Phase Coherent Effects in the Speckle Correlation of Light Scattered from Volume Disordered Media and Randomly Rough Surfaces

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PHASE COHERENT EFFECTS IN THE SPECKLE CORRELATION OF LIGHT SCATTERED FROM VOLUME DISORDERED MEDIA AND RANDOMLY ROUGH SURFACES

by

Bogdan Danila

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Submitted to the
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Bogdan Danila.
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CHAPTER 1
INTRODUCTION

The study of the propagation of light through various types of media is of great importance for applied physics[1-5]. Most transparent media of practical interest are random media, being characterized by some sort of volume disorder resulting in a random space distribution of the refraction index[1,2,6-9]. In addition, another frequently encountered type of disorder occurs when the boundary surface between two otherwise homogeneous media has a rough, highly irregular shape[10-12]. The peculiar features of light propagation through such random media or randomly rough surfaces are best observed in the case of coherent light[6,8-16]. Two characteristics are encompassed by the definition of coherence[1,2,17]. First, coherent light has to be monochromatic namely a superposition of waves having all the same wavelength. The second characteristic is that all component waves have the same phase, i.e. oscillate in unison at any given point. The subject of this dissertation is the study of the angular distributions of coherent light scattered by random media and by randomly rough surfaces, with particular emphasis on the correlations arising between the fluctuations of the intensity patterns.

When coherent light is scattered by an object (a medium delimited by a boundary surface), the linear superposition of the wavelets emerging from different points produces an interference pattern[1,2,17]. If the shape of the surface and the volume distribution of the refraction index are highly regular, this regularity is also reflected by the interference pattern. On the other hand, if the boundary surface is highly irregular or if the medium exhibits volume disorder, the interference
pattern consists of tiny spots distributed at random in space, giving it a granular aspect\[1,2,17,18\]. This granular interference pattern is called speckle. Speckle interferometry has found many applications in engineering\[17,18\]. Some of the more important applications are the measurement of surface roughness\[17\], the measurement of tiny displacements or deformations, and vibration analysis\[17\]. All these applications are based on the fact that the angular distribution of the scattered light, however random, is not uniform and information can be extracted from the shape of the distribution.

In addition, the study of the statistical correlations arising between the scattering intensities corresponding to distinct scattering paths provides more detailed information about the properties of the random surface or medium\[6,8-10,13-16\]. Before proceeding with the definition of the correlation function, we give a brief review of the terms in which the propagation of light is described. When light propagates in a medium, two types of phenomena may occur\[1,2\]: scattering and absorption. The difference between them is that scattering diverts light from its initial direction, while absorption converts it to other forms of energy. On the other hand, scattering may or may not be accompanied by excitation or ionization processes leading to partial absorption\[1,2\]. If no absorption is involved the scattering is called elastic, whereas scattering accompanied by absorption is called inelastic. Absorption is described by means of the attenuation (or extinction) coefficient, defined as the relative decrease in the intensity of light per unit of length in the direction of propagation. Scattering on the other hand, is described by the differential scattering cross section\[1,2\] defined as the ratio between the angular intensity of the scattered wave in a given direction and the intensity of the incident wave. Due to phase coherent effects in scattering, statistical correlations arise under certain conditions
between the fluctuations of the cross sections for two distinct pairs of incident and scattering directions[6,8,10,14]. The speckle correlation function (or speckle correlator) is defined as the variance of the product of the two cross sections about the product of their mean values, or the average of the product of their individual deviations from average[6,10,15].

In the first part of the dissertation, we present results for the speckle correlator computed for volume disordered media by means of numerical simulation[6,15,16]. The scattering cross sections for selected pairs of incidence and scattering directions are computed for a finite set of realizations of the random medium. At the end, estimates of the average cross sections and of the correlation function are calculated for each pair of directions. The scattering is computed in the scalar wave approximation[1,2,6,15], which describes the electromagnetic field by means of a complex scalar function. This approximation is often employed in scattering and diffraction problems in which the changes in polarization are irrelevant, and its validity has been rigorously justified for a variety of optical systems consisting of non-magnetic materials[1,2].

The volume disordered medium is modeled by an array of homogeneous dielectric spheres placed in vacuum[6,15,16]. The centers of the spheres are positioned at random at the vertices of a simple cubic lattice of potential locations, and their radii are small compared to the wavelength of light. Such a medium can be characterized as either periodic on average or homogeneously random, depending on the ratio between the lattice constant and the wavelength of light. If the lattice constant of the virtual array is greater or approximately equal to the wavelength, the system is called periodic on average[6,15,16] whereas if the lattice constant is smaller than the wavelength by at least an order of magnitude, the system behaves
as if the positions of the spheres were sampled using a continuous distribution and therefore is called homogeneously random. In two separate series of simulations, either the dielectric constants or the radii of the spheres are taken to be Gaussianly distributed random variables. When the radii of the spheres are given a random distribution, the sampling process is constrained such that all radii are positive and the spheres do not overlap. The pseudo-random numbers generator used by the sampling routines is described in Ref. [19]. Our focus is on the sensitivity of the speckle correlator to the averages and standard deviations of the radii or dielectric constants distributions. The determination of the standard deviations is particularly important since currently employed non-destructive methods are unable to measure them.

The second part of the dissertation describes theoretical calculations of the scattering cross section and of the speckle correlator in the case of light scattered from randomly rough surfaces [10-12]. Our studies are based on a rough surface model [12] consisting of a plane vacuum-dielectric interface upon which Gaussianly shaped, parallel ridges made from different dielectric materials are placed at random. We are interested in the behavior of the correlation function at frequencies close to the resonant frequency of each ridge material. Previous studies [3-5, 12] have shown that the phase coherent effects in scattering are enhanced in the proximity of the dielectric resonances. The calculations are done using diagrammatic perturbation theory expansion in powers of the surface roughness. The expansion is carried out to the second ladder diagram for the reducible vertex function and to the first maximally crossed diagram for the irreducible vertex function [10-12, 20, 21]. In separate series of calculations, the heights or the widths of the ridges are taken to be Gaussianly distributed. Our goal here was to identify ways of computing the
averages and standard deviations of the heights or widths distributions from differential reflection coefficient or speckle correlations measurements. The next series of computations is dedicated to the study of an interface with identical ridges made at random from two different dielectric materials. In this case, the means of computing the fractional concentration of the two materials are investigated. Finally, the peculiar features exhibited by the \( C^{(1)} \) contribution to the correlation function in the case of periodic on average randomly rough surfaces are investigated.
CHAPTER 2
VOLUME DISORDER. THEORY

2.1 Theory

As discussed above, the random medium consists[6,15,16] of an array of \( N \) dielectric spheres of random radii or dielectric constants, their centers being positioned at random at the vertices of a simple cubic lattice. If \( m \) is the number of vertices per each side of the lattice and \( a \) is the lattice constant, the occupancy ratio for the vertices will be \( \nu = N/m^3 \), and the volume concentration of spheres \( n = \nu/a^3 \). The electrical permittivity of the medium as a function of position can be written[15] as the sum of the vacuum dielectric constant, \( \epsilon_{\text{vac}} = 1 \), and a fluctuating term \( \delta \epsilon(\vec{r}) \) which accounts for the change from vacuum due to the presence of the spheres,

\[
\epsilon(\vec{r}) = 1 + \delta \epsilon(\vec{r}).
\]

Let \( \vec{r}_l \), \( R_l \), and \( \epsilon_l \) denote the position vector, radius and dielectric constant of the \( l \)th sphere. In terms of these quantities, the fluctuating term can be expressed as

\[
\delta \epsilon(\vec{r}) = \sum_l (\epsilon_l - 1) S_l(\vec{r} - \vec{r}_l),
\]

where

\[
S_l(\vec{r}) = \begin{cases} 
  1 & \text{when } |\vec{r}| \leq R_l, \\
  0 & \text{when } |\vec{r}| > R_l.
\end{cases}
\]

From the Faraday and Ampère-Maxwell equations[1,2] written for a linear, non-magnetic, source-free medium

\[
\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{H}}{\partial t} = 0, \tag{2.4}
\]

\[
\nabla \times \vec{H} - \frac{\epsilon}{c} \frac{\partial \vec{E}}{\partial t} = 0, \tag{2.5}
\]

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we find
\[ \nabla(\nabla \cdot \vec{E}) - \Delta \vec{E} + \frac{\epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \] (2.6)

On the other hand, it can be shown\[l,2\] that the first term in (2.6), which is responsible for the changes in polarization, vanishes when \( |\nabla \epsilon| \to 0 \) and is negligible compared to the other two terms when the permittivity varies significantly only over distances that are large compared to the wavelength of light. Thus, we are left essentially with a single scalar field \( \psi(\vec{r}, t) \) determined by
\[ \Delta \psi - \frac{\epsilon}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \] (2.7)

Rigorously speaking, our model does not satisfy the condition mentioned above due to the abrupt variations of the dielectric constant at the surface of each sphere. Good results for the optical intensity can be obtained however\[6,15\], by employing the complex scalar field approximation described in Refs.\[1,2\]. Within the framework of this approximation, the electromagnetic field is described in terms of a complex scalar function satisfying Eq. (2.7). The electromagnetic energy density is given by\[1,2\]
\[ W(\vec{r}, t) = \frac{1}{8\pi} \left( \frac{\epsilon(\vec{r})}{c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + |\nabla \psi|^2 \right), \] (2.8)
while the Poynting vector is
\[ \vec{S}(\vec{r}, t) = -\frac{1}{8\pi} \left( \frac{\partial \psi^*}{\partial t} \nabla \psi + \frac{\partial \psi}{\partial t} \nabla \psi^* \right). \] (2.9)

For a monochromatic field, Eq. (2.7) reduces to the Helmholtz equation\[1,2,6\]
\[ \left[ \Delta + k_0^2 \epsilon(\vec{r}|\omega) \right] \psi(\vec{r}|\omega) = 0, \] (2.10)
where \( k_0 = \omega/c \) is the magnitude of the wavevector in free space. Equation (2.10) can be converted to an integral equation by rewriting it as
\[ \left( \Delta + k_0^2 \right) \psi(\vec{r}|\omega) = -k_0^2 \delta \epsilon(\vec{r}|\omega) \psi(\vec{r}|\omega) \] (2.11)
and treating it as a non-homogeneous equation, which is solved\cite{1,2,6,15} by using
the retarded Green’s function\cite{1,2}

\[ G^{(+)}(\mathbf{r}|\mathbf{r}') = \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \]  

The Green’s function (2.12) is the solution of

\[ (\Delta + k_0^2) G^{(+)}(\mathbf{r}|\mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}') \]  

with retarded wave boundary conditions. Specifically, we find that

\[ \psi(\mathbf{r}) = \psi_{in}(\mathbf{r}) + \frac{k_0^2}{4\pi} \int_V d^3r' G^{(+)}(\mathbf{r}|\mathbf{r}') \delta(\mathbf{r}' - \mathbf{r}) \psi(\mathbf{r}'), \]  

where \( \psi_{in}(\mathbf{r}) \) is the incident wave, solution of Eq. (2.10) for \( \delta \epsilon = 0 \), and \( V \) is the volume of the region containing the spheres. Using (2.2), Eq. (2.14) can be re-written as

\[ \psi(\mathbf{r}) = \psi_{in}(\mathbf{r}) + \frac{k_0^2}{4\pi} \sum_i (\epsilon_i - 1) \int_V d^3r' G^{(+)}(\mathbf{r}|\mathbf{r}') S_i(\mathbf{r}' - \mathbf{r}) \psi(\mathbf{r}'). \]  

This is a Fredholm equation of the first kind\cite{22}, which can be solved by rewriting it as a matrix equation. In order to find an approximate solution for Eq. (2.15) using only finite matrices, we use the fact that the radii of the spheres are small compared to the wavelength of light\cite{6,15}. In this limit \( \psi(\mathbf{r}') \) may be considered constant throughout the volume of each sphere, hence

\[ \psi(\mathbf{r}) = \psi_{in}(\mathbf{r}) + \frac{k_0^2}{4\pi} \sum_i (\epsilon_i - 1) \psi(\mathbf{r}_i) \int_V d^3r' G^{(+)}(\mathbf{r}|\mathbf{r}') S_i(\mathbf{r}' - \mathbf{r}_i). \]  

If for each term of the sum in (2.16) we make a separate substitution \( \mathbf{r}' - \mathbf{r}_i = \mathbf{u} \) we find, using (2.3), that

\[ \psi(\mathbf{r}) = \psi_{in}(\mathbf{r}) + \frac{k_0^2}{4\pi} \sum_i (\epsilon_i - 1) \psi(\mathbf{r}_i) \int_{\mathbf{V}_i} d^3u G^{(+)}(\mathbf{r}|\mathbf{r}_i + \mathbf{u}). \]  

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where the volume $V_i$ of the $i$th sphere is defined by $|\vec{u}| \leq R_i$. In particular, this equation holds for any point $\vec{r}_k$ that is the center of the $k$th sphere, such that

$$\psi(\vec{r}_k) = \psi_{in}(\vec{r}_k) + \frac{k_0^2}{4\pi} \sum_l (\epsilon_l - 1) \psi(\vec{r}_l) \int_{V_l} d^3u \, G^{(+)}(\vec{r}_k|\vec{r}_l + \vec{u}).$$

(2.18)

Denoting $(\epsilon_l - 1) \frac{k_0^2}{4\pi} \int_{V_l} d^3u \, G(\vec{r}_k|\vec{r}_l + \vec{u}) = M_{kl}$, we have

$$\psi(\vec{r}_k) = \psi_{in}(\vec{r}_k) + \sum_l M_{kl} \psi(\vec{r}_l), \quad (\forall) k = 1, N.$$  

(2.19)

or in matrix notation,

$$\vec{\psi} = \vec{\psi}_{in} + \tilde{M} \vec{\psi}.$$  

(2.20)

The solution of the system (2.20) can be formally written as

$$\vec{\psi} = (\tilde{I} - \tilde{M})^{-1} \vec{\psi}_{in}.$$  

(2.21)

By substituting the values $\psi(\vec{r}_l)$ given by (2.21) into (2.17), we find

$$\psi(\vec{r}) = \psi_{in}(\vec{r}) + \sum_{k,l} J_k(\vec{r}) (\tilde{I} - \tilde{M})^{-1}_{kl} \psi_{in}(\vec{r}_l),$$  

