Stratification and Domination in Graphs and Digraphs

Ralucca M. Gera
Western Michigan University

Follow this and additional works at: https://scholarworks.wmich.edu/dissertations

Part of the Applied Mathematics Commons, and the Mathematics Commons

Recommended Citation
STRATIFICATION AND DOMINATION
IN GRAPHS AND DIGRAPHS

by
Ralucca M. Gera

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics

Western Michigan University
Kalamazoo, Michigan
April 2005
In this thesis we combine the idea of stratification with the one of domination in graphs and digraphs, respectively.

A graph is 2-stratified if its vertex set is partitioned into two classes, where the vertices in one class are colored red and those in the other class are colored blue. Let \( F \) be a 2-stratified graph rooted at some blue vertex \( v \). An \( F \)-coloring of a graph \( G \) is a red-blue coloring of the vertices of \( G \) in which every blue vertex \( u \) belongs to a copy of \( F \) rooted at \( u \). The \( F \)-domination number \( \gamma_F(G) \) is the minimum number of red vertices in an \( F \)-coloring of \( G \).

Initially, \( F \)-domination is studied where \( F \) is the red-blue-blue path of order 3 rooted at the blue end-vertex. The \( F \)-domination numbers of some well-known classes of graphs are determined. Bounds for the \( F \)-domination number of a connected graph are established in terms of its order, clique number, diameter, girth, and maximum degree. Characterizations for connected graphs of order \( n \) with \( F \)-domination number \( n \) or 1 are presented.

A set \( S \) of vertices in a graph \( G \) is a dominating set for \( G \) if every vertex not in \( S \) is adjacent to some vertex in \( S \). The domination number \( \gamma(G) \) of the graph \( G \) is the minimum number of vertices in a dominating set for \( G \). A set \( S \) of vertices in a graph \( G \) is an open dominating set if every vertex of \( G \) is adjacent to at least one vertex of \( S \). The minimum cardinality of an open dominating set is the open domination number \( \gamma_0(G) \) of \( G \). A set \( S \) of vertices in \( G \) is a 2-step dominating set if for every vertex \( u \in V(G) - S \), there exists a path of length 2 from \( u \) to some vertex in \( S \). The 2-step
domination number $\gamma_{22}(G)$ is the minimum cardinality of any 2-step dominating set of $G$. A set $S$ of vertices in $G$ is a restrained dominating set of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V(G) - S$. The restrained domination number $\gamma_o(G)$ is the minimum cardinality of a restrained dominating set of $G$. We study the relationships between $F$-domination and these four domination parameters. It is shown in Chapter 3 that (1) for each pair $a, b$ of positive integers, there exists a connected graph $G$ such that $\gamma(G) = a$ and $\gamma_F(G) = b$; (2) for each pair $a, b$ of positive integers with $a \geq 2$, there exists a connected graph $G$ such that $\gamma_0(G) = a$ and $\gamma_F(G) = b$; (3) for each pair $a, b$ of positive integers with $a \leq b$, there exists a connected graph $G$ such that $\gamma_{22}(G) = a$ and $\gamma_F(G) = b$ if and only if $(a, b) \neq (1, i)$ for some $i \geq 2$; and (4) for each pair $a, b$ of positive integers with $a \leq b$, there exists a connected graph $G$ with $\gamma_F(G) = a$ and $\gamma_r(G) = b$. In Chapter 4, we show that a triple $(A, B, C)$ of positive integers with $A \leq B \leq 2A$ and $B \geq 2$ is realizable as the domination number, open domination number, and $F$-domination number, respectively, for some connected graph if and only if $(A, B, C) \neq (k, k, C)$ for integers $k$ and $C$ with $C > k \geq 2$.

As with 2-stratified graphs, a digraph is 2-stratified if its vertex set is partitioned into two classes. For a 2-stratified digraph $H$ rooted at some blue vertex, the $H$-domination number $\gamma_H(D)$ of an digraph $D$ is defined as expected. In Chapter 5, $H$-domination is studied where $H$ is the red-red-blue directed path of order 3. We study relationships between the $H$-domination number $\gamma_H$ and both the domination number $\gamma$ and open domination $\gamma_o$ in digraphs. It is shown that $\gamma(D) \leq \gamma_H(D) \leq \gamma_o(D) \leq \left\lceil \frac{3\omega(D)}{2} \right\rceil$ for every oriented graph $D$. A pair $a, b$ of positive integers with $a \leq b$ and $b \geq 2$ is realizable as the $H$-domination number and open domination number, respectively, for some connected oriented graph if and only if $(a, b) = (2, 3)$ or $3 \leq a \leq b \leq \left\lceil \frac{3\omega(D)}{2} \right\rceil$. A digraph $D$ is $r$-regular, where $r$ is a positive integer, if $\text{id} v = \text{od} v = r$ for every vertex $v$ of $D$. Upper and lower bounds are established for the $H$-domination number of an $r$-regular oriented graph in terms of $r$ and its order.
INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.
ACKNOWLEDGEMENTS

First and foremost I would like to thank my advisor, Dr. Ping Zhang, for her wisdom and patience with me, and also for knowing how to guide me along. Her continuous moral support during these years, enthusiasm, insight and help are greatly appreciated. Without her support I would not have been where I am today.

I am also very grateful to the members of my committee: Professors Gary Chartrand, Clifton Ealy, Allen Schwenk and my outside reader, Professor Garry Johns, for their suggestions and many hours spent on my dissertation. I especially want to thank the professors and students involved in the Graph Theory seminar for their support when I started presenting. The experience to study under the tutelage of respected and acknowledged scholars in the field provided me with the rare opportunity to steep myself in a rich research tradition that serves as a trajectory for my career aspirations. Not only was I exposed to the complex systems of graph theory but also to a world that requires discipline, patience, and perseverance.

I would also like to thank the faculty and staff of the Mathematics department of Western Michigan University for their devotion to their students, and for making my life a little easier on a great number of occasions. Also, I would have been at a loss without many other professors’ advice, comments and suggestions throughout my dissertation process. This includes Professors Yousef Alavi, Joseph Buckley, Paul Eenigenburg, Terrell Hodge, Niloufer Mackey, John Martino, Tabitha Mingus, John Petro, Jeffrey Strom, Arthur White and Margo Chapman.

Special thanks go to my family. My parents, Stephanie and Petru, my husband, Florin, and my brother Raz, for their understanding and support during these years. I would especially want to thank my parents for stressing the importance of higher education, and for doing everything they could to help me achieve it.

Ralucca M. Gera
TABLE OF CONTENTS

ACKNOWLEDGEMENTS ...................................................... ii

LIST OF FIGURES .............................................................. v

1 Introduction ................................................................. 1
   1.1 Motivation and Background ..................................... 1
   1.2 Basic Definitions and Notation ............................... 3
   1.3 Some Known Results .............................................. 6

2 $F_3$-Domination in Graphs ............................................. 10
   2.1 Introduction ........................................................ 10
   2.2 Preliminary Results .............................................. 14
   2.3 $F$-Domination Numbers of Some Well-known Graphs .... 17
   2.4 Joins and Cartesian Products of Graphs .................... 27
   2.5 Bounds for $F$-Domination Numbers of Graphs .......... 33
   2.6 Graphs with Given Order and $F$-Domination Number .... 46

3 Comparing Domination Parameters ................................. 62
   3.1 Introduction ....................................................... 62
   3.2 $F$-Domination and Domination ............................... 67
   3.3 $F$-Domination and Open Domination ....................... 72
   3.4 $F$-Domination and 2-Step Domination ..................... 76
   3.5 $F$-Domination and Restrained Domination ................ 82

4 Realizable Triples ......................................................... 88
   4.1 Introduction ....................................................... 88
   4.2 Two Special Realizable Triples ............................... 90
   4.3 Realizable Triples of Type I ................................... 95
   4.4 Realizable Triples of Type II ................................. 103
Table of Contents - Continued

4.5 Realizable Triples of Type III ................................................................. 113

5 Stratified Domination in Oriented Graphs .............................................. 127
  5.1 Introduction ......................................................................................... 127
  5.2 $H$-Domination and Standard Dominations ....................................... 133
  5.3 $H$-Domination in Regular Oriented Graphs ...................................... 145

6 Topics for Further Study ................................................................. 162
  6.1 Extremal Problems in $F$-Domination ............................................. 162
  6.2 Realizable Triples in $H$-Domination .............................................. 164
  6.3 Lower and Upper $H$-Domination Numbers ................................... 168

REFERENCES ......................................................................................... 174
# LIST OF FIGURES

1. Two 2-stratified graphs $F$ and $F'$ .......................................................... 3  
2. An $F$-coloring of a graph ................................................................................. 4  
3. A minimum $H_i$-dominating set ($i = 1, 2, 3$) for a graph $G$ .................... 5  
4. Showing the converse of Observation 1.2 is false ........................................... 6  
5. Illustrating Observation 1.3 .............................................................................. 6  
6. The five 2-stratified graphs $P_3$ ..................................................................... 7  
7. The 2-stratified graph $F$ and a copy $F'$ of $F$ ................................................ 10  
8. A red-blue coloring of the Petersen graph $P$ ................................................... 11  
9. The Heawood graph $H$ .................................................................................. 13  
10. The McGee graph $M$ ..................................................................................... 13  
11. Three graphs with $F$-domination number 1 .................................................. 15  
12. Minimum $F$-colorings of $G_1$ and $G_1 \times K_2$ .......................................... 30  
13. The graph $G_1$ in Proposition 2.37 for $k = 5$ ................................................. 44  
14. The graph $G_2$ in Proposition 2.37 for $k = 5$ ................................................. 45  
15. A red-blue coloring of the graph $G_k$ ............................................................. 47  
16. The graphs $H_1$ and $H_2$ fixed at $u$ .............................................................. 48  
17. Graphs with $F$-domination number 1 ............................................................. 50  
18. Realizable pairs $k, n$ for $3 \leq n \leq 6$ .............................................................. 54  
19. The graph $S_{a,b}$ ............................................................................................ 57  
20. The graphs $T_{a,b}$ and $T'_{a,b}$ ......................................................................... 58  
21. The graph $G$ for $k = 5$ .................................................................................. 66  
22. The graph $L_a$ ............................................................................................... 68  
23. The graph $G$ in Case 2 for proving (i) ........................................................... 69  
24. The graph $H$ in proving (ii) ........................................................................... 72  

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
<table>
<thead>
<tr>
<th>Page</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>The graphs $H_1$ and $H_2$</td>
<td>84</td>
</tr>
<tr>
<td>26</td>
<td>The graphs $H_1$ and $H_2$</td>
<td>91</td>
</tr>
<tr>
<td>27</td>
<td>The graph $G$ for $k = 2$ and $\ell = 1$ in Subcase 2.1</td>
<td>92</td>
</tr>
<tr>
<td>28</td>
<td>The graph $F_1$ in Subcase 2.3</td>
<td>93</td>
</tr>
<tr>
<td>29</td>
<td>The graph $G$ in Subcase 2.3 for $A = 3$</td>
<td>94</td>
</tr>
<tr>
<td>30</td>
<td>The graph $G$ for $A = 2, 3, 4$</td>
<td>104</td>
</tr>
<tr>
<td>31</td>
<td>The graph $L_\alpha$</td>
<td>107</td>
</tr>
<tr>
<td>32</td>
<td>The graph $G$ for $A = 2, 3, 4$</td>
<td>114</td>
</tr>
<tr>
<td>33</td>
<td>The graph $G$ for $A = 2, 3, 4$</td>
<td>115</td>
</tr>
<tr>
<td>34</td>
<td>Two 2-stratified oriented graphs of $\overline{P}_2$</td>
<td>128</td>
</tr>
<tr>
<td>35</td>
<td>Six 2-stratifications of $\overline{P}_3$</td>
<td>129</td>
</tr>
<tr>
<td>36</td>
<td>An $H$-coloring of an oriented graph</td>
<td>130</td>
</tr>
<tr>
<td>37</td>
<td>The oriented graph $\overline{C}_4$</td>
<td>135</td>
</tr>
<tr>
<td>38</td>
<td>The out-corona $Cor(\overline{C}_8)$ of the 8-cycle</td>
<td>135</td>
</tr>
<tr>
<td>39</td>
<td>The oriented graph $D_3$ for $k = 4$</td>
<td>137</td>
</tr>
<tr>
<td>40</td>
<td>The oriented graph $D_4$</td>
<td>138</td>
</tr>
<tr>
<td>41</td>
<td>The oriented graph $D$ for $a = 8$ and $b = 11$</td>
<td>143</td>
</tr>
<tr>
<td>42</td>
<td>The oriented graph $D$</td>
<td>149</td>
</tr>
<tr>
<td>43</td>
<td>A 2-regular oriented graph $D$ of order 7 with $\gamma_H(D) = 3$</td>
<td>149</td>
</tr>
<tr>
<td>44</td>
<td>A 2-regular oriented graph $D$ of order 8 with $\gamma_H(D) = 4$</td>
<td>150</td>
</tr>
<tr>
<td>45</td>
<td>The $r$-regular oriented graph $D$ for $r = 3$</td>
<td>153</td>
</tr>
<tr>
<td>46</td>
<td>The $r$-regular oriented graph $D$ for $k = 4$</td>
<td>155</td>
</tr>
<tr>
<td>47</td>
<td>A minimum $H$-coloring of $\overline{K}_7$</td>
<td>160</td>
</tr>
</tbody>
</table>
1 INTRODUCTION

1.1 Motivation and Background

Dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. The vertices of a graph can be divided into cut-vertices and non-cut-vertices. Equivalently, the vertices of a tree are divided into its leaves and non-leaves. The set of vertices of a graph is partitioned according to the degrees of its vertices. When studying distance, the vertices of a connected graph are partitioned according to their eccentricities. Also, in a connected rooted graph, the vertices are partitioned according to their distances from the root. Perhaps the best known example of this process, however, is graph coloring, where the vertex set of a graph is partitioned into classes each of which is independent in the graph.

A typical Very Large Scale Integrated (VLSI) Circuit chip consists of millions of transistors assembled through layering of various materials in a silicon base. In recent years, advances in VLSI fabrication technology have made it possible to use more than two routing layers for interconnection. In fact, the most popular processors on the market today use three or more layers. In the design of algorithms to solve the multilayer routing problems encountered in this process, it is desirable to use graphs in which the vertices are partitioned into classes. Motivated by these observations, Rashidi [30] defined a graph $G$ to be a stratified graph if its vertex set is partitioned into classes. He studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [8, 12].

Formally then, a graph $G$ whose vertex set has been partitioned is called a stratified graph. If $V(G)$ is partitioned into $k$ subsets, then $G$ is a $k$-stratified graph. The $k$ subsets are called the strata or color classes of $G$. Suppose that the vertex set of a $k$-stratified graph $G$ is partitioned into $k$ subsets $V_1, V_2, \ldots, V_k$. Unlike vertex coloring,
no condition is placed on the subsets $V_i$, $1 \leq i \leq k$. If $G$ is 2-stratified, then commonly the vertices of one color class are colored red and the vertices of the other color class are colored blue. If a $k$-stratified graph $G'$ is constructed from a graph $G$ by partitioning the vertex set of $G$ into $k$ subsets, then $G'$ is called a $k$-stratification of $G$ and $G$ is the underlying graph of $G'$. Two $k$-stratified graphs $G$ and $H$ are isomorphic if there exists a bijective function $\phi : V(G) \to V(H)$ such that (1) $u$ and $v$ are adjacent in $G$ if and only if $\phi(u)$ and $\phi(v)$ are adjacent in $H$ and (2) $x$ and $\phi(x)$ are colored the same for all $x \in V(G)$. The function $\phi$ is then called a color-preserving isomorphism.

Another closely related concept concerns domination in graphs. A vertex is said to dominate itself and each vertex adjacent to it. A set $S$ of vertices in a graph $G$ is called a dominating set for $G$ if every vertex of $G$ is dominated by some vertex in $S$. A familiar application of domination is also a topic of current interest. For example, a city might contain a number of important locations (such as certain buildings, street intersections, bridges, etc.). We can construct a graph $G$ whose vertices represent these locations. If two locations are within a certain specified distance of each other, then we draw an edge between the corresponding vertices. We would like to station security officers at certain locations. Since it is not practical (especially from an economical point of view) to place security officers at every location, our goal is to place as few security officers as possible at some locations so that if a location does not have a security officer stationed there, then there is a security officer stationed within the prescribed distance of the location. That is, no location is left unprotected. What this describes is the domination number $\gamma(G)$ of the graph $G$, which is the minimum number of vertices in a dominating set for $G$. A dominating set of cardinality $\gamma(G)$ is called a minimum dominating set (or simply, a $\gamma$-set) of $G$.

The work of Berge in 1958 [2] and Ore in 1962 [29] caused domination to become a formal theoretical area of graph theory. But domination did not become an active area until 1977 when a survey paper by Cockayne and Hedetniemi [17] was published. The
concept of domination in graphs, with its many variations, is now well studied in graph theory. There has been increased interest in recent years in the study of domination in graphs. In 1998 two books [22, 23] by Haynes, Hedetniemi, and Slater are devoted exclusively to this subject.

1.2 Basic Definitions and Notation

We refer to the book [13] for graph-theoretical notation and terminology not described in this dissertation. Let $F$ be a 2-stratified graph. So each vertex of $F$ is colored red or blue and there is at least one vertex of each color. Designate a blue vertex $v$ of $F$ as the "root" of $F$. Then $F$ is said to be rooted at $v$. Two rooted 2-stratified graphs $F'$ and $F''$ are considered to be distinct if there is no color-preserving isomorphism from $F'$ to $F''$ that maps the root of $F'$ to the root of $F''$. For example, two distinct 2-stratified graphs $F$ and $F'$ rooted at a blue vertex $v$ are shown in Figure 1.1, where the solid vertices are red vertices and the empty vertices are blue vertices in each graph.

![Figure 1. Two 2-stratified graphs $F$ and $F'$.](image)

In a red-blue coloring of a connected graph $G$, every vertex of $V(G)$ is colored red or blue. It is acceptable if all vertices of $G$ are assigned the same color. If there is at least one vertex of each color, then the red-blue coloring produces a 2-stratification of $G$. Let $F$ be a 2-stratified graph rooted at a blue vertex $v$. By an $F$-coloring of a graph $G$, we mean a red-blue coloring of $G$ such that for every blue vertex $w$ of $G$, there is a copy of $F$ in $G$ with $v$ at $w$. That is, for every blue vertex $w$ of $G$, there exists a 2-stratified subgraph $F'$ of $G$ containing $w$ and a color-preserving isomorphism $\alpha$ from $F$ to $F'$ such that $\alpha(v) = w$. The red-blue coloring of $G$ in which every vertex is colored...
red is vacuously an $F$-coloring for every 2-stratified rooted graph $F$. Suppose that there is an $F$-coloring of a graph $G$ and $w$ is a blue vertex of $G$ that belongs to a copy $F'$ of $F$ rooted at $w$. If $u$ is a red vertex in $F'$, then $w$ is said to be $F$-dominated by $u$. If $c$ is an $F$-coloring of $G$, then the set $R_c$ of all red vertices of $G$ is called an $F$-dominating set of $G$. The 2-stratified graphs were first studied from the point of view of domination by Chartrand, Haynes, Henning, and Zhang in [10, 11].

To illustrate these concepts, consider the 2-stratified graph $F$ and the graph $G$ of Figure 1.2. The red-blue coloring of $G$ given in Figure 1.2 is an $F$-coloring of $G$ since every blue vertex of $G$ belongs to a copy of $F$ rooted at that vertex. For example, the blue vertex $w$ of $G$ belongs to a copy $F'$ of $F$ rooted at $w$, where $V(F') = \{w, x, y, z\}$. Since $x$ and $y$ are red vertices in $F'$ and $F'$ is rooted at $w$, it follows that $w$ is $F$-dominated by each of $x$ and $y$.

![Figure 2. An $F$-coloring of a graph.](image)

Since the red-blue coloring of a graph $G$ in which every vertex of $G$ is colored red is an $F$-coloring of $G$ for every 2-stratified graph $F$, there always exists an $F$-coloring of $G$. The $F$-domination number $\gamma_F(G)$ of $G$ was introduced in [10] as the minimum number of red vertices of $G$ in an $F$-coloring of $G$. By a minimum $F$-coloring of $G$, we mean an $F$-coloring having a minimum number of red vertices, that is, having $\gamma_F(G)$ red vertices. The $F$-coloring of the graph $G$ in Figure 1.2 is, in fact, a minimum $F$-coloring. Therefore, $\gamma_F(G) = 4$ for the graph $G$ of Figure 1.2.

As another example, consider the three 2-stratified graphs $H_1$, $H_2$, and $H_3$ and the graph $G$ of Figure 1.3, where solid vertices denote red vertices, open vertices denote
blue vertices, and the blue vertex labeled $r$ is the root in the 2-stratified graph. Each of the 2-stratified graphs $H_1$, $H_2$, and $H_3$ has the same 2-stratification of the path $P_4$ of order 4 but it is rooted at a different blue vertex. A minimum $H_i$-dominating set of $G$ with exactly $i$ red vertices is also shown in that figure for $i = 1, 2, 3$. Therefore, $\gamma_{H_i}(G) = i$ for $i = 1, 2, 3$.

![Figure 3. A minimum $H_i$-dominating set $(i = 1, 2, 3)$ for a graph $G$.](image)

The concept of $F$-domination was introduced and studied in [10, 11]. The following observations are useful.

**Observation 1.1.** If $F$ is a 2-stratified graph with $r$ red vertices and $G$ is a connected graph of order $n \geq r$, then $r \leq \gamma_F(G) \leq n$.

**Observation 1.2.** Let $F$ be a 2-stratified graph and let $G$ be a connected graph of order $n$. If $G$ does not contain a subgraph isomorphic to the underlying graph of $F$, then

$$\gamma_F(G) = n.$$

The converse of Observation 1.2 is not true. To see this, consider the 2-stratified graph $F$ and the graph $G \cong K_{1,3}$ of Figure 1.4. Certainly, $G$ contains $P_3$ as a subgraph. We show, however, that $\gamma_F(G) = 4$. Let there be given an $F$-coloring $c$ of $G$. If $u$ is colored blue, then $u$ cannot belong to a copy of $F$ rooted at $u$ since there is no path of length 2 with initial vertex $u$. Thus $u$ must be colored red by $c$. Since no two adjacent

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
vertices can be colored blue by \( c \), every vertex of \( G \) must be colored red by \( c \) and so \( \gamma_F(G) = 4 \).

![Diagram](image)

Figure 4. Showing the converse of Observation 1.2 is false.

In the case of ordinary domination, every set of vertices of a graph \( G \) that contains a dominating set of \( G \) is also a dominating set of \( G \). Such is not the case for \( F \)-domination, however.

**Observation 1.3.** There exist a 2-stratified graph \( F \) and a connected graph \( G \) of order \( n \) such that there is an \( F \)-coloring of \( G \) with \( k \) red vertices, where \( k < n \), but there is no \( F \)-coloring with \( k + 1 \) red vertices.

To illustrate Observation 1.3, consider the 2-stratified graph \( F \) and the graph \( G \cong P_4 : v_1, v_2, v_3, v_4 \) of Figure 1.5. The coloring of \( G \) defined by assigning red to \( v_1 \) and \( v_4 \) and blue to \( v_2 \) and \( v_3 \) is an \( F \)-coloring with two red vertices. However, there is no \( F \)-coloring of \( P_4 \) with three red vertices.

![Diagram](image)

Figure 5. Illustrating Observation 1.3.

### 1.3 Some Known Results

Let \( F \) be the 2-stratification of \( K_2 \) rooted at the blue vertex. An \( F \)-coloring of a graph \( G \) is then a red-blue coloring of the vertices of \( G \) with the property that every blue vertex is adjacent to a red vertex. The red vertices of \( G \) therefore constitute a
dominating set of $G$. Hence, $\gamma(G) \leq \gamma_F(G)$. On the other hand, given a $\gamma$-set of $G$ we color the vertices in this set red and all remaining vertices blue. This red-blue coloring of the vertices of $G$ has the property that every blue vertex is adjacent to a red vertex and is therefore an $F$-coloring of $G$ (where $F$ is the 2-stratified $K_2$). Thus, $\gamma_F(G) \leq \gamma(G)$.

These observations yield the following result that appeared in [10].

**Theorem A** If $F$ is the 2-stratified $K_2$, then $\gamma_F(G) = \gamma(G)$ for every graph $G$.

Thus domination in a graph $G$ can be interpreted as a restricted 2-stratification or 2-coloring of $G$, with the red vertices forming a dominating set of $G$. In fact, $F$-domination generalizes not only ordinary domination but other types of domination that have been previously studied (see [10]). Let $F$ be a 2-stratified $P_3$ rooted at a blue vertex $v$. The five possible choices for the graph $F$ are shown in Figure 1.6.

![Figure 6. The five 2-stratified graphs $P_3$.](image)

The following results (Theorems B – E) were established in [10].

**Theorem B** If $G$ is a connected graph of order at least 3, then $\gamma_{F_1}(G) = \gamma(G)$.

A vertex $v$ in a graph $G$ openly dominates each of its neighbors. That is, $v$ dominates the vertices in its neighborhood $N(v)$ but not itself. A set $S$ of vertices in a graph $G$ is an open dominating set if every vertex of $G$ is adjacent to at least one vertex of $S$. In this case, a vertex $v$ in an open dominating set of $G$ is said to openly dominate its neighbors but not itself. The minimum cardinality of an open dominating set is the open domination number $\gamma_0(G)$ of $G$. An open dominating set of cardinality $\gamma_0(G)$ is a minimum open dominating set or a $\gamma_0$-set for $G$. Observe that $\gamma(G) \leq \gamma_0(G)$ for every graph $G$ containing no isolated vertices. The open domination number of a graph $G$
has also been referred to as the total domination number and denoted by $\gamma_t(G)$. Open domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [16] and has now been studied extensively (see [22, 23]).

**Theorem C**  If $G$ is a graph without isolated vertices, then $\gamma_{F_2}(G) = \gamma_o(G)$.

An $F_4$-coloring of $G$ requires that every blue vertex of $G$ is adjacent to both a red and a blue vertex, while $\gamma_{F_4}(G)$ is the minimum number of red vertices required in such a 2-stratification of $G$. Thus, $\gamma_{F_4}(G)$ is the known domination parameter called the restrained domination number $\gamma_r(G)$. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V(G) - S$. Every graph has a restrained dominating set since $V(G)$ is such a set. The restrained domination number $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of $G$. The concept of restrained domination in graphs was introduced and studied by Domke, Hattingh, Hedetniemi, Laskar, and Markus [18]. Clearly, $\gamma_r(G) \geq \gamma(G)$.

**Theorem D**  For every graph $G$, $\gamma_{F_4}(G) = \gamma_r(G)$.

An $F_5$-coloring of $G$ requires that every blue vertex of $G$ is adjacent to (at least) two red vertices, while $\gamma_{F_5}(G)$ is the minimum number of red vertices required in such a 2-stratification of $G$. Thus, $\gamma_{F_5}(G)$ is the well-known domination parameter called the 2-domination number $\gamma_2(G)$ as defined by Fink and Jacobson [20]. A set $S \subseteq V(G)$ is a $k$-dominating set if every vertex not in $S$ is adjacent to at least $k$ vertices in $S$. The $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set of $G$. Observe that $\gamma_k(G) \geq \gamma(G)$ for every graph $G$.

**Theorem E**  For every graph $G$, $\gamma_{F_5}(G) = \gamma_2(G)$.

While the parameters $\gamma_{F_1}(G)$, $\gamma_{F_2}(G)$, $\gamma_{F_4}(G)$, and $\gamma_{F_5}(G)$ are well-known parameters, the parameter $\gamma_{F_3}(G)$ is new. Thus, we study $F_3$-domination in graphs in Chapter 2. In Chapter 3, we compare $F_3$-domination with other well-known domination parameters. In Chapter 4, we determine all triples of positive integers that can
be realized, respectively, as the domination number, the open domination number, and the \( F \)-domination number of some connected graph. Stratified domination in oriented graphs is studied in Chapter 5.
2 F3-Domination in Graphs

2.1 Introduction

We have seen that \( \gamma_{F_1}, \gamma_{F_2}, \gamma_{F_4}, \) and \( \gamma_{F_5} \) are well-known parameters while \( \gamma_{F_3} \) is new. In this chapter we investigate \( F_3 \)-domination in connected graphs. The 2-stratified graph \( F_3 \) rooted at a blue vertex \( v \) is shown in Figure 2.1(a), where the solid vertex is a red vertex and the hollow vertices are blue vertices. For simplification, we write \( F = F_3 \) unless said otherwise.

\[
F = F_3: \begin{array}{c}
\bullet & o & o & v \\
\end{array} \\
F': \begin{array}{c}
\bullet & o & o & v' & v \\
\end{array}
\]

(a) (b)

Figure 7. The 2-stratified graph \( F \) and a copy \( F' \) of \( F \).

Recall that an \( F \)-coloring of a graph \( G \) is a red-blue coloring of the vertices of \( G \) in which every blue vertex \( v \) of \( G \) belongs to a copy of \( F \) rooted at \( v \). Suppose that there is an \( F \)-coloring of a graph \( G \), where a blue vertex \( v \) of \( G \) belongs to a copy \( F' \) of \( F \) rooted at \( v \), say \( V(F') = \{u, v', v\} \), such that \( u \) is red and \( v' \) is blue (see Figure 2.1(b)). Then the blue vertex \( v \) is said to be \( F \)-dominated by the red vertex \( u \). The blue vertex \( v' \) may not be \( F \)-dominated by the red vertex \( u \). Thus, in order for a blue vertex \( v \) to be \( F \)-dominated by a red vertex \( u \), there must be a \( v - u \) blue-blue-red path of length 2 in \( G \). Therefore, for the vertex \( v' \) in \( F' \) in Figure 2.1(b) to be \( F \)-dominated by \( u \), there must be a blue vertex \( w \) in \( G \) adjacent to both \( u \) and \( v' \). In particular, if \( v \) is adjacent to \( u \), then \( v' \) is \( F \)-dominated by \( u \).

The set \( R_c \) of red vertices of a graph \( G \) in an \( F \)-coloring \( c \) of \( G \) is called an \( F \)-dominating set of \( G \). The \( F \)-domination number \( \gamma_F(G) \) of \( G \) is the minimum number of red vertices of \( G \) in an \( F \)-coloring of \( G \). A minimum \( F \)-coloring of \( G \) contains \( \gamma_F(G) \)
red vertices. Since the 2-stratified graph $F$ contains a red vertex,

$$1 \leq \gamma_F(G) \leq n$$

(2.1)

for every connected graph $G$ of order $n$. Moreover, since the order of $F$ is 3, it follows by Observation 1.2 that $\gamma_F(G) = n$ for $n = 1$ or $n = 2$. Therefore, in this chapter, we restrict ourselves to connected graphs of order at least 3.

To illustrate these concepts, we determine the $F$-domination number of the Petersen graph $P$. First, we need some additional definitions and notation. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The radius $\text{rad}(G)$ of $G$ is the minimum eccentricity among the vertices of $G$ and the diameter $\text{diam}(G)$ of $G$ is the maximum eccentricity. A $u-v$ path of length $d(u, v)$ is called a $u-v$ geodesic. Two vertices $u$ and $v$ in a graph $G$ are similar if there exists an automorphism $\phi$ of $G$ such that $\phi(u) = v$. A graph $G$ is vertex-transitive if every two vertices of $G$ are similar. The girth $g(G)$ of a graph $G$ (with cycles) is the length of its shortest cycle. It is known that the Petersen graph $P$ is vertex-transitive and has diameter 2 and girth 5. We are now prepared to determine the $F$-domination number of the Petersen graph $P$.

**Example 2.1.** The Petersen graph $P$ has $F$-domination number 3.

![Petersen graph](image)

Figure 8. A red-blue coloring of the Petersen graph $P$.

Label the vertices of $P$ as shown in Figure 2.2, where a red-blue coloring $c^*$ of $P$
is given by assigning red to $v_1, u_2, v_4$ and blue to the remaining vertices of $P$. Observe that (1) $u_3, u_4, u_5, v_3$ are $F$-dominated by $v_1$, (2) $u_1, v_2$ are $F$-dominated by $v_4$, and (3) $v_5$ is $F$-dominated by $u_2$. Thus every blue vertex $v$ belongs to a copy of $F$ rooted at $v$ and so $c^*$ is an $F$-coloring of $P$ with $R_{c^*} = \{v_1, u_2, v_4\}$. Therefore, $\gamma_F(P) \leq |R_{c^*}| = 3$.

Next we show that $\gamma_F(P) \geq 3$. Assume, to the contrary, that $\gamma_F(P) < 3$. Then $\gamma_F(P) = 1$ or $\gamma_F(P) = 2$. Let there be given a minimum $F$-coloring $c$ of $P$. If $\gamma_F(P) = 1$, then let $R_c = \{x\}$, where $x \in V(P)$. Since $P$ is vertex-transitive, we may assume that $x = v_1$. However no vertex adjacent to $v_1$ is $F$-dominated by $v_1$ since $P$ has girth 5. Thus $\gamma_F(P) = 2$. Let $R_c = \{x, y\}$. We consider two cases, depending on whether $x$ and $y$ are adjacent.

Case 1. $xy \in E(P)$. Since the girth of $P$ is 5, no vertex adjacent to $x$ or $y$ is $F$-dominated by $x$ or $y$.

Case 2. $xy \notin E(P)$. Since $P$ has diameter 2 and the girth of $P$ is 5, there is a unique vertex $z$ in $P$ adjacent to both $x$ and $y$. Again, since the girth of $P$ is 5, it follows that $z$ is not $F$-dominated by $x$ or $y$.

Therefore, $\gamma_F(P) \geq 3$ and so $\gamma_F(P) = 3$.

Since a vertex in a cubic (3-regular) graph $G$ of order $n$ can $F$-dominate at most 7 vertices, it follows that $\gamma_F(G) \geq \lceil n/7 \rceil$. Since the Petersen graph $P$ has girth 5, every vertex of $P$ $F$-dominates exactly 7 vertices. The observation above says that $\gamma_F(P) \geq 2$. We have already seen that $\gamma_F(P) = 3$, however. Another example of a well-known cubic graph is the Heawood graph $H$ shown in Figure 2.3. This graph has girth 6 and order 14. Therefore, $\gamma_F(H) \geq \lceil 14/7 \rceil = 2$. Since the red-blue coloring of $H$ in Figure 2.3 is an $F$-coloring, $\gamma_F(H) \leq 2$ and so $\gamma_F(H) = 2$.

The Heawood graph is a connected cubic graph $G$ of order $n$ for which $\gamma_F(G) = \lceil n/7 \rceil$; while the Petersen graph is a connected cubic graph $G$ of order $n$ for which $\gamma_F(G) > \lceil n/7 \rceil$. Since $\gamma_F(G) - \lceil n/7 \rceil = 1$ for the Petersen graph, this brings up the question of whether there is a connected cubic graph $G$ of order $n$ for which $\gamma_F(G) -$
Figure 9. The Heawood graph $H$.

$[n/7] \geq 2$. For the connected cubic McGee graph $M$ of order 24 and girth 7 shown in Figure 2.4, we then have $\gamma_F(M) \geq [n/7] = 4$. The $F$-coloring in Figure 2.4 shows that $\gamma_F(M) \leq 6$. Whether there is a minimum $F$-coloring of $M$ with 4 or 5 red vertices is not known, however.

Figure 10. The McGee graph $M$.

At first glance, one might think that $F$-domination is the same as the distance domination parameter called $k$-step domination introduced in [26] for $k = 2$. A set $S \subseteq V(G)$ is a $k$-step dominating set if for every vertex $u \in V(G) - S$, there exists a path of length $k$ from $u$ to some vertex in $S$. The $k$-step domination number $\gamma_{\exists k}(G)$ is the min-
imum cardinality of any \( k \)-step dominating set of \( G \). Similarly, for \( F \)-domination every blue vertex must be the initial vertex of some path of length 2 to some red vertex. However, the subtle difference in 2-step domination and \( F \)-domination is that the path of the latter must be a blue-blue-red path, that is, for any blue vertex \( v \), \( F \)-domination requires that a path \( v, u, w \) exists where \( u \) is blue and \( w \) is red. Thus, every \( F \)-dominating set is a 2-step dominating set, but not conversely. Hence, we have the following observation.

**Observation 2.2.** For every graph \( G \),

\[
\gamma_{22}(G) \leq \gamma_F(G). \tag{2.2}
\]

Both equality and strict inequality in (2.2) are possible, as we will see later.

### 2.2 Preliminary Results

In this section, we establish some elementary but useful results.

**Proposition 2.3.** Let \( G \) be a connected graph of order at least 3. Then \( \gamma_F(G) = 1 \) if and only if \( G \) contains a vertex \( u \) such that there is a \( u - v \) path of length 2 in \( G \) for each \( v \in V(G) - \{u\} \).

**Proof.** Let \( G \) be a connected graph of order \( n \geq 3 \) with \( \gamma_F(G) = 1 \). Then there exists an \( F \)-coloring of \( G \) with exactly one red vertex, say \( u \). Let \( v \in V(G) - \{u\} \). Since \( v \) is \( F \)-dominated by \( u \), there is a \( u - v \) path of length 2 in \( G \).

For the converse, assume that \( G \) contains a vertex \( u \) such that there is a \( u - v \) path of length 2 in \( G \) for each vertex \( v \in V(G) - \{u\} \). Then the red-blue coloring defined by assigning red to \( u \) and blue to the remaining vertices of \( G \) is an \( F \)-coloring of \( G \) and so \( \gamma_F(G) = 1 \) by (2.1).

The following two results are immediate consequences of Proposition 2.3.

**Corollary 2.4.** If \( G \) is a connected graph with minimum degree \( \delta(G) \geq 2 \) and \( \text{rad}(G) = 1 \), then \( \gamma_F(G) = 1 \).
The converse of Corollary 2.4 is not true. For example, consider the three graphs $G_1$, $G_2$, and $G_3$ in Figure 2.5. Each of $G_1$, $G_2$, and $G_3$ has $F$-domination number 1, and a minimum $F$-coloring is shown for each in Figure 2.5. Observe that $\delta(G_1) = \text{rad}(G_1) = 2$, $\delta(G_2) = \text{rad}(G_2) = 1$, and $\delta(G_3) = 1$ and $\text{rad}(G_3) = 2$.

\[ \begin{array}{c}
G_1: & \\
& \\
G_2: & \\
\end{array} \]

\[ \begin{array}{c}
G_3: & \\
& \\
& \\
\end{array} \]

Figure 11. Three graphs with $F$-domination number 1.

**Corollary 2.5.** If $G$ is a bipartite graph, then $\gamma_F(G) \geq 2$.

**Proof.** Let $V_1$ and $V_2$ be the partite sets of $G$, and let $v_i \in V_i$ for $i = 1, 2$. Then there is no $v_1 - v_2$ path of length 2 in $G$. This implies that there is no vertex $v$ such that there is a $v - u$ path of length 2 for each $u \in V(G)$. It then follows by Proposition 2.3 that $\gamma_F(G) \geq 2$. □

**Lemma 2.6.** Let $G$ be a connected graph and let $v$ be an end-vertex of $G$ that is adjacent to the vertex $u$ in $G$. If $c$ is an $F$-coloring of $G$ and $u$ is colored red by $c$, then $v$ is also colored red by $c$.

**Proof.** Assume, to the contrary, that there exists an $F$-coloring $c$ of $G$ such that $u$ is colored red and $v$ is colored blue by $c$. Then $v$ is $F$-dominated by some vertex $x \in V(G)$. Since $v$ is adjacent only to $u$, it follows that $v, u, x$ is a blue-blue-red path and so $u$ is blue, which is a contradiction. □

**Proposition 2.7.** If $v$ is an end-vertex of a graph $G$ such that $v$ is adjacent to a vertex of degree 2 in $G$, then $v$ is colored red in any $F$-coloring of $G$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof. Let $u$ be the vertex of degree 2 adjacent to $v$. Assume, to the contrary, that there exists an $F$-coloring $c$ of $G$ such that $v$ is colored blue by $c$. Then $v$ belongs to a copy of $F$ rooted at $v$. Since $\deg v = 1$, it follows that $v$ belongs to a path $P_3 : v, u, x$, where $v$ and $u$ are blue vertices, but $x$ is a red vertex. However, then, the blue vertex $u$ does not belong to any copy of $F$ rooted at $u$, as $x$ is a red vertex and $\deg v = 1$, which is a contradiction. Thus $v$ is colored red by $c$. □

Proposition 2.8. Let there be given an $F$-coloring of a graph $G$. If a blue vertex $v$ is $F$-dominated by a red vertex $u$ that is adjacent to $v$, then $v$ belongs to a triangle in $G$ that contains $u$.

Proof. If a blue vertex $v$ is $F$-dominated by a red vertex $u$, then there is a blue vertex $w$ such that $v$ is adjacent to $w$ and $w$ is adjacent to $u$. Since $u$ is adjacent to $v$, it follows that $v, w, u, v$ form a triangle in $G$. □

By Proposition 2.8, we have the following corollary.

Corollary 2.9. In any $F$-coloring of a triangle-free graph $G$, no blue vertex $v$ of $G$ can be $F$-dominated by an adjacent red vertex.

The complement $\overline{G}$ of a graph $G$ is that graph whose vertex set is $V(G)$ and such that $uv$ is an edge of $\overline{G}$ if and only if $uv$ is not an edge of $G$. Notice that every $F$-coloring of a non-complete graph $G$ is also an $F$-coloring of $G + e$, for each $e \in E(\overline{G})$. This observation yields the following.

Proposition 2.10. If $G$ is a connected graph with nonadjacent vertices $u$ and $v$, then

$$\gamma_F(G + uv) \leq \gamma_F(G).$$

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. As we will see later, the addition of an edge to a connected graph $G$ can result in a graph whose $F$-domination number is the same as that of $G$ or significantly smaller than that of $G$.

As a consequence of Proposition 2.10, we have the following.
Corollary 2.11. If $H$ is a connected graph and $G$ is a connected spanning subgraph of $H$, then

$$\gamma_F(H) \leq \gamma_F(G).$$

In particular, if $T$ is a spanning tree of $H$, then $\gamma_F(H) \leq \gamma_F(T)$.

2.3 $F$-Domination Numbers of Some Well-known Graphs

In this section, we determine the $F$-domination numbers of some well-known graphs, beginning with complete graphs and stars.

Proposition 2.12. For each integer $n \geq 3$, $\gamma_F(K_n) = 1$ and $\gamma_F(K_{1,n-1}) = n$.

Proof. By Proposition 2.3, $\gamma_F(K_n) = 1$. To verify that $\gamma_F(K_{1,n-1}) = n$, let $u$ be the central vertex of $K_{1,n-1}$. Suppose that $c$ is an arbitrary $F$-coloring of $K_{1,n-1}$. Since there is no $u-v$ path of length 2 in $K_{1,n-1}$ for any vertex $v$ in $K_{1,n-1}$, the vertex $u$ must be colored red by $c$. It then follows by Lemma 2.6 that every end-vertex of $K_{1,n-1}$ must also be colored red by $c$. Hence every vertex of $K_{1,n-1}$ is colored red by $c$ and so $|R_c| = n$. Therefore, $\gamma_F(K_{1,n-1}) = n$. □

By Proposition 2.10, the addition of an edge to a connected graph $G$ results in a graph whose $F$-domination number is at most that of $G$. Furthermore, as we have mentioned, the addition of an edge to a connected graph $G$ can result in a graph whose $F$-domination number is the same as that of $G$ or significantly smaller than that of $G$. For example, let $G = K_{1,n-1}$ and so $\gamma_F(G) = n$ by Proposition 2.12. Consider $H = G + e = K_{1,n-1} + e$, where $e$ is an edge in the complement of $K_{1,n-1}$. We claim that $\gamma_F(H) = 1$. To see this, let $u$ be a vertex of degree 2 in $H$. Then the red-blue coloring of $H$ defined by assigning red to $u$ and blue to the remaining vertices of $H$ is an $F$-coloring with exactly one red vertex and so, as claimed, $\gamma_F(H) = 1$. Therefore, adding an edge to a connected graph $G$ can result in a graph whose $F$-domination number is significantly...
smaller than that of $G$. Moreover, by (2.1), $\gamma_F(H) = \gamma_F(H + f) = 1$ for any edge $f$ in the complement of $H$. Thus, adding edges to the graph $H$ results in a graph whose $F$-domination number is the same as that of $H$.

**Proposition 2.13.** Let $G = K_{n_1,n_2,\ldots,n_k}$, where $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $k \geq 2$. Then

$$\gamma_F(G) = \begin{cases} 2 & \text{if } k = 2 \\ 1 & \text{if } k \geq 3. \end{cases}$$

**Proof.** First, suppose that $k = 2$. By Corollary 2.5, $\gamma_F(G) \geq 2$. On the other hand, let $V_1$ and $V_2$ be the partite sets of $G$ and let $v_i \in V_i$ for $i = 1, 2$. Define a red-blue coloring $c$ by assigning red to $v_1$ and $v_2$ and blue to the remaining vertices of $G$. Since each vertex in $V_i$ is $F$-dominated by $v_i$ for $i = 1, 2$, it follows that $c$ is an $F$-coloring of $G$, implying that $\gamma_F(G) \leq 2$. Therefore, $\gamma_F(G) = 2$.

Next, suppose that $k \geq 3$. Let $u$ be a vertex of $G$. Observe that for each $v \in V(G) - \{u\}$, there is a $u - v$ path of length 2 in $G$. It then follows by Proposition 2.3 that $\gamma_F(G) = 1$.

The following observation will be useful.

**Observation 2.14.** If $G = K_{n_1,n_2,\ldots,n_k}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $k \geq 2$, then, $\gamma_F(G + e) = 1$.

Next, we determine the $F$-domination number of a path and a cycle. In order to do this, we first present a lemma.

**Lemma 2.15.** Let there be given an $F$-coloring of a graph $G$ of order $n \geq 4$ that is either a path or a cycle. If $P : v_1, v_2, \ldots, v_n$ is a spanning path of $G$, then

(a) at least one vertex in the set $\{v_i, v_{i+1}, v_{i+2}\}$ is red for all $i$ with $1 \leq i \leq n$, where the addition in the subscripts is performed modulo $n$, and

(b) there is no blue vertex adjacent to two red vertices in $G$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof. We first prove (a). Assume, to the contrary, that for some \( j \), with \( 1 \leq j \leq n-2 \) all three vertices in the set \( \{ v_j, v_{j+1}, v_{j+2} \} \) are colored blue. Since the blue vertex \( v_{j+1} \) belongs to a copy of \( F \) rooted at \( v_{j+1} \), at least one of \( v_{j-1} \) and \( v_{j+3} \) is red, say \( v_{j+3} \). However, then, the blue vertex \( v_{j+2} \) does not belong to any copy of \( F \) rooted at \( v_{j+2} \), which is a contradiction.

Next we prove (b). Assume, to the contrary, that there is a blue vertex \( v \) in \( P \) that is adjacent to two red vertices. Since \( \deg v = 2 \), it follows that \( v \) is not adjacent to any blue vertex. Thus \( v \) does not belong to a copy of \( F \) rooted at \( v \), which is a contradiction. \( \square \)

We first determine the \( F \)-domination number of all paths. Recall for positive integers \( a \) and \( b \) that \( b \mod a \) denotes the remainder obtained when \( b \) is divided by \( a \). Thus \( 0 \leq b \mod a < a - 1 \).

Theorem 2.16. For each positive integer \( n \),

\[
\gamma_F(P_n) = (n + 2) \mod 3 + \lceil n/3 \rceil.
\]

Proof. Let \( P_n : v_1, v_2, \ldots, v_n \) be a path of order \( n \). If \( 1 \leq n \leq 2 \), then \( P_n \) does not contain a copy of \( F \) and so \( \gamma_F(P_n) = n \). Thus the result holds for \( 1 \leq n \leq 2 \). Hence we may assume that \( n \geq 3 \). We consider three cases.

Case 1. \( (n + 2) \mod 3 = 2 \). Then \( n = 3k \) for some integer \( k \geq 1 \). Since it is easy to see that \( \gamma_F(P_3) = 3 \), we may assume that \( n \geq 6 \). Thus \( k \geq 2 \). We first show that \( \gamma_F(P_n) \leq k + 2 \). Define a red-blue coloring \( c^* \) of \( P_n \) by assigning red to \( v_{n-1}, v_n \) and \( v_{3i+1} \) for \( 0 \leq i \leq k - 1 \) and assigning blue to the remaining vertices of \( P_n \). Thus

\[
R_{c^*} = \{ v_{3i+1} : 0 \leq i \leq k - 1 \} \cup \{ v_{n-1}, v_n \}.
\]

Observe that the blue vertex \( v_{3j} \) (\( 1 \leq j \leq k - 1 \)) is adjacent to the blue vertex \( v_{3j-1} \), which is adjacent to the red vertex \( v_{3j-2} \) and that the blue vertex \( v_{3j+2} \) (\( 0 \leq j \leq k - 2 \)) is adjacent to the blue vertex \( v_{3(j+1)} \), which is adjacent to the red vertex \( v_{3(j+1)+1} \). Thus \( c^* \) is an \( F \)-coloring of \( P_n \) and so \( \gamma_F(P_n) \leq |R_{c^*}| = k + 2 \).
Next, we show that $\gamma_F(P_n) \geq k + 2$. Assume, to the contrary, that $\gamma_F(P_n) \leq k + 1$. Let $c$ be a minimum $F$-coloring of $P_n$. By Proposition 2.7, the vertices $v_1$ and $v_n$ must be colored red by $c$. By Lemma 2.15, for each $i$ with $0 \leq i \leq k - 2$, at least one vertex in the set $S_i = \{v_{3i+2}, v_{3i+3}, v_{3i+4}\}$ is colored red by $c$. Since the $k - 1$ sets $S_i$ ($0 \leq i \leq k - 2$) are mutually disjoint, $\gamma_F(P_n) = |R_c| \geq 2 + k - 1 = k + 1$. Therefore, $\gamma_F(P_n) = k + 1$. Hence $v_1$ and $v_n$ are red, exactly one vertex in each set $S_i$ ($0 < i < k - 2$) is colored red, and $v_{n-1}$ is colored blue. Consider $S_{k-2} = \{v_{n-4}, v_{n-3}, v_{n-2}\}$. If $v_{n-4}$ is red, then $v_{n-3}, v_{n-2}, v_{n-1}$ are colored blue, which contradicts Lemma 2.15. If $v_{n-2}$ is red, then the blue vertex $v_{n-1}$ is adjacent to two red vertices, namely $v_{n-2}$ and $v_n$, which again contradicts Lemma 2.15. Therefore, $v_{n-3}$ is red and $v_{n-4}$ and $v_{n-2}$ are blue.

By Lemma 2.15, this forces $v_3$ to be red for $1 \leq i \leq k$. However then, the blue vertex $v_2$ in $S_0$ is adjacent to the two red vertices $v_1$ and $v_3$, which is impossible. Therefore, $\gamma_F(P_n) > k + 2$.

**Case 2.** $(n + 2) \mod 3 = 0$. Then $n = 3k + 1$ for some integer $k \geq 1$. We first show that $\gamma_F(P_n) \leq k + 1$. Define a red-blue coloring $c^*$ of $P_n$ by assigning red to $v_{3i+1}$ for $0 \leq i \leq k$ and assigning blue to the remaining vertices of $P_n$. Thus

$$R_{c^*} = \{v_{3i+1} : 0 \leq i \leq k\}.$$ 

Observe that the blue vertex $v_{3j}$ ($1 \leq j \leq k$) is adjacent to the blue vertex $v_{3j-1}$, which is adjacent to the red vertex $v_{3j-2}$ and that the blue vertex $v_{3j+2}$ ($0 \leq j \leq k - 1$) is adjacent to the blue vertex $v_{3(j+1)}$, which is adjacent to the red vertex $v_{3(j+1)+1}$. Thus $c^*$ is an $F$-coloring of $P_n$ and so $\gamma_F(P_n) \leq |R_{c^*}| = k + 1$.

