6-2000

Rainbow Ramsey Numbers

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RAINBOW RAMSEY NUMBERS

by

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A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of
requirements for the
Degree of Doctor of Philosophy
Department of Mathematics and Statistics

Western Michigan University
Kalamazoo, Michigan
June 2000

RAINBOW RAMSEY NUMBERS

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We investigate a new generalization of the generalized ramsey number for graphs. Recall that the generalized ramsey number for graphs $G_1, G_2, \ldots, G_c$ is the minimum positive integer N such that any coloring of the edges of the complete graph K_N with c colors must contain a subgraph isomorphic to G_i in color i for some i. Bialostocki and Voxman defined $RM(G)$ for a graph G to be the minimum N such that any edge-coloring of K_N with *any* number of colors must contain a subgraph isomorphic to G in which either every edge is the same color (a *monochromatic* G) or every edge is a different color (a *rainbow* G). This number exists if and only if G is acyclic.

Expanding on this definition, we define the *rainbow ramsey number* $RR(G_1, G_2)$ of graphs G_1 and G_2 to be the minimum N such that any edge-coloring of K_N with any number of colors contains either a monochromatic G_1 or a rainbow G_2. This number exists if and only if G_1 is a star *or* G_2 is acyclic. We present upper and lower bounds for $RR(K_{1,n}, K_m)$, $RR(K_n, T_m)$, $RR(K_n, K_{1,m})$, $RR(K_{1,n}, mK_2)$ and $RR(nK_2, K_{1,m})$, where T_m is an arbitrary tree of order m.

We also define the *edge-chromatic ramsey number* $CR(G_1, G_2)$ to be the minimum N such that any edge-coloring of K_N must contain either a monochromatic G_1 or a properly edge-colored G_2. When both are defined, $CR(G_1, G_2) \leq RR(G_1, G_2)$. We consider bounds for $CR(C_n, P_m)$, $CR(K_{1,n}, P_m)$, $CR(P_n, P_m)$,

and the corresponding rainbow ramsey numbers.

These two new ramsey numbers can be further generalized as the $\mathcal{F}$-free ramsey number. For a set of graphs $\mathcal{F}$, an $\mathcal{F}$-free coloring of a graph G is a coloring so that G does not contain any monochromatic subgraph isomorphic to any graph in $\mathcal{F}$. The *$\mathcal{F}$-free ramsey number* of graphs G_1 and G_2, denoted $R_{\mathcal{F}}(G_1, G_2)$, is the minimum N such that every edge-coloring of K_N contains either a monochromatic copy of G_1 or an $\mathcal{F}$-free copy of G_2.

UMI Number: 9973049

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Ann Arbor, MI 48106-1346

ACKNOWLEDGEMENTS

I would particularly like to thank my advisor, Professor Allen Schwenk, for the guidance, advice, and encouragement he has given me, not only as my advisor the last two years, but during my entire time as a student at Western Michigan University. I would also especially like to thank Professor Gary Chartrand, who helped kindle my love for mathematics and continues to show an interest in my mathematical development. Professor Ortrud Oellermann has been remarkably kind and helpful as an outside reader, even when I sent copies of my dissertation by regular mail to Canada and they took two weeks to arrive, leaving her little time to review them. In addition, I would like to acknowledge the other members of my dissertation committee, Professor Art White, Professor Yousef Alavi, and especially Professor Ping Zhang, my second reader.

As both an undergraduate and graduate student, I received a great deal of encouragement and support from Professor John Petro. And I have received valuable guidance in my teaching and career advice from Professor Nil Mackey. Finally, I would certainly not have met deadlines or Graduate College requirements without the help of Margo Chapman, Administrative Secretary for Graduate Programs, and Maryann Bovo.

Thank you also to my husband, Brian, and my parents, Bob and Lois Hansen, who have not always understood why anyone would want a Ph.D. in mathematics but have never failed to be proud and supportive.

Linda Eroh

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INTRODUCTION

Frank Ramsey was actually considering decision problems in formal logic when he proved the theorem which demonstrates the existence of both the traditional and generalized ramsey numbers. In terms of graph theory, the traditional ramsey number $r(n_1, n_2, n_3, \ldots, n_c)$ is the smallest integer N such that any edge-coloring of the complete graph K_N on N vertices with c colors must contain a complete subgraph K_{n_i} on n_i vertices with every edge color i for some i. In generalized ramsey theory, the complete graphs K_{n_i} may be replaced with arbitrary graphs. Many of the results in traditional ramsey theory are asymptotic bounds, though a few specific formulas are known for the generalized ramsey numbers of certain classes of graphs. Recently, Bialostocki and Voxman defined a new generalization allowing the use of an arbitrary number of colors. They considered the diagonal values, that is, values of their number when the two graphs considered are the same. We extend their definition to consider the off-diagonal numbers and other generalizations of the ramsey numbers.

1.1 Background and Basic Definitions

In 1930, in a paper titled "On a Problem of Formal Logic"[13], Frank Ramsey proved the combinatorial result that demonstrates the existence of what would later be called *ramsey numbers.* His result received little attention at the time and was later rediscovered by G. Szekeres and P. Erdös. We state only the finite version of Ramsey's Theorem.

Theorem 1 (Ramsey). *For any positive integers $n_1, n_2, \ldots, n_c$ and.d, there ex-*

ists an integer $N = r_d(n_1, n_2, \ldots, n_c)$ *such that if the d-element subsets of the set* $\{1, 2, 3, \ldots, N\}$ *are colored with c colors, then for some* i, $1 \leq i \leq c$, *there is a subset* $A \subseteq \{1, 2, 3, \ldots, N\}$ *with* n_i *elements such that every d-element subset of* A *is colored with color* i.

We may view the set $\{1, 2, 3, \ldots, N\}$ as the vertices of the complete graph K_N. When $d = 1$, the coloring described in Ramsey's Theorem is a coloring of the vertices of K_N. In this case, Ramsey's Theorem says that for any set of integers $\{n_1, n_2, \ldots, n_k\}$, there is an integer N so that if the vertices of the complete graph K_N are colored with k colors, there must be n_i vertices in color i for some i. Of course, this is simply the Pigeonhole Principle; $N = \sum_{i=1}^{k}(n_i - 1) + 1$ suffices.

The first nontrivial case occurs when $d = 2$. In this case, the coloring described in Ramsey's Theorem may be viewed as a coloring of the edges of the complete graph K_N. From this point of view, Ramsey's Theorem says that for any set of integers $\{n_1, n_2, \ldots, n_c\}$, there is some integer N such that if the edges of K_N are colored with c colors, say colors $\{1, 2, \ldots c\}$, then the resulting graph must contain a complete graph on n_i vertices with every edge colored with color i, for some i. The smallest such integer N is called the *ramsey number* $r(n_1, n_2, \ldots, n_c)$. We will refer to this number here as the *traditional ramsey number.*

When we consider colorings of subsets of order $d \geq 3$, we have ramsey theory for k-uniform hypergraphs with $k \geq 3$. Recall that a *k-uniform hypergraph* is a graph in which edges are replaced by k-element subsets of the vertex set, where k is a constant. Very little work has been done to find ramsey numbers for $d \geq 3$, and no nontrivial hypergraph ramsey numbers are presently known.

We will only prove Ramsey's Theorem in the finite case for $d = 2$, since these are the values which interest us. The following proof is based on the proof in the book by Graham, Rothschild, and Spencer [11, p. 3].

Proof of Ramsey's Theorem for $d = 2$. In any coloring of the edges of K_m with two colors, say red and blue, either at least one edge is red or all of the edges are blue. Thus, $r(2, m) = m$. Similarly, $r(n, 2) = n$.

Now, suppose $r(n, m-1)$ and $r(n-1, m)$ both exist. Let $N = r(n, m-1) + r(n-1, m)$. Consider any coloring of the edges of K_N in red and blue. Let x be an arbitrary vertex of K_N and define

$$U = \{y \in K_N | xy \text{ is red } \}$$

and

$$V = \{y \in K_N | xy \text{ is blue } \}.$$

Since $|U| + |V| + 1 = r(n-1, m) + r(n, m-1)$, either $|U| \geq r(n-1, m)$ or $|V| \geq r(n, m-1)$. Suppose $|U| \geq r(n-1, m)$. Then there is either a blue subgraph K_m or a red subgraph K_{n-1} contained in the subgraph induced by U. If there is a red K_{n-1}, then this graph and the vertex x induce a red K_n. The case $|V| \geq r(n, m-1)$ is similar. Thus, $r(n, m)$ exists for all positive integers n and m and is bounded above by $r(n-1, m) + r(n, m-1)$.

We proceed by induction on the number of colors c. Suppose $r(n_1, n_2, \ldots, n_{c-1})$ exists for all positive integers $n_1, n_2, \ldots, n_{c-1}$, for some $c \geq 3$. Let $N = r(r(n_1, n_2, \ldots, n_{c-1}), n_c)$. For any edge-coloring of K_N with c colors, consider the colors $1, 2, \ldots, c-1$ red and the color c blue. Then there must be either a copy of K_{n_c} in blue, that is, color c, or a copy of $K_{r(n_1, n_2, \ldots, n_{c-1})}$ in red, that is, entirely in colors $1, 2, \ldots, c-1$. In the second case, this "red" graph must contain K_{n_i} in color i for some i, where $1 \leq i \leq c-1$. Thus, $r(n_1, n_2, \ldots, n_c)$ exists for all positive integers $n_1, n_2, \ldots, n_c$. $\square$

As an illustration, we give the proof of the first nontrivial traditional ramsey number.

Theorem 2. *The ramsey number* $r(3,3) = 6$.

Proof. To obtain a coloring of the edges of K_5 with two colors so that the resulting graph does not contain a copy of K_3 in either color, color one 5-cycle red and the remaining 5-cycle blue.

Now suppose the edges of K_6 are colored with two colors, say red and blue. Let v be any vertex. Since v is incident with five edges, there must be some set of three edges incident with v which are colored with the same color, say blue. Suppose u, w and x are the other incident vertices of these three edges. If any one of the edges uw, wx and xu are colored blue, then we have a blue K_3. Otherwise, all three edges are red and form a red K_3. □

Once the ramsey numbers had been viewed in terms of graph theory, it became natural to rewrite the traditional ramsey number as $r(K_{n_1}, K_{n_2}, \ldots K_{n_k})$ and to define $r(G_1, G_2, \ldots G_k)$ for graphs $G_1, G_2, \ldots, G_k$ which are not necessarily complete. This number, known as the *generalized ramsey number*, is defined to be the smallest integer N such that every coloring of the edges of K_N contains a subgraph isomorphic to G_i with every edge colored with color i for some i, $1 \leq i \leq k$. This generalization was first explored in a series of papers by Chvátal and Harary[7]. Harary described his discovery of the generalization:

> In his lecture at my seminar on graph theory, he[Paul Erdös] wrote $G \to F, H$ to mean that every 2-coloring of $E(G)$ contains a green subgraph F or a red subgraph H. He then defined the *ramsey number* $r(m,n)$ as the smallest p such that $K_p \to K_m, K_n$. I proposed at once to rewrite $r(m,n)$ as $r(K_m, K_n)$ and to study the generalized (not only for complete graphs) ramsey number $r(F,H)$, defined of course as the

minimum p such that $K_p \rightarrow F, H$ where graphs F and H have no isolated points. Later we learned that several special cases of $r(F, H)$ were being investigated in Hungary and elsewhere at about the same time.[12]

For more of the early work on generalized ramsey numbers, see the series of papers "Generalized Ramsey Theory for Graphs" by Chvátal and Harary [7]. A good overview of ramsey theory can be found in the book by Graham, Rothschild, and Spencer[11].

1.2 Some Traditional Results

Many of the results concerning the traditional ramsey numbers were demonstrated using the probabilistic method pioneered by Paul Erdös. One early example of this method is the proof of the following lower bound.

Theorem 3 (Erdös). *For any integer $n \geq 3$, the ramsey number*

$$r(n,n) > \lfloor 2^{\frac{n}{2}} \rfloor$$

Proof. Let $N = \lfloor 2^{n/2} \rfloor$. If we label the vertices of K_N, then there are $2^{\binom{N}{2}}$ different colorings of the edges of K_N with two colors, say red and blue. Since n vertices may be chosen in $\binom{N}{n}$ different ways and there are $2^{\binom{N}{2}-\binom{n}{2}}$ different ways to color the remainder of the graph, there are at most

$$\binom{N}{n} 2^{\binom{N}{2}-\binom{n}{2}}$$

different colorings containing a red K_n. By symmetry, then, at most

$$2\binom{N}{n} 2^{\binom{N}{2}-\binom{n}{2}}$$

different colorings contain either a red K_n or a blue K_n. Since

$$\begin{aligned} 2\binom{N}{n}2^{\binom{N}{2}-\binom{n}{2}} &\leq \frac{2N^n}{n!}2^{\binom{N}{2}-\binom{n}{2}} \\ &= \frac{2N^n}{n!}\frac{2^{n/2}}{2^{n^2/2}}2^{\binom{N}{2}} \\ &\leq \frac{2^{n/2+1}}{n!}2^{\binom{N}{2}} \\ &< 2^{\binom{N}{2}} \end{aligned}$$

for $n \geq 3$, there must be some coloring of K_N which does not contain either a red or a blue K_n. □

From the proof of Ramsey's Theorem, it follows that

$$r(n,m) \leq r(n,m-1) + r(n-1,m). \tag{1}$$

Since $r(n,2) = n = \binom{n}{n-1}$ and $r(2,m) = m = \binom{m}{1}$ for any integers n and m greater than 1, the recursion in equation 1 yields the upper bound

$$r(n,m) \leq \binom{n+m-2}{n-1}$$

This bound is approximately

$$r(n,n) \leq \frac{c4^n}{\sqrt{n}}$$

for some constant c.

