	-0.000638	(-0.430, 0.434)	0.0234	0.1721	0.1433	0.1721	10000
	MIR	**0.001740**	**(-0.407, 0.418)**	**0.0214**	**0.1638**	**0.1359**	**0.1638**	**10000**
	CSQM	**0.000321**	**(-0.416, 0.422)**	**0.0219**	**0.1665**	**0.1387**	**0.1665**	**10000**
	CQM	**0.000821**	**(-0.417, 0.421)**	**0.0219**	**0.1665**	**0.1391**	**0.1665**	**10000**
	HL1	0.000968	(-0.379, 0.385)	0.0186	0.1538	0.1289	0.1538	10000
	HL2	0.000910	(-0.378, 0.386)	0.0185	0.1533	0.1283	0.1534	10000
	HL3	0.000966	(-0.377, 0.385)	0.0186	0.1536	0.1284	0.1536	10000
3.0	Mean	0.001512	(-0.348, 0.347)	0.0151	0.1385	0.1167	0.1385	10000
	NM	0.001788	(-0.391, 0.393)	0.0194	0.1567	0.1296	0.1567	10000
	MM	0.001784	(-0.393, 0.396)	0.0195	0.1572	0.1294	0.1572	10000
	MIR	**0.001734**	**(-0.367, 0.374)**	**0.0171**	**0.1474**	**0.1227**	**0.1474**	**10000**
	CSQM	**0.001911**	**(-0.376, 0.373)**	**0.0180**	**0.1506**	**0.1240**	**0.1506**	**10000**
	CQM	**0.001975**	**(-0.373, 0.374)**	**0.0178**	**0.1497**	**0.1235**	**0.1497**	**10000**
	HL1	0.001544	(-0.349, 0.351)	0.0153	0.1389	0.1168	0.1389	10000
	HL2	0.001561	(-0.348, 0.348)	0.0152	0.1386	0.1164	0.1386	10000
	HL3	0.001550	(-0.349, 0.351)	0.0153	0.1387	0.1163	0.1387	10000

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Table D.9: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 19$, $\kappa = 4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	-0.001097	(-0.298, 0.296)	0.0113	0.1195	0.1005	0.1195	10000
	NM	0.000389	(-0.325, 0.333)	0.0138	0.1323	0.1108	0.1323	10000
	MM	0.000825	(-0.325, 0.336)	0.0139	0.1330	0.1116	0.1330	10000
	MIR	**-0.000937**	**(-0.317, 0.315)**	**0.0125**	**0.1250**	**0.1047**	**0.1250**	**10000**
	CSQM	**0.000596**	**(-0.313, 0.321)**	**0.0129**	**0.1283**	**0.1075**	**0.1283**	**10000**
	CQM	**0.000476**	**(-0.311, 0.321)**	**0.0128**	**0.1280**	**0.1077**	**0.1280**	**10000**
	HL1	-0.001139	(-0.298, 0.295)	0.0113	0.1194	0.1001	0.1194	10000
	HL2	-0.000981	(-0.296, 0.295)	0.0112	0.1191	0.1001	0.1191	10000
	HL3	-0.001036	(-0.296, 0.294)	0.0112	0.1192	0.0997	0.1193	10000
5.0	Mean	0.001240	(-0.262, 0.269)	0.0091	0.1074	0.0903	0.1074	10000
	NM	0.001374	(-0.292, 0.299)	0.0111	0.1187	0.0994	0.1187	10000
	MM	0.001286	(-0.292, 0.299)	0.0112	0.1190	0.0996	0.1190	10000
	MIR	**0.001265**	**(-0.273, 0.279)**	**0.0098**	**0.1108**	**0.0918**	**0.1108**	**10000**
	CSQM	**0.001129**	**(-0.281, 0.285)**	**0.0103**	**0.1147**	**0.0968**	**0.1147**	**10000**
	CQM	**0.001047**	**(-0.282, 0.283)**	**0.0102**	**0.1144**	**0.0964**	**0.1144**	**10000**
	HL1	0.001429	(-0.259, 0.266)	0.0089	0.1062	0.0896	0.1062	10000
	HL2	0.001398	(-0.258, 0.265)	0.0089	0.1060	0.0890	0.1060	10000
	HL3	0.001424	(-0.258, 0.265)	0.0089	0.1061	0.0892	0.1061	10000
6.0	Mean	-0.000318	(-0.247, 0.251)	0.0077	0.0982	0.0824	0.0982	10000
	NM	-0.000704	(-0.265, 0.262)	0.0089	0.1064	0.0886	0.1064	10000
	MM	-0.000835	(-0.266, 0.264)	0.0090	0.1067	0.0886	0.1067	10000
	MIR	**0.000247**	**(-0.248, 0.252)**	**0.0080**	**0.0996**	**0.0825**	**0.0996**	**10000**
	CSQM	**-0.000977**	**(-0.256, 0.255)**	**0.0083**	**0.1027**	**0.0862**	**0.1027**	**10000**
	CQM	**-0.001048**	**(-0.252, 0.254)**	**0.0083**	**0.1023**	**0.0860**	**0.1023**	**10000**
	HL1	-0.000282	(-0.237, 0.238)	0.0073	0.0956	0.0797	0.0956	10000
	HL2	-0.000405	(-0.236, 0.238)	0.0072	0.0954	0.0796	0.0954	10000
	HL3	-0.000320	(-0.237, 0.238)	0.0073	0.0955	0.0797	0.0955	10000
7.0	Mean	-0.000258	(-0.228, 0.231)	0.0068	0.0924	0.0775	0.0924	10000
	NM	0.000886	(-0.241, 0.247)	0.0077	0.0991	0.0835	0.0991	10000
	MM	0.000952	(-0.241, 0.247)	0.0078	0.0994	0.0835	0.0994	10000
	MIR	**0.000202**	**(-0.223, 0.227)**	**0.0067**	**0.0920**	**0.0773**	**0.0920**	**10000**
	CSQM	**0.000635**	**(-0.232, 0.236)**	**0.0071**	**0.0953**	**0.0807**	**0.0953**	**10000**
	CQM	**0.000476**	**(-0.232, 0.236)**	**0.0071**	**0.0949**	**0.0802**	**0.0949**	**10000**
	HL1	0.000034	(-0.217, 0.220)	0.0062	0.0888	0.0748	0.0888	10000
	HL2	0.000012	(-0.215, 0.220)	0.0062	0.0885	0.0747	0.0885	10000
	HL3	0.000032	(-0.215, 0.220)	0.0062	0.0886	0.0747	0.0886	10000
8.0	Mean	-0.001461	(-0.214, 0.219)	0.0059	0.0862	0.0717	0.0862	10000
	NM	-0.000899	(-0.226, 0.223)	0.0065	0.0913	0.0777	0.0913	10000
	MM	-0.000874	(-0.227, 0.224)	0.0066	0.0915	0.0778	0.0915	10000
	MIR	**-0.001263**	**(-0.211, 0.208)**	**0.0057**	**0.0844**	**0.0701**	**0.0844**	**10000**
	CSQM	**-0.001007**	**(-0.215, 0.214)**	**0.0061**	**0.0879**	**0.0746**	**0.0879**	**10000**
	CQM	**-0.001073**	**(-0.213, 0.214)**	**0.0060**	**0.0876**	**0.0745**	**0.0876**	**10000**
	HL1	-0.001383	(-0.202, 0.205)	0.0053	0.0818	0.0689	0.0818	10000
	HL2	-0.001350	(-0.201, 0.204)	0.0053	0.0815	0.0689	0.0815	10000
	HL3	-0.001360	(-0.201, 0.205)	0.0053	0.0816	0.0689	0.0816	10000

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Table D.10: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 20$, $\kappa = 0.1,0.2,0.3,0.4,0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	-0.046761	(-2.930, 2.905)	0.8293	1.3545	1.2288	1.3548	10000
	NM	-0.041478	(-2.925, 2.936)	0.8363	1.3628	1.2436	1.3631	9948
	MM	-0.052896	(-2.920, 2.935)	0.8392	1.3660	1.2489	1.3662	9948
	MIR	**-0.013082**	**(-2.935, 2.933)**	**0.8409**	**1.3699**	**1.2633**	**1.3699**	**9948**
	CSQM	**-0.052896**	**(-2.920, 2.935)**	**0.8392**	**1.3660**	**1.2489**	**1.3662**	**9948**
	CQM	**-0.052896**	**(-2.920, 2.935)**	**0.8392**	**1.3660**	**1.2489**	**1.3662**	**9948**
	HL1	-0.040611	(-2.929, 2.914)	0.8310	1.3578	1.2311	1.3580	9999
	HL2	-0.047375	(-2.936, 2.902)	0.8293	1.3551	1.2306	1.3555	10000
	HL3	-0.049125	(-2.927, 2.914)	0.8311	1.3577	1.2328	1.3581	10000
0.2	Mean	0.013201	(-2.799, 2.774)	0.6527	1.1283	0.9335	1.1284	10000
	NM	0.002336	(-2.763, 2.793)	0.6625	1.1401	0.9581	1.1402	9959
	MM	0.016289	(-2.808, 2.806)	0.6669	1.1454	0.9716	1.1456	9959
	MIR	**0.011485**	**(-2.812, 2.792)**	**0.6713**	**1.1506**	**0.9731**	**1.1507**	**9959**
	CSQM	**0.016289**	**(-2.808, 2.806)**	**0.6669**	**1.1454**	**0.9716**	**1.1456**	**9959**
	CQM	**0.016289**	**(-2.808, 2.806)**	**0.6669**	**1.1454**	**0.9716**	**1.1456**	**9959**
	HL1	0.012839	(-2.798, 2.776)	0.6559	1.1324	0.9411	1.1325	9995
	HL2	0.013442	(-2.800, 2.774)	0.6557	1.1324	0.9417	1.1325	9999
	HL3	0.012335	(-2.797, 2.776)	0.6556	1.1319	0.9415	1.1320	9999
0.3	Mean	-0.002546	(-2.610, 2.611)	0.5212	0.9557	0.7463	0.9557	10000
	NM	0.002667	(-2.671, 2.635)	0.5426	0.9845	0.7708	0.9845	9965
	MM	-0.000620	(-2.660, 2.646)	0.5394	0.9822	0.7681	0.9822	9965
	MIR	**-0.002833**	**(-2.669, 2.691)**	**0.5525**	**0.9972**	**0.7783**	**0.9972**	**9965**
	CSQM	**-0.000620**	**(-2.660, 2.646)**	**0.5394**	**0.9822**	**0.7681**	**0.9822**	**9965**
	CQM	**-0.000620**	**(-2.660, 2.646)**	**0.5394**	**0.9822**	**0.7681**	**0.9822**	**9965**
	HL1	-0.004195	(-2.636, 2.601)	0.5258	0.9613	0.7506	0.9613	9995
	HL2	-0.006418	(-2.627, 2.616)	0.5242	0.9595	0.7488	0.9595	10000
	HL3	-0.004729	(-2.635, 2.629)	0.5249	0.9605	0.7487	0.9604	9999
0.4	Mean	0.006585	(-2.305, 2.260)	0.3942	0.7876	0.5927	0.7876	10000
	NM	0.005457	(-2.344, 2.372)	0.4114	0.8104	0.6136	0.8104	9977
	MM	0.007427	(-2.350, 2.338)	0.4098	0.8088	0.6124	0.8088	9977
	MIR	**0.002253**	**(-2.444, 2.396)**	**0.4271**	**0.8318**	**0.6371**	**0.8318**	**9977**
	CSQM	**0.007427**	**(-2.350, 2.338)**	**0.4098**	**0.8088**	**0.6124**	**0.8088**	**9977**
	CQM	**0.007427**	**(-2.350, 2.338)**	**0.4098**	**0.8088**	**0.6124**	**0.8088**	**9977**
	HL1	0.005556	(-2.322, 2.289)	0.3996	0.7955	0.6014	0.7955	9992
	HL2	0.005632	(-2.282, 2.263)	0.3963	0.7914	0.5982	0.7914	9999
	HL3	0.004596	(-2.304, 2.270)	0.3978	0.7932	0.5992	0.7932	9999
0.5	Mean	0.001933	(-1.952, 1.903)	0.2961	0.6522	0.4824	0.6522	10000
	NM	0.001537	(-2.051, 2.013)	0.3169	0.6805	0.4990	0.6805	9982
	MM	0.003351	(-2.018, 1.970)	0.3141	0.6780	0.5027	0.6780	9982
	MIR	**0.008353**	**(-2.102, 2.123)**	**0.3335**	**0.7057**	**0.5228**	**0.7057**	**9982**
	CSQM	**0.003351**	**(-2.018, 1.970)**	**0.3141**	**0.6780**	**0.5027**	**0.6780**	**9982**
	CQM	**0.003351**	**(-2.018, 1.970)**	**0.3141**	**0.6780**	**0.5027**	**0.6780**	**9982**
	HL1	0.002696	(-1.997, 1.919)	0.3021	0.6611	0.4920	0.6611	9992
	HL2	0.002152	(-1.978, 1.915)	0.3007	0.6589	0.4898	0.6589	10000
	HL3	0.002060	(-1.992, 1.917)	0.3018	0.6604	0.4924	0.6604	9999

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Table D.11: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 20$, $\kappa = 1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	-0.003352	(-0.808, 0.806)	0.0805	0.3157	0.2534	0.3157	10000
	NM	-0.001241	(-0.854, 0.872)	0.0918	0.3388	0.2722	0.3389	9997
	MM	-0.002036	(-0.860, 0.876)	0.0929	0.3419	0.2748	0.3419	9997
	MIR	**-0.000502**	**(-0.897, 0.884)**	**0.0982**	**0.3500**	**0.2778**	**0.3500**	**9997**
	CSQM	**-0.002036**	**(-0.860, 0.876)**	**0.0929**	**0.3419**	**0.2748**	**0.3419**	**9997**
	CQM	**-0.002036**	**(-0.860, 0.876)**	**0.0929**	**0.3419**	**0.2748**	**0.3419**	**9997**
	HL1	-0.002652	(-0.837, 0.815)	0.0837	0.3222	0.2555	0.3222	9999
	HL2	-0.002687	(-0.827, 0.816)	0.0831	0.3210	0.2551	0.3210	10000
	HL3	-0.002824	(-0.828, 0.816)	0.0832	0.3212	0.2552	0.3213	10000
1.5	Mean	-0.000790	(-0.554, 0.553)	0.0373	0.2170	0.1796	0.2170	10000
	NM	-0.000545	(-0.598, 0.594)	0.0433	0.2348	0.1948	0.2348	10000
	MM	-0.000789	(-0.608, 0.600)	0.0446	0.2381	0.1971	0.2381	10000
	MIR	**-0.000121**	**(-0.611, 0.610)**	**0.0454**	**0.2397**	**0.1983**	**0.2397**	**10000**
	CSQM	**-0.000789**	**(-0.608, 0.600)**	**0.0446**	**0.2381**	**0.1971**	**0.2381**	**10000**
	CQM	**-0.000789**	**(-0.608, 0.600)**	**0.0446**	**0.2381**	**0.1971**	**0.2381**	**10000**
	HL1	-0.000931	(-0.562, 0.555)	0.0385	0.2207	0.1834	0.2207	10000
	HL2	-0.000945	(-0.560, 0.557)	0.0383	0.2200	0.1839	0.2200	10000
	HL3	-0.000893	(-0.561, 0.557)	0.0384	0.2204	0.1840	0.2204	10000
2.0	Mean	0.000292	(-0.438, 0.429)	0.0240	0.1742	0.1441	0.1742	10000
	NM	0.000364	(-0.479, 0.468)	0.0288	0.1912	0.1599	0.1912	10000
	MM	0.000537	(-0.483, 0.471)	0.0290	0.1920	0.1606	0.1920	10000
	MIR	**0.000024**	**(-0.478, 0.478)**	**0.0288**	**0.1905**	**0.1576**	**0.1905**	**10000**
	CSQM	**0.000537**	**(-0.483, 0.471)**	**0.0290**	**0.1920**	**0.1606**	**0.1920**	**10000**
	CQM	**0.000537**	**(-0.483, 0.471)**	**0.0290**	**0.1920**	**0.1606**	**0.1920**	**10000**
	HL1	0.000356	(-0.445, 0.433)	0.0245	0.1757	0.1447	0.1757	10000
	HL2	0.000418	(-0.439, 0.435)	0.0244	0.1754	0.1448	0.1754	10000
	HL3	0.000427	(-0.443, 0.433)	0.0244	0.1756	0.1454	0.1756	10000
2.5	Mean	0.002325	(-0.376, 0.369)	0.0178	0.1503	0.1275	0.1503	10000
	NM	0.001018	(-0.419, 0.407)	0.0217	0.1665	0.1405	0.1665	10000
	MM	0.000954	(-0.421, 0.407)	0.0219	0.1674	0.1414	0.1674	10000
	MIR	**0.002650**	**(-0.406, 0.403)**	**0.0208**	**0.1624**	**0.1364**	**0.1625**	**10000**
	CSQM	**0.000954**	**(-0.421, 0.407)**	**0.0219**	**0.1674**	**0.1414**	**0.1674**	**10000**
	CQM	**0.000954**	**(-0.421, 0.407)**	**0.0219**	**0.1674**	**0.1414**	**0.1674**	**10000**
	HL1	0.002291	(-0.376, 0.370)	0.0181	0.1515	0.1264	0.1515	10000
	HL2	0.002183	(-0.376, 0.370)	0.0180	0.1512	0.1266	0.1513	10000
	HL3	0.002260	(-0.376, 0.368)	0.0181	0.1513	0.1267	0.1514	10000
3.0	Mean	-0.002276	(-0.332, 0.329)	0.0143	0.1347	0.1137	0.1347	10000
	NM	-0.002135	(-0.369, 0.364)	0.0175	0.1492	0.1250	0.1492	10000
	MM	-0.002408	(-0.369, 0.366)	0.0175	0.1496	0.1254	0.1497	10000
	MIR	**-0.002292**	**(-0.363, 0.356)**	**0.0167**	**0.1447**	**0.1196**	**0.1447**	**10000**
	CSQM	**-0.002408**	**(-0.369, 0.366)**	**0.0175**	**0.1496**	**0.1254**	**0.1497**	**10000**
	CQM	**-0.002408**	**(-0.369, 0.366)**	**0.0175**	**0.1496**	**0.1254**	**0.1497**	**10000**
	HL1	-0.002002	(-0.337, 0.334)	0.0144	0.1355	0.1139	0.1355	10000
	HL2	-0.002106	(-0.333, 0.333)	0.0144	0.1353	0.1128	0.1353	10000
	HL3	-0.002043	(-0.335, 0.333)	0.0144	0.1353	0.1136	0.1354	10000

