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Separable Preference Orders

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SEPARABLE PREFERENCE ORDERS

by

Jonathan K. Hodge

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics

Western Michigan University
Kalamazoo, Michigan
August 2002

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Whenever a decision-maker must express simultaneously his or her preferences on several possibly related issues, the existence of interdependence among these preferences can lead to collective decisions that are unsatisfactory or even paradoxical. Intuitively, an individual's preferences are said to be separable on a subset of issues if they do not depend on the choice of alternatives for issues outside the subset. Here we explore from a mathematical standpoint the properties of separable and nonseparable preference orders.

We begin by formulating a general model of multidimensional preferences and we formally introduce the notions of separability and noninfluentiality. We study the structure of interdependent preferences and explore connections to the previous notions of separability studied in economics.

Next, we examine some of the properties and constructions related to separability. We consider lexicographic and additive orders and use simple tools from set theory to study $k$-majority aggregation of separable preferences. We show that, in contrast to famous results such as Condorcet’s Voting Paradox, the property of separability is preserved by this natural aggregation scheme.

We address the problem of enumerating separable preference orders and introduce the notions of monoseparable, symmetric, preseparable and strongly preseparable preferences. Counting formulas are developed for the latter three of these classes.

Finally, we consider the action of the symmetric group on the set of binary preference orders. We characterize the group of symmetry-preserving permutations and the group of separability-preserving permutations, providing useful insights into the extreme sensitivity to small changes exhibited by separable preferences.
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Acknowledgements

This dissertation is the culmination of an educational journey that would not have been possible without the continual support and encouragement of family, friends, and faculty. To all who have lent a helping hand or a listening ear, I am deeply appreciative.

I especially wish to acknowledge Dr. James Bradley, who introduced me to the joy of mathematics; Dr. Arthur White, who persuaded me to keep pursuing my dreams, even when it wasn't pleasant; and Dr. Allen Schwenk, who was brave enough to venture into unknown territory to help bring my goals to fruition. I could not have asked for a more caring or supportive advisor and I am grateful for all that I have learned from you. Thanks also to the other members of my committee, Dr. Clifton Ealy and Dr. Ping Zhang.

Many thanks to my family for their unfailing love and support. To my parents, for believing in me and selflessly devoting their lives to me. To Jen and Brian, for many good conversations and friendship over the years. To the extended Dauscher clan: it has been wonderful sharing the excitement of the last few years with you. And last, but certainly not least, to Melissa, who has made my life immeasurably rich and has never ceased to love and encourage me. I could not have done any of this without you.

Finally, I would be remiss if I did not acknowledge the Creator of mathematics and of all things for allowing me to take credit for some of His theorems.
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Chapter 1

Introduction

When making decisions that depend on several possibly related criteria, individuals are often required to express simultaneously their preferences on these criteria. A classic example is the referendum election, in which a voter must choose between a yes or no vote for each question or proposal. The predominant method of aggregating votes in such an election is simultaneous voting, in which voters cast ballots for all issues at the same time and votes are counted on an issue-by-issue basis. Thus, under simultaneous voting, a vote of YN in an election on two issues would count as a yes vote for the first issue and a no vote for the second issue. While this method of aggregation seems straightforward, it is not without flaws. To the contrary, whenever even a single voter’s preferences on one issue depend on the outcome of another issue, election results can occur that are at best unsatisfactory and at worst paradoxical (see [3], [4], [16] for examples).

At the heart of this paradoxical behavior lies the concept of separability, which has been studied for many years by economists, political scientists, and mathematicians (Kilgour and Bradley [14] provide a summary of this history). Intuitively, an individual’s preferences are said to be separable on a subset of issues if they do not depend on the choice of alternatives for issues outside the subset. Thus, while preferences among several brands of orange juice may be separable with respect to choices between competing teams in the Superbowl, preferences on two closely related millage proposals in an upcoming election may not be so clear-cut. For example, some voters may be in favor of both proposals but prefer that only one pass, so as not to place an unreasonable financial burden on taxpay-
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ers. Thus, if they knew the outcome of the election on one of the issues beforehand, they would vote the opposite of that outcome on the remaining issue. Unfortunately, because their preferences are nonseparable, they are doomed when voting day comes around. Since simultaneous voting provides no adequate mechanism for expressing interdependent preferences, the votes they cast with the intention of producing a desirable election outcome could just as easily contribute to an outcome which is far from desirable.

To illustrate this phenomenon, consider the following example: Suppose that, in a very democratic family, Mom, Dad, and Ralphie are trying to finalize the details of an upcoming automobile purchase. Due to Ralphie's unfortunate performance in driving school, both of their current vehicles are inoperable. Thus, they need to purchase two new cars, but are having some trouble deciding which two cars to buy. After narrowing the competition down to three choices, they agree to hold a referendum election to make the final decision. Confident in their intent to promote democracy at the most basic levels of government, they meet together on family night and vote on the following three questions:

- **Question 1:** Should we buy the BMW?
- **Question 2:** Should we buy the Ford?
- **Question 3:** Should we buy the Kia?

The ballots are cast without incident, and yet when the votes are tallied, something seems terribly amiss. The outcome of the election is that all three questions pass, each by a margin of 2 votes to 1! Dumbstruck by this strange outcome, but compelled by their commitment to the democratic process, they agree to abide by the result of the election and purchase all three cars. Unfortunately, democracy does not serve them well (at least not in this manifestation). A few months pass, bills begin to accumulate, and Mom and Dad finally realize that, even with Ralphie working overtime at the Burger Hut, they can no longer afford to make the payments on their three new cars. Creditors come knocking at their door, the cars get repossessed, and the happy democratic family finds themselves in a situation that they would have never anticipated! Not only is Dad forced to endure the humiliation of riding to work each day on his son's tricycle, but, to top it all off, the family's credit has been badly damaged. They all agree that it would have been better if
they had bought no cars at all! In fact, anything would have been better than the situation that transpired.

Determined to find out exactly what led to this most unfortunate situation, Mom, Dad, and Ralphie begin to discuss how they voted. Dad is dismayed: he is certain that his once perfect family has become corrupt and that someone must have greedily voted for all three cars. He demands full disclosure of the ballots but is surprised to discover that, contrary to his suspicions, each member of his family voted yes on exactly two of the questions. He voted to buy the BMW and the Ford, his dear wife voted to buy the BMW (for herself) and the Kia (for him), and Ralphie voted to buy the Ford and the Kia (quite cleverly equating lower car payments with a higher allowance). Stupefied, Dad cries out in frustration, “If I had known that you two were going to vote that way, I wouldn’t have voted the way I did! I would have done anything to avoid the mess we’re in now!” And in that very moment, he quite unintentionally stumbles upon a classic paradox of nonseparable preferences and referendum elections. He realizes that the real trouble was not with his family at all, but rather with a subtle, yet fundamental flaw in the referendum election itself. Because his family's preferences on each of the questions in the election were dependent on the outcomes of the other questions, and because they were required to vote on all three questions simultaneously, they had been provided no adequate means for expressing their true desires. As a consequence, the votes they cast with the intention of producing a fair and democratic resolution to their dispute in fact led to the worst possible outcome.

Now we admit that this example is a bit silly and contrived, but it is not too hard to imagine how this same type of voting behavior, and its unfortunate consequences, could arise in a much more serious context. The fact is that nonseparable preferences wreak havoc on referendum elections and the decision-makers who rely on them to democratically solve important real-world problems. In recent years, researchers have begun to pay careful attention to this troublesome issue and have made efforts to better understand nonseparable preferences and their effects on the decision-making process.

Unfortunately, research on alternate voting methods, such as sequential voting (see [14]), has been unable to shed much light on the separability problem. If any conclusion
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has been made, it is that the best kind of preferences to have are those that are separable. Furthermore, Kilgour and Bradley [14] suggest that "one way out of the difficulty [of non-separable preferences] may be to frame questions so as to avoid preference nonseparability." In order to do this, we must first understand the structure of separable and nonseparable preferences both from a theoretical standpoint and as they occur in real-world situations.

Mathematics is a natural setting in which to cultivate understanding. By developing reasonable preference models and studying the mathematical properties of these models, we accomplish two goals. First, we build a theoretical foundation upon which future researchers can better understand the problem of nonseparability and work toward meaningful and applicable solutions. Second, we uncover a surprisingly deep and interesting body of mathematics that is worth studying in its own right.

With that in mind, the goal of this dissertation is to provide a mathematical view of the interesting objects that we call separable preference orders. We do not claim to provide a complete treatment of this emerging area of study. Indeed, there are many questions that remain to be answered, and we have included several ideas for future research both throughout the text in the concluding chapter. We hope that our explorations here will provide a substantive starting point for future investigations.

We begin in Chapter 2 by formulating a general model of multidimensional preferences and we formally introduce the notions of separability and noninfluentiality. We study the structure of interdependent preferences and explore connections to the previous notions of separability studied in economics.

Next, we examine in Chapter 3 some of the properties and constructions related to separability. We consider lexicographic and additive orders and use simple tools from set theory to study k-majority aggregation of separable preferences. We show that, in contrast to famous results such as Condorcet's Voting Paradox, the property of separability is preserved by this natural aggregation scheme.

In Chapter 4, we begin to focus our attention on discrete, and specifically binary, preferences. We consider the problem of enumerating separable preference orders, introducing the notions of monoseparable, symmetric, preseparable and strongly preseparable preferences and exhibit counting formulas for the latter three classes.
Finally, in Chapter 5, we consider the action of the symmetric group on the set of binary preference orders. We characterize the group of symmetry-preserving permutations and the group of separability-preserving permutations, providing useful insights into the extreme sensitivity to small changes exhibited by separable preferences.
Chapter 2

Preference Relations and Separability

2.1 Multidimensional Preferences

We consider the individual preferences associated with a general multistage or multiple-criteria decision making process. We assume that there is a criteria set $Q$ whose elements will be called criteria and upon which we impose no particular structure. For each $q \in Q$, let $X_q$ denote the alternative set for criteria $q$. Then an alternative is an element of the Cartesian product set

$$X_Q = \prod_{q \in Q} X_q.$$ 

For ease of notation, if $x \in X_Q$ and $S \subseteq Q$, then we denote by $x_S$ the projection of $x$ onto $S$, i.e.

$$x_S = (x_q)_{q \in S} \in X_S,$$

where $X_S$ is defined in a manner similar to $X_Q$ above. Furthermore, if $\mathcal{P} = \{S_1, S_2, \ldots, S_n\}$ is a partition of $Q$, then we often write $x = (x_{S_1}, x_{S_2}, \ldots, x_{S_n})$, reordering the criteria if necessary.\(^1\) Specifically, we use the notation $-S$ to denote the complement of $S$ in $Q$ and write $x = (x_S, x_{-S})$.

\(^1\)We allow the parts of $\mathcal{P}$ to be empty, in which case we take the notation $x = (x_{S_1}, x_{S_2}, \ldots, x_{S_n})$ to mean $x = (x_{S_{i_1}}, x_{S_{i_2}}, \ldots, x_{S_{i_m}})$, where $\{i_k : 1 \leq k \leq m\}$ is the set of indices corresponding to the nonempty parts of $\mathcal{P}$. 

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Chapter 2. Preference Relations and Separability

To allow for the most general treatment of preference interdependence, we represent a
decision-maker's preferences by an arbitrary binary relation $\succeq$ on $X_Q$.

2 Historically, it has been common to assume that $\succeq$ is a weak order or a strict partial order (see [8]), though
we presently impose no such restrictions.\footnote{We will occasionally (as in Example 2.2.11) have reason to restrict $\succeq$ and consider only a subset
of $X_Q$, which we will denote $X^*_Q$. In this context, we may replace $X_S$ with $X^*_S$ in all definitions
and theorems, where $X^*_S = \{x_S : x \in X_Q\}$.}

For a relation $\succeq$ on $X_Q$, we let $\succ$ be the relation
declared by $x \succ y \iff x \succeq y$ and $x \neq y$.

Example 2.1.1. Let $Q$ be a finite set representing a number of offices to be filled in simul-
taneous multi-candidate, single-winner elections. Then for each $q \in Q$, $X_q$ is a finite set
representing the candidates for office $q$. In this context, $\succeq$ represents the voter's preference
between election outcomes differing on the choice of candidates for one or more offices.

Example 2.1.2. Let $Q = \{\text{flour, sugar, butter}\}$ and let $X_q$ represent the amounts of in-
gredient $q$ that a baker may put into his cookie recipe. Then each $X_q$ is an interval of
real numbers and $\succeq$ represents the taster's preference between cookies made with differing
combinations of ingredients.

2.2 Separability

Intuitively, a decision-maker's preferences on a subset $S$ of the criteria set are said to be
separable if they do not depend on the choice of alternatives for criteria outside of $S$.

Another way of expressing this idea is to say that the decision-maker's preferences on $S$
are invariant with respect to a change of alternatives on the criteria in $-S$. Formally,

Definition 2.2.1. Let $S \subseteq Q$. Then $S$ is said to be $\succeq$-separable, or separable with respect to
$\succeq$, if whenever two elements $x_S, y_S \in X_S$ have the property that $(x_S, u_S) \succeq (y_S, u_S)$
for some $u_S \in X_S$, then $(x_S, v_S) \succeq (y_S, v_S)$ for all $v_S \in X_S$. A criterion $q \in Q$ is
said to be $\succeq$-separable if $\{q\}$ is $\succeq$-separable. The relation $\succeq$ is said to be separable if each
nonempty $S \subseteq Q$ is $\succeq$-separable.

In addition to Definition 2.2.1, we adopt the convention that both $Q$ and $\emptyset$ are always

\footnote{A relation $\succeq$ on $X_Q$ is a weak order if it is transitive, reflexive and has the property that for
each $x, y \in X_Q$, either $x \succeq y$ or $y \succeq x$. It is a strict partial order if and only if it is transitive and
irreflexive ($x \not\succeq x$ for each $x \in X_Q$). See [15] for details.}

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Chapter 2. Preference Relations and Separability

\(\succeq\)-separable. We also assume, unless otherwise indicated, that all subsets under consideration are nonempty and proper.

Our definition of separability is based on the definition given by Kilgour and Bradley in [14], which is originally due to Yu [21]. Kilgour and Bradley also note that the first study of separability within the context of referendum elections seems to be a 1977 paper by Schwartz [19]. According to economist W.M. Gorman, the earliest notions of separability can be traced back to Leontief in 1948.

We should note that, until 1998, research on separable preferences in referendum elections focused exclusively on the separability of individual criteria. Hodge and Bradley [12] seem to have been the first to apply Yu's more general definition to this particular setting. To distinguish the single-criterion formulation of separability from the general version above, we make the following definition:

**Definition 2.2.2.** A relation \(\geq\) on \(X_Q\) is said to be monoseparable if each \(q \in Q\) is \(\succeq\)-separable.

Yet another way of expressing the separability of a subset \(S\) with respect to \(\geq\) is to say that the relation on \(X_S\) induced by \(\geq\) is well-defined; that is, it does not depend on the choice of alternatives for criteria in \(-S\). Formally, for each \(S \subseteq Q\), let \(\geq_s\) denote the relation on \(X_S\) defined by

\[
x_S \geq_s y_S \iff (x_S, u_{-S}) \geq (y_S, u_{-S}) \quad \text{for all } u_{-S} \in X_{-S}.
\]

Similarly, let \(\nless\) denote the relation defined by

\[
x_S \less y_S \iff (x_S, u_{-S}) \less (y_S, u_{-S}) \quad \text{for all } u_{-S} \in X_{-S}.
\]

It is important to note that \(\less\) does not denote the negation of \(\geq_s\). Rather, it denotes the relation induced on \(X_S\) by \(\less\). Rephrasing Definition 2.2.1, we see that if \(S\) is separable and \((x_S, u_{-S}) \geq (y_S, u_{-S})\) for some \(x_S, y_S \in X_S\) and some \(u_{-S} \in X_{-S}\), then it must be the case that \(x_S \geq_s y_S\). Thus,

**Proposition 2.2.3.** \(S \subseteq Q\) is separable if and only if, for each \(x_S, y_S \in X_S\), either \(x_S \geq_s y_S\) or \(x_S \less y_S\).

\footnote{For ease of notation, we occasionally omit the subscript \(S\) on the induced relation \(\geq_s\), relying on the context to clarify any ambiguity.}
Chapter 2. Preference Relations and Separability

Proof. Suppose $S$ is separable and let $x_S, y_S \in X_S$ be given. If it is not the case that $x_S \not\succeq_S y_S$, then there exists $u_S \in X_S$ such that $(x_S, u_S) \succeq (y_S, u_S)$. But then, by our above comments, it follows that $x_S \succeq_S y_S$. The converse follows immediately from Definition 2.2.1. □

Corollary 2.2.4. $S \subseteq Q$ is $\succeq$-separable if and only if $S$ is $\not\preceq$-separable.

Proof. This follows immediately by noticing that $\succeq$ and $\not\preceq$ are interchangeable in the statement of Proposition 2.2.3. □

Corollary 2.2.5. $\succeq$ is separable if and only if $\not\preceq$ is separable.

Proposition 2.2.6. Let $\succeq$ be a relation on $X_Q$ and let $S \subseteq Q$ be $\succeq$-separable. If $T \subseteq S$ is $\succeq$-separable, then $T$ is $\succeq_S$-separable and $(\succeq_T)_T = \succeq_T$.

Proof. Let $T \subseteq S$ and suppose that $(x_T, u_{S-T}) \succeq_S (y_T, u_{S-T})$ for some $x_T, y_T \in X_T$ and some $u_{S-T} \in X_{S-T}$. Let $v_{S-T} \in X_{S-T}$ be given. Then by the definition of $\succeq_S$, $(x_T, u_{S-T}, w_S) \succeq_S (y_T, u_{S-T}, w_S)$ for all $w_S \in X_S$. But since $T$ is $\succeq$-separable, it follows that $(x_T, v_{S-T}, w_S) \succeq_S (y_T, v_{S-T}, w_S)$ for all $w_S \in X_S$. Thus $(x_T, v_{S-T}) \succeq_S (y_T, v_{S-T})$ and, consequently, $T$ is $\succeq_S$-separable. To show $(\succeq_T)_T = \succeq_T$, we observe that

$$x_T \succeq_T y_T \iff (x_T, u_{S-T}) \succeq_S (y_T, u_{S-T}) \text{ for all } u_{S-T} \in X_{S-T}$$

$$\iff (x_T, u_{S-T}, v_{S-T}) \succeq_S (y_T, u_{S-T}, v_{S-T}) \text{ for all } u_{S-T} \in X_{S-T}, v_{S-T} \in X_S$$

$$\iff (x_T, w_{S-T}) \succeq_S (y_T, w_{S-T}) \text{ for all } w_{S-T} \in X_{S-T}$$

$$\iff x_T \succeq_T y_T.$$

Corollary 2.2.7. Let $\succeq$ be a separable relation on $X_Q$ and let $S \subseteq Q$. Then $\succeq_S$ is a separable relation on $X_S$.

Proof. This is an immediate consequence of Proposition 2.2.6. □

Proposition 2.2.8. Equality is a separable relation on any alternative set.

Proof. Let $S \subseteq Q$ be given and suppose that $(x_S, u_{-S}) = (y_S, u_{-S})$ for some $x_S, y_S \in X_S$ and some $u_{-S} \in X_{-S}$. Then $x_S = y_S$ and so $(x_S, v_{-S}) = (y_S, v_{-S})$ for all $v_{-S} \in X_{-S}$. Since our choice of $S$ was arbitrary, it follows that $=$ is separable. □

Corollary 2.2.9. Inequality is a separable relation on any alternative space.

Proof. This follows directly from Corollary 2.2.5 and Proposition 2.2.8. □
Chapter 2. Preference Relations and Separability

Proposition 2.2.10. A separable relation $\succeq$ on $X_Q$ is either reflexive or irreflexive.

Proof. Let $x, y \in X_Q$ and suppose that $x \succeq x$. Now choose any subset $S$ of $Q$. Since $(x_S, x_{-S}) \succeq (x_S, y_{-S})$ and $S$ is $\succeq$-separable, it follows that $(x_S, y_{-S}) \succeq (x_S, y_{-S})$. But since $-S$ is also $\succeq$-separable, it follows that $(y_S, y_{-S}) \succeq (y_S, y_{-S})$. We have thus shown that if $x \succeq x$ for some $x \in X_Q$, then $y \succeq y$ for all $y \in X_Q$, which establishes our claim. □

Example 2.2.11. Let $Q = \mathbb{R}$ (the real line) and for each $q \in Q$, let $X_q = \mathbb{R}$. Then $X_Q$ is the set of all real-valued functions of a single real variable. Let $X_Q^*$ be the subset of all Lebesgue integrable functions and let $\succeq$ be the weak order on $X_Q^*$ specified by

$$f \succeq g \iff \int_{\mathbb{R}} f \geq \int_{\mathbb{R}} g$$

Now let $S \subseteq \mathbb{R}$ be measurable and let $F = (f_S, h_{-S}), G = (g_S, h_{-S})$ for some $f_S, g_S \in X_S^*$ and some $h_{-S} \in X_{-S}^*$. Suppose also that $F \succeq G$. Then

$$\int_S f_S + \int_{-S} h_{-S} = \int_{\mathbb{R}} F \geq \int_{\mathbb{R}} G = \int_S g_S + \int_{-S} h_{-S}$$

which implies that

$$\int_S f_S \geq \int_S g_S$$

from which it follows that

$$\int_S f_S + \int_{-S} k_{-S} \geq \int_S g_S + \int_{-S} k_{-S}$$

for every $k_{-S} \in X_{-S}^*$. Thus, $S$ is separable. Since our choice of $S$ was arbitrary, we see that every measurable subset of the real numbers is separable with respect to the weak order on $X_Q^*$ induced by the Lebesgue integral.

Example 2.2.12. Let $Q = \{1, 2, \ldots, n\}$ and let $X_i = \{0, 1\}$ for each $1 \leq i \leq n$. Then a relation $\succeq$ on $X_Q$ may be interpreted as an ordering of the $2^n$ possible outcomes of a referendum election on $n$ questions, where a 1 in the $i^{th}$ component of an alternative indicates passage of the $i^{th}$ issue and a 0 indicates failure. Specifically, let $n = 3$ and let $\succeq$ be the linear order on $X_Q$ specified by the binary preference matrix

$$R = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix}$$

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where $x \succ y$ if and only if the row of $R$ corresponding to $x$ appears higher than the row corresponding to $y$. It is easy to see that $\{3\}$ is separable with induced ordering $1 \succ 0$. In contrast, $\{1\}$ is nonseparable since $101 \succ 011$ but $011 \succ 111$. Similarly, $\{2\}$ is nonseparable since $101 \succ 111$ but $011 \succ 001$. Nevertheless, $\{1,2\}$ is separable with induced ordering $10 \succ 01 \succ 00 \succ 11$. Thus, we see that it is possible for a set to be separable even if none of its proper subsets is.

### 2.3 Noninfluentiality

At this point, we hope that the reader has a decent intuitive understanding of the idea of separability. Specifically, we hope that the reader sees that if a subset $S$ of $Q$ is nonseparable with respect to some decision-maker’s preference, then the outcome of the decision-making process on criteria outside of $S$ exerts some sort of influence on the decision-maker’s preference on $S$. In this situation, it seems logical to ask: *On which criteria do the decision-maker’s preferences on $S$ depend?*

We answer this question by identifying the subsets of $Q$ on which the decision-maker’s preferences on $S$ are *not* dependent, i.e. those subsets for which a change of alternatives would not influence the decision-maker’s preferences on $S$. Formally,

**Definition 2.3.1.** Let $S$ and $T$ be disjoint subsets of $Q$. Then $T$ is said to be $\succeq$-noninfluential on $S$, or *noninfluential on $S$ with respect to* $\succeq$, if whenever two elements $x_S, y_S \in X_S$ have the property that $(x_S, u_T, w_{-\langle S \cup T \rangle}) \succeq (y_S, u_T, w_{-\langle S \cup T \rangle})$ for some $(u_T, w_{-\langle S \cup T \rangle}) \in X_S$, then $(x_S, u_T, w_{-\langle S \cup T \rangle}) \succeq (y_S, u_T, w_{-\langle S \cup T \rangle})$ for all $u_T \in X_T$. If $T$ is $\succeq$-noninfluential on $S$ for each nonempty $S \subseteq Q$ with $S \cap T = \emptyset$, then $T$ is said to be $\succeq$-noninfluential, or *noninfluential with respect to* $\succeq$. A criterion $q \in Q$ is said to be $\succeq$-noninfluential if $\{q\}$ is $\succeq$-noninfluential.

As in the definition of separability, we adopt the convention that both $Q$ and $\emptyset$ are always $\succeq$-noninfluential.

**Example 2.3.2.** Consider again the preference matrix from Example 2.2.12. Notice that $\{2\}$ is influential on $\{1\}$ since the induced preference on $\{1\}$ depends on the choice of alternatives on $\{2\}$ (specifically, $1 \succ 0$ given a choice of 0 on $\{2\}$ and $0 \succ 1$ given a choice of 1 on $\{2\}$). Nevertheless, $\{3\}$ is noninfluential on $\{1\}$ since this induced preference is independent of the choice of alternatives on $\{3\}$. Similarly, $\{3\}$ is noninfluential on $\{2\}$ and.
on \{1,2\}, and so it follows that \{3\} is noninfluential. We can show in the same manner that \{1,2\} is noninfluential.