Despite this progress on asymptotic bounds, few actual numbers are known for the traditional ramsey numbers. The situation for generalized ramsey numbers is more promising. One of the better known results in generalized ramsey theory is Chvátal's formula for $r(T, K_n)$ where T is a tree of order m [6]. Given the scarcity of such closed formulas, the proof is surprisingly elegant.

Theorem 4 (Chvátal). *For any tree T of order m, the ramsey number*

$$r(T, K_n) = 1 + (m-1)(n-1)$$

Proof. For the lower bound, color the edges of $n-1$ disjoint copies of K_{m-1} red and the edges between them blue. The resulting graph has $(n-1)(m-1)$ vertices, no connected red subgraph of order m or larger, and no blue K_n.

For the upper bound, let $N = (n-1)(m-1)+1$ and suppose the edges of K_N are colored red and blue. Assume that the resulting graph has no blue K_n as a subgraph, so the subgraph H induced by the red edges does not contain any set of n independent vertices. If we consider H as a graph, any chromatic coloring of the vertices of H can use each color at most $n-1$ times, so at least m different colors must be used. Since H has chromatic number at least m, there must be some subgraph of H with minimum degree at least $m-1$. Otherwise, proceeding by induction on the order of the subgraph, each subgraph of H could be colored with $m-1$ colors by removing a vertex of minimum degree, coloring the remaining subgraph, replacing the removed vertex, and coloring this vertex with some color not used on its neighbors. In this case, H would have chromatic number less than m, which is a contradiction. Since any graph with minimum degree at least $m-1$ contains every tree of order m as a subgraph, T must be a subgraph of H. $\square$

Few other general formulas are known. One such formula which we will use later involves *stripes*, that is, disjoint unions of copies of K_2 [8]. We omit the proof.

Theorem 5 (Cockayne, Lorimer). *If $n_1, n_2, \ldots, n_c$ are positive integers and $n_1 = max(n_1, n_2, \ldots, n_c)$, then*

$$r(n_1K_2, n_2K_2, \ldots, n_cK_2) = n_1 + 1 + \sum_{i=1}^{c}(n_i - 1).$$

1.3 Where the Rainbow Begins

Another important set of problems which are closely related to graphical ramsey theory involve colorings of the integers $\{1, 2, 3, \ldots, N\}$. These problems originated with a result by B. L. van der Waerden in 1927. In the following theorem, we call a sequence *monochromatic* if every integer in the sequence is the same color. Later, we will say that a sequence is *rainbow* if no two integers in the sequence are colored with the same color. See [11, p. 29].

Theorem 6 (van der Waerden). *If the positive integers are colored with two colors, then there is a monochromatic arithmetic progression of any desired length.*

A slight generalization of van der Waerden's result shows that for any positive integers n and k, there is a positive integer $W(n;k)$ such that any coloring of the integers $1, 2, \ldots, W(n;k)$ with k colors must contain a monochromatic arithmetic sequence of length n. A considerable amount of interest has been directed towards discovering these numbers for various values of n and k.

In a paper published in 1979, Erdős and Graham suggested a number of generalizations and new problems related to van der Waerden's Theorem[9]. Among the generalizations, they define $H(n)$ to be the smallest positive integer n such that any coloring of the integers $1, 2, \ldots, H(n)$ with *any* number of colors must contain an arithmetic sequence of length n that is either monochromatic or rainbow.

Bialostocki and Voxman may have been inspired by this generalization when they defined, in [1], the number $RM(G)$ for a graph G. This number is defined as the smallest integer N such that if the edges of the complete graph K_N are colored with *any* number of colors, then the resulting graph must contain a subgraph isomorphic to G in which either every edge is the same color or every edge is a different color. They note that this number exists if and only if G is acyclic. This result follows from a theorem by Erdös and Rado.

For the purposes of the Erdös-Rado Theorem, a *canonical* coloring of either a finite or an infinite complete graph with vertices numbered $1, 2, 3, \ldots$ is any one of four particular edge-colorings. A *monochromatic* coloring is one in which every edge is the same color. In a *minimum* coloring, edge ij is color $min(i, j)$; in a *maximum* coloring, this edge is color $max(i, j)$. For a finite graph, either of these two colorings may be obtained from the other by reversing the order in which the vertices are labelled, but for an infinite graph, they are nonisomorphic. A *rainbow* coloring is an edge-coloring in which every edge is a different color. A graph that is colored with a monochromatic, minimum, maximum, or rainbow coloring is said to be *canonically* colored [11].

We state and prove the Erdös-Rado Theorem only as it applies to finite graphs. For a more general statement, see [11, p. 129].

Theorem 7 (Erdös, Rado). *For any positive integer k, there exists a positive integer N such that any edge-coloring of K_N contains a canonically colored complete subgraph on k vertices.*

Proof. According to Ramsey's Theorem, there is some integer N such that if all of the 4-element subsets of the vertices of K_N are colored with 203 colors, the resulting graph must contain a complete subgraph K_k in which every 4-element subset is the same color. Consider any edge-coloring (coloring of two-element subsets) of K_N. At most six colors are used on the subgraph induced by any set of four vertices $\{a, b, c, d\}$, where we assume $a < b < c < d$. Up to renaming and rearranging colors, there are 203 different ways that a labelled complete subgraph induced by four vertices may be colored. Label each four-element subset of the vertices of K_N with its coloring, up to interchanging colors, so that the four-element subsets are colored with at most 203 colors.

Thus, there must be a complete subgraph of order k, say H, on which all of these four-element subsets are colored with the same color. Label the vertices of H by $v_1, v_2, \ldots v_k$ where $v_i < v_j$ if and only if $i < j$.

Since the subsets $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_5\}$ $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_1, v_4, v_5, v_6\}$, $\{v_2, v_4, v_5, v_6\}$, and $\{v_3, v_4, v_5, v_6\}$ all have the same coloring, the following equations are either all true or all false:

$$\begin{aligned} \text{color } v_1v_2 &= \text{color } v_1v_3 \\ \text{color } v_1v_2 &= \text{color } v_1v_4 \\ \text{color } v_1v_3 &= \text{color } v_1v_4 \\ \text{color } v_2v_3 &= \text{color } v_2v_4 \\ \text{color } v_1v_4 &= \text{color } v_1v_5 \\ \text{color } v_2v_4 &= \text{color } v_2v_5 \\ \text{color } v_3v_4 &= \text{color } v_3v_5 \end{aligned} \tag{2}$$

and, similarly, the set of equations

$$\begin{aligned} \text{color } v_1v_3 &= \text{color } v_2v_3 \\ \text{color } v_1v_4 &= \text{color } v_2v_4 \\ \text{color } v_1v_4 &= \text{color } v_3v_4 \\ \text{color } v_2v_4 &= \text{color } v_3v_4 \end{aligned} \tag{3}$$

are either all true or all false. Notice that the equations

$$\begin{aligned} \text{color } v_1v_2 &= \text{color } v_2v_3 \\ \text{color } v_1v_2 &= \text{color } v_2v_4 \\ \text{color } v_1v_3 &= \text{color } v_3v_4 \\ \text{color } v_2v_3 &= \text{color } v_3v_4 \end{aligned} \tag{4}$$

are either all true or all false. The pair of equations

$$\begin{aligned} \text{color } v_1v_2 &= \text{color } v_3v_4 \\ \text{color } v_1v_2 &= \text{color } v_3v_5 \end{aligned} \tag{5}$$

are either both true or both false, and must be false if the equations in (2) are false. Similarly, the equations

$$\begin{aligned} \text{color } v_2v_3 &= \text{color } v_1v_4 \\ \text{color } v_2v_3 &= \text{color } v_1v_5 \end{aligned} \tag{6}$$

are either both true or both false, and the equations

$$\begin{aligned} \text{color } v_1v_3 &= \text{color } v_2v_4 \\ \text{color } v_1v_3 &= \text{color } v_2v_5 \end{aligned} \tag{7}$$

are either both true or both false, and must all be false if the equations in (2) are false.

Suppose all of the equations (2) and all of the equations in (3) are true. In this case, $\{v_1, v_2, v_3, v_4\}$ is colored monochromatically. Any two nonadjacent edges v_iv_j and v_lv_p in H must be the same color, since $\{v_i, v_j, v_l, v_p\}$ is colored the same as $\{v_1, v_2, v_3, v_4\}$. Also, any two adjacent edges v_iv_j and v_iv_l in H are the same color, since $\{v_i, v_j, v_l, v_p\}$ is colored the same as $\{v_1, v_2, v_3, v_4\}$, where v_p is some other vertex. Thus, H is colored monochromatically.

Suppose the equations in (2) are true and the equations in (3) are false. Then $\{v_1, v_2, v_3, v_4\}$ is colored with the minimum coloring. For any integers $i < j < l$, since $\{v_1, v_2, v_3, v_4\}$ is colored the same as $\{v_i, v_j, v_l, v_p\}$, where v_p is any other vertex (not necessarily having the highest index), the edge v_iv_j must be the same color as the edge v_iv_l and the edge v_jv_l is not the same color as the edge v_iv_l. It follows that H is colored with the minimum coloring.

Similarly, if the equations in (2) are false and the equations in (3) are true, then both $\{v_1, v_2, v_3, v_4\}$ and H are colored with the maximum coloring.

Suppose the sets of equations in (2) and (3) are all false. It follows that the equations in (4) must also be all false. If they were all true, then color $v_1v_2 =$ color $v_2v_3 =$ color $v_3v_4 =$ color v_1v_3. Thus, all of the equations in (2), (3), (4), (5), (6), and (7) are false, so $\{v_1, v_2, v_3, v_4\}$ is rainbow colored. Since any set of vertices $\{v_i, v_j, v_l, v_p\}$ in H must also be rainbow colored, H is also be rainbow-colored. $\square$

Suppose G is an acyclic graph with k vertices. Then any monochromatic copy of K_k contains a monochromatic copy of G, and any minimum-colored, maximum-colored, or rainbow colored copy of K_k contains a rainbow copy of G. Thus, $RM(G)$ exists and is at most the integer N described in the Erdös-Rado Theorem.

However, suppose G contains a cycle $v_1, v_2, \ldots, v_p$. For any integer N, color K_N with the minimum coloring. Consider any subgraph isomorphic to G. Assume without loss of generality that v_1 receives the smallest label of the vertices $v_1, v_2, \ldots, v_p$. Since v_1v_2 and v_1v_p are the same color, this subgraph G is not rainbow colored. However, v_1v_2 and v_2v_3 are different colors, so G is also not monochromatically colored.

Bialostocki and Voxman discovered the following formula for $RM(nK_2)$.

We present a proof similar to theirs[1].

Theorem 8 (Bialostocki, Voxman). *For every positive integer n, the number*

$$RM(nK_2) = n(n-1) + 2$$

Proof. For the lower bound, color the edges of K_{n^2-n+1} as follows. First, partition the vertex set into $n-1$ sets $A_1, A_2, \ldots, A_{n-1}$ where $|A_1| = 2n-1$ and $|A_i| = n-1$ for $2 \leq i \leq n-1$. Color the edges among the vertices in A_i with color i for $1 \leq i \leq n-1$ and color the edges between vertices in A_i and vertices in A_j with color $\max(i,j)$ for $1 \leq i < j \leq n-1$. The resulting graph contains no monochromatic nK_2 and has too few colors to contain a rainbow mK_2.

To demonstrate the upper bound, we proceed by induction on n. For $n = 1$ or $n = 2$, the result is immediate. Assume $n \geq 3$ and $RM((n-1)K_2) = (n-1)(n-2)+2$. Consider any edge-coloring of K_{n^2-n+2}. If fewer than n colors are used, then by Theorem 5, there must be a monochromatic copy of nK_2. Thus, we may assume that at least n colors appear. Choose a set H of n edges, in n different colors, so that H contains as many independent edges as possible. If $|V(H)| = 2n$, then we have a rainbow nK_2, so we may assume that $|V(H)| \leq 2n-1$. Let M be the subgraph of K_{n^2-n+2} induced by the vertices not in H.

If any new color appears on an edge in M which does not appear in H, then we may replace one of the edges in H which is not independent with an edge from M in this new color to obtain a set of n edges in n different colors containing more independent edges than H. This contradicts our choice of H. We may assume that no new color appears in M.