(2.22)

where

$$J_k(\vec{r}) = (\epsilon_k - 1) \frac{k_0^2}{4\pi} \int_{V_k} d^3u \, G^{(+)}(\vec{r}|\vec{r}_k + \vec{u}).$$  

(2.23)

It is obvious from the definition of the elements of matrix $\tilde{M}$, given below Eq. (2.18), that $M_{kl} = J_l(\vec{r}_k)$. The evaluation of the integral in Eq. (2.23) yields[6,15]

$$J_k(\vec{r}) = (\epsilon_k - 1) \left\{ \begin{array}{ll}
(k_0 R_k)^2 j_1(k_0 R_k) \frac{e^{ik_0 |\vec{r} - \vec{r}_k|}}{k_0 |\vec{r} - \vec{r}_k|} & \text{if } \vec{r} \neq \vec{r}_k, \\
e^{ik_0 R_k}(1 - ik_0 R_k) - 1 & \text{if } \vec{r} = \vec{r}_k,
\end{array} \right.$$  

(2.24)

where $R_k$ is the radius of the $k$th sphere and $j_1$ is the spherical Bessel function of index 1. On the other hand, the total wavefield $\psi(\vec{r})$ is the sum of the incident and scattered wavefields,

$$\psi(\vec{r}) = \psi_{in}(\vec{r}) + \psi_{sc}(\vec{r}).$$  

(2.25)
Hence, for the scattered field we have

\[ \psi_{sc}(\vec{r}) = \sum_{k,l} J_k(\vec{r}) \left( \hat{I} - \hat{M} \right)_{kl}^{-1} \psi_{in}(\vec{r}). \]  

(2.26)

Furthermore, for the Green’s function that appears in Eq. (2.24) we can use the far field approximation\[1,2,6,15\]

\[ \frac{e^{ik_0|\vec{r} - \vec{r}_k|}}{k_0|\vec{r} - \vec{r}_k|} \approx \frac{e^{ik_0q \cdot \vec{r}_k}}{k_0r} e^{-ik_0q \cdot \vec{r}_k}, \]  

(2.27)

where \( r \) is the magnitude of the position vector at the point where the field is observed, and \( q \) is the unit vector that gives the direction of propagation of the scattered wave at that point. It should be noted that the far field approximation cannot be used when evaluating the coefficients \( M_{kl} \). If we take the incident wave to be a monochromatic plane wave \( e^{ik_0 \vec{r}} \) we have finally from Eqs. (2.24), (2.26), and (2.27)

\[ \psi_{sc}(\vec{r}) = \frac{e^{ik_0r}}{k_0r} \sum_{k,l} (\epsilon_k - 1)(k_0 R_k)^2 j_1(k_0 R_k) e^{-ik_0q \cdot \vec{r}_k} \left( \hat{I} - \hat{M} \right)_{kl}^{-1} e^{ik_0\hat{r}_l}. \]  

(2.28)

Equation (2.28) describes a spherical outgoing wave with the amplitude modulated as a function of direction, of the form\[23,24\]

\[ \psi_{sc}(\vec{r}) = f(\hat{q}, \hat{k}) \frac{e^{ik_0r}}{r}. \]  

(2.29)

Here \( \hat{k} \) is the direction of the incident wave, \( \hat{q} \) is the direction of scattering, and the scattering amplitude \( f(\hat{q}, \hat{k}) \) is a function of these directions. Using (2.9), we find that the differential scattering cross section is given by\[23,24\]

\[ \frac{d\sigma}{d\Omega}(\hat{q}, \hat{k}) = |f(\hat{q}, \hat{k})|^2. \]  

(2.30)

Specifically, we find

\[ f(\hat{q}, \hat{k}) = \frac{1}{k_0} \sum_{k,l} (\epsilon_k - 1)(k_0 R_k)^2 j_1(k_0 R_k) e^{-ik_0q \cdot \vec{r}_k} \left( \hat{I} - \hat{M} \right)_{kl}^{-1} e^{ik_0\hat{r}_l}. \]  

(2.31)
This formula has been used for the evaluation of the differential cross sections and of the correlation functions defined, for two pairs of incident and scattering directions[6-9,15,16], by

$$C(\mathbf{q}, \mathbf{k} | \mathbf{q}', \mathbf{k}') = \langle \frac{d\sigma}{d\Omega}(\mathbf{q}, \mathbf{k}) \frac{d\sigma}{d\Omega}(\mathbf{q}', \mathbf{k}') \rangle - \langle \frac{d\sigma}{d\Omega}(\mathbf{q}, \mathbf{k}) \rangle \langle \frac{d\sigma}{d\Omega}(\mathbf{q}', \mathbf{k}') \rangle. \quad (2.32)$$

This is the variance of \((\frac{d\sigma}{d\Omega}(\mathbf{q}, \mathbf{k}) \frac{d\sigma}{d\Omega}(\mathbf{q}', \mathbf{k}'))\) about \(\langle \frac{d\sigma}{d\Omega}(\mathbf{q}, \mathbf{k}) \rangle \langle \frac{d\sigma}{d\Omega}(\mathbf{q}', \mathbf{k}') \rangle\). In the absence of a statistical correlation between \(\mathbf{q}, \mathbf{k} \) and \(\mathbf{q}', \mathbf{k}'\), this variance would be zero. The presence of the correlation arises[6,8,14-16] from the phase coherent multiple scattering of light from the dielectric spheres.

The speckle correlation function \(C(\mathbf{q}, \mathbf{k} | \mathbf{q}', \mathbf{k}')\) has been studied theoretically[6-9,14] using the perturbation theory expansion in powers of the volume disorder \(\delta\epsilon\). These studies have shown that the correlation function can be written as a sum of contributions arising in different orders of the perturbation theory expansion,

\[
C = C^{(1)} + C^{(10)} + C^{(1.5)} + C^{(2)} + C^{(3)}. \quad (2.33)
\]

The dominant features of the speckle correlator are the \(C^{(1)}\) and \(C^{(10)}\) short range contributions, which arise in the lowest order of the perturbation expansion[6,8,14,15]. The \(C^{(1)}\) contribution contains two phase-coherent peaks, known as the memory and the time-reversed memory effects. In homogeneously random systems, \(C^{(1)}\) is nonzero only when the condition \(\mathbf{q} - \mathbf{k} - \mathbf{q}' + \mathbf{k}' = 0\) is satisfied, while the condition for \(C^{(10)}\) to be nonzero is \(\mathbf{q} - \mathbf{k} + \mathbf{q}' - \mathbf{k}' = 0\). Other contributions arise in higher orders of the perturbation expansion[6,8,14,15]. They are the intermediate and long range terms, denoted by \(C^{(1.5)}\) and \(C^{(2)}\) respectively, and the infinite range term \(C^{(3)}\). Their magnitudes are much lower than those of the dominant terms. In our simulations, these higher order contributions are masked by the statistical noise[15] and therefore will be neglected.
2.2 Description of the Program

In this section, the operation of the FORTRAN program[15] that was used for the simulations is described in general. The reader is referred to a copy of the program, which appears in Appendix A, for details. All subroutine and program variable names mentioned in this section refer to the program in Appendix A.

The evaluation of the correlation functions is done using two nested loops. Within each cycle of the outer loop a configuration of \( N \) spheres is generated by a subroutine \texttt{coor}, which samples their locations, radii, and dielectric constants. For each sphere, the location is sampled with equal probabilities from the unoccupied vertices the lattice. Next, the radius of the sphere is generated. When the radii of the spheres are randomly distributed, the subroutine checks for overlapping with previously sampled spheres. If any overlapping is detected, the coordinates and the radius of the last sphere are rejected and a new sampling is attempted. At the end, the coordinates, radii, and dielectric constants of the spheres are stored in separate arrays. Once a configuration is established, the differential cross sections are computed within the inner loop using Eqs. (2.30) and (2.31), for a set of pairs of incidence and scattering directions. Two different subroutines, \texttt{psisc} and \texttt{psisd} are used for the computation of \( \frac{d\sigma}{d\Omega}(q,\hat{k}) \) and \( \frac{d\sigma}{d\Omega}(q',\hat{k}') \), respectively. The coefficients of the matrix \( (\hat{I} - \hat{M}) \) are computed by a subroutine \texttt{mat} using (2.24) and supplied to \texttt{psisc} and \texttt{psisd}. Also within this loop are computed the terms of the sums corresponding to the averages that occur in (2.32). For the results presented in the next chapter, the averages were computed for sets of 400 realizations of the random configuration.

The ratio between the wavelength of light and the lattice constant is controlled by adjusting the parameter \( xx \) at the beginning of subroutine \texttt{coor}. This
is an important parameter, since it determines whether the system is periodic on average or homogeneously random. To simulate periodic on average systems a value of $\bar{x} = 1$ was used, whereas for homogeneously random systems the typical value was $\bar{x} = 8$. In order to keep the size of the array constant, the parameter $n_a$ that controls the number of nodes in each dimension must be adjusted accordingly. For our simulations, $n_a$ was adjusted such that the side of the cubic region containing the spheres was always 10 times the wavelength.

The number of spheres $N$ was taken to be much smaller than the number of potential locations. The program runs for configurations containing up to 225 spheres. However, since the computation is very time-consuming and the running time is roughly proportional to $N^2$, some of the runs were made with lower numbers of spheres.

Another important parameter is the ratio between the (average) radius of the particles and the wavelength. We have chosen a value of\cite{15}

$$\frac{\langle R \rangle}{\lambda} = \frac{0.4}{2\pi} \approx 0.06366,$$

which is consistent with the condition that the radius should be much smaller than the wavelength. The volume fraction occupied by the $N$ spheres, given by $f = N \frac{\pi}{750} \left( \frac{\langle R \rangle}{\lambda} \right)^3$, was roughly between $1.0 \times 10^{-4}$ and $2.5 \times 10^{-4}$, corresponding to values of $N$ ranging from 100 to 225.

In separate series of simulations, the radii or the dielectric constants of the spheres were taken to be randomly distributed according to Gaussian distribution laws of the form

$$p(x) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}},$$

where $x_0$ is the average value and $\sigma_x$ the standard deviation of the distribution.
The average radius was always the one defined by Eq. (2.34), and the changes in the shape of the correlation function were studied for values of the standard deviation \( \sigma_R \) ranging from 0 to 0.375\( \langle R \rangle \), both for homogeneously random and for periodic on average media. Even though the radii distribution was truncated by rejecting the negative values, the ratio \( \sigma_R/\langle R \rangle \) was always small enough to ensure that the average and the standard deviation of the distribution are essentially unaffected by the truncation. The average dielectric constant was taken to be \( \langle \epsilon \rangle = -7 \) or \( \langle \epsilon \rangle = -9 \), values that are typical for metals. The changes in the shape of the correlation function were studied for values of the standard deviation \( \sigma_\epsilon \) between 0 and 3, for homogeneously random and for periodic on average media.
CHAPTER 3
VOLUME DISORDER. RESULTS

The correlation functions were scanned along the envelopes of the $C^{(1)}$ and $C^{(10)}$ contributions, which are obtained when $(\vec{q}, \vec{k}, \vec{q}', \vec{k}')$ are varied such that either $\vec{q} - \vec{k} - \vec{q}' + \vec{k}' = 0$ or $\vec{q} - \vec{k} + \vec{q}' - \vec{k}' = 0$ are always satisfied [6,15]. These scans were made for both homogeneously random and periodic on average systems.

The envelope condition for $C^{(1)}$, $\vec{q} - \vec{k} - \vec{q}' + \vec{k}' = 0$, means that $(\vec{q} - \vec{k})$ and $(\vec{q}' - \vec{k}')$ are parallel vectors having the same magnitude. A convenient parameterization for this envelope is obtained by defining [15]

\begin{align*}
\vec{k} &= k_0(\sin \theta, 0, \cos \theta), \quad (3.1) \\
\vec{q} &= k_0(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (3.2) \\
\vec{k}' &= k_0(\sin \theta, 0, -\cos \theta), \quad (3.3) \\
\vec{q}' &= k_0(\sin \theta \cos \phi, \sin \theta \sin \phi, -\cos \theta), \quad (3.4)
\end{align*}

where the polar and azimuthal angles $\theta$ and $\phi$ are defined with respect to the axes of the lattice. The scan must run in principle for all angles $0^\circ \leq \theta \leq 180^\circ$ and $0^\circ \leq \phi \leq 360^\circ$. However, since the correlator is symmetric under reflection through the $x-y$ plane, we need $\theta$ to run only for values up to $90^\circ$. For $C^{(10)}$ to be nonzero, the condition is that $(\vec{q} - \vec{k})$ and $(\vec{q}' - \vec{k}')$ be anti-parallel vectors with the same magnitude. This condition can be fulfilled by defining [15]

\begin{align*}
\vec{k} &= k_0(\sin \theta, 0, \cos \theta), \quad (3.5) \\
\vec{q} &= k_0(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (3.6)
\end{align*}
\[ k' = k_0(-\sin \theta, 0, \cos \theta), \quad (3.7) \]
\[ q' = k_0(-\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta). \quad (3.8) \]

Since the \( C^{(10)} \) correlator is also symmetric under reflexion through the \( x-y \) plane, the results are again plotted only for \( 0^\circ \leq \theta \leq 90^\circ \). The scans of the \( C^{(1)} \) and \( C^{(10)} \) envelopes were made at constant azimuthal angle, for a few representative values \( \phi = 0^\circ, 20^\circ \) and \( 90^\circ \).

In the first set of simulations, all spheres were given the radius defined by Eq. (2.34). Simulations were made for values of \( \sigma_e = 0, 1, 2, \) and \( 3 \). In Figs. 3.1 and 3.2 we present plots of the correlation function along the \( C^{(1)} \) envelope defined by Eqs. (3.1-4), for homogeneously random and for periodic on average media, respectively. The average dielectric constant is \( \langle \epsilon \rangle = -7 \).