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Table D.12: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Spread: $(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{U}(-\pi,\pi)$, Where $\epsilon = 0.1$, Sample Size $n = 20$, $\kappa = 4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	-0.001223	(-0.288, 0.284)	0.0104	0.1145	0.0952	0.1145	10000
	NM	-0.000966	(-0.311, 0.308)	0.0123	0.1250	0.1053	0.1250	10000
	MM	-0.000898	(-0.311, 0.308)	0.0124	0.1254	0.1056	0.1254	10000
	MIR	**-0.000897**	**(-0.304, 0.304)**	**0.0118**	**0.1216**	**0.1020**	**0.1216**	**10000**
	CSQM	**-0.000898**	**(-0.311, 0.308)**	**0.0124**	**0.1254**	**0.1056**	**0.1254**	**10000**
	CQM	**-0.000898**	**(-0.311, 0.308)**	**0.0124**	**0.1254**	**0.1056**	**0.1254**	**10000**
	HL1	-0.001361	(-0.285, 0.282)	0.0103	0.1139	0.0947	0.1139	10000
	HL2	-0.001340	(-0.285, 0.283)	0.0102	0.1136	0.0944	0.1136	10000
	HL3	-0.001301	(-0.286, 0.282)	0.0102	0.1137	0.0943	0.1137	10000
5.0	Mean	0.001598	(-0.255, 0.259)	0.0084	0.1025	0.0851	0.1025	10000
	NM	-0.000086	(-0.270, 0.276)	0.0097	0.1111	0.0939	0.1111	10000
	MM	-0.000234	(-0.270, 0.276)	0.0097	0.1112	0.0940	0.1112	10000
	MIR	**0.001150**	**(-0.268, 0.269)**	**0.0093**	**0.1071**	**0.0886**	**0.1071**	**10000**
	CSQM	**-0.000234**	**(-0.270, 0.276)**	**0.0097**	**0.1112**	**0.0940**	**0.1112**	**10000**
	CQM	**-0.000234**	**(-0.270, 0.276)**	**0.0097**	**0.1112**	**0.0940**	**0.1112**	**10000**
	HL1	0.001376	(-0.250, 0.256)	0.0082	0.1015	0.0836	0.1015	10000
	HL2	0.001258	(-0.251, 0.255)	0.0082	0.1012	0.0837	0.1012	10000
	HL3	0.001315	(-0.251, 0.256)	0.0082	0.1013	0.0836	0.1013	10000
6.0	Mean	-0.000291	(-0.233, 0.236)	0.0071	0.0945	0.0793	0.0945	10000
	NM	-0.000368	(-0.245, 0.245)	0.0078	0.1000	0.0847	0.1000	10000
	MM	-0.000449	(-0.246, 0.246)	0.0079	0.1002	0.0849	0.1002	10000
	MIR	**-0.000194**	**(-0.238, 0.243)**	**0.0075**	**0.0974**	**0.0820**	**0.0974**	**10000**
	CSQM	**-0.000449**	**(-0.246, 0.246)**	**0.0079**	**0.1002**	**0.0849**	**0.1002**	**10000**
	CQM	**-0.000449**	**(-0.246, 0.246)**	**0.0079**	**0.1002**	**0.0849**	**0.1002**	**10000**
	HL1	-0.000572	(-0.228, 0.226)	0.0067	0.0922	0.0775	0.0922	10000
	HL2	-0.000554	(-0.228, 0.226)	0.0066	0.0919	0.0775	0.0919	10000
	HL3	-0.000553	(-0.228, 0.225)	0.0067	0.0920	0.0777	0.0920	10000
7.0	Mean	-0.000666	(-0.221, 0.222)	0.0063	0.0885	0.0739	0.0885	10000
	NM	-0.000581	(-0.226, 0.231)	0.0068	0.0926	0.0774	0.0926	10000
	MM	-0.000577	(-0.227, 0.232)	0.0068	0.0929	0.0776	0.0929	10000
	MIR	**-0.000615**	**(-0.222, 0.224)**	**0.0065**	**0.0896**	**0.0741**	**0.0897**	**10000**
	CSQM	**-0.000577**	**(-0.227, 0.232)**	**0.0068**	**0.0929**	**0.0776**	**0.0929**	**10000**
	CQM	**-0.000577**	**(-0.227, 0.232)**	**0.0068**	**0.0929**	**0.0776**	**0.0929**	**10000**
	HL1	-0.000664	(-0.211, 0.211)	0.0058	0.0853	0.0715	0.0853	10000
	HL2	-0.000714	(-0.211, 0.210)	0.0058	0.0850	0.0715	0.0850	10000
	HL3	-0.000709	(-0.211, 0.210)	0.0058	0.0851	0.0715	0.0851	10000
8.0	Mean	-0.001169	(-0.213, 0.212)	0.0058	0.0854	0.0717	0.0854	10000
	NM	-0.001219	(-0.220, 0.220)	0.0061	0.0881	0.0736	0.0881	10000
	MM	-0.001224	(-0.220, 0.220)	0.0061	0.0883	0.0738	0.0883	10000
	MIR	**-0.000887**	**(-0.210, 0.206)**	**0.0056**	**0.0838**	**0.0697**	**0.0838**	**10000**
	CSQM	**-0.001224**	**(-0.220, 0.220)**	**0.0061**	**0.0883**	**0.0738**	**0.0883**	**10000**
	CQM	**-0.001224**	**(-0.220, 0.220)**	**0.0061**	**0.0883**	**0.0738**	**0.0883**	**10000**
	HL1	-0.001099	(-0.203, 0.199)	0.0052	0.0811	0.0682	0.0811	10000
	HL2	-0.001139	(-0.202, 0.197)	0.0051	0.0809	0.0680	0.0809	10000
	HL3	-0.001096	(-0.202, 0.198)	0.0052	0.0810	0.0680	0.0810	10000

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Appendix E

Contamination on Location

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Table E.1: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	-0.075853	(-2.938, 2.943)	0.8871	1.4263	1.3550	1.4267	10000
	NM	-0.044406	(-2.940, 2.947)	0.8885	1.4286	1.3598	1.4287	9528
	MM	-0.045180	(-2.934, 2.948)	0.8846	1.4235	1.3525	1.4235	9528
	MIR	**-0.065577**	**(-2.953, 2.944)**	**0.8834**	**1.4222**	**1.3422**	**1.4227**	**9528**
	CSQM	**-0.053832**	**(-2.941, 2.936)**	**0.8823**	**1.4207**	**1.3395**	**1.4206**	**9528**
	CQM	**-0.058853**	**(-2.935, 2.946)**	**0.8829**	**1.4209**	**1.3385**	**1.4207**	**9528**
	HL1	-0.112431	(-2.947, 2.914)	0.8876	1.4267	1.3560	1.4275	9863
	HL2	-0.104013	(-2.946, 2.925)	0.8877	1.4271	1.3619	1.4280	10000
	HL3	-0.114408	(-2.947, 2.923)	0.8883	1.4277	1.3596	1.4289	9994
0.2	Mean	0.016475	(-2.869, 2.902)	0.7820	1.2925	1.1604	1.2926	10000
	NM	0.010658	(-2.903, 2.877)	0.7790	1.2901	1.1544	1.2901	9513
	MM	0.023375	(-2.908, 2.864)	0.7755	1.2854	1.1407	1.2853	9513
	MIR	**0.025123**	**(-2.888, 2.858)**	**0.7805**	**1.2889**	**1.1626**	**1.2889**	**9513**
	CSQM	**0.018254**	**(-2.878, 2.867)**	**0.7755**	**1.2838**	**1.1460**	**1.2837**	**9513**
	CQM	**0.014660**	**(-2.874, 2.873)**	**0.7790**	**1.2877**	**1.1582**	**1.2877**	**9513**
	HL1	0.010985	(-2.878, 2.898)	0.7884	1.3001	1.1793	1.3001	9869
	HL2	0.020842	(-2.873, 2.900)	0.7868	1.2987	1.1729	1.2987	10000
	HL3	0.009279	(-2.878, 2.897)	0.7883	1.3007	1.1761	1.3007	9997
0.3	Mean	0.083815	(-2.808, 2.812)	0.6663	1.1433	0.9604	1.1452	10000
	NM	0.090116	(-2.813, 2.829)	0.6708	1.1510	0.9548	1.1523	9537
	MM	0.082370	(-2.800, 2.823)	0.6709	1.1499	0.9622	1.1510	9537
	MIR	**0.083257**	**(-2.796, 2.815)**	**0.6694**	**1.1492**	**0.9595**	**1.1504**	**9537**
	CSQM	**0.080829**	**(-2.824, 2.804)**	**0.6641**	**1.1415**	**0.9517**	**1.1428**	**9537**
	CQM	**0.079776**	**(-2.809, 2.814)**	**0.6655**	**1.1442**	**0.9549**	**1.1451**	**9537**
	HL1	0.088937	(-2.817, 2.827)	0.6754	1.1559	0.9747	1.1574	9873
	HL2	0.086920	(-2.804, 2.829)	0.6723	1.1512	0.9660	1.1528	10000
	HL3	0.090925	(-2.827, 2.827)	0.6737	1.1529	0.9714	1.1546	9995
0.4	Mean	0.078109	(-2.677, 2.687)	0.5657	1.0128	0.8139	1.0151	10000
	NM	0.080759	(-2.685, 2.705)	0.5724	1.0234	0.8217	1.0253	9589
	MM	0.076342	(-2.684, 2.702)	0.5727	1.0222	0.8184	1.0240	9589
	MIR	**0.073974**	**(-2.647, 2.709)**	**0.5715**	**1.0211**	**0.8155**	**1.0229**	**9589**
	CSQM	**0.073165**	**(-2.688, 2.699)**	**0.5636**	**1.0102**	**0.8074**	**1.0121**	**9589**
	CQM	**0.072177**	**(-2.677, 2.693)**	**0.5652**	**1.0131**	**0.8055**	**1.0145**	**9589**
	HL1	0.078658	(-2.695, 2.708)	0.5779	1.0287	0.8201	1.0304	9894
	HL2	0.074978	(-2.687, 2.726)	0.5711	1.0194	0.8090	1.0213	10000
	HL3	0.075104	(-2.695, 2.712)	0.5751	1.0244	0.8156	1.0262	9998
0.5	Mean	0.059386	(-2.505, 2.578)	0.4901	0.9149	0.7215	0.9162	10000
	NM	0.056304	(-2.517, 2.579)	0.4978	0.9257	0.7311	0.9264	9659
	MM	0.052581	(-2.539, 2.536)	0.4953	0.9221	0.7223	0.9227	9659
	MIR	**0.051561**	**(-2.561, 2.562)**	**0.4945**	**0.9217**	**0.7192**	**0.9224**	**9659**
	CSQM	**0.051254**	**(-2.549, 2.532)**	**0.4862**	**0.9100**	**0.7099**	**0.9108**	**9659**
	CQM	**0.051863**	**(-2.550, 2.547)**	**0.4886**	**0.9141**	**0.7148**	**0.9149**	**9659**
	HL1	0.058809	(-2.548, 2.581)	0.5025	0.9315	0.7365	0.9326	9903
	HL2	0.059123	(-2.498, 2.587)	0.4960	0.9230	0.7247	0.9242	10000
	HL3	0.062657	(-2.530, 2.589)	0.4995	0.9273	0.7332	0.9284	9998

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Table E.2: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon=0.1$, Sample Size $n=7$, $\kappa=1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.065908	(-1.366, 1.474)	0.1993	0.5140	0.3910	0.5171	10000
	NM	0.065217	(-1.433, 1.497)	0.2157	0.5405	0.4189	0.5432	9848
	MM	0.069824	(-1.433, 1.525)	0.2174	0.5438	0.4211	0.5466	9848
	MIR	**0.069437**	**(-1.381, 1.504)**	**0.2076**	**0.5269**	**0.4026**	**0.5299**	**9848**
	CSQM	**0.069480**	**(-1.367, 1.479)**	**0.2031**	**0.5206**	**0.4009**	**0.5240**	**9848**
	CQM	**0.068930**	**(-1.376, 1.488)**	**0.2043**	**0.5227**	**0.4013**	**0.5258**	**9848**
	HL1	0.065621	(-1.443, 1.500)	0.2106	0.5320	0.4053	0.5349	9958
	HL2	0.064175	(-1.421, 1.472)	0.2054	0.5235	0.3965	0.5265	10000
	HL3	0.063165	(-1.420, 1.478)	0.2070	0.5262	0.3993	0.5292	9997
1.5	Mean	0.071416	(-0.833, 0.965)	0.0983	0.3527	0.2845	0.3575	10000
	NM	0.071891	(-0.894, 1.045)	0.1133	0.3829	0.3094	0.3871	9964
	MM	0.071837	(-0.906, 1.048)	0.1140	0.3840	0.3107	0.3884	9964
	MIR	**0.072317**	**(-0.871, 1.016)**	**0.1042**	**0.3636**	**0.2935**	**0.3684**	**9964**
	CSQM	**0.071190**	**(-0.855, 0.992)**	**0.1034**	**0.3639**	**0.2950**	**0.3683**	**9964**
	CQM	**0.070691**	**(-0.857, 1.001)**	**0.1049**	**0.3673**	**0.2966**	**0.3714**	**9964**
	HL1	0.073278	(-0.872, 0.991)	0.1041	0.3637	0.2913	0.3687	9985
	HL2	0.072162	(-0.849, 0.991)	0.1013	0.3589	0.2874	0.3637	10000
	HL3	0.073345	(-0.854, 0.990)	0.1027	0.3614	0.2898	0.3663	9999
2.0	Mean	0.075644	(-0.609, 0.765)	0.0589	0.2739	0.2285	0.2800	10000
	NM	0.074385	(-0.671, 0.846)	0.0725	0.3071	0.2567	0.3124	9992
	MM	0.075150	(-0.672, 0.848)	0.0730	0.3078	0.2570	0.3132	9992
	MIR	**0.075988**	**(-0.621, 0.780)**	**0.0620**	**0.2802**	**0.2323**	**0.2866**	**9992**
	CSQM	**0.074821**	**(-0.630, 0.799)**	**0.0638**	**0.2866**	**0.2394**	**0.2922**	**9992**
	CQM	**0.074384**	**(-0.633, 0.798)**	**0.0646**	**0.2881**	**0.2407**	**0.2936**	**9992**
	HL1	0.075807	(-0.640, 0.778)	0.0627	0.2815	0.2304	0.2876	9999
	HL2	0.074252	(-0.626, 0.783)	0.0608	0.2778	0.2299	0.2840	10000
	HL3	0.074594	(-0.630, 0.777)	0.0616	0.2796	0.2305	0.2855	10000
2.5	Mean	0.070146	(-0.510, 0.653)	0.0425	0.2327	0.1957	0.2397	10000
	NM	0.063118	(-0.594, 0.733)	0.0557	0.2671	0.2241	0.2721	9999
	MM	0.063005	(-0.594, 0.734)	0.0558	0.2674	0.2247	0.2724	9999
	MIR	**0.070893**	**(-0.526, 0.662)**	**0.0442**	**0.2369**	**0.1980**	**0.2437**	**9999**
	CSQM	**0.065786**	**(-0.539, 0.672)**	**0.0471**	**0.2453**	**0.2025**	**0.2507**	**9999**
	CQM	**0.067316**	**(-0.539, 0.681)**	**0.0472**	**0.2451**	**0.2027**	**0.2506**	**9999**
	HL1	0.072542	(-0.528, 0.668)	0.0452	0.2395	0.2001	0.2466	10000
	HL2	0.069560	(-0.516, 0.659)	0.0439	0.2364	0.1964	0.2432	10000
	HL3	0.070920	(-0.525, 0.662)	0.0443	0.2374	0.1987	0.2442	10000
3.0	Mean	0.071799	(-0.446, 0.595)	0.0339	0.2076	0.1726	0.2151	10000
	NM	0.066299	(-0.535, 0.677)	0.0452	0.2407	0.2010	0.2463	10000
	MM	0.066670	(-0.535, 0.679)	0.0454	0.2412	0.2014	0.2468	10000
	MIR	**0.070599**	**(-0.455, 0.603)**	**0.0358**	**0.2134**	**0.1784**	**0.2203**	**10000**
	CSQM	**0.067076**	**(-0.483, 0.630)**	**0.0386**	**0.2218**	**0.1867**	**0.2278**	**10000**
	CQM	**0.067367**	**(-0.482, 0.626)**	**0.0389**	**0.2232**	**0.1862**	**0.2285**	**10000**
	HL1	0.072060	(-0.462, 0.604)	0.0359	0.2136	0.1780	0.2213	10000
	HL2	0.070639	(-0.459, 0.602)	0.0354	0.2120	0.1761	0.2196	10000
	HL3	0.071532	(-0.459, 0.608)	0.0357	0.2129	0.1771	0.2204	10000

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Table E.3: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 7$, $\kappa = 4.0, 5.0, 6.0, 7.0, 8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.073341	(-0.349, 0.504)	0.0239	0.1750	0.1476	0.1843	10000
	NM	0.065196	(-0.433, 0.572)	0.0325	0.2038	0.1695	0.2098	10000
	MM	0.065161	(-0.433, 0.572)	0.0325	0.2038	0.1695	0.2098	10000
	MIR	**0.070398**	**(-0.361, 0.512)**	**0.0251**	**0.1793**	**0.1507**	**0.1875**	**10000**
	CSQM	**0.066082**	**(-0.385, 0.531)**	**0.0274**	**0.1870**	**0.1582**	**0.1935**	**10000**
	CQM	**0.066552**	**(-0.386, 0.536)**	**0.0276**	**0.1877**	**0.1572**	**0.1945**	**10000**
	HL1	0.073272	(-0.361, 0.521)	0.0250	0.1796	0.1522	0.1886	10000
	HL2	0.070606	(-0.362, 0.514)	0.0249	0.1786	0.1502	0.1871	10000
	HL3	0.071478	(-0.362, 0.520)	0.0250	0.1791	0.1513	0.1875	10000
5.0	Mean	0.074403	(-0.298, 0.466)	0.0190	0.1558	0.1323	0.1666	10000
	NM	0.061925	(-0.385, 0.522)	0.0261	0.1822	0.1522	0.1879	10000
	MM	0.061916	(-0.385, 0.522)	0.0261	0.1823	0.1523	0.1879	10000
	MIR	**0.069433**	**(-0.313, 0.473)**	**0.0201**	**0.1602**	**0.1342**	**0.1688**	**10000**
	CSQM	**0.063234**	**(-0.334, 0.480)**	**0.0219**	**0.1670**	**0.1410**	**0.1734**	**10000**
	CQM	**0.063918**	**(-0.335, 0.489)**	**0.0220**	**0.1676**	**0.1405**	**0.1743**	**10000**
	HL1	0.073603	(-0.309, 0.478)	0.0200	0.1599	0.1346	0.1700	10000
	HL2	0.069326	(-0.313, 0.474)	0.0199	0.1594	0.1337	0.1681	10000
	HL3	0.070670	(-0.312, 0.477)	0.0199	0.1598	0.1342	0.1688	10000
6.0	Mean	0.075718	(-0.266, 0.440)	0.0163	0.1441	0.1232	0.1560	10000
	NM	0.060111	(-0.345, 0.489)	0.0222	0.1678	0.1408	0.1733	10000
	MM	0.060111	(-0.345, 0.489)	0.0222	0.1678	0.1408	0.1733	10000
	MIR	**0.069878**	**(-0.280, 0.445)**	**0.0172**	**0.1478**	**0.1249**	**0.1572**	**10000**
	CSQM	**0.062521**	**(-0.302, 0.450)**	**0.0187**	**0.1539**	**0.1304**	**0.1607**	**10000**
	CQM	**0.063759**	**(-0.301, 0.458)**	**0.0189**	**0.1545**	**0.1288**	**0.1616**	**10000**
	HL1	0.074217	(-0.275, 0.451)	0.0172	0.1481	0.1254	0.1589	10000
	HL2	0.069552	(-0.279, 0.442)	0.0171	0.1474	0.1251	0.1568	10000
	HL3	0.071218	(-0.277, 0.450)	0.0171	0.1479	0.1253	0.1575	10000
7.0	Mean	0.076902	(-0.243, 0.421)	0.0144	0.1354	0.1153	0.1479	10000
	NM	0.059147	(-0.316, 0.456)	0.0192	0.1556	0.1297	0.1614	10000
	MM	0.059147	(-0.316, 0.456)	0.0192	0.1556	0.1297	0.1614	10000
	MIR	**0.069748**	**(-0.258, 0.426)**	**0.0152**	**0.1385**	**0.1168**	**0.1485**	**10000**
	CSQM	**0.061531**	**(-0.277, 0.424)**	**0.0162**	**0.1429**	**0.1186**	**0.1499**	**10000**
	CQM	**0.062749**	**(-0.276, 0.437)**	**0.0163**	**0.1436**	**0.1187**	**0.1509**	**10000**
	HL1	0.074175	(-0.255, 0.432)	0.0152	0.1392	0.1176	0.1504	10000
	HL2	0.069216	(-0.256, 0.420)	0.0150	0.1379	0.1157	0.1477	10000
	HL3	0.071019	(-0.255, 0.426)	0.0151	0.1384	0.1165	0.1487	10000
8.0	Mean	0.075419	(-0.230, 0.410)	0.0130	0.1290	0.1089	0.1414	10000
	NM	0.053942	(-0.296, 0.432)	0.0170	0.1465	0.1237	0.1518	10000
	MM	0.053942	(-0.296, 0.432)	0.0170	0.1465	0.1237	0.1518	10000
	MIR	**0.066478**	**(-0.245, 0.412)**	**0.0136**	**0.1323**	**0.1115**	**0.1410**	**10000**
	CSQM	**0.056881**	**(-0.267, 0.409)**	**0.0144**	**0.1351**	**0.1134**	**0.1416**	**10000**
	CQM	**0.058388**	**(-0.266, 0.409)**	**0.0146**	**0.1361**	**0.1140**	**0.1429**	**10000**
	HL1	0.072441	(-0.243, 0.422)	0.0139	0.1331	0.1114	0.1437	10000
	HL2	0.066286	(-0.246, 0.404)	0.0135	0.1314	0.1110	0.1406	10000
	HL3	0.068760	(-0.247, 0.412)	0.0137	0.1322	0.1110	0.1420	10000

