Calculation of Electron Scattering for $^4$He in a Continuum Shell Model

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CALCULATION OF ELECTRON SCATTERING FOR $^4$He
IN A CONTINUUM SHELL MODEL

by

Ming Yu

A Thesis
Submitted to the
Faculty of The Graduate Collage
in partial fulfillment of
the requirement for
the degree of Master of Arts
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Calculations for the electroexcitation of the $^4$He, $0^+$ state have been performed within the context of recoil corrected continuum shell model. Approximations previously employed for inelastic scattering with the recoil corrected continuum shell model are eliminated in the work. The $0^+$ state exhausts only 10% of the energy weighted sum rule and previous shell model calculations for the excitation of this state produce about ten times too much strength. This difficulty is alleviated by the use translationally invariant wave functions and the application of continuum boundary conditions for this particle unbound state. The structure of the $0^+$ state is determined by solving the translationally invariant Hamiltonian $H=T+V-T_{c.m.}$ in a $0s^1ns$ basis with the M3Y interaction. The result of the calculation leads to excellent agreement with experiment. These results demonstrate that this state can be understood as a superposition of simple $1p-1h$ excitations in the internal coordinates.
ACKNOWLEDGEMENTS

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Ming Yu
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Calculation of electron scattering for $^4\text{He}$ in a continuum shell model

Yu, Ming, M.A.
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CHAPTER I

INTRODUCTION

There are two fundamental reasons why electron scattering is such a powerful tool for studying nuclear structure. The first is that the basic interaction between the electron and the target nucleus is known. With electron scattering, one can immediately relate the cross section to the transition matrix elements of the local charge and current density operators and thus directly to the structure of the target itself. Of course, the same considerations apply to processes involving real photons, but electrons have the second great advantage that for a fixed energy loss $w$ of the electron, one can vary the three momentum transferred to the nucleus, $q$, the only restriction being that the four momentum transfer be space-like:

$$q_u^2 = q^2 - w^2 > 0$$

With real photons, for a given energy transfer, there is only a single possible momentum transfer since the mass of a real photon is zero:

$$q_u^2 = q^2 - w^2 = 0$$

Thus with electrons, we can study the complete $q^2$ behavior.
of the transition matrix elements and map out the Fourier transforms of the transition charge and current densities. Therefore, one knows the spatial distribution of the transition charge and current densities themselves, and this is currently a source of tremendously rich and unique information on the structure of nuclei.

We are concerned with experiments where we have three variables, the initial and final electron wave number $k_1$, $k_2$ and the scattering angle $\theta$. Instead of these it is more convenient to work with $\theta$, $w$ the energy loss:

$$w = \epsilon_1 - \epsilon_2$$

and $q$, the momentum transfer:

$$q^2 = (k_2 - k_1)^2$$

For a fixed $\theta$, we obtain a typical double differential cross section of the type schematically indicated in Fig.1. Since electrons are very light, they can easily emit real photons as bremsstrahlung as they pass through matter. As they change their velocity while undergoing a collision with the target nucleus, they will also radiate photons. Thus the problem is to relate what is seen in the final spectrometer, $\epsilon_2$ and $\theta$, back to the basic collision with a single nucleus. The calculation of the bremsstrahlung cross section for an electron passing through matter is a completely quantum electrodynamic calculation and is thus (in principle) completely known. The inner bremsstrahlung,
or radiation emitted during the collision with the target nucleus, can also be computed in terms of the quantum electrodynamics of the electron. Thus one can in principle relate what is observed directly back to the fundamental collision with the nucleus. In practice this is sometimes exceedingly complicated. However, since this problem is in principle solved, we shall assume these corrections have been made for the purposes of the present discussion and one has a cross section of the form indicated in Fig.1.

Elastic Scattering

In elastic scattering one has (except for nuclear recoil)

\[ w = 0. \]

The electrons are scattered from the nucleus, leaving the nucleus in its ground state. The cross section for such a scattering from a point nucleus of charge \( Z e \) was first derived by Mott\(^{19} \). Since one expects deviations from point Coulomb scattering when the momentum transfer becomes of the order of the inverse size of the system \( q \sim 1/R, \) electron scattering can be used to measure nuclear sizes. This was first pointed out by Guth\(^{20} \) and by Rose\(^{21} \), Schiff\(^{22,23} \) and Elton\(^{24} \). The first experiments to see this effect clearly were carried out by Lyman\(^{25} \). By varying \( q, \) one measures the Fourier transform of the ground state.
charge distribution and thus the spatial distribution of the charge density itself. If the nucleus has spin 0, one measures the spherical average of the charge distribution in elastic scattering. Many experiments on elastic scattering have been carried out, culminating in the beautiful experiments of Hofstadter and his collaborators. Our best information on nuclear size comes from electron scattering.

Inelastic Scattering

In the next region of the spectrum, \( w > 0 \), one sees spikes corresponding to the excitation of nuclear levels. The first experiments on electron excitation of nuclei to discrete levels are due to Collins and Waldman\(^{26} \) and Wiedenbeck\(^{27} \). The first theoretical discussion of inelastic electron scattering was due to Mamasachlisov\(^{28} \) and Snedden and Touschek\(^{29} \). The multipole analysis of this process is due to Thie\(^{30} \), Schiff\(^{31} \), Robl\(^{32} \), Alder\(^{33} \) and Tassic\(^{34} \).

Quasi-elastic Scattering

The next region of the spectrum is a broad peak which is referred to as the quasi-elastic peak. It corresponds roughly to direct collisions with the individual nucleons in the nucleus. If the nucleons in the nucleus were unbound and at rest, one would expect to see a peak at an energy
loss corresponding to the free kinematics:
\[ w = \frac{q^2}{2M}. \]

Since the nucleons actually are in motion, and we have an energy loss, the peak is displaced and given a width,
\[ w = \frac{-p^2}{2M^*} + \frac{(p-q)^2}{2M^*}. \]

Therefore the peak is at \( w = \frac{q^2}{2M^*} \) and total width is given roughly by:
\[ \frac{qk_F}{M^*} + \frac{q^2}{2M^*} \geq w \geq -\frac{qk_F}{M^*} + \frac{q^2}{2M^*} \]

where \( k_F \) is the Fermi momentum \( (k_F \approx 250 \text{ Mev}) \). A non-interacting fermi gas model therefore predicts \( w \leq (qk_F/M^*) \) + \( (q^2/2M^*) \). Czyz and Gottfried suggested that one should look at the region of larger energy loss since the cross section there must be primarily due to interactions in the nucleus. Hopefully one can obtain information about two-particle correlations from this region; however, one first has to understand the quasi-elastic peak itself in some detail.

**General Electron Scattering**

If one looks only at the final electron in electron scattering, then in the one-photon-exchange approximation the cross section depends on only two form factors \( W_{1,2}(q^2,w) \). The general electron scattering process through
a single photon exchange is indicated in Fig. 2. The initial and final four-momenta of the target are $P_\mu$ and $P'_\mu$ and of the electron are $k_{1\mu}$ and $k_{2\mu}$. The three-momenta and energies are defined by $P_\mu$ and $K_\mu$, where $P_\mu = (P,ie)$ and $K_\mu = (K,ie)$. Thus:

$$P^2_\mu = -M^2 T$$

and since $P'_\mu = P_\mu - q_\mu$, we have:

$$q \cdot P = \frac{1}{2}(q^2_\mu + M^2_f - M^2_T)$$

$$P'^2_\mu = -M^2_f$$

The cross section for this process can be written down from the usual Feynman rules as (Drell and Walecka (1964)):

$$d\sigma = 2Z^2\alpha^2 \frac{dk^2_2}{2\varepsilon_2q^4_\mu} \frac{1}{W_{\mu\nu}T_{\mu\nu}} \frac{1}{[(k_1 \cdot P)^2 - m^2 M^2_f]^h}$$

where

$$W_{\mu\nu} = [(2\pi)^2 \Omega/2^2] \Sigma \delta(4)(P-P'-q)$$

$$<P|J_\nu(0)|P'><P'|J_\mu(0)|P>(E)$$

The $^4$He$(0^+)$ State

In this work we studied the $^4$He structure by inelastic electron scattering. The $T=0$ $0^+$ state of $^4$He is usually treated as the shell model state of 2hw excitation and is a candidate for the breathing mode state. The investigation of excited bound states in the four-nucleon system $^4$He is
particularly interesting, because inelastic electron scattering experiments have shown that this $T=0$ $0^+$ state accounts for a very small percentage of the energy weighted sum rule. Indeed, it has been demonstrated that shell-model calculations which assume a $1s(0s)^{-1} (J = 0, T = 0)$ configuration, over predict the strength by a factor of 20. The inclusion of higher order shell model configurations reduces this factor to between 5 and 10.