Next, we show that $\gamma_F(P_n) \geq k + 1$. Let $c$ be a minimum $F$-coloring of $P_n$. By Lemma 2.15, for each $i$ with $0 \leq i \leq k - 1$, at least one vertex in the set $\{v_{3i+1}, v_{3i+2}, v_{3i+3}\}$ is colored red by $c$. By Proposition 2.7, the vertex $v_n$ must be colored red by $c$. Thus $\gamma_F(P_n) = |R_c| \geq k + 1$. Therefore, $\gamma_F(P_n) = k + 1$.

**Case 3.** $(n + 2) \mod 3 = 1$. Then $n = 3k + 2$ for some integer $k \geq 1$. We first show that $\gamma_F(P_n) \leq k + 2$. Define a red-blue coloring $c^*$ of $P_n$ by assigning red to $v_{n-1},$
v_n, and v_{3i+1} for 0 \leq i \leq k - 1 and assigning blue to the remaining vertices of P_n. Thus

\[ R_{c^*} = \{v_{3i+1} : 0 \leq i \leq k\} \cup \{v_n\}. \]

Observe that the blue vertex v_{3j} (1 \leq j \leq k) is adjacent to the blue vertex v_{3j-1}, which is adjacent to the red vertex v_{3j-2} and that the blue vertex v_{3j+2} (0 \leq j \leq k - 1) is adjacent to the blue vertex v_{3(j+1)}, which is adjacent to the red vertex v_{3(j+1)+1}. Thus c^* is an F-coloring of P_n and so \( \gamma_F(P_n) \leq |R_{c^*}| = k + 2. \)

Next, we show that \( \gamma_F(P_n) \geq k + 2. \) Let c be a minimum F-coloring of P_n. By Proposition 2.7, the vertices v_1 and v_n must be colored red by c. By Lemma 2.15, for each i with 0 \leq i \leq k - 1, at least one vertex in the set \{v_{3i+2}, v_{3i+3}, v_{3i+4}\} is colored red by c. Thus \( \gamma_F(P_n) = |R_c| \geq k + 2. \) Therefore, \( \gamma_F(P_n) = k + 2. \)

With the aid of Lemma 2.15 and Theorem 2.16, we now determine the F-domination number of a cycle.

**Theorem 2.17.** For each integer n \geq 3,

\[ \gamma_F(C_n) = n \ MOD \ 3 + \left\lfloor \frac{n}{3} \right\rfloor. \]

**Proof.** Let C_n : v_1, v_2, \ldots, v_n, v_1 be a cycle of order n. We consider three cases.

**Case 1.** n MOD 3 = 0. Then n = 3k for some integer k \geq 1. We first show that \( \gamma_F(C_n) \leq k. \) Define a red-blue coloring c^* of C_n by assigning red to v_{3i} for 1 \leq i \leq k and blue to the remaining vertices of C_n. Thus \( R_{c^*} = \{v_{3i} : 1 \leq i \leq k\}. \) Observe that the blue vertex v_{3j+1} (0 \leq j \leq k - 1) is adjacent to the blue vertex v_{3j-1}, which is adjacent to the red vertex v_{3j-2}, and that the blue vertex v_{3j+2} (0 \leq j \leq k - 1) is adjacent to the blue vertex v_{3(j+1)}, which is adjacent to the red vertex v_{3j}, where the subscripts are expressed as integers modulo 3k. Thus c^* is an F-coloring of C_n and so \( \gamma_F(C_n) \leq |R_{c^*}| = k. \)

Next, we show that \( \gamma_F(C_n) \geq k. \) Let c be a minimum F-coloring of C_n. By Lemma 2.15, for each i with 0 \leq i \leq k - 1, at least one vertex in the set \{v_{3i+1}, v_{3i+2}, v_{3i+3}\} is colored red by c. Thus \( \gamma_F(C_n) = |R_c| \geq k. \) Therefore, \( \gamma_F(C_n) = k. \)
Case 2. \( n \) MOD 3 = 1. Then \( n = 3k + 1 \) for some integer \( k \geq 1 \). By Proposition 2.10 and Theorem 2.16, \( \gamma_F(C_n) \leq \gamma_F(P_n) = k + 1 \) for \( n = 3k + 1 \). It remains to show that \( \gamma_F(C_n) \geq k + 1 \). Since \( F \) has a red vertex, any \( F \)-coloring of \( C_n \) assigns red to some vertex of \( C_n \). Let \( c \) be a minimum red-blue coloring of \( C_n \). Assume, without loss of generality, that \( v_1 \) is colored red by \( c \). By Lemma 2.15, for each \( i \) with \( 0 \leq i \leq k - 1 \), at least one vertex in the set \( \{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \) is colored red by \( c \). Thus \( \gamma_F(C_n) \geq k + 1 \).

Case 3. \( n \) MOD 3 = 2. Then \( n = 3k + 2 \) for some integer \( k \geq 1 \). By Proposition 2.10 and Theorem 2.16, \( \gamma_F(C_n) \leq \gamma_F(P_n) = k + 2 \) for \( n = 3k + 2 \). It remains to show that \( \gamma_F(C_n) \geq k + 2 \). Since \( F \) has a red vertex, any \( F \)-coloring of \( C_n \) assigns red to some vertex of \( C_n \). Let \( c \) be a minimum red-blue coloring of \( C_n \). Assume, without loss of generality, that \( v_1 \) is colored red by \( c \). By Lemma 2.15, for each \( i \) with \( 0 \leq i \leq k - 1 \), at least one vertex in the set \( S_i = \{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \) is colored red by \( c \). Thus \( \gamma_F(C_n) \geq k + 1 \).

Assume, to the contrary, that \( \gamma_F(C_n) = k + 1 \). Then \( c \) assigns red to \( v_1 \) and to exactly one vertex in \( S_i \) for each \( i \) with \( 0 \leq i \leq k - 1 \), and assigns blue to the remaining vertices of \( G \). Thus \( v_n \) is colored blue by \( c \). Since \( v_n \) is adjacent to the red vertex \( v_1 \), it follows that \( v_n \) must be \( F \)-dominated by \( v_{n-2} \) and so \( v_{n-2} = v_{3(k-1)+3} \in S_{k-1} \) is red. Since exactly one vertex in each set \( S_i \) (\( 0 \leq i \leq k - 2 \)) is red and \( v_{3(k-1)+3} \) is the red vertex in \( S_{k-1} \), it follows by Lemma 2.15 that \( c \) assigns red to the vertex \( v_{3i+3} \) in \( S_i \) for \( 0 \leq i \leq k - 1 \). Therefore,

\[
R_c = \{v_1\} \cup \{v_{3i+3} : 0 \leq i \leq k - 1\}.
\]

However then, the blue vertex \( v_2 \) is adjacent to two red vertices, namely \( v_1 \) and \( v_3 \), which contradicts Lemma 2.15.

For each positive integer \( n \), the \( n \)-cube \( Q_n \) is that graph whose vertex set is the set of \( n \)-bit strings (that is, the set of \( n \)-tuples \( (a_1, a_2, \cdots, a_n) \), where \( a_i \in \{0,1\} \) for \( 1 \leq i \leq n \)) such that two vertices of \( Q_n \) are adjacent if they differ in exactly one coordinate. Recall that \( |V(Q_n)| = 2^n, n \geq 1 \). We present a lower bound for \( \gamma_F(Q_n) \).
Theorem 2.18. For each positive integer \( n \),

\[
\gamma_F(Q_n) \geq 2 \left\lfloor \frac{2^{n-1}}{1 + \binom{n}{2}} \right\rfloor.
\]

Proof. The \( n \)-cube \( Q_n \) is an \( n \)-regular bipartite graph. Suppose that \( V_1 \) and \( V_2 \) are the two partite sets of \( Q_n \). Observe that a vertex in \( V_i \) (\( i = 1, 2 \)) can only \( F \)-dominate a vertex in \( V_i \). We show that every vertex in \( Q_n \) can \( F \)-dominate at most \( 1 + \binom{n}{2} \) vertices in \( Q_n \). Since \( Q_n \) is vertex-transitive, it suffices to show that \( x = (0, 0, \cdots, 0) \) can \( F \)-dominate at most \( 1 + \binom{n}{2} \) vertices in \( Q_n \). Without loss of generality, assume that \( x \in V_1 \) and so \( V_1 \) consists of all vertices of \( Q_n \) whose coordinates have an even number of 1's; that is, \( V_1 \) consists of all vertices \( y \) of \( Q_n \) for which \( d(x, y) \) is even. Furthermore, \( x \) can only \( F \)-dominate itself and those vertices in \( V_1 \) exactly two of whose coordinates are 1; that is, \( x \) can only \( F \)-dominate those vertices \( y \) for which \( d(x, y) = 2 \). Since there are \( \binom{n}{2} \) vertices in \( V_1 \) exactly two of whose coordinates are 1, it follows that \( x \) can \( F \)-dominate at most \( 1 + \binom{n}{2} \) vertices in \( V_1 \).

Let \( c \) be a minimum \( F \)-coloring of \( Q_n \). Since

(1) a vertex in \( V_i \) (\( i = 1, 2 \)) can only \( F \)-dominate a vertex in \( V_i \),

(2) a vertex in \( V_i \) (\( i = 1, 2 \)) can \( F \)-dominate at most \( 1 + \binom{n}{2} \) vertices in \( V_i \), and

(3) there are \( 2^{n-1} \) vertices in each \( V_i \) (\( i = 1, 2 \)),

it follows that \( R_c \) contains at least \( \left\lfloor \frac{2^{n-1}}{1 + \binom{n}{2}} \right\rfloor \) vertices in each partite set of \( Q_n \). Therefore,

\[
\gamma_F(G) = |R_c| \geq 2 \left\lfloor \frac{2^{n-1}}{1 + \binom{n}{2}} \right\rfloor,
\]

as desired. \( \square \)

The lower bound in Theorem 2.18 is attainable for some small values of \( n \), as we show next.

Example 2.19. \( \gamma_F(Q_n) = 2 \) for \( n = 1, 2, 3 \), and \( \gamma_F(Q_n) = 4 \) for \( n = 4, 5, 6 \).
Proof. We have seen that $\gamma_F(Q_1) = \gamma_F(Q_2) = 2$. For $n = 3$, the red-blue coloring of $Q_3$ that assigns red to each vertex in

$\{(0,0,0), (1,1,1)\}$

and blue to the remaining vertices of $Q_3$ is an $F$-coloring of $Q_3$. Note that $(0,0,0)$ and $(1,1,1)$ belong to different partite sets of $Q_3$.

For $n = 4$, the red-blue coloring of $Q_4$ that assigns red to each vertex in

$\{(0,0,0,0), (1,1,1,1), (1,0,0,0), (0,1,1,1)\}$

and blue to the remaining vertices of $Q_4$ is an $F$-coloring of $Q_4$. Note that $(0,0,0,0)$ and $(1,1,1,1)$ belong to the same partite set of $Q_4$ and $(1,0,0,0)$ and $(0,1,1,1)$ belong to the other partite set of $Q_4$.

For $n = 5$, the red-blue coloring of $Q_5$ that assigns red to each vertex in

$\{(0,0,0,0,0), (1,1,1,1,1), (1,0,0,0,0), (0,1,1,1,1)\}$

and blue to the remaining vertices of $Q_5$ is an $F$-coloring of $Q_5$. Note that $(0,0,0,0,0)$ and $(1,1,1,1,1)$ belong to the same partite set of $Q_5$ and $(1,1,1,1,1)$ and $(0,0,0,0,0,0)$ belong to the other partite set of $Q_5$.

For $n = 6$, the red-blue coloring of $Q_6$ that assigns red to each vertex in

$\{(0,0,0,0,0,0), (1,1,1,1,1,1), (1,0,0,0,0,0,0), (0,1,1,1,1,1,1)\}$

and blue to the remaining vertices of $Q_6$ is an $F$-coloring of $Q_6$. Note that $(0,0,0,0,0,0)$ and $(1,1,1,1,1,1)$ belong to the same partite set of $Q_6$ and $(1,0,0,0,0,0,0)$ and $(0,1,1,1,1,1)$ belong to the other partite set of $Q_6$.

The result then follows by Theorem 2.18.

By Example 2.19, if $1 \leq n \leq 6$, then

$$\gamma_F(Q_n) = 2 \left[ \frac{2^{n-1}}{1 + \binom{n}{2}} \right]$$

On the other hand, the inequality in Theorem 2.18 is strict when $n = 7$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Theorem 2.20. \( \gamma_F(Q_7) = 8 \).

Proof. First, we show that

\[
S = \{(0,0,0,0,0,0,0), (0,0,0,1,1,1,1), (1,1,1,0,0,0,1), (1,1,1,1,1,1,0),
(1,1,1,1,1,1,1), (1,1,1,0,0,0,0), (0,0,0,1,1,1,0), (0,0,0,0,0,0,1)\}
\]

is an \( F \)-dominating set of \( Q_7 \). Let \( V_1 \) and \( V_2 \) be the two partite sets of \( Q_7 \), where

\[
(0,0,0,0,0,0,0) \in V_1.
\]

Therefore,

\[
S_1 = \{(0,0,0,0,0,0,0), (0,0,0,1,1,1,1), (1,1,1,0,0,0,1), (1,1,1,1,1,1,0)\} \subseteq V_1
\]

and

\[
S_2 = \{(1,1,1,1,1,1,1), (1,1,1,0,0,0,0), (0,0,0,1,1,1,0), (0,0,0,0,0,0,1)\} \subseteq V_2.
\]

Every vertex of \( V_1 \) either contains no 1s, two 1s, four 1s, or six 1s as coordinates. Clearly, \((0,0,0,0,0,0,0)\) \( F \)-dominates itself. For every vertex \( v \) of \( Q_7 \) containing exactly two 1s as coordinates, there is path of length 2 from \((0,0,0,0,0,0,0)\) to \( v \) that avoids the vertex \((0,0,0,0,0,0,1)\). Thus \((0,0,0,0,0,0,0)\) \( F \)-dominates \( v \). Let \( w \) be a vertex of \( V_1 \) containing six 1s as coordinates. If the 7th coordinate of \( w \) is 0, then \( w \in S \) and so \( w \) \( F \)-dominates itself. If either the 4th, 5th, or 6th coordinate of \( w \) is 0, then \((1,1,1,0,0,0,1)\) \( F \)-dominates \( w \); while if either the 1st, 2nd, or 3rd coordinate of \( w \) is 0, then \((0,0,0,1,1,1,1)\) \( F \)-dominates \( w \).

Next, let \( u \) be a vertex of \( V_1 \) containing exactly four 1s as coordinates. If \( u = (0,0,0,1,1,1,1) \) or \( u = (1,1,1,0,0,0,1) \), then \( u \) \( F \)-dominates itself. If the four coordinates that are 1s are among the first six coordinates of \( u \), then \((1,1,1,1,1,1,0)\) \( F \)-dominates \( u \). Hence we may assume that the 7th coordinate of \( u \) is 1, but \( u \neq (0,0,0,1,1,1,1) \) and \( u \neq (1,1,1,0,0,0,1) \). If exactly two of the 4th, 5th, and 6th coordinates of \( u \) are 1s, then \((0,0,0,1,1,1,1)\) \( F \)-dominates \( u \); while if exactly two of the 1st,
2nd, and 3rd coordinates of $u$ are 1s, then $(1,1,1,0,0,0,1)$ $F$-dominates $u$. Therefore, every vertex in $V_1$ is $F$-dominated by some vertex in $S_1$.

Similarly, every vertex of $V_2$ either contains no 0s, two 0s, four 0s, or six 0s as coordinates. The vertex $(1,1,1,1,1,1,1)$ $F$-dominates itself. Let $v$ be a vertex of $V_2$ containing exactly two 0s as coordinates. Then there is path of length 2 from $(1,1,1,1,1,1,1)$ to $v$ that avoids the vertex $(1,1,1,1,1,1,0)$ and so $(1,1,1,1,1,1,1)$ $F$-dominates $v$. Let $w$ be a vertex of $V_2$ containing exactly six 0s as coordinates. If $w = (0,0,0,0,0,0,1)$, then $w$ $F$-dominates itself. If the 4th, 5th, or 6th coordinate of $w$ is 1, then $(0,0,0,1,1,1,0)$ $F$-dominates $w$. If the 1st, 2nd, or 3rd coordinate of $w$ is 1, then $(1,1,1,0,0,0,0)$ $F$-dominates $w$.

Next, let $u$ be a vertex of $V_2$ containing exactly four 0s as coordinates. If $u = (1,1,1,0,0,0,0)$ or $u = (0,0,0,1,1,1,0)$, then $u$ $F$-dominates itself. If the three coordinates of $u$ that are 1s include the 7th coordinate, then $(0,0,0,0,0,0,1)$ $F$-dominates $u$. Hence we may assume that the 7th coordinate of $u$ is 0, but $u \neq (1,1,1,0,0,0,0)$ and $u \neq (0,0,0,1,1,1,0)$. If exactly two of the 4th, 5th, and 6th coordinates of $u$ are 1s, then $(0,0,0,1,1,1,0)$ $F$-dominates $u$; while if exactly two of the 1st, 2nd, and 3rd coordinates of $u$ are 1s, then $(1,1,1,0,0,0,0)$ $F$-dominates $u$. Therefore, every vertex in $V_2$ is $F$-dominated by some vertex in $S_2$.

Since $S$ is an $F$-dominating set of $Q_7$, it follows that $\gamma_F(Q_7) \leq 8$. By Theorem 2.18, $\gamma_F(Q_7) \geq 2\left[\frac{26}{1+(2)}\right] = 6$. Hence $6 \leq \gamma_F(Q_7) \leq 8$.

Let $T$ be a minimum $F$-dominating set of $Q_7$. By the proof of Theorem 2.18, every partite set of $Q_7$ contains at least three vertices of $T$. We claim that every partite set of $Q_7$ must contain four vertices of $T$. Assume, to the contrary, that $V_1$ contains exactly three vertices of $T$. Since $Q_7$ is vertex-transitive, we may assume that $x = (0,0,0,0,0,0,0)$ is a vertex of $T$, where $|V_1 \cap T| = 3$. Let

$$T_1 = V_1 \cap T = \{x, y, z\}.$$ 

Thus each of $y$ and $z$ contains exactly two 1s, exactly four 1s, or exactly six 1s as
coordinates. Observe that a vertex in $V_1$ can only be $F$-dominated by a vertex in $V_1$. Let $U$ be the set of vertices in $V_2$ that contain exactly four 1s as coordinates. Then

$$|U| = \binom{7}{4} = 35.$$  

Since $x$ $F$-dominates no vertex in $U$, every vertex of $U$ must be $F$-dominated by $y$ or $z$. We make the following observations concerning $y$.

(a) If $y$ contains exactly two 1s as coordinates, then $y$ $F$-dominates $\binom{5}{2} = 10$ vertices in $U$.

(b) If $y$ contains exactly four 1s as coordinates, then $y$ $F$-dominates itself and 12 other vertices in $U$, for a total 13 vertices of $U$.

(c) If $y$ contains exactly six 1s as coordinates, then $y$ $F$-dominates $\binom{7}{3} = 15$ vertices in $U$.

Thus $y$ can $F$-dominate at most 15 vertices in $U$. Similarly, $z$ can $F$-dominate at most 15 vertices in $U$. However, $|U| = 35$ and so $T_1$ cannot $F$-dominate every vertex in $V_1$, which is a contradiction.

Hence, every partite set of $Q_7$ must contain four vertices of $T$, as claimed. Therefore, $\gamma_F(Q_7) = |T| \geq 8$ and so $\gamma_F(Q_7) = 8$. □

Although $\gamma_F(Q_n)$ is unknown for $n \geq 8$, it is likely that $\gamma_F(Q_n) \geq \left\lceil \frac{2^{n-1}}{1+\binom{2}{2}} \right\rceil$ increases as $n$ increases.

2.4 Joins and Cartesian Products of Graphs

In this section, we first determine the $F$-domination number of the join of two graphs.

**Proposition 2.21.** For every graph $G$ of order $n$,

$$\gamma_F(G + K_1) = \begin{cases} n + 1 & \text{if } G \cong K_n \\ 1 & \text{otherwise.} \end{cases}$$
Proof. If $G \cong \overline{K}_n$, then $G \times K_1 \cong K_{1,n}$ and so $\gamma_F(G + K_1) = n + 1$ by Proposition 2.12. If $G \not\cong \overline{K}_n$, then $n \geq 2$ and $G$ has at least one edge $e$. Thus $G + K_1$ contains $K_{1,n} + e$ as a spanning subgraph. Since $\gamma_F(K_{1,n} + e) = 1$, it then follows by Corollary 2.11 that $\gamma_F(G + K_1) = 1$. □

Proposition 2.22. For every two nontrivial graphs $G_1$ and $G_2$,

$$\gamma_F(G_1 + G_2) = \begin{cases} 2 & \text{if } G_1 \text{ and } G_2 \text{ are empty,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose that the order of $G_1$ is $r$ and the order of $G_2$ is $s$. Then $r, s \geq 2$. If $G_1$ and $G_2$ are both empty, then $G_1 + G_2 \cong K_{r,s}$ and so $\gamma_F(G_1 + G_2) = 2$ by Proposition 6.11. On the other hand, if at least one of $G_1$ and $G_2$ is not empty, say $G_1$ is not empty, then let $e \in E(G_1)$ and so $G$ contains $K_{r,s} + e$ as a spanning subgraph. It then follows by Corollary 2.11 and Observation 2.14 that $\gamma_F(G_1 + G_2) = 1$. □

Corollary 2.23. If $G = C_n + K_1$ is the wheel of order $n + 1 \geq 4$, then $\gamma_F(G) = 1$.

We now turn to the cartesian product of a graph and $K_2$.

Proposition 2.24. For every connected graph $G$,

$$\gamma_F(G \times K_2) \leq 2\gamma_F(G). \tag{2.3}$$

Proof. Let $G'$ be the other copy of $G$ in $G \times K_2$, where $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(G') = \{u'_1, u'_2, \ldots, u'_n\}$, and $u_iu'_i \in E(G \times K_2)$ for $1 \leq i \leq n$. If $c$ is a minimum $F$-coloring of $G$ and $c'$ is a minimum $F$-coloring of $G'$, then $c$ and $c'$ produce a minimum $F$-coloring of $G \times K_2$ with $R_c \cup R_{c'}$ as the set of red vertices and so the result follows. □

Equality in (2.3) holds if $G$ is a complete graph. We have seen that $\gamma_F(K_n) = 1$ for $n \geq 3$. Next, we show that $\gamma_F(K_n \times K_2) = 2$ for every integer $n \geq 3$.

Proposition 2.25. For each integer $n \geq 3$,

$$\gamma_F(K_n \times K_2) = 2.$$
Proof. Observe that $\gamma_F(K_2) = 2$. Since $\gamma_F(K_n) = 1$ ($n \geq 3$), it follows by Proposition 2.24 that $\gamma_F(K_n \times K_2) \leq 2$. On the other hand, assume, to the contrary, that $\gamma_F(G) = 1$ and let $c$ be an $F$-coloring of $G$ with exactly one red vertex, say $u_1$. However, since there is no $u_1 - v_1$ path of length 2 in $K_n \times K_2$ it follows that $v_1$ is not $F$-dominated by $u_1$, which is a contradiction. Therefore, $\gamma_F(G) = 2$. □

Strict inequality in (2.3) can also hold. In fact, more can be said.

**Proposition 2.26.** There exist graphs $G_1$, $G_2$, and $G_3$ such that

(a) $\gamma_F(G_1) > \gamma_F(G_1 \times K_2)$,

(b) $\gamma_F(G_2) = \gamma_F(G_2 \times K_2)$,

(c) $\gamma_F(G_3) < \gamma_F(G_3 \times K_2)$.

Proof. We first prove (a). Let $P_5 : v_1, v_2, \ldots, v_5$ be a path of order 5. Then the graph $G_1$ of order $n \geq 6$ is obtained from $P_5$ by adding $n - 5 \geq 1$ new vertices $u_i$ ($1 \leq i \leq n - 5$) and joining each $u_i$ to $v_4$. Let $U = \{u_i : 1 \leq i \leq n - 5\}$. We show that $\gamma_F(G_1) = 3$ and $\gamma_F(G_1 \times K_2) = 2$.

Since the red-blue coloring that assigns red to the vertices of the set $\{v_1, v_2, v_5\}$ and blue to the remaining vertices of $G_1$ is an $F$-coloring of $G_1$, it follows that $\gamma_F(G_1) \leq 3$.

To show that $\gamma_F(G_1) \geq 3$, let $c$ be a minimum $F$-coloring of $G_1$. Since

(1) $v_1$ is an end vertex adjacent to a vertex of degree 2 in $G_1$, it follows that $v_1$ must be colored red by $c$,

(2) $v_2$ can only be $F$-dominated by itself or by $v_4$, and so at least one of $v_2$ and $v_4$ must be colored red by $c$,

(3) $v_5$ can only be $F$-dominated by a vertex in $\{v_3, v_5\} \cup U$ and so at least one vertex of the set $\{v_3, v_5\} \cup U$ must be colored red by $c$, and
(4) the sets \{v_1\}, \{v_2, v_4\} and \{v_3, v_5\} \cup U are mutually disjoint, it follows that \(\gamma_F(G_1) \geq 3\). Therefore \(\gamma_F(G_1) = 3\).

Next, we show that \(\gamma_F(G_1 \times K_2) = 2\). Observe that \(G_1 \times K_2\) is a bipartite graph and so \(\gamma_F(G_1 \times K_2) \geq 2\). On the other hand, let \(H_1'\) and \(H_1''\) be two copies of \(G_1\) in \(G_1 \times K_2\), where \(v_3 \in V(H_1')\) and \(v_3' \in V(H_1'')\) such that \(v_3v_3' \in E(G_1 \times K_2)\). Then the red-blue coloring that assigns red to the vertices of the set \{v_3, v_3'\} and blue to the remaining vertices of \(G_1 \times K_2\) is an \(F\)-coloring of \(G_1 \times K_2\), implying that \(\gamma_F(G_1 \times K_2) \leq 2\) and so \(\gamma_F(G_1 \times K_2) = 2\).

We next prove \((b)\). Let \(P_6 : v_1, v_2, \ldots, v_6\) be a path of order 6. Then the graph \(G_2\) is obtained from \(P_6\) by adding \(n - 6 \geq 1\) new vertices \(u_i\) \((1 \leq i \leq n - 6)\) and joining each \(u_i\) to \(v_5\). Let \(U = \{u_i : 1 \leq i \leq n - 6\}\). We show that
\[
\gamma_F(G_2) = \gamma_F(G_2 \times K_2) = 4.
\]

Since the red-blue coloring that assigns red to the vertices in the set \{\(v_1, v_2, v_3, v_6\}\) and blue to the remaining vertices of \(G_2\) is an \(F\)-coloring of \(G_2\), it follows that \(\gamma_F(G_2) \leq 4\).
To show that $\gamma_F(G_2) \geq 4$, let $c$ be a minimum $F$-coloring of $G_2$. Again, $v_4 \in R_c$. Since $v_6$ is only $F$-dominated by a vertex in $\{v_4, v_6\} \cup U$, it follows that $R_c$ contains at least one vertex in $\{v_4, v_6\} \cup U$. Observe that $v_2$ can only be $F$-dominated by $v_2$ or $v_4$ and so $R_c$ contains at least one vertex in $\{v_2, v_4\}$.

If $v_2 \in R_c$, then $v_3$ can only be $F$-dominated by a vertex in the set $\{v_3, v_5\}$. First, assume that $v_3 \notin R_c$. Since the four sets $\{v_1\}, \{v_3\}, \{v_4\}$, and $\{v_5\} \cup U$ are mutually disjoint and $R_c$ contains at least one vertex in each set, $\gamma_F(G_2) \geq 4$. Thus we may assume that $v_3 \notin R_c$ and $v_5 \in R_c$. Since the four sets $\{v_1\}, \{v_3\}, \{v_4\}$, and $\{v_5\} \cup U$ are mutually disjoint and $R_c$ contains at least one vertex in each set, $\gamma_F(G_2) \geq 4$. If $v_2 \notin R_c$, then $v_4 \in R_c$. This implies that every vertex in $\{v_3, v_5\} \cup U$ belongs to $R_c$ and so $\{v_1, v_3, v_5, v_6\} \cup U \subseteq R_c$. Therefore $\gamma_F(G_2) \geq 4$ and so $\gamma_F(G_2) = 4$.

We next show that $\gamma_F(G_2 \times K_2) = 4$. Let $H_2$ and $H_2'$ be copies of $G_2$, where $V(H_2) = \{u_i, v_j : 1 \leq i \leq n - 6, 1 \leq j \leq 6\}$, $V(H_2') = \{u'_i, v'_j : 1 \leq i \leq n - 6, 1 \leq j \leq 6\}$, and $u'_iu_i, v'_jv_j \in E(G_2 \times K_2)$ for $1 \leq i \leq n - 6$ and $1 \leq j \leq 6$. Let $U = \{u_i : 1 \leq i \leq n - 6\}$ and $U' = \{u'_i : 1 \leq i \leq n - 6\}$. Since the red-blue coloring that assigns red to the vertices of the set $\{v_1, v'_1, u_4, u'_4\}$ and blue to the remaining vertices of $G_2 \times K_2$ is an $F$-coloring of $G_2 \times K_2$, it follows that $\gamma_F(G_2 \times K_2) \leq 4$. To show that $\gamma_F(G_2 \times K_2) \geq 4$, let $c$ be a minimum $F$-coloring of $G_2 \times K_2$. Since

1. $v_6$ is only $F$-dominated by a vertex of the set $\{v_4, v_5, v_6\} \cup U$,
2. $v'_6$ is only $F$-dominated by a vertex of the set $\{v'_4, v_5, v'_6\} \cup U'$,
3. $v_1$ is only $F$-dominated by a vertex of the set $\{v_1, v'_2, v_3\}$,
4. $v'_1$ is only $F$-dominated by a vertex of the set $\{v'_1, v_2, v'_3\}$, and
5. the sets $\{v_4, v'_5, v_6\} \cup U$, $\{v'_4, v_5, v'_6\} \cup U'$, $\{v_1, v'_2, v_3\}$ and $\{v'_1, v_2, v'_3\}$ are mutually disjoint,

it follows that $\gamma_F(G_2 \times K_2) \geq 4$ and so $\gamma_F(G_2 \times K_2) = 4$. Thus $\gamma_F(G_2) = 4 = \gamma_F(G_2 \times K_2)$, as desired.
Finally, we prove (c). Let $P_7: v_1, v_2, \ldots, v_7$ be a path of order 7. Then the graph $G_3$ is obtained from $P_7$ by adding $n - 7 \geq 1$ new vertices $u_i$ ($1 \leq i \leq n - 7$) and joining each $u_i$ to $v_6$. Let $U = \{u_i : 1 \leq i \leq n - 7\}$. We show

$$\gamma_F(G_3) = 3 \text{ and } \gamma_F(G_3 \times K_2) = 4.$$ 

Since the red-blue coloring that assigns red to the vertices in the set $\{v_1, v_4, v_7\}$ and blue to the remaining vertices of $G_3$ is an $F$-coloring of $G_3$, it follows that $\gamma_F(G_3) \leq 3$. We next show that $\gamma_F(G_3) \geq 3$. Let $c$ be a minimum $F$-coloring of $G_3$. Since

1. $v_1$ is an end vertex adjacent to a vertex of degree 2 in $G_3$, it follows that $v_1$ must be colored red by $c$,

2. $v_2$ is only $F$-dominated by itself or by $v_4$, so at least one of $v_2$ and $v_4$ must be colored red by $c$,

3. $v_7$ is only $F$-dominated by itself, by $v_5$ or by an end vertex of the set $U$, so at least one vertex of the set $\{v_5, v_7\} \cup U$ must be colored red by $c$, and

4. the sets $\{v_1\}, \{v_2, v_4\}$ and $\{v_5, v_7\} \cup U$ are mutually disjoint,

it follows that $\gamma_F(G_3) \geq 3$. Therefore $\gamma_F(G_3) = 3$.

We next show that $\gamma_F(G_3 \times K_2) = 4$. Let $H_3$ and $H'_3$ be two copies of $G_3$ in $G_3 \times K_2$ with $V(H_3) = \{u_i, v_j : 1 \leq i \leq n - 7, 1 \leq j \leq 7\}$ and $V(H'_3) = \{u'_i, v'_j : 1 \leq i \leq n - 7, 1 \leq j \leq 7\}$, where $u'_iu_i, v'_jv_j \in E(G_3 \times K_2)$ for $1 \leq i \leq n - 7$ and $1 \leq j \leq 7$. Let $U = \{u_i : 1 \leq i \leq n - 7\}$ and $U' = \{u'_i : 1 \leq i \leq n - 7\}$, $n \geq 8$.

Since the red-blue coloring that assigns red to the vertices of the set $\{v_1, v'_1, v_5, v'_5\}$ and blue to the remaining vertices of $G_3 \times K_2$ is an $F$-coloring of $G_3 \times K_2$, it follows that $\gamma_F(G_3 \times K_2) \leq 4$. Let $c$ be a minimum $F$-coloring of $G_3 \times K_2$. Since

1. $v_7$ is only $F$-dominated by a vertex in the set $\{v_5, v_6, v_7\} \cup U$,

2. $v'_7$ is only $F$-dominated by a vertex in the set $\{v'_5, v_6, v'_7\} \cup U'$,
(3) \(v_1\) is only \(F\)-dominated by a vertex in the set \(\{v_1, v'_2, v_3\}\),

(4) \(v'_1\) is only \(F\)-dominated by a vertex in the set \(\{v'_1, v_2, v'_3\}\),

(5) the sets \(\{v_5, v'_6, v_7\} \cup U\), \(\{v'_5, v_6, v'_7\} \cup U'\), \(\{v_1, v'_2, v_3\}\) and \(\{v'_1, v_2, v'_3\}\) are mutually disjoint.

It follows that \(\gamma_F(G_3 \times K_2) \geq 4\) and so \(\gamma_F(G_3 \times K_2) = 4\).

2.5 Bounds for \(F\)-Domination Numbers of Graphs

In this section, we establish bounds for the \(F\)-domination number of a connected graph \(G\) in terms of some well-known graphical parameters. The clique number \(\omega(G)\) of a graph \(G\) is the maximum order among the complete subgraphs of \(G\). Thus for a connected graph \(G\) of order \(n \geq 2\), \(\omega(G) = 2\) if and only if \(G\) is triangle-free; while \(\omega(G) = n\) if and only if \(G\) is complete. For a connected graph \(G\) that is neither triangle-free nor complete, we present an upper bound for \(\gamma_F(G)\) in terms of its order and clique number.

Proposition 2.27. If \(G\) is a connected graph of order \(n \geq 4\) having clique number \(\omega(G)\), where \(3 \leq \omega(G) \leq n - 1\), then

\[
\gamma_F(G) \leq n - \omega(G).
\]

Proof. Let \(K_\omega\) be a complete subgraph of order \(\omega(G)\) in \(G\). Since \(G\) is not complete, there exist vertices \(x, y, z\) in \(G\), where \(x, y \in V(K_\omega)\) and \(z \notin V(K_\omega)\), such that \(z\) is adjacent to \(x\) but not to \(y\). Let \(U = (V(K_\omega) - \{y\}) \cup \{z\}\). Then \(|U| = \omega(G)\). Define a red-blue coloring \(c\) of \(G\) by assigning blue to each vertex in \(U\) and red to the remaining vertices of \(G\). Since each blue vertex is \(F\)-dominated by the red vertex \(y\), it follows that \(c\) is an \(F\)-coloring and so

\[
\gamma_F(G) \leq |R_c| = n - |U| = n - \omega(G),
\]
as desired. □

The upper bound in Proposition 2.27 is attainable for every connected graph $G$ of order $n \geq 4$ with $\omega(G) = n - 1$. That is, if $\omega(G) = n - 1$, then $\gamma_F(G) = n - \omega(G) = 1$. To see that the upper bound in Proposition 2.27 is attainable for a connected graph $G$ of order $n \geq 4$ with $\omega(G) = n - 2$, let $G$ be the graph obtained from $K_{n-2}$ and $P_2: x, y$ of path of length 1 by joining $x$ to a vertex $v$ of $K_{n-2}$. So $\omega(G) = n - 2$. Let $u \in V(K_{n-2}) - \{v\}$. Then the red-blue coloring $c$ defined by assigning red to $u$ and $y$ and blue to the remaining vertices of $G$ is a minimum $F$-coloring of $G$ and $\gamma_F(G) = 2$. Therefore,

$$\gamma_F(G) = n - \omega(G).$$

However, the upper bound in Proposition 2.27 is not attainable if $\omega(G) = n - 3$ for $n \geq 6$, as we show next.

**Proposition 2.28.** If $G$ is a connected graph of order $n \geq 6$ with $\omega(G) = n - 3$, then

$$\gamma_F(G) \leq 2.$$

**Proof.** Let $G$ be a connected graph of order $n \geq 6$ with $\omega(G) = n - 3 \geq 3$ and let $K_{n-3}$ be a complete subgraph of $G$. Suppose that $V(G) = V(K_{n-3}) \cup \{x, y, z\}$. Let $H = \langle \{x, y, z\} \rangle$ be the subgraph induced by $\{x, y, z\}$. We consider three cases.

**Case 1.** $H \cong K_3$. Since $G$ is connected, each of the vertices $x, y, z$ is adjacent to some vertex of $V(K_{n-3})$, say $xx', yy', zz' \in E(G)$, where $x', y', z' \in V(K_{n-3})$ and $x', y', z'$ may not be distinct. Let $U = \{x', y', z'\}$. If $V(K_{n-3}) - U \neq \emptyset$, then let $v \in V(K_{n-3}) - U$. Then the red-blue coloring of $G$ defined by assigning red to $v$ and blue to the remaining vertices of $G$ is an $F$-coloring of $G$, implying that $\gamma_F(G) = 1$. If $V(K_{n-3}) - U = \emptyset$, then $n = 6$ and $|U| = 3$. Then the red-blue coloring of $G$ defined by assigning red to $x$ and $x'$ and blue to the remaining vertices of $G$ is a minimum $F$-coloring of $G$ and so $\gamma_F(G) = 2$.

**Case 2.** $H \cong P_2 \cup K_1$. We may assume that $xy \in E(G)$. Since $G$ is connected, at least one of $x$ and $y$ is adjacent to some vertex of $V(K_{n-3})$ and $z$ is adjacent to some

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
vertex of \( V(K_{n-3}) \). Assume, without loss of generality, that \( xx', zz' \in E(G) \), where \( x', z' \in V(K_{n-3}) \) and \( x', z' \) may not be distinct. Let \( v \in V(K_{n-3}) - \{x', z'\} \). Then the red-blue coloring of \( G \) defined by assigning red to \( v \) and \( y \) and blue to the remaining vertices of \( G \) is an \( F \)-coloring of \( G \). Therefore, \( \gamma_F(G) \leq 2 \).

**Case 3.** \( H \cong P_3 \) or \( H \cong C_3 \). First, assume that \( H \cong P_3 : x, y, z \). Since \( G \) is connected, at least one of \( x, y, z \) is adjacent to some vertex in \( V(K_{n-3}) \). If \( x \) is adjacent to a vertex \( x' \) in \( V(K_{n-3}) \), then define a red-blue coloring by assigning red to \( x' \) and \( z \) and blue to the remaining vertices of \( G \). Similarly, if \( z \) is adjacent to a vertex \( z' \) in \( V(K_{n-3}) \), then define a red-blue coloring by assigning red to \( x \) and \( z' \) and blue to the remaining vertices of \( G \). If \( y \) is adjacent to a vertex \( y' \) in \( V(K_{n-3}) \), then define a red-blue coloring by assigning red to \( v \) and \( z \), where \( v \in V(K_{n-3}) - \{y'\} \) and blue to the remaining vertices of \( G \). In each case, the red-blue coloring is an \( F \)-coloring of \( G \) and so \( \gamma_F(G) < 2 \).

Next, assume that \( H \cong C_3 : x, y, z, x \). Again, at least one of \( x, y, z \) is adjacent to some vertex in \( V(K_{n-3}) \), say \( z \) is adjacent to \( z' \). Then the red-blue coloring of \( G \) that assigns red to \( x \) and \( z' \), and blue to the remaining vertices of \( G \) is an \( F \)-coloring and so \( \gamma_F(G) \leq 2 \).

Observe that if \( G \) is the graph obtained from the path \( P_4 : u, v, w, x \) by adding a new vertex \( y \) and joining \( y \) to \( w \) and \( x \), then the order of \( G \) is \( n = 5 \), the clique number of \( G \) is \( \omega(G) = 3 \), and the \( F \)-domination number is \( \gamma_F(G) = 2 \), where \( \{u, x\} \) is a minimum \( F \)-dominating set. Thus, \( \gamma_F(G) = n - \omega(G) \). On the other hand, if \( G \) is a connected graph of order \( n \geq 6 \) and \( \omega(G) = 3 \), then \( \gamma_F(G) \neq n - \omega(G) \), as we show next.

**Proposition 2.29.** If \( G \) is a connected graph of order \( n \geq 6 \) and \( \omega(G) = 3 \), then

\[
\gamma_F(G) \leq n - 4.
\]

**Proof.** Let \( C : u, v, w, u \) be a triangle in \( G \). Since \( G \) is connected, there is a vertex in \( V(G) - V(C) \) that is adjacent to some vertex in \( C \). Thus, at least one vertex on \( C \) has degree 3 or more. We consider two cases.
Case 1. At least two vertices on $C$ have degree 3 or more, say $\deg u \geq 3$ and $\deg v \geq 3$. Let $u', v' \in V(G) - V(C)$ such that $uu', vv' \in E(G)$. Then the red-blue coloring that assigns blue to each vertex in $\{u, u', v, v'\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ with exactly $n - 4$ red vertices. Thus $\gamma_F(G) \leq n - 4$.

Case 2. Exactly one vertex on $C$ has degree 3 or more, say $\deg u > 3$ and $\deg u'' = \deg u''' = 2$. We consider two subcases.

Subcase 2.1. $\deg u > 4$. Let $u_1, u_2 \in V(G) - V(C)$ such that $uu_i \in E(G)$ for $i = 1, 2$. Then the red-blue coloring that assigns blue to each vertex in $\{u, u_1, u_2, v\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ with exactly $n - 4$ red vertices. Thus $\gamma_F(G) \leq n - 4$.

Subcase 2.2. $\deg u = 3$. Let $u_1 \in V(G) - V(C)$ such that $uu_1 \in E(G)$. Since (1) $G$ is a connected graph of order $n \geq 6$ and (2) $\deg v = \deg w = 2$, there $u_2, u_3 \in V(G) - V(C)$ such that $u_1u_2 \in E(G)$ and either $u_1u_3 \in E(G)$ or $u_2u_3 \in E(G)$. If $u_1u_3 \in E(G)$, then the red-blue coloring that assigns blue to each vertex in $\{u, u_1, u_2, v\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ with exactly $n - 4$ red vertices. If $u_2u_3 \in E(G)$, then the red-blue coloring that assigns blue to each vertex in $\{u, u_2, v, w\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ with exactly $n - 4$ red vertices. In each case, $\gamma_F(G) \leq n - 4$. □

In order to present upper bounds for the $F$-domination number of a connected graph in terms of some other well-known graphical parameters, we first present a lemma, that generalizes Corollary 2.11.

**Lemma 2.30.** If $G$ is a connected graph of order at least 3 and $H$ is a connected subgraph of $G$, then

$$\gamma_F(G) + |V(H)| \leq |V(G)| + \gamma_F(H).$$

**Proof.** Let $c^*$ be a minimum $F$-coloring of $H$. We extend $c^*$ to an $F$-coloring $c$ of $G$ by assigning red to each vertex in $V(G) - V(H)$. Thus,

$$\gamma_F(G) \leq |R_c| = |V(G) - V(H)| + |R_{c^*}| = |V(G)| - |V(H)| + \gamma_F(H),$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
as desired.

We have seen that $\gamma_F(K_n) = 1$ for $n \geq 3$. For a connected graph $G$ that is not complete, we present an upper bound for $\gamma_F(G)$ in terms of its order and diameter.

**Proposition 2.31.** If $G$ is a connected graph of order $n \geq 3$ and diameter $d$ with $2 \leq d \leq n - 1$, then

$$\gamma_F(G) \leq (d \pmod 3) + n - \left\lfloor \frac{2(d+1)}{3} \right\rfloor. \quad (2.4)$$

**Proof.** Since the diameter of $G$ is $d$, it follows that $P_{d+1}$ is a subgraph of $G$. By Lemma 2.30,

$$\gamma_F(G) \leq n - (d + 1) + \gamma_F(P_{d+1}).$$

We consider three cases.

**Case 1.** $d \pmod 3 = 0$. Then $(d+3) \pmod 3 = 0$. It follows by Theorem 2.16 that $\gamma_F(P_{d+1}) = \left\lceil \frac{d+1}{3} \right\rceil$. So

$$\gamma_F(G) \leq n - (d + 1) + \left\lfloor \frac{d + 1}{3} \right\rfloor = n - \left\lfloor \frac{2(d+1)}{3} \right\rfloor.$$

**Case 2.** $d \pmod 3 = 1$. Then $(d+3) \pmod 3 = 1$. Now $\gamma_F(P_{d+1}) = \left\lceil \frac{d+1}{3} \right\rceil + 1$ by Theorem 2.16. Hence

$$\gamma_F(G) \leq n - (d + 1) + \left( \left\lceil \frac{d + 1}{3} \right\rceil + 1 \right) = n - \left\lfloor \frac{2(d+1)}{3} \right\rfloor + 1.$$

**Case 3.** $d \pmod 3 = 2$. Then $(d+3) \pmod 3 = 2$. Then $\gamma_F(P_{d+1}) = \left\lceil \frac{d+1}{3} \right\rceil + 2$ by Theorem 2.16. Therefore,

$$\gamma_F(G) \leq n - (d + 1) + \left( \left\lceil \frac{d + 1}{3} \right\rceil + 2 \right) = n - \left\lfloor \frac{2(d+1)}{3} \right\rfloor + 2,$$

which gives the desired result. □

Note that for a fixed integer $n \geq 3$, the upper bound in (2.4) is attainable for $d = 2$ or $d = n - 1$. For example, let $n \geq 3$. For $d = 2$, let $G = K_{1,n-1}$, while for $d = n - 1$, let $G = P_n$, where $n \geq 3$. In each case, equality holds. In fact, Proposition 2.31 is
Theorem 2.16 for \( G = P_n \). Furthermore, the upper bound in (2.4) is also attainable for some other values of \( d \). For example, let \( G \) be the graph obtained from the path \( P_{3k+1} : v_1, v_2, \ldots, v_{3k+1}, k \geq 2 \), by adding one vertex \( u \) together with the edge \( uv_4 \). Then the order of \( G \) is \( n = 3k + 2 \) and the diameter of \( G \) is \( d = 3k = n - 2 \). We show that

\[
\gamma_F(G) = n - \left\lfloor \frac{2(d+1)}{3} \right\rfloor + d \mod{3} = (3k + 2) - \left\lfloor \frac{2(3k + 1)}{3} \right\rfloor + (3k) \mod{3} = k + 2.
\]

By Proposition 2.31, \( \gamma_F(G) \leq k + 2 \). To show that \( \gamma_F(G) \geq k + 2 \), let \( c \) be a minimum \( F \)-coloring of \( G \). First, we claim for each \( i \) with \( 2 \leq i \leq k \), that \( R_c \) contains at least one vertex in each set \( S_i = \{v_{3i-1}, v_{3i}, v_{3i+1}\} \); for otherwise, assume to the contrary, that there is an integer \( j \) \((2 \leq j \leq k)\) such that all three vertices in the set \( S_j \) are colored blue. Since the blue vertex \( v_{3j} \) belongs to a red-blue-blue path rooted at \( v_{3j} \), it follows that at least one of \( v_{3j-2} \) and \( v_{3j+2} \) is colored red, say the former. However then, the blue vertex \( v_{3j-1} \) is not \( F \)-dominated by any red vertex, a contradiction. Therefore, \( R_c \) contains at least one vertex in each set \( S_i \) for \( 2 \leq i \leq k \), as claimed. Furthermore, \( v_1 \) is an end-vertex adjacent to a vertex of degree 2 and so \( v_1 \) must be colored red by \( c \) by Proposition 2.7. Since \( v_2 \) is only \( F \)-dominated by \( v_2 \) or \( v_4 \), either \( v_2 \) or \( v_4 \) is colored red by \( c \). If \( v_4 \) is colored red by \( c \), then \( u \) must be colored red and so \( \gamma_F(G) \geq k + 2 \). If \( v_2 \) is colored red by \( c \) and \( v_4 \) is colored blue by \( c \), then \( v_3 \) is only \( F \)-dominated by a vertex of the set \( \{u, v_3, v_5\} \). If \( u \) is colored red by \( c \) or \( v_3 \) is colored red by \( c \), then \( \gamma_F(G) \geq k + 2 \). Thus, we may assume that \( u \) and \( v_3 \) are both blue and \( v_5 \) is colored red by \( c \). We show that \( \gamma_F(G) \geq k + 2 \) in this case. Assume, to the contrary, that \( \gamma_F(G) = k + 1 \). This implies that \( R_c \) contains exactly one vertex in \( S_i \) for \( 2 \leq i \leq k \). Since \( v_5 \) is colored red, it forces the vertex \( v_{3i-1} \) to be the only red vertex in \( S_i \) for \( 2 \leq i \leq k \). In particular, \( v_{3k+1} \) in \( S_k \) is blue, which contradicts Proposition 2.7. Therefore, \( \gamma_F(G) = k + 2 \).

Recall that the girth \( g(G) \) of a graph (with cycles) is the length of a shortest cycle in \( G \). With the aid of Lemma 2.30, we are able to present an upper bound for \( \gamma_F(G) \) of a connected graph \( G \) in terms of its order and girth.
Proposition 2.32. If $G$ is a connected graph of order $n \geq 3$ with girth $g(G) \geq 3$, then

$$\gamma_F(G) \leq n - 2 \left\lfloor \frac{g(G)}{3} \right\rfloor.$$  

Proof. Let $C$ be a cycle of length $g(G)$ in $G$. It then follows by Theorem 2.17 and Lemma 2.30 that

$$\gamma_F(G) \leq |V(G) - V(C)| + \gamma_F(C)$$

$$= (n - g(G)) + \left\lfloor \frac{g(G)}{3} \right\rfloor + g(G) \mod 3$$

$$= n - 2 \left\lfloor \frac{g(G)}{3} \right\rfloor,$$

as desired. □

The upper bound in Proposition 2.32 cannot be improved since if $G = C_n$, where $n \geq 3$, then equality holds. In fact, if $G = C_n$, then Proposition 2.32 is Theorem 2.17. On the other hand, for a connected graph that is not a cycle, the upper bound in Theorem 2.17 can be improved.