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Table E.4: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 8$, $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.026061	(-2.938, 2.950)	0.8887	1.4281	1.3630	1.4281	10000
	NM	0.041354	(-2.957, 2.948)	0.8935	1.4351	1.3653	1.4351	9987
	MM	0.012939	(-2.964, 2.948)	0.8915	1.4323	1.3598	1.4323	9987
	MIR	**0.020391**	**(-2.919, 2.946)**	**0.8951**	**1.4355**	**1.3642**	**1.4355**	**9987**
	CSQM	**0.012939**	**(-2.964, 2.948)**	**0.8915**	**1.4323**	**1.3598**	**1.4323**	**9987**
	CQM	**0.012939**	**(-2.964, 2.948)**	**0.8915**	**1.4323**	**1.3598**	**1.4323**	**9987**
	HL1	0.044756	(-2.937, 2.935)	0.8938	1.4328	1.3716	1.4331	9986
	HL2	0.036882	(-2.927, 2.932)	0.8930	1.4320	1.3650	1.4320	10000
	HL3	0.049398	(-2.937, 2.929)	0.8938	1.4324	1.3715	1.4325	9998
0.2	Mean	0.063212	(-2.851, 2.880)	0.7626	1.2655	1.1221	1.2663	10000
	NM	0.068717	(-2.901, 2.887)	0.7714	1.2778	1.1357	1.2782	9984
	MM	0.057709	(-2.892, 2.875)	0.7716	1.2774	1.1436	1.2781	9984
	MIR	**0.053210**	**(-2.870, 2.897)**	**0.7799**	**1.2885**	**1.1399**	**1.2888**	**9984**
	CSQM	**0.057709**	**(-2.892, 2.875)**	**0.7716**	**1.2774**	**1.1436**	**1.2781**	**9984**
	CQM	**0.057709**	**(-2.892, 2.875)**	**0.7716**	**1.2774**	**1.1436**	**1.2781**	**9984**
	HL1	0.060273	(-2.871, 2.865)	0.7701	1.2748	1.1353	1.2752	9991
	HL2	0.062818	(-2.863, 2.858)	0.7669	1.2707	1.1331	1.2712	10000
	HL3	0.058682	(-2.871, 2.858)	0.7675	1.2714	1.1356	1.2718	9999
0.3	Mean	0.077567	(-2.813, 2.787)	0.6552	1.1294	0.9454	1.1303	10000
	NM	0.075798	(-2.819, 2.828)	0.6709	1.1501	0.9621	1.1512	9989
	MM	0.073332	(-2.814, 2.796)	0.6659	1.1427	0.9597	1.1434	9989
	MIR	**0.078344**	**(-2.768, 2.787)**	**0.6684**	**1.1442**	**0.9633**	**1.1457**	**9989**
	CSQM	**0.073332**	**(-2.814, 2.796)**	**0.6659**	**1.1427**	**0.9597**	**1.1434**	**9989**
	CQM	**0.073332**	**(-2.814, 2.796)**	**0.6659**	**1.1427**	**0.9597**	**1.1434**	**9989**
	HL1	0.082525	(-2.790, 2.798)	0.6618	1.1384	0.9599	1.1400	9997
	HL2	0.080377	(-2.781, 2.797)	0.6595	1.1346	0.9544	1.1360	10000
	HL3	0.080144	(-2.794, 2.795)	0.6599	1.1357	0.9583	1.1372	9997
0.4	Mean	0.074953	(-2.630, 2.641)	0.5377	0.9771	0.7803	0.9789	10000
	NM	0.084669	(-2.610, 2.718)	0.5577	1.0028	0.8065	1.0051	9981
	MM	0.077577	(-2.651, 2.654)	0.5511	0.9953	0.7955	0.9973	9981
	MIR	**0.072294**	**(-2.663, 2.673)**	**0.5607**	**1.0069**	**0.7993**	**1.0085**	**9981**
	CSQM	**0.077577**	**(-2.651, 2.654)**	**0.5511**	**0.9953**	**0.7955**	**0.9973**	**9981**
	CQM	**0.077577**	**(-2.651, 2.654)**	**0.5511**	**0.9953**	**0.7955**	**0.9973**	**9981**
	HL1	0.067856	(-2.576, 2.694)	0.5472	0.9887	0.7881	0.9904	9993
	HL2	0.069761	(-2.600, 2.651)	0.5440	0.9849	0.7864	0.9866	9999
	HL3	0.067980	(-2.591, 2.680)	0.5455	0.9865	0.7856	0.9881	9998
0.5	Mean	0.077894	(-2.414, 2.536)	0.4548	0.8675	0.6681	0.8701	10000
	NM	0.072475	(-2.477, 2.546)	0.4766	0.8952	0.6902	0.8972	9990
	MM	0.077914	(-2.486, 2.518)	0.4713	0.8891	0.6861	0.8916	9990
	MIR	**0.090568**	**(-2.461, 2.556)**	**0.4741**	**0.8915**	**0.6864**	**0.8947**	**9990**
	CSQM	**0.077914**	**(-2.486, 2.518)**	**0.4713**	**0.8891**	**0.6861**	**0.8916**	**9990**
	CQM	**0.077914**	**(-2.486, 2.518)**	**0.4713**	**0.8891**	**0.6861**	**0.8916**	**9990**
	HL1	0.076285	(-2.431, 2.554)	0.4636	0.8795	0.6742	0.8817	9996
	HL2	0.077522	(-2.405, 2.545)	0.4595	0.8744	0.6679	0.8767	10000
	HL3	0.077871	(-2.434, 2.541)	0.4612	0.8766	0.6673	0.8788	9997

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Table E.5: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 8$, $\kappa = 1.0, 1.5, 2.0, 2.5, 3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.067270	(-1.277, 1.355)	0.1793	0.4821	0.3644	0.4851	10000
	NM	0.069496	(-1.372, 1.474)	0.1974	0.5106	0.3841	0.5143	9993
	MM	0.070188	(-1.355, 1.453)	0.1938	0.5060	0.3856	0.5096	9993
	MIR	**0.065969**	**(-1.353, 1.450)**	**0.1981**	**0.5121**	**0.3914**	**0.5153**	**9993**
	CSQM	**0.070188**	**(-1.355, 1.453)**	**0.1938**	**0.5060**	**0.3856**	**0.5096**	**9993**
	CQM	**0.070188**	**(-1.355, 1.453)**	**0.1938**	**0.5060**	**0.3856**	**0.5096**	**9993**
	HL1	0.065520	(-1.318, 1.397)	0.1882	0.4971	0.3778	0.4999	9997
	HL2	0.066545	(-1.298, 1.388)	0.1846	0.4912	0.3722	0.4941	9999
	HL3	0.065904	(-1.306, 1.409)	0.1858	0.4933	0.3748	0.4961	10000
1.5	Mean	0.074409	(-0.753, 0.887)	0.0818	0.3222	0.2647	0.3282	10000
	NM	0.074347	(-0.802, 0.952)	0.0947	0.3464	0.2779	0.3522	10000
	MM	0.074098	(-0.805, 0.946)	0.0932	0.3455	0.2794	0.3513	10000
	MIR	**0.078478**	**(-0.801, 0.953)**	**0.0922**	**0.3413**	**0.2754**	**0.3471**	**10000**
	CSQM	**0.074098**	**(-0.805, 0.946)**	**0.0932**	**0.3455**	**0.2794**	**0.3513**	**10000**
	CQM	**0.074098**	**(-0.805, 0.946)**	**0.0932**	**0.3455**	**0.2794**	**0.3513**	**10000**
	HL1	0.076544	(-0.760, 0.901)	0.0858	0.3298	0.2692	0.3356	10000
	HL2	0.076405	(-0.764, 0.899)	0.0848	0.3281	0.2673	0.3342	10000
	HL3	0.076119	(-0.761, 0.901)	0.0856	0.3292	0.2687	0.3354	10000
2.0	Mean	0.074704	(-0.563, 0.723)	0.0510	0.2549	0.2125	0.2621	10000
	NM	0.070489	(-0.608, 0.764)	0.0592	0.2755	0.2307	0.2811	10000
	MM	0.070276	(-0.610, 0.769)	0.0599	0.2774	0.2319	0.2831	10000
	MIR	**0.076043**	**(-0.583, 0.751)**	**0.0561**	**0.2672**	**0.2207**	**0.2740**	**10000**
	CSQM	**0.070276**	**(-0.610, 0.769)**	**0.0599**	**0.2774**	**0.2319**	**0.2831**	**10000**
	CQM	**0.070276**	**(-0.610, 0.769)**	**0.0599**	**0.2774**	**0.2319**	**0.2831**	**10000**
	HL1	0.075324	(-0.564, 0.729)	0.0529	0.2599	0.2147	0.2668	10000
	HL2	0.074409	(-0.558, 0.721)	0.0523	0.2583	0.2161	0.2650	10000
	HL3	0.074549	(-0.557, 0.729)	0.0526	0.2591	0.2151	0.2659	10000
2.5	Mean	0.071055	(-0.471, 0.611)	0.0369	0.2174	0.1817	0.2248	10000
	NM	0.062037	(-0.525, 0.663)	0.0441	0.2378	0.1994	0.2430	10000
	MM	0.062195	(-0.529, 0.662)	0.0442	0.2380	0.1993	0.2433	10000
	MIR	**0.074182**	**(-0.480, 0.636)**	**0.0396**	**0.2248**	**0.1885**	**0.2329**	**10000**
	CSQM	**0.062195**	**(-0.529, 0.662)**	**0.0442**	**0.2380**	**0.1993**	**0.2433**	**10000**
	CQM	**0.062195**	**(-0.529, 0.662)**	**0.0442**	**0.2380**	**0.1993**	**0.2433**	**10000**
	HL1	0.071254	(-0.474, 0.623)	0.0386	0.2220	0.1844	0.2294	10000
	HL2	0.069406	(-0.476, 0.624)	0.0382	0.2213	0.1855	0.2284	10000
	HL3	0.070050	(-0.476, 0.627)	0.0384	0.2219	0.1847	0.2291	10000
3.0	Mean	0.074277	(-0.392, 0.556)	0.0294	0.1953	0.1652	0.2035	10000
	NM	0.067540	(-0.457, 0.606)	0.0360	0.2151	0.1801	0.2213	10000
	MM	0.067457	(-0.458, 0.606)	0.0361	0.2153	0.1802	0.2215	10000
	MIR	**0.075237**	**(-0.408, 0.576)**	**0.0315**	**0.2022**	**0.1711**	**0.2101**	**10000**
	CSQM	**0.067457**	**(-0.458, 0.606)**	**0.0361**	**0.2153**	**0.1802**	**0.2215**	**10000**
	CQM	**0.067457**	**(-0.458, 0.606)**	**0.0361**	**0.2153**	**0.1802**	**0.2215**	**10000**
	HL1	0.074460	(-0.401, 0.566)	0.0306	0.1989	0.1689	0.2068	10000
	HL2	0.073513	(-0.403, 0.567)	0.0306	0.1986	0.1684	0.2064	10000
	HL3	0.073626	(-0.402, 0.567)	0.0307	0.1992	0.1688	0.2069	10000

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Table E.6: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon=0.1$, Sample Size $n=8$, $\kappa=4.0,5.0,6.0,7.0,8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.073785	(-0.329, 0.482)	0.0213	0.1650	0.1393	0.1752	10000
	NM	0.061721	(-0.381, 0.527)	0.0262	0.1832	0.1530	0.1889	10000
	MM	0.061752	(-0.381, 0.527)	0.0262	0.1832	0.1529	0.1888	10000
	MIR	**0.073752**	**(-0.348, 0.497)**	**0.0228**	**0.1709**	**0.1439**	**0.1804**	**10000**
	CSQM	**0.061752**	**(-0.381, 0.527)**	**0.0262**	**0.1832**	**0.1529**	**0.1888**	**10000**
	CQM	**0.061752**	**(-0.381, 0.527)**	**0.0262**	**0.1832**	**0.1529**	**0.1888**	**10000**
	HL1	0.072469	(-0.342, 0.490)	0.0221	0.1683	0.1422	0.1776	10000
	HL2	0.070898	(-0.345, 0.490)	0.0221	0.1680	0.1409	0.1770	10000
	HL3	0.071275	(-0.344, 0.491)	0.0222	0.1684	0.1414	0.1774	10000
5.0	Mean	0.071875	(-0.287, 0.451)	0.0172	0.1484	0.1245	0.1583	10000
	NM	0.058443	(-0.345, 0.481)	0.0216	0.1653	0.1375	0.1709	10000
	MM	0.058437	(-0.345, 0.481)	0.0216	0.1653	0.1375	0.1709	10000
	MIR	**0.069682**	**(-0.301, 0.467)**	**0.0186**	**0.1547**	**0.1311**	**0.1634**	**10000**
	CSQM	**0.058437**	**(-0.345, 0.481)**	**0.0216**	**0.1653**	**0.1375**	**0.1709**	**10000**
	CQM	**0.058437**	**(-0.345, 0.481)**	**0.0216**	**0.1653**	**0.1375**	**0.1709**	**10000**
	HL1	0.068510	(-0.301, 0.461)	0.0181	0.1518	0.1268	0.1602	10000
	HL2	0.067119	(-0.301, 0.459)	0.0180	0.1513	0.1261	0.1595	10000
	HL3	0.067446	(-0.300, 0.459)	0.0181	0.1516	0.1263	0.1598	10000
6.0	Mean	0.075981	(-0.255, 0.425)	0.0151	0.1395	0.1191	0.1517	10000
	NM	0.060703	(-0.310, 0.452)	0.0184	0.1528	0.1276	0.1595	10000
	MM	0.060703	(-0.310, 0.452)	0.0184	0.1528	0.1276	0.1595	10000
	MIR	**0.072344**	**(-0.270, 0.437)**	**0.0162**	**0.1443**	**0.1226**	**0.1548**	**10000**
	CSQM	**0.060703**	**(-0.310, 0.452)**	**0.0184**	**0.1528**	**0.1276**	**0.1595**	**10000**
	CQM	**0.060703**	**(-0.310, 0.452)**	**0.0184**	**0.1528**	**0.1276**	**0.1595**	**10000**
	HL1	0.071891	(-0.269, 0.432)	0.0157	0.1422	0.1210	0.1526	10000
	HL2	0.070137	(-0.267, 0.427)	0.0157	0.1420	0.1198	0.1517	10000
	HL3	0.070666	(-0.268, 0.429)	0.0157	0.1421	0.1201	0.1521	10000
7.0	Mean	0.074525	(-0.234, 0.403)	0.0133	0.1305	0.1098	0.1427	10000
	NM	0.056647	(-0.283, 0.424)	0.0161	0.1430	0.1212	0.1487	10000
	MM	0.056647	(-0.283, 0.424)	0.0161	0.1430	0.1212	0.1487	10000
	MIR	**0.069737**	**(-0.246, 0.418)**	**0.0143**	**0.1349**	**0.1138**	**0.1449**	**10000**
	CSQM	**0.056647**	**(-0.283, 0.424)**	**0.0161**	**0.1430**	**0.1212**	**0.1487**	**10000**
	CQM	**0.056647**	**(-0.283, 0.424)**	**0.0161**	**0.1430**	**0.1212**	**0.1487**	**10000**
	HL1	0.068985	(-0.244, 0.407)	0.0138	0.1329	0.1130	0.1426	10000
	HL2	0.066973	(-0.248, 0.400)	0.0138	0.1327	0.1120	0.1417	10000
	HL3	0.067563	(-0.247, 0.402)	0.0138	0.1330	0.1126	0.1421	10000
8.0	Mean	0.073090	(-0.218, 0.387)	0.0117	0.1222	0.1038	0.1346	10000
	NM	0.053023	(-0.261, 0.399)	0.0141	0.1336	0.1122	0.1390	10000
	MM	0.053023	(-0.261, 0.399)	0.0141	0.1336	0.1122	0.1390	10000
	MIR	**0.067946**	**(-0.227, 0.399)**	**0.0126**	**0.1266**	**0.1069**	**0.1365**	**10000**
	CSQM	**0.053023**	**(-0.261, 0.399)**	**0.0141**	**0.1336**	**0.1122**	**0.1390**	**10000**
	CQM	**0.053023**	**(-0.261, 0.399)**	**0.0141**	**0.1336**	**0.1122**	**0.1390**	**10000**
	HL1	0.066628	(-0.225, 0.391)	0.0122	0.1250	0.1059	0.1345	10000
	HL2	0.064483	(-0.228, 0.383)	0.0122	0.1248	0.1057	0.1335	10000
	HL3	0.065039	(-0.227, 0.385)	0.0122	0.1250	0.1059	0.1340	10000

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Table E.7: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 19$, $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.114894	(-2.917, 2.919)	0.8278	1.3519	1.2310	1.3529	10000
	NM	0.110049	(-2.911, 2.899)	0.8250	1.3485	1.2276	1.3495	9817
	MM	0.117378	(-2.923, 2.904)	0.8293	1.3543	1.2279	1.3556	9817
	MIR	**0.090792**	**(-2.917, 2.927)**	**0.8320**	**1.3565**	**1.2395**	**1.3581**	**9817**
	CSQM	**0.119002**	**(-2.931, 2.898)**	**0.8277**	**1.3527**	**1.2278**	**1.3541**	**9817**
	CQM	**0.119812**	**(-2.927, 2.903)**	**0.8275**	**1.3524**	**1.2327**	**1.3540**	**9817**
	HL1	0.103837	(-2.915, 2.902)	0.8287	1.3525	1.2402	1.3535	9989
	HL2	0.097716	(-2.929, 2.900)	0.8303	1.3550	1.2414	1.3558	10000
	HL3	0.102945	(-2.928, 2.902)	0.8298	1.3542	1.2397	1.3552	9990
0.2	Mean	0.060808	(-2.796, 2.779)	0.6455	1.1168	0.9280	1.1178	10000
	NM	0.071508	(-2.812, 2.794)	0.6527	1.1265	0.9333	1.1278	9829
	MM	0.060169	(-2.776, 2.798)	0.6539	1.1279	0.9283	1.1290	9829
	MIR	**0.066249**	**(-2.798, 2.782)**	**0.6532**	**1.1265**	**0.9229**	**1.1278**	**9829**
	CSQM	**0.061288**	**(-2.785, 2.773)**	**0.6515**	**1.1245**	**0.9304**	**1.1254**	**9829**
	CQM	**0.061956**	**(-2.779, 2.790)**	**0.6512**	**1.1242**	**0.9296**	**1.1251**	**9829**
	HL1	0.059241	(-2.755, 2.811)	0.6488	1.1205	0.9262	1.1217	9988
	HL2	0.064432	(-2.762, 2.798)	0.6491	1.1206	0.9234	1.1220	9998
	HL3	0.061757	(-2.758, 2.807)	0.6490	1.1206	0.9247	1.1219	9994
0.3	Mean	0.083457	(-2.490, 2.551)	0.4855	0.9069	0.6968	0.9091	10000
	NM	0.084500	(-2.513, 2.602)	0.4970	0.9228	0.7151	0.9251	9810
	MM	0.083272	(-2.536, 2.571)	0.4963	0.9228	0.7131	0.9251	9810
	MIR	**0.080438**	**(-2.511, 2.586)**	**0.5017**	**0.9275**	**0.7225**	**0.9296**	**9810**
	CSQM	**0.083376**	**(-2.561, 2.542)**	**0.4931**	**0.9183**	**0.7048**	**0.9205**	**9810**
	CQM	**0.083374**	**(-2.543, 2.559)**	**0.4927**	**0.9176**	**0.7050**	**0.9199**	**9810**
	HL1	0.082112	(-2.544, 2.539)	0.4948	0.9192	0.7124	0.9210	9994
	HL2	0.083162	(-2.512, 2.538)	0.4918	0.9155	0.7089	0.9175	10000
	HL3	0.082871	(-2.525, 2.539)	0.4936	0.9178	0.7119	0.9197	9999
0.4	Mean	0.073282	(-2.198, 2.209)	0.3671	0.7479	0.5525	0.7503	10000
	NM	0.072511	(-2.274, 2.346)	0.3847	0.7722	0.5748	0.7744	9872
	MM	0.070959	(-2.263, 2.299)	0.3882	0.7779	0.5847	0.7801	9872
	MIR	**0.071247**	**(-2.247, 2.310)**	**0.3867**	**0.7750**	**0.5753**	**0.7769**	**9872**
	CSQM	**0.072203**	**(-2.223, 2.292)**	**0.3841**	**0.7722**	**0.5740**	**0.7746**	**9872**
	CQM	**0.072803**	**(-2.228, 2.275)**	**0.3836**	**0.7716**	**0.5716**	**0.7741**	**9872**
	HL1	0.073371	(-2.238, 2.240)	0.3734	0.7570	0.5599	0.7591	9995
	HL2	0.071962	(-2.224, 2.262)	0.3714	0.7540	0.5584	0.7563	10000
	HL3	0.072233	(-2.234, 2.252)	0.3727	0.7557	0.5599	0.7579	9997
0.5	Mean	0.069876	(-1.730, 1.787)	0.2667	0.6112	0.4619	0.6135	10000
	NM	0.067521	(-1.814, 1.877)	0.2855	0.6391	0.4841	0.6418	9912
	MM	0.070352	(-1.812, 1.876)	0.2856	0.6396	0.4869	0.6419	9912
	MIR	**0.068582**	**(-1.860, 1.844)**	**0.2849**	**0.6377**	**0.4836**	**0.6399**	**9912**
	CSQM	**0.069868**	**(-1.824, 1.837)**	**0.2810**	**0.6325**	**0.4764**	**0.6349**	**9912**
	CQM	**0.069658**	**(-1.793, 1.848)**	**0.2802**	**0.6312**	**0.4725**	**0.6337**	**9912**
	HL1	0.069341	(-1.727, 1.820)	0.2723	0.6197	0.4673	0.6225	9997
	HL2	0.067589	(-1.726, 1.813)	0.2706	0.6172	0.4666	0.6198	10000
	HL3	0.068745	(-1.740, 1.799)	0.2706	0.6171	0.4653	0.6198	9998