Similarly, if every color which appears in H also appears in M, then we may replace some edge in H which is not independent with an edge in the same color from M. Again, we would have a set of n edges in n different colors with more independent edges than H, which contradicts our choice of H. Thus, we may assume that the colors which appear in M are a proper subset of the colors which appear in H.

Suppose $|V(H)| \leq 2n-2$. Then $|V(M)| \geq n^2 - 3n + 4 = (n-1)(n-2)+2$. By the inductive hypothesis, we know that M must contain either a monochromatic $(n-1)K_2$ or a rainbow $(n-1)K_2$. If M contains a monochromatic $(n-1)K_2$, then we may add an edge in the same color from H to obtain a monochromatic nK_2. If M contains a rainbow $(n-1)K_2$, then we may add an edge from H in some new color to obtain a rainbow nK_2.

Thus, we may assume that $|V(H)| = 2n-1$. The structure of H is determined, up to interchanging colors. We may assume that H contains $n-2$ independent edges x_iy_i, where edge x_iy_i is color i, for $1 \leq i \leq n-2$, and two adjacent edges uv and vw, where uv is color $n-1$ and vw is color n. If M contains any edges in colors other than $1, 2, \ldots, n-3$ and $n-2$, then we have a rainbow nK_2. For any $z \in V(M)$, if edge uz is a new color or color $n-1$, then we have a rainbow nK_2; we may assume that all such edges are colored with colors $1, 2, 3, \ldots, n-2$ and n. If no such edge is color n, then the subgraph induced by $M \cup \{u\}$ is colored with colors $1, 2, 3, \ldots, n-3$ and $n-2$. Since this subgraph contains $RM((n-1)K_2) = (n-1)(n-2)+2$ vertices, it must contain a monochromatic $(n-1)K_2$ in one of the colors $1, 2, 3, \ldots, n-2$. We may add an edge from H to obtain a monochromatic nK_2. Thus, we may assume that ua is color n for some vertex $a \in V(M)$. Similarly, every edge wz for $z \in V(M)$ must be in one of the colors $1, 2, 3, \ldots n-1$, and for some $b \in V(M)$, edge wb is color $n-1$. If $a \neq b$, then we have a rainbow nK_2; assume $a = b$.

Now, every edge vz, for $z \in V(M)$ and $z \neq a$, must be colored with one of the colors $1, 2, 3, \ldots, n-2$, or else we have a rainbow nK_2. We may assume without loss of generality that there is some vertex $c \in V(M)$ such that vc is color 1. For any $z \in V(M)$, with $z \neq c$ and $z \neq a$, edges x_1z and y_1z cannot be colored with colors n or $n-1$ or any new color, or we would have a rainbow nK_2. Thus, we may assume that all such edges are colored with the colors $1, 2, \ldots, n-2$.

Consider the subgraph induced by $V(M) \cup \{x_1, y_1\} - \{c\}$. All of the edges on this subgraph are colored with colors $1, 2, \ldots n-2$, so there are not enough colors for a rainbow $(n-1)K_2$. However, there are $RM((n-1)K_2) = (n-1)(n-2)+2$ vertices in this set, so there must be a monochromatic $(n-1)K_2$ in one of the colors $1, 2, \ldots,$ or $n-2$. We may add the appropriate edge from H to obtain a monochromatic nK_2. $\square$

RAINBOW RAMSEY NUMBER

Next we consider a slight generalization of Bialostocki and Voxman's definition. We define the *rainbow ramsey number* $RR(G_1, G_2)$ to be the least positive integer N such that if the edges of K_N are colored with any number of colors, the resulting graph must contain either a subgraph isomorphic to G_1 all of whose edges are the same color or a subgraph isomorphic to G_2 all of whose edges are different colors. Notice that this definition is not symmetric in G_1 and G_2, that is, we have no reason to expect $RR(G_1, G_2)$ and $RR(G_2, G_1)$ to be the same number. (The issue of symmetry is explored further in chapter 5.)

For simplicity, we will say that a graph is *monochromatic* if all of its edges are colored the same color, and we will say that a graph is *rainbow* if all of its edges are colored different colors.

2.1 Existence of Rainbow Ramsey Numbers

We next determine for which graphs G_1 and G_2 the rainbow ramsey number exists. The existence theorem follows quickly from the Erdös-Rado Theorem, but we will present instead a constructive proof independent of this theorem which also yields upper bounds.

First, we need a simple but useful lemma:

Lemma 1. *If $RR(G_1, G_2)$ exists, H_1 is a subgraph of G_1, and H_2 is a subgraph of G_2, then $RR(H_1, H_2)$ also exists and $RR(H_1, H_2) \leq RR(G_1, G_2)$.*

This lemma follows from the fact that any graph which contains a monochromatic copy of G_1 must also contain a monochromatic copy of its subgraph H_1 and any graph which contains a rainbow copy of G_2 must also contain a rainbow copy of H_2. In the statement of the following theorem, a *forest* is an acyclic graph.

Theorem 9. *The rainbow ramsey number $RR(G_1, G_2)$ exists if and only if G_1 is a star or G_2 is a forest.*

Proof. We will consider four cases. The first case demonstrates indirectly that if $RR(G_1, G_2)$ exists, then G_1 is a star or G_2 is a forest. The remaining three cases show the converse; case 2 serves as a lemma for case 3.

***Case 1.* G_1 is not a star and G_2 is not a forest** For any integer N, label the vertices of K_N with the integers $1, 2, \ldots, N$, and color edge ij with color $min(i, j)$. For any color i, every edge of color i is incident with vertex i. Thus, any monochromatic subgraph must be a star.

Suppose that K_N contains a rainbow subgraph isomorphic to G_2. Since G_2 is not a forest, it must contain some cycle C_k. Thus, K_N contains a rainbow subgraph isomorphic to C_k. Let $v_1, v_2, \ldots, v_k$ be the labels of the vertices of this cycle. We may assume without loss of generality that $v_1 \leq v_i$ for each i, $2 \leq i \leq k$. But then by the definition of the coloring, edge $v_k v_1$ and edge $v_1 v_2$ are both colored with the same color v_1. We have a contradiction; there is no rainbow subgraph isomorphic to G_2.

Since this minimum coloring may be used for any integer N, the rainbow ramsey number does not exist in this case.

Case 2. **$G_1 = K_n$ and $G_2 = K_{1,m}$ for some positive integers n and m.** This case serves as a lemma for Case 3. Let

$$\begin{aligned} N &= \frac{(m-1)^{(n-2)(m-1)+2} - 1}{m-2} \\ &= \sum_{i=0}^{(n-2)(m-1)+1} (m-1)^i \end{aligned}$$

Color the edges of K_N with any number of colors. Choose an arbitrary vertex v_1. If m or more colors appear on the edges adjacent to v_1, then we have a rainbow copy of $K_{1,m}$. Otherwise, at most $m-1$ colors appear, so there must be at least

$$\frac{N-1}{m-1} = \sum_{i=0}^{(n-2)(m-1)} (m-1)^i$$

edges incident with v_1 which are colored with the same color, say color 1. Keep only these edges and the vertices W_1 incident with them, and ignore the remainder of the graph.

Now, choose any vertex v_2 from W_1. Again, if m or more colors appear among the edges between v_2 and the other vertices of W_1, then there is a rainbow copy of $K_{1,m}$. Otherwise, at most $m-1$ colors appear, so there exists a set of at least

$$\frac{|W_1|-1}{m-1} \geq \sum_{i=0}^{(n-2)(m-1)-1} (m-1)^i$$

edges in the same color, say color 2, between v_2 and the other vertices of W_1. Notice that colors 1 and 2 are not necessarily distinct.

Continuing in this fashion, we may assume that we have a sequence of vertices $v_1, v_2, \ldots, v_{(n-2)(m-1)+2}$ such that every edge v_iv_j, for $1 \leq i < j \leq (n-2)(m-1)+2$, is color i, where the colors i and j are not necessarily distinct for $i \neq j$. If m or more of these colors, say colors $i_1, i_2, \ldots, i_m$, are distinct, then vertex $v_{(n-2)(m-1)+2}$ is the central vertex of a rainbow $K_{1,m}$, with endvertices $v_{i_1}, v_{i_2}, \dots, v_{i_m}$. Otherwise, there are at most $m-1$ distinct colors appearing in this subgraph. Thus, of the $(n-2)(m-1)+1$ colors appearing, there must be a set of at least $n-1$ colors which are identical, say $i_1 = i_2 = \dots = i_{n-1}$. In this case, the subgraph generated by the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_{n-1}}$ and the vertex $v_{(n-2)(m-1)+2}$ is a monochromatic complete subgraph of order n.

Case 3. $G_1 = K_n$ **and** G_2 **is a tree of order** m **for some integers** n **and** m

We will proceed by induction on the order m of the tree. Since a tree of order 2 or 3 is necessarily a star, the base case is included in Case 2.

Suppose for some integer m that the rainbow ramsey number $RR(K_n, T)$ exists for any tree T of order $m-1$. Let T' be a tree of order m with an endvertex v adjacent to a vertex u. Let $RR(K_n, T' - \{v\}) = M$. From case 2, we know that the rainbow ramsey number $RR(K_n, K_{1,m-1})$ exists; suppose it is N.

Consider the complete graph K_{NM} on NM vertices. Suppose the edges are colored arbitrarily with any number of colors. We may divide the vertices of K_{NM} into N disjoint sets of M vertices each. The subgraph generated by each set of M vertices must contain either a monochromatic copy of K_n or a rainbow copy of $T' - v$. If any monochromatic copy of K_n appears, we are done, so we may assume that we have N rainbow copies of $T' - v$. Let $u_1, u_2, \dots u_N$ be the corresponding copies of the vertex u. Now, the graph generated by the u_i must contain either a monochromatic copy of K_n or a rainbow copy of $K_{1,m-1}$. If it contains a monochromatic K_n, the proof is complete. Suppose u_i is the central vertex of a rainbow copy of $K_{1,m-1}$. One of the $m-1$ different colors in this star must be different from any of the $m-2$ colors appearing in the ith rainbow copy of $T' - \{v\}$. Thus, we may add the edge in this color to the ith rainbow $T' - \{v\}$ to produce a rainbow copy of T'.

Since the rainbow ramsey number exists when G_1 is complete and G_2 is a tree, Lemma 1 implies that it exists for any graph G_1 provided G_2 is a forest.

Case 4. $G_1 = K_{1,n}$ **and** $G_2 = K_m$ **for some positive integers** n **and** m For convenience in what follows, we will use the *falling factorial* notation. If m is an integer, the falling factorial

$$m^{(k)} = m(m-1)\dots(m-k+1) = \frac{m!}{(m-k)!}.$$

Notice that $m^{(k)}$ behaves asymptotically like m^k.

Choose the integer N so that

$$N \geq 3 + \frac{(n-1)(m+1)^{(4)}}{8} \tag{8}$$

and color the edges of K_N arbitrarily. Assume that there is no monochromatic copy of $K_{1,n}$ in K_N. We will show that there must be a rainbow copy of K_m. Notice that the total number of different copies of K_m in K_N is $\binom{N}{m}$.

We wish to bound the number of copies of K_m that are not rainbow. First, consider the number of copies of K_m that contain two adjacent edges, say uv and uw, which are the same color. There are N choices for the vertex u. Suppose there are a_i edges of color i incident with u, where $1 \leq i \leq k$. Then $\sum_{i=1}^{k} a_i = N - 1$, where $1 \leq a_i \leq n-1$ for each i, and the number of different choices for v and w is $\sum_{i=1}^{k} \binom{a_i}{2}$. The maximum occurs when each a_i is as large as possible, so there are at most

$$\sum_{i=1}^{(N-1)/(n-1)} \binom{n-1}{2} = \frac{N-1}{n-1}\binom{n-1}{2} \qquad .$$

choices for v and w. Then there are $\binom{N-2}{m-3}$ choices for the remaining vertices of K_m. Thus, there are at most

$$N\frac{N-1}{n-1}\binom{n-1}{2}\binom{N-3}{m-3}$$

copies of K_m of this type.

Now, consider copies of K_m in which two nonadjacent edges are the same color. There are $\binom{N}{2}$ choices for the first edge, and $N-2$ ways to choose an endpoint for the second edge. This vertex is incident with no more than $n-1$ edges which are the same color as the first edge and not adjacent to that edge. Since neither the order in which the edges are chosen nor the order in which the endpoints of the second edge are chosen is important, we are counting each pair of edges at least 4 times this way. Thus, there are at most $N(N-1)(N-2)(n-1)/8$ ways to choose two nonadjacent edges of the same color, and $\binom{N-4}{m-4}$ ways to choose the remaining vertices of K_m. The edge-colored copy of K_N can contain at most

$$\frac{N(N-1)(N-2)(n-1)}{8}\binom{N-4}{m-4}$$

copies of K_m of this type.