Proposition 2.33. If $G$ is a connected graph of order $n \geq 3$ and girth $g$ such that $3 \leq g < n$, then

$$\gamma_F(G) \leq n - 2 \left\lfloor \frac{g}{3} \right\rfloor - 1.$$  

Proof. Let $C : v_0, v_1, \ldots, v_{g-1}, v_0$ be a cycle in $G$ whose length is $g$, where $3 \leq g < n$. Then $V(G) - V(C) \neq \emptyset$. Since $G$ is connected, there is $u \in V(G) - V(C)$ such that $u$ is adjacent to some vertex in $C$, say $u v_0 \in E(G)$. Define a red-blue coloring $c$ of $G$ that assigns blue to the vertices in the set

$$\{u\} \cup \{v_{3i}, v_{3i+1} : 0 \leq i \leq \left\lfloor g/3 \right\rfloor - 1\},$$

and red to the remaining vertices of $G$, where the subscripts of the vertices are expressed as integers modulo $g$. Since

(1) $v_{3i}$ belongs to the red-blue-blue path $v_{3i+2}, v_{3i+1}, v_{3i}$,
(2) \( v_{3i+1} \) belongs to the red-blue-blue path \( v_{3i-1}, v_{3i}, v_{3i+1} \), and

(3) \( u \) belongs to the red-blue-blue path \( v_{g-1}, v_0, u \),

it follows that \( c \) is an \( F \)-coloring of \( G \) and

\[
|R_c| = n - \left( 1 + 2 \left\lfloor \frac{g}{3} \right\rfloor \right) = n - 2 \left\lfloor \frac{g}{3} \right\rfloor - 1.
\]

Therefore, \( \gamma_F(G) \leq |R_c| = n - 2 \left\lfloor \frac{g}{3} \right\rfloor - 1. \)

The upper bound in Theorem 2.33 cannot be improved. To see this, let \( G \) be the graph obtained from the \((3k + 3)\)-cycle \( C_{3k+3} : v_0, v_1, \ldots, v_{3k+2}, v_0 \), where \( k \geq 1 \), by adding a new vertex \( u \) and joining \( u \) to \( v_0 \). Thus the order of \( G \) is \( n = 3k + 4 \) and the girth of \( G \) is \( g = 3k + 3 \). We show that

\[
\gamma_F(G) = n - 2 \left\lfloor \frac{g}{3} \right\rfloor - 1 = k + 1.
\]

By Theorem 2.33, it suffices to show that \( \gamma_F(G) \geq k + 1 \). Let \( c \) be a minimum \( F \)-coloring of \( G \). We claim, for each \( i \) with \( 0 \leq i \leq k \), that at least one vertex in the set \( S_i = \{v_{3i}, v_{3i+1}, v_{3i+2}\} \) must be colored red, where the subscripts of the vertices are expressed as integers modulo \( 3k + 3 \). Assume, to the contrary, that there is an integer \( j \) with \( 0 \leq j \leq k \) such that all three vertices in \( S_j \) are colored blue. Observe that the blue vertex \( v_{3j+1} \) must belong to a red-blue-blue path rooted at \( v_{3j+1} \) for \( 0 \leq j \leq k \). If \( j = 0 \), then at least one of the three vertices \( u, v_{3k+2}, \) and \( v_3 \) must be colored red. If \( u \) or \( v_{3k+2} \) is colored red, then \( v_0 \) is not \( F \)-dominated by any red vertex in \( R_c \); while if \( v_3 \) is colored red, then \( v_2 \) is not \( F \)-dominated by any red vertex in \( R_c \). If \( j \geq 1 \), then at least one of the two vertices \( v_{3j-1} \) and \( v_{3j+3} \) must be colored red, say the former. However then, the blue vertex \( v_{3j} \) is not \( F \)-dominated by any red vertex, which is impossible. Therefore, \( R_c \) contains at least one vertex in each set \( S_i \) for \( 0 \leq i \leq k \). Since the \( k + 1 \) sets \( S_i \) (\( 0 \leq i \leq k \)) are mutually disjoint, \( \gamma_F(G) = |R_c| \geq k + 1 \).

The following is an immediate consequence of Propositions 2.32 and 2.33.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Corollary 2.34. If $G$ is a connected graph of order $n \geq 3$ with girth $g(G) \geq 3$, then

$$\gamma_F(G) \leq n - 2 \left\lfloor \frac{g(G)}{3} \right\rfloor.$$  \hspace{1cm} (2.5)

Furthermore, equality holds in (2.5) if and only if $G = C_n$.

By Corollary 2.11 and Theorems 2.16 and 2.17, we obtain upper bounds for certain classes of graphs.

Corollary 2.35. Let $G$ be a connected graph of order $n \geq 3$.

(a) If $G$ contains a Hamiltonian path, then

$$\gamma_F(G) \leq \left\lfloor \frac{n}{3} \right\rfloor + (n + 2) \mod 3.$$  

(b) If $G$ is Hamiltonian, then

$$\gamma_F(G) \leq \left\lfloor \frac{n}{3} \right\rfloor + n \mod 3.$$  

In Corollary 2.35 equality holds in (a) if $G = P_n$; while equality holds in (b) if $G = C_n$. In fact, equality also holds in each case for other connected graphs that are not paths or cycles, as we show next.

(a) For an integer $k \geq 3$, let $G$ be the graph obtained from the path $P_{3k+1}$: $v_0$, $v_1$, ..., $v_{3k}$ of order $n = 3k + 1$ by adding the edge $v_2v_{3k-2}$. Hence $G$ has a Hamiltonian path, namely $P_{3k+1}$. We show that

$$\gamma_F(G) = \left\lfloor \frac{n}{3} \right\rfloor + (n + 2) \mod 3 = k + 1.$$  

By Corollary 2.35, $\gamma_F(G) \leq k + 1$. To show that $\gamma_F(G) \geq k + 1$, let $c$ be a minimum $F$-coloring of $G$. We claim that, for each $i$ with $1 \leq i \leq k - 2$, at least one vertex in the set $S_i = \{v_{3i}, v_{3i+1}, v_{3i+2}\}$ is colored red by $c$; for otherwise, there is an integer $j$ ($1 \leq j \leq k - 2$) such that all three vertices in the set $S_j$ are colored blue. Since the blue vertex $v_{3j+1}$ belongs to a red-blue-blue path rooted at $v_{3j+1}$, it follows that at
least one of \( v_{3j-1} \) and \( v_{3j+3} \) is colored red, say the former. However then, the blue vertex \( v_{3j} \) is not \( F \)-dominated by any red vertex, a contradiction. Therefore, \( R_c \) contains at least one vertex in each set \( S_i \) for \( 1 \leq i \leq k-2 \). Furthermore, \( v_0 \) and \( v_{3k} \) must be colored red by \( c \) by Proposition 2.7. Since \( v_{3k-1} \) is only \( F \)-dominated by \( v_2 \), \( v_{3k-3} \) or \( v_{3k-1} \), it follows that \( R_c \) contains at least one vertex in \( \{v_2, v_{3k-3}, v_{3k-1}\} \). Since the \( k+1 \) sets \( S_i \) (\( 1 \leq i \leq k-2 \)), \( \{v_0\} \), \( \{v_{3k}\} \), and \( \{v_2, v_{3k-3}, v_{3k-1}\} \) are disjoint, \( R_c \) must contain at least one vertex in each set, \( \gamma_F(G) = |R_c| \geq k+1 \). Therefore, \( \gamma_F(G) = k+1 \).

(b) For an integer \( k \geq 3 \), let \( G \) be the graph obtained from the cycle \( C_{3k} \): \( v_1, v_2, \ldots, v_{3k}, v_1 \) of order \( n = 3k \) by adding the edge \( v_1v_4 \). Then \( G \) is Hamiltonian. We show that

\[
\gamma_F(G) = \lfloor n/3 \rfloor + n \mod 3 = \lfloor 3k/3 \rfloor + (3k) \mod 3 = k.
\]

By Corollary 2.35, \( \gamma_F(G) \leq k \). To show \( \gamma_F(G) \geq k \), let \( c \) be a minimum \( F \)-coloring of \( G \). We claim that, for each \( i \) with \( 2 \leq i \leq k-1 \), at least one vertex in the set \( S_i = \{v_{3i}, v_{3i+1}, v_{3i+2}\} \) is colored red by \( c \). Assume, to the contrary, that there is an integer \( j \) (\( 2 \leq j \leq k-1 \)) such that all three vertices in the set \( S_j \) are colored blue. Since the blue vertex \( v_{3j+1} \) belongs to a red-blue-blue path rooted at \( v_{3j+1} \), it follows that at least one of \( v_{3j-1} \) and \( v_{3j+3} \) is colored red, say the former. However then, the blue vertex \( v_{3j} \) is not \( F \)-dominated by any red vertex, a contradiction. Therefore \( R_c \) contains at least one vertex in each set \( S_i \) for \( 2 \leq i \leq k-1 \). Since the vertex \( v_2 \) is only \( F \)-dominated by a vertex in the set \( V_2 = \{v_2, v_4, v_{3k}\} \) and the vertex \( v_3 \) is only \( F \)-dominated by a vertex in the set \( V_3 = \{v_1, v_3, v_5\} \), it follows that \( R_c \) contains at least one vertex in each of \( V_2 \) and \( V_3 \). Since the \( k \) sets \( S_i \) (\( 2 \leq i \leq k-1 \)), \( V_2 \), and \( V_3 \) are mutually disjoint and \( R_c \) must contain at least one vertex in each set, \( \gamma_F(G) = |R_c| \geq k \). Therefore, \( \gamma_F(G) = k \).

Next, we present an upper bound for the \( F \)-domination number of a nonstar graph in terms of its order and maximum degree.
Theorem 2.36. If $G$ is a connected graph of order $n \geq 3$ that is not a star, then

$$\gamma_F(G) \leq n - \Delta(G).$$

Proof. Assume first that $\Delta(G) = n - 1$. Since $G$ is not a star, $G$ contains a spanning subgraph isomorphic to $K_{1,n-1} + e$. Since $\gamma_F(K_{1,n-1} + e) = 1$, it then follows by Proposition 2.11 that $\gamma_F(G) = 1$ and so

$$\gamma_F(G) = n - \Delta(G).$$

Next, assume that $2 \leq \Delta(G) = \Delta < n - 1$. Let $x \in V(G)$ such that $\deg x = \Delta$. Suppose that $N(x) = \{x_1, x_2, \ldots, x_\Delta\}$. Since $\Delta < n - 1$ and $G$ is connected, there exists $y \in V(G) \setminus N[x]$ such that $y$ is adjacent to some vertex in $N(x)$, say $yx_\Delta \in E(G)$. Define a red-blue coloring $c$ of $G$ by assigning blue to every vertex in $N[x] \setminus \{x_1\}$ and red to the remaining vertices of $G$. Since the blue vertex $x$ is $F$-dominated by $y$ and each blue vertex $x_i$ ($2 \leq i \leq \Delta$) is $F$-dominated by $x_1$, it follows that $c$ is an $F$-coloring of $G$. Therefore,

$$\gamma_F(G) \leq |R_c| = n - |N[x] \setminus \{x_1\}| = n - \Delta(G),$$

as desired. \qed

The upper bound in Theorem 2.36 cannot be improved. In fact, for each integer $k \geq 2$, there is a connected graph $G$ of order $n = 2k$ with $\Delta(G) = k$ such that $\gamma_F(G) = k$ and so $\gamma_F(G) = n - \Delta(G)$. To construct this graph $G$, we start with the star $K_{1,k}$ with $V(K_{1,k}) = \{u_1, u_2, \ldots, u_k, w\}$, where $k \geq 2$ and $w$ is the central vertex of $K_{1,k}$. Then the graph $G$ is obtained from the graph $K_{1,k}$ by adding $k - 1$ new vertices $v_i$, ($1 \leq i \leq k - 1$) and joining each $v_i$ to the corresponding $u_i$ ($1 \leq i \leq k - 1$). Then the order of $G$ is $2k$ and the maximum degree of $G$ is $\Delta(G) = k$. We show that $\gamma_F(G) = k$. By Theorem 2.36, it suffices to show that $\gamma_F(G) \geq k$. Let $c$ be a minimum $F$-coloring of $G$. By Proposition 2.7, $v_i \in R_c$ for $1 \leq i \leq k - 1$ and so $\gamma_F(G) \geq k - 1$. Assume, to the contrary, that $\gamma_F(G) = k - 1$. Then $R_c = \{v_i : 1 \leq i \leq k - 1\}$. However then,
$u_i$ ($1 \leq i \leq k$) is not $F$-dominated by any vertex in $R_e$, which is a contradiction. Thus $\gamma_F(G) \geq k$ and so $\gamma_F(G) = k$. Therefore, $\gamma_F(G) = n - \Delta(G)$.

We have seen that if $G_1 = K_{1,n-1}$ and $G_2 = K_{1,n-1} + e$, where $n \geq 3$, then $\Delta(G_i) = n - 1$ for $i = 1, 2$, but $\gamma_F(G_1) = n$ and $\gamma_F(G_2) = 1$. Thus $G_1$ and $G_2$ have the same order and the same maximum degree but $\gamma_F(G_1)$ and $\gamma_F(G_2)$ are very different. Certainly, $K_{1,n-1}$ and $K_{1,n-1} + e$ both contain cut-vertices. Next, we show that this is also true for 2-connected graphs.

**Proposition 2.37.** For each integer $k \geq 5$, there exist 2-connected graphs $G_1$ and $G_2$ of order $3k + 1$ and maximum degree $k$ such that

$$\gamma_F(G_1) = 1 \text{ and } \gamma_F(G_2) = k.$$  

**Proof.** Let $W_k = C_k + K_1$ be the wheel, where $C_k : v_1, v_2, \ldots, v_k, v_1$ and $V(K_1) = \{u\}$. First, we construct $G_1$ from $W_k$ and $k$ copies $P(i) : x_i, y_i$ of the path of order 2 ($1 \leq i \leq k$) by joining (1) $x_1$ and $y_k$ to $v_1$ and (2) each $x_i$ and $y_{i-1}$ to $v_i$ for $2 \leq i \leq k$. Then $|V(G_1)| = 3k + 1$ and $\Delta(G_1) = k$. Observe that $\{u\}$ is a minimum $F$-dominating set. Thus $\gamma_F(G_1) = 1$.

![Figure 13. The graph $G_1$ in Proposition 2.37 for $k = 5$.](image)

Next, we construct $G_2$ from the wheel $W_k$ by subdividing each edge on the cycle $C_k$ twice so that each edge $v_i v_{i+1}$ is replaced by the path $v_i, x_i, y_i, v_{i+1}$ for $1 \leq i \leq k - 1$ and the edge $v_k v_1$ is replaced by the path $v_k, x_k, y_k, v_1$. Thus $|V(G_2)| = 3k + 1$ and
\[ \Delta(G_2) = k. \] We show that \( \gamma_F(G_2) = k. \)

\[ G_2 : \]

Figure 14. The graph \( G_2 \) in Proposition 2.37 for \( k = 5. \)

Since the red-blue coloring that assigns red to the vertices in the set \( \{x_1, x_2, \ldots, x_k\} \) and blue to the remaining vertices of \( G_2 \) is an \( F \)-coloring of \( G_2 \), it follows that \( \gamma_F(G_2) \leq k. \) To show \( \gamma_F(G_2) \geq k \), let \( c \) be a minimum \( F \)-coloring of \( G \). We consider two cases.

**Case 1.** \( u \) is colored red by \( c \). Since each vertex \( v_i \) is only \( F \)-dominated by a vertex in the set \( V_i = \{v_i, x_{i-1}, y_i\} \), it follows that \( R_c \) contains at least one vertex in each set \( V_i \). Furthermore, the sets \( V_i \) \((1 \leq i \leq k)\) are mutually disjoint. Therefore, \( |R_c| \geq k \) and so \( \gamma_F(G_2) = |R_c| \geq k. \)

**Case 2.** \( u \) is colored blue by \( c \). Since each vertex \( x_i \) is only \( F \)-dominated by a vertex in the set \( X_i = \{v_{i+1}, x_i, y_{i-1}\} \), it follows that \( R_c \) contains at least one vertex in each set \( X_i \). Furthermore, the sets \( X_i \) \((1 \leq i \leq k)\) are mutually disjoint. Therefore, \( |R_c| \geq k \) and so \( \gamma_F(G_2) = |R_c| \geq k. \)

Therefore, \( \gamma_F(G_2) = k \) as desired. \( \Box \)

There also exist 2-connected graphs \( H_1 \) and \( H_2 \) of the same order and minimum degree such that \( \gamma_F(H_1) \) and \( \gamma_F(H_2) \) are very different. For example, let \( H_1 = C_n \) and \( H_2 = W_n - uv \), where \( n \geq 4 \) and \( u \) is the vertex of degree \( n \) in \( W_n \) and \( v \) is any other...
vertex in $H_2$. Then $\delta(H_1) = \delta(H_2) = 2$, but $\gamma_F(C_n) = \left\lfloor \frac{n}{3} \right\rfloor + n \mod 3$ and $\gamma_F(W_n) = 1$. Thus the difference between $\gamma_F(C_n)$ and $\gamma_F(W_n)$ can be as large as one wishes if $n$ is sufficiently large.

2.6 Graphs with Given Order and $F$-Domination Number

We have seen that if $G$ is a connected graph of order $n$, then $1 \leq \gamma_F(G) \leq n$. First, we present a characterization of connected graphs $G$ of order $n \geq 3$ with $\gamma_F(G) = n$.

**Theorem 2.38.** Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_F(G) = n$ if and only if $G = K_{1,n-1}$.

**Proof.** By Proposition 2.12, $\gamma_F(K_{1,n-1}) = n$. Thus, it remains to verify the converse. Let $G$ be a connected graph of order $n \geq 3$ such that $G \neq K_{1,n-1}$. We consider two cases.

*Case 1.* $G$ is a tree. Since $G \neq K_{1,n-1}$, it follows that $\text{diam}(G) \geq 3$ and so $G$ contains a path $P : v_1, v_2, v_3, v_4$ of length 3. Then the red-blue coloring defined by assigning blue to $v_2$ and $v_3$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ with $n - 2$ red vertices. Thus $\gamma_F(G) \leq n - 2$.

*Case 2.* $G$ is not a tree. Then $G$ contains cycles and so the girth $g(G)$ of $G$ is at least 3. By Proposition 2.32, $\gamma_F(G) \leq n - 2$. □

The proof of Theorem 2.38 gives the following corollary.

**Corollary 2.39.** There is no connected graph $G$ of order $n \geq 3$ with $F$-domination number $n - 1$.

Next, we characterize connected graphs with $F$-domination number 1. In order to do this, we first establish some preliminary results.

**Proposition 2.40.** If $G$ is a connected graph with $\gamma_F(G) = 1$ and $u$ is the only red vertex in a minimum $F$-coloring of $G$, then $e(u) \leq 2$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof. Let $v$ be any vertex in $G$. If $v = u$, then $d(u, v) = 0$. If $v \neq u$, then $v$ is $F$-dominated by $u$. Thus there is a $v - u$ path of length 2 in $G$ and so $d(u, v) \leq 2$. Therefore, $d(u, v) \leq 2$ for every vertex $v$ in $G$, implying that $e(u) \leq 2$. □

Since $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$ for every connected graph $G$, the following is an immediate consequence of Proposition 2.40.

**Corollary 2.41.** If $G$ is a connected graph with $\gamma_F(G) = 1$, then $\text{rad}(G) \leq 2$ and so $\text{diam}(G) \leq 4$.

The bound in Corollary 2.41 is sharp. In fact, there is an infinite class of connected graphs $G$ with $\gamma_F(G) = 1$, $\text{rad}(G) = 2$, and $\text{diam}(G) = 4$. For example, let $k \geq 2$ be an integer. For each integer $i$ with $1 \leq i \leq k$, let $H_i$ be a copy of $K_4 - e$ and let $u_i$ be a vertex of degree 2 in $H_i$. The graph $G_k$ is obtained from the graphs $H_i$ ($1 \leq i \leq k$) by identifying the vertices $u_i$ ($1 \leq i \leq k$) and labeling the identified vertex by $u$. Then $\text{rad}(G_k) = 2$ and $\text{diam}(G_k) = 4$. Since the red-blue coloring of $G_k$ defined by assigning red to the vertex $u$ and blue to the remaining vertices of $G_k$ is an $F$-coloring of $G_k$, it follows that $\gamma_F(G_k) = 1$.

![Figure 15. A red-blue coloring of the graph $G_k$.](image)

By Proposition 2.40 and Corollary 2.41, if $G$ is a connected graph with $\gamma_F(G) = 1$, then $\text{diam}(G) \leq 4$ and there is a vertex $u$ with $e(u) \leq 2$. Next, we present a characterization of connected graphs with $F$-domination number 1. For each of the two graphs $H_1$ and $H_2$ in Figure 2.10, we designate the vertex $u$ of $H_i$ ($i = 1, 2$) as its fixed vertex. Thus $H_1$ and $H_2$ are said to be fixed at $u$.
Theorem 2.42. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_F(G) = 1$ if and only if $G$ contains a vertex $u$ with $e(u) \leq 2$ such that for each vertex $v$ in $V(G) - \{u\}$,

(a) if $d(u, v) = 1$, then $v$ belongs to a subgraph $H_1$ of $G$ fixed at $u$,

(b) if $d(u, v) = 2$, then $v$ belongs to a subgraph $H_2$ of $G$ fixed at $u$,

where $H_1$ and $H_2$ are the graphs shown in Figure 2.10.

Proof. First, assume that $\gamma_F(G) = 1$. Then there is an $F$-coloring of $G$ with exactly one red vertex $u$ and so $e(u) \leq 2$ by Proposition 2.40. Let $v \in V(G) - \{u\}$. If $d(u, v) = 1$, then $v$ is adjacent to $u$ and is $F$-dominated by $u$. It then follows by Proposition 2.8 that $v$ belongs to a subgraph $H_1$ fixed at $u$. If $d(u, v) = 2$, then there is a red-blue-blue $u - v$ path $u, v', v$ of length 2 in $G$. Since the blue vertex $v'$ is adjacent to $u$ and is $F$-dominated by $u$, it follows that $v'$ belongs to a subgraph $H_1$ fixed at $u$. Therefore, $v$ belongs to a subgraph $H_2$ containing $v'$ that is fixed at $u$.

For the converse, assume that $G$ contains a vertex $u$ with $e(u) \leq 2$ such that each vertex $v$ in $V(G) - \{u\}$ satisfies (a) or (b). Then the red-blue coloring of $G$ defined by assigning red to $u$ and blue to the remaining vertices of $G$ is an $F$-coloring of $G$. Therefore, $\gamma_F(G) = 1$. □

The following is a consequence of Theorem 2.42.

Corollary 2.43. A nontrivial connected graph $G$ has $\gamma_F(G) = 1$ if and only if $G$ contains a vertex $u$ such that $N(u)$ is an open dominating set of $G$.

Proof. Assume that $\gamma_F(G) = 1$. Then by Theorem 2.42 there exists a vertex $u \in V(G)$ such that, for each vertex $v$ in $V(G) - \{u\}$, either
(a) $d(v, u) = 1$, and $v$ belongs to a subgraph $H_1$ of $G$ in Figure 2.10 fixed at $u$ or,

(b) $d(v, u) = 2$, and $v$ belongs to a subgraph $H_2$ of $G$ in Figure 2.10 fixed at $u$.

We show that $N(u)$ is an open dominating set of the graph $G$. Let $v \in V(G)$. We consider two cases.

Case 1. $v \in N(u)$. Then $d(v, u) = 1$ and so $v$ belongs to a subgraph $H_1$ of $G$ in Figure 2.10. Thus $v$ is adjacent to some vertex in $N(u)$, and so $v$ is openly dominated by some vertex in $N(u)$.

Case 2. $v \notin N(u)$. Since $u$ is openly dominated by any vertex in $N(u)$, we may assume that $v \neq u$. Then $d(v, u) = 2$. Thus $v$ is adjacent to some vertex in $N(u)$, and so $v$ is openly dominated by some vertex in $N(u)$.

For the converse, assume that $G$ contains a vertex $u$ such that $N(u)$ is an open dominating set. We show that $\gamma_F(G) = 1$ by showing that $\{u\}$ is an $F$-dominating set of $G$. Let $v \in V(G) - \{u\}$. We consider two cases.

Case 1. $v \in N(u)$. Then $d(v, u) = 1$. Since $N(u)$ is an open dominating set, it follows that $v$ is openly dominated by some vertex $v_1 \in N(u)$. Thus $v$ is adjacent to $v_1$ that is adjacent to $u$. Since $v \in N(u)$, it follows that $vu \in E(G)$ and so $v$ belongs to a copy $H_1 = \langle u, v, v_1 \rangle$ of $G$ fixed at $u$.

Case 2. $v \notin N(u)$. Then $d(v, u) \geq 2$. Since $N(u)$ is an open dominating set, it follows that $v$ is openly dominated by some vertex $v_2 \in N(u)$. So $d(v, u) = 2$ with $u, v_2, v$ a path of order 2. Since $v_2$ is openly dominated by some vertex $v_3 \in N(u)$, it follows that $\{v_2v_3, v_2u, v_3u\} \subseteq E(G)$. Thus $v$ belongs to a copy of $H_2$ of $G$ fixed at $u$.

It then follows by Theorem 2.42 that $\gamma_F(G) = 1$. □

Figure 2.11 shows two connected graphs with with $F$-domination number 1. By Corollary 2.11, if $H$ is a graph containing a spanning subgraph with $F$-domination number 1, then $\gamma_F(H) = 1$.

Next, we present two sufficient conditions for a graph $G$ with $F$-domination number 1 to have the property that each vertex of $G$ forms an $F$-dominating set of $G$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
In order to do this, we first make a useful observation.

**Observation 2.44.** If \( G \) is a graph such that for every two distinct vertices \( u \) and \( v \) in \( G \) there is a \( u-v \) path of length 2 in \( G \), then every vertex of \( G \) forms an \( F \)-dominating set of \( G \).

**Theorem 2.45.** If \( G \) is a graph of order \( n \geq 3 \) such that

\[
\deg u + \deg v \geq n + 1
\]

for every two distinct vertices \( u \) and \( v \) in \( G \), then every vertex of \( G \) forms an \( F \)-dominating set of \( G \).

**Proof.** By Observation 2.44, it suffices to show for every two distinct vertices \( u \) and \( v \) of \( G \), that there is a \( u-v \) path of length 2 in \( G \). Let \( W = V(G) - \{u, v\} \). Then \( |W| = n - 2 \). If there exists \( w \in W \) such that \( uw, vw \in E(G) \), then we have the desired result. Assume, to the contrary, that every vertex \( w \in W \) is adjacent to at most one of \( u \) and \( v \). If \( u \) and \( v \) are adjacent, then

\[
\deg u + \deg v \leq |W| + 2 = (n - 2) + 2 = n;
\]

while if \( u \) and \( v \) are nonadjacent, then

\[
\deg u + \deg v \leq |W| = n - 2.
\]

In each case, \( \deg u + \deg v \leq n \), which is a contradiction. \( \square \)

The lower bound in (2.6) cannot be improved. For example, let \( G \) be the graph of order \( n \) obtained from \( K_{n-1} \) by adding the pendant edge \( uv \), where \( u \in V(K_{n-1}) \).
and \( v \) is the added new vertex. Then \( \deg u + \deg v = n \). Observe that \{u\} is not an \( F \)-dominating set of \( G \) since \( u \) cannot \( F \)-dominate \( v \).

**Corollary 2.46.** If \( G \) is a graph of order \( n \geq 3 \) with

\[
\delta(G) \geq \frac{n + 1}{2},
\]  

then every vertex of \( G \) forms an \( F \)-dominating set of \( G \).

The lower bound in (2.7) cannot be improved. For example, let \( n \) be an even integer and \( G = K_{\frac{n}{2}} \times K_2 \). Then \( \delta(G) = n/2 \) but \( \gamma_F(G) = 2 \).

**Theorem 2.47.** If \( G \) is a graph of order \( n \geq 3 \) and size

\[
m \geq \binom{n-1}{2} + 2,
\]  

then every vertex of \( G \) forms an \( F \)-dominating set of \( G \).

**Proof.** Let \( u \) and \( v \) be two distinct vertices of \( G \). We show that there is a \( u - v \) path of length 2 in \( G \). Let \( W = V(G) - \{u, v\} \). Then \( |W| = n - 2 \). If there exists \( w \in W \) such that \( uw, vw \in E(G) \), then we have the desired result. Assume, to the contrary, that every vertex \( w \in W \) is adjacent to at most one of \( u \) and \( v \). Thus, there are at most \( n - 2 \) edges between the two sets \( \{u, v\} \) and \( W \). Furthermore there are at most \( \binom{n-2}{2} \) edges in the subgraph \( (W) \) induced by \( W \) and at most one edge between \( u \) and \( v \). Thus,

\[
m \leq \binom{n-2}{2} + (n-2) + 1 = \binom{n-1}{2} + 1,
\]  

which is a contradiction. \( \square \)

The lower bound in (2.8) cannot be improved. For example, let \( G \) be the graph of order \( n \) obtained from \( K_{n-1} \) by adding the pendant edge \( uv \), where \( u \in V(K_{n-1}) \) and \( v \) is the added new vertex. Then the size of \( G \) is \( m = \binom{n-1}{2} + 1 \). We have seen that \{u\} is not an \( F \)-dominating set of \( G \).

By Corollary 2.5, there are no nontrivial trees with \( F \)-domination number 1. The following result characterizes all trees of order \( n \) with \( F \)-domination number \( n \) or 2. A
double star is a tree of diameter 3. Two vertices $u$ and $v$ in a connected graph $G$ are called antipodal vertices of $G$ if $d(u, v) = \text{diam}(G)$.

**Theorem 2.48.** If $T$ is a tree of order $n \geq 3$, then

$$2 \leq \gamma_F(T) \leq n.$$  \hfill (2.9)

Moreover,

(a) $\gamma_F(T) = n$ if and only if $T$ is a star,

(b) $\gamma_F(T) = 2$ if and only if $T$ is a double star.

**Proof.** By Corollary 2.5 and (2.1), we have $2 \leq \gamma_F(T) \leq n$ for every tree of order $n \geq 3$. Moreover, by Theorem 2.38, $\gamma_F(T) = n$ if and only if $T$ is a star and so (a) holds.

It remains to verify (b). First, we show that if $T$ is a double star, then $\gamma_F(T) = 2$. Let $x$ and $y$ be two antipodal vertices of $T$. The red-blue coloring of $T$ defined by assigning red to $x$ and $y$ and blue to the remaining vertices of $T$ is an $F$-coloring of $G$ with exactly two red vertices. Thus $\gamma_F(T) = 2$ by (2.9).

For the converse, assume that $T$ is a tree that is not a double star. Then $\text{diam } T \neq 3$. If $\text{diam } T \leq 2$, then $T$ is a star and so $\gamma_F(T) = n \geq 3$. Thus, we may assume that $\text{diam } T \geq 4$. We show that $\gamma_F(T) \geq 3$. Assume, to the contrary, that $\gamma_F(T) \leq 2$. Then $\gamma_F(T) = 2$ by (2.9). Let $c$ be a minimum $F$-coloring of $T$ with $R_c = \{x, y\}$. Let $d(x, y) = k$ and let $P : x = x_0, x_1, x_2, \ldots, x_k = y$ be an $x - y$ path of length $k$ in $T$. We consider four cases.

**Case 1.** $k = 1$. Then $x$ and $y$ are adjacent. Since $T \neq P_2$, at least one of $x$ and $y$ is not an end-vertex of $T$. Assume, without loss of generality, that $x$ is not an end-vertex of $T$, and so $x$ is also adjacent to a blue vertex $v$. Since $T$ is a tree, $T$ is triangle-free. It then follows by Corollary 2.9 that $v$ cannot be $F$-dominated by $x$. Hence $v$ is $F$-dominated by $y$. Thus $v$ is adjacent to a blue vertex $v'$ that is adjacent to $y$. However then $x, v, v', y, x$ is a 4-cycle in the tree $T$, which is impossible.
Case 2. \( k = 2 \). Then the blue vertex \( x_1 \) in \( P \) is adjacent to both \( x \) and \( y \). Since \( x \) and \( y \) are the only red vertices in \( G \), it follows that \( x_1 \) is \( F \)-dominated by a red vertex that is adjacent to \( x_1 \), which contradicts Corollary 2.9.

Case 3. \( k = 3 \). Then \( P : x = x_0, x_1, x_2, x_3 = y \) is a path of length 3. Since \( \text{diam} T \geq 4 \), it follows that \( V(T) - V(P) \neq \emptyset \). We claim that every vertex in \( V(T) - V(P) \) is adjacent to either \( x_1 \) or \( x_2 \). Assume, to the contrary, that there is \( v \in V(T) - V(P) \) such that \( v \) is adjacent to neither \( x_1 \) nor \( x_2 \). Suppose, without loss of generality, that \( v \) is \( F \)-dominated by \( x \). Then \( v \) is adjacent to a blue vertex \( v' \in \{x_1, x_2\} \) that is adjacent to \( x \). Since \( v' \) is adjacent to \( x \), it follows by Corollary 2.9 that \( v' \) cannot be \( F \)-dominated by \( x \) and so \( v' \) is \( F \)-dominated by \( y \). Thus \( v' \) is adjacent to a blue vertex \( v'' \) that is adjacent to \( y \). This implies that \( x, v', v'', y, x_2, x_1, x \) is a cycle in the tree \( T \), which is impossible. Therefore, as claimed, every vertex in \( V(T) - V(P) \) is adjacent to either \( x_1 \) or \( x_2 \). Moreover, \( x \) is adjacent to \( x_1 \) and \( y \) is adjacent to \( x_2 \). Therefore, every vertex in \( V(T) - \{x_1, x_2\} \) is adjacent to either \( x_1 \) or \( x_2 \), implying that \( T \) is a double star with central vertices \( x_1 \) and \( x_2 \). This contradicts our assumption that \( T \) is not a double star.

Case 4. \( k \geq 4 \). Then \( d(x_1, y) \geq 3 \). One the other hand, since \( x_1 \) is adjacent to \( x \), it follows that \( x_1 \) cannot be \( F \)-dominated by \( x \) and so \( x_1 \) is \( F \)-dominated by \( y \). This implies that \( d(x_1, y) \leq 2 \), which is a contradiction. \( \square \)

Next, we consider the question of which pairs \( k, n \) of positive integers with \( 1 \leq k \leq n \) can be realized as the \( F \)-domination number and the order of some connected graph, respectively. We have seen in Corollary 2.39 that there is no connected graph \( G \) of order \( n \geq 3 \) with \( \gamma_F(G) = n - 1 \).

For each \( n \in \{3, 4, 5, 6\} \), there exists a connected graph \( G \) of order \( n \) with \( \gamma_F(G) = k \) for each \( k \) with \( 1 \leq k \leq n \) and \( k \neq n - 1 \), as Figure 2.12 shows. Moreover, if \( G \) is a connected graph of order \( n = 1 \) or \( n = 2 \), then \( G \) does not contain \( F \) as a subgraph and so \( \gamma_F(G) = n \). Thus, we have the following.

**Proposition 2.49.** For each pair \( k, n \) of integers with \( 1 \leq k \leq n, k \neq n - 1 \), and
Figure 18. Realizable pairs $k, n$ for $3 \leq n \leq 6$.

$1 \leq n \leq 6$, there exists a connected graph $G$ of order $n$ with $\gamma_F(G) = k$. 

We mentioned in Observation 1.3 that there exists a 2-stratified graph $F'$ and a connected graph $G$ of order $n$ such that there is an $F'$-coloring of $G$ with $k$ red vertices, where $k < n$, but there is no $F'$-coloring with $k + 1$ red vertices. In the case of the 2-stratified graph $F = F_3$, observe that

(1) there is no $F$-coloring of $P_6$ with 3 red vertices,

(2) there is an $F$-coloring of $P_6$ with 4 red vertices,

(3) there is no $F$-coloring of $P_6$ with 5 red vertices,

(4) there is an $F$-coloring of $P_6$ with 6 red vertices.

This illustrates the possible erratic behavior of $F$-colorings.

By Proposition 2.49, we need only be concerned with graphs of order $n \geq 7$.

**Theorem 2.50.** There is no connected graph $G$ of order $n \geq 7$ with $\gamma_F(G) = n - 2$.

**Proof.** Assume, to the contrary, that there exists a connected graph $G$ of order $n \geq 7$ such that $\gamma_F(G) = n - 2$. Let $c$ be a minimum $F$-coloring of $G$ and let $x$ and $y$ be the two blue vertices of $G$. Necessarily, $x$ and $y$ are adjacent in $G$. Suppose that $x$
is $F$-dominated by a red vertex $v$ and $y$ is $F$-dominated by a red vertex $u$. Thus $u$ is adjacent to $x$ and $v$ is adjacent to $y$.

First, we claim that each vertex in $R_c - \{u, v\}$ is adjacent to neither $x$ nor $y$; for otherwise, there exists $w \in R_c - \{u, v\}$ such that $w$ is adjacent to $x$ or $y$, say the former, and the red-blue coloring defined by assigning blue to $x, y, w$ and red the remaining vertices of $G$ is an $F$-coloring with $n - 3$ red vertices, and so $\gamma_F(G) \leq n - 3$, which is a contradiction. Therefore, as claimed, each vertex in $R_c - \{u, v\}$ is adjacent to neither $x$ nor $y$.

Next, we claim that each vertex in $R_c - \{u, v\}$ is adjacent to either $u$ or $v$. Since $n \geq 7$, and $G$ has exactly two blue vertices, it follows that $|R_c - \{u, v\}| \geq 3$. Assume, to the contrary, that there is a red vertex $w \in R_c - \{u, v\}$ such that $wv$ and $wu$ are not edges of $G$. Since $G$ is connected, there is a $u - w$ path or $v - w$ path in $G$, say the former. Suppose that $P : u = u_0, u_1, \ldots, u_k = w$ is a $u - w$ path in $G$. If $k \geq 3$, then $G$ contains the path $v, y, x, u_1, u_2, u_3$ of order 7 as a subgraph. Then the red-blue coloring that assigns blue to the vertices in $\{u_1, u_2, x, y\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ and so $\gamma_F(G) \leq n - 4$, which is a contradiction. Therefore $k \leq 2$. Since $w$ is not adjacent to $u$, it follows that $k = 2$. Then $G$ contains

$$P_6 : v, y, x, u, u_1, w$$

as a subgraph. Let

$$Z = R_c - \{u, u_1, v, w\} = V(G) - \{u, u_1, v, w, x, y\}.$$

We now verify three claims.

**Claim 1:** No vertex in $Z$ is adjacent to $v$ or $w$.

**Proof of Claim 1.** Assume, to the contrary, that there is a vertex $z \in Z$ that is adjacent to $v$ or to $w$, say the former. Then $G$ contains the path $z, v, y, x, u, u_1, w$ of order 7 as a subgraph. Thus the red-blue coloring that assigns blue to the vertices in $\{u, u_1, v, y\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ and so $\gamma_F(G) \leq n - 4$, which is a contradiction. This completes the proof of Claim 1.
Claim 2: No vertex in $Z$ is adjacent to $u_1$.

Proof of Claim 2. Assume, to the contrary, that there is a vertex $z \in Z$ that is adjacent to $u_1$. Then $G$ contains the subgraph obtained from the path $v, y, x, u, u_1, w$ of order 6 by adding the pendant edge $u_1z$. Then the red-blue coloring that assigns blue to the vertices in $\{u, u_1, w\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ and so $\gamma_F(G) \leq n - 3$, a contradiction. This completes the proof of Claim 2.

Claim 3: No vertex in $Z$ is adjacent to $u$.

Proof of Claim 3. Assume, to the contrary, that there is a vertex $z \in Z$ that is adjacent to $u$. Then $G$ contains the subgraph obtained from the path $v, y, x, u, u_1, w$ of order 6 by adding the pendant edge $uz$. Then the red-blue coloring that assigns blue to the vertices $\{u, u_1, x\}$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ and so $\gamma_F(G) \leq n - 3$, a contradiction. This completes the proof of Claim 3.

Since no vertex of $R_c - \{u, v\}$ is adjacent to a vertex of the set $\{x, y\}$, it then follows by Claims 1-3 that no vertex of $Z = V(G) - \{u, v, u_1, w, x, y\}$ is adjacent to any vertex in $\{u, v, u_1, w, x, y\}$. This implies that $G$ is disconnected, which is impossible. Therefore, each vertex in $R_c - \{u, v\}$ is adjacent to either $u$ or $v$.

If $u = v$, then $u, x, y, u$ is a triangle in $G$. Since $G$ is connected and the order of $G$ is at least 7, there is at least one vertex, say $w$, such that $uw \in E(G)$. Then the red-blue coloring that assigns blue to $u, w, y$ and red to the remaining vertices of $G$ is an $F$-coloring of $G$ with $n - 3$ red vertices. Thus $\gamma_F(G) \leq n - 3$, which is contradiction. Therefore, $u \neq v$ and there is a path $P: u, x, y, v$ in $G$, where $u, v \in R_c$. We consider two cases.

Case 1. Every vertex in $R_c - \{u, v\}$ is adjacent to $u$ or every vertex in $R_c - \{u, v\}$ is adjacent to $v$, say the former. Let $w \in R_c - \{u, v\}$. Then the red-blue coloring that assigns red to $v, w, y$ and blue to the remaining vertices of $G$ is an $F$-coloring of $G$ with three red vertices. Thus $\gamma_F(G) \leq 3 < 4 \leq n - 3$.

Case 2. Case 1 does not occur. Let $W_1$ be the set of vertices in $R_c - \{u, v\}$ that
are adjacent to \( u \) and let \( W_2 \) be the set of vertices in \( R_c - \{u, v\} \) that are adjacent to \( v \). Then \( W_1 \neq \emptyset \) and \( W_2 \neq \emptyset \). Since \( n \geq 7 \), at least one of \( W_1 \) and \( W_2 \) contains at least two vertices, say \(|W_1| \geq 2\). Let \( w \in W_1 \). Then the red-blue coloring defined by assigning blue to each vertex in \( \{u, w, x\} \) and red to the remaining vertices of \( G \) is an \( F \)-coloring of \( G \) with \( n - 3 \) red vertices. Therefore, \( \gamma_F(G) \leq n - 3 \).

**Conjecture 2.51.** There is no connected graph \( G \) of order \( n \geq 10 \) with \( \gamma_F(G) = n - 3 \).

Next we show that for every pair \( k, n \) of integers with \( 1 \leq k \leq \lfloor \frac{2n}{3} \rfloor \) and \( n \geq 7 \), there is a connected graph \( G \) of order \( n \) such that \( \gamma_F(G) = k \). In order to do this, we first establish some preliminary results. Let \( a, b \) be integers with \( a \geq b \geq 3 \) and let \( Sa,b \) be the double star with central vertices \( u \) and \( x \) with \( \deg u = a \) and \( \deg x = b \). Let \( N(u) = \{v_1, v_2, \ldots, v_{a-2}\} \) and \( N(x) = \{u, z_1, z_2, \ldots, z_{b-1}\} \).

![Figure 19. The graph \( Sa,b \).](image)

The graph \( Ta,b \) is the tree constructed from \( Sa,b \) by subdividing each of the edges \( ux \) and \( xz_i \) \((1 \leq i \leq b - 1)\) such that (1) \( ux \) is replaced by \( uw \) and \( wx \) and (2) \( xz_i \) is replaced by \( xy_i \) and \( y_i z_i \) for \( 1 \leq i \leq b - 1 \). Furthermore, let

\[
T_{a,b} = Ta,b - z_{b-1}.
\]

Then the order of \( Ta,b \) is \( n = a + 2b \); while the order of \( T_{a,b}' \) is \( n' = a + 2b - 1 \).

**Lemma 2.52.** For each pair \( a, b \) of integers with \( a \geq b \geq 3 \),

\[
\gamma_F(T_{a,b}) = 2b \quad \text{and} \quad \gamma_F(T_{a,b}') = 2b - 1.
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 20. The graphs $T_{a,b}$ and $T'_{a,b}$.

Proof. Let

\[ V = \{v_1, v_2, \ldots, v_{a-2}\} \]
\[ Y = \{y_1, y_2, \ldots, y_{b-1}\} \]
\[ Z = \{z_1, z_2, \ldots, z_{b-1}\} \]
\[ Z' = Z - \{z_{b-1}\} = \{z_1, z_2, \ldots, z_{b-2}\}. \]

Then $|V| = a - 2$, $|Y| = |Z| = b - 1$, and $|Z'| = b - 2$. We first show that $\gamma_F(T_{a,b}) = 2b$. To show $\gamma_F(T_{a,b}) \leq 2b$, let

\[ S^* = \{v, x\} \cup Y \cup Z. \]

Define a red-blue coloring $c^*$ of $T_{a,b}$ by assigning red to the vertices of the set $S^*$ and blue to the remaining vertices of the graph $G$. Observe that (1) each vertex in $V$ is adjacent to the blue vertex $u$ and $u$ is adjacent to the red vertex $v$, (2) the blue vertex $w$ is adjacent to the blue vertex $u$ and $u$ is adjacent to the red vertex $v$, and (3) the blue vertex $u$ is adjacent to the blue vertex $w$ and $w$ is adjacent to the red vertex $x$. Thus, the red-blue coloring $c^*$ is an $F$-coloring of $T_{a,b}$ and so

\[ \gamma_F(T_{a,b}) \leq |S^*| = 2 + 2(b - 1) = 2b. \]
We next show that $\gamma_F(T_{a,b}) \geq 2b$. Let $c$ be a minimum $F$-coloring of $T_{a,b}$. Then $Z \subseteq R_c$ by Proposition 2.7. We consider two cases.

**Case 1.** $x$ is colored red by $c$. Then $Y \subseteq R_c$. Thus $\{x\} \cup Y \cup Z \subseteq R_c$ and so

$$\gamma_F(T_{a,b}) = |R_c| \geq 1 + 2(b-1) = 2b - 1.$$ 

Assume, to the contrary, that $\gamma_F(G) = 2b - 1$. Then $R_c = \{x\} \cup Y \cup Z$. However then, $w$ is not $F$-dominated by any vertex of $R_c$, a contradiction.

**Case 2.** $x$ is colored blue by $c$. Since the vertex $u$ is only $F$-dominated by $u$ or by $x$ and $x$ is blue, it follows that $u$ is colored red by $c$. Thus $\{v\} \cup V \subseteq R_c$ by Lemma 2.6. Hence $\{u, v\} \cup V \cup Z \subseteq R_c$ and so

$$\gamma_F(T_{a,b}) = |R_c| \geq 2 + (a-2) + (b-1) = a + b - 1 \geq 2b - 1.$$ 

Assume to the contrary, that $\gamma_F(T_{a,b}) = 2b - 1$. Then $a = b$ and $R_c = \{u, v\} \cup V \cup Z$. However then, $w$ is not $F$-dominated by any vertex of $R_c$, contradiction. Thus in each case, $\gamma_F(T_{a,b}) \geq 2b$ and so $\gamma_F(T_{a,b}) = 2b$.

Next, we show that $\gamma_F(T'_{a,b}) = 2b - 1$. To show $\gamma_F(T'_{a,b}) \leq 2b - 1$, let

$$S' = \{v, x\} \cup Y \cup Z'.$$

Define a red-blue coloring $c'$ of $T'_{a,b}$ by assigning red to the vertices of the set $S'$ and blue to the remaining vertices of the graph $T'_{a,b}$. Observe that (1) each vertex in $V$ is adjacent to the blue vertex $u$ and $u$ is adjacent to the red vertex $v$, (2) the blue vertex $w$ is adjacent to the blue vertex $u$ and $u$ is adjacent to the red vertex $v$, and (3) the blue vertex $u$ is adjacent to the blue vertex $w$ and $w$ is adjacent to the red vertex $x$. Thus, the red-blue coloring $c'$ is an $F$-coloring of $T'_{a,b}$ and so

$$\gamma_F(T'_{a,b}) \leq |S'| = 2 + (b-1) + (b-2) = 2b - 1.$$ 

We next show that $\gamma_F(T'_{a,b}) \geq 2b - 1$. Let $c$ be a minimum $F$-coloring of $T'_{a,b}$. Then $Z' \subseteq R_c$ by Proposition 2.7. Again, we consider two cases.
Case 1. $x$ is colored red by $c$. Then $Y \subseteq R_c$ and so \( \{x\} \cup Y \cup Z' \subseteq R_c \). Thus

\[
\gamma_F(T_{a,b}') = |R_c| \geq 1 + (b - 1) + (b - 2) = 2b - 2.
\]

Assume, to the contrary, that $\gamma_F(T_{a,b}') = 2b - 2$. Then $R_c = \{x\} \cup Y \cup Z'$. However then, $w$ is not $F$-dominated by any vertex of $R_c$, a contradiction.

Case 2. $x$ is colored blue by $c$. Since the vertex $u$ is only $F$-dominated by $u$ or by $x$ and $x$ is blue, it follows that $u$ is colored red by $c$. Thus $\{u\} \cup V \subseteq R_c$ by Lemma 2.6. Hence $\{u, v\} \cup V \cup Z' \subseteq R_c$ and so

\[
\gamma_F(T_{a,b}') = |R_c| \geq 2 + (a - 2) + (b - 2) = a + b - 2 \geq 2b - 2.
\]

Assume to the contrary, that $\gamma_F(T_{a,b}') = 2b - 2$. Then $a = b$ and $R_c = \{u, v\} \cup V \cup Z'$. However then, $w$ is not $F$-dominated by any vertex of $R_c$, a contradiction. Therefore, in each case $\gamma_F(T_{a,b}') \geq 2b - 1$ and so $\gamma_F(T_{a,b}') = 2b - 1$.

Thus by Lemma 2.52, the order $n$ of $T_{a,b}$ and the order $n'$ of $T_{a,b}'$ are

\[
n = a + \gamma_F(T_{a,b}) \quad \text{and} \quad n' = a + \gamma_F(T_{a,b}').
\]

We are now prepared to show that every pair $k, n$ of integers with $1 < k < \left\lceil \frac{2n}{3} \right\rceil$ and $n \geq 7$ is realizable as the $F$ domination number and order, respectively, of some connected graph.

**Theorem 2.53.** Let $n \geq 7$. For each integer $k$ with $1 \leq k \leq \left\lceil \frac{2n}{3} \right\rceil$, there is a connected graph $G$ of order $n$ such that $\gamma_F(G) = k$.

**Proof.** We first consider small values of $k$, where $1 \leq k \leq 4$. If $k = 1$, let $G = K_n$ and $\gamma_F(G) = 1$. If $k = 2$, let $G = S_{n-2,2}$ be the double star and $\gamma_F(G) = 2$. If $k = 3$, let $G$ be the graph obtained from $P_4: u, w, x, y$ by adding $n - 4$ new vertices $v_1, v_2, \ldots, v_{n-4}$ and joining each vertex $v_i$ to $u$ for $1 \leq i \leq n - 4$. Then $\gamma_F(G) = 3$ as $\{v_1, x, y\}$ is a minimum $F$-dominating set of $G$. If $k = 4$, let $G$ be the graph obtained from $P_4: u, w, x, y$ by adding $n - 4$ new vertices $v_1, v_2, \ldots, v_{n-5}, z$ and joining (1) each
vertex \( v_i \) to \( u \) for \( 1 \leq i \leq n - 5 \) and (2) the vertex \( z \) to \( y \). Then \( \gamma_F(G) = 4 \) as \( \{v_1, x, y, z\} \) is a minimum \( F \)-dominating set of \( G \).

Now let \( k \) be an integer with \( 5 \leq k \leq \lfloor \frac{2n}{3} \rfloor < n \). It then follows by Corollary 2.39 and Theorem 2.50 that \( n - k \geq 3 \). If \( k \) is even, then let \( G = T_{n-k,k/2} \), where \( T_{a,b} \) was defined in Lemma 2.52. Then the order of \( G \) is \( (n-k) + 2\frac{k}{2} = n \) and \( \gamma_F(G) = k \). If \( k \) is odd, then let \( G = T'_{n-k,(k+1)/2} \). By Lemma 2.52, the order of \( G \) is \( (n-k) + 2\frac{k+1}{2} - 1 = n \) and \( \gamma_F(G) = k \).

We conclude this section with the following conjecture.