The exact nature of the $0^+$ state has been somewhat puzzling and has led to speculations that the shell model is inappropriate for describing this system. Two other types of calculations have been performed for this state. A resonating group calculation with a central interaction and bound state approximation was reported in ref 5. The resulting form factor was approximately three times larger than observed. A calculation employing hyperspherical harmonics was reported in ref. 6. Here good agreement with the experimental form factor was obtained. Since the calculated state turned out to be a pure hyperradial excitation, the authors concluded that it was a collective excitation of the ground state which would, therefore contain multiparticle-multihole state.

In this work it is shown that it is possible to describe this state in a $1p$-$1h$ shell model context, provided one employs a realistic interaction,
translationally invariant wave functions, and proper boundary conditions. The recoil corrected continuum shell model\textsuperscript{7)} (RCCSM) incorporates these three important conditions.
CHAPTER II

THEORY

Inelastic Scattering

Let us turn to some general considerations of inelastic scattering to discrete nuclear levels. The selection rules on the nuclear matrix elements follow immediately from the angular momentum and parity properties of the multipole operators. We have the angular momentum selection rule:

$$| J_f - J_i | \leq J \leq J_f + J_i$$

For spin zero nuclei there is therefore a unique multipole that leads to excitation. In the case of monopole transitions, $0^+ \rightarrow 0^+$, we have a particularly simple form:

$$\frac{d\sigma}{d\Omega}(0^+ \rightarrow 0^+) = \frac{Z^2 \sigma_m}{1 + 2\epsilon_1 \sin^2(\theta/2)/M_t} \cdot \frac{4\pi}{Z^2} \cdot \left| \langle 0^+ | M_0(q) | 0^+ \rangle \right|^2 \cdot q$$

Recoil Corrections

So far we have simply considered the nucleus to be a fixed current and charge source that scatters the electron. The kinematic corrections that must be taken into account
because the nucleus can recoil are contained in the following equation:

\[
\frac{d\sigma}{d\Omega} = \frac{4\pi \sigma_H}{(1+2\epsilon_1 \sin^2(\theta/2)/M_f)} \left( \sum_{J=0}^{\infty} \frac{q^4}{2J_j + 1} \right) \left( \left( \frac{q^2}{2} + \tan^2 \frac{\theta}{2} \right) \sum_{J=1}^{\infty} \frac{1}{2J_j + 1} \right) \left( \left\langle |J_f||M^j_s(q)||J_i>\right| \right)^2 + \left( \frac{q^2_{\mu}}{2q^2} \right) \left( \left( \frac{q^2}{2} + \tan^2 \frac{\theta}{2} \right) \sum_{J=1}^{\infty} \frac{1}{2J_j + 1} \right) \left( \left\langle |J_f||T_{jm}(q)||J_i>\right| \right)^2
\]

\[
\frac{1}{\sigma_H} = \frac{\alpha^2 \cos^2(\theta/2)}{4\epsilon_1^2 \sin^4(\theta/2)}
\]

The largest correction is simply the density of the final state factor:

\[
(1 + \frac{2\epsilon_1 \sin^2(\theta/2)}{M_f})^{-1}
\]

The covariant analysis of the vertex and cross section for electron excitation of a nucleon \( \frac{1}{2} + \) to any one of the isobar states has been carried out by Bjorken and Walecka\(^\text{(36)}\). This is certainly a case where the target recoil is important. The cross section has the form:

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\epsilon_1^2 \sin^4(\theta/2) \cos^2(\theta/2)} \left( \frac{1}{1+(2\epsilon_1/M_f)\sin^2(\theta/2)} \right)
\]

\[
\cdot \left( \left| f_\mu |^2 + \frac{q^2_{\mu}}{2q^2} \right| + \frac{M^2_r}{M^2_f} \tan^2 \frac{\theta}{2} \right) (|f_\mu|^2 + |f_\mu|^2)
\]
where $f_0, f_+, f_-$ are the transition (multipole) amplitudes and $q^*$ is the nucleon momentum in the rest frame of the isobar. Apart from replacing $1 \rightarrow (M_R^2/M^2)$ in the coefficient of $\tan^2(\theta/2)$, and $q^2 \rightarrow q^{*2}$, this formula is identical with eqn.(2.4)

The question of recoil corrections in the matrix elements of the transition current and charge densities themselves is a more difficult one. It is possible to say a few simple things. Ehrenberg et al.$^{37}$ and independently Atkinson$^{38}$ have pointed out that the recoil convection current of the whole nucleus cannot interact with the transverse electromagnetic potential generated by the electron, since the interaction goes as $\mathbf{A}(q) \cdot (\mathbf{P} - \mathbf{P}') = q \cdot \mathbf{A}(q)$, which vanishes. Foldy et al.$^{39}$ have solved the problem of two particles, one non-relativistic, spinless, and of finite size while the other is a point Dirac interacting through the Breit Hamiltonian. They conclude that (to order (electron energy)/(nuclear mass)) the effect of dynamic recoil terms is to rotate the scattering amplitude vectors in the complex plane without changing their magnitudes, (a result which is independent of the shape and size of the nuclear charge distribution). To this order, their cross section is only affected by the kinematic recoil corrections.
The continuum shell model wave functions used in this work are obtained with R-matrix techniques. In this approach configuration space is divided into an internal region and an external region in which there is no polarizing interaction between the neutron and the target nucleus. The boundary is characterized by channel radii $a_c$ which in principle may vary for different $c$; however, we shall assume $a_c = a$ is constant for all $c$. Using the Bloch $f$-operator, the wave function for the system which satisfies the equation

$$ (H - E) \psi = 0, \quad \ldots (2.6) $$

may be written for the internal region $r \leq a$ in the form

$$ \psi = (H + f + E)^{-1}f \psi \quad \ldots (2.7) $$

The operator $f$ is defined by

$$ f = \frac{\hbar^2}{2ma} \sum \phi \phi_c \phi_c \delta(r-a) \left( \frac{d}{dr} r - b_c \right) \phi_c, \quad \ldots (2.8) $$

where the $\phi_c$ are surface functions as defined in ref.19 which are orthogonal when integrated over all coordinates except the radial variational variable $r$ (denoted by the rounded brackets)

$$ (\phi_c | \phi_{c'} ) = \delta_{cc'} \quad \ldots (2.8) $$

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The parameters $b_c$, are constants and $m$ is the reduced mass of the system. In terms of a complete set of basis functions $u_i$ (not necessarily orthogonal over the internal region) $\psi$ may be written for $r \leq a$
$$\psi = \sum |u_i>(A^{-1})_{ij}<u_j|\psi\rangle,$$
where
$$A_{ij} = \langle u_i|H+E|u_j\rangle = \int_0^a (u_i|A|u_j)r^i dr.$$

For large separation distances between the neutron and target nucleus
$$\psi \approx \sum (I_c\phi_c\delta_{cc'} - U_{cc'}O_{cc'}\phi_{c'}),$$
where $c$ defines the incident channel, $I_c$ and $O_c$ are unit flux incoming and outgoing radial functions and $U_{cc'}$ is the collision matrix. Assuming that $\psi$ is asymptotic at $r = a$ and projecting on to the surface function $\phi_c$, gives the set of coupled equation for $r = a$:
$$I_c\delta_{cc'} - U_{cc'}O_{cc'} = \Sigma R_{c'c''}[\langle rI_c|\delta_{cc''} - U_{cc''}(rO_{cc''})\rangle - b_{c''}(I_c\delta_{cc''} - U_{cc''}O_{cc''})],$$
where
$$R_{c'c''} = \frac{\hbar^2 a}{2m} \Sigma (\phi_{c''}|u_i>(A^{-1})_{ij}<u_j|\phi_{c''})$$
$$\ldots (2.9)$$
and the quantities $|\phi_c; u_i\rangle$ are radial functions at $r = a$.