**Lemma 2.3.3.** Let \( S \) and \( T \) be disjoint subsets of \( Q \).

(i) If \( T \) is noninfluential on \( S \), then \( T \) is noninfluential on each \( S' \subseteq S \).

(ii) \( T \) is noninfluential if and only if \( T \) is noninfluential on \(-T\).

**Proof.** (i) Suppose that \( T \) is noninfluential on \( S \) and let \( S' \subseteq S \) be given. Let \( x_{S'}, y_{S'} \in X_{S'} \) and \((w_{S-S'}, u_T, w_{-(S \cup T)}) \in X_{-S'} \) be such that

\[
(x_{S'}, w_{S-S'}, u_T, w_{-(S \cup T)}) \geq (y_{S'}, w_{S-S'}, u_T, w_{-(S \cup T)})
\]

and let \( v_T \in X_T \) be given. Since \( T \) is noninfluential on \( S \) and \((x_{S'}, w_{S-S'}, y_{S'}, w_{S-S'}) \in X_S \), it follows that

\[
(x_{S'}, w_{S-S'}, v_T, w_{-(S \cup T)}) \geq (y_{S'}, w_{S-S'}, v_T, w_{-(S \cup T)}).
\]

Since our choice of \( v_T \) was arbitrary, we have established that \( T \) is noninfluential on \( S' \).

(ii) The forward implication follows immediately from Definition 2.3.1. For the converse, suppose that \( T \) is noninfluential on \(-T\) and let \( S \subseteq Q \) be disjoint from \( T \). Then \( S \subseteq -T \), and so (i) implies that \( T \) is noninfluential on \( S \). Since our choice of \( S \) was arbitrary, it follows that \( T \) is noninfluential. \( \square \)

**Proposition 2.3.4.** Let \( S \subseteq Q \). Then \( S \) is separable if and only if \(-S\) is noninfluential.

**Proof.** By Lemma 2.3.3 above, it suffices to show that \( S \) is separable if and only if \(-S\) is noninfluential on \( S \). This follows immediately from Definitions 2.2.1 and 2.3.1. \( \square \)

### 2.4 Set Theoretic Properties

In the past, economists such as W.M. Gorman have studied the notion of separability from within the context of utility theory. A question that was of interest to them, and will be of interest to us, is that of whether the property of separability is preserved by certain set operations. In order to consider previous results pertinent to this question, we must first introduce some terminology.

Let \( S, T \subseteq Q \). Then we say that \( S \) and \( T \) overlap if all of \( S \cap T, S - T, \) and \( T - S \) are nonempty. \( S \) is said to be essential with respect to \( \succeq \) if there exists \( u_{-S} \in X_{-S} \) and \( x_S \),
Chapter 2. Preference Relations and Separability

ys ∈ Xs such that (xs, u–s) ≥ (ys, u–s). S is said to be strictly essential with respect to ≥ if, for all u–s ∈ X–s, there exist xs, ys ∈ Xs such that (xs, u–s) ≥ (ys, u–s). We denote by S ∆ T the symmetric difference of S and T, defined precisely as S ∆ T = (S − T) ∪ (T − S).

Finally, we denote by S ≥ the collection of all ≥-separable subsets of Q and by N ≥ the collection of all ≥-noninfluential subsets of Q.

The following theorem is due to Gorman [10].

**Theorem 2.4.1.** Suppose ≥ is a continuous5 weak order on XQ, and that for each q ∈ Q, Xq is topologically separable and arc-connected. Suppose further that S, T ⊆ Q overlap and that either S − T or T − S is strictly essential. If S and T are ≥-separable, then S ∪ T, S ∩ T, S − T, T − S, and S ∆ T are all ≥-separable and strictly essential.

Kilgour and Bradley [14] demonstrate that when the alternative sets are not arc-connected (as is the case in a referendum election), Gorman’s Theorem may fail. Specifically, they provide an example in which both S and T are separable and yet none of S ∪ T, S − T, T − S, or S ∆ T is separable. It seems that the only portion of Gorman’s Theorem that holds in general is the following:

**Proposition 2.4.2.** S ≥ is closed under finite intersections.

**Proof.** Suppose S1, S2 ∈ S ≥, and let S = S1 ∩ S2, T1 = S1 − S, T2 = S2 − S, and 

\[ T = -(S1 ∪ S2). \]

Suppose also that (xS, uT1, uT2, uT) ≥ (yS, uT1, uT2, uT) for some u ∈ X−S. Choose v ∈ X−S. Since S1 is separable and (vT1, uT) ∈ X−S1, it follows that 

\[ (xS, uT1, vT2, uT) ≥ (yS, uT1, vT2, uT). \]

But since S2 is separable and (uT1, vT) ∈ X−S2, we have that 

\[ (xS, vT1, vT2, vT) ≥ (yS, vT1, vT2, vT). \]

Because our choice of v was arbitrary, it follows that S = S1 ∩ S2 is separable. Thus, S ≥ is closed under the binary operation of set intersection. An easy induction argument then shows that S ≥ is closed under all finite intersections.

**Corollary 2.4.3.** N ≥ is closed under finite unions.

**Proof.** Let S1, S2, ..., Sn ∈ N ≥. Then by Proposition 2.3.4, −S1, −S2, ..., −Sn ∈ S ≥. But by De’Morgan’s Laws and Proposition 2.4.2, it follows that 

\[ -(S1 ∪ S2 ∪ ... ∪ Sn) = -S1 ∩ -S2 ∩ ... ∩ -Sn ∈ S ≥. \]

Thus, by Proposition 2.3.4, S1 ∪ S2 ∪ ... ∪ Sn ∈ N ≥.

---

5≥ is continuous if the sets \{y ∈ XQ : x ≥ y\} and \{y ∈ XQ : y ≥ x\} are closed for each x ∈ XQ.
Given Proposition 2.4.2 and Corollary 2.4.3, \( S_\geq \) and \( N_\geq \) can be viewed as monoids\(^6\) under the binary operations of set intersection and set union, respectively. Consequently,

**Proposition 2.4.4.** The map \( \varphi : S_\geq \to N_\geq \) defined by \( \varphi(S) = -S \) is a monoid isomorphism. In particular, \( S_\geq \cong N_\geq \).

**Proof.** Proposition 2.3.4 establishes that \( \varphi \) is well-defined. By De'Morgan's Laws,

\[
\varphi(S_1 \cap S_2) = -(S_1 \cap S_2) = -S_1 \cup -S_2 = \varphi(S_1) \cup \varphi(S_2),
\]

and so \( \varphi \) is a homomorphism. Now \( \varphi \) is surjective since \( \varphi(-S) = S \) for all \( S \subseteq Q \). Furthermore, \( \varphi \) is injective since \( \varphi(S_1) = \varphi(S_2) \iff -S_1 = -S_2 \iff S_1 = S_2. \) Thus, \( \varphi \) is an isomorphism, as desired. \( \square \)

We close this section by mentioning two open questions that are closely related to the above results. The first concerns the extent to which Gorman's Theorem may fail and the second deals with a potential generalization of Proposition 2.4.2.

**Open Question.** (due to Allen Schwenk) Let \( X_Q \) be finite and let \( S \) be a subset of \( \mathcal{P}(Q) \) that contains both \( Q \) and \( \emptyset \) and that is closed under finite intersections. Does there exist a relation \( \succeq \) on \( X_Q \) such that \( A \subseteq Q \) is separable if and only if \( A \subseteq S \)?

**Open Question.** Is \( S_\geq \) closed under arbitrary (i.e. infinite) intersections?

### 2.5 An Application to Referendum Elections

**Proposition 2.5.1.** Suppose \( Q \) is finite. Then \( \succeq \) is separable if and only if each \( q \in Q \) is noninfluential.

**Proof.** Suppose that \( \succeq \) is separable. Then for each \( q \in Q \), \(-\{q\}\) is separable. Thus, by Proposition 2.3.4, \( q \) is noninfluential. For the converse, suppose that each \( q \in Q \) is noninfluential and let \( S \subseteq Q \) be given. Then

\[
-S = \bigcup_{q \notin S} \{q\}.
\]

Since each \( \{q\} \) is noninfluential and the union is taken over a finite number of sets, Corollary 2.4.3 implies that \(-S\) is noninfluential. Thus, by Proposition 2.3.4, \( S \) is separable. Since our choice of \( S \) was arbitrary, it follows that \( \succeq \) is separable. \( \square \)

\(^6\)A monoid is a set \( S \) together with an associative binary operation \( * \) such that \( S \) contains an element \( e \), called an identity, for which \( a * e = e * a = a \) for all \( a \in S \).
Proposition 2.5.1 provides a mechanism for identifying whether or not a voter’s preferences are separable. This method could be implemented through a simple telephone survey by asking questions such as: “If you knew beforehand the outcome of the election on proposal X, would that affect your preferences on the outcome of one or more of the other proposals in the election?” If a respondent answers no to all such questions (one for each proposal in the referendum), then we can state conclusively that his or her preferences are separable. If a respondent answers yes to one of the questions, then follow-up questions could be asked to help pinpoint the occurrence of nonseparability and influentiality. We note that, for a referendum election on \( n \) proposals, the method of Proposition 2.5.1 requires only \( n \) questions to be asked of the voter. The responses to these \( n \) questions can provide information about all \( 2^n \) subsets of \( Q \).
Chapter 3

Properties Related To Separability

3.1 Basic Definitions and Properties

We pause now to formalize some of the notation introduced in the previous chapter.

**Definition 3.1.1.** Let $X_Q$ be a countable alternative space and let $\succ$ be a linear order on $X_Q$. The sequence $(x_k)$ of elements of $X_Q$ defined by

$$x_i \succ x_j \iff i < j$$

is called the *order sequence* corresponding to $\succ$. Equivalently, $\succ$ is said to be the linear order corresponding to $(x_k)$. The element $x_1$ is said the be the *leader* of $\succ$.

In certain settings, we are primarily interested in the case where preferences are binary in nature, i.e. where $X_q = \{0, 1\}$ for each $q \in Q$. We distinguish this context from other more general settings by saying that the alternative space $X_Q$ is *binary*. A specific case that is of particular interest to us is when $Q$ is finite (say $|Q| = n$), $X_Q$ is binary, and preference is represented by a linear order $\succ$ on $X_Q$ (this is the model historically used in the context of referendum elections, as in [2], [3], [4], [12], [14], [16]). In this case, we say that $\succ$ is a *binary preference order* on $Q$. The binary preference order $\succ$ is said to be *normalized* if its leader is $(1, 1, \ldots, 1)$ and if $(1, 0, 0, \ldots, 0) \succ (0, 1, 0, \ldots, 0) \succ \ldots \succ (0, 0, \ldots, 0, 1)$.

We will often use the binary preference matrix model of Example 2.2.12, defined formally below, as a convenient way of representing binary preference orders. It is clear that the natural correspondence between binary preference orders and binary preference matrices is bijective.
Definition 3.1.2. Let $Q = \{1, 2, \ldots, n\}$, let $X_Q$ be binary, and let $\succ$ be a linear order on $X_Q$ with order sequence $(x_k)$. Then the $2^n \times n$ matrix $(a_{i,j})$ having the $a_{i,j}$ entry equal to the $j^{th}$ component of $x_i$, i.e.

$$a_{i,j} = [x_i]_j,$$

is called the binary preference matrix corresponding to $\succ$. Equivalently, $\succ$ is said to be the linear order corresponding to $(a_{i,j})$.

The binary preference matrix $(a_{i,j})$ is said to be normalized if its corresponding linear order is normalized. It is clear that every binary preference order can be obtained from a normalized binary preference order by reordering the criteria and replacing a subset of the components of each alternative with their bitwise complements. Thus, every binary preference matrix can be obtained from a normalized binary preference matrix by permuting and/or taking bitwise complements of the columns.

The following result is due to Bradley and Kilgour [2].

Proposition 3.1.3.

(i) If $|Q| = 2$, then there is exactly 1 separable normalized binary preference order on $Q$.

(ii) If $|Q| = 3$, then there are exactly 2 separable normalized binary preference orders on $Q$.

(iii) If $|Q| = 4$, then there are exactly 14 separable normalized binary preference orders on $Q$.

The binary preference matrices corresponding to the orders in Proposition 3.1.3 are listed below.

| $|Q| = 2$: |
|---|
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3.2 Unions, Intersections, and Aggregation

Definition 3.2.1. Let \( \mathcal{R} = \{ \succeq_\alpha \mid \alpha \in A \} \) be a collection of relations on \( X_Q \) for some index set \( A \).

(i) The union of the relations in \( \mathcal{R} \), denoted \( \bigcup_{\alpha \in A} \succeq_\alpha \), is the relation \( \succeq \) defined by

\[
x \succeq y \iff x \succeq_\alpha y \text{ for some } \alpha \in A.
\]

(ii) The intersection of the relations in \( \mathcal{R} \), denoted \( \bigcap_{\alpha \in A} \succeq_\alpha \), is the relation \( \succeq' \) defined by

\[
x \succeq' y \iff x \succeq_\alpha y \text{ for all } \alpha \in A.
\]

Proposition 3.2.2. The union of any collection of separable relations on \( X_Q \) is separable.
Proof. Let $\mathcal{R} = \{\succeq_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of separable relations on $X_Q$ and let $\succeq = \bigcup_{\alpha \in \mathcal{A}} \succeq_\alpha$. Now let $S \subseteq Q$ be given and suppose that $(x_S, u_S) \succeq (y_S, u_S)$ for some $x_S, y_S \in X_S$ and some $u_S \in X_{-S}$. Then it must be the case that, for some $\alpha \in \mathcal{A}$, $(x_S, u_S) \succeq_\alpha (y_S, u_S)$. But then the separability of $\succeq_\alpha$ implies that $(x_S, v_S) \succeq_\alpha (y_S, v_S)$ for all $v_S \in X_{-S}$, which certainly implies that $(x_S, v_S) \succeq (y_S, v_S)$ for all $v_S \in X_{-S}$. Since our choice of $S$ was arbitrary, it follows that $\succeq = \bigcup_{\alpha \in \mathcal{A}} \succeq_\alpha$ is separable. $\square$

Corollary 3.2.3. The intersection of any collection of separable relations on $X_Q$ is separable.

Proof. This follows from Corollary 2.2.5 and Proposition 3.2.2 by observing that $\bigcap_{\alpha \in \mathcal{A}} \succeq_\alpha$ is the negation of $\bigcup_{\alpha \in \mathcal{A}} \not\succeq_\alpha$. $\square$

Corollary 3.2.4. If $\succeq$ is separable, then so is $\succ$.

Proof. This follows immediately from Corollary 3.2.3 and Corollary 2.2.9 since $\succ$ is the intersection of $\succeq$ and $\not\succeq$. $\square$

We conclude this section by applying the above results to the problem of aggregating separable preferences. Suppose that $n$ decision-makers are attempting to combine their individual preferences into a collective preference relation that somehow represents their overall views on the issues at hand. A natural method of accomplishing this task would be to choose some cutoff value, say $k$ (where $k$ is an integer with $1 \leq k \leq n$), and then construct the collective preference relation so that the group prefers $x$ to $y$ if and only if at least $k$ of its members prefer $x$ to $y$.\footnote{In the recent literature, this aggregation scheme has been referred to as \textit{optimal majority} aggregation or \textit{k-majority} aggregation. We prefer the latter, as it more specifically describes the method by which the collective preference relation is obtained. Also, optimal majority aggregation assumes that the value chosen for $k$ is somehow optimal with regard to the decision being made, a restriction that is not necessary for our investigations. See [11] or [17] for more details.} Note that, if $k = \lceil n/2 \rceil + 1$, this method corresponds to standard majority aggregation. If $k = n$, then each member can singlehandedly block a preference between any two alternatives. Similarly, if $k = 1$, then each member can singlehandedly force a preference between any two alternatives. Of course, in this situation, another member could also force the opposite of this preference, in which case the method would be indecisive.

Condorcet demonstrated in 1785 that this seemingly natural method of aggregating preferences can lead to some rather unexpected and paradoxical outcomes. Specifically, he...
showed that even when all individual preferences are transitive, the aggregate preference relation obtained by majority rule may be intransitive. This observation became known as Condorcet's paradox and is a standard example of the difficulties inherent in preference aggregation. Fortunately, and somewhat surprisingly, this undesirable behavior does not occur when we consider the effect of aggregation on the property of separability. In fact, Corollary 3.2.5 below shows that, under the aggregation scheme described above, separability of individual preferences is preserved in the resulting collective preference relation.

**Corollary 3.2.5.** Let \( R = \{\succeq_1, \succeq_2, \ldots, \succeq_n\} \) be a collection of separable relations on \( X_Q \) and, for each positive integer \( k \leq n \), let \( \succeq_{(k)} \) be the relation defined by

\[
x \succeq_{(k)} y \iff x \succeq_i y \text{ for at least } k \text{ of the } \succeq_i \in R.
\]

Then \( \succeq_{(k)} \) is separable.

**Proof.** The result follows from Proposition 3.2.2 and Corollary 3.2.3 by observing that

\[
\succeq_{(k)} = \bigcup_{S \subseteq R, |S| = k} \left( \bigcap_{\succeq_i \in S} \succeq_i \right).
\]

\[\square\]

### 3.3 Lexicographic Sums

**Definition 3.3.1.** Let \( Q_1 \) and \( Q_2 \) be disjoint criteria sets and let \( \succeq_1, \succeq_2 \) be relations on \( X_{Q_1} \) and \( X_{Q_2} \), respectively. The lexicographic sum of \( \succeq_1 \) and \( \succeq_2 \) is the relation \( \succeq_1 \oplus \succeq_2 \) on \( X_{Q_1 \cup Q_2} \) defined by

\[
(x_{Q_1}, x_{Q_2}) (\succeq_1 \oplus \succeq_2) (y_{Q_1}, y_{Q_2}) \iff x_{Q_1} \succeq_1 y_{Q_1} \text{ or } (x_{Q_1} = y_{Q_1} \text{ and } x_{Q_2} \succeq_2 y_{Q_2}).
\]

**Proposition 3.3.2.** If \( \succeq_1 \) and \( \succeq_2 \) are separable relations on \( X_{Q_1} \) and \( X_{Q_2} \), respectively, then \( \succeq_1 \oplus \succeq_2 \) is a separable relation on \( X_{Q_1 \cup Q_2} \).

**Proof.** Let \( \succeq = \succeq_1 \oplus \succeq_2 \) and let \( S \subseteq Q_1 \cup Q_2 \) be given. Define \( S_1 = Q_1 \cap S \) and \( S_2 = Q_2 \cap S \). Then \( S = S_1 \cup S_2 \) and \( -S = (Q_1 - S_1) \cup (Q_2 - S_2) \). Now suppose that

\[
(x_{S_1}, x_{S_2}, u_{Q_1 - S_1}, u_{Q_2 - S_2}) \succeq (y_{S_1}, y_{S_2}, u_{Q_1 - S_1}, u_{Q_2 - S_2})
\]

for some \( x, y \in X_S \) and some \( u \in X_{-S} \). Let \( v \in X_{-S} \). We must show that

\[
(x_{S_1}, x_{S_2}, u_{Q_1 - S_1}, u_{Q_2 - S_2}) \succeq (y_{S_1}, y_{S_2}, u_{Q_1 - S_1}, u_{Q_2 - S_2}).
\]

(\textit{f})

By Definition 3.3.1 above, one of the following must occur:
Chapter 3. Properties Related To Separability

(i) \((x_{S_1}, u_{Q_1-S_1}) \succ_1 (y_{S_1}, u_{Q_1-S_1})\), in which case the separability of \(\succ_1\) implies that \((x_{S_1}, u_{Q_1-S_1}) \succ_1 (y_{S_1}, u_{Q_1-S_1})\), which then implies \(\dagger\); or

(ii) \((x_{S_1}, u_{Q_1-S_1}) = (y_{S_1}, u_{Q_1-S_1})\) and \((x_{S_2}, u_{Q_2-S_2}) \succ_2 (y_{S_2}, u_{Q_2-S_2})\), in which case \(x_{S_1} = y_{S_1}\), and so \((x_{S_1}, v_{(Q_1-S_1)}) = (y_{S_1}, v_{(Q_1-S_1)})\). The separability of \(\succ_2\) then implies that \((x_{S_2}, u_{Q_2-S_2}) \succ_2 (y_{S_2}, u_{Q_2-S_2})\), which implies \(\dagger\).

Since each case implies \(\dagger\), we have shown that \(S\) is \(\succ\)-separable. Since our choice of \(S\) was arbitrary, it follows that \(X\) is separable. \(\square\)

A partial converse follows:

**Proposition 3.3.3.** Let \(\succ_1\) and \(\succ_2\) be relations on \(X_{Q_1}\) and \(X_{Q_2}\), respectively. If \(\succ_1 \oplus \succ_2\) is separable, then both \(\succ_1\) and \(\succ_2\) are separable.

**Proof.** Let \(\succ = \succ_1 \oplus \succ_2\). By Corollaries 2.2.7 and 3.2.4, it suffices to show that \(\succ_1 = \succ_{Q_1}\) and that \(\succ_2 = \succ_{Q_2}\). For the former, notice that

\[
x \succ_1 y \iff (x, u) \succ (y, u) \text{ for all } u \in X_{Q_2} \iff x \succ_{Q_1} y.
\]

Similarly, for the latter we have

\[
x \succ_2 y \iff (u, x) \succ (u, y) \text{ for all } u \in X_{Q_1} \iff x \succ_{Q_2} y.
\]

\(\square\)

**Proposition 3.3.4.** The lexicographic sum is an associative binary operation. Specifically, let \(Q_1, Q_2, Q_3\) be disjoint criteria sets and let \(\succ_i\) be a relation on \(X_{Q_i}\) for \(i = 1, 2, 3\). Then

\[
(\succ_1 \oplus \succ_2) \oplus \succ_3 = \succ_1 \oplus (\succ_2 \oplus \succ_3).
\]

**Proof.** Let \(Q = Q_1 \cup Q_2 \cup Q_3\) and let \((x_1, x_2, x_3), (y_1, y_2, y_3) \in X_Q\), where \(x_i, y_i \in X_{Q_i}\) for \(i = 1, 2, 3\). It is straightforward to verify that both \((\succ_1 \oplus \succ_2) \oplus \succ_3\) and \(\succ_1 \oplus (\succ_2 \oplus \succ_3)\) are equal to the relation \(\succ\) on \(X_Q\) defined by

\[
(x_1, x_2, x_3) \succ (y_1, y_2, y_3) \iff x_1 \succ_1 y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \succ_2 y_2 \text{ or } x_1 = y_1, x_2 = y_2, \text{ and } x_3 \succ_3 y_3.
\]

\(\square\)

In light of Proposition 3.3.4, we will allow ourselves to omit parenthesis when taking multiple lexicographic sums. For example, we will write \(\succ_1 \oplus \succ_2 \oplus \succ_3\) rather than \((\succ_1 \oplus \succ_2) \oplus \succ_3\). Note that the operation of taking lexicographic sums is not commutative. Indeed, the order in which the component relations are summed plays an essential role in Definition 3.3.1.
Proposition 3.3.5. Let $Q_1$ and $Q_2$ be disjoint finite criteria sets, let $\succ_1$ be a binary preference order on $Q_1$ with order sequence $(x_k)$ and height\footnote{The height of a linear order on a set $S$ is simply the cardinality of $S$.} $m$, and let $\succ_2$ be a binary preference order on $Q_2$ with order sequence $(y_k)$ and height $n$. Let $(z_k)$ be the order sequence corresponding to $\succ_1 \oplus \succ_2$. Then, for all integers $i$ and $j$ with $0 < i < m - 1$ and $1 \leq j \leq n$,

$$z_{in+j} = (x_{i+1}, y_j).$$

Proof. Let $\succ = \succ_1 \oplus \succ_2$. Observe that, by the definition of the lexicographic sum,

$$(x_k, y_l) \succ (x_{i+1}, y_j) \iff k < i + 1$$

or $k = i + 1$ and $l < j$.

Since there are $in$ elements of the first type and $j - 1$ elements of the second type, exactly $in + j - 1$ elements precede $(x_{i+1}, y_j)$ in the order sequence for $\succ$, as desired. \hfill \Box

We conclude this section by examining a special class of orders that will be of significant use to us in later chapters.

Definition 3.3.6. Let $Q = \{1, 2, \ldots, n\}$ and let $\succ$ be a linear order on $X_Q$. If, for each $q \in Q$, there exists a linear order $\succ_q$ on $X_q$ such that

$$\succ = \succ_{\sigma(1)} \oplus \succ_{\sigma(2)} \oplus \cdots \oplus \succ_{\sigma(n)}$$

for some $\sigma \in S_n$, then $\succ$ is said to be a lexicographic order on $Q$. The permutation $\sigma$ is called the importance permutation of the order.

Note that, in the case that $X_Q$ is binary, a lexicographic order on $Q$ is uniquely determined by its importance permutation and its leader. Thus,

Proposition 3.3.7. Let $|Q| = n$ and let $X_Q$ be binary. Then there are exactly $2^n \cdot n!$ distinct lexicographic orders on $Q$.

Also notice that, since every linear order on a single criterion is vacuously separable, Proposition 3.3.2 implies the following:

Corollary 3.3.8. Every lexicographic order is separable.