**Conjecture 2.54.** If \( G \) is a connected graph of order \( n \geq 7 \) that is not a star, then

\[
\gamma_F(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor.
\]

It suffices to show that Conjecture 2.54 is true for trees.
3 Comparing Domination Parameters

3.1 Introduction

Recall that a vertex \( v \) in a graph \( G \) is said to dominate itself and each vertex adjacent to it. That is, \( v \) dominates the vertices in the closed neighborhood \( N[v] \) of \( v \). A set \( S \) of vertices in a graph \( G \) is called a dominating set of \( G \) if every vertex of \( G \) is dominated by some vertex in \( S \). The minimum cardinality of a dominating set in \( G \) is the domination number \( \gamma(G) \) of \( G \). A dominating set of cardinality \( \gamma(G) \) is called a minimum dominating set. Since the vertex set of a graph is always a dominating set, the domination number is defined for every graph. If \( G \) is a graph of order \( n \), then

\[
1 \leq \gamma(G) \leq n.
\]

A graph \( G \) of order \( n \) has domination number 1 if and only if \( G \) contains a vertex \( v \) of degree \( n - 1 \), in which case the set \( \{v\} \) is a minimum dominating set; while \( \gamma(G) = n \) if and only if \( G \cong K_n \), in which case \( V(G) \) is the unique minimum dominating set. Certainly, this is not the case for \( F \)-domination. We have seen that there are connected graphs \( G \) of order \( n \) with \( \gamma_F(G) = 1 \) and containing no vertices of degree \( n - 1 \) (for example, the graph in Figure 2.9). Moreover, by Theorem 2.38, the star \( K_{1,n-1} \) is the only connected graph of order \( n \geq 2 \) with \( \gamma_F(G) = n \). Notice that the \( F \)-domination number of a graph is the sum of the \( F \)-domination numbers of its components.

A vertex \( v \) in a graph \( G \) openly dominates each of its neighbors. That is, \( v \) openly dominates the vertices in its (open) neighborhood \( N(v) \) but does not openly dominate itself. A set \( S \) of vertices in a graph \( G \) is an open dominating set if every vertex of \( G \) is adjacent to at least one vertex of \( S \). Therefore, a graph \( G \) contains an open dominating set if and only if \( G \) contains no isolated vertices. The minimum cardinality of an open dominating set is the open domination number \( \gamma_o(G) \) of \( G \). An open dominating set of cardinality \( \gamma_o(G) \) is a minimum open dominating set for \( G \). The open domination
number of a graph $G$ is also referred to by some as the total domination number and in which case is denoted by $\gamma_t(G)$. Since no vertex in an open dominating set can be openly dominated by itself, it follows that every open dominating set of $G$ contains at least two vertices and so

$$2 \leq \gamma_o(G) \leq n \quad (3.1)$$

for every graph $G$ of order $n$ containing no isolated vertices. The lower bound can be attained if and only if $G$ contains a spanning double star; while the upper bound is attained if and only if for $G \cong kK_2$, where $n = 2k$.

For a graph $G$ containing no isolated vertices, let $S = \{u_1, u_2, \ldots, u_k\}$ be a minimum dominating set of $G$ and let $S' = \{v_1, v_2, \ldots, v_k\}$, where $v_i \in N(u_i)$ for $1 \leq i \leq k$. Then the set $S \cup S'$ is an open dominating set of $G$. Furthermore, every open dominating set in $G$ is also a dominating set. Therefore,

$$\gamma(G) \leq \gamma_o(G) \leq 2\gamma(G) \quad (3.2)$$

for every graph $G$ containing no isolated vertices. On the other hand, we next show that every pair $a, b$ of positive integers with $a < b < 2a$ and $b \geq 2$ can be realized as the domination number and the open domination number of some connected graph containing no isolated vertices. First, we make an observation.

**Observation 3.1.** Let $G$ be a connected graph. If $v$ is an end-vertex that is adjacent to a vertex $u$ in $G$, then every dominating set of $G$ contains at least one of $u$ and $v$.

The *corona* $cor(G)$ of a graph $G$ is that graph obtained from $G$ by adding a pendant edge to each vertex of $G$. Thus, if $G$ has order $n$, then $cor(G)$ has order $2n$. The following is a consequence of Observation 3.1.

**Proposition 3.2.** Let $G$ be a nontrivial connected graph of order $n$. Then

$$\gamma(cor(G)) = \gamma_o(cor(G)) = n.$$
Proof. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $H = \text{cor}(G)$ with

$$V(H) = \{u_1, u_2, \ldots, u_n\} \cup V(G)$$

where $v_i u_i \in E(H)$ for $1 \leq i \leq n$. Since $V(G)$ is a dominating set of $H$, it follows that $\gamma(H) \leq n$. Moreover, by Observation 3.1, every dominating set of $H$ must contain at least one of $u_i$ and $v_i$ for $1 \leq i \leq n$, implying that $\gamma(G) \geq n$. Therefore, $\gamma(H) = n$.

Because $G$ is a nontrivial connected graph, $V(G)$ is also an open dominating set of $H$ and so $\gamma_0(H) \leq n = \gamma(H)$. It then follows by (3.2) that $\gamma_0(H) = \gamma(H) = n$. □

We are prepared to present a realization result dealing with the domination and the open domination numbers of a connected graph.

**Theorem 3.3.** Let $a$ and $b$ be positive integers with $a \leq b \leq 2a$, and $b \geq 2$. There exists a connected graph $G$ such that $\gamma(G) = a$ and $\gamma_0(G) = b$.

**Proof.** If $a = b$, then the result follows by Proposition 3.2. Thus, we may assume that $a < b$. We consider two cases.

*Case 1. $a < b < 2a$. Then $b \geq 2$. There are two subcases.*

*Subcase 1.1. $b = 2a - 1$. Let $F_i : x_i, y_i, z_i$ $(1 \leq i \leq a - 1)$ be a copy of $P_3$. The graph $G$ is obtained from the graph $P_2 : s, t$ together with the graphs $F_i$ $(1 \leq i \leq a - 1)$ by joining $t$ to each $x_i$ $(1 \leq i \leq a - 1)$. Since the set

$$\{t\} \cup \{y_i : 1 \leq i \leq a - 1\}$$

is a minimum dominating set and

$$\{t\} \cup \{x_i, y_i : 1 \leq i \leq a - 1\}$$

is a minimum open dominating set,

$$\gamma(G) = a \quad \text{and} \quad \gamma_0(G) = 2(a - 1) + 1 = 2a - 1 = b.$$ 

*Subcase 1.2. $b < 2a - 1$. Then $2a - b - 1 \geq 1$. Let $F_i : u_i, v_i$ $(1 \leq i \leq 2a - b - 1)$ be a copy of $P_2$, and let $H_j : x_j, y_j, z_j$ $(1 \leq j \leq b - a)$ be a copy of $P_3$. The graph $G$ is
obtained from the graph $P_2 : s, t$ together with the graphs $F_i$ ($1 \leq i \leq 2a - b - 1$) and $H_j$ $(1 \leq j \leq b - a)$ by joining $t$ to each $u_i$ ($1 \leq i \leq 2a - b - 1$) and each $x_j$ ($1 \leq j \leq b - a$).

Since the set

$$\{t\} \cup \{u_i : 1 \leq i \leq 2a - b - 1\} \cup \{y_j : 1 \leq j \leq b - a\}$$

is a minimum dominating set and

$$\{t\} \cup \{u_i : 1 \leq i \leq 2a - b - 1\} \cup \{x_j, y_j : 1 \leq j \leq b - a\}$$

is a minimum open dominating set, $\gamma(G) = a$ and $\gamma_0(G) = b$.

**Case 2.** $b = 2a$. If $a = 1$, then let $G = P_2$ and so $\gamma(G) = 1$ and $\gamma_0(G) = 2$. Thus we may assume that $a > 1$. Let $F_i : x_i, y_i, z_i$ ($1 \leq i \leq a - 1$) be a copy of $P_3$. The graph $G$ is obtained from the graph $P_3 : r, s, t$ together with the graphs $F_i$ ($1 \leq i \leq a - 1$) by joining the vertex $t$ to each vertex $x_i$ ($1 \leq i \leq a - 1$). Since the set

$$\{s\} \cup \{y_i : 1 \leq i \leq a - 1\}$$

is a minimum dominating set, and the set

$$\{s, t\} \cup \{x_i, y_i : 1 \leq i \leq a - 1\}$$

is a minimum open dominating set, $\gamma(G) = a$ and that $\gamma_0(G) = b$. $\square$

We have seen that an $F$-dominating set of a connected graph $G$ need not be a dominating set of $G$ (for example, the graph of Figure 2.3). On the other hand, a dominating set of $G$ need not be an $F$-dominating set of $G$ either (for example, $G$ is a star of order at least 3). For a nontrivial connected graph $G$ without isolated vertices, there are three possibilities related to the three parameters $\gamma(G)$, $\gamma_0(G)$, and $\gamma_F(G)$, namely

1. $\gamma(G) \leq \gamma_F(G) \leq \gamma_0(G)$,
2. $\gamma(G) \leq \gamma_0(G) \leq \gamma_F(G)$,
3. $\gamma_F(G) \leq \gamma(G) \leq \gamma_0(G)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Next, we show that it is possible for these three parameters to be equal. This result will be useful later in this and the next chapter.

**Proposition 3.4.** For each integer \( k \geq 2 \), there exists a connected graph \( G \) such that

\[
\gamma(G) = \gamma_f(G) = \gamma_o(G) = k.
\]

**Proof.** For \( k = 2 \), let \( G \) be the double star, where \( x \) and \( y \) are the central vertices of \( G \). Then \( \{x, y\} \) is both a minimum dominating set and a minimum open dominating set of \( G \) and so \( \gamma(G) = \gamma_o(G) = 2 \). Furthermore, by Theorem 2.48, \( \gamma_f(G) = 2 \) as well. Therefore,

\[
\gamma(G) = \gamma_f(G) = \gamma_o(G) = 2.
\]

Let \( k \geq 3 \). For each integer \( i \) with \( 1 \leq i \leq k - 1 \), let \( F_i : v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{14} \) be a copy of \( C_4 \). The graph \( G \) is obtained from the graphs \( F_i \) (\( 1 \leq i \leq k - 1 \)) by identifying vertices \( v_{4i} \) (\( 1 \leq i \leq k - 1 \)) and labeling the identified vertex by \( v \). The graph \( G \) is shown in Figure 3.1 for \( k = 5 \).

![Figure 21. The graph G for k = 5.](image)

Since \( \{v\} \cup \{v_{1i} : 1 \leq i \leq k - 1\} \) is both a minimum dominating set and a minimum open dominating set, it follows that \( \gamma(G) = \gamma_o(G) = k \). It remains to show that \( \gamma_f(G) = k \) as well. Let

\[
S = \{v\} \cup \{v_{1i} : 1 \leq i \leq k - 1\}.
\]
Since the red-blue coloring of \( G \) defined by assigning red to every vertex in \( S \) and blue to the remaining vertices of \( G \) is an \( F \)-coloring of \( G \), it follows that \( \gamma_F(G) \leq |S| = k \).

To show that \( \gamma_F(G) \geq k \), let \( c \) be a minimum \( F \)-coloring of \( G \). We consider two cases, according to whether \( v \) is colored blue or red by \( c \).

**Case 1.** \( v \) is colored blue by \( c \). Then every vertex \( v_{2i} (1 \leq i \leq k - 1) \) must be colored red by \( c \) since \( v_{2i} (1 \leq i \leq k - 1) \) is only \( F \)-dominated by itself or by \( v \), implying that \( |R_c| \geq k - 1 \) and so \( \gamma_F(G) \geq k - 1 \). Assume to the contrary that \( \gamma_F(G) = k - 1 \). Then \( R_c = \{ v_{2i} : 1 \leq i \leq k - 1 \} \). However then, \( v_{3i} \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Thus \( \gamma_F(G) \geq k \).

**Case 2.** \( v \) is colored red by \( c \). Then at least one vertex in each set \( \{ v_{1i}, v_{3i} \} \) \((1 \leq i \leq k - 1) \) must be colored red by \( c \) since \( v_{1i} \) and \( v_{3i} \) can only be \( F \)-dominated by each other. Since \( v \) is also colored red, \( \gamma_F(G) \geq k \).

In this chapter, we first study the relationships between \( F \)-domination and standard domination and then \( F \)-domination and open domination in Sections 3.2 and 3.3, respectively. We also investigate the relationships between \( F \)-domination and 2-step domination and between \( F \)-domination and restrained domination in Sections 3.4 and 3.5, respectively.

### 3.2 \( F \)-Domination and Domination

In this section, we show that every pair \( a, b \) of positive integers can be realized as the domination and \( F \)-domination numbers of some connected graph. First, we consider an useful graph. For each integer \( \alpha \geq 1 \), let \( L_\alpha \) be the graph obtained from the 4-cycle \( w, x, y, z, w \) by (1) adding \( \alpha \) new vertices \( w_1, w_2, \ldots, w_\alpha \) and joining each \( w_i \) \((1 \leq i \leq \alpha) \) to \( w \) and (2) adding \( \alpha \) new vertices \( y_1, y_2, \ldots, y_\alpha \) and joining each \( y_i \) \((1 \leq i \leq \alpha) \) to \( y \).

**Lemma 3.5.** For each integer \( \alpha \geq 1 \),

\[
\gamma(L_\alpha) = 2, \quad \gamma_0(L_\alpha) = 3, \quad \text{and} \quad \gamma_F(L_\alpha) = \alpha + 2.
\]
Proof. Since \( \{w, y\} \) is a minimum dominating set and \( \{w, x, y\} \) is a minimum open dominating set, it follows that \( \gamma(L_\alpha) = 2 \) and \( \gamma_0(L_\alpha) = 3 \). It remains to show that \( \gamma_F(L_\alpha) = \alpha + 2 \).

We first show that \( \gamma_F(L_\alpha) \leq \alpha + 2 \). Let \( S = \{w, x\} \cup \{w_i : 1 \leq i \leq \alpha\} \). Define a red-blue coloring \( c^* \) of \( L_\alpha \) by assigning red to each vertex in \( S \) and blue to the remaining vertices of \( L_\alpha \). Observe that (i) the blue vertex \( y \) is adjacent to the blue vertex \( z \) and \( z \) is adjacent to the red vertex \( w \), (ii) the blue vertex \( z \) is adjacent to the blue vertex \( y \) and \( y \) is adjacent to the red vertex \( x \), and (iii) the blue vertex \( y_i \) (\( 1 \leq i \leq \alpha \)) is adjacent to the blue vertex \( y \) and \( y \) is adjacent to the red vertex \( x \). Thus, every blue vertex \( v \) of \( G \) belongs to a copy of \( F \) rooted at \( v \), implying that \( c^* \) is an \( F \)-coloring of \( L_\alpha \). Therefore, \( \gamma_F(L_\alpha) \leq |R_{c^*}| = |S| = \alpha + 2 \).  

Next, we show that \( \gamma_F(L_\alpha) \geq \alpha + 2 \). Assume, to the contrary, that \( \gamma_F(L_\alpha) \leq \alpha + 1 \). Let there be given a minimum \( F \)-coloring \( c \) of \( L_\alpha \). Observe that \( y \) is only \( F \)-dominated by itself or by \( w \) and \( w \) is only \( F \)-dominated by itself or by \( y \). Thus at least one of \( w \) and \( y \) belongs to \( R_c \). By Lemma 2.6, if \( w \in R_c \), then \( w_i \in R_c \) for \( 1 \leq i \leq \alpha \). Similarly, if \( y \in R_c \), then \( y_i \in R_c \) for \( 1 \leq i \leq \alpha \). Thus \( |R_c| \geq \alpha + 1 \). Since \( \gamma_F(L_\alpha) \leq \alpha + 1 \), it follows that \( |R_c| = \alpha + 1 \). Then, we may assume, without loss of generality, that \( R_c = \{w, w_1, w_2, \ldots, w_\alpha\} \). However then, the blue vertex \( x \) does not belong to any copy of \( F \) rooted at \( x \), which is a contradiction. Therefore, \( \gamma_F(L_\alpha) = \alpha + 2 \). □

Theorem 3.6. Let \( a \) and \( b \) be positive integers with \( a \leq b \).

(i) There exists a connected graph \( G \) such that

\[ \gamma_0(G) = a \]
\( \gamma(G) = a \) and \( \gamma_F(G) = b; \)

(ii) There exists a connected graph \( H \) such that

\[ \gamma_F(H) = a \text{ and } \gamma(H) = b. \]

**Proof.** First, we verify (i). Suppose that \( a = 1 \). If \( b = 1 \), let \( G = K_3 \); while if \( b \geq 2 \), let \( G = K_{1,b-1} \). Then \( \gamma(G) = 1 \) and \( \gamma_F(G) = b \) for each integer \( b \geq 1 \). Hence the result holds for \( a = 1 \). Thus, we may assume that \( a \geq 2 \). By Proposition 3.4, the result holds if \( a = b \). Therefore we may assume that \( a < b \) and so \( b - a > 0 \). There are two cases.

**Case 1.** \( a = 2 \). Then \( b \geq 3 \). Let \( G \) be the graph \( L_{b-2} \) in Lemma 3.5. Then \( \gamma(G) = 2 \) and \( \gamma_F(G) = b \).

**Case 2.** \( a \geq 3 \). We begin with a double star \( T \) whose central vertices are \( u \) and \( v \) such that

\[ N(u) = \{u_1, u_2, \ldots, u_{b-a}, v\} \text{ and } N(v) = \{u, v_1, v_2, \ldots, v_{a+b-1}\}. \]

Then the graph \( G \) is obtained from \( T \) by (1) subdividing the edge \( uv \) with a new vertex \( x \) and (2) adding \( a - 2 \) new vertices \( w_1, w_2, \ldots, w_{a-2} \) and joining each \( w_i \) to \( v_i \) for \( 1 \leq i \leq a - 2 \). Let \( W = \{w_1, w_2, \ldots, w_{a-2}\} \).

![Figure 23. The graph \( G \) in Case 2 for proving (i).](image)

First, we show that \( \gamma(G) = a \). Since \( \{u, v, v_1, v_2, \ldots, v_{a-2}\} \) is a dominating set of \( G \), it follows that \( \gamma(G) \leq a \). Next, we show that \( \gamma(G) \geq a \). Let \( S \) be a minimum
dominating set of $G$. Since each vertex $w_i$ ($1 \leq i \leq a - 2$) is only dominated by itself or by $u_i$, it follows that $S$ must contain at least one vertex from each set $\{v_i, w_i\}$ for each $i$ with $1 \leq i \leq a - 2$. Also, each vertex $u_j$ ($1 \leq j \leq b - a$) is only dominated by $u_j$ or by $u$. Thus, either $u_j \in S$ for all $j$ with $1 \leq j \leq b - a$ or $u \in S$. Furthermore, each vertex $v_t$ ($a - 1 \leq t \leq a + b - 1$) is only dominated by $v_t$ or by $v$. Thus either $v_t \in S$ for each $t$ with $a - 1 \leq t \leq a + b - 1$ or $v \in S$. This implies that

$$\gamma(G) = |S| \geq (a - 2) + 2 = a.$$  

Next, we show that $\gamma_F(G) = b$. Let

$$S_0 = (N[u] - \{x\}) \cup \{v_{a+b-1}\} \cup W.$$  

Then $|S_0| = (b - a + 1) + 1 + (a - 2) = b$. Since the red-blue coloring that assigns red to each vertex in $S_0$ and blue to the remaining vertices of $G$ is an $F$-coloring with $b$ red vertices, $\gamma_F(G) \leq b$. To show that $\gamma_F(G) \geq b$, let $c$ be a minimum $F$-coloring of $G$. We make four observations:

1. By Proposition 2.7, $W \subset R_c$.

2. If $v$ is colored red by $c$, then $v_i$ is colored red by $c$ for $1 \leq i \leq a + b - 1$ and so $\gamma_F(G) \geq a + b - 1 > b$, a contradiction. Thus $v$ must be colored blue by $c$.

3. If $u$ is colored blue by $c$, then $u$ is not $F$-dominated by any red vertex in $R_c$ since $u$ is only $F$-dominated by $v$ or by $u$ and $v$ is colored blue by $c$ as shown in (2). Thus, $u$ must be colored red by $c$.

4. By (3), each vertex $u_j \in R_c$ for $1 \leq j \leq b - a$.

It then follows by (1)-(4) that $N[u] - \{x\} \cup W \subseteq R_c$ and so

$$|R_c| \geq (b - a + 1) + (a - 2) = b - 1.$$  

Assume, to the contrary, that $\gamma_F(G) = b - 1$. Then $R_c = (N[u] - \{x\}) \cup W$. However then $v_{a+b-1}$ is not $F$-dominated by any vertex in $R_c$, which is a contradiction. Therefore, $\gamma_F(G) \geq b$ and so $\gamma_F(G) = b$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Next, we verify (ii). First, suppose that \( a = 1 \). If \( b = 1 \), let \( H = K_3 \) and so \( \gamma_F(H) = \gamma(H) = 1 \). Thus we may assume that \( b \geq 2 \).

For the integers \( b \) and \( i \) with \( b \geq 2 \) and \( 1 \leq i \leq b \), let \( F_i \) be a copy of \( K_4 - e \) with \( V(F_i) = \{u_i, v_i, x_i, y_i\} \) such that \( \deg u_i = \deg v_i = 2 \) and \( \deg x_i = \deg y_i = 3 \). The graph \( H \) is obtained from the graphs \( F_i (1 \leq i \leq b) \) by identifying all the vertices \( u_i \) and calling the new vertex \( u \). We first show that \( \gamma(H) = b \). Let

\[
S_0 = \{x_i : 1 \leq i \leq b\}.
\]

Since \( S_0 \) is a dominating set of \( H \), it follows that \( \gamma(H) \leq |S_0| = b \). On the other hand, since \( u_i \) is only adjacent to \( x_i \) and \( y_i \) for each \( i \) \((1 \leq i \leq b)\), at least one vertex in each set \( \{x_i, y_i, u_i\} \) must belong to any dominating set of \( H \) and so \( \gamma(H) \geq b \). Next we show that \( \gamma_F(H) = 1 \). Since the red-blue coloring that assigns red to vertex \( u \) and blue to the remaining vertices of \( H \) is an \( F \)-coloring, it follows by (2.1) that \( \gamma_F(H) = 1 \).

Now let \( a \geq 2 \). By Proposition 3.4, the result holds if \( a = b \). Thus we may assume that \( a < b \). Thus \( b - a > 0 \) and so \( b - a + 2 \geq 3 \). We start with the graph \( W_{b-a+2} = C_{b-a+2} + K_1 \), where \( C_{b-a+2} : y_1, y_2, \ldots, y_{b-a+2}, y_1 \), and \( x \) is the vertex of degree \( b - a + 2 \) in \( W_{b-a+2} \). For each \( i \) with \( 1 \leq i \leq a - 1 \), let \( F_i : s_i, t_i \) be a copy of \( P_2 \). Then the graph \( H \) is obtained from the graphs \( F_i (1 \leq i \leq a - 1) \) and \( W_{b-a+2} \) by (1) adding \( a - 1 \) new edges \( s_i y_1 \) \((1 \leq i \leq a - 1)\) and (2) adding \( b - a + 1 \) new vertices \( z_2, z_3, \ldots, z_{b-a+2} \) and joining each vertex \( z_j \) to \( y_j \) for \( 2 \leq j \leq b - a + 2 \). Let

\[
T = \{t_1, t_2, \ldots, t_{a-1}\}.
\]

We first show that \( \gamma_F(H) = a \). Since the red-blue coloring that assigns red to each vertex in \( \{x\} \cup T \) and blue to the remaining vertices of \( H \) is an \( F \)-coloring with \( a \) red vertices, \( \gamma_F(H) \leq a \). To show that \( \gamma_F(H) \geq a \), let \( c \) be a minimum \( F \)-coloring of \( H \). By Proposition 2.7, we have \( T \subseteq R_c \) and so \( \gamma_F(H) \geq a - 1 \). Assume, to the contrary, that \( \gamma_F(H) = a - 1 \). Hence \( R_c = T \). However then, no blue vertex (different from \( y_1 \)) is \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Thus \( \gamma_F(H) = a \).
Next we show that \( \gamma(H) = b \). Let

\[
S_0 = \{s_1, s_2, \ldots, s_{a-1}, y_2, \ldots, y_{b-a+2}\}.
\]

Since \( S_0 \) is a dominating set of \( H \), it follows that \( \gamma(H) \leq |S_0| = b \). To show that \( \gamma(H) \geq b \), let \( S \) be a minimum dominating set of \( H \). Then \( S \) contains at least one vertex from each set \( \{s_i, t_i\} \) for \( 1 \leq i \leq a - 1 \) and at least one vertex from each set \( \{y_j, z_j\} \) for \( 2 \leq j \leq b - a + 2 \). Thus

\[
\gamma(H) = |S| \geq (a - 1) + (b - a + 1) = b.
\]

Therefore, \( \gamma(H) = b \). \( \square \)

### 3.3 F-Domination and Open Domination

We have seen in Section 3.2 that every pair \( a, b \) of positive integers can be realizable as the domination and F-domination numbers of some connected graph. There is a similar result for the open domination number and F-domination number, as we show in this section. Recall that \( \gamma_o(G) \geq 2 \) for every nontrivial connected graph \( G \).

**Theorem 3.7.** Let \( a \) and \( b \) be positive integers with \( a \leq b \).

(i) If \( a \geq 2 \), then there exists a connected graph \( G \) such that

\[
\gamma_o(G) = a \text{ and } \gamma_F(G) = b;
\]
(ii) If \( b \geq 2 \), then there exists a connected graph \( H \) such that
\[
\gamma_F(H) = a \quad \text{and} \quad \gamma_o(H) = b.
\]

**Proof.** By Proposition 3.4, the result holds if \( a = b \). Thus we may assume that \( a < b \). We first verify (i). If \( a = 2 \), then let \( G = K_{1,b-1} \) and so \( \gamma_o(G) = 2 \) and \( \gamma_F(G) = b \). If \( a = 3 \), then let \( G = L_{b-2} \) and so \( \gamma_o(G) = 3 \) and \( \gamma_F(G) = b \) by Lemma 3.5. Thus, we may assume that \( a \geq 4 \). We begin with a double star \( T \) whose central vertices are \( u \) and \( v \) such that
\[
N(u) = \{u_1, u_2, \ldots, u_{b-a+1}, v\} \quad \text{and} \quad N(v) = \{u, v_1, v_2, \ldots, v_{a+b-1}\}.
\]
Then the graph \( G \) is obtained from \( T \) by (1) subdividing the edge \( uv \) with a new vertex \( x \) and (2) adding \( a-3 \) new vertices \( w_1, w_2, \ldots, w_{a-3} \) and joining each \( w_i \) to \( v_i \) for \( 1 \leq i \leq a-3 \). Let \( W = \{w_1, w_2, \ldots, w_{a-3}\} \).

First, we show that \( \gamma_o(G) = a \). Let
\[
S_o = \{u, x, v, v_1, v_2, \ldots, v_{a-3}\}.
\]
Since \( S_o \) is an open dominating set of \( G \), it follows that \( \gamma_o(G) \leq a \). To show that \( \gamma_o(G) \geq a \), let \( S \) be a minimum open dominating set of \( G \). Since each \( w_i \) (\( 1 \leq i \leq a-3 \)) is only openly dominated by \( v_i \) (\( 1 \leq i \leq a-3 \)) and each \( u_j \) (\( 1 \leq j \leq b-a+1 \)) is only openly dominated by \( u \), we have that \( v_i \in S \) for \( 1 \leq i \leq a-3 \) and \( u \in S \). Similarly, since each \( v_t \) (\( a-2 \leq t \leq a+b-1 \)) is only openly dominated by \( v \), we have \( v \in S \). Thus \( \gamma_o(G) = |S| \geq a-1 \). Assume, to the contrary, that \( \gamma_o(G) = a-1 \). Then
\[
S = \{u, v, v_1, v_2, \ldots, v_{a-3}\}.
\]
However then, \( u \) is not openly dominated by any vertex in \( S \), which is a contradiction. Therefore, \( \gamma_o(G) \geq a \) and so \( \gamma_o(G) = a \).

Next, we show that \( \gamma_F(G) = b \). Let
\[
S' = (N[u] - \{x\}) \cup W \cup \{v_{a+b-1}\}.
\]
By the proof of Theorem 3.6(i), the red-blue coloring that assigns red to each vertex of $S'$ and blue to the remaining vertices of $G$ is a minimum $F$-coloring with $b$ red vertices. Therefore, $\gamma_F(G) = (b - a + 2) + (a - 3) + 1 = b$.

Next, we verify (ii). First, let $a = 1$. For the integers $b$ and $i$ with $b \geq 2$ and $1 \leq i \leq b - 1$, let $F_i$ be a copy of $K_4 - e$ with $V(F_i) = \{u_i, v_i, x_i, y_i\}$ such that $\deg u_i = \deg v_i = 2$ and $\deg x_i = \deg y_i = 3$. The graph $H$ is obtained from the graphs $F_i (1 \leq i \leq b - 1)$ by identifying all the vertices $u_i$ and calling the new vertex $u$.

We first show that $\gamma_0(H) = b$. Let

$$S_0 = \{u\} \cup \{x_i : 1 \leq i \leq b - 1\}.$$  

Since $S_0$ is an open dominating set in $H$, it follows that $\gamma_0(H) \leq |S_0| = b$. On the other hand, let $S$ be a minimum open dominating set of $H$. Since $v_i$ is only adjacent to $x_i$ and $y_i$ for each $i (1 \leq i \leq b - 1)$, at least one vertex in each set $\{x_i, y_i, v_i\}$ must belong to any open dominating set of $H$ and so $\gamma_0(H) \geq b - 1$. Assume, to the contrary, that $\gamma_0(H) = b - 1$. Then

$$S = \{w_i : 1 \leq i \leq b - 1\} \subseteq \{x_i, y_i, v_i : 1 \leq i \leq b - 1\},$$

where $w_i \in \{x_i, y_i, v_i\}$ for each $i$ with $1 \leq i \leq b - 1$. However then, $w_i$ is not openly dominated by any vertex in $S$, which is a contradiction. Thus $\gamma_0(H) \geq b$. Next we show that $\gamma_F(H) = 1$. Since the red-blue coloring that assigns red to vertex $u$ and blue to the remaining vertices of $H$ is an $F$-coloring, it follows by (2.1) that $\gamma_F(H) = 1$.

Now let $2 \leq a < b$. We consider two cases.

Case 1. $b = a + 1$. For each integer $i$ with $1 \leq i \leq a - 1$, let $F_i : u_i, v_i, w_i$ be a copy of the path $P_3$ and let $C_3 : x, y, z, x$ be a copy of a 3-cycle. Then the graph $H$ is obtained from the graphs $F_i (1 \leq i \leq a - 1)$ and $C_3$ by (1) identifying the vertices $u_i (1 \leq i \leq a - 1)$ and calling the new vertex $u$ and (2) joining the vertex $u$ to $x$.

We first show that $\gamma_F(H) = a$. Let

$$S_0 = \{x\} \cup \{w_i : 1 \leq i \leq a - 1\}.$$
Since the red-blue coloring that assigns red to each vertex in \( S_0 \) and blue to the remaining vertices of \( H \) is an \( F \)-coloring with \( a \) red vertices, \( \gamma_F(H) \leq a \).

To show that \( \gamma_F(H) \geq a \), let \( c \) be a minimum \( F \)-coloring of \( H \). By Proposition 2.7, each end-vertex in \( w_i \) (1 \( \leq i \leq a - 1 \)) must be colored red by \( c \) and so \( \gamma_F(H) \geq a - 1 \). Assume to the contrary that \( \gamma_F(H) = a - 1 \). Then

\[
R_c = \{ w_i : 1 \leq i \leq a - 1 \}.
\]

However then, \( y \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Thus \( \gamma_F(H) \geq a \).

Next, we show that \( \gamma_o(H) = a + 1 = b \). Let

\[
S_1 = \{ u, x \} \cup \{ v_i : 1 \leq i \leq a - 1 \} = N[u].
\]

Since \( S_1 \) is an open dominating set in \( H \), it follows that \( \gamma_o(H) \leq |S_1| = a + 1 \). On the other hand, since \( w_i \) (1 \( \leq i \leq a - 1 \)) is only openly dominated by \( v_i \), it follows that \( v_i \in S \) for all 1 \( \leq i \leq a - 1 \) and so \( \gamma_o(H) \geq a - 1 \). Also, since each \( v_i \) is only openly dominated by \( w_i \) or by \( u \), it follows that either \( w_i \in S \) (1 \( \leq i \leq a - 1 \)) or \( u \in S \). This implies that

\[
\gamma_o(H) \geq (a - 1) + 1 = a.
\]

Assume, to the contrary, that \( \gamma_o(H) = a \). Let \( S \) be a minimum open dominating set of \( H \). Then \( S \subseteq \{ u \} \cup \{ v_i, w_i : 1 \leq i \leq a - 1 \} \). However then, \( y \) is not openly dominated by any vertex of \( S \), which is a contradiction. Hence \( \gamma_o(H) \geq a + 1 \).

**Case 2.** \( b \geq a + 2 \). Then \( b - a + 1 \geq 3 \). We start with the graph

\[
W_{b-a+1} = C_{b-a+1} + K_1,
\]

where \( C_{b-a+1} : v_1, y_2, \ldots, y_{b-a+1}, y_1 \) and \( x \) is the vertex of degree \( b - a + 1 \) in \( W_{b-a+1} \). For each \( i \) with 1 \( \leq i \leq a - 1 \), let \( F_i : s_i, t_i \) be a copy of \( P_2 \). Then the graph \( H \) is obtained from the graphs \( F_i \) (1 \( \leq i \leq a - 1 \)) and \( W_{b-a+1} \) by (1) adding \( a - 1 \) new edges \( s_i y_1 (1 \leq i \leq a - 1) \) and (2) adding \( b - a \) new vertices \( z_2, z_3, \ldots, z_{b-a+1} \) and joining each
$z_j$ ($2 \leq j \leq b - a + 1$) to $y_i$. Let

$$T = \{t_1, t_2, \ldots, t_{a-1}\}$$

$$S^* = \{s_1, s_2, \ldots, s_{a-1}\}$$

$$Y = \{y_1, y_2, y_3, \ldots, y_{b-a+1}\}.$$

We first show that $\gamma_F(H) = a$. By the proof of Theorem 3.6(ii), the red-blue coloring that assigns red to each vertex of the set $\{x\} \cup T$ and blue to the remaining vertices of $H$ is a minimum $F$-coloring with $a$ red vertices. Therefore, $\gamma_F(H) = a$.

Next, we show that $\gamma_o(H) = b$. Since $S^* \cup Y$ is an open dominating set of $H$, it follows that

$$\gamma_o(H) \leq |S^* \cup Y| = (a - 1) + (b - a + 1) = b.$$

To show that $\gamma_o(H) \geq b$, observe that every open dominating set of $H$ contains $S^* \cup (Y - \{y_1\})$. On the other hand, $s_1$ is not openly dominated by any vertex in $S^* \cup (Y - \{y_1\})$ and so $S^* \cup (Y - \{y_1\})$ is not an open dominating set of $H$. Therefore,

$$\gamma_o(H) \geq |S^* \cup (Y - \{y_1\})| + 1 = (a - 1) + (b - a) + 1 = b.$$

Therefore, $\gamma_o(H) = b$. □

### 3.4 $F$-Domination and 2-Step Domination

Recall that a set $S$ of vertices in a connected graph $G$ is a $k$-step dominating set if for every vertex $u \in V(G) - S$, there exists a path of length $k$ from $u$ to some vertex in $S$. The $k$-step domination number $\gamma_{2k}(G)$ is the minimum cardinality of any $k$-step dominating set of $G$. These concepts were introduced and studied by Lawson, Haynes, and Boland in [26]. As described in [26], applications of this domination arise naturally in graphical models of networks of broadcasting stations, in which a vertex (station) can communicate with other vertices (stations) at a certain distance. Then $\gamma_{2k}(G)$ is the smallest number of stations which can broadcast to all other stations by sending

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
messages via all paths of length \( k \) of which they are end-vertices. The following three results were established in [26].

**Theorem 3.8.** If \( G \) is a connected graph of order \( n \), then

\[
\frac{n}{\Delta^2(G) - \Delta(G) + 1} \leq \gamma_{32}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]

**Theorem 3.9.** If \( H \) is a spanning subgraph of a graph \( G \), then \( \gamma_{3k}(G) \leq \gamma_{3k}(H) \).

**Theorem 3.10.** If \( G \) is a triangle-free graph and \( \text{diam}(G) = 2 \), then

\[
\gamma_{32}(G) = \gamma(G).
\]

In the case when \( k = 2 \), a set \( S \) of vertices in \( G \) is a 2-step dominating set if for every vertex \( u \in V(G) - S \), there exists a path of length 2 from \( u \) to some vertex in \( S \). The 2-step domination number \( \gamma_{32}(G) \) is the minimum cardinality of any 2-step dominating set of \( G \). We mentioned in Chapter 2 that at first glance, one might think that \( F \)-domination is the same as the 2-step domination. However, there is a subtle difference between 2-step domination and \( F \)-domination. Let \( R_c \) be the set of red vertices of an \( F \)-coloring \( c \) of \( G \). Then every blue vertex in \( V(G) - R_c \) is \( F \)-dominated by a red vertex in \( R_c \). Thus every blue vertex must be the initial vertex of some path of length 2 to some red vertex in \( R_c \). This implies that \( R_c \) is a 2-step dominating set of \( G \). Thus, as indicated in (2.2),

\[
\gamma_{32}(G) \leq \gamma_F(G).
\]

However, the difference between 2-step domination and \( F \)-domination is that the path of the later must be a blue-blue-red path, that is, for any blue vertex \( v \), \( F \)-domination requires that a path \( v, u, w \) exists where \( u \) is blue and \( w \) is red. Therefore, not every 2-step dominating set is an \( F \)-dominating set. As mentioned in Chapter 2, it is possible that

\[
\gamma_{32}(G) = \gamma_F(G) \text{ or } \gamma_{32}(G) < \gamma_F(G)
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
for a connected graph $G$. We verify this now, beginning with the first one.

**Proposition 3.11.** For each positive integer $k$, there exists a graph $G$ with

$$\gamma_F(G) = \gamma_{22}(G) = k.$$ 

**Proof.** For $k = 1$, let $G = K_3$. Then $\gamma_F(G) = \gamma_{22}(G) = 1$. Thus, we may assume that $k \geq 2$. For each integer $i$ with $1 \leq i \leq k - 1$, let $F_i : v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}, v_{1i}$ be a copy $C_6$. The graph $G$ is obtained from the graphs $F_i$ ($1 \leq i \leq k - 1$) by identifying vertices $v_{6i}$ ($1 \leq i \leq k - 1$) and labeling the identified vertex by $v$.

We first show that $\gamma_F(G) = k$. Let

$$S_0 = \{v\} \cup \{v_{3i} : 1 \leq i \leq k - 1\}.$$ 

Since the red-blue coloring of $G$ defined by assigning red to every vertex in $S_0$ and blue to the remaining vertices of $G$ is an $F$-coloring of $G$, it follows that $\gamma_F(G) \leq |S_0| = k$.

To show that $\gamma_F(G) \geq k$, let $c$ be a minimum $F$-coloring of $G$. First, observe that at least one vertex in each set $V_i = \{v_{1i}, v_{2i}, v_{3i}\}$ ($1 \leq i \leq k - 1$) must be colored red by $c$ since any blue vertex $v_{3i}$ can be only $F$-dominated by a red vertex in $V_i$. Thus, $|R_c| \geq k - 1$. If $v$ is also colored red by $c$, then $\gamma_F(G) = |R_c| \geq k$. Thus, we may assume that $v$ is colored blue by $c$. Then at least one vertex in the set $\{v_{2i}, v_{4i}\}$ ($1 \leq i \leq k - 1$) must be colored red by $c$ since a blue vertex $v_{2i}$ ($1 \leq i \leq k - 1$) is only $F$-dominated by itself, by $v$ or by $v_{4i}$. This implies that

$$\gamma_F(G) = |R_c| \geq 2(k - 1) = k + (k - 2) \geq k.$$ 

Therefore, $\gamma_F(G) = k$.

Thus, since $\gamma_{22}(G) \leq \gamma_F(G)$, it follows that $\gamma_{22}(G) \leq k$.

Next we show that $\gamma_{22}(G) \geq k$. Let $S$ be a minimum 2-step dominating set. Since the vertex $v_{3i}$ ($1 \leq i \leq k - 1$) is only 2-step dominated by a vertex in the set $V_i = \{v_{1i}, v_{3i}, v_{2i}\}$, it follows that $S$ contains at least one vertex in each set $V_i$ for $1 \leq i \leq k - 1$. Thus $\gamma_{22}(G) = |S| \geq k - 1$. Assume, to the contrary, that $|S| = k - 1$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Then $S \subseteq \bigcup_{i=1}^{k} V_i$. However then, no vertex in $\{v_{3i}, v_{4i} : 1 \leq i \leq k - 1\}$ is 2-step dominated by any vertex in $S$, which is a contradiction. Therefore, $\gamma_{32}(G) = |S| \geq k$ and so $\gamma_{32}(G) = k$.

We have seen that $\gamma_F(K_{1,n-1}) = n$ and $\gamma_{32}(K_{1,n-1}) = 2$ for $n \geq 3$. Thus $\gamma_F(G) - \gamma_{32}(G)$ can be arbitrarily large if $G = K_{1,n-1}$ and $n$ is large. In fact, more can be said. Next, we determine which pairs $a, b$ of integers with $1 \leq a \leq b$ can be realized as the 2-step domination number and $F$-domination number of some connected graph, respectively. In order to do this, we first present the following result.

**Proposition 3.12.** Let $G$ be a connected graph. Then

$$\gamma_{32}(G) = 1 \text{ if and only if } \gamma_F(G) = 1.$$  

**Proof.** Since $1 \leq \gamma_{32}(G) \leq \gamma_F(G)$, it follows that if $\gamma_F(G) = 1$, then $\gamma_{32}(G) = 1$. For the converse, let $G$ be a connected graph with $\gamma_{32}(G) = 1$. Let $\{v\}$ be a 2-step dominating set of $G$. Define a red-blue coloring $c^*$ that assigns red to the vertex $v$ and blue to each vertex in $V(G) - \{v\}$. Since $\{v\}$ is a 2-step dominating set of $G$, for each $u \in V(G) - \{v\}$, there is an $u - v$ path of length 2 in $G$, implying that there is a blue-blue-red $u - v$ path of length 2 in $G$ rooted at $u$ for all $u \in V(G)$. Thus $u$ is $F$-dominated by $v$. Therefore, $c^*$ is an $F$-coloring and so $\gamma_F(G) = 1$. □

Observe that $\gamma_{32}(K_n) = \gamma_F(K_n) = 1$ for each $n \geq 3$ and so there exist connected graphs $G$ for which $\gamma_{32}(G) = \gamma_F(G) = 1$.

We are now prepared to determine which pairs $a, b$ of integers with $1 \leq a \leq b$ can be realized as the 2-step domination number and $F$-domination number of some connected graph, respectively.

**Theorem 3.13.** For each pair $a, b$ of positive integers with $a \leq b$, there exists a connected graph $G$ such that $\gamma_{32}(G) = a$ and $\gamma_F(G) = b$ if and only if

$$(a, b) \notin \{(1, i) : i \geq 2\}.$$
Proof. By proposition 3.12, there exists no connected graph $G$ such that $\gamma_{32}(G) = 1$ and $\gamma_F(G) = i$ for $i \geq 2$. Thus it remains to verify the converse. By Proposition 3.11, we may assume that $a < b$. Let $a$ and $b$ be positive integers with $a < b$ such that $(a, b) \notin \{(1, i) : i \geq 2\}$. Then $a \geq 2$. We consider two cases, according to whether $a = 2$ or $a \geq 3$.

Case 1. $a = 2$. Then $b \geq 3$. Let $G = L_{b-2}$ described in Lemma 3.5. Then $\gamma_{32}(G) = 2$ and $\gamma_F(G) = b$.

Case 2. $a \geq 3$. For each integer $i$ with $1 \leq i \leq a - 2$, let

$$F_i : v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}, v_i$$

be a copy of the $6$-cycle $C_6$. Let $G$ be the graph obtained from a copy of the graph $L_{b-a}$, as described in Lemma 3.5, and the $a - 2$ copies of $F_i$ by identifying all the vertices $v_{6i}$ and $w$ and labeling the identified vertex by $v$. Let

$$W = \{w_1, w_2, \ldots, w_{b-a}\} \text{ and } Y = \{y_1, y_2, \ldots, y_{b-a}\}.$$ 

Then $|W| = |Y| = b - a$. We first show that $\gamma_{32}(G) = a$. Let

$$S_1 = \{v, x\} \cup \{v_{3i} : 1 \leq i \leq a - 2\}.$$ 

For every vertex $u \in V(G) - S_1$, there exists $u' \in S_1$ such that $d(u, u') = 2$, which implies that there is a path of length two from $u$ to $u'$. Thus $S_1$ is a 2-step dominating set of $G$ and so $\gamma_{32}(G) \leq |S_1| = a$.

Next we show that $\gamma_{32}(G) \geq a$. Let $S_{32}$ be a minimum 2-step dominating set. Since the vertex $v_{3i}$ ($1 \leq i \leq a - 2$) is only 2-step dominated by a vertex in the set $U_i = \{v_{1i}, v_{2i}, v_{3i}\}$, it follows that $S_{32}$ contains at least one vertex in each set $U_i$ for $1 \leq i \leq a - 2$. Moreover, each vertex in $Y$ is only 2-step dominated by a vertex in $\{x, z\} \cup Y$. Thus $S_{32}$ contains at least one vertex in $\{x, z\} \cup Y$. Hence $\gamma_{32}(G) = |S_{32}| \geq a - 1$.

Assume, to the contrary, that $|S_{32}| = a - 1$. Then

$$S_{32} \subseteq (\{x, z\} \cup Y) \cup (\bigcup_{i=1}^{a-2} U_i).$$
However then, no vertex in the set \(\{v_{2i}, v_{4i} : 1 \leq i \leq a - 2\}\) is 2-step dominated by any vertex in \(S_{22}\), which is a contradiction. Therefore, \(\gamma_{22}(G) \geq a\) and so \(\gamma_{22}(G) = a\).

Now, we show \(\gamma_F(G) = b\). Let

\[S^* = S_1 \cup W.\]

Define a red-blue coloring \(c^*\) of \(G\) by assigning red to each vertex in \(S^*\) and blue to the remaining vertices of \(G\). Observe that (i) the blue vertex \(y\) is adjacent to the blue vertex \(z\) and \(z\) is adjacent to the red vertex \(v\), (ii) the blue vertex \(z\) is adjacent to the blue vertex \(y\) and \(y\) is adjacent to the red vertex \(x\), (iii) the blue vertex \(y_j\) \((1 \leq j \leq b - a)\) is adjacent to the blue vertex \(y\) and \(y\) is adjacent to the red vertex \(x\), (iv) the blue vertex \(v_{1,i}\) \((1 \leq i \leq a - 2)\) is adjacent to the blue vertex \(v_{2,i}\) and \(v_{2,i}\) is adjacent to the red vertex \(v_{3,i}\), (v) the blue vertex \(v_{2,i}\) \((1 \leq i \leq a - 2)\) is adjacent to the blue vertex \(v_{1,i}\) and \(v_{1,i}\) is adjacent to the red vertex \(v\), (vi) the blue vertex \(v_{4,i}\) \((1 \leq i \leq a - 2)\) is adjacent to the blue vertex \(v_{5,i}\) and \(v_{5,i}\) is adjacent to the red vertex \(v\), (vii) the blue vertex \(v_{5,i}\) \((1 \leq i \leq a - 2)\) is adjacent to the blue vertex \(v_{4,i}\) and \(v_{4,i}\) is adjacent to the red vertex \(v_{3,i}\). Thus, every blue vertex \(u\) of \(G\) belongs to a copy of \(F\) rooted at \(u\), implying that \(c^*\) is an \(F\)-coloring of \(G\). Therefore, \(\gamma_F(G) \leq |R_{c^*}| = |S^*| = b\).

Next, we show that \(\gamma_F(G) \geq b\). Let there be given a minimum \(F\)-coloring \(c\) of \(G\). First, we make three observations.

(a1) For each integer \(i\) \((1 \leq i \leq a - 2)\), the vertex \(v_{3i}\) is only \(F\)-dominated by a vertex in the set \(A_i = \{v_{1i}, v_{3i}, v_{5i}\}\) and so \(R_c\) contains at least one vertex in \(A_i\).

(a2) Each vertex in \(Y\) is only \(F\)-dominated by a vertex in the set \(\{x, z\} \cup Y\). Thus \(R_c\) contains at least one vertex in \(\{x, z\} \cup Y\).

(a3) The vertex \(y\) is only \(F\)-dominated by a vertex of the set \(\{v, y\}\) and so \(R_c\) contains at least one vertex in \(\{v, y\}\). By Lemma 2.6, if \(v \in R_c\), then \(w_j \in R_c\) for \(1 \leq j \leq b - a\), while if \(y \in R_c\), then \(y_j \in R_c\) for \(1 \leq j \leq b - a\). Thus either \(\{v\} \cup W \subseteq R_c\) or \(\{y\} \cup Y \subseteq R_c\).
By (a3), we consider two cases, according to whether \( \{v\} \cup W \subseteq R_c \) or \( \{y\} \cup Y \subseteq R_c \).

**Case i.** \( \{v\} \cup W \subseteq R_c \). Since \( A_i \ (1 \leq i \leq a-2) \), \( \{x, z\} \cup Y \), and \( \{v\} \cup W \) are mutually disjoint, it follows by (a1) and (a2) that
\[
|R_c| \geq (a-2) + 1 + |\{v\} \cup W| = (a-1) + (b-a+1) = b.
\]

**Case ii.** \( \{y\} \cup Y \subseteq R_c \). In this case, the two sets \( \{x, z\} \cup Y \) and \( \{y\} \cup Y \) are not disjoint. Since \( A_i \ (1 \leq i \leq a-2) \) and \( \{y\} \cup Y \) are mutually disjoint, it follows by (a1) that
\[
|R_c| \geq (a-2) + |\{y\} \cup Y| = (a-2) + (b-a+1) = b - 1.
\]
Assume, to the contrary, that \( |R_c| = b - 1 \). Thus \( R_c \) consists of \( \{y\} \cup Y \) and exactly one vertex from each set \( A_i \) for \( 1 \leq i \leq a-1 \). However then, no vertex in the set
\[
\{v_{2,i}, v_{4,i} : 1 \leq i \leq a-2\}
\]
is \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Thus \( |R_c| \geq b \) and so \( |R_c| = b \).

3.5 *F*-Domination and Restrained Domination

For a graph \( G \), recall that a set \( S \subseteq V(G) \) is a **restrained dominating set** of \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \) and to a vertex in \( V(G) - S \). Every graph has a restrained dominating set since \( V(G) \) is such a set. The **restrained domination number** \( \gamma_r(G) \) is the minimum cardinality of a restrained dominating set of \( G \). The concept of restrained domination in graphs was introduced and studied by Domke, Hattingh, Hedetniemi, Laskar, and Markus [18].

As described in [18], one possible application of the concept of restrained domination is that of prisoners and guards. We use a graph \( G \) to model a prison such that the vertex set of \( G \) consists of the positions of the prisoners and the guards in the prison, and an edge is placed between two vertices if the corresponding individuals can be observed by each other. Then each vertex in a restrained dominating set corresponds to a position of a guard and each vertex not in a restrained dominating set corresponds to a position of a prisoner. In this case, each prisoner’s position is observed by a guard’s
Let $R$ be a minimum restrained dominating set of a graph $G$ of order $n$. Since every vertex in $V(G) - R$ must be adjacent to a vertex in $R$, it follows that $R$ is also a dominating set of $G$ and so

$$1 \leq \gamma(G) \leq \gamma_r(G) \leq n.$$ 

The following result characterizes all connected graphs of order $n$ with restrained domination number $1$ and $n$ (see [18]).

**Theorem 3.14.** Let $G$ be a connected graph of order $n$. Then

(a) $\gamma_r(G) = 1$ if and only if $G \cong K_1 + H$ for some graph $H$ without isolated vertices;

(b) $\gamma_r(G) = n$ if and only if $G$ is a star.