The transition amplitude for elastic scattering is obtained.
from the appropriate collision matrix element $U_{cc}$ by the relation

$$T_{cc} = T_{(l_1)} = \frac{1}{2} i(1 - U_{cc}).$$

Recoil Corrected Continuum Shell Model

The nuclear continuum shell model is reformulated in such a manner that target recoil is taken fully into account. The reformulation is achieved by employing dynamical R-matrix discretization based on intrinsic harmonic oscillator expansion functions.

The continuum shell model has been applied to a wide variety of problems in atomic and nuclear physics. As in the conventional shell model, the basic ansatz is an expansion of the wave function in terms of products of single-particle functions. This ansatz is very flexible and is also very convenient for such operations as the calculation of matrix elements and the proper treatment of the Pauli principle: but it is not, in general, compatible with the requirement that the center of mass coordinate be separable. It follows that shell-model eigenstates generally do not correspond to pure states of internal motion. The great advantage of RCCSM was its ability to provide matrix elements of translationally invariant operators in the internal coordinates by calculating matrix elements in normal shell model coordinates, $x_i$, with a fixed...
The RCCSM in the lp-lh approximations has been very successful in describing low energy nucleon scattering phenomena. It allows one to employ two-nucleon potentials which contain central, spin-orbit, and tensor components. In this work we employ the M3Y interactions. This g-matrix interaction was itself derived from standard realistic interactions and has been used with considerable success in theoretical descriptions of scattering and reaction processes involving medium weight nuclei. By assumption, the Hamiltonian contains only operators which act on the internal coordinates. Therefore, the kinetic energy take the form:

\[ T = (2m)^{-1} \sum P_i^2 - T_{c.m} \]

\[ = A_c [2m(A_c+1)]^{-1} \sum P_i^2 - [m(A_c+1)]^{-1} \sum P_i P_j \]

Where \( A_c \) is the number of nucleons in the target. For the purpose of calculating the many-body matrix element, it is convenient to combine the two-body terms from the kinetic energy with the corresponding terms in the two-body interaction. The model employs the translationally invariant Hamiltonian:

\[ H = T + V \]

\[ = (2m)^{-1} \sum P_i^2 - T_{c.m} + \sum v_{ij} \quad \ldots (2.10) \]

where the two-body interaction is the Coulomb potential plus nuclear forces. It is reasonable to expect that the
$O_2^+$ of $^4$He close to the thresholds has a [3-bodies + 1-body] - like structure.
CHAPTER III

CALCULATION OF MATRIX ELEMENT

A particle of mass $M$ moving in an attractive harmonic oscillator potential satisfies the Schrödinger equation

$$\frac{\hbar^2}{2M} \left( -\nabla^2 + \frac{1}{M \omega^2 r^2} \right) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi)$$

Proper boundary conditions are imposed by R-matrix techniques at a matching radius of $a_c = 7.2$ fm. A smooth joining to Coulomb functions is accomplished by allowing particle excitations up to $2n + 1 = 14$, where $n$ begins at zero. The oscillator constant, $\nu = \hbar^2 / M \omega$, is chosen as 0.36 fm$^{-1}$ to produce the mean-squared radius of $^3\text{H}$.

Previous attempts to describe inelastic scattering of pion and electrons in the context of the recoil corrected continuum shell model have treated the $^4\text{He}$ ground state as a pure $\text{Os}^4$ and considered only the coordinate $r_4$ in Fig.3. This is because the wave function of the coordinate $\xi_3$ is readily available. However, one sees that if the coordinates $r_1$, $r_2$, and $r_3$ will also move slightly with respect to the center of mass. This constitutes a target recoil or center of mass correction which was omitted from

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previous work. In the present work all coordinates are considered as well as ns0s⁻¹ correlations in the ground-state wave function.

In a [3-bodies + 1-body] - like structure, a core has been constructed. The spin quantum number of the core is:

\[ S_c = \left( \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2} \]

The orbital quantum number of the core is zero \((l_c = 0)\).

Therefore:

\[ j_c = l_c + S_c = \frac{1}{2} \]

For the excited particle, its spin quantum number is \(\frac{1}{2}\). The excited particle coupled to the core:

\[ |\frac{1}{2}, (1, \frac{1}{2})j, J_f M_f > \]

Therefore, one is looking at a transition between an initial state and a final state of the same form:

\[ \Phi_{i,f} = a_0 |0s^t>|S = 0> + \sum a_n,\epsilon |0s^t(\epsilon_1, \epsilon_2)|ns_\epsilon R>|S = 0>, \ldots \]  

with components of 0s⁺ and ns0s⁻¹ correlations. Now we try to express this wavefunction in L-S coupling space:

\[ |\frac{1}{2}, (1, \frac{1}{2})j, J_f M_f > = (-1)^{j_1 - J} |(1, \frac{1}{2})j, \frac{1}{2}; J_f M_f > \]

\[ = (-1)^{j_1 - J} \sum (-1)^{j_1 + J + \frac{s}{2}} sj \left\{ \begin{array}{c} 1 \frac{1}{2} j \frac{1}{2} \end{array} \right\} |l, (\frac{1}{2}, \frac{1}{2})s, J_f M_f > \]

Where \(\Sigma\): is the summation from \(s = 0\) to \(s = 1\).

The base of the L-S coupling space:
\( \mathcal{A}_r | l, (\frac{1}{2}, \frac{1}{2}) s, J_f M_f > = \mathcal{A}_r \left\{ n l r (\epsilon_3) \left[ \frac{1}{2}, \frac{1}{2} \right]^{0} (1) \right\} \Omega_s (R) \)

where

\[ \mathcal{A}_r = \frac{1}{2(1+d)} (1 - P_{3d}) \]

Therefore the basis

\[ \phi = \frac{1}{2(1+d)} (1 - P_{3d}) \phi_{nl}^a (\epsilon_3) \Omega_s^a (\epsilon_2) \Omega_s (\epsilon_1) \mid s=0 > \Omega_s (R) \]

\[ = \frac{1}{2(d+1)} \left( \phi_{nl}^a (\epsilon_3) \Omega_s^a (\epsilon_2) \Omega_s (\epsilon_1) + \phi_{nl}^a (4r_3/3) \Omega_s^a (3r_4/2 + r_3/2) \Omega_s (\epsilon_1) \right) \mid s=0 > \Omega_s (R) \]

\[ = \frac{1}{2(d+1)} (D + E). \]

Therefore

\[ < \phi_f | r_0 | \phi_i > = \frac{1}{2(d_f+1)} \left( \frac{2(d_f+1)}{2(d_f+1)} \right) \left( <D|r_0|D> + <D|r_0|E> \right. \]

\[ + \left. <E|r_0|D> + <E|r_0|E> \right) \]

where \( d = <D|E> \).

The more explicit forms and mathematical derivation are in the appendix.

