Resolvability in Graphs

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RESOLVABILITY IN GRAPHS

by

Varaporn Saenpholphat

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics

Western Michigan University
Kalamazoo, Michigan
December 2002
The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in $G$ and a vertex $v$ of $G$, the code of $v$ with respect to $W$ is the $k$-vector $c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set $W$ is a resolving set for $G$ if distinct vertices have distinct codes. A resolving set containing a minimum number of vertices is a basis for $G$. The dimension $\text{dim}(G)$ is the number of vertices in a basis for $G$. A resolving set $W$ of $G$ is connected if the subgraph $\langle W \rangle$ induced by $W$ is a connected subgraph of $G$. The minimum cardinality of a connected resolving set $W$ in a graph $G$ is the connected resolving number $\text{cr}(G)$. A connected resolving set of cardinality $\text{cr}(G)$ is called a $\text{cr}$-set of $G$. We study the relationships between $\text{cr}$-sets and bases in a nontrivial connected graph $G$. The connected resolving numbers of some well-known classes of graphs are determined.

It is shown, for a pair $a, b$ of integers with $1 \leq a \leq b$, that there exists a connected graph $G$ with $\text{dim}(G) = a$ and $\text{cr}(G) = b$ if and only if $(a, b) \notin \{(1, k) : k \geq 2\}$.

For a triple $a, b, n$ of integers with $1 \leq a \leq b \leq n$, there exists a connected graph $G$ of order $n$ such that $\text{dim}(G) = a$ and $\text{cr}(G) = b$ if and only if (i) $a = b = 1$, (ii) $a \in \{n - 2, n - 1\}$ and $b = n - 1$, or (iii) $2 \leq a \leq b \leq n - 2$. We show that there exists a graph with a unique $\text{cr}$-set of cardinality $k$ for every integer $k \geq 2$. Minimal and forcing connected resolving sets in graphs are studied.
graph $H$ is a resolving graph if there is a graph $G$ with a minimum connected resolving set $W$ such that the subgraph $\langle W \rangle$ of $G$ induced by $W$ is isomorphic to $H$. It is shown that every connected graph is a resolving graph.

For a set $S$ of vertices in a connected graph $G$ and a vertex $v$ of $G$, the distance $d(v,S)$ between $v$ and $S$ is defined as $d(v,S) = \min \{d(v,x) : x \in S\}$. For an ordered $k$-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex $v$ of $G$, the code of $v$ with respect to $\Pi$ is defined as the $k$-vector $c_{\Pi}(v) = (d(v,S_1), d(v,S_2), \ldots, d(v,S_k))$. The partition $\Pi$ is called a resolving partition for $G$ if distinct vertices of $G$ have distinct codes with respect to $\Pi$. Resolving partitions with prescribed properties are studied.

For edges $e$ and $f$ in a connected graph $G$, the distance $d(e,f)$ between $e$ and $f$ is the minimum nonnegative integer $k$ for which there exists a sequence $e = e_0, e_1, \ldots, e_k = f$ of edges of $G$ such that $e_i$ and $e_{i+1}$ are adjacent for $i = 0, 1, \ldots, k - 1$. For an edge $e$ of $G$ and a subgraph $F$ of $G$, the distance between $e$ and $F$ is $d(e,F) = \min \{d(e,f) : f \in E(F)\}$. For an ordered $k$-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph $G$ and $e \in E(G)$, the $\mathcal{D}$-code of $e$ is the $k$-vector $c_{\mathcal{D}}(e) = (d(e,G_1), d(e,G_2), \ldots, d(e,G_k))$. The decomposition $\mathcal{D}$ is a resolving decomposition for $G$ if every two distinct edges of $G$ have distinct $\mathcal{D}$-codes. Resolving decompositions with prescribed properties are investigated.
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ACKNOWLEDGEMENTS

First of all, I would like to take this opportunity to command Professor Ping Zang. Her dedication of being my advisor has been tremendous affect on my carrier. Her help and support are endlessly valuable to me. Her inspiration has led me to discover my interests and dissertation, Graph Theory.

Second, I would like to thank the members of my committee Professors Gary Chartrand, Allen Schwenk, Donald VanderJagt, and Arthur White. Now, I would like to extend my thanks to Professor Donald VanderJagt for being an outstanding outside reader. Next, I personally thank you Professor Gary Chartrand as my second reader. Professor Gary Chartrand was not only sacrificed his family time to read my research, but he truly expressed his interests and enthusiasm on my research. Also, Professor Jay Wood, chair of Mathematics Department, please accept my thanks for all supports.

Third, I would like to thank The Royal Thai Government and Srinakharinwirot University for giving me the scholarship through out the years to complete my Master and Ph.D. programs.

Fourth, my sister, Tassanee, and my parents, Suphot and Yukyun, I love you all and would like to thank you for every phone calls on the weekend to support and listen to my concerns. My achievements are yours!

Last but not least, I would like to send a special thank you to Phu Nguyen. Through tough time, he is always there to conform and kindly offer his assistance.

Varaporn Saenpholphat
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Chapter 1

Introduction

1.1 Background

The distance \( d(u, v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \( u - v \) path in \( G \). For an ordered set \( W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G) \) and a vertex \( v \) of \( G \), we refer to the \( k \)-vector

\[
c_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))
\]

as the code of \( v \) with respect to \( W \). The set \( W \) is called a resolving set for \( G \) if distinct vertices have distinct codes. A resolving set containing a minimum number of vertices is a minimum resolving set or a basis for \( G \). The metric dimension \( \dim(G) \) is the number of vertices in a basis for \( G \). For example, consider the graph \( G \) shown in Figure 1.1. The ordered set \( W_1 = \{v_1, v_3\} \) is not a resolving set for \( G \) since \( c_{W_1}(v_2) = (1, 1) = c_{W_1}(v_4) \), that is, \( G \) contains two vertices with the same code.

![Figure 1.1: Resolving sets in a graph \( G \)](image)

On the other hand, \( W_2 = \{v_1, v_2, v_3\} \) is a resolving set for \( G \) since the codes for the vertices of \( G \) with respect to \( W_2 \) are
\[ \begin{align*}
\mathbf{c}_{W_2}(v_1) &= (0, 1, 1), \\
\mathbf{c}_{W_2}(v_2) &= (1, 0, 1), \\
\mathbf{c}_{W_2}(v_3) &= (1, 1, 0), \\
\mathbf{c}_{W_2}(v_4) &= (1, 2, 1), \\
\mathbf{c}_{W_2}(v_5) &= (2, 1, 1).
\end{align*} \]

However, \(W_2\) is not a minimum resolving set for \(G\) since \(W_3 = \{v_1, v_2\}\) is also a resolving set. The codes for the vertices of \(G\) with respect to \(W_3\) are

\[ \begin{align*}
\mathbf{c}_{W_3}(v_1) &= (0, 1, 1), \\
\mathbf{c}_{W_3}(v_2) &= (1, 0, 1), \\
\mathbf{c}_{W_3}(v_3) &= (1, 1, 0), \\
\mathbf{c}_{W_3}(v_4) &= (1, 2, 1), \\
\mathbf{c}_{W_3}(v_5) &= (2, 1, 1).
\end{align*} \]

Since no single vertex constitutes a resolving set for \(G\), it follows that \(W_3\) is a minimum resolving set for this graph \(G\).

The inspiration for these concepts stems from chemistry. As described in [3], a fundamental mathematical problem in the study of chemical structures is that of providing a mathematical classification of chemical compounds. In such a situation, the compounds are modeled by mathematical objects and the compounds are studied by means of these mathematical objects. Such studies are useful to the understanding of chemical structures.

Chemical compounds can be naturally represented by graphs. The atoms of a molecule are represented by vertices, and the edges of the graph represent the valence bonds between a pair of atoms. For example, a methane molecule has the formula \(CH_4\), where \(C\) indicates a carbon atom and \(H_4\) indicates four atoms of hydrogen. A methane molecule can be represented by the graph shown in Figure 1.2.

![Figure 1.2: Representing a methane molecule by a graph](image)

With this representation, it is possible to show that compounds with the same molecular formula can differ structurally. For example, both ethyl alcohol and
dimethyl ether have the atomic formula $C_2H_6O$ (where $O$ represents an atom of oxygen) even though they have dissimilar chemical properties. The chemical graphs of these compounds are shown in Figure 1.3.

Two chemical compounds may be chemically and structurally different yet behave similarly in certain types of reactions. The reason for this similarity lies in certain common substructures that exist within these compounds. If a set $S$ of atoms has been identified in two compounds and their distances relative to some ordered set $T$ of atoms are the same for both compounds, then these compounds are considered similar or equivalent.

An additional restriction may be placed on the set $T$. Let $v$ be an element (vertex) of $S$. We can associate with $v$ an ordered list of $k$ numbers that gives the distances from $v$ to each of the vertices in $T$. Finding a smallest ordered set $T$ such that the ordered lists associated with every two distinct vertices of $S$ are distinct has applications to classification problems in pharmaceutical chemistry, as described in [3, 27]. Although a chemical compound can be represented naturally by a graph, we desire a more symbolic representation of a chemical compound. In order to describe such a representation for chemical compounds, it suffices to describe one for graphs in general. We do this by means of distance and introducing the resolving set in graphs.

The idea of a resolving set (and of a minimum resolving set) has appeared in the literature previously. In [32], and later in [33], Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph $G$ as its location number $\text{loc}(G)$.
Slater described the usefulness of these ideas when working with U.S. sonar and Coast Guard Loran (Long range aids to navigation) stations. Following Slater [22, 23, 24], we can think of a resolving set as a set $W$ of vertices in a graph $G$ so that each vertex in $G$ is uniquely determined by its distances to the vertices of $W$.

To illustrate this concept, we consider a somewhat simplified example. Suppose that a certain facility consists of five rooms $R_1, R_2, R_3, R_4, R_5$ (shown in Figure 1.4). The distance between rooms $R_1$ and $R_3$ is 2 and the distance between $R_2$ and $R_4$ is also 2. The distance between all other pairs of distinct rooms is 1. The distance between a room and itself is 0. Suppose that a certain (red) sensor is placed in one of the rooms. If a fire should take place in one of the rooms, then the sensor is able to detect the distance from the room with the red sensor to the room containing the fire. Suppose, for example, that the sensor is placed in $R_1$. If a fire occurs in $R_3$, then the sensor alerts us that a fire has occurred in a room at distance 2 from $R_1$; that is, the fire is in $R_3$ since $R_3$ is the only room at distance 2 from $R_1$. If the fire is in $R_1$, then the sensor indicates that the fire has occurred in a room at distance 0 from $R_1$; that is, the fire is in $R_1$. However, if the fire is in any of the other three rooms, then the sensor tells us that there is a fire in a room at distance 1 from $R_1$. But with this information, we cannot tell exactly in which room the fire has occurred. In fact, there is no room in which the (red) sensor can be placed to identify the exact location of a fire in every instance.

![Figure 1.4: A facility consisting of five rooms](image)

On the other hand, if we place the red sensor in $R_1$ and a blue sensor in $R_2$, and a fire occurs in $R_4$, say, then the red sensor in $R_1$ tells us that there is a fire
in a room at distance 1 from $R_1$, while the blue sensor tells us that the fire is in a room at distance 2 from $R_2$, that is, $R_4$ has the code $(1, 2)$. Since the codes are distinct for all rooms, the minimum number of sensors required to detect the exact location of any fire is two. Even though "2" is the answer, care must be taken as to where the two sensors are placed. For example, we cannot place sensors in $R_1$ and $R_3$ since, in this case, the codes of $R_2$, $R_4$, and $R_5$ are all $(1, 1)$, and we cannot distinguish the precise location of the fire.

The facility that we have just described can be represented by a graph, whose vertices are the rooms (see Figure 1.5). (Notice that this is actually the same graph as that in Figure 1.1, except for the way in which the vertices are labeled.) For the graph $G$ of Figure 1.5 then, the dimension of $G$ is 2 and $\{R_1, R_2\}$ is a minimum resolving set for $G$.

![Figure 1.5: A graph representing a facility consisting of five rooms](image)

Independently, Harary and Melter [26] discovered the concept of a location number as well but used the term metric dimension, rather than location number. Recently, these concepts were rediscovered by Johnson [27, 28] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. He and his coauthors [3] used the term resolving set for locating set and used metric dimension for location number the terminology that we have adopted. We refer to the book [10] for graphical-theoretical notation and terminology not described in this dissertation.

1.2 Some Known Results on the Dimension of a Graph

The dimensions of some well-known classes of graphs have been determined in [3, 26, 32, 33]. We state these in the next two results.

**Theorem A** Let $G$ be a connected graph of order $n \geq 2$. 

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CHAPTER 1. INTRODUCTION

(a) Then \( \dim(G) = 1 \) if and only if \( G = P_n \), the path of order \( n \).

(b) Then \( \dim(G) = n - 1 \) if and only if \( G = K_n \), the complete graph of order \( n \).

(c) For \( n \geq 3 \), \( \dim(C_n) = 2 \), where \( C_n \) is the cycle of order \( n \).

(d) For \( n \geq 4 \), \( \dim(G) = n - 2 \) if and only if \( G = K_{r,s} \) \( (r, s \geq 1) \), \( G = K_r + K_s \) \( (r \geq 1, s \geq 2) \), or \( G = K_r + (K_1 \cup K_s) \) \( (r, s \geq 1) \).

In order to present a formula for the dimension of a tree that is not a path, we need some additional definitions and notation. A vertex of degree at least 3 in a graph \( G \) is called a major vertex. An end-vertex \( u \) of \( G \) is said to be a terminal vertex of a major vertex \( v \) of \( G \) if \( d(u, v) < d(u, w) \) for every other major vertex \( w \) of \( G \). The terminal degree \( \text{ter}(v) \) of a major vertex \( v \) is the number of terminal vertices of \( v \). A major vertex \( v \) of \( G \) is an exterior major vertex of \( G \) if it has positive terminal degree. Let \( \sigma(G) \) denote the sum of the terminal degrees of the major vertices of \( G \) and let \( \text{ex}(G) \) denote the number of exterior major vertices of \( G \). In fact, \( \sigma(G) \) is the number of end-vertices of \( G \). For example, the tree \( G \) of Figure 1.6 has four major vertices, namely, \( v_1, v_2, v_3, v_4 \). The terminal vertices of \( v_1 \) are \( u_1 \) and \( u_2 \), the terminal vertices of \( v_3 \) are \( u_3, u_4, \) and \( u_5 \), and the terminal vertices of \( v_4 \) are \( u_6 \) and \( u_7 \). The major vertex \( v_2 \) has no terminal vertex and so \( v_2 \) is not an exterior major vertex of \( G \). Thus \( G \) has three exterior major vertices \( v_1, v_3, \) and \( v_4 \), where \( \text{ter}(v_1) = 2, \text{ter}(v_3) = 3, \) and \( \text{ter}(v_4) = 2 \). Therefore, \( \sigma(G) = 7 \) and \( \text{ex}(G) = 3 \).

**Theorem B** If \( T \) is a tree that is not a path, then \( \dim T = \sigma(T) - \text{ex}(T) \).

In fact, a characterization of bases in a tree \( T \) that is not a path has been established in [11].

**Theorem C** Let \( T \) be a tree of order \( n \geq 4 \) that is not a path having \( p \) exterior major vertices \( v_1, v_2, \ldots, v_p \). For \( 1 \leq i \leq p \), let \( u_{i1}, u_{i2}, \ldots, u_{ik_i} \) be the terminal vertices of \( v_i \), and let \( P_{ij} \) be the \( v_i - u_{ij} \) path \( (1 \leq j \leq k_i) \). Suppose that \( W \) is a
Figure 1.6: A graph with three exterior major vertices

set of vertices of $T$. Then $W$ is a resolving set of $T$ if and only if $W$ contains at least one vertex from each of the paths $P_{ij} - v_i$ ($1 \leq j \leq k_i$ and $1 \leq i \leq p$) with at most one exception for each $i$ with $1 \leq i \leq p$. Moreover, $W$ is a basis of $T$ if and only if $W$ contains exactly one vertex from each of the paths $P_{ij} - v_i$ ($1 \leq j \leq k_i$ and $1 \leq i \leq p$) with exactly one exception for each $i$ with $1 \leq i \leq p$.

This topic has been studied further in [2, 16]. Resolving sets that satisfy certain prescribed properties have been studied in [1, 5, 30, 31].

1.3 Some Known Results on the Partition Dimension of a Graph

For a set $S$ of vertices of $G$ and a vertex $v$ of $G$, the distance $d(v, S)$ between $v$ and $S$ is defined as

$$d(v, S) = \min\{d(v, x) : x \in S\}.$$ 

For an ordered $k$-partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$ and a vertex $v$ of $G$, the code of $v$ with respect to $\Pi$ is defined as the $k$-vector

$$c_\Pi(v) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k)).$$ 

The partition $\Pi$ is called a resolving partition for (or of) $G$ if the distinct vertices of $G$ have distinct codes with respect to $\Pi$. The minimum $k$ for which there is a resolving $k$-partition of $V(G)$ is the partition dimension $\text{pd}(G)$ of $G$. A resolving partition of $V(G)$ containing $\text{pd}(G)$ elements is called a minimum resolving partition. The resolving partition and partition dimension of a graph were introduced and studied in [12, 13], where the following results were established.
CHAPTER 1. INTRODUCTION

Theorem D  If G is a nontrivial connected graph, then $pd(G) \leq \dim(G) + 1$.

Theorem E  Let G be a connected graph of order $n \geq 2$. Then

(a) $pd(G) = 2$ if and only if $G = P_n$.

(b) $pd(G) = n$ if and only if $G = K_n$.

(c) $pd(G) = n - 1$ if and only if $G \in \{K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-2})\}$, $n \geq 3$.

Resolving partitions that satisfy certain prescribed properties have been studied. A partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$ is independent if the subgraph $\langle S_i \rangle$ induced by $S_i$ is independent in $G$ for each $i$ ($1 \leq i \leq k$). If $\Pi$ is a resolving independent partition of $V(G)$, then, by coloring the vertices in $S_i$ by $i$ ($1 \leq i \leq k$), we obtain a proper coloring $c$ of $G$ with $k$ colors that distinguishes all vertices of $G$ in terms of their distances from the color classes. Thus, such a coloring $c$ of a connected graph $G$ is called a resolving-coloring. A minimum resolving-coloring uses a minimum number of colors and this number is the resolving-chromatic number $\chi_r(G)$ of $G$. In [6, 7], a resolving-coloring is referred to as a locating-coloring and the resolving-chromatic number as the locating-chromatic number.

In [6, 7], resolving-chromatic number numbers of some well-known classes of graphs are determined and bounds for the resolving-chromatic number of a connected graph are established in terms of its order and diameter. Also, all connected graphs of order $n$ with resolving-chromatic number $n$ or $n - 1$ are determined. Since every resolving-coloring is a coloring, $2 \leq \chi(G) \leq \chi_r(G)$ for each connected graph $G$ of order at least 2. It is shown in [6] that for each pair $a, b$ of integers with $2 \leq a \leq b$, there exists a connected graph with chromatic number $a$ and resolving-chromatic number $b$.

1.4 Some Known Results on the Decomposition Dimension of a Graph

A decomposition of a graph $G$ is a collection of subgraphs of $G$, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A
decomposition into \( k \) subgraphs is a \( k \)-decomposition. A decomposition \( \mathcal{D} = \{G_1, G_2, \ldots, G_k\} \) is ordered if the ordering \( (G_1, G_2, \ldots, G_k) \) has been imposed on \( \mathcal{D} \). If each subgraph \( G_i \) (\( 1 \leq i \leq k \)) is isomorphic to a graph \( H \), then \( \mathcal{D} \) is called an \( H \)-decomposition of \( G \). Decompositions of graphs have been the subject of many studies.

For edges \( e \) and \( f \) in a connected graph \( G \), the distance \( d(e, f) \) between \( e \) and \( f \) is the minimum nonnegative integer \( k \) for which there exists a sequence \( e = e_0, e_1, \ldots, e_k = f \) of edges of \( G \) such that \( e_i \) and \( e_{i+1} \) are adjacent for \( i = 0, 1, \ldots, k - 1 \). Thus \( d(e, f) = 0 \) if and only if \( e = f \), \( d(e, f) = 1 \) if and only if \( e \) and \( f \) are adjacent, and \( d(e, f) = 2 \) if and only if \( e \) and \( f \) are nonadjacent edges that are adjacent to a common edge of \( G \). Also, this distance equals the standard distance between vertices \( e \) and \( f \) in the line graph \( L(G) \). For an edge \( e \) of \( G \) and a subgraph \( F \) of \( G \), we define the distance between \( e \) and \( F \) as

\[
d(e, F) = \min_{f \in E(F)} d(e, f).
\]

Let \( \mathcal{D} = \{G_1, G_2, \ldots, G_k\} \) be an ordered \( k \)-decomposition of a connected graph \( G \). For \( e \in E(G) \), the \( \mathcal{D} \)-code (or simply the code) of \( e \) is the \( k \)-vector

\[
c_{\mathcal{D}}(e) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k)).
\]

Hence exactly one coordinate of \( c_{\mathcal{D}}(e) \) is 0, namely the \( i \)-th coordinate if \( e \in E(G_i) \). The decomposition \( \mathcal{D} \) is said to be a resolving decomposition for \( G \) if every two distinct edges of \( G \) have distinct \( \mathcal{D} \)-codes. The minimum \( k \) for which \( G \) has a resolving \( k \)-decomposition is its decomposition dimension \( \dim_d(G) \). A resolving decomposition of \( G \) with \( \dim_d(G) \) elements is a minimum resolving decomposition for \( G \). These concepts were first introduced in [4] and studied further in [18, 25]. If \( G \) is a nontrivial connected graph, then \( \dim_d(G) \geq 2 \). Also, if \( G \) is a connected graph of order \( n \geq 5 \), then \( \dim_d(G) \leq n \). It was shown in [4] the path \( P_n \) of order \( n \geq 3 \) is the only connected graph of order \( n \) with decomposition dimension 2. An upper bound for the decomposition dimension of the complete graph \( K_n \) of order \( n \geq 3 \) has been established in [4].
CHAPTER 1. INTRODUCTION

**Theorem F**  Let $G$ be a connected graph of order $n \geq 3$. Then $\text{dim}_d(G) = 2$ if and only if $G = P_n$.

**Theorem G**  For every integer $n \geq 3$,

$$
\text{dim}_d(K_n) \leq \lfloor \frac{2n+5}{3} \rfloor = \begin{cases} 
2n/3 + 1 & \text{if } n \equiv 0 \pmod{3}, \\
(2n + 1)/3 + 1 & \text{if } n \equiv 1 \pmod{3}, \\
(2n + 2)/3 + 1 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
$$

The bound in Theorem G was improved by Nakamigawa [29] to

$$
\text{dim}_d(K_n) \leq (1/2 + o(1)) \log^2 n.
$$

Küngen and West [25] improved this upper bound to $(3.2 + o(1)) \log n$. 

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Chapter 2

Connected Resolving Sets in Graphs

2.1 Introduction

In general, a connected graph $G$ can have many resolving sets. In this chapter, we consider those resolving sets whose vertices are located "close" to one another. A resolving set $W$ of $G$ is *connected* if the subgraph $\langle W \rangle$ induced by $W$ is a connected subgraph of $G$. The minimum cardinality of a connected resolving set $W$ in a graph $G$ is the *connected resolving number* $cr(G)$. A connected resolving set of cardinality $cr(G)$ is called a *$cr$-set* of $G$. Since the vertex set $V(G)$ of $G$ is a resolving set and $\langle V(G) \rangle = G$ is connected, $cr(G)$ is defined for every connected graph $G$. Since every connected resolving set is a resolving set, $\dim(G) \leq cr(G)$ for all connected graphs $G$.

To illustrate this concept, consider the graph $G$ of Figure 2.1. The set $W = \{u, v\}$ is a basis for $G$ and so $\dim(G) = 2$. The codes for the vertices of $G$ with respect to $W$ are $c_W(u) = (0, 2)$, $c_W(v) = (2, 0)$, $c_W(w) = (1, 2)$, $c_W(x) = (1, 1)$, $c_W(y) = (2, 1)$. Since $\langle \{u, v\} \rangle$ is disconnected, $W$ is not a connected resolving set. On the other hand, the set $W' = \{u, v, x\}$ is a connected resolving set. The codes for the vertices of $G$ with respect to $W'$ are $c_{W'}(u) = (0, 2, 1)$, $c_{W'}(v) = (2, 0, 1)$, $c_{W'}(w) = (1, 2, 1)$, $c_{W'}(x) = (1, 1, 0)$, $c_{W'}(y) = (2, 1, 1)$. Since $G$ contains no connected 2-element resolving set, that is, a resolving set consisting of two adjacent
vertices, $cr(G) = 3$.

![Graph G](image)

Figure 2.1: A graph $G$ with $\dim(G) = 2$ and $cr(G) = 3$

The example just presented also illustrates an important point. When determining whether a given set $W$ of vertices of a graph $G$ is a resolving set for $G$, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of $G$ whose distance from $w$ is 0. We first make a few observations that will be of use on several occasions.

**Observation 2.1** Every superset of a resolving set of a connected graph $G$ is also a resolving set of $G$.

**Observation 2.2** If a connected graph $G$ contains a set $S$ of $p \geq 2$ vertices of $G$ such that $d(u, x) = d(v, x)$ for each pair $u, v$ of vertices of $S$ and every $x \in V(G) - \{u, v\}$, then every resolving set of $G$ contains at least $p - 1$ vertices of $S$.

Two vertices $u$ and $v$ of a connected graph $G$ are defined to be **distance similar** if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$.

**Proposition 2.3** Two vertices $u$ and $v$ in a connected graph are distance similar if and only if (1) $uv \notin E(G)$ and $N(u) = N(v)$ or (2) $uv \in E(G)$ and $N[u] = N[v]$.

**Proof.** Suppose that $u$ and $v$ are distance similar vertices of a connected graph $G$. We consider two cases.

**Case 1.** $uv \notin E(G)$. Assume, to the contrary, that $N(u) \neq N(v)$. Then there exists $x \in N(u)$ such that $x \notin N(v)$ or there exists $y \in N(v)$ such that
y \notin N(u)$, say the former. However, then, $d(u, x) = 1$ and $d(v, x) > 1$, which is a contradiction.

**Case 2.** $uv \in E(G)$. Assume, to the contrary, that $N[u] \neq N[v]$. Since $u, v \in N[u] \cap N[v]$, there exists $x \in N[u] - \{u, v\}$ such that $x \notin N[v]$ or there exists $y \in N[v] - \{u, v\}$ such that $y \notin N[u]$, say the former. However, then, $d(u, x) = 1$ and $d(v, x) > 1$, which is a contradiction.

For the converse, $u$ and $v$ are two vertices in $G$ that satisfy (1) or (2). In either case, $N(u) = N(v)$. We show that $u$ and $v$ are distance similar vertices in $G$. Let $w \in V(G) - \{u, v\}$. Suppose that $d(u, w) = d_1$ and $d(v, w) = d_2$. Assume, without loss of generality, that $d_1 \leq d_2$. Let $P : u = u_0, u_1, \ldots, u_{d_1} = w$ be a $u - w$ path in $G$ of length $d_1$. Since $N(u) = N(v)$, it follows that $v$ is adjacent to $u_1$. Joining $v$ to the vertex $u_1$ in the path $P - u$, we obtain a $v - w$ path $Q : v, u_1, \ldots, u_{d_1} = w$ of length $d_1$ in $G$. Thus $d_2 \leq d_1$ and so $d_1 = d_2$. Therefore, $u$ and $v$ are distance similar vertices of $G$. 

**Proposition 2.4** Distance similarity in a graph $G$ is an equivalence relation on $V(G)$.

**Proof.** It is clear that distance similarity in a graph is reflexive and symmetric. Thus it remains to show that distance similarity is also transitive. Assume that $u, v, w \in V(G)$ such that $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$ and $d(v, y) = d(w, y)$ for all $y \in V(G) - \{v, w\}$. We show that $d(u, z) = d(w, z)$ for all $z \in V(G) - \{u, w\}$. Let $z \in V(G) - \{u, w\}$. If $z \neq v$, then $z \in V(G) - \{u, v, w\}$ and so $d(u, z) = d(v, z) = d(w, z)$. Thus we may assume that $z = v$. Since $w \in V(G) - \{u, v\}$, it follows that $d(u, w) = d(v, w)$. On the other hand, since $u \in V(G) - \{v, w\}$, we have $d(v, u) = d(w, u)$. Therefore, $d(u, v) = d(w, v)$, as desired.

Let $V_1, V_2, \ldots, V_k$ be the $k$ ($\geq 1$) distinct distance similar equivalence classes of $V(G)$. By Observation 2.2, if $W$ be a resolving set of $G$, then $W$ contains at least $|V_i| - 1$ vertices from each equivalence class $V_i$ for all $i$ with $1 \leq i \leq k$. Thus we have the following.
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Observation 2.5  Let $G$ be a nontrivial connected graph of order $n$. If $G$ has $k$ distance similar equivalence classes, then $\dim(G) \geq n - k$ and so $cr(G) \geq n - k$

Note that for a given integer $n \geq 4$, the bounds in Observation 2.5 are sharp for most values of $k$. Moreover, strict inequality can hold for most values of $k$ in each case as well, as we will see in Section 2.8.

Observation 2.6  Let $G$ be a connected graph. Then $\dim(G) = cr(G)$ if and only if $G$ contains a connected basis.

2.2  Comparison of $cr$-Sets and Bases in Graphs

In this section, we study the relationship between $cr$-sets and bases in a nontrivial connected graph $G$. By Observation 2.1, if $W$ and $W'$ are two sets of vertices of a graph $G$ such that $W$ is a resolving set of $G$ and $W \subseteq W'$, then $W'$ is also a resolving set of $G$. Therefore, if $W$ is a basis of $G$ such that $\langle W \rangle$ is disconnected, then surely there is a smallest superset $W'$ of $W$ for which $\langle W' \rangle$ is connected. This suggests the following question: For each basis $W$ of a nontrivial connected graph $G$, does there exist a $cr$-set $W'$ of $G$ such that $W \subseteq W'$? We show that this question has a negative answer.

Proposition 2.7  There is an infinite class of connected graphs $G$ such that some $cr$-sets of $G$ contain a basis of $G$ and others contain no basis of $G$.

Proof.  Let $G$ be the graph obtained from the 4-cycle $u_1, u_2, u_3, u_4, u_1$ by adding the $k$ ($\geq 2$) new vertices $v_1, v_2, \ldots, v_k$ and joining each $v_i$ ($1 \leq i \leq k$) to $u_1$ and $u_4$. The graph $G$ is shown in Figure 2.2. Let $V = \{v_1, v_2, \ldots, v_k\}$. Since $W = \{u_2\} \cup (V - \{v_k\})$ is a basis of $G$, it follows that $\dim(G) = k$.

Next we show that $cr(G) = k + 1$. By Observation 2.2, every resolving set of $G$ contains at least $k - 1$ vertices in $V$. Thus every connected resolving set must contain at least one vertex from $\{u_1, u_4\}$. However, if $S$ is a set of vertices of $G$ consisting of $k - 1$ vertices from $V$ and one vertex from $\{u_1, u_4\}$, say $S = \ldots.$
(\(V - \{v_k\}\) \(\cup \{u_1\}\), then \(c_S(u_2) = c_S(v_k)\) and so \(S\) is not a resolving set. Therefore, \(G\) contains no connected basis and so \(cr(G) \geq k + 1\) by Observation 2.6. On the other hand, \(S_1 = \{u_1, u_2\} \cup (V - \{v_k\})\) is a connected resolving set of cardinality \(k + 1\) and so \(cr(G) \leq k + 1\). Therefore, \(cr(G) = k + 1\).

Observe that the cr-set \(S_1\) contains the basis \(W = \{u_2\} \cup (V - \{v_k\})\) of \(G\). On the other hand, let \(S_2 = \{u_1\} \cup V\). Observe that \(c_{S_2}(u_2) = (1, 2, 2, \ldots), c_{S_2}(u_3) = (2, 2, 2, \ldots),\) and \(c_{S_2}(u_4) = (1, 1, 1, \ldots)\). Thus \(S_2\) is also a cr-set of \(G\). Since every basis of \(G\) contains exactly \(k - 1\) vertices from \(V\) and exactly one vertex from \(\{u_2, u_3\}\), it follows that \(S_2\) contains no basis of \(G\).

Proposition 2.7 suggests yet another question. For each connected graph \(G\), does there exist some cr-set \(W'\) and some basis \(W\) of \(G\) with \(W \subseteq W'\)? We show that even this question has a negative answer as well by presenting a stronger result.

**Theorem 2.8**  There is an infinite class of connected graphs \(G\) such that every cr-set of \(G\) is disjoint from every basis of \(G\).

**Proof.** For two integers \(p, q \geq 3\), let \(G\) be that graph obtained from two odd cycles \(C_{2p+1} : u_0, u_1, u_2, \ldots, u_p, u'_p, u'_{p-1}, \ldots, u'_1, u_0\) and \(C_{2q+1} : v_0, v_1, v_2, \ldots, v_q, v'_q, v'_{q-1}, \ldots, v'_1, v_0\) by (1) identifying the vertex \(u_0\) of \(C_{2p+1}\) with the vertex \(v_0\) of \(C_{2q+1}\), denoting the identified vertex by \(x\), and (2) adding a pendant edge \(xy\). The graph \(G\) is shown in Figure 2.3.

First we make an observation. Let \(U = \{u_1, u_2, \ldots, u_p\}, U' = \{u'_1, u'_2, \ldots, u'_q\}\),
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Figure 2.3: The graph $G$ in the proof of Theorem 2.8 for $p = 4$ and $q = 3$

$u_p', u_3', u_2', u_1', v_1', v_2', \ldots, v_q'$, $V = \{v_1, v_2, \ldots, v_q\}$, and $V' = \{v_1', v_2', \ldots, v_q'\}$. If $S$ is a resolving set of $G$, then $S$ contains at least one vertex from each of $U \cup U'$ and $V \cup V'$. Otherwise, if $S \subseteq U \cup U' \cup \{x, y\}$, then $c_S(v_i) = c_S(v'_i)$. If $S \subseteq V \cup V' \cup \{x, y\}$, then $c_S(u_i) = c_S(u'_i)$. In either case, $S$ is not a resolving set, which is a contradiction.

Let $W_1 = \{u_p, v_q\}, W_2 = \{u_p', v_q\}, W_3 = \{u_p, v_q'\}$, and $W_4 = \{u_p', v_q'\}$. Since $G$ is not a path (which is the only nontrivial connected graph of dimension 1) and each of the sets $W_i$ ($1 \leq i \leq 4$) is a resolving set, $\dim(G) = 2$. We show next that the sets $W_i$ ($1 \leq i \leq 4$) are the only bases of $G$. Assume, to the contrary, that $G$ contains a basis $W$ that is distinct from all $W_i$ for $1 \leq i \leq 4$. Let $W = \{s, t\}$.

By the observation above, $W$ contains exactly one vertex from each of $U \cup U'$ and $V \cup V'$, say $s \in U \cup U'$ and $t \in V \cup V'$. We consider two cases.

Case 1. $s = u_p$ or $s = u'_p$. Assume, without loss of generality, that $s = u_p$. Since $W \neq W_i$ for $1 \leq i \leq 4$, it follows that $t = v_j$ or $t = v'_j$ for $1 \leq j \leq q - 1$. If $t = v_j$ for $1 \leq j \leq q - 1$, then $d(v'_i, t) = d(y, t)$. Since $d(v'_i, s) = d(y, s)$, it follows that $c_W(v'_i) = c_W(y)$, a contradiction. If $t = v'_j$ for $1 \leq j \leq q - 1$, then $d(v_1, t) = d(y, t)$. Since $d(v_1, s) = d(y, s)$, it follows that $c_W(v_1) = c_W(y)$, a contradiction.

Case 2. $s = u_i$ or $s = u'_i$, where $1 \leq i \leq p - 1$. Assume, without loss of generality, that $s = u_i$ for some $i$ with $1 \leq i \leq p - 1$. Then $t = v_j$ or $t = v'_j$ for $1 \leq j \leq q$. If $t = v_j$ for $1 \leq j \leq q - 1$, then $c_W(v'_i) = c_W(y)$. If $t = v'_j$ for $1 \leq j \leq q - 1$, then $c_W(v_1) = c_W(y)$. If $t = v_q$ or $t = v'_q$, then $c_W(u'_i) = c_W(y)$. Thus in each case, a contradiction is produced.
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Therefore, $W$ is not a basis of $G$ and so the sets $W_i$ (1 $\leq$ $i$ $\leq$ 4) are the only bases of $G$.

Next we show that $cr(G) = 5$. Since $S_0 = \{u_1, u'_1, v_1, v'_1, x\}$ is a connected resolving set, $cr(G)$ $\leq$ 5. Assume, to the contrary, that $cr(G)$ $\leq$ 4. Since $W_i$ (1 $\leq$ $i$ $\leq$ 4) are the only bases of $G$ and none of $W_i$ (1 $\leq$ $i$ $\leq$ 4) are connected, $G$ contains no connected basis. Thus $cr(G)$ $\geq$ 3 by Observation 2.6. Let $S$ be a $cr$-set of $G$. Then $|S| = 3$ or $|S| = 4$. We consider these two cases.

Case 1. $|S| = 3$. Let $N[x] = \{u_1, u'_1, v_1, v'_1, x, y\}$ be the closed neighborhood of $x$. By the observation above, $S$ contains at least one vertex from each of $V \cup V'$ and $U \cup U'$. This implies that $x \in S$ and $S \subseteq N[x]$. Since $\langle N[x]\rangle = K_{1,5}$ and $cr(K_{1,5}) = 5$, it follows that $S$ is not a connected resolving set of $G$, a contradiction.

Case 2. $|S| = 4$. Again, $S$ contains at least one vertex from each of $V \cup V'$ and $U \cup U'$. An argument similar to that used in Case 1 shows that if $S \subseteq N[x]$, then $S$ is not a connected resolving set for $G$. Thus $S \not\subseteq N[x]$. Since $|S| = 4$ and $S$ is connected, $S$ must contain $x$ and exactly one vertex from each of $\{u_1, u'_1\}$, $\{v_1, v'_1\}$, and $\{u_2, u'_2, v_2, v'_2\}$. Assume, without loss of generality, that $S = \{u_1, u'_1, x, v_1\}$. However, then $c_S(u'_1) = c_S(v'_1)$, which is a contradiction.

Therefore, $cr(G) = 5$. Moreover, we claim that each $cr$-set $S$ is a subset of $\{u_1, u'_1, v_1, v'_1, x, y\}$. Assume, to the contrary, that there is a $cr$-set $S'$ such that $S'$ is not a subset of $\{u_1, u'_1, v_1, v'_1, x, y\}$. Since (1) $S'$ is connected, (2) $|S'| = 5$, and (3) $S'$ must contain at least one vertex from each of $V \cup V'$ and $U \cup U'$, it follows that $S'$ is a subset of $\{u_1, u_2, u_3, u'_1, u'_2, u'_3, v_1, v_2, v_3, v'_1, v'_2, v'_3, x, y\}$. Since $|S'| = 5$, it follows that $S'$ contains at most one vertex from $\{u_3, u'_3, v_3, v'_3\}$. Suppose first that $S'$ contains no vertex from $\{u_3, u'_3, v_3, v'_3\}$. Assume, without loss of generality, that $S' = \{u_2, u_1, v_1, x, y\}$ or $S' = \{u_2, u_1, v_1, v_2, x\}$. However, in each case, $c_{S'}(u'_1) = c_{S'}(v'_1)$, which is a contradiction. Thus $S'$ contains exactly one vertex from $\{u_3, u'_3, v_3, v'_3\}$, say $u_3 \in S'$. Then $S' = \{u_3, u_2, u_1, v_1, x\}$ or $S' = \{u_3, u_2, u_1, v'_1, x\}$. However, then, either $c_{S'}(u'_1) = c_{S'}(v'_1)$ or $c_{S'}(v_1) = c_{S'}(y)$, which is a contradiction. Therefore, each $cr$-set $S$ is a subset of $\{u_1, u'_1, v_1, v'_1, x, y\}$, as claimed. Since each basis $W$ of $G$ is a subset of $\{u_p, u'_p, v_q, v'_q\}$ and
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p, q ≥ 3, it follows that every cr-set of G is disjoint from every basis of G.

The graph G constructed in the proof of Theorem 2.8 can be modified to obtain an even stronger result. First we need an additional definition. Let X and Y be two sets of vertices in a connected graph G. The distance between X and Y is defined as

\[ d(X, Y) = \min\{d(x, y) \mid x \in X \text{ and } y \in Y\}. \]

Corollary 2.9 For each positive integer N, there is an infinite class of connected graphs G such that d(W, S) ≥ N for every basis W of G and every cr-set S of G.

Proof. Let G be the graph constructed in the proof of Theorem 2.8 for p ≥ q ≥ \(\max\{3, N + 1\} \). Since each basis W of G is a subset of \(\{u_p, u'_p, v_p, v'_p\} \) and each cr-set S is a subset of \(\{u_1, u'_1, v_1, v'_1, x, y\} \), it follows that d(W, S) ≥ p − 1 ≥ N, as desired.

The following three results give relationships between cr-sets and bases in some well-known classes of graphs, namely complete graphs, complete bipartite graphs, cycles, and trees.

Proposition 2.10 If G is a complete graph of order at least 3 or a complete bipartite graph that is not a star, then a set W of vertices of G is a basis of G if and only if W is a cr-set of G.

Proof. If G = \(K_n\) for \(n \geq 3\), every set of \(n - 1\) vertices in G form a basis and a cr-set of G. Let G = \(K_{r,s}\) with partite sets \(V_1\) and \(V_2\), where \(|V_1| = r\) and \(|V_2| = s\) and \(2 \leq r \leq s\). Then every basis W of G contains exactly \(r - 1\) vertices from \(V_1\) and exactly \(s - 1\) vertices from \(V_2\). Since \(<W>\) is connected, W is a cr-set. By Observation 2.6, we have \(\dim(G) = \sigma(G)\), which implies that every cr-set of G is a basis.

Proposition 2.11 For a cycle \(C_n\) of order \(n \geq 4\), every cr-set of \(C_n\) is a basis of \(C_n\).
Proof. Since $\text{dim}(C_n) = \text{cr}(C_n) = 2$ for $n \geq 4$ and every $cr$-set of $C_n$ consists of two adjacent vertices, every $cr$-set is also a basis. ■

The converse of Proposition 2.11 is not true for $n \geq 5$ since some basis of $C_n$ consists of two nonadjacent vertices of $C_n$ and therefore, it is not a $cr$-set of $C_n$.

By Theorem C, we have the following result whose proof is straightforward and is therefore omitted.

Proposition 2.12 If $T$ is a tree that is not a path, then every $cr$-set of $T$ contains a basis of $T$ as a proper subset.

2.3 The Connected Resolving Numbers of Some Well-Known Graphs

If $G$ is a connected graph of order $n$, then every set of $n - 1$ vertices of $G$ is a resolving set of $G$. Moreover, every nontrivial connected graph $G$ contains a vertex $v$ that is not a cut-vertex and so $V(G) - \{v\}$ is a connected resolving set for $G$. Thus

$$1 \leq \text{cr}(G) \leq n - 1$$

(2.1)

for all connected graphs $G$ of order $n \geq 3$. The lower and upper bounds in (2.1) are both sharp. For example, if $G = P_n$, then each end vertex of $G$ is a connected basis for $G$ and so $\text{cr}(G) = 1$. Since $P_n$ is the only connected graph of order $n \geq 2$ with dimension 1 by Theorem A, we have the following.

Corollary 2.13 Let $G$ be a connected graph of order $n \geq 2$. Then $\text{cr}(G) = 1$ if and only if $G = P_n$.

If $G = C_n$, where $n \geq 3$, then any two adjacent vertices of $G$ form a connected basis for $G$, and so we have the following.

Proposition 2.14 If $G = C_n$ for $n \geq 3$, then $\text{cr}(G) = 2$.

Next, we show that the complete graph $K_n$ and the star $K_{1,n-1}$ are the only connected graphs of order $n \geq 4$ having connected resolving number $n - 1$. 

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Theorem 2.15  Let $G$ be a connected graph of order $n \geq 4$. Then $cr(G) = n - 1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.

Proof. Since $\dim(K_n) = n - 1$ and every induced subgraph of $K_n$ is connected, $cr(K_n) = \dim(K_n) = n - 1$ by Observation 2.6. For $G = K_{1,n-1}$, let $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$, where $v$ is the central vertex of $G$. By Observation 2.2, every $cr$-set of $G$ contains at least $n - 2$ end-vertices and so $cr(G) \geq n - 2$. On the other hand, the subgraph induced by any $n - 2$ end-vertices of $G$ is the graph $\overline{K}_{n-2}$, which is not connected, and so $cr(G) \geq n - 1$. Hence $cr(G) = n - 1$.

Next we show that if $G$ is a connected graph of order $n \geq 4$ that is neither a complete graph nor a star, then $cr(G) \leq n - 2$. To do this, it suffices to show the following stronger statement: If $G$ is connected graph of order $n \geq 4$ that is neither a complete graph nor a star, then $G$ contains distinct vertices $u, v, w_1, w_2$ such that $V(G) - \{u, v\}$ is a connected resolving set and (1) $u$ is adjacent to $w_1$ and (2) $v$ is adjacent to $w_2$ but not to $w_1$. We proceed by induction on the order $n$ of $G$. For $n = 4$, the graphs $G_i$ $(1 \leq i \leq 4)$ of Figure 2.4 are only connected graphs order 4 that are different from $K_4$ or $K_{1,3}$. For each $i$ $(1 \leq i \leq 4)$, the vertices $u, v, w_1, w_2$ are shown in Figure 2.4 and $W = V(G_i) - \{u, v\} = \{w_1, w_2\}$ is a connected resolving set in $G_i$. Moreover, $u$ is adjacent to $w_2$ but not to $w_1$. Thus the statement is true for $n = 4$. Assume that the statement is true for $n - 1 \geq 4$.

![Figure 2.4: Graphs $G_i$ $(1 \leq i \leq 4)$](image)

Let $G$ be a connected graph of order $n \geq 5$ that is not $K_n$ or $K_{1,n-1}$ and let $x$ be vertex of $G$ such that $G' = G - x$ is connected and $G' \neq K_{n-1}, K_{1,n-2}$. 

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By the induction hypothesis, $G'$ contains distinct vertices $u, v, w_1, w_2$ such that $W' = V(G') - \{u, v\}$ is a connected resolving set and $u$ is adjacent to $w_1$ and $v$ is adjacent to $w_2$ but not to $w_1$. Since $G$ is connected, $x$ is adjacent to some vertex in $G$. We consider two cases.

**Case 1. $x$ is adjacent to at least one vertex in $W'$.** Let $W = W' \cup \{x\} = V(G) - \{u, v\}$. Then $\langle W \rangle$ is connected. Since $d(u, w_1) = 1$ and $d(v, w_1) \geq 2$, it follows that $c_W(u) \neq c_W(v)$ and so $W$ is a connected resolving set of $G$. Moreover, $u$ is adjacent to $w_1$ and $v$ is adjacent to $w_2$ but not to $w_1$.

**Case 2. $x$ is adjacent to no vertex in $W'$.** There are three subcases.

**Subcase 2.1. $x$ is adjacent to both $u$ and $v$.** Let $W = W' \cup \{x\} = W'(u, v)$. Then $\langle W \rangle$ is connected. Since $d(u, w_1) = 1$ and $d(v, w_1) \geq 2$, it follows that $c_W(u) \neq c_W(v)$ and so $W$ is a connected resolving set of $G$. Moreover, $u$ is adjacent to $w_1$ and $x$ is adjacent to $v$ but not to $w_1$.

**Subcase 2.2. $x$ is adjacent to $u$ but not to $v$.** Let $W = W' \cup \{u\} = V(G) - \{v, x\}$. Then $\langle W \rangle$ is connected. Since $d(v, w_2) = 1$ and $d(x, w_2) \geq 3$, it follows that $c_W(v) \neq c_W(x)$ and so $W$ is connected resolving set of $G$. Moreover, $v$ is adjacent to $w_2$ and $x$ is adjacent to $u$ but not to $w_2$.

**Subcase 2.3. $x$ is adjacent to $v$ but not to $u$.** Let $W = W' \cup \{v\} = V(G) - \{u, x\}$. Then $\langle W \rangle$ is connected. Since $d(u, w_1) = 1$ and $d(x, w_1) \geq 3$, it follows that $c_W(u) \neq c_W(x)$ and $W$ is connected resolving set of $G$. Moreover, $u$ is adjacent to $w_1$ and $x$ is adjacent to $v$ but not to $w_1$.

Thus, in either case, $G$ contains a connected resolving set of cardinality $n - 2$. Therefore, $cr(G) \leq n - 2$.

We now determine the connected resolving numbers of complete $k$-partite graphs ($k \geq 2$) that are not stars.

**Proposition 2.16** For $k \geq 2$, let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete $k$-partite graph that is not a star. Let $n = n_1 + n_2 + \cdots + n_k$ and $\ell$ be the number of one's in
Then
\[
\text{cr}(G) = \begin{cases} 
    n - k & \text{if } \ell = 0 \\
    n - k + \ell - 1 & \text{if } \ell \geq 1.
\end{cases}
\]

**Proof.** Assume that \(1 \leq n_1 \leq n_2 \leq \cdots \leq n_k\). For each \(i\) with \(1 \leq i \leq k\), let \(V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}\) be a partite set of \(G\). We consider two cases.

**Case 1.** \(\ell = 0\). Then \(n_i \geq 2\) for all \(1 \leq i \leq k\). Since \(G\) has \(k\) distinct distance similar equivalence classes, namely \(V_1, V_2, \ldots, V_k\), it follows from Observation 2.5 that \(\text{cr}(G) \geq n - k\). On the other hand, let \(W = \bigcup_{i=1}^{k} \{V_i - \{v_{i1}\}\}\). Since \(W\) is a resolving set of \(G\) and \(\langle W \rangle = K_{n_{1-1}, n_{2-1}, \ldots, n_{k-1}}\) is connected, \(W\) is a connected resolving set and so \(\text{cr}(G) \leq |W| = n - k\). Thus \(\text{cr}(G) = n - k\).

**Case 2.** \(\ell \geq 1\). Then \(n_i = 1\) for all \(1 \leq i \leq \ell\) and \(n_i \geq 2\) for all \(\ell + 1 \leq i \leq k\). Let \(U_1 = \bigcup_{i=1}^{\ell} V_i = \{v_{11}, v_{21}, \ldots, v_{\ell1}\}\) and \(U_j = V_{\ell+j-1}\) for all \(j\) with \(2 \leq j \leq k - \ell + 1\). Then \(U_1, U_2, \ldots, U_{k-\ell+1}\) are \(k - \ell + 1\) distinct distance similar equivalence classes and so \(\text{cr}(G) \geq n - (k - \ell + 1) = n - k + \ell - 1\) by Observation 2.5. On the other hand, let
\[
W = \{v_{21}, \ldots, v_{\ell1}\} \cup \left( \bigcup_{i=\ell+1}^{k} \{V_i - \{v_{i1}\}\} \right).
\]
Then \(\langle W \rangle\) is connected. Since \(d(v_{11}, w) = 1\) for all \(w \in W\), \(d(v_{i1}, w) = 1\) if \(w \in W - \{V_i - \{v_{i1}\}\}\) and \(d(v_{i1}, w) = 2\) if \(w \in V_i - \{v_{i1}\}\) for all \(i\) with \(\ell + 1 \leq i \leq k\), it follows that \(W\) is a resolving set. Thus \(W\) is a connected resolving set and so \(\text{cr}(G) \leq |W| = n - k + \ell - 1\). Therefore, \(\text{cr}(G) = |W| = n - k + \ell - 1\). ■

Using Theorem C, we are able to characterize the \(\text{cr}\)-sets in a tree \(T\). Again, we omit the proof of the following result since it is routine.

**Theorem 2.17** Let \(T\) be a tree of order \(n \geq 4\) that is not a path having \(p\) exterior major vertices \(v_1, v_2, \ldots, v_p\). For \(1 \leq i \leq p\), let \(u_{i1}, u_{i2}, \ldots, u_{ik_i}\) be the terminal vertices of \(v_i\), and let \(P_{ij}\) be the \(v_i - u_{ij}\) path \((1 \leq j \leq k_i)\). Suppose that \(W\) is a set of vertices of \(T\). Then \(W\) is a \(\text{cr}\)-set of \(T\) if and only if
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(a) $W$ contains exactly one vertex from each of the paths $P_{ij} - v_i$, $1 \leq j \leq k_i$ and $1 \leq i \leq p$, with exactly one exception for each $i$ with $1 \leq i \leq p$.

(b) For each pair $i, j$ with $1 \leq j \leq k_i$ and $1 \leq i \leq p$, if $x_{ij} \in W$, then $x_{ij}$ is adjacent to $v_i$ in the path $P_{ij}$.

(c) $W$ contains all vertices in the paths between any two vertices described in (b).

Corollary 2.18 Let $T$ be a tree of order $n \geq 4$ that is not a path having $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of $v_i$, where $d(v_i, u_{ij}) = \ell_{ij}$ ($1 \leq j \leq k_i$). Then

$$cr(T) = n + \dim(T) - \sum_{i,j} \ell_{ij}.$$  

2.4 Graphs With Prescribed Connected Resolving Number

We have seen that if $G$ is a connected graph of order $n$ with $cr(G) = k$, then $1 \leq k \leq n - 1$. In fact, every pair $k, n$ of integers with $1 \leq k \leq n - 1$ is realizable as the connected resolving number and order of some graph, as we show next.

Theorem 2.19 For each pair $k, n$ with $1 \leq k \leq n - 1$, there is a connected graph $G$ of order $n$ with connected resolving number $k$.

Proof. For $k = 1$, let $G = P_n$; for $k = 2$, let $G = C_n$. By Corollary 2.13 and Proposition 2.14, we have the desired result. For $k \geq 3$ let $G$ be that graph obtained from the path $P_{n-k+1} : u_1, u_2, \ldots, u_{n-k+1}$ by adding the $k-1$ new vertices $v_i$ ($1 \leq i \leq k-1$) and joining each $v_i$ to $u_1$. Then the order of $G$ is $n$ and $cr(G) = k$ by Corollary 2.18.

If $G$ is connected graph with $\dim(G) = a$ and $cr(G) = b$, then $a \leq b$. Next we show that every pair $a, b$ of integers with $2 \leq a \leq b$ is realizable as the dimension and connected resolving number of some connected graph.
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Theorem 2.20  For every pairs \( a, b \) of integers with \( 2 \leq a \leq b \), there exists a connected graph \( G \) such that \( \dim(G) = a \) and \( \text{cr}(G) = b \).

Proof. For \( b = a \geq 2 \), let \( G = K_{a+1} \); while for \( b = a + 1 \), let \( G = K_{1,a+1} \). Then the graph \( G \) has the desired properties by Theorems A, B, and 2.15. So we may assume that \( b \geq a + 2 \). Let \( G \) be obtained from the path \( P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2} \) of order \( b - a + 2 \) by adding the \( a \) new vertices \( v_1, v_2, \ldots, v_a \) and the \( a \) edges \( v_i u_2 \) \( (1 \leq i \leq a - 1) \) and \( v_a u_{b-a+1} \). It then follows from Theorem B and Corollary 2.18 that \( \dim(G) = a \) and \( \text{cr}(G) = b \), as desired. \( \blacksquare \)

By Theorem A, the path \( P_n \) of order \( n \geq 2 \) is the only nontrivial connected graph with dimension 1. Since \( \text{cr}(P_n) = 1 \), there is no connected graph \( G \) with \( \dim(G) = 1 \) and \( \text{cr}(G) = k \) for some integer \( k \neq 1 \). Thus the following is a consequence of Theorem 2.20.

Corollary 2.21  For positive integers \( a, b \) with \( a \leq b \), there exists a connected graph \( G \) with \( \dim(G) = a \) and \( \text{cr}(G) = b \) if and only if \( (a, b) \notin \{(1, k) : k \geq 2\} \).

Next, we determine those triples \( a, b, n \) of positive integers that can be realized as the dimension, connected resolving number, and order, respectively, of some connected graph.

