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Relativistic and Non-Relativistic Nuclear Charge Form Factors

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RELATIVISTIC AND NON-RELATIVISTIC NUCLEAR CHARGE FORM FACTORS

by

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This thesis presents an examination that the two nuclear charge form factors, relativistic and non-relativistic, are equivalent approximately in a appreciable range. The relativistic one is contributed by both vector charge density and tensor charge density. And the non-relativistic charge form factor is contributed by the nuclear densities from experimental electron scattering cross section,these densities are fundamental to much of nuclear structure physics.

The Dirac equation is used to calculate the vector charge density and tensor density, because the charge density from the analysis of the experimental data may be compared with proton charge densities obtained from various models of nuclear structure. The standard procedure for calculating charge densities from the Schroedinger equation is no longer correct if nucleons obey the Dirac equation.

This work only explored the case of a single state (ground state) of a single particle.

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CHAPTER I

INTRODUCTION

The impulse approximation for the construction of the nuclear optical potential had been very extensively developed within the context of Schroedinger dynamics. The leading term of the optical potential can be approximated as

$$V_{opt}(E,\vec{q}) = t(E,\vec{q})\rho(\vec{q}). \tag{1.1}$$

Here, t is the nucleon-nucleon scattering matrix, $\vec{q}$ is the momentum transfer, and $\rho(\vec{q})$ is a nuclear form factor. Truncating the multiple scattering series expansion for the optical potential at its leading term, Equation (1.1), is called the impulse approximation. Nonrelativistically,

$$\rho(\vec{q}) \propto \int d\vec{r} e^{i\vec{q}\cdot\vec{r}} \rho(\vec{r}) \tag{1.2}$$

where $\rho(\vec{r})$ is the nuclear density. Electron scattering has been used to determine

$$F_{ch}(q) \propto \int d\vec{r} e^{i\vec{q}\cdot\vec{r}} \rho_{ch}(\vec{r}). \tag{1.3}$$

Dirac phenomenology is based on a relativistic form of the impulse aproximation. Instead of solving the Schroedinger equation $[-\frac{1}{2M}\nabla^2 + V_{opt}(r)]\Psi = E\Psi$, one can solve the Dirac equation $[-i\not{\partial} + M - V_{opt}]\Psi = E\Psi$. It has been found that the optical potential which is needed in the Dirac equation to describe high energy nucleon-nucleon scattering is considerally simpler in form than the one required for the Schroedinger equation. This encourages us to view the nucleon as a "Dirac particle" (i.e. as one whose wave function satisfies an appropriate Dirac equation), and to explore the consequences of this assumption. The most direct test will be the predicted charge form factors.

The elastic scattering of electrons from nuclei has a long history. One obtains a charge form factor from the knowledge of the differential cross section. From this form factor one determines the charge density. (For heavy nuclei, the Born approximation

is inadequate and one must solve the Dirac equation for the electron to determine the charge distribution.) The charge density obtained from the analysis of the experimental data may be compared with proton charge densities obtained from various models of nuclear structure. The standard procedure for calculating charge densities from the Schroedinger equation is no longer correct if nucleons obey the Dirac equation. Instead of the charge density as in equation (1.3), there are two densities which contribute to the charge form factor.

$$F_{ch}(q^2) = \int d\vec{r} j_0(qr) F_1^p(q^2) \rho_V(r) + \frac{q}{2M_N} \int d\vec{r} j_1(qr) F_2^p(q^2) \rho_T(r). \qquad (1.4)$$

Here ρ_V and ρ_T can be determined from the exact form of the Dirac wave function (see Chapter II).

The usual nonrelatistic form factor is given by

$$F_{ch}(q^2) = G_E^p(q^2) \int d\vec{r} j_0(qr) \rho_V(r) \qquad (1.5)$$

where $G_E^p(q^2) = F_1^p + \frac{q^2}{2M_N} F_2^p$. Equation (1.5) has been used to extract nuclear densities from experimental electron scattering cross section and these densities are fundamental to much of nuclear structure physics. If equation (1.4) and (1.5) are both correct (over some as yet undetermined kinematic range), then one must have

$$-\int j_1(qr) \rho_T(r) d\vec{r} \approx \frac{q}{2M_N} \int j_0(qr) \rho_V(r) d\vec{r}. \qquad (1.6)$$

Equation (1.6) was first pointed out by Celenza et al., 1985. These authors did not derive it and, indeed, it does not seem to follow from the Dirac equation.

In the present work, equation (1.6) will be examined both analytically and numerically to determine the range of its validity. We conclude that equation (1.6) holts surprisingly well for single particle states for $q \leq 1.8 fm^{-1}$. For medium mass nuclei like ^{48}Ca, equation (1.6) is approximately valid for $q < 1.0 fm^{-1}$.