It was noticed in [10] that $\gamma_F(G) = \gamma_r(G)$. Furthermore, every $F_4$-coloring of a graph $G$ is also an $F$-coloring (or $F_3$-coloring) of $G$ and so $\gamma_F(G) \leq \gamma_{F_4}(G)$. Therefore, we have the following.

**Theorem 3.15.** For every graph $G$, $\gamma_F(G) \leq \gamma_r(G)$.

By Theorem 3.15, if $G$ is a graph with $\gamma_F(G) = a$ and $\gamma_r(G) = b$, then $a \leq b$. Next, we show that every pair $a, b$ of positive integers with $a \leq b$ is realizable as the $F$-domination number and the restrained domination number of some connected graph, respectively. In order to do this, we first introduce the graphs $H_1$ and $H_2$ in Figure 3.5. Observe that

1. $\gamma_F(H_1) = 1$ and $\gamma_r(H_1) = 2$, where $\{u_0\}$ is a minimum $F$-dominating set of $H_1$ and $\{u_0, u_3\}$ is a minimum restrained dominating set of $H_1$;
(2) $\gamma_F(H_2) = 1$ and $\gamma_r(H_2) = 2$, where $\{v_2\}$ is a minimum $F$-dominating set and $\{v_2, v_3\}$ is a minimum restrained dominating set of $H_2$.

Figure 25. The graphs $H_1$ and $H_2$.

**Theorem 3.16.** For each pair $a, b$ of positive integers with $a \leq b$, there exists a connected graph $G$ with $\gamma_F(G) = a$ and $\gamma_r(G) = b$.

**Proof.** If $a = b$, then let $G = K_{1,a-1}$, where $a \geq 1$. Since $\gamma_F(G) = \gamma_r(G) = a$ by Theorem 3.14, the result holds if $a = b$. Thus, we may assume that $1 \leq a < b$. We consider two cases, according to whether $a = 1$ or $a \geq 2$. In each case, we construct a connected graph $G$ with $\gamma_F(G) = a$ and $\gamma_r(G) = b$.

**Case 1.** $a = 1$. For each $i$ with $1 \leq i \leq b - 1$, let $F_i$ be a copy of $H_1$ in Figure 3.5, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}$, where $u_{i,p}$ corresponds to $u_{p}$ in $H_1$ for $0 \leq p \leq 6$. The graph $G$ is then obtained from the graphs $F_i$ ($1 \leq i \leq b - 1$) by identifying all vertices $u_{i,0}$ and labeling the identified vertex $v$.

Since $\{v\}$ is an $F$-dominating set of $G$, it follows that $\gamma_F(G) = 1$. Thus, it remains to show that $\gamma_r(G) = b$. Observe that

$$S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq b - 1\}$$

is a restrained dominating set of $G$ and so

$$\gamma_r(G) \leq |S| = 1 + (b - 1) = b.$$ 

To show $\gamma_r(G) \geq b$, let $R$ be a minimum restrained dominating set of $G$. We consider two subcases.
Subcase 1.1. \( v \in R \). Since \( u_{i,3} \) is only adjacent to \( u_{i,2} \) and \( u_{i,4} \), it follows that \( R \) contains at least one vertex in each set \( \{u_{i,2}, u_{i,3}, u_{i,4}\} \) for \( 1 \leq i \leq b - 1 \). Hence \( \gamma_r(G) = |R| \geq 1 + (b - 1) = b \).

Subcase 1.2. \( v \notin R \). Observe that

1. each \( u_{i,1} \) is only adjacent to \( v \) and \( u_{i,2} \),
2. each \( u_{i,6} \) is only adjacent to \( v \) and \( u_{i,5} \), and
3. \( \{u_{i,1}, u_{i,2}\} \cap \{u_{i,5}, u_{i,6}\} = \emptyset \).

Since \( v \notin R \), it follows that \( R \) contains at least one vertex in each set \( \{u_{i,1}, u_{i,2}\} \) and \( \{u_{i,5}, u_{i,6}\} \) for \( 1 \leq i \leq b - 1 \). Since \( b \geq 2 \), it follows that
\[
\gamma_F(G) = |R| \geq 2(b - 1) = b + 2 \geq b.
\]

Thus \( \gamma_r(G) = b \).

Case 2. \( a \geq 2 \). For each \( i \) with \( 1 \leq i \leq b - a \), let \( F_i \) be a copy of \( H_1 \) in Figure 3.5, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). For each \( j \) with \( 1 \leq j \leq a - 1 \), let \( G_j \) be a copy of \( H_2 \) in Figure 3.5 with \( \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\} \), where \( v_{j,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). The graph \( G \) is then obtained from the graphs \( F_i \) and \( G_j \), where \( 1 \leq i \leq b - a \) and \( 1 \leq j \leq a - 1 \), by identifying all vertices \( u_{i,0} \) and \( v_{j,3} \) and labeling the identified vertex \( v \).

We first show that \( \gamma_F(G) = a \). Since \( \{v\} \cup \{v_{j,2} : 1 \leq j \leq a - 1\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq a \). On the other hand, let \( c \) be a minimum \( F \)-coloring of \( G \). If \( v \in R_c \), then \( v_{j,1} \) can be \( F \)-dominated only by some vertex in \( V(G_j) - \{v\} \) for \( 1 \leq j \leq a - 1 \). This implies that \( R_c \) contains at least one vertex from each set \( V(G_j) - \{v\} \) for \( 1 \leq j \leq a - 1 \). Hence \( \gamma_F(G) = |R_c| \geq 1 + (a - 1) = a \). Thus, we may assume \( v \notin R_c \).

Observe that

1. each \( u_{i,3} \) is only \( F \)-dominated by a vertex of the set \( X_i = \{u_{i,1}, u_{i,3}, u_{i,4}, v\} \),
2. each \( u_{i,4} \) is only \( F \)-dominated by a vertex of \( Y_i = \{u_{i,2}, u_{i,4}, u_{i,6}, v\} \), and
(3) \((X_i - \{v\}) \cap (Y_i - \{v\}) = \emptyset\).

Since \(v \not\in R_c\), it follows that \(R_c\) contains at least one vertex in each set \(X_i\) and \(Y_i\) for \(1 \leq i \leq b - a\). Furthermore, \(v_{j,0}\) is only \(F\)-dominated by a vertex in each set \(\{v, v_{j,0}, v_{j,1}, v_{j,2}\}\). Since \(v \not\in R_c\), it follows that \(R_c\) must contain at least one vertex from each set \(V(G_j) - \{v\}\) for \(1 \leq j \leq a - 1\). Hence

\[
\gamma_F(G) = |R_c| \geq 2(b - a) + (a - 1) = 2b - a - 1
\]

\[
= b + (b - a - 1) \geq b > a,
\]

since \(b - a - 1 \geq 0\) as \(b > a\), which is a contradiction. Thus \(\gamma_F(G) = a\).

Next we show that \(\gamma_r(G) = b\). Observe that

\[
R^* = \{v\} \cup \{u_{i,3} : 1 \leq i \leq b - a\} \cup \{v_{j,2} : 1 \leq j \leq a - 1\}
\]

is a restrained dominating set of \(G\) and so

\[
\gamma_r(G) \leq |R^*| = 1 + (b - a) + (a - 1) = b.
\]

To show \(\gamma_r(G) \geq b\), let \(R\) be a minimum restrained dominating set of \(G\). We consider two subcases.

**Subcase 2.1.** \(v \in R\). Since \(v_{j,2}\) is only adjacent to \(v_{j,0}\) and \(v_{j,1}\), it follows that \(R\) must contain at least one vertex in each set \(V(G_j) - \{v\}\) for \(1 \leq j \leq a - 1\). Furthermore, \(u_{i,3}\) is only adjacent to \(u_{i,2}\) and \(u_{i,4}\). Thus \(R\) must contain at least one vertex in each set \(\{u_{i,2}, u_{i,3}, u_{i,4}\}\) for \(1 \leq i \leq b - a\). Therefore,

\[
\gamma_r(G) = |R| \geq 1 + (a - 1) + (b - a) = b.
\]

**Subcase 2.2.** \(v \not\in R\). Observe that

1. each \(u_{i,1}\) is only adjacent to \(v\) and \(u_{i,2}\),
2. each \(u_{i,6}\) is only adjacent to \(v\) and \(u_{i,5}\), and
3. \(\{u_{i,1}, u_{i,2}\} \cap \{u_{i,5}, u_{i,6}\} = \emptyset\).
Since \( v \notin R \), it follows that \( R \) must contain at least one vertex in each set \( \{u_{i,1}, u_{i,2}\} \) and at least one vertex in each set \( \{u_{i,5}, u_{i,6}\} \) for \( 1 \leq i \leq b - a \). Furthermore, \( v_{j,0} \) is only adjacent to \( v_{j,1} \) and \( v_{j,2} \), so \( R \) must contain at least one vertex in each set \( V(G_j) - \{v\} \) for \( 1 \leq j \leq a - 1 \). Again, since \( b - a - 1 \geq 0 \), it follows that

\[
\gamma_F(G) = |R| \geq 2(b - a) + (a - 1) = b + (b - a - 1) \geq b.
\]

Thus \( \gamma_r(G) = b \).

We summarize the main results we have established in this chapter as follows:

1. For each pair \( a, b \) of positive integers with \( b \geq 2 \), there exists a connected graph \( G \) such that \( \gamma(G) = a \) and \( \gamma_0(G) = b \).

2. For each pair \( a, b \) of positive integers, there exists a connected graph \( G \) such that \( \gamma(G) = a \) and \( \gamma_F(G) = b \).

3. For each pair \( a, b \) of positive integers with \( a \geq 2 \), there exists a connected graph \( G \) such that \( \gamma_0(G) = a \) and \( \gamma_F(G) = b \).

4. For each pair \( a, b \) of positive integers with \( a \leq b \), there exists a connected graph \( G \) such that \( \gamma_{F2}(G) = a \) and \( \gamma_F(G) = b \) if and only if \( (a, b) \notin \{(1, i) : i \geq 2\} \).

5. For each pair \( a, b \) of positive integers with \( a \leq b \), there exists a connected graph \( G \) with \( \gamma_F(G) = a \) and \( \gamma_r(G) = b \).
4 REALIZABLE TRIPLES

4.1 Introduction

Recall that for a nontrivial connected graph $G$, the following are possible:

I. $\gamma_F(G) \leq \gamma(G) \leq \gamma_0(G)$,

II. $\gamma(G) \leq \gamma_0(G) \leq \gamma_F(G)$,

III. $\gamma(G) \leq \gamma_F(G) \leq \gamma_0(G)$.

Also, we have seen the following realization results established in Chapter 3:

1. For each pair $a, b$ of positive integers with $b \geq 2$, there exists a connected graph $G$ such that $\gamma(G) = a$ and $\gamma_0(G) = b$.

2. For each pair $a, b$ of positive integers, there exists a connected graph $G$ such that $\gamma(G) = a$ and $\gamma_F(G) = b$.

3. For each pair $a, b$ of positive integers with $a \geq 2$, there exists a connected graph $G$ such that $\gamma_0(G) = a$ and $\gamma_F(G) = b$.

This gives rise to the following natural question.

**Problem 4.1.** For which triples $(A, B, C)$ of positive integers, does there exist a connected graph $G$ such that $\gamma(G) = A$, $\gamma_0(G) = B$, and $\gamma_F(G) = C$?

Since for every connected graph $G$,

$$\gamma(G) \leq \gamma_0(G) \leq 2\gamma(G) \text{ and } \gamma_0(G) \geq 2,$$

it follows that no triple $(A, B, C)$ of positive integers with $A > B$, or $B > 2A$, or $B = 1$ can be realized, respectively, as the domination number, the open domination number,
and the $F$-domination number of any connected graph. For this reason, by a \textit{triple}, we mean an ordered triple $(A,B,C)$ of positive integers with

$$A \leq B \leq 2A \text{ and } B \geq 2.$$  \hspace{1cm} (4.1)

Furthermore, we define a triple $(A,B,C)$ to be \textit{realizable} if there exists a connected graph $G$ such that

$$\gamma(G) = A, \gamma_o(G) = B, \text{ and } \gamma_F(G) = C.$$  \hspace{1cm} (4.2)

Certainly, not every triple is realizable. In fact, there are infinitely many triples that not realizable, as we show next.

\textbf{Proposition 4.2.} \textit{There is no connected graph $G$ such that}

$$\gamma(G) = \gamma_o(G) = 2 \text{ and } \gamma_F(G) \geq 3.$$ 

\textit{Therefore, no triple $(2,2,C)$ is realizable for any integer $C \geq 3$.}

\textbf{Proof.} Let $G$ be a connected graph of order $n$ such that $\gamma(G) = \gamma_o(G) = 2$. Then there exist two adjacent vertices $x,y \in V(G)$ such that $\{x,y\}$ is an open dominating set. Thus, each vertex $v \in V(G) - \{x,y\}$ is either adjacent to $x$ or adjacent to $y$. Since $\gamma(G) = 2$, it follows that neither $\{x\}$ nor $\{y\}$ is a dominating set. This implies that there is at least one vertex in $V(G) - \{x,y\}$ that is adjacent to $x$ and at least one vertex in $V(G) - \{x,y\}$ that is adjacent to $y$. Therefore, $\deg x \geq 2$ and $\deg y \geq 2$. Therefore, $G$ has a spanning subgraph that is isomorphic to the double star $T$ whose central vertices are $x$ and $y$. By Lemma 2.30, we have that $\gamma_F(G) \leq \gamma_F(T)$. Furthermore, by Theorem 2.48, $\gamma_F(T) = 2$, and so $\gamma_F(G) \leq 2$. As a consequence, each triple $(2,2,C)$ is not realizable for every integer $C \geq 3$. \hfill $\Box$

On the other hand, there are infinitely many realizable triples. We have seen in Proposition 3.4 that for each integer $k \geq 2$, there exists a connected graph $G$ such that

$$\gamma(G) = \gamma_F(G) = \gamma_o(G) = k.$$ 

Therefore, we have the following.
Corollary 4.3. Every triple \((k, k, k)\) is realizable for each integer \(k \geq 2\).

By Proposition 4.2 and Corollary 4.3, there are infinitely many nonrealizable triples and infinitely many realizable triples. In this chapter, we investigate which triples are realizable. To simplify notation, we classify all triples under consideration into the following three types:

A triple \((A, B, C)\) is of type I if \(C < A < B\);

A triple \((A, B, C)\) is of type II if \(A < B < C\);

A triple \((A, B, C)\) is of type III if \(A < C < B\).

4.2 Two Special Realizable Triples

In this section, we show that every triple \((A, B, C)\) with \(A = 1\) or \(C = 1\) is realizable, beginning with the case \(A = 1\). Observe that if \(A = 1\), then \(B = 2\).

Proposition 4.4. Every triple \((1, 2, C)\) is realizable.

Proof. For \(C = 1\), let \(G = K_3\). Then \(\gamma(G) = 1\), \(\gamma_\sigma(G) = 2\), and \(\gamma_F(G) = 1\).

For \(C \geq 2\), let \(G = K_{1,C-1}\). Then \(\gamma(G) = 1\), \(\gamma_\sigma(G) = 2\), and \(\gamma_F(G) = C\). \(\square\)

Next, we show that every triple \((A, B, 1)\) is realizable. In order to do this, we first recall the graphs \(H_1\) and \(H_2\) in Figure 4.1 (which also appeared in Section 3.5). Observe that

\[
\begin{align*}
\gamma(H_1) &= 2, \quad \gamma_\sigma(H_1) = 3, \quad \text{and} \quad \gamma_F(H_1) = 1; \\
\gamma(H_2) &= 1, \quad \gamma_\sigma(H_2) = 2, \quad \text{and} \quad \gamma_F(H_2) = 1.
\end{align*}
\]

The graphs \(H_1\) and \(H_2\) will play an important role in constructing connected graphs with prescribed domination number, open domination number, and \(F\)-domination number.

Theorem 4.5. Every triple \((A, B, 1)\) is realizable.
Proof. For \( A = 1 \) and \( B = 2 \), let \( G = K_3 \) and so \( \gamma_F(G) = \gamma(G) = 1 \) and \( \gamma_0(G) = 2 \). Thus we may assume that \( A \geq 2 \). We consider two cases, according to whether \( A = B \) or \( A \neq B \).

Case 1. \( A = B \). If \( A = B = 2 \), let \( G \) be the graph obtained from the graph \( K_4 - e \) by adding a new vertex and joining this new vertex to a vertex of degree 2 in \( K_4 - e \). Then \( \gamma_F(G) = 1 \) and \( \gamma(G) = \gamma_0(G) = 2 \). Now let \( A = B \geq 3 \). Let \( s > A \) be an integer and consider the graph \( P_s + K_1 \), where \( P_s : u_1, u_2, \ldots, u_s \) and \( u \) is the vertex in \( P_s + K_1 \) with \( \text{deg} \ u = s \). Then the graph \( G \) is obtained from \( P_s + K_1 \) by adding \( A - 1 \) new vertices \( v_1, v_2, \ldots, v_{A-1} \) and joining each \( v_i \) to \( u_i \) for \( 1 \leq i \leq A - 1 \). Since \( N(u) \) is an open dominating set of \( G \), it follows by Corollary 2.43 that \( \gamma_F(G) = 1 \). Since \( \{u, u_1, u_2, \ldots, u_{A-1}\} \) is a minimum dominating and minimum open dominating set of \( G \), it follows that \( \gamma(G) = \gamma_0(G) = A \).

Case 2. \( A < B \leq 2A - 2 \). We consider three subcases.

Subcase 2.1. \( A < B \leq 2A - 2 \). Let \( B = A + k \), where \( k \geq 1 \), and let \( \ell = A - k - 1 \). Since \( B = A + k \leq 2A - 2 \), it follows that \( \ell \geq 1 \).

Consider the graphs \( H_1 \) and \( H_2 \) in Figure 4.1. For each \( i \) with \( 1 \leq i \leq k \), let \( F_i \) be a copy of \( H_1 \) with \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). For each \( j \) with \( 1 \leq j \leq \ell \), let \( G_j \) be a copy of \( H_2 \) with \( V(G_j) = \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\} \), where \( v_{j,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). The graph \( G \) is then obtained from the graphs \( F_i \) and \( G_j \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq \ell \) by identifying all vertices \( u_{i,0} \) and \( v_{j,0} \) and labeling the identified vertex \( u \) (see Figure 4.2 for the graph \( G \)).
when \( k = 2 \) and \( \ell = 1 \).

![Graph G for k=2 and \( \ell = 1 \) in Subcase 2.1.](image)

Figure 27. The graph \( G \) for \( k = 2 \) and \( \ell = 1 \) in Subcase 2.1.

Observe that \( N(v) \) is an open dominating set of \( G \). Thus \( \gamma_F(G) = 1 \) by Corollary 2.43. Since

\[
S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq k\} \cup \{v_{j,1} : 1 \leq j \leq \ell\}
\]

is a minimum dominating set of \( G \), it follows that

\[
\gamma(G) = |S| = 1 + k + \ell = 1 + k + (A - k - 1) = A.
\]

Furthermore, because

\[
S_0 = S \cup \{u_{i,2} : 1 \leq i \leq k\}
\]

is a minimum open dominating set of \( G \), it follows that

\[
\gamma_0(G) = |S_0| = |S| + k = A + k = B.
\]

**Subcase 2.2.** \( B = 2A - 1 \). For each \( i \) with \( 1 \leq i \leq A - 1 \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1 such that \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). Then the graph \( G \) is obtained from the graphs \( F_i, 1 \leq i \leq A - 1 \), by identifying all vertices \( u_{i,0} \) and labeling the identified vertex by \( v \). Again, \( \gamma_F(G) = 1 \) and \( \{v\} \) is the minimum \( F \)-dominating set. Since \( S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq A - 1\} \) is a minimum dominating set of \( G \) and \( S \cup \{u_{i,2} : 1 \leq i \leq A - 1\} \) is a minimum open dominating set of \( G \), it follows that \( \gamma(G) = |S| = A \) and \( \gamma_0(G) = |S| + (A - 1) = 2A - 1 = B \).

**Subcase 2.3.** \( B = 2A \). If \( A = 1 \) and \( B = 2 \), then the graph \( H_2 \) of Figure 4.1 has the desired property. Thus we may assume that \( A \geq 2 \). Let \( p \geq 2 \) be an integer. For
each integer $i$ with $1 \leq i \leq A - 1$, let $F_i$ be the graph obtained the path $u_i, y_i, v_i$ by adding $2p$ new vertices $r_{i,j}$ $(1 \leq j \leq 2p)$ and joining (1) each vertex $r_{i,j}$ $(1 \leq j \leq p)$ to $u_i$ and $y_i$ and (2) each vertex $r_{i,j}$ $(p + 1 \leq j \leq 2p)$ to $y_i$ and $v_i$. The graph $F_i$ is shown in Figure 4.3.

![Figure 28. The graph $F_i$ in Subcase 2.3.](image)

Then the graph $G$ is obtained from the $A - 1$ graphs $F_i$ $(1 \leq i \leq A - 1)$ and the path $P : z, w, x, w'$ of order 4 by (1) adding the edge $xz$ and (2) joining each of the two vertices $w$ and $z$ to each vertex in $\{u_i, v_i\}$ for $1 \leq i \leq A - 1$. The graph $G$ is shown in Figure 4.4 for $A = 3$. Since

$$N(w) = \{x, z\} \cup \{u_i, v_i : 1 \leq i \leq A - 1\}$$

is an open dominating set of $G$, it follows by Corollary 2.43 that $\gamma_F(G) = 1$. It remains to show that $\gamma(G) = A$ and $\gamma_o(G) = B$.

We first show that $\gamma(G) = A$. Since the set

$$\{x\} \cup \{y_i : 1 \leq i \leq A - 1\}$$

is a dominating set of $G$, it follows that $\gamma(G) \leq A$. On the other hand, let $S$ be a minimum dominating set of $G$. For each integer $i$ with $1 \leq i \leq A - 1$, let

$$R_i = \{r_{i,j} : 1 \leq j \leq p\} \quad \text{and} \quad R'_i = \{r_{i,j} : p + 1 \leq j \leq 2p\}.$$ 

Observe that

(a1) since $w'$ is the end-vertex of $G$ and $w'$ is adjacent to $x$, the set $S$ contains at least one vertex in $\{w', x\}$, and
Figure 29. The graph $G$ in Subcase 2.3 for $A = 3$.

(a2) since $y_i$ is only dominated by a vertex in the set

$$A_i = \{u_i, v_i, y_i\} \cup R_i \cup R'_i,$$

the set $S$ must contain at least one vertex in each set $A_i$.

The $A$ sets $\{w', x\}$ and $A_i$ ($1 \leq i \leq A - 1$) are pairwise disjoint. It then follows by (a1) and (a2) that $S$ contains at least $A$ distinct vertices of $G$ and so $\gamma(G) \geq A$. Therefore, $\gamma(G) = A$.

Next, we show that $\gamma_o(G) = B$. Since $N(w)$ is an open dominating set of $G$, it follows that $\gamma_o(G) \leq |N(w)| = 2A$. On the other hand, let $S_o$ be a minimum open dominating set of $G$. First, we verify the following claim.

Claim. For each integer $i$ with $1 \leq i \leq A - 1$, the set $S_o$ must contain at least two vertices in each set $A_i$ in (4.3).

Proof of Claim. Assume, to the contrary, that $S_o$ contains at most one vertex in $A_i$ for some $i$ with $1 \leq i \leq A - 1$. Observe that each vertex in $R_i$ is only openly dominated by a vertex in $B_i = \{u_i, y_i\}$ and so $S_o$ must contain at least one vertex in $B_i$. Similarly, each vertex in $R'_i$ is only openly dominated by a vertex in $C_i = \{v_i, y_i\}$ and so $S_o$ must contain at least one vertex in $C_i$. Since $B_i \cup C_i = \{u_i, v_i, y_i\} \subseteq A_i$, it follows that $S_o$ contains at least one vertex in $A_i$. Hence $S_o$ contains exactly one vertex in $A_i$. Because
$B_i \cap C_i = \{y_i\}$, the vertex $y_i$ is the only vertex of $A_i$ that belongs to $S_0$. However, $y_i$ is only openly dominated by a vertex in $A_i - \{y_i\}$, implying that $y_i$ is not openly dominated by any vertex in $S_0$, which is a contradiction.

This completes the proof of the claim. Therefore, $S_0$ must contain at least two vertices in each set $A_i$ for $1 \leq i \leq A - 1$. Moreover, the end-vertex $w'$ is only openly dominated by $x$ and $x$ is only openly dominated by a vertex in the set $V(P) - \{x\} = \{w, w', z\}$. Thus $S_0$ must contain at least two vertices in $V(P)$. Since the $A$ subsets $V(P)$ and $A_i$ ($1 \leq i \leq A - 1$) of $V(G)$ are pairwise disjoint, $S_0$ contains at least $2A$ distinct vertices of $G$ and so $\gamma_0(G) = |S_0| \geq 2A$. Therefore, $\gamma_0(G) = 2A = B$. □

4.3 Realizable Triples of Type I

In this section, we show that every triple of type I is realizable.

**Theorem 4.6.** Every triple $(A, B, C)$ of positive integers with

$$1 \leq C < A < B < 2A$$

is realizable.

**Proof.** For each such triple $(A, B, C)$, we construct a connected graph $G$ that satisfies (4.2). First, we outline our proof. By Theorem 4.5, we may assume that $C \geq 2$. Furthermore, by Corollary 4.3, the result holds for $A = B = C$. Thus it suffices to consider the following three cases:

**Case I.** $2 \leq C < A < B \leq 2A$;

**Case II.** $2 \leq C < A = B$;

**Case III.** $2 \leq C = A < B \leq 2A$.

In each case, we construct a connected graph $G$ with the properties described in (4.2).

**Case I.** $2 \leq C < A < B \leq 2A$. Let

$$A = C + k \text{ and } B = C + \ell.$$
Since $C < A < B \leq 2A$, it follows that $1 \leq k < \ell \leq C + 2k$. We consider three cases, according to whether $k < \ell < 2k$, $2k \leq \ell < C + 2k$, or $\ell = C + 2k$.

**Case 1.** $k < \ell < 2k$. For each $i$ with $1 \leq i \leq \ell - k$, let $F_i$ be a copy of $H_1$ in Figure 4.1, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,\delta}\}$, where $u_{i,p}$ corresponds to $u_p$ in $H_1$ for $0 \leq p \leq 6$. For each $j$ with $1 \leq j \leq 2k - \ell$, let $G_j$ be a copy of $H_2$ in Figure 4.1 with $\{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}$, where $v_{j,q}$ corresponds to $v_q$ in $H_2$ for $0 \leq q \leq 3$. For each $t$ with $1 \leq t \leq \ell - 1$, let $I_t$ be a copy of $H_2$ in Figure 4.1 with $\{w_{t,0}, w_{t,1}, w_{t,2}, w_{t,3}\}$, where $w_{t,q}$ corresponds to $v_q$ in $H_2$ for $0 \leq q \leq 3$. The graph $G$ is then obtained from the graphs $F_i$, $G_j$ and $I_t$ ($1 \leq i < \ell - k$, $1 \leq j < 2k - \ell$, and $1 \leq t < \ell - 1$) by identifying all vertices $u_{i,0}$, $v_{j,0}$ and $w_{t,0}$ and labeling the identified vertex $v$.

We first show that $\gamma_F(G) = C$. Since $\{v\} \cup \{w_{t,2} : 1 \leq t \leq C - 1\}$ is an $F$-dominating set, $\gamma_F(G) \leq C$. On the other hand, let $c$ be a minimum $F$-coloring of $G$. If $v \in R_c$, then $w_{t,1}$ can be $F$-dominated only by some vertex in $V(I_t) - \{v\}$ for $1 \leq t \leq C - 1$. This implies that $R_c$ contains at least one vertex from each set $V(I_t) - \{v\}$ for $1 \leq t \leq C - 1$. Hence $\gamma_F(G) = |R_c| \geq 1 + (C - 1) = C$. Thus, we may assume $v \notin R_c$.

Observe that

1. each $u_{i,3}$ is only $F$-dominated by a vertex of the set $X_i = \{u_{i,1}, u_{i,4}, u_{i,5}, v\}$,

2. each $u_{i,4}$ is only $F$-dominated by a vertex of $Y_i = \{u_{i,2}, u_{i,4}, u_{i,5}, v\}$, and

3. $(X_i - \{v\}) \cap (Y_i - \{v\}) = \emptyset$.

Since $v \notin R_c$, it follows that $R_c$ contains at least two vertices in each set $V(F_i) - \{v\}$ ($1 \leq i \leq \ell - k$). Because $v_{j,3}$ is only $F$-dominated by a vertex in $\{v_{j,1}, v_{j,2}, v_{j,3}, v\}$ and $v \notin R_c$, it follows that at least one vertex in $\{v_{j,1}, v_{j,2}, v_{j,3}\}$ must be in $R_c$ for $1 \leq j \leq 2k - \ell$. Furthermore, $w_{t,0}$ is only $F$-dominated by a vertex in $\{v, w_{t,0}, w_{t,1}, w_{t,2}\}$. Since $v \notin R_c$, it follows that $R_c$ must contains at least one vertex from $V(I_t) - \{v\}$ for $1 \leq t \leq C - 1$. Hence

$$\gamma_F(G) = |R_c| \geq 2(\ell - k) + (2k - \ell) + (C - 1) = \ell + C - 1 > C,$$
as \( \ell > k \geq 1 \), which is impossible. Thus \( v \in R_c \) and so \( \gamma_F(G) = C \).

Next we show that \( \gamma(G) = A \) and \( \gamma_0(G) = B \). Observe that

\[
S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq \ell - k\} \cup \{v_{j,1} : 1 \leq j \leq 2k - \ell\} \cup \{w_{t,1} : 1 \leq t \leq C - 1\}
\]

is a minimum dominating set of \( G \) and so

\[
\gamma(G) = |S| = 1 + (\ell - k) + (2k - \ell) + (C - 1) = C + k = A.
\]

Furthermore, the set

\[
S_0 = S \cup \{u_{i,2} : 1 \leq i \leq \ell - k\}
\]

is a minimum open dominating set of \( G \) and so

\[
\gamma_0(G) = |S_0| = |S| + \ell - k = (C + k) + \ell - k = C + \ell = B.
\]

**Case 2.** \( 2k \leq \ell < C + 2k \). For each \( i \) with \( 1 \leq i \leq \ell - k \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). For each \( j \) with \( 1 \leq j \leq C + 2k - \ell - 1 \), let \( G_j \) be a copy of \( H_2 \) in Figure 4.1 with \( \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\} \), where \( v_{j,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). The graph \( G \) is then obtained from the graphs \( F_i \) and \( G_j \) for \( 1 \leq i \leq \ell - k \) and \( 1 \leq j \leq C + 2k - \ell - 1 \) by (1) identifying all vertices \( u_{i,0} \) and \( v_{j,0} \) and labeling the identified vertex \( v \) and (2) adding \( C - 1 \) new vertices \( w_t \) (\( 1 \leq t \leq C - 1 \)) and joining each \( w_t \) to \( v \).

We first show that \( \gamma_F(G) = C \). Since \( \{v\} \cup \{w_t : 1 \leq t \leq C - 1\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq C \). On the other hand, let \( c \) be a minimum \( F \)-coloring of \( G \). If \( v \in R_c \), then \( w_t \in R_c \) for \( 1 \leq t \leq C - 1 \) and so \( \gamma_F(G) = |R_c| \geq C \). Thus, we may assume that \( v \notin R_c \). Observe that

1. each \( u_{i,3} \) is only \( F \)-dominated by a vertex of the set \( X_i = \{u_{i,1}, u_{i,3}, u_{i,5}, v\} \),
2. each \( u_{i,4} \) is only \( F \)-dominated by a vertex of \( Y_i = \{u_{i,2}, u_{i,4}, u_{i,6}, v\} \), and
3. \( (X_i - \{v\}) \cap (Y_i - \{v\}) = \emptyset \).
Thus at least two vertices of each $F_i$ ($1 \leq i \leq \ell - k$) must be colored red by $c$. Since $v_{j,3}$ is only $F$-dominated by itself, by $v_{j,2}$, or by $v$ and $v \notin R_c$, it follows that $v_{j,2} \in R_c$ or $v_{j,3} \in R_c$ for $1 \leq j \leq C + 2k - \ell - 1$. Thus,

$$\gamma_F(G) = |R_c| \geq 2(\ell - k) + (C + 2k - \ell - 1) = C + \ell - 1 > C$$

as $\ell > k \geq 1$, which is impossible. Thus $v \in R_c$ and so $\gamma_F(G) = C$.

To show that $\gamma(G) = A$ and $\gamma_o(G) = B$, observe that

$$S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq \ell - k\} \cup \{v_{j,1} : 1 \leq j \leq C + 2k - \ell - 1\}$$

is a minimum dominating set of $G$ and so

$$\gamma(G) = |S| = 1 + (\ell - k) + (C + 2k - \ell - 1) = C + k = A.$$

Furthermore, since the set

$$S_o = S \cup \{u_{i,2} : 1 \leq i \leq \ell - k\}$$

is a minimum open dominating set of $G$, it follows that

$$\gamma_o(G) = |S_o| = |S| + \ell - k = C + \ell = B.$$

**Case 3.** $\ell = C + 2k$. In this case, $B = 2A$. We construct a connected graph $G$ with the desired properties in the following four steps.

1. Let $p \geq 2$ be an integer. For each integer $i$ with $1 \leq i \leq A - C + 1$, let $F_i$ be the graph obtained the path $u_i, y_i, v_i$ by (1) adding $2p$ new vertices $r_{i,j}$ ($1 \leq j \leq 2p$), (2) joining each vertex $r_{i,j}$ ($1 \leq j \leq p$) to $u_i$ and $y_i$, and (3) joining each vertex $r_{i,j}$ ($p + 1 \leq j \leq 2p$) to $y_i$ and $v_i$.

2. The graph $F^*$ is then obtained from the $A - C + 1$ copies of $F_i$ and a new vertex $x$ by (1) joining $x$ to $y_1$ and to each vertex in the set $\{u_i, v_i : 1 \leq i \leq A - C + 1\}$ and (2) joining $v_i$ to $u_{i+1}$ for all $i$ with $1 \leq i \leq A - C + 1$ and $v_{A-C+1}$ to $u_1$.

3. For each integer $t$ with $1 \leq t \leq C - 1$, let $G_t : w_{t,1}, w_{t,2}, w_{t,3}$ be a copy of $P_3$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
(4) Then the graph $G$ is obtained from the graphs $F^*$ and $G_t$ ($1 \leq t \leq C - 1$) by joining each $w_{t,1}$ ($1 \leq t \leq C - 1$) to $x$.

We first show that $\gamma_F(G) = C$. Since $\{x\} \cup \{w_{t,3} : 1 \leq t \leq C - 1\}$ is an $F$-dominating set, $\gamma_F(G) \leq C$. To show that $\gamma_F(G) \geq C$, let $c$ be a minimum $F$-coloring of $G$. By Proposition 2.7, $w_{t,3} \in R_c$ and so $\gamma_F(G) \geq C - 1$. Assume, to the contrary, that $\gamma_F(G) = C - 1$. Then $R_c = \{w_{t,3} : 1 \leq t \leq C - 1\}$. However then, no vertex of the graph $F^*$ is $F$-dominated by any vertex in $R_c$, which is a contradiction. Therefore, $\gamma_F(G) = C$.

Next, we show that $\gamma(G) = A$ and $\gamma_0(G) = 2A$. Observe that

$$S = \{w_{t,2} : 1 \leq t \leq C - 1\} \cup \{y_i : 1 \leq i \leq A - C + 1\}$$

is a minimum dominating set of $G$ and so $\gamma(G) = |S| = A$. Since the set

$$S_0 = \{w_{t,1}, w_{t,2} : 1 \leq t \leq C - 1\} \cup \{u_i, v_i : 1 \leq i \leq A - C + 1\}$$

is a minimum open dominating set of $G$, it follows that $\gamma_0(G) = |S_0| = 2A$.

Case II. $2 \leq C < A = B$. Let

$$A = C + k,$$

where $k \geq 1$.

First, assume that $C = 2$. For each $i$ with $1 \leq i \leq A - 1$, let $G_i$ be a copy of $H_2$ in Figure 4.1 with $\{v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}\}$, where $v_{i,q}$ corresponds to $v_q$ in $H_2$ for $0 \leq q \leq 3$. The graph $G$ is then obtained from the graphs $G_i$ for $1 \leq i \leq A - 1$ by identifying all vertices $v_{i,0}$ and labeling the identified vertex $v$ and adding one new vertex $u$ together with the edge $uv$.

We first show that $\gamma_F(G) = 2$. Since $\{v, u\}$ is an $F$-dominating set, $\gamma_F(G) \leq 2$. On the other hand, let $c$ be a minimum $F$-coloring of $G$. If $v \in R_c$, then $u \in R_c$ since $u$ is an end-vertex that is adjacent to $v$. Thus $|R_c| \geq 2$. So we may assume that $v \notin R_c$. Then, for each $i$ with $1 \leq i \leq A - 1$, the vertex $v_{i,3}$ is only $F$-dominated by itself or by $v_{i,2}$. Hence $R_c$ must contain at least one vertex in each set $\{v_{i,2}, v_{i,3}\}$ for $1 \leq i \leq A - 1$ and so $|R_c| \geq A - 1 \geq C = 2$. Therefore, $\gamma_F(G) = 2$.
Next, observe that

\[ S = \{v\} \cup \{x_{i,1} : 1 \leq i \leq A - 1\} \]

is both a minimum dominating set and a minimum open dominating set of \(G\). Therefore,

\[ \gamma(G) = \gamma_o(G) = |S| = 1 + (A - 1) = A. \]

Now assume that \(C \geq 3\). For each \(i\) with \(1 \leq i \leq C - 2\), let \(X_i\) be the graph obtained from the 5-cycle \(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,5}, x_{i,1}\) by adding a new vertex \(x_{i,0}\) and joining \(x_{i,0}\) to \(x_{i,1}, x_{i,3}\), and \(x_{i,4}\). For each integer \(j\) with \(1 \leq j \leq A - C + 1\), let \(G_j\) be a copy of \(H_2\) in Figure 4.1 with \(\{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}\), where \(v_{j,q}\) corresponds to \(v_q\) in \(H_2\) for \(0 \leq q \leq 3\). Now, let \(G\) be the graph obtained from the graphs \(X_i\) and \(G_j\) for \(1 \leq i \leq C - 2\) and \(1 \leq j \leq A - C + 1\) by (1) identifying all vertices \(x_{i,0}\) and \(v_{j,0}\) and labeling the identified vertex by \(v\) and (2) adding a new vertex \(u\) and the edge \(uv\).

We first show that \(\gamma_F(G) = C\). Since \(\{v, u\} \cup \{x_{i,1} : 1 \leq i \leq C - 2\}\) is an \(F\)-dominating set, \(\gamma_F(G) \leq C\). On the other hand, let \(c\) be a minimum \(F\)-coloring of \(G\). Assume first that \(v \in R_c\). Then \(u \in R_c\) since \(u\) is an end-vertex that is adjacent to the red vertex \(v\). Since \(v \in R_c\), each \(x_{i,1}\) can only be \(F\)-dominated by a vertex in \(V(X_i) - \{v\}\) for \(1 \leq i \leq C - 2\). This implies that \(R_c\) contains at least one vertex from each set \(V(X_i) - \{v\}\) for \(1 \leq i \leq C - 2\). Thus \(|R_c| \geq 2 + (C - 2) = C\). Assume next that \(v \notin R_c\). Then each vertex \(x_{i,2}\) (\(1 \leq i \leq C - 2\)) can only be \(F\)-dominated by a vertex in the set \(V(X_i) - \{v\}\). So \(R_c\) contains at least one vertex from each set \(V(X_i) - \{v\}\) for \(1 \leq i \leq C - 2\). Furthermore, since \(v \notin R_c\), each \(v_{j,3}\) can only be \(F\)-dominated by a vertex in \(V(G_j) - \{v\}\) for \(1 \leq j \leq A - C + 1\). Thus \(R_c\) contains at least one vertex from each set \(V(G_j) - \{v\}\) for \(1 \leq j \leq A - C + 1\). This implies that

\[ \gamma_F(G) = |R_c| \geq (C - 2) + (A - C + 1) = A - 1 \geq C. \]

Therefore, \(\gamma_F(G) = C\). Furthermore, since

\[ S = \{v\} \cup \{x_{i,1} : 1 \leq i \leq C - 2\} \cup \{v_{j,1} : 1 \leq j \leq A - C + 1\} \]
is both a minimum dominating set and a minimum open dominating set of \( G \), it follows that \( \gamma(G) = \gamma_o(G) = |S| = A \).

**Case III.** \( 2 \leq C = A < B \leq 2A \). Let

\[ B = A + \ell, \text{ where } 1 \leq \ell \leq A. \]

We consider two cases, according to whether \( 1 \leq \ell < A \), or \( \ell = A \).

**Case 1.** \( 1 \leq \ell < A \). If \( A = 2 \) and \( B = 3 \), then let \( G \) be the graph obtained from the graph \( H_1 \) by adding a new vertex \( u \) and the edge \( u_0u \). Then \( \{u, u_0\} \) is a minimum \( F \)-dominating set, \( \{u_0, u_4\} \) is a minimum dominating set and \( \{u_0, u_4, u_5\} \) is a minimum open dominating set. Therefore, \( \gamma(G) = \gamma_F(G) = 2 \) and \( \gamma_o(G) = 3 \). Thus, we can consider that \( A \geq 3 \). For each \( i \) with \( 1 \leq i \leq \ell \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). For each \( j \) with \( 1 \leq j \leq A - \ell - 1 \), let \( G_j \) be a copy of \( H_2 \) in Figure 4.1 with \( \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\} \), where \( v_{j,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). The graph \( G \) is then obtained from the graphs \( F_i \) and \( G_j \) for \( 1 \leq i \leq \ell \) and \( 1 \leq j \leq A - \ell - 1 \) by (1) identifying all vertices \( u_{i,0} \) and \( v_{j,0} \) and labeling the identified vertex \( v \) and (2) adding \( A - 1 \) new vertices \( w_t \) \( (1 \leq t \leq A - 1) \) and joining each \( w_t \) to \( v \).

We first show that \( \gamma_F(G) = A \). Since \( \{v\} \cup \{w_t : 1 \leq t \leq A - 1\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq A \). On the other hand, let \( c \) be a minimum \( F \)-coloring of \( G \). If \( v \in R_c \), then \( w_t \in R_c \) for \( 1 \leq t \leq A - 1 \) and so \( \gamma_F(G) = |R_c| \geq A \). Thus, we may assume that \( v \notin R_c \). Observe that

1. each \( u_{i,3} \) is only \( F \)-dominated by a vertex of the set \( X_i = \{u_{i,1}, u_{i,3}, u_{i,5}, v\} \),

2. each \( u_{i,4} \) is only \( F \)-dominated by a vertex of \( Y_i = \{u_{i,2}, u_{i,4}, u_{i,6}, v\} \), and

3. \( (X_i \setminus \{v\}) \cap (Y_i \setminus \{v\}) = \emptyset \).

Thus at least two vertices of each \( F_i \) \( (1 \leq i \leq \ell) \) must be colored red by \( c \). Since \( v_{j,3} \) is only \( F \)-dominated by a vertex in \( \{v_{j,1}, v_{j,2}, v_{j,3}, v\} \) and \( v \notin R_c \), it follows that at least one vertex in \( \{v_{j,1}, v_{j,2}, v_{j,3}\} \) must be in \( R_c \) for \( 1 \leq j \leq A - \ell - 1 \). Thus,
\[ \gamma_F(G) = |R_c| \geq 2\ell + (A - \ell - 1) = A + \ell - 1 \geq A \]

as \( \ell > 0 \). Thus \( \gamma_F(G) \geq A \).

To show that \( \gamma(G) = A \) and \( \gamma_o(G) = B \), observe that

\[ S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq \ell\} \cup \{v_{j,1} : 1 \leq j \leq A - \ell - 1\} \]

is a minimum dominating set of \( G \) and so

\[ \gamma(G) = |S| = 1 + \ell + (A - \ell - 1) = A. \]

Furthermore, since the set

\[ S_o = S \cup \{u_{i,2} : 1 \leq i \leq \ell\} \]

is a minimum open dominating set of \( G \), it follows that

\[ \gamma_o(G) = |S_o| = |S| + \ell = A + \ell = B. \]

Case 2. \( \ell = A \). In this case, \( B = 2A \). We construct a connected graph \( G \) with the desired properties in the following three steps.

(1) Let \( p \geq 2 \) be an integer. Let \( F^* \) be the graph obtained from the path \( u, y, v \) by (a) adding \( 2p \) new vertices \( r_i (1 \leq i \leq 2p) \), (b) joining each vertex \( r_i (1 \leq i \leq p) \) to \( u \) and \( y \), (c) joining each vertex \( r_i (p + 1 \leq i \leq 2p) \) to \( y \) and \( v \), and (d) adding a new vertex \( x \) and joining \( x \) to each vertex in \( \{u, v, y\} \).

(2) For each integer \( j \) with \( 1 \leq j \leq A - 1 \), let \( G_j : w_{j,1}, w_{j,2}, w_{j,3} \) be a copy of \( P_3 \).

(3) Then the graph \( G \) is obtained from the graphs \( F^* \) and \( G_j (1 \leq j \leq A - 1) \) by joining each \( w_{j,1} (1 \leq j \leq A - 1) \) to \( x \).

We first show that \( \gamma_F(G) = A \). Since \( \{x\} \cup \{w_{j,3} : 1 \leq j \leq A - 1\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq A \). To show that \( \gamma_F(G) \geq A \), let \( c \) be a minimum \( F \)-coloring of \( G \). By Proposition 2.7, \( w_{j,3} \in R_c \) and so \( \gamma_F(G) \geq A - 1 \). Assume, to the contrary,
that $\gamma_F(G) = A - 1$. Then $R_c = \{w_{j,3} : 1 \leq j \leq A - 1\}$. However then, no vertex of the graph $F^*$ is $F$-dominated by any vertex in $R_c$, which is a contradiction. Therefore, $\gamma_F(G) = A$.

Next, we show that $\gamma(G) = A$ and $\gamma_0(G) = 2A$. Observe that

$$S = \{y\} \cup \{w_{j,2} : 1 \leq j \leq A - 1\}$$

is a minimum dominating set of $G$ and so $\gamma(G) = |S| = A$. Since the set

$$S_0 = \{x, y\} \cup \{w_{j,1}, w_{j,2} : 1 \leq j \leq A - 1\}$$

is a minimum open dominating set of $G$, it follows that $\gamma_0(G) = |S_0| = 2A$. □

4.4 Realizable Triples of Type II

Recall that a triple $(A, B, C)$ is of type II if $A < B < C$. In this section we determine which triples of type II are realizable. By Proposition 4.2, each triple $(2, 2, C)$ of type II is nonrealizable for $C > 3$. In fact, more can be said.

**Proposition 4.7.** Let $k \geq 2$ be an integer. If $G$ is a connected graph with $\gamma(G) = \gamma_0(G) = k$, then $\gamma_F(G) \leq k$ and so no triple $(k, k, C)$ is realizable for $C > k$.

**Proof.** Let $G$ be a connected graph with $\gamma(G) = \gamma_0(G) = k$ and let $S$ be a minimum open dominating set of $G$. Necessarily $S$ is also a minimum dominating set.

Let $v_1 \in S$. Since $S$ is a minimum dominating set and $G$ is connected, there exists $u_1 \in S$ such that $u_1$ is dominated by $v_1$. Since $S$ is a minimum dominating set and $|S| = k \geq 2$, there is $u_2 \in S$ that is not dominated by $v_1$. Let $v_2 \in S$ such that $v_2$ dominates $u_2$. If $k \geq 3$, then there is $u_3 \in S$ that is not dominated by any vertex in $\{v_1, v_2\}$. Let $v_3 \in S$ such that $v_3$ dominates $u_3$. Continuing in this manner, we arrive at the set $U = \{v_1, v_2, \ldots, v_k\}$. (Note that $S = \{v_1, v_2, \ldots, v_k\}$ and $u_i$ and $v_i$ are adjacent for $1 \leq i \leq k$).

We claim that $U$ is an $F$-dominating set of $G$. Let $x \in V(G)$. If $x = u_i$ for $1 \leq i \leq k$, then $x$ is $F$-dominated by itself. If $x = v_i$ for some $i$ ($1 \leq i \leq k$), then since
$S$ is a minimum open dominating set of $G$, there is a $v_j \in S$ that is adjacent to $u_i$ (and $v_j$ is adjacent to $u_j$). Then $u_i$ is $F$-dominated by $u_j$. Otherwise, $x \notin U \cup S$. Since $S$ is a dominating set, $x$ is adjacent to some vertex $v_i$ ($1 \leq i \leq k$) (and $v_i$ is adjacent to $u_i$). Then $x$ is $F$-dominated by $u_i$. Thus, $U$ is an $F$-dominating set of $G$, as claimed. Hence $\gamma_F(G) \leq |U| = k$. Therefore, $(k, k, C)$ is nonrealizable for any $C > k$. □

**Theorem 4.8.** Every triple $(A, B, C)$ of type II with $B = 2A$ is realizable.

**Proof.** By Proposition 4.4, every triple $(1, 2, C)$ is realizable for each positive integer $C$. Thus we may assume that $A \geq 2$. Let $P : v_1, v_2, \ldots, v_{3A-2}$ be a path of order $3A - 2$ and let $G$ be the caterpillar obtained from $P$ by adding $C - A - 1 \geq 1$ pendant edges at each vertex $v_{3i+1}$ for $0 \leq i \leq A - 1$. For $A = 2, 3, 4$, the graph $G$ is drawn in Figure 4.5.

**Figure 30.** The graph $G$ for $A = 2, 3, 4$.

For each vertex $v_{3i+1}$ ($0 \leq i \leq A - 1$), let $W_i = N(v_{3i+1}) - V(P)$. We show that $\gamma_F(G) = C$. Since

$$S = W_0 \cup \{v_1\} \cup \{v_{3i+2} : 0 \leq i \leq A - 2\} \cup \{w_{A-1}\},$$

where $w_{A-1} \in W_{A-1}$, is an $F$-dominating set, $\gamma_F(G) \leq |S| = C$. To show that $\gamma_F(G) \geq C$, let $c$ be a minimum $F$-coloring of $G$.

First, we show that if $v_1 \in R_c$, then $|R_c| \geq C$. Suppose that $v_1 \in R_c$. Then necessarily, $W_0 \subseteq R_c$. We verify the following two claims.
Claim 1. At least one vertex in \( \{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1} \) must be red for each \( i \) with \( 0 \leq i \leq A - 3 \). Assume, to the contrary, that each vertex in \( \{v_{3j+2}, v_{3j+3}, v_{3j+4}\} \cup W_{j+1} \) is blue for some \( j \) with \( 0 \leq j \leq A - 3 \). Then a vertex in \( W_{j+1} \) can only be \( F \)-dominated by \( v_{3j+5} \) and so \( v_{3j+5} \in R_c \). However then, \( v_{3j+4} \) is not \( F \)-dominated by any vertex in \( R_c \), a contradiction. Therefore, at least one vertex in \( \{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1} \) is red for \( 0 \leq j \leq A - 3 \).

Claim 2. At least two vertices in \( \{v_{3A-4}, v_{3A-3}, v_{3A-2}\} \cup W_{A-1} \) must be red. Since \( u_{A-1} \in W_{A-1} \) is only \( F \)-dominated by \( v_{3A-3} \) or by a vertex in \( W_{A-1} \), either \( v_{3A-3} \) is red or some vertex in \( W_{A-1} \) is red. Furthermore, since \( v_{3A-2} \) is only \( F \)-dominated by \( v_{3A-3} \) or by \( v_{3A-4} \), it follows that \( v_{3A-2} \in R_c \) or \( v_{3A-4} \in R_c \). Therefore, at least two vertices in \( \{v_{3A-4}, v_{3A-3}, v_{3A-2}\} \cup W_{A-1} \) are red.