Theorem 2.22  Let \( a, b, n \) be integers with \( n \geq 5 \). Then there exists a connected graph \( G \) of order \( n \) such that \( \dim(G) = a \) and \( \text{cr}(G) = b \) if and only if \( a, b, n \) satisfy one of the following:

\[(a) \quad a = b = 1, \]
\[(b) \quad b = n - 1 \text{ and } a \in \{n - 2, n - 1\}, \]
\[(c) \quad 2 \leq a \leq b \leq n - 2. \]

Proof. If \( G = P_n \), then \( \dim(G) = \text{cr}(G) = 1 \) by Theorem A and Corollary 2.13. By Theorem 2.15, if \( G \) is a connected graph of order \( n \geq 3 \), then \( \text{cr}(G) = n - 1 \).
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if and only if $G = K_n$ or $G = K_{1,n-1}$. Since $\dim(K_n) = n - 1$ and $\dim(K_{1,n-1}) = n - 2$, there is no connected graph $G$ of order $n \geq 5$ such that $\cr(G) = n - 1$ and $1 \leq \dim(G) \leq n - 3$. Therefore, if $G$ is a connected graph of order $n$ with $\dim(G) = a$ and $\cr(G) = b$, then $a, b$, and $n$ must satisfy one of (a), (b), and (c).

It remains to verify the converse. If $a = b = 1$, then let $G = P_n$ and the result is true. Next, assume that $b = n - 1$. For $a = n - 1$, let $G = K_n$; while for $a = n - 2$, let $G = K_{1,n-1}$. Since $\dim(K_n) = \cr(K_n) = n - 1$, $\dim(K_{1,n-1}) = n - 2$, and $\cr(K_{1,n-1}) = n - 1$, the result holds for $b = n - 1$. Now assume that $2 \leq a \leq b \leq n - 2$. We consider two cases.

Case 1. $a = b$. Let $G$ be the graph obtained from the complete graph $K_{a+1}$, where $V(K_{a+1}) = \{u_1, u_2, \ldots, u_{a+1}\}$, and the path $P_{n-a-1} : v_1, v_2, \ldots, v_{n-a-1}$ by joining $u_{a+1}$ with $v_1$. Then the order of $G$ is $n$. By Observation 2.2, every resolving set contains at least $a$ vertices from $V(K_{a+1})$ and so $\cr(G) \geq \dim(G) \geq a$. On the other hand, let $W = \{u_1, u_2, \ldots, u_a\}$. Since $c_W(u_{a+1}) = (1, 1, \ldots, 1)$, $c_W(v_i) = (i+1, i+1, \ldots, i+1)$ for all $i$ with $1 \leq i \leq n-a-1$, and $\langle W \rangle = K_a$, it follows that $W$ is a connected resolving set. Thus $\dim(G) \leq |W| = a$ and $\cr(G) \leq |W| = a$. Therefore, $\dim(G) = \cr(G) = a$.

Case 2. $a < b$. If $b = a + 1$, let $G$ be the graph obtained from the path $P_{n-a} : u_1, u_2, \ldots, u_{n-a}$ by adding the $a$ new vertices $v_1, v_2, \ldots, v_a$ and joining each $v_i$ ($1 \leq i \leq a$) to the end-vertex $u_1$ in $P_{n-a}$. Then the order of $G$ is $n$ and by Theorem B and Corollary 2.18, $\dim(G) = a$ and $\cr(G) = a + 1 = b$. If $b \geq a + 2$, let $G$ be the graph obtained from the path $P_{n-a} : u_1, u_2, \ldots, u_{n-a}$ by adding the $a$ new vertices $v_1, v_2, \ldots, v_a$ and joining each $v_i$ ($1 \leq i \leq a - 1$) to $u_2$ and joining $v_a$ to $u_{b-a+1}$. Then the order of $G$ is $n$, $\dim(G) = a$, and $\cr(G) = b$ by Theorem B and Corollary 2.18.

2.5 Graphs with Unique $\cr$-Sets

In general, the graphs we've seen have had several $\cr$-sets. In this section we show that for every integer $k \geq 2$, there exists a graph with a unique $\cr$-set of cardinality $k$. For positive integers $d$ and $n$ with $d < n$, define $f(n, d)$ as the least positive integer $k$ such that $k + d^k \geq n$. It was shown [3] that if $G$ is a
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connected graph of order \( n \geq 2 \) and diameter \( d \), then \( \text{dim}(G) \geq f(n, d) \). Since \( \text{cr}(G) \geq \text{dim}(G) \) for every graph \( G \), we have the following.

**Corollary 2.23** For a connected graph \( G \) of order \( n \geq 2 \) and diameter \( d \),

\[
\text{cr}(G) \geq \text{dim}(G) \geq f(n, d).
\]

We now show that for each integer \( k \geq 2 \), there exists a graph \( G \) containing a unique \( \text{cr} \)-set of cardinality \( k \). The graph in the following proof is a modification of the one constructed in [11].

**Theorem 2.24** For \( k \geq 2 \), there exists a graph with a unique \( \text{cr} \)-set of cardinality \( k \).

**Proof.** Let \( G_1 = K_{2^k} \) with vertex set \( U = \{u_0, u_1, \ldots, u_{2^k-1}\} \) and let \( G_2 = K_k \) with vertex set \( W = \{w_{k-1}, w_{k-2}, \ldots, w_0\} \). Then the graph \( G \) is obtained from \( G_1 \) and \( G_2 \) by adding edges between \( U \) and \( W \) as follows. Let each integer \( j \) (\( 0 \leq j \leq 2^k - 1 \)) be expressed in its base 2 (binary) representation. Thus, each such \( j \) can be expressed as a sequence of \( k \) coordinates, that is, a \( k \)-vector, where the rightmost coordinate represents the value (either 0 or 1) in the \( 2^0 \) position, the coordinate to its immediate left is the value in the \( 2^1 \) position, etc. For integers \( i \) and \( j \), with \( 0 \leq i \leq k - 1 \) and \( 0 \leq j \leq 2^k - 1 \), we join \( w_i \) and \( u_j \) if and only if the value in the \( 2^i \) position in the binary representation of \( j \) is 1. For \( k = 3 \), the edges joining \( W \) and \( U \) in the graph \( G \) just constructed are shown in Figure 2.5.

We first show that \( W \) is a \( \text{cr} \)-set of \( G \). Since \( G \) has diameter 2 and order \( k + 2^k \), it follows by Corollary 2.23 that \( \text{cr}(G) \geq k \). Also, since \( G \) has diameter 2, the distance between every two distinct vertices of \( G \) is 1 or 2. We claim that \( W \) is a connected resolving set for \( G \). Since \( \langle W \rangle \) is complete, it suffices to show that \( W \) is a resolving set. To do this we need only show that the vertices of \( U \) have distinct codes with respect to \( W \). The code for each \( u_j \) (\( 0 \leq j \leq 2^k - 1 \)) can be expressed as

\[
c_W(u_j) = (2 - a_{k-1}, 2 - a_{k-2}, \ldots, 2 - a_0)
\]
where $a_m$ ($0 \leq m \leq k-1$) is the value in the $2^m$ position of the binary representation of $j$. Since the binary codes $a_{k-1}a_{k-2}\cdots a_1a_0$ are distinct for the vertices of $U$, their codes $(2-a_{k-1}, 2-a_{k-2}, \ldots, 2-a_0)$ are distinct as well. Hence $W$ is a resolving set of $G$ and $\text{cr}(G) \leq k$. Thus $\text{cr}(G) = k$. Since $W$ is a connected resolving set and $|W| = k$, we conclude that $W$ is a $cr$-set for $G$, as claimed.

It remains only to show that $G$ has no other $cr$-set. First, we make an observation. If $U'$ is a subset of $U$, then $|U'| = k$ and so $|U - U'| = 2^k - k \geq 2$. Since the distance between every two distinct vertices of $U$ is 1, there are at least two vertices having the same codes with respect to $U'$ and so $U'$ is not a resolving set. Hence every $cr$-set of $G$ contains at least one vertex of $W$. In what follows, it is now useful to reorder the set $W$ as $W = \{w_0, w_1, \ldots, w_{k-1}\}$, namely $w_0$, being in position 0, $w_1$ in position 1, etc. However, in any representation, we continue to refer to $2^0$ positions, $2^1$ positions, etc., listed from the right. Suppose that $S = W' \cup U'$, where $W' \subseteq W$, $U' \subseteq U$, $|W'| = k - j$, and $|U'| = j$, where $1 \leq j \leq k - 1$. We now order the set $S$ by placing each vertex of $W'$ in the same position it occupied in $W$ and where the elements of $U'$ are ordered arbitrarily to occupy the vacant positions of $S$. Let $w \in W - W'$, where $w$ occupied position $i$ in $W$. If $u \in U - U'$, then the code of $u$ with respect to $S$ has 1 in the $2^i$ position. In fact, since $|W - W'| = |U'| = j$, every vertex $u \in U - U'$ has 1 in each of $j$
specific coordinates in its code with respect to $S$. So there are $2^{k-j}$ distinct codes of the vertices of $U - U'$, and there exactly $2^j$ vertices of each code. If $j = 1$, then there are two vertices of $U - U'$ with the same code with respect to $S$. If $j \geq 2$, then $2^j - j \geq 2$ and for each of $2^{k-j}$ distinct codes of the vertices of $U$ with respect to $W'$, there are at least two vertices of $U - U'$ with the same code with respect to $S$. This is a contradiction.

This result can be extended to the following.

**Theorem 2.25** For every pair $r, k$ of integers with $k \geq 2$ and $0 \leq r \leq k$, there exists a connected graph $G$ such that $\text{cr}(G) = k$ and exactly $r$ vertices of $G$ belong to every $\text{cr}$-set of $G$.

**Proof.** For $r = 0$, let $G = K_{k+1}$. Since every $k$ vertices of $G$ form a $\text{cr}$-set, no vertex of $G$ belong to every $\text{cr}$-set. Thus $\text{cr}(G) = k$ and $r = 0$. For $r = 1$, let $G = K_{1,k}$. Since every $\text{cr}$-set consists of the central vertex $v$ of $G$ and any $k-1$ end-vertices of $G$, it follows that $v$ belongs to every $\text{cr}$-set of $G$ and no other vertex belongs to every $\text{cr}$-set of $G$. Hence $\text{cr}(G) = k$ and $r = 1$.

For $r = k$, the graph $G$ constructed in the proof of Theorem 2.24 has a unique $\text{cr}$-set $W$ containing $k$ vertices. Thus the $k$-vertices in $W$ are the only vertices of $G$ belonging to every $\text{cr}$-set of $G$. Thus $G$ has the desired properties. For $r = k - 1 \geq 2$, take the construction of the graph in the proof of Theorem 2.24 for $|W| = k - 1$ and take two copies of $u_{2^k-1}$, say $x$ and $y$, each of which has the same neighborhood as $u_{2^k-1}$. Then the resulting graph $G$ has connected resolving number $k$. Moreover, $G$ has exactly two $\text{cr}$-sets $W_1 = W \cup \{x\}$ and $W_2 = W \cup \{y\}$. Thus the $k - 1$ vertices in $W$ are the only vertices of $G$ belonging to every $\text{cr}$-set of $G$. Thus $r = k - 1 = \text{cr}(G) - 1$.

For $2 \leq r \leq k - 2$, let $G$ be the graph obtained from the path $P_r : u_1, u_2, \ldots, u_r$ by adding the $k - r + 2$ new vertices $v_1, v_2, \ldots, v_{k-r+2}$ and joining (1) each of $v_1$ and $v_2$ to $u_1$ and (2) each of $v_i$, where $3 \leq i \leq k - r + 2$, to $u_r$. Then $\text{cr}(G) = k$. By observation 2.2, every $\text{cr}$-set of $G$ contains at least one vertex from $\{v_1, v_2\}$ and at least $k - r - 1 \geq 1$ vertices from $\{v_3, v_4, \ldots, v_{k-r+2}\}$. In order to form a connected resolving set, the $r$ vertices of $P_r$ must belong to every $\text{cr}$-set of $G$.  

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Moreover, if \( v \in V(G) - V(P_r) \), then \( v \) is an end-vertex of \( G \) and there is a cr-set of \( G \) does not contain \( v \) and so \( v \) does not belong to every cr-set of \( G \). Therefore, exactly \( r \) vertices of \( G \) belong to every cr-set of \( G \).

2.6 Minimal Connected Resolving Sets in a Graph

A resolving set \( W \) of a nontrivial connected \( G \) is a minimal resolving set if no proper subset of \( W \) is a resolving set. The maximum cardinality of a minimal resolving set is the upper dimension \( \text{dim}^+(G) \). A minimal resolving set of cardinality \( \text{dim}^+(G) \) is an upper basis for \( G \). Since every minimum resolving set is a minimal resolving set for \( G \), it follows that \( \text{dim}(G) \leq \text{dim}^+(G) \). These concepts were introduced and studied in [11]. Similarly, a connected resolving set \( W \) of \( G \) is a minimal connected resolving set if no proper subset of \( W \) is a connected resolving set. The maximum cardinality of a minimal connected resolving set is the upper connected resolving number \( \text{cr}^+(G) \).

To illustrate these concepts, consider the graph \( G = P_3 \times P_4 \) of Figure 2.6. Since \( W = \{u_1, v_1, w_1\} \) is a cr-set for \( G \), it follows that \( \text{cr}(G) = 3 \). Now let \( W' = \{v_1, v_2, v_3, w_3, w_4\} \). The codes for the vertices of \( V(G) - W' \) with respect to \( W' \) are

\[
\begin{align*}
  c_{W'}(u_1) &= (1, 2, 3, 4, 5) \\
  c_{W'}(u_2) &= (2, 1, 2, 3, 4) \\
  c_{W'}(u_3) &= (3, 2, 1, 2, 3) \\
  c_{W'}(u_4) &= (4, 3, 2, 3, 2) \\
  c_{W'}(u_5) &= (3, 2, 1, 2, 1) \\
  c_{W'}(w_1) &= (1, 2, 3, 2, 3) \\
  c_{W'}(w_2) &= (2, 1, 2, 1, 2)
\end{align*}
\]

Thus \( W' \) is a connected resolving set as well. Let \( W_1 = W' - \{v_1\} \), \( W_2 = W' - \{v_2\} \), \( W_3 = W' - \{v_3\} \), \( W_4 = W' - \{w_3\} \), and \( W_5 = W' - \{w_4\} \). Since \( c_{W_1}(u_2) = c_{W}(v_1) \), \( c_{W_2}(u_3) = c_{W_3}(v_4) \), and \( \langle W_i \rangle \) is not connected in \( G \) for \( i = 2, 3, 4 \). No set \( W_i \) (\( 1 \leq i \leq 5 \)) is a connected resolving set. Hence \( W' \) is a minimal connected resolving set and so \( \text{cr}^+(G) \geq 5 \). By a case-by-case analysis, it can be shown that there is no minimal connected resolving set of cardinality 6 or more. Hence \( W' \) is an upper cr-set and \( \text{cr}^+(G) = 5 \).
The preceding example suggests a question. Once it has been shown that $W'$ is a minimal connected resolving set (of cardinality 5) for the graph $G$ of Figure 2.6 and there is no minimal connected resolving set of cardinality 6, doesn’t this show that $cr^+(G) = 5$? The answer is no. In fact, it is possible that a graph contains no minimal connected resolving set of cardinality $k$ for some positive integer $k$ but has a minimal connected resolving set of cardinality greater than $k$, as we show next.

**Theorem 2.26** For each integer $k \geq 7$, there exists a connected graph $G$ such that $G$ contains no minimal connected resolving set of cardinality $k$, but $G$ contains a minimal connected resolving set of cardinality $k + 1$.

**Proof.** Let $G$ be the graph shown in Figure 2.7. Let $U = \{u_1, u_2, u_3, u_4\}$, $U' = \{u'_1, u'_2, u'_3, u'_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, $V' = \{v'_1, v'_2, v'_3, v'_4\}$, and $Y = \{y_1, y_2, \ldots, y_{k-5}\}$. By an argument similar to the one used in the proof of Theorem 2.8, we can show that $cr(G) = k - 1$ and every minimum connected resolving set of $G$ is one of the sets $S_0, S_1, S_2$ defined as

$$
S_0 = (Y - \{y\}) \cup \{u_1, u'_1, v_1, v'_1, x\}, \text{ where } y \in Y,
$$
$$
S_1 = Y \cup \{u_1, u'_1, v, x\}, \text{ where } v \in \{v_1, v'_1\},
$$
$$
S_2 = Y \cup \{u, v_1, v'_1, x\}, \text{ where } u \in \{u_1, u'_1\}.
$$

Moreover,

(1) every connected resolving set of $G$ contains $x$,
(2) every connected resolving set of $G$ contains at least one vertex from each of the sets \{u_1, u'_1\} and \{v_1, v'_1\},

(3) every resolving set of $G$ contains at least $k - 6$ vertices from $Y$ by Observation 2.2.

Claim 1 The graph $G$ contains no minimal connected resolving set of cardinality $k$.

Proof of Claim 1. Assume, to the contrary, that $S$ is a minimal connected resolving set of $G$ with $k$ elements. By (1)–(3), we may assume that

$$T = (Y - \{y_1\}) \cup \{u_1, v_1, x\} \subseteq S.$$  

Let $S' = S - T$. Then $|S'| = 3$. Observe that

$$c_T(u'_1) = c_T(v'_1) = c_T(y_1) = (2, 2, \ldots, 2, 1),$$
$$c_T(u'_2) = c_T(v'_2) = (3, 3, \ldots, 3, 2),$$
$$c_T(u'_3) = c_T(v'_3) = (4, 4, \ldots, 4, 3).$$

Assume first that $S' \subseteq (U \cup U')$ or $S' \subseteq (V \cup V')$, say the former. Since (i) $d(v'_1, u) = d(y_1, u)$ for all $u \in U \cup U'$, (ii) $d(v'_1, s) = d(y_1, s)$ for all $s \in S$, and (iii) $c_T(u'_1) = c_T(y_1)$, it follows that $c_S(v'_1) = c_S(y_1)$, which contradicts the fact that $S$ is a resolving set. Thus $S'$ contains at most two vertices from each of $U \cup U'$ and
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$V \cup V'$. We consider three cases, according to whether $S'$ contains exactly two vertices from $U \cup U'$, or exactly one vertex from $U \cup U'$, or no vertex from $U \cup U'$.

Case 1. $S'$ contains exactly two vertices from $U \cup U'$. Since $S$ is connected and $u_1 \in S$, there are three subcases.

Subcase 1.1. $u_2, u_3 \in S'$. Then $S' = \{u_2, u_3, z\}$, where $z \in \{y_1\} \cup V \cup V'$. Since $S$ is connected and $v_1 \in S$, it follows that $z \in \{y_1, v_2, v_1\}$. If $z = y_1$, then $c_{S'}(u'_1) = c_{S'}(v'_1) = (3, 4, 2)$. Since $c_T(u'_1) = c_T(v'_1)$, it follows that $c_S(u'_1) = c_S(v'_1)$, a contradiction. If $z = v_2$, then $c_{S'}(u'_1) = c_{S'}(v'_1) = (3, 4, 3)$. Since $c_T(u'_1) = c_T(v'_1)$, it follows that $c_S(u'_1) = c_S(v'_1)$, a contradiction. If $z = v'_1$, then $c_{S'}(u'_1) = c_{S'}(v_1) = (3, 4, 2)$. Since $c_T(u'_1) = c_T(v_1)$, it follows that $c_S(u'_1) = c_S(y_1)$, a contradiction.

Subcase 1.2. $u_2, u'_1 \in S'$. Then $S' = \{u_2, u'_1, z\}$, where $z \in \{y_1, v_2, v'_1\}$. If $z = y_1$, then the connected resolving set $S_1$ is a proper subset of $S$, which is a contradiction. If $z = v_2$, then $c_{S'}(v'_1) = c_{S'}(y_1) = (3, 2, 3)$. Since $c_T(v'_1) = c_T(y_1)$, it follows that $c_S(v'_1) = c_S(y_1)$, a contradiction. If $z = v'_1$, then the connected resolving set $S_0$ is a proper subset of $S$, which is a contradiction.

Subcase 1.3. $u'_1, u'_2 \in S'$. Then $S' = \{u'_1, u'_2, z\}$, where $z \in \{y_1, v_2, v'_1\}$. If $z = y_1$, then the connected resolving set $S_1$ is a proper subset of $S$, which is a contradiction. If $z = v_2$, then $c_{S'}(v'_1) = c_{S'}(y_1) = (2, 3, 3)$. Since $c_T(v'_1) = c_T(y_1)$, it follows that $c_S(v'_1) = c_S(v_1)$, a contradiction. If $z = v'_1$, then the connected resolving set $S_0$ is a proper subset of $S$, which is a contradiction.

Case 2. $S'$ contains exactly one vertex $u$ from $U \cup U'$. Since $S$ is connected and $u_1 \in S$, it follows that $u = u_2$ or $u = u'_1$. We consider these two subcases.

Subcase 2.1. $u = u_2$. There are five subcases.

Subcase 2.1.1. $S' = \{u_2, y_1, v_2\}$. Then $c_S(u'_1) = c_S(v'_1)$, a contradiction.

Subcase 2.1.2. $S' = \{u_2, y_1, v'_1\}$. Then the connected resolving set $S_2$ is a proper subset of $S$, which is a contradiction.

Subcase 2.1.3. $S' = \{u_2, v_2, v_3\}$. Then $c_S(u'_1) = c_S(v_1)$, a contradiction.

Subcase 2.1.4. $S' = \{u_2, v_2, v'_1\}$. Then $c_S(u'_1) = c_S(y_1)$, a contradiction.

Subcase 2.1.5. $S' = \{u_2, u'_1, v_2\}$. Then $c_S(u'_1) = c_S(y_1)$, a contradiction.
Subcase 2.2. $u = u'_i$. There are five subcases.

Subcase 2.2.1. $S' = \{u'_1, y_1, v_2\}$. Then the connected resolving set $S_1$ is a proper subset of $S$, which is a contradiction.

Subcase 2.2.2. $S' = \{u'_1, y_1, v'_1\}$. Then the connected resolving set $S_0$ is a proper subset of $S$, which is a contradiction.

Subcase 2.2.3. $S' = \{u'_1, v_2, v_3\}$. Then $c_S(v'_1) = c_S(y_1)$, a contradiction.

Subcase 2.2.4. $S' = \{u'_1, v_2, v'_1\}$. Then the connected resolving set $S_0$ is a proper subset of $S$, which is a contradiction.

Subcase 2.2.5. $S' = \{u'_1, v'_1, v'_2\}$. Then the connected resolving set $S_0$ is a proper subset of $S$, which is a contradiction.

Case 3. $S'$ contains no vertex from $U \cup U'$. Thus $S'$ contains $y_1$ and exactly two vertices from $V \cup V'$. An argument similar to one used in Case 1 will show that Case 3 is impossible as well.

This completes the proof of Claim 1.

Claim 2 The graph $G$ contains a minimal connected resolving set of cardinality $k + 1$.

Proof of Claim 2. Let $S^* = \{u_1, x\} \cup V \cup Y$. Then $|S^*| = k + 1$. Since $S^*$ is a connected resolving set of $G$ with $k + 1$ elements, it remains to show that $S^*$ is minimal; that is, no proper subset of $S^*$ is a connected resolving set of $G$. It suffices to show that for every $w \in S^*$, the set $S^* - \{w\}$ not a connected resolving set of $G$. Let $T^* = S^* - \{w\}$. If $w \in \{x, v_1, v_2, v_3\}$, then $\langle T^* \rangle$ is not connected. If $w = u_1$, then $T^*$ contains no vertex of $\{u_1, u'_1\}$ and so $T^*$ is not a connected resolving set of $G$ by (2). Thus we may assume that $w = v_4$, or $w = y_i$ for some $i$ with $1 \leq i \leq k - 5$. If $w = v_4$, then $c_{T^*}(u'_1) = c_{T^*}(v'_1)$; while if $w = y_i$, say $w = y_1$, then $c_{T^*}(u'_1) = c_{T^*}(y_1)$. In either case, $T^*$ is not a resolving set of $G$, which is a contradiction. Therefore, $S^*$ is a minimal connected resolving set of cardinality $k + 1$. This completes the proof of Claim 2.

Therefore, the graph $G$ contains no minimal connected resolving set of cardinality $k$, but $G$ contains a minimal connected resolving set of cardinality $k + 1$. 

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as desired. ■

Certainly, if $G$ is a nontrivial connected graph of order $n$, then

$$1 \leq cr(G) \leq cr^+(G) \leq n - 1.$$  

The graph $G$ described in Theorem 2.26 shows that there is no "Intermediate Value Theorem" for minimal connected resolving sets, that is, if $k$ is an integer such that $cr(G) < k < cr^+(G)$, then there need not exist a minimal connected resolving set of cardinality $k$ in $G$.

Next we show that $cr^+(G) = cr(G)$ for some well-known graphs.

**Theorem 2.27** Let $G$ be a nontrivial connected graph. If $G$ is a complete graph, a cycle, a complete $k$-partite graph ($k \geq 2$), or a tree, then $cr^+(G) = cr(G)$.

**Proof.** We will only verify that $cr^+(T) = cr(T)$ for all nontrivial trees $T$ since the proofs for others are routine. If $T$ is a path, then $cr^+(T) = cr(T) = 1$. Thus we may assume that $T$ is not a path. Assume, to the contrary, that there is a nontrivial tree $T$ that is not a path such that $cr^+(T) > cr(T)$. Let $T$ have order $n \geq 4$ and $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of $v_i$, and let $P_{ij}$ be the $v_i - u_{ij}$ path ($1 \leq j \leq k_i$). Let $S$ be an upper cr-set with $|S| = cr^+(T)$. Since $S$ is a resolving set, it follows by Theorem C that $S$ contains at least one vertex from each of the paths $P_{ij} - v_i$ ($1 \leq j \leq k_i$ and $1 \leq i \leq p$) with at most one exception for each $i$ with $1 \leq i \leq p$. Let $S'$ be a subset $S$ such that $S'$ consists of all those vertices required by Theorem C. Since $\langle S \rangle$ is connected, $S$ must contain all vertices of $T$ belonging to each $x-y$ path for all $x, y \in S'$. On the other hand, by Theorem 2.17, for a set $W$ of vertices of $T$, $W$ is a cr-set of $T$ if and only if (a) $W$ contains exactly one vertex from each of the paths $P_{ij} - v_i$, $1 \leq j \leq k_i$ and $1 \leq i \leq p$, with exactly one exception for each $i$ with $1 \leq i \leq p$, (b) for each pair $i, j$ with $1 \leq j \leq k_i$ and $1 \leq i \leq p$, if $x_{ij} \in W$, then $x_{ij}$ is adjacent to $v_i$ in the path $P_{ij}$, and (c) $W$ contains all vertices in the paths between any two vertices described in (b). This implies that if $|S| > cr(T)$, then $S$ contains a cr-set as a proper subset, which is a contradiction. ■
Thus, we have the following.

(a) If $G = K_n$ for $n \geq 3$ or $G = K_{1,n-1}$ for $n \geq 4$, then $\alpha^+(G) = \alpha(G) = n - 1$.

(b) If $G = P_n$ for $n \geq 2$, then $\alpha^+(G) = \alpha(G) = 2$.

(c) If $G = C_n$ for $n \geq 4$, then $\alpha^+(G) = \alpha(G) = 2$.

(d) For $k \geq 2$, let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete $k$-partite graph that is not a star. Let $n = n_1 + n_2 + \cdots + n_k$ and $\ell$ be the number of 1's in $\{n_i : 1 \leq i \leq k\}$. Then

$$\alpha^+(G) = \alpha(G) = \begin{cases} n - k & \text{if } \ell = 0 \\ n - k + \ell - 1 & \text{if } \ell \geq 1. \end{cases}$$

(e) Let $T$ be a tree that is not a path, having order $n \geq 4$ and $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of $v_i$ and let $\ell_{ij} = d(v_i, u_{ij})$ ($1 \leq j \leq k_i$). Then

$$\alpha^+(T) = \alpha(T) = n + \dim(T) - \sum_{i,j} \ell_{ij}.$$ 

We have seen that in Theorem 2.15 that $K_n$ and $K_{1,n-1}$ are the only connected graphs of order $n \geq 4$ with connected resolving number $n - 1$. In fact, this is also true for the upper connected numbers of graphs, as we show next.

**Theorem 2.28** Let $G$ be a connected graph of order $n \geq 4$. Then $\alpha^+(G) = n - 1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.

**Proof.** We have seen that $\alpha^+(G) = n - 1$ if $G = K_n$ or $G = K_{1,n-1}$. It remains to verify the converse. We show that if $G$ is a connected graph of order $n \geq 4$ that is neither a complete graph nor a star, then $\alpha^+(G) \leq n - 2$. To do this, it suffices to show the following stronger statement: If $G$ is a connected graph of order $n \geq 4$ that is neither a complete graph nor a star, then, for each $u \in V(G)$ for which
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$V(G) - u$ is a connected resolving set, there exist two distinct vertices $v$ and $w$ in $G - u$ such that (1) $V(G) - \{u, v\}$ is a connected resolving set for $G$ and (2) $w$ is adjacent to exactly one of $u$ and $v$ in $G$. We proceed by induction on the order $n$ of $G$. For $n = 4$, the graphs $G_i$ $(1 \leq i \leq 4)$ of Figure 2.8 are the only connected graphs order 4 that are different from $K_4$ or $K_{1,3}$. For each $i$ $(1 \leq i \leq 4)$, the vertices $u, v, w$ are shown in Figure 2.8 and $W = V(G_i) - \{u, v\}$ is a connected resolving set in $G_i$. Note that for $G_3$ and $G_4$ there are two possible choices (see Figure 2.8) for $u$ such that $V(G_3) - \{u\}$ and $V(G_4) - \{u\}$ are connected resolving sets in $G_3$ and $G_4$, respectively. Moreover, $w$ is adjacent to exactly one of $u$ and $v$. Thus the statement is true for $n = 4$. Assume that the statement is true for $n - 1 \geq 4$.

Let $G$ be a connected graph of order $n \geq 5$ that is not $K_n$ or $K_{1,n-1}$ and let $x$ be vertex of $G$ such that $V(G) - x$ is a connected resolving set of $G$. Let $G' = G - x$. We consider three cases.

Case 1. $G' = K_{n-1}$. Since $G \neq K_n$, there are distinct vertices $v, w, y$ in $G'$ such that $x$ is adjacent to $y$ but not to $w$. Then $\langle V(G) - \{v, x\} \rangle = K_{n-2}$, $d_G(x, w) = 2$, and $d_G(v, w) = 1$. This implies that $V(G) - \{v, x\}$ is a connected resolving set of $G$. Moreover, $w$ is adjacent to exactly one of $v$ and $x$ in $G$.

Case 2. $G' = K_{1,n-2}$. Since $G \neq K_{1,n-1}$, there exist two end-vertices $v$ and $w$ in $G'$. Then $\langle V(G) - \{v, x\} \rangle = K_{n-2}$, $d_G(x, w) = 2$, and $d_G(v, w) = 1$. This implies that $V(G) - \{v, x\}$ is a connected resolving set of $G$. Moreover, $w$ is adjacent to exactly one of $v$ and $x$ in $G$.

Figure 2.8: Graphs $G_i$ $(1 \leq i \leq 4)$, $G'_3$ and $G'_4$
w of $G'$ such that $x$ is adjacent to at least one of $v$ and $w$, say $x$ is adjacent to $w$. Then $\langle V(G) - \{v, x\} \rangle = K_{1,n-2}$, $d_G(x, w) = 1$, and $d_G(v, w) = 2$. This implies that $V(G) - \{v, x\}$ is a connected resolving set of $G$. Moreover, $w$ is adjacent to exactly one of $v$ and $x$ in $G$.

Case 3. $G' \neq K_{n-1}$ and $G' \neq K_{1,n-2}$. Let $u \in V(G')$ such that $G' - u$ is connected. Therefore, $V(G') - \{u\}$ is a connected resolving set for $G'$. By the induction hypothesis, there exist two distinct vertices $v$ and $w$ in $G' - u$ such that (1) $V(G') - \{u, v\}$ is a connected resolving set for $G'$ and (2) $w$ is adjacent to exactly one of $u$ and $v$ in $G'$. There are two subcases.

Subcase 3.1. $w$ is adjacent to $u$. Then $w$ is not adjacent to $v$. If $x$ is adjacent to $w$ in $G$, then $d_G(x, w) = 1$ and $d_G(v, w) \geq 2$, implying that $V(G) - \{v, x\}$ is a resolving set of $G$. Since $V(G') - \{u, v\}$ is a connected resolving set for $G'$, it follows that $\langle V(G') - \{u, v\} \rangle$ is connected. Moreover, $u$ is adjacent to $w$. Hence $\langle V(G) - \{v, x\} \rangle$ is connected. Therefore, $V(G) - \{v, x\}$ is a connected resolving set of $G$. Also, $w$ is adjacent to exactly one of $v$ and $x$. If $x$ is not adjacent to $w$ in $G$, then $d_G(x, w) \geq 2$ and $d_G(u, w) = 1$, $V(G) - \{u, x\}$ is a resolving set of $G$. Since $G - u - x = G' - u$ is connected, it follows that $V(G) - \{u, x\}$ is a connected resolving set of $G$. Moreover, $w$ is adjacent to exactly one of $u$ and $x$.

Subcase 3.2. $w$ is adjacent to $v$. Then $w$ is not adjacent to $u$. If $x$ is adjacent to $w$ in $G$, then $w$ is adjacent to exactly one of $x$ and $u$. Moreover, an argument similar to one used in Subcase 3.1 shows that $V(G) - \{u, x\}$ is a connected resolving set of $G$. So we may assume that $x$ is not adjacent to $w$ in $G$. If there is $z \in V(G) - \{u, v, w, x\}$ such that $uz \in E(G)$, then $V(G) - \{v, x\}$ is a connected resolving set of $G$ and $w$ is adjacent to exactly one of $v$ and $x$. Thus we may assume that there is no vertex in $V(G) - \{u, v, w, x\}$ that is adjacent to $w$ in $G$. If there is $w' \in V(G) - \{u, v, x\}$ such that $xw' \in E(G)$, then $V(G) - \{u, x\}$ is a connected resolving set of $G$. Moreover, $w'$ is adjacent to exactly one of $u$ and $x$. Hence we may assume that there is no edge between $\{u, x\}$ and $V(G) - \{u, v, x\}$. If $|V(G) - \{u, v, x\}| = 2$, or $|V(G) - \{u, v, x\}| = 3$, then there exists $w'' \in V(G) - \{u, v, x\}$ such that
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$w''$ is adjacent to $w$. Since $d_G(w'', w) = 1$ and $d_G(x, w) \geq 2$, it follows that $V(G) - \{w'', x\}$ is a connected resolving set of $G$. Moreover, $w$ is adjacent to exactly one of $w''$ and $x$ in $G$. Thus, we may assume that $|V(G) - \{u, v, x\}| \geq 4$. By the induction hypothesis, $\langle V(G') - \{u, v\}\rangle = \langle V(G) - \{u, v, x\}\rangle$ is connected in $G'$. Then there is $u', v' \in V(G) - \{u, v, x\}$ such that $\langle V(G) - \{u, v, x, u'\}\rangle$ is connected and $u'$ is adjacent to $v'$. Thus $\langle V(G) - \{x, u'\}\rangle$ is connected. Since $d_G(x, v') \geq 2$ and $d_G(u', v') = 1$, it follows that $V(G) - \{x, u'\}$ is a connected resolving set of $G$. Moreover, $v'$ is adjacent to exactly one of $u'$ and $x$.

Therefore, in all cases, $cr^+(G) \leq n - 2$, as desired. ■

Note that every graph $G$ encountered thus far has the property that either $cr^+(G) = cr(G)$ or $cr^+(G) - cr(G) \leq 2$. This might lead one to believe that $cr^+(G)$ and $cr(G)$ are close for every connected graph $G$. However, this is not the case. Indeed, as we next show, every pair $a, b$ of integers with $2 \leq a \leq b$ is realizable as the connected resolving number and upper connected resolving number of some graph.

Theorem 2.29  For every pair $a, b$ of integers with $2 \leq a \leq b$, there exists a connected graph $G$ with $cr(G) = a$ and $cr^+(G) = b$.

Proof. We consider two cases, according to whether $a = 2$ or $a \geq 3$.

Case 1. $a = 2$. Let $G = P_b \times P_2$, where $u_1, u_2, \ldots, u_b$ and $v_1, v_2, \ldots, v_b$ are two copies of $P_b$ in $G$. Since $\{u_1, v_1\}$ is a $cr$-set of $G$, it follows that $cr(G) = 2$.

We now show that $cr^+(G) = b$. If $b = 2$, then $G = C_4$ and $cr^+(C_4) = 2$. Thus we may assume that $b \geq 3$. Let $U = \{u_1, u_2, \ldots, u_b\}$. We show that $U$ is a minimal connected resolving set of $G$. Since $U$ is a resolving set and $\langle U \rangle = P_b$, it follows that $U$ is a connected resolving set of $G$. It remains to show that $U$ is minimal. Assume, to the contrary, that $U$ is not minimal. Then there exists a proper subset $S$ in $U$ such that $S$ is a connected resolving set of $G$. Since $S$ is connected, $\langle S \rangle = P_s$ is a subpath in $\langle U \rangle$, where $s = |S|$. This implies that at most one of $u_1$ and $u_b$ belongs to $S$, say $u_1 \notin S$ and $u_b \in S$. Then $c_S(u_1) = c_S(v_2), \ldots, c_S(u_b) = c_S(v_1)$.
which is a contradiction. Therefore, \( U \) is a minimal connected resolving set of \( G \) and so \( \text{cr}^+(G) \geq |U| = b \).

Next we show that \( \text{cr}^+(G) \leq b \). Assume, to the contrary, that \( \text{cr}^+(G) \geq b + 1 \). Let \( W \) be an upper \( \text{cr} \)-set of \( G \) with \( |W| = \text{cr}^+(G) \geq b + 1 \). Then there exists \( i \) with \( 1 \leq i \leq b \) such that \( \{u_i, v_i\} \subseteq W \). Since \( \{u_1, v_1\} \) and \( \{u_b, v_b\} \) are \( \text{cr} \)-sets of \( G \), it follows that \( 2 \leq i \leq b - 1 \). Since \( W \) is connected, at least one vertex in \( \{u_{i-1}, u_{i+1}, v_{i-1}, v_{i+1}\} \) belong to \( W \), say \( u_{i-1} \in W \). However, the proper subset \( \{u_i, u_{i-1}, v_i\} \) of \( W \) is a connected resolving set of \( G \), which is a contradiction. Therefore, \( \text{cr}^+(G) = b \), as claimed.

Case 2. \( a \geq 3 \). Let \( u_1, u_2, \ldots, u_{b-a+2} \) and \( v_1, v_2, \ldots, v_{b-a+2} \) be two copies of \( P_{b-a+2} \) in \( P_{b-a+2} \times P_2 \) and \( K_a \) a complete graph with \( V(K_a) = \{w_1, w_2, \ldots, w_a\} \). Let \( G \) be the graph obtained from the graphs \( P_b \times P_2 \) and \( K_a \) by identifying the vertices \( u_1 \) and \( w_1 \) and denoting the identified vertex by \( u_1 \). Since \( \{u_1, v_1, w_3, w_4, \ldots, w_a\} \) is a \( \text{cr} \)-set of \( G \), it follows that \( \text{cr}(G) = a \).

Next we show that \( \text{cr}^+(G) = b \). Let

\[
W = \{u_1, u_2, \ldots, u_{b-a+2}, w_3, w_4, \ldots, w_a\},
\]

Then \( W \) is a connected resolving set of \( G \). We show, in fact, that \( W \) is minimal. By Observation 2.2, every \( \text{cr} \)-set contains at least \( a - 2 \) vertices from \( \{w_2, w_3, w_4, \ldots, w_a\} \). Thus \( W - \{w_i\} \) is not a resolving set for all \( 3 \leq i \leq a \). For each \( j \) with \( 1 \leq j \leq b - a + 1 \), since \( \langle W - \{u_j\} \rangle \) is not connected, \( W - \{u_j\} \) is not a connected resolving set. Moreover, \( c_{W - \{u_{b-a+2}\}}(u_{b-a+2}) = c_{W - \{u_{b-a+1}\}}(u_{b-a+1}) \), implying that \( W - \{u_{b-a+2}\} \) is not a resolving set. Thus no proper subset of \( W \) is a connected resolving set and so \( W \) is minimal. Therefore, \( \text{cr}^+(G) \geq |W| = b \).

We now show that \( \text{cr}^+(G) \leq b \). Assume, to the contrary, that \( \text{cr}^+(G) \geq b + 1 \). Let \( W' \) be an upper \( \text{cr} \)-set of \( G \) with \( |W'| = \text{cr}^+(G) \). Since \( W' \) is a resolving set, \( W' \) contains at least \( a - 2 \) vertices from \( \{w_2, w_3, w_4, \ldots, w_a\} \). Assume, without loss of generality, that \( \{w_3, w_4, \ldots, w_a\} \subseteq W' \). If \( W' \subseteq V(K_a) \), then \( c_{W'}(u_2) = c_{W'}(v_1) \), which is impossible. Thus \( W' \) contains at least one vertex from \( V(P_{b-a+2} \times P_2) - \{u_1\} \). Since \( W' \) is connected, it follows that \( u_1 \in W' \). If \( u_1 \in W' \), then the \( \text{cr} \)-set \( \{u_1, v_1, w_3, w_4, \ldots, w_a\} \) is a proper subset of \( W' \), which
is a contradiction. Thus \( v_i \notin W' \). Since \( |W'| \geq b + 1 \), it follows that \( W' \) contains \( \{u_i, v_i\} \) for some \( i \) with \( 2 \leq i \leq a - b + 2 \). Let \( i_0 \) be the smallest integer such that \( \{u_{i_0}, v_{i_0}\} \subseteq W' \) and let

\[
S = \{u_1, u_2, \ldots, u_{i_0}, v_{i_0}, w_3, w_4, \ldots, w_a\}.
\]

Then \( S \) is a connected resolving set. If \( 2 \leq i_0 \leq b - a + 1 \), then \( |S| = (i_0 + 1) + (a - 2) \leq (b - a + 2) + (a - 2) = b \). Thus \( S \) is a proper subset of \( W' \), which is a contradiction. If \( i_0 = b - a + 2 \), then \( S' = S - \{v_{i_0}\} \subseteq W' \) is a connected resolving set \( G \). Since \( S' \) is a proper subset of \( W' \), a contradiction is produced. Therefore, \( cr^+(G) \leq b \).

Note that the paths are the only connected graphs with connected resolving number \( 1 \). Since the upper connected resolving number of all paths is also \( 1 \), the following corollary is an immediate consequence of Theorem 2.29.

**Corollary 2.30** Let \( a \) and \( b \) be integers with \( 1 \leq a \leq b \). Then there exists a connected graph \( G \) with \( cr(G) = a \) and \( cr^+(G) = b \) if and only if \( (a, b) \neq (1, i) \) for all \( i \geq 2 \).

### 2.7 Forcing cr-Sets in a Graph

For every \( cr \)-set \( W \) of \( G \), there is always some subset \( S \) of \( W \) that determines \( W \) as the unique \( cr \)-set containing \( S \). Such "forcing subsets" will now be considered. More formally, for a \( cr \)-set \( W \) of \( G \), a subset \( S \) of \( W \) with the property that \( W \) is the unique \( cr \)-set containing \( S \) is called a forcing subset of \( W \). The forcing number \( f_G(W, cr) \) of \( W \) in \( G \) is the minimum cardinality of a forcing subset for \( W \); while the forcing connected resolving number (or, simply, the forcing \( cr \)-number) \( f(G, cr) \) of \( G \) is the smallest forcing number among all \( cr \)-sets of \( G \). Since the connected resolving number is understood in this context, we write \( f_G(W) \) for \( f_G(W, cr) \) and \( f(G) \) for \( f(G, cr) \). Hence if \( G \) is a nontrivial connected graph with \( f(G) = a \) and \( cr(G) = b \), then \( 0 \leq a \leq b \) and there exists a \( cr \)-set \( W \) of cardinality \( b \) containing a forcing subset of cardinality \( a \).
For example, a cr-set $W$ in the graph $G$ of Figure 2.1 consists of $x$ and one vertex from each set $\{u, w\}$ and $\{v, y\}$. Thus $W$ is not the unique cr-set containing any one of its vertices. On the other hand, the vertex $x$ belong to every cr-set of $G$ and, consequently, $W$ is the unique cr-set containing $W - \{x\}$ and so $f_c(W) = 2$. Since this is true for all cr-set of $G$, it follows that $f(G) = 2$ and so $f(G) < cr(G) = 3$ for the graph $G$ of Figure 2.1. This example also illustrates the following lemma.

**Lemma 2.31** Let $G$ be a nontrivial connected graph. If $S$ is a set of vertices of $G$ that is contained in every cr-set of $G$, then

$$f(G) \leq cr(G) - |S|.$$ 

**Proof.** Let $W$ be a cr-set of $G$. Since $W$ must contain $S$, it follows that $W$ is the unique cr-set containing $W - S$. This implies that $f_c(W) \leq |W - S| = cr(G) - |S|$. Therefore, $f(G) \leq cr(G) - |S|$. 

Note that the upper bound in Lemma 2.31 is sharp and that strict inequality can hold as well.

It is immediate that $f(G) = 0$ if and only if $G$ has a unique cr-set. If $G$ has no unique cr-set but contains a vertex belonging to only one cr-set, then $f(G) = 1$. Moreover, if for every cr-set $W$ of $G$ and every proper subset $S$ of $W$, the set $W$ is not the unique cr-set containing $S$, then $f(G) = cr(G)$. We summarize these observations below.

**Lemma 2.32** For a graph $G$,

(a) the forcing cr-number $f(G) = 0$ if and only if $G$ has a unique cr-set,

(b) $f(G) = 1$ if and only if $G$ has at least two distinct cr-set but some vertex of $G$ belongs to exactly one cr-set, and

(c) $f(G) = cr(G)$ if and only if no cr-set of $G$ is the unique cr-set containing any of its proper subsets.
Forcing concepts have been studied for several parameters in graph theory, including the chromatic number [17], the graph reconstruction number [21], the dimension of a graph [15], and geodetic concepts [14]. Also, many invariants arising from the study of forcing in graph theory offer subjects for new and applicable research. A survey of graphical forcing parameters is discussed in [20].

We now determine the forcing cr-numbers of these graphs.

Proposition 2.33  If $G = K_n, C_n, K_{r,s}$, where $n \geq 3$ and $2 \leq r \leq s$, then

$$f(G) = cr(G).$$

Proof. Assume that $G$ is the complete graph $K_n$ of order $n \geq 3$. Since every set $W$ of $n - 1$ vertices in $K_n$ is a cr-set of $K_n$, it follows that $W$ is not the unique cr-set containing any of its proper subset. By Lemma 2.32, $f(K_n) = cr(K_n)$. Next assume that $G$ is a cycle $C_n$ of order $n \geq 4$. Then every pair $u, v$ of adjacent vertices forms a cr-set of $C_n$. Thus there is no cr-set of $C_n$ that is the unique cr-set containing any of its proper subset. Thus $f(C_n) = cr(C_n)$ by Lemma 2.32. Finally assume that $G = K_{r,s}$ ($2 \leq r \leq s$) whose the partite sets are $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$. Then by Observation 2.2 every cr-set $W$ of $G$ has the form $W = W_1 \cup W_2$, where $W_i \subseteq V_i$ ($i = 1, 2$) with $|W_1| = r - 1 \geq 1$ and $|W_2| = s - 1 \geq 1$. Assume, without loss of generality, that $W = V(G) - \{u_r, v_s\}$. Let $S$ be a proper subset of $W$. Then $S = S_1 \cup S_2$, where $S_i \subseteq W_i$ ($i = 1, 2$) such that $|S_1| \leq r - 2$ or $|S_2| \leq s - 2$, say $|S_1| \leq r - 2$. Thus there is $u_i \in W$, where $1 \leq i \leq r - 1$, such that $u_i \notin S_1$. Then $W' = (W - \{u_i\}) \cup \{u_r\}$ is a cr-set of $G$ containing $S$. Since $W' \neq W$, it follows that $W$ is not the unique cr-set containing $S$. Therefore, $f(G) = cr(G)$. ■

The following corollary is an immediate consequence of Theorem C.

Corollary 2.34  Let $T$ be a tree that is not a path. Then every cr-set $W$ of $T$ contains a basis $W'$ of $T$ as a proper subset. Moreover, $W$ is the unique cr-set of $G$ that contains $W'$.

We are now prepared to determine the forcing cr-number of a tree.
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Proposition 2.35  If \( T \) be a tree of order at least 2, then \( f(T) = \dim(T) \).

Proof. If \( T \) is a path, then each of its end-vertices forms a \( cr \)-set and so \( f(T) \geq 1 \) by Lemma 2.32. On the other hand, \( f(T) \leq cr(T) = 1 \). Thus \( f(T) = 1 \). We now assume that \( T \) is a tree that is not a tree, having order \( n \geq 4 \) and \( p \) exterior major vertices \( v_1, v_2, \ldots, v_p \). For \( 1 \leq i \leq p \), let \( u_{i1}, u_{i2}, \ldots, u_{ik_i} \) be the terminal vertices of \( v_i \) and let \( P_{ij} \) be the \( v_i - u_{ij} \) path \( (1 \leq j \leq k_i) \) in \( T \). For each pair \( i, j \) with \( 1 \leq i \leq p \) and \( 1 \leq j \leq k_i \), let \( x_{ij} \) be the vertex in \( P_{ij} \) such that \( x_{ij} \) is adjacent to \( v_i \). Suppose that \( W \) is a \( cr \)-set of \( T \). We show that \( f_T(W) = \dim(G) \).

By Corollary 2.34, the \( cr \)-set \( W \) contains a basis \( W' \) of \( T \) as a proper subset and \( W \) is the unique \( cr \)-set containing \( W' \). Thus \( f_T(W) \leq |W'| = \dim(T) \). Next we show that \( f_T(W) \geq \dim(T) \). Let \( S \subseteq W \) with \( |S| < \dim(T) \). Then \( W' - S \neq \emptyset \). Let \( x \in W' - S \). Then there exist integers \( i \) and \( j \) with \( 1 \leq i \leq p \) and \( 1 \leq j \leq k_i \) such that \( x \) is a vertex from the path \( P_{ij} \) in \( T \), say \( x \) is a vertex from \( P_{11} \) in \( T \). Since \( W \) is a \( cr \)-set, it follows that \( x = x_{11} \) that is adjacent \( v_1 \). Since \( x \in W' \), it follows that \( \text{ter}(v_1) = k_1 \geq 2 \). Assume, without loss of generality, that for each \( j \) with \( 1 \leq j \leq k_1 - 1 \), the vertex \( x_{1j} \) from \( P_{1j} \) in \( T \) belongs to \( W' \). Moreover, there is no vertex of \( P_{1,k_1} - v_1 \) that belongs to \( W' \). Then, by Theorem C, we see that \( W'' = (W' - \{x_{11}\}) \cup \{x_{1k_1}\} \) is a basis of \( T \), where \( x_{1k_1} \) is the vertex in the path \( P_{1,k_1} - v_1 \) that is adjacent to \( v_1 \). Let \( W^* \) consist of \( W'' \) and all vertices in the paths between any two vertices in \( W'' \). Again, by Theorem C, \( W^* \) is a \( cr \)-set containing \( S \). Since \( W \neq W^* \), it follows that \( W \) is not the unique basis containing \( S \), implying that \( f_G(W) \geq \dim(G) \). Thus \( f_G(W) = \dim(G) \) for all \( cr \)-sets \( W \) in \( G \). Therefore, \( f(T) = \dim(T) = \sigma(T) - \text{ex}(T) \).

By Proposition 2.35 and Theorem A, for a star \( K_{1,n-1} \), where \( n \geq 4 \),

\[
f(K_{1,n-1}) = \dim(K_{1,n-1}) = cr(K_{1,n-1}) - 1 = n - 2.
\]

We have already noted that if \( G \) is a nontrivial connected graph with \( f(G) = a \) and \( cr(G) = b \), then \( 0 \leq a \leq b \) and \( b \geq 1 \). We now determine which pairs \( a, b \) of integers with \( 0 \leq a \leq b \) and \( b \geq 1 \) are realizable as the forcing \( cr \)-number and connected resolving number of some nontrivial connected graph. Recall that for \( k \geq 2 \), there exists a connected graph of \( cr \)-number \( k \) with a unique \( cr \)-set.
Theorem 2.36  Let $a$ and $b$ be integers with $0 \leq a \leq b$ and $b \geq 1$. Then there exists a connected graph $G$ with $f(G) = a$ and $cr(G) = b$ if and only if $(a, b)(a, b) \neq (0, 1)$.

Proof. Observe that nontrivial paths are the only nontrivial connected graphs with $cr$-number 1. Since the forcing $cr$-number of all nontrivial paths is also 1 by Proposition 2.35, there is no connected graph $G$ with $f(G) = 0$ and $cr(G) = 1$.

It remains to verify the converse. If $a = 0$, then by Theorem 2.24, there is a connected graph $G$ of $cr$-number $b$ with a unique $cr$-set for each integer $b \geq 2$. Thus $f(G) = 0$ by Lemma 2.32. So we may assume that $a \geq 1$. Let $a = b \geq 1$.

For $a = b = 1$, let $G = P_n$ for some $n \geq 2$ and so $f(G) = cr(G) = 1$. For $b = a \geq 2$, let $G = K_{a+1}$ and so $f(G) = cr(G) = a$. Let $b = a + 1$. If $a = 1$, then $b = 2$. Consider the graph $G$ of Figure 2.9. The set $W = \{u, v\}$ is a $cr$-set of $G$ and so $cr(G) = 2$. Since $\{u, w\}$ is also a $cr$-set, $f(G) \geq 1$ by Lemma 2.32. We show that $W$ is the unique $cr$-set of $G$ containing $v$. Assume, to the contrary, that $W$ is not the unique $cr$-set of $G$ containing $v$. Let $W'$ be a $cr$-set of $G$ containing $v$ such that $W' \neq W$. Since $\langle W'\rangle$ is connected and containing $v$, it follows that $W' = \{v, x\}$. However, $cw'(w) = cw'(y) = (2, 1)$, which is a contradiction. Thus $W$ is the unique $cr$-set of $G$ containing $v$ and so $f_G(W) = 1$. Therefore, $f(G) = 1$ and $cr(G) = 2$ for the graph $G$ of Figure 2.9. If $a \geq 2$, then let $G = K_{1,a+1}$ and $f(G) = a$ and $cr(G) = a + 1 = b$ by (2.2).

![Figure 2.9: A graph $G$ with $f(G) = 1$ and $cr(G) = 2$](image)

We now assume that $a \geq 1$ and $b \geq a + 2$. Let $r = b - a$. Then $2 \leq r \leq b - 1$.

First we construct a connected graph $H$ of order $r + 2r$ with $V(H) = U \cup V$, where $U = \{u_0, u_1, \cdots, u_{2r-1}\}$ and the ordered set $V = \{v_{r-1}, v_{r-2}, \cdots, v_0\}$ are disjoint. The induced subgraphs $\langle U \rangle$ and $\langle V \rangle$ of $H$ are complete. To define the adjacencies
between \( V \) and \( U \), let each integer \( j \) \((0 \leq j \leq 2^r - 1)\) be expressed in its base 2 (binary) representation. Thus, each such \( j \) can be expressed as a sequence of \( r \) coordinates, that is, an \( r \)-vector, where the rightmost coordinate represents the value (either 0 or 1) in the \( 2^0 \) position, the coordinate to its immediate left is the value in the \( 2^1 \) position, etc. In particular, \( 2^{r-1} = (1, 1, \ldots, 1) \). For integers \( i \) and \( j \), with \( 0 \leq i \leq r - 1 \) and \( 0 \leq j \leq 2^r - 1 \), we join \( u_i \) and \( u_j \) if and only if the value in the \( 2^i \) position in the binary representation of \( j \) is 1. Therefore, \( u_{2^{r-1}} \) is adjacent to every vertex in \( V \). The structure of \( H \) is based on one given in the proof of Theorem 2.24, where it was shown that \( H \) has \( cr \)-number \( r \) and \( V \) is the unique basis of \( H \).