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Table E.8: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon=0.1$, Sample Size $n=19$, $\kappa=1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.069103	(-0.681, 0.833)	0.0694	0.2964	0.2442	0.3018	10000
	NM	0.069943	(-0.736, 0.882)	0.0818	0.3235	0.2670	0.3283	9988
	MM	0.068888	(-0.756, 0.898)	0.0842	0.3286	0.2703	0.3336	9988
	MIR	**0.067959**	**(-0.740, 0.879)**	**0.0785**	**0.3142**	**0.2554**	**0.3195**	**9988**
	CSQM	**0.069235**	**(-0.736, 0.893)**	**0.0807**	**0.3207**	**0.2623**	**0.3261**	**9988**
	CQM	**0.069405**	**(-0.733, 0.887)**	**0.0803**	**0.3202**	**0.2609**	**0.3255**	**9988**
	HL1	0.070287	(-0.686, 0.851)	0.0709	0.2989	0.2470	0.3043	9999
	HL2	0.070389	(-0.680, 0.848)	0.0705	0.2980	0.2459	0.3034	10000
	HL3	0.070347	(-0.684, 0.849)	0.0708	0.2986	0.2457	0.3039	10000
1.5	Mean	0.074300	(-0.427, 0.584)	0.0328	0.2047	0.1724	0.2134	10000
	NM	0.073996	(-0.490, 0.651)	0.0411	0.2296	0.1936	0.2368	9999
	MM	0.073060	(-0.492, 0.656)	0.0419	0.2314	0.1945	0.2386	9999
	MIR	**0.076200**	**(-0.460, 0.621)**	**0.0370**	**0.2168**	**0.1796**	**0.2253**	**9999**
	CSQM	**0.073009**	**(-0.474, 0.633)**	**0.0391**	**0.2235**	**0.1890**	**0.2309**	**9999**
	CQM	**0.072978**	**(-0.470, 0.622)**	**0.0389**	**0.2228**	**0.1863**	**0.2302**	**9999**
	HL1	0.074888	(-0.427, 0.588)	0.0334	0.2067	0.1754	0.2154	10000
	HL2	0.074716	(-0.428, 0.586)	0.0333	0.2063	0.1751	0.2150	10000
	HL3	0.074733	(-0.427, 0.587)	0.0334	0.2066	0.1754	0.2153	10000
2.0	Mean	0.071608	(-0.341, 0.477)	0.0215	0.1649	0.1373	0.1747	10000
	NM	0.067968	(-0.393, 0.539)	0.0282	0.1897	0.1590	0.1970	10000
	MM	0.067927	(-0.394, 0.539)	0.0284	0.1900	0.1590	0.1973	10000
	MIR	**0.072728**	**(-0.360, 0.503)**	**0.0240**	**0.1746**	**0.1456**	**0.1841**	**10000**
	CSQM	**0.068063**	**(-0.379, 0.517)**	**0.0262**	**0.1829**	**0.1531**	**0.1906**	**10000**
	CQM	**0.068135**	**(-0.380, 0.514)**	**0.0260**	**0.1822**	**0.1530**	**0.1899**	**10000**
	HL1	0.072035	(-0.346, 0.486)	0.0219	0.1662	0.1390	0.1762	10000
	HL2	0.071616	(-0.345, 0.487)	0.0219	0.1660	0.1383	0.1759	10000
	HL3	0.071779	(-0.346, 0.488)	0.0219	0.1661	0.1390	0.1760	10000
2.5	Mean	0.071225	(-0.271, 0.425)	0.0155	0.1407	0.1189	0.1519	10000
	NM	0.065323	(-0.333, 0.473)	0.0208	0.1637	0.1381	0.1710	10000
	MM	0.065296	(-0.333, 0.473)	0.0209	0.1638	0.1382	0.1712	10000
	MIR	**0.071058**	**(-0.297, 0.444)**	**0.0172**	**0.1484**	**0.1250**	**0.1588**	**10000**
	CSQM	**0.065053**	**(-0.319, 0.457)**	**0.0193**	**0.1574**	**0.1321**	**0.1653**	**10000**
	CQM	**0.064920**	**(-0.321, 0.454)**	**0.0192**	**0.1570**	**0.1324**	**0.1649**	**10000**
	HL1	0.071150	(-0.280, 0.430)	0.0158	0.1419	0.1192	0.1528	10000
	HL2	0.070592	(-0.281, 0.427)	0.0158	0.1419	0.1187	0.1526	10000
	HL3	0.070879	(-0.280, 0.428)	0.0158	0.1419	0.1189	0.1527	10000
3.0	Mean	0.075590	(-0.230, 0.391)	0.0123	0.1260	0.1074	0.1396	10000
	NM	0.066784	(-0.294, 0.435)	0.0171	0.1484	0.1261	0.1573	10000
	MM	0.066832	(-0.294, 0.435)	0.0171	0.1485	0.1262	0.1574	10000
	MIR	**0.074724**	**(-0.250, 0.405)**	**0.0137**	**0.1323**	**0.1123**	**0.1452**	**10000**
	CSQM	**0.067109**	**(-0.279, 0.421)**	**0.0158**	**0.1426**	**0.1217**	**0.1522**	**10000**
	CQM	**0.067248**	**(-0.276, 0.419)**	**0.0157**	**0.1421**	**0.1203**	**0.1518**	**10000**
	HL1	0.074791	(-0.232, 0.389)	0.0126	0.1269	0.1080	0.1403	10000
	HL2	0.074196	(-0.234, 0.388)	0.0126	0.1270	0.1081	0.1402	10000
	HL3	0.074506	(-0.232, 0.389)	0.0126	0.1270	0.1081	0.1403	10000

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Table E.9: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 19$, $\kappa = 4.0, 5.0, 6.0, 7.0, 8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.074372	(-0.186, 0.348)	0.0090	0.1071	0.0904	0.1227	10000
	NM	0.062342	(-0.244, 0.392)	0.0128	0.1267	0.1052	0.1362	10000
	MM	0.062395	(-0.244, 0.393)	0.0128	0.1267	0.1052	0.1363	10000
	MIR	**0.070570**	**(-0.204, 0.355)**	**0.0102**	**0.1135**	**0.0951**	**0.1268**	**10000**
	CSQM	**0.062966**	**(-0.234, 0.381)**	**0.0118**	**0.1219**	**0.1017**	**0.1319**	**10000**
	CQM	**0.063251**	**(-0.233, 0.376)**	**0.0117**	**0.1215**	**0.1011**	**0.1316**	**10000**
	HL1	0.071592	(-0.195, 0.346)	0.0093	0.1081	0.0907	0.1226	10000
	HL2	0.070855	(-0.196, 0.349)	0.0093	0.1082	0.0913	0.1224	10000
	HL3	0.071197	(-0.195, 0.349)	0.0093	0.1081	0.0913	0.1225	10000
5.0	Mean	0.072770	(-0.155, 0.310)	0.0071	0.0958	0.0821	0.1125	10000
	NM	0.057381	(-0.212, 0.343)	0.0099	0.1123	0.0943	0.1206	10000
	MM	0.057377	(-0.212, 0.343)	0.0099	0.1123	0.0943	0.1206	10000
	MIR	**0.066985**	**(-0.175, 0.322)**	**0.0080**	**0.1014**	**0.0860**	**0.1143**	**10000**
	CSQM	**0.058065**	**(-0.201, 0.328)**	**0.0092**	**0.1083**	**0.0912**	**0.1173**	**10000**
	CQM	**0.058409**	**(-0.199, 0.328)**	**0.0091**	**0.1081**	**0.0914**	**0.1173**	**10000**
	HL1	0.068663	(-0.164, 0.309)	0.0073	0.0970	0.0824	0.1115	10000
	HL2	0.067712	(-0.166, 0.308)	0.0073	0.0971	0.0824	0.1111	10000
	HL3	0.068206	(-0.164, 0.309)	0.0073	0.0971	0.0822	0.1113	10000
6.0	Mean	0.071970	(-0.145, 0.296)	0.0061	0.0883	0.0737	0.1062	10000
	NM	0.053568	(-0.199, 0.317)	0.0085	0.1041	0.0873	0.1123	10000
	MM	0.053568	(-0.199, 0.317)	0.0085	0.1041	0.0873	0.1123	10000
	MIR	**0.064068**	**(-0.164, 0.303)**	**0.0069**	**0.0936**	**0.0789**	**0.1070**	**10000**
	CSQM	**0.053696**	**(-0.189, 0.304)**	**0.0079**	**0.1001**	**0.0838**	**0.1086**	**10000**
	CQM	**0.053757**	**(-0.188, 0.304)**	**0.0078**	**0.0998**	**0.0842**	**0.1085**	**10000**
	HL1	0.065881	(-0.155, 0.293)	0.0063	0.0895	0.0744	0.1043	10000
	HL2	0.064966	(-0.156, 0.292)	0.0063	0.0895	0.0746	0.1040	10000
	HL3	0.065404	(-0.156, 0.293)	0.0063	0.0895	0.0745	0.1042	10000
7.0	Mean	0.073050	(-0.128, 0.283)	0.0055	0.0831	0.0695	0.1026	10000
	NM	0.051974	(-0.185, 0.298)	0.0074	0.0967	0.0813	0.1048	10000
	MM	0.051974	(-0.185, 0.298)	0.0074	0.0967	0.0813	0.1048	10000
	MIR	**0.063427**	**(-0.151, 0.288)**	**0.0061**	**0.0880**	**0.0740**	**0.1016**	**10000**
	CSQM	**0.051931**	**(-0.176, 0.288)**	**0.0068**	**0.0933**	**0.0780**	**0.1018**	**10000**
	CQM	**0.051996**	**(-0.174, 0.287)**	**0.0068**	**0.0931**	**0.0778**	**0.1018**	**10000**
	HL1	0.065515	(-0.142, 0.278)	0.0056	0.0841	0.0699	0.0996	10000
	HL2	0.064391	(-0.142, 0.277)	0.0056	0.0841	0.0702	0.0991	10000
	HL3	0.064946	(-0.141, 0.277)	0.0056	0.0841	0.0700	0.0993	10000
8.0	Mean	0.072663	(-0.115, 0.270)	0.0048	0.0780	0.0650	0.0982	10000
	NM	0.048884	(-0.170, 0.278)	0.0065	0.0908	0.0773	0.0985	10000
	MM	0.048884	(-0.170, 0.278)	0.0065	0.0908	0.0773	0.0985	10000
	MIR	**0.061075**	**(-0.137, 0.269)**	**0.0053**	**0.0820**	**0.0686**	**0.0954**	**10000**
	CSQM	**0.048983**	**(-0.161, 0.266)**	**0.0059**	**0.0869**	**0.0734**	**0.0951**	**10000**
	CQM	**0.049031**	**(-0.160, 0.266)**	**0.0059**	**0.0865**	**0.0727**	**0.0949**	**10000**
	HL1	0.063574	(-0.127, 0.265)	0.0049	0.0785	0.0647	0.0939	10000
	HL2	0.062326	(-0.129, 0.263)	0.0049	0.0784	0.0654	0.0933	10000
	HL3	0.062927	(-0.129, 0.264)	0.0049	0.0784	0.0651	0.0936	10000

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Table E.10: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 20$, $\kappa = 0.1, 0.2, 0.3, 0.4, 0.5$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
0.1	Mean	0.054419	(-2.933, 2.911)	0.8182	1.3374	1.2289	1.3377	10000
	NM	0.050147	(-2.906, 2.897)	0.8211	1.3411	1.2336	1.3412	9960
	MM	0.056823	(-2.905, 2.904)	0.8211	1.3416	1.2190	1.3418	9960
	MIR	**0.076856**	**(-2.927, 2.918)**	**0.8299**	**1.3534**	**1.2399**	**1.3533**	**9960**
	CSQM	**0.056823**	**(-2.905, 2.904)**	**0.8211**	**1.3416**	**1.2190**	**1.3418**	**9960**
	CQM	**0.056823**	**(-2.905, 2.904)**	**0.8211**	**1.3416**	**1.2190**	**1.3418**	**9960**
	HL1	0.060798	(-2.920, 2.923)	0.8194	1.3397	1.2301	1.3402	9991
	HL2	0.058211	(-2.918, 2.925)	0.8194	1.3392	1.2280	1.3395	9999
	HL3	0.060643	(-2.901, 2.925)	0.8189	1.3384	1.2293	1.3389	10000
0.2	Mean	0.035177	(-2.756, 2.808)	0.6308	1.0990	0.9113	1.0996	10000
	NM	0.045981	(-2.819, 2.759)	0.6447	1.1173	0.9229	1.1175	9958
	MM	0.043155	(-2.780, 2.732)	0.6421	1.1129	0.9248	1.1131	9958
	MIR	**0.041592**	**(-2.754, 2.805)**	**0.6530**	**1.1254**	**0.9388**	**1.1260**	**9958**
	CSQM	**0.043155**	**(-2.780, 2.732)**	**0.6421**	**1.1129**	**0.9248**	**1.1131**	**9958**
	CQM	**0.043155**	**(-2.780, 2.732)**	**0.6421**	**1.1129**	**0.9248**	**1.1131**	**9958**
	HL1	0.030722	(-2.752, 2.766)	0.6346	1.1035	0.9139	1.1040	9997
	HL2	0.034730	(-2.754, 2.792)	0.6336	1.1022	0.9125	1.1028	10000
	HL3	0.028888	(-2.752, 2.782)	0.6348	1.1038	0.9123	1.1043	9999
0.3	Mean	0.067770	(-2.502, 2.559)	0.4795	0.9006	0.6924	0.9018	10000
	NM	0.069043	(-2.595, 2.589)	0.4984	0.9260	0.7113	0.9271	9976
	MM	0.066755	(-2.507, 2.563)	0.4910	0.9148	0.7093	0.9161	9976
	MIR	**0.070645**	**(-2.584, 2.645)**	**0.5145**	**0.9451**	**0.7399**	**0.9472**	**9976**
	CSQM	**0.066755**	**(-2.507, 2.563)**	**0.4910**	**0.9148**	**0.7093**	**0.9161**	**9976**
	CQM	**0.066755**	**(-2.507, 2.563)**	**0.4910**	**0.9148**	**0.7093**	**0.9161**	**9976**
	HL1	0.071213	(-2.543, 2.564)	0.4886	0.9128	0.7025	0.9139	9996
	HL2	0.070249	(-2.540, 2.567)	0.4858	0.9090	0.6995	0.9102	9999
	HL3	0.070725	(-2.553, 2.566)	0.4868	0.9102	0.7011	0.9113	9998
0.4	Mean	0.073848	(-2.155, 2.102)	0.3494	0.7252	0.5429	0.7277	10000
	NM	0.076014	(-2.217, 2.208)	0.3685	0.7502	0.5597	0.7523	9979
	MM	0.075088	(-2.170, 2.166)	0.3651	0.7468	0.5606	0.7495	9979
	MIR	**0.076419**	**(-2.286, 2.281)**	**0.3809**	**0.7686**	**0.5785**	**0.7715**	**9979**
	CSQM	**0.075088**	**(-2.170, 2.166)**	**0.3651**	**0.7468**	**0.5606**	**0.7495**	**9979**
	CQM	**0.075088**	**(-2.170, 2.166)**	**0.3651**	**0.7468**	**0.5606**	**0.7495**	**9979**
	HL1	0.069559	(-2.129, 2.256)	0.3578	0.7370	0.5530	0.7393	9996
	HL2	0.072150	(-2.133, 2.230)	0.3572	0.7358	0.5528	0.7383	9999
	HL3	0.070757	(-2.136, 2.258)	0.3577	0.7365	0.5519	0.7389	9999
0.5	Mean	0.069017	(-1.585, 1.764)	0.2522	0.5913	0.4488	0.5936	10000
	NM	0.071970	(-1.666, 1.820)	0.2687	0.6155	0.4696	0.6184	9981
	MM	0.069442	(-1.721, 1.803)	0.2709	0.6192	0.4693	0.6216	9981
	MIR	**0.064638**	**(-1.790, 1.903)**	**0.2851**	**0.6393**	**0.4786**	**0.6410**	**9981**
	CSQM	**0.069442**	**(-1.721, 1.803)**	**0.2709**	**0.6192**	**0.4693**	**0.6216**	**9981**
	CQM	**0.069442**	**(-1.721, 1.803)**	**0.2709**	**0.6192**	**0.4693**	**0.6216**	**9981**
	HL1	0.069843	(-1.607, 1.767)	0.2585	0.6004	0.4539	0.6032	9999
	HL2	0.068700	(-1.636, 1.747)	0.2572	0.5981	0.4502	0.6008	10000
	HL3	0.069109	(-1.615, 1.755)	0.2577	0.5992	0.4526	0.6019	9999

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Table E.11: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\mathrm{vM}(0,\kappa)+\epsilon\mathrm{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon=0.1$, Sample Size $n=20$, $\kappa=1.0,1.5,2.0,2.5,3.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
1.0	Mean	0.072749	(-0.655, 0.801)	0.0655	0.2867	0.2325	0.2928	10000
	NM	0.070466	(-0.721, 0.862)	0.0758	0.3101	0.2517	0.3148	9999
	MM	0.070526	(-0.735, 0.880)	0.0772	0.3131	0.2556	0.3177	9999
	MIR	**0.071993**	**(-0.734, 0.868)**	**0.0784**	**0.3130**	**0.2530**	**0.3184**	**9999**
	CSQM	**0.070526**	**(-0.735, 0.880)**	**0.0772**	**0.3131**	**0.2556**	**0.3177**	**9999**
	CQM	**0.070526**	**(-0.735, 0.880)**	**0.0772**	**0.3131**	**0.2556**	**0.3177**	**9999**
	HL1	0.072746	(-0.662, 0.814)	0.0675	0.2910	0.2346	0.2973	10000
	HL2	0.072465	(-0.666, 0.814)	0.0671	0.2903	0.2340	0.2964	10000
	HL3	0.072653	(-0.664, 0.815)	0.0673	0.2906	0.2340	0.2968	10000
1.5	Mean	0.075217	(-0.433, 0.579)	0.0317	0.2009	0.1686	0.2095	10000
	NM	0.072452	(-0.484, 0.639)	0.0389	0.2228	0.1848	0.2299	10000
	MM	0.072732	(-0.486, 0.643)	0.0395	0.2243	0.1864	0.2315	10000
	MIR	**0.077554**	**(-0.452, 0.614)**	**0.0359**	**0.2133**	**0.1759**	**0.2221**	**10000**
	CSQM	**0.072732**	**(-0.486, 0.643)**	**0.0395**	**0.2243**	**0.1864**	**0.2315**	**10000**
	CQM	**0.072732**	**(-0.486, 0.643)**	**0.0395**	**0.2243**	**0.1864**	**0.2315**	**10000**
	HL1	0.074920	(-0.437, 0.590)	0.0323	0.2028	0.1700	0.2113	10000
	HL2	0.074872	(-0.435, 0.586)	0.0323	0.2028	0.1705	0.2112	10000
	HL3	0.074925	(-0.434, 0.589)	0.0322	0.2027	0.1704	0.2112	10000
2.0	Mean	0.075277	(-0.329, 0.475)	0.0202	0.1600	0.1328	0.1708	10000
	NM	0.070325	(-0.376, 0.518)	0.0255	0.1805	0.1498	0.1893	10000
	MM	0.070235	(-0.378, 0.519)	0.0257	0.1813	0.1511	0.1900	10000
	MIR	**0.077624**	**(-0.353, 0.492)**	**0.0226**	**0.1698**	**0.1431**	**0.1807**	**10000**
	CSQM	**0.070235**	**(-0.378, 0.519)**	**0.0257**	**0.1813**	**0.1511**	**0.1900**	**10000**
	CQM	**0.070235**	**(-0.378, 0.519)**	**0.0257**	**0.1813**	**0.1511**	**0.1900**	**10000**
	HL1	0.076041	(-0.332, 0.476)	0.0206	0.1617	0.1342	0.1726	10000
	HL2	0.075754	(-0.332, 0.473)	0.0205	0.1615	0.1345	0.1725	10000
	HL3	0.075869	(-0.333, 0.473)	0.0205	0.1616	0.1341	0.1725	10000
2.5	Mean	0.074473	(-0.261, 0.411)	0.0146	0.1367	0.1154	0.1493	10000
	NM	0.068636	(-0.310, 0.455)	0.0191	0.1566	0.1337	0.1660	10000
	MM	0.068590	(-0.310, 0.455)	0.0192	0.1567	0.1335	0.1661	10000
	MIR	**0.074790**	**(-0.277, 0.430)**	**0.0164**	**0.1452**	**0.1222**	**0.1569**	**10000**
	CSQM	**0.068590**	**(-0.310, 0.455)**	**0.0192**	**0.1567**	**0.1335**	**0.1661**	**10000**
	CQM	**0.068590**	**(-0.310, 0.455)**	**0.0192**	**0.1567**	**0.1335**	**0.1661**	**10000**
	HL1	0.073998	(-0.263, 0.410)	0.0148	0.1381	0.1170	0.1505	10000
	HL2	0.073615	(-0.263, 0.411)	0.0149	0.1381	0.1170	0.1504	10000
	HL3	0.073843	(-0.263, 0.410)	0.0148	0.1381	0.1169	0.1505	10000
3.0	Mean	0.075375	(-0.221, 0.380)	0.0118	0.1228	0.1040	0.1367	10000
	NM	0.067845	(-0.270, 0.426)	0.0158	0.1421	0.1195	0.1517	10000
	MM	0.067884	(-0.270, 0.426)	0.0158	0.1421	0.1196	0.1517	10000
	MIR	**0.074668**	**(-0.239, 0.397)**	**0.0133**	**0.1303**	**0.1099**	**0.1432**	**10000**
	CSQM	**0.067884**	**(-0.270, 0.426)**	**0.0158**	**0.1421**	**0.1196**	**0.1517**	**10000**
	CQM	**0.067884**	**(-0.270, 0.426)**	**0.0158**	**0.1421**	**0.1196**	**0.1517**	**10000**
	HL1	0.074067	(-0.225, 0.381)	0.0121	0.1241	0.1046	0.1373	10000
	HL2	0.073560	(-0.226, 0.379)	0.0121	0.1242	0.1049	0.1372	10000
	HL3	0.073786	(-0.225, 0.380)	0.0121	0.1241	0.1043	0.1373	10000