CHAPTER II

DIRAC PARTICLE IN AN INFINITE SCALAR SQUARE WELL

In order to study equation (1.6) of Chapter I, we will solve the Dirac equation for a simple potential and form the relevant densities ρ_V and ρ_T. The equation is

$$[\vec{\alpha}\cdot\vec{p}+\beta(M+V_s)]\Psi = E\Psi \tag{2.1}$$

with the potential

$$V_s = \begin{cases} 0 & \text{if } r < a \\ \infty & \text{if } r > a. \end{cases} \tag{2.2}$$

The 4×4 matries in equation (2.1), have the following forms:

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \tag{2,3}$$

$$\beta = \alpha^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.4}$$

So, the equation (2.1) will be

$$\begin{bmatrix} M+V_s & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -M-V_s \end{bmatrix}\Psi = E\Psi. \tag{2.5}$$

In analogy with the corresponding nonrelativistic result, take the wave function

$$\Psi^j_{lm} = \begin{pmatrix} \frac{iG}{r}[Y^l\chi^{\frac{1}{2}}]^j_m \\ \frac{F}{r}\vec{\sigma}\cdot\vec{r}[Y^l\chi^{\frac{1}{2}}]^j_m \end{pmatrix}. \tag{2.6}$$

Substitute this wave function into (2.5):

$$(M+V_s)\frac{iG}{r}[Y^l\chi^{\frac{1}{2}}]^j_m + \vec{\sigma}\cdot\vec{p}\frac{F}{r}\vec{\sigma}\cdot\hat{r}[Y^l\chi^{\frac{1}{2}}]^j_m = E\frac{iG}{r}[Y^l\chi^{\frac{1}{2}}]^j_m \tag{2.7}$$

$$\vec{\sigma}\cdot\vec{p}\frac{iG}{r}[Y^l\chi^{\frac{1}{2}}]^j_m + (-M-V_s)\frac{F}{r}\vec{\sigma}\cdot\hat{r}[Y^l\chi^{\frac{1}{2}}]^j_m = E\frac{F}{r}\vec{\sigma}\cdot\hat{r}[Y^l\chi^{\frac{1}{2}}]^j_m. \tag{2.8}$$

We know

$$\vec{\sigma}\cdot\hat{r}[Y^l\chi^{\frac{1}{2}}]^j_m = -[Y^{l'}\chi^{\frac{1}{2}}]^j_m \tag{2.9}$$

where l' is the "other" orbital angular momentum which can couple to $\frac{1}{2}$ to give j. For example if $l = 2$, $j = \frac{2}{5}$, then $l' = 3$. So we can now find the following form (Bjorken and Drell, 1964):

$$\vec{\sigma}\cdot\vec{p}\frac{F}{r}\vec{\sigma}\cdot\hat{r}[Y^l\chi^{\frac{1}{2}}]^j_m = -\vec{\sigma}\cdot\vec{p}\frac{F}{r}[Y^{l'}\chi^{\frac{1}{2}}]^j_m = [-i\frac{F'}{r} - i\kappa_{l'j}\frac{F}{r^2}][Y^l\chi^{\frac{1}{2}}]^j_m \tag{2.10}$$

where

$$\kappa = \begin{cases} -(l+1) = -(j+\frac{1}{2}) & \text{if } j = l+\frac{1}{2} \\ +l = +(j+\frac{1}{2}) & \text{if } j = l-\frac{1}{2}. \end{cases} \tag{2.11}$$

Substitute (2.10) and $\kappa_{l'j} = -\kappa_{lj}$ into (2.7) and (2.8). We can have the two following basic equations:

$$(E - M - V_s)G = -F' + \frac{\kappa_{lj}}{r}F \tag{2.12}$$

$$(E + M + V_s)F = G' + \frac{\kappa_{lj}}{r}G. \tag{2.13}$$

Now we need to solve the wave functions G and F. From (2.10)

$$G = \frac{1}{E - M - V_s}[-F' + \frac{\kappa}{r}F] \tag{2.14}$$

$$G' = \frac{1}{E - M - V_s}[-F'' + \frac{\kappa}{r}F' - \frac{\kappa}{r^2}F] \tag{2.15}$$

where

$$\kappa = \kappa_{lj}. \tag{2.16}$$

Substitute (2.14) and (2.15) into (2.13)

$$F'' + [E^2 - (M + V_s)^2 - \frac{\kappa(\kappa-1)}{r^2}]F = 0. \tag{2.17}$$

From (2.13)

$$F = \frac{1}{E + M + V_s}[G' + \frac{\kappa}{r}G] \tag{2.18}$$

$$F' = \frac{1}{E + M + V_s}[G'' + \frac{\kappa}{r}G' - \frac{\kappa}{r^2}G]. \tag{2.19}$$