Since \( \{v_1\} \cup W_0 \subseteq R_c \), it then follows by Claims 1 and 2 that

\[
\gamma_F(G) = |R_c| \geq 1 + |W_0| + (A - 2) + 2 = 1 + (C - A - 1) + (A - 2) + 2 = C.
\]

Therefore, if \( v_1 \in R_c \), then \( |R_c| \geq C \). Next, assume that \( v_1 \notin R_c \). We now consider two cases.

Case 1. Suppose that \( v_{3i+1} \in R_c \) for some \( i \) (\( 0 \leq i \leq A - 1 \)). Let \( j \) be the smallest integer \( i \) such that \( v_{3i+1} \in R_c \). If \( j = 0 \), then \( v_1 \in R_c \) and we have seen that \( |R_c| \geq C \). Hence, we may assume that \( 1 \leq j \leq A - 1 \). Thus \( v_{3j+1} \in R_c \) and \( W_j \subseteq R_c \). If \( j < A - 1 \) and \( j \leq i \leq A - 2 \), then an argument similar to the situation where \( v_1 \in R_c \) shows that at least one vertex in \( \{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1} \) must be red. We now show that if \( 0 \leq i \leq j - 1 \), then some vertex in \( \{v_{3i+1}, v_{3i+2}, v_{3i+3}\} \cup W_i \) is red. If \( j \geq 2 \), then we first consider \( v_{3i+1} \), where \( 1 \leq i < j \). Thus \( v_{3i+1} \) is blue and is either \( F \)-dominated by \( v_{3i+3} \) or by \( v_{3i-1} \). If \( v_{3i+1} \) is \( F \)-dominated by \( v_{3i+3} \), then \( v_{3i+3} \in R_c \). If \( v_{3i+1} \) is \( F \)-dominated by \( v_{3i-1} \), then \( v_{3i-1} \in R_c \) and \( v_{3i} \) is blue. Hence either \( v_{3i+2} \in R_c \) or \( w_i \in R_c \) for some \( w_i \in W_i \). For \( i = 0 \), the blue vertex \( v_1 \) can only be \( F \)-dominated by \( v_3 \) and the blue vertex \( v_2 \) can only be \( F \)-dominated by a vertex in \( W_0 \). Thus at least two vertices in
\{v_1, v_2, v_3\} \cup W_0 \text{ must be red, which implies that}

\[ \gamma_F(G) = |R_c| \geq 2 + (j - 1) + 1 + (C - A - 1) + (A - 2 - j + 1) = C. \]

**Case 2.** Suppose that \( v_{3i+1} \) is blue for every integer \( i \) (0 \leq i \leq A - 1). We claim that \( v_{3i+1} \) is blue and \( v_{3i+3} \) is red for every integer \( i \) (0 \leq i \leq A - 2). We verify this by induction. First, because \( v_1 \) is blue, \( v_1 \) can only be \( F \)-dominated by \( v_3 \) and so \( v_3 \in R_c \). In addition, this says that \( v_2 \) is blue and so some vertex in \( W_0 \) is red. Assume that \( v_{3k+1} \) is blue and \( v_{3k+3} \) is red, where 0 \leq k < A - 2. By the assumption in Case 2, \( v_{3k+4} \) is blue. Since \( v_{3k+4} \) is blue and \( v_{3k+3} \) is red, \( v_{3k+4} \) can only be \( F \)-dominated by \( v_{3k+6} \) and so \( v_{3k+6} \) is red. This verifies the claim. Thus \( v_{3(A-2)+3} = v_{3A-3} \) is red. If \( v_{3A-3} \) is blue, then \( v_{3A-2} \) is not \( F \)-dominated by any vertex. Hence \( v_{3A-2} \in R_c \) and so \( W_{A-1} \subseteq R_c \) as well. Therefore,

\[ \gamma_F(G) = |R_c| \geq (A - 1) + 1 + (C - A - 1) = C, \]

as desired.

Furthermore, since \( \{v_{3i+1} : 0 \leq i \leq A - 1\} \) is a minimum domination set and for \( w_i \in W_i \) with 0 \leq i \leq A - 1, \( \{v_{3i+1}, w_i : 0 \leq i \leq A - 1\} \) is a minimum open domination set, \( \gamma(G) = A \) and \( \gamma_0(G) = 2A = B. \)

It remains to consider those triples \((A, B, C)\) of type II with \( B \neq 2A \). First, we establish some additional definitions and notation. For each integer \( \alpha \geq 1 \), let \( L_\alpha \) be the graph obtained from the 4-cycle \( w, x, y, z, w \) by (1) adding \( \alpha \) new vertices \( w_1, w_2, \ldots, w_\alpha \) and joining each \( w_i \) (1 \leq i \leq \alpha) to \( w \) and (2) adding \( \alpha \) new vertices \( y_1, y_2, \ldots, y_\alpha \) and joining each \( y_i \) (1 \leq i \leq \alpha) to \( y \). (Note that the graph \( L_\alpha \) also appeared in Chapter 3 and is redrawn in Figure 4.6.) By Lemma 3.5, we have \( \{w, y\} \) is a minimum dominating set, \( \{w, x, y, w\} \) is a minimum open dominating set, and \( \{w, x\} \cup \{w_i : 1 \leq i \leq \alpha\} \) is a minimum \( F \)-dominating set. Therefore,

\[ \gamma(L_\alpha) = 2, \quad \gamma_0(L_\alpha) = 3, \quad \text{and} \quad \gamma_F(L_\alpha) = \alpha + 2 \quad (4.4) \]

for every integer \( \alpha \geq 1 \).
Theorem 4.9. Let \((A, B, C)\) be a triple of type II such that \(B \neq 2A\). If \((A, B, C) \neq (k, k, C)\) where \(C > k > 2\), then \((A, B, C)\) is realizable.

**Proof.** Since \(B \neq 2A\), it follows that \(A \geq 2\). By Corollary 4.3 and Proposition 4.7, it suffices to consider the following two cases:

**Case I.** \(2 \leq A < B < C\);

**Case II.** \(2 \leq A < B = C\).

In each case, we construct a connected graph with the properties in (4.2).

**Case I.** \(2 \leq A < B < C\). Note that \(C \geq B + 1 \geq 3\). If \(A = 2\), then \(B = 3\), and \(C \geq 4\). Then the graph \(L_{C-2}\) has the desired properties by (4.4). Thus, we may assume that \(A \geq 3\), and so \(B \geq 4\) and \(C \geq 5\). Let

\[ B = A + k \text{ and } C = A + \ell. \]

Since \(A < B < 2A\) and \(B < C\), it follows that \(1 \leq k < A\) and \(k < \ell\). We consider three cases, according to whether \(k = 1\), \(2 \leq k \leq A - 2\), or \(k = A - 1\).

**Case 1.** \(k = 1\). For each \(i\) with \(1 \leq i \leq A - 2\), let \(G_i\) be a copy of \(H_2\) in Figure 4.1 with \(\{v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}\}\), where \(v_{i,q}\) corresponds to \(v_q\) in \(H_2\) for \(0 \leq q \leq 3\). Let \(G\) be the graph obtained from a copy of the graph \(L_{C-2}\) and the \(A - 2\) copies of \(G_i\) by identifying all the vertices \(v_{i,0}\) and \(w\) and calling the new vertex \(v\).

We first show that \(\gamma_F(G) \leq C\). Let

\[ S = \{v, x\} \cup \{w_j : 1 \leq j \leq C - 2\}. \]
Define a red-blue coloring $c^*$ of $G$ by assigning red to each vertex in $S$ and blue to the remaining vertices of $G$. Observe that (i) the blue vertex $y$ is adjacent to the blue vertex $z$ and $z$ is adjacent to the red vertex $v$, (ii) the blue vertex $z$ is adjacent to the blue vertex $y$ and $y$ is adjacent to the red vertex $x$, and (iii) the blue vertex $y_j (1 \leq j \leq C - 2)$ is adjacent to the blue vertex $y$ and $y$ is adjacent to the red vertex $x$. Thus, every blue vertex $u$ of $G$ belongs to a copy of $F$ rooted at $u$, implying that $c^*$ is a $F$-coloring of $G$. Therefore, $\gamma_F(G) \leq |R_c| = |S| = C$.

Next, we show that $\gamma_F(G) \geq C$. Assume, to the contrary, that $\gamma_F(G) \leq C - 1$. Let there be given a minimum $F$-coloring $c$ of $G$. Observe that the vertex $y$ is only $F$-dominated by a vertex of the set $\{v, y\}$. Thus at least one of $v$ and $y$ belongs to $R_c$. By Lemma 2.6, if $v \in R_c$, then $w_j \in R_c$, for $1 \leq j \leq C - 2$. Thus $|R_c| \geq C - 1$ and so $\gamma_F(G) \geq C - 1$. Therefore, $\gamma_F(G) = C - 1$ and $R_c = \{v, w_1, w_2, \ldots, w_{C-2}\}$. Then the blue vertex $x$ does not belong to a copy of $F$ rooted at $x$, which is a contradiction. Similarly, if $y \in R_c$, then $y_j \in R_c$, for $1 \leq j \leq C - 2$. Thus $|R_c| \geq C - 1$ and so $\gamma_F(G) \geq C - 1$. Therefore, $\gamma_F(G) = C - 1$ and $R_c = \{y, y_1, y_2, \ldots, y_{C-2}\}$. Then the blue vertex $x$ does not belong to a copy of $F$ rooted at $x$, which is a contradiction. Therefore, $\gamma_F(G) = C$.

Next we show that $\gamma(G) = A$ and $\gamma_0(G) = B$. Observe that

$$S = \{v, y\} \cup \{v_i : 1 \leq i \leq A - 2\}$$

is a minimum dominating set of $G$ and so

$$\gamma(G) = |S| = 2 + (A - 2) = A.$$ 

Furthermore, the set

$$S_o = S \cup \{x\}$$

is a minimum open dominating set of $G$ and so

$$\gamma_0(G) = |S_o| = |S| + 1 = B.$$ 

Case 2. $2 \leq k \leq A - 2$. For each $i$ with $1 \leq i \leq k - 1$, let $F_i$ be a copy of $H_1$ in Figure 4.1, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}$, where $u_{i,p}$ corresponds to $u_p$ in $H_1$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
for $0 \leq p \leq 6$. For each $j$ with $1 \leq j \leq A - k - 1$, let $G_j$ be a copy of $H_2$ in Figure 4.1 with $\{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}$, where $v_{j,q}$ corresponds to $v_q$ in $H_2$ for $0 \leq q \leq 3$. The graph $G$ is then obtained from the graph $L_{C-2}$ and the graphs $F_i, G_j$ for $1 \leq i \leq k - 1$ and $1 \leq j \leq A - k - 1$ by identifying all vertices $u_{i,0}, u_{i,0}$ and $w$ and labeling the identified vertex $v$.

Since $\{v, x\} \cup \{w_t : 1 \leq t \leq C - 2\}$ is an $F$-dominating set, $\gamma_F(G) \leq C$. On the other hand, let $c$ be a red-blue coloring of $G$. Since $y$ is only $F$-dominated by itself or by $v$, it follows that either $v \in R_c$ or $y \in R_c$. Thus either $W_1 = \{v\} \cup \{w_t : 1 \leq t \leq C - 2\} \subseteq R_c$ or $W_2 = \{y\} \cup \{w_t : 1 \leq t \leq C - 2\} \subseteq R_c$, and so $\gamma_F(G) \geq C - 1$. Assume, to the contrary, that $\gamma_F(G) = C - 1$. Then either $R_c = W_1$ or $R_c = W_2$. In each case, $x$ is not $F$-dominated by any vertex in $R_c$, which is a contradiction. Therefore $\gamma_F(G) = C$.

Since $S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\} \cup \{v_{j,1} : 1 \leq j \leq A - k - 1\}$ is a minimum dominating set of $G$, it follows that

$$\gamma(G) = |S| = 2 + (k - 1) + (A - k - 1) = A.$$ 

Furthermore, the set $S_o = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\}$ is a minimum open dominating set of $G$, it follows that

$$\gamma_o(G) = |S_o| = |S| + 1 + (k - 1) = A + k = B.$$ 

**Case 3.** $k = A - 1$. Then $B = A + k = 2k + 1$ and $C = A + \ell = k + \ell + 1$.

For each $i$ with $1 \leq i \leq k - 1$, let $F_i$ be a copy of $H_1$ in Figure 4.1, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}$, where $u_{i,p}$ corresponds to $u_p$ in $H_1$ for $0 \leq p \leq 6$. The graph $G$ is then obtained from the graph $L_{C-2}$ and the graphs $F_i$ for $1 \leq i \leq k - 1$ by identifying all vertices $u_{i,0}$ and $w$ and labeling the identified vertex $v$.

Since $\{v, x\} \cup \{w_j : 1 \leq j \leq C - 2\}$ is an $F$-dominating set, $\gamma_F(G) \leq C$. On the other hand, let $c$ be a red-blue coloring of $G$. Since $y$ is only $F$-dominated by itself or by $v$, 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
it follows that either \( v \in R_c \) or \( y \in R_c \). Thus either \( W_1 = \{v\} \cup \{w_j : 1 \leq j \leq C - 2\} \subseteq R_c \) or \( W_2 = \{y\} \cup \{y_j : 1 \leq j \leq C - 2\} \subseteq R_c \), and so \( \gamma_F(G) \geq C - 1 \). Assume, to the contrary, that \( \gamma_F(G) = C - 1 \). Then either \( R_c = W_1 \) or \( R_c = W_2 \). In each case, \( x \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Therefore \( \gamma_F(G) = C \).

Since
\[
S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\}
\]
is a minimum dominating set of \( G \), it follows that
\[
\gamma(G) = |S| = 2 + (k - 1) = k + 1 = A.
\]
Furthermore, the set
\[
S_0 = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\}
\]
is a minimum open dominating set of \( G \), it follows that
\[
\gamma_0(G) = |S_0| = |S| + 1 + (k - 1) = (k + 1) + 1 + (k - 1) = 2k + 1 = B.
\]

**Case II.** \( 2 \leq A < B = C \). If \( A = 2 \), then \( B = 3 \) and the graph \( L_{B-2} \) has the desired properties by (4.4). Thus, we may assume that \( A \geq 3 \). Let
\[
B = A + k.
\]
Since \( A < B < 2A \), it follows that \( 1 \leq k < A \). We consider three cases, according to whether \( k = 1, 2 \leq k \leq A - 2 \), or \( k = A - 1 \).

**Case 1.** \( k = 1 \). For each \( i \) with \( 1 \leq i \leq A - 2 \), let \( G_i \) be a copy of \( H_2 \) in Figure 4.1 with \( \{v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}\} \), where \( v_{i,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). Let \( G \) be the graph obtained from a copy of the graph \( L_{B-2} \) and the \( A - 2 \) copies of \( G_i \) by identifying all the vertices \( v_{i,0} \) and \( w \) and calling the new vertex \( v \).

We first show that \( \gamma_F(G) \leq B \). Let
\[
S = \{v, x\} \cup \{w_j : 1 \leq j \leq B - 2\}.
\]
Define a red-blue coloring $c^*$ of $G$ by assigning red to each vertex in $S$ and blue to the remaining vertices of $G$. Observe that (i) the blue vertex $y$ is adjacent to the blue vertex $z$ and $z$ is adjacent to the red vertex $v$, (ii) the blue vertex $z$ is adjacent to the blue vertex $y$ and $y$ is adjacent to the red vertex $x$, and (iii) the blue vertex $y_j$ $(1 \leq j \leq B - 2)$ is adjacent to the blue vertex $y$ and $y$ is adjacent to the red vertex $x$. Thus, every blue vertex $u$ of $G$ belongs to a copy of $F$ rooted at $u$, implying that $c^*$ is an $F$-coloring of $G$. Therefore, $\gamma_F(G) \leq |R_c| = |S| = B$.

Next, we show that $\gamma_F(G) \geq B$. Assume, to the contrary, that $\gamma_F(G) \leq B - 1$. Let there be given a minimum $F$-coloring $c$ of $G$. Observe that the vertex $y$ is only $F$-dominated by a vertex of the set $\{v, y\}$. Thus at least one of $v$ and $y$ belongs to $R_c$. By Lemma 2.6, if $v \in R_c$, then $w_j \in R_c$, for $1 \leq j \leq B - 2$. Thus $|R_c| \geq B - 1$ and so $\gamma_F(G) \geq B - 1$. Therefore, $\gamma_F(G) = B - 1$. If $R_c = \{v, w_1, w_2, \ldots, w_{B-2}\}$, then the blue vertex $x$ does not belong to a copy of $F$ rooted at $x$, which is a contradiction. Similarly, if $y \in R_c$, then $y_j \in R_c$, for $1 \leq j \leq B - 2$. Thus $|R_c| \geq B - 1$ and so $\gamma_F(G) \geq B - 1$. Therefore, $\gamma_F(G) = B - 1$. If $R_c = \{y, y_1, y_2, \ldots, y_{B-2}\}$, then the blue vertex $x$ does not belong to a copy of $F$ rooted at $x$, which is a contradiction. Therefore, $\gamma_F(G) = B$.

Next we show that $\gamma(G) = A$ and $\gamma_o(G) = B$. Observe that

$$S = \{v, y\} \cup \{u_i : 1 \leq i \leq A - 2\}$$

is a minimum dominating set of $G$ and so

$$\gamma(G) = |S| = 2 + (A - 2) = A.$$ 

Furthermore, the set

$$S_o = S \cup \{x\}$$

is a minimum open dominating set of $G$ and so

$$\gamma_o(G) = |S_o| = |S| + 1 = B.$$ 

Case 2. $2 \leq k \leq A - 2$. For each $i$ with $1 \leq i \leq k - 1$, let $F_i$ be a copy of $H_1$ in Figure 4.1, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}$, where $u_{i,p}$ corresponds to $v_p$ in $H_1$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
for $0 \leq p \leq 6$. For each $j$ with $1 \leq j \leq A - k - 1$, let $G_j$ be a copy of $H_2$ in Figure 4.1 with \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}, where $v_{j,q}$ corresponds to $v_q$ in $H_2$ for $0 \leq q \leq 3$. The graph $G$ is then obtained from the graph $L_{6-2}$ and the graphs $F_i, G_j$ for $1 \leq i \leq k - 1$ and $1 \leq j \leq A - k - 1$ by identifying all vertices $u_{i,0}, v_{j,0}$ and $w$ and labeling the identified vertex $v$.

Since $\{v, x\} \cup \{w_t : 1 \leq t \leq B - 2\}$ is an $F$-dominating set, $\gamma_F(G) \leq B$. On the other hand, let $c$ be a red-blue coloring of $G$. Since $y$ is only $F$-dominated by itself or by $v$, it follows that either $v \in R_c$ or $y \in R_c$. Thus either $W_1 = \{v\} \cup \{w_t : 1 \leq t \leq B - 2\} \subseteq R_c$ or $W_2 = \{y\} \cup \{y_t : 1 \leq t \leq B - 2\} \subseteq R_c$, and so $\gamma_F(G) \geq B - 1$. Assume, to the contrary, that $\gamma_F(G) = B - 1$. Then either $R_c = W_1$ or $R_c = W_2$. In each case, $x$ is not $F$-dominated by any vertex in $R_c$, which is a contradiction. Therefore $\gamma_F(G) = B$. Since

\[
S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\} \cup \{v_{j,1} : 1 \leq j \leq A - k - 1\}
\]
is a minimum dominating set of $G$, it follows that

\[
\gamma(G) = |S| = 2 + (k - 1) + (A - k - 1) = A.
\]

Furthermore, the set

\[
S_0 = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\}
\]
is a minimum open dominating set of $G$, it follows that

\[
\gamma_0(G) = |S_0| = |S| + 1 + (k - 1) = A + k = B.
\]

Case 3. $k = A - 1$. Then $B = A + k = 2k + 1$. For each $i$ with $1 \leq i \leq k - 1$, let $F_i$ be a copy of $H_1$ in Figure 4.1, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}$, where $u_{i,p}$ corresponds to $u_p$ in $H_1$ for $0 \leq p \leq 6$. The graph $G$ is then obtained from the graph $L_{6-2}$ and the graphs $F_i$ for $1 \leq i \leq k - 1$ by identifying all vertices $u_{i,0}$ and $w$ and labeling the identified vertex $v$.

Since $\{v, x\} \cup \{w_j : 1 \leq j \leq B - 2\}$ is an $F$-dominating set, $\gamma_F(G) \leq B$. On the other hand, let $c$ be a red-blue coloring of $G$. Since $y$ is only $F$-dominated by itself or by $v$,
it follows that either \( v \in R_c \) or \( y \in R_c \). Thus either \( W_1 = \{v\} \cup \{w_j : 1 \leq j \leq B - 2\} \subseteq R_c \) or \( W_2 = \{y\} \cup \{y_j : 1 \leq j \leq B - 2\} \subseteq R_c \), and so \( \gamma_F(G) \geq B - 1 \). Assume, to the contrary, that \( \gamma_F(G) = B - 1 \). Then either \( R_c = W_1 \) or \( R_c = W_2 \). In each case, \( x \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Therefore \( \gamma_F(G) = B \).

Since
\[
S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\}
\]
is a minimum dominating set of \( G \), it follows that
\[
\gamma(G) = |S| = 2 + (k - 1) = k + 1 = A.
\]
Furthermore, the set
\[
S_o = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\}
\]
is a minimum open dominating set of \( G \), it follows that
\[
\gamma_o(G) = |S_o| = |S| + 1 + (k - 1) = (k + 1) + 1 + (k - 1) = 2k + 1 = B.
\]
This completes the proof. \( \square \)

Combining Proposition 4.7 and Theorems 4.8 and 4.9, we have the following.

**Corollary 4.10.** A triple \((A, B, C)\) of type II is realizable if and only if \((A, B, C) \neq (k, k, C)\) for any integers \(k\) and \(C\) with \(C > k \geq 2\).

### 4.5 Realizable Triples of Type III

Recall that a triple \((A, B, C)\) is of type III if \( A \leq C \leq B \). In this section we show that every triple \((A, B, C)\) is of type III is realizable, beginning with those triples for which \( B = 2A \).

**Theorem 4.11.** Every triple \((A, B, C)\) of type III with \( B = 2A \) is realizable.

**Proof.** By Proposition 4.4, \((1,2,1)\) and \((1,2,2)\) are realizable. Thus, we may assume that \( A \geq 2 \). First, suppose that \( A = C \). Let \( G \) be the graph obtained from the
cycle $C_{3,A} : v_1, v_2, \ldots, v_{3A}, v_1$ by adding the pendant edge $u_iv_{3i+1}$ for $0 \leq i \leq A - 1$. We show that $\gamma_{F}(G) = A$. Since $\{v_{3i+2} : 0 \leq i \leq A - 1\}$ is an $F$-dominating set, $\gamma_{F}(G) \leq A$.

Now let $c$ be a minimum $F$-coloring of $G$. We claim that $R_c$ contains at least one vertex in $V_i = \{u_i, v_{3i+1}, v_{3i+2}, v_{3i+3}\}$ for each $i$ with $0 \leq i \leq A - 1$. Assume, to the contrary, that $R_c \cap V_j = \emptyset$ for some $j$ ($0 \leq j \leq A - 1$). Then $u_j$ can only be $F$-dominated by $v_{3j}$, where $j$ is expressed as an integer modulo $A$ and so $v_{3j} \in R_c$. However then, the blue vertex $v_{3j+1}$ is not $F$-dominated by any vertex in $R_c$, a contradiction. Therefore, $R_c$ contains at least one vertex in $V_i = \{u_i, v_{3i+1}, v_{3i+2}, v_{3i+3}\}$ for $0 \leq i \leq A - 1$, implying that $\gamma_{F}(G) = |R_c| \geq A$. Therefore, $\gamma_{F}(G) = A$. Observe that $S = \{v_{3i+1} : 0 \leq i \leq A - 1\}$ is a minimum dominating set and $S \cup \{u_i : 0 \leq i \leq A - 1\}$ is a minimum open dominating set; so $\gamma(G) = A$ and $\gamma_0(G) = 2A$.

Next, suppose that $A < C$. If $C \geq A + 2$, then $C - A - 1 \geq 1$. Let $G$ be the graph constructed in Theorem 4.8, that is, let $G$ be the caterpillar obtained from the path $P : v_1, v_2, \ldots, v_{3A-2}$ of order $3A - 2$ by adding $C - A - 1 \geq 1$ pendant edges at each vertex $v_{3i+1}$ for $0 \leq i \leq A - 1$. For $A = 2, 3, 4$, the graph $G$ is drawn in Figure 4.7.

![Figure 32. The graph $G$ for $A = 2, 3, 4$.](image)

For each vertex $v_{3i+1}$ ($0 \leq i \leq A - 1$), let $W_i = N(v_{3i+1}) - V(P)$. For $w_{A-1} \in W_{A-1}$,

$$ S = W_0 \cup \{v_1\} \cup \{v_{3i+2} : 0 \leq i \leq A - 2\} \cup \{w_{A-1}\}, $$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
is a minimum $F$-dominating set by the proof of Theorem 4.8. Thus $\gamma_F(G) = |S| = C$. Furthermore, since $\{v_{3i+1} : 0 \leq i \leq A - 1\}$ is a minimum domination set and for $w_i \in W_i$ with $0 \leq i \leq A - 1$, the set $\{v_{3i+1}, w_i : 0 \leq i \leq A - 1\}$ is a minimum open domination set. Therefore, $\gamma(G) = A$ and $\gamma_0(G) = 2A = B$.

Thus, we may assume that $C = A + 1$. Let $P : v_1, v_2, \ldots, v_{3A-2}$ be a path of order $3A - 2$ and let $H$ be the caterpillar obtained from $P$ by adding three pendant edges at each vertex $v_{3i+1}$ for $0 \leq i \leq A - 1$. For each vertex $v_{3i+1}$ ($0 \leq i \leq A - 1$), let $W_i = N(v_{3i+1}) - V(P)$. The graph $G$ is then obtained from $H$ by joining two vertices in $W_{A-1}$. For $A = 2, 3, 4$, the graph $G$ is drawn in Figure 4.8.

![Figure 33. The graph $G$ for $A = 2, 3, 4$.](image)

First, we show that $\gamma_F(G) = A + 1$. For $w_i \in W_i$ for $i = 0, A - 1$, where $\deg w_{A-1} = 2$, $S = \{w_0, w_{A-1}\} \cup \{v_{3i} : 1 \leq i \leq A - 1\}$ is an $F$-dominating set and so $\gamma_F(G) \leq |S| = A + 1$. To show that $\gamma_F(G) \geq A + 1$, let $c$ be a minimum $F$-coloring of $G$.

First, we show that if $v_1 \in R_c$, then $|R_c| > A + 1 = C$. Suppose that $v_1 \in R_c$. Then necessarily, $W_0 \subseteq R_c$. We verify the following claim.

**Claim.** At least one vertex in $\{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1}$ must be red for each $i$ with $0 \leq i \leq A - 2$. Assume, to the contrary, that each vertex in $\{v_{3j+2}, v_{3j+3}, v_{3j+4}\} \cup W_{j+1}$ is blue for each $j$ with $0 \leq j \leq A - 2$. Then $\gamma_0(G) \leq 2A = B$. This contradicts the assumption that $\gamma_0(G) = 2A = B$. Therefore, at least one vertex in $\{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1}$ must be red for each $i$ with $0 \leq i \leq A - 2$. This completes the proof.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
$W_{j+1}$ is blue for some $j$ with $0 \leq j \leq A - 2$. First suppose that $0 \leq j \leq A - 3$. Then a vertex in $W_{j+1}$ can only be $F$-dominated by $v_{3j+5}$ and so $v_{3j+5} \in R_c$. However then, $v_{3j+4}$ is not $F$-dominated by any vertex in $R_c$, a contradiction. Next suppose that $j = A - 2$. Then $w_{A-1} \in W_{A-1}$ can only be $F$-dominated by $v_{3A-3}$ or by a vertex in $W_{A-1}$ and so either $v_{3A-3}$ is red or some vertex in $W_{A-1}$ is red.

Since $\{v_1\} \cup W_0 \subseteq R_c$, it then follows by the claim

$$\gamma_F(G) = |R_c| \geq 1 + |W_0| + (A - 1)$$
$$= 1 + 3 + (A - 1) = A + 3 > A + 1 = C.$$ 

Therefore, if $v_1 \in R_c$, then $|R_c| > C$. Since this is impossible, it follows that $v_1$ is blue.

We now consider two cases.

**Case 1.** $v_{3i+1} \in R_c$ for some $i$ $(1 \leq i \leq A - 1)$. Let $j$ be the smallest integer $i$ such that $v_{3i+1} \in R_c$. Thus $v_{3j+1} \in R_c$ and $W_j \subseteq R_c$. If $j < A - 1$ and $j \leq i \leq A - 2$, then an argument similar to the situation where $v_1 \in R_c$ shows that at least one vertex in $\{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1}$ must be red. We now show that if $0 \leq i \leq j - 1$, then some vertex in $\{v_{3i+1}, v_{3i+2}, v_{3i+3}\} \cup W_i$ is red. If $j \geq 2$, then we first consider $v_{3i+1}$, where $1 \leq i < j$. Thus $v_{3i+1}$ is blue and is either $F$-dominated by $v_{3i+3}$ or by $v_{3i-1}$. If $v_{3i+1}$ is $F$-dominated by $v_{3i+3}$, then $v_{3i+3} \in R_c$. If $v_{3i+1}$ is $F$-dominated by $v_{3i-1}$, then $v_{3i-1} \in R_c$ and $v_{3i}$ is blue. Hence either $v_{3i+2} \in R_c$ or $w_i \in R_c$ for some $w_i \in W_i$. For $i = 0$, the blue vertex $v_1$ can only be $F$-dominated by $v_3$ and the blue vertex $v_2$ can only be $F$-dominated by a vertex in $W_0$. Thus at least two vertices in $\{v_1, v_2, v_3\} \cup W_0$ must be red, which implies that

$$\gamma_F(G) = |R_c| \geq 2 + (j - 1) + 1 + 3 + (A - 2 - j + 1) = A + 4 > A + 1 = C.$$ 

Thus Case 1 cannot occur.

**Case 2.** $v_{3i+1}$ is blue for every integer $i$ $(0 \leq i \leq A - 1)$. We claim that $v_{3i+1}$ is blue and $v_{3i+3}$ is red for every integer $i$ $(0 \leq i \leq A - 2)$. We verify this by induction. First, because $v_1$ is blue, $v_1$ can only be $F$-dominated by $v_3$ and so $v_3 \in R_c$. In addition,
this says that $v_2$ is blue and some vertex in $W_0$ is red. Assume that $v_{3k+1}$ is blue and $v_{3k+3}$ is red, where $0 \leq k < A - 2$. By the assumption in Case 2, $v_{3k+4}$ is blue. Since $v_{3k+4}$ is blue and $v_{3k+3}$ is red, $v_{3k+4}$ can only be $F$-dominated by $v_{3k+6}$ and so $v_{3k+6}$ is red. This verifies the claim. Thus $v_3(A-2)+3 = v_3A-3$ is red. Since $v_3A-2$ can only be $F$-dominated by $v_3A-2$ or by a vertex of degree 2 in $W_{A-1}$, it follows that either $v_3A-2$ is red or a vertex of degree 2 in $W_{A-1}$ is red. Therefore,

$$\gamma_F(G) = |R_e| \geq 2 + (A - 2) + 1 = A + 1 = C,$$

as desired.

Furthermore, since $\{v_{3i+1} : 0 \leq i \leq A - 1\}$ is a minimum domination set and $\{v_{3i+1}, w_i : 0 \leq i \leq A - 1\}$, where $w_i \in W_i$, is a minimum open domination set, it follows that $\gamma(G) = A$ and $\gamma_0(G) = 2A = B$. □

Theorem 4.12. Every triple $(A, B, C)$ of type III with

$$2 \leq A \leq C \leq B < 2A$$

is realizable.

Proof. First, we make an observation. If $(A, B, C)$ is a triple with

$$2 \leq A < C < B < 2A,$$

then $A \geq 3$. To see this, assume, to the contrary, that $A = 2$. Then $B < 2A = 4$ and so $B = 3$. This implies that $2 < C < 3$, which is impossible.

Also, by Corollary 4.3, the result holds for $A = B = C$. Therefore, it suffices to consider following three cases:

Case I. $3 \leq A < C < B < 2A$;

Case II. $2 \leq A < C = B < 2A$;

Case III. $2 \leq A = C < B < 2A$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
In each case, we construct a connected graph with the desired properties.

**Case I.** \(3 \leq A < C < B < 2A\). Let

\[ C = A + k \quad \text{and} \quad B = A + \ell. \]

Since \(A < C < B < 2A\), it follows that \(1 \leq k < \ell < A\). Thus, we consider two cases, according to whether \(2 \leq \ell \leq A - 2\), or \(\ell = A - 1\).

**Case 1.** \(2 \leq \ell \leq A - 2\). For each \(i\) with \(1 \leq i \leq \ell - 1\), let \(F_i\) be a copy of \(H_i\) in Figure 4.1, where \(V(F_i) = \{u_i,0, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}\), where \(u_{i,p}\) corresponds to \(u_p\) in \(H_1\) for \(0 \leq p \leq 6\). For each \(j\) with \(1 \leq j \leq A - \ell - 1\), let \(G_j\) be a copy of \(H_2\) in Figure 4.1 with \(\{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}\), where \(v_{j,q}\) corresponds to \(v_q\) in \(H_2\) for \(0 \leq q \leq 3\). The graph \(G\) is then obtained from the graph \(L_{C - 2}\) and the graphs \(F_i, G_j\) for \(1 \leq i \leq \ell - 1\) and \(1 \leq j \leq A - \ell - 1\) by identifying all vertices \(u_{i,0}, v_{j,0}\) and \(w\) and labeling the identified vertex \(v\).

Since \(\{v, x\} \cup \{w_t : 1 \leq t \leq C - 2\}\) is an \(F\)-dominating set, \(\gamma_F(G) \leq C\). On the other hand, let \(c\) be a red-blue coloring of \(G\). Since \(y\) is only \(F\)-dominated by itself or by \(v\), it follows that either \(v \in R_c\) or \(y \in R_c\). Thus either \(W_1 = (v) \cup \{w_t : 1 \leq t \leq C - 2\} \subseteq R_c\) or \(W_2 = (y) \cup \{y_t : 1 \leq t \leq C - 2\} \subseteq R_c\), and so \(\gamma_F(G) \geq C - 1\). Assume, to the contrary, that \(\gamma_F(G) = C - 1\). Then either \(R_c = W_1\) or \(R_c = W_2\). In each case, \(x\) is not \(F\)-dominated by any vertex in \(R_c\), which is a contradiction. Therefore \(\gamma_F(G) = C\).

Since

\[ S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq \ell - 1\} \cup \{v_{j,1} : 1 \leq j \leq A - \ell - 1\} \]

is a minimum dominating set of \(G\), it follows that

\[ \gamma(G) = |S| = 2 + (\ell - 1) + (A - \ell - 1) = A. \]

Furthermore, the set

\[ S_o = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq \ell - 1\} \]

is a minimum open dominating set of \(G\), it follows that
\[ \gamma_0(G) = |S_0| = |S| + 1 + (\ell - 1) = A + \ell = B. \]

**Case 2.** \( \ell = A - 1 \). Then \( B = A + \ell = 2\ell + 1 \) and \( C = A + k = k + \ell + 1 \).

For each \( i \) with \( 1 \leq i \leq \ell - 1 \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{ u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6} \} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). The graph \( G \) is then obtained from the graph \( L_{C-2} \) and the graphs \( F_i \) for \( 1 \leq i \leq \ell - 1 \) by identifying all vertices \( u_{i,0} \) and \( w \) and labeling the identified vertex \( v \).

Since \( \{ v, x \} \cup \{ w_j : 1 \leq j \leq C - 2 \} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq C \). On the other hand, let \( c \) be a red-blue coloring of \( G \). Since \( y \) is only \( F \)-dominated by itself or by \( v \), it follows that either \( v \in R_c \) or \( y \in R_c \). Thus either \( W_1 = \{ v \} \cup \{ w_j : 1 \leq j \leq C - 2 \} \subseteq R_c \) or \( W_2 = \{ y \} \cup \{ y_j : 1 \leq j \leq C - 2 \} \subseteq R_c \), and so \( \gamma_F(G) \geq C - 1 \). Assume, to the contrary, that \( \gamma_F(G) = C - 1 \). Then either \( R_c = W_1 \) or \( R_c = W_2 \). In each case, \( x \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Therefore \( \gamma_F(G) = C \).

Since
\[ S = \{ v, y \} \cup \{ u_{i,3} : 1 \leq i \leq \ell - 1 \} \]
is a minimum dominating set of \( G \), it follows that
\[ \gamma(G) = |S| = 2 + (\ell - 1) = \ell + 1 = A. \]

Furthermore, the set
\[ S_o = S \cup \{ x \} \cup \{ u_{i,2} : 1 \leq i \leq \ell - 1 \} \]
is a minimum open dominating set of \( G \), it follows that
\[ \gamma_0(G) = |S_0| = |S| + 1 + (\ell - 1) = (\ell + 1) + 1 + (\ell - 1) = 2\ell + 1 = B. \]

**Case II.** \( 2 \leq A < C = B \). If \( A = 2 \), then \( B = 3 \) and the graph \( L_1 \) has the desired properties by (4.4). Thus, we may assume that \( A \geq 3 \). Let
\[ B = A + k. \]

Since \( A < B < 2A \), it follows that \( 1 \leq k < A \). We consider three cases, according to whether \( k = 1, 2 \leq k \leq A - 2 \), or \( k = A - 1 \).
Case 1. \( k = 1 \). For each \( i \) with \( 1 \leq i \leq A - 2 \), let \( G_i \) be a copy of \( H_2 \) in Figure 4.1 with \( \{ v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3} \} \), where \( v_{i,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). Let \( G \) be the graph obtained from a copy of the graph \( L_{B-2} \) and the \( A - 2 \) copies of \( G_i \) by identifying all the vertices \( v_{i,0} \) and \( w \) and calling the new vertex \( v \).

We first show that \( \gamma_{F}(G) \leq B \). Let

\[
S = \{ v, x \} \cup \{ w_j : 1 \leq j \leq B - 2 \}.
\]

Define a red-blue coloring \( c^* \) of \( G \) by assigning red to each vertex in \( S \) and blue to the remaining vertices of \( G \). Observe that (i) the blue vertex \( y \) is adjacent to the blue vertex \( z \) and \( z \) is adjacent to the red vertex \( v \), (ii) the blue vertex \( z \) is adjacent to the blue vertex \( y \) and \( y \) is adjacent to the red vertex \( x \), and (iii) the blue vertex \( y_j \) (\( 1 \leq j \leq B - 2 \)) is adjacent to the blue vertex \( y \) and \( y \) is adjacent to the red vertex \( x \). Thus, every blue vertex \( u \) of \( G \) belongs to a copy of \( F \) rooted at \( u \), implying that \( c^* \) is an \( F \)-coloring of \( G \). Therefore, \( \gamma_{F}(G) \leq |R_c| = |S| = B \).

Next, we show that \( \gamma_{F}(G) \geq B \). Assume, to the contrary, that \( \gamma_{F}(G) < B - 1 \). Let there be given a minimum \( F \)-coloring \( c \) of \( G \). Observe that the vertex \( y \) is only \( F \)-dominated by a vertex of the set \( \{ v, y \} \). Thus at least one of \( v \) and \( y \) belongs to \( R_c \).

By Lemma 2.6, if \( v \in R_c \), then \( w_j \in R_c \), for \( 1 \leq j \leq B - 2 \). Thus \( |R_c| \geq B - 1 \) and so \( \gamma_{F}(G) \geq B - 1 \). Therefore, \( \gamma_{F}(G) = B - 1 \). If \( R_c = \{ v, w_1, w_2, \ldots, w_{B-2} \} \), then the blue vertex \( x \) does not belong to a copy of \( F \) rooted at \( x \), which is a contradiction. Similarly, if \( y \in R_c \), then \( y_j \in R_c \), for \( 1 \leq j \leq B - 2 \). Thus \( |R_c| \geq B - 1 \) and so \( \gamma_{F}(G) \geq B - 1 \). Therefore, \( \gamma_{F}(G) = B - 1 \). If \( R_c = \{ y, y_1, y_2, \ldots, y_{B-2} \} \), then the blue vertex \( x \) does not belong to a copy of \( F \) rooted at \( x \), which is a contradiction. Therefore, \( \gamma_{F}(G) = B \).

Next we show that \( \gamma(G) = A \) and \( \gamma_0(G) = B \). Observe that

\[
S = \{ v, y \} \cup \{ v_{i,1} : 1 \leq i \leq A - 2 \}
\]

is a minimum dominating set of \( G \) and so

\[
\gamma(G) = |S| = 2 + (A - 2) = A.
\]
Furthermore, the set
\[ S_o = S \cup \{x\} \]
is a minimum open dominating set of \( G \) and so
\[ \gamma_o(G) = |S_o| = |S| + 1 = B. \]

**Case 2.** \( 2 \leq k \leq A - 2 \). For each \( i \) with \( 1 \leq i \leq k - 1 \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). For each \( j \) with \( 1 \leq j \leq A - k - 1 \), let \( G_j \) be a copy of \( H_2 \) in Figure 4.1 with \( \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\} \), where \( v_{j,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). The graph \( G \) is then obtained from the graph \( L_{B-2} \) and the graphs \( F_i, G_j \) for \( 1 \leq i \leq k - 1 \) and \( 1 \leq j \leq A - k - 1 \) by identifying all vertices \( u_{i,0}, v_{j,0} \) and \( w \) and labeling the identified vertex \( v \).

Since \( \{v, x\} \cup \{w_t : 1 \leq t \leq B - 2\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq B \). On the other hand, let \( c \) be a red-blue coloring of \( G \). Since \( y \) is only \( F \)-dominated by itself or by \( v \), it follows that either \( v \in R_c \) or \( y \in R_c \). Thus either \( W_1 = \{v\} \cup \{w_t : 1 \leq t \leq B - 2\} \subseteq R_c \) or \( W_2 = \{y\} \cup \{y_t : 1 \leq t \leq B - 2\} \subseteq R_c \), and so \( \gamma_F(G) \geq B - 1 \). Assume, to the contrary, that \( \gamma_F(G) = B - 1 \). Then either \( R_c = W_1 \) or \( R_c = W_2 \). In each case, \( x \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Therefore \( \gamma_F(G) = B \).

Since
\[ S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\} \cup \{v_{j,1} : 1 \leq j \leq A - k - 1\} \]
is a minimum dominating set of \( G \), it follows that
\[ \gamma(G) = |S| = 2 + (k - 1) + (A - k - 1) = A. \]

Furthermore, the set
\[ S_o = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\} \]
is a minimum open dominating set of \( G \), it follows that
\[ \gamma_o(G) = |S_o| = |S| + 1 + (k - 1) = A + k = B. \]
Case 3. \( k = A - 1 \). Then \( B = A + k = 2k + 1 \). For each \( i \) with \( 1 \leq i \leq k - 1 \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). The graph \( G \) is then obtained from the graph \( L_{B-2} \) and the graphs \( F_i \) for \( 1 \leq i \leq k - 1 \) by identifying all vertices \( u_{i,0} \) and \( w \) and labeling the identified vertex \( v \).

Since \( \{v, x\} \cup \{w_j : 1 \leq j \leq B - 2\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq B \). On the other hand, let \( c \) be a red-blue coloring of \( G \). Since \( y \) is only \( F \)-dominated by itself or by \( v \), it follows that either \( v \in R_c \) or \( y \in R_c \). Thus either \( W_1 = \{v\} \cup \{w_j : 1 \leq j \leq B - 2\} \subseteq R_c \) or \( W_2 = \{y\} \cup \{w_j : 1 \leq j \leq B - 2\} \subseteq R_c \), and so \( \gamma_F(G) \geq B - 1 \). Assume, to the contrary, that \( \gamma_F(G) = B - 1 \). Then either \( R_c = W_1 \) or \( R_c = W_2 \). In each case, \( x \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Therefore \( \gamma_F(G) = B \).

Since 
\[
S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\}
\]
is a minimum dominating set of \( G \), it follows that
\[
\gamma(G) = |S| = 2 + (k - 1) = k + 1 = A.
\]
Furthermore, the set 
\[
S_o = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\}
\]
is a minimum open dominating set of \( G \), it follows that
\[
\gamma_o(G) = |S_o| = |S| + 1 + (k - 1) = (k + 1) + 1 + (k - 1) = 2k + 1 = B.
\]

Case III. \( 2 \leq A = C < B \). If \( A = 2 \), then \( B = 3 \). Let \( G \) be obtained from the graph \( K_4 - e \) by adding a path \( P_2 \) and joining a vertex of \( P - 2 \) to a vertex of degree two in \( K_4 - e \). Then \( \gamma(G) = \gamma_F(G) = 2 \) and \( \gamma_o(G) = 3 \). So we consider the case when \( A \geq 3 \). Note that \( B \geq A + 1 \geq 4 \). We consider three cases, according to whether \( k = 1 \), \( 2 \leq k \leq A - 2 \), or \( k = A - 1 \).

Case 1. \( k = 1 \). For each \( i \) with \( 1 \leq i \leq A - 2 \), let \( G_i \) be a copy of \( H_2 \) in Figure 4.1 with \( \{u_{i,0}, u_{i,1}, u_{i,2}, v_i,3\} \), where \( v_{i,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). Let
\( G \) be the graph obtained from a copy of the graph \( L_{A-2} \) and the \( A-2 \) copies of \( G_i \) by identifying all the vertices \( v_{i,0} \) and \( w \) and calling the new vertex \( v \).

We first show that \( \gamma_F(G) \leq A \). Let

\[
S = \{v, x\} \cup \{w_j \mid 1 \leq j \leq A-2\}.
\]

Define a red-blue coloring \( c^* \) of \( G \) by assigning red to each vertex in \( S \) and blue to the remaining vertices of \( G \). Observe that (i) the blue vertex \( y \) is adjacent to the blue vertex \( z \) and \( z \) is adjacent to the red vertex \( v \), (ii) the blue vertex \( z \) is adjacent to the blue vertex \( y \) and \( y \) is adjacent to the red vertex \( x \), and (iii) the blue vertex \( y_j \) (\( 1 \leq j \leq A-2 \)) is adjacent to the blue vertex \( y \) and \( y \) is adjacent to the red vertex \( x \). Thus, every blue vertex \( u \) of \( G \) belongs to a copy of \( F \) rooted at \( u \), implying that \( c^* \) is a \( F \)-coloring of \( G \). Therefore, \( \gamma_F(G) \leq |R_c| = |S| = A \).

Next, we show that \( \gamma_F(G) \geq A \). Assume, to the contrary, that \( \gamma_F(G) < A - 1 \). Let there be given a minimum \( F \)-coloring \( c \) of \( G \). Observe that the vertex \( y \) is only \( F \)-dominated by a vertex of the set \( \{v, y\} \). Thus at least one of \( v \) and \( y \) belongs to \( R_c \).

By Lemma 2.6, if \( v \in R_c \), then \( w_j \in R_c \), for \( 1 \leq j \leq A-2 \). Thus \( |R_c| \geq A - 1 \) and so \( \gamma_F(G) \geq A - 1 \). Therefore, \( \gamma_F(G) = A - 1 \). If \( R_c = \{v, w_1, w_2, \ldots, w_{A-2}\} \), then the blue vertex \( x \) does not belong to a copy of \( F \) rooted at \( x \), which is a contradiction. Similarly, if \( y \in R_c \), then \( y_j \in R_c \), for \( 1 \leq j \leq A-2 \). Thus \( |R_c| \geq A - 1 \) and so \( \gamma_F(G) \geq A - 1 \).

Therefore, \( \gamma_F(G) = A - 1 \). If \( R_c = \{y, y_1, y_2, \ldots, y_{A-2}\} \), then the blue vertex \( x \) does not belong to a copy of \( F \) rooted at \( x \), which is a contradiction. Therefore, \( \gamma_F(G) = A \).

Next we show that \( \gamma(G) = A \) and \( \gamma_o(G) = B \). Observe that

\[
S = \{v, y\} \cup \{v_{i,1} \mid 1 \leq i \leq A-2\}
\]

is a minimum dominating set of \( G \) and so

\[
\gamma(G) = |S| = 2 + (A - 2) = A.
\]

Furthermore, the set

\[
S_o = S \cup \{x\}
\]
is a minimum open dominating set of \( G \) and so

\[
\gamma_0(G) = |S_0| = |S| + 1 = B.
\]

**Case 2.** \( 2 \leq k \leq A - 2 \). For each \( i \) with \( 1 \leq i \leq k - 1 \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \) corresponds to \( u_p \) in \( H_1 \) for \( 0 \leq p \leq 6 \). For each \( j \) with \( 1 \leq j \leq A - k - 1 \), let \( G_j \) be a copy of \( H_2 \) in Figure 4.1 with \( \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\} \), where \( v_{j,q} \) corresponds to \( v_q \) in \( H_2 \) for \( 0 \leq q \leq 3 \). The graph \( G \) is then obtained from the graph \( L_{A-2} \) and the graphs \( F_i, G_j \) for \( 1 \leq i \leq k - 1 \) and \( 1 \leq j \leq A - k - 1 \) by identifying all vertices \( u_{i,0}, v_{j,0} \) and \( w \) and labeling the identified vertex \( v \).

Since \( \{v, x\} \cup \{u_t : 1 \leq t \leq A - 2\} \) is an \( F \)-dominating set, \( \gamma_F(G) \leq A \). On the other hand, let \( c \) be a red-blue coloring of \( G \). Since \( y \) is only \( F \)-dominated by itself or by \( v \), it follows that either \( v \in R_c \) or \( y \in R_c \). Thus either \( W_1 = \{v\} \cup \{w_t : 1 \leq t \leq A - 2\} \subseteq R_c \) or \( W_2 = \{y\} \cup \{v_t : 1 \leq t \leq A - 2\} \subseteq R_c \), and so \( \gamma_F(G) \geq A - 1 \). Assume, to the contrary, that \( \gamma_F(G) = A - 1 \). Then either \( R_c = W_1 \) or \( R_c = W_2 \). In each case, \( x \) is not \( F \)-dominated by any vertex in \( R_c \), which is a contradiction. Therefore \( \gamma_F(G) = A \).

Since

\[
S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\} \cup \{v_{j,1} : 1 \leq j \leq A - k - 1\}
\]

is a minimum dominating set of \( G \), it follows that

\[
\gamma(G) = |S| = 2 + (k - 1) + (A - k - 1) = A.
\]

Furthermore, the set

\[
S_0 = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\}
\]

is a minimum open dominating set of \( G \), it follows that

\[
\gamma_0(G) = |S_0| = |S| + 1 = A + k = B.
\]

**Case 3.** \( k = A - 1 \). Then \( B = A + k = 2k + 1 \). For each \( i \) with \( 1 \leq i \leq k - 1 \), let \( F_i \) be a copy of \( H_1 \) in Figure 4.1, where \( V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\} \), where \( u_{i,p} \)
corresponds to $u_p$ in $H_1$ for $0 \leq p \leq 6$. The graph $G$ is then obtained from the graph $L_{A-2}$ and the graphs $F_i$ for $1 \leq i \leq k - 1$ by identifying all vertices $u_{i,0}$ and $w$ and labeling the identified vertex $v$.

Since $\{v, x\} \cup \{w_j : 1 \leq j \leq A-2\}$ is an $F$-dominating set, $\gamma_F(G) \leq A$. On the other hand, let $c$ be a red-blue coloring of $G$. Since $y$ is only $F$-dominated by itself or by $v$, it follows that either $v \in R_c$ or $y \in R_c$. Thus either $W_1 = \{v\} \cup \{w_j : 1 \leq j \leq A-2\} \subseteq R_c$ or $W_2 = \{y\} \cup \{y_j : 1 \leq j \leq A-2\} \subseteq R_c$, and so $\gamma_F(G) \geq A - 1$. Assume, to the contrary, that $\gamma_F(G) = A - 1$. Then either $R_c = W_1$ or $R_c = W_2$. In each case, $x$ is not $F$-dominated by any vertex in $R_c$, which is a contradiction. Therefore $\gamma_F(G) = A$.