The speckle correlator for \( \phi = 0^\circ \) appears as the sum of two contributions exhibiting quite different behavior. First, we have a "background" contribution, constant as a function of \( \theta \) and increasing steadily by about two orders of magnitude as \( \sigma_e \) is varied from 0 to 3. The second contribution is a complicated function of both \( \theta \) and \( \sigma_e \), but its average over all angles \( \theta \) is approximately constant. There are important differences in the shape of this contribution between the homogeneously random and the periodic on average systems. In particular, the plot corresponding to the periodic on average case exhibits two peaks related to Bragg scattering. No fast increasing background is seen in the case of the \( \phi = 20^\circ \) and \( \phi = 90^\circ \) scans. There is however an important feature present in all these cases, namely an increase in the amplitude of the correlation function at small angles \( \theta \) as \( \sigma_e \) increases. In addition, the \( \phi = 90^\circ \) scan for the periodic on average case exhibits a peak at \( \theta = 75^\circ \) and a broad plateau centered at \( \theta = 90^\circ \), whose amplitude also increases with \( \sigma_e \). These last features are also associated with Bragg scattering.
Figure 3.1: Plots of the angular speckle correlator along the $C^{(1)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a homogeneously random medium with $(\epsilon) = -7$. Each plot exhibits the results for four different values of $\sigma_c$. 

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Figure 3.2: Plots of the angular speckle correlator along the $C^{(1)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a periodic on average medium with $\langle \epsilon \rangle = -7$. Each plot exhibits the results for four different values of $\sigma_\epsilon$. 

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Figure 3.3: Plots of the angular speckle correlator along the $C^{(1)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a homogeneously random medium with $(\epsilon) = -9$. Each plot exhibits the results for four different values of $\sigma_\epsilon$. 

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Figure 3.4: Plots of the angular speckle correlator along the $C^{(1)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a periodic on average medium with $\langle \epsilon \rangle = -9$. Each plot exhibits the results for four different values of $\sigma_e$. 

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Results for $\langle \varepsilon \rangle = -9$ are presented in Figs. 3.3 and 3.4, for homogeneously random and for periodic on average media, respectively. The amplitude of the correlation function in this case is significantly larger than for $\langle \varepsilon \rangle = -7$, but no significant differences in shape are seen.

The second part of the study involved varying the standard deviation of the radii distribution $\sigma_R$ while the value of the dielectric constant is fixed, $\varepsilon = -9$. Different scans of the $C^{(1)}$ envelope are presented in Figs. 3.5 and 3.6, both for homogeneously random and for periodic on average systems. These scans were made at the same angles $\phi = 0^\circ$, $20^\circ$, and $90^\circ$, with $\theta$ running from $0^\circ$ to $90^\circ$.

Again we see the "background" of the $\phi = 0^\circ$ plots increasing as $\sigma_R$ increases, as well as the increase in the amplitude of the correlation function around $\theta = 0^\circ$ in the case of the $\phi = 20^\circ$ and $90^\circ$ plots. For periodic on average media, the amplitude of the broad plateau centered at $\theta = 90^\circ$ increases much faster compared to the other features of the plot than in the case of random $\sigma_\varepsilon$. This raises the possibility of discerning between the systems exhibiting random dielectric constants and those exhibiting random radii.

Finally, in Fig. 3.7 we present plots of the $C^{(10)}$ contribution for homogeneously random arrays of spheres of Gaussianly distributed random radii. The dielectric constant of the spheres is $\varepsilon = -9$. The envelope scans of the $C^{(10)}$ envelope are very similar to those for $C^{(1)}$, exhibiting the same type of "background" behavior at $\phi = 0^\circ$. The only differences worth noting are in the shape of the $\theta$-depending contribution in the $\phi = 0^\circ$ plots.

From the results presented above, we conclude that the best way to determine the standard deviations of the dielectric constants or radii distributions is from the ratio between the amplitude of the speckle correlation function at small angles $\theta$ and
Figure 3.5: Plots of the angular speckle correlator along the $C^{(1)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a homogeneously random medium with $\epsilon = -9$. Each plot exhibits the results for four different values of $\sigma_R$. 

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Figure 3.6: Plots of the angular speckle correlator along the $C^{(1)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a periodic on average medium with $\epsilon = -9$. Each plot exhibits the results for four different values of $\sigma_R$. 

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Figure 3.7: Plots of the angular speckle correlator along the $C^{(10)}$ envelope as functions of $\theta$, for a) $\phi = 0^\circ$, b) $\phi = 20^\circ$ and c) $\phi = 90^\circ$ for a homogeneously random medium with $\epsilon = -9$. Each plot exhibits the results for four different values of $\sigma_R$.
the amplitude at some intermediate value, say $\theta = 15^\circ$. These measurements have to be carried out for values of the azimuthal angle $\phi$ between $10^\circ$ and $90^\circ$. In the case of periodic on average systems, the standard deviations can be also determined from the ratio between the amplitude of the correlation function at large angles ($\theta \approx 90^\circ$) and the amplitude at some intermediate value of $\theta$. 
In this chapter we discuss the case of one-dimensional disorder[10-12] occurring at
the interface between vacuum and a homogeneous dielectric. The random surface
consists of an array of parallel, Gaussian cylinder shaped, dielectric ridges placed at
random on a planar vacuum-dielectric interface[12]. The $x_3$ axis is perpendicular to
the interface, the $x_2$ axis is parallel to the axes of symmetry of the ridges, and the
$x_1 - x_3$ plane is the scattering plane. The positions of the ridges on the interface
are not correlated and the average number of ridges per unit of length in the $x_1$
direction is small enough that the likelihood of a strong overlapping between any
two ridges is neglegable. The outer surface is described by a random profile function
$x_3 = \zeta(x_1)$ with
\[
\zeta(x_1) = \sum_i A_i e^{-\left(\frac{x_1 - x_{1i}}{R_i}\right)^2},
\]
(4.1)
where $A_i$, $R_i$, and $x_{1i}$ are parameters characterizing the height, width, and center
position of ridge $i$.

The dielectric constants of the substrate and ridge materials are denoted by
$\epsilon_0$ and $\epsilon_1$ respectively. The substrate material is taken to be CdS while the ridges
are taken to be ZnS, GaAs, or NaF. All these materials exhibit a characteristic
Reststrahl resonance. The functional dependence on frequency of their electrical
permittivities is of the form
\[
\epsilon(\omega) = \epsilon_\infty \frac{\omega_L^2 - \omega^2 - i\frac{\omega}{\tau}}{\omega_T^2 - \omega^2 - i\frac{\omega}{\tau}},
\]
(4.2)
where $\epsilon_\infty$, $\omega_L$, $\omega_T$, and $\tau$ are constants characterizing a given material. The resonant
frequencies of the $\varepsilon_1$ ridge materials are in the range $\Re\varepsilon_0(\omega) < -1$, which is required for the propagation of plasmon-polaritons along the substrate (CdS) surface. The surface plasmon-polaritons are scattered by the randomly positioned ridges. The resonance of the ridge material increases their scattering cross sections, thus enhancing the phase-coherent multiple scattering effects. The resonance also results in a stronger coupling of the bulk electromagnetic modes above and below the surface to the plasmon-polaritons propagating along the planar interface.

The model described above has been employed in Ref.[12], where the scattering cross sections are calculated for interfaces with identical ridges. We extend the study to somewhat more realistic models, by allowing the heights or the widths of the ridges to be Gaussianly distributed random variables. The case of identically shaped ridges made at random from two different dielectric materials is also studied. In addition, we compute the $C^{(1)}$, $C^{(10)}$, and $C^{(1.5)}$ contributions to the speckle correlation function.

In the following, averages over the ensemble of realizations of the form[12,20,21]

$$\langle \tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\tilde{\zeta}(k_3)\ldots \rangle$$

are computed. Here $\tilde{\zeta}(k)$ is the Fourier transform of the profile function. However, the fundamental quantity is the average of the product of two Fourier coefficients. As for the products of more than two coefficients, we apply the following factorization rules[20,21]:

1. The average of the product of an odd number of coefficients vanishes.

2. The average of the product of an even number of coefficients is expressed as the sum of all possible distinct combinations of products of pair averages.

The scattering cross sections and the correlation functions computed for random surfaces contain[10-12] both specular terms, like those arising in the case of
reflection from a flat surface, and continuously varying non-specular terms corresponding to diffuse scattering. The diffuse contribution contains the phase-coherent effects\cite{8,10-12,14}, which are easiest to observe experimentally. Since we are interested primarily in these phase coherent effects, in the following we limit our study to the diffuse scattering components and neglect the specular terms.

The surface is illuminated by a monochromatic, \( p \)-polarized electromagnetic plane wave incident from vacuum. The general form of the \( x_2 \) component of the magnetic field is given by\cite{12}

\[
\left[ \Delta + \left( \frac{\omega}{c} \right)^2 \right] H_2(x_1, x_3|\omega) = 0 \tag{4.3}
\]

for \( x_3 > \zeta(x_1) \),

\[
\left[ \Delta + \epsilon_1(x_1|\omega) \left( \frac{\omega}{c} \right)^2 \right] H_2(x_1, x_3|\omega) = 0 \tag{4.4}
\]

for \( 0 < x_3 \leq \zeta(x_1) \) and

\[
\left[ \Delta + \epsilon_0(\omega) \left( \frac{\omega}{c} \right)^2 \right] H_2(x_1, x_3|\omega) = 0 \tag{4.5}
\]

for \( x_3 \leq 0 \). Here \( \epsilon_0 \) and \( \epsilon_1 \) are the electrical permittivities of the substrate and ridge materials, respectively. Using Green’s theorem and the extinction theorem\cite{1,2,10-12,24}, we find that the solution satisfying the boundary conditions at infinity is given, in the three regions, by\cite{10-12}

\[
H_{2\text{vac}}(x_1, x_3) = e^{ikx_1 - i\alpha_\nu(k,\omega)x_3} + \int \frac{dq}{2\pi} R(q|k) e^{iqx_1 + i\alpha_\nu(q,\omega)x_3} \tag{4.6}
\]

\[
H_{2\text{id}}(x_1, x_3) = \int \frac{dq}{2\pi} \left[ A(q|k) e^{iqx_1 + i\alpha_1(q,\omega)x_3} + B(q|k) e^{iqx_1 - i\alpha_1(q,\omega)x_3} \right] \tag{4.7}
\]

\[
H_{2\text{sub}}(x_1, x_3) = \int \frac{dq}{2\pi} S(q|k) e^{iqx_1 - i\alpha_0(q,\omega)x_3} \tag{4.8}
\]

In Eqs. (4.6-8) \( k \) and \( q \) are the horizontal components of the incident and scattered wavevectors, respectively. The functions \( \alpha_\nu, \alpha_1, \) and \( \alpha_0 \) which give the vertical
components of the wavevectors are defined by

\[ \alpha_i(p, \omega) = \sqrt{\varepsilon_i(\omega) \left( \frac{\omega}{c} \right)^2 - p^2}, \]

with the real and imaginary parts taken to be positive and obviously \( \epsilon_v = 1 \) for vacuum. The scattering and transmission amplitudes, denoted by \( R(q|k) \) and \( S(q|k) \) respectively, are obtained by applying the electromagnetic boundary conditions to the solutions of the Maxwell equations above the surface, inside the ridges, and in the substrate material. In doing this, the validity of the Rayleigh hypothesis is assumed.

At each of the two surfaces, the matching solutions must satisfy the following boundary conditions[1,2,24]:

\[ H^<_{\parallel}(x_1, \zeta(x_1)) = H^>_{\parallel}(x_1, \zeta(x_1)), \]

\[ \varepsilon^< \tilde{E}^<(x_1, \zeta(x_1)) \cdot \hat{n} = \varepsilon^> \tilde{E}^>(x_1, \zeta(x_1)) \cdot \hat{n}, \]

\[ \tilde{E}^<(x_1, \zeta(x_1)) \times \hat{n} = \tilde{E}^>(x_1, \zeta(x_1)) \times \hat{n}, \]

where \( \hat{n} = \hat{n}(x_1) \) is the normal at the surface. For each surface, one of the first two conditions is redundant since they impose the same constraints on the scattering amplitudes. By matching the boundary conditions (4.10-12) we find that the scattering and transmission amplitudes \( R(q|k) \) and \( S(q|k) \) are given by[10-12]

\[ R(q|k) = 2\pi \delta(q - k) R_0(k) - 2i G_0(q) T(q|k) G_0(k) \alpha_v(k), \]

\[ S(q|k) = 2\pi \delta(q - k) S_0(k) - 2i G_0(q) U(q|k) G_0(k) \alpha_v(k). \]

Here \( R_0(k) \) and \( S_0(k) \) are the Fresnel coefficients for the reflection and transmission of \( p \)-polarized light at a flat dielectric surface and are given by

\[ R_0(k) = \frac{\varepsilon_0(\omega) \alpha_v(k, \omega) - \alpha_0(k, \omega)}{\varepsilon_0(\omega) \alpha_v(k, \omega) + \alpha_0(k, \omega)}, \]

\[ S_0(k) = \frac{2\varepsilon_0(\omega) \alpha_v(k, \omega)}{\varepsilon_0(\omega) \alpha_v(k, \omega) + \alpha_0(k, \omega)}. \]
while \( G_0(k) \) is the surface-polariton Green’s function for a flat surface and is given by

\[
G_0(k) = \frac{i\epsilon_0}{\epsilon_0\alpha_0(k) + \alpha_0(k)}.
\]  

(4.17)

The scattering matrix \( T(q|k) \) satisfies the equation[10-12]

\[
T(q|k) = V(q|k) + \int \frac{dp}{2\pi} V(q|p)G_0(p)T(p|k),
\]  

(4.18)

where \( V(q|k) \) is the scattering potential. To leading order in the amplitude of the profile function \( \zeta \), the scattering potential is given by

\[
V(q|k) = \frac{\epsilon_1 - 1}{\epsilon_1\epsilon_0}\left[\epsilon_0 2qk - \epsilon_1\alpha_0(q)\alpha_0(k)\right]\zeta(q - k).
\]  

(4.19)

The transmission matrix \( U(q|k) \) is the solution of[12]

\[
U(q|k) = V'(q|k) + \int \frac{dp}{2\pi} V'(q|p)G_0(p)T(p|k),
\]  

(4.20)

with the transmission potential given by

\[
V'(q|k) = \frac{\epsilon_1 - 1}{\epsilon_1\epsilon_0}\left[\epsilon_0 2qk + \epsilon_1\alpha_0(q)\alpha_0(k)\right]\zeta(q - k).
\]  

(4.21)

For the purpose of computing the scattering cross section and the correlation function it is convenient to introduce the surface polariton Green’s function for the rough surface, \( G(q|k) \), defined as the solution of the Dyson equation[20,21]

\[
G(q|k) = 2\pi\delta(q - k)G_0(k) + G_0(q) \int \frac{dp}{2\pi} V(q|p)G(p|k).
\]  

(4.22)

Let us denote the integral in the last term of Eq. (4.22) by \( \Lambda(q|k) \). Multiplying (4.22) by \( V(r|q) \) and integrating over \( q \) we find that \( \Lambda(r|k) \) satisfies the equation

\[
\Lambda(r|k) = V(r|k)G_0(k) + \int \frac{dq}{2\pi} V(r|q)G_0(q)\Lambda(q|k),
\]  

(4.23)
which coincides with the equation for $T(q|k)G_0(k)$ obtained from (4.18). This allows us to write

$$G(q|k) = 2\pi \delta(q - k)G_0(k) + G_0(q)T(q|k)G_0(k).$$  \hspace{1cm} (4.24)

Eq. (4.24), together with (4.13-14) and (4.20), can be used to express the scattering and transmission amplitudes in terms of the surface polariton Green's function $G(q|k)$.