Thus, there are at most

$$
\begin{aligned}
& N\frac{N-1}{n-1}\binom{n-1}{2}\binom{N-3}{m-3}+\frac{N(N-1)(N-2)(n-1)}{8}\binom{N-4}{m-4} \\
= & \binom{N}{m}\left[\frac{(n-2)m^{(3)}}{2(N-2)}+\frac{(n-1)m^{(4)}}{8(N-3)}\right] \\
< & \binom{N}{m}\left[\frac{(n-1)m^{(3)}}{2(N-3)}+\frac{(n-1)m^{(4)}}{8(N-3)}\right] \\
= & \binom{N}{m}\left[\frac{(n-1)m^{(3)}(4+m-3)}{8(N-3)}\right] \\
= & \binom{N}{m}\left[\frac{(n-1)(m+1)^{(4)}}{8(N-3)}\right] \\
\leq & \binom{N}{m}
\end{aligned}
$$

nonrainbow copies of K_m in K_N, which means that there must be at least one rainbow copy. The last inequality follows from equation 8.

We know from Lemma 1 that since the rainbow ramsey number $RR(K_{1,n}, K_m)$ exists, the number $RR(K_{1,n}, G_2)$ also exists for any graph G_2 of order m. □

This proof immediately produces the upper bounds

$$
\begin{aligned}
RR(K_n, K_{1,m}) &\leq \sum_{i=0}^{(n-2)(m-1)+1} (m-1)^i \\
RR(K_n, T_m) &\leq \prod_{j=2}^{m-1}\left(\sum_{i=1}^{(n-2)(j-1)+1} (j-1)^i\right) \\
RR(K_{1,n}, K_m) &\leq 3+\frac{(n-1)(m+1)^{(4)}}{8}
\end{aligned}
$$

where T_m is an arbitrary tree of order m. The second bound can be improved. In Case 3, we actually only need $(m-1)(N-1)+M$ vertices to force $N = RR(K_n, K_{1,m-1})$ copies of $T-\{v\}$. We can force the copies of $T-\{v\}$ one at a time, removing the vertices of one copy before forcing the next one. Thus,

$$
RR(K_n, T_m) \leq (m-1)\left(\sum_{i=0}^{(n-2)(m-2)+1} (m-2)^i - 1\right) + RR(K_n, T_{m-1}).
$$

Since $RR(K_n, T_3) \leq n$, the improved upper bound is

$$RR(K_n, T_m) \leq \sum_{j=3}^{m-1} j \left(\sum_{i=0}^{(n-2)(j-1)+1} (j-1)^i - 1 \right) + n.$$

Rainbow ramsey numbers also have a strong relationship with generalized ramsey numbers, as the following theorem illustrates. In the theorem, $r(G; m-1) = r(G, G, \ldots, G)$, the generalized ramsey number for a monochromatic graph G when a complete graph is edge-colored with $m-1$ colors.

Theorem 10. *For any positive integer $m \geq 2$ and any graph G,*

$$r(G; m-1) \leq RR(G, mK_2) \leq r(G; m-1) + 2(m-1).$$

Proof. For the lower bound, suppose $N = r(G; m-1)$. Consider a coloring of K_{N-1} with $m-1$ colors that does not contain any monochromatic copy of G. Since the graph is colored with fewer than m colors, it also cannot contain any rainbow copy of mK_2. Thus, $RR(G, mK_2) \geq N$.

For the upper bound, let $M = r(G; m-1) + 2(m-1)$. Consider any coloring of the edges of K_M. If fewer than $m-1$ colors are used, then there must be a monochromatic copy of G. Choose an edge in, say, color 1, and remove the two vertices incident with this edge. If fewer than $m-1$ colors are used on the remaining K_{M-2}, there must be a monochromatic G; so there must be some edge in a color other than 1. Remove the vertices incident with this edge. We may repeat this argument until we have removed the vertices incident with $m-1$ independent edges in $m-1$ different colors, leaving a complete graph on $r(G; m-1)$ vertices. If the complete graph induced by these $r(G; m-1)$ vertices is edge-colored with $m-1$ or fewer colors, then it contains a monochromatic copy of G. Otherwise, it contains at least m colors, including a color distinct from the colors used on the $m-1$ edges already removed. In that case, we have a rainbow copy of mK_2. Thus, $RR(G, mK_2) \leq M$. □

2.2 Lower Bounds

The preceding existence proof provides rough upper bounds on the rainbow ramsey numbers. In this section, we will present some general lower bounds.

Theorem 11. *For any positive integers $n \geq 3$ and $m \geq 3$ and any tree T_m of order m, $RR(K_n, T_m) \geq (n-1)^{m-2}+1$.*

Proof. Let $N = (n-1)^{m-2}$. We may view the vertices of K_N as represented by the set of $(m-2)$-tuples whose entries are elements of $\{ 1, 2, \ldots n-1 \}$. Color the edge between two $(m-2)$-tuples with color i if the first position in which their entries differ is position i. Since only $m-2$ colors are used, no subgraph of K_N can form a rainbow T_m. Suppose any n different vertices are chosen. Let j be the index of the first entry in which some pair of these $(m-2)$-tuples differs. Thus, the first $j-1$ entries are identical for all n vertices. Now, some pair of these vertices differ in the jth entry, but since there are only $n-1$ choices for this entry, some pair must be the same in the jth entry. Thus, at least one edge is color j and at least one edge is a color strictly greater than j. These vertices cannot form a monochromatic K_n. □

We can obtain an alternative lower bound by generalizing the proof of Erdös's bound $r(n,n) > \lfloor 2^{n/2} \rfloor$ in Theorem 3.

Theorem 12. *For any positive integers m and n which satisfy $4 \leq m \leq (n!)^{2/(n+2)} + 2$, and any tree T of order m, the rainbow ramsey number*

$$RR(K_n, T) > \lfloor (m-2)^{n/2} \rfloor.$$

Proof. Let $N = \lfloor (m-2)^{n/2} \rfloor$. If we color the edges of K_N with $m-2$ or fewer colors, there are $(m-2)^{N(N-1)/2}$ different colorings. For any set of n vertices, there are $(m-2)^{N(N-1)/2-n(n-1)/2}$ different colorings of K_N in which these n vertices form a monochromatic K_n in color 1. Thus, the number of nonidentical colorings of K_N which contain a monochromatic K_n in color 1 is at most

$$\begin{aligned} & \binom{N}{n}(m-2)^{N(N-1)/2-n(n-1)/2} \\ < \; & \frac{N^n}{n!}(m-2)^{N(N-1)/2-n(n-1)/2} \\ \leq \; & (m-2)^{N(N-1)/2}\left[\frac{(m-2)^n}{(n!)^2}\right]^{1/2} \\ \leq \; & \frac{1}{m-2}(m-2)^{N(N-1)/2} \end{aligned}$$

where the last inequality holds because $m \leq (n!)^{2/(n+2)} + 2$. Since the same argument holds for each of the $m-2$ colors, there are strictly less than $(m-2)^{N(N-1)/2}$ colorings of K_N which contain a monochromatic subgraph on n vertices. There must be some coloring with no such subgraph. Since only $m-2$ colors are used, this graph also cannot contain a rainbow subgraph isomorphic to T. □

We should note here that the condition $m \leq (n!)^{2/(n+2)} + 2$ is not unreasonable. For $n \geq 8$, $(n!)^{2/(n+2)} \geq n$, so the bound above holds for $m \leq n+2$.

Theorem 13. *For any integers $n \geq 3$ and $m \geq 3$, the rainbow ramsey number*

$$RR(K_{1,n}, K_m) \geq \begin{cases} (n-1)\binom{\frac{m^2-m-2}{2}}{} + 2 & \text{if } n \text{ is odd or } m \cong 0 \text{ or } 1 \pmod 4 \\ (n-1)\binom{\frac{m^2-m-2}{2}}{} + 1 & \text{otherwise} \end{cases}$$

Proof. Let $N = (n-1)\left(\frac{m^2-m-2}{2}\right) + 1$. If n is odd, then factor K_N into $(N-1)/2$ hamiltonian cycles. Color $(n-1)/2$ of the hamiltonian cycles with each color. The resulting graph contains no monochromatic $K_{1,n}$ and strictly fewer than $\binom{m}{2}$ colors.

Assume that n is even.. If $m \cong 0, 1 \pmod 4$, then N is even. Thus, K_N can be 1-factored, and $n-1$ 1-factors colored with each color. If $m \cong 2, 3 \pmod 4$, then let $N' = (n-1)\left(\frac{m^2-m-2}{2}\right)$. Since N' is even, the complete graph $K_{N'}$ can be decomposed into $(n-1)\left(\frac{m^2-m-2}{2}\right) - 1$ 1-factors. Color $(n-1)$ 1-factors with each color except for the last and color $(n-2)$ 1-factors in the last color to obtain a coloring with no monochromatic $K_{1,n}$ and no rainbow K_m. □

2.3 Stars

In the simplest case, when both graphs are stars, we have the following closed formula for the rainbow ramsey number.

Theorem 14. *The rainbow ramsey number* $RR(K_{1,n}, K_{1,m}) = (n-1)(m-1)+2$.

Proof. Suppose the edges of $K_{(n-1)(m-1)+2}$ are colored with any number of colors. Consider any vertex v. Then there are $(n-1)(m-1)+1$ edges incident with v, so either m or more different colors appear on these edges, or some set of n of these edges are the same color. Thus, we have either a rainbow $K_{1,m}$ or a monochromatic $K_{1,n}$.

We must also show that $K_{(n-1)(m-1)+1}$ may be colored so that neither graph appears. If $(n-1)(m-1)+1$ is even, then $K_{(n-1)(m-1)+1}$ can be factored into 1-factors. There are $(n-1)(m-1)$ of these 1-factors; color $n-1$ of them with each color to obtain a coloring with no monochromatic $K_{1,n}$ and no rainbow $K_{1,m}$.

If $(n-1)(m-1)+1$ is odd and $n-1$ is even, then $K_{(n-1)(m-1)+1}$ can be factored into $(n-1)(m-1)/2$ hamiltonian cycles. Color $(n-1)/2$ of these cycles with each color to obtain the desired coloring.

Finally, if $(n-1)(m-1)+1$ is odd and $n-1$ is also odd, color $K_{(n-1)(m-1)+1}$

as follows. For convenience, set $N = (n-1)(m-1)+1$. Label the vertices of K_N by $\{x\} \cup \{v_{i,j} | 1 \le i \le m-1, 1 \le j \le n-1\}$. For each i, with $1 \le i \le m-1$, color the edges of the complete graph induced by $\{x, v_{i,1}, v_{i,2}, \ldots v_{i,n-1}\}$ with color i. For each i and j with $i \ne j$, color the edges joining the vertices $\{v_{i,1}, v_{i,2}, v_{i,3}, \ldots v_{i,n-1}\}$ with the vertices $\{v_{j,1}, v_{j,2}, v_{j,3}, \ldots v_{j,n-1}\}$ with some new color. Thus, the edges in any given color induce a subgraph isomorphic to either K_n or $K_{n-1,n-1}$, neither of which contain $K_{1,n}$. Exactly $m-1$ colors appear at each vertex, so there is no rainbow $K_{1,m}$. □

2.4 Rainbow Ramsey Numbers and Matchings

We will call a 1-regular graph a *matching.* Notice that any 1-regular graph consists of n disjoint copies of the complete graph on 2 vertices, for some integer n. Such a graph is commonly denoted by nK_2.

In Theorem 5, Cockayne and Lorimer presented a formula for the standard ramsey number for such graphs:

$$r(n_1K_2, n_2K_2, \ldots n_cK_2) = n_1 + 1 + \sum_{i=1}^{c} n_i - 1. \tag{9}$$

In particular, if $n_1 = n_2 = \ldots = n_c$, we have

Corollary 1. *If n is any positive integer, then*

$$r(nK_2, nK_2, \ldots nK_2) = (c+1)(n-1)+2$$

A graph colored with c or fewer colors cannot possibly contain a rainbow copy of $(c+1)K_2$. If the graph is colored with $c+1$ or more colors, then such a subgraph is possible. Thus, taking $m = c+1$,

$$RR(nK_2, mK_2) \ge r(nK_2, nK_2, \ldots nK_2) = m(n-1)+2 \ .$$

We may easily see the inequality $RR(nK_2, mK_2) \geq m(n-1)+2$ directly. Color the graph $K_{m(n-1)+1}$ as follows. Color all of the edges of a subgraph isomorphic to K_{2n-1} with color 1. Choose $n-1$ additional vertices and color all of the edges among these vertices and between these vertices and those already colored with color 2. For each color $i = 3, 4, \dots m-1$, choose $n-1$ additional vertices and color the edges among those vertices and between those vertices and the part of the graph already colored with color i. The resulting graph has $2n-1+(m-2)(n-1) = m(n-1)+1$ vertices and contains no set of n independent edges in the same color. Since only $m-1$ colors appear, it also cannot contain a set of m independent edges in different colors.

In the case when $m = n$, Bialostocki and Voxman showed that this inequality is in fact an equality, in Theorem 8.