Now the graph \( G \) is obtained from \( H \) by adding the \( a \) new vertices \( x_1, x_2, \ldots, x_a \) such that each \( x_i \) \((1 \leq i \leq a)\) has the same neighborhood as \( u_{2^{r-1}} \) in \( V \) and the induced subgraph \((U \cup \{x_1, x_2, \ldots, x_a\})\) is complete. This graph \( G \) is a modification of the graph in Figure 2.5. An argument similar to the one used in the proof of Theorem 2.24 shows that \( cr(G) = b \). Moreover, every \( cr \)-set of \( G \) consists of \( V \) and \( a \) vertices from \( \{u_{2^{r-1}}, x_1, x_2, \ldots, x_a\} \). Thus, it remains to show that \( f(G) = a \). Since every \( cr \)-set of \( G \) contains \( V \), it then follows from Lemma 2.31 that \( f(G) \leq cr(G) - |V| = b - (b - a) = a \). Next we show that \( f(G) \geq a \). Assume, to the contrary, that \( f(G) < a \). Then there exists a \( cr \)-set \( W \) of \( G \) such that \( f_G(W) = f(G) < a \). This implies that \( W \) contains a forcing subset \( S \) with \( |S| \leq a - 1 \). Without loss of generality, assume that \( W = V \cup X \) with \( X = \{x_1, x_2, \ldots, x_a\} \). Since \( |S| \leq a - 1 \), there exists \( x \in W \cap X \) such that \( x \notin S \). Then \( W' = (W - \{x\}) \cup \{u_{2^{r-1}}\} \) is a \( cr \)-set of \( G \) containing \( S \) and distinct from \( W \). Hence \( W \) is not the unique \( cr \)-set containing \( S \), a contradiction. Thus \( f_G(W) = a \). This is true for every \( cr \)-set \( W \) in \( G \) and so \( f(G) = a \). Therefore, \( f(G) = a \) and \( cr(G) = b \), as desired.

### 2.8 Resolving Graphs

A connected graph \( H \) is a **resolving graph** if there is a graph \( G \) with a minimum connected resolving set \( W \) such that the subgraph \( \langle W \rangle \) of \( G \) induced by \( W \) is isomorphic to \( H \). As we see in this section, the condition that a graph is a resolving graph is not a demanding one. In order to do this, we first present a lemma.
**Lemma 2.37**  If $U$ is a distance similar equivalence class of a connected graph $G$, then $U$ is either independent in $G$ or in $\overline{G}$.

**Proof.** Assume, to the contrary, that there is a connected graph $G$ having a distance similar equivalence class $U$ such that $U$ is neither independent in $G$ nor in $\overline{G}$. Then there exist $x, y, z \in U$ such that $x$ is adjacent to $y$ but $x$ is not adjacent to $z$ in $G$. Thus $d_G(x, y) = 1$ and $d_G(x, z) \geq 2$, contradicting the fact that $y, z \in U$. $\blacksquare$

**Theorem 2.38**  Every connected graph is a resolving graph.

**Proof.** Let $H$ be a connected graph of order $n$. Since each end-vertex of a nontrivial path is a minimum connected resolving set, the result holds for $n = 1$. Thus we may assume that $n \geq 2$. Let $H_1$ and $H_2$ be two copies of $H$ with $V(H_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(H_2) = \{v'_1, v'_2, \ldots, v'_n\}$, where $v_i$ corresponds $v'_i$ for $1 \leq i \leq n$. Let $G$ be the graph obtained from $H_1$ and $H_2$ by, for each $i$ with $1 \leq i \leq n$,

1. joining $v'_i$ to every vertex in $N_{H_1}(v_i)$ and
2. joining $v'_i$ to $v_i$ if and only if $v_i$ belongs to an independent distance similar equivalence class $U$ in $H_1$ with $|U| \geq 2$.

Thus, for each $i$ with $1 \leq i \leq n$, the vertices $v_i$ and $v'_i$ have the same neighborhood in $G$ and so $v_i$ and $v'_i$ are distance similar in $G$. Let $V_i = \{v_i, v'_i\}$ for each $i$ with $1 \leq i \leq n$. We first verify the following claim.

**Claim:** The sets $V_1, V_2, \ldots, V_n$ are the only distance similar equivalence classes of $G$.

**Proof of Claim.** Assume, to the contrary, that $U$ is a distance similar equivalence class of $G$ such that $U \neq V_i$ for all $i$ with $1 \leq i \leq n$. Since $V(G) = V_1 \cup V_2 \cup \ldots \cup V_n$, there exists some integer $i$ with $1 \leq i \leq n$ such that $V_i \subseteq U$. Assume, without loss of generality, that $V_1 \subseteq U$. Since $U \neq V_1$,
it follows that there is $x \in U - V_1$, say $x \in V(H_1)$. Let $x'$ be the vertex in $H_2$ that corresponds to $x$ in $H_1$. Since $x$ and $x'$ are distance similar, it follows that \( \{x, x'\} \subseteq U \). Thus $v_1, v'_1, x, x'$ are distance similar in $G$, where $v_1, x \in V(H_1)$ and $v'_1, x' \in V(H_2)$. We consider two cases.

Case 1. $v_1$ is not adjacent to $x$ in $H_1$. Since $v_1$ and $x$ distance similar in $G$, it follows that $v_1$ and $x$ are distance similar in $H_1$. Thus, by Lemma 2.37, the vertices $v_1$ and $x$ belong to an independent distance similar equivalence class $U$ in $H_1$ with $|U| \geq 2$. By (2) the vertex $v'_1$ is adjacent to $v_i$ but not to $x$. Thus $d_G(v_1, v'_1) = 1$ and $d_G(v'_1, x) \geq 2$, which is a contradiction.

Case 2. $v_1$ is adjacent to $x$ in $H_1$. By Lemma 2.37, the vertices $v_1$ and $x$ belong to a distance similar equivalence class $U$ in $H_1$ such that $(U)$ is complete in $G$. Since $x \in N_{H_1}(v_1)$, it then follows from (1) that $v'_1$ is adjacent to $x$ but not to $v_1$ in $G$. Thus $d_G(v_1, v'_1) \geq 2$ and $d_G(v'_1, x) = 1$, which is a contradiction.

Therefore, the sets $V_i$ (1 $\leq i \leq n$) are the only distance similar equivalence classes of $G$, as claimed. Thus if $x, y \in V(H_1)$ or $x, y \in V(H_2)$, then $x, y$ belong to distinct distance similar equivalence classes in $G$.

Let $W = V(H_1)$. Next, show that $W$ is a minimum connected resolving set of $G$. By Observation 2.2, every minimum connected resolving set of $G$ contains at least one vertex from each distance similar equivalence class $V_i$ for all $i$ with $1 \leq i \leq n$. Thus $cr(G) \geq n$. Since $(W) = H_1$ is connected and $|W| = n$, it suffices to show that $W$ is a resolving set of $G$. Let $u, v \in V(G) - W = V(H_2)$. Then $u$ and $v$ belong to distinct distance similar equivalence classes of $G$. Thus there exists $x \in V(G) - \{u, v\}$ such that $d_G(u, x) \neq d_G(v, x)$. Assume first that $x \in W$. Since $d_G(u, x) \neq d_G(v, x)$, it follows that $c_W(u) \neq c_W(v)$. Next assume that $x \in V(H_2)$. Then let $x'$ be the vertex in $W$ that is corresponding to $x$ in $G$. Since $\{x, x'\}$ is a distance similar equivalence class in $G$, it follows that $d_G(x, u) = d_G(x', u)$ and $d_G(x, v) = d_G(x', v)$, implying that $c_W(u) \neq c_W(v)$. Therefore, $W$ is a resolving set of $G$ and so $W$ is a minimum resolving set of $G$. Since $(W) = H_1 = H$, it follows that $H$ is a resolving graph. ■
To illustrate Theorem 2.38, consider the graph $H$ shown in Figure 2.10. Note that $H$ has four distinct distance similar equivalence classes, namely, $\{v_1, v_2\}$, $\{v_3\}$, $\{v_4\}$, and $\{v_5, v_6\}$. Note that $\{v_1, v_2\}$ is the only independent distance similar equivalence class of cardinality at least 2 in $G$. The graph $G$ (also shown in Figure 2.10) is constructed by the method described in the proof of Theorem 2.38. Let $W = V(H_1)$. The codes for the vertices of $G$ with respect to $W$ are

$$c_W(v'_1) = (1, 2, 1, 2, 3, 3), \quad c_W(v'_2) = (2, 1, 1, 2, 3, 3),$$
$$c_W(v'_3) = (1, 1, 2, 1, 2, 2), \quad c_W(v'_4) = (2, 2, 1, 2, 1, 1),$$
$$c_W(v'_5) = (3, 3, 2, 1, 2, 1), \quad c_W(v'_6) = (3, 3, 2, 1, 1, 2).$$

By the proof of Theorem 2.38, $W$ is a minimum resolving set of $G$ of $G$ with $\langle W \rangle = H_1 = H$. Therefore, $H$ is a resolving graph.

We have seen in Observation 2.5 that if $G$ is a nontrivial connected graph of order $n$ having $k$ distance similar equivalence classes, then $\dim(G) \geq n - k$ and so $cr(G) \geq n - k$. In Section 2.1, we mentioned, for a given integer $n \geq 4$, that the lower bounds in Observation 2.5 are sharp for most values of $k$ and that strict inequality can hold for most values of $k$ in each case. We are now prepared to verify this.
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First, we consider the sharpness of the bounds. Since \( cr(G) \geq \dim(G) \geq 1 \) for all connected graphs \( G \) of order \( n \geq 4 \), the lower bound \( n - k \) for both \( cr(G) \) and \( \dim(G) \) cannot be attained when \( k = n \). The graph \( G = P_n \) is the only connected graph of order \( n \geq 4 \) for which \( cr(G) = \dim(G) = 1 \). Furthermore, \( P_n \) has \( n \) distance similar equivalence classes. Thus, the lower bound \( n - k \) for \( cr(G) \) and \( \dim(G) \) cannot be sharp if \( k = n - 1 \). This lower bound is sharp, however, for all integers \( k \) with \( 2 \leq k \leq n - 2 \), as we now show.

**Proposition 2.39**  
For each pair \( n, k \) of integers with \( 2 \leq k \leq n - 2 \) and \( n \geq 4 \),

(a) there exists a connected graph \( G \) of order \( n \) having \( k \) distance similar equivalence classes such that \( cr(G) > \dim(G) = n - k \),

(b) there exists a connected graph \( H \) of order \( n \) having \( k \) distance similar equivalence classes such that \( cr(H) = \dim(H) = n - k \).

**Proof.** We first verify (a). For \( k = 1 \), let \( G = K_n \). Then \( G \) has only one distance similar equivalence class and \( \dim(G) = n - 1 \) by Theorem A. For \( k = 2 \), let \( G = K_{1,n-1} \). Then \( G \) has two distance similar equivalence classes and \( \dim(G) = n - 2 \) by Theorem A. For \( 3 \leq k \leq n - 2 \), let \( G \) be the graph obtained from the path \( P_k : u_1, u_2, \ldots, u_k \) by adding \( n - k \) new vertices \( v_1, v_2, \ldots, v_{n-k} \) and joining (1) \( v_1 \) to \( u_1 \) and \( u_2 \) and (2) each of \( v_i \) (\( 2 \leq i \leq n-k \)) to \( u_{i-1} \). Then \( G \) has \( k \) distance similar equivalence classes, namely \( \{u_1, v_1\}, \{u_i\}, 2 \leq i \leq k - 1, \) and \( \{u_k, v_2, v_3, \ldots, v_{n-k}\} \). Since \( W = \{v_1, v_2, \ldots, v_{n-k}\} \) is a minimum resolving set of \( G \), it follows that \( \dim(G) = |W| = n - k \). Moreover, every minimum resolving set of \( G \) contains at least one vertex from \( \{u_1, v_1\} \) and at least \( n - k - 1 \) vertices from \( \{u_k, v_2, v_3, \ldots, v_{n-k}\} \) and so no minimum resolving set of \( G \) is connected. It then follows by Observation 2.6 that \( cr(G) > \dim(G) = n - k \) and so (a) holds.

Next, we verify (b). For \( k = 1 \), let \( H = K_n \). Then \( H \) has only one distance similar equivalence class and \( cr(H) = n - 1 \) by Theorem 2.15. For \( 2 \leq k \leq \lfloor n/2 \rfloor \), let \( H = K_{n_1,n_2,\ldots,n_k} \) be a complete \( k \)-partite graph, where \( n = n_1 + n_2 + \ldots, n_k \) and \( n_i \geq 2 \) for \( 1 \leq i \leq k \). Then \( H \) has \( k \) distance similar equivalence classes and \( cr(G) = n - k \) by Proposition 2.16. For \( \lfloor n/2 \rfloor \leq k \leq n - 2 \), let \( F = \)}
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\((n - k - 1)K_2 + \overline{K}_2\), where \(V((n - k - 1)K_2) = \{w_i, w'_i : 1 \leq i \leq n - k - 1\}\), 
\(E((n - k - 1)K_2) = \{w_iw'_i : 1 \leq i \leq n - k - 1\}\), and \(V(\overline{K}_2) = \{v_1, v_2\}\), and let \(P_{2k-n} : u_1, u_2, \ldots, u_{2k-n}\) be a path of order \(2k - n\). The graph \(H\) is obtained from \(F\) and \(P_{2k-n}\) by joining \(u_1\) to both \(v_1\) and \(v_2\). Then \(H\) has \(k\) distance similar equivalence classes, namely \(\{w_i, w'_i\}, 1 \leq i \leq n - k - 1, \{v_1, v_2\}, \{u_i\}, 1 \leq i \leq 2k - n\). Since \(W^* = \{w_i : 1 \leq i \leq n - k - 1\} \cup \{v_1\}\) is a minimum connected resolving set of \(H\), it follows that \(cr(H) = |W^*| = n - k\). Therefore, \(cr(H) = \dim(H) = n - k\) by Observation 2.5.

We have already seen that the inequalities \(cr(G) > n - k\) and \(\dim(G) > n - k\) are not only possible for \(k = n\) and \(k = n - 1\), but, in fact, are necessary. The only graph \(G\) of order \(n \geq 4\) for which \(k = 1\) is \(K_n\); however, \(cr(K_n) = \dim(K_n) = n - 1\), and so \(cr(G) = \dim(G) = n - k\) is required if \(k = 1\). For \(k = 2\), \(\dim(G) = n - k\) as well since the strict inequality \(\dim(G) > n - k\) in this case would imply that \(\dim(G) = n - 1\) and that \(G = K_n\), which is impossible. For all other values of \(k\) in the range \(1 \leq k \leq n\), the inequalities \(cr(G) > n - k\) and \(\dim(G) > n - k\) can occur, however. We show this now.

**Proposition 2.40** Let \(n, k\) be positive integers with \(k \leq n\) and \(n \geq 4\).

(a) For \(k \geq 3\), there exists a connected graph \(G\) of order \(n\) having \(k\) distance similar equivalence classes such that \(\dim(G) > n - k\).

(b) For \(k \geq 2\), there exists a connected graph \(H\) of order \(n\) having \(k\) distance similar equivalence classes such that \(cr(H) > n - k\).

**Proof.** We first verify (a). For \(k = n\), let \(G = P_n\). Then \(G\) has \(n\) distance similar equivalence classes and \(\dim(G) = 1 > n - k = 0\). For \(3 \leq k \leq n - 1\), let \(G\) be the graph obtained from the complete graph \(K_{n-k+1}\) and the path \(P_{k-1} : v_1, v_2, \ldots, v_{k-1}\) by joining \(v_i\) to every vertex in \(K_{n-k+1}\). Thus \(G\) has \(k\) distance similar equivalence classes, namely \(V(K_{n-k+1})\) and \(\{v_i\}, 1 \leq i \leq k - 1\). Since \(V(K_{n-k+1})\) is a minimum resolving set of \(G\), it follows that \(\dim(G) = n - k + 1 > n - k\) and so (a) holds.
Next, we verify (b). For \( k = n \), let \( H = P_n \). Then \( H \) has \( n \) distance similar equivalence classes and \( cr(H) = 1 > n - k = 0 \). For \( k = 2 \), let \( H = K_{1,n-1} \). Then \( H \) has two distance similar equivalence classes and \( cr(H) = n - 1 > n - k = n - 2 \) by Theorem 2.15. For \( 3 \leq k \leq n - 1 \), let \( H \) be the graph \( G \) described in (a). Thus \( H \) has \( k \) distance similar equivalence classes. Since \( V(K_{n-k+1}) \) is connected, it follows by Observation 2.6 that \( cr(H) = \dim(H) = n - k + 1 > n - k \) and so (b) holds. 

\[ \blacksquare \]
Chapter 3

Connected Resolving Partitions in Graphs

3.1 Introduction

Let $G$ be a nontrivial connected graph. Recall that a partition $\Pi$ of $V(G)$ is called a resolving partition for $G$ if distinct vertices of $G$ have distinct codes with respect to $\Pi$, and the minimum $k$ for which there is a resolving $k$-partition of $V(G)$ is the partition dimension $\text{pd}(G)$ of $G$. A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is connected if the subgraph $\langle S_i \rangle$ induced by each subset $S_i$ ($1 \leq i \leq k$) is connected in $G$. The minimum $k$ for which there is a connected resolving $k$-partition of $V(G)$ is the connected partition dimension $\text{cpd}(G)$ of $G$. A connected resolving partition of $V(G)$ containing $\text{cpd}(G)$ elements is called a minimum connected resolving partition (or cr-partition) of $V(G)$. If $G$ is a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the $n$-partition $\{S_1, S_2, \ldots, S_n\}$, where $S_i = \{v_i\}$ for $1 \leq i \leq n$, is a connected resolving partition for $G$. Thus $\text{cpd}(G)$ is defined for every nontrivial connected graph $G$. Certainly, every connected resolving partition of a connected graph is a resolving partition. Thus if $G$ is a connected graph of order $n \geq 2$, then

$$2 \leq \text{pd}(G) \leq \text{cpd}(G) \leq n. \quad (3.1)$$

To illustrate these concepts, consider the graph $G$ in Figure 3.1. Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{u_1, v\}$, $S_2 = \{u_2, w\}$, and $S_3 = \{u_3, x, y\}$. The partition
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Π is shown in Figure 3.1(a). Then $c_\Pi(u_1) = (0, 1, 2)$, $c_\Pi(u_2) = (1, 0, 2)$, $c_\Pi(u_3) = (1, 1, 0)$, $c_\Pi(v) = (0, 1, 1)$, $c_\Pi(w) = (1, 0, 1)$, $c_\Pi(x) = (1, 2, 0)$, $c_\Pi(y) = (2, 1, 0)$. So Π is a resolving partition for $G$. Since there is no resolving 2-partition in $G$, it follows that $pd(G) = 3$. Since the subgraph $\langle S_3 \rangle$ induced by $S_3$ is disconnected in $G$, it follows that Π is not connected. On the other hand, $\Pi' = \{S_1', S_2', S_3', S_4'\}$, where $S_1' = \{u_1, v\}$, $S_2' = \{u_2, w\}$, $S_3' = \{u_3\}$, and $S_4' = \{x, y\}$ is the connected resolving partition of $V(G)$ shown in Figure 3.1(b). By a case-by-case analysis, it can be shown that $\Pi'$ is a $cr$-partition of $G$ and so $cpd(G) = 4$. Thus $pd(G) < cpd(G)$ for the graph $G$ of Figure 3.1.

![Graph G with pd(G) = 3 and cpd(G) = 4](image)

Figure 3.1: A graph $G$ with $pd(G) = 3$ and $cpd(G) = 4$

This example also illustrates an important point. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving partition of $V(G)$. If $u \in S_i$ and $v \in S_j$, where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$, then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, S_i) = 0$ and $d(u, S_j) \neq 0$. Thus, when determining whether a given partition $\Pi$ of $V(G)$ is a resolving partition for a graph $G$, we need only verify that the vertices of $G$ belonging to the same element in $\Pi$ have distinct codes with respect to $\Pi$. The following observations are useful. Recall that two vertices $u$ and $v$ in a connected graph are distance similar if $d(u, x) = d(v, x)$ for all $x \in V(G) \setminus \{u, v\}$. 

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Observation 3.1 Let $G$ be a nontrivial connected graph. If $\Pi$ is a resolving partition of $V(G)$ and $u$ and $v$ are two distance similar vertices of $G$, then $u$ and $v$ belong to distinct elements of $\Pi$. In particular, if $G$ contains a vertex that is adjacent to $k$ end-vertices of $G$, then $pd(G) \geq k$ and $cpd(G) \geq k$.

Observation 3.2 Let $G$ be a connected graph. Then $pd(G) = cpd(G)$ if and only if $G$ contains a minimum resolving partition that is connected.

3.2 Refinements of a Resolving Partition of a Graph

Let $\Pi$ be a partition of $V(G)$. A partition $\Pi'$ of $V(G)$ is called a refinement of $\Pi$ if each element of $\Pi'$ is a subset of some element of $\Pi$. For the graph $G$ of Figure 3.1, the partition $\Pi'$ of $V(G)$ shown in Figure 3.1(b) is a connected refinement of the partition $\Pi$ of $V(G)$ shown in Figure 3.1(a). We have seen that the refinement $\Pi'$ of the resolving partition $\Pi$ for the graph $G$ of Figure 3.1 is also a resolving partition for $G$. This is no coincidence, as we show that every refinement of a resolving partition of a connected graph $G$ is also a resolving partition of $G$.

Proposition 3.3 Let $G$ be a nontrivial connected graph and let $\Pi$ be a resolving partition of $G$. If $\Pi'$ is refinement of $\Pi$, then $\Pi'$ is also a resolving partition of $G$.

Proof. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ and $\Pi' = \{S'_1, S'_2, \ldots, S'_\ell\}$, where $k \leq \ell$, such that each set $S'_i$ ($1 \leq i \leq \ell$) is a subset of $S_j$ for some $j$ with $1 \leq j \leq k$. Let $u$ and $v$ be two distinct vertices of $G$. We show that $c_{\Pi'}(u) \neq c_{\Pi'}(v)$. Since $\Pi$ is a resolving partition of $G$, it follows that $c_{\Pi}(u) \neq c_{\Pi}(v)$. Thus $d(u, S_j) \neq d(v, S_j)$ for some $j$ with $1 \leq j \leq k$, say $d(u, S_1) \neq d(v, S_1)$. If $S_1$ is an element of $\Pi'$, then $d(u, S_1) \neq d(v, S_1)$ and so $c_{\Pi'}(u) \neq c_{\Pi'}(v)$.

Thus we may assume that

$$S_1 = S'_{i_1} \cup S'_{i_2} \cup \ldots \cup S'_{i_h},$$

where $1 \leq i_1 < i_2 < \ldots < i_h \leq \ell$ and $h \geq 2$. Observe that at least one of $u$ and $v$ does not belong to $S_1$, for otherwise, $d(u, S_1) = 0 = d(v, S_1)$. We consider two cases.
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Case 1. Exactly one of u and v is in $S_1$, say $u \in S_1$ and $v \notin S_1$. Thus $u \in S_{1_p}'$ for some $p$ with $1 \leq p \leq h$ and so $d(u, S_{1_p}') = 0$. Since $v \notin S_1$, it follows that $v \notin S_{1_p}'$ and so $d(v, S_{1_p}') \neq 0$. Hence $c_{\Pi'}(u) \neq c_{\Pi'}(v)$.

Case 2. $u, v \notin S_1$. Let $x, y \in S_1$ such that $d(u, S_1) = d(u, x)$ and $d(v, S_1) = d(v, y)$, where say $d(u, x) < d(v, y)$. If $x, y \in S_{1_p}'$ for some $p$ with $1 \leq p \leq h$, then $d(u, S_{1_p}') = d(u, x) < d(v, y) = d(v, S_{1_p}')$, implying that $c_{\Pi'}(u) \neq c_{\Pi'}(v)$. If $x \in S_{1_p}'$ and $y \in S_{1_q}'$, where $1 \leq p \neq q \leq h$, then $d(u, S_{1_p}') = d(u, x) < d(v, y) \leq d(v, S_{1_q}')$, again, implying that $c_{\Pi'}(u) \neq c_{\Pi'}(v)$.

Therefore, $\Pi'$ is a resolving partition of $G$.

According to Proposition 3.3, if we are given a minimum resolving partition $\Pi$ of a connected graph $G$, then we can find a connected resolving partition $\Pi'$ of $G$, where $\Pi'$ is a refinement of $\Pi$. Indeed, the partition each of whose elements consist of a single vertex has this property. However, $\Pi'$ need not be a minimum connected resolving partition for $G$. In fact, it may be the case that no minimum connected resolving partition is a refinement of any minimum resolving partition of $V(G)$. For example, consider the graph $G$ of Figure 3.2.

Figure 3.2: A graph $G$ with $\text{pd}(G) = 3$ and $\text{cpd}(G) = 5$

Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{u_1, v_1, w_1\}$, $S_2 = \{u_2, v_2, w_2\}$ and $S_3 = \{u_3,$

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Thus $\Pi$ is a resolving partition of $G$. Since $G$ is not a path, it then follows by Theorem E that $\Pi$ is a minimum resolving partition of $V(G)$ and so that $\text{pd}(G) = 3$. Using a case-by-case analysis, we can show that every minimum resolving partition of $V(G)$ has the same structure as $\Pi$. Thus, a connected refinement of any minimum resolving partition of $V(G)$ contains at least 7 elements. On the other hand, let $\Pi^* = \{S_1^*, S_2^*, S_3^*, S_4^*, S_5^*\}$, where $S_i^* = \{u_1, u_2, u_3, v_i, w_i\}$ and each set $S_i^*$ contains exactly one vertex of $V(G) - S_i^*$ for $2 \leq i \leq 5$. Then $\Pi^*$ is a connected partition of $V(G)$ as shown in Figure 3.2(b). Observe that

\[
\begin{align*}
c_{\Pi^*}(u_1) &= (0, 1, 2, 2), \\
c_{\Pi^*}(u_2) &= (0, 2, 2, 1, 1), \\
c_{\Pi^*}(u_3) &= (0, 2, 2, 2, 2), \\
c_{\Pi^*}(v_1) &= (0, 2, 2, 3, 3), \\
c_{\Pi^*}(w_1) &= (0, 3, 3, 2, 2).
\end{align*}
\]

Thus $\Pi^*$ is a connected resolving partition of $V(G)$ and so $\text{cpd}(G) \leq |\Pi^*| = 5$. In fact, $G$ contains no connected resolving partition of cardinality 4 and so $\text{cpd}(G) = 5$. Therefore, no minimum connected resolving partition of $V(G)$ is a refinement of any minimum resolving partition of $G$.

### 3.3 Preliminary Results

We have seen that if $G$ is a connected graph of order $n \geq 2$, then $2 \leq \text{cpd}(G) \leq n$. We now present improved upper and lower bounds for the connected partition dimension of a connected graph in terms of its order and diameter. For integers $n$ and $d$ with $1 \leq d < n$, we define $g(n, d)$ as the least positive integer $k$ for which $kd^{k-1} \geq n$. Thus $g(n, 1) = n$ for all $n \geq 2$.

**Theorem 3.4** If $G$ is a connected graph of order $n \geq 3$ and diameter $d$, then

\[
g(n, d) \leq \text{cpd}(G) \leq n - d + 1.
\]
Proof. We first establish the upper bound. If \( d = 1 \), then \( G = K_n \) and \( \text{cpd}(K_n) = n \) by Observation 3.1. On the other hand, \( g(n, 1) = n \) and \( n - d + 1 = n \). So the result is true for \( d = 1 \). Thus we may assume that \( d \geq 2 \). Let \( u \) and \( v \) be vertices of \( G \) for which \( d(u, v) = d \) and let \( u = v_1, v_2, \ldots, v_{d+1} = v \) be a \( u - v \) path of length \( d \). Assume that \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Then the partition \( \Pi = \{S_1, S_2, \ldots, S_{n-d+1}\} \) of \( V(G) \), where \( S_1 = \{v_1, v_2, \ldots, v_d\} \) and \( S_i = \{v_{i+d-1}\} \) for \( 2 \leq i \leq n - d + 1 \) is a connected resolving \((n - d + 1)\)-partition of \( V(G) \). Therefore, \( \text{cpd}(G) \leq n - d + 1 \).

Next we verify the lower bound. Suppose that \( \text{cpd}(G) = k \) and that \( \Pi \) is a connected resolving \( k \)-partition of \( V(G) \). Since (1) each code of a vertex with respect to \( \Pi \) is a \( k \)-vector whose coordinates are nonnegative integers not exceeding \( d \) with exactly one zero coordinate and (2) all \( n \) codes are distinct, it follows that \( kd^{k-1} \geq n \). Thus \( g(n, d) \leq k = \text{cpd}(G) \).

Note that the upper and lower bounds given in Theorem 3.4 can be attained. Consider the graphs \( G_1 \) and \( G_2 \) of Figure 3.3. It can be verified that \( \text{cpd}(G_i) = 3 \) for \( i = 1, 2 \). A \( cr \)-partition in each of \( G_1 \) and \( G_2 \) is also shown in Figure 3.3. The graph \( G_1 \) has order \( n = 5 \) and \( \text{diam } G_1 = 3 \). Thus \( \text{cpd}(G_1) = 3 = n - \text{diam } G_1 + 1 \), attaining the upper bound. On the other hand, The graph \( G_2 \) has order \( n = 9 \) and \( \text{diam } G_2 = 4 \). Since \( g(9, 4) = 3 \), it follows that \( \text{cpd}(G_2) = g(9, 4) \), attaining the lower bound.

![Figure 3.3: The graphs \( G_1 \) and \( G_2 \)](image)

An argument similar to the one used in the proof of Theorem 3.4 establishes the following result for the partition dimension of a graph.
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**Theorem 3.5** If $G$ is a connected graph of order $n \geq 3$ and diameter $d$, then

$$g(n, d) \leq pd(G) \leq n - d + 1.$$ 

It was shown in [13] that for each integer $n \geq 2$, the path $P_n$ of order $n$ is the only connected graph of order $n$ having partition dimension 2 and that the complete graph $K_n$ is the only connected graph of order $n$ having partition dimension $n$. We show that this is also true for the connected partition dimension of a connected graph.

**Proposition 3.6** Let $G$ be a connected graph of order $n \geq 2$. Then

(a) $\text{cpd}(G) = 2$ if and only if $G = P_n$,

(b) $\text{cpd}(G) = n$ if and only if $G = K_n$.

**Proof.** We first verify (a). Let $P_n : v_1, v_2, \ldots, v_n$, where $n \geq 2$, and let $\Pi = \{S_1, S_2\}$ be the partition of $V(P_n)$ with $S_1 = \{v_1\}$ and $S_2 = \{v_2, v_3, \ldots, v_n\}$. Then $(S_1) = K_1$ and $(S_2) = P_{n-1}$ are connected in $P_n$. Since $c_{\Pi}(v_1) = (0, 1)$ and $c_{\Pi}(v_i) = (i - 1, 0)$ for $2 \leq i \leq n$, it follows that $\Pi$ is a cr-partition of $P_n$ and so $\text{cpd}(P_n) = 2$ by (3.1). For the converse, if $G$ is a connected graph of order $n \geq 2$ with $\text{cpd}(G) = 2$. Then $pd(G) = 2$ by (3.1) and so $G = P_n$, which establishes (a).

Next we verify (b). We have seen that $\text{cpd}(K_n) = n$. On the other hand, if $G$ is not a complete graph, then $\text{diam} G \geq 2$. It then follows from Theorem 3.4 that $\text{cpd}(G) \leq n - 1$.

It was shown in [13] that $pd(K_{r,s}) = r + 1$ if $r = s$ and $pd(K_{r,s}) = \max\{r, s\}$ if $r \neq s$. We now show that this is also true for the connected partition dimension of $K_{r,s}$.

**Proposition 3.7** For positive integers $r, s$,

$$\text{cpd}(K_{r,s}) = \begin{cases} 
  r + 1 & \text{if } r = s \\
  \max\{r, s\} & \text{if } r \neq s.
\end{cases}$$
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Proof. Let $G = K_{r,s}$ with partite sets $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$. By Observation 3.2, it suffices to show that $G$ contains a minimum resolving partition that is connected. For $r = s$, let $\Pi = \{S_1, S_2, \ldots, S_{r+1}\}$, where $S_i = \{u_i, v_i\}$ ($1 \leq i \leq r - 1$), $S_r = \{u_r\}$, and $S_{r+1} = \{v_r\}$. Since $\Pi$ is a connected resolving $(r+1)$-partition of $V(G)$, it follows that if $r = s$, then $\text{cpd}(K_{r,s}) = r + 1$. For $r \neq s$, assume, without loss of generality, that $r > s$. Let $\Pi = \{S_1, S_2, \ldots, S_r\}$, where $S_i = \{u_i, v_i\}$ ($1 \leq i \leq s$) and $S_i = \{u_i\}$ ($s + 1 \leq i \leq r$). Since $\Pi$ is a connected resolving $r$-partition of $V(G)$, it follows that $\text{cpd}(K_{r,s}) = r = \max\{r, s\}$.

Thus, if $G$ is a path, a complete graph, or a complete bipartite graph, then $\text{pd}(G) = \text{cpd}(G)$. This observation yields the following.

Corollary 3.8 For each integer $k \geq 2$, there is a connected graph $G$ with

$$\text{pd}(G) = \text{cpd}(G) = k.$$ 

Note that every graph $G$ encountered thus far has the property that $\text{cpd}(G) - \text{pd}(G) \leq 2$. This might lead one to believe that $\text{cpd}(G)$ and $\text{pd}(G)$ are close for every connected graph $G$. However, this is not the case. In fact, as we will see in the next section, the difference $\text{cpd}(G) - \text{pd}(G)$ can be arbitrarily large.

3.4 Connected Partition Dimensions of Trees

The partition dimensions of some special types of trees that are not paths were studied in [12]. There is no general formula, however, for the partition dimension of a tree that is not a path. In this section we present a formula for the connected partition dimension of a tree that is not a path. Recall that a vertex of degree at least 3 in a connected graph $G$ is called a major vertex of $G$. An end-vertex $u$ of $G$ is said to be a terminal vertex of a major vertex $v$ of $G$ if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $G$. The terminal degree $\text{ter}(v)$ of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ of $G$ is an exterior major vertex of $G$ if it has positive terminal degree. Then $\sigma(G)$ denotes the sum
of the terminal degrees of the major vertices of $G$ and $ex(G)$ denotes the number of exterior major vertices of $G$.

We first present a lemma that provides a lower bound for the connected partition dimension of a connected graph $G$ in terms of $\sigma(G)$ and $ex(G)$.

**Lemma 3.9**  If $G$ is a connected graph, then

$$\text{cpd}(G) \geq \sigma(G) - \text{ex}(G) + 1.$$  

**Proof.** Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a connected resolving partition of $G$. Suppose that $G$ contains $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For each $i$ with $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \ldots, u_{ik}$ be the terminal vertices of $v_i$. For each $i$ with $1 \leq i \leq p$, let $P_{ij}$ be the $v_i - u_{ij}$ path in $G$ for $1 \leq j \leq k_i$ and let $x_{ij}$ be a vertex in $P_{ij}$ that is adjacent to $v_i$. Then let $Q_{ij}$ be the $x_{ij} - u_{ij}$ subpath of $P_{ij}$ for $1 \leq i \leq p$ and $1 \leq j \leq k_i$.

Without loss of generality, assume that $v_1 \in S_1$. We claim that at least one vertex $a_{1j}$ from the path $Q_{1j}$ ($1 \leq j \leq k_1$) such that (1) the vertices $a_{1j}$ ($1 \leq j \leq k_1$) belong to distinct elements in $\Pi$ and (2) $a_{1j} \notin S_1$ for $1 \leq j \leq k_1$ with at most one exception. Assume, to the contrary, that this is not the case. We may assume that $V(Q_{11})$ and $V(Q_{12})$ are contained in the same element of $\Pi$. Since $d(x_{11}, v) = d(x_{12}, v)$ for all $v \in V(G) - ((V(Q_{11}) \cup V(Q_{12})))$, it follows that $c_{\Pi}(x_{11}) = c_{\Pi}(x_{12})$, which is a contradiction. Thus assume, without loss of generality, that $a_{1j} \in S_j$ for $2 \leq j \leq k_1$. Since $\Pi$ is a connected partition of $V(G)$ and $v_1 \in S_1$, no vertex in $V(G) - \left(\bigcup_{j=2}^{k_1} V(Q_{1j})\right)$ belongs to $S_j$ for $2 \leq j \leq k_1$; for otherwise, the subgraph $\langle S_j \rangle$ cannot be connected in $G$. On the other hand, the vertex $a_{11}$ is either in $S_1$ or in $S_t$ for some integer $t$ with $k_1 + 1 \leq t \leq k$. In either case,

$$k \geq k_1 = (k_1 - 1) + 1.$$  

Next, we consider the exterior major vertex $v_2$. Since $v_2 \notin S_j$ for all $j$ with $2 \leq j \leq k_1$, we assume that $v_2 \in S_\ell$, where $\ell = 1$ or $k_1 + 1 \leq \ell \leq k$. Similarly, at least one vertex $a_{2j}$ from the path $Q_{2j}$ ($1 \leq j \leq k_2$) such that (1) the vertices $a_{2j}$ ($1 \leq j \leq k_2$) belong to distinct elements in $\Pi$ and (2) $a_{2j} \notin S_\ell$ for $1 \leq j \leq k_2$ with
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at most one exception. Thus, we may assume that \( a_{2j} \in S_{j+k_1-1} \) for \( 2 \leq j \leq k_2 \) and \( j + k_1 - 1 \neq \ell \). Then no vertex in \( V(G) - \left( \bigcup_{j=2}^{k_2} V(Q_{2j}) \right) \) belongs to \( S_{j+k_1-1} \) for all \( j \) with \( 2 \leq j \leq k_2 \). Note that either \( S_1 = S_\ell \), or \( S_1 \neq S_\ell \). In either case, all elements \( S_i \), where \( 1 \leq i \leq k_1 + k_2 - 1 \), are distinct elements in \( \Pi \). Thus

\[
k \geq k_1 + k_2 - 1 = (k_1 - 1) + (k_2 - 1) + 1.
\]

Continuing this procedure to the remaining exterior major vertices of \( G \), we obtain

\[
k \geq \left( \sum_{i=1}^{p} (k_i - 1) \right) + 1 = \sigma(G) - \text{ex}(G) + 1.
\]

Therefore, \( \text{cpd}(G) \geq \sigma(G) - \text{ex}(G) + 1 \).

We are now prepared to present a formula for the connected partition dimension of a tree that is not a path.

**Theorem 3.10** If \( T \) is a tree that is not a path, then

\[
\text{cpd}(T) = \sigma(T) - \text{ex}(T) + 1.
\]

**Proof.** By Lemma 3.9, \( \text{cpd}(T) \geq \sigma(T) - \text{ex}(T) + 1 \). Thus it remains to show that \( \text{cpd}(T) \leq \sigma(T) - \text{ex}(T) + 1 \). Let

\[
k = \sigma(T) - \text{ex}(T) + 1.
\]

Suppose that \( T \) contains \( p \) exterior major vertices \( v_1, v_2, \ldots, v_p \). For each \( i \) with \( 1 \leq i \leq p \), let \( u_{i1}, u_{i2}, \ldots, u_{ik_i} \) be the terminal vertices of \( v_i \). For each \( i \) with \( 1 \leq i \leq p \), let \( P_{ij} \) be the \( v_i - u_{ij} \) path in \( T \) for \( 1 \leq j \leq k_i \) and let \( x_{ij} \) be a vertex in \( P_{ij} \) that is adjacent to \( v_i \). Then let \( Q_{ij} \) be the \( x_{ij} - u_{ij} \) subpath of \( P_{ij} \) for \( 1 \leq i \leq p \) and \( 1 \leq j \leq k_i \).

Let \( U = \{v_1, u_{11}, u_{21}, \ldots, u_{p1}\} \) and let \( T_1 \) be the subtree of \( T \) of smallest size such that \( T_1 \) contains \( U \). Let \( S_0 = V(T_1) \) and \( S_{ij} = V(Q_{ij}) \) for \( 1 \leq i \leq p \) and \( 2 \leq j \leq k_i \). Define a \( k \)-partition \( \Pi \) of \( V(T) \) by

\[
\Pi = \{S_0, S_{12}, S_{13}, \ldots, S_{1k_1}, S_{22}, S_{23}, \ldots, S_{2k_2}, \ldots, S_{p2}, S_{p3}, \ldots, S_{pk_p}\}.
\]
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Then II is connected. We now show that II is a resolving partition of $V(T)$. Note that it suffices to show that the vertices of $T$ belonging to same element of II have distinct codes with respect to II. Let $x, y \in V(T)$. We consider two cases.

Case 1. $x, y \in S_0$. Then $d(x, S_{ij}) = d(x, x_{ij})$ and $d(y, S_{ij}) = d(y, x_{ij})$ for all pairs $i, j$ with $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Let

$$B = \{x_{ij} : 1 \leq i \leq p \text{ and } 2 \leq j \leq k_i\}.$$ 

By Theorem C, the set $B$ is a resolving set of $T$ and so $c_B(x) \neq c_B(y)$. Observe that the first coordinate in each of $c_\Pi(x)$ and $c_\Pi(y)$ is 0, the remaining $k - 1$ coordinates of $c_\Pi(x)$ are exactly those of $c_B(x)$, and the remaining $k - 1$ coordinates of $c_\Pi(y)$ are exactly those of $c_B(y)$. Since $c_B(x) \neq c_B(y)$, it follows that $c_\Pi(x) \neq c_\Pi(y)$.

Case 2. $x, y \in S_{ij}$, where $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Then $d(x, S_0) = d(x, v_i)$ and $d(y, S_0) = d(y, v_i)$. Since $x$ and $y$ are two distinct vertices in the $x_{ij} - u_{ij}$ path $Q_{ij}$, it follows that $d(x, v_i) \neq d(y, v_i)$ and so $d(x, S_0) \neq d(y, S_0)$. Therefore, $c_\Pi(x) \neq c_\Pi(y)$.

Therefore, II is a connected resolving $k$-partition of $V(T)$ and so

$$\text{cpd}(T) \leq k = \sigma(T) - \text{ex}(T) + 1,$$

as desired.

To illustrate Theorem 3.10, we consider the tree $T$ of Figure 3.4 with $\sigma(T) = 7$ and $\text{ex}(T) = 3$. The subtree $T_1$ of $T$ that contains $U = \{v_1, u_{11}, u_{21}, u_{31}\}$ and the four subpaths $Q_{12}, Q_{22}, Q_{32}, Q_{33}$ are shown in Figure 3.4, where the subgraph $T_1$ of $T$ is drawn in bold. By Theorem 3.10 the 5-partition $\Pi = \{S_0, S_{12}, S_{22}, S_{32}, S_{33}\}$ of $V(T)$ is a $cr$-partition and so $\text{cpd}(T) = 5 = \sigma(T) - \text{ex}(T) + 1$.

3.5 Graphs With Prescribed Partition Dimension

and Connected Partition Dimension

We have seen that if $G$ is a nontrivial connected graph with $\text{pd}(G) = a$ and $\text{cpd}(G) = b$, then $2 \leq a \leq b$. Also, by Theorems E and 3.6, the nontrivial paths are the only nontrivial connected graphs with partition dimension 2 and
connected partition dimension 2. With the aid of Theorem 3.10, we are now able to show that every pair $a, b$ of integers with $3 \leq a \leq b$ is realizable as the partition dimension and the connected partition dimension, respectively, of some connected graph.

**Theorem 3.11**  For every pair $a, b$ of integers with $3 \leq a \leq b$, there is a connected graph $G$ with $\text{pd}(G) = a$ and $\text{cpd}(G) = b$.

**Proof.** For $a = b$, let $G = K_{1,a}$. Then $\text{pd}(G) = \text{cpd}(G) = a$ by Theorems E and 3.7. Thus we may assume that $3 \leq a < b$. Let $G$ be the graph obtained from the path $P_{b-a+1} : u_1, u_2, \ldots, u_{b-a+1}$ of order $b - a + 1$ by adding $2b - a$ new vertices $v_i$ ($1 \leq i \leq a$) and $w_j, w'_j$ ($1 \leq j \leq b - a$) and joining (1) each vertex $v_i$ to $u_1$ for all $i$ with $1 \leq i \leq a$ and (2) each of $w_j$ and $w'_j$ to $u_{j+1}$ for all $j$ with $1 \leq j \leq b - a$. Thus $G$ is a tree that is not a path.

First, we show that $\text{cpd}(G) = b$. Since $\sigma(G) = 2(b-a)+a$ and $\text{ex}(G) = b-a+1$, it then follows by Theorem 3.10 that

$$\text{cpd}(G) = \sigma(G) - \text{ex}(G) + 1 = [2(b-a)+a] - (b-a+1) + 1 = b.$$
Next, we show that pd(G) = a. Since the vertex u_1 is adjacent to a end-vertices, it follows that pd(G) ≥ a by Observation 3.1. On the other hand, let Π = \{S_1, S_2, \ldots, S_a\} be a partition of V(G), where

\begin{align*}
S_1 &= V(P_{b-a+1}) \cup \{v_1, w_1, w_2, \ldots, w_{b-a}\}, \\
S_2 &= \{v_2, w'_1, w'_2, \ldots, w'_{b-a}\}, \\
S_i &= \{u_i\} \text{ for } 3 \leq i \leq a.
\end{align*}

Observe that

\begin{align*}
c_Π(u_i) &= (0, 1, i, \ldots) \text{ for } 1 \leq i \leq b - a + 1, \\
c_Π(v_1) &= (0, 2, 2, \ldots), \\
c_Π(w_i) &= (0, 2, i + 2, \ldots) \text{ for } 1 \leq i \leq b - a, \\
c_Π(v_2) &= (1, 0, 2, \ldots), \\
c_Π(w'_i) &= (1, 0, i + 2, \ldots) \text{ for } 1 \leq i \leq b - a.
\end{align*}

Thus Π is a resolving partition of V(G) and so pd(G) ≤ |Π| = a. Therefore, pd(G) = a, as desired.

\section*{3.6 Graphs With Connected Partition Dimension n - 1}

We have seen that the complete graph \(K_n\) of order \(n \geq 2\) is the only connected graph of order \(n\) with connected partition dimension \(n\). Thus, if \(G\) is a connected graph of order \(n \geq 3\) that is not a complete graph, then cpd(\(G\)) ≤ \(n - 1\). It was shown in [13] that the graphs \(K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-2})\) are the only connected graphs of order \(n \geq 3\) with partition dimension \(n - 1\). Applying the same technique used in [13], we now show that those graphs are also the only connected graphs of order \(n \geq 3\) with connected partition dimension \(n - 1\). In order to do this, we first present a corollary which is an immediate consequence of Theorem 3.4.

\textbf{Corollary 3.12} \quad \textit{If } G \textit{ is a connected graph of order } n \geq 3 \textit{ and } \text{cpd}(G) = n - 1, \textit{ then } \text{diam } G = 2.
Theorem 3.13  Let $G$ be a connected graph of order $n \geq 3$. Then $\text{cpd}(G) = n - 1$ if and only if $G$ is one of the graphs $K_{1,n-1}$, $K_n - e$, $K_1 + (K_1 \cup K_{n-2})$.

Proof. It is routine to verify that the graphs mentioned in the theorem have connected partition dimension $n - 1$. For the converse, assume that $G$ is a connected graph of order $n \geq 3$ with connected partition dimension $n - 1$. By Corollary 3.12, it follows that the diameter of $G$ is 2. Suppose first that $G$ is bipartite. Since the diameter of $G$ is 2, it follows that $G = K_{r,s}$ for some integers $r$ and $s$ with $n = r + s \geq 3$. By Proposition 3.7, it follows that $G = K_{1,n-1}$.

We now suppose that $G$ is not bipartite. Let $Y$ be the vertex set of a maximum clique of $G$. We show that $|Y| \geq 3$. Since $G$ is not bipartite, $G$ contains an odd cycle. Let $C_{2t+1}$ be the smallest odd cycle in $G$. Since the diameter of $G$ is 2, it follows that $C_{2t+1}$ is $C_3$ or $C_5$. Suppose first that $C_{2t+1} = C_5 : v_1, v_2, v_3, v_4, v_5, v_1$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_4\}$, $S_3 = \{v_5\}$, and $S_i$ (for $4 \leq i \leq n - 2$) consists of a single vertex of $V(G) - \{v_1, v_2, v_3, v_4, v_5\}$. Then each induced subgraph $\langle S_i \rangle$ is connected for all $i$ with $1 \leq i \leq n - 2$. Since $c_{\Pi}(v_1) = (0, 2, 1, \ldots)$, $c_{\Pi}(v_2) = (0, 2, 2, \ldots)$, and $c_{\Pi}(v_3) = (0, 1, 2, \ldots)$, it follows that $\Pi$ is a connected resolving $(n - 2)$-partition of $V(G)$, contradicting $\text{cpd}(G) = n - 1$. Therefore, $C_{2t+1} = C_3$. Since $G$ contains $K_3$ as a subgraph, it follows that $|Y| \geq 3$.

Let $U = V(G) - Y$. Since $G$ is not complete, $|U| \geq 1$. Assume first that $|U| = 1$. Then $G = K_s + (K_1 \cup K_t)$ for some integers $s$ and $t$. Since $G$ is connected and $G$ is not complete, $s \geq 1$ and $t \geq 1$. Let $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V(K_t) = \{v_1, v_2, \ldots, v_t\}$, and $V(K_1) = \{w\}$. We consider two cases.

Case 1. $s \geq t$. Let $\Pi = \{S_1, S_2, \ldots, S_{s+1}\}$, where $S_1 = \{u_i, v_i\}$ ($1 \leq i \leq t$), $S_i = \{u_i\}$ ($t + 1 \leq i \leq s$), and $S_{s+1} = \{w\}$. Since $d(u, w) = 1$ for $u \in V(K_s)$ and $d(v, w) = 2$ for $v \in V(K_t)$, it follows that $\Pi$ is a connected resolving $(s + 1)$-partition of $V(G)$. Hence $\text{cpd}(G) \leq s + 1$. By Observation 3.1, $\text{cpd}(G) \geq s$. However, $\text{cpd}(G) \neq s$, for otherwise $s = n - 1$ and $G = K_n$. Therefore, $\text{cpd}(G) = s + 1$. Since $\text{cpd}(G) = n - 1$, it follows that $s = n - 2$ and $t = 1$. Therefore, $G = K_{n-2} + (K_1 \cup K_1) = K_n - e$. 

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Case 2. \( s < t \). Let \( \Pi = \{S_1, S_2, \ldots, S_{t+1}\} \), where \( S_i = \{u_i, v_i\} \) (1 \( \leq i \leq s \)), \( S_i = \{v_i\} \) (s + 1 \( \leq i \leq t \)), and \( S_{t+1} = \{w\} \), is a connected resolving partition of \( V(G) \). Thus \( \text{cpd}(G) \leq t + 1 \). By Observation 3.1, \( \text{cpd}(G) \geq t \). However, \( \text{cpd}(G) \neq t \), for otherwise \( t = n - 1 \) and \( s = 0 \), implying that \( G \) is disconnected. Therefore, \( \text{cpd}(G) = t + 1 \). Since \( \text{cpd}(G) = n - 1 \), we have \( t = n - 2 \) and \( s = 1 \). Therefore,

\[
G = K_1 + (K_1 \cup K_{n-2}).
\]

Next we assume that \( |U| \geq 2 \). We claim first that \( U \) is an independent set of vertices. Suppose that this is not the case. Then \( U \) contains two adjacent vertices \( u \) and \( w \). Because of the defining property of \( Y \), there exist \( v \in Y \) such that \( uv \notin E(G) \) and \( v' \in Y \) such that \( wv' \notin E(G) \), where \( v \) and \( v' \) are not necessarily distinct. We also consider these two cases.

Case 1. There exists a vertex \( v \in Y \) such that \( uv, wv \notin E(G) \). We consider two subcases.

Subcase 1.1. There exists a vertex \( x \in Y \) that is adjacent to exactly one of \( u \) and \( w \), say \( u \). Since \( |Y| \geq 3 \), there exists a vertex \( y \in Y \) that is distinct from \( v \) and \( x \). Thus \( G \) contains the subgraph shown in Figure 3.5, where dashed lines indicate that the given edge is not present.

Figure 3.5: The subgraph of \( G \) in Subcase 1.1

Let \( \Pi = \{S_1, S_2, \ldots, S_{n-2}\} \), where \( S_1 = \{u, w\} \), \( S_2 = \{v, x\} \), \( S_3 = \{y\} \), and each of remaining sets \( S_i \) (4 \( \leq i \leq n - 2 \)) consists of exactly one vertex from \( V(G) - \{u, w, y, x, v\} \). Then \( \langle S_i \rangle \) is connected for all \( i \) with 1 \( \leq i \leq n - 2 \). Since \( c_\Pi(u) = (0, 1, \ldots), c_\Pi(v) = (2, 0, \ldots), c_\Pi(w) = (0, 2, \ldots) \), and \( c_\Pi(x) = (1, 0, \ldots) \), it follows that \( \Pi \) is a connected resolving \((n - 2)\)-partition of \( V(G) \), contradicting the fact that \( \text{cpd}(G) = n - 1 \). Thus this subcase cannot occur.
Subcase 1.2. Every vertex of $Y$ is adjacent to both $u$ and $w$ or to neither $u$ nor $w$. If $u$ and $w$ are adjacent to every vertex in $Y - \{v\}$, then the vertices of $(Y - \{v\}) \cup \{u, w\}$ are pairwise adjacent, contradicting the defining property of $Y$. Thus, there exists a vertex $y \in Y$ with $y$ distinct from $v$ such that $y$ is adjacent to neither $u$ nor $w$. Since the diameter of $G$ is 2, there is a vertex $x$ of $G$ that is adjacent to both $u$ and $v$ and a vertex $z$ of $G$ that is adjacent to both $y$ and $w$. Since $x$ and $z$ are not necessarily distinct and they do not necessarily belong to $Y$, we have two subcases.

Subcase 1.2.1. $x = z$. Then $G$ contains the subgraph shown in Figure 3.6. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{x, y, w\}$, $S_2 = \{u\}$, $S_3 = \{v\}$, and each of the remaining sets $S_i$ (4 ≤ $i$ ≤ $n - 2$) consists of only one vertex from $V(G) - \{u, w, y, x, v\}$. Then $\langle S_i \rangle$ is connected for all $i$ with 1 ≤ $i$ ≤ $n - 2$. Since $c_{\Pi}(x) = (0, 1, 1, \ldots)$, $c_{\Pi}(y) = (0, 2, 1, \ldots)$, and $c_{\Pi}(w) = (0, 1, 2, \ldots)$, it follows that $\Pi$ is a connected resolving $(n - 2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n - 1$.

Subcase 1.2.2. $x \neq z$. Then $G$ contains the subgraph shown in Figure 3.7. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{u\}$, $S_2 = \{w\}$, $S_3 = \{x, y\}$, $S_4 = \{v, z\}$, and each of the remaining sets $S_i$ (5 ≤ $i$ ≤ $n - 2$) consists of only one vertex from $V(G) - \{v, u, w, x, y, z\}$. Then $\langle S_i \rangle$ is connected for all $i$ with 1 ≤ $i$ ≤ $n - 2$. Since $c_{\Pi}(x) = (1, a, 0, \ldots)$, $c_{\Pi}(y) = (2, 2, 0, \ldots)$, $c_{\Pi}(v) = (2, 2, 1, 0, \ldots)$, and $c_{\Pi}(z) = (b, 1, 1, 0, \ldots)$, where each of $a$ and $b$ is either 1 or 2, it follows that $\Pi$ is a connected resolving $(n - 2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n - 1$. Thus Subcase 1.2 and, in fact, Case 1 cannot occur.
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Figure 3.7: The subgraph of $G$ in Subcase 1.2.2

Case 2. There exist distinct vertices $v$ and $v'$ in $Y$ such that $uv, uv' \notin E(G)$.

For each vertex $y_0$ of $Y$, $y_0$ is adjacent to at least one of $u$ and $w$, for otherwise, we have the conditions of Case 1. Necessarily, then $vw, v'u \in E(G)$. Since $|Y| \geq 3$, there exists a vertex $y$ in $Y$ distinct from $v$ and $v'$. Also, at least one of the edges $yu$ and $yw$ must be present in $G$, say $yu \in E(G)$. Thus $G$ contains the subgraph shown in Figure 3.8.