The poles of the single-particle Green’s functions correspond to the modes of propagation of the surface polaritons, and their functional dependence on $\omega$ gives the dispersion relation. Using the definition of $\alpha_0(p)$ given by equation (4.9), the Green’s function for the flat surface (4.17) can be rewritten as

$$G_0(p) = \frac{i\epsilon_0}{1 - \epsilon_0^2} \frac{\epsilon_0 \sqrt{(\omega/c)^2 - p^2} - \sqrt{\epsilon_0^2 (\omega/c)^2 - p^2}}{2\omega \sqrt{\epsilon_0/c + 1}} \left( \frac{1}{p - \frac{\omega}{c} \sqrt{\epsilon_0/c + 1}} - \frac{1}{p + \frac{\omega}{c} \sqrt{\epsilon_0/c + 1}} \right).$$  \hspace{1cm} (4.25)

Using this form we see that the dispersion relation for the surface polaritons has two branches, at $p(\omega) = \pm \frac{\omega}{c} \sqrt{\epsilon_0(\omega)/c + 1}$, corresponding to propagation in opposite directions along the $x_1$ axis. The average single-particle Green’s function for the rough surface $G(p)$, which is introduced in the next chapter, has a similar expression.

As will be seen, when solving for $G(q|k)$ it is sometimes convenient to retain only the residues in the Laurent expansion for $G(p)$. 

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CHAPTER 5
SURFACE DISORDER. THE SCATTERING CROSS SECTION

5.1 Scattering Efficiency

In vacuum, the Poynting vector is given by\[1,2,24\]
\[
\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} = \frac{c}{4\pi} \vec{H} \times (\hat{p} \times \vec{H}),
\]
(5.1)

where \(\hat{p}\) denotes the unit vector that gives the direction of propagation of either the incident or the scattered wave,
\[
\hat{p} = \frac{\nu}{\nu}(p, 0, \pm \alpha_v(p, \omega)).
\]
(5.2)

Consequently, the time-averaged normal component of the Poynting vector for the incident wave can be written as\[10-12\]
\[
S^{(\text{in})}_n(k, \omega) = -\frac{c^2}{8\pi\nu}|H_2^{(\text{in})}(k, \omega)|^2 \alpha_v(k, \omega) = -\frac{c^2}{8\pi\nu}\alpha_v(k, \omega).
\]
(5.3)

For the scattered wave we have\[1,2,24\]
\[
\vec{E}^{(sc)}(k, \omega) = \frac{c}{\omega} \int \frac{dq}{2\pi} R(q|k) (\alpha_v(q, \omega), 0, -q) e^{i\nu x_1 + i\alpha_v(q, \omega)x_3},
\]
(5.4)

hence the time-averaged normal component of the Poynting is
\[
S^{(sc)}_n(x_1, x_3|k, \omega) = \frac{c^2}{8\pi\nu} \int \frac{dq}{2\pi} \int \frac{dq'}{2\pi} R(q|k) R^*(q'|k) \times \alpha_v(q', \omega) e^{-i(q-q')x_1} e^{-i(\alpha_v(q', \omega)-\alpha_v(q, \omega)x_3}.
\]
(5.5)

By averaging \(S^{(sc)}_n(x_1, x_3|k, \omega)\) over the whole length \(L\) of the surface in the \(x_1\) direction, we find
\[
\langle S^{(sc)}_3(k, \omega) \rangle = \frac{c^2}{8\pi\omega L} \int \frac{dq}{2\pi} |R(q|k)|^2 \alpha_v(q, \omega).
\]
(5.6)
The efficiency by which the incident wave is converted into scattered waves with the horizontal component of the wavevector lying between \( q \) and \( q + dq \) is given by the scattering efficiency, defined by

\[
I(q|k) = \frac{1}{S_3^{(in)}} \frac{d\langle S_3^{(ref)} \rangle}{dq} = \frac{1}{2\pi L \alpha_v(k, \omega)} |R(q|k)|^2.
\]  

(5.7)

By averaging the scattering efficiency (5.7) a second time over the ensemble of realizations of the surface, we have

\[
I(q|k) = \frac{1}{2\pi L \alpha_v(k)} \langle |R(q|k)|^2 \rangle.
\]  

(5.8)

Using Eqs. (4.13) and (4.24) we find that

\[
R(q|k) = -2\pi\delta(q - k) - 2i\alpha_v(k)G(q|k),
\]  

(5.9)

and from here

\[
|R(q|k)|^2 = 2\pi\delta(q - k) [L - 4\alpha_v(k)\Re G(k|k)] + 4\alpha_v^2(k)|G(q|k)|^2.
\]  

(5.10)

The first term in the right-hand side of Eq. (5.10) is the specular term that we are going to neglect. By combining Eqs. (5.8) and (5.10) we obtain the scattering efficiency for diffuse scattering in terms of the surface polariton Green’s function for the rough surface,

\[
I(q|k)_{\text{diff}} = \frac{2}{\pi L} \alpha_v(q)\alpha_v(k) \langle |G(q|k)|^2 \rangle_{\text{diff}}.
\]  

(5.11)

The average that appears in the right-hand side of Eq. (5.11) is a particular solution of the Bethe-Salpeter equation

\[
\langle G^*(q|k)G(q'|k') \rangle = 2\pi\delta(q - k)G^*(k)2\pi\delta(q' - k')G(k') + G^*(q)G(q')
\]  

\[
\times \int \frac{dr}{2\pi} \int \frac{ds}{2\pi} \langle \Gamma(q, r|q', s) \rangle \langle G^*(r|k)G(s|k') \rangle,
\]  

(5.12)

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where $2\pi \delta(q-k)G(k)$ is the average of the single-particle Green’s function for the rough surface, and $\Gamma(q,r|p,s)$ is the irreducible four-vertex function. To lowest order in $\zeta$, the average of the irreducible four-vertex function is given by [8,10-12]

$$\langle \Gamma(q,r|p,s) \rangle = \langle V(q|r)V^*(p|s) \rangle - \langle V(q|r) \rangle \langle V^*(p|s) \rangle.$$  

(5.13)

By substituting Eq. (4.19) into (5.13) we find

$$\langle \Gamma(q,r|p,s) \rangle = \langle v(q|r)v^*(p|s)\zeta(q-r)\zeta^*(p-s) \rangle - \langle v(q|r)\zeta(q-r) \rangle \langle v^*(p|s)\zeta^*(p-s) \rangle,$$

(5.14)

where we have denoted

$$v(q|k) = \frac{\epsilon_1 - 1}{\epsilon_1 \epsilon_0} \left[ \epsilon_0^2 q k - \epsilon_1 \alpha_0(q) \alpha_0(k) \right].$$

(5.15)

### 5.2 Evaluation of the Irreducible Vertex Functions

The Fourier transform of the profile function is obtained as in Ref. [12], by discretizing the $x_1$ axis and allowing the Gaussian ridges to be centered only at the vertices of a one-dimensional lattice given by $x_{1n} = n\Delta x$, $n \in \mathbb{Z}$. Let us denote by $c_n$ the occupation index of vertex $n$ for a given instance of the surface. Since the statistical properties of the surface are invariant at translations and $c_n$ can only be 0 or 1, we have $<c_n> = <c_n^2> = c_0$ and obviously $\mathcal{N} = \frac{c_0}{\Delta x}$.

We will consider first the case of an interface with ridges characterized by random geometrical parameters but made from the same material. Consequently, only the Fourier transforms in (5.14) have to be averaged. The profile function (4.1) can be rewritten in terms of the occupation indices $c_n$ as

$$\zeta(x_1) = \sum_n c_n A_n e^{-\left(\frac{x_1 - n\Delta x}{\bar{R}_n}\right)^2},$$

(5.16)
and its Fourier transform is given by

$$\hat{\zeta}(p) = \sqrt{\pi} \sum_{n} c_{n} A_{n} R_{n} e^{-\frac{p^{2} R_{n}^{2}}{4}} e^{-i p \Delta x}. \quad (5.17)$$

For the average of the product of two Fourier coefficients we find, using (5.17),

$$\langle \hat{\zeta}(p) \hat{\zeta}(q) \rangle = \pi \sum_{m,n} K_{mn}(p,q) e^{-i(m p + n q) \Delta x}, \quad (5.18)$$

where

$$K_{mn}(p,q) = \langle c_{m} c_{n} A_{m} A_{n} R_{m} R_{n} e^{-\frac{p^{2} R_{m}^{2}}{4}} e^{-\frac{q^{2} R_{n}^{2}}{4}} \rangle. \quad (5.19)$$

In evaluating the averages in (5.19) we will make the hypothesis that the fluctuations of the width and height of a given ridge as well as the fluctuations of the parameters of any two distinct ridges are not correlated. Hence, for $m = n$ we can write

$$K_{mm}(p,q) = \langle c_{m} c_{m} A^{2} A^{2} \rangle \langle R^{2} e^{-\frac{p^{2} R^{2}}{4}} \rangle \langle R^{2} e^{-\frac{q^{2} R^{2}}{4}} \rangle, \quad (5.20)$$

while for $m \neq n$ we have

$$K_{mn}(p,q) = c_{0} \langle A^{2} \rangle \langle R^{2} e^{-\frac{p^{2} R^{2}}{4}} \rangle \langle R^{2} e^{-\frac{q^{2} R^{2}}{4}} \rangle, \quad (5.21)$$

where the averages are computed using the probability density functions that characterize the ensemble of realizations of the surface.

From (5.18-21) and using the Fourier expansion of the Dirac function on the \((-\pi, \pi)\) interval

$$\sum_{m} e^{i m p} = 2 \pi \delta(p), \quad (5.22)$$

we find

$$\langle \hat{\zeta}(p) \rangle = \sqrt{\pi} c_{0} \frac{\langle A \rangle \langle R \rangle}{\Delta x} 2 \pi \delta(p), \quad (5.23)$$

$$\langle \hat{\zeta}(p) \hat{\zeta}(q) \rangle = \frac{\pi}{\Delta x} \left[ c_{0}^{2} \langle A^{2} \rangle \langle R^{2} e^{-\frac{p^{2} R^{2}}{4}} \rangle - c_{0}^{2} \langle A \rangle^{2} \langle R^{2} e^{-\frac{p^{2} R^{2}}{4}} \rangle^{2} \right] 2 \pi \delta(p + q)$$

$$+ \pi \left( c_{0} \frac{\langle A \rangle \langle R \rangle}{\Delta x} \right)^{2} 2 \pi \delta(p) 2 \pi \delta(q). \quad (5.24)$$
Finally, by substituting (5.23-24) into (5.14) and taking into account the fact that $\zeta(p) = \zeta(-p)$ we find that the average irreducible vertex function is given by [10-12]

$$
(\Gamma(q, r|p, s)) = 2\pi\delta(q - r - p + s)\Gamma_0(q, r|p, s), \tag{5.25}
$$

with

$$
\Gamma_0(q, r|p, s) = v(q|r)v^*(p|s)\frac{\pi}{\Delta x} \left[ c_0(A^2)\langle R^2 e^{-(q-r)^2 R^2_4} \rangle - c_0^2(A)\langle R e^{-(q-r)^2 R^2_4} \rangle^2 \right]. \tag{5.26}
$$

Let us consider next the model of an interface with identical ridges, each ridge being made at random from a material belonging to a set of $N$ dielectric materials. Let $\epsilon_i$ denote the permittivity of ridge material $i$, and $P_i$ be the probability for a ridge to be made from that material. In this case we start by computing the scattering potential for a single ridge, centered at node $n$ and made from material $i(n)$, which is given by

$$
V_n(q|k) = \sqrt{\pi c_n} R e^{-(q-k)^2 R^2_4} e^{-in(q-k)\Delta x} v_i^{(n)}(q|k), \tag{5.27}
$$

where we have denoted

$$
v_i^{(q|k)} = \frac{\epsilon_i - 1}{\epsilon_i^2 - 1} [\epsilon_0 q k - \epsilon_i \alpha_0(q)\alpha_0(k)]. \tag{5.28}
$$

The total scattering potential will be the sum of all potentials from individual ridges,

$$
V(q|k) = \sum_n V_n(q|k). \tag{5.29}
$$

Using (5.27) and (5.29), the average irreducible vertex function can be written as

$$
(\Gamma(q, r|p, s)) = \pi A^2 R^2 e^{-(q-r)^2 R^2_4 + (p-s)^2 R^2_4} \times \sum_{m,n} \langle c_m c_n v_i^{(m)}(q|r) v_i^{(n)*}(p|s) \rangle e^{-im(q-r)\Delta x} e^{in(p-s)\Delta x}. \tag{5.30}
$$
Since the set of probabilities \( \{P_i\} \) is the same for all vertices, and the fluctuations of the parameters corresponding to different vertices are not correlated, the averages in (5.30) will take the form

\[
\langle e_m^2 v^{i(m)}(q|r)v^{i(m)*}(p|s) \rangle = c_0 \langle v(q|r)v^*(p|s) \rangle
\]

for \( m = n \), and

\[
\langle c_m c_n v^{i(m)}(q|r)v^{i(n)*}(p|s) \rangle = c_0^2 \langle v(q|r)\rangle \langle v(p|s) \rangle^*
\]

for \( m \neq n \), where

\[
\langle v(q|r) \rangle = \sum_i v^i(q|r) P_i,
\]

\[
\langle v(q|r)v^*(p|s) \rangle = \sum_i v^i(q|r)v^{i*}(p|s) P_i.
\]

Finally, by using (5.22), we find that the irreducible four-vertex function has the same form as in (5.25), with

\[
\Gamma_0(q, r|p, s) = \pi \frac{A^2 R^2}{\Delta x} e^{-(q-r)^2 \frac{R^2}{2\Delta x}} \left[ c_0 \langle v(q|r)v^*(p|s) \rangle - c_0^2 \langle v(q|r)\rangle \langle v(p|s) \rangle^* \right].
\]

### 5.3 Evaluation of the Green’s Functions

By substituting (5.25) in the Bethe-Salpeter equation (5.12) written specifically for \((q, k) = (q', k')\)

\[
\langle |G(q|k)|^2 \rangle = 2\pi \delta(q - k)L|G(k)|^2
\]

\[
+ |G(q)|^2 \int \frac{dr}{2\pi} \int \frac{ds}{2\pi} \langle \Gamma(q, r|q, s) \rangle \langle G^*(r|k)G(s|k) \rangle,
\]

it follows that [10-12]

\[
\langle |G(q|k)|^2 \rangle = 2\pi \delta(q - k)L|G(k)|^2 + |G(q)|^2 \int \frac{ds}{2\pi} K(q|s) \langle |G(s|k)|^2 \rangle,
\]

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where $K(q|k) = \Gamma_0(q, k|q, k)$.