We should mention that Corollary 3.3.8 has been proved in the past (see [13], for example), but follows here as a result of a more general theory of lexicographic sums than
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has been previously studied. We will see shortly that this corollary is also implied by the upcoming Propositions 3.4.4 and 3.4.8.

Of special interest to us will be the normalized lexicographic order, i.e. the lexicographic order with leader \((1,1,\ldots,1)\) and importance permutation equal to the identity permutation. We call this special order the standard lexicographic order on \(Q\), denoted \(\succ_{\text{lex}}\). The standard lexicographic order has the useful property that the \(k\)th row of the binary preference matrix corresponding to \(\succ_{\text{lex}}\) is, loosely speaking, the binary expansion of \(2^n - k\) (with leading zeros possibly added). More precisely,

**Proposition 3.3.9.** Let \(|Q| = n \geq 2\) and let \((a_{ij})\) be the binary preference matrix corresponding to \(\succ_{\text{lex}}\). Then, for each positive integer \(i \leq 2^n\),

\[
\sum_{j=1}^{n} a_{ij} \cdot 2^{n-j} = 2^n - i
\]

**Proof.** We proceed by induction. For \(n = 2\), the result can be easily verified by examining the 4 rows of the standard lexicographic order on 2 criteria. Now suppose that it is true for some integer \(n \geq 2\). For each positive integer \(k\), let \(\succ_k\) denote the standard lexicographic order on \(k\) criteria, so that \(\succ_{n+1} = \succ_n \oplus \succ_1\). Let \((a_{ij}), (b_{ij})\) be the binary preference matrices corresponding to \(\succ_n\) and \(\succ_{n+1}\), respectively. Proposition 3.3.5 implies that

\[
b_{2i-1,j} = b_{2(i-1)+1,j} = \begin{cases} a_{ij} & \text{if } 1 \leq j \leq n; \\ 1 & \text{if } j = n + 1. \end{cases}
\]

Similarly,

\[
b_{2i,j} = b_{2(i-1)+2,j} = \begin{cases} a_{ij} & \text{if } 1 \leq j \leq n; \\ 0 & \text{if } j = n + 1. \end{cases}
\]

Thus, by the induction hypothesis,

\[
\sum_{j=1}^{n+1} b_{2i-1,j} \cdot 2^{n+1-j} = 1 + \sum_{j=1}^{n} a_{ij} \cdot 2^{n+1-j} = 1 + 2(2^n - i) = 2^{n+1} - (2i - 1)
\]

and

\[
\sum_{j=1}^{n+1} b_{2i,j} \cdot 2^{n+1-j} = \sum_{j=1}^{n} a_{ij} \cdot 2^{n+1-j} = 2(2^n - i) = 2^{n+1} - 2i,
\]

as desired. □
3.4 Additive Orders

For this section, we assume that $Q$ is a finite criteria set.

Definition 3.4.1. Let $\succeq$ be a weak order on $X_Q$. A value function for $\succeq$ is a function $v : X_Q \rightarrow \mathbb{R}$ such that, for all $x, y \in X_Q$, $x \succeq y \iff v(x) \geq v(y)$. The relation $\succeq$ is said to be additive if, for each $q \in Q$, there exists a value function $v_q : X_q \rightarrow \mathbb{R}$ such that $v(x) = \sum_{q \in Q} v_q(x_q)$ is a value function for $\succeq$.

In recent literature (see [2], [14]), examples of additive orders have typically involved linear value functions, that is, value functions of the form $v(x) = \sum_{q \in Q} c_q x_q$, where $c_q \in \mathbb{R}$ for each $q \in Q$ (of course, in this case it is necessary for each $X_q$ to be a subset of $\mathbb{R}$). If $\succeq$ is an additive order that admits such a linear value function, we say that $\succeq$ is linearly additive. In light of this distinction, it seems reasonable to ask: Are there additive orders that are not linearly additive? The answer to this question is yes, though we must look beyond the binary preference orders to find them (see Proposition 3.4.3 below).

Take, for example, the additive order $\succeq$ on $X_{\{1,2\}}$ with value function $v(x) = x_1 + x_2^2$, where $X_1 = X_2 = \{-1, 2, -3\}$. Let $\sim$ denote the indifference relation\footnote{The indifference relation $\sim$ corresponding to a weak order $\succeq$ is the relation defined by $x \sim y \iff x \succeq y$ and $y \succeq x$. Notice that, if $\succeq$ is an additive weak order with value function $v$, then $x \sim y \iff v(x) = v(y)$.} corresponding to $\succeq$. Then $(-3, -3) \sim (2, 2)$, since

$$v((-3, -3)) = -3 + 9 = 6 = 2 + 4 = v((2, 2))$$

and $(-1, 2) \sim (2, -1)$, since

$$v((-1, 2)) = -1 + 4 = 3 = 2 + 1 = v(2, -1).$$

By this token, any linear value function $v^*(x) = c_1 x_1 + c_2 x_2$ for $\succeq$ would necessarily satisfy

$$-3c_1 - 3c_2 = v^*((-3, -3)) = v^*((2, 2)) = 2c_1 + 2c_2$$

and

$$-c_1 + 2c_2 = v^*((-1, 2)) = v^*((2, -1)) = 2c_1 - c_2.$$ 

But some simple algebra shows that the only solution to this system of equations is $c_1 = c_2 = 0$. If this were the case, then we would have $x \sim y$ (i.e. $x \succeq y$ and $y \succeq x$) for every
\( x, y \in X_Q \). But this clearly does not occur, for, by our above calculations, \((-3, 3) \succeq (-1, 2)\) but \((-1, 2) \not\preceq (-3, 3)\). Thus, there cannot exist a linear value function for \(\succeq\); it follows that \(\succeq\) is not linearly additive.

Proposition 3.4.3 below establishes our previous claim that we must look beyond the binary preference orders to find examples of additive orders that are not linearly additive.

**Lemma 3.4.2.** Let \(\succeq_1\) and \(\succeq_2\) be additive orders on \(X_Q\) with respective value functions \(v\) and \(w\). If \(v - w\) is a constant function, then \(\succeq_1 = \succeq_2\).

**Proof.** Suppose that, for some \(c \in \mathbb{R}\), \(v(x) - w(x) = c\) for all \(x \in X_Q\). Then

\[
v(x_1) \geq v(x_2) \iff w(x_1) + c \geq w(x_2) + c \iff w(x_1) \geq w(x_2),
\]

and so \(x_1 \succeq_1 x_2 \iff x_1 \succeq_2 x_2\). \(\square\)

**Proposition 3.4.3.** Let \(Q = \{1, 2, \ldots, n\}\), let \(X_q = \{a_q, b_q\} \subset \mathbb{R}\) for each \(q \in Q\), and let \(\succeq\) be a weak order on \(X_Q\). Then \(\succeq\) is additive if and only if \(\succeq\) is linearly additive.

**Proof.** The reverse implication is immediate. To prove the forward implication, assume that \(\succeq\) is additive and let \(v(x) = \sum_{q=1}^{n} v_q(x_q)\) be a value function for \(\succeq\). Notice that, for each \(q \in Q\) and for each \(x_q \in X_q\) (recall that \(X_q = \{a_q, b_q\}\)), we have

\[
v_q(x_q) = \frac{v_q(b_q) - v_q(a_q)}{b_q - a_q} (x_q - a_q) + v_q(a_q).
\]

Now let

\[
c_q = \frac{v_q(b_q) - v_q(a_q)}{b_q - a_q}.
\]

Then

\[
v(x) = \sum_{q=1}^{n} c_q x_q + \sum_{q=1}^{n} (v_q(a_q) - c_q a_q).
\]

Since the latter sum is a constant, Lemma 3.4.2 implies that \(w(x) = \sum_{q=1}^{n} c_q x_q\) is a value function for \(\succeq\). \(\square\)

Many authors have studied the connections between separability and additivity. We state below some of their main findings. The first result has appeared in several locations; its proof may be found in either [14] or [21]. The second is due to Gorman [10].

**Proposition 3.4.4.** Let \(\succeq\) be a additive order on \(X_Q\). Then \(\succeq\) is separable.
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**Proposition 3.4.5.** Suppose that, for each $q \in Q$, $q$ is strictly essential and $X_q$ is topologically separable and arc-connected. Suppose also that $\succeq$ is a continuous weak order on $X_Q$. Then $\succ$ is additive if and only if $\succeq$ is separable.

It is known that Proposition 3.4.5 may fail when the alternative sets are not arc-connected. The example we provide below to demonstrate this fact is due to Bradley and Kilgour [2].

**Example 3.4.6.** Let $Q = \{1, 2, 3, 4, 5\}$ and let $\succ$ be any binary preference order on $X_Q$ such that

$$
(0, 1, 1, 0, 1) \succ (1, 0, 0, 1, 0) \\
(1, 0, 0, 0, 1) \succ (0, 1, 1, 0, 0) \\
(0, 0, 1, 1, 0) \succ (0, 1, 0, 0, 1) \\
(0, 1, 0, 0, 0) \succ (0, 0, 1, 0, 1).
$$

To show that $\succ$ is not additive, it suffices, by Proposition 3.4.3, to show that $\succ$ is not linearly additive. Now if there did exist a linear value function $v(x) = \sum_{q=1}^{5} c_q x_q$ for $\succ$, we could conclude the following:

$$
c_2 + c_3 + c_5 > c_1 + c_4 \\
c_1 + c_5 > c_2 + c_3 \\
c_3 + c_4 > c_2 + c_5 \\
c_2 > c_3 + c_5
$$

But adding the first three inequalities yields $c_3 + c_5 > c_2$, which is a contradiction to the fourth inequality.

It is worth mentioning that this example may be easily modified to demonstrate the existence of a separable but non-additive order on $n$ criteria for every integer $n \geq 5$. Specifically, let $Q = \{1, 2, 3, \ldots, n\}$ and let $\succ$ be any binary preference order on $X_Q$ such that

$$
(0, 1, 1, 0, 1, \ldots, 0) \succ (1, 0, 0, 1, 0, \ldots, 0) \\
(1, 0, 0, 0, 1, \ldots, 0) \succ (0, 1, 1, 0, 0, \ldots, 0) \\
(0, 0, 1, 1, 0, \ldots, 0) \succ (0, 1, 0, 0, 1, \ldots, 0) \\
(0, 1, 0, 0, 0, \ldots, 0) \succ (0, 0, 1, 0, 1, \ldots, 0).
$$
In the same manner as above, it can be shown that $\succ$ is not additive. The following proposition, which appears in [2], shows that Example 3.4.6 is the smallest counterexample to Proposition 3.4.5.

**Proposition 3.4.7.** Let $|Q| \leq 4$. Then every separable binary preference order on $Q$ is additive.

We conclude this chapter by considering the relationship between additive and lexicographic orders. The following proposition is due to Kilgour [13].

**Proposition 3.4.8.** Every lexicographic order is additive.

Kilgour also demonstrates that the converse of this proposition does not hold; that is, there exist additive orders that are not lexicographic.
Chapter 4

Preseparable Orders

In this chapter, we consider some combinatorial problems related to the notion of separability. We introduce preseparable extensions as a mechanism for merging two separable preference orders in a manner that preserves the separability structure of the component orders. We consider the effect of executing a finite sequence of these extensions and explore counting problems related to the resulting orders.

4.1 Preseparable Extensions

Definition 4.1.1. Let $Q_1$ and $Q_2$ be disjoint criteria sets and let $\succ_1$ and $\succ_2$ be linear orders on $X_{Q_1}$ and $X_{Q_2}$, respectively. A preseparable extension of $\succ_1$ by $\succ_2$ is a linear order $\succ$ on $X_{Q_1} \cup X_{Q_2}$ such that, for $i = 1, 2$,

$$x_{Q_i} \succ_i y_{Q_i} \implies (x_{Q_i}, u_{-Q_i}) \succ (y_{Q_i}, u_{-Q_i}) \text{ for all } u_{-Q_i} \in X_{-Q_i}.$$

If $Q_2$ consists of a single element, then the extension is said to be simple. If $Q = \{1, 2, \ldots, n\}$ is a finite criteria set, then a linear order $\succ$ on $X_Q$ is said to be preseparable if there exists a sequence $\succ_1, \succ_2, \ldots, \succ_n = \succ$ such that, for each positive integer $i \leq n - 1$, $\succ_i$ is a linear order on $\{1, 2, \ldots, i\}$ and $\succ_{i+1}$ is a simple preseparable extension of $\succ_i$.

Note that if $\succ$ is a preseparable order with the corresponding sequence $\succ_1, \succ_2, \ldots, \succ_n$ of preseparable extensions, then each $\succ_i$ is also preseparable.

Lemma 4.1.2. Let $\succ$ be a preseparable extension of $\succ_1$ by $\succ_2$. If, for $i = 1$ or $i = 2$, $(x_{Q_i}, u_{-Q_i}) \succ (y_{Q_i}, u_{-Q_i})$ for some $u_{-Q_i} \in X_{-Q_1}$, then $x_{Q_i} \succ_i y_{Q_i}$.
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Proof. We prove the contrapositive. Suppose that $x_{Q_i} \not\succ y_{Q_i}$. Because $\succ$ is a linear order, it must be that $y_{Q_i} \succ x_{Q_i}$. But then $(y_{Q_i}, u_{-Q_i}) \succ (x_{Q_i}, u_{-Q_i})$ for all $u_{-Q_i} \in X_{-Q_i}$, from which it follows that $(x_{Q_i}, u_{-Q_i}) \not\succ (y_{Q_i}, u_{-Q_i})$ for all $u_{-Q_i} \in X_{-Q_i}$. □

Proposition 4.1.3. Preseparable extensions preserve separability. Specifically, let $\succ_1$ and $\succ_2$ be linear orders on $X_{Q_1}$ and $X_{Q_2}$, respectively, and suppose that $\succ$ is a preseparable extension of $\succ_1$ by $\succ_2$. If, for $i = 1$ or $i = 2$, $S \subseteq Q_i$ is $\succ_i$-separable, then $S$ is $\succ$-separable and $\succ = (\succ_i)_S$.

Proof. Suppose that $S \subseteq Q_i$ is $\succ_i$-separable and that $(x_S, u_{Q_i}, u_{-Q_i}) \succ (y_S, u_{Q_i}, u_{-Q_i})$ for some $x_S, y_S \in X_S$ and some $u \in X_{-S}$. Lemma 4.1.2 implies that $(x_S, u_{Q_i-S}) \succ_i (y_S, u_{Q_i-S})$. But then the $\succ_i$-separability of $S$ implies that $(x_S, u_{Q_i-S}) \succ_i (y_S, u_{Q_i-S})$ for all $u_{Q_i-S} \in X_{Q_i-S}$. Thus, by Definition 4.1.1, it follows that $(x_S, u_{Q_i-S}, u_{-Q_i}) \succ (y_S, u_{Q_i-S}, u_{-Q_i})$ for all $v \in X_{-Q_i}$ and so $S$ is $\succ$-separable. To show $\succ = (\succ_i)_S$, we observe that

$$x_S \succ y_S \iff (x_S, u_{Q_i-S}, u_{-Q_i}) \succ (y_S, u_{Q_i-S}, u_{-Q_i}) \text{ for all } u \in X_{-S} \iff (x_S, u_{Q_i-S}) \succ_i (y_S, u_{Q_i-S}) \text{ for all } u_{Q_i-S} \in X_{Q_i-S} \iff x_S \succ_i y_S.$$

Corollary 4.1.4. Let $\succ_1, \succ_1^*$ be linear orders on $X_{Q_1}$, let $\succ_2, \succ_2^*$ be linear orders on $X_{Q_2}$, and suppose that $\succ$ is a preseparable extension of $\succ_1$ by $\succ_2$ and that $\succ^*$ is a preseparable extension of $\succ_1^*$ by $\succ_2^*$. If either $\succ_1 \neq \succ_1^*$ or $\succ_2 \neq \succ_2^*$, then $\succ \neq \succ^*$.

Proof. We prove the contrapositive. Thus, suppose that $\succ = \succ^*$. Then $\succ_1 = \succ_1^*$ and $\succ_2 = \succ_2^*$. But Proposition 4.1.3 then implies that $\succ_1 = \succ_1^*$ and $\succ_1 = \succ_2^*$, as desired. □

Proposition 4.1.3 shows that preseparable extensions preserve the separability structure of their component orders. Since preseparable orders are obtained through successive simple preseparable extensions and since any linear order on a single criterion is vacuously separable, it follows that every preseparable order is monoseparable. The following example demonstrates that, while monoseparability is a necessary condition for preseparability, it is not sufficient.

Example 4.1.5. Let $\succ$ be the linear order corresponding to the binary preference matrix

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Upon inspection, we see that $\succ$ is separable on each criterion, with induced order $1 \succ 0$. Thus, $\succ$ is monoseparable. Now if $\succ$ were preseparable, then $\succ$ would be a simple preseparable extension of some linear order on $X_{\{q_1,q_2\}}$. But since preseparable extensions preserve separability and $\{q_1,q_2\}$ is $\succ'$-separable with respect to any linear order $\succ'$ on $X_{\{q_1,q_2\}}$, it follows that $\{q_1,q_2\}$ would then be $\succ$-separable. Notice however that this is not the case: since $011 \succ 101$ and yet $100 \succ 010$, $\{q_1,q_2\}$ is not separable. Consequently, $\succ$ cannot be preseparable.

Given our observations in Example 4.1.5, a natural question to ask is: What conditions must be placed on monoseparable orders to obtain preseparable orders? The proposition below answers this question.

**Proposition 4.1.6.** Let $Q = \{1,2,\ldots,n\}$ and let $\succ$ be a linear order on $X_Q$. Then $\succ$ is preseparable if and only if:

(i) $\succ$ is monoseparable, and

(ii) For each $j \in \{1,2,\ldots,n\}$, the set $Q_j = \{1,2,\ldots,j\}$ is $\succ$-separable.

**Proof.** $(\Rightarrow)$ Suppose $\succ$ is preseparable and let $\succ_1, \succ_2, \ldots, \succ_n = \succ$ be the corresponding sequence of simple preseparable extensions. By our above remarks, $\succ$ is monoseparable. Now since each $\succ_j$ is a linear order on $X_{Q_j}$, it follows that each $Q_j$ is $\succ$-j-separable. But since $\succ$ is obtained from $\succ_j$ through a finite sequence of simple preseparable extensions, all of which preserve separability, it follows that $Q_j$ is $\succ$-separable.

$(\Leftarrow)$ For the converse, we proceed by induction. The result is immediate for $n = 1$. Now suppose that every linear order on $X_{Q'}$ having $|Q'| = n - 1$ and satisfying properties (i) and (ii) is preseparable. Let $S = Q_{n-1}$. By Proposition 2.2.6, $\succ_S$ is monoseparable and $Q_j$ is $\succ$-separable for each $j \in \{1,2,\ldots,n-1\}$. Thus, by the inductive hypothesis, $\succ_S$ is preseparable. By Definition 4.1.1, it now suffices to show that $\succ$ is a simple preseparable extension of $\succ_S$ by $\succ_{\{n\}}$. This follows easily since $-S = \{n\}$ and

$$x_S \succ_S y_S \implies (x_S,u_n) \succ (y_S,u_n) \text{ for all } u_n \in X_{\{n\}}$$
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and

\[ x_n \succ_{(n)} y_n \Rightarrow (x_n, u_S) \succ (y_n, u_S) \text{ for all } u_S \in X_S. \]

\[ \square \]

Corollary 4.1.7. If \( Q \) is finite and \( \succ \) is a separable linear order on \( X_Q \), then \( \succ \) is preseparable.

4.2 Counting Binary Preseparable Orders

Definition 4.2.1. A sequence \((a_k)\) of \(n\) \((+1)\)s and \((-1)\)s is called a characteristic sequence if, for each positive integer \(k \leq n\), the \(k\)th partial sum of \((a_k)\) is nonnegative, i.e. \(\sum_{i=1}^{k} a_i \geq 0\). If a characteristic sequence contains the same number of \((+1)\)s as \((-1)\)s, then the sequence is said to be complete. If \(a_{n-k+1} = -a_k\) for each \(k\), then the sequence is said to be symmetric.

Note that a characteristic sequence is one in which, at any point in the sequence, the number of \((+1)\)s appearing up to that point is greater than or equal to the number of \((-1)\)s. Thus, the sequences \((+1, +1, +1, +1, -1, -1), (+1, +1, -1, -1, +1, -1)\) and \((+1, -1, +1, -1, +1, -1)\) are all characteristic sequences, the second being complete (but not symmetric) and the third being symmetric (and complete — in fact, one can easily verify that every symmetric characteristic sequence is complete). On the other hand, neither \((+1, +1, -1, -1, -1, -1)\) nor \((-1, -1, +1, +1, -1, +1)\) nor \((-1, +1, -1, +1, -1, +1)\) is a characteristic sequence.

Proposition 4.2.2. Let \(Q_1 = \{1, 2, \ldots, n\}, Q_2 = \{n+1\}, |X_{Q_1}| = m < \infty, \text{ and } |X_{n+1}| = 2.\]

If \(\succ_1\) is a linear order on \(X_{Q_1}\) and \(\succ_2\) is a linear order on \(X_{n+1}\), then there is a bijection between the set of preseparable extensions of \(\succ_1\) by \(\succ_2\) and the set of complete characteristic sequences of length \(2m\).

Proof. Let \(Q = Q_1 \cup Q_2\). Let the elements of \(X_{n+1}\) be labeled +1 and -1 so that +1 \(\succ_2\) -1.

Let \((x_k)\) be the order sequence corresponding to \(\succ_1\). For each complete characteristic sequence \((a_k)\) of length \(2m\), let \((y_k)\) be the sequence of alternatives in \(X_Q\) defined by

\[ y_k = (x_j, \pm 1) \iff a_k \text{ is the } j^{th} \pm 1 \text{ in } (a_k). \]

Because \((a_k)\) is complete (and hence there are \(m\) \((+1)\)s and \(m\) \((-1)\)s in \((a_k)\)), each element of \(X_Q\) appears in \((y_k)\) exactly once. Thus, let \(\succ\) be the linear order corresponding to \((y_k)\) and notice the following:
(i) \((a_k)\) has all nonnegative partial sums \(\iff (x_j, +1) \succ (x_j, -1)\) for all \(j\).

(ii) For \(s = \pm 1\), \((x_i, s) \succ (x_j, s)\) \(\iff\) the \(i^{th}\) \(s\) appears before the \(j^{th}\) \(s\) in \((a_k)\) \(\iff\) \(i < j \iff x_i \succ x_j\).

These observations establish that the map \((a_k) \mapsto (y_k)\), as defined above, is a well-defined function from the set of all complete characteristic sequences of length \(2n\) onto the set of all preseparable extensions of \(\succ_1\) by \(\succ_2\). We need only to show that this map is injective. To do so, observe that if two characteristic sequences \((a_k)\) and \((b_k)\) differ, say \(a_i \neq b_i\) for some \(i\), then their corresponding linear orders differ: specifically, the \(i^{th}\) most preferred alternatives differ in the \(\{n + 1\}\) component. This proves that our function is bijective, as desired. □

To proceed, we require the following well-known result (paraphrased to suit our terminology and notation). Its proof can be found in Brualdi [6] (pages 252-254).

**Proposition 4.2.3.** The number of complete characteristic sequences of length \(2n\) is the \(n^{th}\) Catalan number \(C_n\), where

\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \]

**Example 4.2.4.** Let \(\succ_1\) and \(\succ_2\) be defined by \(0 \succ_1 1\) and \(1 \succ_2 0\) (so that both \(\succ_1\) and \(\succ_2\) are binary preference orders on a single criterion). Since there are exactly 2 complete characteristic sequences of length 4, there are exactly 2 preseparable extensions of \(\succ_1\) by \(\succ_2\), as shown below:

\[
\begin{align*}
(+1, -1, +1, -1) & \iff (0, 1) \succ (0, 0) \succ (1, 1) \succ (1, 0) \\
(+1, +1, -1, -1) & \iff (0, 1) \succ (1, 1) \succ (0, 0) \succ (1, 0)
\end{align*}
\]

We now have enough machinery to be able to count the number of binary preseparable orders on a given finite criteria set.

**Proposition 4.2.5.** Let \(|Q| = n < \infty\) and suppose that \(X_Q\) is binary. Then the number of distinct preseparable orders on \(X_Q\) is given by the formula

\[ \pi(n) = 2^n \prod_{i=0}^{n-1} C_{2^i}. \]

**Proof.** Let \(Q = \{1, 2, \ldots, n\}\) and let \(Q' = Q - \{n\}\). Every preseparable order on \(X_Q\) is a simple preseparable extension of a preseparable order on \(X_{Q'}\). Thus, to form a preseparable order on \(X_Q\), we must...
(i) choose a preseparable order \( \succ_1 \) on \( X_Q' \),

(ii) choose a linear order \( \succ_2 \) on \( X_q \) (either \( 1 \succ 0 \) or \( 0 \succ 1 \)), and

(iii) form a preseparable extension of \( \succ_1 \) by \( \succ_2 \).