But because they are solutions to the same Hamiltonian at different energies, the states are orthogonal. This orthogonality is crucial in describing the shape of the form factor.
The great advantage of the RCCSM was its ability to provide matrix elements of translationally invariant operators in the internal coordinates by calculating matrix elements in normal shell-model coordinates, $x_i$, with a fixed origin. (Fig. 3.) This was very convenient for operators such as the two-body interaction, the kinetic energy, and transition operators for which a long wavelength approximation could be made. However, at high momentum transfer $q$, operators such as $j^q(q|x_i - R|)$ do not lend themselves to a simple decomposition in terms of the shell-model coordinates, $x_i$. Therefore, the matrix element of interest for the present problem,

$$ M_0 = \langle \Psi_i \| \sum Y_0(x_i) j_0(qx_i) \| \Psi_i \rangle, $$

...(3.2)

must be done explicitly in the relative coordinates. This involves a number of terms, most of which require three-dimensional integral.

Cross sections for excitation of states above particle emission threshold are given by expression

$$ \frac{d^2\sigma}{d\Omega dE} = \frac{1}{2\pi \hbar^2} \sum \frac{\mu_c}{k_c} (d\sigma_{c,J_B}/d\Omega) $$

...(3.3)

where $\mu$ is the nucleon reduced mass, $k_c$ is the nucleon asymptotic relative momentum in the channel $c$, and
\[ \frac{\mathrm{d} \sigma_{c, \theta}}{\mathrm{d} \Omega} \] is a fictitious Born cross section, calculated for nucleon wave functions with flux \( v_c \) in channel \( c \). The index \( c \) stands for \( aJ_c j_l \) with \( J_c \) and \( j \) coupled to \( J_\theta \), where \( J_c \) is the angular momentum of \( ^3\text{H} \) or \( ^3\text{H} \), \( l \) and \( j \) are the nucleon orbital and total angular momentum, and \( a \) distinguishes between \(^3\text{H} \) and \(^3\text{He} \).

Electron scattering data is often reduced to a form factor. The form factor is defined as

\[ F = \frac{(\mathrm{d}\sigma/\mathrm{d}\Omega)}{4\pi\sigma_M}, \quad \ldots(3.4) \]

where

\[ \sigma_M = \frac{1}{2E_0} \frac{\cos^2 \theta}{\sin^4 \theta} \left( 1 + \frac{2E_0 \sin^2 \frac{1}{2} \theta}{M c^2} \right)^2 \]

\[ \ldots(3.5) \]
CHAPTER IV

RESULTS

The first comparison shown in Fig. 4 is the calculated ground-state form factor with the measured form factor. The theoretical curves are multiplied by the finite proton size correction factor, $1/(1+0.0533q^2)^4$. Two theoretical curves are plotted. The dashed line results from assuming a pure $0s^4$ configuration with $v = 0.36 \text{ fm}^{-2}$.

Because of the expansion

$$4\pi F(q) = (1 + \cdots + \langle r^2 \rangle q^2 + \cdots + \langle r^4 \rangle q^4 + \cdots)^2$$

...(4.1)

a comparison of data with calculated results at the first shoulder of the elastic form factor will test agreement between calculated and measured rms radii. The dashed curve clearly shows too large a radius because the oscillator constant was chosen to fit $^3\text{H}$. The solid line, which is the result of including the ground-state correlations, does quite well in the first shoulder region, demonstrating that the M3Y interaction produces an rms radius in agreement with experiment.

Comparison with three measured sets of form factors for the first excited state is made by the following
procedure. This procedure does not eliminate the confusion among data sets. The reduced matrix elements $B(C0,q)$, given in ref.16, are converted to the above definition of form factors via $F(q) = B(C0,q)/4$. The form factors in Ref.17 are converted to the definition in Eq.(2.14) by squaring them, dividing by $4\pi Z^2$, and dividing by a correction factor of 3.2. The cross sections were used in this case because the reported form factors did not appear to be quite consistent with the definitions in Eqs.(3.4) and (3.5).

The calculated form factors are shown in Fig. 5 as compared to the data. The short-dash line is the calculated result, assuming a pure Os$^4$ ground state and discarding Os$^4$ components in the excited state. The dashed line is the calculated form factor with the complete wave functions included. One sees a 30% reduction in strength due to ground-state correlations. The solid line results from finite proton size correction of the dashed line and is the final result. Both the size and shape of the form factor is reasonable.

A three-dimensional graph of the calculation corresponding to the dashed line in Fig.5 is shown in Fig.6. Here it is pointed out that the theoretical points in Fig.5 were obtained by integrating over the energy region, $E_p = 0-1.2$ Mev. This effect would be difficult to estimate since the different experimental papers showed
different shapes and ranges for the background.

Another source of uncertainty that could increase the calculated form factor can be seen in Fig. 6. Here one sees that part of the resonance appears to be cut away. The beginning of this cut coincides with the opening of the neutron threshold. The neutron escapes easily and produces a very broad resonance. An asymmetric shape was predicted earlier from the work of Crone and Werntz. The reduction in strength due to this effect is probably greater in the calculation than reflected in the data for two reasons. First, the M3V interaction places the $0^+$ resonance slightly higher than its observed location, and the neutron threshold is calculated to be 0.69 MeV instead of 0.76 MeV as observed experimentally. This difficulty with the Coulomb energy difference between $^3$H and $^3$He occurs in most binding energy calculations that use only the Coulomb potential to break charge symmetry. The two effects combine to produce a cut that begins approximately 0.2 MeV early in the calculation and therefore steals some strength. Indeed, the later experiments did not report as asymmetric shape to their peaks.

Finally, shown in Table I are the components of the wave function for $E_p = 0.60$ MeV. Here one can see that many configurations contribute to the wave function. This mixins was not due primarily to the interaction, but to satisfying
the continuum boundary conditions.
In conclusion, the inelastic electron scattering form factor for the first excited state of $^4$He appears to be well-described by shell model terms, provided one employs a realistic interaction, translationally invariant wave functions, and proper continuum boundary conditions. The wave function at resonance is not a simple $2\hbar w$ excitation, but requires a linear combination of s-state particle excitations of high oscillator principle quantum number in order to satisfy the boundary conditions. However, because this linear combination is necessary only to produce the correct shape of the wave function for the one-particle excitation, this calculation suggests that the $0^+$ can best be described as a $1p-1h$ excitation and not a collective state.
TABLE I. Percentage of particle configuration in the wave function at $E_p = 6.0$ MeV.

<table>
<thead>
<tr>
<th>Particle orbit</th>
<th>Percentage proton excitation</th>
<th>Percentage neutron excitation</th>
</tr>
</thead>
<tbody>
<tr>
<td>os$^4$</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td>1s</td>
<td>19.2</td>
<td>14.4</td>
</tr>
<tr>
<td>2s</td>
<td>12.1</td>
<td>7.6</td>
</tr>
<tr>
<td>3s</td>
<td>15.0</td>
<td>9.8</td>
</tr>
<tr>
<td>4s</td>
<td>9.8</td>
<td>6.2</td>
</tr>
<tr>
<td>5s</td>
<td>5.7</td>
<td>3.8</td>
</tr>
<tr>
<td>6s</td>
<td>2.0</td>
<td>1.4</td>
</tr>
<tr>
<td>7s</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Fig. 1  A Typical Double Differential Electron Scattering Cross Section

\[ \frac{1}{\sigma_n} \frac{d^4 \sigma}{d\Omega dE} \sigma_n = \frac{4a^2}{q^4} k_0^2 \cos \theta \]

Fig. 2  The General Electron Scattering Process in Lowest Order in \( \alpha \)

\[ q_\mu = (k_2 - k_1)_\mu \cdot (P - P')_\mu \]
Fig. 3 The RCCSM Coordinate System.

Fig. 4 The Elastic Form Factor for $^4$He.
The Dashed Line is the Result of a Calculation with the $0s^4$ Configuration.
The Solid Curve is the Result of a Calculation that includes ground-state correlations. The Data Are from Ref. 13.
Fig. 5 The Inelastic Form Factor for $^4\text{He} (0^+)$.
The Short Dash Line is the Result of a Calculation with the $0s^4$ Configuration for the Ground-State Correlations. The Solid Line is the Same Calculation as the Dash Line, But Corrected for Proton Finite Size. The Data are Represented by Diamonds, Circles, and Crosses are from Ref. 3, 1, and 2, Respectively.