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Table E.12: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Contamination on Location:$(1-\epsilon)\text{vM}(0,\kappa)+\epsilon\text{vM}(\frac{\pi}{4},\kappa)$, Where $\epsilon = 0.1$, Sample Size $n = 20$, $\kappa = 4.0, 5.0, 6.0, 7.0, 8.0$, Based on 10,000 Random Samples.

k	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
4.0	Mean	0.071070	(-0.182, 0.337)	0.0086	0.1053	0.0895	0.1191	10000
	NM	0.059610	(-0.231, 0.361)	0.0115	0.1213	0.1027	0.1295	10000
	MM	0.059619	(-0.231, 0.361)	0.0115	0.1213	0.1026	0.1295	10000
	MIR	**0.068584**	**(-0.202, 0.349)**	**0.0098**	**0.1120**	**0.0940**	**0.1240**	**10000**
	CSQM	**0.059619**	**(-0.231, 0.361)**	**0.0115**	**0.1213**	**0.1026**	**0.1295**	**10000**
	CQM	**0.059619**	**(-0.231, 0.361)**	**0.0115**	**0.1213**	**0.1026**	**0.1295**	**10000**
	HL1	0.068644	(-0.189, 0.336)	0.0088	0.1067	0.0909	0.1192	10000
	HL2	0.067947	(-0.190, 0.336)	0.0089	0.1068	0.0908	0.1190	10000
	HL3	0.068274	(-0.189, 0.337)	0.0088	0.1068	0.0910	0.1191	10000
5.0	Mean	0.073332	(-0.154, 0.310)	0.0069	0.0938	0.0784	0.1110	10000
	NM	0.059253	(-0.200, 0.335)	0.0092	0.1080	0.0907	0.1172	10000
	MM	0.059256	(-0.200, 0.335)	0.0092	0.1080	0.0907	0.1172	10000
	MIR	**0.068793**	**(-0.175, 0.320)**	**0.0079**	**0.1001**	**0.0841**	**0.1141**	**10000**
	CSQM	**0.059256**	**(-0.200, 0.335)**	**0.0092**	**0.1080**	**0.0907**	**0.1172**	**10000**
	CQM	**0.059256**	**(-0.200, 0.335)**	**0.0092**	**0.1080**	**0.0907**	**0.1172**	**10000**
	HL1	0.068916	(-0.162, 0.307)	0.0071	0.0950	0.0790	0.1098	10000
	HL2	0.068218	(-0.163, 0.308)	0.0071	0.0950	0.0797	0.1096	10000
	HL3	0.068573	(-0.163, 0.307)	0.0071	0.0950	0.0792	0.1097	10000
6.0	Mean	0.070805	(-0.140, 0.293)	0.0060	0.0874	0.0738	0.1041	10000
	NM	0.052587	(-0.187, 0.309)	0.0079	0.1000	0.0844	0.1081	10000
	MM	0.052587	(-0.187, 0.309)	0.0079	0.1000	0.0844	0.1081	10000
	MIR	**0.063200**	**(-0.163, 0.300)**	**0.0068**	**0.0932**	**0.0785**	**0.1055**	**10000**
	CSQM	**0.052587**	**(-0.187, 0.309)**	**0.0079**	**0.1000**	**0.0844**	**0.1081**	**10000**
	CQM	**0.052587**	**(-0.187, 0.309)**	**0.0079**	**0.1000**	**0.0844**	**0.1081**	**10000**
	HL1	0.064630	(-0.146, 0.289)	0.0061	0.0884	0.0742	0.1020	10000
	HL2	0.063730	(-0.148, 0.288)	0.0061	0.0884	0.0742	0.1016	10000
	HL3	0.064160	(-0.147, 0.288)	0.0061	0.0884	0.0743	0.1018	10000
7.0	Mean	0.071451	(-0.123, 0.278)	0.0052	0.0821	0.0703	0.0998	10000
	NM	0.050716	(-0.172, 0.290)	0.0068	0.0930	0.0779	0.1011	10000
	MM	0.050716	(-0.172, 0.290)	0.0068	0.0930	0.0779	0.1011	10000
	MIR	**0.062525**	**(-0.148, 0.284)**	**0.0059**	**0.0871**	**0.0735**	**0.0999**	**10000**
	CSQM	**0.050716**	**(-0.172, 0.290)**	**0.0068**	**0.0930**	**0.0779**	**0.1011**	**10000**
	CQM	**0.050716**	**(-0.172, 0.290)**	**0.0068**	**0.0930**	**0.0779**	**0.1011**	**10000**
	HL1	0.063734	(-0.135, 0.273)	0.0054	0.0831	0.0705	0.0970	10000
	HL2	0.062766	(-0.136, 0.271)	0.0054	0.0831	0.0704	0.0965	10000
	HL3	0.063231	(-0.135, 0.273)	0.0054	0.0831	0.0705	0.0967	10000
8.0	Mean	0.072539	(-0.113, 0.268)	0.0047	0.0779	0.0671	0.0973	10000
	NM	0.048818	(-0.160, 0.270)	0.0060	0.0877	0.0734	0.0954	10000
	MM	0.048818	(-0.160, 0.270)	0.0060	0.0877	0.0734	0.0954	10000
	MIR	**0.062626**	**(-0.134, 0.270)**	**0.0053**	**0.0823**	**0.0703**	**0.0959**	**10000**
	CSQM	**0.048818**	**(-0.160, 0.270)**	**0.0060**	**0.0877**	**0.0734**	**0.0954**	**10000**
	CQM	**0.048818**	**(-0.160, 0.270)**	**0.0060**	**0.0877**	**0.0734**	**0.0954**	**10000**
	HL1	0.063177	(-0.126, 0.263)	0.0048	0.0786	0.0675	0.0933	10000
	HL2	0.062092	(-0.126, 0.261)	0.0048	0.0785	0.0671	0.0927	10000
	HL3	0.062612	(-0.126, 0.262)	0.0048	0.0785	0.0673	0.0929	10000

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Appendix F

Wrapped Exponential Distribution

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Table F.1: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Exponential distribution($\lambda = 1$), Mean = atan($1/\lambda$) = 0.7854, Median = 0.6508

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
7	Mean	0.812906	(0.308, 1.538)	0.0491	0.2456	0.2013	0.2441	10000
	NM	0.710687	(0.174, 1.615)	0.0687	0.2946	0.2489	0.2912	9993
	MM	0.711993	(0.168, 1.648)	0.0708	0.2992	0.2514	0.2958	9993
	MIR	**0.806313**	**(0.261, 1.599)**	**0.0593**	**0.2737**	**0.2292**	**0.2893**	**9993**
	CSQM	**0.735053**	**(0.211, 1.575)**	**0.0606**	**0.2773**	**0.2343**	**0.2764**	**9993**
	CQM	**0.747486**	**(0.218, 1.583)**	**0.0611**	**0.2798**	**0.2369**	**0.2799**	**9993**
	HL1	0.839170	(0.273, 1.575)	0.0588	0.2742	0.2320	0.3036	9990
	HL2	0.794032	(0.250, 1.553)	0.0575	0.2724	0.2308	0.2851	10000
	HL3	0.808757	(0.256, 1.567)	0.0578	0.2734	0.2314	0.2907	10000
8	Mean	0.809702	(0.326, 1.472)	0.0420	0.2278	0.1886	0.2268	10000
	NM	0.710575	(0.204, 1.521)	0.0556	0.2656	0.2230	0.2637	10000
	MM	0.711102	(0.201, 1.537)	0.0568	0.2686	0.2265	0.2666	10000
	MIR	**0.830640**	**(0.289, 1.568)**	**0.0538**	**0.2603**	**0.2189**	**0.2892**	**10000**
	CSQM	**0.711102**	**(0.201, 1.537)**	**0.0568**	**0.2686**	**0.2265**	**0.2666**	**10000**
	CQM	**0.711102**	**(0.201, 1.537)**	**0.0568**	**0.2686**	**0.2265**	**0.2666**	**10000**
	HL1	0.809639	(0.283, 1.515)	0.0497	0.2536	0.2174	0.2755	10000
	HL2	0.793724	(0.275, 1.504)	0.0496	0.2529	0.2147	0.2693	10000
	HL3	0.798073	(0.277, 1.510)	0.0499	0.2538	0.2168	0.2715	10000
9	Mean	0.807666	(0.349, 1.420)	0.0369	0.2138	0.1768	0.2129	10000
	NM	0.698945	(0.201, 1.482)	0.0531	0.2584	0.2184	0.2563	9996
	MM	0.699991	(0.197, 1.504)	0.0548	0.2626	0.2213	0.2603	9996
	MIR	**0.791456**	**(0.285, 1.528)**	**0.0490**	**0.2496**	**0.2087**	**0.2634**	**9996**
	CSQM	**0.715745**	**(0.232, 1.457)**	**0.0485**	**0.2471**	**0.2086**	**0.2462**	**9996**
	CQM	**0.723654**	**(0.237, 1.461)**	**0.0486**	**0.2483**	**0.2102**	**0.2480**	**9996**
	HL1	0.810383	(0.302, 1.478)	0.0445	0.2400	0.2054	0.2638	9999
	HL2	0.795244	(0.292, 1.464)	0.0443	0.2390	0.2038	0.2573	10000
	HL3	0.802481	(0.295, 1.471)	0.0446	0.2400	0.2043	0.2608	9999
10	Mean	0.804684	(0.375, 1.390)	0.0331	0.2023	0.1678	0.2014	10000
	NM	0.700776	(0.233, 1.416)	0.0457	0.2406	0.2043	0.2387	10000
	MM	0.700423	(0.229, 1.437)	0.0467	0.2431	0.2066	0.2411	10000
	MIR	**0.812078**	**(0.322, 1.502)**	**0.0453**	**0.2392**	**0.2026**	**0.2618**	**10000**
	CSQM	**0.700423**	**(0.229, 1.437)**	**0.0467**	**0.2431**	**0.2066**	**0.2411**	**10000**
	CQM	**0.700423**	**(0.229, 1.437)**	**0.0467**	**0.2431**	**0.2066**	**0.2411**	**10000**
	HL1	0.808575	(0.323, 1.431)	0.0397	0.2265	0.1932	0.2515	10000
	HL2	0.789103	(0.308, 1.421)	0.0399	0.2267	0.1932	0.2444	10000
	HL3	0.801185	(0.319, 1.425)	0.0395	0.2256	0.1923	0.2478	10000
11	Mean	0.801722	(0.393, 1.353)	0.0298	0.1930	0.1622	0.1923	10000
	NM	0.689908	(0.231, 1.398)	0.0443	0.2371	0.2010	0.2351	10000
	MM	0.689688	(0.226, 1.414)	0.0453	0.2399	0.2034	0.2378	10000
	MIR	**0.792497**	**(0.333, 1.408)**	**0.0380**	**0.2199**	**0.1860**	**0.2392**	**10000**
	CSQM	**0.700346**	**(0.255, 1.379)**	**0.0409**	**0.2286**	**0.1957**	**0.2272**	**10000**
	CQM	**0.705629**	**(0.260, 1.381)**	**0.0408**	**0.2286**	**0.1959**	**0.2277**	**10000**
	HL1	0.801013	(0.338, 1.394)	0.0363	0.2166	0.1857	0.2414	10000
	HL2	0.786846	(0.329, 1.380)	0.0362	0.2162	0.1848	0.2354	10000
	HL3	0.793291	(0.334, 1.384)	0.0363	0.2165	0.1853	0.2381	10000

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Table F.2: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Exponential Distribution(λ = 1), Mean = atan(1/λ) = 0.7854, Median = 0.6508

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
12	Mean	0.799028	(0.405, 1.312)	0.0265	0.1821	0.1524	0.1814	10000
	NM	0.689015	(0.254, 1.318)	0.0376	0.2180	0.1841	0.2168	10000
	MM	0.688604	(0.253, 1.340)	0.0384	0.2200	0.1850	0.2187	10000
	MIR	**0.811893**	**(0.366, 1.400)**	**0.0355**	**0.2122**	**0.1779**	**0.2404**	**10000**
	CSQM	**0.688604**	**(0.253, 1.340)**	**0.0384**	**0.2200**	**0.1850**	**0.2187**	**10000**
	CQM	**0.688604**	**(0.253, 1.340)**	**0.0384**	**0.2200**	**0.1850**	**0.2187**	**10000**
	HL1	0.797343	(0.352, 1.359)	0.0329	0.2060	0.1751	0.2310	10000
	HL2	0.783820	(0.341, 1.340)	0.0328	0.2055	0.1738	0.2254	10000
	HL3	0.790021	(0.347, 1.351)	0.0328	0.2058	0.1745	0.2279	10000
13	Mean	0.798005	(0.422, 1.301)	0.0246	0.1755	0.1475	0.1750	10000
	NM	0.684415	(0.253, 1.313)	0.0369	0.2176	0.1887	0.2160	9999
	MM	0.684231	(0.249, 1.326)	0.0377	0.2197	0.1899	0.2180	9999
	MIR	**0.784521**	**(0.353, 1.370)**	**0.0339**	**0.2076**	**0.1752**	**0.2257**	**9999**
	CSQM	**0.690603**	**(0.277, 1.304)**	**0.0342**	**0.2093**	**0.1810**	**0.2082**	**9999**
	CQM	**0.693712**	**(0.279, 1.298)**	**0.0341**	**0.2090**	**0.1812**	**0.2080**	**9999**
	HL1	0.797385	(0.370, 1.342)	0.0307	0.1992	0.1707	0.2258	10000
	HL2	0.785422	(0.362, 1.333)	0.0305	0.1983	0.1703	0.2202	10000
	HL3	0.791202	(0.365, 1.335)	0.0306	0.1986	0.1701	0.2229	10000
14	Mean	0.797020	(0.431, 1.276)	0.0225	0.1680	0.1409	0.1674	10000
	NM	0.684236	(0.269, 1.279)	0.0330	0.2048	0.1723	0.2033	10000
	MM	0.683499	(0.269, 1.283)	0.0337	0.2067	0.1740	0.2052	10000
	MIR	**0.803566**	**(0.377, 1.375)**	**0.0319**	**0.2017**	**0.1700**	**0.2278**	**10000**
	CSQM	**0.683499**	**(0.269, 1.283)**	**0.0337**	**0.2067**	**0.1740**	**0.2052**	**10000**
	CQM	**0.683499**	**(0.269, 1.283)**	**0.0337**	**0.2067**	**0.1740**	**0.2052**	**10000**
	HL1	0.796619	(0.384, 1.310)	0.0280	0.1899	0.1624	0.2171	10000
	HL2	0.783305	(0.370, 1.303)	0.0281	0.1902	0.1618	0.2119	10000
	HL3	0.788673	(0.376, 1.306)	0.0281	0.1902	0.1617	0.2139	10000
15	Mean	0.795763	(0.439, 1.254)	0.0213	0.1630	0.1363	0.1625	10000
	NM	0.679014	(0.270, 1.282)	0.0331	0.2043	0.1753	0.2028	10000
	MM	0.689977	(0.268, 1.287)	0.0337	0.2061	0.1770	0.2046	10000
	MIR	**0.787465**	**(0.385, 1.320)**	**0.0283**	**0.1892**	**0.1585**	**0.2120**	**10000**
	CSQM	**0.684701**	**(0.285, 1.260)**	**0.0306**	**0.1968**	**0.1672**	**0.1957**	**10000**
	CQM	**0.684979**	**(0.289, 1.261)**	**0.0303**	**0.1961**	**0.1670**	**0.1951**	**10000**
	HL1	0.792224	(0.392, 1.293)	0.0266	0.1847	0.1557	0.2118	10000
	HL2	0.781187	(0.382, 1.283)	0.0265	0.1844	0.1557	0.2070	10000
	HL3	0.786431	(0.387, 1.286)	0.0266	0.1845	0.1548	0.2091	10000
16	Mean	0.796216	(0.452, 1.234)	0.0196	0.1567	0.1314	0.1563	10000
	NM	0.680924	(0.293, 1.232)	0.0293	0.1932	0.1647	0.1921	10000
	MM	0.680892	(0.292, 1.238)	0.0298	0.1948	0.1660	0.1938	10000
	MIR	**0.805647**	**(0.413, 1.321)**	**0.0267**	**0.1842**	**0.1555**	**0.2164**	**10000**
	CSQM	**0.680892**	**(0.292, 1.238)**	**0.0298**	**0.1948**	**0.1660**	**0.1938**	**10000**
	CQM	**0.680892**	**(0.292, 1.238)**	**0.0298**	**0.1948**	**0.1660**	**0.1938**	**10000**
	HL1	0.793082	(0.401, 1.276)	0.0245	0.1775	0.1508	0.2068	10000
	HL2	0.782587	(0.395, 1.267)	0.0244	0.1773	0.1515	0.2018	10000
	HL3	0.787449	(0.397, 1.271)	0.0245	0.1774	0.1506	0.2040	10000

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Table F.3: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Exponential Distribution(λ = 2), Mean = atan($1/\lambda$) = 0.4637, Median = 0.3456