It is clear that (i) may be completed in \( \pi(n - 1) \) distinct ways, and (ii) in 2 distinct ways. Since every linear order on \( X_Q' \) has height \( 2^{n-1} \), Proposition 4.2.3 implies that (iii) may be completed in \( C_{2n-1} \) distinct ways. Furthermore, Corollary 4.1.4 implies that all preseparable orders formed in this manner will be distinct. Thus,

\[
\pi(n) = 2 \cdot \pi(n - 1) \cdot C_{2n-1}.
\]

Clearly \( \pi(1) = 2 \) and so a simple induction argument finishes the proof. \( \square \)

### 4.3 Symmetric Extensions

Let \( X_Q \) be a binary alternative set. For an alternative \( x \in X_Q \), let \( \overline{x} \) denote the bitwise complement of \( x \) (so that if, for example, \( x = (1,0,1) \), then \( \overline{x} = (0,1,0) \)). Definition 4.3.1 and Propositions 4.3.2 and 4.3.3 are generalizations of results originally due to Bradley and Kilgour [2].

**Definition 4.3.1.** Let \( \succeq \) be a relation on a binary alternative set \( X_Q \). Then \( \succeq \) is said to satisfy the **mirror property** if, for all \( x, y \in X_Q \), \( x \succeq y \implies \overline{y} \succeq \overline{x} \).

**Proposition 4.3.2.** Let \( |Q| = n < \infty \) and let \( X_Q \) be binary. Then a linear order \( \succ \) on \( X_Q \) satisfies the mirror property if and only if its order sequence \( (x_\lambda) \) is such that \( x_{2^n-i+1} = \overline{x}_i \) for each positive integer \( i \leq 2^n \).

**Proof.** (\( \Rightarrow \)) Suppose that \( \succ \) satisfies the mirror property and let \( i \leq 2^n \) be given. By definition of the order sequence, \( x_j \succ x_i \iff j < i \). But our assumption that \( \succ \) satisfies the mirror property then implies that \( \overline{x}_j \succ \overline{x}_i \iff i < j \). Since every \( x_k \in (x_\lambda) \) can be written as \( \overline{x}_j \) for some \( j \) (namely, choose the unique \( j \) for which \( x_j \) is the bitwise complement of \( x_k \)), it follows that \( x_k \succ \overline{x}_i \) for exactly \( 2^n - i \) choices of \( k \). Thus, \( \overline{x}_i = x_{2^n-i+1} \), as desired.

(\( \Leftarrow \)) For the converse, suppose that \( x_{2^n-i+1} = \overline{x}_i \) for each positive integer \( i \leq 2^n \). Then

\[
x_i \succ x_j \iff i < j
\]

\[
\iff 2^n - j + 1 < 2^n - i + 1
\]

\[
\iff \overline{x}_j = x_{2^n-j+1} \succ x_{2^n-i+1} = \overline{x}_i.
\]

\( \square \)
Proposition 4.3.3. If \( \succeq \) is a separable relation on a binary alternative set \( X_Q \), then \( \succeq \) satisfies the mirror property.

Proof. Let \( x, y \in X_Q \) and suppose that \( x \succeq y \). If \( x = y \), then Proposition 2.2.10 implies that \( \succeq \) is reflexive, from which it follows that \( y \succeq y = x \). Furthermore, if \( x = \bar{y} \), then \( x = y \), and so \( \bar{y} = x \succeq y = \bar{y} \). Now suppose that \( x \neq y \) and let \( S = \{ q \in Q \mid x_q = y_q \} \). By assumption, \( S \) is a proper, nonempty subset of \( Q \). Also notice that \( x_S = y_S \) and \( x_{-S} = y_{-S} \). Thus,

\[
(y_S, y_{-S}) = (x_S, x_{-S}) = x \succeq y = (y_S, y_{-S}),
\]

from which the separability of \( \succeq \) implies that

\[
\bar{y} = (y_S, y_{-S}) \succeq (y_S, y_{-S}) = (x_S, x_{-S}) = x.
\]

\( \square \)

Example 4.3.4. Let \( \succ \) be the linear order corresponding to the binary preference matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

By Proposition 4.3.2, \( \succ \) clearly satisfies the mirror property. Notice however that \( \{2\} \) is nonseparable with respect to \( \succ \) since \( 111 \succ 101 \) but \( 100 \succ 110 \). Thus, the mirror property, though necessary for separability, is not sufficient. Indeed, we have seen that the mirror property does not even guarantee monoseparability.

Example 4.3.5. (due to Bradley & Kilgour [2]) Let \( \succ \) be the linear order corresponding to the binary preference matrix
where the bottom half of the matrix is completed so that \( \succ \) satisfies the mirror property.

It is straightforward, albeit quite tedious, to verify that:

- \( \succ \) is monoseparable with induced order \( 1 \succ 0 \) on each criterion.
- \( \{1, 2\} \) is \( \succ \)-separable with induced order \( 11 \succ 10 \succ 01 \succ 00 \).
- \( \{1, 2, 3\} \) is \( \succ \)-separable with induced order

\[
\begin{align*}
111 & \succ 110 \succ 101 \succ 011 \succ 100 \succ 010 \succ 001 \succ 000. 
\end{align*}
\]

- \( \{1, 2, 3, 4\} \) is \( \succ \)-separable with induced order

\[
\begin{align*}
1111 & \succ 1110 \succ 1101 \succ 1011 \succ 1100 \succ 1010 \succ 0111 \succ 1100 \succ 1010 \succ 0111 \succ 1000 \succ 0101 \succ 0011 \succ 0110 \succ 1000 \succ 0101 \succ 0011 \succ 0100 \succ 0010 \succ 0001 \succ 0000. 
\end{align*}
\]

Thus, by Proposition 4.1.6, \( \succ \) is preseparable. By construction, \( \succ \) satisfies the mirror property. Nevertheless, \( \{2, 3, 5\} \) is not \( \succ \)-separable since \( 11010 \succ 10111 \) but \( 10101 \succ 11000 \). We see then that, while the mirror property and preseparability are both necessary for separability, even both properties taken together are not sufficient.

**Definition 4.3.6.** Let \( Q_1 \) and \( Q_2 \) be disjoint criteria sets, and suppose that \( |X_{Q_1}| = m \) and \( |X_{Q_2}| = n \). Let \( \succ_1 \) and \( \succ_2 \) be linear orders on \( X_{Q_1} \) and \( X_{Q_2} \), respectively and let \( (x_k) \) and \( (y_k) \) be the order sequences corresponding to \( \succ_1 \) and \( \succ_2 \). A **symmetric extension** of \( \succ_1 \) by
$\succ_2$ is a linear order $\succ$ on $X_{Q_1\cup Q_2}$ such that, for all positive integers $i$, $j$, $k$ and $l$ with $i$, $k \leq m$ and $j$, $l \leq n$, $$ (x_i, y_j) \succ (x_k, y_l) \iff (x_{m-k+1}, y_{n-l+1}) \succ (x_{m-i+1}, y_{n-j+1}). $$

**Proposition 4.3.7.** Let $\succ_1$ and $\succ_2$ be as in Definition 4.3.6. A linear order $\succ$ on $X_{Q_1\cup Q_2}$ is a symmetric extension of $\succ_1$ by $\succ_2$ if and only if its order sequence $(z_k) = ((x_{i_k}, y_{j_k}))$ is such that $z_{mn-p+1} = (x_{m-i_p+1}, y_{n-j_p+1})$ for each positive integer $p \leq mn$.

**Proof.** $(\Rightarrow)$ Suppose that $\succ$ is a symmetric extension of $\succ_1$ by $\succ_2$ and let $p \leq mn$ be given. For each $q \leq mn$, let $z_q^* = (x_{m-i_q+1}, y_{n-j_q+1})$. We wish to show that $z_{mn-p+1} = z_p^*$. By definition of the order sequence, $z_q \succ z_p \iff q < p$. But our assumption that $\succ$ is a symmetric extension of $\succ_1$ by $\succ_2$ then implies that $z_p^* \succ z_q^* \iff q < p$. Since every $z_k \in (z_k)$ can be written as $z_q^*$ for some $q$ (namely, choose the unique $q$ for which $z_q^* = z_k$), it follows that $z_p^* \succ z_k$ for exactly $p - 1$ choices of $k$. Thus, $z_k \succ z_p^*$ for exactly $mn - (p - 1) - 1 = mn - p$ choices of $k$, and so $z_{mn-p+1} = z_p^*$.

$(\Leftarrow)$ For the converse, suppose that $z_{mn-p+1} = z_p^*$ for each $p \leq mn$. We wish to show that $z_p \succ z_q \iff z_q^* \succ z_p^*$. Observe that $$ z_p \succ z_q \iff p < q $$ $$ \iff mn - q + 1 < mn - p + 1 $$ $$ \iff z_q^* = z_{mn-q+1} \succ z_{mn-p+1} = z_p^*, $$ as desired. $\Box$

**Proposition 4.3.8.** Symmetric extensions preserve the mirror property. Specifically, if $\succ_1$ and $\succ_2$ are linear orders on finite binary alternative sets and $\succ$ is a symmetric extension of $\succ_1$ by $\succ_2$, then $\succ$ satisfies the mirror property if and only if both $\succ_1$ and $\succ_2$ satisfy the mirror property.

**Proof.** Let $\succ_1$ and $\succ_2$ have heights $m$ and $n$ and order sequences $(x_k)$ and $(y_k)$, respectively. Let $(z_k) = ((x_{i_k}, y_{j_k}))$ be the order sequence corresponding to $\succ$.

$(\Rightarrow)$ Suppose that $\succ$ satisfies the mirror property and let $i \leq m$ and $j \leq n$ be given. Since $\succ$ is a linear order, there exists an integer $k$ such that $z_k = (x_i, y_j)$. Thus $$ (\overline{x}_i, \overline{y}_j) = \overline{z}_k $$ $$ = z_{mn-k+1} \quad \text{(by Proposition 4.3.2)} $$ $$ = (x_{m-i+1}, y_{n-j+1}) \quad \text{(by Proposition 4.3.7)}, $$
from which it follows that \( \overline{x}_i = x_{m-i+1} \) and \( \overline{y}_j = y_{n-j+1} \). Since our choices of \( i \) and \( j \) were arbitrary, it follows that both \( \succ_1 \) and \( \succ_2 \) satisfy the mirror property.

\((\Leftarrow)\) Suppose that both \( \succ_1 \) and \( \succ_2 \) satisfy the mirror property and let \( \succ \) be a symmetric extension of \( \succ_1 \) by \( \succ_2 \). Then, for each \( k \leq mn \),

\[
\overline{z}_k = (\overline{x}_k, \overline{y}_j_k)
\]

\[
= (x_{m-i_k+1}, y_{n-j_k+1}) \quad \text{(by Proposition 4.3.2)}
\]

\[
= z_{mn-k+1} \quad \text{(by Proposition 4.3.7)}.
\]

Thus \( \succ \) satisfies the mirror property. \( \square \)

**Proposition 4.3.9.** Let \( Q_1 = \{1, 2, \ldots, n\}, Q_2 = \{n + 1\} \), and let \( |X_{Q_1}| = m < \infty \) and \( |X_{n+1}| = 2 \). Let \( \succ_1 \) and \( \succ_2 \) be linear orders on \( X_{Q_1} \) and \( X_{n+1} \), respectively, and let \( \succ \) be a preseparable extension of \( \succ_1 \) by \( \succ_2 \). Then \( \succ \) is a symmetric extension of \( \succ_1 \) by \( \succ_2 \) if and only its characteristic sequence \((a_k)\) is symmetric.

**Proof.** Let the elements of \( X_{n+1} \) be labeled \(+1\) and \(-1\) and let \((x_k), (y_k), \) and \((z_k) = (x_k, y_k)\) be the order sequences corresponding to \( \succ_1 \), \( \succ_2 \), and \( \succ \), respectively. Note that, for each \( k \), \( y_{j_k} = a_k \). Also note that, since \( y_{j_1} = \pm 1 \), \( y_{2-j_2+1} = y_{j_2} \).

\((\Rightarrow)\) Suppose \( \succ \) is a symmetric extension of \( \succ_1 \) by \( \succ_2 \). Then, by Proposition 4.3.7,

\[
z_{2m-k+1} = (x_{m-i_k+1}, y_{2-j_k+1}) = (x_{m-i_k+1}, -y_{j_k}).
\]

Thus, \( a_{2m-k+1} = -y_{j_k} = -a_k \), as desired.

\((\Leftarrow)\) For the converse, assume that \((a_k)\) is symmetric. We wish to show that \( \succ \) is a symmetric extension of \( \succ_1 \) by \( \succ_2 \). By Proposition 4.3.7, it suffices to show that

\[
z_{2m-k+1} = (x_{m-i_k+1}, y_{2-j_k+1}) = (x_{m-i_k+1}, -y_{j_k})
\]

for each \( k \leq 2m \). Choose such a \( k \). We consider two cases, depending on whether \( y_{j_k} = +1 \) or \( y_{j_k} = -1 \). In the first case, we have

\[
z_k = (x_k, +1) \implies a_k \text{ is the } i_k^{th} \text{ (+1) in } (a_k)
\]

\[
\implies a_k \text{ is preceded by } i_k + 1 \text{ (+1)s}
\]

\[
\implies a_{2m-k+1} = -1 \text{ and } a_{2m-k+1} \text{ is followed by } i_k - 1 \text{ (-1)s}
\]

\[
\implies a_{2m-k+1} \text{ is preceded by } m - (i_k - 1) - 1 = m - i_k \text{ (-1)s}
\]

\[
\implies a_{2m-k+1} \text{ is the } (m - i_k + 1)^{th} \text{ (-1)}
\]

\[
\implies z_{2m-k+1} = (x_{m-i_k+1}, -1) = (x_{m-i_k+1}, -y_{j_k}),
\]
as desired. A similar argument holds if $j_k = -1$. Thus, $\succ$ is a symmetric extension of $\succ_1$ by $\succ_2$. □

**Proposition 4.3.10.** Let $Q_1 = \{1, 2, \ldots, n\}$, $Q_2 = \{n + 1\}$, and let $X_{Q_1}$ and $X_{n+1}$ be binary. Let $\succ_1$ and $\succ_2$ be linear orders on $X_{Q_1}$ and $X_{n+1}$, respectively, and let $\succ$ be a preseparable extension of $\succ_1$ by $\succ_2$. Then $\succ$ satisfies the mirror property if and only if $\succ$ is a symmetric extension of $\succ_1$ by $\succ_2$ and both $\succ_1$ and $\succ_2$ satisfy the mirror property.

**Proof.** Suppose that $\succ$ satisfies the mirror property and let $(x_k), (y_k)$, and $(z_k) = (x_{jk}, y_{jk})$ be the order sequences corresponding to $\succ_1$, $\succ_2$, and $\succ$, respectively. Note that $\succ_1$, $\succ_2$ and $\succ$ have heights $2^n$, 2, and $2^{n+1}$. Without loss of generality, assume that $1 \succ_2 0$ (if this is not the case, then we can relabel the elements of $X_{n+1}$ to make it so). Let $(a_k)$ be the characteristic sequence corresponding to $\succ$. Then for each positive integer $k \leq 2^{n+1}$,

$$a_k = \begin{cases} +1 & \text{if } y_{jk} = 1 \\ -1 & \text{if } y_{jk} = 0 \end{cases}$$

But since $\succ$ satisfies the mirror property, we have

$$(x_{ik}, y_{jk}) = x_k = z_{2^{n+1}-k+1} = (x_{i_{2^{n+1}-k+1}}, y_{j_{2^{n+1}-k+1}})$$

and so $y_{j_{2^{n+1}-k+1}} = y_{jk}$. Thus, $a_{2^{n+1}-k+1} = -a_k$, and, by Proposition 4.3.9, it follows that $\succ$ is a symmetric extension of $\succ_1$ by $\succ_2$. Proposition 4.3.8 then implies that both $\succ_1$ and $\succ_2$ satisfy the mirror property. The reverse implication follows immediately from Proposition 4.3.8. □

### 4.4 Counting Strongly Preseparable Orders

Propositions 4.1.7 and 4.3.3 establish that every separable linear order on a finite binary alternative set must be preseparable and satisfy the mirror property. Our goal in this section is to count the orders satisfying these properties. For convenience, if $\succ$ is a linear order on a finite binary alternative set $X_Q$ and $\succ$ satisfies the mirror property, we say that $\succ$ is a symmetric order on $Q$. If $\succ$ is both symmetric and preseparable, we say that $\succ$ is strongly preseparable.

**Proposition 4.4.1.** Let $|Q| = n$ and let $X_Q$ be binary. Then there are exactly $2^{2^n-1} \cdot 2^{n-1}!$ symmetric orders on $Q$. 

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Proof. Let \((x_k)\) be the order sequence corresponding to \(\succ\). Since \(X_Q\) is binary, \(\succ\) has height \(2^n\). Thus, every symmetric order on \(Q\) can be uniquely determined by specifying \(x_1, x_2, \ldots, x_{2^n-1}\) so that \(x_i \neq x_j\) for all \(i, j \leq 2^{n-1}\) (the bottom half of the order sequence is determined by the mirror property). Furthermore, all such choices of \(x_1, x_2, \ldots, x_{2^n-1}\) correspond to symmetric orders. Now \(x_1\) may be chosen in \(2^n\) ways, whereas \(x_2\) may be chosen in \(2^n - 2\) ways (\(x_2\) cannot be equal to \(x_1\) or \(x_1\)), \(x_3\) may be chosen in \(2^n - 4\) ways, and so on. Thus, the number of symmetric orders on \(Q\) is equal to

\[
2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2 = 2^{2^{n-1}} \cdot 2^{n-1}! \tag*{\square}
\]

For each positive integer \(n\), let \(P(n)\) denote the probability that a randomly selected binary preference order on a finite criteria set of cardinality \(n\) is separable.

Corollary 4.4.2. \(P(n) \to 0\) as \(n \to \infty\).

Proof. For each positive integer \(n\), let \(s(n)\) denote the number of separable binary preference orders on a finite criteria set of cardinality \(n\). Note that the total number of binary preference orders on such a criteria set is \(2^n!\). By Propositions 4.3.3 and 4.4.1, \(s(n) \leq 2^{2^{n-1}} \cdot 2^{n-1}!\), and so

\[
P(n) = \frac{s(n)}{2^n!} \leq \frac{2^{2^{n-1}} \cdot 2^{n-1}!}{2^n!} = \frac{2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2}{2^n \cdot (2^n - 1) \cdot (2^n - 2) \cdots 2 \cdot 1} = \frac{1}{(2^n - 1) \cdot (2^n - 3) \cdot (2^n - 5) \cdots 3 \cdot 1},
\]

which approaches 0 (very quickly!) as \(n \to \infty\). \(\square\)

Lemma 4.4.3. Let \(n\) be a positive integer. Then the \(n\)th Catalan number satisfies

\[
C_n = 2 \binom{2n}{n} - \binom{2n + 1}{n}.
\]

Proof. Observe that

\[
2 \binom{2n}{n} - \binom{2n + 1}{n} = \left(2 - \frac{2n + 1}{n + 1}\right) \binom{2n}{n} = \frac{1}{n + 1} \binom{2n}{n} = C_n.
\]

\(\square\)
Proposition 4.4.4. The number of symmetric characteristic sequences of length $2n$ is the $n^{th}$ middle binomial coefficient $W_n$, where

$$W_n = \binom{n}{[n/2]}.$$

Proof. Every symmetric characteristic sequence of length $2n$ is uniquely determined by its first $n$ elements. Furthermore, every characteristic sequence of length $n$ can be extended uniquely to a symmetric characteristic sequence of length $2n$. Thus, it suffices to show that the number of characteristic sequences of length $n$ is $W_n$.

Let $\eta(n, m)$ denote the number of characteristic sequences of length $n$ whose sum is $m$ and observe that $\eta(n, m) = 0$ whenever (i) $n$ and $m$ have different parity, (ii) $m < 0$, or (iii) $n < m$. Also observe that, to form a characteristic sequence of length $n$ whose sum is $m$, we must either append a (+1) to a characteristic sequence of length $n - 1$ whose sum is $m - 1$, or append a (-1) to a characteristic sequence of length $n - 1$ whose sum is $m + 1$. Thus,

$$\eta(n, m) = \eta(n - 1, m - 1) + \eta(n - 1, m + 1).$$

Finally, notice that Proposition 4.2.3 implies that $\eta(2n, 0) = C_n$.

We wish to count the number of characteristic sequences of length $n$. Let $\eta(n)$ denote this quantity and observe that

$$\eta(n) = \sum_{m=0}^{n} \eta(n, m).$$

We now use induction on $n$ to show that $\eta(n) = W_n$. For $n = 1$, our claim is immediate, since the only characteristic sequence of length 1 is the sequence (+1). Now suppose that, for some $n > 1$, we have $\eta(n - 1) = W_{n-1}$. Then

$$\eta(n) = \sum_{m=0}^{n} \eta(n, m)$$

$$= \sum_{m=0}^{n} \eta(n - 1, m - 1) + \sum_{m=0}^{n} \eta(n - 1, m + 1)$$

$$= \eta(n - 1, -1) + \eta(n - 1, 0) + 2 \left( \sum_{m=1}^{n-1} \eta(n - 1, m) \right) + \eta(n - 1, n) + \eta(n - 1, n + 1).$$
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But we observed earlier that \( \eta(n - 1, -1) = \eta(n - 1, n) = \eta(n - 1, n + 1) = 0 \). Thus,

\[
\eta(n) = 2 \sum_{m=0}^{n-1} \eta(n - 1, m) - \eta(n - 1, 0)
\]

\[
= 2 \sum_{m=0}^{n-1} \eta(n - 1, m) - \eta(n - 1, 0)
\]

\[
= 2\eta(n - 1) - \eta(n - 1, 0)
\]

\[
= 2W_{n-1} - \eta(n - 1, 0).
\]

If \( n \) is even, say \( n = 2k \) for some integer \( k \), then \( n - 1 \) is odd and so \( \eta(n - 1, 0) = 0 \). Thus,

\[
\eta(n) = 2W_{n-1} = 2 \left( \binom{2k - 1}{k - 1} \right) = \binom{2k - 1}{k} = \binom{2k}{k} = W_n.
\]

If \( n \) is odd, say \( n = 2k + 1 \) for some integer \( k \), then \( \eta(n - 1, 0) = \eta(2k, 0) = C_k \). In this case, Lemma 4.4.3 implies that

\[
\eta(n) = 2W_{n-1} - C_k = 2 \left( \binom{2k}{k} \right) - C_k = \binom{2k + 1}{k} = W_n.
\]

In either case, \( \eta(n) = W_n \), as desired. \( \square \)

Proposition 4.4.5. Let \( |Q| = n < \infty \) and let \( X_Q \) be binary. Then the number of strongly preseparable orders on \( Q \) is given by the formula

\[
\omega(n) = 2^n \prod_{k=1}^{n-1} \binom{2k}{2k-1}.
\]

Proof. Let \( Q = \{1, 2, \ldots, n\} \) and let \( Q' = Q - \{n\} \). Proposition 4.3.10 establishes that every strongly preseparable order \( \succ \) on \( Q \) is a symmetric preseparable extension of a strongly preseparable order on \( Q' \). Thus, to form a strongly preseparable order on \( Q \), we must:

(i) choose a strongly preseparable order \( \succ_1 \) on \( Q' \),

(ii) choose a linear order \( \succ_2 \) on \( X_n \) (either \( 1 \succ 0 \) or \( 0 \succ 1 \)), and

(iii) form a symmetric preseparable extension of \( \succ_1 \) by \( \succ_2 \).

It is clear that (i) may be completed in \( \omega(n - 1) \) distinct ways, and (ii) in 2 distinct ways. Since every linear order on \( X_{Q'} \) has height \( 2^{n-1} \), Propositions 4.3.9 and 4.4.4 imply that (iii) may be completed in \( W_{2n-1} = \binom{2n-1}{2n-2} \) distinct ways (since we must choose a symmetric characteristic sequence of length \( 2^n \)). Furthermore, Corollary 4.1.4 implies that all strongly preseparable orders formed in this manner will be distinct. Thus,

\[
\omega(n) = 2 \cdot \omega(n - 1) \cdot \binom{2n-1}{2n-2}.
\]

It is clear that \( \omega(1) = 2 \) and so a simple induction argument finishes the proof. \( \square \)
4.5 Other Combinatorial Results

The reader may wonder at this point why we have tackled the problem of counting symmetric, preseparable and strongly preseparable orders, but have avoided the problem of counting additive, separable, and monoseparable orders. Unfortunately, the answer to this question is more pragmatic than it is enlightening. Simply put, the elements of the latter three classes of orders seem to be rather difficult to count.

Consider, for example, the class of monoseparable orders. Kilgour [13] appears to have been the first to observe that the problem of determining the number of monoseparable binary preference orders on a criteria set $Q$ of cardinality $n$ is equivalent to the problem of determining the number of linear extensions of the partial order on $\mathcal{P}(Q)$ induced by inclusion (that is, the partial order $\succeq$ defined by $S_1 \succeq S_2 \iff S_1 \subseteq S_2$). In general, the problem of counting linear extensions of a given partial order is very hard. In fact, Brightwell and Winkler [5] have shown that this problem is \#P-Complete, meaning that it is among the hardest of the problems in the complexity class \#P (the class of counting problems associated with decision problems in NP). The \#P problem corresponding to any NP-Complete problem is NP-Hard, meaning intuitively that it is at least as hard as the hardest NP problems.\(^1\)

Sha and Kleitman [20] dealt specifically with the problem of counting linear extensions of the partial order induced by subset inclusion, arriving at asymptotic upper and lower bounds. Nevertheless, no progress has been made on a closed formula, and, given the complexity of the general problem of counting linear extensions, it seems unlikely that such a formula will be obtained.