We suspect that their result can be generalized as follows:

Conjecture 1. *For every pair of positive integers n and m, where $n \geq 3$ and $m \geq 2$,*

$$RR(nK_2, mK_2) = m(n-1)+2.$$

First, we handle several trivial cases. Any graph with at least one edge must contain both a monochromatic and a rainbow K_2, so $RR(K_2, mK_2) = RR(nK_2, K_2) = 2$. If a graph contains at least n independent edges, then either two of the edges are different colors or all of them are the same color. Thus, $RR(nK_2, 2K_2) = 2n$. Similarly, if a graph contains at least m independent edges, then it must contain either a rainbow mK_2 or a monochromatic $2K_2$. However, a graph with fewer than $2m$ vertices could be colored with every edge a different color to avoid these two graphs. Therefore, $RR(2K_2, mK_2) = 2m$.

Bialostocki and Voxman's proof can be adapted to show Conjecture 1 in the case $m < n$.

Theorem 15. *For any two positive integers n and m, where $2 \leq m < n$,*

$$RR(nK_2, mK_2) = m(n-1) + 2.$$

Proof. We will proceed by induction on m. The formula holds when $m = 2$, as discussed above. For some $m \geq 3$, suppose the edges of $K_{m(n-1)+2}$ are colored with any number of colors. If fewer than m colors are used, then we may apply Corollary 1 with $c = m - 1$ to see that some monochromatic copy of nK_2 must appear. Thus, we may assume without loss of generality that at least m colors are used.

Choose one edge of each of m different colors that appear in such a way that the number of independent edges in this set is maximal. Let H represent these edges and let $V(H)$ represent the vertices incident with these edges. If $|V(H)| = 2m$, then we have a rainbow copy of mK_2 and we are done. Assume that $|V(H)| \leq 2m - 1$.

Let $M = V(K_{m(n-1)+2}) - V(H)$. If there is any color which appears in the graph induced by M and not in H, then the number of independent edges in H is not maximal, which contradicts our choice of H. If *every* color which appears in H also appears in M, then we may choose some color in H which does not appear on an independent edge and replace that edge with an edge of the same color in M to produce a set of representatives of the colors with more independent edges than H. Again, this contradicts our choice of H. Thus, the colors appearing in M must be a proper subset of the set of colors appearing in H.

Since $m < n$, the set M contains at least

$$\begin{aligned} |M| &\geq (n-1)m + 2 - (2m-1) \\ &= nm - 3m + 3 \\ &\geq nm - 2m - n + 1 + 3 \\ &= (n-2)(m-1) + 2 \end{aligned}$$

vertices. Therefore, by the inductive hypothesis, the subgraph generated by M contains either a monochromatic copy of $(n-1)K_2$ or a rainbow copy of $(m-1)K_2$. Since H contains one edge of each color appearing in M and at least one edge of a color not appearing in M, we may add an edge from H to the subgraph in M to produce either a monochromatic nK_2 or a rainbow mK_2. □

Next we will show that the same formula holds for $m = n + 1$. Two of the smaller values must be shown separately.

Theorem 16. *The rainbow ramsey number* $RR(3K_2, 4K_2) = 10$.

Proof. By the coloring described previously, we know that $RR(3K_2, 4K_2) \geq 10$. Suppose the edges of K_{10} are colored with any number of colors. Consider any set of 5 independent edges, say ab, cd, ef, gh and ij. If 4 or more colors appear, or if some color appears at least 3 times, we are done. Without loss of generality, we may assume that the edges ab, cd, ef, gh and ij are colored with colors $1, 1, 2, 2$, and 3, respectively.

Notice that if color 3 is used on any of the edges ac, bd, ad, bc, then it cannot be used on any of the edges eg, fh, eh, fg without creating a monochromatic $3K_2$ in color 3. Thus, we may assume that this color appears on at most one of these sets of four edges. Assume without loss of generality that color 3 does not appear on the edges ac, bd, ad, bc. Notice that color 2 cannot appear on these edges either without creating a monochromatic $3K_2$.

Case 1. **One of the edges ac, bd, ad, bc is some new color.** Suppose without loss of generality that ac is a new color, color 4. Since ac, bd, ef, and ij are independent edges, edge bd must be one of the colors $2, 3$ or 4, or else we have a rainbow $4K_2$.

We may assume that bd is color 4. If the edge ce is any color except 2 or 3, then we have a rainbow $4K_2$, using either ab or bd along with ce, gh, and ij. Similarly, we may assume that df is colored either 2 or 3. If ce and df are the same color, then together with either gh or ij they form a monochromatic $3K_2$. Thus, without loss of generality, ce is color 3 and df is color 2.

By the same argument, one of the edges ag and bh is color 2 and the other is color 3. However, we now have $3K_2$ in color 3.

Case 2. **The edges ac, bd, ad, bc are all color 1.** If any edge from the set of vertices a, b, c, d to the set e, f, g, h is a new color, then we have a rainbow $4K_2$.

Consider the edges ae, cg, bf, and dh, colored in the three colors $1, 2, 3$. If color 1 appears twice, then we have $3K_2$ in color 1. Similarly, if color 3 appears twice, we have a monochromatic $3K_2$. If color 2 appears twice incident with ef or twice incident with gh, then we have $3K_2$ in color 2. We may assume that color 2 appears twice, once incident with the edge ef and once incident with gh. Without loss of generality, edges ae and cg are color 2, edge bf is color 1 and edge dh is color 3.

Consider edge ai. If this edge is in some new color, then ai, cg, bf and dh form a rainbow $4K_2$. If it is color 1, then it forms a monochromatic $3K_2$ along with bf and cd. If it is color 2, then it forms a monochromatic $3K_2$ along with ef and gh. Thus, we may assume without loss of generality that edge ai is color 3. Similarly, we may assume that edge cj is color 3. But then edges ai, cj and dh form a monochromatic $3K_2$. □

Theorem 17. *The rainbow ramsey number $RR(4K_2, 5K_2) = 17$.*

Proof. The lower bound follows from the coloring discussed previously.

Suppose that the edges of K_{17} are colored with any number of colors. If 4 or fewer colors are used, then by Corollary 1, there is a monochromatic subgraph isomorphic to $4K_2$. Thus, we may assume that at least 5 colors are used.

Since $RR(4K_2, 4K_2) = 14 \leq 17$, we may also assume without loss of generality that there is a rainbow subgraph isomorphic to $4K_2$; we will label the colors 1, 2, 3, and 4. Some color 5 must appear somewhere in the graph. If color 5 appears on an edge independent from the edges of the $4K_2$, we are done.

Suppose an edge of color 5 appears incident with two of the edges of the $4K_2$, as shown in Figure 1. Since $RR(3K_2, 3K_2) = 8 \leq 9$, there must be either a monochromatic or a rainbow $3K_2$ on the remaining 9 vertices. If there is a monochromatic $3K_2$ in some new color, then we have a rainbow $5K_2$ in colors 1, 2, 3, 4, and this new color. If there is a monochromatic $3K_2$ in one of the colors 1, 2, 3, 4, or 5, then we may add the appropriate edge to obtain a monochromatic $4K_2$. Thus, we may assume wlog that there is a rainbow $3K_2$, necessarily using three of the four colors 1, 2, 3, and 4. In particular, there is an edge in color 3 or an edge in color 4, so, up to interchanging colors, we may assume that we have a subgraph as shown in Figure 2.

Let $N = V(K_{17}) - \{a, b, c, d, e, f, g, h, i\}$. If N contains an edge in any color other than 1, 2, and 3, then we have a rainbow $5K_2$. Since $|N| = 8 =$

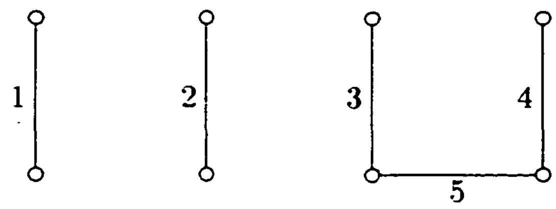

Figure 1. Possible Location for Edge of Color 5 in Theorem 17.

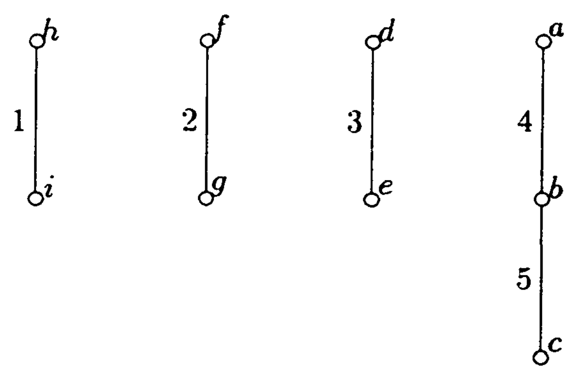

Figure 2. Other Possible Location for Edge of Color 5 in Theorem 17.

$RR(3K_2, 3K_2)$, there must be either a monochromatic $3K_2$ in color 1, 2, or 3 or a rainbow $3K_2$ on colors 1, 2, and 3 on N. If N contains a monochromatic $3K_2$, then we have a monochromatic $4K_2$ in the original graph. Thus, we may assume that N contains three independent edges in colors 1, 2, and 3, respectively. The remaining independent edge in N must be color 1, 2, or 3, say wlog color 1. Without loss of generality, we have the graph shown in Figure 3.

Let $M = V(K_{17}) - \{a, b, c\}$. Since $|M| = 14 = RR(4K_2, 4K_2)$, we may assume wlog that M contains a rainbow $4K_2$. If this $4K_2$ does not contain an edge of color 4 and an edge of color 5, then we may add edge bc or edge ab to obtain a rainbow $5K_2$. Thus, we may assume that an edge of color 4 and an edge of color 5 appear in M.

If the color 4 edge appears anywhere in M besides the edges ng, nf, og, of, pd, pe, qe, and/or qd, then we have a rainbow $5K_2$. Without loss of generality, we may assume that edge ng is color 4.

Consider edge op. If op is color 1, then we have a $4K_2$ in color 1. If op is color 2, 4, or 5, or some new color, then we have a rainbow $5K_2$. Thus, op must be color 3. Similarly, oq, oe, od, fp, fq, fe, and fd must all be color 3.

Consider edge qd. If qd is color 1, we have a monochromatic $4K_2$ in color 1; if qd is color 2, 4, or 5, or some new color, then we have a rainbow $5K_2$. Thus, qd and, similarly, edges qe, pe, and pd must all be color 3.

Now, if any edge on the vertices h, i, j, k, l, and m is color 3, we have a $4K_2$ in color 3. If any one of these edges is color 2, 4, or 5 or some new color, then we have a rainbow $5K_2$. Thus, we may assume that vertices h, i, j, k, l, and m induce a complete graph in color 1.

Finally, consider the six edges hd, ie, jf, ko, lp, and mq. If two or more

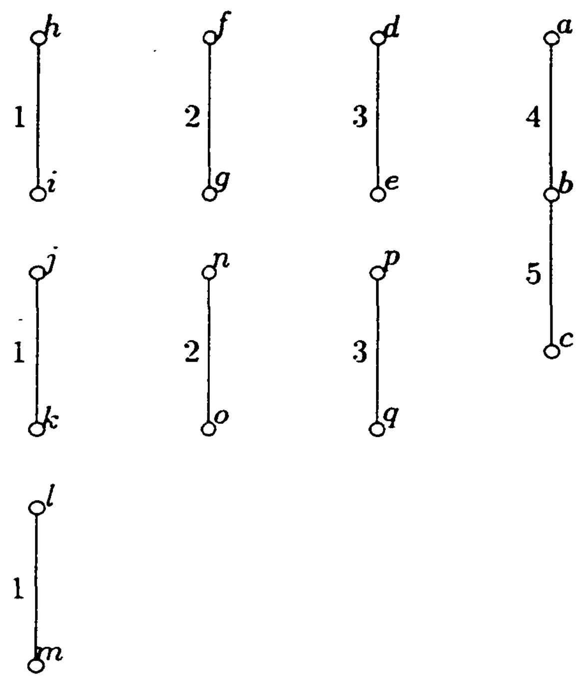

Figure 3. Subgraph Which Must Exist, WLOG, in Theorem 17.

of these edges are color 1 or if two or more are color 3, then we have a monochromatic $4K_2$. If any one of these edges is color 2, 4, or 5, or a new color, then we have a rainbow $5K_2$. There are no other possibilities; we must have either a monochromatic $4K_2$ or a rainbow $5K_2$. □

The proof for $n \geq 5$ and $m = n + 1$ actually shows a slightly more general case. First, we will need a few technical lemmas.