Fig. 6 Form Factor Per MeV As a Function of $q$ and Proton Energy. Calculation Includes Ground-State Correlations But not Proton Finite Size Correction.
APPENDIX

CALCULATION OF MATRIX ELEMENT
In $^4$He, two neutrons and one proton constitute a 3-bodies core. The spin quantum number of the core is

$$S_c = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}$$

The orbital quantum number of the core is zero. ($l_c = 0$)

Therefore

$$j_c = l_c + S_c = \frac{1}{2}$$

For the excited proton coupled to the core one has:

$$|\frac{1}{2}, (1, \frac{1}{2})j, J_f, M_f >.$$ 

Now we try to express this wavefunction in L-S coupling space

$$|\frac{1}{2}, (1, \frac{1}{2})j, J_f, M_f > = (-1)^j \Sigma (-1)^{l_s + j_s} S_j (\frac{1}{2} \frac{1}{2} j)$$

where $\Sigma$ is the summation from $s = 0$ to 1.

The base of the L-S coupling space

$$E_f |1, (\frac{1}{2}, \frac{1}{2})s, J_f, M_f > = E_f n^s (\epsilon_3)[\frac{1}{2}, \frac{1}{2}](1) O_s(R)$$

where

$$E_f = \frac{1}{\sqrt{2(d+1)}}(1 - P_{1j}) = \frac{1}{\sqrt{2(d+1)}} (1 - P_{34})$$

The basis = $E_f \phi_n^6(\epsilon_3) O_s^6(\epsilon_2) O_s(\epsilon_1) |S=0>O_s(R)$. 

Geometrically:

$$r_1' + \epsilon_1 = r_2'$$

$$r_4' = 3\epsilon_3/4$$

$$r_3' + \epsilon_3/4 = 2\epsilon_2'/3$$

Therefore

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The basis = $R_s \phi^8_{nl}(\epsilon_3)O^8_s(\epsilon_2)O^8_s(\epsilon_1)|S=0>0^s(R)$

$$= R_s \phi^8_{nl}(4r_4'/3)O^8_s(r_4'/2+3r_3'/2)O^8_s(\epsilon_1)|S=0>0^s(R)$$

$$+ \phi^8_{nl}(4r_3'/3)O^8_s(r_3'/2+3r_4'/2)O^8_s(\epsilon_1)|S=0>0^s(R).$$

Let

$$D = \phi^8_{nl}(4r_4'/3)O^8_s(r_4'/2+3r_3'/2)O^8_s(\epsilon_1)$$

$$E = \phi^8_{nl}(4r_3'/3)O^8_s(r_3'/2+3r_4'/2)O^8_s(\epsilon_1)$$

Calculation of the Matrix Element:

$$\Phi_{i,f} = a_0|0^S>|S=0> +$$

$$\Sigma a_{nt}(1-P_3)4^t(\epsilon_1,\epsilon_2)ns(\epsilon_3)|0^S(\epsilon_1/\epsilon_2)|S=0>,$$

Matrix Element = $<\Phi_{f}|\Sigma Y_0(\mathbf{r}_i)j_0(q\mathbf{r}_i)|\Phi_{i}>$

Let $\Phi_{i} = a_0|0^S>|S=0> +$

$$\Sigma a_{nt}(1-P_3)4^t(\epsilon_1,\epsilon_2)ns(\epsilon_3)|0^S(\epsilon_1/\epsilon_2)|S=0>,$$

$$\Phi_{f} = b_0|0^S>|S=0> +$$

$$\Sigma b_{nt}(1-P_3)4^t(\epsilon_1,\epsilon_2)ns(\epsilon_3)|0^S(\epsilon_1/\epsilon_2)|S=0>,$$

$$\Sigma Y_0(\mathbf{r}_i)j_0(q\mathbf{r}_i) = Y_0(\mathbf{r}_1)j_0(q\mathbf{r}_1) + Y_0(\mathbf{r}_2)j_0(q\mathbf{r}_2) +$$

$$Y_0(\mathbf{r}_3)j_0(q\mathbf{r}_3) + Y_0(\mathbf{r}_4)j_0(q\mathbf{r}_4)$$

$$= \sigma_1(\mathbf{r}_1) + \sigma_2(\mathbf{r}_2) + \sigma_3(\mathbf{r}_3) + \sigma_4(\mathbf{r}_4)$$

$$<0|\sigma_1|0> = \int 0^s(\epsilon_3)0^s(\epsilon_2)0^s(\epsilon_1)\sigma_1O^s_1(\epsilon_1)O^s_2(\epsilon_2)O^s_3(\epsilon_3)d\epsilon_1d\epsilon_2d\epsilon_3$$

where $\mathbf{r}_1 = -\epsilon_1/2 - \epsilon_2/3 - \epsilon_3/4$.  

$$<0|\sigma_2|0> = \int 0^s(\epsilon_3)0^s(\epsilon_2)0^s(\epsilon_1)\sigma_2O^s_1(\epsilon_1)O^s_2(\epsilon_2)O^s_3(\epsilon_3)d\epsilon_1d\epsilon_2d\epsilon_3$$

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where $r_2 = \frac{\epsilon_1}{2} - \frac{\epsilon_2}{3} - \frac{\epsilon_3}{4}$.

$<0|\sigma_3|0> = \int \phi^{*}(\epsilon_3) \phi^{*}(\epsilon_2) \phi^{*}(\epsilon_1) \sigma_3 O_{\epsilon_3}(\epsilon_3) O_{\epsilon_2}(\epsilon_2) O_{\epsilon_1}(\epsilon_1) \, d\epsilon_1 d\epsilon_2 d\epsilon_3$

where $r_3 = \frac{2\epsilon_2}{3} - \frac{\epsilon_3}{4}$.

$<0|\sigma_4|0> = \int \phi^{*}(\epsilon_3) \phi^{*}(\epsilon_2) \phi^{*}(\epsilon_1) \sigma_4 O_{\epsilon_3}(\epsilon_3) O_{\epsilon_2}(\epsilon_2) O_{\epsilon_1}(\epsilon_1) \, d\epsilon_1 d\epsilon_2 d\epsilon_3$

where $r_4 = 3\epsilon_3 / 4$.

Obviously $<0|\sigma_4|0> = \int \phi^{*}(\epsilon_3) \sigma_4 O_{\epsilon_3}(\epsilon_3) \, d\epsilon_3$.

$|E+D> \text{ and } |0> \text{ are antisymmetric in } 1, 2 \text{ and } 3, 4$. Therefore:

$<E+D|\sigma_1 + \sigma_2|0> = 2<E+D|\sigma_1|0>$

$<E+D|\sigma_3 + \sigma_4|0> = 2<E+D|\sigma_4|0>$

First look at $<D|\sigma_4|0>$:

$<D|\sigma_4|0> = \int \phi^{*}(\epsilon_3) \phi^{*}(\epsilon_2) \phi^{*}(\epsilon_1) \sigma_4 O_{\epsilon_3}(\epsilon_3) O_{\epsilon_2}(\epsilon_2) O_{\epsilon_1}(\epsilon_1) \, d\epsilon_1 d\epsilon_2 d\epsilon_3$