Substitute (2.18) and (2.19) into (2.12)

$$G'' + [E^2 - (M + V_s)^2 - \frac{\kappa(\kappa+1)}{r^2}]G = 0 \tag{2.20}$$

then obtain the equitions (2.17) and (2.20). They are the differential equations of the wave functions. When $j = l + \frac{1}{2}$

$$\kappa = -(l+1). \tag{2.21}$$

Substitute (2.21) into (2.17) and (2.20)

$$G'' + [E^2 - (M + V_s)^2 - \frac{l(l+1)}{r^2}]G = 0 \tag{2.22}$$

$$F'' + [E^2 - (M + V_s)^2 - \frac{(l+1)(l+2)}{r^2}]F = 0 \tag{2.23}$$

When $r < a \quad V_s = 0$, let

$$k^2 = E^2 - M^2 \tag{2.24}$$

and

$$l' = l + 1. \tag{2.25}$$

So the equations (2.22) and (2.23) will have the following forms:

$$G'' + [k^2 - \frac{l(l+1)}{r^2}]G = 0 \tag{2.26}$$

$$F'' + [k^2 - \frac{l'(l'+1)}{r^2}]F = 0. \tag{2.27}$$

Let

$$\begin{cases} G = rR_1 \\ F = rR_2, \end{cases} \tag{2.28}$$

$$\begin{cases} G' = rR_1' + R_1 \\ F' = rR_2' + R_2, \end{cases} \tag{2.29}$$

$$and \begin{cases} G'' = rR_1'' + 2R_1' \\ F'' = rR_2'' + 2R_2'. \end{cases} \tag{2.30}$$

Substitute (2.28), (2.29) and (2.30) into (2.26) and (2.27)

$$R_1'' + \frac{2}{r}R_1' + [k^2 - \frac{l(l+1)}{r^2}]R_1 = 0 \tag{2.31}$$

$$R_2'' + \frac{2}{r}R_2' + [k^2 - \frac{l'(l'+1)}{r^2}R_2 = 0. \tag{2.32}$$

Let

$$\rho = kr. \tag{2.33}$$

So the equations (2.31) and (2.32) will be an instance of Bessel's equation.

$$\frac{d^2R_1}{d\rho^2} + \frac{2}{\rho}\frac{dR_1}{d\rho} + [1 - \frac{l(l+1)}{\rho^2}]R_1 = 0 \tag{2.34}$$

$$\frac{d^2R_2}{d\rho^2} + \frac{2}{\rho}\frac{dR_2}{d\rho} + [1 - \frac{l'(l'+1)}{\rho^2}]R_2 = 0. \tag{2.35}$$

The solutions will be:

$$R_1 \propto j_l(\rho) \, or \, n_l(\rho) \tag{2.36}$$

$$R_2 \propto j_{l'}(\rho) \, or \, n_{l'}(\rho) \tag{2.37}$$

where

$$j_l(\rho) = (\frac{\pi}{2\rho})^{\frac{1}{2}} J_{l+\frac{1}{2}}(\rho) \tag{2.38}$$

$$n_l(\rho) = (-1)^{l+1}(\frac{\pi}{2\rho})^{\frac{1}{2}} J_{-l-\frac{1}{2}}(\rho). \tag{2.39}$$

The j and n are called spherical Bessel functions. The boundary condition requires Ψ to be finite at $r = 0$. So the n_l solutions must be rejected. We have

$$R_1 = Aj_l(\rho) \tag{2.40}$$

$$and \; R_2 = Bj_{l'}(\rho) \tag{2.41}$$

where A and B are constants. So, from (2.28) and (2.33) we have

$$G = rAj_l(\rho) = rAj_l(kr) \tag{2.42}$$

$$and \; F = rBj_{l'}(\rho) = rBj_{l'}(kr). \tag{2.43}$$

For the ground state with $l = 0$, the wave functions will be

$$G = \frac{A}{k} sin(kr) \tag{2.44}$$

$$F = Br[\frac{sin(kr)}{(kr)^2} - \frac{cos(kr)}{kr}] \tag{2.45}$$

when $r > a \quad V = \infty$. The appropriate boundary condition for this potential is $R_1(a) = 0$ (This makes the particle current $\vec{j} \equiv \overline{\Psi}\vec{\gamma}\Psi$ vanish at r=a). So from the boundary condition where $r = a$

$$R_1 = Aj_0(ka) = 0 \tag{2.46}$$

$$j_0(ka) = \frac{1}{ka} sin(ka) = 0. \tag{2.47}$$

So

$$ka = n\pi. \tag{2.48}$$

From (2.24):

$$E^2 = \frac{n^2\pi^2}{a^2} + M^2. \tag{2.49}$$

Now the A and B need to be solved as following:

From (2.13) and $V_s = 0$, we have

$$(E + M)F = G' + \frac{\kappa}{r} G. \tag{2.50}$$

Substitute (2.44) and (2.45) into (2.50), then let $\kappa = -1$ when $(l = 0)$

$$A = -\frac{B}{k}(E + M). \tag{2.51}$$

From normalization

$$\int \Psi^+ \Psi d\vec{r} = 1. \tag{2.52}$$

We have

$$1 = \int_0^a (G^2 + F^2) dr = \int_0^a [Arj_0(kr)]^2 dr + \int_0^a [Brj_1(kr)]^2 dr \tag{2.53}$$

$$= \frac{A^2}{k^2}[\frac{a}{2} - \frac{1}{4k} sin(2ka)] + \frac{B^2 a}{2k^2}[1 - \frac{2sin^2(ka)}{(ka)^2} + \frac{sin(2ka)}{(2ka)}]. \tag{2.54}$$

Substitute (2.48) into (2.54)

$$B = \frac{1}{\sqrt{\frac{a}{2k^4}(E + M)^2 + \frac{a}{2k^2}}}. \tag{2.55}$$

Substitute (2.55) into (2.51)

$$A = -\frac{(E + M)}{k^2} \sqrt{\frac{a}{2k^2}(E + M)^2 + \frac{a}{2}} \equiv -\frac{(E + M)}{k^2} N. \tag{2.56}$$

So, now we have the wave function Ψ as following

$$\Psi = N \begin{pmatrix} -i\frac{(E+M)}{k^2} j_l(\rho)[Y^l \chi^{\frac{1}{2}}]^j_m \\ k j_{l+1}(\rho) \vec{\sigma} \cdot \hat{r} [Y^l \chi^{\frac{1}{2}}]^j_m \end{pmatrix}. \tag{2.57}$$

For ground state

$$\Psi_{l=0} = N \begin{pmatrix} -i\frac{(E+M)}{k^2} j_0(\rho)[Y^0\chi^{\frac{1}{2}}]^{\frac{1}{2}}_m \\ kj_1(\rho)\vec{\sigma}\cdot\hat{r}[Y^0\chi^{\frac{1}{2}}]^{\frac{1}{2}}_m \end{pmatrix}. \tag{2.58}$$

The scalar density

$$\rho_S \equiv \frac{G^2}{r^2} - \frac{F^2}{r^2}. \tag{2.59}$$

The vector density

$$\rho_V \equiv \frac{G^2}{r^2} + \frac{F^2}{r^2}. \tag{2.60}$$

The tensor density

$$\rho_T \equiv \frac{2G \times F}{r^2}. \tag{2.61}$$

For ground state ($\kappa = -1$)

$$ScalarDensity = A^2\frac{sin^2(kr)}{(kr)^2} - B^2[\frac{sin(kr)}{(kr)^2} - \frac{cos(kr)}{(kr)}]^2 \tag{2.62}$$

$$VectorDensity = A^2\frac{sin^2(kr)}{(kr)^2} + B^2[\frac{sin(kr)}{(kr)^2} - \frac{cos(kr)}{(kr)}]^2 \tag{2.63}$$

$$TensorDensity = \frac{2ABsin(kr)[\frac{sin(kr)}{(kr)^2} - \frac{cos(kr)}{(kr)}]}{kr}. \tag{2.64}$$

We will discuss the form factors entering eq(1.6) of Chapter I which arise from these densities in the next chapter. The following figures show the wave function of equation (2.58) and the relations in ρ_S,ρ_V and ρ_T for different a.

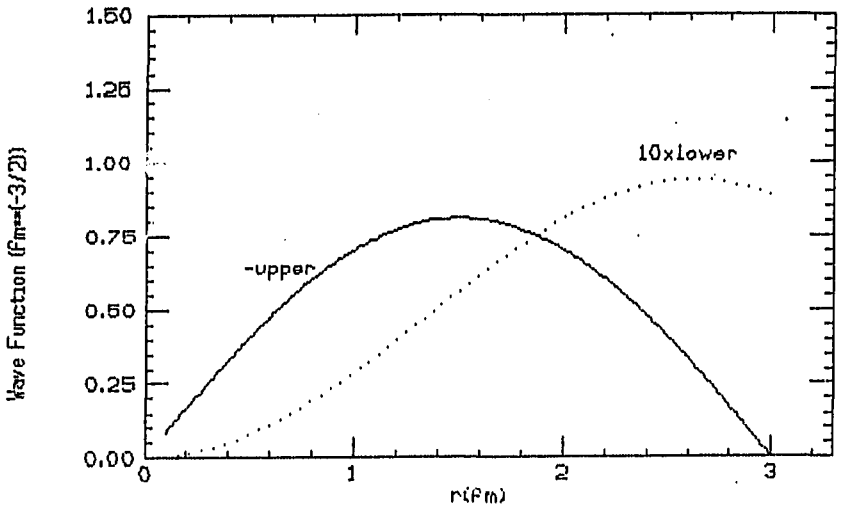

Figure 2.1: The wave function of equation(2.58). $M = 938$ MeV, $\rho = ka$ where k is determined by equation(2.48) with $n = 1$ and $a = 3fm$.