Figure 3.8: The subgraph of $G$ in Case 2

Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{u,w,y\}$, $S_2 = \{v\}$. $S_3 = \{v'\}$, and each of the remaining sets $S_i$ ($4 \leq i \leq n - 2$) consists of only one vertex from $V(G) - \{u,w,y,v,v'\}$. Since $c_\Pi(u) = (0,2,1,\ldots)$, $c_\Pi(w) = (0,1,2,\ldots)$, and $c_\Pi(y) = (0,1,1,\ldots)$, it follows that $\Pi$ is a connected resolving $(n - 2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n - 1$. Therefore, $U$ is an independent set.

Next we claim the $N(u) = N(w)$ for all $u, w \in U$. It suffices to show that if $uv \in E(G)$, then $vw \in E(G)$. Suppose that $uv \in E(G)$ for some vertex $v$ of $G$. Necessarily $v \in Y$. Assume, to the contrary, that $wv \notin E(G)$. Since $Y$ is the
vertex set of a maximum clique, there exists \( y \in Y \) such that \( uy \notin E(G) \). Since \( G \) is connected and \( U \) is independent, \( w \) is adjacent to some vertex of \( Y \). We consider two cases.

**Case A.** \( w \) is adjacent only to \( y \). Since \( w \) and \( y \) are not adjacent to \( u \), it follows that \( d(w,u) = 3 \), which contradicts the fact that the diameter of \( G \) is 2.

**Case B.** There exists a vertex \( x \) in \( Y \) distinct from \( y \) such that \( wx \in E(G) \). Thus \( G \) contains the subgraph shown in Figure 3.9. Let \( \Pi = \{ S_1, S_2, \ldots, S_{n-2} \} \), where \( S_1 = \{ w, x \} \), \( S_2 = \{ u, v \} \), \( S_3 = \{ y \} \), and each of the remaining sets \( S_i \) \((4 \leq i \leq n - 2)\) consists of only one vertex of \( V(G) - \{ u, w, x, v, y \} \). Then \( \langle S_i \rangle \) is connected for all \( 1 \leq i \leq n - 2 \). Since \( c_{\Pi}(u) = (*, 0, 2, \ldots) \), where * is either 1 or 2, \( c_{\Pi}(v) = (1, 0, 1, \ldots) \), \( c_{\Pi}(w) = (0, 2, 1, \ldots) \), and \( c_{\Pi}(z) = (0, 1, 1, \ldots) \), it follows that \( \Pi \) is a connected resolving \((n - 2)\)-partition of \( V(G) \), contradicting the fact that \( \text{cpd}(G) = n - 1 \).

![Figure 3.9: The subgraph of G in Case B](image)

Therefore, \( V(G) = Y \cup U \), where \( \langle Y \rangle \) is complete, \( U \) is independent, \( |Y| \geq 3 \), \( |U| \geq 2 \), and \( N(u) = N(w) \) for all \( u, w \in U \).

Next we show that for each \( u \in U \), there exists at most one vertex of \( Y \) not contained in \( N(u) \). Suppose, to the contrary, that there are two vertices \( x, y \in Y \) not in \( N(u) \). Let \( u \) be a vertex of \( U \) that is distinct from \( u \). Thus \( wx, wy \notin E(G) \). Since \( G \) is connected, there exists \( z \in Y \) such that \( z \in N(u) = N(w) \). Thus \( G \) contains the subgraph shown in Figure 3.10.

Let \( \Pi = \{ S_1, S_2, \ldots, S_{n-2} \} \), where \( S_1 = \{ y, z, w \} \), \( S_2 = \{ u \} \), \( S_3 = \{ x \} \), and each of the remaining sets \( S_i \) \((4 \leq i \leq n - 2)\) consists of only one vertex of \( V(G) - \{ y, z, w, u, x \} \). Since \( c_{\Pi}(y) = (0, 2, 1, \ldots) \), \( c_{\Pi}(z) = (0, 1, 1, \ldots) \), and
$$c_\Pi(w) = (0, 2, 2, \ldots),$$ it follows that $\Pi$ is a connected resolving $(n - 2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n - 1$. Thus, as claimed, for each $u \in U$, there exists at most one vertex of $Y$ not contained in $N(u)$.

Now either $N(u) = Y$ or $N(u) = Y - \{v\}$ for some $v \in Y$. If $N(u) = Y$, then $G = K_s + \overline{K_t}$ for $s = |Y| \geq 3$ and $t = |U| \geq 2$. If $N(u) = Y - \{v\}$, then $G = K_s + (K_1 \cup \overline{K_t})$, where $V(K_1) = \{v\}$, $s = |Y| - 1 \geq 2$, and $t = |U| \geq 2$. However, $K_s + (K_1 \cup \overline{K_t}) = K_s + \overline{K_{t+1}}$. In either case, $G = K_s + \overline{K_t}$, where $t \geq 3$ and so $s \leq n - 3$. We show, in fact, that $\text{cpd}(G) \neq n - 1$ for every such graph $G$. Let $V(K_s) = \{u_1, u_2, \ldots, u_s\}$ and $V(\overline{K_t}) = \{v_1, v_2, \ldots, v_t\}$. We consider three cases.

**Case I.** $s = t$. Let $\Pi = \{S_1, S_2, \ldots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ (1 $\leq i \leq s - 1$), $S_s = \{u_s\}$, and $S_{s+1} = \{v_s\}$. Since $d(u, v_i) = 1$ ($u \in V(K_s)$) and $d(v, u_i) = 2$ ($v \in V(K_t)$), it follows that $\Pi$ is a connected resolving $(s + 1)$-partition of $V(G)$. Hence $\text{cpd}(G) \leq s + 1 \leq n - 3 + 1 = n - 2$.

**Case II.** $s > t$. Let $\Pi = \{S_1, S_2, \ldots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ (1 $\leq i \leq t - 1$), $S_t = \{u_t\}$ ($t + 1 \leq i \leq s$), and $S_{s+1} = \{v_t\}$. Since $d(u, v_i) = 1$ ($u \in V(K_s)$) and $d(v, u_i) = 2$ ($v \in V(K_t)$), it follows that $\Pi$ is a connected resolving $(s + 1)$-partition of $V(G)$. Hence $\text{cpd}(G) \leq s + 1 \leq n - 3 + 1 = n - 2$.

**Case III.** $s < t$. Let $\Pi = \{S_1, S_2, \ldots, S_s\}$, where $S_i = \{u_i, v_i\}$ (1 $\leq i \leq s$) and $S_t = \{v_t\}$ ($s + 1 \leq i \leq t$). Since $\Pi$ is a connected resolving $t$-partition of $V(G)$, it follows that $\text{cpd}(G) \leq t \leq n - 2$.

Consequently, if $\text{cpd}(G) = n - 1$, then $G$ is one of the graphs $K_{1,n-1}$, $K_{n-1}$, $K_1 + (K_1 \cup K_{n-2})$. □
Chapter 4

Acyclic Resolving Partitions in Graphs

4.1 Introduction

We mentioned in Chapter 1 that, for a connected graph $G$, a partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ is independent if the subgraph $\langle S_i \rangle$ induced by $S_i$ is independent in $G$ for each $i$ ($1 \leq i \leq k$). Partitions of $V(G)$ that are both resolving and independent were studied in [6, 7] and called resolving-colorings since the coloring of $G$ obtained by coloring each vertex of $S_i$ ($1 \leq i \leq k$) the color $i$ is a proper coloring. The resolving-chromatic number $\chi_r(G)$ of $G$ is the minimum $k$ for which there is a resolving, independent $k$-partition of $V(G)$.

Every independent set $S$ of vertices in a graph $G$ has the property that $\langle S \rangle$ is acyclic and many results concerning the chromatic number of $G$ have been extended to partitions of $V(G)$ in which each subset induces an acyclic subgraph. It is natural, therefore, to study resolving partitions in which each subset induces an acyclic subgraph. This is the subject of the current chapter.

For a connected graph $G$, a partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is acyclic if the subgraph $\langle S_i \rangle$ induced by $S_i$ is acyclic in $G$ for each $i$ ($1 \leq i \leq k$). The vertex-arboricity $a(G)$ of $G$ is defined in [8, 9] as the minimum $k$ such that $V(G)$ has an acyclic $k$-partition. If an acyclic partition $\Pi$ of $V(G)$ is also a resolving partition, then $\Pi$ is called a resolving acyclic partition of $G$. The minimum $k$ for which $G$
contains a resolving acyclic $k$-partition is the acyclic partition dimension $\text{apd}(G)$ of $G$. Since every resolving acyclic partition is an acyclic partition, $a(G) \leq \text{apd}(G)$ for each connected graph $G$.

To illustrate these concepts, consider the graph $G$ of Figure 4.1(a). Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{x\}$, $S_2 = \{u\}$, and $S_3 = \{v, y, z\}$, as shown in Figure 4.1(b). Then the corresponding codes of vertices of $G$ are $c_\Pi(u) = (1, 0, 1)$, $c_\Pi(v) = (1, 2, 0)$, $c_\Pi(x) = (0, 1, 1)$, $c_\Pi(y) = (1, 1, 0)$, and $c_\Pi(z) = (2, 1, 0)$. Since the codes of the vertices of $G$ with respect to $\Pi$ are distinct, $\Pi$ is a resolving partition of $G$. Because no 2-partition is a resolving partition of $G$, it follows that $\Pi$ is a minimum resolving partition of $G$ and so $\text{pd}(G) = 3$. However, $\Pi$ is not acyclic since $\langle S_3 \rangle = K_3$. On the other hand, let $\Pi' = \{S_1', S_2', S_3', S_4'\}$, where $S_1' = \{x\}$, $S_2' = \{u\}$, $S_3' = \{v, y\}$, and $S_4' = \{z\}$ as shown in Figure 4.1(c). It can be verified that $\Pi'$ is a resolving acyclic partition of $G$ and no 3-partition is a resolving acyclic partition of $G$. Thus $\text{apd}(G) = 4$.

![Figure 4.1: A graph $G$ with $\text{pd}(G) = 3$ and $\text{apd}(G) = 4$](image)

Relationships among $\text{pd}(G)$, $\text{apd}(G)$, and $\chi_r(G)$ are stated next.

**Observation 4.1** For every connected graph $G$ of order $n \geq 2$,

$$2 \leq \text{pd}(G) \leq \text{apd}(G) \leq \chi_r(G) \leq n.$$ 

Since $\chi_r(G) = 5$ for the graph $G$ of Figure 4.1(a), it follows that $\text{pd}(G) < \text{apd}(G) < \chi_r(G)$ for the graph $G$ of Figure 4.1(a).

### 4.2 Refinements of a Resolving Partition of a Graph
By Proposition 3.3, if we are given a minimum resolving partition \( \Pi \) of a connected graph \( G \), then we can find an acyclic refinement \( \Pi^* \) of \( \Pi \), where \( \Pi^* \) is then necessarily a resolving partition of \( G \). Of course, the partition each of whose elements consist of a single vertex has this property. The refinement \( \Pi^* \) need not be a minimum acyclic resolving partition for \( G \). In fact, it may be the case that no minimum acyclic resolving partition is a refinement of any minimum resolving partition of \( V(G) \). For example, consider the graph \( G \) of Figure 4.2. Let \( U = \{u_1, u_2, u_3, u_4\} \), \( V = \{v_1, v_2, v_3, v_4\} \), and \( W = \{w_1, w_2, w_3, w_4\} \). Since \( \Pi_0 = \{U, V, W\} \) is a resolving partition of \( G \) and \( G \) is not a path, it follows by Theorem E that \( \text{pd}(G) = 3 \).

![Figure 4.2: A graph \( G \) with \( \text{pd}(G) = 3 \)](image)

**Claim** The partition \( \Pi_0 = \{U, V, W\} \) is the unique minimum partition of \( V(G) \).

**Proof of Claim** Let \( \Pi = \{S_1, S_2, S_3\} \) be a minimum partition of \( V(G) \). Now, let \( X = \{x_1, x_2, x_3, x_4\} \) be a set of four vertices of \( G \) such that \( \langle X \rangle = K_4 \). First we make two observations.

1. **At most two elements of \( \Pi \) contain the vertices of \( X \).** If this does not occur, then we may assume, without loss of generality, that \( x_1 \in S_1 \), \( x_2 \in S_2 \), and \( x_3, x_4 \in S_4 \). Then \( c_\Pi(x_3) = c_\Pi(x_4) = (1, 1, 0) \), which is a contradiction.
(2) None of $S_1$, $S_2$, and $S_3$ contain exactly three vertices of $X$. For otherwise, assume, without loss of generality, that $x_1, x_2, x_3 \in S_1$ and $x_4 \in S_2$. Since the codes of $x_1, x_2,$ and $x_3$ with respect to $\Pi$ are distinct and the diameter of $G$ is 3, it follows that \{c_\Pi(x_1), c_\Pi(x_2), c_\Pi(x_3)\} \subseteq \{(0,1,1), (0,1,2), (0,1,3)\}.

Since $\Pi$ is a resolving partition, $c_\Pi(x_1), c_\Pi(x_2),$ and $c_\Pi(x_3)$ are distinct and so \{c_\Pi(x_1), c_\Pi(x_2), c_\Pi(x_3)\} = \{(0,1,1), (0,1,2), (0,1,3)\}. We may assume that $c_\Pi(x_1) = (0,1,1)$. Thus $d(x_1,S_3) = 1$ and so $x_1$ is adjacent to some vertex in $S_3$. However, then, $c_\Pi(x_2) = c_\Pi(x_3) = (0,1,2)$, which is a contradiction.

We now show that each of $U$, $V$, and $W$ is a subset of exactly one of $S_1$, $S_2$, and $S_3$. We show this for $U$ only since the proofs for $V$ and $W$ are the same. Assume, to the contrary, that this is not the case. Then either (a) $U \cap S_i \neq \emptyset$ for all $i$ ($1 \leq i \leq 3$) or (b) $U \cap S_i = \emptyset$ for exactly one $i$ ($1 \leq i \leq 3$). By (1), it follows that (a) cannot occur. By (2), we may assume that $|U \cap S_1| = |U \cap S_2| = 2$. This gives rise to three cases.

Case 1. $u_1, u_2 \in S_1$ and $u_3, u_4 \in S_2$. Since (i) $\langle \{u_1, u_4, w_1, w_2\}\rangle = K_4$, (ii) $u_1 \in S_1$, and (iii) $u_4 \in S_2$, it follows from (1) and (2) that either $w_1 \in S_1$ and $w_2 \in S_2$ or $w_1 \in S_2$ and $w_2 \in S_1$, say the former. Thus $u_1, w_1 \in S_1$ and $u_4, w_2 \in S_2$. Again, since $\langle W \rangle = K_4$, $w_1 \in S_1$, and $w_2 \in S_2$, it then follows by (2) that $W \subseteq S_1 \cup S_2$ and $|W \cap S_1| = |W \cap S_2| = 2$. Since each vertex in $S_1$ is adjacent to at least one vertex in $S_2$ and the diameter of $G$ is 3, it follows that \{c_\Pi(v) : v \in S_1\} \subseteq \{(0,1,1), (0,1,2), (0,1,3)\}. Because $\Pi$ is a resolving partition, the four vertices in $S_1$ must have distinct codes, which is a contradiction.

Case 2. $u_1, u_3 \in S_1$ and $u_2, u_4 \in S_2$.

Case 3. $u_1, u_4 \in S_1$ and $u_2, u_3 \in S_2$.

The proofs of Cases 2 and 3 are similar to the proof of Case 1 and is therefore omitted. Thus $U$ belongs to exactly one element of $\Pi$, say $U \subseteq S_1$. Similarly, each of $V$ and $W$ belongs to exactly one element of $\Pi$. Since $S_i \neq \emptyset$ for all $i$ ($1 \leq i \leq 3$), we may assume that $U \subseteq S_1$, $V \subseteq S_2$, and $W \subseteq S_3$. However, this implies that $S_1 = U$, $S_2 = V$, and $S_3 = W$. This completes the proof of the claim.

Next we show that $\text{apd}(G) = 4$. Since $G$ contains no minimum resolving par-
partition that is acyclic, it follows that \( \text{apd}(G) \geq \text{pd}(G) + 1 = 4 \). On the other hand, the partition \( \Pi^* = \{S_1, S_2, S_3, S_4\} \) of \( V(G) \), where 
\[ S_1 = \{u_1, u_2, v_2, v_3, w_2, w_3\}, \]
\[ S_2 = \{u_3, u_4\}, \]
\[ S_3 = \{v_1, v_4\}, \]
and 
\[ S_4 = \{w_1, w_4\}, \]
is an acyclic resolving partition of \( V(G) \). The partition \( \Pi^* \) is shown in Figure 4.3. Thus \( \text{apd}(G) \leq |\Pi^*| = 4 \) and so \( \text{apd}(G) = 4 \). Since any acyclic refinement of a minimum resolving partition of \( V(G) \) contains at least 6 elements, it follows that no minimum acyclic resolving partition of \( V(G) \) is a refinement of any minimum resolving partition of \( G \).

\[ \text{Figure 4.3: A minimum acyclic resolving partition} \ \Pi^* \ \text{of} \ \ V(G) \]

4.3 Bounds for the Acyclic Partition Dimension of a Graph

We have seen that \( 2 \leq \text{apd}(G) \leq n \) for every nontrivial connected graph \( G \) of order \( n \). We can determine all connected graphs of order \( n \geq 2 \) with acyclic partition dimension 2 or \( n \) by an argument similar to the one used in the proof of Proposition 3.6.

**Theorem 4.2** Let \( G \) be a connected graph of order \( n \geq 2 \). Then

(a) \( \text{apd}(G) = 2 \) if and only if \( G = P_n \).

(b) \( \text{apd}(G) = n \) if and only if \( G = K_n \).
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By Theorem 4.2, if \( G \) is a connected graph of order \( n \geq 3 \) that is neither \( P_n \) nor \( K_n \), then

\[
3 \leq \text{apd}(G) \leq n - 1. \tag{4.1}
\]

However, the bounds in (4.1) can be improved. By an argument similar to the one used in the proof of Theorem 3.4, we have the following. Recall that for integers \( n \) and \( d \) with \( 1 \leq d < n \), the number \( g(n, d) \) is the least positive integer \( k \) for which \( kd^{k-1} \geq n \).

**Theorem 4.3** If \( G \) is a connected graph of order \( n \geq 3 \) and diameter \( d \geq 2 \), then

\[
g(n, d) \leq \text{apd}(G) \leq n - d + 1.
\]

Next, we study relationships between the acyclic partition dimensions of a connected graph and other parameters, including arboricity, partition dimension, dimension, and resolving-chromatic number. In particular, we present bounds for the acyclic partition dimension of a graph in terms of these parameters. First, we present two lemmas, whose proofs are straightforward and are therefore omitted.

**Lemma 4.4** If \( G \) is a connected graph containing a vertex that is adjacent to \( k \) end-vertices of \( G \), then \( \text{apd}(G) \geq k \).

**Lemma 4.5** If \( H \) is an induced subgraph of a nontrivial connected graph \( G \), then \( a(H) \leq a(G) \).

In the next three results, we present bounds for the acyclic partition dimension of a connected graph in terms of (1) its resolving-chromatic number, (2) its partition dimension and arboricity, and (3) its dimension and arboricity.

**Theorem 4.6** For every nontrivial connected graph \( G \),

\[
\frac{\chi_r(G)}{2} \leq \text{apd}(G) \leq \chi_r(G).
\]
Proof. The upper bound follows from Observation 4.1. To verify the lower bound, let $G$ be a nontrivial connected graph with $\text{apd}(G) = k$, and let

$$\Pi = \{S_1, S_2, \ldots, S_k\}$$

be a resolving acyclic partition of $V(G)$. If $\Pi$ is independent, then

$$\chi_r(G) = 2^k = k = \text{apd}(G).$$

It then follows by Observation 4.1 that $\text{apd}(G) = \chi_r(G) > \chi_r(G)/2$. Thus we may assume that $\Pi$ is not independent. If an element $S_i$ of $\Pi$, where $1 \leq i \leq k$, is not independent, then $\langle S_i \rangle$ is acyclic and so $\chi(\langle S_i \rangle) = 2$. Hence $S_i$ can be partitioned into two nonempty independent sets, namely, the two color classes of any proper minimum coloring of $\langle S_i \rangle$. Define a partition $\Pi'$ of $V(G)$ from $\Pi$ by (1) partitioning each nonindependent element of $\Pi$ into two independent subsets and (2) retaining each independent element of $\Pi$. Thus $\Pi'$ is an independent partition of $V(G)$ with at most $2k$ elements. Furthermore, $\Pi'$ is a refinement of $\Pi$. By Proposition 3.3, $\Pi'$ is also a resolving partition of $G$. Therefore,

$$\chi_r(G) = |\Pi'| \leq 2k = 2 \text{apd}(G)$$

and so $\text{apd}(G) \geq \chi_r(G)/2$. ■

In order to study the sharpness of the bounds in Theorem 4.6, we first determine the acyclic partition dimension of connected bipartite graphs.

Theorem 4.7 If $G$ is a connected bipartite graph with partite sets of cardinalities $r$ and $s$, then

$$\text{apd}(G) \leq \begin{cases} r + 1 & \text{if } r = s \\ \max\{r, s\} & \text{if } r \neq s. \end{cases} \quad (4.2)$$

Moreover, equality in (4.2) holds if $G = K_{r,s}$.

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of $G$, where $1 \leq r \leq s$, say. First, we assume that $G \neq K_{r,s}$. There are two cases.
Case 1. \( r = s \). Since \( G \neq K_{r,r} \), we may assume that \( u_{r-1}u_r \in E(G) \) and \( u_ru_r \notin E(G) \). If \( r = 2 \), then \( G = P_4 \) and, by Theorem 4.2, \( \text{apd}(G) = 2 < 3 = r+1 \). If \( r \geq 3 \), let

\[
\Pi = \{S_1, S_2, \ldots, S_r\}
\]

be the partition of \( V(G) \) where \( S_i = \{u_i, v_i\} \) for \( 1 \leq i \leq r-2 \), \( S_{r-1} = \{u_{r-1}, v_{r-1}, u_r\} \), and \( S_r = \{v_r\} \). Observe that \( d(u_i, S_r) \) is odd and \( d(v_i, S_r) \) is even for \( 1 \leq i \leq r-2 \). Furthermore, \( d(u_{r-1}, S_r) = 1 \), \( d(v_{r-1}, S_r) \) is even, and \( d(u_r, S_r) \) is odd but different from 1. Thus the codes of vertices of \( G \) with respect to \( \Pi \) are distinct and so \( \Pi \) is a resolving acyclic partition of \( G \). Hence \( \text{apd}(G) \leq |\Pi| = r \).

Case 2. \( r < s \). Since \( G \neq K_{r,s} \), we may assume that \( s \geq 3 \) and \( u_ru_s \in E(G) \) and \( u_ru_{s-1} \notin E(G) \). Let

\[
\Pi = \{S_1, S_2, \ldots, S_s\}
\]

be the partition of \( V(G) \) where \( S_i = \{u_i, v_i\} \) if \( 1 \leq i \leq r-1 \), \( S_r = \{u_r\} \), \( S_i = \{v_i\} \) if \( r+1 \leq i \leq s-1 \), and \( S_s = \{v_{s-1}, v_s\} \). Observe that \( d(u_i, S_r) \) is odd and \( d(v_i, S_r) \) is even for \( 1 \leq i \leq r-1 \). Also, \( d(v_s, S_r) = 1 \) and \( d(v_{s-1}, S_r) \neq 1 \). Thus, the codes of the vertices of \( G \) with respect to \( \Pi \) are distinct. Therefore, \( \Pi \) is a resolving acyclic partition and \( \text{apd}(G) \leq |\Pi| = s \).

Finally, we show that equality in (4.2) holds for \( K_{r,s} \). It was shown in [13] that \( \text{pd}(K_{r,r}) = r+1 \) and \( \text{pd}(K_{r,s}) = s \) if \( r < s \). It then follows by Observation 4.1 that \( \text{apd}(K_{r,r}) \geq r+1 \) and \( \text{apd}(K_{r,s}) \geq s \) if \( r < s \). On the other hand, if \( r = s \), then the partition

\[
\Pi = \{S_1, S_2, \ldots, S_{r+1}\}
\]

where \( S_i = \{u_i, v_i\} \) for \( 1 \leq i \leq r-1 \), \( S_r = \{u_r\} \), and \( S_{r+1} = \{v_r\} \) is a resolving acyclic partition of \( V(K_{r,r}) \); while if \( r < s \), say, then

\[
\Pi' = \{S'_1, S'_2, \ldots, S'_s\}
\]
where \( S_i' = \{u_i, v_i\} \) for \( 1 \leq i \leq r \) and \( S_i' = \{u_i\} \) for \( r + 1 \leq i \leq s \), is a resolving acyclic partition of \( V(K_{r, s}) \). Thus \( \text{apd}(K_{r, r}) = r + 1 \) and \( \text{apd}(K_{r, s}) = s \) if \( r < s \).

By Theorem 4.6, if \( G \) is a connected graph of order \( n \) with \( \text{apd}(G) = a \) and \( \chi_r(G) = b \), then \( a \geq 2 \) and \( b/2 \leq a \leq b \). On the other hand, it is straightforward to show that, for each pair \( a, b \) of integers with \( a \geq 2 \) and \( b/2 < a \leq b \), there exists a connected graph \( G \) such that \( \text{apd}(G) = a \) and \( \chi_r(G) = b \). It was shown in [6] that if \( G \) is a nontrivial complete graph or complete bipartite graph, then \( \chi_r(G) = n \). Thus, if \( a = b \geq 2 \), then the graph \( K_a \) has the property that \( \text{apd}(G) = \chi_r(G) = a \) by Theorem 4.2. If \( a \geq 2 \) and \( b/2 < a < b \), then \( K_{a, b-a} \) has the property that \( \text{apd}(G) = a \) by Theorem 4.7 and \( \chi_r(G) = b \). These observations yield the following.

**Proposition 4.8** For every pair \( a, b \) of integers with \( a \geq 2 \) and \( b/2 < a \leq b \), there exists a connected graph \( G \) with \( \text{apd}(G) = a \) and \( \chi_r(G) = b \).

However, the following question remains open.

**Problem 4.9** Does there exist a connected graph \( G \) such that \( \chi_r(G) = 2 \text{apd}(G) \)?

**Theorem 4.10** For every nontrivial connected graph \( G \),

\[
\text{pd}(G) \leq \text{apd}(G) \leq a(G) \text{pd}(G).
\]

In particular, if \( G \) is a tree, then \( \text{pd}(G) = \text{apd}(G) \).

**Proof.** The lower bound follows from Observation 4.1. To verify the upper bound, let \( G \) be a nontrivial connected graph with partition dimension \( k \) and vertex-arboricity \( a \). Furthermore, let

\[
\Pi = \{S_1, S_2, \ldots, S_k\}
\]

be a resolving partition of \( V(G) \). If \( \Pi \) is acyclic, then \( \Pi \) is an acyclic resolving partition of \( V(G) \) and so

\[
\text{apd}(G) \leq |\Pi| = k = \text{pd}(G) \leq a(G) \text{pd}(G)
\]
since \( a(G) \geq 1 \). Thus we may assume that \( \Pi \) is not acyclic. Let \( a_i = a(\langle S_i \rangle) \) for \( 1 \leq i \leq k \). So \( 1 \leq a_i \leq a \) by Lemma 4.5. If \( S_i \) is not acyclic \( (1 \leq i \leq k) \), then \( a_i \geq 2 \) and \( S_i \) can be partitioned into \( a_i \) nonempty subsets, each of which is acyclic. Define a partition \( \Pi' \) of \( V(G) \) from \( \Pi \) by (1) partitioning each nonacyclic element \( S \) of \( \Pi \) into \( a(\langle S \rangle) \) acyclic subsets of \( S \) and (2) retaining each acyclic element of \( \Pi \). So \( \Pi' \) is an acyclic partition of \( V(G) \) with at most \( \sum_{i=1}^{k} a_i \leq ak \) elements. Moreover, \( \Pi' \) is a refinement of \( \Pi \). By Proposition 3.3, \( \Pi' \) is also a resolving partition of \( G \). Therefore,

\[
\text{apd}(G) \leq |\Pi'| \leq ak = a(G) \text{pd}(G).
\]

In particular, if \( G \) is a tree, then \( a(G) = 1 \) and so \( \text{apd}(G) = \text{pd}(G) \).

**Theorem 4.11** For every connected graph \( G \),

\[
a(G) \leq \text{apd}(G) \leq a(G) + \text{dim}(G).
\]

**Proof.** We have seen that \( \text{apd}(G) \) is bounded below by \( a(G) \) for every connected graph \( G \). Thus it remains only to verify the upper bound. Let \( \text{dim}(G) = k \), let \( W = \{w_1, w_2, \ldots, w_k\} \) be a resolving set of \( G \), and let \( H = G - W \). Suppose that \( \{V_1, V_2, \ldots, V_{a(H)}\} \) is an acyclic partition of \( V(H) \). Then the partition

\[
\Pi = \{S_1, S_2, \ldots, S_k, V_1, V_2, \ldots, V_{a(H)}\}
\]

of \( V(G) \) where \( S_i = \{w_i\}, 1 \leq i \leq k \), is acyclic. Since the \( k \)-vectors \( c_W(v), v \in V(G), \) are distinct, the codes \( c_{\Pi}(v), v \in V(G), \) are distinct as well. It then follows by Lemma 4.5 that

\[
\text{apd}(G) \leq |\Pi| = a(H) + \text{dim}(G) \leq a(G) + \text{dim}(G),
\]

as desired.}

The *girth* of a graph (with cycles) is the length of its shortest cycle. Next, we provide bounds for the acyclic partition dimension of a connected graph in terms of its order and girth.
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Theorem 4.12  If $G$ is a connected graph of order $n \geq 3$ and girth $\ell \geq 3$, then

$$3 \leq \text{apd}(G) \leq n - \ell + 3.$$  

In particular, if $G$ is a cycle of order $n \geq 3$, then $\text{apd}(G) = 3$.

Proof. Since $\ell \geq 3$, it follows that $G$ is not a path and so $\text{apd}(G) \geq 3$ by Theorem 4.2. To verify the upper bound, let $C_{\ell} : v_1, v_2, \ldots, v_{\ell}, v_1$ be a cycle of length $\ell$ in $G$, let $d = \lfloor \ell/2 \rfloor$, and let

$$\Pi = \{S_1, S_2, \ldots, S_{n-\ell+3}\}$$

be the partition of $V(G)$ where $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3, \ldots, v_d\}$, $S_3 = \{v_{d+1}, v_{d+2}, \ldots, v_{\ell}\}$, and each set $S_i$ ($4 \leq i \leq n - \ell + 3$) consists of exactly one vertex in $V(G) - V(C_{\ell})$. Since $C_{\ell}$ is a cycle of smallest length in $G$, it follows that $\langle S_2 \rangle$ and $\langle S_3 \rangle$ are acyclic, implying that $\Pi$ is acyclic. Furthermore,

$$c_\Pi(v_1) = (0, 1, 1, \ldots),$$
$$c_\Pi(v_i) = (i - 1, 0, \min\{i, d - i + 1\}, \ldots) \text{ for } 2 \leq i \leq d,$$
$$c_\Pi(v_i) = (\min\{i - 1, \ell - i + 1\}, \min\{i - d, \ell - i + 2\}, 0, \ldots) \text{ for } d + 1 \leq i \leq \ell.$$

Since the codes of the vertices of $G$ are distinct, $\Pi$ is a resolving acyclic partition of $V(G)$. Thus $\text{apd}(G) \leq |\Pi| = n - \ell + 3$. Observe that if $G$ is a cycle of order $n$, then $\ell = n$ and so $\text{apd}(G) = 3$ by (4.1).

Since the girth of $K_n$ is 3 and the girth of $C_n$ is $n$, by Theorems 4.2 and 4.12, $\text{apd}(G) = n - \ell + 3$ for $G = K_n$ or $G = C_n$. In fact, $K_n$ and $C_n$ are the only connected graphs $G$ of order $n \geq 3$ and girth $\ell \geq 3$ such that $\text{apd}(G) = n - \ell + 3$, as we show next.

Theorem 4.13  Let $G$ be a connected graph of order $n \geq 3$ and girth $\ell \geq 3$. Then $\text{apd}(G) = n - \ell + 3$ if and only if $G = K_n$ or $G = C_n$.

Proof. We have seen that $\text{apd}(G) = n - \ell + 3$ for $G = K_n$ or $G = C_n$. Thus it remains to verify the converse. Assume that $G$ is a connected graph of order
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$n \geq 3$ with girth $\ell \geq 3$ such that $\text{apd}(G) = n - \ell + 3$. If $\ell = 3$, then $\text{apd}(G) = n$ and, by Theorem 4.2, $G = K_n$. Thus we may assume that $\ell \geq 4$. We show in this case that $G = C_n$.

Assume, to the contrary, that $G \neq C_n$. Let $C_\ell : v_1, v_2, \ldots, v_\ell, v_1$ be a smallest cycle in $G$, where $\ell < n$. Since $G$ is connected and $G \neq C_n$, there exists a vertex $v \in V(G) - V(C_\ell)$ such that $v$ is adjacent to a vertex of $C_\ell$, say $vv_1 \in E(G)$. We consider two cases.

Case 1. $\ell = 4$. Then $G$ contains a subgraph obtained from the 4-cycle $v_1, v_2, v_3, v_4, v_1$ by adding an edge $vv_1$. Since $\ell = 4$, it follows that $vv_2, vv_4 \notin E(G)$; while the edge $vv_3$ may or may not be present. Let

$$\Pi = \{S_1, S_2, \ldots, S_{n-\ell+2}\}$$

be the partition of $V(G)$ where $S_1 = \{v, v_1\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_4\}$, and each set $S_i$ ($4 \leq i \leq n-\ell+2$) consists of exactly one vertex of $V(G) - (V(C_\ell) \cup \{v\})$. Then $\Pi$ is acyclic. Since $d(v, S_3) = 2$, $d(v_1, S_3) = 1$, $d(v_2, S_3) = 2$, and $d(v_3, S_3) = 1$, it follows that $c_\Pi(v) \neq c_\Pi(v_1)$ and $c_\Pi(v_2) \neq c_\Pi(v_3)$. Thus $\Pi$ is an acyclic resolving partition of $G$ and so $\text{apd}(G) \leq |\Pi| = n - \ell + 2$, which is a contradiction. Therefore, $G = C_4$.

Case 2. $\ell \geq 5$. Since $C_\ell$ is a smallest cycle in $G$, it follows that $v$ is adjacent exactly one vertex of $C_\ell$. Let $d = \lceil \ell/2 \rceil$ and let

$$\Pi = \{S_1, S_2, \ldots, S_{n-\ell+2}\}$$

be the partition of $V(G)$ where $S_1 = \{v, v_1\}$, $S_2 = \{v_2, v_3, \ldots, v_d\}$, $S_3 = \{v_{d+1}, v_{d+2}, \ldots, v_\ell\}$, and each set $S_i$ ($4 \leq i \leq n-\ell+2$) consists of exactly one vertex of $V(G) - (V(C_\ell) \cup \{v\})$. Since $C_\ell$ is a smallest cycle in $G$, it follows that $\langle S_2 \rangle$ and $\langle S_3 \rangle$ are acyclic and so $\Pi$ is an acyclic partition of $V(G)$. Since

$$c_\Pi(v) = (0, 2, 2, \ldots),$$
$$c_\Pi(v_1) = (0, 1, 1, \ldots),$$
$$c_\Pi(v_i) = (i - 1, 0, \min\{i, d - i + 1\}, \ldots) \text{ for } 2 \leq i \leq d,$$
$$c_\Pi(v_i) = (\min\{i - 1, \ell - i + 1\}, \min\{i - d, \ell - i + 2\}, 0, \ldots) \text{ for } d + 1 \leq i \leq \ell,$$

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it follows that $\Pi$ is a resolving partition of $G$. Thus, $\text{apd}(G) \leq |\Pi| = n - \ell + 2$, which is a contradiction. Therefore, $G = C_n$.

We have seen that if $G$ is a connected graph of order $n \geq 2$, then $2 \leq \text{pd}(G) \leq \text{apd}(G) \leq \chi_r(G) \leq n$. It was shown in [6, 13] that $\text{pd}(K_n) = \chi_r(K_n) = n$ and $\text{pd}(P_n) = \chi_r(P_n) = 2$. Moreover, for $n \geq 3$, $\text{pd}(C_n) = 3$; while $\chi_r(C_n) = 3$ if $n$ is odd and $\chi_r(C_n) = 4$ if $n$ is even. From Theorems 4.2 and 4.12, we then have the following.

Corollary 4.14 If $G = K_n, P_n$ for $n \geq 2$ or $G = C_n$ for each odd integer $n \geq 3$, then

$$\text{pd}(G) = \text{apd}(G) = \chi_r(G).$$

The clique number of a graph is the maximum order among the complete subgraphs of the graph. We now present a lower bound for the acyclic partition dimension of a connected graph in terms of its clique number.

Theorem 4.15 For a connected graph $G$ with the clique number $\omega$,

$$\text{apd}(G) \geq \left\lceil \frac{\omega}{2} \right\rceil + 1.$$  

Moreover, for each integer $\omega \geq 2$, there exists a connected graph $G_\omega$ having clique number $\omega$ such that $\text{apd}(G_\omega) = \lceil \omega/2 \rceil + 1$.

Proof. Let $\text{apd}(G) = k$ and let $\Pi$ be a resolving acyclic $k$-partition of $G$. Furthermore, let $F$ be a complete subgraph of order $\omega$ in $G$. Since $F$ is complete and $\Pi$ is acyclic, every element of $\Pi$ contains at most two vertices of $F$. Hence $\text{apd}(G) \geq \lceil \omega/2 \rceil$. Next we show that $\text{apd}(G) \geq \lceil \omega/2 \rceil + 1$. Since $\text{apd}(G) \geq 2$, the result holds for $\omega = 2$. So we assume that $\omega \geq 3$. Assume, to the contrary, that $\text{apd}(G) = \lceil \omega/2 \rceil$. Let

$$\Pi^* = \{S_1, S_2, \ldots, S_{\lceil \omega/2 \rceil}\}$$

be a resolving acyclic partition of $V(G)$. Thus, each subset $S_i$ ($1 \leq i \leq \lceil \omega/2 \rceil$) contains at least one vertex of $F$. Moreover, at least one set $S_i$ ($1 \leq i \leq \lceil \omega/2 \rceil$)
contains exactly two vertices of $F$. We may assume, without loss of generality, that $S_1$ consists of two distinct vertices $u$ and $v$ of $F$. Since $d(u, S_i) = d(v, S_i) = 1$ for all $i$ with $2 \leq i \leq \lceil \omega/2 \rceil$, it follows that $c_{\Pi^*}(u) = c_{\Pi^*}(v)$, which is a contradiction. Therefore, $\text{apd}(G) \geq \lceil \omega/2 \rceil + 1$.

For each integer $\omega \geq 2$, let $G_\omega$ be the graph obtained from $K_\omega$, where $V(K_\omega) = \{v_1, v_2, \ldots, v_\omega\}$, by adding the $\lfloor \omega/2 \rfloor$ pendant edges $u_i v_i$ for $1 \leq i \leq \lfloor \omega/2 \rfloor$. Then $\omega(G_\omega) = \omega$. We show that $\text{apd}(G_\omega) = \lceil \omega/2 \rceil + 1$. Since $\text{apd}(G_\omega) \geq \lceil \omega/2 \rceil + 1$, it suffices to show that $\text{apd}(G_\omega) \leq \lceil \omega/2 \rceil + 1$. Let $U = \{u_1, u_2, \ldots, u_{\lfloor \omega/2 \rfloor}\}$. There are two cases.

Case 1. $\omega$ is even. Then $\lceil \omega/2 \rceil = \omega/2$. Let

$$\Pi = \{S_1, S_2, \ldots, S_{\frac{\omega}{2} + 1}\}$$

where $S_i = \{v_i, v_{\omega-i+1}\}$ for $1 \leq i \leq \omega/2$ and $S_{\frac{\omega}{2} + 1} = U$. Then $\Pi$ is acyclic. For each $i$ with $1 \leq i \leq \omega/2$, observe that (1) the $\left(\frac{i}{2} + 1\right)$th coordinate of $c_{\Pi}(v_i)$ is 1, (2) the $\left(\frac{i}{2} + 1\right)$th coordinate of $c_{\Pi}(v_{\omega-i})$ is 2, and (3) the $\left(\frac{i}{2} + 1\right)$th coordinate of $c_{\Pi}(u_i)$ is 0, the $i$th coordinate of $c_{\Pi}(u_i)$ is 1, and the remaining coordinates of $c_{\Pi}(u_i)$ are 2. Thus $\Pi$ is a resolving partition of $V(G_\omega)$. Therefore, $\text{apd}(G_\omega) \leq |\Pi| = \omega/2 + 1 = \lceil \omega/2 \rceil + 1$.

Case 2. $\omega$ is odd. Then $\lceil \omega/2 \rceil = (\omega + 1)/2$. Let

$$\Pi' = \{S'_1, S'_2, \ldots, S'_{\frac{\omega}{2} + 3}\}$$

where $S'_i = \{v_i, v_{\omega-i}\}$ for $1 \leq i \leq (\omega - 1)/2$, $S'_{\frac{\omega}{2} + 1} = U$, and $S'_{\frac{\omega}{2} + 3} = \{v_\omega\}$. Then $\Pi'$ is acyclic. For each $i$ with $1 \leq i \leq (\omega - 1)/2$, observe that (1) the $\left(\frac{\omega+1}{2}\right)$th coordinate of $c_{\Pi'}(u_i)$ is 1, (2) the $\left(\frac{\omega+1}{2}\right)$th coordinate of $c_{\Pi'}(v_{\omega-i})$ is 2, and (3) the $\left(\frac{\omega+1}{2}\right)$th coordinate of $c_{\Pi'}(u_i)$ is 0, the $i$th coordinate of $c_{\Pi'}(u_i)$ is 1, and the remaining coordinates of $c_{\Pi'}(u_i)$ are 2. Thus $\Pi'$ is a resolving acyclic partition of $V(G_\omega)$. Therefore, $\text{apd}(G_\omega) \leq |\Pi'| = (\omega + 3)/2 = \lceil \omega/2 \rceil + 1$.

4.4 Graphs with Prescribed Acyclic Partition Dimension

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We have seen that if \( G \) is a nontrivial connected graph of order \( n \) and diameter \( d \), then \( 2 \leq \text{apd}(G) \leq n-d+1 \). This suggests a question: For which triples \( d, k, n \) of integers with \( 2 \leq k \leq n-d+1 \) and \( 1 \leq d \leq n-1 \), does there exist a connected graph \( G \) of order \( n \) and diameter \( d \) such that \( \text{apd}(G) = k \)? We now study those triples.

**Theorem 4.16** For each triple \( d, k, n \) of integers with \( 2 \leq d \leq n-2 \) and \( 3 \leq (n-d+1)/2 \leq k \leq n-d+1 \), there exists a connected graph of order \( n \) having diameter \( d \) and acyclic partition dimension \( k \).

**Proof.** First, assume that \( k = n - d + 1 \). Let \( G \) be the graph obtained from \( K_{n-d+1} \) and the path \( P_{d-1} : v_1, v_2, \ldots, v_{d-1} \) by joining \( v_1 \) to a vertex \( u \) in \( K_{n-d+1} \). Then the order of \( G \) is \( n \) and the diameter of \( G \) is \( d \). We show that \( \text{apd}(G) = n-d+1 \). Since every element of any resolving acyclic partition of \( V(G) \) contains at most one vertex of \( V(K_{n-d+1}) - \{u\} \) by Lemma 3.1, it follows that \( \text{apd}(G) \geq n - d \). Assume, to the contrary, that \( \text{apd}(G) = n - d - 1 \). Let

\[
V(K_{n-d+1}) - \{u\} = \{u_1, u_2, \ldots, u_{n-d}\}
\]

and let \( \Pi = \{S_1, S_2, \ldots, S_{n-d}\} \) be a resolving acyclic partition of \( V(G) \). Assume, without loss of generality, that \( u_i \in S_i \) and \( u \in S_1 \). However, then, \( c_\Pi(u) = (0,1,1,\ldots,1) = c_\Pi(u_1) \), which is a contradiction. Thus, \( \text{apd}(G) \geq n - d + 1 = k \). On the other hand,

\[
\Pi^* = \{S_1^*, S_2^*, \ldots, S_{n-d+1}^*\},
\]

where \( S_i^* = \{u_i\} \) (\( 1 \leq i \leq n - d \)) and \( S_{n-d+1}^* = \{u\} \cup V(P_{d-1}) \), is a resolving acyclic partition of \( V(G) \) and so \( \text{apd}(G) \leq |\Pi^*| = n-d+1 \). Therefore, \( \text{apd}(G) = n - d + 1 = k \).

Next, assume that \( (n-d+1)/2 \leq k \leq n-d \). We consider two cases.

**Case 1.** \( d = 2 \). Then \( (n-1)/2 \leq k \leq n-2 \). If \( (n+1)/2 \leq k \leq n-2 \), then \( k > n - k \) and \( G = K_{k,n-k} \) has the desired property by Theorem 4.7. If \( (n-1)/2 \leq k < (n+1)/2 \), then \( k = n/2 \) if \( n \) is even and \( k = (n-1)/2 \) if \( n \) is odd. We consider these two subcases.
Subcase 1.1. $k = n/2$. Let $G = (K_{k-1} \cup kK_1) + K_1$. Let $V(K_1) = \{u_1\}$, $V(K_{k-1}) = \{u_2, u_3, \ldots, u_k\}$, and $V(kK_1) = \{v_1, v_2, \ldots, v_k\}$. Since $u_1$ is adjacent to $k$ end-vertices, $\text{apd}(G) \geq k$ by Lemma 3.1. On the other hand,

$$\Pi = \{S_1, S_2, \ldots, S_k\},$$

where $S_i = \{u_i, v_i\} \ (1 \leq i \leq k)$, is a resolving acyclic partition of $V(G)$ and so $\text{apd}(G) = k$.

Subcase 1.2. $k = (n - 1)/2$. Let $G = (P_k \cup kK_1) + K_1$. Let $V(K_1) = \{w\}$, $V(P_k) = \{u_1, u_2, \ldots, u_k\}$, and $V(kK_1) = \{v_1, v_2, \ldots, v_k\}$. Since $w$ is adjacent to $k$ end-vertices, $\text{apd}(G) \geq k$ by Lemma 3.1. On the other hand,

$$\Pi = \{S_1, S_2, \ldots, S_k\},$$

where $S_1 = \{u_1, v_1, w\}$ and $S_i = \{u_i, v_i\} \ (2 \leq i \leq k)$, is a resolving acyclic partition of $V(G)$ and so $\text{apd}(G) = k$.

Case 2. $3 \leq d \leq n - 2$. Again, we consider two subcases.

Subcase 2.1. $k = n - d$. Let $G$ be the graph obtained from the path $P_d : u_1, u_2, \ldots, u_d$ by adding the $n - d$ pendant edges $u_1v_i \ (1 \leq i \leq n - d)$. Then the order of $G$ is $n$ and the diameter of $G$ is $d$. Since $u_1$ is adjacent to $n - d$ end-vertices, it then follows by Lemma 3.1 that $\text{apd}(G) \geq n - d$. On the other hand, let

$$\Pi = \{S_1, S_2, \ldots, S_{n-d}\}$$

be the partition of $V(G)$ where $S_1 = \{u_3, v_1\}$, $S_2 = \{v_2\} \cup (V(P_{n-d} - \{u_3\})$, and $S_i = \{v_i\} \text{ for } 3 \leq i \leq n - d$. Since $\Pi$ is a resolving acyclic partition of $G$, it follows that $\text{apd}(G) \leq |\Pi| = n - d$. Thus $\text{apd}(G) = n - d$.

Subcase 2.2. $k \leq n - d - 1$. Since $k \leq n - d - 1$, it follows that $n - d - k + 1 \geq 2$. Let $G$ be the graph obtained from the path $P_{d-1} : u_1, u_2, \ldots, u_{d-1}$ by adding the $n - d + 1$ pendant edges $u_1v_i \ (1 \leq i \leq k)$, $u_2w_1$, and $u_{d-1}w_j \ (2 \leq j \leq n - d - k + 1)$. Then the order of $G$ is $n$ and the diameter of $G$ is $d$. We show that $\text{apd}(G) = k$. 

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Since \( k \geq (n - d + 1)/2 \), it follows that \( k \geq n - d - k + 1 > n - d - k \). Thus \( \text{apd}(G) \geq k \) by Lemma 3.1. On the other hand, let

\[
\Pi = \{S_1, S_2, \ldots, S_k\},
\]

where \( S_1 = \{v_1, w_1\} \), \( S_2 = \{u_2, v_2, w_2\} \), \( S_3 = (V(P_{d-1}) - \{u_2\}) \cup \{v_3, w_3\} \), \( S_i = \{v_i, w_i\} \) for \( 4 \leq i \leq n - d - k + 1 \), and \( S_i = \{v_i\} \) for \( n - d - k + 2 \leq i \leq k \). Since \( G \) is a tree, it suffices to show that \( \Pi \) is a resolving partition of \( G \). Observe that

1. \( c_\Pi(v_1) = (0, 2, \ldots) \) and \( c_\Pi(w_1) = (0, 1, \ldots) \),
2. \( c_\Pi(u_2) = (1, 0, 1, \ldots) \), \( c_\Pi(v_2) = (2, 0, 1, \ldots) \), and \( c_\Pi(w_2) = (2, 0, 2, \ldots) \) if \( d = 3 \) and \( c_\Pi(w_2) = (d - 1, 0, 1, \ldots) \) if \( d \geq 4 \),
3. \( c_\Pi(v_3) = (2, 2, 0, \ldots) \), \( c_\Pi(w_3) = (2, 1, 0, \ldots) \) if \( d = 3 \) and \( c_\Pi(w_3) = (d - 1, 2, 0, \ldots) \) if \( d \geq 4 \), and \( c_\Pi(u_1) = (1,*,0,\ldots) \) and \( c_\Pi(u_i) = (i - 1,*,0,\ldots) \) for \( 3 \leq i \leq d - 1 \),
4. \( c_\Pi(v_i) = (2,*,1,\ldots) \), \( c_\Pi(w_i) = (2,*,2,\ldots) \) if \( d = 3 \) and \( c_\Pi(w_i) = (d - 1,*,1,\ldots) \) if \( d \geq 4 \), where \( * \) represents an irrelevant coordinate for \( 4 \leq i \leq n - d - k + 1 \).

Thus all codes of the vertices of \( G \) are distinct, and so \( \Pi \) is resolving, implying that \( \text{apd}(G) \leq |\Pi| = k \). Therefore, \( \text{apd}(G) = k \).

By Theorem 4.16, for each triple \( d, k, n \) of integers with \( 2 \leq d \leq n - 2 \) and \( 3 \leq (n - d + 1)/2 \leq k \leq n - d + 1 \), there exists a connected graph of order \( n \) having diameter \( d \) and acyclic partition dimension \( k \). On the other hand, for those triples \( d, k, n \) of integers with \( 2 \leq d \leq n - 2 \) and \( 3 \leq k \leq (n - d - 1)/2 \), the following question is open.

**Problem 4.17** For which triples \( d, k, n \) of integers with \( 2 \leq d \leq n - 2 \) and \( 3 \leq k \leq (n - d - 1)/2 \), does there exist a connected graph of order \( n \) having diameter \( d \) and acyclic partition dimension \( k \)?
We have seen that if $G$ is a connected graph with $a(G) = a$ and $apd(G) = b$, then $1 \leq a \leq b$ and $b \geq 2$. Next we study those pairs $a, b$ of integers with $1 \leq a \leq b$ and $b \geq 2$ that are realizable as the vertex-arboricity and acyclic partition dimension of some connected graph. Since trees are the only connected graphs having the vertex-arboricity $1$, it follows that

$$a(T) \neq apd(T)$$

for all trees $T$. Thus we may assume that $a \geq 2$. It is known that if $G$ is a connected graph of order $n$, then $a(G) \leq \lceil n/2 \rceil$. Thus if $apd(G) > \lceil n/2 \rceil$, then $a(G) \neq apd(G)$. On the other hand, we show next that each pair $a, b$ of integers with $2 \leq a \leq b - 1$ is realizable as the vertex-arboricity and acyclic partition dimension of some connected graph.

**Theorem 4.18** For each pair $a, b$ of integers with $2 \leq a \leq b - 1$, there exists a connected graph $G$ with $a(G) = a$ and $apd(G) = b$.

**Proof.** For $a = 2$, let $G = K_{a,b}$. It is known that $a(G) = 2$. Since $a < b$, it follows by Theorem 4.7 that $apd(G) = b$ and so $G$ has the desired properties. For $a \geq 3$, let $G$ be the graph obtained from $K_{2a}$ with $V(K_{2a}) = \{u_1, u_2, \ldots, u_{2a}\}$ by adding the $b + a - 1$ new vertices $v_1, v_2, \ldots, v_a$ and $w_1, w_2, \ldots, w_{b-1}$, joining each vertex $v_i$ to $u_i$ ($1 \leq i \leq a$) and joining each vertex $w_j$ ($1 \leq j \leq b - 1$) to $u_a$. Thus $a(G) = a$.

Next, we show that $apd(G) = b$. Since $u_a$ is adjacent to $b$ end-vertices, namely, $v_a$ and $w_j$ ($1 \leq j \leq b - 1$), it then follows by Lemma 3.1 that $apd(G) \geq b$. On the other hand, let

$$\Pi = \{S_1, S_2, \ldots, S_b\}$$

be a partition of $V(G)$, where $S_i = \{u_i, u_{a+i}, w_i\}$, $1 \leq i \leq a$, $S_i = \{w_i\}$, $a + 1 \leq i \leq b - 1$, and $S_b = \{v_1, v_2, \ldots, v_a\}$. Thus $\Pi$ is acyclic. Notice that

(1) $d(u_i, S_j) = 1$ for $1 \leq i \neq j \leq a$ and $d(u_i, S_b) = 1$, 

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(2) \( d(u_{a+i}, S_j) = 1 \) for \( 1 \leq i \neq j \leq a \) and \( d(u_{a+i}, S_b) = 2 \), and

(3) \( d(w_i, S_j) = 2 \) for \( 1 \leq i \neq j \leq a \) and \( j \neq a \) and \( d(w_i, S_a) = 1 \) for \( 1 \leq i \leq a \).

Since \( a \geq 3 \), it follows that \( c_\Pi(u_i) \), \( c_\Pi(u_{a+i}) \), and \( c_\Pi(w_i) \) are distinct for \( 1 \leq i \leq a \). Furthermore, the \( i \)th coordinate of \( c_\Pi(v_i) \) is 1, the \( b \)th coordinate of \( c_\Pi(v_i) \) is 0, and the remaining coordinates of \( c_\Pi(v_i) \) are 2 or 3, implying that the codes \( c_\Pi(v_i) \), \( 1 \leq i \leq a \), are distinct. Hence \( \Pi \) is a resolving acyclic partition of \( V(G) \) and so \( \text{apd}(G) \leq |\Pi| = b \). Therefore, \( \text{apd}(G) = b \).

Since we know of no connected graph \( G \) with \( a(G) = \text{apd}(G) \), the following question is suggested.

**Problem 4.19** Does there exist a connected graph \( G \) such that \( a(G) = \text{apd}(G) \)?

4.5 Graphs with Acyclic Partition Dimension \( n - 1 \)

In this section, we determine all nontrivial connected graphs of order \( n \) with acyclic partition dimension \( n - 1 \). In order to do this, we first present a consequence of Theorems 4.3 and 4.12.

**Corollary 4.20** If \( G \) is a connected graph of order \( n \geq 3 \) with \( \text{apd}(G) = n - 1 \), then the diameter of \( G \) is 2 and the girth of \( G \) is at most 4.