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
7	Mean	0.469022	(0.195, 0.848)	0.0139	0.1315	0.1104	0.1312	10000
	NM	0.374320	(0.100, 0.837)	0.0181	0.1495	0.1282	0.1473	10000
	MM	0.374471	(0.100, 0.837)	0.0181	0.1496	0.1283	0.1475	10000
	MIR	**0.432501**	**(0.158, 0.846)**	**0.0156**	**0.1394**	**0.1181**	**0.1484**	**10000**
	CSQM	**0.387655**	**(0.125, 0.811)**	**0.0154**	**0.1379**	**0.1181**	**0.1374**	**10000**
	CQM	**0.394389**	**(0.128, 0.812)**	**0.0158**	**0.1397**	**0.1200**	**0.1401**	**10000**
	HL1	0.464048	(0.167, 0.915)	0.0181	0.1500	0.1269	0.1689	10000
	HL2	0.427165	(0.146, 0.848)	0.0162	0.1424	0.1202	0.1499	10000
	HL3	0.437037	(0.152, 0.873)	0.0169	0.1453	0.1229	0.1552	10000
8	Mean	0.469215	(0.214, 0.816)	0.0118	0.1220	0.1039	0.1216	10000
	NM	0.375887	(0.123, 0.773)	0.0141	0.1325	0.1132	0.1312	10000
	MM	0.375856	(0.123, 0.773)	0.0141	0.1326	0.1134	0.1313	10000
	MIR	**0.450631**	**(0.177, 0.836)**	**0.0143**	**0.1345**	**0.1143**	**0.1513**	**10000**
	CSQM	**0.375856**	**(0.123, 0.773)**	**0.0141**	**0.1326**	**0.1134**	**0.1313**	**10000**
	CQM	**0.375856**	**(0.123, 0.773)**	**0.0141**	**0.1326**	**0.1134**	**0.1313**	**10000**
	HL1	0.439456	(0.167, 0.827)	0.0144	0.1347	0.1144	0.1473	10000
	HL2	0.429791	(0.164, 0.813)	0.0139	0.1326	0.1125	0.1423	10000
	HL3	0.432259	(0.163, 0.821)	0.0141	0.1335	0.1137	0.1438	10000
9	Mean	0.468284	(0.221, 0.796)	0.0109	0.1167	0.0979	0.1164	10000
	NM	0.370199	(0.117, 0.773)	0.0143	0.1338	0.1147	0.1320	10000
	MM	0.370255	(0.117, 0.773)	0.0143	0.1339	0.1147	0.1321	10000
	MIR	**0.422549**	**(0.166, 0.804)**	**0.0134**	**0.1288**	**0.1088**	**0.1361**	**10000**
	CSQM	**0.378173**	**(0.137, 0.752)**	**0.0125**	**0.1250**	**0.1065**	**0.1240**	**10000**
	CQM	**0.382151**	**(0.139, 0.754)**	**0.0126**	**0.1253**	**0.1063**	**0.1247**	**10000**
	HL1	0.440234	(0.180, 0.817)	0.0133	0.1290	0.1079	0.1429	10000
	HL2	0.431088	(0.173, 0.800)	0.0130	0.1274	0.1069	0.1380	10000
	HL3	0.435388	(0.177, 0.807)	0.0132	0.1283	0.1073	0.1404	10000
10	Mean	0.467385	(0.230, 0.782)	0.0098	0.1108	0.0934	0.1106	10000
	NM	0.370506	(0.132, 0.735)	0.0119	0.1216	0.1032	0.1205	10000
	MM	0.370477	(0.132, 0.735)	0.0119	0.1216	0.1032	0.1205	10000
	MIR	**0.436225**	**(0.185, 0.806)**	**0.0126**	**0.1255**	**0.1059**	**0.1379**	**10000**
	CSQM	**0.370477**	**(0.132, 0.735)**	**0.0119**	**0.1216**	**0.1032**	**0.1205**	**10000**
	CQM	**0.370477**	**(0.132, 0.735)**	**0.0119**	**0.1216**	**0.1032**	**0.1205**	**10000**
	HL1	0.441135	(0.191, 0.798)	0.0121	0.1227	0.1041	0.1385	10000
	HL2	0.427234	(0.180, 0.779)	0.0116	0.1209	0.1030	0.1316	10000
	HL3	0.436690	(0.189, 0.788)	0.0118	0.1216	0.1026	0.1357	10000
11	Mean	0.468249	(0.245, 0.763)	0.0088	0.1051	0.0880	0.1048	10000
	NM	0.366046	(0.129, 0.721)	0.0118	0.1207	0.1017	0.1196	10000
	MM	0.366086	(0.129, 0.721)	0.0118	0.1208	0.1018	0.1197	10000
	MIR	**0.426532**	**(0.194, 0.750)**	**0.0102**	**0.1133**	**0.0949**	**0.1250**	**10000**
	CSQM	**0.371902**	**(0.146, 0.707)**	**0.0104**	**0.1135**	**0.0956**	**0.1127**	**10000**
	CQM	**0.374809**	**(0.149, 0.708)**	**0.0104**	**0.1135**	**0.0957**	**0.1130**	**10000**
	HL1	0.437126	(0.198, 0.769)	0.0107	0.1158	0.0979	0.1318	10000
	HL2	0.427425	(0.190, 0.756)	0.0104	0.1144	0.0968	0.1266	10000
	HL3	0.431847	(0.194, 0.766)	0.0106	0.1151	0.0972	0.1290	10000

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Table F.4: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Exponential Distribution(λ = 2), Mean = atan($1/\lambda$) = 0.4637, Median = 0.3456

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
12	Mean	0.467337	(0.251, 0.748)	0.0080	0.1005	0.0845	0.1003	10000
	NM	0.365980	(0.145, 0.695)	0.0101	0.1121	0.0941	0.1111	10000
	MM	0.365973	(0.145, 0.696)	0.0101	0.1121	0.0942	0.1111	10000
	MIR	**0.438372**	**(0.210, 0.755)**	**0.0098**	**0.1111**	**0.0937**	**0.1288**	**10000**
	CSQM	**0.365973**	**(0.145, 0.696)**	**0.0101**	**0.1121**	**0.0942**	**0.1111**	**10000**
	CQM	**0.365973**	**(0.145, 0.696)**	**0.0101**	**0.1121**	**0.0942**	**0.1111**	**10000**
	HL1	0.435022	(0.207, 0.749)	0.0097	0.1106	0.0931	0.1270	10000
	HL2	0.426875	(0.202, 0.739)	0.0095	0.1094	0.0916	0.1225	10000
	HL3	0.430395	(0.204, 0.744)	0.0096	0.1100	0.0924	0.1244	10000
13	Mean	0.466765	(0.254, 0.735)	0.0075	0.0967	0.0804	0.0965	10000
	NM	0.362776	(0.141, 0.691)	0.0100	0.1108	0.0917	0.1097	10000
	MM	0.362841	(0.141, 0.691)	0.0100	0.1109	0.0917	0.1098	10000
	MIR	**0.421143**	**(0.196, 0.724)**	**0.0094**	**0.1081**	**0.0915**	**0.1188**	**10000**
	CSQM	**0.366918**	**(0.156, 0.676)**	**0.0090**	**0.1050**	**0.0878**	**0.1042**	**10000**
	CQM	**0.368918**	**(0.159, 0.675)**	**0.0089**	**0.1049**	**0.0882**	**0.1041**	**10000**
	HL1	0.434367	(0.208, 0.736)	0.0091	0.1068	0.0907	0.1238	10000
	HL2	0.427062	(0.204, 0.730)	0.0089	0.1056	0.0897	0.1196	10000
	HL3	0.430546	(0.206, 0.734)	0.0090	0.1062	0.0904	0.1215	10000
14	Mean	0.466073	(0.263, 0.721)	0.0069	0.0932	0.0768	0.0931	10000
	NM	0.361921	(0.151, 0.674)	0.0087	0.1035	0.0876	0.1027	10000
	MM	0.361891	(0.151, 0.675)	0.0087	0.1036	0.0876	0.1027	10000
	MIR	**0.431759**	**(0.212, 0.737)**	**0.0090**	**0.1065**	**0.0901**	**0.1221**	**10000**
	CSQM	**0.361891**	**(0.151, 0.675)**	**0.0087**	**0.1036**	**0.0876**	**0.1027**	**10000**
	CQM	**0.361891**	**(0.151, 0.675)**	**0.0087**	**0.1036**	**0.0876**	**0.1027**	**10000**
	HL1	0.434115	(0.218, 0.720)	0.0084	0.1028	0.0867	0.1207	10000
	HL2	0.424650	(0.212, 0.710)	0.0082	0.1016	0.0856	0.1155	10000
	HL3	0.428196	(0.215, 0.714)	0.0083	0.1020	0.0856	0.1173	10000
15	Mean	0.466947	(0.269, 0.713)	0.0065	0.0908	0.0773	0.0907	10000
	NM	0.361456	(0.150, 0.672)	0.0087	0.1045	0.0885	0.1037	10000
	MM	0.361463	(0.150, 0.672)	0.0087	0.1046	0.0886	0.1037	10000
	MIR	**0.423310**	**(0.216, 0.702)**	**0.0078**	**0.0990**	**0.0836**	**0.1130**	**10000**
	CSQM	**0.364133**	**(0.163, 0.656)**	**0.0079**	**0.0996**	**0.0842**	**0.0989**	**10000**
	CQM	**0.365594**	**(0.165, 0.655)**	**0.0078**	**0.0994**	**0.0845**	**0.0988**	**10000**
	HL1	0.432299	(0.222, 0.710)	0.0078	0.0998	0.0846	0.1181	10000
	HL2	0.425160	(0.217, 0.702)	0.0077	0.0988	0.0835	0.1141	10000
	HL3	0.428532	(0.220, 0.706)	0.0078	0.0993	0.0842	0.1161	10000
16	Mean	0.466017	(0.272, 0.703)	0.0061	0.0875	0.0738	0.0874	10000
	NM	0.359821	(0.163, 0.642)	0.0076	0.0972	0.0830	0.0965	10000
	MM	0.359968	(0.163, 0.642)	0.0076	0.0972	0.0830	0.0965	10000
	MIR	**0.431968**	**(0.226, 0.709)**	**0.0076**	**0.0976**	**0.0835**	**0.1157**	**10000**
	CSQM	**0.359854**	**(0.163, 0.642)**	**0.0076**	**0.0972**	**0.0830**	**0.0965**	**10000**
	CQM	**0.359854**	**(0.163, 0.642)**	**0.0076**	**0.0972**	**0.0830**	**0.0965**	**10000**
	HL1	0.430789	(0.226, 0.696)	0.0073	0.0963	0.0817	0.1142	10000
	HL2	0.424321	(0.223, 0.687)	0.0072	0.0954	0.0807	0.1104	10000
	HL3	0.427397	(0.224, 0.691)	0.0073	0.0958	0.0812	0.1123	10000

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Appendix G

Wrapped Laplace Distribution

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Table G.1: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Laplace Distribution, $\lambda_1 = 2.0$, $\lambda_2 = 0.5$, p = 0.2, Mean = -0.6435 (See Appendix I), Median = -0.6147 (See Appendix I), Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
7	Mean	-0.664337	(-2.272, 1.111)	0.2389	0.5699	0.4262	0.5696	10000
	NM	-0.624486	(-2.350, 1.141)	0.2508	0.5897	0.4427	0.5892	9823
	MM	-0.625056	(-2.300, 1.130)	0.2539	0.5981	0.4606	0.5976	9823
	MIR	**-0.682073**	**(-2.179, 1.158)**	**0.2534**	**0.5939**	**0.4536**	**0.5961**	**9823**
	CSQM	**-0.641736**	**(-2.226, 1.039)**	**0.2416**	**0.5779**	**0.4386**	**0.5772**	**9823**
	CQM	**-0.651496**	**(-2.238, 1.087)**	**0.2440**	**0.5809**	**0.4381**	**0.5802**	**9823**
	HL1	-0.682279	(-2.266, 1.284)	0.2530	0.5912	0.4384	0.5930	9940
	HL2	-0.665603	(-2.252, 1.227)	0.2472	0.5843	0.4393	0.5846	10000
	HL3	-0.671826	(-2.267, 1.241)	0.2504	0.5884	0.4410	0.5891	10000
8	Mean	-0.663855	(-2.183, 0.971)	0.2155	0.5338	0.3996	0.5335	10000
	NM	-0.627508	(-2.291, 1.032)	0.2312	0.5598	0.4227	0.5593	9995
	MM	-0.630077	(-2.245, 1.064)	0.2289	0.5604	0.4297	0.5598	9995
	MIR	**-0.699505**	**(-2.120, 1.214)**	**0.2418**	**0.5771**	**0.4423**	**0.5818**	**9995**
	CSQM	**-0.630077**	**(-2.245, 1.064)**	**0.2289**	**0.5604**	**0.4297**	**0.5598**	**9995**
	CQM	**-0.630077**	**(-2.245, 1.064)**	**0.2289**	**0.5604**	**0.4297**	**0.5598**	**9995**
	HL1	-0.676073	(-2.207, 1.060)	0.2244	0.5487	0.4143	0.5503	9996
	HL2	-0.667855	(-2.189, 1.026)	0.2216	0.5455	0.4124	0.5463	10000
	HL3	-0.670753	(-2.198, 1.076)	0.2234	0.5477	0.4148	0.5488	10000
9	Mean	-0.652076	(-2.095, 0.712)	0.1888	0.4946	0.3673	0.4943	10000
	NM	-0.608393	(-2.171, 0.741)	0.2006	0.5165	0.3943	0.5169	9877
	MM	-0.611741	(-2.151, 0.729)	0.2035	0.5239	0.4094	0.5241	9877
	MIR	**-0.676601**	**(-2.037, 0.999)**	**0.2108**	**0.5309**	**0.4014**	**0.5322**	**9877**
	CSQM	**-0.623659**	**(-2.100, 0.755)**	**0.1955**	**0.5098**	**0.3912**	**0.5093**	**9877**
	CQM	**-0.629788**	**(-2.090, 0.766)**	**0.1974**	**0.5124**	**0.3915**	**0.5116**	**9877**
	HL1	-0.663217	(-2.059, 0.857)	0.1978	0.5089	0.3791	0.5096	9988
	HL2	-0.655022	(-2.087, 0.779)	0.1948	0.5061	0.3836	0.5061	9997
	HL3	-0.658174	(-2.083, 0.819)	0.1957	0.5070	0.3821	0.5072	9990
10	Mean	-0.652655	(-2.039, 0.593)	0.1734	0.4715	0.3530	0.4713	10000
	NM	-0.612891	(-2.080, 0.685)	0.1903	0.4997	0.3782	0.4998	9994
	MM	-0.610140	(-2.102, 0.618)	0.1904	0.5032	0.3866	0.5035	9994
	MIR	**-0.694168**	**(-2.029, 1.047)**	**0.2119**	**0.5308**	**0.3976**	**0.5351**	**9994**
	CSQM	**-0.610140**	**(-2.102, 0.618)**	**0.1904**	**0.5032**	**0.3866**	**0.5035**	**9994**
	CQM	**-0.610140**	**(-2.102, 0.618)**	**0.1904**	**0.5032**	**0.3866**	**0.5035**	**9994**
	HL1	-0.667776	(-2.054, 0.751)	0.1819	0.4860	0.3671	0.4869	9993
	HL2	-0.658888	(-2.051, 0.672)	0.1793	0.4829	0.3674	0.4832	10000
	HL3	-0.664278	(-2.057, 0.706)	0.1805	0.4842	0.3658	0.4848	9999
11	Mean	-0.650408	(-1.902, 0.482)	0.1524	0.4406	0.3400	0.4404	10000
	NM	-0.602310	(-1.979, 0.492)	0.1679	0.4696	0.3676	0.4705	9937
	MM	-0.603219	(-1.971, 0.474)	0.1687	0.4741	0.3775	0.4749	9937
	MIR	**-0.682688**	**(-1.895, 0.677)**	**0.1707**	**0.4711**	**0.3655**	**0.4740**	**9937**
	CSQM	**-0.613869**	**(-1.928, 0.473)**	**0.1613**	**0.4610**	**0.3658**	**0.4610**	**9937**
	CQM	**-0.618973**	**(-1.910, 0.493)**	**0.1618**	**0.4613**	**0.3637**	**0.4611**	**9937**
	HL1	-0.666790	(-1.931, 0.550)	0.1602	0.4542	0.3522	0.4550	9996
	HL2	-0.657605	(-1.919, 0.502)	0.1578	0.4511	0.3525	0.4513	10000
	HL3	-0.662103	(-1.931, 0.541)	0.1596	0.4533	0.3525	0.4539	9999

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Table G.2: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Laplace Distribution, $\lambda_1 = 2.0$, $\lambda_2 = 0.5$, p = 0.2, Mean = -0.6435 (See Appendix I), Median = -0.6147 (See Appendix I), Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
12	Mean	-0.653557	(-1.818, 0.415)	0.1367	0.4150	0.3194	0.4148	10000
	NM	-0.610309	(-1.903, 0.408)	0.1545	0.4468	0.3484	0.4471	9996
	MM	-0.609377	(-1.908, 0.403)	0.1561	0.4530	0.3610	0.4534	9996
	MIR	**-0.702483**	**(-1.866, 0.687)**	**0.1687**	**0.4684**	**0.3582**	**0.4744**	**9996**
	CSQM	**-0.609377**	**(-1.908, 0.403)**	**0.1561**	**0.4530**	**0.3610**	**0.4534**	**9996**
	CQM	**-0.609377**	**(-1.908, 0.403)**	**0.1561**	**0.4530**	**0.3610**	**0.4534**	**9996**
	HL1	-0.665863	(-1.780, 0.491)	0.1420	0.4252	0.3286	0.4262	9994
	HL2	-0.659879	(-1.787, 0.435)	0.1404	0.4238	0.3306	0.4242	10000
	HL3	-0.662489	(-1.782, 0.462)	0.1408	0.4241	0.3294	0.4247	9999
13	Mean	-0.653454	(-1.751, 0.351)	0.1269	0.3986	0.3087	0.3983	10000
	NM	-0.608406	(-1.847, 0.328)	0.1410	0.4260	0.3378	0.4265	9945
	MM	-0.608044	(-1.848, 0.330)	0.1442	0.4346	0.3522	0.4352	9945
	MIR	**-0.690205**	**(-1.810, 0.567)**	**0.1510**	**0.4402**	**0.3395**	**0.4434**	**9945**
	CSQM	**-0.615332**	**(-1.828, 0.334)**	**0.1386**	**0.4244**	**0.3387**	**0.4244**	**9945**
	CQM	**-0.618590**	**(-1.820, 0.343)**	**0.1389**	**0.4247**	**0.3399**	**0.4244**	**9945**
	HL1	-0.665763	(-1.782, 0.395)	0.1324	0.4088	0.3194	0.4097	9998
	HL2	-0.658445	(-1.761, 0.378)	0.1313	0.4077	0.3199	0.4080	10000
	HL3	-0.661226	(-1.769, 0.385)	0.1318	0.4080	0.3188	0.4086	9999
14	Mean	-0.651302	(-1.731, 0.295)	0.1184	0.3855	0.3034	0.3853	10000
	NM	-0.606293	(-1.803, 0.310)	0.1335	0.4133	0.3286	0.4139	9994
	MM	-0.607163	(-1.829, 0.297)	0.1374	0.4230	0.3421	0.4236	9994
	MIR	**-0.699487**	**(-1.814, 0.635)**	**0.1555**	**0.4452**	**0.3431**	**0.4508**	**9994**
	CSQM	**-0.607163**	**(-1.829, 0.297)**	**0.1374**	**0.4230**	**0.3421**	**0.4236**	**9994**
	CQM	**-0.607163**	**(-1.829, 0.297)**	**0.1374**	**0.4230**	**0.3421**	**0.4236**	**9994**
	HL1	-0.664418	(-1.737, 0.337)	0.1234	0.3948	0.3133	0.3959	9997
	HL2	-0.657564	(-1.737, 0.323)	0.1223	0.3941	0.3141	0.3945	10000
	HL3	-0.661282	(-1.738, 0.329)	0.1228	0.3942	0.3137	0.3950	10000
15	Mean	-0.663750	(-1.671, 0.230)	0.1077	0.3652	0.2884	0.3647	10000
	NM	-0.614377	(-1.756, 0.235)	0.1231	0.3968	0.3165	0.3968	9976
	MM	-0.615531	(-1.778, 0.240)	0.1273	0.4070	0.3313	0.4070	9976
	MIR	**-0.702738**	**(-1.764, 0.408)**	**0.1296**	**0.4048**	**0.3154**	**0.4102**	**9976**
	CSQM	**-0.618849**	**(-1.750, 0.232)**	**0.1220**	**0.3965**	**0.3205**	**0.3962**	**9976**
	CQM	**-0.620120**	**(-1.741, 0.230)**	**0.1217**	**0.3954**	**0.3193**	**0.3951**	**9976**
	HL1	-0.677219	(-1.706, 0.255)	0.1135	0.3765	0.2979	0.3782	9999
	HL2	-0.669959	(-1.682, 0.236)	0.1122	0.3753	0.2998	0.3763	10000
	HL3	-0.673891	(-1.698, 0.241)	0.1128	0.3759	0.2989	0.3772	9998
16	Mean	-0.660816	(-1.660, 0.212)	0.1037	0.3591	0.2830	0.3587	10000
	NM	-0.614875	(-1.725, 0.232)	0.1187	0.3900	0.3170	0.3900	9997
	MM	-0.612926	(-1.754, 0.236)	0.1222	0.3985	0.3276	0.3986	9997
	MIR	**-0.711579**	**(-1.750, 0.416)**	**0.1305**	**0.4059**	**0.3138**	**0.4136**	**9997**
	CSQM	**-0.612926**	**(-1.754, 0.236)**	**0.1222**	**0.3985**	**0.3276**	**0.3986**	**9997**
	CQM	**-0.612926**	**(-1.754, 0.236)**	**0.1222**	**0.3985**	**0.3276**	**0.3986**	**9997**
	HL1	-0.673676	(-1.677, 0.247)	0.1084	0.3683	0.2911	0.3701	9999
	HL2	-0.667597	(-1.679, 0.221)	0.1073	0.3672	0.2928	0.3683	10000
	HL3	-0.670584	(-1.682, 0.233)	0.1076	0.3675	0.2916	0.3689	10000