Lemma 2. *Assume that* $RR(nK_2,(m-1)K_2) = (m-1)(n-1)+2$. *Suppose* $K_{m(n-1)+2}$ *is edge-colored with any number of colors. Then either* $K_{m(n-1)+2}$ *contains a monochromatic* nK_2 *or a rainbow* mK_2, *or any set of independent edges in a given color can be extended to a set of* $\lceil \frac{n}{2} \rceil$ *independent edges in that color.*

Proof. Suppose there is a set of k independent edges in the same color, say color

1. Let M be the set of $2k$ vertices incident with these edges. If

$$\begin{aligned} 2k &\leq m(n-1)+2-RR(nK_2,(m-1)K_2) \\ &= m(n-1)+2-[(m-1)(n-1)+2] \\ &= n-1, \end{aligned}$$

then we may assume that there is either a monochromatic nK_2 or a rainbow $(m-1)K_2$ on the remaining vertices. If the rainbow $(m-1)K_2$ does not contain color 1, then we may add an edge in color 1 to produce a rainbow mK_2. Otherwise, the rainbow $(m-1)K_2$ contains an edge in color 1 independent from the edges in M. We may add the vertices incident with this edge to M and repeat the argument. Continuing in this fashion, we can extend the set M until $|M| = 2k$, where $2k > n-1$, that is, until $k > (n-1)/2$. □

We will primarily use this lemma in the following form.

Corollary 2. *Assume $RR(nK_2,(m-1)K_2) = (m-1)(n-1)+2$ and $n \geq 5$. If $K_{m(n-1)+2}$ is edge-colored with any number of colors, then either the graph contains a monochromatic nK_2 or a rainbow mK_2, or any edge or pair of independent edges in a single color can be extended to a set of three independent edges in that color.*

Lemma 3. *Assume that $RR(nK_2,pK_2) = p(n-1)+2$ for every positive integer $p < m$. Suppose $K_{m(n-1)+2}$ is edge-colored with any number of colors and suppose the resulting graph does not contain either a monochromatic nK_2 or a rainbow mK_2. If M is a set of vertices and S is a set of c colors, $c \geq 1$, such that*

(1) there is a set of c independent edges on the vertices of M containing an edge in each color of S and

(2) $|M| \leq c(n-1)$,

then there is an edge in $K_{m(n-1)+2}$ independent of M colored with one of the colors of S.

Proof. Let M be such a set. Since

$$\begin{aligned} |M| &\leq c(n-1) \\ &= (m(n-1)+2) - ((m-c)(n-1)+2) \\ &= (m(n-1)+2) - RR(nK_2, (m-c)K_2), \end{aligned}$$

the remainder of the graph must contain either a monochromatic nK_2 or a rainbow $(m-c)K_2$. If none of the colors of S appear in the rainbow $(m-c)K_2$, then it can be extended to a rainbow mK_2. Thus, we may assume that there is a rainbow $(m-c)K_2$ independent from M containing an edge in one of the colors of S. $\square$

We are now ready to prove the main result. Notice that for $n \geq 5$, we have $n+1 \leq \frac{3}{2}(n-1)$.

Theorem 18. *For $n \geq 5$ and $2 \leq m \leq \frac{3}{2}(n-1)$, the rainbow ramsey number*

$$RR(nK_2, mK_2) = m(n-1)+2$$

Proof. We proceed by strong induction on m, using Theorems 8 and 15 as the base. Thus, we assume that the formula holds for $RR(nK_2, pK_2)$ for all $p < m$ and that $m > n \geq 5$. Suppose $K_{m(n-1)+2}$ is edge-colored with any number of colors. Since $m(n-1)+2 \geq (m-1)(n-1)+2 = RR(nK_2, (m-1)K_2)$, we may assume without loss of generality that there is a rainbow $(m-1)K_2$, say in colors $\{1, 2, \ldots m-1\}$. Now, since $m \leq 2(n-1)$, it follows that there are at least $m(n-1)+2-2(m-1) \geq (m-2)(n-2)+2 = RR((n-1)K_2, (m-2)K_2)$ vertices remaining. If a monochromatic $(n-1)K_2$ appears in a new color, then we may add an edge in this new color to the rainbow $(m-1)K_2$ to produce a rainbow mK_2. If a monochromatic $(n-1)K_2$ appears in one of the colors $1, 2, \ldots m-1$, then this subgraph along with the appropriate edge from the rainbow $(m-1)K_2$ yields a monochromatic nK_2.

Thus, we may assume without loss of generality that a rainbow $(m-2)K_2$ appears, independent from the $(m-1)K_2$. If any new color appears on this $(m-2)K_2$, then we have a rainbow mK_2. Thus, without loss of generality, we may assume that the $(m-2)K_2$ is colored with colors $1, 2, \ldots m-2$.

Since $m \leq (3/2)(n-1)$, there are at least $m(n-1)+2-2(m-1)-2(m-2) \geq (m-3)(n-3)+2 = RR((n-2)K_2, (m-3)K_2)$ vertices remaining. If there is a monochromatic $(n-2)K_2$ on these vertices in one of the colors $1, 2, \ldots m-2$, then we have a monochromatic nK_2. If, on the other hand, there is a monochromatic $(n-2)K_2$ or a rainbow $(m-3)K_2$ containing some new color, then we have a rainbow mK_2. Thus, we may assume, without loss of generality, that we have one of the following three cases.

Case 1 **There is a monochromatic $(n-2)K_2$ in color $m-1$.** Label the vertices as shown in Figure 4, so that edges u_iv_i and w_ix_i are color i for $1 \leq i \leq m-2$.

From corollary 1, if only $m-1$ colors were used to color the edges of $K_{m(n-1)+2}$, then there must be a monochromatic nK_2. Thus, we may assume that there is some new color, say color m, appearing on these vertices. According to corollary 2, we may also assume that this color appears on at least 3 independent edges. If any edge in color m is not an edge u_iw_i, u_ix_i, v_iw_i or v_ix_i for some i,

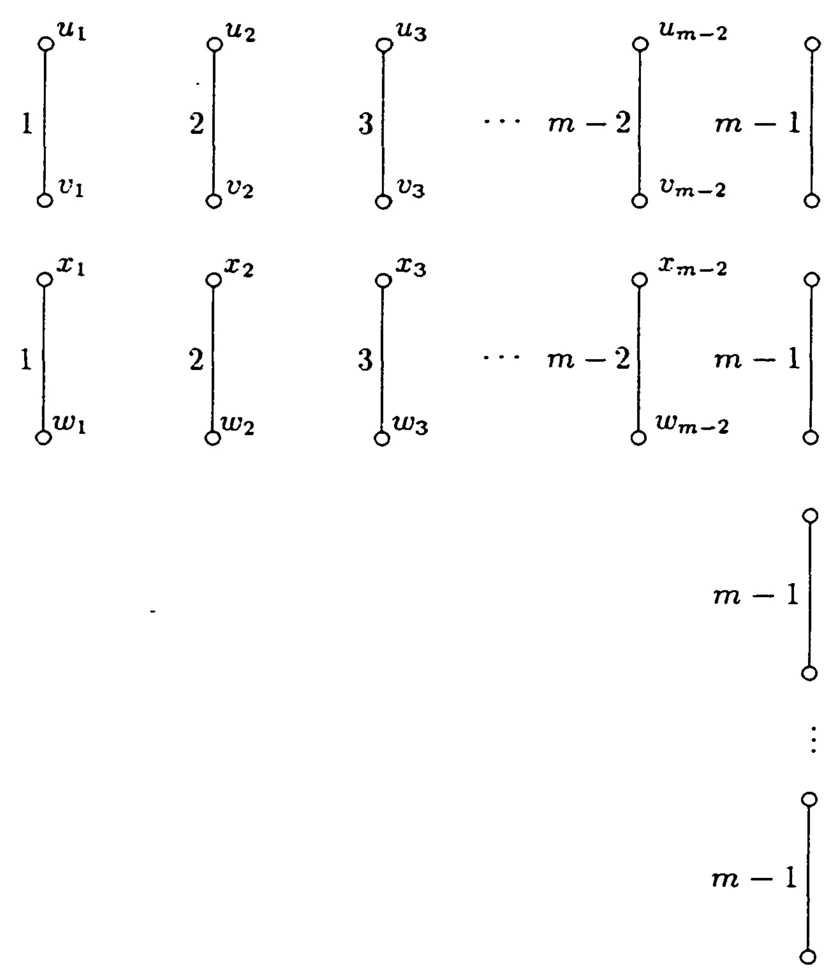

Figure 4. Case 1 of Theorem 18.

$1 \le i \le m-2$, then we have a rainbow mK_2. At most 2 of the 3 independent edges in color m can appear incident with u_i, v_i, w_i and x_i for any given i. Thus, we may assume without loss of generality that edges v_1w_1 and v_2w_2 are color m.

We will proceed by induction. Let

$$M_{\le i} = \{u_j, v_j, w_j, x_j | 1 \le j \le i\}$$

Then the graph induced by $M_{\le 2}$ contains a pair of independent edges in *any* two of the three colors $1, 2$, and m, that is, it contains two independent edges in colors 1 and 2, two independent edges in colors 1 and m, and two independent edges in colors 2 and m.

Suppose, for any i, $1 \le i \le m-2$, that the graph induced by $M_{\le i}$ contains a set of i independent edges in *any* i of the colors $1, 2, \ldots i$, and m. Since $|M_{\le i}| = 4i$, we may apply lemma 3 with $c = i$ and $S = \{1, 2, \ldots i\}$. Since $n \ge 5$, we have $4i \le c(n-1)$. Thus, there must be some edge independent from $M_{\le i}$ in one of the colors $1, 2, \ldots i$. If this edge is not u_jw_j, u_jx_j, v_jw_j or v_jx_j for some j, where $i < j \le m-2$, then we have a rainbow mK_2 using this edge in, say, color k, a matching on $M_{\le i}$ in the colors $\{1, 2, \ldots i, m\} - \{k\}$, and a matching in the remainder of the graph in colors $i+1, i+2, \ldots m-1$. Thus, we may assume without loss of generality that the new edge in color k, $1 \le k \le i$, is the edge $v_{i+1}w_{i+1}$. Let C be any subset of $i+1$ colors from the set $\{1, 2, \ldots i+1, m\}$. If C contains color $i+1$, then the graph induced by $M_{\le i+1}$ contains a set of independent edges in the colors of C, since $M_{\le i}$ contains a set of independent edges in colors $C - \{i+1\}$. If C does not contain color $i+1$, then $C = \{1, 2, \ldots i, m\}$. Since the graph induced by $M_{\le i}$ contains a set of independent edges in colors $\{1, 2, \ldots i, m\} - \{k\}$, the graph induced by $M_{\le i+1}$ contains a set of independent edges in the colors of C.

Continuing inductively, we may assume that $M_{\le m-2}$ contains a set of $m-2$

independent edges in *any* $m-2$ of the colors $\{1,2,\dots m-2,m\}$. If we apply lemma 3 with $c=m-2$ and $S=\{1,2,\dots m-2\}$, then we may assume that there is an edge independent from $M_{\leq m-2}$ in one of the colors $1,2,\dots m-2$. Then this edge, say in color k, an independent edge in color $m-1$, and a set of independent edges in $M_{\leq m-2}$ in colors $\{1,2,\dots m-2,m\}-\{k\}$ form a rainbow mK_2.

Case 2 **There is a rainbow $(m-3)K_2$ not containing color $m-1$.** Without loss of generality, we may assume that there is a subgraph as shown in Figure 5. As in case 1, we may assume that some new color, say m, appears on at least three independent edges. If any edge in this new color is not adjacent to either the edge in color $m-1$ shown in Figure 5 or *both* of the edges of color $m-2$, then we have a rainbow mK_2. Since at most two independent edges can be adjacent to the edge in color $m-1$, we may assume that at least one edge of color m appears adjacent to both edges of color $m-2$.

Let M be the set of vertices incident with the edges of colors $m-2$ and $m-1$ shown in the figure. We may apply lemma 3 with $c=2$ and $S=\{m-2,m-1\}$. Since $6\leq 2(n-1)$ for $n\geq 5$, we may assume that there is an edge in color $m-1$ or color $m-2$ independent from M. If an edge in color $m-2$ appears, then we have a rainbow mK_2; we may assume that an edge in color $m-1$ appears. Let M' be the set of vertices in M along with the two endpoints of this new edge of color $m-1$. Apply lemma 3 to M' with $c=2$ and $S=\{m-2,m-1\}$, since $8\leq 2(n-1)$ for $n\geq 5$. Thus, there must be another edge in color $m-1$ independent from M'.

Now, from corollary 2, we may also assume that there is an edge in color $m-2$ independent from the two edges in that color shown in Figure 5. If this edge is not adjacent to the edge in color $m-1$, then we have a rainbow mK_2.