If \( n = 3 \), then \( G = P_3 \) or \( G = K_3 \). Since \( \text{apd}(P_3) = 2 \) and \( \text{apd}(K_3) = 3 \), it follows that \( P_3 \) is the only connected graph of order 3 with acyclic partition dimension 2. If \( n = 4 \), then, by Theorem 4.2 and (4.1), any connected graph \( G \) of order 4 such that \( G \neq P_4, K_4 \) has \( \text{apd}(G) = 3 \). For \( n \geq 5 \), we have the following characterization. We employ the proof technique used in the proof of Theorem 3.13,

**Theorem 4.21** Let \( G \) be a connected graph of order \( n \geq 5 \). Then \( \text{apd}(G) = n - 1 \) if and only if

\[
G \in S = \{C_4 + K_1, K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-2})\}.
\]
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Proof. It is straightforward to verify that the graphs belonging to the set $S$ have acyclic partition dimension $n - 1$. For the converse, assume that $G$ is a connected graph of order $n \geq 5$ with acyclic partition dimension $n - 1$. If $G$ is bipartite, then the diameter of $G$ is 2 by Corollary 4.20 and so $G = K_{r,s}$ for some integers $r$ and $s$ with $n = r + s \geq 5$. It then follows by Theorem 4.7 that $G = K_{1,n-1}$. If $G$ is not bipartite, let $Y$ be the vertex set of a maximum clique of $G$. Since $G$ is not bipartite, $G$ contains an odd cycle $C$. By Corollary 4.20, the girth of $G$ is at most 4 and so $C = C_3$. Therefore, $|Y| \geq 3$. Let $U = V(G) - Y$ and then $|U| \geq 1$ since $G$ is not complete.

Assume first that $|U| = 1$. Then $G = K_s + (K_1 \cup K_t)$ for some integers $s$ and $t$. Since $G$ is connected and $G$ is not complete, $s \geq 1$ and $t \geq 1$. Let $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V(K_t) = \{v_1, v_2, \ldots, v_t\}$, and $V(K_1) = \{w\}$. If $s \geq t$, let

$$
\Pi = \{S_1, S_2, \ldots, S_{s+1}\},
$$

where $S_i = \{u_i, v_i\}$ (1 ≤ $i$ ≤ $t$), $S_i = \{u_i\}$ ($t + 1 \leq i \leq s$), and $S_{s+1} = \{w\}$. Then $\Pi$ is acyclic. Since $d(u, w) = 1$ for $u \in V(K_s)$ and $d(v, w) = 2$ for $v \in V(K_t)$, it follows that $\Pi$ is a resolving acyclic $(s + 1)$-partition of $V(G)$. Hence $apd(G) \leq s + 1$. By Observation 3.1, $apd(G) \geq s$. However, $apd(G) \neq s$, for otherwise $s = n - 1$ and $G = K_n$. Therefore, $apd(G) = s + 1$. Since $apd(G) = n - 1$, it follows that $s = n - 2$ and $t = 1$. Therefore,

$$
G = K_{n-2} + (K_1 \cup K_1) = K_n - e.
$$

For $s < t$, the partition $\Pi = \{S_1, S_2, \ldots, S_{t+1}\}$, where $S_i = \{u_i, v_i\}$ (1 ≤ $i$ ≤ $s$), $S_i = \{v_i\}$ ($s + 1 \leq i \leq t$), and $S_{t+1} = \{w\}$, is a resolving acyclic partition of $V(G)$. Thus $apd(G) \leq t + 1$. By Observation 3.1, $apd(G) \geq t$. However, $apd(G) \neq t$, for otherwise $t = n - 1$ and $s = 0$, implying that $G$ is disconnected. Therefore, $apd(G) = t + 1$. Since $apd(G) = n - 1$, we have $t = n - 2$ and $s = 1$. Therefore,

$$
G = K_1 + (K_1 \cup K_{n-2}).
$$

Next we assume that $|U| \geq 2$. If $n = 5$, then $|Y| = 3$ and $|U| = 2$. It is routine to verify that $C_4 + K_1$ is the only graph with the desired properties. For $n \geq 6$, we
claim that $U$ is an independent set of vertices. Assume, to the contrary, that $U$
contains adjacent vertices $u$ and $w$. Since $Y$ is the vertex set of a maximum clique
of $G$, there exist $v \in Y$ such that $uv \notin E(G)$ and $v' \in Y$ such that $wv' \notin E(G)$,
where $v$ and $v'$ are not necessarily distinct. We consider these two cases.

Case 1. There exists a vertex $v \in Y$ such that $uv, wv \notin E(G)$. We now
consider two subcases.

Subcase 1.1. There exists a vertex $x \in Y$ that is adjacent to exactly one of $u$
and $w$, say $u$. Since $|Y| \geq 3$, there exists a vertex $y \in Y$ that is distinct from $v$
and $x$. Let

$$\Pi = \{S_1, S_2, \ldots, S_{n-2}\},$$

where $S_1 = \{u, w\}$, $S_2 = \{v, x\}$, $S_3 = \{y\}$, and each of remaining sets $S_i$ ($4 \leq i \leq n-2$) contains exactly one vertex from $V(G) - \{u, v, w, x, y\}$. Then $\langle S_i \rangle$ is acyclic
for all $1 \leq i \leq n-2$. Since $c_\Pi(u)=(0,1,\ldots)$, $c_\Pi(v)=(2,0,\ldots)$, $c_\Pi(w)=(0,2,\ldots)$,
and $c_\Pi(x)=(1,0,\ldots)$, it follows that $\Pi$ is a resolving acyclic $(n-2)$-partition of
$V(G)$, a contradiction.

Subcase 1.2. Every vertex of $Y$ is adjacent to either both $u$ and $w$ or to neither
$u$ nor $w$. If $u$ and $w$ are adjacent to every vertex in $Y - \{v\}$, then the induced
subgraph $\langle (Y - \{v\}) \cup \{u, w\}\rangle$ is complete in $G$, contradicting the defining prop­
erty of $Y$. Thus, there exists a vertex $y \in Y$ such that $y$ is distinct from $v$, and $y$
is adjacent to neither $u$ nor $w$. Since the diameter of $G$ is 2, there is a vertex $x$ of
$G$ that is adjacent to both $u$ and $v$. Let

$$\Pi = \{S_1, S_2, \ldots, S_{n-2}\},$$

where $S_1 = \{x, y, w\}$, $S_2 = \{u\}$, $S_3 = \{v\}$, and each of the remaining sets $S_i$
($4 \leq i \leq n-2$) consists of exactly one vertex from $V(G) - \{u, v, w, x, y\}$. Since
$y$ is not adjacent to $w$, it follows that $\langle S_1 \rangle$ is acyclic and so $\Pi$ is acyclic. Since
$c_\Pi(x)=(0,1,1,\ldots)$, $c_\Pi(y)=(0,2,1,\ldots)$, and $c_\Pi(w)=(0,1,2,\ldots)$, it follows that $\Pi$
is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.
Case 2. There exist distinct vertices $v$ and $v'$ in $Y$ such that $uv, wv' \notin E(G)$. Necessarily, $vw, v'u \in E(G)$. Since $|Y| \geq 3$, there exists a vertex $y \in Y$ distinct from $v$ and $v'$. Also, at least one of the edges $yu$ and $yw$ must be present in $G$, say $yu$. If $yw \notin E(G)$, let
\[ \Pi = \{ S_1, S_2, \ldots, S_{n-2} \}, \]
where $S_1 = \{ u, w, y \}$, $S_2 = \{ v \}$, $S_3 = \{ v' \}$, and each of the remaining sets $S_i$ ($4 \leq i \leq n - 2$) consists of exactly one vertex from $V(G) - \{ u, v, v', w, y \}$. Since $yw \notin E(G)$, it follows that $\Pi$ is acyclic. Because $c_\Pi(u) = (0, 2, 1, \ldots)$, $c_\Pi(w) = (0, 1, 2, \ldots)$, and $c_\Pi(y) = (0, 1, 1, \ldots)$, it follows that $\Pi$ is a resolving acyclic $(n - 2)$-partition of $V(G)$, a contradiction. Thus, we assume that $yw \in E(G)$. Since $n \geq 6$, there exists $x \in V(G) - \{ u, v, v', w, y \}$. We consider two subcases, according to whether $x \in Y$ or $x \in U$.

Subcase 2.1. $x \in Y$. Then $\Pi = \{ S_1, S_2, \ldots, S_{n-2} \}$, where $S_1 = \{ v \}$, $S_2 = \{ v' \}$, $S_3 = \{ x, w \}$, $S_4 = \{ u, y \}$, and each of remaining sets $S_i$ ($5 \leq i \leq n - 2$) consists of exactly one vertex from $V(G) - \{ u, v, v', w, x, y \}$. Thus $\Pi$ is acyclic. Since $c_\Pi(x) = (1, 1, 0, \ldots)$, $c_\Pi(w) = (1, 2, 0, \ldots)$, $c_\Pi(u) = (2, 1, 1, 0, \ldots)$, and $c_\Pi(y) = (1, 1, 1, 0, \ldots)$, it follows that $\Pi$ is a resolving acyclic $(n - 2)$-partition of $V(G)$, a contradiction.

Subcase 2.2. $x \in U$. Then there exists $y' \in Y$ that is not adjacent to $x$, for otherwise, $x \in Y$. We consider four subcases, according to whether $y' \in \{ v, v', y \}$ or $y' \notin \{ v, v', y \}$.

Subcase 2.2.1. $y' = v$. Let $\Pi = \{ S_1, S_2, \ldots, S_{n-2} \}$, where $S_1 = \{ v \}$, $S_2 = \{ u, v' \}$, $S_3 = \{ x, w \}$, $S_4 = \{ y \}$, and each of remaining sets $S_i$ ($5 \leq i \leq n - 2$) consists of exactly one vertex from $V(G) - \{ u, v, v', w, x, y \}$. Since $c_\Pi(u) = (2, 0, \ldots)$, $c_\Pi(v') = (1, 0, \ldots)$, $c_\Pi(x) = (2, *, 0, \ldots)$, where $*$ is either 1 or 2, and $c_\Pi(w) = (1, 1, 0, \ldots)$, it follows that $\Pi$ is a resolving acyclic $(n - 2)$-partition of $V(G)$, a contradiction.

Subcase 2.2.2. $y' = v'$. Let $\Pi = \{ S_1, S_2, \ldots, S_{n-2} \}$, where $S_1 = \{ y \}$, $S_2 = \{ v' \}$, $S_3 = \{ u, w \}$, $S_4 = \{ x, v \}$, and each of remaining sets $S_i$ ($5 \leq i \leq n - 2$) consists
of exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_{\Pi}(u) = (1, 1, 0, \ldots)$, $c_{\Pi}(w) = (1, 2, 0, \ldots)$, $c_{\Pi}(x) = (a, 2, b, 0, \ldots)$, where each of $a$ and $b$ is either 1 or 2, and $c_{\Pi}(v) = (1, 1, 1, \ldots)$, it follows that $\Pi$ is a resolving acyclic $(n - 2)$-partition of $V(G)$, a contradiction.

**Subcase 2.2.3.** $y' = y$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{y\}$, $S_2 = \{v\}$, $S_3 = \{v', x\}$, $S_4 = \{u, w\}$, and each of remaining sets $S_i$ ($5 \leq i \leq n - 2$) consists of exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_{\Pi}(v') = (1, 1, 0, \ldots)$, $c_{\Pi}(x) = (2, *, 0, \ldots)$, $c_{\Pi}(u) = (1, 2, 1, 0, \ldots)$, and $c_{\Pi}(w) = (1, 1, *, \ldots)$, where * is either 1 or 2, it follows that $\Pi$ is a resolving acyclic $(n - 2)$-partition of $V(G)$, a contradiction.

**Subcase 2.2.4.** $y' \notin \{v, v', y\}$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{v\}$, $S_2 = \{v'\}$, $S_3 = \{y'\}$, $S_4 = \{u, y\}$, and each of remaining sets $S_i$ ($5 \leq i \leq n - 2$) consists of exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_{\Pi}(y') = (1, 1, 0, \ldots)$, $c_{\Pi}(w) = (1, 2, 0, \ldots)$, $c_{\Pi}(u) = (2, 1, 1, 0, \ldots)$, and $c_{\Pi}(y) = (1, 1, 1, 0, \ldots)$, it follows that $\Pi$ is a resolving acyclic $(n - 2)$-partition of $V(G)$, a contradiction.

Therefore, in any case, $U$ is an independent set. Next we claim that $N(u) = N(w)$ for all $u, w \in U$. It suffices to show that if $uw \in E(G)$, then $vw \in E(G)$. Suppose that $uv \in E(G)$ for some vertex $v$ of $G$. Necessarily $v \in Y$. Assume, to the contrary, that $vw \notin E(G)$. Since $Y$ is the vertex set of a maximum clique, there exists $y \in Y$ such that $uy \notin E(G)$. Since $G$ is connected and $U$ is independent, $w$ is adjacent to some vertex of $Y$. First, assume that $w$ is adjacent only to $y$. Since $w$ and $y$ are not adjacent to $u$, it follows that $d(w, u) = 3$, which contradicts the fact that the diameter of $G$ is 2. Thus, there exists a vertex $x$ in $Y$ distinct from $y$ such that $wx \in E(G)$. Let

$$\Pi = \{S_1, S_2, \ldots, S_{n-2}\},$$

where $S_1 = \{w, x\}$, $S_2 = \{u, v\}$, $S_3 = \{y\}$, and each of the remaining sets $S_i$ ($4 \leq i \leq n - 2$) consists of exactly one vertex of $V(G) - \{u, w, x, v, y\}$. Then $\langle S_i \rangle$ is acyclic for all $i$ ($1 \leq i \leq n - 2$). Since $c_{\Pi}(u) = (\ast, 0, 2, \ldots)$, where $\ast$ is either 1 or 2, $c_{\Pi}(v) = (1, 0, 1, \ldots)$, $c_{\Pi}(w) = (0, 2, 1, \ldots)$, and $c_{\Pi}(x) = (0, 1, 1, \ldots)$, it follows...
that $\Pi$ is a resolving acyclic $(n-2)$-partition of $V(G)$, contradicting the fact that $\text{apd}(G) = n-1$.

Thus far, we have, for $n \geq 6$, $V(G) = Y \cup U$, where $\langle Y \rangle$ is complete, $U$ is independent, $|Y| \geq 3$, $|U| \geq 2$, and $N(u) = N(w)$ for all $u, w \in U$. Next we show that for each $u \in U$, there exists at most one vertex of $Y$ not contained in $N(u)$. Assume, to the contrary, that there are two vertices $x, y \in Y$ not in $N(u)$. Let $w$ be a vertex of $U$ that is distinct from $u$. Thus $wx, wy \notin E(G)$. Since $G$ is connected, there exists $z \in Y$ such that $z \in N(u) = N(w)$. Let

$$\Pi = \{S_1, S_2, \ldots, S_{n-2}\},$$

where $S_1 = \{y, z, w\}$, $S_2 = \{u\}$, $S_3 = \{x\}$, and each of the remaining sets $S_i$ ($4 \leq i \leq n-2$) consists of exactly one vertex of $V(G) - \{y, z, w, u, x\}$. Since $wy \notin E(G)$, it follows that $\langle S_1 \rangle$ is acyclic and so $\Pi$ is acyclic. Since $c_{\Pi}(y) = (0, 2, 1, \ldots)$, $c_{\Pi}(z) = (0, 1, 1, \ldots)$, and $c_{\Pi}(w) = (0, 2, 2, \ldots)$, it follows that $\Pi$ is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Now either $N(u) = Y$ or $N(u) = Y - \{v\}$ for some $v \in Y$. If $N(u) = Y$, then $G = K_s + \overline{K}_t$ for $s = |Y| \geq 3$ and $t = |U| \geq 2$. If $N(u) = Y - \{v\}$, then $G = K_s + (K_1 \cup \overline{K}_t)$, where $V(K_1) = \{v\}$, $s = |Y| - 1 \geq 2$, and $t = |U| \geq 2$. However,

$$K_s + (K_1 \cup \overline{K}_t) = K_s + \overline{K}_{t+1}.$$

In either case, $G = K_s + \overline{K}_t$, where $t \geq 2$. If $t = 2$, then $G = K_{n-2} + \overline{K}_2 = K_n - e$, which implies that $\text{apd}(G) = n - 1$. Thus we may assume that $t \geq 3$ and so $s \leq n - 3$. Let $V(K_s) = \{u_1, u_2, \ldots, u_s\}$ and $V(\overline{K}_t) = \{v_1, v_2, \ldots, v_t\}$. If $s = t$, let

$$\Pi = \{S_1, S_2, \ldots, S_{s+1}\},$$

where $S_i = \{u_i, v_i\}$ ($1 \leq i \leq s - 1$), $S_s = \{u_s\}$, and $S_{s+1} = \{v_s\}$. Since $d(u, v_s) = 1$, where $u \in V(K_s)$, and $d(v, v_s) = 2$, where $v \in V(K_t)$, it follows that $\Pi$ is a resolving acyclic $(s+1)$-partition of $V(G)$. Hence $\text{apd}(G) \leq s + 1 \leq n - 3 + 1 = n - 2$, which is a contradiction. If $s > t$, let
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\[ \Pi = \{S_1, S_2, \ldots, S_{s+1}\}, \]

where \( S_i = \{u_i, v_i\} \) (1 \( \leq \) i \( \leq \) t - 1), \( S_t = \{u_t\} \) (t \( \leq \) i \( \leq \) s), and \( S_{s+1} = \{v_t\} \). Since \( d(u, v_t) = 1 \), where \( u \in V(K_s) \), and \( d(v, v_t) = 2 \), where \( v \in V(K_t) \), it follows that \( \Pi \) is a resolving acyclic \((s+1)\)-partition of \( V(G) \). Hence \( \text{apd}(G) \leq s + 1 \leq n - 3 + 1 = n - 2 \), which is a contradiction. If \( s < t \), let

\[ \Pi = \{S_1, S_2, \ldots, S_t\}, \]

where \( S_i = \{u_i, v_i\}, 1 \leq i \leq s \), and \( S_i = \{v_i\}, s + 1 \leq i \leq t \). Since \( \Pi \) is a resolving acyclic \( t \)-partition of \( V(G) \), it follows that \( \text{apd}(G) \leq t \leq n - 2 \), which is a contradiction. ■

4.6 Acyclic Partition Dimensions of Cartesian Products of Graphs

Next we show that the acyclic partition dimension of the Cartesian product of \( K_2 \) and a nontrivial connected graph \( H \) is bounded above by \( \text{apd}(H) + a(H) \).

Theorem 4.22  For every nontrivial connected graph \( H \),

\[ \text{apd}(H \times K_2) \leq \text{apd}(H) + a(H) \]

Proof. Let \( G = H \times K_2 \), where \( H_1 \) and \( H_2 \) are the two copies of \( H \) in the construction of \( G \). Suppose that \( \text{apd}(H) = k \) and \( a(H) = a \). Let \( \Pi = \{S_1, S_2, \ldots, S_k\} \) be a resolving acyclic partition of \( V(H_1) \), and let \( \{W_1, W_2, \ldots, W_a\} \) be an acyclic partition of \( V(H_2) \). Then

\[ \Pi^* = \{S_1, S_2, \ldots, S_k, W_1, W_2, \ldots W_a\} \]

is an acyclic partition of \( V(G) \). We show that \( \Pi^* \) is a resolving partition of \( V(G) \). Let \( x \) and \( y \) be vertices of \( G \) such that \( c_{\Pi^*}(x) = c_{\Pi^*}(y) \). We show that \( x = y \). Assume, to the contrary, that \( x \neq y \). We consider three cases.

Case 1. Both \( x \) and \( y \) belong to \( H_1 \). Then \( d_G(x, S_i) = d_{H_1}(x, S_i) \) and \( d_G(y, S_i) = d_{H_1}(y, S_i) \) (1 \( \leq \) i \( \leq \) k). Since \( d_G(x, S_i) = d_G(y, S_i) \) for all \( 1 \leq i \leq k \),

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it follows that $d_{H_1}(x, S_i) = d_{H_1}(y, S_i)$ for $1 \leq i \leq k$ and so $c_\Pi(x) = c_\Pi(y)$, contradicting the fact that $\Pi$ is a resolving acyclic partition of $H_1$.

**Case 2.** Both $x$ and $y$ belong to $H_2$. Let $x'$ and $y'$ be the vertices of $H_1$ that correspond to $x$ and $y$ in $H_2$, respectively. Since $x \neq y$, we have $x' \neq y'$. Notice that

$$d_G(x, S_i) = d_G(x', S_i) + 1 = d_{H_1}(x', S_i) + 1$$

and

$$d_G(y, S_i) = d_G(y', S_i) + 1 = d_{H_1}(y', S_i) + 1 \text{ for } 1 \leq i \leq k.$$ 

Thus

$$d_{H_1}(x', S_i) = d_G(x, S_i) - 1 \text{ and } d_{H_1}(y', S_i) = d_G(y, S_i) - 1 \text{ for } 1 \leq i \leq k.$$ 

Since $d_G(x, S_i) = d_G(y, S_i)$ for $1 \leq i \leq k$, it follows that $d_{H_1}(x', S_i) = d_{H_1}(y', S_i)$ for $1 \leq i \leq k$ and so $c_\Pi(x') = c_\Pi(y')$, a contradiction.

**Case 3.** Either $x \in V(H_1)$ and $y \in V(H_2)$, or $x \in V(H_2)$ and $y \in V(H_1)$, say the former. Suppose that $y \in W_i$, where $1 \leq i \leq a$ and so $d_G(y, W_i) = 0$. However, $x \notin V(H_2)$ and so $x \notin W_i$, which implies that $d_G(x, W_i) > 0$. Thus $c_\Pi(x) \neq c_\Pi(y)$, contradicting the assumption.

Therefore, $\Pi^*$ is a resolving acyclic partition of $V(G)$ and so

$$\text{apd}(G) \leq |\Pi^*| = \text{apd}(H) + a(H),$$

as desired. ■

Equality in Theorem 4.22 can hold. For example, if $H = P_n$, where $n \geq 2$, then $\text{apd}(P_n) = 2$ and $a(P_n) = 1$. By Theorem 4.2, $\text{apd}(P_n \times K_2) \geq 3$. On the other hand, let $H_1 : u_1, u_2, \ldots, u_n$ and $H_2 : v_1, v_2, \ldots, v_n$ be two copies of $P_n$ in $P_n \times K_2$. Since $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = V(H_1)$, $S_2 = \{v_1\}$, and $S_3 = V(H_2) - \{v_1\}$, is a resolving acyclic partition of $V(P_n \times K_2)$, it follows that $\text{apd}(P_n \times K_2) = 3$. Therefore, for $n \geq 2,$
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$$\text{apd}(P_n \times K_2) = 3 = \text{apd}(P_n) + \alpha(P_n).$$

Strict inequality in Theorem 4.22 can also hold. To illustrate this, we study \(\text{apd}(K_n \times K_2)\) for \(n \geq 3\), beginning with \(n = 3, 4\).

**Proposition 4.23** \(\text{apd}(K_3 \times K_2) = 3\) and \(\text{apd}(K_4 \times K_2) = 4\).

**Proof.** For \(n = 3, 4\), let \(H_1\) and \(H_2\) be two copies of \(K_n\) in \(G\), where \(V(H_1) = \{u_1, u_2, \ldots, u_n\}\), \(V(H_2) = \{v_1, v_2, \ldots, v_n\}\), and \(u_iv_i \in E(G)\) for \(1 \leq i \leq n\). For \(n = 3\), let \(\Pi = \{S_1, S_2, S_3\}\), where \(S_1 = \{u_1, u_2, v_1\}\), \(S_2 = \{v_2, v_3\}\), and \(S_3 = \{u_3\}\). Since \(\Pi\) is a resolving acyclic partition, it follows by Theorem 4.2 that \(\text{apd}(K_3 \times K_2) = 3\).

For \(n = 4\), let \(\Pi^* = \{S^*_1, S^*_2, S^*_3, S^*_4\}\) be a partition of \(V(G)\), where \(S^*_1 = \{u_1, u_3\}\), \(S^*_2 = \{u_2, v_2, v_3\}\), \(S^*_3 = \{v_1, v_4\}\), and \(S^*_4 = \{u_4\}\). Since \(\Pi^*\) is a resolving acyclic partition, \(\text{apd}(G) \leq |\Pi^*| = 4\). Assume, to the contrary, that \(\text{apd}(G) = 3\). Let \(\Pi = \{S_1, S_2, S_3\}\) be a minimum resolving partition of \(G\). Since \(\langle S_i \rangle, 1 \leq i \leq 3\), is acyclic, \(S_i\) contains at most two vertices of \(H_1\). Thus at least two of \(S_1, S_2,\) and \(S_3\) contain vertices of \(H_1\).

Suppose that each of \(S_1, S_2,\) and \(S_3\) contains some vertex of \(H_1\). Then assume, without loss of generality, that \(u_i \in S_i\) for \(1 \leq i \leq 3\). Since \(\Pi\) is a partition of \(V(G)\), it follows that \(u_i \notin S_j\) for some \(i\) with \(1 \leq i \leq 3\), say \(u_4 \notin S_1\). Then \(c_\Pi(u_1) = c_\Pi(u_4) = (0, 1, 1)\), a contradiction. Thus exactly two of \(S_1, S_2,\) and \(S_3\) contains vertices of \(H_1\), say \(S_1\) and \(S_2\) contain vertices of \(H_1\). Moreover, each of \(S_1\) and \(S_2\) contains exactly two vertices of \(H_1\). We may assume that \(u_1, u_2 \in S_1\) and \(u_3, u_4 \in S_2\). Similarly, exactly two of \(S_1, S_2,\) and \(S_3\) contains vertices of \(H_2\), each of which contains exactly two vertices of \(H_2\). Since \(S_3 \neq \emptyset\) and \(S_3 \cap V(H_1) = \emptyset\), it follows that \(S_3\) consists of exactly two vertices of \(H_2\). This implies that exactly one of \(S_1\) and \(S_2\) contains the two vertices in \(V(H_2) - S_3\), say \(V(H_2) - S_3 \subset S_1\) and so \(S_2 = \{u_3, u_4\}\). Since \(\langle S_1 \rangle\) is acyclic and \(u_1, u_2 \in S_1\), it follows that \(S_1\) contains at most one of \(v_1\) and \(v_2\). We consider two cases.

**Case 1.** \(S_1\) contains exactly one of \(v_1\) and \(v_2\), say \(v_1 \in S_1\). Thus \(S_1 = \{u_1, u_2, v_1, v_j\}\), where \(j = 3\) or \(j = 4\). If \(S_1 = \{u_1, u_2, v_1, v_3\}\), then \(S_2 = \{u_3, u_4\}\).
and \( S_3 = \{v_2, v_4\} \). Thus \( c_\Pi(u_2) = c_\Pi(v_3) = (0, 1, 1) \), a contradiction. If \( S_1 = \{u_1, u_2, v_1, v_4\} \), then \( S_2 = \{u_3, u_4\} \) and \( S_3 = \{v_2, v_3\} \). Thus \( c_\Pi(u_2) = c_\Pi(v_4) = (0, 1, 1) \), a contradiction.

**Case 2.** \( S_1 \) contains neither \( v_1 \) nor \( v_2 \). Thus \( S_1 = \{u_1, u_2, v_3, v_4\} \), \( S_2 = \{u_3, u_4\} \), and \( S_3 = \{v_1, v_2\} \). Then \( c_\Pi(u_1) = c_\Pi(u_2) = (0, 1, 1) \), a contradiction. ■

Thus \( \text{apd}(K_n \times K_2) = \text{apd}(K_n) = n \) for \( n = 3, 4 \). However, \( \text{apd}(K_n \times K_2) < n \) for \( n \geq 5 \), as we show next.

**Theorem 4.24** If \( n \geq 5 \), then

\[
\text{apd}(K_n \times K_2) \leq \begin{cases} 
\frac{3n}{4} & \text{if } n \equiv 0 \pmod{4} \\
\frac{3(n-1)}{4} + 2 & \text{if } n \equiv 1 \pmod{4} \\
\frac{3(n-2)}{4} + 2 & \text{if } n \equiv 2 \pmod{4} \\
\frac{3(n-3)}{4} + 3 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** Let \( G = K_n \times K_2 \) and let \( H_1 \) and \( H_2 \) be two copies of \( K_n \) in \( G \), where \( V(H_1) = \{u_1, u_2, \ldots, u_n\} \), \( V(H_2) = \{v_1, v_2, \ldots, v_n\} \), and \( u_iv_i \in E(G) \) for \( 1 \leq i \leq n \). We consider four cases.

**Case 1.** \( n \equiv 0 \pmod{4} \). Let \( k = n/4 \) and let \( \Pi = \{S_1, S_2, \ldots, S_{3k}\} \) be a partition of \( V(G) \), where \( S_i = \{u_i, u_{4k-i}\} \) for \( 1 \leq i \leq k \), \( S_{k+i} = \{v_i, v_{k+i}\} \) for \( 1 \leq i \leq k \), \( S_{2k+i} = \{u_{k+i}, u_{3k-i}, v_{2k+i}, v_{4k-i}\} \) for \( 1 \leq i \leq k-1 \), and \( S_{3k} = \{u_{2k}, u_{4k}, v_{3k}, v_{4k}\} \). Thus \( \Pi \) is acyclic. Since \( d(u_i, S_{k+i}) = 1 \) and \( d(u_{4k-i}, S_{k+i}) = 2 \), it follows that \( c_\Pi(u_i) \neq c_\Pi(u_{4k-i}) \) for \( 1 \leq i \leq k \). Since \( d(v_i, S_i) = 1 \) and \( d(v_{k+i}, S_i) = 2 \), it follows that \( c_\Pi(v_i) \neq c_\Pi(v_{k+i}) \) for \( 1 \leq i \leq k \). For each pair \( i, j \) with \( 1 \leq i \neq j \leq k-1 \), observe that

1. \( d(u_{k+i}, S_{k+i}) = 1, d(u_{k+i}, S_i) = 1, d(u_{k+i}, S_j) = 1, \)
2. \( d(u_{3k-i}, S_{k+i}) = 2, d(u_{3k-i}, S_i) = 1, \)
3. \( d(v_{2k+i}, S_{k+i}) = 1, d(v_{2k+i}, S_i) = 2, \) and
(4) \(d(v_{4k-i}, S_{k+i}) = 1, d(v_{4k-i}, S_i) = 1, d(v_{4k-i}, S_j) = 2.\)

Thus \(c_\Pi(u_{k+i}), c_\Pi(u_{3k-i}), c_\Pi(v_{2k+i}),\) and \(c_\Pi(v_{4k-i})\) are distinct for all \(i\). Moreover,

(1) \(d(u_{2k}, S_{2k}) = 1, d(u_{2k}, S_k) = 1, d(u_{2k}, S_j) = 1\) for \(1 \leq j \leq k - 1,\)

(2) \(d(u_{4k}, S_{2k}) = 2, d(u_{4k}, S_k) = 1,\)

(3) \(d(v_{3k}, S_{2k}) = 1, d(v_{3k}, S_k) = 1, d(v_{3k}, S_j) = 2\) for \(1 \leq j \leq k - 1,\) and

(4) \(d(v_{4k}, S_{2k}) = 1, d(v_{4k}, S_k) = 2.\)

Thus \(c_\Pi(u_{2k}), c_\Pi(u_{4k}), c_\Pi(v_{3k}),\) and \(c_\Pi(v_{4k})\) are all distinct. Therefore, \(\Pi\) is a resolving acyclic partition of \(G\) and so

\[\text{apd}(G) \leq |\Pi| = 3k = 3n/4.\]

Case 2. \(n \equiv 1 \pmod{4}\). Let \(k = (n - 1)/4\) and let \(\Pi = \{S_1, S_2, \ldots, S_{3k+2}\}\) be a partition of \(V(G)\), where \(S_i = \{u_i, u_{4k-i}\}\) for \(1 \leq i \leq k, S_{k+i} = \{v_i, v_{k+i}\}\) for \(1 \leq i \leq k\), \(S_{2k+i} = \{u_{k+i}, u_{3k-i}, v_{2k+i}, v_{4k-i}\}\) for \(1 \leq i \leq k-1, S_{3k} = \{u_{2k}, u_{4k}, v_{3k}, v_{4k}\}, S_{3k+1} = \{u_{4k+1}\},\) and \(S_{3k+2} = \{v_{4k+1}\}\). Then \(\Pi\) is acyclic. Since \(d(u_i, S_{k+i}) = 1\) and \(d(u_{4k-i}, S_{k+i}) = 2,\) it follows that \(c_\Pi(u_i) \neq c_\Pi(u_{4k-i})\) for \(1 \leq i \leq k.\) Since \(d(v_i, S_i) = 1\) and \(d(v_{k+i}, S_i) = 2,\) it follows that \(c_\Pi(v_i) \neq c_\Pi(v_{k+i})\) for \(1 \leq i \leq k.\) For each \(i\) with \(1 \leq i \leq k - 1,\) observe that

(1) \(d(u_{k+i}, S_{k+i}) = 1, d(u_{k+i}, S_i) = 1, d(u_{k+i}, S_{3k+1}) = 1,\)

(2) \(d(u_{3k-i}, S_{k+i}) = 2, d(u_{3k-i}, S_i) = 1,\)

(3) \(d(v_{2k+i}, S_{k+i}) = 1, d(v_{2k+i}, S_i) = 2,\) and

(4) \(d(v_{4k-i}, S_{k+i}) = 1, d(v_{4k-i}, S_i) = 1, d(v_{4k-i}, S_{3k+1}) = 2.\)

Thus \(c_\Pi(u_{k+i}), c_\Pi(u_{3k-i}), c_\Pi(v_{2k+i}),\) and \(c_\Pi(v_{4k-i})\) are distinct for all \(i.\) Moreover,

(1) \(d(u_{2k}, S_{2k}) = 1, d(u_{2k}, S_k) = 1, d(u_{2k}, S_{3k+1}) = 1,\)

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(2)  \( d(u_{4k}, S_{2k}) = 2, \ d(u_{4k}, S_k) = 1, \)

(3)  \( d(v_{3k}, S_{2k}) = 1, \ d(v_{3k}, S_k) = 1, \ d(v_{3k}, S_{3k+1}) = 2, \) and

(4)  \( d(v_{4k}, S_{2k}) = 1, \ d(v_{4k}, S_k) = 2. \)

Thus \( c_\Pi(u_{2k}), c_\Pi(u_{4k}), c_\Pi(v_{3k}), \) and \( c_\Pi(v_{4k}) \) are all distinct. Therefore, \( \Pi \) is a resolving acyclic partition of \( G \) and so

\[ \text{apd}(G) \leq |\Pi| = 3k + 2 = \frac{3}{4}(n - 1) + 2. \]

Case 3. \( n \equiv 2 \pmod{4} \). Let \( k = (n - 2)/4 \) and let \( \Pi = \{S_1, S_2, \ldots, S_{3k+2}\} \) be a partition of \( V(G) \), where \( S_i = \{u_i, u_{4k-i}\} \) for \( 1 \leq i \leq k, \ S_{k+i} = \{v_i, v_{k+i}\} \) for \( 1 \leq i \leq k \), \( S_{2k+i} = \{u_{k+i}, u_{4k-i}, v_{2k+i}, v_{4k-i}\} \) for \( 1 \leq i \leq k-1, \ S_{3k} = \{u_{2k}, u_{4k}, v_{3k}, v_{4k}\}, \ S_{3k+1} = \{u_{4k+1}\}, \) and \( S_{3k+2} = \{u_{4k+2}, v_{4k+1}, v_{4k+2}\} \). Then \( \Pi \) is acyclic. Since \( d(u_i, S_{k+i}) = 1 \) and \( d(u_{4k-i}, S_{k+i}) = 2 \), it follows that \( c_\Pi(u_i) \neq c_\Pi(u_{4k-i}) \) for \( 1 \leq i \leq k \). Since \( d(v_i, S_i) = 1 \) and \( d(v_{k+i}, S_i) = 2 \), it follows that \( c_\Pi(v_i) \neq c_\Pi(v_{k+i}) \) for \( 1 \leq i \leq k \). For each \( i \) with \( 1 \leq i \leq k-1 \), observe that

(1)  \( d(u_{k+i}, S_{k+i}) = 1, \ d(u_{k+i}, S_i) = 1, \ d(u_{k+i}, S_{3k+1}) = 1, \)

(2)  \( d(u_{3k-i}, S_{k+i}) = 2, \ d(u_{3k-i}, S_i) = 1, \)

(3)  \( d(v_{2k+i}, S_{k+i}) = 1, \ d(v_{2k+i}, S_i) = 2, \) and

(4)  \( d(v_{4k-i}, S_{k+i}) = 1, \ d(v_{4k-i}, S_i) = 1, \ d(v_{4k-i}, S_{3k+1}) = 2. \)

Thus \( c_\Pi(u_{k+i}), c_\Pi(u_{3k-i}), c_\Pi(v_{2k+i}), \) and \( c_\Pi(v_{4k-i}) \) are distinct for all \( i \). Moreover,

(1)  \( d(u_{2k}, S_{2k}) = 1, \ d(u_{2k}, S_k) = 1, \ d(u_{2k}, S_{3k+1}) = 1, \)

(2)  \( d(u_{4k}, S_{2k}) = 2, \ d(u_{4k}, S_k) = 1, \)

(3)  \( d(v_{3k}, S_{2k}) = 1, \ d(v_{3k}, S_k) = 1, \ d(v_{3k}, S_{3k+1}) = 2, \) and

(4)  \( d(v_{4k}, S_{2k}) = 1, \ d(v_{4k}, S_k) = 2. \)
Thus $c_{\Pi}(u_{2k})$, $c_{\Pi}(u_{4k})$, $c_{\Pi}(v_{3k})$, and $c_{\Pi}(v_{4k})$ are all distinct. Finally,

1. $d(u_{4k+2}, S_1) = 1$, $d(u_{4k+2}, S_{3k+1}) = 1$,
2. $d(v_{4k+1}, S_1) = 2$, $d(v_{4k+1}, S_{3k+1}) = 1$, and
3. $d(v_{4k+2}, S_1) = 2$, $d(v_{4k+2}, S_{3k+1}) = 2$.

Thus $c_{\Pi}(u_{4k+2})$, $c_{\Pi}(v_{4k+1})$, and $c_{\Pi}(v_{4k+2})$ are distinct. Therefore, $\Pi$ is a resolving acyclic partition of $G$ and so

$$\text{apd}(G) \leq |\Pi| = 3k + 2 = \frac{3}{4}(n - 2) + 2.$$ 

\textbf{Case 4.} $n \equiv 3 \pmod{4}$. Let $k = (n - 3)/4$ and let $\Pi = \{S_1, S_2, \ldots, S_{3k+3}\}$ be a partition of $V(G)$, where $S_i = \{u_i, u_{4k-i}\}$ for $1 \leq i \leq k$, $S_{k+i} = \{v_i, v_{k+i}\}$ for $1 \leq i \leq k$, $S_{2k+1} = \{u_{k+i}, u_{3k-i}, v_{2k+i}, v_{4k-i}\}$ for $1 \leq i \leq k-1$, $S_{3k} = \{u_{2k}, u_{4k}, v_{3k}, v_{4k}\}$, $S_{3k+1} = \{u_{4k+1}, u_{4k+2}\}$, $S_{3k+2} = \{v_{4k+3}, v_{4k+1}\}$, and $S_{3k+3} = \{v_{4k+2}, v_{4k+3}\}$. Then $\Pi$ is acyclic. Since $d(u_i, S_{k+i}) = 1$ and $d(u_{4k-i}, S_{k+i}) = 2$, it follows that $c_{\Pi}(u_i) \neq c_{\Pi}(u_{4k-i})$ for $1 \leq i \leq k$. Since $d(v_i, S_i) = 1$ and $d(v_{k+i}, S_i) = 2$, it follows that $c_{\Pi}(v_i) \neq c_{\Pi}(v_{k+i})$ for $1 \leq i \leq k$. For each $i$ with $1 \leq i \leq k - 1$, observe that

1. $d(u_{k+i}, S_{k+i}) = 1$, $d(u_{k+i}, S_i) = 1$, $d(u_{k+i}, S_{3k+1}) = 1$,
2. $d(u_{3k-i}, S_{k+i}) = 2$, $d(u_{3k-i}, S_i) = 1$,
3. $d(u_{2k+i}, S_{k+i}) = 1$, $d(u_{2k+i}, S_i) = 2$, and
4. $d(u_{4k-i}, S_{k+i}) = 1$, $d(u_{4k-i}, S_i) = 1$, $d(u_{4k-i}, S_{3k+1}) = 2$.

Thus $c_{\Pi}(u_{k+i})$, $c_{\Pi}(u_{3k-i})$, $c_{\Pi}(v_{2k+i})$, and $c_{\Pi}(v_{4k-i})$ are distinct for all $i$. Moreover,

1. $d(u_{2k}, S_{2k}) = 1$, $d(u_{2k}, S_k) = 1$, $d(u_{2k}, S_{3k+1}) = 1$,
2. $d(u_{4k}, S_{2k}) = 2$, $d(u_{4k}, S_k) = 1$,
3. $d(v_{3k}, S_{2k}) = 1$, $d(v_{3k}, S_k) = 1$, $d(v_{3k}, S_{3k+1}) = 2$, and

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(4) \( d(v_{4k}, S_{2k}) = 1, d(v_{4k}, S_k) = 2. \)

Thus \( c(n(u_{2k}), c(n(u_{4k}), c(n(v_{3k}), and c(n(v_{4k}) are all distinct. Furthermore, since

(1) \( d(u_{4k+1}, S_{3k+3}) = 2 \) and \( d(u_{4k+2}, S_{3k+3}) = 1, \)
(2) \( d(u_{4k+3}, S_1) = 1 \) and \( d(v_{4k+1}, S_1) = 2, \) and
(3) \( d(v_{4k+2}, S_{3k+1}) = 1 \) and \( d(v_{4k+3}, S_{3k+1}) = 2, \)

it follows that

\[ c(n(u_{4k+1}) \neq c(n(u_{4k+2}), c(n(u_{4k+3}) \neq c(n(v_{4k+1}), and c(n(v_{4k+2}) \neq c(n(v_{4k+3}). \]

Therefore, \( \Pi \) is a resolving acyclic partition of \( G \) and so

\[ \text{apd}(G) \leq |\Pi| = 3k + 3 = \frac{3}{4}(n - 3) + 3. \]

This completes the proof.

Since \( a(K_n) = \lfloor n/2 \rfloor \) for \( n \geq 3 \), it follows by Theorem 4.24 that strict inequality in Theorem 4.22 can hold.

### 4.7 Acyclic Partition Ratios

If \( H \) is an induced subgraph of a graph \( G \), then \( a(H) \leq a(G) \) by Lemma 4.5. Thus

\[ 0 < \frac{a(H)}{a(G)} \leq 1 \] (4.3)

for each induced subgraph \( H \) of a graph \( G \). Let \( G \) be a nontrivial connected graph and \( H \) a connected induced subgraph of \( G \). We define the acyclic partition ratio of \( H \) and \( G \) by

\[ r_a(H, G) = \frac{\text{apd}(H)}{\text{apd}(G)}. \]
Unlike the ratio in (4.3), it need not occur that \( r_a(H,G) \leq 1 \), however. By Theorem 4.21, \( \text{apd}(K_{1,m}) = m \) for all \( m \geq 2 \). Hence for \( G = K_{1,m} \) and \( H = K_2 \), we can make the ratio \( r_a(H,G) \) as small as we wish by choosing \( m \) arbitrarily large. Although this may not be surprising, it may be unexpected that, in fact, we can make \( r_a(H,G) \) as large as we wish, as we now verify.

For \( n \geq 3 \), we label the vertices of the star \( K_{1,2n+1} \) with \( v_0, v_1, v_2, \ldots, v_{2n}, v'_1, v'_2, \ldots, v'_{2n} \), where \( v_0 \) is the central vertex. We then add two new vertices \( x \) and \( x' \) and \( 2^{n+1} \) edges \( xv_i \) and \( x'v'_i \) for \( 1 \leq i \leq 2^n \). Next, we add two sets \( W = \{w_1, w_2, \ldots, w_n\} \) and \( W' = \{w'_1, w'_2, \ldots, w'_n\} \) of vertices, together with the edges \( w_ix \) and \( w'_ix' \) for \( 1 \leq i \leq n \). Finally, we add edges between \( W \) and \( \{v_0, v_1, v_2, \ldots, v_{2n}\} \) so that each of the \( 2^n \) possible \( n \)-tuples of 1s and 2s appears exactly once such that the representations \( (d(v_i, w_1), d(v_i, w_2), \ldots, d(v_i, w_n)) \) are distinct for \( 1 \leq i \leq 2^n \). Similarly, edges are added between \( W' \) and \( \{v'_1, v'_2, \ldots, v'_{2n}\} \) so that \( (d(v'_i, w'_1), d(v'_i, w'_2), \ldots, d(v'_i, w'_n)) \) are distinct for \( 1 \leq i \leq 2^n \). Denote the resulting graph by \( G \). The graph \( G \) for \( n = 3 \) is shown in Figure 4.4.

Let \( \Pi = \{S_1, S_2, \ldots, S_{2n+2}\} \) be the partition of \( V(G) \) where \( S_i = \{w_i\} \) (1 \( \leq i \leq n \) ), \( S_{n+j} = \{w'_j\} \) (1 \( \leq j \leq n \) ), and \( S_{2n+1} = \{x, x'\} \) and \( S_{2n+2} = V(K_{1,2n+1}) \). Since \( \langle S_i \rangle \) is acyclic for \( 1 \leq i \leq 2n + 2 \), it follows that \( \Pi \) is acyclic. Next we show that \( \Pi \) is a resolving partition of \( V(G) \). By construction, \( c_\Pi(v_i) = c_\Pi(v_j) \) implies that \( i = j \) and \( c_\Pi(v'_i) = c_\Pi(v'_j) \) implies that \( i = j \). Moreover,

\[
\begin{align*}
c_\Pi(x) &= (1,1,\ldots,1,4,4,\ldots,4,0,1), \\
c_\Pi(v_i) &= (*,*,\ldots,*,3,3,\ldots,3,1,0), \ 1 \leq i \leq 2^n, \\
c_\Pi(v_0) &= (2,2,\ldots,2,2,2,\ldots,2,2,0), \\
c_\Pi(v'_i) &= (3,3,\ldots,3,*,*,\ldots,*,1,0), \ 1 \leq i \leq 2^n, \\
c_\Pi(x') &= (4,4,\ldots,4,1,1,\ldots,1,0,1),
\end{align*}
\]

where * represents an irrelevant coordinate. Thus \( \Pi \) is a resolving acyclic \((2n+2)\)-partition of \( V(G) \). Observe that \( G \) contains \( H = K_{1,2n+1} \) as an induced subgraph and

\[
\frac{\text{apd}(H)}{\text{apd}(G)} \geq \frac{2^{n+1}}{2n + 2}.
\]
Since 

$$\lim_{n \to \infty} \frac{2^{n+1}}{2n + 2} = \infty,$$

there exist a connected graph $G$ and an induced subgraph $H$ of $G$ such that $r_a(H, G) = \text{apd}(H)/\text{apd}(G)$ is arbitrary large. Therefore, we have the following result.

**Proposition 4.25** \hspace{1em} Let $\epsilon$ and $M$ be two real numbers.

1. There exist a connected graph $G$ and an induced subgraph $H$ of $G$ such that $r_a(H, G) < \epsilon$.

2. There exist a connected graph $G'$ and an induced subgraph $H'$ of $G'$ such that $r_a(H', G') > M$. 

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Chapter 5

Connected Resolving Decompositions

5.1 Introduction

Recall that a decomposition \( \mathcal{D} \) of a nontrivial connected graph \( G \) is said to be a resolving decomposition for \( G \) if every two distinct edges of \( G \) have distinct \( \mathcal{D} \)-codes. The minimum \( k \) for which \( G \) has a resolving \( k \)-decomposition is its decomposition dimension \( \dim_d(G) \). A resolving decomposition \( \mathcal{D} = \{G_1, G_2, \ldots, G_k\} \) of a connected graph \( G \) is connected if each subgraph \( G_i \) (1 \( \leq i \leq k \)) is a connected subgraph in \( G \). The minimum \( k \) for which \( G \) has a connected resolving \( k \)-decomposition is its connected decomposition dimension \( \text{cdim}_d(G) \). A connected resolving decomposition of \( G \) with \( \text{cdim}_d(G) \) elements is called a minimum connected resolving decomposition of \( G \). If \( G \) has \( m \geq 2 \) edges, then the \( m \)-decomposition \( \mathcal{D} = \{G_1, G_2, \ldots, G_m\} \), where each set \( E(G_i) \) (1 \( \leq i \leq m \)) consists of a single edge, is a connected resolving decomposition of \( G \). Thus \( \text{cdim}_d(G) \) is defined for every connected graph \( G \) of size at least 2. Moreover, every connected resolving \( k \)-decomposition is a resolving \( k \)-decomposition, and so

\[
2 \leq \dim_d(G) \leq \text{cdim}_d(G) \leq m
\]

for every connected graph \( G \) of size \( m \geq 2 \).

To illustrate these concepts, consider the graph \( G \) of Figure 5.1. Let \( \mathcal{D}_1 = \{G_1, \)
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$G_2, G_3 \}$, where $E(G_1) = \{e_1, e_5, f_1, f_5, f_4\}$, $E(G_2) = \{e_2, e_3, f_2\}$, and $E(G_3) = \{e_4, e_5, f_3, f_6, f_7\}$. The $D_1$-codes of the edges of $G$ are:

- $c_{D_1}(e_1) = (0,1,2)$
- $c_{D_1}(e_2) = (1,0,2)$
- $c_{D_1}(e_3) = (2,0,1)$
- $c_{D_1}(e_4) = (2,1,0)$
- $c_{D_1}(e_5) = (0,4,1)$
- $c_{D_1}(e_6) = (1,4,0)$
- $c_{D_1}(f_1) = (0,1,1)$
- $c_{D_1}(f_2) = (1,0,1)$
- $c_{D_1}(f_3) = (1,1,0)$
- $c_{D_1}(f_4) = (0,2,1)$
- $c_{D_1}(f_5) = (0,3,1)$
- $c_{D_1}(f_6) = (1,3,0)$
- $c_{D_1}(f_7) = (1,2,0)$

Thus $D_1$ is a resolving decomposition of $G$. By Theorem F, $\dim_d(G) = |D_1| = 3$. However, $D_1$ is not connected since $G_1$ and $G_3$ are not connected subgraphs in $G$. On the other hand, let $D_2 = \{G'_1, G'_2, G'_3, G'_4, G'_5\}$, where $E(G'_1) = \{e_1, f_1\}$, $E(G'_2) = \{e_5, f_4, f_5\}$, $E(G'_3) = \{e_2, e_3, f_2\}$, $E(G'_4) = \{e_4, f_3\}$, and $E(G'_5) = \{e_6, f_6, f_7\}$. Then $D_2$ is a connected resolving decomposition of $G$. But $D_2$ is not minimum since the decomposition $D_3 = \{G'_1, G'_2, G'_3, G'_4\}$, where $E(G'_1) = \{e_1\}$, $E(G'_2) = \{e_3\}$, $E(G'_3) = \{e_5\}$, and $E(G'_4) = E(G) - \{e_1, e_3, e_5\}$, is a connected resolving decomposition of $G$ with fewer elements. Indeed, it can be verified that $D_3$ is a minimum connected resolving decomposition of $G$ and so $\text{cdim}_d(G) = |D_3| = 4$.

![Figure 5.1: A graph $G$ with $\dim_d(G) = 3$ and $\text{cdim}_d(G) = 4$](image)

The example just presented illustrates an important point. Let $D = \{G_1, G_2, \ldots, G_k\}$ be a resolving decomposition of $G$. If $e \in E(G_i)$ and $f \in E(G_j)$, where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$, then $c_D(e) \neq c_D(f)$ since $d(e, G_i) = 0$ and $d(e, G_j) \neq 0$. Thus, when determining whether a given decomposition $D$ of a graph $G$ is a resolving decomposition for $G$, we need only verify that the edges of
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G belonging to the same subgraph in D have distinct D-codes. The following two observations are useful.

Observation 5.1 Let D be a resolving decomposition of G and let \( e_1, e_2 \in E(G) \). If \( d(e_1, f) = d(e_2, f) \) for all \( f \in E(G) - \{e_1, e_2\} \), then \( e_1 \) and \( e_2 \) belong to distinct elements of D.

Observation 5.2 Let G be a connected graph. Then \( \dim_d(G) = \text{cdim}_d(G) \) if and only if G contains a minimum resolving decomposition that is connected.

5.2 Refinements of a Connected Decomposition of a Graph

Let D be a decomposition of a connected graph G. Then a decomposition \( D^* \) of G is called a refinement of D if every element in \( D^* \) is a subgraph of some element of D. For the graph G of Figure 5.1, the decomposition \( D_2 \) of G is a connected refinement of \( D_1 \). We have seen that \( D_1 \) is resolving and its refinement \( D_2 \) is also resolving. Similar to the situation in the resolving partition of a connected graph, this is no coincidence, as we show now.

Theorem 5.3 Let D be a resolving decomposition of a connected graph G. If \( D^* \) is a refinement of D, then \( D^* \) is also a resolving decomposition of G.

Proof. Let \( D = \{G_1, G_2, \ldots, G_k\} \) and \( D^* = \{H_1, H_2, \ldots, H_s\} \) be two decompositions of G, where \( k \leq \ell \), such that each subgraph \( H_i \) \((1 \leq i \leq \ell)\) is a subgraph of \( G_j \) for some \( j \) with \( 1 \leq j \leq k \). Let e and f be distinct edges of G. We show that \( c_{D^*}(e) \neq c_{D^*}(f) \). Since D is a resolving decomposition of G, it follows that \( c_D(e) \neq c_D(f) \). Thus \( d(e, G_j) \neq d(f, G_j) \) for some \( j \) with \( 1 \leq j \leq k \), say \( d(e, G_1) \neq d(f, G_1) \). If \( G_1 \) is an element of \( D^* \), then \( d(e, G_1) \neq d(f, G_1) \) and so \( c_{D^*}(e) \neq c_{D^*}(f) \). Thus we may assume that \( E(G_1) = E(H_{i_1}) \cup E(H_{i_2}) \cup \ldots \cup E(H_{i_s}) \), where \( 1 \leq i_1 < i_2 < \ldots < i_s \leq \ell \) and \( s \geq 2 \). Observe that at least one of e and f does not belong to \( G_1 \), for otherwise, \( d(e, G_1) = 0 = d(f, G_1) \). We consider two cases.
Case 1. Exactly one of \( e \) and \( f \) is in \( G_i \), say \( e \in G \) and \( f \in G \). Thus \( e \in E(H_i) \) for some \( p \) with \( 1 \leq p \leq s \) and so \( d(e, H_i) = 0 \). Since \( f \notin E(G) \), it follows that \( f \notin E(H_i) \) and so \( d(f, H_i) \neq 0 \). Hence \( c_{D^*}(e) \neq c_{D^*}(f) \).

Case 2. \( e, f \notin E(G) \). Let \( e', f' \in E(G) \) such that \( d(e, G) = d(e, e') \) and \( d(f, G) = d(f, f') \), where say \( d(e, e') < d(f, f') \). If \( e', f' \in E(H_i) \) for some \( p \) with \( 1 \leq p \leq s \), then \( d(e, H_i) = d(e, e') < d(f, f') = d(f, H_i) \), implying that \( c_{D^*}(e) \neq c_{D^*}(f) \). If \( e' \in E(H_i) \) and \( f' \in E(H_i) \), where \( 1 \leq p \neq q \leq s \), then \( d(e, H_i) = d(e, e') < d(f, f') \leq d(f, H_i) \), again, implying that \( c_{D^*}(e) \neq c_{D^*}(f) \).

Therefore, \( D^* \) is a resolving decomposition of \( G \).

According to Theorem 5.3, if we are given a minimum resolving decomposition \( D \) of a connected graph \( G \), then we can find a connected resolving decomposition of \( D^* \) of \( G \), where \( D^* \) is a refinement of \( D \). Indeed, the decomposition each of whose elements consists of a single edge has this property. However, \( D^* \) need not be a minimum connected resolving decomposition for \( G \). In fact, it may be the case that no minimum connected resolving decomposition is a refinement of any minimum resolving decomposition of \( G \). For example, consider the graph \( G \) of Figure 5.1. Indeed, using a case-by-case analysis, we can show that the graph \( G \) of Figure 5.1 has two distinct minimum resolving decompositions (up to isomorphic), namely, \( \{G_1, G_2, G_3\} \) and \( \{H_1, H_2, H_3\} \), where \( G_1 = G_2 = P_3 \cup P_4, G_3 = P_4, H_1 = H_2 = P_2 \cup 2P_3, \) and \( H_3 = P_4 \). For example, let \( \mathcal{D} = \{G_1, G_2, G_3\} \), where \( E(G_1) = \{e_1, e_5, f_1, f_5, f_4\} \), \( E(G_2) = \{e_2, e_3, f_2\} \), and \( E(G_3) = \{e_4, e_6, f_3, f_5, f_7\} \), and Let \( \tilde{\mathcal{D}} = \{H_1, H_2, H_3\} \), where \( E(H_1) = \{e_1, e_5, f_1, f_4, f_6\} \), \( E(H_2) = \{e_2, e_3, f_2\} \), and \( E(H_3) = \{e_4, e_5, f_3, f_5, f_7\} \). The decompositions \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \) are shown in Figure 5.2. Since each connected refinement of \( \mathcal{D} \) contains at least five elements, each connected refinement of \( \tilde{\mathcal{D}} \) contains at least seven elements, and \( \text{cdim}_d(G) = 4 \), it follows that no minimum connected resolving decomposition of \( G \) is a refinement of any minimum resolving decomposition of \( G \).