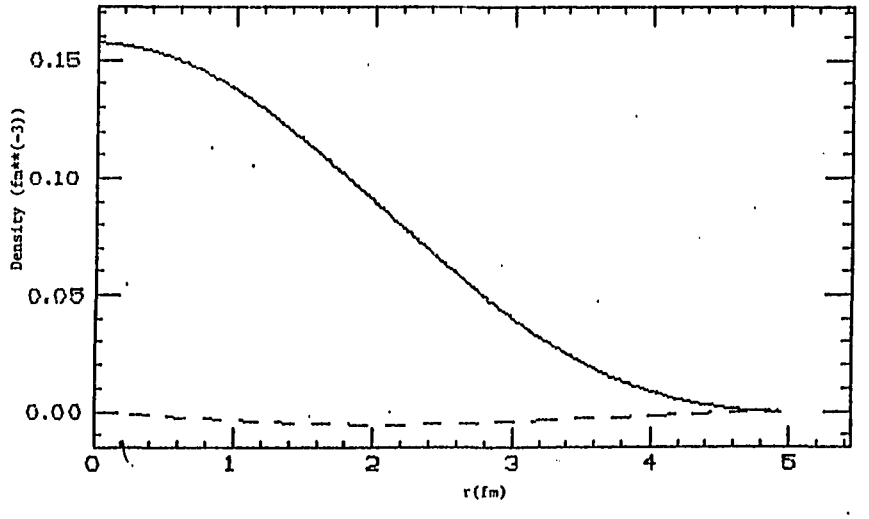

Figure 2.2: The densities of equation's (2.62), (2.63) and (2.64). The solid wave is ρ_S, dots ρ_V and dash ρ_T with $a = 5$fm. In this figure ρ_V is indistinguishable from ρ_S.

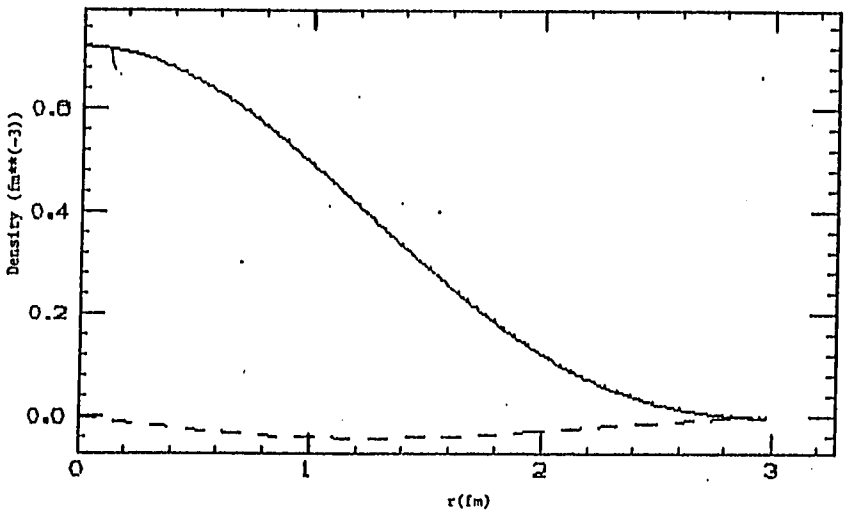

Figure 2.3: Same as figure 2 with $a = 3$fm.

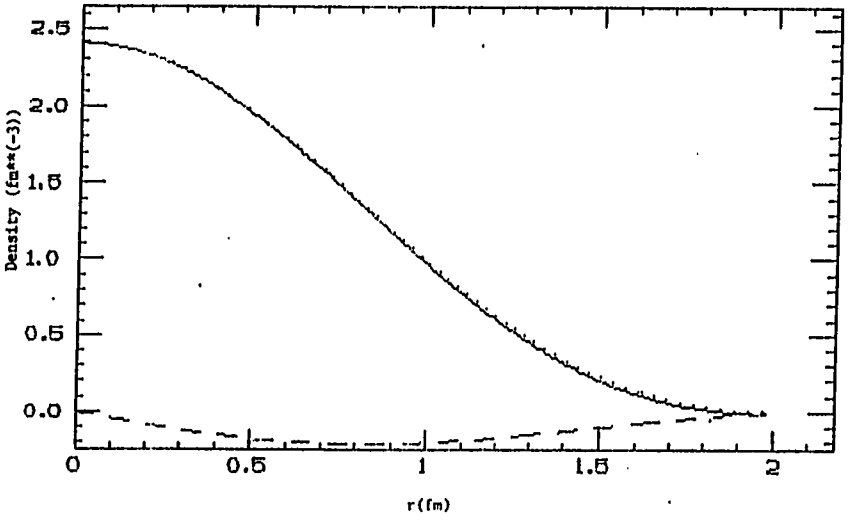

Figure 2.4: Same as figure 2 with $a = 2$fm.