$= \int \phi^{*}(\epsilon_3) \sigma_4 (3\epsilon_3 / 4) O_{\epsilon_3}(\epsilon_3) \, d\epsilon_3$
Secondly look at $<E | \sigma_4 | 0>$, and $E = P_{3,4}D$

$$<E | \sigma_4 | 0> =$$

$$\int \phi_{nl} s(\epsilon_3 / 3 + 8\epsilon_2 / 9) O^s_3(\epsilon_3 + \epsilon_2 / 3) \sigma_4(3\epsilon_3 / 4) O_5(\epsilon_3) O_6(\epsilon_2) d\epsilon_2 d\epsilon_3$$

let

$$X = -\epsilon_3 / 3 + 8\epsilon_2 / 9$$

$$Y = \epsilon_3$$

Therefore

$$\frac{\partial(\epsilon_2, \epsilon_3)}{\partial(X, Y)} = |J| = \frac{9}{8}.$$ 

$$<E | \sigma_4 | 0> =$$

$$\left[ \frac{-8^3}{9} \right] \int \phi_{nl} s(X) O^s_5(3X / 8 + 9Y / 8)$$

$$\sigma_4(3Y / 4) O_5(Y) O_6(9X / 8 + 3Y / 8) dX dY$$

$$= \left[ \frac{-8^3}{9} \right] \int \phi_{nl} s(X) \frac{2V}{3\pi} e^{-1/3V(9/64)(10x^2 + 7x + 12y)}$$

$$\sigma_4(3Y / 4) O_5(Y) dX dY$$

$$= \left[ \frac{-8^3}{9} \right] \int \phi_{nl} s(X) \frac{2V}{3\pi} e^{-3/64V(4y^2 + 4x^2)} e^{-18/64V(x+y)^2}$$

$$\sigma_4(3Y / 4) O_5(Y) dX dY.$$ 

$$e^{-r(a-b)^2} = 4\pi \Sigma (i)^{-1} e^{-r(a^2+b^2)} j_l(2i\sigma) Y_{lm}(a) Y_{lm}^*(b)$$

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\[
<\mathbf{E}\mid \sigma_4\mid 0> = \frac{8^3}{9} \int \phi_{nl}^*(X) \frac{2V}{3\pi} e^{-3/64V(4y^2+4x^2)} \cdot 4\pi \\
\Sigma(i)^{-1} e^{9\sqrt{32(x^2+y^2)}} \cdot j_1(-9ivxy/16) Y_{lm}(x) Y_{lm}^*(y) \\
\cdot \sigma_4(3y/4) O_b(y) \, dx \, dy.
\]

\[
= \frac{8^3}{9} \int \phi_{nl}^*(X) \frac{2V}{3\pi} \Sigma(i)^{-1} e^{15\sqrt{32(x^2+y^2)}}. \\
j_1(-9ivxy/16) Y_{lm}(x) Y_{lm}^*(y) \cdot \sigma_4(3y/4) O_b(y) \, dx \, dy.
\]

and \(\phi_{nl}(x) = R_{nl}(x) Y_{lm}(x)\); \(\sigma_4(3y/4) = \sigma_4(3y/4) \cdot Y_{lm}(y)\)

dy = y^2 \, dy \, d\Omega; \\
dx = x^2 \, dx \, d\Omega

\[
\frac{9}{8} \int \frac{2V}{3\pi} R_{nl}(X) (i)^{-1} e^{15\sqrt{32(x^2+y^2)}} \\
j_1(-9ivxy/16) \cdot \sigma_4(3y/4) O_b(y) y^2 x^2 \, dx \, dy
\]

Therefore \(\langle D+E\mid \sigma_4\mid 0 > = \int \phi_{nl}^*(\epsilon_3) \sigma_4(3\epsilon_3/4) O_b(\epsilon_3) \, d\epsilon_3 + \)

\[
\frac{9}{8} \int \frac{2V}{3\pi} R_{nl}(X) (i)^{-1} e^{15\sqrt{32(x^2+y^2)}} \\
j_1(-9ivxy/16) \cdot \sigma_4(3y/4) O_b(y) y^2 x^2 \, dx \, dy
\]

and \(\langle D+E\mid \sigma_3 + \sigma_4\mid 0 > = 2\langle D+E\mid \sigma_4\mid 0 >\)
Now try $<D+E|\sigma_1|0>$:

$$D = \phi_{nl}(\epsilon_3)O_8(\epsilon_2)O_8(\epsilon_1)$$

$$= \phi_{nl}(\epsilon_3)(2\sqrt[3]{3\pi})^{3/4}(\sqrt{3\pi})^{3/4}e^{-(\epsilon_1/3\epsilon_3^{1/4})^2}$$

and $\epsilon_2^2/3+\epsilon_1^2/4$ is symmetric in $r_1, r_2, r_3$

therefore $D$ is symmetric in $r_1, r_2, r_3$.

$E = P_{34}D$. So, $E$ is symmetric in $r_1, r_2, r_4$

$<D|\sigma_1(r_1)|0> = <D|\sigma_1(r_2)|0> = <D|\sigma_1(r_3)|0>$

and $<E|\sigma_1(r_1)|0> = <E|\sigma_1(r_2)|0> = <E|\sigma_1(r_4)|0>$

therefore $<D+E|\sigma_1|0> = 2<D|\sigma_1|0> = 2<D|\sigma_1|0>$

$<D|\sigma_1(r_3)|0>$

$$= \int O_8^*(\epsilon_3)O_8^*(\epsilon_2)O_8^*(\epsilon_1)\sigma_1O_8(\epsilon_1)O_8(\epsilon_2)O_8(\epsilon_1)d\epsilon_1d\epsilon_2d\epsilon_3$$

where $r_3 = 2\epsilon_2/3 - \epsilon_4/3$.

$r_3$ is not related to $\epsilon_1$

therefore $\int O_8^*(\epsilon_1)O_8(\epsilon_1)d\epsilon_1 = 1$

$<D|\sigma_1(r_3)|0>$
\[
= \int \phi_0^*(\varepsilon_3) O_0^*(\varepsilon_2) \sigma_1(r_3) O_0(\varepsilon_3) O_0(\varepsilon_2) O_0 d\varepsilon_2 d\varepsilon_3
\]

\[
= \int \phi_{nl}^*(-4r_4/3) O_0^*(3r_3/2+r_4/2)
\]

\[
\sigma_1(r_3) O_0(4r_4/3) O_0(3r_3/2+r_4/2) d\varepsilon_2 d\varepsilon_3
\]

let
\[
\varepsilon_2 = -3r_3/2 + r_4/2
\]

\[
\varepsilon_3 = 4r_4
\]

Therefore
\[
\left(\frac{\partial^2}{\partial r_3 \partial r_4}\right) = |J| = 2
\]

\[
<\Phi|\sigma_1(r_3)|0> = 2^3 \int \phi_{nl}^*(-4r_4/3) O_0^*(3r_3/2+r_4/2)
\]

\[
\sigma_1(r_3) O_0(4r_4/3) O_0(3r_3/2+r_4/2) dr_4 dr_3
\]

\[
= 2^3 \int \phi_{nl}^*(4r/3) \left(\frac{2v}{3\pi}\right)^{3/2} e^{-2v/3(3r/2+r/4)^2}
\]

\[
\cdot \sigma_1(r_3) O_0(4r_4/3) dr_3 dr_4.
\]

\[
e^{-r(a-b)} = 4\pi \Sigma (i)^{-1} e^{-r(a+b)^2} j_1(2\pi r(a-b)) Y_{lm}(a) Y_{lm*}(b)
\]

\[
<\Phi|\sigma_1(r_3)|0> = \int 4\pi \cdot 2^3 (i)^{-1} R_{nl}^*(4r/3) \left(\frac{2v}{3\pi}\right)^{3/2} e^{-v/6(3r^2+r^2)}
\]