5.3 Bounds for the Connected Decomposition Dimension of a Graph

We have seen that if \( G \) is a connected graph of size \( m \geq 2 \), then \( 2 \leq \text{cdim}_{d}(G) \leq m \). We characterize those connected graphs \( G \) of size \( m \geq 2 \) such
that $\text{cdim}_d(G) = 2$ and those connected graphs $G$ of size $m \geq 2$ such that $
abla \text{cdim}_d(G) = m$.

**Theorem 5.4** Let $G$ be a connected graph of order $n \geq 3$ and size $m$. Then

(a) $\text{cdim}_d(G) = 2$ if and only if $G = P_n$, and

(b) $\text{cdim}_d(G) = m$ if and only if $G = K_3$ or $G = K_{1,n-1}$.

**Proof.** We first verify (a). Let $P_n : v_1, v_2, \ldots, v_n$ and let $D = \{G_1, G_2\}$ be the decomposition of $P_n$ in which $E(G_1) = \{v_1v_2\}$ and $G_2$ is the path $v_2, v_3, \ldots, v_n$. Thus $D$ is connected. For $2 \leq i \leq n - 1$, the edge $v_i v_{i+1}$ is the unique edge of $G_2$ at distance $i - 1$ from $G_1$. Therefore, $D$ is a connected resolving decomposition of $P_n$ and so $\text{cdim}_d(P_n) = 2$. For the converse, let $G$ be a connected graph of order $n \geq 3$ and $\text{cdim}_d(G) = 2$. By (5.1) $\text{dim}_d(G) = 2$ as well. It then follows by Theorem F that $G = P_n$.

Next we verify (b). It is routine to show that $\text{cdim}_d(K_3) = 3$ and $\text{cdim}_d(K_{1,n-1}) = n - 1$ and so the graphs described in (b) have $\text{cdim}_d(G) = |E(G)|$. For the converse, let $G$ be a connected graph of order $n \geq 3$ and size $m \geq 2$ such that $\text{cdim}_d(G) = m$. If $m = 2$, then $G = P_3$ and $\text{cdim}_d(P_3) = 2$ by (a). If $m = 3$, then $G \in \{P_4, K_3, K_{1,3}\}$. Since $\text{cdim}_d(P_4) = 2$ and $\text{cdim}_d(K_3) = \text{cdim}_d(K_{1,3}) = 3$, it
follows that $G = K_3$ or $G = K_{1,3}$. Now let $G$ be a connected graph of size $m \geq 4$ and let $E(G) = \{e_1, e_2, \ldots, e_m\}$. If $G \neq K_{1,n-1}$, then $G$ contains a path $P_4$ of order 4 with three edges, say $e_1, e_2$, and $e_3$, such that $d(e_1, e_2) = 1$, $d(e_1, e_3) = 2$, and $d(e_2, e_3) = 1$. Then $D = \{G_1, G_2, \ldots, G_{m-1}\}$, where $E(G_1) = \{e_1, e_2\}$ and $E(G_i) = \{e_{i+1}\}$ for $2 \leq i \leq m - 1$, is a connected resolving decomposition of $G$. Thus $\text{cdim}_d(G) \leq |D| = m - 1$.

It was shown in [4] that $\dim_d(K_3) = 3$ and $\dim_d(K_{1,n-1}) = n - 1$. Thus the following corollary is a consequence of (5.1) and Theorem 5.4.

**Corollary 5.5** Let $G$ be a connected graph of order $n \geq 3$ and of size $m$. Then $\dim_d(G) = m$ if and only if $G = K_3$ or $G = K_{1,n-1}$.

In the next two results, we present bounds for $\text{cdim}_d(G)$ of a connected graph $G$ in terms of (1) its size and diameter and (2) its size and girth.

**Theorem 5.6** If $G$ is a connected graph of size $m \geq 2$ and diameter $d$, then

$$1 + \log_{d+1} m \leq \text{cdim}_d(G) \leq m - d + 2.$$  

**Proof.** First, we verify the upper bound. Let $u, v \in V(G)$ such that $d(u, v) = d$ and let $P : u = v_1, v_2, \ldots, v_{d+1} = v$ be a $u - v$ path of length $d$ in $G$. Also, let $E(G) - E(P) = \{e_1, e_2, \ldots, e_{m-d}\}$. Let

$$D = \{G_1, G_2, \ldots, G_{m-d+2}\},$$

where $E(G_i) = \{e_i\}$ for $1 \leq i \leq m - d$, $E(G_{m-d+1}) = \{v_1, v_2\}$, and $E(G_{m-d+2}) = E(P - v_1)$. Then $D$ is a connected decomposition of $G$. Since $d(v_i, v_{i+1}, G_{m-d+1}) = i - 1$ for $2 \leq i \leq d$, it follows that $D$ is a resolving decomposition of $G$. Therefore, $\text{cdim}_d(G) \leq |D| = m - d + 2$.

Next, we verify the lower bound. Let $D$ be a connected resolving $k$-decomposition of $G$. Since (1) exactly one coordinate of the code of an edge in $G$ with respect to $D$ is 0 and there are $k$ choices for the zero coordinate in the code of an edge, (2) each of the $k - 1$ nonzero coordinates of the code of an edge is a positive integer
not exceeding $d$, and (3) all codes of the $m$ edges of $G$ are distinct, it follows that $kd^{k-1} \geq m$. Certainly, for each pair $d, k$ of positive integers, $(d + 1)^k > kd^{k-1}$. Thus, if $G$ is a nontrivial connected graph of size $m$, diameter $d$, and connected decomposition dimension $k$, then $(d + 1)^k > kd^{k-1} \geq m$, and so $\text{cdim}_d(G) > \log_{d+1} m$. Therefore, $\text{cdim}_d(G) \geq \log_{d+1} m + 1$ since $\text{cdim}_d(G)$ is an integer.

If $d = 1$, then $G = K_n$ for some $n \geq 3$. Since $\text{dim}(K_n) = \text{cdim}(K_n) = m$, where $m = \binom{n}{2}$ is the size of $K_n$, it follows that the upper bound in Theorem 5.6 is not sharp for $d = 1$. If $d = 2$, then $G = K_{1,m}$ is the only graph with $\text{cdim}_d(G) = m - d + 2 = m$ by Theorem 5.4. Thus we may assume that $m \geq d \geq 3$. If $m = d$, then $G = P_{m+1}$ and $\text{cdim}_d(G) = 2 = m - d + 2$. If $m \geq d + 1$, let $G$ be the graph obtained from the path $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ by adding the $m - d \geq 1$ new vertices $v_1, v_2, \ldots, v_{m-d}$ and joining each of these vertices to $u_d$. Then the diameter of $G$ is $d$ and size of $G$ is $m$. Moreover, it can be verified that $\text{cdim}_d(G) = m - d + 2$. Thus the upper bound in Theorem 5.6 is sharp for $d \geq 2$.

By an argument similar to the one used in the proof of Theorem 5.6, we have the corresponding result for the decomposition dimension of a connected graph; that is, if $G$ is a connected graph of size $m \geq 2$ and diameter $d$, then

$$1 + \log_{d+1} m \leq \text{dim}_d(G) \leq m - d + 2.$$ 

**Theorem 5.7** If $G$ is a connected graph of size $m \geq 3$ and girth $\ell \geq 3$, then

$$3 \leq \text{cdim}_d(G) \leq m - \ell + 3.$$ 

Moreover, $\text{cdim}_d(G) = m - \ell + 3$ if and only if $G$ is a cycle of order at least 3.

**Proof.** Since $\ell \geq 3$, it follows that $G$ is not a path and so $\text{cdim}_d(G) \geq 3$ by Theorem 5.4. It remains to verify the upper bound. If $\ell = 3$, then $\text{cdim}_d(G) \leq m$ by (5.1) and so the upper bound holds. Thus we may assume that $\ell \geq 4$. Let $C_\ell : v_1, v_2, \ldots, v_\ell, v_1$ be a cycle of length $\ell$ in $G$, let $d = \lfloor \ell/2 \rfloor$, and let

$$D = \{G_1, G_2, \ldots, G_{m-\ell+3}\}$$
be a decomposition of $G$, where $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4, \ldots, v_dv_{d+1}\}$, $E(G_3) = \{v_{d+1}v_{d+2}, v_{d+2}v_{d+3}, \ldots, v_{d+1}v_1\}$, and each of $G_i$ ($4 \leq i \leq m - \ell + 3$) contains exactly one edge in $E(G) - E(C_\ell)$. Thus $\mathcal{D}$ is connected. Furthermore, since

- $c_D(v_1v_2) = (0, 1, 1, \ldots)$,
- $c_D(v_iv_{i+1}) = (i - 1, 0, \min\{i, d - i + 1\}, \ldots)$ for $2 \leq i \leq d$,
- $c_D(v_{d+1}v_{d+2}) = (d, 1, 0, \ldots)$,
- $c_D(v_iv_{i+1}) = (\ell - i + 1, \min\{i - d, \ell - i + 2\}, 0, \ldots)$ for $d + 2 \leq i \leq \ell - 1$,
- $c_D(v_\ell v_1) = (1, 2, 0, \ldots)$,

it follows that the $D$-codes of edges of $G$ are distinct. Thus $\mathcal{D}$ is a connected resolving decomposition of $G$ and so $\text{cdim}_d(G) \leq |\mathcal{D}| = m - \ell + 3$.

If $G$ is a cycle $C_n$ of order $n \geq 3$, then $\ell = m = n$ and so $\text{cdim}_d(G) = 3$. For the converse, let $G \neq C_n$ be a connected graph of order $n \geq 3$, size $m \geq 3$, and girth $\ell \geq 3$ and let $C_\ell : v_1, v_2, \ldots, v_\ell, v_1$ be a smallest cycle in $G$, where $\ell < n$. Since $G$ is connected and $G \neq C_\ell$, it follows that $m \geq 4$ and there exists a vertex $v \in V(G) - V(C_\ell)$ such that $v$ is adjacent to a vertex of $C_\ell$, say $vv_1 \in E(G)$. We consider three cases.

**Case 1.** $\ell = 3$. Then $G$ contains an induced subgraph $H_1$ of Figure 5.3(a), where dashed lines indicate that the given edges may or may not be present. Let

$$\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\},$$

where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, $E(G_3) = \{v_1v_3\}$, and each of $G_i$ ($4 \leq i \leq m - \ell + 2$) contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Since $d(vv_1, G_2) = 2$ and $d(v_1v_2, G_2) = 1$, it follows that $\mathcal{D}$ is a connected resolving decomposition of $G$ and so $\text{cdim}_d(G) \leq |\mathcal{D}| = m - \ell + 2$.

**Case 2.** $\ell = 4$. Then $G$ contains an induced subgraph $H_2$ of Figure 5.3(b), where the dashed line indicate that the given edge may or may not be present. Let

$$\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\},$$

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where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, $E(G_3) = \{v_1v_4, v_3v_4\}$, and each of $G_i$ $(4 \leq i \leq m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Since $d(vv_1, G_2) = 2$, $d(v_1v_2, G_2) = 1$, $d(v_1v_4, G_2) = 2$, and $d(v_3v_4, G_2) = 1$, it follows that $\mathcal{D}$ is a connected resolving decomposition of $G$ and so $\text{cdim}_d(G) \leq |\mathcal{D}| = m - \ell + 2$.

Case 3. $\ell \geq 5$. Since $C_\ell$ is a smallest cycle in $G$, it follows that $v$ is adjacent exactly one vertex of $C_\ell$. Let $d = \lfloor \ell/2 \rfloor$ and let

$$\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$$

be a decomposition of $G$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4, \ldots, v_dv_{d+1}\}$, $E(G_3) = \{v_{d+1}v_{d+2}, v_{d+2}v_{d+3}, \ldots, v_{\ell-1}v_\ell, v_\ell v_1\}$, and each of $G_i$ $(4 \leq i \leq m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Thus $\mathcal{D}$ is connected. Since

\begin{align*}
    c_D(vv_1) &= (0, 2, 2, \ldots), \\
    c_D(v_1v_2) &= (0, 1, 1, \ldots), \\
    c_D(v_i v_{i+1}) &= (i - 1, 0, \min\{i, d - i + 1\}, \ldots) \text{ for } 2 \leq i \leq d, \\
    c_D(v_{d+1}v_{d+2}) &= (d, 1, 0, \ldots), \\
    c_D(v_i v_{i+1}) &= (\ell - i + 1, \min\{i - d, \ell - i + 2\}, 0, \ldots) \text{ for } d + 2 \leq i \leq \ell - 1, \\
    c_D(v_\ell v_1) &= (1, 2, 0, \ldots),
\end{align*}

it follows that $\mathcal{D}$ is a connected resolving decomposition of $G$. Thus $\text{cdim}_d(G) \leq |\mathcal{D}| = m - \ell + 2$. 

Figure 5.3: The subgraphs $H_1$ and $H_2$
Therefore, \( \text{cdim}_d(G) \leq m - \ell + 2 \) if \( G \neq C_n \), as desired.

Next, we present an upper bound for \( \text{cdim}_d(G) \) of a connected graph \( G \) in terms of its order. In order to do this, we first present a lemma.

**Lemma 5.8** Let \( G \) be a connected graph of order \( n \geq 5 \), let \( T \) be a spanning tree of \( G \) with \( E(T) = \{e_1, e_2, \ldots, e_{n-1}\} \), and let \( H = G - E(T) \). Then the decomposition \( \mathcal{D} = \{F_1, F_2, \ldots, F_{n-1}, H\} \), where \( E(F_i) = \{e_i\} \) for \( 1 \leq i \leq n - 1 \), is a resolving decomposition of \( G \).

**Proof.** Let \( e \) and \( f \) be two edges of \( G \). If \( e \) and \( f \) belong to distinct elements of \( \mathcal{D} \), then \( c_D(e) \neq c_D(f) \). Thus we may assume that \( e \) and \( f \) belong to the same element \( H \) in \( \mathcal{D} \). We show that \( c_D(e) \neq c_D(f) \). Let \( e = uv \), let \( P \) be the unique \( u - v \) path in \( T \), and let \( u' \) and \( v' \) be the vertices on \( P \) adjacent to \( u \) and \( v \), respectively. If \( f \) is adjacent to at most one of \( uu' \) and \( vv' \), then either \( d(e, uu') \neq d(f, uu') \) or \( d(e, vv') \neq d(f, vv') \), and so \( c_D(e) \neq c_D(f) \). Hence we may assume that \( f \) is adjacent to both \( uu' \) and \( vv' \). We consider two cases, according to whether \( u' = v' \) or \( u' \neq v' \).

**Case 1.** \( u' = v' \). Then \( f \) is incident with the vertex \( u' \). Since \( n \geq 5 \) and \( T \) is a spanning tree, there is a vertex \( x \in V(G) - \{u, v, u'\} \) such that \( x \) is adjacent in \( T \) with exactly one of \( u, v \) and \( u' \). If \( u' x \in E(T) \), then \( d(f, u' x) = 1 = d(e, u' x) \); otherwise, \( d(e, ux) = 1 \neq 2 = d(f, ux) \) or \( d(e, vx) = 1 \neq 2 = d(f, vx) \), according to whether \( ux \) or \( vx \) is an edge of \( T \). So \( c_D(e) \neq c_D(f) \).

**Case 2.** \( u' \neq v' \). Then we may assume that \( f \) is incident with \( u' \). Let \( g \) be an edge of \( T \) distinct from \( uu' \) that is incident with \( u' \). Then \( d(e, g) = 2 \neq 1 = d(f, g) \). Thus \( c_D(e) \neq c_D(f) \).

For a connected graph \( G \), let 

\[
m_k(G) = \min\{k(G - T) : T \text{ is a spanning tree of } G\},
\]

where \( k(G - T) \) is the number of components of \( G - T \).

**Theorem 5.9** If \( G \) is a connected graph of order \( n \geq 5 \), then 

\[
\text{cdim}_d(G) \leq n + m_k(G) - 1.
\]
Proof. If \( G \) is a tree of order \( n \), then \( m_k(G) = 0 \). Since the size of \( G \) is \( n - 1 \), it follows by (5.1) that \( \text{cdim}_d(G) \leq n - 1 \) and so the result is true for a tree. Thus we may assume that \( G \) is a connected graph that is not a tree. Suppose that \( m_k(G) = k \). Let \( T \) be a spanning tree of \( G \) such that \( k(G - T) = k \), where \( E(T) = \{e_1, e_2, \ldots, e_{n-1}\} \) and \( H_1, H_2, \ldots, H_k \) are \( k \) components of \( G - T \). Let

\[
\mathcal{D}_0 = \{F_1, F_2, \ldots, F_{n-1}, H_1, H_2, \ldots, H_k\},
\]

where \( E(F_i) = \{e_i\} \) for \( 1 \leq i \leq n - 1 \). Then \( \mathcal{D}_0 \) is a connected decomposition of \( G \) with \( n + k - 1 \) elements. Since \( \mathcal{D}_0 \) is a refinement of the decomposition \( \mathcal{D} \) described in Lemma 5.8, it follows by Theorem 5.3 that \( \mathcal{D}_0 \) is a resolving decomposition of \( G \). Therefore,

\[
\text{cdim}_d(G) \leq |\mathcal{D}_0| = n + k - 1 = n + m_k(G) - 1,
\]

which is the desired result.

Note that the upper bound in Theorem 5.9 is attainable for stars. On the other hand, strict inequality in Theorem 5.9 can hold as well. For example, the graph \( G \) of Figure 5.4 has order \( n = 8 \) and \( m_k(G) = 2 \). Since \( \mathcal{D} = \{G_1, G_2, G_3\} \), where \( E(G_1) = \{e_1, e_2, e_3, e_5, e_7, e_8, e_9\} \), \( E(G_2) = \{e_4\} \), and \( E(G_3) = \{e_6\} \), is a connected resolving decomposition of \( G \), it then follows by Theorem 5.4 that \( \text{cdim}_d(G) = 3 \). Therefore, \( \text{cdim}_d(G) < n + m_k(G) - 1 \) for the graph of Figure 5.4.

![Figure 5.4: A graph \( G \) with \( \text{cdim}_d(G) < n + m_k(G) - 1 \)](image)

5.4 Connected Decomposition Dimensions of Trees

The decomposition dimensions of trees that are not paths have been studied in [4, 19], where bounds for the decomposition dimensions of trees that are not paths have been determined. However, there is no general formula for the decomposition
dimension of a tree that is not a path. In this section, we establish a formula for
the connected decomposition dimension of a tree that is not a path. In order to
do this, we first present a useful lemma. Some additional definitions are needed.

For an ordered set \( W = \{e_1, e_2, \ldots, e_k\} \) of edges in a connected graph \( G \) and
an edge \( e \) of \( G \), the \( k \)-vector
\[
c_W(e) = (d(e, e_1), d(e, e_2), \ldots, d(e, e_k))
\]
is referred to as the code of \( e \) with respect to \( W \). For a cut-vertex \( v \) in a connected
graph \( G \) and a component \( H \) of \( G - v \), the subgraph \( H \) and the vertex \( v \) together
with all edges joining \( v \) and \( V(H) \) in \( G \) is called a branch of \( G \) at \( v \). For a bridge
\( e \) in a connected graph \( G \) and a component \( F \) of \( G - e \), the subgraph \( F \) together
the bridge \( e \) is called a branch of \( G \) at \( e \). For two edges \( e = u_1u_2 \) and \( f = v_1v_2 \) in
\( G \), an \( e-f \) path in \( G \) is a path with its initial edge \( e \) and terminal edge \( f \).

**Lemma 5.10** Let \( T \) be a tree that is not a path such that \( T \) has order \( n \geq 4 \)
and \( p \) exterior major vertices \( v_1, v_2, \ldots, v_p \). For \( 1 \leq i \leq p \), let \( u_{i1}, u_{i2}, \ldots, u_{ik_i} \) be
the terminal vertices of \( v_i \), let \( P_{ij} \) be the \( v_i - u_{ij} \) path \( (1 \leq j \leq k_i) \), and let \( x_{ij} \) be
a vertex in \( P_{ij} \) that is adjacent to \( v_i \). Let
\[
W = \{u_ix_{ij} : 1 \leq i \leq p \text{ and } 2 \leq j \leq k_i\}.
\]
Then \( c_W(e) \neq c_W(f) \) for each pair \( e, f \) of distinct edges of \( T \) that are not edges of
\( P_{ij} \) for \( 1 \leq i \leq p \) and \( 2 \leq j \leq k_i \).

**Proof.** Let \( e \) and \( f \) be two edges of \( T \) that are not edges of \( P_{ij} \) for \( 1 \leq i \leq p \)
and \( 2 \leq j \leq k_i \). We consider two cases.

**Case 1.** \( e \) lies on some path \( P_{i1} \) for some \( i \) with \( 1 \leq i \leq p \). There are two
subcases.

**Subcase 1.1.** There is an edge \( w \in W \) such that \( f \) lies on the \( e-w \) path or
\( e \) lies on the \( f-w \) path. Then either \( d(f, w) < d(e, w) \) or \( d(e, w) < d(f, w) \). In
either case, \( c_W(e) \neq c_W(f) \).

**Subcase 1.2.** Every path between \( f \) and an edge of \( W \) does not contain \( e \) and
every path between \( e \) and an edge of \( W \) does not contain \( f \). Necessarily, then \( f \)
lies on some path $P_{\ell_1}$ in $T$ for some $1 \leq \ell \leq p$. Observe that $i \neq \ell$, for otherwise, there exists $w \in W$ such that either $f$ lies on an $e - w$ path or $e$ lies on an $f - w$ path. Since $v_i$ and $v_\ell$ are exterior major vertices, it follows that $\deg v_i \geq 3$ and $\deg v_\ell \geq 3$. Thus there exist a branch $B_1$ at $v_i$ that does not contain $x_{i_1}$ and a branch $B_2$ at $v_\ell$ that does not contain $x_{\ell_1}$. Necessarily, each of $B_1$ and $B_2$ must contain an edge of $W$. Let $w_1$ and $w_2$ be two edges in $W$ such that $w_i$ belongs to $B_i$ for $i = 1, 2$. If $d(e, w_2) \neq d(f, w_2)$, then $c_W(e) \neq c_W(f)$. Thus we may assume that $d(e, w_2) = d(f, w_2)$. However, then $d(e, w_1) < d(f, w_1)$, again implying that $c_W(e) \neq c_W(f)$.

**Case 2.** $e$ lies on no path $P_{\ell_1}$ for all $i$ with $1 \leq i \leq p$. Then there are at least two branches at $e$, say $B_1^*$ and $B_2^*$, each of which contains some exterior major vertex of terminal degree at least 2. Thus each branch $B_i^*$ ($i = 1, 2$) contains an edge in $W$. Let $w_i^* \in W$ such that $w_i^*$ belongs to $B_i^*$ for $i = 1, 2$. First, assume that $f \in E(B_1^*)$. Then the $f - w_2^*$ path of $T$ contains $e$. So $d(e, w_2^*) < d(f, w_2^*)$, implying that $c_W(e) \neq c_W(f)$. Next, assume that $f \notin E(B_1^*)$. Then the $f - w_1^*$ path of $T$ contains $e$. Thus $d(e, w_1^*) < d(f, w_1^*)$ and so $c_W(e) \neq c_W(f)$.

We are now prepared to establish a formula for the connected decomposition dimension of a tree that is not a path.

**Theorem 5.11** *If $T$ is a tree that is not a path, then*

\[
\text{cdim}_d(T) = \sigma(T) - \text{ex}(T) + 1.
\]

**Proof.** Suppose that $T$ contains $p$ exterior major vertices $v_1, v_2, \ldots, v_p$. For each $i$ with $1 \leq i \leq p$, let $u_{i_1}, u_{i_2}, \ldots, u_{i_k}$ be the terminal vertices of $v_i$. For each pair $i, j$ of integers with $1 \leq i \leq p$ and $1 \leq j \leq k_i$, let $P_{ij}$ be the $v_i - u_{ij}$ path in $T$ and let $x_{ij}$ be a vertex in $P_{ij}$ that is adjacent to $v_i$. Let $D$ be a connected resolving decomposition of $T$. First we verify the following claim.

**Claim.** For each fixed exterior major vertex $v_i$ ($1 \leq i \leq p$), there is at least one edge $e_{ij}$ from each path $P_{ij}$ ($1 \leq j \leq k_i$) such that the $k_i$ edges $e_{ij}$ ($1 \leq j \leq k_i$) of $T$ belong to distinct elements in $D$. 

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**Proof of Claim.** Assume, to the contrary, that this is not the case. Since each element in \( \mathcal{D} \) is connected, we assume, without loss of generality, that \( P_{i1} \) and \( P_{i2} \) are contained in the same element of \( \mathcal{D} \). However, then, \( d(v_i x_{i1}, e) = d(v_i x_{i2}, e) \) for all \( e \in E(G - (P_{i1} \cup P_{i2})) \), and so \( c_{\mathcal{D}}(v_i x_{i1}) = c_{\mathcal{D}}(v_i x_{i2}) \), which is a contradiction. Therefore, for each fixed \( i \) with \( 1 \leq i \leq p \), the \( k_i \) edges \( e_{ij} \in E(P_{ij}) \) \( (1 \leq j \leq k_i) \) belong to distinct elements in \( \mathcal{D} \), as claimed.

First, we show that \( \text{cdim}_d(T) \geq \sigma(T) - \text{ex}(T) + 1 \). Let
\[
\mathcal{D} = \{G_1, G_2, \ldots, G_t\}
\]
be a minimum connected resolving decomposition of \( T \). Let \( V = \{v_1, v_2, \ldots, v_p\} \) be the set of the exterior major vertices of \( T \). First, assume that \( p = 1 \). Since the \( k_1 \) edges \( e_{1j} \in E(P_{1j}) \) \( (1 \leq j \leq k_1) \) belong to distinct elements in \( \mathcal{D} \), it follows that
\[
\text{cdim}_d(G) \geq k_1 = \sigma(T) - \text{ex}(T) + 1.
\]
Thus we may assume that \( p \geq 2 \). We proceed by the following steps:

**Step 1.** Since \( p \geq 2 \), there exists an exterior major vertex \( v_i \) with \( 1 \leq i \leq p \) such that \( \deg v_i = k_i + 1 \). Start with such an exterior major vertex, say \( v_1 \) with \( \deg v_1 = k_1 + 1 \). Since the \( k_1 \) edges \( e_{1j} \in E(P_{1j}) \) \( (1 \leq j \leq k_1) \) belong to distinct elements in \( \mathcal{D} \), we may assume, without loss of generality, that \( e_{1j} \in E(G_j) \) for \( 1 \leq j \leq k_1 \). Thus
\[
\text{cdim}_d(G) = |\mathcal{D}| \geq k_1 = (k_1 - 1) + 1.
\]

**Step 2.** Consider an exterior major vertex \( v \in V - \{v_1\} \) such that the \( v_1 - v \) path in \( T \) contains no other exterior major vertices in \( V - \{v_1, v\} \). We may assume that \( v = v_2 \). Then the \( k_2 \) edges \( e_{2j} \in E(P_{2j}) \) \( (1 \leq j \leq k_2) \) belong to distinct elements in \( \mathcal{D} \). We claim that at most one of the edges \( e_{2j} \) \( (1 \leq j \leq k_2) \) belongs to the elements \( G_1, G_2, \ldots, G_{k_1} \) of \( \mathcal{D} \). Assume, to the contrary, that two edges in \( \{e_{2j} : 1 \leq j \leq k_2\} \) belong to \( G_1, G_2, \ldots, G_{k_1} \), say \( e_{21} \) and \( e_{22} \) belong to \( G_1, G_2, \ldots, G_{k_1} \). Since \( e_{21} \) and \( e_{22} \) belong to distinct elements in \( \mathcal{D} \), it follows that \( e_{21} \) and \( e_{22} \) belong to two distinct elements of \( G_1, G_2, \ldots, G_{k_1} \), say \( e_{21} \in E(G_1) \) and \( e_{22} \in E(G_2) \).
and \( e_{2j} \in E(G_2) \). However, then, either \( G_1 \) or \( G_2 \) must be disconnected, which is a contradiction. Hence, as claimed, at most one of the edges \( e_{2j} \) \((1 \leq j \leq k_2)\) belongs to the elements \( G_1, G_2, \ldots, G_{k_1} \) in \( \mathcal{D} \). Then assume, without loss of generality, that \( e_{2j} \in E(G_{j+k_1}) \) for \( 1 \leq j \leq k_2 - 1 \). Thus \( G_1, G_2, \ldots, G_{k_1}, G_{k_1+1}, \ldots, G_{k_1+k_2-1} \) must be distinct elements of \( \mathcal{D} \), implying that

\[
\text{cdim}_d(G) = |\mathcal{D}| \geq k_1 + k_2 - 1 = (k_1 - 1) + (k_2 - 1) + 1.
\]

If \( p = 2 \), then \( k_1 + k_2 - 1 = \sigma(T) - \text{ex}(T) + 1 \) and the proof is complete; Otherwise, we continue to the next step.

**Step 3.** Consider an exterior major vertex \( v \in V - \{v_1, v_2\} \) such that the \( v_1 - v \) path in \( T \) contains no other exterior major vertices in \( V - \{v_1, v_2\} \). We may assume that \( v = v_3 \). Then the \( k_3 \) edges \( e_{3j} \in E(P_{3j}) \) \((1 \leq j \leq k_3)\) belong to distinct elements in \( \mathcal{D} \). Again, we claim that at most one of the edges \( e_{3j} \in E(P_{3j}) \) \((1 \leq j \leq k_3)\) belongs to some element \( G_i \) of \( \mathcal{D} \), where \( 1 \leq i \leq k_1 + k_2 - 1 \). Assume, to the contrary, that two edges in \( \{e_{3j} : 1 \leq j \leq k_2\} \) belong to \( G_s \) and \( G_t \), respectively, where \( 1 \leq s < t \leq k_1 + k_2 - 1 \). If \( 1 \leq s < t \leq k_1 \) or \( k_1 + 1 \leq s < t \leq k_1 + k_2 - 1 \), then at least one of \( G_s \) and \( G_t \) must be disconnected, which is impossible. On the other hand, if \( 1 \leq s \leq k_1 \) and \( k_1 + 1 \leq t \leq k_1 + k_2 - 1 \), then, since \( G_s \) and \( G_t \) are connected, there must be a cycle in \( T \), which is again impossible. Thus, we may assume, without loss of generality, that \( e_{3j} \in E(G_{k_1+k_2-1+j}) \) for \( 1 \leq j \leq k_3 - 1 \). Hence all subgraphs \( G_i \) \((1 \leq i \leq k_1 + k_2 + k_3 - 2)\) are distinct elements of \( \mathcal{D} \) and so

\[
\text{cdim}_d(G) = |\mathcal{D}| \geq k_1 + k_2 + k_3 - 2 = (k_1 - 1) + (k_2 - 1) + (k_3 - 1) + 1.
\]

We continue this procedure to the remaining exterior major vertices in \( V - \{v_1, v_2, v_3\} \) and repeat the argument similar to the one in the previous step until we exhaust all vertices in \( V \). Then we obtain

\[
\text{cdim}_d(G) = |\mathcal{D}| \geq \left( \sum_{i=1}^{p} (k_i - 1) \right) + 1 = \sigma(G) - \text{ex}(G) + 1.
\]

Next we show that \( \text{cdim}_d(T) \leq \sigma(T) - \text{ex}(T) + 1 \). Let
$k = \sigma(T) - \text{ex}(T) + 1.$

Let $f_{ij} = u_ix_{ij}$ for $1 \leq i \leq p$ and $1 \leq j \leq k_i$. Let $U = \{u_1, u_11, u_21, \ldots, u_p1\}$ and let $T_0$ be a subtree of $T$ of smallest size such that $T_0$ contains $U$. Now, let

$$D = \{T_0, P_{12}, P_{13}, \ldots, P_{1k_1}, P_{22}, P_{23}, \ldots, P_{2k_2}, \ldots, P_{p2}, P_{p3}, \ldots, P_{pk_p}\}.$$

Certainly, $D$ is a connected $k$-decomposition of $T$. We show that $D$ is a resolving decomposition of $T$. It suffices to show that the edges of $T$ belonging to same element of $D$ have distinct $D$-codes. Let $e, f \in E(T)$. We consider two cases.

**Case 1.** $e, f \in E(T_0)$. Then $d(e, P_{ij}) = d(e, f_{ij})$ and $d(f, P_{ij}) = d(f, f_{ij})$ for all pairs $i, j$ with $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Let

$$W = \{f_{ij} : 1 \leq i \leq p \text{ and } 2 \leq j \leq k_i\}.$$

By Lemma 5.10, $c_W(e) \neq c_W(f)$. Observe that the first coordinate in each of $c_D(e)$ and $c_D(f)$ is 0, the remaining $k - 1$ coordinates of $c_D(e)$ are exactly those of $c_W(e)$, and the remaining $k - 1$ coordinates of $c_D(f)$ are exactly those of $c_W(f)$. Since $c_W(e) \neq c_W(f)$, it follows that $c_D(e) \neq c_D(f)$.

**Case 2.** $e, f \in E(P_{ij})$, where $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Then $d(e, T_0) = d(e, f_{ii1})$ and $d(f, T_0) = d(f, f_{ii1})$. Since $e$ and $f$ are two distinct edges in the path $P_{ij}$, it follows that $d(e, f_{ii1}) \neq d(f, f_{ii1})$ and so $d(e, T_0) \neq d(f, T_0)$. Therefore, $c_D(e) \neq c_D(f)$.

Therefore, $D$ is a connected resolving $k$-decomposition of $T$ and so $\text{cdim}_d(T) \leq k = \sigma(T) - \text{ex}(T) + 1$, as desired.

By Theorems 3.10 and 5.11, if $T$ is a tree that is not a path, then

$$\text{cpd}(T) = \text{cdim}_d(T) = \sigma(T) - \text{ex}(T) + 1.$$ 

On the other hand, we have seen in Lemma 3.9 that

$$\text{cpd}(G) \geq \sigma(G) - \text{ex}(G) + 1$$

for every connected graph $G$. However, this is not true in general for connected decomposition dimensions of graphs. To see this, consider the graphs $F$ and $H$ of
CHAPTER 5. CONNECTED RESOLVING DECOMPOSITIONS

Figure 5.5. Notice that \( \sigma(F) = 3 \), \( \text{ex}(F) = 1 \), \( \sigma(H) = 8 \), and \( \text{ex}(H) = 3 \). Thus \( \sigma(F) - \text{ex}(F) + 1 = 3 \) and \( \sigma(H) - \text{ex}(H) + 1 = 6 \).

On the other hand, let

\[ D_1 = \{ F_1, F_2, F_3, F_4 \}, \]

where \( E(F_1) = \{ u_1 u_2, u_1 u_7, u_2 u_3 \} \), \( E(F_2) = \{ u_1 u_6, u_1 u_8, u_5 u_6 \} \), \( E(F_3) = \{ u_3 u_4, u_4 u_5 \} \), and \( E(F_4) = \{ u_1 u_9 \} \) be a connected decomposition of \( F \); and let

\[ D_2 = \{ H_1, H_2, H_3, H_4, H_5 \}, \]

where \( E(H_1) = \{ v_1 v_2, v_1 x_1, v_2 v_3, v_3 v_4, v_4 y_1 \} \), \( E(H_2) = \{ v_1 v_6, v_1 x_2, v_4 v_5, v_4 y_2, v_5 v_6, v_5 w_1 \} \), \( E(H_3) = \{ v_1 x_3 \} \), \( E(H_4) = \{ v_4 y_3 \} \), and \( E(H_5) = \{ v_5 w_2 \} \) be a connected decomposition of \( H \). The decompositions \( D_1 \) of \( F \) and \( D_2 \) of \( H \) are shown in Figure 5.5. Since \( D_1 \) and \( D_2 \) are resolving, it follows that \( \text{cdim}_d(F) \leq 4 \) and \( \text{cdim}_d(H) \leq 5 \). In fact, it can be verified that \( \text{cdim}_d(F) = 4 \) and \( \text{cdim}_d(H) = 5 \). Therefore,

\[ \text{cdim}_d(F) > \sigma(F) - \text{ex}(F) + 1, \]

\[ \text{cdim}_d(H) < \sigma(H) - \text{ex}(H) + 1. \]
Moreover, for the graph $G$ of Figure 5.1,
\[ \text{cdim}_d(G) = \sigma(G) - \text{ex}(G) + 1 = 4. \]

### 5.5 Graphs With Connected Decomposition Dimension $m - 1$

In this section, we establish a characterization of connected graphs of size $m \geq 3$ having connected decomposition dimension $m - 1$. For $n \geq 4$, let $T_n$ be the tree of order $n$ obtained from the path $P_3$ by adding $n - 3$ pendant edges at an end-vertex of $P_3$. The tree $T_n$ is shown in Figure 5.6. In particular, $T_4 = P_4$.

**Theorem 5.12** Let $G$ be a connected graph of size $m \geq 3$. Then $\text{cdim}_d(G) = m - 1$ if and only if $G$ is one of the graphs in Figure 5.6.

![Figure 5.6: The graphs $C_4$, $(K_1 \cup K_2) + K_1$, and $T_n$ in Theorem 5.12](image)

**Proof.** It is routine to verify that the graphs stated in the theorem have connected decomposition dimension $m - 1$. For the converse, assume that $G$ is a connected graph of size $m \geq 3$ and connected decomposition dimension $m - 1$. If $m = 3$, then $G \in \{P_4, K_3, K_{1,3}\}$. Since $\text{cdim}_d(P_4) = 2$ and $\text{cdim}_d(K_3) = \text{cdim}_d(K_{1,3}) = 3$ by Theorem 5.4, it follows that $P_4 = T_4$ is the only graph with the desired property. If $m = 4$, then
\[ G \in \{C_4, (K_1 \cup K_2) + K_1, K_{1,4}, P_5, T_5\}. \]

Since $\text{cdim}_d(G) = 3 = m - 1$ if $G \in \{C_4, (K_1 \cup K_2) + K_1, T_5\}$, it follows by Theorem 5.4 that $C_4, (K_1 \cup K_2) + K_1$, and $T_5$ are the only connected graphs with the desired property for $m = 4$. 

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We now assume that \( m \geq 5 \). First, suppose that \( G \) is not a tree. Let \( \ell \) be the girth of \( G \). If \( \ell \geq 5 \), then \( \text{cdim}_d(G) \leq m - 2 \) by Theorem 5.7. Thus \( \ell = 4 \) or \( \ell = 3 \). We consider these two cases.

Case 1. \( \ell = 4 \). Let \( C_4 : v_1, v_2, v_3, v_4, v_1 \) be a cycle of length 4 in \( G \). Since \( G \) is connected and \( m \geq 5 \), there exists a vertex \( v \) not in \( C \) such that \( v \) is adjacent to a vertex of \( C \), say \( v \) is adjacent to \( v_1 \). Since \( \ell = 4 \), it follows that \( v \) is not adjacent to \( v_i \) for \( i = 2,4 \). Let \( D = \{G_1, G_2, \ldots, G_{m-2}\} \), where \( E(G_1) = \{vv_1, v_1v_2\} \), \( E(G_2) = \{v_1v_4, v_3v_4\} \), \( E(G_3) = \{v_2v_3\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m-2) \) consists of exactly one edge in \( E(G) - (E(C) \cup \{vv_1\}) \). Then \( D \) is connected. Since \( c_D(vv_1) = (0,1,2,\ldots) \), \( c_D(v_1v_2) = (0,1,1,\ldots) \), \( c_D(v_1v_4) = (1,0,2,\ldots) \), and \( c_D(v_3v_4) = (2,0,1,\ldots) \), it follows that \( D \) is a connected resolving decomposition of \( G \) and so \( \text{cdim}_d(G) \leq |D| = m - 2 \), which is a contradiction.

Case 2. \( \ell = 3 \). If the order of \( G \) is 4, then \( G = K_4 - e \) or \( G = K_4 \). Since \( \text{cdim}_d(K_4 - e) = 3 \) and \( \text{cdim}_d(K_4) = 4 \), it follows that \( \text{cdim}_d(G) = m - 2 \) for all connected graphs \( G \) of order 4 and size \( m \geq 5 \). Thus we may assume that \( n \geq 5 \). Let \( C : v_1, v_2, v_3, v_1 \) be a 3-cycle in \( G \). Then there exists a vertex \( v \) not in \( C \) such that \( v \) is adjacent to a vertex of \( C \), say \( vv_1 \in E(G) \). Since \( n \geq 5 \) and \( G \) is connected, there exists a vertex \( w \in V(G) - \{v, v_1, v_2, v_3\} \) such that \( w \) is adjacent to at least one vertex in \( \{v, v_1, v_2, v_3\} \). We consider three subcases.

Subcase 2.1. \( w \) is adjacent to \( v \). Let \( D = \{G_1, G_2, \ldots, G_{m-2}\} \), where \( E(G_1) = \{v_1v, vw\} \), \( E(G_2) = \{v_1v_2, v_2v_3\} \), \( E(G_3) = \{v_1v_3\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m-2) \) consists of exactly one edge in \( E(G) - (E(C) \cup \{vv_1\}) \). Since \( c_D(v_1v_2) = (1,0,\ldots) \), \( c_D(v_2v_3) = (2,0,\ldots) \), \( c_D(v_1v) = (0,1,\ldots) \), and \( c_D(v_3v_4) = (0,2,\ldots) \), it follows that \( D \) is a connected resolving decomposition of \( G \) and so \( \text{cdim}_d(G) \leq |D| = m - 2 \), which is a contradiction.

Subcase 2.2. \( w \) is adjacent to \( v_1 \). Let \( D = \{G_1, G_2, \ldots, G_{m-2}\} \), where \( E(G_1) = \{v_1v, v_1v_2\} \), \( E(G_2) = \{v_1v_3, v_1w\} \), \( E(G_3) = \{v_2v_3\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m-2) \) consists of exactly one edge in \( E(G) - (E(C) \cup \{vv_1\}) \). Since \( c_D(v_1v) = (0,1,2,\ldots) \), \( c_D(v_1v_2) = (0,1,1,\ldots) \), \( c_D(v_1v_3) = (1,0,1,\ldots) \), and \( c_D(v_1w) = (1,0,2,\ldots) \), it follows that \( D \) is a connected resolving decomposition of \( G \) and so \( \text{cdim}_d(G) \leq |D| = m - 2 \), a contradiction.
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Subcase 2.3. \( w \) is adjacent to \( v_2 \) or to \( v_3 \), say \( w \) is adjacent to \( v_2 \). Let \( D = \{G_1, G_2, \ldots, G_{m-2}\} \), where \( E(G_1) = \{v_1v_2, v_2w\} \), \( E(G_2) = \{v_1v_3, v_1w\} \), \( E(G_3) = \{v_2v_3\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m-2) \) consists of exactly one edge in \( E(G) - (E(C) \cup \{v_1v, v_2w\}) \). Since \( c_D(v_1v_2) = (0, 1, 1 \ldots) \), \( c_D(v_2w) = (0, 2, 1, \ldots) \), \( c_D(v_1v_3) = (1, 0, 1, \ldots) \), and \( c_D(v_1v) = (1, 0, 2, \ldots) \), it follows that \( D \) is a connected resolving decomposition of \( G \) and so \( \text{cdim}_d(G) \leq |D| = m - 2 \), again, a contradiction.

Thus, \( G \) is a tree of size \( m \geq 5 \). Since \( \text{cdim}_d(P_n) = 2 \) for \( n \geq 3 \), it follows that \( G \) is not a path. Furthermore, by Theorem 5.6, the diameter \( d \) of \( G \) is at most 3. If \( d = 2 \), then \( G \) is a star and so \( \text{cdim}_d(G) = m \). Thus \( d = 3 \) and \( G \) is a double star. Let \( u \) and \( v \) be the two central vertices of \( G \); that is, \( u \) and \( v \) are not end-vertices of \( G \). If \( \text{deg}u \geq 3 \) and \( \text{deg}v \geq 3 \), then \( u \) and \( v \) are exterior major vertices of \( G \) and so \( \text{ex}(G) = 2 \). Since \( \sigma(G) = m - 1 \), it follows by Theorem 5.11 that \( \text{cdim}_d(G) = (m - 1) - 2 + 1 = m - 2 \), which is a contradiction. Thus exactly one of \( u \) and \( v \) has degree 3 or more. Therefore, \( G = T_n \), as desired.

\[ \square \]

5.6 Graphs With Connected Decomposition Dimension \( m-2 \)

In this section we present a characterization of connected graphs of size \( m \geq 4 \) having connected decomposition dimension \( m - 2 \). For \( n \geq 5 \), let \( H_n = (K_2 \cup (n - 3)K_1) + K_1 \). For \( n \geq 6 \), let \( X_n \) denote any double star of order \( n \) with two central vertices of degree at least 3. For \( n \geq 5 \), let \( Y_n \) be the graph obtained from \( P_4 \) by adding \( n - 4 \) pendant edges at an end-vertex of \( P_4 \), and let \( Z_n \) be the graph obtained from \( P_5 : v_1, v_2, v_3, v_4, v_5 \) by adding \( n - 5 \) pendant edges at \( v_3 \). In particular, \( Y_5 = Z_5 = P_5 \). The graphs \( H_n, X_n, Y_n, \) and \( Z_n \) are shown in Figure 5.7.

Theorem 5.13 Let \( G \) be a connected graph of size \( m \geq 4 \). Then \( \text{cdim}_d(G) = m - 2 \) if and only if \( G \) is one of the graphs in Figure 5.7.

Proof. It is routine to verify that each graph \( G \) in Figure 5.7 has connected decomposition dimension \( m - 2 \), where \( m \) is the size of \( G \). For the converse, assume that \( G \) is a connected graph of order \( n \geq 4 \), size \( m \geq 4 \), and connected decomposition dimension \( m - 2 \).
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If $n = 4$ and $m \geq 4$, then $G \in \{C_4, (K_2 \cup K_1) + K_1, K_4, K_4 - e\}$. Since $\text{cdim}_d(K_4) = 4$, and $\text{cdim}_d(K_4 - e) = 3$, it follows by Theorem 5.12 that $K_4 = F_1$ and $K_4 - e = F_2$ are the only graphs with the desired property for $n = 4$. If $n = 5$ and $m = 4$, then $G \in \{K_{1,4}, P_5, T_5\}$, where $T_5$ is the graph of Figure 5.6 for $n = 5$. By Theorems 5.4 and 5.12, $P_5$ is the only graph with the desired property. If $n = 5$ and $m = 5$, then $G \in \{F_i : 3 \leq i \leq 6\} \cup \{H_5\}$. Since $\text{cdim}_d(F_i) = \text{cdim}_d(H_5) = 3 = m - 2$ for $3 \leq i \leq 6$, the graphs $F_i$, $3 \leq i \leq 6$, and $H_5$ are the only graphs with the desired property for $n = 5$ and $m = 5$.

We now assume that $n \geq 5$ and $m \geq 6$. We may assume that $G$ is not one of the graphs of Figure 5.7. First, suppose that $G$ is not a tree. Let $\ell$ be the girth of $G$. Since $\text{cdim}_d(G) = m - 2$, it follows by Theorem 5.7 that $3 \leq \ell \leq 5$. We consider three cases, according to whether $\ell = 5$, $\ell = 4$, or $\ell = 3$.

Case 1. $\ell = 5$. Let $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ be a 5-cycle in $G$. Since $m \geq 6$ and $C_5$ is a smallest cycle in $G$, there exists a vertex $v \in V(G) - V(C_5)$ such that $v$ is adjacent exactly one vertex of $C_5$, say $v$ is adjacent to $v_1$. Let $D = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of $G$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4\}$, $E(G_3) = \{v_4v_5, v_5v_1\}$, and each of $E(G_i) (4 \leq i \leq m - 3)$ consists

Figure 5.7: The graphs $F_i (1 \leq i \leq 6)$, $H_n$, $X_n$, $Y_n$, and $Z_n$ in Theorem 5.13
of exactly one edge in \( E(G) - (E(C_3) \cup \{vv_1\}) \). Thus \( \mathcal{D} \) is connected. Since 
\[
c_D(vv_1) = (0, 2, 1, \ldots), \quad c_D(v_1v_2) = (0, 1, 1, \ldots), \quad c_D(v_2v_3) = (1, 0, 2, \ldots), \quad c_D(v_3v_4) = (2, 0, 1, \ldots), \quad c_D(v_4v_5) = (2, 1, 0, \ldots), \quad \text{and} \quad c_D(v_5v_1) = (1, 2, 0, \ldots),
\]
it follows that \( \mathcal{D} \) is a connected resolving decomposition of \( G \). Thus \( \text{cdim}_d(G) \leq |\mathcal{D}| = m - 3 \), which is a contradiction.

**Case 2.** \( \ell = 4 \). Let \( C_4 : v_1, v_2, v_3, v_4, v_1 \) be a 4-cycle in \( G \). Since \( n \geq 5 \), there exists a vertex \( v \in V(G) - V(C_4) \) such that \( v \) is adjacent one vertex of \( C_4 \), say, \( v \) is adjacent to \( v_1 \). Since \( m \geq 6 \), it follows that \( G \) contains an edge \( f \) such that \( f \notin E(C_4) \cup \{vv_1\} \) and \( f \) is adjacent to some edge in \( E(C_4) \cup \{vv_1\} \). Thus \( G \) must contain a subgraph that is isomorphic to one of the graphs \( A_i \) \((1 \leq i \leq 5) \) in Figure 5.8.

![Figure 5.8: The graphs A_i (1 ≤ i ≤ 5)](image)

**Subcase 2.1.** \( G \) contains a subgraph that is isomorphic to \( A_1 \). Let \( \mathcal{D} = \{G_1, G_2, \ldots, G_{m-3}\} \) be a decomposition of \( G \), where \( E(G_1) = \{vv_3\} \), \( E(G_2) = \{vv_1, v_1v_2\} \), \( E(G_3) = \{v_1v_4, v_2v_3, v_3v_4\} \), and each of \( E(G_i) \) \((4 \leq i \leq m - 3)\) consists of exactly one edge in \( E(G) - (E(C_4) \cup \{vv_1, vv_3\}) \). Thus \( \mathcal{D} \) is connected. Since 
\[
c_D(vv_1) = (1, 0, 1, \ldots), \quad c_D(v_1v_2) = (2, 0, 1, \ldots), \quad c_D(v_1v_4) = (2, 1, 0, \ldots), \quad c_D(v_2v_3) = (1, 1, 0, \ldots), \quad \text{and} \quad c_D(v_3v_4) = (1, 2, 0, \ldots),
\]
it follows that \( \mathcal{D} \) is a resolving decomposition of \( G \).

**Subcase 2.2.** \( G \) contains a subgraph that is isomorphic to \( A_2 \). Let \( \mathcal{D} = \{G_1, G_2, \ldots, G_{m-3}\} \) be a decomposition of \( G \), where \( E(G_1) = \{vv_1, v_1v_4\} \) , \( E(G_2) = \{vv_1, v_1v_4\} \), \( E(G_3) = \{v_1v_4, v_2v_3, v_3v_4\} \), and each of \( E(G_i) \) \((4 \leq i \leq m - 3)\) consists of exactly one edge in \( E(G) - (E(C_4) \cup \{vv_1, vv_3\}) \). Thus \( \mathcal{D} \) is connected. Since 
\[
c_D(vv_1) = (1, 0, 1, \ldots), \quad c_D(v_1v_2) = (2, 0, 1, \ldots), \quad c_D(v_1v_4) = (2, 1, 0, \ldots), \quad c_D(v_2v_3) = (1, 1, 0, \ldots), \quad \text{and} \quad c_D(v_3v_4) = (1, 2, 0, \ldots),
\]
\( \{v_4w, v_3v_4\} \) \( E(G_3) = \{v_1v_2, v_2v_3\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m - 3) \) consists of exactly one edge in \( E(G) - (E(C_4) \cup \{vu_1, vu_4\}) \). Thus \( D \) is connected. Since 
\[
\begin{align*}
c_D(vu_1) &= (0, 2, 1, \ldots), \\
c_D(v_1v_4) &= (0, 1, 1, \ldots), \\
c_D(v_4w) &= (1, 0, 2, \ldots), \\
c_D(v_3v_4) &= (1, 0, 1, \ldots), \\
c_D(v_1v_2) &= (1, 2, 0, \ldots), \quad \text{and} \\
c_D(v_2v_3) &= (2, 1, 0, \ldots), \end{align*}
\]
it follows that \( D \) is a resolving decomposition of \( G \).

Subcase 2.3. \( G \) contains a subgraph that is isomorphic to \( A_3 \). Let \( D = \{G_1, G_2, \ldots, G_{m-3}\} \) be a decomposition of \( G \), where \( E(G_1) = \{vu_1, vu_4\} \), \( E(G_2) = \{v_1v_2, v_2v_3\} \) \( E(G_3) = \{v_3w, v_3v_4\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m - 3) \) consists of exactly one edge in \( E(G) - (E(C_4) \cup \{vu_1, v_3w\}) \). Thus \( D \) is connected. Since 
\[
\begin{align*}
c_D(vu_1) &= (0, 1, 2, \ldots), \\
c_D(v_1v_4) &= (0, 1, 1, \ldots), \\
c_D(v_1v_2) &= (1, 0, 2, \ldots), \\
c_D(v_2v_3) &= (2, 0, 1, \ldots), \\
c_D(v_3w) &= (2, 1, 0, \ldots), \quad \text{and} \\
c_D(v_3v_4) &= (1, 1, 0, \ldots), \end{align*}
\]
it follows that \( D \) is a resolving decomposition of \( G \).

Subcase 2.4. \( G \) contains a subgraph that is isomorphic to \( A_4 \). Let \( D = \{G_1, G_2, \ldots, G_{m-3}\} \) be a decomposition of \( G \), where \( E(G_1) = \{vu_1, vu_4\} \), \( E(G_2) = \{v_1v_2, v_2v_3\} \) \( E(G_3) = \{v_3w, v_4v_1\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m - 3) \) consists of exactly one edge in \( E(G) - (E(C_4) \cup \{vu_1, v_3w\}) \). Thus \( D \) is connected. Since 
\[
\begin{align*}
c_D(vu_1) &= (0, 2, 1, \ldots), \\
c_D(v_1v_2) &= (0, 1, 1, \ldots), \\
c_D(v_2v_3) &= (1, 0, 2, \ldots), \\
c_D(v_3v_4) &= (2, 0, 1, \ldots), \\
c_D(v_1v_4) &= (1, 1, 0, \ldots), \quad \text{and} \\
c_D(v_1w) &= (1, 2, 0, \ldots), \end{align*}
\]
it follows that \( D \) is a resolving decomposition of \( G \).