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Table G.3: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Laplace Distribution, $\lambda_1 = 0.8$, $\lambda_2 = 1.25$, p = 0.61, Mean = 0.2213 (See Appendix I), Median = 0.3955 (See Appendix I), Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
7	Mean	0.230860	(-0.895, 1.456)	0.1501	0.4352	0.3289	0.4351	10000
	NM	0.210007	(-0.949, 1.564)	0.1571	0.4410	0.3207	0.4794	9903
	MM	0.210560	(-0.940, 1.553)	0.1579	0.4434	0.3244	0.4822	9903
	MIR	**0.236826**	**(-0.995, 1.480)**	**0.1646**	**0.4617**	**0.3559**	**0.4831**	**9903**
	CSQM	**0.218370**	**(-0.918, 1.457)**	**0.1505**	**0.4337**	**0.3246**	**0.4673**	**9903**
	CQM	**0.222412**	**(-0.952, 1.451)**	**0.1553**	**0.4433**	**0.3361**	**0.4737**	**9903**
	HL1	0.238794	(-0.996, 1.505)	0.1642	0.4605	0.3548	0.4820	9972
	HL2	0.230044	(-0.951, 1.472)	0.1554	0.4441	0.3362	0.4709	10000
	HL3	0.233839	(-0.988, 1.476)	0.1592	0.4509	0.3426	0.4754	9999
8	Mean	0.222885	(-0.893, 1.381)	0.1380	0.4149	0.3130	0.4149	10000
	NM	0.206020	(-0.927, 1.436)	0.1438	0.4194	0.3087	0.4578	9998
	MM	0.207609	(-0.908, 1.458)	0.1451	0.4222	0.3125	0.4607	9998
	MIR	**0.239107**	**(-1.034, 1.473)**	**0.1671**	**0.4660**	**0.3597**	**0.4848**	**9998**
	CSQM	**0.207609**	**(-0.908, 1.458)**	**0.1451**	**0.4222**	**0.3125**	**0.4607**	**9998**
	CQM	**0.207609**	**(-0.908, 1.458)**	**0.1451**	**0.4222**	**0.3125**	**0.4607**	**9998**
	HL1	0.228461	(-0.932, 1.405)	0.1479	0.4324	0.3291	0.4584	10000
	HL2	0.223649	(-0.919, 1.396)	0.1434	0.4241	0.3223	0.4530	10000
	HL3	0.225126	(-0.920, 1.405)	0.1453	0.4275	0.3262	0.4557	10000
9	Mean	0.227374	(-0.732, 1.273)	0.1154	0.3781	0.2935	0.3781	10000
	NM	0.201261	(-0.772, 1.318)	0.1202	0.3809	0.2814	0.4247	9954
	MM	0.204624	(-0.776, 1.347)	0.1221	0.3841	0.2810	0.4271	9954
	MIR	**0.241020**	**(-0.864, 1.325)**	**0.1357**	**0.4154**	**0.3253**	**0.4363**	**9954**
	CSQM	**0.211146**	**(-0.738, 1.298)**	**0.1165**	**0.3766**	**0.2837**	**0.4147**	**9954**
	CQM	**0.214552**	**(-0.744, 1.298)**	**0.1185**	**0.3809**	**0.2899**	**0.4163**	**9954**
	HL1	0.235032	(-0.791, 1.279)	0.1240	0.3945	0.3114	0.4194	9998
	HL2	0.228849	(-0.770, 1.275)	0.1194	0.3855	0.2999	0.4137	9999
	HL3	0.231279	(-0.774, 1.278)	0.1214	0.3895	0.3044	0.4165	9997
10	Mean	0.227825	(-0.689, 1.188)	0.0994	0.3495	0.2736	0.3495	10000
	NM	0.209724	(-0.697, 1.246)	0.1044	0.3530	0.2636	0.3949	10000
	MM	0.208261	(-0.692, 1.254)	0.1041	0.3528	0.2660	0.3963	10000
	MIR	**0.243486**	**(-0.874, 1.307)**	**0.1327**	**0.4087**	**0.3207**	**0.4295**	**10000**
	CSQM	**0.208261**	**(-0.692, 1.254)**	**0.1041**	**0.3528**	**0.2660**	**0.3963**	**10000**
	CQM	**0.208261**	**(-0.692, 1.254)**	**0.1041**	**0.3528**	**0.2660**	**0.3963**	**10000**
	HL1	0.233171	(-0.738, 1.198)	0.1067	0.3648	0.2867	0.3913	9997
	HL2	0.228996	(-0.715, 1.195)	0.1033	0.3574	0.2794	0.3871	10000
	HL3	0.231298	(-0.717, 1.188)	0.1048	0.3606	0.2819	0.3888	9998
11	Mean	0.232300	(-0.611, 1.149)	0.0917	0.3359	0.2626	0.3358	10000
	NM	0.207683	(-0.634, 1.192)	0.0957	0.3393	0.2571	0.3838	9962
	MM	0.208613	(-0.636, 1.209)	0.0972	0.3420	0.2590	0.3864	9962
	MIR	**0.246995**	**(-0.713, 1.215)**	**0.1094**	**0.3699**	**0.2923**	**0.3917**	**9962**
	CSQM	**0.212899**	**(-0.622, 1.183)**	**0.0928**	**0.3336**	**0.2550**	**0.3757**	**9962**
	CQM	**0.215119**	**(-0.616, 1.180)**	**0.0939**	**0.3363**	**0.2605**	**0.3767**	**9962**
	HL1	0.237500	(-0.674, 1.173)	0.1001	0.3515	0.2768	0.3790	9998
	HL2	0.234200	(-0.631, 1.160)	0.0959	0.3439	0.2688	0.3729	10000
	HL3	0.236002	(-0.653, 1.166)	0.0977	0.3472	0.2722	0.3753	9999

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Table G.4: Mean, NM, MM, MIR, CSQM, CQM, HL1, HL2, and HL3 for Wrapped Laplace Distribution, λ_1 = 0.8, λ_2 = 1.25, p = 0.61, Mean = 0.2213 (See Appendix I), Median = 0.3955 (See Appendix I), Based on 10,000 Random Samples.

Sample Size	Measure	Point Est	95 % CL	CV	Mean Abs Dev	Median Abs Dev	Mean Abs Error	n
12	Mean	0.227262	(-0.590, 1.086)	0.0851	0.3236	0.2559	0.3235	10000
	NM	0.206452	(-0.593, 1.106)	0.0881	0.3255	0.2513	0.3708	9999
	MM	0.206406	(-0.601, 1.129)	0.0892	0.3283	0.2539	0.3738	9999
	MIR	**0.247597**	**(-0.716, 1.184)**	**0.1068**	**0.3668**	**0.2892**	**0.3871**	**9999**
	CSQM	**0.206406**	**(-0.601, 1.129)**	**0.0892**	**0.3283**	**0.2539**	**0.3738**	**9999**
	CQM	**0.206406**	**(-0.601, 1.129)**	**0.0892**	**0.3283**	**0.2539**	**0.3738**	**9999**
	HL1	0.232414	(-0.632, 1.112)	0.0910	0.3354	0.2649	0.3644	9998
	HL2	0.228861	(-0.620, 1.098)	0.0886	0.3304	0.2601	0.3616	10000
	HL3	0.230295	(-0.624, 1.105)	0.0900	0.3332	0.2626	0.3634	10000
13	Mean	0.228718	(-0.559, 1.044)	0.0794	0.3138	0.2509	0.3138	10000
	NM	0.204978	(-0.570, 1.086)	0.0829	0.3164	0.2449	0.3641	9984
	MM	0.205317	(-0.593, 1.111)	0.0848	0.3196	0.2460	0.3672	9984
	MIR	**0.246932**	**(-0.661, 1.164)**	**0.0997**	**0.3551**	**0.2848**	**0.3758**	**9984**
	CSQM	**0.208799**	**(-0.575, 1.093)**	**0.0805**	**0.3115**	**0.2430**	**0.3565**	**9984**
	CQM	**0.210428**	**(-0.574, 1.087)**	**0.0808**	**0.3127**	**0.2424**	**0.3556**	**9984**
	HL1	0.234454	(-0.577, 1.077)	0.0851	0.3257	0.2625	0.3546	9998
	HL2	0.230694	(-0.570, 1.074)	0.0827	0.3205	0.2545	0.3516	10000
	HL3	0.232821	(-0.568, 1.080)	0.0837	0.3227	0.2583	0.3527	9999
14	Mean	0.231151	(-0.507, 1.032)	0.0732	0.2985	0.2342	0.2984	10000
	NM	0.208951	(-0.541, 1.074)	0.0767	0.3010	0.2311	0.3478	9997
	MM	0.208221	(-0.553, 1.084)	0.0774	0.3024	0.2314	0.3499	9997
	MIR	**0.255083**	**(-0.652, 1.146)**	**0.0970**	**0.3476**	**0.2753**	**0.3680**	**9997**
	CSQM	**0.208221**	**(-0.553, 1.084)**	**0.0774**	**0.3024**	**0.2314**	**0.3499**	**9997**
	CQM	**0.208221**	**(-0.553, 1.084)**	**0.0774**	**0.3024**	**0.2314**	**0.3499**	**9997**
	HL1	0.236502	(-0.535, 1.069)	0.0779	0.3091	0.2424	0.3386	9998
	HL2	0.232815	(-0.523, 1.064)	0.0757	0.3045	0.2377	0.3363	10000
	HL3	0.234584	(-0.523, 1.064)	0.0767	0.3065	0.2409	0.3372	10000
15	Mean	0.233901	(-0.459, 0.996)	0.0662	0.2866	0.2333	0.2866	10000
	NM	0.207615	(-0.497, 1.049)	0.0712	0.2914	0.2283	0.3404	9987
	MM	0.208579	(-0.511, 1.074)	0.0734	0.2953	0.2320	0.3447	9987
	MIR	**0.251969**	**(-0.559, 1.076)**	**0.0818**	**0.3202**	**0.2606**	**0.3423**	**9987**
	CSQM	**0.212045**	**(-0.480, 1.041)**	**0.0693**	**0.2875**	**0.2250**	**0.3339**	**9987**
	CQM	**0.213734**	**(-0.479, 1.040)**	**0.0692**	**0.2879**	**0.2258**	**0.3323**	**9987**
	HL1	0.238426	(-0.485, 1.024)	0.0702	0.2961	0.2402	0.3261	9998
	HL2	0.235370	(-0.479, 1.016)	0.0685	0.2919	0.2357	0.3239	10000
	HL3	0.236984	(-0.481, 1.017)	0.0694	0.2941	0.2374	0.3250	9999
16	Mean	0.227700	(-0.466, 0.983)	0.0634	0.2764	0.2162	0.2763	10000
	NM	0.204700	(-0.485, 1.009)	0.0654	0.2775	0.2142	0.3312	9999
	MM	0.203977	(-0.485, 1.020)	0.0667	0.2802	0.2160	0.3344	9999
	MIR	**0.250352**	**(-0.588, 1.108)**	**0.0838**	**0.3206**	**0.2519**	**0.3435**	**9999**
	CSQM	**0.203977**	**(-0.485, 1.020)**	**0.0667**	**0.2802**	**0.2160**	**0.3344**	**9999**
	CQM	**0.203977**	**(-0.485, 1.020)**	**0.0667**	**0.2802**	**0.2160**	**0.3344**	**9999**
	HL1	0.233130	(-0.492, 1.018)	0.0676	0.2856	0.2241	0.3216	10000
	HL2	0.230623	(-0.482, 1.002)	0.0660	0.2821	0.2198	0.3198	10000
	HL3	0.231678	(-0.484, 1.008)	0.0667	0.2836	0.2221	0.3206	10000

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Appendix H

Algorithm for Computing the MIR, CSQM, & CQM

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Step 1: Check number of observations (n) is odd or even and put all observ ations in to range $[-\pi, \pi)$ by adjusting for multiple of 2π. The R codes *rangey* is provided to put data in to range $[-\pi, \pi)$.

Step 2: Sort observations.

Step 3: Identify all potential median observations within $[-\pi, \pi)$ range

1. If sample size, n, is odd, each observation is the potential median

$$x_i = pm_i$$

where pm_i stands for potential median.

2. When n is even, the potential median is the circular mean of two adjacent observations. For any $i = 1, \ldots, n-1$

$$pm_i = \frac{(x_{(i)} + x_{(i+1)})}{2} \tag{H.1}$$

For the last observation connected to the first observation in the circle, we need to put this mid-point in the right position, i.e.,

$$pm_n = \frac{(x_{(1)} + x_{(n)})}{2} + \pi, \qquad \text{if } pm_n < \pi \tag{H.2}$$

$$pm_n = \frac{(x_{(1)} + x_{(n)})}{2} - \pi, \qquad \text{if } pm_n \geq \pi. \tag{H.3}$$

The R codes *evenmd1* is provided to calculate all potential medians for even sample size.

Step 4: Find the median that satisfies the concept of Mardia median. i.e., the majority of data is closer to median than to anti median by using the concept of *cosine function*

$$\sum_{i=1}^{n}(\cos(x_i - \hat{\theta}_i) > \frac{(n-1)}{2}) \tag{H.4}$$

and equal split by using the concept of *sine function.*

$$\sum_{i=1}^{n} \text{sgn}(\sin(x_i - \hat{\theta}_i)) = 0. \tag{H.5}$$

where $\hat{\theta}_i = pm_i$ (see H.2 - H.3)

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Step 5A: As a result we obtain semi-circle. The (linear) median for the left is the first (lower) quantile (Q_L) while the (linear) median for the right is the third (upper) quantile (Q_R). The circular mean of Q_L and Q_R is the proposed MIR estimator of the median direction on the circle. The R code *mir1* calculates the MIR median.

Step 5B: The average (circ.mean) of the two symmetrically placed order statistics closed to the Mardia median is the proposed central quasi median estimator of the median direction on the circle. The R code *quasim* calculates the central quasi median.

Step 5C: The average (circ.mean) of the two symmetrically placed order statistics included Mardia median is the proposed central smoothed quasi median estimator of the median direction on the circle. The R code *qmd* calculates the central smoothed quasi median.

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Appendix I

Computation for Wrapped Laplace Distribution

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To compute the mean and median for wrapped Laplace distribution, by using mixture of two exponential distribution, we first need to use equation (1.41) to obtain p and λ.

Mean and median in Table G.1 - G.2 are obtained as follow:

Suppose $\lambda = 1$ and $\kappa = 2$, therefore $\lambda_1 = \lambda\kappa = 2$, $\lambda_2 = \lambda/\kappa = 1/2$, and $p = \frac{1}{\kappa^2+1} = 1/5$.

Recall the sample mean direction is:

$$\hat{\bar{\alpha}} = \begin{cases} \arctan\frac{1}{\lambda\kappa} - \arctan\frac{\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \arctan\frac{1}{\lambda\kappa} - \arctan\frac{\kappa}{\lambda}, & \kappa > 1. \end{cases} \tag{I.1}$$

Therefore

$$\begin{aligned} \hat{\bar{\alpha}} &= 2\pi + \arctan\frac{1}{\lambda\kappa} - \arctan\frac{\kappa}{\lambda}, \quad \text{for} \kappa > 1. \\ &= 2\pi + \arctan\frac{1}{2} - \arctan(2) \\ &= 2\pi + 0.463648 - 1.107149 \\ &= 5.639684 \quad (\text{mod } 2\pi) \\ &= \mathbf{-0.6435} \end{aligned}$$

The sample median direction is:

$$\hat{\xi}_0 = \begin{cases} \xi^*, & \lambda > 0, 0 < \kappa < 1 \\ \xi^* + \pi, & \lambda > 0, \kappa > 1, \end{cases} \tag{I.2}$$

with $\xi \in [0, \pi]$ such that

$$\frac{1}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\xi}}{1+e^{-\lambda\kappa\pi}} + \kappa^2 \frac{e^{\lambda\xi/\kappa}}{1+e^{\lambda\pi/\kappa}}\right) = \frac{1}{2} \tag{I.3}$$

where λ and κ must satisfied (1.41).

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Therefore from equation (1.49), we get

$$
\begin{aligned}
\frac{1}{1+4}\left(\frac{e^{-2x}}{1+e^{-2\pi}}+\frac{4e^{x/2}}{1+e^{\pi/2}}\right) &= \frac{1}{2} \\
\left(\frac{e^{-2x}}{1+e^{-2\pi}}+\frac{4e^{x/2}}{1+e^{\pi/2}}\right) &= \frac{5}{2} \\
0.998136e^{-2x}+0.688412e^{x/2} &= \frac{5}{2} \\
e^{x/2}(0.998136e^{-4}+0.688412) &= \frac{5}{2} \\
e^{x/2} &= \frac{5}{2*0.706693} \\
&= 3.537602 \\
x/2 &= \ln(3.537602) \\
&= 1.263449 \\
x &= 2*1.263449 \\
&= 2.526898
\end{aligned}
$$

$$
\begin{aligned}
\text{so median} &= 2.526898+\pi = 5.668491 \\
&= 5.668491 \pmod{2\pi} \\
&= \mathbf{-0.6147}
\end{aligned}
$$

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Mean and median in Table G.3 - G.4 are obtained as follow:

suppose $\lambda = 1$ and $\kappa = 0.8$

therefore $\lambda_1 = \lambda\kappa = 0.8$

$\lambda_2 = \lambda/\kappa = 1/0.8 = 1.25$, and

$p = \frac{1}{\kappa^2+1} = 0.61$

the sample mean direction is:

$$\begin{aligned}
\hat{\bar{\alpha}} &= \arctan\frac{1}{\lambda\kappa} - \arctan\frac{\kappa}{\lambda}, \text{ for} \kappa < 1. \\
&= \arctan\frac{1}{0.8} - \arctan(0.8) \\
&= \arctan(1.25) - \arctan(0.8) \\
&= 0.896055 - 0.674741 \\
&= \mathbf{0.2213}
\end{aligned}$$

The sample median direction is:

$$\begin{aligned}
\frac{1}{1+0.64}\left(\frac{e^{-0.8x}}{1+e^{-0.8\pi}} + \frac{0.64e^{x/0.8}}{1+e^{\pi/0.8}}\right) &= \frac{1}{2} \\
\frac{e^{-0.8x}}{1+e^{-0.8\pi}} + \frac{0.64e^{x/0.8}}{1+e^{\pi/0.8}} &= 0.82 \\
e^{x/0.8}(0.925067e^{-0.64} + 0.012366) &= 0.82 \\
e^{x/0.8} &= 1.639517 \\
x/0.8 &= \ln(1.639517) \\
x &= 0.8 * 0.494402 \\
&= \mathbf{0.3955}
\end{aligned}$$

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Appendix J

R Codes

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The R system implements a dialect of the S language that was developed at AT&T Bell Laboratories by Rick Becker, John Chambers and Allan Wilks. Versions of R are available, at no cost. It is available through the Comprehensive. R Archive Network (CRAN). Go to http://cran.r-project.org/, and find the nearest mirror site.

R includes a number of packages in its library. The base package is automatically loaded at the beginning of the session. For working on the Circular Statistics we need to load CircStats and MASS at the beginning of working. For example:

```
Library (MASS)       # Loads the MASS package
Library (CircStats)  # Loads the CircStats package
```

Below are some of the R functions written for simulations. The input is vector x (in radians).

1. ave.ang3, this function calculates circular mean direction
```
ave.ang3←function(a, digits=options()$digits, na.value=NaN)
{y ← sum(sin(a))
x ← sum(cos(a))
ifelse(zapsmall(x,digits) ==0 & zapsmall(y,digits)==0,
na.value, atan(y,x))}
```

2. evenmd1, this function calculates all potential medians for even samples.
```
evenmd1←function(x){
n←length(x)
x←rangey(x)
x←sort(x)
evenmd←c()
for(i in 1:n-1){
evenmd[i]←(x[i]+x[i+1])/2}
for(i in 1:n){
s1←(x[1]+x[n])/2+pi
s2← (x[1]+x[n])/2-pi
emd←ifelse(s1 < pi,s1,s2)}
evenmd←c(evenmd,emd)
evenmd}
```

3. rangey, this function put data in to range $-\pi, \pi$.
```
rangey←function(x){
```


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```
#transform data in to range(−π, π)
y←ifelse(x < -pi,x+2*pi,x)
y1←ifelse(y > pi,y-2*pi,y)
return(y1)}
```

4. MIR, this function give the sample MIR.

```
mir1←function(x){
mm←cmedM(x)
lenx←length(x)
x←rangey(x)
x←sort(x)
upperq←c()
lowerq←c()
for(i in 1:lenx){
newx[i]←x[i]-mm
newx[i]←rangey(newx[i])
upperq[i]←ifelse(round(newx[i],10) ≤ 0, newx[i],9999)
lowerq[i]←ifelse(round(newx[i],10) ≥ 0, newx[i],9999)}
}
upperq←upperq[upperq≠ 9999]
lowerq←lowerq[lowerq ≠ 9999]
q1←median(lowerq)
q3←median(upperq)
y←c(q1,q3)
mir←circ.mean(y)
mir←mir+mm
mir←rangey(mir)
mir}
```

5. CQM, this function give the sample central quasi medians.