Since
\[ S = \{v, y\} \cup \{u_{i,3} : 1 \leq i \leq k - 1\} \]
is a minimum dominating set of $G$, it follows that
\[ \gamma(G) = |S| = 2 + (k - 1) = k + 1 = A. \]

Furthermore, the set
\[ S_0 = S \cup \{x\} \cup \{u_{i,2} : 1 \leq i \leq k - 1\} \]
is a minimum open dominating set of $G$, it follows that
\[ \gamma_0(G) = |S_0| = |S| + 1 + (k - 1) = (k + 1) + 1 + (k - 1) = 2k + 1 = B. \]

This completes the proof. \(\square\)

Combining Theorems 4.11 and 4.12, we have the following.

**Corollary 4.13.** Every triple of type III is realizable.

We summarize the main results we have established in this chapter as follows:

1. Theorem 4.6: Every triple of type I is realizable.

2. Corollary 4.10: A triple $(A, B, C)$ of type II is realizable if and only if $(A, B, C) \neq (k, k, C)$ for any integers $k$ and $C$ with $C > k \geq 2$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Combining these three results, we have the main result of this chapter.

**Theorem 4.14.** A triple \((A, B, C)\) is realizable if and only if \((A, B, C) \neq (k, k, C)\) for any integers \(k\) and \(C\) with \(C > k \geq 2\).
5 Stratified Domination in Oriented Graphs

5.1 Introduction

If a digraph $D$ has the property that for each pair $u, v$ of distinct vertices of $D$, at most one of $(u, v)$ and $(v, u)$ is an arc of $D$, then $D$ is an oriented graph. An oriented graph $D$ can also be obtained by assigning a direction to each edge of some graph $G$. In this case, the digraph $D$ is then referred to as an orientation of the graph $G$ and $G$ is the underlying graph of $D$. An oriented graph $D$ whose vertex set has been partitioned is called a stratified oriented graph. If $\{V_1, V_2, \ldots, V_k\}$ is a partition of $V(D)$, then $D$ is a $k$-stratified oriented graph. The sets $V_1, V_2, \ldots, V_k$ are also called the strata (or the color classes) of $D$. As with $k$-stratified graphs, if $k = 2$, then we ordinarily color the vertices of $V_1$ red and the vertices of $V_2$ blue.

For a connected digraph $D$, a red-blue coloring of $D$ is a coloring in which every vertex is colored red or blue. It is acceptable if all vertices of $D$ are colored the same. If there is at least one vertex of each color, then the red-blue coloring of $D$ produces a 2-stratification of $D$.

Let $F$ be a (connected) 2-stratified oriented graph rooted at some blue vertex. An $F$-coloring of an oriented graph $D$ is a red-blue coloring of the vertices of $D$ in which every blue vertex $v$ belongs to a copy $F'$ of $F$ rooted at $v$ in $D$. In this case, $v$ is said to be $F$-dominated by some red vertex in $F'$. A red vertex is $F$-dominated by itself. The $F$-domination number $\gamma_F(D)$ is the minimum number of red vertices in an $F$-coloring of $D$. The set of red vertices in an $F$-coloring $c$ of $D$ is also called an $F$-dominating set of $D$ and is denoted by $R_c$. If $|R_c| = \gamma_F(D)$, then $c$ is a minimum $F$-coloring of $D$ and $R_c$ is a minimum $F$-dominating set of $D$. Therefore, if $F$ has $r$ red vertices, then

$$r \leq \gamma_F(D) \leq n$$

(5.1)

for every oriented graph $D$ of order $n \geq r$. 127
Observation 5.1. Let $F$ be a 2-stratified oriented graph. If $D$ is an oriented graph of order $n$ such that $D$ has no subdigraph isomorphic to $F$, then

$$\gamma_F(D) = n.$$ 

If $F$ is a connected 2-stratified oriented graph of order 2, then $F$ is one of the 2-stratifications of $\overline{P}_2$ in Figure 5.1. In each case, the $F$-domination number is a well-known domination parameter, as we show next.

$$F_1 : \bullet \rightarrow \circ \quad F_2 : \circ \rightarrow \bullet$$

Figure 34. Two 2-stratified oriented graphs of $\overline{P}_2$.

Let $D$ be an oriented graph. A vertex $v$ is said to dominate (or out-dominate) itself together with all vertices adjacent from $v$. A set $S \subseteq V(D)$ is a dominating set for $D$ if every vertex in $D$ is dominated by some vertex in $S$. The domination number $\gamma(D)$ is the minimum cardinality of a dominating set in $D$. A dominating set of cardinality $\gamma(D)$ is called a minimum dominating set of $D$, or simply a $\gamma$-set for $D$. We first establish a result that is not unexpected.

Proposition 5.2. For the 2-stratified oriented graph $F_1$ of $\overline{P}_2$,

$$\gamma(D) = \gamma_{F_1}(D)$$

for every oriented graph $D$.

Proof. Let $D$ be an oriented graph. We first show that $\gamma_{F_1}(D) \leq \gamma(D)$. Let $S$ be a minimum dominating set of $D$. Define a red-blue coloring $c$ of $D$ by coloring each vertex in $S$ red and the remaining vertices blue. We show that $c$ is an $F_1$-coloring of $D$. Let $v \in V(D) - S$ be an arbitrary blue vertex. Since $S$ is a dominating set of $D$, the vertex $v$ is dominated by some vertex $u \in S$, that is, $v$ is adjacent from $u$. Since $u$ is colored red by $c$, it follows that $v$ belongs to a copy of $F_1$ rooted at $v$. Thus $c$ is an $F_1$-coloring using $|S|$ red vertices. Therefore, $\gamma_{F_1}(D) \leq |S| = \gamma(D)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Next, we show that $\gamma(D) \leq \gamma_{F_1}(D)$. Let there be given a minimum $F_1$-coloring $c'$ and let $R_{c'}$ be the set of red vertices of $D$ assigned by $c'$. We show that $R_{c'}$ is a dominating set of $D$. If $v \in V(D) - R_{c'}$, then $v$ is a blue vertex and so $v$ belongs to a copy of $F_1$ rooted at $v$; that is, $v$ is adjacent from some red vertex in $R_{c'}$. So $v$ is dominated by some vertex $R_{c'}$, implying that $R_{c'}$ is a dominating set of $D$. Therefore, $\gamma(D) \leq |R_{c'}| = \gamma_{F_1}(D)$.

The converse $D^*$ of an oriented graph $D$ has the same vertex set as $D$ and the arc $(u, v)$ is in $D^*$ if and only if the arc $(v, u)$ is in $D$.

**Proposition 5.3.** For the $2$-stratified oriented graph $F_2$ of $\bar{P}_2$,

$$\gamma_{F_2}(D) = \gamma_{F_1}(D^*)$$

for every oriented graph $D$.

By Proposition 5.3, we need not be concerned with studying the parameter $\gamma_{F_2}$ since the $F_2$-domination number of an oriented graph always equals the $F_1$-domination number of its converse. Also, since the $F_1$-domination number of an oriented graph equals the well-studied out-domination number, we proceed to consider $F$-domination for $2$-stratified oriented graphs of higher order. This leads us to $2$-stratified paths of order 3. There are six of these, as shown in Figure 5.2.

![Figure 35. Six 2-stratifications of $\bar{P}_3$.](image)

In this chapter, we study the $2$-stratified oriented graph $H_1$. Necessarily, the only blue vertex in $H_1$ is the root of $H_1$. To simplify the notation, we write $H = H_1$. Since
$H$ has two red vertices, it follows by (5.1) that if $D$ is a digraph of order $n \geq 2$, then

$$2 \leq \gamma_H(D) \leq n.$$  \hfill (5.2)

We now illustrate $H$-domination for the tournament $T$ in Figure 5.3. Recall that for a vertex $v$ in an oriented graph, the number of vertices to which a vertex $v$ is adjacent is the *outdegree* of $v$ and is denoted by $\text{od} v$. The number of vertices from which $v$ is adjacent is the *indegree* of $v$ and is denoted by $\text{id} v$. Thus the *degree* of $v$ is $\text{deg} v = \text{od} v + \text{id} v$.

![Figure 36. An $H$-coloring of an oriented graph.](image)

Consider the red-blue coloring $c'$ of $T$ that assigns red to $v_1, v_2$, and $v_4$ and blue to $v_3$ and $v_5$, as shown in Figure 5.3. Observe that

1. the blue vertex $v_3$ is the terminal vertex of the directed red-red-blue path $P_3 : v_2, v_3, v_5$, and
2. the blue vertex $v_5$ is the terminal vertex of the directed red-red-blue path $P_3 : v_2, v_4, v_5$.

Thus $c'$ is an $H$-coloring of $T$ with three red vertices and so $\gamma_H(T) \leq 3$. Therefore, either $\gamma_H(T) = 2$ or $\gamma_H(T) = 3$. We claim that $\gamma_H(T) = 3$. First, $\gamma_H(T) \geq 2$ by (5.2). Assume, to the contrary, that $\gamma_H(T) = 2$. Let $c$ be a minimum $H$-coloring of $T$, where $x$ and $y$ are the two red vertices of $T$. Necessarily, $x$ and $y$ are adjacent vertices in $T$. Assume, without loss of generality, that $(x, y)$ is an arc of $T$. Since $c$ is an $H$-coloring...
of $T$, for every blue vertex $z$, the digraph $T$ must contain the directed path $x, y, z$ and so $\text{od}_T y \geq 3$. Since every vertex of $T$ has outdegree 2, this is impossible. Therefore, as claimed, $\gamma_H(T) = 3$.

The argument just used to show that $\gamma_H(T) \neq 2$ for the digraph $T$ of Figure 5.3 gives us the following result.

**Proposition 5.4.** An oriented graph $D$ of order $n \geq 3$ has $\gamma_H(D) = 2$ if and only if $D$ contains a vertex $v$ with $\text{id} v = 1$ and $\text{od} v = n - 2$.

We now introduce some additional definitions that will be useful in what follows. Let $c$ be an $H$-coloring of an oriented graph $D$ and let $w$ be a blue vertex of $D$. Necessarily, $w$ is the terminal vertex of a directed red-red-blue path $\overrightarrow{P}_3$, say $\overrightarrow{P}_3 : u, v, w$. In this case, we say that $w$ is $H$-dominated by $v$ or that $v$ $H$-dominates $w$. That is, in this context, every blue vertex $w$ of $D$ is $H$-dominated by some red vertex $v$ and so $w$ is adjacent from $v$, which, in turn, is adjacent from another red vertex. Consequently, if a red vertex $v$ $H$-dominates a blue vertex $w$, then $v$ is adjacent to $w$ and adjacent from another red vertex.

The following two observations are useful.

**Observation 5.5.** Let $v$ be a vertex in an oriented graph $D$.

(a) If $\text{od} v = 0$, then $v$ cannot dominate or $H$-dominate any other vertex in $D$.

(b) If $\text{id} v = 0$, then $v$ can neither $H$-dominate nor be $H$-dominated by any other vertex in $D$.

**Observation 5.6.** Let $v$ be a vertex in an oriented graph $D$ with $\text{id} v = 1$ and let $c$ be an $H$-coloring of $D$.

(a) If $(u, v)$ is an arc of $D$, then at least one of $u$ and $v$ must be colored red by $c$.

(b) If $(u, v)$ is an arc of $D$, $\text{id} u = 1$, and $(w, u)$ is an arc of $D$, then at least two of $u$, $v$, and $w$ must be colored red by $c$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The following is an immediate consequence of Observation 5.5.

**Corollary 5.7.** Let $I$ be the set of all vertices of an oriented graph $D$ with indegree 0. Then

(a) $I$ belongs to every dominating set of $D$, and

(b) $I \subseteq R_c$ for every $H$-coloring $c$ of $D$.

Recall that if $D$ is a nontrivial oriented graph of order $n$ and $\gamma_H(D) = k$, then $2 \leq k \leq n$. Next, we show that every pair $k, n$ of integers with $2 \leq k \leq n$ is realizable as the $H$-domination number and the order of some connected oriented graph, respectively.

**Proposition 5.8.** For each pair $k, n$ of integers with $2 \leq k \leq n$, there exists a connected oriented graph $D$ of order $n$ with $\gamma_H(D) = k$.

**Proof.** Let $\overrightarrow{K}_{1,k-1}$ be the orientation of the star $K_{1,k-1}$ whose vertex set is $\{u, v_1, v_2, \ldots, v_{k-1}\}$, where $u$ is the central vertex of $K_{1,k-1}$ and $(u, v_i) \in E(\overrightarrow{K}_{1,k-1})$ for $1 \leq i \leq k-1$. Construct the oriented graph $D$ from $\overrightarrow{K}_{1,k-1}$ by adding $n - k$ new vertices $w_1, w_2, \ldots, w_{n-k}$ together with the $n - k$ new arcs $(v_i, w_i)$ for $1 \leq i \leq n - k$. Then the order of $D$ is $n$.

We show that $\gamma_H(D) = k$. Define a red-blue coloring $c$ of $D$ by assigning red to every vertex of $V(\overrightarrow{K}_{1,k-1})$, and blue to the remaining vertices of $D$. Then $c$ is an $H$-coloring of $D$ with $k$ red vertices, so $\gamma_H(D) \leq k$. Next, we show that $\gamma_H(D) \geq k$. Let there be given a minimum $H$-coloring $c$ of $D$ and let $R_c$ be the set of red vertices of $c$. Since $\text{id} u = 0$, it follows that $u \in R_c$. Also, since every vertex $v_i$ $(1 \leq i \leq k - 1)$ is adjacent from only a vertex with indegree 0, it follows that $v_i$ is not $H$-dominated by any vertex distinct from $v_i$. Therefore, $v_i \in R_c$ for $1 \leq i \leq k - 1$. Thus $V(\overrightarrow{K}_{1,k-1}) \subseteq R_c$ and so $\gamma_H(D) = |R_c| \geq k$. Therefore, $\gamma_H(D) = k$. \qed
5.2 $H$-Domination and Standard Dominations

In this section, we compare $H$-domination number with two well-known domination parameters in digraphs, namely domination and open domination.

A vertex $v$ is said to *openly dominate* (or *openly out-dominate*) all vertices adjacent from $v$. A set $S \subseteq V(D)$ is an *open dominating set* for $D$ if every vertex in $D$ is openly dominated by some vertex in $S$. The *open domination number* $\gamma_o(D)$ is the minimum cardinality of an open dominating set in $D$. An open dominating set of cardinality $\gamma_o(D)$ is called a *minimum open dominating set* of $D$, or simply a $\gamma_o$-set for $D$.

The following observations are useful.

**Observation 5.9.** Let $D$ be an oriented graph. Then the open domination number $\gamma_o(D)$ is defined for an oriented graph $D$ if and only if $\text{id} x > 1$ for every vertex $x$ in $D$.

**Observation 5.10.** If $D$ is an oriented graph with $\gamma(D) = 1$, then $\gamma_o(D)$ is not defined.

**Proposition 5.11.** If $S$ is an open dominating set of an oriented graph $D$, then $|S| \geq 3$ and the subdigraph $\langle S \rangle$ induced by $S$ contains a directed cycle.

**Proof.** We first show that $|S| \geq 3$. Since no vertex openly dominates itself, $S$ contains at least two elements. Assume, to the contrary, that $S$ contains exactly two elements $u$ and $v$, say. Then $u$ is openly dominated by $v$ and $v$ is openly dominated by $u$. Hence $(u,v)$ and $(v,u)$ both are arcs of $D$, a contradiction.

Next, we show that $\langle S \rangle$ contains a directed cycle. Assume, to the contrary, that $\langle S \rangle$ contains no directed cycles. Let $\overrightarrow{P}_k : v_1, v_2, \ldots, v_k$ be a longest directed path in $\langle S \rangle$, where $k \geq 2$. Then $v_1$ is not adjacent from any vertex in $S$ that does not lie on $\overrightarrow{P}_k$ since $\overrightarrow{P}_k$ is a longest directed path. Moreover, since $\langle S \rangle$ contains no directed cycles, $v_1$ is not adjacent from any vertex on $\overrightarrow{P}_k$. This implies that $v_1$ is not adjacent from any vertex in $S$ and so $v_1$ is not openly dominated by any vertex in $S$. Therefore, $S$ is not an open dominating set of $D$, which is a contradiction. \(\Box\)
Corollary 5.12. For every oriented graph $D$ for which $\gamma_o(D)$ is defined,

$$\gamma_o(D) \geq 3.$$

We now show that if $D$ is any oriented graph for which $\gamma_o(D)$ is defined, then $\gamma_H(D)$ is bounded above by $\gamma_o(D)$ and bounded below by $\gamma(D)$.

Proposition 5.13. Let $D$ be an oriented graph such that $\text{id} x \geq 1$ for every $x \in V(D)$. For the 2-stratification $H = H_1$ of $\overline{P}_3$,

$$\gamma(D) \leq \gamma_H(D) \leq \gamma_o(D).$$

Proof. We first show $\gamma(D) \leq \gamma_H(D)$. Let there be given a minimum $H$-coloring $c$ of $D$ and let $R_c$ be the set of the red vertices of $D$ assigned by $c$. We show that $R_c$ is a dominating set of $D$. If $v \in V(D) - R_c$, then $v$ is a blue vertex and $v$ belongs to a copy of $H$ rooted at $v$. This implies that $v$ is adjacent from a red vertex in $R_c$, and so $v$ is dominated by some vertex in $R_c$. Therefore, $R_c$ is a dominating set of $D$ and so $\gamma(D) \leq |R_c| = \gamma_H(D)$.

Next, we show that $\gamma_H(D) \leq \gamma_o(D)$. Let $S$ be a minimum open dominating set of $D$. Define a coloring $c$ by assigning red to each vertex in $S$ and blue to each vertex in $V(D) - S$. We show that $c$ is an $H$-coloring. Let $v \in V(D) - S$ be a blue vertex. Since $S$ is an open dominating set of $D$, it follows that $v$ is openly dominated by some vertex $u \in S$ and so $v$ is adjacent from the red vertex $u$. Moreover, $u$ is also openly dominated by some vertex $w \in S$ and so $u$ is adjacent from some red vertex $w$. Hence $v$ belongs to a copy of $H$ with vertex set $\{u, v, w\}$. Therefore, $c$ is an $H$-coloring of $D$ and so $\gamma_H(D) \leq |S| = \gamma_o(D)$.

The two inequalities in Proposition 5.13 can both be strict. For example, for the oriented graph $\overline{C}_4$ of Figure 5.4, we have $\gamma(\overline{C}_4) = 2$, $\gamma_H(\overline{C}_4) = 3$, and $\gamma_o(\overline{C}_4) = 4$.

On the other hand, both inequalities in Proposition 5.13 can be equalities. In order to show this, we first establish some additional definitions. For each integer $k \geq 3$, let
Figure 37. The oriented graph $\overrightarrow{C_4}$.

$\overrightarrow{C_k} : u_1, u_2, \ldots, u_k, u_1$

be the directed $k$-cycle. Then the out-corona $\text{Cor}(\overrightarrow{C_k})$ of $\overrightarrow{C_k}$ is the oriented graph obtained from $\overrightarrow{C_k}$ by adding $k$ new vertices $v_1, v_2, \ldots, v_k$ and $k$ new arcs $(u_i, v_i)$ for $1 \leq i \leq k$.

Figure 38. The out-corona $\text{Cor}(\overrightarrow{C_8})$ of the 8-cycle.

**Lemma 5.14.** For each $k \geq 3$, there exists a connected oriented graph $D$ such that

$$\gamma(D) = \gamma_{H}(D) = \gamma_{o}(D) = k.$$  

**Proof.** For $k \geq 3$, let $D = \text{Cor}(\overrightarrow{C_k})$. Since $S = \{u_1, u_2, \ldots, u_k\}$ is a dominating set of $D$, it follows that $\gamma(D) \leq k$. On the other hand, every vertex $v_i$ is only dominated by itself or by $u_i$ for $1 \leq i \leq k$ and so every dominating set of $D$ contains at least one of $u_i$ and $v_i$ for $1 \leq i \leq k$. Therefore, $S$ is a minimum dominating set of $D$ and so $\gamma(D) = k$. Consequently, $\gamma_{o}(D) \geq k$. Since $S$ is also an open dominating set of $D$, it follows that $\gamma_{o}(D) \leq k$ and so by Proposition 5.13, $\gamma(D) = \gamma_{H}(D) = \gamma_{o}(D)$. 

We now show that all possible combinations of equalities and strict inequalities are possible in the two inequalities in Proposition 5.13.
Proposition 5.15. For the 2-stratification $H = H_1$ of $\tilde{F}_3$, there exist

(a) infinitely many connected oriented graphs $D_1$ such that $\gamma(D_1) < \gamma_H(D_1) < \gamma_o(D_1)$,

(b) infinitely many connected oriented graphs $D_2$ such that $\gamma(D_2) = \gamma_H(D_2) = \gamma_o(D_2)$,

(c) infinitely many connected oriented graphs $D_3$ such that $\gamma(D_3) = \gamma_H(D_3) < \gamma_o(D_3)$, and

(d) infinitely many connected oriented graphs $D_4$ such that $\gamma(D_4) < \gamma_H(D_4) = \gamma_o(D_4)$.

Proof. We have already seen that $\gamma(C_4) < \gamma_H(C_4) < \gamma_o(C_4)$. We now show that there are infinitely many oriented graphs $D_1$ such that $\gamma(D_1) < \gamma_H(D_1) < \gamma_o(D_1)$. For a positive integer $k$, let $D_1 \cong C_{6k}$. If $v$ is a blue vertex in an $H$-coloring of $D_1$, then the vertex $u$ adjacent to $v$ and the vertex $w$ adjacent to $u$ must be red. Thus at least two-thirds of the vertices of $D_1$ are red and so $\gamma_H(D_1) \geq 4k$. Since the red-blue coloring of $D_1$ that assigns red to $v_i$ if $i \equiv 0 \pmod{3}$, and blue to the remaining vertices of $D_1$ is an $H$-coloring of $D_1$, it follows that $\gamma_H(D_1) \leq 4k$. Therefore, $\gamma_H(D_1) = 4k$. Since each vertex $v$ of $D_1$ is openly dominated by a unique vertex of $D_1$ (namely the vertex $u$ adjacent to $v$), it follows that the only open dominating set of $D_1$ is $V(D_1)$ and so $\gamma_o(D_1) = 6k$. Finally, since each vertex of $D_1$ only dominates two vertices of $D_1$, it follows that $\gamma(D_1) \geq 3k$. However,

$$\{v_i : i \equiv 0 \pmod{2}\}$$

is a dominating set and so $\gamma(D_1) = 3k$. Therefore, $\gamma(D_1) < \gamma_H(D_1) < \gamma_o(D_1)$, which verifies (a).

That (b) holds is a consequence of Lemma 5.14. It therefore only remains to verify (c) and (d). We start with (c).

Let $k \geq 2$ be an integer and for each integer $i$ with $1 \leq i \leq k - 1$, let $G_i : u_i, v_i, w_i, u_i$ be a directed 3-cycle. Let $D_3$ be the oriented graph obtained from the digraphs $G_i$ ($1 \leq i \leq k - 1$) by identifying the vertices $w_i$ for $1 \leq i \leq k - 1$ and labeling
the identified vertex by \( w \). Observe that the order of \( D_3 \) is \( 2k - 1 \). (For \( k = 4 \), the oriented graph \( D_3 \) is shown in Figure 5.6.) We show that

\[
\gamma(D_3) = \gamma_H(D_3) = k \quad \text{and} \quad \gamma_0(D_3) = k + 1.
\]

We first show that \( \gamma(D_3) = k \). Since

\[
S' = \{w, u_1, u_2, \ldots, u_{k-1}\}
\]

is a dominating set, \( \gamma(D_3) \leq k \). Next we show that \( \gamma(D_3) \geq k \). For each integer \( i \) with \( 1 \leq i \leq k - 1 \), the vertex \( v_i \) is only dominated by itself and \( u_i \). Thus every dominating set of \( D_3 \) must contain at least one vertex of each set \( \{u_i, v_i\} \) for \( 1 \leq i \leq k - 1 \). Since each of \( u_i \) and \( v_i \) dominates two vertices, any set \( \{x_1, x_2, \ldots, x_{k-1}\} \), where \( x_i \in \{u_i, v_i\} \) for \( 1 \leq i \leq k - 1 \), dominates at most \( 2k - 2 \) distinct vertices, it follows that \( \gamma(D_3) \geq k \) and so \( \gamma(D_3) = k \).

![Figure 39](image)

**Figure 39.** The oriented graph \( D_3 \) for \( k = 4 \).

Next, we show that \( \gamma_H(D_3) = k \). Define a red-blue coloring \( c \) of \( D_3 \) by assigning the color red to each vertex in the set \( S' \) described in (5.3) and blue otherwise. For each \( i \) with \( 1 \leq i \leq k - 1 \), the blue vertex \( v_i \) is adjacent from the red vertex \( u_i \), which is adjacent from the red vertex \( w \). Thus each blue vertex \( v_i \) belongs to a copy of \( H \) rooted at \( v_i \) for \( 1 \leq i \leq k - 1 \). Hence \( c \) is an \( H \)-coloring of \( D_3 \) and so \( \gamma_H(D_3) \leq |S'| = k \). It then follows by Proposition 5.13 that \( \gamma_H(D_3) = k \).

Finally, we show that \( \gamma_0(D_3) = k + 1 \). Since \( S' \cup \{v_1\} \) is an open dominating set, \( \gamma_0(D_3) \leq k + 1 \). To show that \( \gamma_0(D_3) \geq k + 1 \), observe that

(1) the vertex \( w \) is only openly dominated by some vertex \( v_i \) for \( 1 \leq i \leq k - 1 \),
(2) each vertex $u_i$ ($1 \leq i \leq k - 1$) is only openly dominated by $w$, and

(3) each vertex $v_i$ ($1 \leq i \leq k - 1$) is only openly dominated by $u_i$.

Thus every open dominating set of $D_3$ must contain at least one vertex in \{ $v_1, v_2, \ldots, v_{k-1}$\}, the vertex $w$, and every vertex $u_i$ for $1 \leq i \leq k - 1$, implying that

$$\gamma_o(D_3) \geq 1 + 1 + (k - 1) = k + 1.$$  

Therefore, $\gamma_o(D_3) = k + 1$. This completes the proof of (c).

We now verify (d). For $k \geq 2$, let $D_4$ be the oriented graph obtained from the directed 3-cycle $C_3 : u, v, w, u$ by adding

(1) the $k - 1$ new vertices $u_1, u_2, \ldots, u_{k-1}$ and the $k - 1$ arcs $(u, u_i)$ for $1 \leq i \leq k - 1$,

(2) the $k - 2$ new vertices $x_1, x_2, \ldots, x_{k-2}$ and the $k - 2$ arcs $(u_j, x_j)$ for $1 \leq j \leq k - 2$, and

(3) the vertex $y$ and the arc $(w, y)$.

The oriented graph $D_4$ is shown in Figure 5.7.

![Figure 5.7](image)

Figure 5.7. The oriented graph $D_4$.

We show that

$$\gamma(D_4) = k \text{ and } \gamma_H(D_4) = \gamma_o(D_4) = k + 1.$$  

First, we show that $\gamma(D_4) = k$. Since

$$S'' = \{u, u_1, u_2, \ldots, u_{k-2}, w\}$$  

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
is a dominating set, $\gamma(D_4) \leq |S''| = k$. On the other hand, let $S$ be a minimum dominating set of $D_4$. Observe that

(i) for each $j$ with $1 \leq j \leq k - 2$, the vertex $x_j$ is dominated only itself and $u_j$,

(ii) the vertex $u_{k-1}$ is dominated only by itself and by $u$, and

(iii) the vertex $y$ is dominated only by itself and $w$.

Thus $S$ contains at least one vertex from each of the sets $\{u_j, x_j\}$ ($1 \leq j \leq k - 2$), $\{u_{k-1}, u\}$, and $\{y, w\}$. This implies that

$$\gamma(D_4) = |S| \geq (k - 2) + 2 = k.$$ 

Therefore, $\gamma(D_4) = k$.

Next, we show that $\gamma_H(D_4) = k + 1$. Define a red-blue coloring $c$ of $D_4$ by assigning red to each vertex in $S'' \cup \{v\}$ and blue otherwise. Since

(1) each blue vertex $x_j$ ($1 \leq j \leq k - 2$) is $H$-dominated by the red vertex $u_j$,

(2) the blue vertex $u_{k-1}$ is $H$-dominated by the red vertex $u$, and

(3) the blue vertex $y$ is $H$-dominated by the red vertex $w$,

it follows that $c$ is an $H$-coloring with $k + 1$ red vertices and so

$$\gamma_H(D_4) \leq k + 1.$$ 

Next we show that $\gamma_H(D_4) \geq k + 1$. Since $x = 1$ for every vertex $x$ of $D$, it follows by Observation 5.6 that the pairwise disjoint sets $T_i = \{u_i, x_i\}$ ($1 \leq i \leq k - 2$), and $T_{k-1} = \{u_{k-1}, u\}$ satisfy Observation 5.6(a) and $T_k = \{v, w, y\}$ satisfies Observation 5.6(b). Therefore, $\gamma_H(D_4) \geq k + 1$ and so $\gamma_H(D_4) = k + 1$.

By Proposition 5.13, $\gamma_0(D_4) \geq k + 1$. Since $S'' \cup \{v\}$ is an open dominating set, $\gamma_0(D_4) = k + 1$, completing the proof of (d). □

In the proof of Proposition 5.15, we showed that $\gamma_H(\bar{C}_{6k}) = 4k$. Later we will determine $\gamma_H(\bar{C}_n)$ for an arbitrary integer $n \geq 3$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Next, we show that every pair $a, b$ of positive integers with $a \leq b$ and $b \geq 2$ is realizable as the domination number and $H$-domination number of some connected oriented graph $D$. For positive integers $r$ and $s$, let $K_{r,s}$ be the oriented complete bipartite graph with partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ such that

$$E(K_{r,s}) = \{(u_i, v_j) : 1 \leq i \leq r, 1 \leq j \leq s\}.$$

**Corollary 5.16.** For every pair $a, b$ of positive integers with $a \leq b$ and $b \geq 2$, there exists a connected oriented graph $D$ such that $\gamma(D) = a$ and $\gamma_H(D) = b$.

**Proof.** By Proposition 5.15(c), the result holds if $a = b$. Thus, we may assume that $1 \leq a < b$. Let $D = K_{a,b-a}$, whose partite sets are $U$ and $V$ with $|U| = a$ and $|V| = b - a$. Observe that $\text{id}_U = 0$ for each vertex $u \in U$ and so each vertex in $U$ is dominated only by itself. Thus $U$ belongs to every dominating set of $D$. Since $U$ is a dominating set of $D$, it follows that $\gamma(D) = a$. Furthermore, $D$ does not contain the directed $F_3$ as a subdigraph and so $\gamma_H(D) = |V(D)| = b$. \qed

Although every pair $a, b$ of positive integers with $a \leq b$ and $b \geq 2$ is realizable as the domination number and $H$-domination number of some connected oriented graph, this is not the case for the $H$-domination number $\gamma_H$ and open domination number $\gamma_o$. However, we characterize all pairs $a, b$ of integers with $2 \leq a \leq b$ that are realizable as the $H$-domination number and open domination number of some connected oriented graph, beginning with those pairs $a, b$, where $a = 2$.

**Proposition 5.17.** Let $D$ be a connected oriented graph of order $n \geq 3$ for which $\gamma_o(D)$ exists. If $\gamma_H(D) = 2$, then $\gamma_o(D) = 3$.

**Proof.** Let $D$ be a connected oriented graph of order $n \geq 3$ with $\gamma_H(D) = 2$ and let $c$ be a minimum $H$-coloring of $D$ such that $R_c = \{x, y\}$. Thus either $(x, y) \in E(D)$ or $(y, x) \in E(D)$, say the former. This implies that every blue vertex $z \in V(D) - \{x, y\}$ is $H$-dominated by $y$. Hence $(y, z) \in E(D)$ for each $z \in V(D) - \{x, y\}$. Since $\gamma_o(D)$ exists, it follows by Observation 5.9 that $\text{id}_v \geq 1$ for every vertex $v$ of $D$. Since $(x, y) \in$
it follows that \((z', x) \in E(D)\) for some \(z' \in V(D) - \{x, y\}\). Then \(\{x, y, z'\}\) is an open dominating set of \(D\) and so \(\gamma_o(D) \leq 3\). It then follows by Proposition 5.11 that \(\gamma_o(D) = 3\).

We saw in Proposition 5.13 that if \(D\) is a connected oriented graph for which \(\gamma_o(D)\) exists, then \(\gamma_o(D)\) is bounded below by \(\gamma_H(D)\). We now show that \(\gamma_o(D)\) is bounded above by \(\frac{3\gamma_H(D)}{2}\).

**Theorem 5.18.** If \(D\) is a connected oriented graph for which \(\gamma_o(D)\) exists, then

\[
\gamma_o(D) \leq \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor.
\]

**Proof.** Let \(c\) be a minimum \(H\)-coloring of \(D\) and let \(R_c\) be the set of red vertices in \(D\). Furthermore, let \(R_1\) be the subset of \(R_c\) consisting of all vertices that \(H\)-dominate at least one blue vertex in \(D\) and let \(R_2 = R_c - R_1\). (Note that \(R_2\) may be empty.) Thus \(\gamma_H(D) = |R_c| = |R_1| + |R_2|\). We consider two cases.

**Case 1.** \(|R_2| \leq |R_1|\). Then \(|R_2| \leq \frac{1}{2}\gamma_H(D)\). Observe that every blue vertex in \(D\) is adjacent from some red vertex in \(R_1\) and so is openly dominated by some vertex in \(R_1\). Also, every red vertex in \(R_1\) is adjacent from some red vertex in \(R_c\) and so is openly dominated by some vertex in \(R_c\). Thus every vertex in \(V(D) - R_2\) is openly dominated by some vertex in \(R_c\). Since \(\gamma_o(D)\) exists, every vertex in \(D\) has positive indegree. For each vertex \(x \in R_2\), let \(y_x \in V(D)\) such that \(y_x\) is adjacent to \(x\), that is, \(x\) is openly dominated by \(y_x\). Let

\[
Y = \{y_x : x \in R_2\}.
\]

Then \(|Y| \leq |R_2|\) and every vertex in \(R_2\) is openly dominated by some vertex in \(Y\). Hence \(R_c \cup Y\) is an open dominating set of \(D\) and so

\[
\gamma_o(D) \leq |R_c \cup Y| \leq |R_c| + |Y| \leq |R_c| + |R_2| \leq \gamma_H(D) + \frac{1}{2}\gamma_H(D) = \frac{3\gamma_H(D)}{2}.
\]

**Case 2.** \(|R_1| \leq |R_2|\). Then \(|R_1| \leq \frac{1}{2}\gamma_H(D)\). Observe that every blue vertex in \(D\) is openly dominated by some vertex in \(R_1\). For each vertex \(v \in R_1\), let \(w_v \in V(D)\)
such that \(w_v\) is adjacent to \(v\) and let \(W = \{w_v : v \in R_1\}\). Then \(|W| \leq |R_1|\) and every vertex in \(R_1\) is openly dominated by some vertex in \(W\). Again, for each vertex \(x \in R_2\), let \(y_x \in V(D)\) such that \(y_x\) is adjacent to \(x\), that is, \(x\) is openly dominated by \(y_x\). Let \(Y = \{y_x : x \in R_2\}\). Then \(|Y| \leq |R_2|\) and every vertex in \(R_2\) is openly dominated by some vertex in \(Y\). Hence \(R_1 \cup W \cup Y\) is an open dominating set of \(D\) and so

\[
\gamma_o(D) \leq |R_1 \cup W \cup Y| \leq |R_1| + |W| + |Y|
\]

\[
\leq |R_1| + |R_1| + |R_2| = |R_1| + |R_c|
\]

\[
\leq \frac{1}{2} \gamma_H(D) + \gamma_H(D) = \frac{3\gamma_H(D)}{2}.
\]

Therefore, \(\gamma_o(D) \leq \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor\), as desired. \(\Box\)

By Proposition 5.17, if \(D\) is a connected oriented graph of order \(n \geq 3\) for which \(\gamma_o(D)\) exists and \(\gamma_H(D) = 2\), then \(\gamma_o(D) = 3\). Thus, there is no connected oriented graph with \(H\)-domination number 2 and open domination number 4 or more. On the other hand, we show that every pair \(a, b\) of integers with \(3 < a < b < \frac{3a}{2}\) is realizable as the \(H\)-domination number and open domination number of some connected oriented graph.

**Theorem 5.19.** For every pair \(a, b\) of integers with \(3 < a < b < \frac{3a}{2}\), there exists a connected oriented graph \(D\) such that \(\gamma_H(D) = a\) and \(\gamma_o(D) = b\).

**Proof.** By the proof of Proposition 5.15(c), if \(b = a\) or \(b = a + 1\), then there exists a connected oriented graph \(D\) such that \(\gamma_H(D) = a\) and \(\gamma_o(D) = b\). Thus, we may assume that \(b \geq a + 2\). Let

\[
k = b - a - 1 \text{ and } \ell = 3a - 2b + 1.
\]

Since \(b \geq a + 2\) and \(b \leq \left\lfloor \frac{3a}{2} \right\rfloor\), it follows that \(k \geq 1\) and \(\ell \geq 1\).

For each \(i\) with \(1 \leq i \leq k + \ell\), let \(G_i\) be a copy of \(C_3 : v_{i,1}, v_{i,2}, v_{i,3}, v_{i,1}\). First, we construct an oriented graph \(D'\) obtained from the first \(\ell\) copies of of \(G_i\) (\(1 \leq i \leq \ell\)) by identifying all the vertices \(v_{i,1}\) (\(1 \leq i \leq \ell\)) and labeling the identified vertex by \(v_1\). Then we construct the oriented graph \(D\) from \(D'\) and the \(k\) copies of \(G_j\) \((\ell + 1 \leq j \leq \ell + k)\) by
adding a new vertex \( v \) and the \( k + 1 \) new arcs \((v_1, v)\) and \((v_{j,1}, v)\) for \( \ell + 1 \leq j \leq \ell + k \).

(Figure 5.8 shows the oriented graph \( D \) when \( a = 8 \) and \( b = 11 \), in which case \( k = 2 \) and \( \ell = 3 \).)

Figure 41. The oriented graph \( D \) for \( a = 8 \) and \( b = 11 \).

We show that \( \gamma_H(D) = a \) and \( \gamma_o(D) = b \). Let

\[
\begin{align*}
V_1 &= \{v_j\} \cup \{v_{j,1} : \ell + 1 \leq j \leq \ell + k\}, \\
V_2 &= \{v_t,t : 1 \leq t \leq \ell + k\}, \\
V_3 &= \{v_{i,3} : 1 \leq i \leq \ell - 1\}, \\
V_3' &= \{v_{\ell,3}\} \cup \{v_{j,3} : \ell + 1 \leq j \leq \ell + k\}.
\end{align*}
\]

We first show that \( \gamma_H(D) = a \). Since the set \( V_1 \cup V_3 \cup V_3' \) is an \( H \)-dominating set,

\[
\gamma_H(D) \leq |V_1 \cup V_3 \cup V_3'| = (1 + k) + (\ell + k) = 2k + \ell + 1 = 2(b - a - 1) + (3a - 2b + 1) + 1 = a.
\]

To show that \( \gamma_H(D) \geq a \), let \( c \) be a minimum \( H \)-coloring of \( D \) and let \( R_c \) be the set of red vertices of \( D \). By Observation 5.6,

(a) \( R_c \) contains at least one vertex in \( \{v_{i,2}, v_{i,3}\} \) for \( 1 \leq i \leq \ell \);

(b) \( R_c \) contains at least two vertices in \( \{v_{j,1}, v_{j,2}, v_{j,3}\} \) for \( \ell + 1 \leq j \leq \ell + k \).
Thus $|R_c| \geq 2k + \ell$. We claim that $|R_c| > 2k + \ell$. Assume, to the contrary, that $|R_c| = 2k + \ell$. Then $R_c$ contains exactly one vertex in $\{v_{i,2}, v_{i,3}\}$ for $1 \leq i \leq \ell$ and exactly two vertices in $\{v_{j,1}, v_{j,2}, v_{j,3}\}$ for $\ell + 1 \leq j \leq \ell + k$. In particular, $v_1 \notin R_c$. Observe that the only directed path $\bar{P}_3$ in $D$ having $v_1$ as a terminal vertex is $v_{i,2}, v_{i,3}, v_1$ for some $i$ with $1 \leq i \leq \ell$. Since $R_c$ contains exactly one vertex in $\{v_{i,2}, v_{i,3}\}$ for all $i$ with $1 \leq i \leq \ell$, it follows that $v_1$ is not $H$-dominated by $R_c$, which is a contradiction. Therefore,

$$\gamma_H(D) = |R_c| > 2k + \ell + 1 = a$$

and so $\gamma_H(D) = a$.

Now we show that $\gamma_0(D) = b$. First we show that $\gamma_0(D) \leq b$. Since the set $V_1 \cup V_2 \cup V_3$ is an open dominating of $D$, it follows that

$$\gamma_0(D) \leq |V_1 \cup V_2 \cup V_3| = 3k + (\ell + 2) = 3(b - a - 1) + (3a - 2b + 1) + 2 = b.$$ 

To show $\gamma_0(D) \geq b$, let $S_0$ be a minimum open dominating set. Observe that

(i) for each $i$ with $1 \leq i \leq \ell$, the vertex $v_{i,3}$ is only openly dominated by $v_{i,2}$. Thus $v_{i,2} \in S_0$ for $1 \leq i \leq \ell$,

(ii) for each $i$ with $1 \leq i \leq \ell$, the vertex $v_{i,2}$ is only openly dominated by $v_1$ and so $v_1 \in S_0$, and

(iii) the vertex $v_1$ is only openly dominated by some vertex $v_{i,3}$, where $1 \leq i \leq \ell$.

It follows by (i)–(iii) that $S_0$ must contain at least $\ell + 2$ vertices in the set

$$\{v_1\} \cup \left( \bigcup_{i=1}^{\ell} V(G_i) \right).$$

Furthermore, for each $j$ with $\ell + 1 \leq j \leq \ell + k$,

(1) the vertex $v_{j,1}$ can only be openly dominated by $v_{j,3}$,
(2) the vertex \( v_{j,2} \) can only be openly dominated by \( v_{j,1} \), and

(3) the vertex \( v_{j,3} \) can only be openly dominated by \( v_{j,2} \).

This implies that \( \{v_{i,1}, v_{i,2}, v_{i,3}\} \subseteq S_\ell \) for \( \ell + 1 \leq i \leq \ell + k \). Therefore,

\[
\gamma_0(D) = |S_\ell| \geq (3k + (\ell + 2)) = 3(b - a - 1) + (3a - 2b + 3) = b.
\]

This completes the proof. \( \square \)

Combining Proposition 5.17 and Theorems 5.18 and 5.19, we have the following characterization of those pairs \( a, b \) of integers with \( 2 < a < b \) that are realizable as the \( \gamma \)-domination number and open domination number of some connected oriented graph.

**Corollary 5.20.** Let \( a \) and \( b \) be integers with \( 2 < a < b \). Then there exists a connected oriented graph \( D \) such that \( \gamma_H(D) = a \) and \( \gamma_0(D) = b \) if and only if

\[
(a, b) = (2, 3) \quad \text{or} \quad 3 < a < b \leq \left\lfloor \frac{3a}{2} \right\rfloor.
\]

5.3 \( H \)-Domination in Regular Oriented Graphs

A connected oriented graph \( D \) is said to be \( r \)-regular if

\[
id v = \od v = r
\]

for some nonnegative integer \( r \) and for every \( v \in V(D) \). If \( D \) is a connected \( r \)-regular oriented graph of order \( n \), then the underlying graph of \( D \) is \( 2r \)-regular. Thus \( n \geq 2r + 1 \) and so \( r \leq \left\lfloor \frac{n-1}{2} \right\rfloor \). In this section, we investigate \( H \)-domination in \( r \)-regular oriented graphs.

We have seen in (5.2) that if \( D \) is a connected oriented graph of order \( n \geq 2 \), then \( 2 \leq \gamma_H(D) \leq n \). Moreover, by Proposition 5.8, for each pair \( k, n \) of integers with \( 2 \leq k \leq n \), there exists a connected oriented graph \( D \) of order \( n \) with \( \gamma_H(D) = k \). However, this is not the case for \( r \)-regular connected oriented graphs; that is, for fixed
positive integers $r$ and $n$ with $n \geq 2r + 1$, there are pairs $k, n$ of integers with $2 \leq k \leq n$ such that there is no connected $r$-regular oriented graph of order $n$ with $\gamma_H(D) = k$, as we will see in this section.

If $r = 1$, then the directed $n$-cycle $C_n$ is the only connected 1-regular oriented graph of order $n \geq 3$. In Proposition 5.15, we showed that $\gamma_H(C_{6k}) = 4k$ for each positive integer $k$. We now determine the $H$-domination number of $C_n$ for every integer $n \geq 3$.

**Theorem 5.21.** For each integer $n \geq 3$,

$$\gamma_H(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

**Proof.** We first show that $\gamma_H(C_n) \leq \left\lceil \frac{2n}{3} \right\rceil$. Let

$$C_n : v_1, v_2, \ldots, v_n, v_1.$$

Then $n = 3k + t$, where $k \geq 1$ and $0 \leq t \leq 2$. Define a red-blue coloring $\mathcal{c}'$ of $C_n$ by assigning blue to each vertex in the set

$$\{v_{3i} : 1 \leq i \leq k\}$$

and red to the remaining vertices of $C_n$. Since each blue vertex $v_{3i}$ belongs to a copy of $H$ rooted at $v_{3i}$ in $C_n$, namely the directed path: $v_{3i-2}, v_{3i-1}, v_{3i}$, it follows that $\mathcal{c}'$ is an $H$-coloring of $C_n$ with exactly $2k + t$ red vertices. Hence

$$\gamma_H(C_n) \leq 2k + t.$$

Observe that

$$\left\lceil \frac{2n}{3} \right\rceil = \left\lceil \frac{6k + 2t}{3} \right\rceil = 2k + \left\lceil \frac{2t}{3} \right\rceil$$

$$= \begin{cases} 
2k & \text{if } t = 0 \\
2k + 1 & \text{if } t = 1 \\
2k + 2 & \text{if } t = 2 
\end{cases}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
and so \([2n/3] = 2k + t\). Therefore,

\[ \gamma_H(C_n) \leq \left\lceil \frac{2n}{3} \right\rceil. \]

Next, we show that \(\gamma_H(C_n) \geq [2n/3]\). Certainly, at least one vertex of \(C_n\) is colored blue in a minimum \(H\)-coloring of \(C_n\). Assume, without loss of generality, that \(v_{3k}\) is colored blue. By Observation 5.6, at least two vertices in each set \(\{v_{3i-2}, v_{3i-1}, v_{3i}\}\), \(1 \leq i \leq k\), are colored red. Furthermore, since \(v_{3k}\) is blue, it follows by Observation 5.6 that \(v_{3k+1}\) and \(v_{3k+2}\) are red (where the addition in \(3k + 1\) and \(3k + 2\) is done modulo \(n\)). Hence at least \(2k + t\) vertices of \(C_n\) are colored red. Thus

\[ \gamma_H(C_n) \geq 2k + t = \left\lceil \frac{2n}{3} \right\rceil. \]

Therefore, \(\gamma_H(C_n) = [2n/3]\). \(\square\)

We now turn to \(r\)-regular connected oriented graphs for \(r \geq 2\). We first establish some definitions. For a vertex \(u\) of an oriented graph \(D\), the \emph{out-neighborhood} \(N^+(u)\) and \emph{in-neighborhood} \(N^-(u)\) of \(u\) are defined by

\[ N^+(u) = \{x: (u,x) \in E(D)\} \quad \text{and} \quad N^-(u) = \{x: (x,u) \in E(D)\}. \]

The following lemma is useful.

**Lemma 5.22.** Let \(D\) be a connected oriented graph of order \(n\) and let \(\Delta\) be the maximum outdegree among all vertices of \(D\) with positive indegree. Then

\[ \gamma_H(D) \leq n - \Delta. \]

**Proof.** Let \(y\) be a vertex of \(D\) with \(\text{od } y = \Delta\) and \(\text{id } y \geq 1\). Then the red-blue coloring of \(D\) that assigns blue to each vertex in \(N^+(y)\) and red to remaining vertices of \(D\) is an \(H\)-coloring of \(D\) with \(n - \Delta\) red vertices. Therefore, \(\gamma_H(D) \leq n - \Delta\). \(\square\)

**Proposition 5.23.** Let \(r \geq 2\) be an integer. If \(D\) is a connected \(r\)-regular oriented graph of order \(n\), then

\[ 3 \leq \gamma_H(D) \leq n - r. \] (5.4)
Proof. By Lemma 5.22, $\gamma_H(D) \leq n - r$. Thus it remains to establish the lower bound in (5.4). Assume, to the contrary, that there exists a connected $r$-regular oriented graph $D$ such that $\gamma_H(D) = 2$. Let $c$ be a minimum $H$-coloring of $D$ with $R_c = \{x, y\}$. Then either $(x, y)$ or $(y, x)$ is an arc in $D$, say the former. Thus $y$ $H$-dominates all $r = n - 2$ blue vertices of $D$. Since $n \geq 2r + 1$, it follows that $r + 2 \geq 2r + 1$ and so $r \leq 1$, which is a contradiction. Therefore, $\gamma_H(D) \geq 3$. □

The lower bound of the inequality (5.4) is sharp. In order to show this, we first introduce an additional definition. Let $K_{r,s,t}$ be the complete 3-partite graph with partite sets $V_1, V_2, V_3$ with $|V_1| = r, |V_2| = s$, and $|V_3| = t$ and let $\tilde{K}_{r,s,t}$ denote the orientation of $K_{r,s,t}$ such that each arc of $\tilde{K}_{r,s,t}$ is an arc from a vertex in $V_1$ to a vertex in $V_2$, or an arc from a vertex in $V_2$ to a vertex in $V_3$, or an arc from a vertex in $V_3$ to a vertex in $V_1$.

Theorem 5.24. For each integer $r \geq 2$, there is a connected $r$-regular oriented graph $D$ such that

$$\gamma_H(D) = 3.$$  

Proof. We begin with the directed 3-cycle $\tilde{C}_3 : u, v, w, u$ and the oriented graph $\tilde{K}_{r-1,r-1,r-1}$ with partite sets $X, Y, Z$, where then $|X| = |Y| = |Z| = r - 1$. Then the oriented graph $D$ is constructed from $\tilde{C}_3$ and $\tilde{K}_{r-1,r-1,r-1}$ by adding the arcs

$$(u, x), (v, y), (w, z), (x, y), (y, w), \text{ and } (z, u)$$

for all $x \in X$, $y \in Y$, and $z \in Z$. Then $D$ is a connected $r$-regular oriented graph of order $3r$. The oriented graph $D$ is shown in Figure 5.9.

We show that $\gamma_H(D) = 3$. Let $c$ be a red-blue coloring that assigns red to the vertices in $\tilde{C}_3$ and blue to the remaining vertices of $D$. Since

(1) each blue vertex $x \in X$ belongs to the red-red-blue path $w, u, x,$

(2) each blue vertex $y \in Y$ belongs to the red-red-blue path $u, v, y,$ and

(3) each blue vertex $z \in Z$ belongs to the red-red-blue path $v, w, z,$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
it follows that $c$ is an $H$ coloring of $D$ with exactly three red vertices. Thus $\gamma_H(D) \leq 3$ and so $\gamma_H(D) = 3$ by Proposition 5.23.