We suspect that the problems of counting separable and additive orders are also quite difficult, though this suspicion is presently nothing more than an educated guess. Thus, we leave the following for future research:

**Open Question.** Is it possible to find a closed formula for the number of separable (or additive) binary preference orders on a given finite criteria set?

\(^1\)Formally, "a problem is in \#P if there is a nondeterministic, polynomial-time Turing machine that, for each instance $I$ of the problem, has a number of accepting computations that is exactly equal to the number of distinct solutions for instance $I$ (quoted from [1])."
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The table below summarizes the known combinatorial results pertaining to binary preference orders on small criteria sets.

| $|Q|$: | 2   | 3   | 4   | 5   | 6   |
|------|-----|-----|-----|-----|-----|
| Monoseparable | 8   | 384 | 26,886,144 | ?   | ?   |
| Preseparable    | 8   | 224 | 641,000   | $\sim 4.5 \times 10^{13}$ | $\sim 5.0 \times 10^{30}$ |
| Symmetric       | 8   | 384 | 10,321,920 | $\sim 1.4 \times 10^{18}$ | $\sim 1.1 \times 10^{45}$ |
| Strongly Preseparable | 8   | 96  | 13,440   | $\sim 3.5 \times 10^{8}$ | $\sim 4.2 \times 10^{17}$ |
| Separable       | 8   | 96  | 5,376    | ?   | ?   |
| Additive        | 8   | 96  | 5,376    | ?   | ?   |
| Lexicographic   | 8   | 48  | 384      | 3,840 | 46,080 |

Table 4.1: Counts of certain classes of orders on small criteria sets.

4.6 Summary of Related Properties

We conclude this chapter by summarizing our results concerning the various properties related to separability. Recall that we have shown the following inclusions between the various classes of preference orders on a finite criteria set:

lexicographic $\subseteq$ additive $\subseteq$ separable

$\subseteq$ strongly preseparable $\subseteq$ preseparable $\subseteq$ monoseparable.

For binary preferences on a criteria set having at least 5 elements, all of these inclusions are proper. For smaller criteria sets, we note the following:

- When the criteria set has 4 elements, additivity and separability are equivalent and all other inclusions are proper.

- When the criteria set has 3 elements, separability, strong preseparability and additivity are equivalent and all other inclusions are proper.

- When the criteria set has 2 elements, we have the following result:

**Proposition 4.6.1.** Let $|Q| = 2$ and let $\succ$ be a binary preference order on $Q$. The following are equivalent:
Chapter 4. Preseparable Orders

(i) \( \succ \) is lexicographic
(ii) \( \succ \) is additive
(iii) \( \succ \) is separable
(iv) \( \succ \) is strongly preseparable
(v) \( \succ \) is preseparable
(vi) \( \succ \) is monoseparable
(vii) \( \succ \) is symmetric

Proof. By our above remarks, (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) \( \Rightarrow \) (vi). Since \(|Q| = 2\), the only proper subsets of \( Q \) are singletons. Thus, separability and monoseparability are equivalent. But since the only normalized separable preference order on 2 criteria is the standard lexicographic order, all separable preference orders on \( Q \) are lexicographic. Thus, all monoseparable preference orders on \( Q \) are lexicographic, and so (vi) \( \Rightarrow \) (i). We now show (vii) \( \Leftrightarrow \) (i). Notice that, by Proposition 4.4.1, there are exactly 8 symmetric orders on \( Q \). But this is exactly the number of lexicographic orders on \( Q \), by Proposition 3.3.7. Since every lexicographic order on \( Q \) is separable and thus symmetric, and it follows that \( \succ \) is symmetric if and only if \( \succ \) is lexicographic, which completes the proof. \( \square \)
Chapter 5

Separability and The Symmetric Group

5.1 The Canonical Action

For this chapter, we assume that $Q$ is a finite criteria set, say $Q = \{1, 2, \ldots, n\}$, and that $X_Q$ is binary. Unless otherwise noted, we assume also that $n \geq 2$. Let $O_n$ denote the set of all linear orders on $X_Q$, i.e. the set of all binary preference orders on $n$ criteria. We wish to define an action of the symmetric group $S_{2^n}$ on the set $O_n$. Thus, let $\succ \in O_n$ be given and let $(x_k)$ be its corresponding order sequence. For any $\sigma \in S_{2^n}$, define $\sigma(\succ)$ to be the linear order whose corresponding order sequence $(y_k)$ is given by $y_{\sigma(k)} = x_k$, or more conveniently, $y_k = x_{\sigma^{-1}(k)}$. Observe that $\sigma(\succ)$ is well-defined since $\sigma$ is a permutation and is thus bijective. We claim that the map $(\sigma, \succ) \mapsto \sigma(\succ)$ is indeed a group action of $S_{2^n}$ on $O_n$.

To verify this claim, first notice that, if we denote by $1$ the identity permutation, then $1(\succ)$ is the the linear order whose order sequence $(y_k)$ is given by

$$y_k = x_{1^{-1}(k)} = x_{1(k)} = x_k.$$

Thus $1(\succ) = \succ$. We now must show that, for all $\sigma, \tau \in S_{2^n}$, $(\sigma\tau)(\succ) = \sigma(\tau(\succ))$. Recall that $(\sigma\tau)(\succ)$ is the linear order whose order sequence $(y_k)$ is given by

$$y_k = x_{(\sigma\tau)^{-1}(k)} = x_{(\tau^{-1}\sigma^{-1})(k)} = x_{\tau^{-1}(\sigma^{-1}(k))}.$$
Let $\succ' = \tau(\succ)$ and let $(w_k)$ be the order sequence corresponding to $\succ'$. Then, by definition, $w_k = x^{-1}(k)$. Now let $(z_k)$ be the order sequence corresponding to $\sigma(\succ') = \sigma(\tau(\succ))$. Then

$$z_k = w_{\sigma^{-1}(k)} = x_{\tau^{-1}(\sigma^{-1}(k))} = y_k,$$

which implies that $(y_k) = (z_k)$. Thus $(\sigma \tau)(\succ) = \sigma(\tau(\succ))$, as desired.

We call this action the **canonical action** of $S_{2^n}$ on $O_n$. It is straightforward to verify that the canonical action is both faithful and transitive. Moreover, the stabilizer of any order $\succ \in O_n$ is trivial.  

**5.2 Symmetry-Preserving Permutations**

Recall that, if $\succ$ is a symmetric order on $X_Q$ with corresponding order sequence $(x_k)$, then $\overline{x}_k = x_{2^n-k+1}$ for each $1 \leq k \leq 2^n$. For convenience and to be consistent with this notation, we define $\overline{k} = 2^n - k + 1$, so that $\succ$ is symmetric if and only if $\overline{x}_k = x_\overline{k}$ for each $k$.

**Definition 5.2.1.** A permutation $\sigma \in S_{2^n}$ is said to **preserve symmetry** if, whenever $\succ$ is a symmetric order on $X_Q$, then $\sigma(\succ)$ is also symmetric.

We denote by $\overline{S}_{2^n}$ the set of symmetry-preserving permutations in $S_{2^n}$. Similarly, we denote by $\overline{O}_n$ the collection of all symmetric orders on $X_Q$.

**Proposition 5.2.2.** The set $\overline{S}_{2^n}$ of symmetry-preserving permutations is a subgroup of $S_{2^n}$.

**Proof.** The identity permutation clearly preserves symmetry, and so $1 \in \overline{S}_{2^n}$. Since $\overline{S}_{2^n}$ is a finite subset of $S_{2^n}$, it now suffices to show that $\overline{S}_{2^n}$ is closed under function composition. Thus, let $\sigma$, $\tau \in \overline{S}_{2^n}$ and let $\succ$ be a symmetric order on $X_Q$. Since $\tau$ preserves symmetry, $\tau(\succ)$ is symmetric. But since $\sigma$ preserves symmetry, $(\sigma \tau)(\succ) = \sigma(\tau(\succ))$ is then symmetric, as desired. \hfill $\square$

**Proposition 5.2.3.** A permutation $\sigma \in S_{2^n}$ preserves symmetry if and only if

$$\sigma(r) = s \implies \sigma(\overline{r}) = \overline{s}$$

for all $1 \leq r, s \leq 2^n$.

---

1Recall that a group action $(G, A) \rightarrow A$ is said to be **faithful** if its kernel is the identity, i.e. if $\{\sigma \in G \mid \sigma(a) = a \text{ for all } a \in A\} = \{1\}$. The action is said to be **transitive** if it has only one orbit, i.e. if, for all $a, b \in A$, there exists $\sigma \in G$ such that $\sigma(a) = b$. The **stabilizer** $G_a$ of an element $a \in A$ is the set of all $\sigma \in G$ such that $\sigma(a) = a$, i.e. $G_a = \{\sigma \in G \mid \sigma(a) = a\}$. Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
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Proof. Let \( \succ \) be any symmetric order on \( X_Q \), and let \( (x_k) \) and \( (y_k) \) be the order sequences corresponding to \( \succ \) and \( \sigma(\succ) \), respectively.

(\( \Rightarrow \)) Suppose that \( \sigma \) preserves symmetry and that \( \sigma(r) = s \) for some \( r \) and \( s \). By the definition of the canonical action, \( y_s = y_{\sigma(r)} = x_r \). But since \( \sigma \) preserves symmetry, both \( \succ \) and \( \sigma(\succ) \) are symmetric. Thus, \( y_r = y_s = x_r = x_s \), which implies that \( \sigma(\overline{\phi}) = \overline{s} \), as desired.

(\( \Leftarrow \)) For the converse, suppose that \( \sigma(r) = s \implies \sigma(\overline{\phi}) = \overline{s} \) for all \( r \) and \( s \). Then for each \( k \), \( \sigma^{-1}(k) = \sigma^{-1}(\overline{k}) \) and so

\[
\overline{y}_k = \overline{x}_k = \overline{x}_{\sigma^{-1}(k)} = \overline{x}_{\sigma^{-1}(\overline{k})} = \overline{y}_k,
\]

which implies that \( \sigma(\succ) \) is symmetric. Since our choice of \( \succ \) was arbitrary, it follows that \( \sigma \) preserves symmetry. \( \square \)

Proposition 5.2.4. Let \( \succ \) be symmetric. Then \( \sigma(\succ) \) is symmetric if and only if \( \sigma \) preserves symmetry.

Proof. The reverse implication is immediate. Thus, assume that \( \sigma(\succ) \) is symmetric and assume, to the contrary, that \( \sigma \) does not preserve symmetry. Let \( (x_k) \) and \( (y_k) \) be the order sequences corresponding to \( \succ \) and \( \sigma(\succ) \), respectively. By Proposition 5.2.3, there exist \( r \) and \( s \) such that \( \sigma(r) = s \) and \( \sigma(\overline{\phi}) \neq \overline{s} \). But then, by the definition of the canonical action, \( y_s = x_r \) and \( y_s \neq x_r \). Since both \( \succ \) and \( \sigma(\succ) \) are symmetric, this then implies that \( \overline{y}_s = \overline{x}_s \neq \overline{x}_r = \overline{y}_r \), a contradiction. \( \square \)

Proposition 5.2.5. If \( \sigma \) preserves symmetry and \( \sigma(\succ) \) is symmetric, then \( \succ \) is symmetric.

Proof. Suppose that \( \sigma \) preserves symmetry and that \( \sigma(\succ) \) is symmetric. By Proposition 5.2.2, \( \sigma^{-1} \) preserves symmetry. But then it must be the case that \( \succ = \sigma^{-1}(\sigma(\succ)) \) is symmetric, as desired. \( \square \)

Note that the contrapositive of Proposition 5.2.5 says that symmetry-preserving permutations must also preserve asymmetry. Specifically, we know that if \( \sigma \) preserves symmetry and \( \succ \) is asymmetric, then \( \sigma(\succ) \) must also be asymmetric.

Proposition 5.2.6. There is a bijection between the set \( \overline{S}_{2^n} \) of symmetry-preserving permutations and the set \( \overline{O}_n \) of all symmetric orders on \( X_Q \). In particular,

\[
|\overline{S}_{2^n}| = |\overline{O}_n| = 2^{2^n-1} \cdot 2^{n-1}!
\]
Proof. Define a map $\varphi : S_{2^n} \to S_n$ by $\varphi(\sigma) = \sigma(\tau_{\text{lex}})$. Now $\varphi$ is injective since the stabilizer of $\tau_{\text{lex}}$ is trivial. To see that $\varphi$ is surjective, let $\succ$ be any symmetric order on $X_Q$ and choose $\sigma \in S_{2^n}$ so that $\sigma(\tau_{\text{lex}}) = \tau_{\text{lex}}$ (the transitivity of the canonical action implies that such a $\sigma$ exists). Since both $\tau_{\text{lex}}$ and $\tau$ are symmetric, Proposition 5.2.4 implies that $\sigma$ preserves symmetry. Thus, $\sigma^{-1}$ preserves symmetry and

$$\varphi(\sigma^{-1}) = \sigma^{-1}(\tau_{\text{lex}}) = \sigma^{-1}(\sigma(\tau_{\text{lex}})) = \tau_{\text{lex}},$$

as desired. It follows that $\varphi$ is a bijection and so Proposition 4.4.1 implies that $|S_{2^n}| = 2^{2^{n-1}} \cdot 2^{n-1}!$. \(\square\)

Proposition 5.2.7. Let $\sigma \in S_{2^n}$ have cycle decomposition $\sigma = \tau_1 \tau_2 \cdots \tau_q$. Then $\sigma$ preserves symmetry if and only if, for each $i \leq q$, either

(i) $\tau_i = (x_1, x_2, \ldots, x_j, x_1, x_2, \ldots, x_j)$, where $x_k \neq x_l$ for all $k, l \leq j$; or

(ii) $\tau_i = (x_1, x_2, \ldots, x_j)$, where $x_k \neq x_l$ for all $k, l \leq j$ and $\bar{\tau}_i = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_j) = \tau_p$ for some $p \neq i$.

Proof. ($\Rightarrow$) Suppose that $\sigma = \tau_1 \tau_2 \cdots \tau_q$ preserves symmetry and let $\tau_i = (x_1, x_2, \ldots, x_m)$ be given. Let $j + 1$ be the smallest index such that $x_{j+1} = \bar{x}_l$ for some $l < j + 1$. If there is no such index, then (ii) occurs by Proposition 5.2.3 and the proof is finished. Thus, assume that such an index exists. We claim that $l = 1$.

Suppose, to the contrary, that $l \neq 1$. Since $\sigma(x_j) = \bar{x}_l$, Proposition 5.2.3 implies that $\sigma(\bar{x}_l) = x_l$. But since $l \neq 1$, this implies that $\bar{x}_l = x_{l-1}$, from which it follows that $x_j = x_{l-1}$ with $l + 1 < j$, a contradiction to the minimality of $j + 1$.

Thus, $l = 1$ and so Proposition 5.2.3 now implies that $\tau_i = (x_1, x_2, \ldots, x_j, x_1, x_2, \ldots, x_j)$ with $x_k \neq x_l$ for all $k, l \leq j$, which is exactly case (i).

($\Leftarrow$) For the converse, suppose that, for each $i \leq q$, either (i) or (ii) holds. It is easy to see, in this case, that $\sigma(r) = s \implies \sigma(\bar{s}) = \bar{r}$ for all $r$ and $s$. Thus, Proposition 5.2.3 implies that $\sigma$ preserves symmetry, as desired. \(\square\)

For a cycle $\tau = (x_1, x_2, \ldots, x_m)$ we define the dual of $\tau$, denoted $\bar{\tau}$, by $\bar{\tau} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$, as in Proposition 5.2.7. If $\bar{\tau} = \tau$ (as in (i) above) then we say that $\tau$ is self-dual. Using this terminology, we see that if $\sigma$ preserves symmetry and $\tau$ appears in the cycle decomposition of $\sigma$, then $\tau$ is either self-dual or $\bar{\tau}$ also appears in the cycle decomposition of $\sigma$. Since self-dual cycles necessarily have even length, we conclude the following:
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Corollary 5.2.8. If the cycle decomposition of \( \sigma \) contains an odd number of cycles of odd length, then \( \sigma \) does not preserve symmetry.

Another obvious corollary of Proposition 5.2.7 is this:

Corollary 5.2.9. The transposition \((a, b)\) preserves symmetry if and only if it is self-dual.

There is a natural way to extend the notion of the dual of a cycle to an arbitrary permutation \( \sigma \) in \( S_{2^n} \). Specifically, if \( \sigma = \tau_1 \tau_2 \cdots \tau_q \), where \( \tau_1, \tau_2, \ldots, \tau_n \) are disjoint cycles, we define the dual of \( \sigma \) by \( \bar{\sigma} = \bar{\tau}_1 \bar{\tau}_2 \cdots \bar{\tau}_q \). Equivalently, \( \bar{\sigma} = \gamma \sigma \gamma^{-1} \), where

\[
\gamma = \left(x_1, \bar{x}_1\right)\left(x_2, \bar{x}_2\right) \cdots \left(x_{2^n-1}, \bar{x}_{2^n-1}\right).
\]

Using this latter definition, we observe that the map \( \sigma \mapsto \bar{\sigma} \) is a homomorphism, since

\[
\sigma \tau \mapsto \gamma \sigma \tau \gamma^{-1} = (\gamma \sigma \gamma^{-1})(\gamma \tau \gamma^{-1}) = \bar{\sigma} \bar{\tau} \text{ for all } \sigma, \tau \in S_{2^n}.
\]

Proposition 5.2.10. The group \( \overline{S}_{2^n} \) of symmetry-preserving permutations is isomorphic to the group of symmetries of a \( 2^{n-1} \)-dimensional hypercube.

Proof. Let \( S \) denote the group of symmetries of a \( 2^{n-1} \)-dimensional hypercube. Any such symmetry can be obtained by permuting the \( 2^{n-1} \) dimensions of the hypercube and then reflecting over a subset of these dimensions. A straightforward argument thus shows that

\[
S \cong \left[Z_2\right]^{2^n-1} \times S_{2^{n-1}} = Z_2 \wr S_{2^{n-1}},
\]

where \( Z_2 \) denotes the cyclic group of 2 elements (see [9]). We claim that \( \overline{S}_{2^n} \) is isomorphic to this same group. Specifically, define \( S^1 = \langle (a, \bar{a}) \mid 1 \leq a \leq 2^{n-1} \rangle \) and \( S^2 = \langle \sigma \bar{\sigma} \mid \sigma \in S_{2^{n-1}} \rangle \). By Proposition 5.2.7, both \( S^1 \) and \( S^2 \) are subgroups of \( \overline{S}_{2^n} \). We will show that \( S^1 \cong \left[Z_2\right]^{2^{n-1}} \), \( S^2 \cong S_{2^{n-1}} \), and that \( S = S^1 \rtimes S^2 \).

To show that \( S^1 \cong \left[Z_2\right]^{2^{n-1}} \), we note that \( S^1 \) is an abelian group of order \( 2^{2^{n-1}} \) and that each nonidentity element of \( S^1 \) has order 2. Up to isomorphism, there is only one such group, namely \([Z_2]^{2^{n-1}}\), as desired.

Next, note that in order to define a homomorphism \( \varphi : S^2 \to S_{2^{n-1}} \), it suffices to specify the action of \( \varphi \) on the generators of \( S^2 \) and then show that \( \varphi \) preserves multiplication among these generators. Thus, we define \( \varphi \) by \( \varphi(\sigma \bar{\sigma}) = \sigma \). For all \( \sigma, \tau \in S_{2^{n-1}} \), the cycle decompositions of \( \sigma \) and \( \tau \) are disjoint and so \( \sigma \) and \( \tau \) commute. Thus,

\[
\varphi(\sigma \bar{\sigma} \tau) = \varphi(\sigma \tau \bar{\sigma}) = \varphi((\sigma \tau)(\bar{\sigma} \bar{\tau})) = \sigma \tau = \varphi(\sigma \bar{\sigma}) \varphi(\tau \bar{\tau}),
\]
and therefore $\varphi$ is a homomorphism. It is clear that $\varphi$ is bijective and so we have shown that $S^2 \cong S_{2n-1}$.

It remains to show that $S = S^1 \times S^2$. First, we claim that $S^1$ is a normal subgroup of $\overline{S}_{2n}$. To verify this claim, let $\sigma = (x_1, \overline{x}_1)(x_2, \overline{x}_2) \cdots (x_m, \overline{x}_m) \in S^1$ and let $\tau \in \overline{S}_{2n}$. Then

$$\tau \sigma \tau^{-1} = (\tau(x_1), \tau(\overline{x}_1))(\tau(x_2), \tau(\overline{x}_2)) \cdots (\tau(x_m), \tau(\overline{x}_m)).$$

But since $\tau$ preserves symmetry, $\tau(x_i) = \tau(\overline{x}_i)$ for all $i$. Thus

$$\tau \sigma \tau^{-1} = (\tau(x_1), \tau(x_1))(\tau(x_2), \tau(x_2)) \cdots (\tau(x_m), \tau(x_m)) \in S^1,$$

which proves that $S^1$ is normal in $\overline{S}_{2n}$. From this it follows that $S^1 S^2$ is a subgroup of $\overline{S}_{2n}$. But $S^1 \cap S^2 = 1$, and so $S^1 S^2 \cong S^1 \times S^2$. Furthermore, Proposition 5.2.6 implies that

$$|S^1 S^2| = |S^1||S^2| = 2^{2n-1} \cdot 2^{n-1} = |\overline{S}_{2n}|.$$

Thus $\overline{S}_{2n} = S^1 \times S^2$, which concludes the proof. $\Box$

### 5.3 Separability-Preserving Permutations

We now turn our attention to the set of permutations that preserve the property of separability.

**Definition 5.3.1.** A permutation $\sigma \in \overline{S}_{2n}$ is said to preserve separability if, whenever $\succ$ is a separable order on $X_Q$, then $\sigma(\succ)$ is also separable.

---

2We confess that our notation here is a bit ambiguous. By $S^1 \times S^2$, we mean the semidirect product of $S_1$ and $S_2$ with respect to the homomorphism $\alpha : S^2 \to \text{Aut}(S^1)$, where $\alpha$ maps $\gamma \in S^2$ to the automorphism of conjugation by $\gamma$ on $S^1$. By $[Z_2]^{2^{n-1}} \times S_{2n-1}$ we mean the semidirect product of $Z_2 \times Z_2 \times \cdots \times Z_2$ and $S_{2n-1}$ with respect to the homomorphism $\beta : S_{2n-1} \to \text{Aut}([Z_2]^{2^{n-1}})$ defined by $(\beta(\sigma))(x_1, x_2, \ldots, x_m) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)})$. By definition, this semidirect product is the wreath product of $Z_2$ and $S_{2n-1}$, which we denote by $Z_2 \wr S_{2n-1}$. Note that elements of $S^1$ have the form $(i_1, \overline{i}_1)(i_2, \overline{i}_2) \cdots (i_m, \overline{i}_m)$ for $1 \leq i_k \leq 2^{n-1}$. Thus, we can explicitly specify the isomorphism $\psi : S^1 \to [Z_2]^{2^{n-1}}$ by sending the element $(i_1, \overline{i}_1)(i_2, \overline{i}_2) \cdots (i_m, \overline{i}_m)$ to the vector having 1's in the $i_k$ components and 0's elsewhere. Under this isomorphism and the isomorphism $\varphi : S^2 \to S_{2n-1}$ specified in our proof, it is straightforward to verify that the homomorphisms $\alpha$ and $\beta$ induce the same semidirect product. We also note that we do need to use this more complicated machinery (instead of the simpler direct product) to describe the group. Indeed, since $S^2$ is not normal in $\overline{S}_{2n}$ (this can be easily verified), it follows that $\overline{S}_{2n} \cong S^1 \times S^2 \not\cong S^1 \times S^2$.

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We denote by $S^*_n$ the set of separability-preserving permutations. Similarly, we denote by $O^*_n$ the collection of all separable orders on $X_Q$.

**Proposition 5.3.2.** The set $S^*_n$ of separability-preserving permutations is a subgroup of $S_2^n$.

**Proof.** Let $\sigma \in S^*_n$ and choose some $\succ \in O^*_n$. Since $\sigma$ preserves separability and $\succ$ is separable, it follows that $\sigma(\succ)$ is separable. But then Proposition 4.3.3 implies that both $\succ$ and $\sigma(\succ)$ are symmetric. It then follows from Proposition 5.2.4 that $\sigma$ preserves symmetry. Thus, $\sigma \in S_2^n$ and so we have shown that $S^*_n \subseteq S_2^n$. The proof that $S^*_n$ is a subgroup is analogous to the proof of Proposition 5.2.2. □

**Proposition 5.3.3.** If $\sigma$ preserves separability and $\sigma(\succ)$ is separable, then $\succ$ is separable.

**Proof.** The result is immediate, since $\succ = \sigma^{-1}(\sigma(\succ))$ and, by Proposition 5.3.2, $\sigma^{-1}$ preserves separability. □

We note here (analogous to our remarks following Proposition 5.2.5), that Proposition 5.3.2 implies that a separability-preserving permutation also preserves nonseparability. Unfortunately, not all of the nice properties of $S_2^n$ carry over to their analogs in $S^*_n$. For example, consider the action of the permutation $\sigma = (2, 4, 7, 5)$ on $\succ_{lex}$:

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\xrightarrow{\sigma}
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\xrightarrow{\sigma}
\begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

Notice that both $\succ_{lex}$ and $\sigma(\succ_{lex})$ are separable, and yet $\sigma(\sigma(\succ_{lex})) = \sigma^2(\succ_{lex})$ is nonseparable. It follows that $\sigma$ does not preserve separability.