From (5.36) it is straightforward to show that the diffuse part of $\langle |G(q|k)|^2 \rangle$ is given by

$$\langle |G(q|k)|^2 \rangle_{\text{diff}} = L|G(q)|^2 |G(k)|^2 \tau(q|k),$$

(5.37)

where $\tau(q|k)$ is the reducible vertex function for the diffuse scattering and satisfies the equation

$$\tau(q|k) = K(q|k) + \int \frac{ds}{2\pi} K(q|s)|G(s)|^2 \tau(s|k).$$

(5.38)

This integral equation can be solved[8-12,20,21] by the method of iterations, using $\tau(q|k) = K(q|k)$ as the lowest order approximation. In the following we use only the first iteration of (5.38), which corresponds to the summation of first two ladder diagrams, and write

$$\tau(q|k) = K(q|k) + \int \frac{ds}{2\pi} K(q|s)|G(s)|^2 K(s|k).$$

(5.39)

To these ladder-diagram terms we must add[8-12,20,21] the contribution of the first maximally-crossed diagram to the irreducible vertex function. This diagram contains intermediate states contributing to the localization of the surface polaritons[25]. The irreducible vertex function corresponding to these states, which will be denoted by $\langle \Lambda(q, r|p, r) \rangle$, can be expressed [10-12] in terms of the irreducible vertex function from Eq. (5.13) as

$$\langle \Lambda(q, r|p, s) \rangle = 2\pi \delta(q - r - p + s) \Lambda_0(q, r|p, s),$$

(5.41)

where

$$\Lambda_0(q, r|p, s) = \int \frac{du}{2\pi} \Gamma_0(q, u|q + s - u, s) G(u) G^*(q + s - u) \Gamma_0(u, r|p, q + s - u).$$

(5.42)
By using a procedure similar to the one that was employed to derive (5.37-38) from (5.35) and retaining only the lowest order approximation, we find that the reducible vertex function for the maximally crossed diagram is given by [10-12]

\[ \tau^{(c)}(q|k) = \Lambda_0(q,k|q,k). \] (5.43)

Combining Eqs. (5.39) and (5.43) we find the total reducible vertex function

\[ \tau^{(t)}(q|k) = K(q|k) + \int \frac{ds}{2\pi} K(q|s)|G(s)|^2 K(s|k) + \int \frac{ds}{2\pi} \Gamma_0(q,s)|q + k - s,k \]
\[ \times G(s)G^*(q + k - s)\Gamma_0(s,k|q,q + k - s) \] (5.44)

which, together with (5.11) and (5.37), gives the scattering efficiency

\[ I(q|k)_{\text{diff}} = \frac{2}{\alpha_0(q)\alpha_0(k)}|G(q)|^2|G(k)|^2\tau^{(t)}(q|k). \] (5.45)

To conclude our theoretical discussion of the scattering efficiency, we have to compute the average single-particle Green's function \( G(k) \) that appears in Eqs. (5.44) and (5.45). The Dyson equation written for \( G(k) \) yields[12,20,21]

\[ G(k) = \frac{1}{G_0^{-1}(k) - \Sigma(k)}, \] (5.46)

where \( \Sigma(k) \) is the self-energy correction, solution of the integral equation[12,20,21]

\[ \langle \mathcal{V}(q|k) \rangle + \int \frac{dp}{2\pi} G(p)\langle \mathcal{V}(q|p)\mathcal{V}(p|k) \rangle = 0, \] (5.47)

with

\[ \mathcal{V}(q|k) = V(q|k) - 2\pi \delta(q - k)\Sigma(k). \] (5.48)

To the lowest order in \( \Sigma(k) \), the system formed by Eqs. (5.46-48) yields

\[ \langle \mathcal{V}(q|k) \rangle - 2\pi \delta(q - k)\Sigma(k) + \int \frac{dp}{2\pi} \frac{\langle \mathcal{V}(q|p)\mathcal{V}(p|k) \rangle}{G_0^{-1}(p) - \Sigma(p)} = 0. \] (5.49)
The averages that appear in Eq. (5.49) can now be evaluated. Using (5.23), we find

$$\langle V(q|k) \rangle = 2\pi\delta(q - k)V_0(k), \quad (5.50)$$

with

$$V_0(k) = \sqrt{\pi c_0} \frac{\langle A \rangle \langle R \rangle}{\Delta x} v(k|k) \quad (5.51)$$

if the widths or the heights of the ridges are randomly distributed, or

$$V_0(k) = \sqrt{\pi c_0} \frac{AR}{\Delta x} \langle v(k|k) \rangle \quad (5.52)$$

in the case of random ridge materials. Here $v(q|k)$ and $\langle v(q|k) \rangle$ are those defined by (5.15) and (5.33), respectively. For $\langle V(q|p)V(p|k) \rangle$, we find

$$\langle V(q|p)V(p|k) \rangle \approx 2\pi\delta(q - k)\tilde{\Gamma}_0(p, k|p, k) \quad (5.53)$$

where $\tilde{\Gamma}_0(q, r|p, s)$ can be computed using the same procedures that were employed for (5.26) or (5.34) earlier in this chapter, leading to similar expressions.

Finally, by substituting (5.50) and (5.53) into (5.49), we have

$$\Sigma(k) = V_0(k) + \int \frac{dp}{2\pi} \frac{\tilde{\Gamma}_0(p, k|p, k)}{G^{-1}_0(p) - \Sigma(p)}, \quad (5.54)$$

which can be solved by iteration, using $\Sigma(k) = V_0(k)$ as the lowest order approximation.

### 5.4 Results for the Scattering Cross Section

The particular forms of the averages in the expression (5.26) of $\Gamma_0$ depend on the actual distribution functions that are chosen to describe the fluctuations of the ridge heights $A_i$ and widths $R_i$. Since the right-hand side of (5.26) is a simple function of $<A>$ and $<A^2>$, it follows that most of the information contained in the distribution function of $A$ is irrelevant. All we need to know is its average and
standard deviation. On the other hand, the dependence of (5.26) on $R$ is much more complicated, and detailed knowledge of the distribution function $p(R)$ is required. In our studies, the ridge widths are taken to be distributed according to a truncated Gaussian law, defined by

$$p(R) = \begin{cases} \frac{1}{a \sqrt{2\pi}} e^{-\frac{(R-R_0)^2}{2a^2}} & \text{for } R \geq 0, \\
0 & \text{for } R < 0. \end{cases} \quad (5.55)$$

Here $R_0$ and $a$ are the mean and the standard deviation of the complete Gaussian distribution. The transcendental function $\text{erf}(x)$ is a primitive of the Gaussian probability density function, defined by

$$\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt. \quad (5.56)$$

The averages in (5.26) can be evaluated using the truncated Gaussian distribution (5.55) and the definition (5.56), yielding

$$\langle R e^{-p^2 \frac{R^2}{a^2}} \rangle = a^2 \rho_1^2(p) e^{-\frac{R_0^2}{2a^2}} + \sqrt{2\pi} \rho_1(p) e^{-p^2 \frac{R_0^2}{a^2}} \left[ 1 - \text{erf}\left( \frac{-R_1(p)}{a\sqrt{2}} \right) \right], \quad (5.57)$$

$$\langle R^2 e^{-p^2 \frac{R^2}{a^2}} \rangle = a^2 \rho_2^2(p) \frac{\rho_2(p) e^{-\frac{R_0^2}{2a^2}} + \sqrt{2\pi} \left[ 1 + \rho_2(p) \right] e^{-p^2 \frac{R_0^2}{a^2}} \left[ 1 - \text{erf}\left( \frac{-R_2(p)}{a\sqrt{2}} \right) \right]}{\sqrt{2\pi} \left[ 1 - \text{erf}\left( \frac{-R_0}{a\sqrt{2}} \right) \right]},$$

where the functions $\rho_1$ and $\rho_2$ are defined by

$$\rho_1(p) = \frac{R_0}{\sqrt{1 + p^2 a^2}},$$

$$\rho_2(p) = \frac{R_0}{\sqrt{1 + p^2 a^2}}. \quad (5.58)$$

Equations (5.44-46) together with (5.54) and (5.26) or (5.34) are used for the evaluation of the differential reflection coefficient, defined by

$$\frac{\partial R}{\partial \theta_s} = \frac{\omega}{2\pi c} \cos \theta_s I(q|k)_{\text{diff}}, \quad (5.59)$$
for different ridge materials. The substrate is in each case CdS, while the ridges are taken to be made from ZnS, NaF, or combinations with different proportions of ZnS and NaF. The parameters characterizing the functional dependence (4.2) on \( \omega \) of each dielectric can be found in Table 6.1[12]. All constants except for \( \epsilon_\infty \) are given in units of cm\(^{-1}\). The geometrical parameters of the interface are expressed in units of \( \frac{\varepsilon}{\omega_{T1}^{2}} \), where \( \omega_{T1}^{2} \) is the resonant frequency of the ridge material. We have chosen a single non-random value of the ridge height \( A = 0.05 \), while the widths of the ridges are Gaussianly distributed with \( \langle R \rangle = 1 \) and the variance parameter \( \alpha \) taking different values. The value of the lattice parameter is \( \Delta x = 0.1 \)[12].

The behavior of the differential reflection coefficient as a function of the variance parameter \( \alpha \) was studied for values of \( \alpha \) between 0 and 1 and for normalized frequencies \( f = \frac{\omega}{\omega_{T1}^{2}} \) approaching the resonance value \( f = 1 \) from above and below. In Fig. 5.1 we present plots of the differential reflection coefficient for NaF computed as a function of \( \theta_s \), for \( \theta_t = 10^\circ \) and different values of \( f \) and \( \alpha \). There are obvious changes in the shape of the function when \( \alpha \) is varied while \( f \) is kept constant. Similar changes in shape are revealed in Fig. 5.2 for ZnS. In general, the values of the reflection coefficient corresponding to scattering angles in a range close to the angle of specular reflection seem to be more sensitive to the variation of \( \alpha \).

A more detailed analysis shows that, for both materials, there are changes in the height of the enhanced backscattering peak. This peak, centered at \( \theta_{s0} = \theta_t + \pi \), is accounted for by the maximally crossed diagram term in (5.44). As the standard deviation of the widths distribution increases, the height of the backscattering peak decreases. This shows that the phase coherent effects leading to the enhanced backscattering phenomenon are weakened by the increased disorder in the system. The width of the backscattering peak for a given material and frequency is approx-
Figure 5.1: Differential reflection coefficient for NaF, plotted for (a) $f = 0.980$, (b) $f = 0.988$, and (c) $f = 0.996$, for $a = 0$, 0.5 and 1

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Figure 5.2: Differential reflection coefficient for ZnS, plotted for (a) $f = 0.980$, (b) $f = 0.988$, and (c) $f = 0.996$, for $a = 0, 0.5$ and 1.

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Table 5.1: Widths of the enhanced backscattering peak at different frequencies

<table>
<thead>
<tr>
<th></th>
<th>$f = 0.980$</th>
<th>$f = 0.988$</th>
<th>$f = 0.996$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZnS</td>
<td>16.4°</td>
<td>24.1°</td>
<td>34.4°</td>
</tr>
<tr>
<td>NaF</td>
<td>2.61°</td>
<td>3.09°</td>
<td>4.35°</td>
</tr>
</tbody>
</table>

Table 5.1: Widths of the enhanced backscattering peak at different frequencies

Imately constant. Several values of the widths are presented in Table 5.1.

In Fig. 5.3 we present plots of the amplitude of the backscattering peak versus the standard deviation parameter $a$. Since the shape of the peak fits neither Gaussian nor Lorentzian patterns with enough accuracy, the height is defined simply by subtracting the value of a linear background from the reflection coefficient at $\theta_s = \theta_{s0}$:

$$H = \frac{\partial R}{\partial \theta_s}(\theta_{s0}) - \frac{1}{2} \left[ \frac{\partial R}{\partial \theta_s}(\theta_{s0} - 25^\circ) + \frac{\partial R}{\partial \theta_s}(\theta_{s0} + 25^\circ) \right]. \quad (5.60)$$

All plots in Fig. 5.3 fit approximately the same pattern, scaled by a global factor which depends on frequency and material only. These results prove that, for an interface with ridges of random widths, it is possible to determine not only the average but also the variance of the distribution from measurements of the differential reflection coefficient.