In the proof of Theorem 5.24, the connected $r$-regular oriented graph $D$ for which $\gamma_H(D) = 3$ has order $3r$. If $r = 2$, for example, this provides a connected 2-regular oriented graph $D$ of order 6 with $\gamma_H(D) = 3$. In fact, the oriented graph $D$ of Figure 5.10 is also a connected 2-regular oriented graph with $\gamma_H(D) = 3$, but $D$ has order 7. There is no such oriented graph of order 8 however, as we now show.

Figure 42. The oriented graph $D$.

Figure 43. A 2-regular oriented graph $D$ of order 7 with $\gamma_H(D) = 3$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proposition 5.25. If $D$ is a connected 2-regular oriented graph of order $n \geq 8$, then
\[ \gamma_H(D) \geq 4. \]

Proof. Assume, to the contrary, that $\gamma_H(D) = 3$. Let $c$ be a minimum $H$-coloring of $D$ with $R_c = \{x, y, z\}$. Since the order of $D$ is at least 8, it follows that $D$ has at least 5 blue vertices. Since $D$ is 2-regular, each red vertex can $H$-dominate at most two blue vertices of $D$. Thus there are two red vertices, each of which $H$-dominates exactly two blue vertices. We may assume, without loss of generality, that each of $x$ and $y$ $H$-dominates exactly two blue vertices. This implies that both $x$ and $y$ are adjacent from $z$ and so $z$ cannot $H$-dominate any blue vertex. Therefore, at least one blue vertex is not $H$-dominated by any vertex in $R_c$, a contradiction. \[ \square \]

The oriented graph $D$ in Figure 5.11 is a connected 2-regular oriented graph of order 8 with $\gamma_H(D) = 4$. The red-blue coloring shown in the figure is an $H$-coloring of $D$, implying that $\gamma_H(D) \leq 4$. It then follows by Proposition 5.25 that $\gamma_H(D) = 4$.

![Figure 44. A 2-regular oriented graph $D$ of order 8 with $\gamma_H(D) = 4$.](image)

Proposition 5.25 can, in fact, be generalized.

Theorem 5.26. Let $r \geq 2$ be an integer. If $D$ is a connected $r$-regular oriented graph of order $n \geq 2r + 1$, then
\[ \gamma_H(D) \geq \min \left\{ \frac{n + r}{1 + r} \cdot \frac{n}{r} \right\}. \]
Proof. Let $D$ be a connected $r$-regular oriented graph of order $n \geq 2r + 1$ with $\gamma_H(D) = k$. By Proposition 5.23, we know that $\gamma_H(D) \leq n - r$ and so we may assume that $k < n$. Let $c$ be a minimum $H$-coloring of $D$ and let $R_c$ be the set of the red vertices of $c$ in $D$. Furthermore, let $R_1$ be the subset of $R_c$ consisting of all vertices that $H$-dominate at least one blue vertex in $D$ and let $R_2 = R_c - R_1$. Thus $k = |R_c| = |R_1| + |R_2|$. Let

$$B = V(D) - R_c$$

be the set of the blue vertices of $c$ in $D$. We consider two cases.

Case 1. $R_2 = \emptyset$. Since $D$ is $r$-regular, each vertex in $R_1$ can $H$-dominate at most $r$ blue vertices. Furthermore, since each vertex $v$ in $R_1$ $H$-dominates at least one blue vertex and $R_2 = \emptyset$, it follows that $v$ must also be adjacent from some red vertex $u$ in $R_1$. Hence there are at least $k$ arcs in $D$, both of whose incident vertices are red. This implies that $R_1$ can $H$-dominate at most $kr - k$ blue vertices and so $|B| \leq kr - k$. Therefore,

$$n = |B| + |R_c| \leq (kr - k) + k = kr$$

and so $k \geq n/r$.

Case 2. $R_2 \neq \emptyset$. Since each blue vertex can only be $H$-dominated by some red vertex in $R_1$ and each vertex in $R_1$ can $H$-dominate at most $r$ blue vertices,

$$|B| \leq |R_2|r = (|R_c| - |R_2|)r.$$  

Since $|R_2| \geq 1$, it follows that

$$n = |B| + |R_c| \leq (|R_c| - |R_2|)r + |R_c|$$

$$\leq (|R_c| - 1)r + |R_c| = (k - 1)r + k$$

$$= k(r + 1) - r.$$  

Hence $k \geq (n + r)/(r + 1)$.  

\[\square\]
Since \((n + r)/(1 + r) \leq n/r\) if \(n \geq r^2\), the following is a consequence of Theorem 5.26.

**Corollary 5.27.** Let \(r \geq 2\) be an integer. If \(D\) is a connected \(r\)-regular oriented graph of order \(n \geq r^2\), then

\[
\gamma_H(D) \geq \frac{n + r}{1 + r}.
\]

The lower bound in Theorem 5.26 (or in Corollary 5.27) is sharp. In fact, more can be said.

**Proposition 5.28.** For each integer \(r \geq 2\), there exist an integer \(n \geq r^2\) and a connected \(r\)-regular oriented graph \(D\) of order \(n\) such that

\[
\gamma_H(D) = \frac{n + r}{1 + r} = \min \left\{ \frac{n + r}{1 + r}, \frac{n}{r} \right\}.
\]

**Proof.** Let \(K_{1,r} : u, v_1, v_2, \ldots, v_r\) be the star of order \(r + 1\) centered at \(u\), that is, \(\deg u = r\). Recall that \(\bar{K}_{1,r}\) is the orientation of \(K_{1,r}\) such that \((u, v_i)\) is an arc in \(\bar{K}_{1,r}\) for \(1 \leq i \leq r\). For each \(j\) with \(1 \leq j \leq r\), let \(D_j\) be a copy of \(\bar{K}_{1,r}\), where

\[ V(D_j) = \{v_j, w_{j,1}w_{j,2}, \ldots, w_{j,r}\} \text{ and } \od v_j = r. \]

Then the digraph \(D\) is obtained from the digraphs \(D_j\) \((1 \leq j \leq r)\) by adding

1. the arcs \((w_{j,1}, u)\) for \(1 \leq j \leq r\),
2. the arcs \((w_{j,1}, v_i)\) for \(1 \leq i, j \leq r\) and \(i \neq j\),
3. the arcs \((w_{j,t}, w_{j,1})\) for \(1 \leq j \leq r\) and \(2 \leq t \leq r\), and
4. the arcs \((w_{r,s}, w_{1,t})\) and \((w_{j,s}, w_{j+1,t})\) for \(1 \leq j \leq r - 1, 2 \leq s, t \leq r\).

The digraph \(D\) for \(r = 3\) is shown in Figure 5.12.

Observe that

1. \(N^+(u) = \{v_i : 1 \leq i \leq r\}\) and \(N^-(u) = \{w_{j,1} : 1 \leq j \leq r\}\);
2. for $1 \leq i \leq r$,
   \[ N^+(v_i) = \{w_{i,t} : 1 \leq t \leq r\}, \]
   \[ N^-(v_i) = \{u\} \cup \{w_{j,1} : 1 \leq j \leq r, j \neq i\}; \]

3. for $1 \leq j \leq r$,
   \[ N^+(w_{j,1}) = \{u\} \cup \{v_i : 1 \leq i \leq r, i \neq j\}, \]
   \[ N^-(w_{j,1}) = \{v_j\} \cup \{w_{j,t} : 2 \leq t \leq r\}; \]

4. for $2 \leq t \leq r$,
   \[ N^+(w_{1,t}) = \{w_{1,1}\} \cup \{w_{2,s} : 2 \leq s \leq r\}, \]
   \[ N^-(w_{1,t}) = \{v_1\} \cup \{w_{r,s} : 2 \leq s \leq r\}, \]
\[ N^+(w_{r,t}) = \{w_{r,1}\} \cup \{w_{1,s} : 2 \leq s \leq r\}, \]
\[ N^-(w_{r,t}) = \{v_r\} \cup \{w_{r-1,s} : 2 \leq s \leq r\}; \]

5. for \(2 \leq j \leq r - 1\) and \(2 \leq t \leq r,\)
\[ N^+(w_{j,t}) = \{w_{j,1}\} \cup \{w_{j+1,s} : 2 \leq s \leq r\}, \]
\[ N^-(w_{j,t}) = \{v_j\} \cup \{w_{j-1,s} : 2 \leq s \leq r\}. \]

Hence \(D\) is a connected \(r\)-regular oriented graph of order \(n = r^2 + r + 1\). Since
\[ \frac{n + r}{r + 1} = \frac{(r^2 + r + 1) + r}{r + 1} = r + 1 \]
and \(n \geq r^2\), it follows by Corollary 5.27 that
\[ \gamma_H(D) \geq r + 1. \]

On the other hand, let \(c\) be the red-blue coloring of \(D\) with
\[ R_c = \{u\} \cup \{v_j : 1 \leq j \leq r\}. \]
Since each blue vertex \(w_{j,t}\) \((1 \leq j, t \leq r)\) belongs to the red-red-blue path \(u, v_j, w_{j,t}\), it follows that \(c\) is an \(H\)-coloring of \(D\) and so
\[ \gamma_H(D) \leq |R_c| = r + 1. \]
Therefore,
\[ \gamma_H(D) = r + 1 = \frac{n + r}{1 + r}, \]
as desired. \(\square\)

**Proposition 5.29.** For each integer \(r \geq 2\), there exist an integer \(n\) with \(2r + 1 \leq n \leq r^2\) and a connected \(r\)-regular oriented graph \(D\) of order \(n\) for which
\[ \gamma_H(D) = \frac{n}{r} = \min \left\{ \frac{n + r}{1 + r}, \frac{n}{r} \right\}. \]

**Proof.** Let \(k\) be an integer with \(3 \leq k \leq r\) and let

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
be a directed $k$-cycle. For each $i$ with $1 \leq i \leq k$, let $D_i \cong K_{r-1}$ be the empty digraph of order $r-1$ with

$$V(D_i) = W_i = \{w_{ij} : 1 \leq j \leq r-1\}.$$  

Then the oriented graph $D$ is constructed from $\bar{C}$ and $D_i$ ($1 \leq i \leq k$) by adding

1. the arcs $(w_{1,s}, w_{k,t})$ for $1 \leq s, t \leq r-1$ and $(w_{i,s}, w_{i-1,t})$ for $2 \leq i \leq k$ and $1 \leq s, t \leq r-1$,

2. the arcs $(v_1, w_{1,j})$ and $(w_{1,j}, v_k)$ for $1 \leq j \leq r-1$ and $(v_i, w_{i,j})$ and $(w_{i,j}, v_{i-1})$ for $2 \leq i \leq k$ and $1 \leq j \leq r-1$.

Figure 46. The $r$-regular oriented graph $D$ for $k = 4$.

Then the order of $D$ is $n = k(r - 1) + r = kr$. Furthermore,

1. $N^+(v_1) = \{v_2\} \cup W_1$ and $N^-(v_1) = \{v_k\} \cup W_2$,

2. for $2 \leq i \leq k-1$,

$$N^+(v_i) = \{v_{i+1}\} \cup W_i$$

and

$$N^-(v_i) = \{v_{i-1}\} \cup W_{i+1}.$$
3. $N^+(v_k) = \{v_1\} \cup W_k$ and $N^-(v_k) = \{v_{k-1}\} \cup W_1$.

4. for $1 \leq j \leq r - 1$,
   
   $N^+(w_{i,j}) = \{v_k\} \cup W_k$ and $N^-(w_{i,j}) = \{v_1\} \cup W_2$,

5. for $2 \leq i \leq k - 1$ and $1 \leq j \leq r - 1$,
   
   $N^+(w_{i,j}) = \{v_{i-1}\} \cup W_{i-1}$ and $N^-(w_{i,j}) = \{v_1\} \cup W_{i+1}$,

6. for $1 \leq j \leq r - 1$,
   
   $N^+(w_{k,j}) = \{v_{k-1}\} \cup W_{k-1}$ and $N^-(w_{k,j}) = \{v_k\} \cup W_1$.

Hence $D$ is a connected $r$-regular oriented graph of order $n = kr$. Thus

$$k = \frac{n}{r}.$$

Since $n = kr \leq r^2$, it follows by Theorem 5.26 that

$$\gamma_H(D) \geq \frac{n}{r} = k.$$

Therefore, it suffices to show that

$$\gamma_H(D) \leq k.$$

Let $c$ be the red-blue coloring that assigns red to the vertices of $C$ and blue to the remaining vertices of $D$. For each $i$ and $j$ ($1 \leq i \leq k$, $1 \leq j \leq r - 1$), where $i$ is expressed as an integer modulo $k$, since each blue vertex $w_{i,j} \in W_i$ belongs to the red-red-blue path $v_{i-1}, v_i, w_{i,j}$, it follows that $c$ is an $H$-coloring of $D$ with exactly $k$ red vertices. Thus $\gamma_H(D) \leq k$ and so $\gamma_H(D) = k$. □

As a final note, we mention that when $k = r$ in the proof of Proposition 5.29, the digraph $D$ constructed has order $n = r^2$ and therefore,

$$\gamma_H(D) = r = \frac{n}{r} = \frac{n + r}{1 + r}.$$

We have seen in Proposition ?? that if $D$ is a connected $r$-regular oriented graph of order $n$, then $\gamma_H(D)$ is bounded above by $n - r$. This upper bound can be improved, as we show next.
Proposition 5.30. Let \( r \geq 2 \) be an integer. If \( D \) is a connected \( r \)-regular oriented graph of order \( n \), then

\[
\gamma_H(D) \leq n - \left\lfloor \frac{3r}{2} \right\rfloor + 1. \tag{5.5}
\]

Proof. Let \( D \) be a connected \( r \)-regular oriented graph and \( x, y \in V(D) \) such that \( (x, y) \in E(D) \). Let

\[
N^+(y) = \{y_1, y_2, \ldots, y_r\} \quad \text{and} \quad N^+[y] = N^+(y) \cup \{y\}.
\]

Furthermore, let

\[
D_{x,y} = (N^+(y) \cup \{x\})
\]

be the subdigraph induced by \( N^+(y) \cup \{x\} \). Since the order of \( D_{x,y} \) is \( r + 1 \), the size of \( D_{x,y} \) is at most \( \frac{r(r+1)}{2} \) and so

\[
\sum_{v \in V(D_{x,y})} \text{od}_{D_{x,y}} v \leq \frac{r(r+1)}{2}, \tag{5.6}
\]

where \( \text{od}_{D_{x,y}} v \) is the outdegree of \( v \) in \( D_{x,y} \). Let

\[
k = \min\{\text{od}_{D_{x,y}} v : v \in V(D_{x,y})\}.
\]

By (5.6), \( k \leq r/2 \) and so

\[
k \leq \left\lfloor \frac{r}{2} \right\rfloor.
\]

We consider two cases.

Case 1. \( k = \text{od}_{D_{x,y}} y_i \) for some \( 1 \leq i \leq r \), say \( k = \text{od}_{D_{x,y}} y_1 \). Since \( \text{od}_{Dy_1} = r \), it follows that \( y_1 \) is adjacent to at least \( \lceil r/2 \rceil \) vertices that are not in \( D_{x,y} \). Let \( Z \) be the set of vertices of maximum cardinality such that

\[
Z \subseteq N^+(y_1) \quad \text{and} \quad Z \cap V(D_{x,y}) = \emptyset.
\]

Then \( |Z| \geq \lceil r/2 \rceil \). Define the red-blue coloring \( c \) of \( D \) that assigns blue to each vertex in

\[
(N^+(y) - \{y_1\}) \cup Z = \{y_2, y_3, \ldots, y_r\} \cup Z
\]

and red to the remaining vertices of \( D \). Since
(i) each blue vertex \( y_i \) \((2 \leq i \leq r)\) is \( H \)-dominated by \( y \), and

(ii) each blue vertex in \( Z \) is \( H \)-dominated by \( y_1 \),

it follows that \( c \) is an \( H \)-coloring of \( D \) with

\[
R_c = V(D) - (\{y_2, y_3, \cdots , y_r\} \cup Z).
\]

Therefore,

\[
\gamma_H(D) \leq |R_c| = n - [(r - 1) + |Z|] \\
\leq n - (r - 1 + \left\lceil \frac{r}{2} \right\rceil) = n - \left\lceil \frac{3r}{2} \right\rceil + 1.
\]

**Case 2.** \( k = od_{D_x,y} x \) and \( od_{D_x,y} y_i > k \) for all \( i \) with \( 1 \leq i \leq r \). Thus \( x \) is adjacent to at least \( \lceil r/2 \rceil \) vertices that are not in \( D_{x,y} \). Since \((x, y) \in E(D)\), it follows that \( x \) is adjacent to at least \( \lceil r/2 \rceil - 1 \) vertices that are not in \( V(D_{x,y}) \cup \{y\} \). Let \( X \) be the set of vertices of maximum cardinality such that

\[
X \subseteq N^+(x) \text{ and } X \cap (V(D_{x,y}) \cup \{y\}) = \emptyset.
\]

Then \(|X| \geq \lceil r/2 \rceil - 1\). We consider two subcases.

**Subcase 2.1.** There exists \( z \notin N^+(y) \) such that \((z, x) \in E(D)\). Define a red-blue coloring \( c \) by assigning blue to each vertex in

\[
N^+(y) \cup X = \{y_1, y_2, \cdots , y_r\} \cup X
\]

and red to the remaining vertices of \( D \). Since

(i) each blue vertex in \( N^+(y) \) is \( H \)-dominated by \( y \), and

(ii) each blue vertex in \( X \) is \( H \)-dominated by \( x \),

it follows that \( c \) is an \( H \)-coloring of \( D \) with \( R_c = V(D) - (N^+(y) \cup X) \). Therefore,

\[
\gamma_H(D) \leq |R_c| = n - (r + |X|) \\
\leq n - \left( r + \left\lceil \frac{r}{2} \right\rceil - 1 \right) = n - \left\lceil \frac{3r}{2} \right\rceil + 1.
\]
Subcase 2.2. Subcase 2.1 does not occur. This implies that $N^-(x) \subseteq N^+(y)$.

Since $id_x = r$ and $|N^+(y)| = r$, it follows that $N^-(x) = N^+(y)$ and so $|X| = r - 1$.

Define the red-blue coloring $c$ of $D$ that assigns blue to each vertex in

$$(N^+(y) - \{y_1\}) \cup X = \{y_2, y_3, \ldots, y_r\} \cup X$$

and red to the remaining vertices of $D$. Since

(i) each blue vertex $y_i$ ($2 \leq i \leq r$) is $H$-dominated by $y$, and

(ii) each blue vertex in $X$ is $H$-dominated by $x$,

it follows that $c$ is an $H$-coloring of $D$ with

$$R_c = V(D) - (\{y_2, y_3, \ldots, y_r\} \cup X).$$

Since $r \geq 2$, it follows that

$$\gamma_H(D) \leq |R_c| = n - [r - 1] + |X|$$

$$= n - 2(r - 1) \leq n - \left\lfloor \frac{3r}{2} \right\rfloor + 1,$$

as desired. \qed

Both equality and strict inequality in (5.5) are possible. In order to show this, we present an additional result. For each integer $r \geq 2$, let $K_{2r+1}$ be the complete graph of order $2r + 1$ with

$$V(K_{2r+1}) = \{v_0, v_1, \ldots, v_{2r}\}.$$

Define the orientation $\vec{K}_{2r+1}$ of $K_{2r+1}$ by

$$E(\vec{K}_{2r+1}) = \{(v_i, v_{i+j}) : 0 \leq i \leq 2r, 1 \leq j \leq r\},$$

where the subscripts are expressed as the integers $0, 1, 2, \ldots, 2r$ modulo $2r + 1$. Observe that $\vec{K}_{2r+1}$ is $r$-regular.
Proposition 5.31. For each integer \( r \geq 2 \),
\[ \gamma_H(\tilde{K}_{2r+1}) = 3. \]

Proof. Let \( D = \tilde{K}_{2r+1} \). First, we show that \( \gamma_H(D) \geq 3 \). Assume, to the contrary, that \( \gamma_H(D) = 2 \). Let \( c \) be an \( H \)-coloring of \( D \) with exactly two red vertices, say \( x \) and \( y \). Assume, without loss of generality, that \((x, y) \in E(D)\). Since \( y \) is the only red vertex that is adjacent from a red vertex, \( y \) is the only red vertex that can \( H \)-dominate the blue vertices of \( D \). Since \( \text{od}_y = r \), it follows that \( y \) can \( H \)-dominate at most \( r \) blue vertices of \( D \). On the other hand, \( D \) has \( 2r - 1 \) blue vertices. Since \( 2r - 1 > r \) for \( r \geq 2 \), it follows that \( c \) is not an \( H \)-coloring of \( D \), a contradiction.

We next show that \( \gamma_H(D) \leq 3 \). Let \( c^* \) be the red-blue coloring that assigns red to the vertices of the set \( \{v_0, v_1, v_{r+1}\} \) and blue to the remaining vertices of \( D \). Since each blue vertex \( v_i \) (\( 2 \leq i \leq r \)) is adjacent from the red vertex \( v_1 \), and \( v_1 \) is adjacent from the red vertex \( v_0 \), it follows that \( v_i \) is \( H \)-dominated by \( v_1 \). Since each blue vertex \( v_j \) (\( r + 2 \leq j \leq 2r \)) is adjacent from the red vertex \( v_{r+1} \), and \( v_{r+1} \) is adjacent from the red vertex \( v_1 \), it follows that \( v_j \) is \( H \)-dominated by \( v_{r+1} \). Hence \( c^* \) is an \( H \)-coloring of \( D \) with three red vertices and so \( \gamma_H(D) \leq 3 \). Therefore, \( \gamma_H(D) = 3 \). \( \square \)

Figure 47. A minimum \( H \)-coloring of \( \tilde{K}_7 \).

Observe that the 2-regular tournament \( \tilde{K}_5 \) has order \( n = 5 \) and the 3-regular
tournament $\bar{K}_7$ has order $n = 7$. In each case, $n - \lfloor 3r/2 \rfloor + 1 = 3$ and $\gamma_H(\bar{K}_5) = \gamma_H(\bar{K}_7) = 3$. Therefore, the equality in (5.5) holds for $\bar{K}_5$ and $\bar{K}_7$. On the other hand, the 4-regular tournament $\bar{K}_9$ has order $n = 9$. In this case, $n - \lfloor 3r/2 \rfloor + 1 = 4$; while $\gamma_H(\bar{K}_9) = 3$. Hence the strict inequality in (5.5) holds for $\bar{K}_9$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
6 TOPICS FOR FURTHER STUDY

6.1 Extremal Problems in $F$-Domination

For positive integers $n$ and $m$ with $2 \leq n - 1 \leq m \leq \binom{n}{2}$, let $G(n, m)$ be the set of all connected graphs of order $n$ and size $m$. For the 2-stratified graph $F$ defined in Chapter 2, define

$$
\gamma_F(n, m) = \min \{ \gamma_F(G) : G \in G(n, m) \}
$$

$$
\Gamma_F(n, m) = \max \{ \gamma_F(G) : G \in G(n, m) \}.
$$

Since $F$ contains exactly one red vertex,

$$
1 \leq \gamma_F(n, m) \leq \Gamma_F(n, m) \leq n.
$$

We have seen that if $G$ is a connected graph $G$ of order $n \geq 3$, then $\gamma_F(G) \neq n - 1$. Thus

$$
\gamma_F(n, m) \neq n - 1 \text{ and } \Gamma_F(n, m) \neq n - 1
$$

for all integers $n, m$ with $2 \leq n - 1 \leq m \leq \binom{n}{2}$. Now the question is: For each fixed pair $n, m$ of integers with $2 \leq n - 1 \leq m \leq \binom{n}{2}$, how large or how small can $\gamma_F(n, m)$ and $\Gamma_F(n, m)$ be?

The values of $\gamma_F(n, m)$ can be determined. In order to show this, we first present a result, which is a consequence of Proposition 2.10,

**Corollary 6.1.** If $n$, $m_1$, and $m_2$ are integers with $2 \leq n - 1 \leq m_1 \leq m_2 \leq \binom{n}{2}$, then

$$
\gamma_F(n, m_1) \geq \gamma_F(n, m_2) \text{ and } \Gamma_F(n, m_1) \geq \Gamma_F(n, m_2).
$$

By Corollary 6.1, the sequences $\{\gamma_F(n, m)\}$ and $\{\Gamma_F(n, m)\}$ are nonincreasing with respect to $m$ for each fixed $n$. 

162
Theorem 6.2. Let \( n \) and \( m \) be integers with \( 2 \leq n - 1 \leq m \leq \binom{n}{2} \). Then
\[
\gamma_F(n, m) = \begin{cases} 
3 & \text{if } n = 3 \text{ and } m = 2 \\
2 & \text{if } m = n - 1 \text{ and } n \geq 4 \\
1 & \text{if } n = m = 3 \text{ or } 4 \leq n \leq m \leq \binom{n}{2}
\end{cases}
\]

Proof. Note that \( P_3 \) is the only graph of order 3 and size 2 and \( K_3 \) is the only graph of order 3 and size 3. Since \( \gamma_F(P_3) = 3 \) and \( \gamma_F(K_3) = 1 \), it follows that
\( \Gamma_F(3, 2) = 3 \) and \( \gamma_F(3, 3) = 1 \).

Now let \( n \geq 4 \). First, assume that \( m = n - 1 \). By Theorem 2.48, \( \gamma_F(n, n - 1) \geq 2 \).

On the other hand, let \( T \) be a double star. Then \( \gamma_F(T) = 2 \) by Theorem 2.48. This implies that \( \gamma_F(n, n - 1) \leq 2 \). Therefore, \( \gamma_F(n, n - 1) = 2 \).

Next, assume that \( n \leq m \leq \binom{n}{2} \). Let \( C_3 : v_1, v_2, v_3, v_1 \) be a cycle of order 3 and let \( G \) be the graph obtained from \( C_3 \) by adding \( n - 3 \) new vertices \( u_1, u_2, \ldots, u_{n-3} \) and joining each vertex \( u_i \) \( (1 \leq i \leq n - 3) \) to \( v_1 \). Then \( G \) is a unicyclic graph of order \( n \) and size \( m = n \). The red-blue coloring of \( G \) defined by assigning red to \( v_2 \) and blue to the remaining vertices of \( G \) is an \( F \)-coloring of \( G \) with exactly one red vertex. Therefore, \( \gamma_F(G) = 1 \) and so \( \gamma_F(n, n) = 1 \). It then follows by Corollary 6.1 that \( \gamma_F(n, m) = 1 \) for all \( n, m \) with \( n \leq m \leq \binom{n}{2} \).

It remains to determine \( \Gamma_F(n, m) \). Since \( \gamma_F(K_n) = 1 \) and \( \gamma_F(K_{1,n-1}) = n \) for \( n \geq 3 \), we have the following.

Observation 6.3. For \( n \geq 3 \), \( \Gamma_F(n, n - 1) = n \) and \( \Gamma_F(n, \binom{n}{2}) = 1 \).

We have seen in Theorem 2.47 that if \( G \) is a graph of order \( n \geq 3 \) and size \( m \geq \binom{n-1}{2} + 2 \), then every vertex of \( G \) forms an \( F \)-dominating set of \( G \). Thus each of such graph has \( F \)-domination number 1. Therefore, we have the following.

Proposition 6.4. Let \( n \geq 3 \) be an integer. If \( m \geq \binom{n-1}{2} + 2 \), then \( \Gamma_F(n, m) = 1 \).

We plan to investigate \( \Gamma_F(n, m) \) for \( n \leq m \leq \binom{n-1}{2} + 1 \).
6.2 Realizable Triples in $H$-Domination

Let $H$ be the 2-stratified oriented graph defined in Chapter 5. We have seen that the $H$-domination number of a connected oriented graph is intermediate to its domination number and open domination number. For a connected oriented graph $D$, the condition id $x \geq 1$ for every $x \in V(D)$ assures us that $\gamma_o(D)$ is defined. Of course, $\gamma(D)$ and $\gamma_H(D)$ are defined for all oriented graphs $D$. This suggests the following question.

**Problem 6.5.** For which triples $A, B, C$ of positive integers with $A \leq B \leq C$, does there exist a connected oriented graph $D$ such that $\gamma(D) = A$, $\gamma_H(D) = B$, and $\gamma_o(D) = C$?

We have also seen that for every connected oriented graph $D$ for which $\gamma_o(D)$ exists,

$$1 \leq \gamma(D) \leq \gamma_H(D) \leq \gamma_o(D) \leq \min \left\{ 2A, \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor \right\} \text{ and } \gamma_o(D) \geq 3.$$

For this reason, by a *triple*, we mean (in this section) an ordered triple $(A, B, C)$ of positive integers with

$$A \leq B \leq C \leq \min \left\{ 2A, \left\lfloor \frac{3B}{2} \right\rfloor \right\} \text{ and } C \geq 3. \quad (6.1)$$

Furthermore, we define a triple $(A, B, C)$ to be *realizable* if there is a connected oriented graph $D$ such that

$$\gamma(D) = A, \quad \gamma_H(D) = B, \quad \text{and} \quad \gamma_o(D) = C. \quad (6.2)$$

Certainly, there are infinitely many triples that are not realizable. For example, by Theorem 5.10, every triple $(1, B, C)$ is nonrealizable for all positive integers $B$ and $C$. Also, $(2, 2, 4)$ is nonrealizable by Proposition 5.17. On the other hand, there are infinitely many realizable triples, as we show next.

**Proposition 6.6.** For each triple $A, B, C$ of positive integers with

$$3 \leq A < B < C \leq \min \left\{ 2A, \left\lfloor \frac{B}{2} \right\rfloor + A - 1 \right\}, \quad (6.3)$$

...
there exists a connected oriented graph $D$ such that

$$\gamma(D) = A, \gamma_H(D) = B, \text{ and } \gamma_0(D) = C.$$  \hfill (6.4)

**Proof.** For a triple $A, B, C$ of positive integers satisfying (6.3), let

$$k = B - A, \ell = C - B, \text{ and } s = 2A + B - 2C - 1.$$ 

It then follows by (6.3) that $k$, $\ell$, and $s$ are positive integers.

For each $i$ with $1 \leq i \leq k$, let $G_i$ be a directed 4-cycle $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,1}$. For each $j$ with $1 \leq j \leq \ell$, let $H_j$ be the oriented graph obtained from the directed 3-cycle: $v_{j,1}, v_{j,2}, v_{j,3}, v_{j,1}$ by adding a new vertex $v_{j,0}$ and a new arc $(v_{j,1}, v_{j,0})$. For each $t$ with $1 \leq t \leq s$, let $I_t$ be the directed path $y_{t,0}, y_{t,1}, y_{t,2}$ of order 3. Then $D$ is the oriented graph obtained from the $k$ copies of $G_i$, the $\ell$ copies of $H_j$, and the $s$ copies of $I_t$ by identifying all the vertices $x_{i,1}$ ($1 \leq i \leq k$), $v_{j,0}$ ($1 \leq j \leq \ell$), and $y_{t,0}$ ($1 \leq t \leq s$) and labeling the identified vertex $v$.

We show that $D$ has the desired properties described in (6.4). Let

$$
\begin{align*}
X_2 &= \{x_{i,2} : 1 \leq i \leq k\}, \\
X_3 &= \{x_{i,3} : 1 \leq i \leq k\}, \\
V &= \{v_{j,1}, v_{j,3} : 1 \leq j \leq \ell\}, \\
V_2 &= \{v_{j,2} : 1 \leq j \leq \ell\}, \\
T &= \{y_{t,1} : 1 \leq t \leq s\}.
\end{align*}
$$

We first show that $\gamma(D) = A$. Since the set $V \cup X_3 \cup T \cup \{v\}$ is a dominating set,

$$\gamma(D) \leq |V \cup X_3 \cup T \cup \{v\}| = 2\ell + k + s + 1 = A.$$ 

To show that $\gamma(D) \geq A$, let $S$ be a minimum dominating set of $D$.

For each $i$ ($1 \leq i \leq k$), since

(i) each vertex $x_{i,2}$ is only dominated by $v = x_{i,1}$ or $x_{i,2}$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
(ii) each vertex $x_{i,3}$ is only dominated by $x_{i,2}$ or $x_{i,3}$, and

(iii) each vertex $x_{i,4}$ is only dominated by $x_{i,3}$ or $x_{i,4}$,

it follows that no single vertex can dominate the vertices in the set $V(G_i)$. Thus at least two vertices in the set $V(G_i)$ must belong to $S$ for $1 \leq i \leq k$.

For each $j$ with $1 \leq j \leq \ell$, since

1. every vertex in $V(H_j) - \{v_{j,0}\}$ is only dominated by a vertex in $V(H_j) - \{v_{j,0}\}$, and

2. no single vertex in $V(H_j) - \{v_{j,0}\}$ can dominate all vertices in $V(H_j) - \{v_{j,0}\}$,

it follows that $S$ must contain at least two vertices in the set $V(H_j) - \{v_{j,0}\}$ for each $j$ with $1 \leq j \leq \ell$.

For each $t$ with $1 \leq t \leq s$, since each vertex $y_{t,2}$ can only be dominated by $y_{t,1}$ or by $y_{t,2}$, it follows that at least one vertex in $V(H_t) - \{y_{t,0}\}$ must belong to $S$ for $1 \leq t \leq s$.

Therefore,

$$\gamma(D) = |S| \geq 2k + 2\ell + s \geq 2\ell + k + 1 + s = A,$$

and so $\gamma(D) = 2\ell + k + 1 + s = A$.

Next, we show that $\gamma_H(D) = B$. Observe that the set $V \cup X_2 \cup X_3 \cup T \cup \{v\}$ is an $H$-dominating set and so

$$\gamma_H(D) \leq |V \cup X_2 \cup X_3 \cup T \cup \{v\}| = 2\ell + 2k + s + 1 = B.$$ 

To show that $\gamma_H(D) \geq B$, let $c$ be a minimum $H$-coloring of $D$.

For each integer $i (1 \leq i \leq k)$, we show that at least two vertices in $V(G_i) - \{x_{i,1}\}$ are colored red by $c$. For $1 \leq i \leq k$, since $x_{i,4}$ is only $H$-dominated by $x_{i,3}$ or by itself, we consider these two cases.

Case 1. $x_{i,4}$ is $H$-dominated by $x_{i,3}$. Then $x_{i,2}$ and $x_{i,3}$ must be colored red by $c$, and so at least two vertices in each set $V(G_i) - \{x_{i,1}\}$ are in $R_c$, where $1 \leq i \leq k$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Case 2. \( x_{i,A} \) is \( H \)-dominated by itself. Then \( x_{i,A} \) is colored red by \( c \) for all \( 1 \leq i \leq k \). Moreover, \( x_{i,3} \) is only \( H \)-dominated by itself, or by \( x_{i,2} \), thus at least one vertex in each set \( \{x_{i,2}, x_{i,3}\} \) (\( 1 \leq i \leq k \)) is colored red by \( c \).

Therefore, at least two vertices in \( V(G_i) - \{x_{i,1}\} \) (\( 1 \leq i \leq k \)) are colored red by \( c \).

For each \( j \) with \( 1 \leq j \leq \ell \), observe that a vertex in \( V(H_j) - \{v_{j,0}\} \) is only \( H \)-dominated by a vertex in \( V(H_j) - \{v_{j,0}\} \) and no single vertex in \( V(H_j) - \{v_{j,0}\} \) can \( H \)-dominate all vertices in \( V(H_j) - \{v_{j,0}\} \). It follows that \( R_c \) must contain at least two vertices in the set \( V(H_j) - \{v_{j,0}\} \) for \( 1 \leq j \leq \ell \).

For each \( t \) with \( 1 \leq t \leq s \), the vertex \( y_{t,2} \) is only \( H \)-dominated by itself, or by \( y_{t,1} \). Thus at least one of \( y_{t,1} \) and \( y_{t,2} \) must be colored red by \( c \). If \( y_{t,2} \) is colored red, then \( y_{t,1} \) is only \( H \)-dominated by itself or by \( v \), and so at least one of \( v \) and \( y_{t,1} \) must be colored red. If \( y_{t,2} \) is colored blue, then \( y_{t,1} \) and \( v \) must be colored red. Thus in either case, at least two vertices in \( \{v, y_{t,1}, y_{t,2}\} \) must be colored red for \( 1 \leq t \leq s \). This implies that at least \( s + 1 \) vertices in \( \sum_{i=1}^{s} V(H_i) \) must be red.

Therefore,

\[
\gamma_H(D) \geq 2k + 2\ell + (s + 1) = B
\]

and so \( \gamma(D) = B \).

Finally, we show that \( \gamma_o(D) = C \). Since the set \( V \cup V_2 \cup X_2 \cup X_3 \cup T \cup \{v\} \) is an open dominating set of \( D \), it follows that

\[
\gamma_o(D) \leq |V \cup V_2 \cup X_2 \cup X_3 \cup T \cup \{v\}| = 3\ell + 2k + s + 1 = C.
\]

To show that \( \gamma_o(D) \geq C \), let \( S_o \) be a minimum open dominating set of \( D \).

For each \( i \) (\( 1 \leq i \leq k \)), since the vertex \( x_{i,p} \), where \( 2 \leq p \leq 4 \), is only openly dominated by the vertex \( x_{i,p-1} \), it follows that \( \{x_{i,1} = v, x_{i,2}, x_{i,3}\} \subseteq S_o \). Therefore, \( v \) and at least two vertices in the set \( V(G_i) - \{x_{i,1}\} \) must belong to \( S_o \) for all \( i \) with \( 1 \leq i \leq k \).
For each \( j \) with \( 1 \leq j \leq \ell \), the vertex \( v_{j,p} \) is only openly dominated by \( v_{j,p-1} \), where \( 1 \leq p \leq 3 \), where the second subscript of the vertex is expressed as an integer modulo 3. This implies that \( \{v_{j,1}, v_{j,2}, v_{j,3}\} \subseteq S_o \) for all \( j \) with \( 1 \leq j \leq \ell \). Hence at least three vertices in \( V(H_j) - \{v_{j,0}\} \) must also belong to \( S_o \) for \( 1 \leq j \leq \ell \).

For each \( t \) with \( 1 \leq t \leq s \), each vertex \( y_{t,2} \) is only openly dominated by \( y_{t,1} \).

Hence \( y_{t,1} \) belongs to \( S_o \) for \( 1 \leq t \leq s \).

Therefore,

\[
\gamma(D) \geq |S_o| = 2k + 3\ell + s + 1 = C,
\]

and so \( \gamma(D) = 2k + 3\ell + s + 1 = C \). \( \square \)

We plan to determine all realizable triples.

### 6.3 Lower and Upper \( H \)-Domination Numbers

Let \( H \) be the 2-stratified oriented graph defined in Chapter 5. For a connected graph \( G \), the lower \( H \)-domination number \( \gamma_H(G) \) and upper \( H \)-domination number \( \Gamma_H(G) \) of a graph \( G \) are defined by

\[
\gamma(G : H) = \min \{\gamma_H(D) : D \text{ is an orientation of } G\}
\]

\[
\Gamma(G : H) = \max \{\gamma_H(D) : D \text{ is an orientation of } G\}
\]

Since \( H \) contains two red vertices,

\[
2 \leq \gamma(G : H) \leq \Gamma(G : H) \leq n \tag{6.5}
\]

for every connected graph \( G \) of order \( n \). First, we present characterizations for (1) nontrivial connected graphs whose lower \( H \)-domination number is 2 and (2) nontrivial connected graphs of \( n \) whose upper \( H \)-domination number is \( n \).

**Proposition 6.7.** Let \( G \) be a nontrivial connected graph of order \( n \). Then

\[
\gamma(G : H) = 2 \text{ if and only if } \Delta(G) = n - 1.
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof. First, assume that $\gamma(G : H) = 2$. Then there exists an orientation $D$ of $G$ for which $\gamma_H(D) = 2$. Let $c$ be a minimum $H$-coloring of $D$ with $R_c = \{x, y\}$. Necessarily, $x$ and $y$ are adjacent. Assume, without loss of generality, that $(x, y) \in E(D)$. Since $y$ is the only red vertex that is adjacent from a red vertex, $y$ is the only red vertex of $D$ that can $H$-dominate the blue vertices of $D$. Since $c$ is an $H$-coloring of $D$, it follows that every blue vertex is $H$-dominated by $y$ and so $y$ is adjacent to every vertex in $V(D) - \{x\}$. Therefore, $y$ is adjacent to every vertex in $G$ and so $\deg y = n - 1$ and so $\Delta(G) = n - 1$.

For the converse, assume that $\Delta(G) = n - 1$. Let $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$, where $\deg v = n - 1$. Define an orientation $D^*$ of $G$ such that $(v_1, v) \in E(D^*)$ and $(v, v_i) \in E(D^*)$ for $2 \leq i \leq n - 1$. Since the red-blue coloring $c^*$ of $D^*$ with $R_{c^*} = \{v, v_1\}$ is an $H$-coloring of $D^*$, it follows that $\gamma_H(D^*) = 2$. Therefore, $\gamma(G : H) = 2$. □

Next, we characterize nontrivial connected graphs of $n$ whose upper $H$-domination number is $n$. An oriented graph $D$ is antidirected if every vertex of $D$ has outdegree 0 or indegree 0.

**Proposition 6.8.** Let $D$ be a connected oriented graph of order $n \geq 2$. Then

\[ \gamma_H(D) = n \] if and only if $D$ is antidirected.

Proof. If $D$ is antidirected, then $D$ does not contain $H$ as a subdigraph and so $\gamma_H(D) = n$. For the converse, suppose that $D$ is not antidirected. Then there exists a vertex $v$ such that $\text{id}_v \geq 1$ and $\text{od}_v \geq 1$. Let $u$ and $w$ be two vertices of $D$ such that $u, v, w$ is a directed path of order 3 in $D$. Define a coloring $c$ of $D$ by assigning red to each vertex in $V(D) - \{w\}$ and blue to $w$. Since $c$ is a $H$-coloring of $D$ with $n - 1$ red vertices, it follows that $\gamma_H(D) \leq n - 1$. □

It is known that only bipartite graphs have antidirected orientations, as we state next (see [9]).

**Proposition 6.9.** A graph $G$ has an antidirected orientation if and only if $G$ is bipartite.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The following is a consequence of Propositions 6.8 and 6.9, which characterizes all connected graphs $G$ of order $n$ for which $\Gamma(G : H) = n$.

**Corollary 6.10.** Let $G$ be a connected graph of order $n$. Then

$$\Gamma(G : H) = n \text{ if and only if } G \text{ is bipartite.}$$

We have seen in (6.5) that $\gamma(G : H) \leq \Gamma(G : H)$ for every connected graph $G$. Certainly, it is possible that $\gamma(G : H) < \Gamma(G : H)$, as we show next.

**Proposition 6.11.** Let $r, s$ be integers with $1 \leq r \leq s$. Then $\Gamma(K_{r,s} : H) = n$ and

$$\gamma(K_{r,s} : H) = \begin{cases} 2 & \text{if } r = 1 \\ 3 & \text{if } r \geq 2. \end{cases}$$

**Proof.** The fact that $\Gamma(K_{r,s} : H) = n$ is an immediate consequence of Corollary 6.10. Thus it remains to determine $\gamma(K_{r,s} : H)$. Since $\gamma(K_{1,s} : H) = 2$ by Proposition 6.7, we may assume that $r \geq 2$. Again, by Proposition 6.7, $\gamma(K_{r,s} : H) \geq 3$. To show that $\gamma(K_{r,s} : H) \leq 3$, we construct an orientation $D$ of $K_{r,s}$ for which $\gamma_H(D) = 3$.

Let the partite sets of $K_{r,s}$ be

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } V = \{v_1, v_2, \ldots, v_s\}$$

and let $D$ be an orientation of $K_{r,s}$ such that

$$\{(u_1, v_1), (v_1, u_2)\} \cup \{(v_1, u_i) : 3 \leq i \leq r\} \cup \{(u_2, v_j) : 2 \leq j \leq s\} \subseteq E(D).$$

We show that $\gamma_H(D) = 3$. Define a red-blue coloring $c$ of $D$ by assigning red to $u_1, v_1, u_2$ and blue to the remaining vertices of $D$. Since (1) the blue vertex $u_i$ ($3 \leq i \leq r$) is adjacent from the red vertex $v_1$, which is adjacent from the red vertex $u_1$ and (2) the blue vertex $v_j$ ($2 \leq j \leq s$) is adjacent from the red vertex $u_2$, which is adjacent from the red vertex $v_1$, it follows that $c$ is an $H$-coloring of $D$ and so $\gamma_H(D) \leq 3$. \hfill \Box

By Proposition 6.11, $\gamma(K_{r,s} : H) < \Gamma(K_{r,s} : H)$ if $r + s \geq 4$. In fact, this can be extended to all connected bipartite graphs.
Proposition 6.12. If $G$ is a connected bipartite graph of order $n \geq 3$, then

$$\gamma(G : H) < \Gamma(G : H).$$

Proof. If $G$ is a connected bipartite graph of order $n \geq 3$, then $\Gamma(G : H) = n$ by Corollary 6.10. On the other hand, let $P_k : u_1, u_2, \ldots, u_k$ be a longest path in $G$. Then $k \geq 3$. We define an orientation $D$ of $G$ by orienting the path $P_k$ as a directed path $\vec{P}_k : u_1, u_2, \ldots, u_k$, and orienting the remaining edges of $G$ arbitrarily. Then the red-blue coloring $c$ that assigns blue to the vertex $u_3$ and red to the remaining vertices of $D$ is an $H$-coloring, and so $\gamma_H(D) \leq n - 1$. Therefore, $\gamma(G : H) \leq n - 1$ and so $\gamma(G : H) < \Gamma(G : H)$. \qed

Proposition 6.12 brings up a natural question:

Problem 6.13. Is it true that

$$\gamma(G : H) < \Gamma(G : H)$$

for every connected graph of order $n \geq 4$?

Problem 6.14. (Intermediate Value Problem) Let $G$ be a connected graph of order $n \geq 4$ such that $\gamma(G : H) < \Gamma(G : H)$. For which integers $k$ with $\gamma(G : H) < k < \Gamma(G : H)$, does there exist an orientation $D$ of $G$ such that $\gamma_H(D) = k$?

Problems 6.13 and 6.14 suggest the following concept. For a connected graph $G$, the $H$-domination spectrum of $G$ is the set

$${\gamma}_H(D) : D \text{ is an orientation of } G \}.$$

As an example, we now determine the $H$-domination spectrum of a complete bipartite graph.

Proposition 6.15. Let $K_{r,s}$ be the complete bipartite graph with $1 \leq r \leq s$. Then

$${\gamma}_H(D) = \begin{cases} 
\{2,3,\ldots,1+s\} & \text{if } r = 1 \\
\{3,4,\ldots,r+s\} & \text{if } r \geq 2.
\end{cases}$$
Proof. First, assume that $r = 1$. Let $V(K_{1,s}) = \{x, y_1, y_2, \ldots, y_s\}$, where $x$ is the central vertex of $K_{1,s}$. For each integer $k$ with $2 \leq k \leq s+1$, we show that there is an orientation $D_k$ of $K_{1,s}$ such that $\gamma_H(D_k) = k$. By Corollary 6.10 and Proposition 6.11, the result holds for $k = 2$ and $k = 1 + s$. Thus, we may assume that $3 \leq k \leq s$. Let $D_k$ be the orientation of $K_{1,s}$ such that
\[ E(D_k) = \{(y_i, x) : 1 \leq i \leq k - 1\} \cup \{(x, y_j) : k \leq j \leq s\}. \]
Let $c'$ be the red-blue coloring of $D_k$ such that
\[ R_{c'} = \{x\} \cup \{y_i : 1 \leq i \leq k - 1\}. \]
Since every blue vertex $y_j$ ($k \leq j \leq s$) belongs to the red-red-blue directed path $y_1, x, y_j$, it follows that $c'$ is an $H$-coloring of $D_k$ with $k$ red vertices, implying that $\gamma_H(D_k) \leq k$.

On the other hand, let $c$ be a minimum $H$-coloring of $D_k$. Since $\text{id} y_i = 0$ for $1 \leq i \leq k-1$, every vertex $y_i \in R_c$ and so $|R_c| \geq k - 1$. Assume, to the contrary, that $\gamma_H(D_k) = k - 1$. Then $R_c = \{y_i : 1 \leq i \leq k - 1\}$, which is impossible since no two vertices in $R_c$ are adjacent. Therefore, $\gamma_H(D_k) = k$.

Next, assume that $r \geq 2$. Let the bipartite sets of $K_{r,s}$ be
\[ \{u_1, u_2, \ldots, u_r\} \text{ and } \{v_1, v_2, \ldots, v_s\}. \]
By Proposition 6.7, $\gamma_H(D) \geq 3$ for every orientation $D$ of $K_{r,s}$. For each integer $k$ with $3 \leq k \leq r + s$, we show that there is an orientation $D_k$ of $K_{r,s}$ such that $\gamma_H(D_k) = k$. By Corollary 6.10 and Proposition 6.11, the result holds if $k = 3$ or $k = r + s$. Thus, we may assume that $4 \leq k \leq r + s - 1$. Let $p$ and $q$ be integers with $2 \leq p \leq r$ and $1 \leq q \leq s - 1$ such that $p + q = k$ and let $D_k$ be the orientation of $K_{r,s}$ such that
\begin{enumerate}
  \item[(a)] $(v_j, u_1) \in E(D_k)$ for $1 \leq j \leq q$ and $(u_1, v_j) \in E(D_k)$ for $q + 1 \leq j \leq s$;
  \item[(b)] for $2 \leq i \leq p$, $(u_i, v_j) \in E(D_k)$ for $1 \leq j \leq s$; and
  \item[(c)] for $p + 1 \leq i \leq r$, $(v_j, u_i) \in E(D)$ for $1 \leq j \leq q$ and $(u_i, v_j) \in E(D_k)$ for $q + 1 \leq j \leq s$.
\end{enumerate}
Let $c^*$ be the red-blue coloring of $D_k$ such that

$$R_{c^*} = \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q\}.$$ 

Observe that

(i) the blue vertex $u_i$ ($p + 1 \leq i \leq r$) is adjacent from the red vertex $v_1$ and $v_1$ is adjacent from the red vertex $u_2$, that is, $u_i$ is $H$-dominated by $v_1$, and

(ii) the blue vertex $v_j$ ($q + 1 \leq j \leq s$) is adjacent from the red vertex $u_1$ and $u_1$ is adjacent from the red vertex $v_1$, that is, $v_j$ is $H$-dominated by $u_1$.

Thus $c^*$ is an $H$-coloring of $D_k$ and so

$$\gamma_H(D_k) \leq |R_{c^*}| = p + q = k.$$ 

To show that $\gamma_H(D_k) \geq k$, let $c$ be a minimum $H$-coloring of $D_k$.

(1) Since $id u_i = 0$ for $2 \leq i \leq p$, it follows by Observation 5.5(b) that $u_i \in R_c$.

(2) Since each vertex $v_j$ ($1 \leq j \leq q$) is only adjacent from $u_i$ ($2 \leq i \leq p$) and $id u_i = 0$, it again follows by Observation 5.5(b) that $v_j \in R_c$.

(3) Since each vertex $v_j$ ($q + 1 \leq j \leq s$) can only be $H$-dominated by itself, $u_1$, or a vertex $u_i$ for $p + 1 \leq i \leq r$, it follows that $R_c$ contains at least one vertex in

$$\{u_1, u_{p+1}, u_{p+2}, \ldots, u_r\} \cup \{v_{q+1}, v_{q+2}, \ldots, v_s\}.$$ 

Combining (1) – (3), we have

$$\gamma_H(D_k) = |R_c| \geq (p - 1) + q + 1 = p + q = k.$$ 

Therefore, $\gamma_H(D_k) = k$. 

We plan to investigate the $H$-domination spectra of some well-known classes of graphs such as $K_n, C_n, P_n$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
REFERENCES