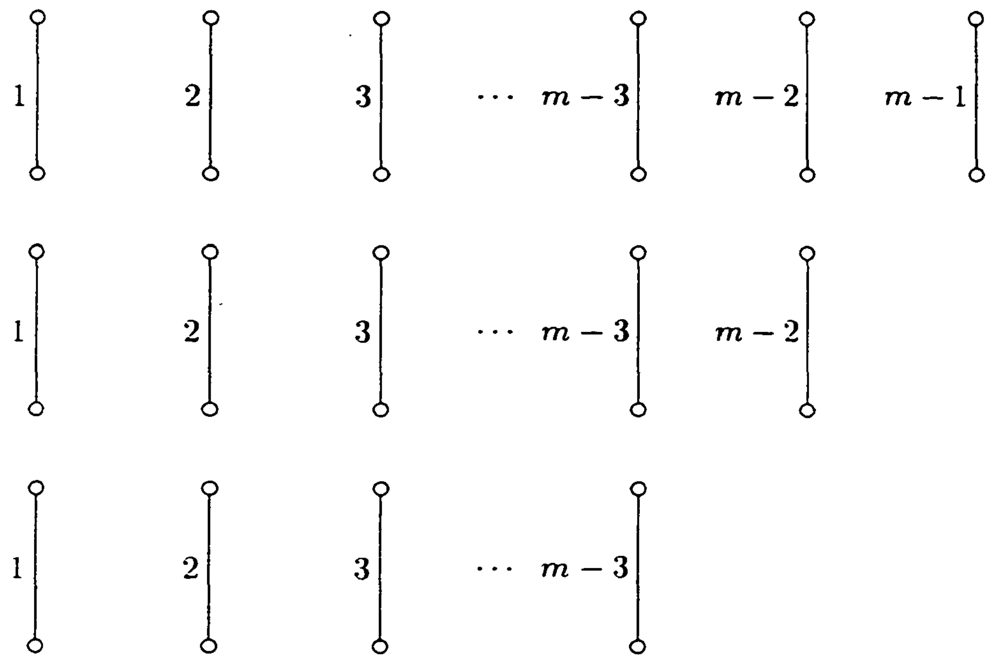

Figure 5. Case 2 of Theorem 18.

So we may assume that there is an edge in color $m-2$ adjacent to the edge of color $m-1$. Since there are two independent edges in $V(K_N)-M$ in color $m-1$, there is an edge in color $m-1$ independent from this new edge in color $m-2$. Consider these two edges in colors $m-1$ and $m-2$, respectively, and the edge of color m. If there is still a set of $m-3$ independent edges in colors $1, 2, \ldots m-3$ on the remainder of the graph, then we have a rainbow mK_2.

Since we are using three vertices of $V(K_N)-M$, it is possible that these three vertices are incident with three different edges in the same color, say color $m-3$. Let L be the set of vertices in M along with the 6 vertices adjacent to the edges in color $m-3$. We may apply lemma 3 to L with $S = \{m-3, m-2, m-1\}$. Since $12 \leq 3(n-1)$ for $n \geq 5$, there must be some edge independent from L in one of these three colors. Observe that with this edge and the edges in L, we can

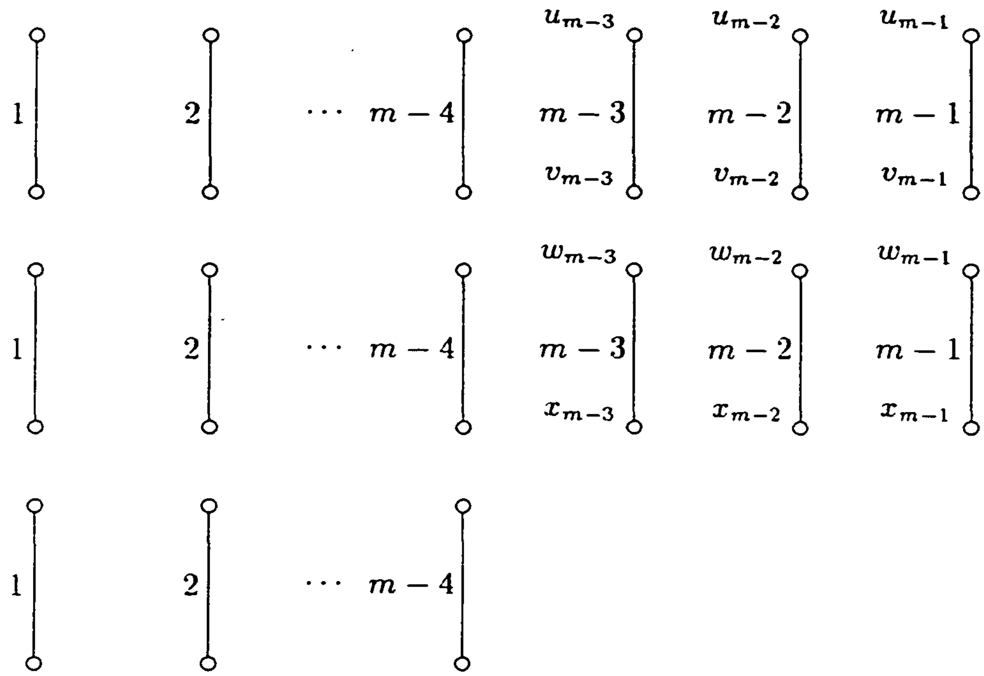

Figure 6. Case 3 of Theorem 18.

obtain an independent set of edges in colors $m-3, m-2, m-1$ and m. There must be an independent set of edges in colors $1, 2, \ldots m-4$ on the vertices remaining, so we have a rainbow mK_2.

Case 3 **There is a rainbow $(m-3)K_2$ containing color $m-1$.** We may assume that we have the graph shown in Figure 6, with edges u_iv_i and w_ix_i in color i, for $i = m-3, m-2, m-1$.

As in the previous two cases, we may assume that there is some new color, say color m, appearing on at least three independent edges. If any edge in color m is not one of the edges u_iw_i, u_ix_i, v_iw_i or v_ix_i for $i = m-3, m-2$, or $m-1$, then we have a rainbow mK_2. Since at most two independent edges can be chosen from $\{u_iw_i, u_ix_i, v_iw_i, v_ix_i\}$ for each i, we may assume without loss of generality that edges $v_{m-2}w_{m-2}$ and $v_{m-1}w_{m-1}$ are color m.

Let $M = \{u_{m-2}, v_{m-2}, w_{m-2}, x_{m-2}, u_{m-1}, v_{m-1}, w_{m-1}, x_{m-1}\}$. If we apply lemma 3 to M with $c = 2$ and $S = \{m-2, m-1\}$, we have some edge in color $m-2$ or $m-1$ independent from M. If this edge is not one of the edges $u_{m-3}w_{m-3}$, $u_{m-3}x_{m-3}$, $v_{m-3}w_{m-3}$ or $v_{m-3}x_{m-3}$, then we have a rainbow mK_2. Assume wolog that edge $v_{m-3}w_{m-3}$ is color $m-2$ or $m-1$. Let $M' = \{u_i, v_i, w_i, x_i | i = m-3, m-2, m-1\}$, and let $S = \{m-3, m-2, m-1\}$. According to lemma 3, there is some edge in one of the colors $m-3, m-2, m-1$ independent from M'. Thus, there is a rainbow mK_2. □

We have seen that the formula

$$RR(nK_2, mK_2) = m(n-1) + 2$$

from Conjecture 1 holds for $m \leq \frac{3}{2}(n-1)$. In general, for $n \geq 2$, we have

$$m(n-1) + 2 \leq RR(nK_2, mK_2) \leq 2(n-1)m$$

The lower bound was discussed previously. Notice that the upper bound holds for $n = 2$ and for $m = 1$ provided $n \geq 2$. For any $n \geq 3$ and $m \geq 2$, suppose $RR(nK_2, (m-1)K_2) \leq 2(n-1)(m-1)$ and $RR((n-1)K_2, mK_2) \leq 2(n-2)m$. Consider any edge-coloring of $K_{2(n-1)m}$. If the resulting graph does not contain a rainbow mK_2, then without loss of generality it must contain a monochromatic $(n-1)K_2$. If we remove these $2(n-1)$ vertices, there are $2(n-1)(m-1)$ vertices remaining. Thus, there is either a monochromatic nK_2 or a rainbow $(m-1)K_2$ on the remaining vertices. Without loss of generality, then, we have a monochromatic $(n-1)K_2$, say in color c, and a disjoint rainbow $(m-1)K_2$. Either the rainbow $(m-1)K_2$ contains an edge in color c or it does not. If it contains an edge in color

c, then this edge along with the monochromatic $(n-1)K_2$ form a monochromatic nK_2. Otherwise, an edge in color c from the $(n-1)K_2$ may be added to the rainbow $(m-1)K_2$ to produce a rainbow mK_2.

2.5 Matchings and Stars

Next, we consider the rainbow ramsey number when one of our graphs is a matching and the other is a star. In the case of a monochromatic star and a rainbow matching, the following upper and lower bounds meet to give a formula for an infinite number of parameters n and m. First, we present the lower bound.

Theorem 19. *For any positive integers n and m, provided that n is odd or m is even, the rainbow ramsey number $RR(K_{1,n}, mK_2) \geq (n-1)(m-1)+2$. If n is even and m is odd, then $RR(K_{1,n}, mK_2) \geq (n-1)(m-1)+1$.*

Proof. Let $N = (n-1)(m-1)+1$. If n is odd, then N is also odd, and K_N can be factored into hamiltonian cycles. Color $(n-1)/2$ of the hamiltonian cycles with each color. The resulting graph contains no monochromatic $K_{1,n}$ and fewer than m colors.

If n and m are both even, then N is even. In this case, K_N can be factored into 1-factors. Color $n-1$ 1-factors with each color to obtain a coloring with neither a monochromatic $K_{1,n}$ nor a rainbow mK_2. If n is even and m is odd, then $N-1$ is even. Thus, K_{N-1} can be factored into $N-2$ 1-factors. Color $n-1$ 1-factors with each color, with only $n-2$ 1-factors in the last color. Again, only $m-1$ colors are used, and each color appears at most $n-1$ times at any given vertex. □

In the corresponding upper bound, we employ the standard convention that

$$\binom{k}{2} = 0$$

if $k < 2$.

Theorem 20. *For any positive integers n and m, the rainbow ramsey number*

$$RR(K_{1,n}, mK_2) \leq (n-1)(m-1) + 2 + \binom{m-n+3}{2}$$

.

Proof. Let $N = (n-1)(m-1)+2+\binom{m-n+3}{2}$. Suppose the edges of K_N are colored so that there is no monochromatic subgraph isomorphic to $K_{1,n}$. We will show that there must be a rainbow copy of mK_2. Since $RR(K_{1,n}, K_{1,m}) = (n-1)(m-1)+2$, we may assume without loss of generality that there is a rainbow copy of $K_{1,m}$. Temporarily remove these $m+1$ vertices from the graph. Notice that there are

$$(n-1)(m-1)+2+\binom{m-n+3}{2} - (m+1) \geq (n-1)(m-2)+2+\binom{m-n+2}{2}$$

vertices remaining.

Continuing inductively for $i = m-1, m-2, \ldots 1$, suppose we have $(n-1)(i-1) + 2 + \binom{i-n+3}{2} \geq RR(K_{1,n}, K_{1,i})$ vertices. We may assume wlog that there is a rainbow copy of $K_{1,i}$. If we remove these $i+1$ vertices, we are left with $(n-1)(i-1)+2-(i+1)+\binom{i-n+3}{2}$ vertices. If $i-n+2 \leq 0$, that is, if $i-n+3 \leq 1$, then $\binom{i-n+3}{2} = 0$. Thus, there are

$$\begin{aligned}
& (n-1)(i-1)+2-(i+1)+\binom{i-n+3}{2} \\
= \; & (n-1)(i-1)+2-(i+1) \\
= \; & (n-1)(i-2)+2+(n-i-2) \\
\geq \; & (n-1)(i-2)+2+\binom{i-n+2}{2}
\end{aligned}$$

.

vertices left over. If $i - n + 2 > 0$, then $\binom{i-n+3}{2} = \sum_{j=1}^{i-n+2} j$, so we have

$$\begin{aligned} & (n-1)(i-1) + 2 - (i+1) + \sum_{j=1}^{i-n+2} j \\ = \; & (n-1)(i-1) + 2 - (i+1) + (i-n+2) + \sum_{j=1}^{i-n+1} j \\ = \; & (n-1)(i-2) + 2 + (n-i-2) + (i-n+2) + \sum_{j=1}^{i-n+1} j \\ = \; & (n-1)(i-2) + 2 + \sum_{j=1}^{i-n+1} j \\ = \; & (n-1)(i-2) + 2 + \binom{i-n+2}{2} \end{aligned}$$

vertices remaining.

Thus, we may assume that there are vertex-disjoint rainbow copies of $K_{1,i}$ for $i = 1, 2, \ldots m$, in which the same color could appear in *different* stars. Choose a collection of edges as follows. Take the edge in $K_{1,1}$. Now, one of the two edges in $K_{1,2}$ must be some new color, so take that edge. For each i, $K_{1,i}$ contains one more edge than we have previously chosen, so we may take an edge in some new color. Thus, there is a rainbow copy of mK_2. □

Thus, we have a formula when $n \geq m + 2$ and n is odd or m is even. For other values of n and m, we have upper and lower bounds. It is possible that neither bound is sharp. For instance, for $n = m = 3$, the previous two theorems yield $6 \leq RR(K_{1,3}, 3K_2) \leq 9$. The actual value lies strictly between these bounds.

Theorem 21. $RR(K_{1,3}, 3K_2) = 7$.