We see then that, while the property of preserving symmetry is global in the sense that if $\sigma$ preserves the symmetry of one symmetric order, then $\sigma$ preserves the symmetry of all symmetric orders, the property of preserving separability is more localized. Specifically, we see that it is possible for a permutation to preserve the separability of one separable order while failing to preserve the separability of another. To formalize this local property, we make the following definition:
Definition 5.3.4. Let $\succ$ be a separable order on $X_Q$. A permutation $\sigma \in S_{2^n}$ is said to (locally) preserve the separability of $\succ$ if $\sigma(\succ)$ is separable.

An obvious consequence of this definition is the following:

Proposition 5.3.5. A permutation $\sigma \in S_{2^n}$ preserves separability if and only if $\sigma$ preserves the separability of each $\succ \in \mathcal{O}_n^*$.

For a separable order $\succ$, we denote by $S_{2^n}^\succ$ the set of permutations which preserve the separability of $\succ$. In the special case where $\succ = \succ_{lex}$, we write $S_{2^n}^{\text{lex}}$ instead.

Notice that if $\sigma$ preserves the separability of some separable preference order $\succ$, then both $\succ$ and $\sigma(\succ)$ are symmetric (by Proposition 4.3.3), from which it follows by Proposition 5.2.4 that $\sigma$ preserves symmetry. Thus, $S_{2^n}^\succ \subseteq S_{2^n}$ for each $\succ \in \mathcal{O}_n^*$.

Is $S_{2^n}^\succ$ a subgroup of $S_{2^n}$? We know, by the previous example, that for $\succ = \succ_{lex}$, the answer to this question is no (since $\sigma$ preserves the separability of $\succ_{lex}$ and $\sigma^2$ does not). On the other hand, it would not be entirely unreasonable to think that we might be able to find some separable order $\succ$ for which things turn out differently and for which $S_{2^n}^\succ$ is indeed a subgroup of $S_{2^n}$. The reality of the situation is that, for $n \geq 3$, no such order exists: For each $\succ \in \mathcal{O}_n^*$, $S_{2^n}^\succ$ fails to be a group. We must, however, delay the proof of this claim until we have built up some stronger machinery (the result and its proof are presented near the end of this chapter as Proposition 5.4.37).

Proposition 5.3.6. Let $\succ \in \mathcal{O}_n^*$. Then $|S_{2^n}^\succ| = |\mathcal{O}_n^*|$. 

Proof. We prove the result by exhibiting a bijection between $S_{2^n}^\succ$ and $\mathcal{O}_n^*$. Let $\varphi : S_{2^n}^\succ \rightarrow \mathcal{O}_n^*$ be the map defined by $\varphi(\sigma) = \sigma(\succ)$. Since $\succ$ is separable and $\sigma$ preserves the separability of $\succ$, $\varphi$ is well-defined. To see that $\varphi$ is injective, recall that:

\[ \sigma_1(\succ) = \sigma_2(\succ) \iff \sigma_2^{-1}\sigma_1 \text{ stabilizes } \succ \iff \sigma_2^{-1}\sigma_1 = 1 \iff \sigma_1 = \sigma_2. \]

To see that $\varphi$ is surjective, note that, for any $\succ' \in \mathcal{O}_n^*$, there exists $\sigma \in S_{2^n}$ such that $\sigma(\succ) = \succ'$ (by the transitivity of the canonical action). But since $\succ$ and $\succ'$ are both separable, it follows that $\sigma \in S_{2^n}^\succ$. Thus $\varphi$ is surjective, as desired. $\square$

Corollary 5.3.7. For all $\succ_1, \succ_2 \in \mathcal{O}_n^*$, $|S_{2^n}^{\succ_2^{-1}}| = |S_{2^n}^{\succ_2}|$. 

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Chapter 5. Separability and The Symmetric Group

5.4 The Structure of $S^*_2$

We now turn our attention to determining the structure of the group $S^*_2$. As we observed earlier, this group does not behave as nicely as $S^*_2$. As such, it will be considerably more difficult to characterize. We will give the punchline first and then build up the theory necessary to prove our claims.

Let $V_4$ denote the Klein 4-group and let $D_4$ denote the group of symmetries of the square (the dihedral group of degree 4). Recall that $V_4 \cong Z_2 \times Z_2$.

**Theorem 5.4.1.**

(i) For $n = 2$, $S^*_2 = S^*_4 \cong D_4$.

(ii) For $n = 3$, $S^*_2 = S^*_8 \cong V_4 \times S_3$.

(iii) For $n \geq 4$, $S^*_2 \cong V_4$.

The proof of Theorem 5.4.1 is, for the most part, combinatorial in nature. The cases for $n \leq 4$ are verified by hand and/or computer using the machinery of Proposition 3.1.3. An induction argument then completes the proof.

We should point out that we actually prove a stronger claim than that which is asserted by Theorem 5.4.1. Specifically, we show that $\sigma \in S^*_2$ if and only if its inversion number $\text{inv}(\sigma)$ belongs to the set $\{0, 1, (\binom{n}{2}) - 1, \binom{n}{2}\}$. We then prove that this set of inversion numbers corresponds uniquely to a set of permutations isomorphic to $V_4$.

The glue that holds the induction together is the claim that if $S^*_2$ contains a permutation $\sigma$ for which $1 < \text{inv}(\sigma) < (\binom{n}{2}) - 1$, then $S^*_2 - 1$ contains a permutation $\sigma'$ for which $1 < \text{inv}(\sigma') < (\binom{n-1}{2}) - 1$. The proof of this fact is lengthy and, at times, tedious. Nevertheless, it is not overly complicated and the patient reader should have no trouble with the details. Of course, one can skip many of the proofs of the intermediate results that follow and still gain a good understanding of the machinery used to prove the main theorem.

We begin by developing some tools that will allow us to translate results about $S^*_2$ into results about $S^*_2 - 1$.

Let $\succ \in C^*_n$ and let $(x_\lambda)$ be the order sequence corresponding to $\succ$. Then we define a map $p_i : C^*_n \to C^*_{n-1}$, called the $i^{th}$ projection map on $C^*_n$, by $p_i(\succ) = \succ_{Q-\{s_i\}}$. This map is well-defined by Corollary 2.2.7.
Lemma 5.4.2. Let \( \succ \in \mathcal{O}_{n-1}^* \) and let \( \succ_1 \in \mathcal{O}_1 \) (so either \( 1 \succ_1 0 \) or \( 0 \succ_1 1 \)). Then

(i) \( p_1(\succ_1 \oplus \succ) = \succ \).

(ii) \( p_n(\succ \oplus \succ_1) = \succ \).

Proof. This follows directly from Definition 3.3.1. \( \square \)

For a fixed \( \succ \in \mathcal{O}_n^* \), the projection maps defined above induce maps \( s_i : S_{2^n} \to S_{2^n-1}^* \) in the following manner: For \( \sigma \in S_{2^n}^* \), let \( \sigma' \in S_{2^n-1}^* \) be the unique permutation for which \( \sigma'(p_i(\succ)) = p_i(\sigma(\succ)) \). Since \( \succ \) is separable and \( \sigma \) preserves separability, \( \sigma' \) is well-defined. Now define \( s_i \) to be the map which sends \( \sigma \) to \( \sigma' \).

Definition 5.4.3. Let \( \sigma \in S_m \) and let \( a \) and \( b \) be integers such that \( 1 \leq a < b \leq m \). We say that \( \sigma \) inverts the pair \( (a, b) \) if \( \sigma(a) > \sigma(b) \). We denote by \( \text{inv}(\sigma) \) the number of distinct pairs inverted by \( \sigma \).

Note that, for \( \sigma \in S_m \), \( 0 \leq \text{inv}(\sigma) \leq \binom{2^n}{2} \). Furthermore, \( \text{inv}(\sigma) = 0 \) if and only if \( \sigma = 1 \) and \( \text{inv}(\sigma) = \binom{2^n}{2} \) if and only if \( \sigma = \tau_r \), where \( \tau_r \) is the reflection permutation, defined by \( \tau_r(a) = m - a + 1 = \bar{a} \) for all \( a \).

Proposition 5.4.4. Let \( \succ \in \mathcal{O}_n^* \) have order sequence \( (x_k) \), let \( \sigma \in S_{2^n}^* \), and let \( i \leq n \) be given. Let \( (y_k) \) be the order sequence corresponding to \( p_i(\succ) \) and let \( a, b \leq 2^n \) be such that \( x_a = (y_c, j) \) and \( x_b = (y_d, j) \) for some \( c, d \leq 2^{n-1} \) and some \( j \in \{0, 1\} \). Then \( \sigma \) inverts \( (a, b) \) if and only if \( s_i(\sigma) \) inverts \( (c, d) \).

Proof. Let \( x_a = (y_c, j) \) and \( x_b = (y_d, j) \) be as above. Without loss of generality, assume that \( x_a \succ x_b \), so that \( a < b \). Let \( \succ_i = p_i(\succ) \), let \( \succ' = \sigma(\succ) \), and let \( \succ'_i = p_i(\sigma(\succ)) = p_i(\sigma(\succ')) \). It follows from the definition of \( p_i(\succ) \) that \( y_c \succ_i y_d \). Now observe that

\[
\sigma \text{ inverts } (a, b) \iff \sigma(a) > \sigma(b)
\iff (y_d, j) = x_b \succ' x_a = (y_c, j)
\iff y_d \succ'_i y_c.
\]

Note that \( s_i \) depends on our choice of \( \succ \). This notational ambiguity will cause no real difficulties, as the choice of \( \succ \) should be made clear by the context in which \( s_i \) appears.

In this definition, our pairs are unordered, in the sense that we do not distinguish between \( (a, b) \) and \( (b, a) \). Our convention will be to list the smaller of \( a \) and \( b \) first whenever their relative sizes are known.
But since \( s_i(\sigma) \) is the unique permutation for which
\[
  s_i(\sigma)(\gamma_i) = s_i(\sigma)(p_i(\gamma)) = p_i(\sigma(\gamma)) = p_i(\gamma') = \gamma_i,
\]
it follows that \( \sigma \) inverts \((a,b)\) if and only if \( y_d \succ y_c \), if and only if \( s_i(\sigma) \) inverts \((c,d)\). \( \square \)

Lemma 5.4.5. The permutation \( \sigma \) inverts \((a,b)\) if and only if one of the following conditions holds:

(i) \( \tau \) inverts \((a,b)\) and \( \sigma \) does not invert \((\tau(b),\tau(a))\); or

(ii) \( \tau \) does not invert \((a,b)\) and \( \sigma \) inverts \((\tau(a),\tau(b))\).

Proof. If either (i) or (ii) holds, then \( \sigma(\tau(a)) > \sigma(\tau(b)) \), as desired. Now suppose conversely that \( \tau \sigma \) inverts \((a,b)\). Then \( \sigma(\tau(a)) > \sigma(\tau(b)) \). If \( \tau \) inverts \((a,b)\), then \( \tau(b) < \tau(a) \), which implies that \( \sigma \) does not invert \((\tau(b),\tau(a))\). On the other hand, if \( \tau \) does not invert \((a,b)\), then \( \tau(a) < \tau(b) \), which implies that \( \sigma \) inverts \((\tau(a),\tau(b))\). Since \( \tau \) must either invert or not invert the pair \((a,b)\), it follows that either (i) or (ii) must occur. \( \square \)

Lemma 5.4.6. Let \( \sigma \in S_{2^n} \). Then \( \text{inv}(\tau,\sigma) = \binom{2^n}{2} - \text{inv}(\sigma) \).

Proof. Since \( \tau \) inverts all possible pairs, Lemma 5.4.5 implies that \( \tau \sigma \) inverts \((a,b)\) if and only if \( \sigma \) does not invert \((a,b)\). The result then follows from the fact that there are \( \binom{2^n}{2} \) possible pairs. \( \square \)

Lemma 5.4.7. If \( \sigma \) inverts \((a,b)\), then \( \sigma^{-1} \) inverts \((\sigma(b),\sigma(a))\).

Proof. Suppose \( \sigma \) inverts \((a,b)\). Then \( \sigma(b) < \sigma(a) \) and \( \sigma^{-1}(\sigma(b)) = b > a = \sigma^{-1}(\sigma(a)) \), which implies that \( \sigma^{-1} \) inverts \((\sigma(b),\sigma(a))\). \( \square \)

The following proposition implies that a permutation is uniquely determined by the set of pairs that it inverts.\(^5\)

Proposition 5.4.8. If \( \sigma \neq \tau \), then there exists a pair \((a,b)\) such that \((a,b)\) is inverted by exactly one of \( \sigma \) or \( \tau \).

Proof. Suppose \( \sigma \neq \tau \). Then \( \sigma \tau^{-1} \neq 1 \). By our comments following Definition 5.4.3, this implies that \( \sigma \tau^{-1} \) inverts some pair \((a,b)\). Thus, by Lemma 5.4.5, it must be the case that either

(i) \( \tau^{-1} \) inverts \((a,b)\) and \( \sigma \) does not invert \((\tau^{-1}(b),\tau^{-1}(a))\); or

\(^{5}\)This set of pairs is sometimes referred to as the inversion table of the permutation.
(ii) $\tau^{-1}$ does not invert $(a, b)$ and $\sigma$ inverts $(\tau^{-1}(a), \tau^{-1}(b))$.

If (i) occurs, then Lemma 5.4.7 implies that $\tau$ inverts $(\tau^{-1}(b), \tau^{-1}(a))$ (and $\sigma$ does not). If (ii) occurs, the same lemma implies that $\tau$ does not invert $(\tau^{-1}(a), \tau^{-1}(b))$ (and $\sigma$ does).

In either case, the pair $(\tau^{-1}(a), \tau^{-1}(b))$ is inverted by exactly one of $\sigma$ or $\tau$. □

Proposition 5.4.9. Let $\succ_1 \in \mathcal{O}_{n-1}^*$ and let $\succ_2 \in \mathcal{O}_1^*$. Then, for any $\sigma \in S_{2^n}^*$,

(i) The permutation $s_1(\sigma)$ induced by $p_1(\succ_2 \oplus \succ_1)$ does not depend on the choice of $\succ_1$. Furthermore, $s_1(\sigma) \in S_{2^n}^*$.

(ii) The permutation $s_2(\sigma)$ induced by $p_2(\succ_1 \oplus \succ_2)$ does not depend on the choice of $\succ_1$. Furthermore, $s_2(\sigma) \in S_{2^n}^*$.

Proof. We will prove (i) and leave the analogous proof of (ii) to the reader. Assume, without loss of generality, that $1 \succ_2 0$ and let $(x_k), (y_k)$ be the order sequences corresponding to $\succ_1$ and $\succ_2 \oplus \succ_1$, respectively. Let $\sigma \in S_{2^n}$ be given and let $\sigma' = s_1(\sigma) \in S_{2^n}$. Then $\sigma'$ is the unique permutation for which $\sigma'(p_1(\succ_2 \oplus \succ_1)) = p_1(\sigma(\succ_2 \oplus \succ_1))$. But by Lemma 5.4.2, $p_1(\succ_2 \oplus \succ_1) = \succ_1$ and so $\sigma'$ is the unique permutation for which $\sigma'(\succ_1) = p_1(\sigma(\succ_2 \oplus \succ_1))$.

Thus, for any pair $(a, b)$, we have

\[
\sigma' \text{ inverts } (a, b) \iff x_b \sigma'(\succ_1) x_a \\
\iff x_b p_1(\sigma(\succ_2 \oplus \succ_1)) x_a \\
\iff (1, x_b) \sigma(\succ_2 \oplus \succ_1) (1, x_a) \text{ and } (0, x_b) \sigma(\succ_2 \oplus \succ_1) (0, x_a) \\
\iff y_b \sigma(\succ_2 \oplus \succ_1) y_a \text{ and } y_b+2^{n-1} \sigma(\succ_2 \oplus \succ_1) y_a+2^{n-1} \\
\text{(by Proposition 3.3.5)} \\
\iff \sigma \text{ inverts both } (a, b) \text{ and } (a+2^{n-1}, b+2^{n-1}).
\]

Thus, the inversion table for $\sigma'$ is completely determined by the inversion table for $\sigma$, which clearly does not depend on the choice of $\succ_1$. By Proposition 5.4.8, $\sigma'$ is uniquely determined by its inversion table. Consequently, $\sigma' = s_1(\sigma)$ is uniquely determined by $\sigma$ alone and hence does not depend on the choice of $\succ_1$.

To see that $\sigma' = s_1(\sigma) \in S_{2^n}^*$, observe that, for any $\succ_1 \in \mathcal{O}_{n-1}^*$, $\sigma(\succ_2 \oplus \succ_1)$ is separable and thus, by Corollary 2.2.7, $p_1(\sigma(\succ_2 \oplus \succ_1))$ is separable. But then, by Lemma 5.4.2 and the definition of $s_1(\sigma)$, we have

\[
\sigma'(\succ_1) = \sigma'(p_1(\succ_2 \oplus \succ_1)) = p_1(\sigma(\succ_2 \oplus \succ_1)),
\]
which we just observed to be separable. Since \(\sigma_1\) was chosen arbitrarily, it follows that 
\[\sigma' = s_1(\sigma) \in S_{2^n - 1}^*.\]

**Lemma 5.4.10.** If \(\sigma\) inverts \((a, b)\) and \(a < c < b\), then \(\sigma\) inverts either \((a, c)\) or \((c, b)\).

**Proof.** Suppose, to the contrary, that \(\sigma\) inverts neither \((a, c)\) nor \((c, b)\). Then 
\[\sigma(a) < \sigma(c) < \sigma(b),\]
a contradiction to the assumption that \(\sigma\) inverts \((a, b)\). \(\square\)

**Lemma 5.4.11.** If \(\sigma\) inverts \((a, b)\) and \((b, c)\), then \(\sigma\) inverts \((a, c)\).

**Proof.** If \(\sigma\) inverts \((a, b)\) and \(\sigma\) inverts \((b, c)\), then \(\sigma(a) > \sigma(b) > \sigma(c)\), as desired. \(\square\)

**Lemma 5.4.12.** Let \(\sigma \in S_{2^n}\). If \(\sigma\) inverts \((a, b)\), then \(\sigma\) inverts \((\bar{b}, \bar{a})\).

**Proof.** Suppose \(\sigma\) inverts \((a, b)\). Then \(\sigma(a) > \sigma(b)\), which implies that 
\[\sigma(\bar{b}) = \overline{\sigma(b)} > \overline{\sigma(a)} = \sigma(\bar{a}),\]
as desired. \(\square\)

**Proposition 5.4.13.** Let \(\sigma \in S_{2^n}\). If \(\text{inv}(\sigma) = 1\), then \(\sigma = (2^{n-1}, 2^{n-1} + 1)\).

**Proof.** Suppose \(\text{inv}(\sigma) = 1\) and let \((a, b)\) be the pair inverted by \(\sigma\). By Lemma 5.4.10, \(b = a + 1\), for otherwise there would exist \(c\) with \(a < c < b\) and hence another inversion. By Lemma 5.4.12, \(\sigma\) also inverts \((\bar{b}, \bar{a}) = (\bar{a} + 1, \bar{a})\), which implies that \(a = \bar{a} + 1 = 2^n - a\). Thus, \(a = 2^{n-1}\) and the pair inverted by \(\sigma\) is exactly \((2^{n-1}, 2^{n-1} + 1)\). Since the permutation \((2^{n-1}, 2^{n-1} + 1)\) inverts this pair and no others, Proposition 5.4.8 implies that 
\[\sigma = (2^{n-1}, 2^{n-1} + 1)\]. \(\square\)

We call the permutation \((2^{n-1}, 2^{n-1} + 1)\) the **central transposition**, denoted by \(\tau_c\).

**Lemma 5.4.14.** If \(\text{inv}(\sigma) = \binom{2^n}{2} - 1\), then \(\sigma = \tau_r \tau_c\).

**Proof.** If \(\text{inv}(\sigma) = \binom{2^n}{2} - 1\), then \(\text{inv}(\tau_r \sigma) = 1\) (by Lemma 5.4.6). Thus, \(\tau_r \sigma = \tau_c\), which implies that \(\sigma = \tau_r \tau_c\). \(\square\)

**Definition 5.4.15.** Let \(\succ \in \mathcal{O}_n\) and let \((x_k)\) be the order sequence corresponding to \(\succ\). The 4-tuple \((a, b, c, d)\), where \(a < b\) and \(c < d\), is said to be a **complete set** of indices (with respect to \(\succ\)) if for some \(S \subset Q\), there exist \(y, z \in X_S\) and \(u, v \in X_{\neg S}\) such that 
\[
x_a = (y, u) \quad x_c = (y, v) \\
x_b = (z, u) \quad x_d = (z, v)
\]
Proposition 5.4.16. Let $\sigma \in S_{2^n}$ and let $\succ \in O_n^*$. Then $\sigma$ preserves the separability of $\succ$ if and only if, for each complete set $\langle a, b, c, d \rangle$, $\sigma$ inverts either both or neither of the pairs $\langle a, b \rangle, \langle c, d \rangle$.

Proof. ($\Rightarrow$) We prove the contrapositive. Thus, suppose that there exists a complete set $\langle a, b, c, d \rangle$ such that $\sigma$ inverts exactly one of $\langle a, b \rangle$ and $\langle c, d \rangle$. Without loss of generality, assume that $\langle a, b \rangle$ is the inverted pair. Let $S \subset Q$ and $x_a, x_b, x_c, x_d$ be as in Definition 5.4.15 and let $\succ' = \sigma(\succ)$. Then $x_b \succ' x_a$ (since $x_a \succ x_b$) and $x_c \succ' x_d$. This, however, implies that $S$ is not $\succ'$-separable, since $z \succ' y$ given a choice of $u$ on $X_S$ but $y \succ' z$ given a choice of $v$ on $X_S$. Thus $\succ' = \sigma(\succ)$ is nonseparable and so $\sigma$ does not preserve the separability of $\succ$.

($\Leftarrow$) For the converse, suppose that $\sigma$ does not preserve the separability of $\succ$; that is, suppose that $\succ' = \sigma(\succ)$ is not separable. Then for some $S \subset Q$, there exist $y, z \in X_S$ and $u, v \in X_{-S}$ such that $(z, u) \succ' (y, u)$ and $(y, v) \succ' (z, v)$. But since $\succ$ is separable, either $(y, u) \succ (z, u)$ and $(y, v) \succ (z, v)$ or $(z, u) \succ (y, u)$ and $(z, v) \succ (y, v)$. Without loss of generality, assume the former. Let $(x_k)$ be the order sequence corresponding to $\succ$ and let $a, b, c, d$ be the indices such that

$$x_a = (y, u), \quad x_c = (y, v),$$

$$x_b = (z, u), \quad x_d = (z, v).$$

Then $\langle a, b, c, d \rangle$ is a complete set. Furthermore $\sigma$ inverts $\langle a, b \rangle$ (since $x_b \succ' x_a$) but does not invert $\langle c, d \rangle$ (since $x_c \succ' x_d$).

Corollary 5.4.17. Let $\sigma \in S_{2^n}$ and let $\succ_1, \succ_2 \in O_n^*$. If $\succ_1$ and $\succ_2$ have the same complete sets, then $\sigma$ preserves the separability of $\succ_1$ if and only if $\sigma$ preserves the separability of $\succ_2$.

Corollary 5.4.18. A permutation $\sigma$ belongs to $S_{2^n}$ if and only if $\sigma$ preserves the separability of each normalized binary preference order in $O_n^*$.

Proof. The forward implication is immediate. For the converse, suppose that $\sigma$ preserves the separability of each normalized binary preference order in $O_n^*$. For any $\succ \in O_n^*$ there exists a normalized binary preference order $\succ' \in O_n^*$ such that $\succ$ can be obtained from $\succ'$ by reordering the criteria and replacing a subset of the components of each alternative with their bitwise complements. Clearly this process preserves the complete sets of $\succ'$ and so $\succ$ and $\succ'$ have the same complete sets. Since $\sigma$ preserves the separability of $\succ'$, Corollary 5.4.17 implies that $\sigma$ preserves the separability of $\succ$. Since $\succ$ was chosen arbitrarily, it follows that $\sigma \in S_{2^n}^*$. 

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Corollary 5.4.19. The central transposition \( \tau_c \) belongs to \( S_{2^n}^* \).

Proof. Let \( \succ \in C_n^* \) and let \( (x_k) \) be the order sequence corresponding to \( \succ \). Then \( \succ \in C_n^* \) and so \( x_{2^n-1} = x_{2^n-1+1} \). Thus, there does not exist a complete set \( (a, b, c, d) \) for which \( a = 2^n-1 \) and \( b = 2^n-1 + 1 \) or for which \( c = 2^n-1 \) and \( d = 2^n-1 + 1 \). Since \( (2^n-1, 2^n-1 + 1) \) is the only pair inverted by \( \tau_c \), Proposition 5.4.16 implies that \( \tau_c \) preserves the separability of \( \succ \). Since our choice of \( \succ \) was arbitrary, it follows that \( \tau_c \in S_{2^n}^* \). □

Corollary 5.4.20. The reflection permutation \( \tau_r \) belongs to \( S_{2^n}^* \).