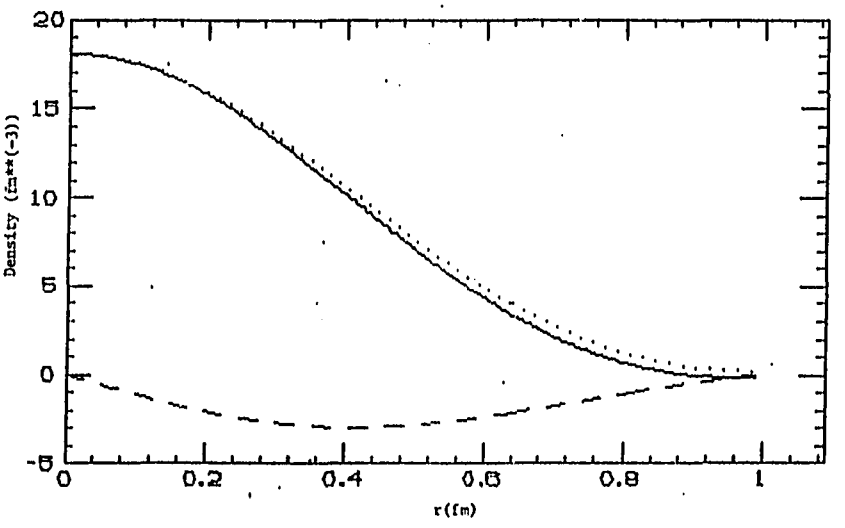

Figure 2.5: Same as figure 2 with $a = 1$fm.

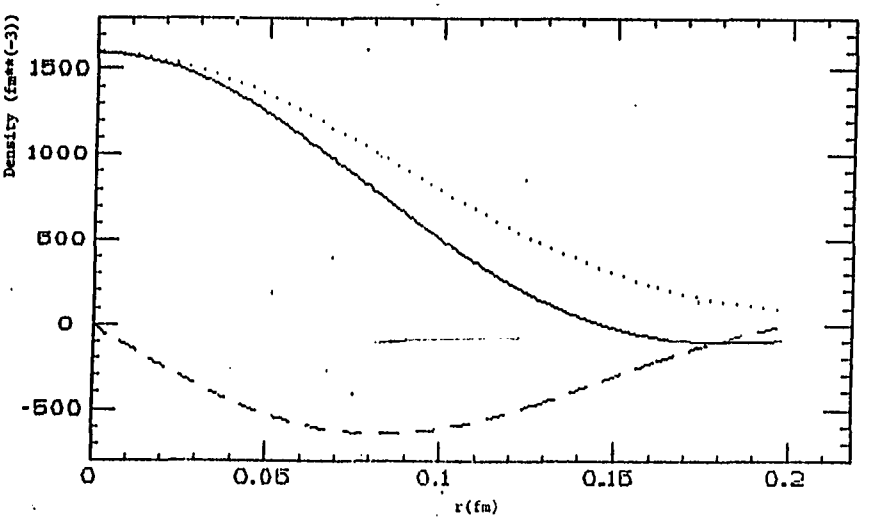

Figure 2.6: Same as figure 2 with $a = 0.2$fm.

CHAPTER III

DIRAC FORM FACTOR

We have discussed the Dirac Equation for a single particle in the ground state of an infinite scalar square well in Chapter II. Now we use this wave function in eq(1.6) of Chapter I. The necessary Integrations are performed with a Gaussian Algorithm with 273 points (Stroud and Secrest, 1966). We present below the following ratio of eq(1.6):

$$R = \frac{\int j_1(qr)\rho_T(r)d\vec{r}}{-\frac{q}{2M_N}\int j_0(qr)\rho_V(r)d\vec{r}} \tag{3.1}$$