```
qmd←function(x){
mm←cmedM(x)
x←rangey(x)
x←sort(x)
lenx←length(x)
upperq←c()
lowerq←c()
for(i in 1:lenx){
newx[i]←x[i]-mm
newx[i]←x[i]-mm
```


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```
newx[i]←rangey(newx[i])
lowerq[i]←ifelse(round(newx[i],10) < 0,newx[i],9999)
upperq[i]←ifelse(round(newx[i],10) > 0,newx[i],9999)
}
lowerq←lowerq[lowerq≠ 9999]
upperq←upperq[upperq≠ 9999]
qR←max(lowerq)
qL←min(upperq)
y←c(qL,qR)
cqmd←circ.mean(y)
cqmd←cqmd+mm
cqmd←rangey(cqmd)
cqmd}
```

6. CSQM, this function give the sample central smooth quasi medians.

```
qmd←function(x){
mm←cmedM(x)
x←rangey(x)
x←sort(x)
lenx←length(x)
upperq←c()
lowerq←c()
for(i in 1:lenx){
newx[i]←x[i]-mm
newx[i]←x[i]-mm
newx[i]←rangey(newx[i])
lowerq[i]←ifelse(round(newx[i],10) < 0,newx[i],9999)
upperq[i]←ifelse(round(newx[i],10) > 0,newx[i],9999)
}
lowerq←lowerq[lowerq≠ 9999]
upperq←upperq[upperq≠ 9999]
qR←max(lowerq)
qL←min(upperq)
y←c(qL,0.0,qR)
qmd←circ.mean(y)
qmd←qmd+mm
qmd←rangey(qmd)
qmd}
```

7. simumat4, this function simulate data from wrapped exponential dis-

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tribution.
```
simumat4←function(n,scale,M){
res←matrix(nrow=n,ncol=M)
res1←c()
for(i in 1:M){
#Converts exponential variates to wrapped exponential variates
res1←rexp(n,scale)%% (2*pi)
res[,i]←res1}
res}
```

8. cmd, this function calculates circular mean absolute deviation
```
cmd←function(x,est){
cmd←pi-mean(abs(pi-(abs(rangey(x-est)))))
cmd}
```

9. cmeda, this function calculates circular median absolute deviation
```
cmeda←function(x,est){
cmeda←pi-median(abs(pi-abs(rangey(x-est))))
cmeda}
```

10. cme, this function calculates circular mean absolute error
```
cme←function(x,parameter){
cme←pi-mean(abs(pi-(abs(rangey(x-parameter)))))
cme}
```

11. cmee, this function calculates circular median absolute error
```
cmee←function(x,parameter){
cmee←pi-median(abs(pi-abs(rangey(x-parameter))))
cmee}
```

12. test, this function calculates 9 location estimators on the circle
```
test←function(x,pmean,pmedian){
temp1←cmean1(x)
temp2←cnm1(x)
temp3←cmm1(x)
cmedM.na←is.na(apply(x,2,cmedM))
newx←x[,cmedM.na==FALSE]
temp4←MIR1(newx)
```


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```
temp8←qmd1(newx)
temp9←quasi(newx)
temp5←HL1(x)
temp6←HL2(x)
temp7←HL3(x)
list(temp1,temp2,temp3,temp4,temp8,temp9,temp5,temp6,temp7)}
```

13. cmean1, this function calculates circular mean, CI, circular variance, circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
cmean1←function(x,pmean){
a1←apply(x,2,ave.ang3)
a1←sort(a1)
n1←length(a1)
lowerb1←a1[n1*.025]
upperb1←a1[n1*.975]
est1← ave.ang3(a1)
rho1← est.rho(a1)
vari1←1-rho1
cmd1←cmd(a1,est1)
cmeda1←cmeda(a1,est1)
cmae1←pi-mean(abs(pi-(abs(a1-pmean))))
list(cmean=est1,rho1=rho1,varience1=vari1,CMD1=cmd1,CMEDA1
=cmeda1,CMAE1=cmae1,lowerb1=lowerb1,upperb1=upperb1,n1=n1)}
```

14. cnm1, this function calculates circular new median, CI, circular variance, circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
cnm1←function(x,pmedian){
a2←apply(x,2,cmed)
a2←sort(a2)
n2←length(a2)
lowerb2←a2[n2*0.025]
upperb2←a2[n2*.975]
est2← ave.ang3(a2)
rho2← est.rho(a2)
vari2←1-rho2
cmd2←cmd(a2,est2)
cmeda2←cmeda(a2,est2)
cmae2←pi-mean(abs(pi-(abs(a2-pmedian))))
```


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```
list(cnm=est2,rho2=rho2,varience2=vari2,CMD2=cmd2,CMEDA2
=cmeda2,CMAE2=cmae2,lowerb2=lowerb2,upperb2=upperb2,n2=n2)}
```

15. cmm1, this function calculates circular Madia median, CI, circular variance, circular mean absolute deviation circular median absolute deviation and circular mean absolute error.

```
cmm1←function(x,pmedian){
a3←apply(x,2,cmedM)
a3←sort(a3)
n3←length(a3)
lowerb3←a3[n3*.025]
upperb3←a3[n3*.975]
est3← ave.ang3(a3)
rho3← est.rho(a3)
vari3←1-rho3
cmd3←cmd(a3,est3)
cmeda3←cmeda(a3,est3)
cmae3←pi-mean(abs(pi-(abs(a3-pmedian))))
list(cmm=est3,rho3=rho3,varience3=vari3,CMD3=cmd3,CMEDA3
=cmeda3,CMAE3=cmae3,lowerb3=lowerb3,upperb3=upperb3,n3=n3)}
```

16. MIR1, this function calculates circular MIR, CI, circular variance , circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
MIR1←function(x,pmedian){
a4←apply(x,2,mir1)
a4←sort(a4)
n4←length(a4)
lowerb4←a4[n4*.025]
upperb4←a4[n4*.975]
est4← ave.ang3(a4)
rho4← est.rho(a4)
vari4←1-rho4
cmd4←cmd(a4,est4)
cmeda4←cmeda(a4,est4)
cmae4←pi-mean(abs(pi-(abs(a4-pmedian))))
list(MIR=est4,rho4=rho4,varience4=vari4,CMD4=cmd4,CMEDA4
=cmeda4,CMAE4=cmae4,lowerb4=lowerb4,upperb4=upperb4,n4=n4)}
```


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17. qmd1, this function calculates circular smoothed quasi medians, CI, circular variance, circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
qmd1←function(x,pmedian){
a8←apply(x,2,qmd)
a8←sort(a8)
n8←length(a8)
lowerb8←a8[n8*.025]
upperb8←a8[n8*.975]
est8← ave.ang3(a8)
rho8← est.rho(a8)
vari8←1-rho8
cmd8←cmd(a8,est8)
cmeda8←cmeda(a8,est8)
cmae8←pi-mean(abs(pi-(abs(a8-pmedian))))
list(quasiM=est8,rho8=rho8,varience8=vari8,CMD8=cmd8, CMEDA8=
=cmeda8,CMAE8=cmae8,lowerb8=lowerb8,upperb8=upperb8,n8=n8)}
```

18. quasi, this function calculates circular quasi medians, CI, circular variance, circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
quasi←function(x,pmedian){
a9←apply(x,2,quasim)
a9←sort(a9)
n9←length(a9)
lowerb9←a8[n9*.025]
upperb9←a9[n9*.975]
est9←ave.ang3(a9)
rho9←est.rho(a9)
vari9←1-rho9
cmd9←cmd(a9,est9)
cmeda9←cmeda(a9,est9)
cmae9←pi-mean(abs(pi-(abs(a9-pmedian))))
list(quasim=est9,rho9=rho9,varience9=vari9,CMD9=cmd9,CMEDA9=
cmeda9,CMAE9=cmae9,lowerb9=lowerb9,upperb9=upperb9,n9=n9)}
```

19. HL1, this function calculates circular HL1, CI, circular variance , circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
HL1←function(x,pmedian)}
```


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```
a5←apply(x,2,wppcircmed)
a5←sort(a5)
n5←length(a5)
lowerb5←a5[n5*.025]
upperb5←a5[n5*.975]
est5← ave.ang3(a5)
rho5← est.rho(a5)
vari5←1-rho5
cmd5←cmd(a5,est5)
cmeda5←cmeda(a5,est5)
cmae5←pi-mean(abs(pi-(abs(a5-pmedian))))
list(HL1=est5,rho5=rho5,varience5=vari5,CMD5=cmd5,CMEDA5
=cmeda5,CMAE5=cmae5,lowerb5=lowerb5,upperb5=upperb5,n5=n5)}
```

20. HL2, this function calculates circular HL2, CI, circular variance , circular mean absolute deviation,circular median absolute deviation and circular mean absolute error.

```
HL2←function(x,pmedian){
a6←apply(x,2,wppcircmed2)
a6←sort(a6)
n6←length(a6)
lowerb6←a6[n6*.025]
upperb6←a6[n6*.975]
est6← ave.ang3(a6)
rho6← est.rho(a6)
vari6←1-rho6
cmd6←cmd(a6,est6)
cmeda6←cmeda(a6,est6)
cmae6←pi-mean(abs(pi-(abs(a6-pmedian))))
list(HL2=est6,rho6=rho6,varience6=vari6,CMD6=cmd6,CMEDA6
=cmeda6,CMAE6=cmae6,lowerb6=lowerb6,upperb6=upperb6,n6=n6)}
```

21. HL3, this function calculates circular HL3, CI, circular variance , circular mean absolute deviation, circular median absolute deviation and circular mean absolute error.

```
HL3←function(x,pmedian){
a7←apply(x,2,wppcircmeda)
a7←sort(a7)
n7←length(a7)
lowerb7←a7[n7*.025]
```


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```
upperb7←a7[n7*.975]
est7← ave.ang3(a7)
rho7← est.rho(a7)
vari7←1-rho7
cmd7←cmd(a7,est7)
cmeda7←cmeda(a7,est7)
cmae7←pi-mean(abs(pi-(abs(a7-pmedian))))
list(HL3=est7,rho7=rho7,varience7=vari7,CMD7=cmd7,CMEDA7
=cmeda7,CMAE7=cmae7,lowerb7=lowerb7,upperb7=upperb7,n7=n7)}
```

22. simumat10, this function simulates data from wrapped Laplace distribution.

```
simumat10←function(n,scale1,scale2,p,M){
res←matrix(nrow=n,ncol=M)
res1←c()
for(i in 1:M){
#determines number of mixtures using binomial distribution
nummix←rbinom(1,n,p)
#Converts exponential variates to wrapped exponential variates
r1←rexp(n-nummix,scale1)%%(2*pi)
#Converts exponential variates to wrapped negative exponential variates
r2←rexp(nummix,scale2)
r2←-r2
r2←r2%%(2*pi)
res1←c(r1,r2)
res[,i]←sort(res1)}
res}
```


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Appendix K

Proof of Theorem 2 and Equation 1.6

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Theorem 2 (Cabrera, Maguluri and Singh, 1994)

If f has a continuous second derivative f'' in a neighborhood of M. then the variance of $(1/\sqrt{2})$-median has the following common expression for an even as well as odd sample size n, i.e.

$$E(M_n - M)^2 = \frac{1}{4nf^2(M)} - \frac{3}{4n^2 f^2(M)} + 0(1/n^2). \qquad \text{(K.1)}$$

Proof

Let F be any distribution function with density f. Let V_n denote the sample median for the uniform distribution. Then by the quantile transformation, we can write $M_n - M$ as

$$M_n - M = F^{-1}(V_n) - F^{-1}(\frac{1}{2}).$$

Here F^{-1} is the left continuous version of the inverse function. We can write $E(M_n - M)^2$ as

$$E(M_n - M)^2 = E(M_n - M)^2 I(|V_n - V| \le \epsilon) + E(M_n - M)^2 I(|V_n - V| > \epsilon) \;=\; I + II$$

By Cauchy-Schwarz inequality, the term II is bounded by

$$\{E(M_n - M)^4\}^{1/2}\{P(|V_n - V| > \epsilon)\}^{1/2}.$$

But $E(M_n - M)^4 < \infty$ for n large by condition (iii)(see for example Bickel, 1967) and $P(|V_n - V| > \epsilon) \to 0$ exponentially (see Inequality 1 in Shorack and Wellner, 1986, p. 453). Thus II is $0(n^{-3})$.

Now let look at I. Using Taylor series expansion, we have

$$\begin{aligned} F^{-1}(V_n) - F^{-1}(\frac{1}{2}) &= \frac{1}{f(M)}(V_n - \frac{1}{2}) - \frac{f'(M)}{2f^3(M)}(V_n - \frac{1}{2})^2 + \\ &+ \frac{1}{6}\left(\frac{-(f''f^2)(M) + 3(f'^2 f)(M)}{f^6(M)}\right)(V_n - \frac{1}{2})^3 + R_n, \end{aligned}$$

Where the contribution of R_n to I is $\frac{c}{n^3} + 0(n^{-3})$ for both n odd and even. Hence the MSE of M_n is given by

$$\begin{aligned} E(M_n - M)^2 &= \frac{1}{f^2(M)}E(V_n - \frac{1}{2})^2 - \frac{f'(M)}{f^4(M)}E(V_n - \frac{1}{2})^3 + \qquad \text{(K.2)} \\ &+ \frac{1}{f^6(M)}\left(\frac{(f'(M))}{4} + \frac{(3f'^3 - f''f)(M)}{3}\right)E(V_n - \frac{1}{2})^4 + \\ &+ \frac{c}{n^3} + 0(n^{-3}). \end{aligned}$$

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Since $E(V_n - \frac{1}{2})^3 = 0$, we need to compute only $E(V_n - \frac{1}{2})^2$ and $E(V_n - \frac{1}{2})^4$, to express the variance up to terms of order $\frac{1}{n^3}$. Now for $n = 2m+1, V_n = U_{(m+1)}$ and for $n = 2m, V_n = \frac{1}{2}(U_{(m)} + U_{(m+1)})$. where $U_{(i)}$ is the ith order statistic from the uniform distribution. Using the formula (3.1.7) in David (1970). we have

$$E(V_n - \frac{1}{2})^2 = \frac{1}{4(2m+3)} = \frac{1}{8m} - \frac{3}{16m^2} + \frac{9}{32m^3} + 0\left(\frac{1}{m^3}\right) \qquad \text{(K.3)}$$

for $n = 2m+1$, and

$$E(V_n - \frac{1}{2})^2 = \frac{m}{4(m+1)(2m+1)} = \frac{1}{8m} - \frac{3}{16m^2} + \frac{7}{32m^3} + 0\left(\frac{1}{m^3}\right) \qquad \text{(K.4)}$$

for $n = 2m$. Next considering $E(V_n - \frac{1}{2})^4$, we have

$$E(V_n - \frac{1}{2})^4 \doteq \begin{cases} \frac{36}{4^4m^3} - \frac{48}{4^4m^3} = \frac{-12}{4^4m^3} & \text{for } n = 2m+1, \\ \frac{12}{4^4m^3} - \frac{24}{2^4m^3} = \frac{-12}{4^4m^3} & \text{for } n = 2m. \end{cases} \qquad \text{(K.5)}$$

This prove by substitute (K.3), (K.4) and (K.5) in (K.2).

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Equation 1.6 (Hodges and Lehmann, 1967)

Proof

Recall Hodges and Lehmann proposed a *Quasi Medians* as an average of two order statistics of the same order when counting is done from both ends:

$$M_r = \begin{cases} (Y_{m+1-r} + Y_{m+1+r})/2 & \text{if } n = 2m+1 \\ (Y_{m-r} + Y_{m+1+r})/2 & \text{if } n = 2m, \end{cases} \tag{K.6}$$

where $Y_1 < Y_2 < ... < Y_n$ are the ordered observations, and r is fixed, and $2r < n$. It turns out that up to terms of order $1/n^2$

$$\text{Var}(M_r) = \begin{cases} \frac{1}{4f^2 n} - \frac{1}{16 f^2 n^2}(g + 8r + 8), & \text{if } n = odd \\ \frac{1}{4f^2 n} - \frac{1}{16 f^2 n^2}(g + 8r + 12), & \text{if } n = even \end{cases} \tag{K.7}$$

(r is assumed fixed, $2r < n$, n is large, and $f = f(0)$ and $g = f''(0)/f^3(0)$). Putting successively $n = 2m+1, 2m, 2m-1$ and using the fact that $1/(4m+2) \approx (1 - 1/2m)/4m$, etc., we obtain,

$$\text{Var}(M_r) = \begin{cases} \frac{1}{8mf^2} - \frac{1}{64m^2 f^2}(g + 8r + 12) & n = 2m, 2m+1 \\ \frac{1}{8mf^2} - \frac{1}{64m^2 f^2}(g + 8(r-1) + 12) & n = 2m - 1. \end{cases} \tag{K.8}$$

Distribution and Variance of Quasi Medians

Let $X_1, \ldots, X_n$ be identically, independently distributed with distribution F whose symmetric density f has a continuous second derivative at the origin. We are concerned with an asymptotic expansion of the distribution and variance of the quasi medians M_r defined by (K.6). Similar results for the median and other order statistics were obtained by a slightly different method by David and Johnson (1954). (See also the discussion in [4], Section 2.1.)

Consider the case $n = 2m$. The joint density of Y_{m-r} and Y_{m+r+1} is, up to a constant factor, given by

$$h(x, y) \sim [F(x)]^{a-r}[F(y) - F(x)]^{2r}[1 - F(y)]^{a-r} f(x) f(y) \tag{K.9}$$

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where $a = m - 1$. By using the expansion

$$F(x) \sim \frac{1}{2} + xf + \frac{1}{6}x^3 f'', \quad 1 - F(y) \sim \frac{1}{2} - yf - \frac{1}{6}y^3 f'' \tag{K.10}$$

where $f = f(0), f'' = f''(0)$, and introducing $s = 2a(y-x)f, t = \sqrt{2a}(y+x)f$, we find that to terms of order $1/a$, and up to an additive constant,

$$a \log\{F(x)[1-F(y)]\} = -s - \frac{t^2}{2} + \frac{1}{a}\left[\frac{-s^2}{4} - \frac{gt^4}{48} - \frac{t^4}{8} - \frac{gst^2}{16} - \frac{st^2}{2}\right] \tag{K.11}$$

where $g = f''/f^3$. Similarly,

$$[F(y) - F(x)]^{2r} = s^{2r}\left[1 + \frac{rgt^2}{8a}\right]$$

$$\frac{f(x)f(y)}{f^2} = 1 + \frac{gt^2}{2a}$$

$$\{4F(x)[1 - F(y)]\}^{-r} = 1 + \frac{rs}{a} + \frac{rt^2}{2a}, \tag{K.12}$$

all up to a constant factor and to terms of order $1/a$. The joint density of s and t (considered as random variables), which is proportional to $h(x, y)$, may therefore be written, again up to a constant factor, as

$$\chi^2_{2r+1}(s)\phi(t)\left[1 + \frac{1}{a}R(s,t)\right] \tag{K.13}$$

where χ^2_{2r+1} and ϕ denote the χ^2-density with $2r + 1$ degrees of freedom and the standard normal density respectively, and where

$$R(s,t) = rs + \frac{rt^2}{2} + \frac{(r+1)gt^2}{8} - \frac{gst^2}{16} - \frac{gt^4}{48} - \frac{s^2}{4} - \frac{st^2}{2} - \frac{t^4}{8}. \tag{K.14}$$

This shows that to first order terms, s and t are independently distributed as χ^2_{2r+1} and standard normal respectively, but that to terms of order $1/a$ these variables are no longer independent.

We next need to integrate, in order to get an expansion first for the density of t, and then for its variance. These formal operations will require further assumptions on F, since for example the variance of the median need not even exist. In the present paper we do not provide rigorous justification for these integrations, so that the rest of the argument is heuristic. We hope to return to this mathematical problem at a later time.

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Integrating (K.13) with respect to s gives the density of t as

$$K\phi(t)\left[1+\frac{1}{a}(A+Bt^2+Ct^4)\right] \tag{K.15}$$

with

$$A=\frac{(r-1)(2r+1)}{2},\quad B = \frac{g-8(r+1)}{16},\quad C = \frac{-(g+6)}{48} \tag{K.16}$$

To determine K, one may integrate (K.15) from $-\infty$ to ∞. Setting the integral equal to one yields

$$K=1-\frac{A+B+3C}{a}$$

and hence for the density of t, up to terms of order $1/a$,

$$p(t)=\phi(t)\left[1-\frac{1}{a}(B+3C+Bt^2+Ct^4)\right]. \tag{K.17}$$

Integrating once more, we find for the variance of t

$$\mathrm{Var}(t)=\int t^2p(t)dt = 1+\frac{2}{a}(B+6C). \tag{K.18}$$

Since $M_r = t/2f\sqrt{2a}$, the variance of M_r is obtained by dividing the right hand side of (K.18) by $8af^2$; on substituting the value of B and C from (K.16) and replacing a by $m-1$, we found formula (K.8) and (K.7) respectively. For $n=2m$. The case of odd n requires only minor changes.

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