Proof. For any complete set \( (a, b, c, d) \), \( \tau_r \) inverts both \( (a, b) \) and \( (c, d) \). Thus, by Proposition 5.4.16, \( \tau_r \) preserves the separability of any \( \succ \in C_n^* \), from which it follows that \( \tau_r \in S_{2^n}^* \). □

Corollary 5.4.21. \( S_{2^n}^* \) contains a subgroup isomorphic to \( V_4 \).

Proof. Since \( |\tau_r| = |\tau_c| = |\tau_c \tau_r| = 2 \), the subgroup \( S = \{1, \tau_c, \tau_r, \tau_c \tau_r\} \) contains exactly three nonidentity elements, all of which have order 2. Thus, \( S \cong V_4 \). □

Let \( \sigma \in S_{2^n} \) and let \( (a, b) \) be some pair. If both \( a \) and \( b \) are even, then we say that the pair \( (a, b) \) is even. Similarly, if both \( a \) and \( b \) are odd, then we say that \( (a, b) \) is odd.

If \( a, b \leq 2^n-1 \), then we call \( (a, b) \) a top-half pair. Similarly, if \( a, b > 2^n-1 \), then we call \( (a, b) \) a bottom-half pair. The pair \( (2^n-1, 2^n-1 + 1) \) is called the central pair and all other pairs are said to be non-central.

We denote by \( \text{inve}(\sigma) \), \( \text{invo}(\sigma) \), \( \text{inv}_{t}(\sigma) \), and \( \text{inv}_{b}(\sigma) \) the respective numbers of even, odd, top-half, and bottom-half pairs inverted by \( \sigma \). Notice that

\[
0 \leq \text{inve}(\sigma), \text{invo}(\sigma), \text{inv}_{t}(\sigma), \text{inv}_{b}(\sigma) \leq \binom{2^n-1}{2}.
\]

We now continue our journey toward the proof of Theorem 5.4.1 with a long string of technical lemmas. These lemmas help to establish Proposition 5.4.34, a critical ingredient in our proof.

Lemma 5.4.22. Let \( \sigma \in \overline{S}_{2^n} \). Then \( \text{inve}(\sigma) = \text{invo}(\sigma) \) and \( \text{inv}_{t}(\sigma) = \text{inv}_{b}(\sigma) \).

Proof. First observe that if \( (a, b) \) is an even pair, then \( (\overline{b}, \overline{a}) \) is an odd pair (and vice versa). Similarly, if \( (a, b) \) is a top-half pair, then \( (\overline{b}, \overline{a}) \) is a bottom-half pair (and vice versa). Now let \( \sigma \in \overline{S}_{2^n} \) and suppose that \( \sigma \) inverts \( (a, b) \). Then, by Lemma 5.4.12, \( \sigma \) also inverts \( (\overline{b}, \overline{a}) \). Consequently, the map \( (a, b) \mapsto (\overline{b}, \overline{a}) \) is a bijection between the set of even pairs inverted by \( \sigma \) and the set of odd pairs inverted by \( \sigma \). It is also a bijection between the set of top-half pairs inverted by \( \sigma \) and the set of bottom-half pairs inverted by \( \sigma \). □
Lemma 5.4.23. Let $\sigma \in \overline{S}_2^n$ and suppose that $\text{inv}(\sigma) \geq 2$. Then either $\text{inv}_e(\sigma) > 0$ or $\text{inv}_t(\sigma) > 0$.

Proof. Suppose, to the contrary, that both $\text{inv}_e(\sigma) = 0$ and $\text{inv}_t(\sigma) = 0$. Then for each pair $(a, b)$ inverted by $\sigma$, it must be that $a \leq 2^{n-1}$, $b > 2^{n-1}$, and $a$ and $b$ have opposite parity. Since $\text{inv}(\sigma) \geq 2$, there exists a non-central pair $(a, b)$ inverted by $\sigma$. Since the pair is non-central and $a$ and $b$ have opposite parity, we have that $b - a > 2$ and so $a + 1 < b - 1$. Let $c = a + 1$ and $d = b - 1$. Then $a < c < d < b$ and exactly one of the following conditions is satisfied:

(i) $a, c, d \leq 2^{n-1}$ and $b > 2^{n-1}$
(ii) $a, c \leq 2^{n-1}$ and $d, b > 2^{n-1}$
(iii) $a \leq 2^{n-1}$ and $c, d, b > 2^{n-1}$.

In case (i), we note that $\sigma$ does not invert $(a, c)$ since it is a top-half pair and $\text{inv}_t(\sigma) = 0$. Furthermore, $\sigma$ does not invert $(c, b) = (a + 1, b)$ since $a + 1$ and $b$ have the same parity and $\text{inv}_e(\sigma) = \text{inv}_o(\sigma) = 0$. But $\sigma$ does invert $(a, b)$ and so we have that $\sigma(c) < \sigma(b) < \sigma(a) < \sigma(c)$, a contradiction. Similar contradictions can be reached for cases (ii) and (iii). \qed

Lemma 5.4.24. Let $\sigma \in \overline{S}_2^n$, where $n \geq 5$, and suppose that $\text{inv}(\sigma) \leq \frac{1}{2} \left( \binom{2^n}{2} \right)$. Then either $\text{inv}_e(\sigma) < \left( \binom{2^n-1}{2} \right) - 1$ or $\text{inv}_t(\sigma) < \left( \binom{2^n-1}{2} \right) - 1$.

Proof. We prove the contrapositive. Suppose that $\text{inv}_e(\sigma), \text{inv}_t(\sigma) \geq \left( \binom{2^n-1}{2} \right) - 1$. Let $S_e, S_o, S_t$, and $S_b$ be the sets of even, odd, top-half, and bottom-half pairs inverted by $\sigma$. Then $|S_e|, |S_t| \geq \left( \binom{2^n-1}{2} \right) - 1$ and, by Lemma 5.4.22, $|S_o|, |S_b| \geq \left( \binom{2^n-1}{2} \right) - 1$. Notice that $S_e \cap S_o = S_t \cap S_b = 0$. Notice also that $|S_e \cap S_t| = |S_o \cap S_b| \leq \left( \binom{2^{n-2}}{2} \right)$ and that $|S_e \cap S_b| = |S_o \cap S_t| \leq \left( \binom{2^{n-2}}{2} \right)$.
Thus,
\[
\text{inv}(\sigma) \geq |S_e \cup S_o \cup S_t \cup S_b|
\]
\[
= |S_e| + |S_o| + |S_t| - |S_e \cap S_t| - |S_o \cap S_t| - |S_e \cap S_b| - |S_o \cap S_b|
\]
\[
\geq 4 \left( \binom{2^{n-1}}{2} - 1 \right) - 4 \left( \binom{2^{n-2}}{2} \right)
\]
\[
= 2(2^{n-1})(2^{n-1} - 1) - 2(2^{n-2})(2^{n-2} - 1) - 4
\]
\[
= 2^{2n-1} - 2^n - 2^{2n-3} + 2^{n-1} - 4
\]
\[
= 2^{2n-1} - 2^n - 2^{2n-3} - 4
\]
\[
= (2^{2n-2} - 2^n) + (2^{2n-2} - 2^n - 2^{2n-3} - 4)
\]
\[
= \frac{1}{2} \left( \binom{2^n}{2} + 2^{2n-3} - (2^{n-2} + 4) \right)
\]
\[
> \frac{1}{2} \left( \binom{2^n}{2} + 2^{2n-3} - 2^{n-1} \right) \quad \text{(since } n \geq 5)\]
\[
= \frac{1}{2} \left( \binom{2^n}{2} + 2^{n-1}(2^{n-2} - 1) \right)
\]
\[
> \frac{1}{2} \left( \binom{2^n}{2} \right). \quad \square
\]

Lemma 5.4.25. Let \( \sigma \in S_{2^n} \), where \( n \geq 4 \).

(i) If \( \text{inv}_e(\sigma) \geq \left( \binom{2^{n-1}}{2} - 1 \right) \), then \( \text{inv}_t(\sigma) \geq 5 \).

(ii) If \( \text{inv}_t(\sigma) \geq \left( \binom{2^{n-1}}{2} - 1 \right) \), then \( \text{inv}_e(\sigma) \geq 5 \).

Proof. We prove (i) and leave the analogous proof of (ii) to the reader. Suppose that \( \text{inv}_e(\sigma) \geq \left( \binom{2^{n-1}}{2} - 1 \right) \). Then there is at most one even pair that is not inverted by \( \sigma \). Since \( \binom{2^{n-1}}{2} \) of the top-half pairs are also even, \( \sigma \) must invert at least \( \binom{2^{n-1}}{2} - 1 \) top-half pairs. But \( n \geq 4 \), and so it follows that \( \text{inv}_t(\sigma) \geq \left( \binom{2^{n-2}}{2} \right) - 1 \geq \binom{4}{4} - 1 = 5 \), as desired. \( \square \)

Lemma 5.4.26. For all odd integers \( a \) and \( b \) with \( a < b \leq 2^n - 1 \), \( (a, b, a + 1, b + 1) \) is a complete set with respect to \( \succ_{\text{lex}} \).

Proof. Let \( \succ_{n-1} \) be the standard lexicographic order on \( n-1 \) criteria and let \( \succ_1 \) be the linear order on one criterion specified by \( 1 \succ_1 0 \). Then \( \succ_{\text{lex}} = \succ_{n-1} \oplus \succ_1 \). Let \((y_k), (x_k)\) be the order sequences corresponding to \( \succ_{\text{lex}} \) and \( \succ_{n-1} \), respectively. Then, by Proposition 3.3.5, \( y_{2k-1} = (x_k, 1) \) and \( y_{2k} = (x_k, 0) \) for each \( k \leq 2^{n-1} \). Now let \( a = 2i - 1 \) and let

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\[ b = 2j - 1, \text{ where } i < j. \text{ Then} \]

\[ y_a = (x_i, 1) \quad y_{a+1} = (x_i, 0) \]
\[ y_b = (x_j, 1) \quad y_{b+1} = (x_j, 0) \]

and so \( (a, b, a + 1, b + 1) \) is a complete set. \( \square \)

**Lemma 5.4.27.** Let \( \succ_{lex} \) be the standard lexicographic order on \( n \) criteria. Then for all \( a < b \leq 2^{n-1}, \ (a, b, a + 2^{n-1}, b + 2^{n-1}) \) is a complete set with respect to \( \succ_{lex} \).

**Proof.** The proof is analogous to that of Lemma 5.4.26. \( \square \)

**Lemma 5.4.28.** Let \( \sigma \in S_{2^n} \). If \( \text{inv}_a(\sigma) = 1 \) and \( (a, b) \) is the unique odd pair inverted by \( \sigma \), then \( a + b = 2^n \).

**Proof.** Suppose that \( \text{inv}_a(\sigma) = 1 \) and let \( (a, b) \) be the unique odd pair inverted by \( \sigma \). By Lemma 5.4.26, \( (a, b, a + 1, b + 1) \) is a complete set with respect to \( \succ_{lex} \). But since \( \sigma \) preserves the separability of \( \succ_{lex} \), Proposition 5.4.16 implies that \( \sigma \) inverts \( (a + 1, b + 1) \). By Lemma 5.4.12, \( \sigma \) also inverts \( (b + 1, a + 1) \), which is an odd pair since both \( a + 1 \) and \( b + 1 \) are even. But \( (a, b) \) is the unique odd pair inverted by \( \sigma \) and so it must be that \( (b + 1, a + 1) = (a, b) \). Thus

\[ a = b + 1 = 2^n - (b + 1) + 1 = 2^n - b, \]

as desired. \( \square \)

**Lemma 5.4.29.** Let \( \sigma \in S_{2^n} \). If \( \text{inv}_t(\sigma) = 1 \) and \( (a, b) \) is the unique top-half pair inverted by \( \sigma \), then \( a + b = 2^n - 1 + 1 \).

**Proof.** Suppose that \( \text{inv}_t(\sigma) = 1 \) and let \( (a, b) \) be the unique top-half pair inverted by \( \sigma \). By Lemma 5.4.27, \( (a, b, a + 2^{n-1}, b + 2^{n-1}) \) is a complete set with respect to \( \succ_{lex} \). Thus \( \sigma \) must also invert \( (a + 2^{n-1}, b + 2^{n-1}) \) and \( (b + 2^{n-1}, a + 2^{n-1}) \). But \( (b + 2^{n-1}, a + 2^{n-1}) \) is a top-half pair and so it must be that

\[ a = b + 2^{n-1} = 2^n - (b + 2^{n-1}) + 1 = 2^n - b + 1. \]

**Lemma 5.4.30.** Let \( \succ \in C_n \) and let \( \succ_1 \in C_1 \). If \( (a, a + 1, b, b + 1) \) is a complete set with respect to \( \succ \), then \( (2a, 2a + 1, 2b, 2b + 1) \) is a complete set with respect to \( \succ \oplus \succ_1 \).

**Proof.** Without loss of generality, assume that \( 1 \succ 0 \). Let \( (x_k), (y_k) \) be the order sequences corresponding to \( \succ \) and \( \succ \oplus \succ_1 \), respectively. If \( (a, a + 1, b, b + 1) \) is a complete set with

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respect to $\succ$, then there exists $S \subset Q$ and $y, z \in X_S, u, v \in X_S$ such that

$$x_a = (y, u) \quad x_b = (y, v)$$

$$x_{a+1} = (z, u) \quad x_{b+1} = (z, v).$$

Now by Proposition 3.3.5, we have

$$x_{2a} = (y, u, 0) \quad x_{2b} = (y, v, 0)$$

$$x_{2a+1} = (z, u, 1) \quad x_{2b+1} = (z, v, 1).$$

But then $(2a, 2a + 1, 2b, 2b + 1)$ is a complete set with respect to $\succ \oplus \succ_1$, as desired. □

Lemma 5.4.31. Let $\sigma \in S_{2n}$, where $n \geq 4$. If $\sigma$ inverts $(2^{n-2}, 2^{n-2} + 1)$, then there is a top-half pair $(a, b) \neq (2^{n-2}, 2^{n-2} + 1)$ such that $\sigma$ inverts $(a, b)$.

Proof. It suffices to show that, for each $n \geq 4$, there exists an order $\succ \in O^*_n$ such that $(2^{n-2}, 2^{n-2} + 1, 2^{n-1} - 2^{n-4}, 2^{n-1} - 2^{n-4} + 1)$ is a complete set with respect to $\succ$. Indeed, once we have established this fact, our result follows directly from Proposition 5.4.16, since $(2^{n-1} - 2^{n-4}, 2^{n-1} - 2^{n-4} + 1)$ is a top-half pair.

We proceed by induction. For $n = 4$, consider the separable order $\succ$ corresponding to the normalized binary preference matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Notice that $(4, 5, 7, 8)$ is a complete set with respect to $\succ$, as desired. Now suppose that our claim is true for some $n \geq 4$ and let $\succ' \in O^*_n$ be chosen so that $(2^{n-2}, 2^{n-2} + 1, 2^{n-1} - 2^{n-4}, 2^{n-1} - 2^{n-4} + 1)$ is a complete set with respect to $\succ'$. Let $\succ_1$ be the linear order on one criterion specified by $1 \succ_1 0$. Then, by Lemma 5.4.30, $(2^{n-1}, 2^{n-1} + 1, 2^n - 2^{n-3}, 2^n - 2^{n-3} + 1)$ is a complete set with respect to $\succ' \oplus \succ_1$.

Proposition 3.3.2 implies that $\succ' \oplus \succ_1 \in O^*_{n+1}$, which completes the proof. □
Lemma 5.4.32. Let \( \sigma \in S_{2^n} \), where \( n \geq 4 \). Then \( \text{inv}_t(\sigma) \neq 1 \).

Proof. Suppose that \( \text{inv}_t(\sigma) = 1 \) and let \( \langle a, b \rangle \) be the unique top-half pair inverted by \( \sigma \). Then, by Lemma 5.4.29, \( a + b = 2^{n-1} + 1 \). We consider two cases:

Case 1. \( b = a + 1 \). Then \( a = 2^{n-2} \) and \( b = 2^{n-2} + 1 \). Consequently, by Lemma 5.4.31, there exists a top-half pair \( \langle c, d \rangle \neq \langle a, b \rangle \) that is inverted by \( \sigma \). This, however, is a contradiction to the assumption that \( \text{inv}_t(\sigma) = 1 \).

Case 2. \( b > a + 1 \). Then \( a < b - 1 < b \). Since \( \sigma \) inverts \( \langle a, b \rangle \), Lemma 5.4.10 implies that \( \sigma \) inverts either \( \langle a, b - 1 \rangle \) or \( \langle b - 1, b \rangle \), both of which are top-half pairs. This too is a contradiction to the assumption that \( \text{inv}_t(\sigma) = 1 \). \( \square \)

Lemma 5.4.33. Let \( \sigma \in S_{2^n} \), where \( n \geq 4 \). If \( \text{inv}_o(\sigma) = 1 \), then \( \text{inv}_t(\sigma) > 1 \).

Proof. Suppose that \( \text{inv}_o(\sigma) = 1 \) and let \( \langle a, b \rangle \) be the unique odd pair inverted by \( \sigma \). Then, by Lemma 5.4.28, \( a + b = 2^n \). If \( b \neq a + 2 \), then there exists some odd integer \( c \) such that \( a < c < b \). But then Lemma 5.4.10 implies that \( \sigma \) inverts either \( \langle a, c \rangle \) or \( \langle c, b \rangle \), a contradiction to the assumption that \( \text{inv}_o(\sigma) = 1 \). Thus, it must be the case that \( b = a + 2 \), which implies that \( a = 2^{n-1} - 1 \) and \( b = 2^{n-1} + 1 \). Let \( \succ_{\text{lex}} \) be the standard lexicographic order on \( n \) criteria and let \( (x_k) \), \( (y_k) \) be the order sequences corresponding to \( \succ_{\text{lex}} \) and \( \tau_c(\succ_{\text{lex}}) \), respectively. Then, by Proposition 3.3.9,

\[
\begin{align*}
y_{2^{n-1}-1} &= x_{2^{n-1}-1} = (1, 0, 0, \ldots, 0, 1) \\
y_{2^{n-1}+1} &= x_{2^n} = (1, 0, 0, \ldots, 0, 0) \\
y_1 &= x_1 = (1, 1, 1, \ldots, 1) \\
y_2 &= x_2 = (1, 1, 1, \ldots, 1, 0).
\end{align*}
\]

Thus, \( (1, 2^{n-1} - 1, 2^{n-1} + 1) \) is a complete set with respect to \( \tau_c(\succ_{\text{lex}}) \). Since \( \sigma \in S_{2^n} \) and \( \sigma \) inverts \( \langle a, b \rangle = (2^{n-1} - 1, 2^{n-1} + 1) \), Proposition 5.4.16 implies that \( \sigma \) inverts \( (1, 2) \), which implies that \( \text{inv}_t(\sigma) \geq 1 \). Lemma 5.4.32 then makes this inequality strict. \( \square \)

Proposition 5.4.34. If \( \sigma \in S_{2^n} \) and \( 2 \leq \text{inv}_o(\sigma) \leq (2^n - 2) \), then either \( 1 < \text{inv}_o(\sigma) < (2^{n-1}) - 1 \) or \( 1 < \text{inv}_t(\sigma) < (2^{n-1}) - 1 \).

Proof. Suppose, to the contrary, that neither \( 1 < \text{inv}_o(\sigma) < (2^{n-1}) - 1 \) nor \( 1 < \text{inv}_t(\sigma) < (2^{n-1}) - 1 \). Then one of the following must occur:

(i) \( \text{inv}_o(\sigma), \text{inv}_t(\sigma) \leq 1 \)
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(ii) \( \text{inv}_o(\sigma) \leq 1 \) and \( \text{inv}_l(\sigma) \geq \binom{2^{n-1}}{2} - 1 \)

(iii) \( \text{inv}_o(\sigma) \geq \binom{2^{n-1}}{2} - 1 \) and \( \text{inv}_l(\sigma) \leq 1 \)

(iv) \( \text{inv}_o(\sigma), \text{inv}_l(\sigma) \geq \binom{2^{n-1}}{2} - 1 \)

Lemmas 5.4.24 and 5.4.25 rule out cases (ii) - (iv). Thus, it must be the case that both \( \text{inv}_o(\sigma), \text{inv}_l(\sigma) \leq 1 \). Lemma 5.4.32 implies that \( \text{inv}_l(\sigma) = 0 \). But then Lemma 5.4.23 implies that \( \text{inv}_o(\sigma) = 1 \). This, however, is a contradiction, since, by Lemma 5.4.33, \( \text{inv}_o(\sigma) = 1 \Rightarrow \text{inv}_l(\sigma) > 1 \). Thus, it must be the case that either \( 1 < \text{inv}_o(\sigma) < \binom{2^{n-1}}{2} - 1 \) or \( 1 < \text{inv}_l(\sigma) < \binom{2^{n-1}}{2} - 1 \), as desired. \( \Box \)

We are now (finally!) ready to tie all of the pieces together and prove Theorem 5.4.1.

Proof of Theorem 5.4.1. For \( n = 2 \), separability and symmetry are equivalent by Proposition 4.6.1. Thus, \( S_2^* = \overline{S}_4 \), which is, by Proposition 5.2.10, isomorphic to \( D_4 \), the group of symmetries of the square.

For \( n = 3 \) and \( n = 4 \), we use a brute-force method to calculate \( S_{2n}^* \). Specifically, we note that, by Corollary 5.4.18, a permutation \( \sigma \) belongs to \( S_{2n}^* \) if and only if \( \sigma \) preserves the separability of each normalized binary preference order in \( \mathcal{O}_n^* \), that is, if and only if \( \sigma \in S_{2n}^* \) for each normalized \( \triangleright \in \mathcal{O}_n^* \). Proposition 3.1.3 provides explicit descriptions of all such orders. Furthermore, by taking permutations and/or bitwise complements of the columns of these normalized binary preference orders, we can explicitly determine all of the elements of \( \mathcal{O}_n^* \). We then calculate \( S_{2n}^* \) for each normalized \( \triangleright \in \mathcal{O}_n^* \), using Proposition 5.3.6 to note that

\[
S_{2n}^* = \bigcap_{\triangleright' \in \mathcal{O}_n^*} \{ \sigma \in S_{2n} : \sigma(\triangleright) = \triangleright' \}.
\]

Finally, we obtain \( S_{2n}^* \) by taking the intersection of these \( S_{2n}^* \).

The case for \( n = 3 \) can be done by hand (albeit somewhat tediously), since only 2 sets of permutations must be intersected. The result is a group of 24 permutations, generated by \( \sigma = (18)(27)(36)(45) \), \( \tau = (18)(26)(37) \), and \( \gamma = (12)(36)(45)(78) \). Notice that \( \sigma \tau = (23)(45)(67) \), \( \sigma \gamma = (17)(28) \), and \( \tau \gamma = (167)(283)(45) \). Thus,

\[
\langle \sigma, \tau, \gamma \mid \sigma^2 = \tau^2 = \gamma^2 = (\sigma \tau)^2 = (\sigma \gamma)^2 = (\tau \gamma)^6 = 1 \rangle
\]

is a presentation for \( S_6^* \). By Coxeter and Moser [7], it follows that \( S_6^* \cong Z_2 \times D_6 \cong V_4 \times S_3 \).

The situation is slightly more complicated for \( n = 4 \), since we are now intersecting 14 sets, each containing 5376 permutations. While this would be nearly impossible by hand,
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a computer has no problem accomplishing the task. After a few minutes, we find that $S^*_{16} = \{1, \tau_c, \tau_r, \tau_c \tau_r\} \cong V_4$ (see Appendix A for more detail).

Indeed, our claim is that $S^*_{2n} = \{1, \tau_c, \tau_r, \tau_c \tau_r\} \cong V_4$ for all $n \geq 4$. We use induction to prove this claim.

We have already established the base case and so all that remains is the inductive step. Thus, suppose that, for some $n \geq 5$, $S^*_{2n-1} = \{1, \tau_c, \tau_r, \tau_c \tau_r\}$ and suppose also that $S^*_{2n} \neq \{1, \tau_c, \tau_r, \tau_c \tau_r\}$. Corollary 5.4.21 implies that $\{1, \tau_c, \tau_r, \tau_c \tau_r\} \subset S^*_n$. Thus, it must be the case that this containment is proper. By our comments following Definition 5.4.3 and by Proposition 5.4.13 and Lemma 5.4.14, there exists $\sigma \in S^*_{2n}$ with $2 \leq \inv{\sigma} \leq \left(\frac{2n}{2}\right) - 2$. Furthermore, we may assume, by Lemma 5.4.6, that $2 \leq \inv{\sigma} \leq \frac{3}{2}\left(\frac{2n}{2}\right)$. But then Proposition 5.4.34 implies that either $1 < \inv{r(\sigma)} < \left(\frac{2n-1}{2}\right)$ or $1 < \inv{l(\sigma)} < \left(\frac{2n-1}{2}\right)$.