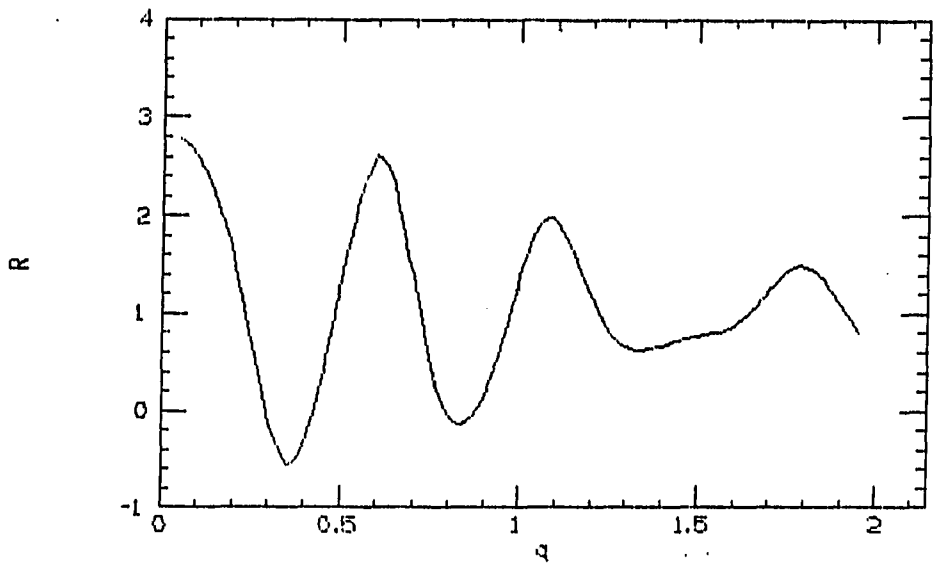

Figure 3.1: The ratio of eq(3.1) with a = 1fm.

We observe from fig 3.1-3.4 that eq(1.6) is satisfied on the average, with the curious exception that the asymptotic ratio R is closer to 1.2 than 1.0. We conclude that use

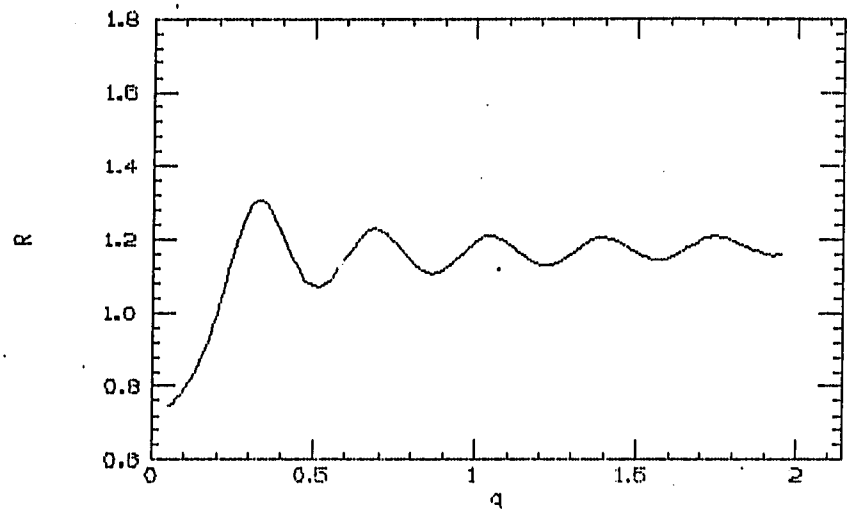

Figure 3.2: Same as figure 3.1 with a = 2fm.

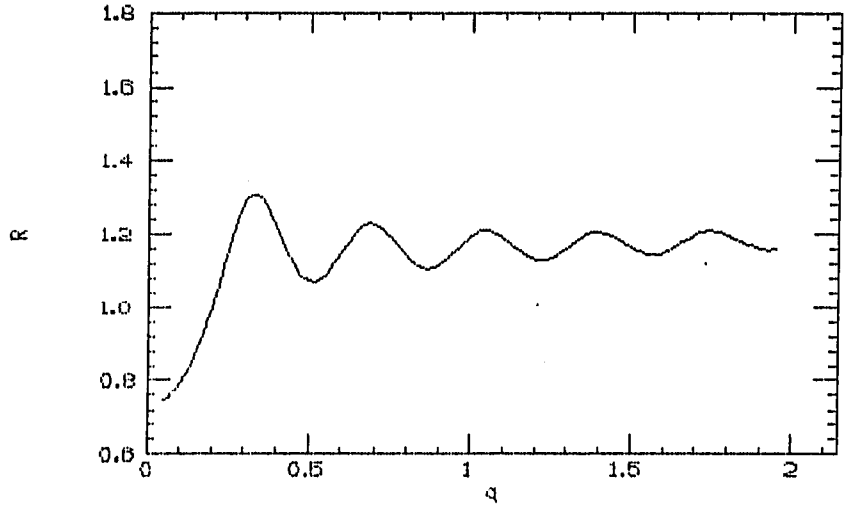

Figure 3.3: Same as figure 3.1 with a = 3fm.