The results for ZnS-NaF mixtures are presented in Fig. 5.4. The differential reflection coefficients for different proportions of the mixture are compared, each plot corresponding to a given frequency. The shape of the function describing the differential reflection coefficient is seen to transform continuously as the concentration of ZnS varies from 0 to 100%. A plot of the backscattering peak height as a function of the concentration of ZnS is presented in Fig. 5.5. The results show that for interfaces with ridges made from two different materials it is possible to determine the fractional concentration that characterizes the system.
Figure 5.3: Height of the backscattering peak as a function of $a$ for NaF (a) and ZnS (b)
Figure 5.4: Differential reflection coefficient for different mixtures ZnS-NaF, plotted for (a) $f = 0.980$, (b) $f = 0.988$, and (c) $f = 0.996$
Figure 5.5: Height of the backscattering peak as a function of the ZnS concentration
CHAPTER 6
CORRELATION FUNCTIONS

When many-body perturbation theory is applied\[8,20,21\] to the multiple scattering of light from randomly rough surfaces, different correlation terms arise as calculations are carried out to higher orders of the perturbation parameter, which is the amplitude of the profile function $\zeta$. In the first two sections of this chapter we present the theory and the results for the $C^{(1)}$ and $C^{(10)}$ short range contributions to the correlation function. Both these contributions are of order $\zeta^4$. The next section deals with the medium range $C^{(1.5)}$ contribution of order $\zeta^6$. The $C^{(2)}$ and $C^{(3)}$ contributions, of orders $\zeta^8$ and $\zeta^{12}$ respectively, have been shown\[10\] to be of much lower amplitude and will be neglected. Unlike $C^{(1)}$ and $C^{(10)}$, the higher order contributions are no longer constrained by delta functions in momentum space. In the last section we present calculations and results for the case of periodic on average randomly rough surfaces.

6.1 Theory for $C^{(1)}$ and $C^{(10)}$

We define the correlation function as the variance of $(I(q|k)I(q'|k'))$ about $(I(q|k))$(I(q'|k')). From this definition and from Eq. (5.7), we find

$$C(q,k|q',k') = \frac{1}{(2\pi L)^2} \frac{\alpha_v(q)\alpha_v(q')}{\alpha_v(k)\alpha_v(k')} \left[ |R(q|k)|^2 |R(q'|k')|^2 - \langle |R(q|k)|^2 \rangle \langle |R(q'|k')|^2 \rangle \right].$$

By using (5.9) to rewrite $R(q|k)$ in terms of the surface polariton Green's function $G(q|k)$ and neglecting the specular terms, Eq. (6.1) becomes

$$C(q,k|q',k') = \left( \frac{2}{\pi L} \right)^2 \frac{\alpha_v(q)\alpha_v(k)\alpha_v(q')\alpha_v(k')}{\alpha_v(k)\alpha_v(k')}$$

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According to the factorization rules\cite{20,21} stated at the beginning of Chapter 5, the terms of leading order in \( \zeta \) from \( \langle |G(q|k)\rangle^2|G(q'|k')\rangle^2 \) can be obtained by using the approximation

\[
\langle |G(q|k)\rangle^2|G(q'|k')\rangle^2 \approx \langle |G(q|k)\rangle^2\langle |G(q'|k')\rangle^2 + \langle G^*(q|k)G(q'|k') \rangle \langle G(q|k)G^*(q'|k') \rangle + \langle G^*(q|k)G^*(q'|k') \rangle \langle G(q|k)G(q'|k') \rangle.
\]  

(6.3)

This factorization should suffice for our treatment of the phase-coherent properties of the speckle correlator. From (6.2) and (6.3) we find\cite{10-12}

\[
C(q,k|q',k') = F(q,k|q',k') \left[ \langle G^*(q|k)G(q'|k') \rangle \langle G(q|k)G^*(q'|k') \rangle + \langle G^*(q|k)G^*(q'|k') \rangle \langle G(q|k)G(q'|k') \rangle \right],
\]  

(6.4)

with

\[
F(q,k|q',k') = \left( \frac{2}{\pi L} \right)^2 \alpha_v(q)\alpha_v(k)\alpha_v(q')\alpha_v(k').
\]  

(6.5)

The averages that appear in (6.4) are solutions of Bethe-Salpeter equations \cite{8-12,20,21}. For \( \langle G^*(q|k)G(q'|k') \rangle \), we have

\[
\langle G^*(q|k)G(q'|k') \rangle = 2\pi \delta(q-k)G^*(k)2\pi \delta(q'-k')G(k') + G^*(q)G(q')
\]

\[
\times \int \frac{dr}{2\pi} \int \frac{ds}{2\pi} \langle \Gamma(q,r|q',s) \rangle \langle G^*(r|k)G(s|k') \rangle,
\]  

(6.6)

where for the irreducible vertex function \( \langle \Gamma(q,r|q',s) \rangle \) we can use the approximation (5.13). By adding up the contributions from the first two ladder diagrams and the first maximally-crossed diagram\cite{10-12}, we find

\[
\langle G^*(q|k)G(q'|k') \rangle = 2\pi \delta(q-k)G^*(k)2\pi \delta(q'-k')G(k') + 2\pi \delta(q-k-q'+k')
\]

\[
\times G^*(q)G^*(k)G(q')G(k')L(q,k|q',k'),
\]  

(6.7)
where
\[ \mathcal{L}(q, k|q', k') = \Gamma_0(q, k|q', k') + \int \frac{ds}{2\pi} \Gamma_0(q, q - q' + s|q', s) G^*(q - q' + s) G(s) \Gamma_0(q - q' + s, k|s, k') + \int \frac{ds}{2\pi} \Gamma_0(q, s|q + k' - s, k') G^*(q + k' - s) G(s) \Gamma_0(s, k|q', q + k' - s). \]

For \( \langle G(q|k)G(q'|k') \rangle \), the Bethe-Salpeter equation takes the form [10-12]
\[ \langle G(q|k)G(q'|k') \rangle = 2\pi \delta(q - k) G(k) 2\pi \delta(q' - k') G(k') + G(q) G(q') \times \int \frac{dr}{2\pi} \int \frac{ds}{2\pi} \langle \tilde{\Gamma}(r|q', s) \rangle \langle G(r|k) G(s|k') \rangle \]
leading, after the summation of the same terms as in (6.7), to
\[ \langle G(q|k)G(q'|k') \rangle = 2\pi \delta(q - k) G(k) 2\pi \delta(q' - k') G(k') + 2\pi \delta(q - k + q' - k') \times G(q) G(q') G(k') G(k') \mathcal{U}(q, k|q', k') \]
with
\[ \mathcal{U}(q, k|q', k') = \tilde{\Gamma}_0(q, k|q', k') \]
\[ + \int \frac{ds}{2\pi} \tilde{\Gamma}_0(q, q + q' - s|q', s) G(q + q' - s) G(s) \tilde{\Gamma}_0(q + q' - s, k|s, k') + \int \frac{ds}{2\pi} \tilde{\Gamma}_0(q, s|k' - q + s, k') G(k' - q + s) G(s) \tilde{\Gamma}_0(s, k|q', k' - q + s). \]

The average irreducible vertex function \( \langle \tilde{\Gamma} \rangle \) is defined by
\[ \langle \tilde{\Gamma}(q, r|p, s) \rangle = (V(q, r)V(p, s)) - \langle V(q, r) \rangle \langle V(p, s) \rangle = 2\pi \delta(q - r + p - s) \tilde{\Gamma}_0(q, r|p, s), \]
where \( \tilde{\Gamma}_0 \) can be computed using the same procedures that were employed for \( \Gamma_0 \) in Chapter 5.

By retaining only the terms corresponding to diffuse scattering we have, from Eqs. (6.4), (6.7), and (6.10),
\[ C(q, k|q', k') \approx C^{(1)}(q, k|q', k') + C^{(10)}(q, k|q', k') \]
where

\begin{align}
C^{(1)}(q, k|q', k') &= 2\pi L\delta(q - k - q' + k')H(q, k|q', k')|\mathcal{L}(q, k|q', k')|^2, \quad (6.14) \\
C^{(10)}(q, k|q', k') &= 2\pi L\delta(q - k + q' - k')H(q, k|q', k')|\mathcal{U}(q, k|q', k')|^2, \quad (6.15)
\end{align}

and

\[ H(q, k|q', k') = F(q, k|q', k')|G(q)G(k)G(q')G(k')|^2. \quad (6.16) \]

### 6.2 Results for $C^{(1)}$ and $C^{(10)}$

Equations (6.14-16) together with (6.8), (6.11), and (5.26) or (5.34) are used for the evaluation of the correlation functions for different ridge systems placed on a CdS substrate. The ridge materials are ZnS, GaAs, or NaF. These materials were chosen\[12\] such that the resonant frequencies of the ridges are higher than the resonant frequency of the CdS substrate, and the condition $\Re \varepsilon_0(\omega^\text{ridge}) < -1$ is satisfied. This allows for the propagation of surface plasmon-polaritons along the substrate surface. The dielectric resonance of the ridge material increases the surface polariton scattering cross section. As a result, the intensity of the phase-coherent multiple scattering increases by about two orders of magnitude. Table 6.1 shows the parameters characterizing the functional dependence on $\omega$ for each dielectric\[12\].

All constants except for $\varepsilon_\infty$ are in units of $\text{cm}^{-1}$. As in the case of the differential reflection coefficient, the average occupancy number was taken to be $c_0 = 0.001$ while $\frac{\omega_{\text{rad}}}{c}A = 0.05$, $\frac{\omega_{\text{rad}}}{c}\langle R \rangle = 1$, and $\omega_{\text{rad}}\Delta x = 0.1[12]$.

The horizontal components of the wavevectors can be written as

\begin{align}
q &= \frac{\omega}{c} \sin \theta, \quad q' = \frac{\omega}{c} \sin \theta' \\
k &= \frac{\omega}{c} \sin \theta, \quad k' = \frac{\omega}{c} \sin \theta'
\end{align}

\[ (6.17) \]
Table 6.1: Parameters characterizing the functional dependence on frequency of the dielectric constants of the ridge materials

<table>
<thead>
<tr>
<th>Material</th>
<th>$\varepsilon_\infty$</th>
<th>$\omega_f/2\pi c$</th>
<th>$\omega_L/2\pi c$</th>
<th>$1/2\pi c\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZnS</td>
<td>5.7</td>
<td>282.0</td>
<td>352.0</td>
<td>6.77</td>
</tr>
<tr>
<td>GaAs</td>
<td>11.0</td>
<td>268.7</td>
<td>292.1</td>
<td>2.40</td>
</tr>
<tr>
<td>NaF</td>
<td>1.75</td>
<td>262.0</td>
<td>431.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

where $(\theta_i, \theta_s)$ and $(\theta_i', \theta_s')$ are the two pairs of incidence and scattering angles.

Consequently, (6.5) becomes

$$F(\theta_s, \theta_i | \theta_s', \theta_i') = \left( \frac{2}{\pi L} \right)^2 \left( \frac{\omega}{c} \right)^4 \cos \theta_i \cos \theta_s \cos \theta_i' \cos \theta_s'. \quad (6.18)$$

The correlation functions are computed along the envelopes of $C^{(1)}$ and $C^{(10)}$ for $\theta_i = 20^\circ$, $\theta_s = -10^\circ$, and $\theta_i'$ running from $-90^\circ$ to $90^\circ$. The behavior of the correlation functions is studied for frequencies above and below the resonant frequency of the ridge material, where the increase of the surface polariton scattering cross section occurs.

In a first series of calculations, the ridges are taken to be identical and made from the same material. In this case, the average irreducible vertex function is given by (5.26), with $A$ and $R$ fixed. The results are presented in Figs. 6.1 and 6.2 for $C^{(1)}$ and $C^{(10)}$, respectively. The correlation functions experience a dramatic increase in amplitude as the frequency approaches the resonant frequency of the ridge material. However, to facilitate comparison between the results obtained for different frequencies, the plots have been renormalized to have unit area below each curve. The relative amplitude increase of the speckle correlator is more significant than that of the differential reflection coefficient[12], and confirms the enhancement of the phase coherent effects in the proximity of the resonant frequency. Since the $C^{(10)}$ envelope essentially mirrors the behavior of $C^{(1)}$ but without the phase-
Figure 6.1: Plots of the correlation function along the $C^{(1)}$ envelope for NaF, GaAs, and ZnS ridges at different frequencies
Figure 6.2: Plots of the correlation function along the $C^{(10)}$ envelope for GaAs and NaF ridges at different frequencies
coherent peaks, its study is omitted in the remainder of this section.

The case of ridge systems with random width is presented next. Here the average irreducible vertex function is given by (5.26) with $R$ distributed according to the truncated Gaussian law (5.55). The $C^{(1)}$ correlation functions for several systems of random width NaF ridges characterized by different standard deviation parameters are plotted in Fig. 6.3. As in the case of the differential reflection coefficient, there are changes in the shape of the correlation function when the standard deviation of the widths distribution is varied at constant frequency. In general, the slowly varying background (accounted for by the first ladder diagram) is seen to increase as the standard deviation $\sigma$ increases, while the amplitude of the memory and time-reversed memory peaks (accounted for by the second ladder and the first maximally crossed diagrams respectively) decreases. The widths of both peaks are approximately constant. Unlike the enhanced backscattering peak of the differential reflection coefficient, the peaks of the speckle correlator fit Gaussian patterns. Plots of the amplitudes and widths of the memory and time-reversed memory peaks as functions of $\sigma$ at different frequencies are presented in Fig. 6.4. The results show that the heights of these peaks can be used as indicators of the standard deviation of the ridge width distribution.

Finally, an interface formed of identical ridges made at random from two materials is considered. In this case, the average irreducible vertex function is given by (5.34). For the results presented below, the two materials are NaF and GaAs. The fractional concentration of NaF is denoted by $P$. The concentration of GaAs is then $1 - P$. The behavior of the $C^{(1)}$ envelope plotted versus $\theta_s'$ is studied as a function of the fractional concentration $P$. Plots are made for two frequencies, one below and another one above the resonant frequency of NaF.
Figure 6.3: Plots of the correlation function along the $C^{(1)}$ envelope for an interface with random width NaF ridges, for different values of the frequency and the standard deviation of the width distribution
Figure 6.4: Plots of the height and width of the memory and time-reversed memory peaks as functions of the standard deviation of the width distribution at different frequencies.
Figure 6.5: Plots of the correlation function along the $C^{(1)}$ envelope for an interface with NaF-GaAs ridges at different values of the fractional concentration of NaF.
Figure 6.6: Plots of the relative amplitude of the coherent peaks versus the fractional concentration of NaF, for NaF-ZnS and NaF-GaAs ridge systems.
\[ P = 0 \quad P = 0.5 \quad P = 1 \]

<table>
<thead>
<tr>
<th></th>
<th>( f_L )</th>
<th>( f_U )</th>
<th>( f_L )</th>
<th>( f_U )</th>
<th>( f_L )</th>
<th>( f_U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NaF-ZnS</td>
<td>0.987</td>
<td>1.029</td>
<td>0.985</td>
<td>1.032</td>
<td>0.984</td>
<td>1.036</td>
</tr>
<tr>
<td>NaF-GaAs</td>
<td>0.988</td>
<td>1.014</td>
<td>0.986</td>
<td>1.024</td>
<td>0.984</td>
<td>1.036</td>
</tr>
</tbody>
</table>

Table 6.2: Upper and lower boundaries of the frequency ranges where the amplitude of the phase-coherent peaks is greater than 5\% of the background.