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\[ j_1(-ivr_3 r_4) \cdot \sigma_1(r_3) \sigma_0(4r_4/3) r_3^2 r_4^2 \, dr_3 dr_4. \]

and \[ j_1(x) = \left(\frac{-x}{2}\right)^l \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+\nu+1)} \left(\frac{-x}{2}\right)^{2j} \]

\[ j_1(-x) = \left(\frac{-x}{2}\right)^l \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j+\nu+1)} \left(\frac{-x}{2}\right)^{2j} \]

\[ = (-1)^l j_1(x) \]

\[ <D|\sigma_1(r_3) | 0 > = \int 4\pi \cdot 2^3 (-i)^{-1} R_{nl}^* (4x/3) \frac{2\nu}{3\pi} e^{-\nu/6(9r^2+r_4^2)} \]

\[ j_1(i vr_3 r_4) \cdot \sigma_1(r_3) \sigma_0(4r_4/3) r_3^2 r_4^2 \, dr_3 dr_4. \]

\[ <E+D|\sigma_1(r_3) | 0 > = 2<D|\sigma_1 | 0 > = \]

\[ 2 \int 4\pi \cdot 2^3 (-i)^{-1} R_{nl}^* (4x/3) \frac{2\nu}{3\pi} e^{-\nu/6(9r^2+r_4^2)} \]

\[ j_1(i vr_3 r_4) \cdot \sigma_1(r_3) \sigma_0(4r_4/3) r_3^2 r_4^2 \, dr_3 dr_4. \]

\[ <E+D|\sigma_1(r_3) + \sigma_2(r_3) | 0 > = 2<E+D|\sigma_1 | 0 > = \]

\[ 4 \int 4\pi \cdot 2^3 (-i)^{-1} R_{nl}^* (4x/3) \frac{2\nu}{3\pi} e^{-\nu/6(9r^2+r_4^2)} \]
\[ j_1(ivr_3 r_4) \cdot \sigma_1(r_3) O_s(4r_4/3) r_3^2 r_4^2 \, dr_3 dr_4. \]

Therefore \( <\Phi | \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 | 0> = -\frac{j}{1} \frac{1}{J^{d+1}} \).

\[
= \int 4 \cdot O^a(\epsilon_3) \sigma_4 O_s(\epsilon_3) d\epsilon_3 + \int R_{nl}^* (\epsilon_3) \sigma_4 (3\epsilon_3/4) O_s(\epsilon_3) \epsilon_3^2 d\epsilon_3 \\
+ \frac{9}{8} \left( \frac{2v}{3\pi} \right)^3 \int (\frac{2v}{3\pi})^{3/2} 4\pi R_{nl}^* (x)^{(i)\cdot l} e^{15v/32(x^2+y^2)} \]

\[
j_1(-9ivxy/16) \cdot \sigma_4 (3y/4) O_s(y) y^2 x^2 dx dy + \\
4 \int 4\pi \cdot 2^3 (-i)^l R_{nl}^* (4x/3) (\frac{2v}{3\pi})^{3/2} e^{-v/6(9r_1+r_4)} \]

\[
j_1(ivr_3 r_4) \cdot \sigma_1(r_3) O_s(4r_4/3) r_3^2 r_4^2 \, dr_3 dr_4. \]

\[ <0 | \sigma_1 | D+E> = <D+E | \sigma_1 | 0>. \]

Now try \( <D | \sigma_1 | D> + <E | \sigma_1 | D> + <D | \sigma_1 | E> + <E | \sigma_1 | E>. \)

\[ <D | \sigma_4 | D> = \int \phi_{nl}^* (\epsilon_3) O^*(\epsilon_2) O^*(\epsilon_1) \sigma_4 \phi_{nl} (\epsilon_1) O_s(\epsilon_2) O_s(\epsilon_1) d\epsilon_1 d\epsilon_2 d\epsilon_3 \]

\[ = \int \phi_{nl}^* (\epsilon_3) \sigma_4 (3\epsilon_3/4) \phi_{nl} (\epsilon_3) d\epsilon_3. \]
\[
\langle E | \sigma_4 | D \rangle = \int \phi_{nl}^*(\epsilon_3) O_s^*(\epsilon_2) \sigma_4(r_4) \phi_{nl}(\epsilon_3) O_s(\epsilon_2) d\epsilon_2 d\epsilon_3 \\
= \int \phi_{nl}^*(4r_3/3) O_s^*(3r_4/2+r_3/2) \sigma_4(r_4) \phi_{nl}(4r_4/3) O_s(3r_3/2+r_4/2) d\epsilon_2 d\epsilon_3
\]

let \( r_3 = 2\epsilon_2 / 3 - \epsilon_3 / 3 \) \( x = 8\epsilon_2 / 9 - \epsilon_3 / 3 \)  
\( r_4 = 3\epsilon_3 / 4 \) \( y = \epsilon_3 \)

Therefore

\[
\frac{\partial(\epsilon_2, \epsilon_3)}{\partial(x, y)} = \left| \sigma \right|^{-1} = \frac{9}{8}
\]

\[
\langle E | \sigma_4 | D \rangle = \int \phi_{nl}^*(x) \left( \frac{2V}{3\pi} \right)^{3/2} e^{-V/3(3r/2+r/4)^2} \\
\cdot \sigma_4(y) \phi_{nl}(y) O_s(9x/8+y/2) d\epsilon_3 d\epsilon_2.
\]

\[
= \left( \frac{9}{8} \right)^3 \int \phi_{nl}^*(x) \left( \frac{2V}{3\pi} \right)^{3/2} e^{-3V/64(3y^2x)^1} \\
\cdot \sigma_4(3y/4) \phi_{nl}(y) e^{-3V/64(3y^2x)^1} dxdy.
\]

\[
= 4\pi \left( \frac{9}{8} \right)^3 \int \phi_{nl}^*(x) \left( \frac{2V}{3\pi} \right)^{3/2} e^{-15V/32(y^2+x^2)} (i)^1
\]

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\[ \cdot j_1(9ivxy/16) \cdot \sigma_4(3y/4) \phi_{nl}(y) x^2 y^2 dx dy. \]

\[ <D | \sigma_4 | E> = \int \phi_{nl}^*(\epsilon_3) O^*(\epsilon_2) O^*(\epsilon_1) \cdot \]

\[ \sigma_n(x_4) \phi_{nl}(\epsilon_3) O_\delta(\epsilon_2) O_\delta(\epsilon_1) d\epsilon_1 d\epsilon_2 d\epsilon_3 \]

\[ = \left( \frac{9}{8} \right)^3 \frac{1}{4\pi} \int R_{nl}^*(y) \left( \frac{2v}{3\pi} \right)^{3/2} e^{-15v/32(y^4+x^4)} (i)^l \]

\[ j_1(9ivxy/16) \cdot \sigma_4(3y/4) R_{nl}(x) x^2 y^2 dx dy. \]

\[ <E | \sigma_4 | E> = \int \phi_{nl}^*(4x_3/3) O^*(3x_4/2+x_3/2) O^*(\epsilon_1) \cdot \]

\[ \sigma_n(x_4) \phi_{nl}^*(4x_3/3) O^*(3x_4/2+x_3/2) O^*(\epsilon_1) d\epsilon_1 d\epsilon_2 d\epsilon_3 \]

\[ = \left( \frac{9}{8} \right)^3 4\pi \int R_{nl}^*(x) \left( \frac{2v}{3\pi} \right)^{3/2} e^{-15v/32(3y^4+x^4)} (i)^l \]

\[ (-1)^l j_1(9ivxy/16) \cdot \sigma_4(3y/4) \phi_{nl}(x) x^2 y^2 dx dy. \]

Therefore \[ <\phi | \sigma_3+\sigma_4 | R > \phi_{nl}(\epsilon_3) O_\delta(\epsilon_2) O_\delta(\epsilon_1) | S=0 > O_\delta(R) > \]

\[ = - \frac{1}{2} \frac{1}{\sqrt{1+d_i}} \frac{1}{\sqrt{1+d_f}} \frac{j}{1} \left( \int \phi_{nl}^*(\epsilon_3) \sigma_4(3\epsilon_3/4) \phi_{nl}(\epsilon_3) d\epsilon_3 \right) \]

\[ + 4\pi \left( \frac{9}{8} \right)^3 \int R_{nl}^*(x) \left( \frac{2v}{3\pi} \right)^{3/2} e^{-15v/32(y^4+x^4)} (i)^l \]