Subcase 2.5. \( G \) contains a subgraph that is isomorphic to \( A_5 \). Since \( \ell = 4 \), it follows that \( v_1w, vv_2, vv_4 \notin E(G) \). If \( vv_3 \in E(G) \), then \( G \) contains a subgraph that is isomorphic to \( A_1 \), and so the result follows by Subcase 2.1. If \( v_2w \in E(G) \) or \( v_4w \in E(G) \), then \( G \) contains a subgraph that is isomorphic to \( A_2 \), and so the result follows by Subcase 2.2. If \( v_3w \in E(G) \), then \( G \) contains a subgraph that is isomorphic to \( A_3 \), and so the result follows by Subcase 2.3. Thus we may assume that none of \( vv_2, vv_3, vv_4, v_1w, v_2w, v_3w, v_4w \) is an edge of \( G \). Let \( D = \{G_1, G_2, \ldots, G_{m-3}\} \) be a decomposition of \( G \), where \( E(G_1) = \{vu\} \), \( E(G_2) = \{vu_1, vu_2\} \) \( E(G_3) = \{v_1v_4, v_2v_3, v_3v_4\} \), and each of \( E(G_i) \) \( (4 \leq i \leq m - 3) \) consists of exactly one edge in \( E(G) - (E(C_4) \cup \{vu_1, v_1w\}) \). Thus \( D \) is connected. Since 
\[
\begin{align*}
c_D(vu_1) &= (1, 0, 1, \ldots), \\
c_D(v_1v_2) &= (2, 0, 1, \ldots), \\
c_D(v_2v_3) &= (3, 1, 0, \ldots), \\
c_D(v_1v_4) &= (2, 1, 0, \ldots), \quad \text{and} \\
c_D(v_3v_4) &= (3, 2, 0, \ldots), \end{align*}
\]
it follows that \( D \)
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is a resolving decomposition of $G$.

Thus, in each case, $G$ has a connected resolving decomposition with $m - 3$ elements, and so $\text{cdim}_d(G) \leq m - 3$, which is a contradiction.

Case 3. $\ell = 3$. Let $C_3 : v_1, v_2, v_3, v_1$ be a 3-cycle in $G$. Since $G$ is not one of the graphs in Figure 5.7, it follows that $G \neq H_n$. Since $n \geq 5$, there exist $v, w \in V(G) - V(C_3)$ such that the subgraph induced by $\{v_1, v_2, v_3, v, w\}$ is a connected subgraph of $G$. This fact together with $m \geq 6$ implies that $G$ must contain a subgraph that is isomorphic to one of the graphs $B_i$ ($1 \leq i \leq 10$) in Figure 5.9.

We proceed by cases. In each of the following subcases, we construct a connected resolving decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-3}\}$ of $G$ by choosing $G_1, G_2, G_3$ such that $|E(G_1)| + |E(G_2)| + |E(G_3)| = 6$ and such that each $G_i$ ($4 \leq i \leq m-3$) contains exactly one edge from $E(G) - (E(G_1) \cup E(G_2) \cup E(G_3))$.

Subcase 3.1. $G$ contains $B_1$. Let $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, and

\[ B_1 : \]
\[ B_2 : \]
\[ B_3 : \]
\[ B_4 : \]
\[ B_5 : \]
\[ B_6 : \]
\[ B_7 : \]
\[ B_8 : \]
\[ B_9 : \]
\[ B_{10} : \]

Figure 5.9: The graphs $B_i$ ($1 \leq i \leq 10$)
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$E(G_3) = \{v_1v_3, v_1v, v_1w, vw_3\}$. Then $c_D(v_1v_3) = (1, 1, 0, \ldots), c_D(v_1v) = (1, 2, 0, \ldots), c_D(vw) = (2, 2, 0, \ldots)$, and $c_D(wv_3) = (2, 1, 0, \ldots)$.

**Subcase 3.2.** $G$ contains $B_2$. Let $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_1v_3, vv_1\}$, and $E(G_3) = \{v_2v_3, v_2w, wv_3\}$. Then $c_D(v_1v_3) = (1, 0, 1, \ldots), c_D(v_1v) = (1, 0, 2, \ldots), c_D(v_2v_3) = (1, 1, 0, \ldots), c_D(v_2w) = (1, 2, 0, \ldots)$, and $c_D(wv_3) = (2, 1, 0, \ldots)$.

**Subcase 3.3.** $G$ contains $B_3$. Let $E(G_1) = \{v_1v_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, and $E(G_3) = \{v_1v_2, v_2w, vv_1\}$. Then $c_D(vv_1) = (0, 2, 1, \ldots), c_D(v_1v_3) = (0, 1, 1, \ldots), c_D(v_1v_2) = (1, 1, 0, \ldots), c_D(v_2w) = (2, 1, 0, \ldots)$, and $c_D(wv_3) = (1, 1, 0, \ldots)$.

**Subcase 3.4.** $G$ contains $B_4$. If $vv_2 \in E(G)$ or $v_3w \in E(G)$, then $G$ contains $B_3$ and so the result follows by Subcase 3.3. Thus we may assume that $vv_2 \notin E(G)$ and $v_3w \notin E(G)$. Similarly, we may assume that $wv_2 \notin E(G)$ and $v_3w \notin E(G)$. Let $E(G_1) = \{vv\}$, $E(G_2) = \{v_1v_1, v_1v_2\}$, and $E(G_3) = \{v_2v_3, v_3v_1, vv_1\}$. Then $c_D(vv_1) = (1, 0, 1, \ldots), c_D(v_1v_2) = (2, 0, 1, \ldots), c_D(v_2v_3) = (3, 1, 0, \ldots), c_D(v_3v_1) = (2, 1, 0, \ldots)$, and $c_D(wv_3) = (1, 1, 0, \ldots)$.

**Subcase 3.5.** $G$ contains $B_5$. Let $E(G_1) = \{vv_1\}$, $E(G_2) = \{v_2w, v_1v_2\}$, and $E(G_3) = \{v_1v_3, v_2v_3, v_3x\}$. Then $c_D(v_2w) = (2, 0, 1, \ldots), c_D(v_1v_2) = (1, 0, 1, \ldots), c_D(v_1v_3) = (1, 1, 0, \ldots), c_D(v_2v_3) = (2, 1, 0, \ldots)$, and $c_D(v_3x) = (2, 2, 0, \ldots)$.

**Subcase 3.6.** $G$ contains $B_6$. Let $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3x\}$, and $E(G_3) = \{v_1v_3, v_1w\}$. Then $c_D(vv_1) = (0, 2, 1, \ldots), c_D(v_1v_2) = (0, 1, 1, \ldots), c_D(v_2v_3) = (1, 0, 1, \ldots), c_D(v_3x) = (2, 0, 1, \ldots), c_D(v_1v_3) = (1, 1, 0, \ldots)$, and $c_D(v_1w) = (1, 2, 0, \ldots)$.

**Subcase 3.7.** $G$ contains $B_7$. Let $E(G_1) = \{vv_1, v_1v_3\}$, $E(G_2) = \{v_1v_2, v_2v_3\}$, and $E(G_3) = \{wx, v_3w\}$. Then $c_D(vv_1) = (0, 1, 2, \ldots), c_D(v_1v_3) = (0, 1, 1, \ldots), c_D(v_1v_2) = (1, 0, 2, \ldots), c_D(v_2v_3) = (1, 0, 1, \ldots), c_D(wx) = (2, 2, 0, \ldots)$, and $c_D(v_3w) = (1, 1, 0, \ldots)$.

**Subcase 3.8.** $G$ contains $B_8$. If there is an edge joining one vertex in $\{v_2, v_3\}$ and one vertex in $\{w, x\}$, then $G$ contains at least one of $B_1, B_2,$ and $B_7$ and so the result follows by Subcases 3.1, 3.2, or 3.7. Thus we may assume that there is no edge between $\{v_2, v_3\}$ and $\{w, x\}$. Let $E(G_1) = \{wx, vv, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, and $E(G_3) = \{v_1v_3\}$. Then $c_D(wx) = (0, *, 3, \ldots)$, where * is either 3 or
Subcase 3.9. G contains $B_9$. If there is an edge joining one vertex in \{v_2, v_3\} and one vertex in \{v, x\}, then $G$ contains $B_1$ or $B_2$ and so the result follows by Subcases 3.1 or 3.2. Thus we may assume that there is no edge between \{v_2, v_3\} and \{v, x\}. Let $E(G_1) = \{vw, v_1v, v_1v_2\}$, $E(G_2) = \{vx\}$, and $E(G_3) = \{v_1v_3, v_2v_3\}$. Then $c_D(vw) = (0, 1, 2, \ldots)$, $c_D(vv_1) = (0, 1, 1, \ldots)$, $c_D(v_1v_2) = (0, 2, 1, \ldots)$, $c_D(v_1v_3) = (1, 2, 0, \ldots)$, and $c_D(v_2v_3) = (1, 3, 0, \ldots)$.

Subcase 3.10. G contains $B_{10}$. If there is an edge joining one vertex in \{v_2, v_3\} and one vertex in \{v, w\}, then $G$ contains $B_1$ or $B_3$ and so the result follows by Subcases 3.1 or 3.3. Thus we may assume that there is no edge between \{v_2, v_3\} and \{v, w\}. Let $E(G_1) = \{vw, vv_1, vv_3\}$, $E(G_2) = \{vx, v_1v_2\}$, and $E(G_3) = \{v_2v_3\}$. Then $c_D(vw) = (0, 2, 3, \ldots)$, $c_D(vv_1) = (0, 1, 2, \ldots)$, $c_D(v_1v_3) = (0, 1, 1, \ldots)$, $c_D(v_2v_3) = (1, 0, 2, \ldots)$, and $c_D(v_1v_2) = (1, 0, 1, \ldots)$.

Thus, in each subcase above, $G$ has a connected resolving decomposition with $m - 3$ elements, and so so $\text{cdim}_d(G) \leq m - 3$, which is a contradiction.

Therefore, $G$ is a tree of order $n \geq 5$ and size $m \geq 6$. Let $d$ be the diameter of $G$. By Theorem 5.4, $G$ is neither a path nor a star and so $d \geq 3$. On the other hand, by Theorem 5.6, $d \leq 4$. Thus, $d = 3$ or $d = 4$. We consider these two cases.

Case 1. $d = 3$. Then $G$ is a double star. Let $u$ and $v$ be the two central vertices of $G$. Since $G$ is not a star, at least one of $u$ and $v$ has degree 3 or more. On the other hand, if exactly one of $u$ and $v$ has degree 3 or more, then $\text{cdim}_d(G) = m - 1$ by Theorem 5.12. Therefore, $G = X_n$ as shown in Figure 5.7.

Case 2. $d = 4$. Let $P_5 : v_1, v_2, v_3, v_4, v_5$ be a path of length 4 in $G$. Since $G \neq P_5$, at least one of the vertices $v_2, v_3, v_4$ has degree 3 or more. We claim that $G = Y_n$ or $G = Z_n$ in Figure 5.7 in this case. Assume, to the contrary, that this is not true. Then $G$ contains a subgraph that is isomorphic to one of the graphs $T_1$, $T_2$, and $T_3$ in Figure 5.10.

If $G$ contains the subgraph that is isomorphic to $T_1$, then let $D = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of $G$, where $E(G_1) = \{v_2v\}$, $E(G_2) = \{v_3w\}$, $E(G_3) = E(P_5)$, and each of $E(G_i) (4 \leq i \leq m - 3)$ consists of exactly one edge in
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Figure 5.10: The graphs $T_i \ (1 \leq i \leq 3)$

$E(G) - (E(P_5) \cup \{v_2v, v_3w\})$. If $G$ contains the subgraph that is isomorphic to $T_2$, then let $D = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of $G$, where $E(G_1) = \{vv_2\}$, $E(G_2) = \{v_4w\}$, $E(G_3) = E(P_5)$, and each of $E(G_i) \ (4 \leq i \leq m-3)$ consists of exactly one edge in $E(G) - (E(P_5) \cup \{v_2v, v_4w\})$. If $G$ contain the subgraph that is isomorphic to $T_3$, then let $D = \{G_1, G_2, \ldots, G_{m-3}\}$ be a decomposition of $G$, where $E(G_1) = \{v_1v_2, v_2v_3\}$, $E(G_2) = \{v_3v_4, v_4v_5\}$, $E(G_3) = \{v_3v, vw\}$, and each of $E(G_i) \ (4 \leq i \leq m-3)$ consists of exactly one edge in $E(G) - (E(P_5) \cup \{v_3v, vw\})$. In each case, it can be verified that $D$ is a connected resolving decomposition of $G$ and so $\text{cdim}_d(G) \leq |D| = m - 3$, which is a contradiction. Therefore, $G = Y_n$ or $G = Z_n$, as claimed.

5.7 Graphs With Prescribed Connected Decomposition Dimension

We have seen that if $G$ is a connected graph of size at least 2 with $\dim_d(G) = a$ and $\text{cdim}_d(G) = b$, then $2 \leq a \leq b$. Furthermore, the paths of order at least 3 are the only connected graphs $G$ of size at least 2 with $\dim_d(G) = \text{cdim}_d(G) = 2$. Thus there is no connected graph $G$ with $\dim_d(G) = 2$ and $\text{cdim}_d(G) > 2$. On the other hand, every pair $a, b$ of integers with $3 \leq a \leq b$ is realizable as the decomposition dimension and connected decomposition dimension, respectively, of some connected graph. In order to show this, we first present a lemma.

Lemma 5.14 Let $G$ be a connected graph that is not a star. If $G$ contains a vertex that is adjacent to $k \geq 1$ end-vertices, then

$$\text{cdim}_d(G) \geq \dim_d(G) \geq k + 1.$$
Proof. By Observation 5.1, \( \dim_d(G) \geq k \). Next we show that \( \dim_d(G) \neq k \). Assume, to the contrary, that \( \dim_d(G) = k \). Let \( D = \{G_1, G_2, \ldots, G_k \} \) be a resolving decomposition of \( G \). Let \( v \) be a vertex of \( G \) that is adjacent to \( k \) end-vertices \( v_1, v_2, \ldots, v_k \). Let \( e_i = vv_i \), where \( 1 \leq i \leq k \). By Observation 5.1, the \( k \) edges \( e_i \) (\( 1 \leq i \leq k \)) belong to distinct elements of \( D \). Without loss of generality, assume that \( e_i \in E(G_i) \) for \( 1 \leq i \leq k \). Since \( G \) is not a star, there exists a vertex \( w \) distinct from \( v_i \) (\( 1 \leq i \leq k \)) such that \( w \) is adjacent to \( v \) and \( w \) is not an end-vertex of \( G \). We may assume the edge \( e = vw \) belongs to \( G_1 \). However, then, \( c_D(e) = c_D(e_1) = (0, 1, 1, \ldots, 1) \), which is a contradiction. Thus \( \dim_d(G) \geq k + 1 \). The fact that \( cdim_d(G) \geq k + 1 \) follows by (5.1).

Theorem 5.15 For every pair \( a, b \) of integers with \( 3 \leq a \leq b \), there exists a connected graph \( G \) such that \( \dim_d(G) = a \) and \( cdim_d(G) = b \).

Proof. For \( a = b \geq 3 \), let \( G = K_{1,a} \). Since \( \dim_d(K_{1,a}) = cdim_d(K_{1,a}) = a \), the result holds for \( a = b \). Thus we may assume that \( a < b \). We consider two cases, according to whether \( a = 3 \) or \( a \geq 4 \).

Case 1. \( a = 3 \). For each \( i \) with \( 1 \leq i \leq b - 1 \), let \( T_i \) be the tree obtained from the path \( P_i : u_{i1}, u_{i2}, \ldots, u_{ii} \) of order \( i \) by adding two new vertices \( u_i^* \) and \( u_i^* \) and joining \( u_i \) and \( u_i^* \) to \( v_i \). Then the graph \( G \) is obtained from the graphs \( T_i \) (\( 1 \leq i \leq b - 1 \)) by adding edges \( u_{i1}u_{i+1,1} \) for \( 1 \leq i \leq b - 2 \). The graph \( G \) is shown in Figure 5.11 for \( b = 5 \). Since \( G \) is a tree with \( \sigma(G) = 2(b - 1) \) and \( ex(G) = b - 1 \), it follows by Theorem 5.11 that \( cdim_d(G) = b \). It remains to show that \( \dim_d(G) = 3 \). Let \( D = \{G_1, G_2, G_3\} \), where \( E(G_1) = \{u_{i1}v_{i1}\} \), \( E(G_2) = \{u_{i1}v_{ii} : 2 \leq i \leq b - 1\} \), and \( E(G_3) = E(G) - (E(G_1) \cup E(G_2)) \). We show that \( D \) is a resolving decomposition of \( G \). Observe that

\[
\begin{align*}
c_D(u_{i1}v_{i1}) &= (2i - 1, 0, 1) \text{ for } 2 \leq i \leq b - 1, \\
c_D(u_{i1}^{*}v_{i1}) &= (1, 3, 0), \\
c_D(v_{i1}v_{21}) &= (1, 2, 0), \\
c_D(v_{i1}v_{i+1,1}) &= (i, i, 0) \text{ for } 2 \leq i \leq b - 2, \\
c_D(u_{i1}v_{i,j+1}) &= (i + j - 1, i - j, 0) \text{ for } j \leq i, 2 \leq i \leq b - 1, 1 \leq j \leq b - 2,
\end{align*}
\]
$c_D(u_i^* u_{ii}) = (2i - 1, 1, 0)$ for $2 \leq i \leq b - 1$.

Since all $D$-codes of the edges of $G$ are distinct, $D$ is a resolving decomposition of $G$ and so $\dim_d(G) \leq |D| = 3$. By Theorem F, $\dim_d(G) = 3$.

![Figure 5.11: A graph $G$ in Case 1 for $b = 5$](image)

**Case 2.** $a \geq 4$. Let $G$ be the graph obtained from the path $P_{b-a+4}$: $u_1, u_2, \ldots, u_{b-a+4}$ of order $b-a+4$ by adding

1. $a - 2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ and joining each vertex $v_i$ ($1 \leq i \leq a-2$) to the vertex $u_2$,
2. a new vertex $u_{a-1}$ and joining $u_{a-1}$ to the vertex $u_{b-a+3}$, and
3. $2(b-a)$ new vertices $w_3, w_3^*, w_4, w_4^*, \ldots, w_{b-a+2}, w_{b-a+2}^*$ and joining $w_j$ and $w_j^*$ to the vertex $u_j$ for $3 \leq j \leq b-a+2$.

Since $G$ is a tree with

$$\sigma(G) = (a - 1) + 2(b-a+1) = 2b - a + 1 \quad \text{and} \quad ex(G) = b - a + 2,$$

it follows by Theorem 5.11 that $c\dim_d(G) = b$. Next we show that $\dim_d(G) = a$. Since $u_2$ is adjacent to $a-1$ end-vertices and $T$ is not a star, it then follows by Lemma 5.14 that $\dim_d(G) \geq a$. On the other hand, let
\[ \mathcal{D} = \{G_1, G_2, \ldots, G_a\}, \]

where \( E(G_1) = E(P_{b-a+4}) \cup \{u_i w_i : 3 \leq i \leq b - a + 2\} \), \( E(G_2) = \{u_2v_1\} \cup \{u_i w_i^* : 3 \leq i \leq b - a + 2\} \), \( E(G_3) = \{u_{b-a+3}v_{a-1}\} \), and \( E(G_i) = \{u_2v_{i-2}\} \) for \( 4 \leq i \leq a \). It can be verified that \( \mathcal{D} \) is a resolving decomposition of \( G \), and so \( \dim_d(G) \leq |\mathcal{D}| = a \). Therefore, \( \dim_d(G) = a \), as desired.

5.8 Connected Decomposition Dimension Ratios

It is often the case that if \( f \) is a parameter defined in terms of the edges of a connected graph \( G \) of size \( m \), then \( 0 < f(G) \leq m \). There are numerous instances in the literature of two such parameters \( f_1 \) and \( f_2 \) being studied, where \( 0 < f_1(G) \leq f_2(G) \leq m \) for every graph \( G \). A common problem concerns whether every two integers \( a \) and \( b \) with \( 0 < a \leq b \) are realizable as the values of \( f_1 \) and \( f_2 \), respectively, for some graph. We saw an example of this in Theorem 5.15, where \( f_1 = \dim_d \) and \( f_2 = \cdim_d \). Normally, a considerably more challenging problem involves, for a given integer \( m \geq 2 \), determining those pairs \( a, b \) of integers with \( 0 < a \leq b \leq m \) for which there exists a graph \( G \) of size \( m \) such that \( f_1(G) = a \) and \( f_2(G) = b \). Often, only partial results of this nature exist. Of course, if for some pair \( a, b \) of integers with \( 0 < a \leq b \leq m \), say, there exists a graph \( G \) of size \( m \) such that \( f_1(G) = a \) and \( f_2(G) = b \), then \( 0 < a/m \leq b/m \leq 1 \). In this case, we say that the rational numbers \( a/m \) and \( b/m \) are realizable as the \( f_1 \)-ratio and \( f_2 \)-ratio, respectively, of some graph. This suggests a new, less restrictive problem when considering such pairs \( f_1, f_2 \) of parameters.

For a connected graph \( G \) of size \( m \geq 2 \), the decomposition dimension ratio \( r_{\dim}(G) \) of \( G \) and the connected decomposition dimension ratio \( r_{\cd}(G) \) of \( G \) are defined as

\[
\begin{align*}
  r_{\dim}(G) &= \frac{\dim_d(G)}{m} \quad \text{and} \quad r_{\cd}(G) = \frac{\cdim_d(G)}{m}.
\end{align*}
\]
Since \(2 \leq \dim_d(G) \leq \cdim_d(G) \leq m\) for every connected graphs \(G\) of size \(m \geq 2\), it follows that

\[
0 < r_{\dim}(G) \leq r_{\cdim}(G) \leq 1.
\]

By Theorem 5.4 and Corollary 5.5, we have the following.

**Proposition 5.16** Let \(G\) be a connected graph of size \(m \geq 2\). Then

\[
r_{\dim}(G) = 1 \text{ if and only if } r_{\cdim}(G) = 1.
\]

In Theorem 5.15, there is no mention of the size of a graph having decomposition dimension \(a\) and connected decomposition dimension \(b\). On the other hand, it is routine to verify that every graph described in Theorem 5.12 has size \(m\) and decomposition dimension \(m - 1\). Thus, if \(a\) and \(m\) are integers with \(2 \leq a \leq m - 2\), then there is no connected graph \(G\) of size \(m\) such that \(\dim_d(G) = a\) and \(\cdim_d(G) = m - 1\). Furthermore, it can be verified, for the graphs described in Theorem 5.13, that

1. \(\dim_d(F_i) = m - 2\) for \(1 \leq i \leq 6\),
2. \(\dim_d(H_n) = m - 2\) for \(n \geq 5\),
3. \(\dim_d(X_n) = \max\{\deg u, \deg v\} + 1\), where \(u\) and \(v\) are the central vertices of \(X_n\) for \(n \geq 6\),
4. \(\dim_d(Y_n) = m - 2\) for \(n \geq 5\),
5. \(\dim_d(Z_n) = m - 2\) if \(n = 5, 6\) and \(\dim_d(Z_n) = m - 3\) if \(n \geq 7\),

where the integer \(m\) is the size of the graph under consideration. Hence, if \(a\) and \(m\) are integers with \(2 \leq a \leq \lfloor (m - 1)/2 \rfloor\) and \(m \geq 7\), then there is no connected graph \(G\) of size \(m\) such that \(\dim_d(G) = a\) and \(\cdim_d(G) = m - 2\). Therefore, there exist infinitely many triples \(a, b, m\) of integers with \(2 \leq a \leq b \leq m\) that are not realizable as the decomposition dimension, connected decomposition dimension, and size of any connected graph. However, we show that every pair \(s, t\) of rational numbers with \(0 < s \leq t < 1\) is realizable as the decomposition dimension ratio and connected decomposition dimension ratio for some connected graph.
Theorem 5.17  For each pair $s, t$ of rational numbers with $0 < s \leq t < 1$, there is a connected graph $G$ such that $r_{\text{dim}}(G) = s$ and $r_{\text{cd}}(G) = t$.

Proof. Let $s = s_1/s_2$ and $t = t_1/t_2$, where $s_1, s_2, t_1, t_2$ be positive integers. Let $a, b \geq 20$ be integers such that $as_2 = bt_2$. Since $0 < s \leq t$, it follows that

$$0 < \frac{as_1}{as_2} \leq \frac{bt_1}{bt_2}.$$  

Because $as_2 = bt_2$, we obtain $0 < as_1 \leq bt_1$. Let

$$bt_1 = kas_1 + k_0,$$

where $k \geq 1$ and $0 \leq k_0 \leq as_1$. Since $b \geq 20$ and $k \geq 1$, it follows that $kb t_2 \geq 20$ and $kas_1 \leq kbt_1$. We construct a connected graph $G$ of size $kbt_2$ such that $\text{dim}_d(G) = kas_1$ and $\text{cdim}_d(G) = kbt_1$. There are two cases.

Case 1. $0 \leq k_0 \leq 5$. Let

$$N = k(k_0 + 4) - 2 \geq 2 \text{ and } L = k[b(t_2 - t_1) - 2k_0 - 8] - 6 \geq 2.$$  

Furthermore, let $P : v_1, v_2, \ldots, v_N$ be a copy of a path of order $N$ and $Q : w_1, w_2, \ldots, w_L$ be a copy of a path of order $L$. Then the graph $G$ is obtained from $P$ and $Q$ by adding

(1) $kas_1 - 1$ new vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,kas_1 - 1}$ and joining each of these vertices to the vertex $v_1$,

(2) $kas_1 - 2$ new vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,kas_1 - 2}$ and joining each of these vertices to the vertex $v_i$ for every $i$ with $2 \leq i \leq k$,

(3) two new vertices $u_{i,1}, u_{i,2}$ and joining these two vertices to the vertex $v_i$ for each $i$ with $k + 1 \leq i \leq N$, and

(4) the edge $u_{N,1}w_1$.  

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Then the size of $G$ is

$$m = |E(P)| + |E(Q)| + (kas_1 - 1) + (k - 1)(kas_1 - 2) + 2(N - k) + 1$$

$$= (N - 1) + (L - 1) + (kas_1 - 1) + (k - 1)(kas_1 - 2) + 2(N - k) + 1$$

$$= kbt_2 - kbt_1 + k^2as_1 + kk_0$$

$$= kbt_2 - kbt_1 + k(bt_1 - k_0) + kk_0 = kbt_2 = kas_2.$$ 

Since $ex(G) = N = k(k_0 + 4) - 2$ and

$$\sigma(G) = (kas_1 - 1) + (k - 1)(kas_1 - 2) + 2(N - k),$$

it then follows by Theorem 5.11 that

$$\text{cdim}_d(G) = k^2as_1 + k_0k = k(bt_1 - k_0) + kk_0 = kbt_1.$$ 

Thus, it remains to show that $\text{dim}_d(G) = kas_1$. Since $v_1$ is adjacent to $kas_1 - 1$ end-vertices in $G$, it follows by Theorem 5.14 that $\text{dim}_d(G) \geq kas_1$. On the other hand, let $D = \{G_1, G_2, \ldots, G_{kas_1}\}$ be a decomposition of $G$, where $E(G_1) = E(P) \cup E(Q) \cup \{u_iv_i : 1 \leq i \leq N\} \cup \{u_{N,1}w_1\}$, $E(G_2) = \{u_{i,2}v_i : 1 \leq i \leq N-1\}$, $E(G_j) = \{u_{ij}v_i : 1 \leq i \leq k\}$ for $3 \leq j \leq kas_1 - 2$, $E(G_{kas_1-1}) = \{u_{1,kas_1-1}v_1\}$, and $E(G_{kas_1}) = \{u_{N,2}v_N\}$. Since

$$d(u_{ij}v_i, G_{kas_1-1}) = i \text{ for } 1 \leq i \leq N \text{ and } 1 \leq j \leq kas_1 - 2,$$

$$d(v_iw_{i+1}, G_{kas_1-1}) = i \text{ for } 1 \leq i \leq N - 1,$$

$$d(u_{N,1}w_1, G_{kas_1-1}) = N + 1,$$

$$d(w_iw_{i+1}, G_{kas_1-1}) = N + 1 + i \text{ for } 1 \leq i \leq L - 1,$$

$$d(u_{i,1}v_i, G_{kas_1}) = N + 1 - i \text{ for } 1 \leq i \leq N,$$

$$d(v_iw_{i+1}, G_{kas_1}) = N - i \text{ for } 1 \leq i \leq N - 1,$$

it follows that $D$ is a resolving decomposition of $G$, implying that $\text{dim}_d(G) \leq |D| = kas_1$. Therefore, $\text{dim}_d(G) = kas_1$.

Case 2. $5 < k_0 < as_1$. Let $N = 4k + 2 \geq 6$ and let $L = kb(t_2 - t_1) - 2(4k+1) \geq 10$. Furthermore, let $P : v_1, v_2, \ldots, v_N$ be a path of order $N$ and $Q : w_1, w_2, \ldots, w_L$ be a path of order $L$. Then the graph $G$ is obtained from $P$ and $Q$ by adding
CHAPTER 5. CONNECTED RESOLVING DECOMPOSITIONS

(1) \(k a s_1 - 1\) new vertices \(u_{1,1}, u_{1,2}, \ldots, u_{1,k a s_1 - 1}\) and joining each of these vertices to the vertex \(v_1\),

(2) \(k a s_1 - 2\) new vertices \(u_{i,1}, u_{i,2}, \ldots, u_{i,k a s_1 - 2}\) and joining each of these vertices to the vertex \(v_i\) for every \(i\) with \(2 \leq i \leq k\),

(3) two new vertices \(u_{i,1}, u_{i,2}\) and joining these two vertices to the vertex \(v_i\) for every \(i\) with \(k + 1 \leq i \leq 4k\),

(4) \(k k_0 - 2\) new vertices \(u_{N-1,1}, u_{N-1,2}, \ldots, u_{N-1,k k_0 - 2}\) and joining each of these vertices to the vertex \(v_{N-1}\),

(5) two new vertices \(u_{N,1}, u_{N,2}\) and joining these two vertices to the vertex \(v_N\), and

(6) the edge \(u_{N,1}w_1\).

Then the size of \(G\) is

\[
m = |E(P)| + |E(Q)| + (k a s_1 - 1) + (k - 1)(k a s_1 - 2) + 2(3k + 1) + (k k_0 - 2) + 1
\]

\[
= (N - 1) + (L - 1) + (k a s_1 - 1) + (k - 1)(k a s_1 - 2) + 6k + k k_0 + 1
\]

\[
= k^2 a s_1 + k k_0 + k b t_2 - k b t_1 = k b t_2 = k a s_2.
\]

Since \(e x(G) = N = 4k + 2\) and

\[
\sigma(G) = (k a s_1 - 1) + (k - 1)(k a s_1 - 2) + 2(3k + 1) + (k k_0 - 2),
\]

it follows by Theorem 5.11 that

\[
\text{cdim}_d(G) = k^2 a s_1 + k k_0 = k b t_1.
\]

Thus it remains to show that \(\text{dim}_d(G) = k a s_1\). Since \(\text{ter}(v_1) > \text{ter}(v_i) > \text{ter}(v_N)\) for all \(2 \leq i \leq N - 1\), it follows by Theorem 5.14 that

\[
\text{dim}_d(G) \geq \text{ter}(v_1) + 1 = (k a s_1 - 1) + 1 = k a s_1.
\]
On the other hand, let $\mathcal{D} = \{G_1, G_2, \ldots, G_{\text{kas}_1}\}$ be a decomposition of $G$, where

\begin{align*}
E(G_1) &= E(P) \cup E(Q) \cup \{u_{ij}v_i : 1 \leq i \leq N\} \cup \{u_{N,1}w_1\}, \\
E(G_2) &= \{v_{ij}^2v_i : 1 \leq i \leq N - 1\}, \\
E(G_j) &= \{u_{ij}v_i : 1 \leq i \leq k, i = N - 1\} \text{ for } 3 \leq j \leq k_{k_0} - 1, \\
E(G_j) &= \{u_{ij}v_i : 1 \leq i \leq k\} \text{ for } k_{k_0} - 1 \leq j \leq k_{a_1} - 2, \\
E(G_{\text{kas}_1 - 1}) &= \{u_1, \ldots, v_{\text{kas}_1 - 1}\}, \text{ and } E(G_{\text{kas}_1}) = \{u_{N,2}v_N\}. \text{ Since}
\end{align*}

\begin{align*}
d(u_{ij}v_i, G_{\text{kas}_1 - 1}) &= i \text{ for } 1 \leq i \leq k, i = N_1, 1 \leq j \leq k_{a_1} - 2, \\
d(v_i, v_{i+1}, G_{\text{kas}_1 - 1}) &= i \text{ for } 1 \leq i \leq N - 1, \\
d(u_{N,1}w_1, G_{\text{kas}_1 - 1}) &= N + 1, \\
d(u_{i+1}, v_i, G_{\text{kas}_1}) &= N + 1 + i \text{ for } 1 \leq i \leq L - 1, \\
d(u_{i+1}, v_i, G_{\text{kas}_1}) &= N + 1 - i \text{ for } 1 \leq i \leq N, \\
d(v_i, v_{i+1}, G_{\text{kas}_1}) &= N - i \text{ for } 1 \leq i \leq N - 1,
\end{align*}

it follows that $\mathcal{D}$ is a resolving decomposition of $G$, implying that $\dim_d(G) \leq |\mathcal{D}| = \text{kas}_1$. Thus $\dim_d(G) = \text{kas}_1$.

Hence, in either case, we construct a connected graph $G$ of size $kbt_2$ such that $\dim_d(G) = \text{kas}_1$ and $\text{cdim}_d(G) = kbt_1$. Therefore,

$$r_{\dim}(G) = \frac{\text{kas}_1}{\text{kas}_2} = \frac{s_1}{s_2} = s \quad \text{and} \quad r_{\text{cd}}(G) = \frac{\text{kbt}_1}{\text{kbt}_2} = \frac{t_1}{t_2} = t,$$

as desired. $\blacksquare$
Chapter 6

Independent Resolving Decompositions

6.1 Introduction

A decomposition $D = \{G_1, G_2, \ldots, G_k\}$ of a connected graph $G$ is called independent if $E(G_i)$ is independent for each $i$ ($1 \leq i \leq k$) in $G$. This concept can be considered from an edge coloring point of view. Recall that a proper edge coloring (or simply, an edge coloring) of a nonempty graph $G$ is an assignment $c$ of colors (positive integers) to the edges of $G$ so that adjacent edges are colored differently, that is, $c : E(G) \rightarrow \mathbb{N}$ is a mapping such that $c(e) \neq c(f)$ if $e$ and $f$ are adjacent edges of $G$. The minimum $k$ for which there is an edge coloring of $G$ using $k$ distinct colors is called the edge chromatic number $\chi_e(G)$ of $G$. If $D = \{G_1, G_2, \ldots, G_k\}$ is an independent decomposition of a graph $G$, then by assigning color $i$ to all edges in $G_i$ ($1 \leq i \leq k$), we obtain an edge coloring of $G$ using $k$ distinct colors. On the other hand, if $c$ is an edge coloring of a connected graph $G$, using the colors $1, 2, \cdots, k$ for some positive integer $k$, then $c(e) \neq c(f)$ for adjacent edges $e$ and $f$ in $G$. Equivalently, $c$ produces a decomposition $D$ of $E(G)$ into color classes (independent sets) $C_1, C_2, \cdots, C_k$, where the edges of $C_i$ are colored $i$ for $1 \leq i \leq k$. Thus, for an edge $e$ in a graph $G$, the $k$-vector

$$c_D(e) = (d(e, C_1), d(e, C_2), \cdots, d(e, C_k))$$
is called the color code (or simply the code) \( c_D(e) \) of \( e \). If distinct edges of \( G \) have distinct color codes, then \( c \) is called a resolving edge coloring (or independent resolving decomposition) of \( G \). Thus a resolving edge coloring of \( G \) is an edge coloring that distinguishes all edges of \( G \) in terms of their distances from the resulting color classes. A minimum resolving edge coloring uses a minimum number of colors, and this number is the resolving edge chromatic number \( \chi_{re}(G) \) of \( G \).

Suppose that \( E(G) = \{e_1, e_2, \ldots, e_m\} \), where \( m \geq 2 \). By assigning the color \( i \) to \( e_i \) for \( 1 \leq i \leq m \), we obtain a resolving edge coloring of \( G \). Thus \( \chi_{re}(G) \) is defined for every graph \( G \). Moreover, every resolving edge coloring is an edge coloring and every resolving edge coloring is a resolving decomposition. Therefore,

\[
2 \leq \max\{\dim_d(G), \chi_e(G)\} \leq \chi_{re}(G) \leq m \tag{6.1}
\]

for each connected graph \( G \) of size \( m \geq 2 \).

To illustrate these concepts, consider the graph \( G \) of Figure 6.1. Let \( D_1 = \{G_1, G_2, G_3\} \) be the decomposition of \( G \), where \( E(G_1) = \{v_1v_2, v_2v_5\} \), \( E(G_2) = \{v_2v_3, v_2v_6, v_3v_6\} \), and \( E(G_3) = \{v_3v_4, v_3v_5\} \). Since \( D_1 \) is a resolving decomposition of \( G \), it follows by Theorem F that \( \dim_d(G) = 3 \). Define an edge coloring \( c \) of \( G \) by assigning the color 1 to \( v_1v_2 \) and \( v_3v_5 \), the color 2 to \( v_2v_5 \) and \( v_3v_6 \), the color 3 to \( v_2v_3 \), and the color 4 to \( v_2v_6 \) and \( v_3v_4 \). The coloring \( c \) is shown in Figure 6.1. Since \( c \) is a minimum edge coloring of \( G \), it follows that \( \chi_e(G) = 4 \). However, \( c \) is not a resolving edge coloring. To see this, let \( D_2 = \{C_1, C_2, C_3, C_4\} \) be the decomposition of \( G \) into color classes resulting from \( c \), where the edges in \( C_i \) are colored \( i \) by \( c \). Then \( c_{D_2}(v_2v_5) = (1, 0, 1, 1) = c_{D_2}(v_3v_6) \). On the other hand, define an edge coloring \( c^* \) of \( G \) by assigning the color 1 to \( v_1v_2 \) and \( v_3v_5 \), the color 2 to \( v_2v_3 \), the color 3 to \( v_2v_5 \) and \( v_3v_4 \), the color 4 to \( v_2v_6 \) and \( v_3v_4 \), and the color 5 to \( v_3v_6 \). The coloring \( c^* \) is shown in Figure 6.1. Since \( c^* \) is an resolving edge coloring and \( G \) has no resolving edge coloring with 4 colors, \( \chi_{re}(G) = 5 \).

The following three observations are useful to us, the first of which is a consequence of (6.1) and the fact the edge chromatic number \( \chi_e(G) \) of a graph \( G \) is bounded below by the maximum degree \( \Delta(G) \) of \( G \). The edge independence number \( \beta_1(G) \) of a graph \( G \) is the maximum cardinality of an independent set of edges in \( G \).
CHAPTER 6. INDEPENDENT RESOLVING DECOMPOSITIONS

FIGURE 6.1: A graph $G$ with $\dim_G(G) = 3$, $\chi_e(G) = 4$, and $\chi_{re}(G) = 5$

**Observation 6.1**  For every graph $G$, $\chi_{re}(G) \geq \Delta(G)$.

**Observation 6.2**  If $G$ is a graph of size $m \geq 2$, then $\chi_{re}(G) \geq \lfloor m/\beta_1(G) \rfloor$.

**Observation 6.3**  Let $G$ be a connected graph. Then $\dim_d(G) = \chi_{re}(G)$ if and only if $G$ contains a minimum resolving decomposition, each of whose elements is independent in $E(G)$.

According to Theorem 5.3, if we are given a minimum resolving decomposition $\mathcal{D}$ of a connected graph $G$, then we can find an independent resolving decomposition $\mathcal{D}^*$ of $G$, where $\mathcal{D}^*$ is a refinement of $\mathcal{D}$. Indeed, the decomposition each of whose elements consists of a single edge has this property. However, a refinement of a minimum resolving decomposition of a graph $G$ need not be a minimum independent resolving decomposition for $G$. In fact, it may be the case that no minimum independent resolving decomposition is a refinement of any minimum resolving decomposition of $G$. For example, consider the graph $G$ of Figure 6.2. The edge coloring $c : E(G) \rightarrow \mathbb{N}$ of $G$ defined by $c(f_1) = 1$, $c(e_1) = c(f_3) = 2$, $c(e_2) = 3$, and $c(e_3) = c(f_2) = 4$, is a minimum edge coloring of $G$. Since $c$ is also resolving, it follows that $\chi_e(G) = \chi_{re}(G) = 4$.

Using a case-by-case analysis, we can show that the graph $G$ of Figure 6.2 has two distinct minimum resolving decompositions (up to isomorphism). The
two distinct minimum resolving decompositions of $G$ of Figure 6.2 are $D_1 = \{G_1, G_2, G_3\}$ and $D_2 = \{H_1, H_2, H_3\}$, where $G_1 = K_{1,3}$, $G_2 = G_3 = K_2$, $H_1 = P_3$, $H_2 = K_3$, and $H_3 = K_4$. For example, let $D_1 = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_2, e_3, f_1\}$, $E(G_2) = \{f_2\}$, and $E(G_3) = \{f_3\}$, and let $D_2 = \{H_1, H_2, H_3\}$, where $E(H_1) = \{e_1, f_1\}$, $E(H_2) = \{e_2, e_3, f_3\}$, and $E(H_3) = \{f_2\}$. The decompositions $D_1$ and $D_2$ are shown in Figure 6.3. Since each independent refinement of $D_i$, where $i = 1, 2$, contains at least six elements and $\chi_{re}(G) = 4$, it follows that no minimum independent resolving decomposition of $G$ is a refinement of any minimum resolving decomposition of $G$.

The following observation will be useful to us.

**Observation 6.4** Let $D$ be an independent resolving decomposition of a connected graph $G$. If $D^*$ is a refinement of $D$, then $D^*$ is also an independent resolving decomposition of $G$.

### 6.2 Resolving Edge Chromatic Numbers of Paths, Stars, and Cycles

In this section, we determine the resolving edge chromatic numbers of paths, stars, and cycles. We begin with paths.
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Proposition 6.5  For \( n \geq 3 \),

\[
\chi_{re}(P_n) = \begin{cases} 
2 & \text{if } n = 3, \\
3 & \text{if } n \geq 4.
\end{cases}
\]

Proof. Let \( P_n : v_1, v_2, \ldots, v_n \) be the path of order \( n \geq 3 \), where \( e_i = v_iv_{i+1} \) for \( 1 \leq i \leq n - 1 \). Clearly, \( \chi_{re}(P_3) = 2 \). Let \( n \geq 4 \). First, we show that \( \chi_{re}(P_n) \geq 3 \). Assume, to the contrary, that \( \chi_{re}(P_n) = 2 \). Let \( c \) be a resolving edge coloring of \( P_n \) using two colors. Since adjacent edges are colored differently by \( c \), we may assume that \( c(e_i) = 1 \) for odd \( i \) and \( c(e_i) = 2 \) for even \( i \), where \( 1 \leq i \leq n - 1 \) and let \( D = \{C_1, C_2\} \) be the decomposition of \( P_n \) into the edge color classes resulting from \( c \). Since \( c_D(e_1) = (0,1) = c_D(e_3) \), it follows that \( c \) is not resolving, which is a contradiction. Thus \( \chi_{re}(P_n) \geq 3 \). Next, we show that \( \chi_{re}(P_n) \leq 3 \). Assign the color 1 to \( e_1 \), the color 2 to \( e_i \) if \( i \) is even, and the color 3 to \( e_i \) if \( i \) is odd and \( i \geq 3 \). Since \( d(e_1, e_i) = i - 1 \) for \( 2 \leq i \leq n - 1 \), it follows that this edge coloring is resolving, and so \( \chi_{re}(P_n) \leq 3 \). Therefore, \( \chi_{re}(P_n) = 3 \) for \( n \geq 4 \).

The following corollary is a consequence of Proposition 6.5 and (6.1).

Corollary 6.6  Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( \chi_{re}(G) = 2 \) if and only if \( G = P_3 \).

Proof. By Proposition 6.5, \( \chi_{re}(P_3) = 2 \). Thus it remains to verify the converse. Assume that \( G \) is a connected graph of order \( n \geq 3 \) with \( \chi_{re}(G) = 2 \). It then follows by (6.1) that \( \dim_d(G) = 2 \) and so \( G = P_n \). Therefore, \( n = 3 \) by Proposition 6.5.

Corollary 6.7  Let \( G \) be a connected graph of size \( m \geq 3 \). Then

\[ 3 \leq \chi_{re}(G) \leq m. \]

By Observation 6.1 and Corollary 6.7, we have the following.

Proposition 6.8  For \( n \geq 3 \), \( \chi_{re}(K_{1,n-1}) = n - 1 \).
Next, we determine the resolving edge chromatic numbers of cycles.

**Theorem 6.9**  For $n \geq 3$,

$$\chi_{re}(C_n) = \begin{cases} 
3 & \text{if } n \text{ is odd,} \\
4 & \text{if } n \text{ is even.}
\end{cases}$$

**Proof.** Let $C_n : v_1, v_2, \ldots, v_n, v_1$ be the cycle of order $n \geq 3$. Let $e_i = v_i v_{i+1}$ for $1 \leq i \leq n - 1$ and let $e_n = v_n v_1$. We consider two cases.

**Case 1.** $n \geq 3$ is odd. Assign the color 1 to $e_1$, the color 2 to $e_i$ if $i$ is even, and the color 3 to $e_i$ if $i \geq 3$ and $i$ is odd. Let $D$ be the decomposition of $C_n$ resulted from this coloring. By Corollary 6.7, it suffices to show that this is a resolving edge coloring. We consider two subcases.

**Subcase 1.1.** $n = 4k + 1$, where $k \geq 1$. For $1 \leq i \leq k$, $c_D(e_{2i}) = (2i - 1, 0, 1)$ and for $k + 1 \leq i \leq 2k$, $c_D(e_{2i}) = (4k - 2i + 2, 0, 1)$. Also, for $1 \leq i \leq k$, $c_D(e_{2i+1}) = (2i, 1, 0)$ and for $k + 1 \leq i \leq 2k$, $c_D(e_{2i+1}) = (4k - 2i + 1, 1, 0)$. Since the codes $c_D(e_i)$ are distinct, this coloring is a resolving edge coloring.

**Subcase 1.2.** $n = 4k + 3$, where $k \geq 0$. For $1 \leq i \leq k + 1$, $c_D(e_{2i}) = (2i - 1, 0, 1)$ and for $k + 2 \leq i \leq 2k + 1$, $c_D(e_{2i}) = (4k - 2i + 4, 0, 1)$. Also, for $1 \leq i \leq k$, $c_D(e_{2i+1}) = (2i, 1, 0)$ and for $k + 1 \leq i \leq 2k + 1$, $c_D(e_{2i+1}) = (4k - 2i + 3, 1, 0)$. Since the codes $c_D(e_i)$ are distinct, this coloring is a resolving edge coloring.

Therefore, $\chi_{re}(C_n) = 3$ if $n \geq 3$ is odd by Corollary 6.7.

**Case 2.** $n \geq 4$ is even. Assign the color 1 to $e_1$, the color 2 to $e_2$, the color 3 to $e_i$ if $i \geq 3$ and $i$ is odd, and the color 4 to $e_i$ if $i \geq 4$ and $i$ is even. We show that this is a resolving edge coloring of $C_n$, thereby verifying that $\chi_{re}(C_n) \leq 4$. Let $D$ be the decomposition of $C_n$ resulted from this coloring.

**Subcase 2.1.** $n = 4k$, where $k \geq 1$. For $1 \leq i \leq k$, $c_D(e_{2i+1}) = (2i, 2i - 1, 0, 1)$, while for $k + 1 \leq i \leq 2k - 1$, $c_D(e_{2i+1}) = (4k - 2i, 4k - 2i + 1, 0, 1)$. For $2 \leq i \leq k$, $(e_{2i}) = (2i - 1, 2i - 2, 1, 0)$; while for $k + 1 \leq i \leq 2k$, $c_D(e_{2i}) = (4k - 2i + 1, 4k - 2i - 1, 0, 1)$.
2i + 2, 1, 0). Since the codes $c_D(e_i)$ are distinct, this coloring is a resolving edge coloring.

**Subcase 2.2.** $n = 4k + 2$, where $k \geq 1$. For $1 \leq i \leq k$, $c_D(e_{2i+1}) = (2i, 2i - 1, 0, 1)$, while for $k + 1 \leq i \leq 2k$, $c_D(e_{2i+1}) = (4k - 2i + 2, 4k - 2i + 3, 0, 1)$. For $1 \leq i \leq k + 1$, $(e_{2i}) = (2i - 1, 2i, 1, 0)$; while for $k + 1 \leq i \leq 2k + 1$, $c_D(e_{2i}) = (4k - 2i + 3, 4k - 2i + 4, 1, 0)$. Since the codes $c_D(e_i)$ are distinct, this coloring is a resolving edge coloring.

It remains only to show that $\chi_{re}(C_n) \geq 4$ if $n$ is even. Assume, to the contrary, this is not the case. Then there exists a resolving edge coloring $c$ of $C_n$ that uses three colors, say, 1, 2, 3, for some even $n \geq 4$. At least one of the colors, say 2, is used to color an even number $t$ of edges of $C_n$, where $2 \leq t \leq n/2$. As we proceed cyclically about $C_n$, beginning with $e_1$, let $e_{i_1}, e_{i_2}, \ldots, e_{i_t}$ be the edges on $C_n$ that are colored 2. Since no two of these edges are adjacent, it follows that for each integer $j$ with $1 \leq j \leq t$, the interval

$$I_j = \{e_{i_j+1}, e_{i_j+2}, \ldots, e_{i_{j+1}-1}\}$$

of integers (subscripts computed modulo $n$) is nonempty.

First, we claim that no interval has odd cardinality that is three or more; for assume, to the contrary, that some interval $I_j$ contains an odd number of (three or more) edges. Without loss of generality, assume that $e_{i_j+1}$ and $e_{i_{j+1}-1}$ are colored 1. However,

$$c_D(e_{i_j+1}) = c_D(e_{i_{j+1}-1}) = (0, 1, 1),$$

which contradicts the fact that $c$ is a resolving edge coloring of $C_n$.

Second, we claim that no interval has even cardinality; for assume, to the contrary, that there are intervals containing an even number of edges. Since $C_n$ has even length, there must be an even number of intervals containing an even number of edges. Let $I_j$ and $I_k$ be two distinct intervals containing an even number of edges. Assume, without loss of generality, that $e_{i_{j+1}}$ is colored 1. Exactly one of $e_{i_k+1}$ and $e_{i_{k+1}-1}$ is colored 1, say the former. Then
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\[ c_D(e_{i,j+1}) = c_D(e_{i,k+1}) = (0, 1, 1), \]
a contradiction. Consequently, all \( t = n/2 \) intervals contain exactly one edge. Necessarily, there is a smallest integer \( i_j \) (1 \( \leq j \leq n/2 \)) such that \( e_{i,j-1} \) and \( e_{i,j+1} \) are colored differently, say 1 and 3, respectively. Necessarily, there is an integer \( i_k > i_j \) such that \( e_{i,k-1} \) is colored 3 and \( e_{i,k+1} \) is colored 1. However, then

\[ c_D(e_{i,j}) = c_D(e_{i,k}) = (1, 0, 1), \]
producing a contradiction.

Therefore, \( \chi_{re}(C_n) = 4 \) if \( n \) is even. \( \blacksquare \)

6.3 Bounds for the Resolving Edge Chromatic Number of a Graph

With the aid of Proposition 6.5 and Theorem 6.9, we are able to establish upper bounds for the resolving edge chromatic number of a connected graph in terms of (1) its size and diameter and (2) its size and girth.

Theorem 6.10 \( \text{If } G \text{ is a connected graph of size } m \geq 3 \text{ and diameter } d, \text{ then} \)

\[ \chi_{re}(G) \leq m - d + 3. \]

Moreover, \( \chi_{re}(G) = m - d + 3 \) if and only if \( G \) is a path of size \( m \geq 3 \).

Proof. If \( d \leq 3 \), then \( \chi_{re}(G) \leq m \leq m - d + 3 \) by (6.1). Thus, we may assume that \( d \geq 4 \). Let \( u \) and \( v \) be vertices of \( G \) for which \( d(u, v) = d \), and let \( P_{d+1} : u = v_1, v_2, \ldots, v_{d+1} = v \) be a \( u - v \) path of length \( d \) in \( G \). Let \( e_i = v_iv_{i+1} \) for \( 1 \leq i \leq d \). Let \( E(G) - E(P_{d+1}) = \{f_1, f_2, \ldots, f_{m-d}\} \). Assign the color 1 to \( e_1 \), the color 2 to \( e_i \) if \( i \) is even, the color 3 to \( e_i \) if \( i \) is odd, and the color \( j + 3 \) to \( f_j \) for \( 1 \leq j \leq m - d \). By the proof of Proposition 6.5, this is a resolving edge coloring of \( G \) with \( m - d + 3 \) colors. Thus \( \chi_{re}(G) \leq m - d + 3 \).

If \( G = P_n \), where \( n \geq 4 \), then \( m = d = n - 1 \) and \( \chi_{re}(P_n) = 3 \) by Proposition 6.5. Therefore, \( \chi_{re}(G) = m - d + 3 \) if \( G \) is a path of size \( m \geq 3 \). It remains to
verify the converse. Assume, to the contrary, that there is a connected graph $G$ of size $m \geq 3$ and diameter $d$ such that $X_{re}(G) = m - d + 3$, but $G$ is not a path. If $d \leq 2$, then $m - d + 3 \geq m + 1$. Since $X_{re}(G) \leq m$ by (6.1), a contradiction is produced. Thus we may assume that $d \geq 3$. Let $u$ and $v$ be two vertices of $G$ with $d(u, v) = d$ and let

$$P : u = v_1, v_2, \ldots, v_{d+1} = v$$

be a $u - v$ path of length $d$. Since (1) $G$ is connected, (2) $P$ is a shortest $u - v$ path, and (3) $G$ is not a path, it follows that there exists $e \in E(G) - E(P)$ such that (i) $e$ is incident with neither $u$ nor $v$ and (ii) $e$ is incident with exactly one vertex in $V(P) - \{u, v\}$. Let $e = v_jx$, where $x \notin V(P)$ and $j \in \{2, 3, \ldots, d\}$. Define

$$Q_1 : x, v_j, v_{j-1}, \ldots, v_1 \quad \text{and} \quad Q_2 : x, v_j, v_{j+1}, \ldots, v_{d+1}$$

be the paths obtained, respectively, from the $v_1 - v_j$ subpath of $P$ by joining $x$ to $v_j$ and from the $v_j - v_{d+1}$ subpath of $P$ by joining $x$ to $v_j$ (see Figure 6.4). We consider two cases.

Case 1. There exists no induced cycle $C$ in $G$ such that $C$ contains the edge $e$ and an $x - y$ subpath of $Q_i$ for any $y \in V(Q_i) - \{x, v_j\}$ and for any $i \in \{1, 2\}$. Define an edge coloring $c$ of $G$ by assigning the color 1 to $v_i v_{i+1}$ if $i$ is odd and $1 \leq i \leq d$, the color 2 to $v_i v_{i+1}$ if $i$ is even and $1 \leq i \leq d$, the color 3 to $e$, and distinct colors from $\{4, 5, \ldots, m - d + 2\}$ to the remaining $m - d - 1$ edges in $E(G) - (E(P) \cup \{e\})$. The colors assigned by $c$ to the edges in $E(P) \cup \{e\}$ are shown in Figure 6.5 for $d = 9$ and $j = 5$. Let
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\[ D = \{ C_1, C_2, \ldots, C_{m-d+2} \} \]

be the decomposition of \( G \) resulting from \( c \). Since

\[
\begin{align*}
  d(v_j+i, v_{j+i+1}, C_3) &= i + 1 \quad \text{for } 0 \leq i \leq d - j, \\
  d(v_{j-i}, v_{j-i+1}, C_3) &= i \quad \text{for } 1 \leq i \leq j - 1,
\end{align*}
\]

the codes \( c_D(v_i v_{i+1}), 1 \leq i \leq d \), are distinct. Therefore, \( c \) is a resolving edge coloring of \( G \) using \( m - d + 2 \) colors. Therefore, \( \chi_{re}(G) \leq m - d + 2 \), which is a contradiction.

![Figure 6.5: The colors of edges in \( E(P) \cup \{e\} \) in Case 1](image)

Case 2. There exists an induced cycle \( C \) in \( G \) such that \( C \) contains the edge \( e \) and an \( x - y \) subpath of \( Q_i \) for some \( y \in V(Q_i) - \{x, v_j\} \) and for some \( i \in \{1, 2\} \).

Without loss of generality, assume that \( C \) is an induced cycle in \( G \) such that \( C \) contains an \( x - y \) subpath of \( Q_1 \) for some \( y \in V(Q_1) - \{x, v_j\} \). Let \( f \) be the edge of \( C \) that is adjacent to \( e \) but not on \( P \). We consider two subcases.