The results for NaF-GaAs ridge systems (rescaled to unit area) are presented in Fig. 6.5. The \( C^{(1)} \) correlation functions for different proportions of the mixture are compared, each plot corresponding to a given frequency. The shape of the correlation function is seen to transform continuously as the concentration of NaF varies from 0 to 100\%. Plots of the ratio of the phase-coherent peak amplitude to the background contribution versus the NaF concentration are presented in Fig. 6.6, for NaF-ZnS and NaF-GaAs ridge systems. The relative peak amplitude is seen to depend almost linearly on the fractional concentration. The upper and lower boundaries of the frequency ranges where the amplitude of the phase-coherent peaks is greater than 5\% of the background are given in Table 6.2. The results show that, for interfaces with identical ridges made at random from two dielectric materials, the heights of the phase-coherent peaks can be used (at least in principle) to determine the fractional concentration that characterizes the system. In practice, the determination of the fractional concentration from the amplitudes of the phase-coherent peaks could be rendered impossible by the relatively large experimental uncertainties associated with speckle correlations measurements.

6.3 Study of the \( C^{(1.5)} \) Surface Polariton Effects

We now turn to the study of the next higher order \( \tilde{C}^6 \) contribution to the perturbation theory expansion of the speckle correlation function in terms of the surface
disorder. This contribution is denoted by $C^{(1.5)}$ and encompasses multiple scattering processes represented by the simplest completely connected diagrams contributing to the four-particle Green's functions that define the speckle correlator. A list of the most important diagrams contributing to $C^{(1.5)}$ is given in Fig. 2 of Ref. 10. One such diagram is reproduced in Fig. 6.7. The contribution from this diagram can be written as

$$D^{(1.5)}_a(q,k|q',k') = |G(q)G(k)G(q')G(k')|^2$$

$$\times \int \frac{dr}{2\pi} \int \frac{ds}{2\pi} \langle \Gamma(r,q,s)|G^*(r)G(s)\rangle \langle \Gamma(r,k|q',k')|\Gamma(q',k'|s,k)\rangle.$$  \hspace{1cm} (6.19)

The subscript $a$ on the left hand side of (6.19) is a reminder of the fact that this is only one of the contributions to the total $C^{(1.5)}$. Using (5.25) to express $<\Gamma>$, we find

$$C^{(1.5)}_a(q,k|q',k') = L F(q,k|q',k')|G(q)G(k)G(q')G(k')|^2|G(k-k'+q')|^2$$

$$\times |\Gamma_0(k-k'+q',k|q',k')|^2 \Gamma_0(q,k-k'+q'|q,k-k'+q'),$$  \hspace{1cm} (6.20)

where $L$ is the length of the surface in the $x_1$ direction.

A general characteristic of the diagrams contributing to $C^{(1.5)}$ is that, for average irreducible vertex functions of the form (5.25), the momenta of the internal Green's function lines must be equal. This gives rise to factors of the form $|G(s)|^2$, where $s = k-k'+q'$, $q-q'+k'$, etc. In addition, the incident and scattering angles are no longer constrained by delta functions in momentum space, but can take any value between $-\pi/2$ and $\pi/2$.

When plotted as a function of the incident and scattered wavevectors, the $C^{(1.5)}$ contribution exhibits a series of peaks arising from the poles in $|G(s)|^2$ at $s = \pm K_{sp}$, where $K_{sp}$ is the surface polariton momentum. The width of these peaks
Table 6.3: Normalization constants for $C^{(1,5)}$ at different frequencies below resonance

<table>
<thead>
<tr>
<th>$f$</th>
<th>ZnS</th>
<th>$f$</th>
<th>GaAs</th>
<th>$f$</th>
<th>NaF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.960</td>
<td>0.0128</td>
<td>0.968</td>
<td>5.500e-4</td>
<td>0.968</td>
<td>2.390e-4</td>
</tr>
<tr>
<td>0.964</td>
<td>0.0290</td>
<td>0.972</td>
<td>1.344e-3</td>
<td>0.972</td>
<td>6.698e-4</td>
</tr>
<tr>
<td>0.968</td>
<td>0.0688</td>
<td>0.976</td>
<td>3.659e-3</td>
<td>0.976</td>
<td>2.148e-3</td>
</tr>
<tr>
<td>0.972</td>
<td>0.1726</td>
<td>0.980</td>
<td>0.01146</td>
<td>0.980</td>
<td>8.213e-3</td>
</tr>
<tr>
<td>0.976</td>
<td>0.4596</td>
<td>0.984</td>
<td>0.04349</td>
<td>0.984</td>
<td>0.04019</td>
</tr>
<tr>
<td>0.980</td>
<td>1.2988</td>
<td>0.988</td>
<td>0.2161</td>
<td>0.988</td>
<td>0.2874</td>
</tr>
<tr>
<td>0.984</td>
<td>3.8607</td>
<td>0.992</td>
<td>1.5697</td>
<td>0.992</td>
<td>4.0394</td>
</tr>
</tbody>
</table>

Figure 6.7: A typical diagram contributing to the $C^{(1,5)}$ term of the speckle correlator

is proportional to the imaginary part of the self-energy correction to the pole of the single-particle Green's function.

Following the parameterization used in Ref. 10, we have studied the $C^{(1,5)}$ correlation function for fixed $\theta_i = -20^\circ$, $\theta_s = -10^\circ$, and $\theta'_s = -30^\circ$ with $\theta'_s$ running from $-90^\circ$ to $90^\circ$. The geometrical parameters $\Lambda$, $\langle R \rangle$, $\Delta x$, and $c_0$ as well as the ridge and substrate materials are the same as those used in our treatment of $C^{(1)}$ and $C^{(10)}$ earlier in this chapter. When the ridge width is random, its distribution

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is given by (5.55). All diagrams given in Fig. 2 of Ref. 10 are included in our calculation. The average single-particle Green's function is computed using (5.26) with the self-energy correction $\Sigma$ given, in the coherent potential approximation, by a single iteration of (5.54). The irreducible vertex functions are computed to the lowest order in $\tilde{\xi}$ using (5.13).

In Fig. 6.8 results are presented illustrating the behavior of $C^{(1,5)}$ as a function of the frequency of light for systems of identical ridges made from ZnS, GaAs, and NaF placed on a CdS substrate. For each ridge material, a set of frequencies is chosen in the neighborhood of the resonant frequency. The amplitudes of all peaks exhibited by $C^{(1,5)}$ are seen to increase as the frequency approaches the resonant frequency from above and below. Due to the significant changes in amplitude, the plots presented here are normalized to have unit area below each curve. The normalization constants for the curves corresponding to frequencies below resonance are given in Table 6.3. The peaks also become wider with increasing frequency (both below and above the resonance) and the positions of their centers shift. The widening effect is due to the increase of $\Re \epsilon_0$ as a function of frequency for frequencies above the CdS resonance. This leads to the increase of the imaginary part of the poles in the single-particle Green's function $G(s)$. The widening is more pronounced for low absolute values of $\Re \epsilon_0$. This explains the large width of the peaks in the case of ZnS, whose resonant frequency is close to the root of $\Re \epsilon_0(\omega)$. The frequency shifts of the peak positions are explained by the dependence of $K_{sp}$ on $\omega$.

The behavior of $C^{(1,5)}$ as a function of the standard deviation of the ridge width distribution for $\langle R \rangle = 1$ is displayed in Fig. 6.9. Two frequencies are chosen for each of ZnS, GaAs, and NaF ridge materials, one below and another one above resonance. The standard deviation parameter $a$ is given values between 0 and 1.
Figure 6.8: Plots of the $C^{(1,5)}$ contribution to the correlation function vs scattering angle $\theta'$ for ZnS, GaAs, and NaF ridges at frequencies approaching the resonant frequency of the ridge material from below and above.
Figure 6.9: Plots of the $O^{(1.5)}$ contribution to the correlation function vs scattering angle $\theta_s'$ for ZnS, GaAs, and NaF ridges at different values of the standard deviation of the width distribution.
The plots in Fig. 6.9 show an increase in the amplitude of the leftmost peak as the standard deviation increases, while the amplitudes of the other peaks decrease.

6.4 Periodic on Average Randomly Rough Surfaces

In this section we present calculations and results for periodic on average randomly rough surfaces. As in the case of the volume disorder, the \( C^{(1)} \) contribution to the correlation function of a periodic on average surface exhibits features that are not present when the surface is homogeneously random. These new features include contributions to the slowly varying background, as well as a series of peaks arising from phase-coherent effects.

The calculations are based on the rough surface model described at the beginning of Chapter 5 and used in the previous sections of this chapter. We have limited our study to periodic on average arrays of identical ridges. To account for the average periodicity however, the lattice parameter \( \Delta x \) was given values comparable to the wavelength of light at resonance \( (2\pi \) in units of \( c/\omega^{rd} \)), much larger than the value of \( \Delta x = 0.1 \) that was used in the calculations for homogeneously random arrays. Moreover, larger values of the average occupancy ratio \( c_0 \) are needed to keep the average distance between two ridges \( \Delta x/c_0 \) at reasonably low values.

From a mathematical point of view, the new features in the speckle correlator arise from contributions to \( < \zeta > \) corresponding to the harmonics of the average profile function. These contributions are either absent or negligible in the case of a homogeneously random array. If the argument \( p \) is no longer restricted to \( (-\pi, \pi) \), Eq. (5.22) must be replaced by

\[
\sum_k e^{ikp} = \sum_n 2\pi \delta(p - 2n\pi). \tag{6.21}
\]
Hence the average of the Fourier transform of the profile function becomes

\[ \langle \hat{\zeta}(p) \rangle = \sqrt{\pi} c_0 \frac{AR}{\Delta x} e^{-p^2 \frac{R^2}{4}} \sum_k 2\pi \delta \left(p - \frac{2k\pi}{\Delta x}\right), \quad (6.22) \]

while the average of the product of two Fourier coefficients is

\[ \langle \hat{\zeta}(p)\hat{\zeta}(q) \rangle = \pi A^2 R^2 e^{-(p^2+q^2) \frac{R^2}{4}} \left[ (c_0 - c_0^2) \frac{2\pi}{\Delta x} \sum_k \delta \left(p + q - \frac{2k\pi}{\Delta x}\right) \right. \\
\left. + \left(c_0 \frac{2\pi}{\Delta x}\right)^2 \sum_{k,l} \delta \left(p - \frac{2k\pi}{\Delta x}\right) \delta \left(q - \frac{2l\pi}{\Delta x}\right) \right]. \quad (6.23) \]

Combining (6.22) and (6.23) with Eq. (5.14) written for the case of identical ridges we find

\[ \langle \Gamma(q,r|p,s) \rangle = \sum_n \Gamma_n(q,r|p,s) 2\pi \delta \left(q - r - p + s - \frac{2n\pi}{\Delta x}\right), \quad (6.24) \]

where

\[ \Gamma_n(q,r|p,s) = \nu(q,r)\nu^*(p,s)\pi(c_0 - c_0^2) \frac{A^2 R^2}{\Delta x} e^{-(q-r-\frac{n\pi}{\Delta x})^2} e^{-\frac{n^2}{2} \left(\frac{\pi R}{\Delta x}\right)^2}. \quad (6.25) \]

From (6.24), using the same procedure that was used in the derivation of (6.14), we find that the \(C^{(1)}\) contribution to the correlation function can be written as

\[ C^{(1)}(q,k|q',k') = H(q,k|q',k') \sum_n 2\pi L \delta \left(q - k - q' + k' - \frac{2n\pi}{\Delta x}\right) |\mathcal{L}_n(q,k|q',k')|^2, \quad (6.26) \]

where

\[ \mathcal{L}_n(q,k|q',k') = \Gamma_n(q,k|q',k') + \sum_m \left[ \int \frac{ds}{2\pi} \Gamma_m(q,q - q' + s - \frac{2m\pi}{\Delta x}) \times G^*(q - q' + s - \frac{2m\pi}{\Delta x}) G(s) \Gamma_{n-m}(q - q' + s - \frac{2m\pi}{\Delta x},k|s,k') \\
+ \int \frac{ds}{2\pi} \Gamma_m(q,s|q + k' - s - \frac{2m\pi}{\Delta x},k') G^*(q + k' - s - \frac{2m\pi}{\Delta x}) G(s) \times \Gamma_{n-m}(s,k|q',q + k' - s - \frac{2m\pi}{\Delta x}) \right]. \quad (6.27) \]
Figure 6.10: Plots of the $C^{(1)}$ contribution to the correlation function for a periodic on average array of GaAs ridges on a CdS substrate. Each plot corresponds to a different value of the lattice parameter $\Delta x$

The sum in (6.26) runs in principle over all integer values of $n$ but in light of (6.17) the constraints expressed by means of the $\delta$ functions cannot be satisfied but for a finite set of values of $n$. For the $n$-th term in the expansion (6.26) to be meaningful in the case of a given ridge system, $\theta'_i$ must exist for at least some open subinterval in the range of $\theta_s$. By imposing this condition, we find that $n$ can take only the integer values in the interval $\alpha(\sin \theta_s - \sin \theta_i - 2) < n < \alpha(\sin \theta_s - \sin \theta_i + 2)$, where $\alpha = \frac{\omega}{\omega_c^2a} \Delta x$. The center positions of the various peaks can be found from an analysis of the pole approximations[10,11] of the integrals in (6.27). Thus, the integrals arising from the second ladder diagram exhibit peaks at $q' = q - \frac{2mn\pi}{\Delta x}$.
Figure 6.11: Plots of the $n = -4$ to $n = 2$ terms and of the total $C^{(1)}$ for $\Delta x = 12.5$ at different frequencies. All curves have been normalized to unit area.
while those arising from the first maximally crossed diagram exhibit peaks at $q' = -k + \frac{2(m-n)\pi}{\Delta x}$. Since $\Delta x$ is expressed in units of $c/\omega_{T}^{id}$, we see from (6.17) that the positions of most peaks depend on frequency. Moreover, the only second ladder diagram terms contributing peaks in (6.27) are those for which $m$ lies in the interval $\alpha(\sin \theta_s - 1) < m < \alpha(\sin \theta_s + 1)$. Likewise, for the maximally crossed diagram terms the constraints are such that $\alpha(\sin \theta_i - 1) < m - n < \alpha(\sin \theta_i + 1)$.