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\[ j_1\left(\frac{9ivxy}{16}\right) \cdot \sigma_4\left(\frac{3y}{4}\right) \phi_{nl}(y) x^2 y^2 \, dx \, dy. \]

\[ + \frac{9}{8} \left(\frac{9i}{8}\right)^3 \frac{1}{4\pi} \int R_{nl}^*(y) \left(\frac{9i}{8}\right)^{3/2} e^{-15v/32(y^2+x^2)} (i)^l \]

\[ j_1\left(\frac{9ivxy}{16}\right) \cdot \sigma_4\left(\frac{3y}{4}\right) R_{nl}(x) x^2 y^2 \, dx \, dy. \]

\[ + \frac{9}{8} \left(\frac{9i}{8}\right)^3 4\pi \int R_{nl}^*(x) \left(\frac{9i}{8}\right)^{3/2} e^{-15v/32(3y^2+x^2)} (i)^{-l} \]

\[ (-1)^l j_1\left(\frac{9ivxy}{16}\right) \cdot \sigma_4\left(\frac{3y}{4}\right) \phi_{nl}(x) x^2 y^2 \, dx \, dy \].

where

\[ \phi_f = -\frac{1}{\sqrt{2}} \frac{j_1}{l_i} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+d_f}} \{ \text{D+E } \} S=0 \geq O_s(R) \]

\[ \Phi_i = -\frac{1}{\sqrt{2}} \frac{j_1}{l_i} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+d_i}} \{ \text{D+E } \} S=0 \geq O_s(R) \]

and \( l_i = 2l_i + 1 = 1; \quad j_i = 2j_i + 1 = 2 \)

Then for \( <\Phi_f | \sigma_1 + \sigma_2 | \Phi_i> \)

\[ 2 <\Phi_f | \sigma_1 | K_7 \phi_{nl}^* (\epsilon_3) O^0_6 (\epsilon_2) O_5 (\epsilon_1) | S=0 > O_s(R) > \]

\[ <D | \sigma_1 (r_i) | D> \]

\[ = \int \phi_{nl}^* (\epsilon_3) O_5 (\epsilon_2) O_6 (\epsilon_1) j_1 \phi_{nl} (\epsilon_3) O^0_6 (\epsilon_2) O_5 (\epsilon_1) \, d\epsilon_1 d\epsilon_2 d\epsilon_3 \]
\[
\langle E | \sigma_1 | E \rangle = \int \phi_\text{nl}(4x_4/3)^* O^* (3r_3^2+2r_4^2) O^*(\epsilon_1) \cdot \\
\sigma_1(r_3) \phi_\text{nl}(4x_4/3)^* O^* (3r_3^2+2r_4^2) O(\epsilon_1) d\epsilon_1 d\epsilon_2 d\epsilon_3 \\
= \int \phi_\text{nl}(4x_4/3)^* O^* (3r_3^2+2r_4^2) \cdot \\
\sigma_1(r_3) \phi_\text{nl}(4x_4/3)^* O^* (3r_3^2+2r_4^2) d\epsilon_1 d\epsilon_2 d\epsilon_3 \\
\frac{\partial}{\partial (r_3, r_4)} (\epsilon_2, \epsilon_3) = 2 \\
= 2^3 \int \phi_\text{nl}(4x_4/3)^* (\frac{2v}{3\pi})^{3/2} e^{-2v/3(r_3^2+2r_4^2)^{1/2}} \cdot \\
\sigma_1(r_3) \phi_\text{nl}(4x_4/3) dr_3 dr_4. \\
= 4\pi \cdot 2^3 (-i)^{-1} \int R_\text{nl}(4x/3)^* (\frac{2v}{3\pi})^{3/2} e^{-v/6(r_3^2+r_4^2)} \cdot \\
j_l (ivar_3^2) \cdot \sigma_1(r_3) \phi_\text{nl}(4x_4/3) r_3^2 r_4^2 dr_3 dr_4. \\
\text{and } \langle D | \sigma_1+\sigma_2 | D \rangle = 2\langle D | \sigma_1 | D \rangle$

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\[
4\pi \cdot 2^4 (i)^{-1} \int R_{nl}^* \left( \frac{4\pi}{3} \left( \frac{2}{9\pi^2} \right)^{3/2} e^{-\nu/6(9\rho^2 + \rho^2)} \right)^2 \\
\epsilon_1 (i\nu r_3 r_4) \cdot \phi_{n,l} (4\pi/3) r_3^2 r_4^2 dr_3 dr_4.
\]

calculate \( \langle E|\sigma_1|E \rangle \):

\( E \) is symmetric in \( r_1, r_2, r_4 \)

\( \sigma_1(r_1) = \sigma_1(r_2) = \sigma_3(r_4) \).

\[
\langle E|\sigma_1|E \rangle = \int \phi_{n,l}^* (4\pi/3) O^* (3r_4/2 + r_3/2) O^* (\epsilon_1).
\]

\[
\sigma_1(r_4) \phi_{n,l} (4\pi/3) O (3r_4/2 + r_3/2) O (\epsilon_1) d\epsilon_1 d\epsilon_2 d\epsilon_3
\]

\[
= 4\pi \cdot 2^3 (i)^{-1} \int R_{nl}^* \left( \frac{4\pi}{3} \left( \frac{2}{9\pi^2} \right)^{3/2} e^{-\nu/6(9\rho^2 + \rho^2)} \right)^2 \\
(-1)^l j_1 (i\nu r_3 r_4) \cdot \sigma_1(r_3) \phi_{n,l} (4\pi/3) r_3^2 r_4^2 dr_3 dr_4.
\]

Now try \( \langle D|\sigma_1|E \rangle \) \( D \) and \( E \) are all symmetric in \( r_1, r_2 \)

therefore \( \langle D|\sigma_1 + \sigma_2|E \rangle = 2\langle D|\sigma_1|E \rangle \)

let \( x = r_3 + r_4 \)

\( \epsilon_1 = x - z \)

\( y = \epsilon_3 \)

\( \epsilon_2 = 3x/2 - 3y/4 \)

\( z = 2\epsilon_1 \)

\( \epsilon_3 = y \)
Therefore
\[
\frac{\partial (\epsilon)}{\partial (x)} = |J| = \frac{3}{2}
\]

\[
\langle D|\sigma_1|E\rangle = \int \phi_{n_1}^{*}(y) O_{s}^{*}(3x/2-3y/4)[O_{b}(x+z)]^2 \cdot \\
\sigma_1(x) \phi_{n_1}(4x/3-y) O_s(x/2+3y/4) \, dx \, dy \, dz
\]

\[
= \left( \frac{3}{2} \right)^3 \frac{1}{4\pi} \int R_{n_1}^{*}(y) \left( \frac{v^2}{3\pi} \right)^{3/2} e^{-v(3y^2/8+4y^2/3+z^2/2-x\cdot y/2)} \\
\times \left( j_t(ivxz) \cdot j_t(qz/2) \right) R_{n_1}(|4x/3-y|) z^2 \, dx \, dy \, dz.
\]

and \[\langle E|\sigma_1|D\rangle\]

\[
= \left( \frac{3}{2} \right)^3 \frac{1}{4\pi} \int R_{n_1}^{*}(y) \left( \frac{v^2}{3\pi} \right)^{3/2} e^{-v(3y^2/8+4y^2/3+z^2/2-x\cdot y/2)} \\
\times \left( j_t(ivxz) \cdot j_t(qz/2) \right) R_{n_1}^*(|4x/3-y|) z^2 \, dx \, dy \, dz.
\]
BIBLIOGRAPHY

1. T. deForest, Jr. and J.D. Walecka, Advances in Physics, VOL. 15, No. 57, 1 (1966)


17. R. F. Frosch, Nucl. Phys. A110, 657 (1968)
22. Schiff, L.I. Microwave Laboratory, Stanford University Report 102 (1949)
25. Lyman, E.M., Hanson, A.O., and Scott, M.B., Phys. Rev. 84, 626 (1951)
31. Schiff, L.I., Ibid. 96, 765 (1954)

38. Atkinson, R., (1963) (private communication)