Suppose that $1 < \inv{r(\sigma)} < \left(\frac{2n-1}{2}\right) - 1$ and let $\succ_1$ be the linear order on one criterion specified by $1 \succ_1 0$. Then for any $\succ \in \mathcal{O}^n_{n-1}$, the permutation $\sigma' = s_n(\sigma)$ induced by $p_n(\succ \oplus \succ_1)$ is an element of $S^*_{2n-1}$ (by Proposition 5.4.9). On the other hand, Propositions 3.3.9 and 5.4.4 establish a bijection between the odd pairs inverted by $\sigma$ and all pairs inverted by $\sigma'$, and so $\inv{\sigma'} = \inv{\sigma}$. This, however, is a contradiction, since $1 < \inv{r(\sigma)} < \left(\frac{2n-1}{2}\right) - 1$ and $\sigma' \in S^*_{2n-1}$, which is equal to $\{1, \tau_c, \tau_r, \tau_c \tau_r\}$ by the induction hypothesis.

A similar contradiction is obtained if we assume that $1 < \inv{l(\sigma)} < \left(\frac{2n-1}{2}\right) - 1$ (in this case we consider the permutation $\sigma' = s_1(\sigma)$ induced by $p_1(\succ_1 \oplus \succ)$). Since each case leads to contradiction, our assumption that $S^*_{2n} \neq \{1, \tau_c, \tau_r, \tau_c \tau_r\}$ must be false.

The surprising conclusion of Theorem 5.4.1 (namely that $S^*_n$ contains only 4 elements for $n \geq 4$) is the last in a trilogy of disturbing results about separable preferences. In Chapter 1, we gave an example of the paradoxical behavior that can occur when preferences are nonseparable. We also recalled previous research asserting that the best way out of the problem of nonseparability may be to avoid the very possibility of nonseparable preferences. Unfortunately, Corollary 4.4.2 indicates that this could be a formidable task, as the vast majority of preference orderings are nonseparable. We now know also that even if we can force all preferences to be separable, most changes to these preferences (even very small modifications such as the interchanging of two non-central, adjacent alternatives) have the potential to introduce nonseparability. Thus, an unfortunate practical implication of our research is the following:
Separable preferences, which seem to be essential to effective multiple-criteria decision-making, are rare and extremely sensitive to small changes.

Recall our earlier claim that, for \( n > 3 \), there does not exist a separable order \( \succ \in C_n^* \) for which \( S_2^n \) is a subgroup of \( \overline{S}_2^n \). We now prove this result.

**Lemma 5.4.35.** For every \( n \geq 3 \), \( S_2^{lex} \) is not a subgroup of \( \overline{S}_2^n \).

**Proof.** Let \( n \geq 3 \) be given and let \( (y_k) \) be the order sequence corresponding to \( \succ_{lex} \). Let

\[
\sigma = (2, 2 + 2^{n-1} - 1)(4, 4 + 2^{n-1} - 1) \cdots (2^{n-1}, 2^{n-1} + 2^{n-1} - 1).
\]

Proposition 3.3.9 implies that, for every \( x \in X_{Q-(1,n)} \), \( y_m = (1, x, 0) \) if and only if \( m = 2k \) for some \( k \leq 2^{n-2} \). Now let \( y_{2k} = (1, x, 0) \), as above. We claim that \( y_{2k+2^{n-1}-1} = (0, x, 1) \).

To prove this, it suffices to show that

\[
2^n - (2k + 2^{n-1} - 1) = 0 \cdot 2^{n-1} + \sum_{j=2}^{n-1} x_j \cdot 2^{n-j} + 1 \cdot 2^0 = \sum_{j=2}^{n-1} x_j \cdot 2^{n-j} + 1.
\]

Now we know that

\[
2^n - 2k = 2^{n-1} + \sum_{j=2}^{n-1} x_j \cdot 2^{n-j},
\]

and so

\[
\sum_{j=2}^{n-1} x_j \cdot 2^{n-j} = 2^n - 2k - 2^{n-1}.
\]

Thus

\[
\sum_{j=2}^{n-1} x_j \cdot 2^{n-j} + 1 = 2^n - 2k - 2^{n-1} + 1 = 2^n - (2k + 2^{n-1} - 1),
\]

as desired.

It follows from our above observations that \( \sigma \) interchanges in \( (y_k) \) all pairs of elements of \( X_Q \) of the form \( ((1, x, 0), (0, x, 1)) \) for \( x \in X_{Q-(1,n)} \). Clearly, the effect of this action is to interchange the first and the \( n \)th criteria, while maintaining the original order specified by \( \succ_{lex} \). Thus, \( \sigma(\succ_{lex}) \) is separable, and so \( \sigma \in S_2^{lex} \).

Now let \( \gamma = \tau_c \sigma \). Then \( \gamma(\succ_{lex}) = \tau_c(\sigma(\succ_{lex})) \) is separable since \( \tau_c \in S_2^n \). Thus, \( \gamma \in S_2^{lex} \). Notice also that

\[
\gamma = (2^{n-1}, 2^{n-1} + 1)(2, 2 + 2^{n-1} - 1)(4, 4 + 2^{n-1} - 1) \cdots (2^{n-1}, 2^{n-1} + 2^{n-1} - 1)
\]

\[
= (2, 2^{n-1}, 2^n - 1, 2^{n-1} + 1)(4, 4 + 2^{n-1} - 1) \cdots (2^{n-1} - 2, 2^{n-1} - 2 + 1)
\]
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and so \( \gamma^2 = (2^{n-1}, 2^{n-1} + 1)(2, 2^n - 1) = \tau_c(2, 2^n - 1) \). We claim that \( \gamma^2 \notin S_{2^n}^{S_{2^n}} \). Since \( \tau_c \in S_{2^n} \), it suffices to show that \( (2, 2^n - 1) \notin S_{2^n}^{S_{2^n}} \). Let \((z_k)\) be the order sequence corresponding to \( \gamma = (2, 2^n - 1)(\gamma_{1_{\text{lex}}}^+) \). Then

\[
\begin{align*}
z_1 &= y_1 = (1, 1, \ldots, 1, 1) \\
z_2 &= y_{2^n - 1} = (0, 0, \ldots, 0, 1) \\
z_2^{n-1-1} &= y_{2^n-1-1} = (1, 0, \ldots, 0, 1) \\
z_{2^n-1+1} &= y_{2^n-1+1} = (0, 1, \ldots, 1, 1).
\end{align*}
\]

Thus, \((1, 1, \ldots, 1, 1) \succ (0, 1, \ldots, 1, 1)\) but \((0, 0, \ldots, 0, 1) \succ (1, 0, \ldots, 0, 1)\), and so \( \gamma = (2, 2^n - 1)(\gamma_{1_{\text{lex}}}^+) \) is not separable. It follows that \( \gamma^2 \notin S_{2^n}^{S_{2^n}} \) and so \( S_{2^n}^{S_{2^n}} \) is not a subgroup of \( S_{2^n} \).

\[\square\]

**Lemma 5.4.36.** Let \( \gamma_1^+ \succ \gamma_2^+ \in \mathcal{O}_n^+ \). If \( S_{2^n}^{\gamma_1^+} \) is a subgroup of \( S_{2^n} \), then so is \( S_{2^n}^{\gamma_2^+} \).

**Proof.** Suppose that \( S_{2^n}^{\gamma_1^+} \) is a subgroup of \( S_{2^n} \). We wish to show that \( S_{2^n}^{\gamma_2^+} \) is also a subgroup of \( S_{2^n} \). Since \( S_{2^n}^{\gamma_2^+} \) is finite and \( 1 \in S_{2^n}^{\gamma_2^+} \), it suffices to show that \( S_{2^n}^{\gamma_2^+} \) is closed under its operation. To this end, choose \( \tau_1, \tau_2 \in S_{2^n}^{\gamma_1^+} \) and let \( \sigma \in S_{2^n} \) be such that \( \sigma(\gamma_1) = \gamma_2 \). Since \( \sigma(\gamma_1) = \gamma_2 \in \mathcal{O}_n^+ \), it follows that \( \sigma \in S_{2^n}^{\gamma_1^+} \). Thus, \( \sigma^{-1} \in S_{2^n}^{\gamma_1^+} \). Now

\[\tau_1 \sigma(\gamma_1) = \tau_1(\sigma(\gamma_1)) = \tau_1(\gamma_2) \in \mathcal{O}_n^+ \]

since \( \tau_1 \in S_{2^n}^{\gamma_2^+} \), from which it follows that \( \tau_1 \sigma \in S_{2^n}^{\gamma_1^+} \). Similarly, \( \tau_2 \sigma \in S_{2^n}^{\gamma_1^+} \). But then

\[\tau_1 \tau_2 \sigma = (\tau_1 \sigma)(\sigma^{-1})(\tau_2 \sigma) \in S_{2^n}^{\gamma_1^+} \]

which implies that \( \tau_1 \tau_2(\gamma_2) = \tau_1 \tau_2(\gamma_1) \in \mathcal{O}_n^+ \). It follows that \( \tau_1 \tau_2 \in S_{2^n}^{\gamma_2^+} \), as desired. \[\square\]

**Proposition 5.4.37.** Let \( n \geq 3 \) and let \( \gamma \in \mathcal{O}_n^+ \). Then \( S_{2^n}^{\gamma} \) is not a subgroup of \( S_{2^n} \).

**Proof.** If \( S_{2^n}^{\gamma} \) were a subgroup of \( S_{2^n} \), then, by Lemma 5.4.36, \( S_{2^n}^{S_{2^n}} \) would also be a subgroup of \( S_{2^n} \). This, however, is a contradiction to Lemma 5.4.35. \[\square\]

We remark here that, for \( n = 2 \), \( S_{2^n}^{\tau_c} \) is always a subgroup of \( S_{2^n} \). In fact for every \( \gamma \in \mathcal{O}_2^+ \), \( S_{2^n}^{\gamma} = S_{2} \). To see this, notice that \( S_{2} \subseteq S_{2}^{\gamma} \) for every \( \gamma \in \mathcal{O}_2^+ \). Also notice that \( |S_{2}^{\gamma}| = |S_{2}^{\gamma}| = 8 = |S_{2}^{\gamma}| \) by Theorem 5.4.1 and by Propositions 3.3.7, 4.6.1, and 5.3.6. Thus, \( S_{2}^{\gamma} = S_{2} \), as desired.
Chapter 6

Questions for Future Research

We have mentioned a few open questions within the preceding text; we now take a moment to discuss in more detail some of the possible directions that future research in this area might take.

In Chapter 2, we observed that some of Gorman’s nice set-theoretic and structural properties may fail to occur when the alternative sets are not arc-connected (as is the case in a referendum election). In Chapter 3, we noted in this context the existence of preference orders that are separable but not additive. This was also in contrast to a theorem of Gorman, one which again required arc-connectedness of the alternative sets. Thus, it would seem reasonable to explore the extent to which weaker versions of Gorman’s theorems might hold in a discrete setting.

In Chapter 4, we mentioned that several combinatorial questions remain unanswered. Counting separable and additive preference orders seems to be a difficult and interesting problem. We sense that further investigations into the action of the symmetric group on the set of separable preference orders may provide some tools to help us attack these and other related enumeration problems. We note here that a characterization of the set of permutations that preserve the separability of the standard lexicographic order could quickly lead to a formula for the number of separable preference orders. Along these lines, we imagine that it would be quite useful to study also the structure of the groups of permutations that preserve other properties, such as monoseparability, preseparability, and additivity.
Appendix A

Computer Code and Output

In this appendix, we discuss briefly the methods used to calculate $S_{2^n}$ for $n = 4$. We also provide the Visual Basic code and a portion of the resulting Microsoft Access crosstab query used for this calculation.

A.1 The Separable Group Calculator

A Visual Basic program was used to calculate $S_{2^n}$ for $n = 4$. The interface for this program is shown below in Figure A.1.

The calculation requires three steps.

1. First, each of the 14 normalized separable binary preference matrices are input into the program. The user enters each matrix into the provided text box and then clicks the “Generate Matrices” button. This button calls a subroutine (cmdAddMatrix.Click) that generates and stores in an array all of the preference matrices differing from the input matrix by permutations and/or bitwise complements of columns. Thus, after entering each of the 14 normalized separable binary preference matrices, the program has generated and stored all 5,376 separable binary preference matrices.

2. Next, the user clicks the “Calculate Group!” button, calling a subroutine (cmdGroup.Click) that writes to a Microsoft Access database the information necessary to calculate $S_{2^n}$. Specifically, for each pair $(\succ, \succ')$, where $\succ$ is a normalized separable preference order and $\succ'$ is any separable preference order, the program calculates the permutation
Figure A.1: The Separable Group Calculator interface

σ for which σ(>) = >' and stores the triple (σ, >, >') as a record in the database.
Note that a permutation σ preserves separability if and only if, for each normalized
> ∈ O_n^*, there exists >' ∈ O_n^* such that the triple (σ, >, >') occurs as a record of the
generated database.

3. To complete the calculation, the user clicks the “Mark Separability Preserving Permutations” button, calling a subroutine(cmdMark_Click) that marks the records corresponding to permutations for which the condition described in (ii) holds.

The detailed code for this process is listed below.

```
' Declarations and constants
'-----------------------------------------------------------------------

Const nQuest = 4
Type PrefMatrix
    Row(1 To 16, 1 To 4) As Boolean
End Type

Dim SepMatrices() As PrefMatrix
Dim OrigMatrices() As PrefMatrix
Dim Perms() As String
```
Appendix A. Computer Code and Output

Dim nMatrices As Integer
Dim oldMatrix() As Boolean
Dim origMatrix() As Boolean
Dim curMatrix() As Boolean
Dim colPerms() As Integer
Dim nPerms As Integer

Sub cmdAddMatrix_Click()
' This subroutine generates all separable preference matrices corresponding to a given normalized matrix.
'
    Dim cnt As Integer
    If Not InitializeMatrixO Then MsgBox "Error!": Exit Sub
    ReDim Preserve OrigMatrices(l To nMatrices + 1) As PrefMatrix
    ReDim Preserve SepMatrices(l To (nMatrices + 1) * 384) As PrefMatrix
    nMatrices = nMatrices + 1
    For l = 1 To 2 ^ nQuest
        For m = 1 To nQuest
            OrigMatrices(nMatrices).Row(l, m) = origMatrix(l, m)
        Next m
    Next l
    cnt = (nMatrices - 1) * 384 + 1
    For k = 1 To nPerms
        For j = 0 To 2 ^ nQuest - 1
            curRow = ""
            ResetCurMatrix
            ColPerm k
            ColComp j
            For l = 1 To 2 ^ nQuest
                For m = 1 To nQuest
                    SepMatrices(cnt).Row(l, m) = curMatrix(l, m)
                Next m
            Next l
            cnt = cnt + 1
        Next j
    Next k
End Sub

Private Function InitializeMatrixO As Boolean
' This function converts the text entered in txtMatrix to the data type of a preference matrix (PrefMatrix)
'
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Dim curPos As Long, oldPos As Long
Dim curRow As String, curRowNum As Integer
Dim curCol As Integer

Dim permString As String

On Error GoTo errhand1

ReDim origMatrix(1 To 2 ^ nQuest, 1 To nQuest) As Boolean
ReDim oldMatrix(1 To 2 ^ nQuest, 1 To nQuest) As Boolean
ReDim curMatrix(1 To 2 ^ nQuest, 1 To nQuest) As Boolean

nPerms = Factorial(nQuest)
ReDim colPerms(1 To nPerms, 1 To nQuest) As Integer

permString = "4321,3421,3241,3214,4231,2431,2341,2314,4213,2143,2134,
4312,3412,3142,3124,4132,1432,1342,1324,4123,1423,1243,1234"

For i = 1 To nPerms
    For j = 1 To nQuest
        colPerms(i, j) = Mid(permString, (nQuest + 1) * (i - 1) + j, 1)
    Next j
Next i

oldPos = 1
curPos = 1
curRowNum = 1

Do While curPos <> 0
    curPos = InStr(oldPos, txtMatrix.Text, Chr(13))
    If curPos <> 0 Then
        curRow = Mid(txtMatrix.Text, oldPos, curPos - oldPos)
        oldPos = curPos + 2
    Else
        curRow = Right(txtMatrix.Text, Len(txtMatrix.Text) - oldPos + 1)
    End If
    For curCol = 1 To nQuest
        origMatrix(curRowNum, curCol) = -Val(Mid(curRow, curCol, 1))
        oldMatrix(curRowNum, curCol) = origMatrix(curRowNum, curCol)
    Next curCol
    curRowNum = curRowNum + 1
Loop

InitializeMatrix = True

Exit Function errhand1:
    MsgBox "An error has occurred while attempting to initialize the matrix."
    Exit Function
End Function
Private Sub ResetCurMatrix()
' This routine clears the data in the current matrix.

    For a = 1 To 2 ^ nQuest
        For b = 1 To nQuest
            curMatrix(a, b) = origMatrix(a, b)
        Next b
    Next a
End Sub

Private Sub SetOldMatrix()
' This routine writes the current matrix to the oldMatrix variable so that actions can be performed on it while retaining its original data.

    For c = 1 To 2 ^ nQuest
        For d = 1 To nQuest
            oldMatrix(c, d) = curMatrix(c, d)
        Next d
    Next c
End Sub

Private Sub colPerm(ByVal pNum As Long)
' This subroutine permutes the columns of the current matrix according to the permutation number pNum.

    SetOldMatrix
    For i = 1 To nQuest
        For j = 1 To 2 ^ nQuest
            curMatrix(j, i) = oldMatrix(j, colPerms(pNum, i))
        Next j
    Next i
End Sub

Private Sub ColComp(ByVal colMask As Integer)
' This subroutine takes the bitwise complement of a subset of the criteria set determined by colMask.

    For i = 1 To nQuest

If colMask And 2 ^ (nQuest - i) Then
    For j = 1 To 2 ^ nQuest
        curMatrix(j, i) = Not curMatrix(j, i)
    Next j
End If
Next i

Private Sub cmdGroup_Click()
'--------------------------------------------------------
' This subroutine populates the database used to
' calculate the group of separability-preserving
' permutations.
'--------------------------------------------------------

Dim db As Database
Dim tabPerms As Recordset

Dim nMatrices As Integer
Dim nOrig As Integer
nMat = UBound(SepMatrices)
NOrig = UBound(OrigMatrices)

Dim curMatrixPerm(1 To 2 ^ nQuest)
Dim curCycleDecomp As String
Dim match As Boolean, curStart As Integer, curPos As Integer

Dim curPerm As String, curPermCount As Integer

Set db = DBEngine.OpenDatabase(txtFile.Text)
Set tabPerms = db.OpenRecordset("Perms", dbOpenTable)

For i = 1 To nMat 'For each separable matrix
    For j = 1 To nOrig 'For each normalized separable matrix
        For k = 1 To 2 ^ nQuest 'Find out which permutation is required
            For l = 1 To 2 ^ nQuest ' to take normalized matrix j
                match = True 'to separable matrix i.
                For m = 1 To nQuest
                    If SepMatrices(i).Row(k, m) <> OrigMatrices(j).Row(l, m) Then
                        match = False
                        Exit For
                    End If
                Next m
                If match Then
                    curMatrixPerm(k) = 1
                    Exit For
                End If
            Next l
        Next k
    Next j
Next i
'Now find the cycle decomposition of the permutation.

curCycleDecomp = ""
curStart = 1
curPos = 1

Do
  If curCycleDecomp = "" Then
    curStart = 1
  Else
    curStart = 0
    For p = 1 To 2 ^ nQuest
      If InStr(curCycleDecomp, "( & p & ",") = 0
          And InStr(curCycleDecomp, ", & p & ")") = 0
          And InStr(curCycleDecomp, ", & p & ",") = 0
          And InStr(curCycleDecomp, "( & p & ")") = 0 Then
            curStart = p
            Exit For
      End If
    Next p
    If curStart = 0 Then Exit Do
  End If

  For p = 1 To 2 ^ nQuest
    If curMatrixPerm(p) = curStart Then curPos = p: Exit For
  Next p

  If curPos = curStart Then
    curCycleDecomp = curCycleDecomp & "( & curPos & ")"
  Else
    curCycleDecomp = curCycleDecomp & "( & curStart & ", & curPos
    Do While curPos <> curStart
      For p = 1 To 2 ^ nQuest
        If curMatrixPerm(p) = curPos Then curPos = p: Exit For
      Next p
      If curPos = curStart Then
        curCycleDecomp = curCycleDecomp & ")"
      Else
        curCycleDecomp = curCycleDecomp & ", & curPos
      End If
    Loop
  End If
Loop

'Write the information to the database as a new record.

tabPerms.AddNew
  tabPerms!Sepmatrix = i
  tabPerms!origMatrix = j
  tabPerms!Perm = curCycleDecomp
Private Sub cmdMark_Click()

'---------------------------------------------------------------------
'This subroutine marks the separability-preserving
'permutations in the database created by
'cmdGroup_Click.
'---------------------------------------------------------------------

Set db = DBEngine.OpenDatabase(txtFile.Text)
Set tabPerms = db.Execute("SELECT Sepmatrix, origMatrix, Perm, Preserves
FROM Perms ORDER BY Perm")

tabPerms.MoveFirst
curPermCount = 0
curPerm = tabPerms!Perm

cnt = 0
nOrig = UBound(OrigMatrices)

Do While Not tabPerms.EOF
If tabPerms!Perm = curPerm Then
    curPermCount = curPermCount + 1
Else
    curPermCount = 1
    curPerm = tabPerms!Perm
End If
If curPermCount = nOrig Then
    tabPerms.Edit
    tabPerms!Preserves = True
    tabPerms.Update
End If
    tabPerms.MoveNext
    cnt = cnt + 1
Loop

db.Close

End Sub

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A.2 The Crosstab Query

We should point out that the program used to calculate $S_n^*$ for $n = 4$ generated a wealth of data that could prove to be useful in future research. Specifically, the program calculated exactly which permutations preserve the separability of each normalized separable preference order on 4 criteria. Thus, it seems that one could learn a great deal by examining these data and searching for patterns and correlations. We do not take up this task here, but we do present two examples of the sort of information that can be garnered from this type of investigation.

Table A.2 below lists the first 24 (of 59,488) rows of a query run by Microsoft Access to cross-tabulate and count, for each $\sigma \in S_{2^n}$, the number of normalized separable preference orders whose separability is preserved by $\sigma$. These rows correspond to all permutations in $S_{2^n}$ that preserve the separability of at least 8 of the 14 normalized separable preference orders. For ease of notation, we number the normalized separable preference matrices 1 through 14, where numbers 1 through 7 correspond to the first row of matrices for the $|Q| = 4$ case on page 18 and numbers 8 through 14 correspond to the second row.

Notice that, of the approximately 21 trillion (16!) permutations in $S_{16}$, only 59,488 preserve the separability of even one separable preference order. Also notice that only 24 of these preserve the separability of at least 8 of the 14 normalized separable preference orders. A question that we can easily answer by looking at the results of our crosstab query is: \textit{Given an integer \( n \) with \( 1 \leq n \leq 14 \), how many permutations in $S_{16}$ preserve the separability of at least \( n \) of the 14 normalized separable orders on 4 criteria?} The answer to this question for different values of \( n \) is summarized in Table A.2.
Appendix A. Computer Code and Output

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>preserves the separability of...</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12,13,14</td>
<td>14</td>
</tr>
<tr>
<td>(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12,13,14</td>
<td>14</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 13)(5, 12)(6, 11)(7, 10)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12,13,14</td>
<td>14</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 13)(5, 12)(6, 11)(7, 10)(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12,13,14</td>
<td>14</td>
</tr>
<tr>
<td>(4, 5)(8, 9)(12, 13)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>12</td>
</tr>
<tr>
<td>(4, 5)(12, 13)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>12</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 12)(5, 13)(6, 11)(7, 10)(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>12</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 12)(5, 13)(6, 11)(7, 10)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>12</td>
</tr>
<tr>
<td>(1, 12)(3, 6)(7, 10)(8, 9)(11, 14)(15, 16)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 2)(3, 6)(7, 10)(11, 14)(15, 16)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 2)(3, 6)(4, 5)(7, 10)(8, 9)(11, 14)(12, 13)(15, 16)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 2)(3, 6)(4, 5)(7, 10)(11, 14)(12, 13)(15, 16)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 15)(2, 16)(3, 11)(4, 13)(5, 12)(6, 14)(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 15)(2, 16)(3, 11)(4, 12)(5, 13)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 15)(2, 16)(3, 11)(4, 12)(5, 13)(6, 14)(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 15)(2, 16)(3, 11)(4, 12)(5, 13)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 15)(2, 16)(3, 11)(4, 12)(5, 13)(6, 14)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(6, 7)(8, 9)(10, 11)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(6, 7)(10, 11)</td>
<td>1,2,3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 13)(5, 12)(6, 10)(7, 11)(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>8</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 12)(5, 13)(6, 10)(7, 11)(8, 9)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>8</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 13)(5, 12)(6, 10)(7, 11)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>8</td>
</tr>
<tr>
<td>(1, 16)(2, 15)(3, 14)(4, 12)(5, 13)(6, 10)(7, 11)</td>
<td>1,2,3,4,5,6,7,8,9,10,11,12</td>
<td>8</td>
</tr>
</tbody>
</table>

Table A.1: Permutations that preserve the separability of at least 8 normalized separable preference orders.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7, 8</th>
<th>9-12</th>
<th>13, 14</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>59488</td>
<td>13024</td>
<td>1436</td>
<td>1084</td>
<td>80</td>
<td>64</td>
<td>24</td>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

Table A.2: The number of permutations that preserve the separability of at least $n$ normalized separable preference orders.

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Bibliography


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