In the following we present results for $C^{(1)}$ as a function of $\theta_s'$ for a GaAs/CdS interface. These results are for $\theta_i = 20^\circ$ and $\theta_s = -10^\circ$. The values of the geometrical parameters are $A = 0.1$ and $R = 1$ in units of $c/\omega_{T}^{id}$ while the average occupancy ratio is $c_0 = 0.5$. The plots in Fig. 6.10 show the changes in the shape of the correlation function when the value of the lattice parameter $\Delta x$ is varied between 5 and 12.5. All plots are for the same frequency $\omega/\omega_{T}^{id} = 0.976$. Each term on the right-hand side of Eq. (6.26) has its own interval of non-zero contribution, which explains the cusps exhibited by the curves in Fig. 6.10. For $\Delta x$ sufficiently high, these intervals will cover the whole range of $\theta_s'$ between $-\pi/2$ and $\pi/2$. In Fig. 6.11 we present results detailing the contribution of each term in (6.26) at various frequencies. The lattice parameter here is $\Delta x = 12.5$ in units of $c/\omega_{T}^{id}$. With the choice of parameters presented above, we find that the non-zero contributions to $C^{(1)}$ come from the $n = -4$ through $n = 2$ terms in (6.26), while the significant values for $m$ are -2 through 1 for the second ladder diagram terms and $n - 1$ through $n + 2$ for the maximally crossed diagram terms. The first seven plots in Fig. 6.11 are for the $n = -4$ through $n = 2$ terms, while the last plot shows the sum of all contributions. The center positions of the peaks (computed for $\omega = \omega_{T}^{id}$) are at $\theta_s' = -57.64^\circ$, $-42.56^\circ$, $-20^\circ$, $-10^\circ$, $9.24^\circ$, $19.21^\circ$, $41.55^\circ$, and $56.27^\circ$. The relative amplitude increase of these peaks compared to the first ladder diagram background suggests that they
represent localization phenomena due to the phase coherent multiple scattering of the surface polaritons.
CHAPTER 7
CONCLUSIONS

In the first part of the dissertation, the speckle correlation functions for a series of non-magnetic volume-disordered media are computed by means of numerical simulations. The media are characterized by piecewise constant distributions of the electrical permittivity, with spherical regions of different permittivities immersed into a background medium. The dielectric permittivity of the background medium is essentially that of the free space. Both periodic on average and homogeneously random space distributions of the spheres are considered. The electrical permittivities or the radii of the spheres are randomly distributed. The results show that it is possible to determine the standard deviations of either the radii or the electrical permittivities distributions from speckle correlations measurements. This is true for homogeneously random as well as for periodic on average media.

The differential reflection coefficients and speckle correlation functions in the case of light scattered by randomly rough dielectric surfaces are studied in the second part of the dissertation. The random surfaces are modeled by arrays of Gaussian shaped, parallel dielectric ridges placed upon a planar vacuum-dielectric interface. The results show that if the heights or the widths of the ridges are randomly distributed, it is possible to determine the averages and the standard deviations of their distributions from differential reflection coefficient or speckle correlations measurements in the diffusely scattered light. In addition, if the ridges are identically shaped but made at random from two different materials, differential reflection coefficient measurements can be used to determine the fractional concentration of the
two materials. Finally, the speckle correlation function in the case of random sur-
faces consisting of periodic on average arrays of identical ridges are computed. The
speckle correlator is seen to exhibit a whole new series of features, including slowly
varying background contributions as well as phase coherent peaks.
APPENDIX A
PROGRAM LISTING

implicit none
integer i, k, l, n, na, ns, ntot, indx(225)
real pi, phi, theta, s1, s2, em, se, rm, sr
real x(3,225), e(225), r(225), c(225)
real corr(7,90), sitot(7,90), s2tot(7,90)
complex atot, a(225,225), ai(225,225)
common pi
pi=4.*atan(1.)
n=100
na=10
ns=7
ntot=40
em=-9.
rm=0.40
sr=0.
open (file='angles', unit=18, access='sequential',
& form='formatted', status='old')
read (18,*) phi
if (phi.lt.0.) then
   close(unit=18)
   stop
endif
phi=phi*pi/180.
se=0.
do k=1,ns
   write(*,*) 'se=', se
   do i=1,90
      sitot(k,i)=0.
      s2tot(k,i)=0.
      corr(k,i)=0.
   enddo
   do l=1,ntot
      write(*,*) l
      call coor(l,n,na,na*na*na,x)
      call epsnrad(l,n,em,se,e,rm,sr,r,c)
      do i=1,90

theta=float(i)*pi/180.
call psisc(n,indx,theta,phi,x,e,r,c,a,ai,atot)
s1=cabs(atot)**2
sitot(k,i)=sitot(k,i)+s1
call psisd(n,indx,theta,phi,x,e,r,c,a,ai,atot)
s2=cabs(atot)**2
s2tot(k,i)=s2tot(k,i)+s2
corr(k,i)=corr(k,i)+s1*s2
enddo
dndo
s1=float(n*ntot)
s2=s1*float(n)
do i=1,90
    s1tot(k,i)=s1tot(k,i)/s1
    s2tot(k,i)=s2tot(k,i)/s1
    corr(k,i)=corr(k,i)/s2
    corr(k,i)=corr(k,i)-s1tot(k,i)*s2tot(k,i)
enddo
se=se+0.5
enddo
2 format(f7.2, 7(2x,el2.5))
do i=1,90
    write(19,2) float(i), (sitot(k,i), k=1,ns)
    write(20,2) float(i), (s2tot(k,i), k=1,ns)
    write(21,2) float(i), (corr(k,i), k=1,ns)
enddo
goto 5
stop
end

subroutine coor(l,n,na,na3,x)
implicit none
integer i, j, k, l, n, na, na3, nat, nt, ic, idum, itype(na3)
real pi, xx, xq, ran3, x(3,n), y(3,na3)
common pi
idum=-144*l
nt=0
xx=1.
xx=2.*pi/xx
do i=1,na
do j=1,na
do k=1,na
nt=nt+1
y(1,nt)=xx*float(i)
y(2,nt)=xx*float(j)
y(3,nt)=xx*float(k)
itype(nt)=0
enddo
enddo
ic=0
do 20 i=1,100000
  xq=ran3(idum)
  nat=int(float(na3)*xq)+1
  if(itype(nat).ne.0) goto 20
  ic=ic+1
  itype(nat)=1
  if(ic.eq.n) goto 25
20 continue
25 k=0
do 21 i=1,na3
  if(itype(i).eq.0) goto 21
  k=k+1
  x(1,k)=y(1,i)
  x(2,k)=y(2,i)
  x(3,k)=y(3,i)
21 continue
return
end

subroutine epsnrad(1,n,em,se,e,rm,sr,r,c)
implicit none
integer l, i, n, idum
real e(n), em, se, r(n), rm, sr, c(n)
real pi, ro, ran3
common pi
idum=-124*l
do i=1,n
  ro=sqrt(-2.*log(ran3(idum)))
e(i)=em+se*ro*cos(2.*pi*ran3(idum))
r(i)=rm+sr*ro*sin(2.*pi*ran3(idum))
c(i)=sin(r(i))-r(i)*cos(r(i))
enddo
return

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subroutine psisc(n, indx, theta, phi, x, e, r, c, a, ai, at)
implicit none
integer i, j, n, indx(n)
real pi, theta, phi
real x(3,n), xi(3), xs(3), e(n), r(n), c(n)
complex q, at, a(n,n), ai(n,n), fazl(n), faz2(n)
common pi
q=cmplx(0.,1.)
xi(1)=sin(theta)
xi(2)=0.
xi(3)=cos(theta)
xs(1)=sin(theta)*cos(phi)
xs(2)=sin(theta)*sin(phi)
xs(3)=cos(theta)
do i=1,n
   fazl(i)=cmplx(0.,0.)
faz2(i)=cmplx(0.,0.)
do j=1,3
   fazl(i)=fazl(i)+xi(j)*x(j,i)
faz2(i)=faz2(i)-xs(j)*x(j,i)
endo
dolo fazl(i)=cexp(q*fazl(i))
faz2(i)=cexp(q*faz2(i))
do endo
call mat(n,x,e,r,c,a)
call inv(n,indx,a,ai)
at=0.
do i=1,n
   do j=1,n
at=at+(e(i)-1.)*c(i)*faz2(i)*ai(i,j)*fazl(j)
endo
do
endo return
end

subroutine psisd(n, indx, theta, phi, x, e, r, c, a, ai, at)
implicit none
integer i, j, n, indx(n)
real pi, theta, phi
real x(3,n), xi(3), xs(3), e(n), r(n), c(n)
complex q, at, a(n,n), ai(n,n), faz1(n), faz2(n)
c
common pi
q=cmplx(0.,1.)

xi(1)=sin(theta)
xi(2)=0.
xi(3)=-cos(theta)
xs(1)=sin(theta)*cos(phi)
xs(2)=sin(theta)*sin(phi)
xs(3)=-cos(theta)
do i=1,n
    faz1(i)=cmplx(0.,0.)
    faz2(i)=cmplx(0.,0.)
do j=1,3
        faz1(i)=faz1(i)+xi(j)*x(j,i)
        faz2(i)=faz2(i)-xs(j)*x(j,i)
    enddo
    faz1(i)=cexp(q*faz1(i))
    faz2(i)=cexp(q*faz2(i))
endo
call mat(n,x,e,r,c,a)
call inv(n,indx,a,ai)
at=0.
do i=1,n
do j=1,n
    at=at+(e(i)-1.)*c(i)*faz2(i)*ai(i,j)*faz1(j)
endo
return
c
end

subroutine mat(n,x,e,r,c,a)
imPLICIT none
integer i, k, l, n
real pi, s, x(3,n), e(n), r(n), c(n)
complex q, a1, a(n,n)
common pi
q=cmplx(0.,1.)
do k=1,n
do l=1,n
    if(k.eq.1) then
        a1=q*r(l)
        a1=cexp(a1)*(1.-ai)-1.
endo
a(k,1) = 1. - a1*(e(l)-1.)
else
  a(k,1) = 0.
  s = 0.
  do i = 1, 3
    s = s + (x(i,k) - x(i,1))**2
  enddo
  s = sqrt(s)
  a1 = c(l)*cexp(q*s)/s
  a(k,1) = -a1*(e(l)-1.)
endif
enddo
endo
dendo
return
end

c subroutine inv(n, indx, a, ai)
  implicit none
  integer i, j, n, indx(n)
  real d
  complex a(n,n), ai(n,n)
  do i = 1, n
    do j = 1, n
      ai(i,j) = 0.
    enddo
    ai(i,i) = 1.
  enddo
  call ludcmp(n, a, indx, d)
  do j = 1, n
    call lubksb(n, a, indx, ai(1,j))
  enddo
  return
end

c subroutine ludcmp(n, a, indx, d)
  implicit none
  integer n, nmax, i, imax, j, k, indx(n)
  real aamax, tiny, d
  complex sum, dum, a(n,n)
  parameter (nmax=500, tiny=1.e-20)
  real vv(nmax)
  d = 1.

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doi=1,n
aamax=0.
doj=1,n
if(cabs(a(i,j)).gt.aamax) aamax=cabs(a(i,j))
enddo
if(aamax.eq.0.) pause 'singular matrix in ludcmp'
vvid=1./aamax
enddo
doj=1,n
doi=1,j-1
sum=a(i,j)
dok=1,i-1
  sum=sum-a(i,k)*a(k,j)
enddo
a(i,j)=sum
enddo
aamax=0.
doi=j,n
sum=a(i,j)
dok=1,j-1
  sum=sum-a(i,k)*a(k,j)
enddo
a(i,j)=sum
dum=vvid*cabs(sum)
if(cabs(dum).ge.aamax) then
  imax=i
  aamax=dum
endif
enddo
if(j.ne.imax) then
  do k=1,n
    dum=a(imax,k)
    a(imax,k)=a(j,k)
    a(j,k)=dum
  enddo
d=-d
vv(imax)=vv(j)
endif
indx(j)=imax
if(cabs(a(j,j)).eq.0.) a(j,j)=tiny
if(j.ne.n) then
dum=1./a(j,j)
enddo

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do i=j+1,n
    a(i,j)=a(i,j)*dum
endo
endif
endo
return
end

subroutine lubksb(n,a,indx,b)
implicit none
integer i, ii, j, ll, n, indx(n)
complex sum, a(n,n), b(n)
ii=0
do i= l,n
    ll=indx(i)
    sum=b(ll)
    b(ll)=b(i)
    if(ii.ne.0) then
        do j=ii,i-1
            sum=sum-a(i,j)*b(j)
        enddo
    else
        if(cabs(sum).ne.0.) ii=i
    endif
    b(i)=sum
endo
do i=n,1,-1
    sum=b(i)
    do j=i+1,n
        sum=sum-a(i,j)*b(j)
    enddo
    b(i)=sum/a(i,i)
endo
return
end

real function ran3(idum)
integer idum, mbig, mseed, mz, mj, mk, ma(55)
integer i, iff, ii, inext, inextp, k
real fac
parameter (mbig=1000000000,mseed=161803398,mz=0,fac=1./mbig)
save iff, inext, inextp, ma

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data iff /O/.
if(idum.lt.0.or.iff.eq.0) then
  iff=1
  mj=mseed-iabs(idum)
  mj=mod(mj,mbig)
  ma(55)=mj
  mk=1
  do i=1,54
    ii=mod(21*i,55)
    ma(ii)=mk
    mk=mj-mk
    if(mk.lt.mz) mk=mk+mbig
    mj=ma(ii)
  enddo
  do k=1,4
    do i=1,55
      ma(i)=ma(i)-ma(1+mod(i+30,55))
      if(ma(i).lt.mz) ma(i)=ma(i)+mbig
    enddo
  enddo
  inext=0
  inextp=31
  idum=1
endif
inext=inext+1
if(inext.eq.56) inext=1
inextp=inextp+1
if(inextp.eq.56) inextp=1
mj=ma(inext)-ma(inextp)
if(mj.lt.mz) mj=mj+mbig
ma(inext)=mj
ran3=mj*fac
return
end
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