Subcase 2.1. The edge \( f \) is adjacent to an edge of \( P \). Since \( P \) is a shortest \( u - v \) path, it follows that either (1) \( f = v_{j-1} x \) or \( f = v_{j-2} x \) or (2) \( f = v_{j+1} x \) or \( f = v_{j+2} x \). We may assume that (1) occurs (see Figure 6.6). Then \( C \) is either a 3-cycle or a 4-cycle. In this subcase, the edge coloring \( c \) of \( G \) described in Case 1 is also a resolving edge coloring of \( G \) with \( m - d + 2 \) colors. The colors assigned by \( c \) to the edges in \( E(P) \cup \{e\} \) are shown in Figure 6.6 for \( d = 9 \) and \( j = 5 \). Therefore, \( \chi_{re}(G) \leq m - d + 2 \), which is a contradiction.

Subcase 2.2. The edge \( f \) is not adjacent to any edge of \( P \). Define an edge coloring \( c \) of \( G \) by assigning the color 1 to \( v_1 v_2 \), the color 2 to \( f \) and \( v_i v_{i+1} \) if \( i \) is even and \( 1 \leq i \leq d \), the color 3 to \( v_i v_{i+1} \) if \( i \) is odd and \( 1 \leq i \leq d \), and distinct colors from \( \{4, 5, \ldots, m - d + 2\} \) to the remaining \( m - d - 1 \) edges in
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Figure 6.6: The colors of edges in \( E(P) \cup \{e\} \) in Subcase 2.1

\( E(G) - (E(P) \cup \{f\}) \). The colors assigned by \( c \) to the edges in \( E(P) \cup \{f\} \) are shown in Figure 6.7 for \( d = 9 \) and \( j = 5 \). Since \( f \) is not adjacent to any edge of \( P \), it follows that \( c \) is an edge coloring of \( G \). Let

\[ D = \{C_1, C_2, \ldots, C_{m-d+2}\} \]

be the decomposition of \( G \) resulting from \( c \). Since

\[
\begin{align*}
    d(v_i v_{i+1}, C_1) &= i - 1 \quad \text{for } 2 \leq i \leq d, \\
    d(v_i v_{i+1}, C_3) &= 1 \quad \text{for } 2 \leq i \leq d, \\
    d(f, C_3) &= 2,
\end{align*}
\]

it follows that \( c \) is a resolving edge coloring of \( G \) using \( m - d + 2 \) colors. Thus \( \chi_{re}(G) \leq m - d + 2 \), producing a contradiction.

Figure 6.7: The colors of edges in \( E(P) \cup \{f\} \) in Subcase 2.2

Therefore, if \( G \) is a connected graph of size \( m \geq 3 \) and diameter \( d \) with resolving edge chromatic number \( m - d + 3 \), then \( G \) is a path.  

\[ \Box \]
Chapter 6. Independent Resolving Decompositions

Theorem 6.11 If G is a connected graph of size m and girth \( \ell \), where \( m \geq \ell \geq 3 \), then

\[
\chi_{re}(G) \leq \begin{cases} 
  m - \ell + 3 & \text{if } \ell \text{ is odd}, \\
  m - \ell + 4 & \text{if } \ell \text{ is even}.
\end{cases}
\]

Proof. Let \( C_\ell : v_1, v_2, \ldots, v_\ell, v_1 \) be a cycle of order \( \ell \geq 3 \) in \( G \), where \( e_i = v_i v_{i+1} \) for \( 1 \leq i \leq \ell - 1 \) and \( e_\ell = v_1 v_\ell \). We consider two cases.

Case 1. \( \ell = 2k + 1 \), where \( k \geq 1 \). Let \( E(G) - E(C_\ell) = \{ f_1, f_2, \ldots, f_{m-\ell} \} \). Define an edge coloring \( c' : E(G) \to \mathbb{N} \) by \( c'(e_1) = 1, c'(e_i) = 2 \) if \( 2 \leq i \leq \ell \) and \( i \) is even, \( c'(e_i) = 3 \) if \( 3 \leq i \leq \ell \) and \( i \) is odd, and \( c'(f_i) = i + 3 \) for each \( i \) with \( 1 \leq i \leq m - \ell \). Let

\[
D' = \{ C'_1, C'_2, \ldots, C'_{m-\ell+3} \}
\]

be the decomposition of \( G \) into color classes resulting from \( c' \). Observe that if \( i \) is even, then \( c_{D'}(e_i) = (i - 1, 0, 1, \ldots) \) for \( 2 \leq i \leq k + 1 \) and \( c_{D'}(e_i) = (2k - i + 2, 0, 1, \ldots) \) for \( k + 2 \leq i \leq 2k + 1 \). If \( i \) is odd, then \( c_{D'}(e_i) = (i - 1, 1, 0, \ldots) \) for \( 2 \leq i \leq k + 1 \) and \( c_{D'}(e_i) = (2k - i + 2, 1, 0, \ldots) \) for \( k + 2 \leq i \leq 2k + 1 \). Hence the codes \( c_{D'}(e_i) \), \( 1 \leq i \leq 2k + 1 \), are distinct and so \( c' \) is a resolving edge coloring with \( m - \ell + 3 \) colors. Therefore, \( \chi_{re}(G) \leq m - \ell + 3 \), as desired.

Case 2. \( \ell = 2k \), where \( k \geq 2 \). Let \( E(G) - E(C_\ell) = \{ f_1, f_2, \ldots, f_{m-\ell} \} \). Define an edge coloring \( c : E(G) \to \mathbb{N} \) by \( c(e_1) = 1, c(e_2) = 2, c(e_i) = 3 \) if \( 3 \leq i \leq \ell \) and \( i \) is odd, \( c(e_i) = 4 \) if \( 4 \leq i \leq \ell \) and \( i \) is even, and \( c(f_i) = i + 4 \) for each \( i \) with \( 1 \leq i \leq m - \ell \). Let

\[
D = \{ C_1, C_2, \ldots, C_{m-\ell+4} \}
\]

be the decomposition of \( G \) into color classes resulting from \( c \). Observe that if \( i \) is odd, then \( c_{D}(e_i) = (i - 1, i - 2, 0, 1, \ldots) \) for \( 3 \leq i \leq k + 1 \) and \( c_{D}(e_i) = (2k - i + 1, 2k - i + 2, 0, 1, \ldots) \) for \( k + 2 \leq i \leq 2k - 1 \). If \( i \) is even, then \( c_{D}(e_i) = (i - 1, i - 2, 1, 0, \ldots) \) for \( 4 \leq i \leq k + 1 \) and \( c_{D}(e_i) = (2k - i + 1, 2k - i + 2, 1, 0, \ldots) \) for \( k + 2 \leq i \leq 2k \). Hence the codes \( c_{D}(e_i) \), \( 1 \leq i \leq 2k \), are distinct and so \( c \) is a
resolving edge coloring with \( m - \ell + 4 \) colors. Therefore, \( \chi_{re}(G) \leq m - \ell + 4 \), as desired.

Note that if \( G = C_n \) for some odd integer \( n \), then \( \ell = m = n \) and so \( \chi_{re}(G) = m - \ell + 3 \). However, the odd cycles are not the only connected graphs \( G \) of size \( m \geq 3 \) having girth \( \ell \geq 3 \) and \( \chi_{re}(G) = m - \ell + 3 \). For example, let \( G \) be the graph of Figure 6.8. It can be verified that \( \chi_{re}(G) = 4 \). A minimum resolving edge coloring of \( G \) is also shown in Figure 6.8. Since the size of \( G \) is 6 and the girth of \( G \) is 5, it follows that \( \chi_{re}(G) = 4 = m - \ell + 3 \).

![Figure 6.8: A graph G with \( \chi_{re}(G) = 4 = m - \ell + 3 \)](image)

On the other hand, the even cycles are the only connected graphs \( G \) of size \( m \) and girth \( \ell \geq 3 \) such that \( \chi_{re}(G) = m - \ell + 4 \).

**Theorem 6.12** Let \( G \) be a connected graph of size \( m \geq 3 \) and girth \( \ell \geq 3 \). Then \( \chi_{re}(G) = m - \ell + 4 \) if and only if \( G = C_n \) for some even \( n \geq 4 \).

**Proof.** If \( G = C_n \), where \( n \geq 4 \), then \( \ell = m = n \). Since \( \chi_{re}(C_n) = 4 \) for even \( n \geq 4 \) by Theorem 6.9, it follows that \( \chi_{re}(G) = m - \ell + 4 \) if \( G \) is an even cycle.

To verify the converse, let \( G \) be a connected graph of size \( m \geq 3 \) and girth \( \ell \geq 3 \) such that \( \chi_{re}(G) = m - \ell + 4 \). Suppose that \( G \neq C_n \). Let \( C_\ell : v_1, v_2, \ldots, v_\ell, v_1 \) be a cycle of order \( \ell \) in \( G \), where \( \ell < n \). Let \( e_i = v_iv_{i+1} \) for \( 1 \leq i \leq \ell - 1 \) and \( e_\ell = v_\ell v_1 \). If \( \ell \) is odd, then \( \chi_{re}(G) \leq m - \ell + 3 < m - \ell + 4 \) by Theorem 6.11.

Thus we may assume that \( \ell = 2k \geq 4 \) is even. Since \( G \) contains the cycle \( C_\ell \) and \( G \neq C_\ell \), it follows that \( m \geq \ell + 1 \). Because \( G \) is connected, there exists an
edge \( e \in E(G) - E(C_t) \) that is incident with (exactly) one vertex in \( C_t \), say \( v_1 \).

Since \( C_t \) is the smallest cycle of \( G \), it follows that \( e \) is adjacent to exactly the two edges \( e_1 \) and \( e_2 \) of \( C_t \). Let \( E(G) - (E(C_t) \cup \{e\}) = \{f_1, f_2, \ldots, f_{m-t-1}\} \). Define an edge coloring \( c \) of \( G \) by assigning the color 1 to \( e_1 \), the color 2 to \( e_2 \), the color 3 to \( e \) and \( e_i \) for odd \( i \geq 3 \), the color 4 to \( e \) for even \( i \geq 4 \), and the color \( j + 4 \) to \( f_j \) for \( 1 \leq j \leq m - \ell - 1 \). Let \( D \) be the decomposition of \( G \) into color classes resulting from \( c \). Observe that \( c_D(e_1) = (1, 2, 0, 1, \ldots) \) if \( i \) is odd, then \( c_D(e_i) = (i-1, i-2, 0, 1, \ldots) \) for \( 3 \leq i \leq k+1 \) and \( c_D(e_i) = (2k-i+1, 2k-i+2, 0, 1, \ldots) \) for \( k+2 \leq i \leq 2k-1 \). If \( i \) is even, then \( c_D(e_i) = (i-1, i-2, 1, 0, \ldots) \) for \( 4 \leq i \leq k+1 \) and \( c_D(e_i) = (2k-i+1, 2k-i+2, 1, 0, \ldots) \) for \( k+2 \leq i \leq 2k \). Thus \( D \) is a resolving decomposition of \( G \). Therefore, \( \chi_{re}(G) \leq |D| = m - \ell + 3 < m - \ell + 4 \), which is a contradiction. 

Next we establish bounds for the resolving edge chromatic number of a connected graph in terms of (1) its order and edge chromatic number and (2) its decomposition dimension and edge chromatic number.

**Proposition 6.13** If \( G \) is a connected graph of order \( n \geq 5 \), then 
\[
\chi_e(G) \leq \chi_{re}(G) \leq n + \chi_e(G) - 1.
\]

**Proof.** The lower bound follows by (6.1). To verify the upper bound, let \( m \) be the size of \( G \). If \( G \) is a tree of order \( n \), then \( m = n - 1 \). Since \( \chi_{re}(G) \leq m \), the result is true for a tree. Thus we may assume that \( G \) is a connected graph that is not a tree. Let \( T \) be a spanning tree of \( G \) with \( E(T) = \{e_1, e_2, \ldots, e_{n-1}\} \). Let \( H = (E(G) - E(T)) \) be the subgraph induced by \( E(G) - E(T) \). Then \( H \) is a nonempty subgraph of \( G \). Let \( \chi_e(H) = k \) and let \( H_1, H_2, \ldots, H_k \) be the decomposition of \( H \) into the color classes resulting from a minimum edge coloring of \( H \). Now let
\[
D = \{F_1, F_2, \ldots, F_{n-1}, H\} \text{ and } D^* = \{F_1, F_2, \ldots, F_{n-1}, H_1, H_2, \ldots, H_k\},
\]
where \( E(F_i) = \{e_i\} \) for \( 1 \leq i \leq n - 1 \). Since \( D \) is a resolving decomposition of \( G \) by Lemma 5.8 and \( D^* \) is a refinement of \( D \), it follows by Theorem 5.3 that
$D^*$ is a resolving decomposition of $G$ as well. Thus $D^*$ is a resolving independent decomposition of $G$, and so

$$\chi_{re}(G) \leq |D^*| = n + k - 1 = n + \chi_e(H) - 1 \leq n + \chi_e(G) - 1,$$

as desired.

Let $G$ be a connected graph of order $n \geq 5$ that is not a tree and let

$$k = \min\{\chi_e((E(G) - E(T))) : T \text{ is a spanning tree of } G\}.$$  \hspace{1cm} (6.2)

The following corollary is an immediate consequence of the proof of Proposition 6.13.

**Corollary 6.14** If $G$ is a connected graph of order $n \geq 5$, then

$$\chi_{re}(G) \leq n + k - 1$$

where $k$ is defined in (6.2).

It is known that $\Delta(G) \leq \chi_e(G) \leq \Delta(G) + 1$ for every nonempty graph $G$, where the upper bound is due to Vizing [34]. Thus, if $G$ is a connected graph of order $n \geq 5$, then

$$\Delta(G) \leq \chi_{re}(G) \leq n + \Delta(G).$$  \hspace{1cm} (6.3)

However, just how good are the upper bounds in Proposition 6.13, Corollary 6.14, and (6.3) are not known.

### 6.4 Graphs with Prescribed Resolving Edge Chromatic Number

We have seen that if $G$ is a connected graph of size $m \geq 3$, then $3 \leq \chi_{re}(G) \leq m$. In this section, we determine all connected graphs whose resolving edge chromatic number is one of these extremes. We begin by determining all the connected graphs $G$ with $\chi_{re}(G) = 3$. In order to do this, we first present some preliminary results.
Lemma 6.15  If $T$ is a tree with $\Delta(T) = 3$ and having exactly one vertex of degree 3, then $\chi_{re}(T) = 3$.

Proof. Since $\Delta(T) = 3$, it follows by Observation 6.1 that $\chi_{re}(T) \geq 3$. Thus, it remains to show that $\chi_{re}(T) \leq 3$. Suppose that $x$ is the only vertex of degree 3 in $T$. Then we may assume that $T$ is the graph obtained from the paths

$$P_{k_1} : u_1, u_2, \ldots, u_{k_1}, \quad P_{k_2} : v_1, v_2, \ldots, v_{k_2}, \quad P_{k_3} : w_1, w_2, \ldots, w_{k_3}$$

by adding the vertex $x$ and three edges $xu_1, xv_1, xw_1$, where $k_1, k_2,$ and $k_3$ are positive integers. Define an edge coloring $c$ of $T$ by assigning (1) the color 1 to the edges $xu_1, u_iu_{i+1}$ for even $i$ with $2 \leq i \leq k_1 - 1$, and $v_jv_{j+1}$ for odd $j$ with $1 \leq j \leq k_2 - 1$, (2) the color 2 to the edges $xv_1, u_iu_{i+1}$ for odd $i$ with $1 \leq i \leq k_1 - 1$, $v_jv_{j+1}$ for even $j$ with $2 \leq j \leq k_2 - 1$, and $w_\ell w_{\ell+1}$ for odd $\ell$ with $1 \leq \ell \leq k_3 - 1$, and (3) the color 3 to the edges $xw_1, w_\ell w_{\ell+1}$ for even $\ell$ with $2 \leq \ell \leq k_3 - 1$. The edge coloring $c$ of $T$ is illustrated in Figure 6.9 for $k_1 = 5, k_2 = 4, \text{ and } k_3 = 5$.

Figure 6.9: Illustrating the edge coloring $c$ in $T$

Let $D = \{C_1, C_2, C_3\}$ be a decomposition of $T$ resulting from $c$. Since (a) $c_D(xu_1) = (0, 1, 1)$, (b) $c_D(u_iu_{i+1}) = (0, 1, i+1)$, where $i$ is even and $2 \leq i \leq k_1 - 1$, and (c) $c_D(v_jv_{j+1}) = (0, 1, j+1)$, where $j$ is odd and $1 \leq j \leq k_2 - 1$, it follows that all edges in $C_1$ have distinct color codes. Since (a) $c_D(xv_1) = (1, 0, 1)$, (b) $c_D(u_iu_{i+1}) = (1, 0, i+1)$, where $i$ is odd and $1 \leq i \leq k_1 - 1$, (c) $c_D(v_jv_{j+1}) = (1, 0, j+1)$, where $j$ is even and $2 \leq j \leq k_2 - 1$, and (d) $c_D(w_\ell w_{\ell+1}) = (\ell+1, 0, 1)$, where odd $\ell$ and $1 \leq \ell \leq k_3 - 1$, it follows that all edges in $C_2$ have distinct color codes.
Finally, \( c_D(xw_1) = (1, 1, 0) \) and \( c_D(w_\ell w_{\ell+1}) = (\ell + 1, 1, 0) \), where \( \ell \) is even and \( 2 \leq \ell \leq k_3 - 1 \), implying that all edges in \( C_3 \) have distinct color codes. Hence, \( c \) is a resolving edge coloring of \( T \) using three colors and so \( \chi_{re}(T) \leq 3 \). Therefore, \( \chi_{re}(T) = 3 \).

A connected graph containing exactly one cycle is called a unicyclic graph.

**Lemma 6.16** If \( G \) is a unicyclic graph with \( \Delta(G) = 3 \) and having an even cycle and exactly one vertex of degree 3, then \( \chi_{re}(G) = 3 \).

**Proof.** Since \( \Delta(G) = 3 \), it follow from Observation 6.1 that \( \chi_{re}(G) \geq 3 \). Thus, it remains to show that \( \chi_{re}(G) \leq 3 \). Suppose that \( G \) contains an even cycle of order \( k \geq 4 \). Let \( C_k : u_1, u_2, \ldots, u_k, u_1 \) be an even cycle of order \( k \) and \( P_b : v_1, v_2, \ldots, v_b \) be a path of order \( b \geq 1 \). Then we may assume that \( G \) is that graph obtained from \( C_k \) and \( P_b \) by adding an edge \( u_1v_1 \). Define an edge coloring \( c \) of \( G \) by assigning (1) the color 1 to the edges \( u_iu_{i+1} \) for odd \( i \) with \( 1 \leq i \leq k - 1 \), (2) the color 2 to the edges \( u_1v_1 \) and \( v_jv_{j+1} \) for even \( j \) with \( 2 \leq j \leq b - 1 \), and (3) the color 3 to the edges \( u_iu_{i+1} \) for even \( i \) with \( 2 \leq i \leq k \), where the subscripts are expressed as integers module \( k \), and \( v_jv_{j+1} \) for odd \( j \) with \( 1 \leq j \leq b - 1 \). The edge coloring \( c \) is illustrated in Figure 6.10 for \( k = 10 \) and \( b = 4 \).

Figure 6.10: Illustrating the edge coloring \( c \) in \( G \)

Let \( D = \{C_1, C_2, C_3\} \) be a decomposition of \( G \) resulting from \( c \). Let \( k = 2a \), where \( a \geq 2 \). Since \( d(u_iu_{i+1}, C_2) = i \) for odd \( i \) with \( 1 \leq i \leq a \) and \( d(u_iu_{i+1}, C_2) = k - i + 1 \) for odd \( i \) with \( a + 1 \leq i \leq k - 1 \), it follows that all edges in \( C_1 \) have distinct color codes. Since \( d(u_1v_1, C_1) = 1 \) and \( d(v_jv_{j+1}, C_1) = j + 1 \), where \( j \) is even and

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2 \leq j \leq b - 1$, it follows that all edges in $C_2$ have distinct color codes. Finally, $c_D(u_iu_{i+1}) = (1, i, 0)$ for even $i$ with $1 \leq i \leq a$, $c_D(u_iu_{i+1}) = (1, k + 1 - i, 0)$ for even $i$ with $a + 1 \leq i \leq k$, and $c_D(u_ju_{j+1}) = (j + 1, 1, 0)$, where $j$ is odd and $1 \leq j \leq b - 1$. Thus all edges in $C_3$ have distinct color codes. Hence $c$ is a resolving edge coloring of $G$ using three colors and so $\chi_{re}(G) \leq 3$. Therefore, $\chi_{re}(G) = 3$. 

**Lemma 6.17** If $G$ is a unicyclic graph with $\Delta(T) = 3$ and having an odd cycle and exactly one vertex of degree 3, then $\chi_{re}(G) = 4$.

**Proof.** We may assume that $G$ is obtained from the cycle $C_k$: $u_1, u_2, \ldots, u_k$, $u_1$ and a path $P_0 : v_1, v_2, \ldots, v_b$ by adding the edge $u_1v_1$. First, we show that $\chi_{re}(G) \leq 4$. Let $k = 2a + 1$ for some $a \geq 1$. Define an edge coloring $c$ of $G$ by assigning (1) the color 1 to $u_1v_1$ and $u_iv_{i+1}$ for even $i$ with $2 \leq i \leq b - 1$, (2) the color 2 to the edges $u_{i+1}u_{i+2}$ for odd $i$ with $1 \leq i \leq a$ and the edges $u_ju_{j+1}$ for even $j$ with $a + 2 \leq i \leq k - 1$, (3) the color 3 to the edge $u_1u_k$, the edges $u_{i+1}u_{i+2}$ for even $i$ with $2 \leq i \leq a$, the edges $u_ju_{j+1}$ for odd $j$ with $a + 2 \leq i \leq k - 2$, and the edges $v_{i+1}v_{i+2}$ for odd $i$ with $1 \leq i \leq b - 1$, and (4) the color 4 to the edge $u_{a+1}u_{a+2}$. The edge coloring $c$ is illustrated in Figure 6.11 for $k = 9$ and $b = 4$.

![Figure 6.11: Illustrating the edge coloring $c$ in $G$](image)

Let $D = \{C_1, C_2, C_3, C_4\}$ be the decomposition of the graph $G$ resulting from $c$. Since $d(u_1v_1, C_2) = 1$ and $d(u_iv_{i+1}, C_2) = i + 1$ for even $i$ with $2 \leq i \leq b - 1$, it follows that all edges in $C_1$ have distinct color codes. Since $d(u_{i+1}u_{i+2}, C_1) = i$ for odd $i$ with $1 \leq i \leq a$ and $d(u_ju_{j+1}, C_1) = k + 1 - j$ for even $j$ with $a + 2 \leq i \leq k - 1$, it follows that all edges in $C_2$ have distinct color codes.
Finally, since (a) $c_p(u_iu_{i+1}) = (i, 1, 0, a + 1 - i)$ for even $i$ with $2 \leq i \leq a$, (b) $c_p(u_ju_{j+1}) = (k + 1 - j, 1, 0, j - a - 1)$ for odd $j$ with $a + 2 \leq i \leq k - 2$, (c) $c_p(u_1u_k) = (1, 1, 0, k - a - 1)$, and (d) $c_p(u_iu_{i+1}) = (1, i + 1, 0, a + 1 + i)$ for odd $i$ with $1 \leq i \leq b - 1$, it follows that all edges in $G$ have distinct color codes. Thus, $c$ is a resolving edge coloring of $G$ using 4 colors and so $\chi_{re}(G) \leq 4$.

Next, we show that $\chi_{re}(G) > 4$. Since $\Delta(G) = 3$, it follow from Observation 6.1 that $\chi_{re}(G) \geq 3$. Assume, to the contrary, that $\chi_{re}(G) = 3$. Let $c'$ be a resolving edge coloring of $G$ using three colors and let $D' = \{C'_1, C'_2, C'_3\}$ be the decomposition of $G$ resulting from $c'$. Assume, without loss of generality, that $u_1v_1 \in C'_1$, $u_1u_2 \in C'_2$, and $u_1u_{2a+1} \in C'_3$. Then $c_{D'}(u_1v_1) = (0, 1, 1)$, $c_{D'}(u_1u_2) = (1, 0, 1)$, and $c_{D'}(u_1u_k) = (1, 1, 0)$. If $k = 3$, then $u_2u_3 \in C'_1$. However, then $c_{D'}(u_2u_3) = (0, 1, 1) = c_{D'}(u_1v_1)$, which is a contradiction. So $\chi_{re}(G) > 4$ for $k = 3$. Thus we may assume that $k \geq 5$. Since $u_1u_2 \in C'_2$, it follows that $u_2u_3 \in C'_1$ or $u_2u_3 \in C'_3$. We consider these two cases.

**Case 1.** $u_2u_3 \in C'_1$. Notice that no three consecutive edges in $E(C'_k) - \{u_1u_k\}$ are colored by three different colors, namely, the colors 1, 2, 3. Otherwise, assume, to the contrary, that $e_1, e_2, e_3$ are consecutive edges in $E(C'_k) - \{u_1u_k\}$ such that $e_1, e_2, e_3$ are colored differently by the colors 1, 2, and 3, say $e_i \in C'_1$ for $1 \leq i \leq 3$. Then $c_{D'}(e_2) = (1, 0, 1) = c_{D'}(u_1u_2)$, which is a contradiction. Since $\chi_{re}(G) = 3$, there are only two colors that can be assigned to the edges in $E(C'_k) - \{u_1u_k\}$. Since $u_1u_2 \in C'_2$ and $u_2u_3 \in C'_1$, it follows that $u_iu_{i+1} \in C'_2$ for odd $i$ with $1 \leq i \leq k - 2$ and $u_iu_{i+1} \in C'_1$ for even $i$ with $2 \leq i \leq k - 1$. Since $u_1u_k \in C'_3$, it follows that $c_{D'}(u_{k-1}u_k) = (0, 1, 1)$. Since $c_{D'}(u_1v_1) = (0, 1, 1)$ as well, a contradiction is produced.

**Case 2.** $u_2u_3 \in C'_3$. Similarly, no three consecutive edges in $E(C'_k) - \{u_1u_k\}$ are colored by three different colors and so there are only two colors that can be assigned to the edges in $E(C'_k) - \{u_1u_k\}$. Since $u_1u_2 \in C'_2$ and $u_2u_3 \in C'_3$, it follows that $u_iu_{i+1} \in C'_2$ for odd $i$ with $1 \leq i \leq k - 2$ and $u_iu_{i+1} \in C'_3$ for even $i$ with $2 \leq i \leq k - 1$. However, then, the adjacent edges $u_1u_k$ and $u_{k-1}u_k$ are both colored by 3, which is a contradiction.

Therefore, $\chi_{re}(G) = 4$. 

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Let $T$ be set of all trees $T$ with $\Delta(T) = 3$ having exactly one vertex of degree 3 and let $U$ be set of all unicyclic graphs $G$ with $\Delta(G) = 3$ having an even cycle and exactly one vertex of degree 3. The following corollary is a consequence of Lemmas 6.15 – 6.17.

**Corollary 6.18** Let $G$ be a connected graph with $\Delta(G) = 3$ such that $G$ contains exactly one vertex of degree 3. Then $\chi_{re}(G) = 3$ if and only if $G \in T \cup U$.

**Proof.** By Lemmas 6.15 and 6.16, if $G \in T \cup U$, then $\chi_{re}(G) = 3$. For the converse, assume that $G$ is a connected graph with $\Delta(G) = 3$ such that $G$ contains exactly one vertex of degree 3 and $\chi_{re}(G) = 3$. If $G$ is a tree, then $G \in T$; while if $G$ is not a tree, then $G$ contains exactly one cycle. It then follows from Lemmas 6.16 and 6.17 that $G \in U$. □

We are prepared to establish a characterization of connected graphs $G$ of order $n \geq 4$ with $\chi_{re}(G) = 3$.

**Theorem 6.19** Let $G$ be a connected graph of order $n \geq 4$. Then $\chi_{re}(G) = 3$ if and only if (a) $G = P_n$, (b) $G = C_n$, where $n$ is odd, or (c) $G \in T \cup U$.

**Proof.** By Proposition 6.5, Theorem 6.9, and Corollary 6.18, each of the graphs described in (a), (b), and (c) has resolving edge chromatic number 3. For the converse, assume that $G$ is a connected graph of $n \geq 4$ with $\chi_{re}(G) = 3$. Let $D = \{C_1, C_2, C_3\}$ be a decomposition of $G$ resulting from a minimum resolving edge coloring of $G$. Since $\chi_{re}(G) = 3$, it follows by Observation 6.1 that $\Delta(G) = 2$ or $\Delta(G) = 3$. If $\Delta(G) = 2$, then $G = P_n$, where $n \geq 4$, or $G = C_n$, where $n \geq 4$ is odd, and so (a) or (b) holds by Proposition 6.5 and Theorem 6.9. Thus we may assume that $\Delta(G) = 3$. We first verify the following claim.

**Claim:** The graph $G$ contains exactly one vertex of degree 3.

**Proof of Claim:** Assume, to the contrary, that $G$ contains two distinct vertices $u$ and $v$ such that $\deg u = \deg v = 3$. We consider two cases, depending on whether $u$ and $v$ are adjacent.
Case 1. *u and v are adjacent.* Let $e = uv$. Let $e_1$ and $e_2$ be the two edges in $G$ that are distinct from $e$ and incident with $u$, and let $f_1$ and $f_2$ be the two edges in $G$ that are distinct from $e$ and incident with $v$. Then $G$ contains a subgraph that is isomorphic to one of the three graphs in Figure 6.12. Note that \{c(uv), c(e_1), c(e_2)\} = \{c(uv), c(f_1), c(f_2)\} = \{1, 2, 3\}. Thus, in each case, either $c_D(e_1) = c_D(f_1)$ or $c_D(e_1) = c_D(f_2)$, which is a contradiction.

![Figure 6.12: Three graphs in Case 1](image)

Case 2. *u and v are not adjacent.* Let $e_1, e_2, e_3$ be the edges in $G$ that are incident with $u$ and let $f_1, f_2, f_3$ be the edges in $G$ that are incident with $v$. Then $G$ contains a subgraph that is isomorphic to one of the four graphs in Figure 6.13. Note that \{c(e_1), c(e_2), c(e_3)\} = \{c(f_1), c(f_2), c(f_3)\} = \{1, 2, 3\}. By a case-by-case analysis, it can be verified that there exist $e \in \{e_1, e_2, e_3\}$ and $f \in \{f_1, f_2, f_3\}$ such that $c_D(e) = c_D(f)$, which is a contradiction.

This completes the proof of claim. Thus $G$ contains exactly one vertex of degree 3. It then follows by Corollary 6.18 that $G \in \mathcal{T} \cup \mathcal{U}$ and so (c) holds.

Next, we establish a characterization for connected graphs of size $m \geq 2$ with resolving edge chromatic number $m$. First, we present a lemma.

**Lemma 6.20** For $G = K_4 - e$ or $G = K_4$, $\chi_{re}(G) = |E(G)|$.

**Proof.** We label the edges of $K_4 - e$ and $K_4$ as shown in Figure 6.14. First, we show that $\chi_{re}(K_4 - e) = 5$. Since the size of $K_4 - e$ is 5, it follows by (6.1) that $\chi_{re}(K_4 - e) \leq 5$. Assume to the contrary, that $\chi_{re}(K_4 - e) \leq 4$. It then follows by Observation 6.4 that $K_4 - e$ has a resolving edge coloring $c$ using four colors. Let $\mathcal{D} = \{C_1, C_2, C_3, C_4\}$ be the decomposition of $K_4 - e$ resulting from

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c. Without loss of generality, assume that $|C_1| = 2$. Since $e_5$ is adjacent to every edge in $K_4 - e$, it follows that $e_5 \notin C_1$; so $C_1 = \{e_1, e_3\}$ or $C_1 = \{e_2, e_4\}$, say the former. However, then $c_{D'}(e_1) = c_{D'}(e_3)$, which is a contradiction. Therefore, $\chi_{re}(K_4 - e) = 5$.

Next we show that $\chi_{re}(K_4) = 6$. Certainly, $\chi_{re}(K_4) \leq 6$ since the size of $K_4$ is 6. Assume to the contrary, that $\chi_{re}(K_4) \leq 5$. Again, by Observation 6.4, let $c'$ be a resolving edge coloring of $K_4$ using five colors and let $D' = \{C'_1, C'_2, C'_3, C'_4, C'_5\}$ be the decomposition of $K_4$ resulting from $c'$. We may assume that $|C'_1| = 2$ and that $C'_1 = \{e, f\}$, where $e$ and $f$ are nonadjacent edges in $K_4$. Thus $\{e, f\} = \{e_1, e_3\}$,
\{e, f\} = \{e_2, e_4\}, or \{e, f\} = \{e_5, e_6\}. By symmetry, we may assume that \(C'_1 = \{e, f\} = \{e_1, e_3\}\) and so \(C'_2 = \{e_2\}, C'_3 = \{e_4\}, C'_4 = \{e_5\}, \) and \(C'_5 = \{e_6\}\). Then \(c_D(e_1) = (0, 1, 1, 1, 1) = c_D(e_3),\) which is a contradiction. Therefore, \(\chi_{re}(K_4) = 6.\)

\[\blacklozenge\]

**Theorem 6.21** Let \(G\) be a connected graph of order \(n \geq 3\) and size \(m\).

1. If \(n = 3\) or \(n = 4\), then \(\chi_{re}(G) = m,\)
2. If \(n \geq 5\), then \(\chi_{re}(G) = m\) if and only if \(G = K_{1,n-1}\)

**Proof.** We first verify (a). If \(n = 3\), then \(G \in S_1 = \{P_3, K_3\}\). If \(n = 4\), then

\[G \in S_2 = \{P_4, K_{1,3}, P_3 \cup K_1, C_4, K_4 - e, K_4\}\]

With the aid of Propositions 6.5, 6.8, Theorem 6.9, and Lemma 6.20, it is straightforward to verify that each graph in \(S_1 \cup S_2\) has resolving chromatic number equal to its size and so (a) holds.

Next we verify (b). By Proposition 6.8, \(\chi_{re}(K_{1,n-1}) = n - 1.\) Thus it remains to show that if \(G\) is a connected graph of order \(n \geq 5\) and size \(m\) that is not a star, then \(\chi_{re}(G) \leq m - 1.\) Since \(G \neq K_{1,n-1}\), where \(n \geq 5\), it follows that \(G\) contains a path \(P : v_1, v_2, v_3, v_4\) of order 4. Let \(e_i = v_i v_{i+1}\) for \(i = 1, 2, 3.\) Since \(n \geq 5\) and \(G\) is connected, there exists \(v \in V(G) - V(P)\) such that \(v\) is adjacent to at least one vertex of \(P.\) We consider two cases.

**Case 1.** \(v\) is adjacent to \(v_1\) or \(v_4,\) say the former. Define an edge coloring \(c\) of \(G\) by assigning the color 1 to \(e_1\) and \(e_2,\) the color 2 to \(vv_1,\) the color 3 to \(e_3,\) and distinct colors from \(\{4, 5, \ldots, m - 1\}\) to the remaining \(m - 4\) edges in \(E(G) - (E(P) \cup \{vv_1\}).\) Let \(D = \{C_1, C_2, \ldots, C_{m-1}\}\) be the decomposition of \(G\) resulting from the edge coloring \(c.\) Since \(d(e_1, C_2) = 1\) and \(d(e_3, C_2) = 2\) or \(d(e_3, C_2) = 3,\) it follows that \(c\) is a resolving edge coloring of \(G\) and so \(\chi_{re}(G) \leq m - 1.\)

**Case 2.** \(v\) is adjacent to \(v_2\) or \(v_3,\) say the later. Define an edge coloring \(c\) of \(G\) by assigning the color 1 to \(e_1\) and \(e_3,\) the color 2 to \(vv_3,\) the color 3 to \(e_2,\) and distinct
colors from \{4, 5, \ldots, m-1\} to the remaining \(m-4\) edges in \(E(G) - (E(P) \cup \{vu_3\})\).

Let \(D = \{C_1, C_2, \ldots, C_{m-1}\}\) be the decomposition of \(G\) resulting from the coloring \(c\). Since \(d(e_1, C_2) = 2\) or \(d(e_1, C_2) = 3\) and \(d(e_3, C_2) = 1\), it follows that \(c\) is a resolving edge coloring of \(G\) and so \(\chi_{re}(G) \leq m - 1\).

Therefore, the star \(K_{1,n-1}\) is the only connected graph of order \(n \geq 5\) and size \(m\) with resolving edge chromatic number \(m\) and so (b) holds. ■

Combining (1) and (2) in Theorem 6.21, we have the following.

**Corollary 6.22** Let \(G\) be a connected graph of order \(n \geq 3\) and size \(m \geq 2\). Then \(\chi_{re}(G) = m\) if and only if \(G\) is one the graphs in Figure 6.15.

![Graphs](image)

**Figure 6.15:** The graphs described in Corollary 6.22

By Corollary 6.7 if \(G\) is a connected graph of size \(m \geq 3\), then \(3 \leq \chi_{re}(G) \leq m\). Next we show that every pair \(k, m\) of integers with \(3 \leq k \leq m\) is realizable as the resolving edge chromatic number and size of some connected graph.

**Theorem 6.23** For each pair \(k, m\) of integers with \(3 \leq k \leq m\), there exists a connected graph \(G\) of size \(m\) with \(\chi_{re}(G) = k\).

**Proof.** By Theorems 6.19 and 6.21, the result is true if \(k = 3\) or \(k = m\). Thus we may assume that \(3 < k < m\). Let \(G\) be the graph obtained from the path \(P : v_1, v_2, \ldots, v_{m-k+2}\) of order \(m - k + 2\) by adding \(k - 1\) pendant edges \(u_iv_i\) for
1 \leq i \leq k - 1. Then the size of G is m. Since the edge \( v_1v_2 \) is adjacent to \( k - 1 \) pendant edges, it follows by Observation 6.1 that \( \chi_{re}(G) \geq k \). On the other hand, define an edge coloring \( c \) of \( G \) by assigning the color 1 to \( u_i v_1 \) and \( u_i v_{i+1} \) if \( i \) is even and \( 2 \leq i \leq m - k + 1 \), the color \( i \) to \( u_i v_1 \) for \( 2 \leq i \leq k - 1 \), and the color \( k \) to \( u_i v_{i+1} \) if \( i \) is odd and \( 1 \leq i \leq m - k + 1 \). Let \( D = \{C_1, C_2, \ldots, C_k\} \) be the decomposition of \( G \) resulting from the edge coloring \( c \) of \( G \). Since (1) \( d(u_1 v_1, C_2) = 1 \), (2) \( d(u_i v_{i+1}, C_2) = i \) for \( 1 \leq i \leq m - k + 1 \), and (3) the edges \( u_i v_1 \) and \( v_1 v_2 \) are colored differently, it follows that the color codes \( c_D(e), e \in E(G) \), are distinct. Thus \( c \) is a resolving edge coloring of \( G \) using \( k \) colors and so \( \chi_{re}(G) \leq k \). Therefore, \( \chi_{re}(G) = k \).

6.5 On the Resolving Edge Chromatic Number of a Complete Graph

In this section, we provide bounds for the resolving edge chromatic number of a complete graph. We have seen that \( \chi_e(K_3) = \chi_{re}(K_3) = 3 \) and \( \chi_{re}(K_n) \geq \chi_e(K_n) \) for \( n \geq 4 \). In fact, \( \chi_{re}(K_n) \geq \chi_e(K_n) + 1 \) for \( n \geq 4 \), as we show next. It is known that \( \chi_e(K_n) = n - 1 \) if \( n \) is even and \( \chi_e(K_n) = n \) if \( n \) is odd. Moreover, the edge independence number \( \beta_1(K_n) = [n/2] \) for each \( n \geq 3 \).

Proposition 6.24  For every integer \( n \geq 4 \),

\[
\chi_{re}(K_n) \geq \chi_e(K_n) + 1 = \begin{cases} 
n & \text{if } n \text{ is even} \\
n + 1 & \text{if } n \text{ is odd.}
\end{cases}
\]

Proof. Assume, to the contrary, that \( \chi_{re}(K_n) = \chi_e(K_n) \). Let \( c \) be a minimum resolving edge coloring of \( K_n \) and let \( D = \{G_1, G_2, \ldots, G_{\chi_e(K_n)}\} \) be the decomposition of \( G \) resulting from \( c \). We consider two cases.

Case 1. \( n \) is even. Then \( \chi_e(K_n) = n - 1 \). Since the size of \( K_n \) is \( (n - 1)n/2 \) and \( \beta_1(K_n) = n/2 \), it follows that color \( i \) (\( 1 \leq i \leq n - 1 \)) is assigned to exactly \( n/2 \) independent edges of \( K_n \) and so \( |E(G_i)| = n/2 \geq 2 \) for \( 1 \leq i \leq n - 1 \). Let \( e, f \in E(G_i) \) for some \( i \) with \( 1 \leq i \leq n - 1 \), say \( e, f \in E(G_1) \). If each of \( e \) and \( f \) is adjacent to at least one edge in \( G_i \) for all \( i \) with \( 2 \leq i \leq n - 1 \), then
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c_D(e) = c_D(f) = (0, 1, 1, \ldots, 1), a contradiction. Thus we may assume that at least
one of e and f is not adjacent to any edges in G_i for some i with 2 \leq i \leq n - 1, say
e is not adjacent to any edges in G_2. Let e = uv. Then G_2 \subseteq K_n - \{u, v\} = K_{n-2}.
Since \beta_1(K_{n-2}) = (n - 2)/2, it follows that |E(G_2)| \leq (n - 2)/2, which is a
contradiction.

Case 2. n is odd. Then \chi_e(K_n) = n. Since the size of K_n is (n - 1)n/2 and
\beta_1(K_n) = (n - 1)/2, it follows that |E(G_i)| = (n - 1)/2 \geq 2 for 1 \leq i \leq n. Let
e, f \in E(G_1). If each of e and f is adjacent to at least one edge in G_i for all i
with 2 \leq i \leq n, then c_D(e) = c_D(f) = (0, 1, 1, \ldots, 1), a contradiction. Thus we
may assume that at least one of e and f is not adjacent to any edges in G_i for
some i with 2 \leq i \leq n, say e is not adjacent to any edges in G_2. Let e = uv.
Then G_2 \subseteq K_n - \{u, v\} = K_{n-2}. Since \beta_1(K_{n-2}) = (n - 3)/2, it follows that
|E(G_2)| \leq (n - 3)/2, which is a contradiction.

Since \chi_{re}(K_4) = 6 by Lemma 6.21, strict inequality in Proposition 6.24 holds
for n = 4. On the other hand, we have equality in Proposition 6.24 for n = 5.
Let V(K_5) = \{v_1, v_2, \ldots, v_5\} and define an edge coloring c of K_5 by assigning
the color 1 to v_1v_2 and v_3v_5, the color 2 to v_1v_3 and v_2v_4, the color 3 to v_1v_4,
the color 4 to v_1v_5 and v_2v_3, the color 5 to v_2v_5 and v_3v_4, and the color 6 to
v_4v_5. The coloring c is shown in Figure 6.16. Let \mathcal{D} = \{C_1, C_2, \ldots, C_6\} be the
decomposition of K_5 resulting from c. Then d(v_1v_2, C_6) = 2, d(v_3v_5, C_6) = 1,
d(v_1v_3, C_6) = 2, d(v_2v_4, C_6) = 1, d(v_1v_5, C_6) = 1, d(v_2v_3, C_6) = 2, d(v_2v_5, C_3) = 2,
and d(v_3v_4, C_3) = 1. Thus the color codes c_D(e), e \in E(K_5), are distinct and so c
is a resolving edge coloring of K_5 using 6 colors. Thus \chi_{re}(K_5) \leq 6. It follows by
Proposition 6.24 that \chi_{re}(K_5) = 6. Therefore, \chi_{re}(K_5) = \chi_e(K_5) + 1 and so we
have equality in Proposition 6.24 for n = 5.

Next, we present an upper bound for \chi_{re}(K_n) in terms of n for n \geq 3.

Theorem 6.25 For every integer n \geq 3,

\chi_{re}(K_n) \leq \lceil (5n - 3)/3 \rceil = \begin{cases} 2n/3 + (n - 1) & \text{if } n \equiv 0 \pmod{3} \\ (2n + 1)/3 + (n - 1) & \text{if } n \equiv 1 \pmod{3} \\ (2n + 2)/3 + (n - 1) & \text{if } n \equiv 2 \pmod{3}. \end{cases}
**Figure 6.16: A minimum edge coloring of $K_5$**

**Proof.** Since $\chi_{re}(K_3) = 3$, $\chi_{re}(K_4) = 6$, and $\chi_{re}(K_5) = 6$, the result is true for $3 \leq n \leq 5$. For $n \geq 6$, let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. We define a decomposition $\mathcal{D}$ of $K_n$ based on whether $n$ is congruent to 0, 1, or 2 modulo 3. Thus, there are three cases.

**Case 1.** $n \equiv 0 \pmod{3}$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{(2n/3)+1}\}$, where

$$E(G_{2i+1}) = \{v_{3i}v_{3i+1}\} \quad \text{and} \quad E(G_{2i+2}) = \{v_{3i+1}v_{3i+2}\}$$

for $0 \leq i \leq n/3 - 1$, and let $G_{(2n/3)+1}$ consist of the remaining edges of $K_n$ that do not belong to any other elements in $\mathcal{D}$. Figure 6.17 shows the decomposition $\mathcal{D}$ for $n = 9$. The subgraphs $G_1, G_2, \ldots, G_6$ are indicated in Figure 6.17. All edges not shown belong to $G_7$. Observe that $\{G_1, G_2, \ldots, G_{2n/3}\}$ is actually a $K_2$-decomposition of the factor $(n/3)P_3$ of $K_n$.

We show that $\mathcal{D}$ is a resolving decomposition of $K_n$. Observe that every edge of $G_{(2n/3)+1}$ either joins (a) a vertex that belongs to $G_i$ only and a vertex that belongs to $G_j$ only, where $1 \leq i \neq j \leq 2n/3$, (b) a vertex that belongs to $G_i$ only and a vertex that belongs to both $G_j$ and $G_{j+1}$ for some $i$ and $j$ with $\{i\} \cap \{j, j+1\} = \emptyset$ or (c) a vertex that belongs to both $G_i$ and $G_{i+1}$ and a vertex that belongs to both $G_j$ and $G_{j+1}$ for some $i$ and $j$ with $\{i, i+1\} \cap \{j, j+1\} = \emptyset$. Furthermore, each edge of $G_{(2n/3)+1}$ satisfies exactly one of (a), (b), and (c). Thus $\mathcal{D}$ is a resolving decomposition of $K_n$. Define an independent refinement $\mathcal{D}^*$ of $\mathcal{D}$ by (1) retaining each subgraph $G_i$ in $\mathcal{D}$ for $1 \leq i \leq 2n/3$ and (2) decomposing $G_{(2n/3)+1}$ into $n-1$
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Figure 6.17: The decomposition $\mathcal{D}$ for $n = 9$

independent subgraphs. Notice that (2) is possible since $\Delta(G_{(2n/3)+1}) = n - 2$ and so $\chi_e(G_{(2n/3)+1}) \leq n - 1$. Thus

$$\mathcal{D}^* = \{G_1, G_2, \ldots, G_{2n/3}, H_1, H_2, \ldots, H_{n-1}\},$$

where $\{H_1, H_2, \ldots, H_{n-1}\}$ is an independent decomposition $G_{(2n/3)+1}$. Since $D^*$ is a refinement of the resolving decomposition $D$ of $K_n$, it then follows by Observation 6.4 that $\mathcal{D}^*$ is a resolving independent decomposition of $K_n$ and so $\chi_{re}(K_n) \leq |D^*| = 2n/3 + (n - 1)$.

Case 2. $n \equiv 1 \pmod{3}$. We proceed as in Case 1 with $\mathcal{D} = \{G_1, G_2, \ldots, G_{(2n+1)/3}, G_{(2n+2)/3}\}$, where $\{G_1, G_2, \ldots, G_{(2n+1)/3}\}$ is a $K_2$-decomposition of the factor $[(n - 4)/3]P_3 \cup K_{1,3}$. Figure 6.18(a) shows the decomposition $\mathcal{D}$ for $n = 10$. The subgraphs $G_1, G_2, \ldots, G_7$ are indicated in the figure. All edges not shown belong to $G_8$. An argument similar to that of Case 1 shows that $\mathcal{D}$ is a resolving decomposition. Define an independent refinement $\mathcal{D}^*$ of $\mathcal{D}$ by (1) retaining each subgraph $G_i$ in $\mathcal{D}$ for $1 \leq i \leq (2n + 1)/3$ and (2) decomposing $G_{(2n+4)/3}$ into $n - 1$ independent subgraphs. Then $\mathcal{D}^*$ is a resolving independent decomposition of $K_n$ and so $\chi_{re}(K_n) \leq |D^*| = (2n + 1)/3 + (n - 1)$.

Case 3. $n \equiv 2 \pmod{3}$. Again we proceed as in the previous two cases, where $\mathcal{D} = \{G_1, G_2, \ldots, G_{(2n+2)/3}, G_{(2n+3)/3}\}$ such that $\{G_1, G_2, \ldots, G_{(2n+2)/3}\}$ is a $K_2$-decomposition of the factor $[(n - 5)/3]P_3 \cup K_{1,4}$. Figure 6.18(b) shows the decomposition $\mathcal{D}$ for $n = 11$. The subgraphs $G_1, G_2, \ldots, G_8$ are indicated in the

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Figure 6.18: The decompositions $D$ for $n = 10$ and for $n = 11$ figure. All edges not shown belong to $G_9$. Again, an argument similar to that used in Case 1 shows $D$ is a resolving decomposition. Then we define an independent refinement $D^*$ of $D$ by (1) retaining each subgraph $G_i$ in $D$ for $1 \leq i \leq (2n+2)/3$ and (2) decomposing $G_{(2n+5)/3}$ into $n-1$ independent subgraphs. Thus $\chi_{re}(K_n) \leq |D^*| = (2n+2)/3 + (n-1)$.

If $n = 4$, then $\lceil(5n-3)/3\rceil = 6$ and $\chi_{re}(K_4) = 6$ as well. Thus the upper bound in Theorem 6.25 is attained when $n = 4$. Since $\chi_{re}(K_5) = 6$ and $\lceil(5n-3)/3\rceil = 8$ for $n = 5$, we have strict inequality in Theorem 6.25 when $n = 5$.

There are several open questions here:

(1) Just how good are the bounds for $\chi_{re}(K_n)$ given in Proposition 6.24 and Theorem 6.25?

(2) Is there, in fact, a formula for $\chi_{re}(K_n)$?

(3) One might think that $\chi_{re}(K_n) \leq \chi_{re}(K_{n+1})$ for all $n \geq 3$. Is this the case?
Chapter 7

Topics for Further Study

7.1 Independent Resolving Sets in Graphs

In Chapter 2, we studied connected resolving sets in graphs, where, recall, a resolving set \( W \) is connected if \( \langle W \rangle \) is connected in \( G \). A resolving set \( W \) of a connected graph \( G \) is independent if \( \langle W \rangle \) is independent in \( G \). The cardinality of a minimum independent resolving set is the independent resolving number \( \text{ir}(G) \). Not all graphs have an independent resolving set and so \( \text{ir}(G) \) is not defined for all graphs \( G \). We plan to study the problem of determining those graphs \( G \) for which \( \text{ir}(G) \) is defined. Furthermore, when \( \text{ir}(G) \) is defined, we would like to compute this number in many cases or at least provide bounds.

Among all resolving sets \( W \) of \( G \) for which the size of \( \langle W \rangle \) is minimum, let \( S \) be one of minimum cardinality. Such a set \( S \) is called a size resolving set of \( G \). The cardinality of \( S \) is called the size resolving number \( \text{sr}(G) \). Notice that the sets \( S \) such that the size of \( \langle S \rangle \) is 0 only occur in those graphs \( G \) for which \( \text{ir}(G) \) is defined. We plan to study \( \text{sr}(G) \) as well.

7.2 Resolving \( H \)-Partitions in Graphs

Let \( H \) be a set of graphs. For a connected graph \( G \), a resolving partition \( \Pi = \{S_1, S_2, \ldots, S_k\} \) of \( V(G) \) is called an \( H \)-partition of \( G \) if the subgraph \( \langle S_i \rangle \) induced by each subset \( S_i \) (\( 1 \leq i \leq k \)) is isomorphic to some graph in \( H \). The
minimum \( k \) for which \( G \) has a resolving \( \mathcal{H} \)-partition is its \( \mathcal{H} \)-partition dimension \( \text{pd}_\mathcal{H}(G) \). Observe that if \( \mathcal{H} \) is the set of all empty graphs, then \( \mathcal{H} \)-partition is independent partition, as described in Section 1.3. On the other hand, if \( \mathcal{H} \) is the set of all connected graphs, then \( \mathcal{H} \)-partition of \( G \) is connected partition, as described in Chapter 3. Moreover, if \( \mathcal{H} \) is the set of all forests (acyclic graphs), then \( \mathcal{H} \)-partition is acyclic resolving partition, as described in Chapter 4. For some choices of \( \mathcal{H} \), the parameter \( \text{pd}_\mathcal{H}(G) \) is not defined. We plan to study the \( \mathcal{H} \)-partition dimensions of graphs both in terms of existence and its value when it exists for various choices of \( \mathcal{H} \).

### 7.3 Acyclic Resolving Decompositions in Graphs

In Chapter 5, we studied connected resolving decompositions in graphs; while in Chapter 6, we studied independent resolving decompositions in graphs. As in the case of partitions in a connected graph \( G \), we define a resolving decomposition \( \mathcal{D} = \{G_1, G_2, \ldots, G_k\} \) of \( G \) to be acyclic if each subgraph \( G_i \) (\( 1 \leq i \leq k \)) is acyclic in \( G \). The minimum \( k \) for which \( G \) has an acyclic resolving \( k \)-decomposition is its acyclic decomposition dimension \( \text{adim}_d(G) \). Since every independent resolving decomposition is acyclic and every acyclic resolving decomposition is resolving, it follows that \( \dim_d(G) \leq \text{adim}_d(G) \leq \chi_{re}(G) \) for every connected graph \( G \). Clearly, an acyclic resolving decomposition may not be connected and a connected resolving decomposition may not be acyclic. Thus it is possible that \( \text{adim}_d(G) \leq \text{cdim}_d(G) \) or \( \text{cdim}_d(G) \leq \text{adim}_d(G) \) for a graph \( G \). We plan to study \( \text{adim}_d(G) \) and the relationships between \( \text{adim}_d(G) \) and \( \dim_d(G) \), \( \text{cdim}_d(G) \), or \( \chi_{re}(G) \).

### 7.4 Resolving \( \mathcal{H} \)-Decompositions in Graphs

Again, let \( \mathcal{H} \) be a set of graphs. A resolving decomposition \( \mathcal{D} = \{G_1, G_2, \ldots, G_k\} \) of a connected graph \( G \) is called an \( \mathcal{H} \)-decomposition of \( G \) if each subgraph \( G_i \) (\( 1 \leq i \leq k \)) is isomorphic to some graph in \( \mathcal{H} \). The minimum \( k \) for which \( G \) has a resolving \( \mathcal{H} \)-decomposition is its \( \mathcal{H} \)-decomposition dimension \( \text{dim}_\mathcal{H}(G) \). Similarly, if \( \mathcal{H} \) is the set of all connected graphs, then \( \mathcal{H} \)-decomposition of \( G \) is connected.
decomposition, as described in Chapter 5; if $\mathcal{H}$ is the set of all empty graphs, then $\mathcal{H}$-decomposition is independent decomposition, as described in Chapter 6; if $\mathcal{H}$ is the set of all acyclic graphs, then $\mathcal{H}$-decomposition is acyclic resolving decomposition of $G$, as described in Section 7.3. Certainly, for some choices of $\mathcal{H}$, the parameter $\text{dim}_\mathcal{H}(G)$ is not defined. We are interested in choosing $\mathcal{H}$ to be the set of all stars or the set of all cycles and determining those graphs for which $\text{dim}_\mathcal{H}(G)$ is defined. Furthermore, when $\text{dim}_\mathcal{H}(G)$ is defined, we would like to compute this number or at least establish bounds.
Bibliography