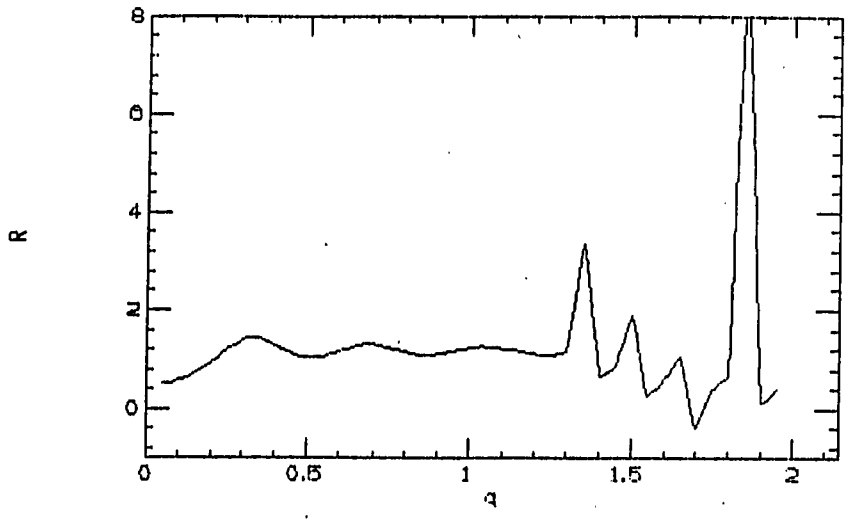

Figure 3.4: Same as figure 3.1 with a = 5fm.

of eq(1.5) to interpret single particle charge form factor is not seriously in error.

In a typical experiment one is sampling all of the nucleons in the nucleus. For example, ^{48}Ca has 20 electrically charged particles and 48 particles possessing a magnetic moment. All 48 particles will contribute to the charge density. We have used the computer code HPLUS2.FOR(HARTREE) written by Horowitz and Serot(1981) to estimate the relativisitic scalar, vector and tensor densities for this nucleus. The assumed potential is not a simple square well, but a finite well given by solving a set of coupled equations for nucleons and mesons. These densities have been commonly used for nuclear reaction calculations. Figure 3.5 shows that eq(1.6) holds approximately for $q < 1.0 fm^{-1}$. Except again, the ratio R is closer to 1.2 than 1.0. The elastic electron scattering form factor has been measured at much higher values of q. We conclude that eq(1.5) connot be used to determine a charge form factor from such experiments, but rather that eq(1.4) must be used together with other experiments (such as elastic hadron scattering) to determine both ρ_V and ρ_T independently.

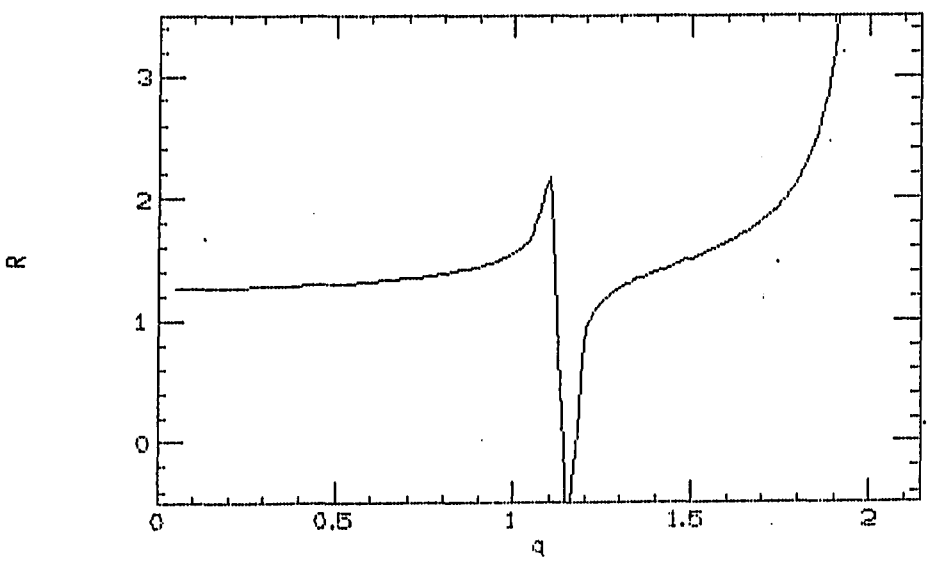

Figure 3.5: The ratio of eq(6) for ^{48}Ca.

CHAPTER IV

CONCLUSION

The equation 6 of Chapter I could not be derived analytically in this work. Indeed, our numerical work shows that this equation is only approximately valid. It holds well for single particle states for $q < 1.8fm^{-1}$. When interpreting elastic electron scattering experiments at $q > 1.0fm^{-1}$, the analysis will have to take into account the relativistic motion of nucleons in nuclei in order to extract useful information about the distribution of mass and electric charge at the correspondingly small distances.

Calculations of the scalar density and vector density with the Dirac equation produced an interesting result that the difference between the two becomes appreciable for small radius of the potential field. As attention was focused mostly on vector density which contributes to the charge form factor, the above result remains to be explained by further work. The Dirac equation was used in the computations because the standard procedure with the Schroedinger equation is not valid in this case.

This work explored the case of a single state. For further work, it should be interesting to investigate the multiple state situation. According to my own experience with the computations performed, if higher precision is desired, it may be helpful to use more points when performing integrations with Gaussian algorithms.

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