Proof. Figure 7 shows a coloring of K_6 containing neither a monochromatic $K_{1,3}$ nor a rainbow $3K_2$.

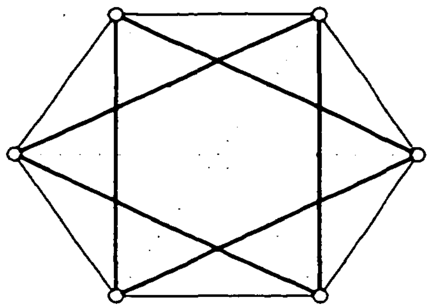

Figure 7. Coloring of K_6 Showing $RR(K_{1,3}, 3K_2) \geq 7$.

Suppose the edges of K_7 are colored so that no monochromatic $K_{1,3}$ appears. Take out any edge ab. From Theorem 20, we know that $RR(K_{1,3}, 2K_2) \leq 5$, so we may assume without loss of generality that there are two independent edges in different colors, say colors 1 and 2, on the remaining 5 vertices. If the edge we removed is in a color other than color 1 or 2, then we are done. We may assume that we have three independent edges ab, cd, and ef, colored with colors 1, 1, and 2 respectively. Let x be the other vertex. If any one of the edges ax, bx, cx, or dx is a color other than color 1 or 2, then we have a rainbow $3K_2$. If any color appears more than twice at x, we have a monochromatic $K_{1,3}$. Thus, we may assume that two of these four edges are color 1 and the other two are color 2. There are only two cases up to symmetry.

Case 1 **Suppose edges ax and bx are color 1, and edges cx and dx are color 2.** Consider the four edges ac, ad, bc and bd. If any one of these edges is color 1, we have a $K_{1,3}$ in color 1. If any one is a new color, then that edge, edge ef, and either ax or bx forms a rainbow $3K_2$. Thus, we may assume that all four are color 2; but then we have a copy of $K_{1,3}$ in color 2 centered at vertex c.

Case 2 **Suppose edges ax and cx are color 1 and edges bx and dx are color 2.** Consider the two edges ae and ce. If either edge is color 1, we have a $K_{1,3}$ in color 1 centered at a or c. If either is a new color, then along with edges bx and cd or edges dx and ab, this edge forms a rainbow $3K_2$. Therefore, we may assume that both ae and ce are color 2; but then, again, we have a copy of $K_{1,3}$ in color 2 centered at vertex e. □

When $m - n + 3$ is large, the following upper bound is often better.

Theorem 22. *For any positive integers n and m,*

$$RR(K_{1,n}, mK_2) \leq (n+1)(m-1) + 2.$$

If $(n+1)(m-1) \geq 2m+1$ (for instance, $n \geq 2$ and $m \geq 4$ or $n \geq 3$ and $m \geq 3$), then we may improve the bound above to

$$RR(K_{1,n}, mK_2) \leq (n+1)(m-1).$$

Proof. Notice that $RR(K_{1,1}, mK_2) = 2$ for any m, since any coloring of $K_{1,1}$ is monochromatic. We may assume that $n \geq 2$.

We will proceed by induction on m. If $m = 1$, then any copy of mK_2 is rainbow-colored, and $RR(K_{1,n}, mK_2) = 2$.

For any positive integers $n \geq 2$ and $m \geq 2$, assume that $RR(K_{1,n}, (m-1)K_2) \leq (n+1)(m-2) + 2 \leq (n+1)(m-1)$. Consider any edge-coloring of K_N, where $N = (n+1)(m-1)$ if $(n+1)(m-1) \geq 2m+1$ and $N = (n+1)(m-1) + 2$ otherwise. By the inductive hypothesis, we may assume that there is a rainbow copy of $(m-1)K_2$. Label the vertices of this matching $x_1, x_2, \ldots x_{m-1}$ and $y_1, y_2, \ldots, y_{m-1}$, so that edge x_iy_i is color i for $1 \leq i \leq m-1$.

First, suppose $(n+1)(m-1) < 2m+1$. Since $n \geq 2$, we still have $N = (n+1)(m-1)+2 > 2(m-1)$. Thus, there is some vertex w distinct from the vertices $\{x_1, y_1, x_2, y_2, \ldots, x_{m-1}, y_{m-1}\}$. Consider the $N - 2(m-1) - 1 = (n-1)(m-1)+1$ edges incident with w and not incident with $\{x_1, y_1, x_2, y_2, \ldots, x_{m-1}, y_{m-1}\}$. Either some color appears n times on these edges, producing a monochromatic $K_{1,n}$, or some new color appears. The edge in this new color, along with the edges $x_i y_i$ for $1 \leq i \leq m-1$, forms a rainbow mK_2.

Suppose $(n+1)(m-1) \geq 2m+1$. In this case, $K_{(n+1)(m-1)}$ contains some other edge uv independent from the edges $x_1y_1, x_2y_2, \ldots, x_{m-1}y_{m-1}$. If uv is colored with a new color, we are done. We may assume without loss of generality that uv is colored with color 1. Now, since $(n+1)(m-1) \geq 2m+1$, there is some other vertex, say w. Consider the edges wz, where z is distinct from the vertices $\{w, x_2, y_2, x_3, y_3, \ldots, x_{m-1}, y_{m-1}\}$. There are $(n+1)(m-1) - 2(m-2) - 1 = (n-1)(m-1)+1$ such edges. Either n of these edges are the same color, so we have a monochromatic copy of $K_{1,n}$ with central vertex w, or m different colors appear on these edges. Thus, one of these edges must be in a new color m distinct from $1, 2, \ldots, m-1$. This edge can be adjacent with at most one of uv or x_1y_1; assume wlog that it is not adjacent to x_1y_1. Then this edge along with the edges x_iy_i, $1 \leq i \leq m-1$, forms a rainbow mK_2. □

Suppose, instead, that we consider monochromatic matchings and rainbow stars. We have a lower bound on the order of $2nm$.

Theorem 23. *For any positive integers $n \geq 2$ and $m \geq 3$, the rainbow ramsey number*

$$RR(nK_2, K_{1,m}) \geq (2m-3)(n-1)+1.$$

Proof. Let $N = (2m-3)(n-1)$. Divide the vertices of K_N into $2m-3$ subsets of order $n-1$ each, say $S_1, S_2, \ldots S_{2m-3}$. Color every edge within S_i with color i. Color the edges from S_i to $S_{i+1}, S_{i+2}, \ldots S_{i+m-2}$ with color i for $1 \leq i \leq 2m-3$, where the indices are taken modulo $2m-3$. Since there are exactly $m-2$ sets $S_{i+m-1}, S_{i+m}, \ldots S_{i+2m-4}$ joined to S_i by colors other than i and every other set is joined to S_i with color i, any vertex in S_i is incident with at most $m-1$ colors. Thus, there is no rainbow $K_{1,m}$. Any edge in color i is incident with one of the $n-1$ vertices in S_i, so there are at most $n-1$ independent edges in any given color. Hence, there are no monochromatic subgraphs isomorphic to nK_2. □

For the upper bound, we first demonstrate the following recursive result.

Lemma 4. *For any positive integers n and m, where $m \geq 2$ and $n \geq 2$, the rainbow ramsey number $RR(nK_2, K_{1,m}) \leq m+2(m-1)(n-2)+RR(nK_2, K_{1,m-1})$.*

Proof. Let $N = m + 2(m-1)(n-2) + RR(nK_2, K_{1,m-1})$. Consider any coloring of the edges of K_N. There must be either a monochromatic copy of nK_2 or a rainbow copy of $K_{1,m-1}$; assume that there is a rainbow copy of $K_{1,m-1}$, with edges in colors $1, 2, \ldots, m-1$. If we remove the m vertices incident with this subgraph, there are $2(m-1)(n-1) + RR(nK_2, K_{1,m-1})$ vertices remaining. Thus, we may assume without loss of generality that there is another rainbow $K_{1,m-1}$.

If these two disjoint rainbow copies of $K_{1,m-1}$ have no colors in common, then consider the edge e between their central vertices in K_N. At least one of the copies of $K_{1,m-1}$ does not contain any edges in the same color as e, so this copy along with e forms a rainbow copy of $K_{1,m}$. Thus, we may assume without loss of generality that the two rainbow copies of $K_{1,m-1}$ share a color, say color 1.

Now, remove the m vertices incident with the first rainbow $K_{1,m-1}$ and the 2 vertices incident with the edge in color 1 in the other star. There are $2((m-1)(n-2)-1)+RR(nK_2, K_{1,m-1})$ vertices remaining, so we may assume that there is another rainbow copy of $K_{1,m-1}$. Without loss of generality, this copy contains an edge colored with one of the colors $1, 2, \ldots, m-1$. Remove the two vertices incident with this edge.

If we continue in this fashion, we have a rainbow copy of $K_{1,m-1}$ in colors $1, 2, \ldots, m-1$ and $(m-1)(n-2)$ disjoint independent edges colored with the same $m-1$ colors. There are $RR(nK_2, K_{1,m-1})$ vertices remaining, so we may assume without loss of generality that there is another rainbow copy of $K_{1,m-1}$. If this copy does not contain any of the colors $1, 2, \ldots, m-1$, then consider the edge e between its central vertex and the central vertex of the other rainbow $K_{1,m-1}$. In this case, at least one of the rainbow copies of $K_{1,m-1}$ does not contain any edge in the same color as edge e, so this $K_{1,m-1}$ and edge e form a rainbow $K_{1,m}$.

Otherwise, the new rainbow $K_{1,m-1}$ contains an edge in one of the colors $1, 2, \ldots, m-1$. We have $(m-1)(n-2)+1$ such edges independent of the rainbow $K_{1,m-1}$ in colors $1, 2, \ldots, m-1$. Some color must appear $n-1$ times, plus once in the rainbow $K_{1,m-1}$, to form a monochromatic nK_2. □

Since $RR(nK_2, K_{1,2}) = 2n$, the lemma above yields the upper bound

$$RR(nK_2, K_{1,m}) \leq 2n + (3+4+\ldots+m) + 2(2+3+\ldots+(m-1))(n-2).$$

If we simplify, we obtain the following upper bound, on the order of m^2n.

Theorem 24. *For any integers m and n, where $m \geq 2$ and $n \geq 2$, the rainbow ramsey number $RR(nK_2, K_{1,m}) \leq m(m-1)n - \frac{1}{2}(3m+1)(m-2)$.*

GENERALIZATIONS OF THE RAINBOW RAMSEY NUMBER

If we view the rainbow ramsey number in a more general context, several related numbers are naturally defined, including the edge-chromatic ramsey number and the $\mathcal{F}$-free ramsey number.

3.1 Edge-Chromatic Ramsey Number

The *edge-chromatic ramsey number* $CR(G_1, G_2)$ is the minimum integer N such that if the edges of K_N are colored with any number of colors, then the resulting graph contains either a subgraph isomorphic to G_1 with every edge the same color or a subgraph isomorphic to G_2 with no two adjacent edges the same color, that is, properly colored. It is immediate that $CR(G_1, G_2) \leq RR(G_1, G_2)$ for any graphs G_1 and G_2. The existence proof for the edge-chromatic ramsey number is essentially the same as the proof for the rainbow ramsey numbers, so we omit it here. The edge-chromatic ramsey number $CR(G_1, G_2)$ exists if and only if G_1 is a star or G_2 is acyclic.

Naturally, if G_2 is a star $K_{1,m}$ or a triangle C_3, then $CR(G_1, G_2) = RR(G_1, G_2)$. In order to compare these two numbers, we next consider bounds and formulas for both numbers for several classes of graphs.

3.2 Bounds for Cycles and Paths

It is not hard to show that $RR(C_n, P_2) = CR(C_n, P_2) = 2$ and $RR(C_n, P_3) = CR(C_n, P_3) = n$ for any $n \geq 3$. However, for longer paths, the two parameters differ significantly.

Theorem 25. *For any integer $m \geq 2$, $CR(C_3, P_m) = m$.*

Proof. The result is immediate for $m = 2$ and $m = 3$. We proceed by induction on m.

Suppose for some $m \geq 4$, we know that $CR(C_3, P_{m-1}) = m - 1$. Let the edges of K_m be colored arbitrarily. We will assume that K_m contains no monochromatic triangle C_3. Then necessarily K_m contains a properly colored subgraph isomorphic to P_{m-1}. Suppose this path has vertices $v_1, v_2, \ldots v_{m-1}$, where the edge v_iv_{i+1} is color c_i, for $1 \leq i \leq m-2$. Then $c_i \neq c_{i+1}$ for $1 \leq i \leq m-3$, but otherwise the colors need not be distinct. Let x be the vertex of K_m not on this path. If xv_1 is not color c_1, then $x, v_1, v_2, \ldots v_{m-1}$ is a properly colored path on m vertices. We may assume that xv_1 is color c_1.

Suppose xv_i is color c_i for some i. If xv_{i+1} is also color c_i, then x, v_i, and v_{i+1} form a monochromatic triangle. If xv_{i+1} is some other color besides c_{i+1}, then $v_1, v_2, \ldots v_i, x, v_{i+1}, \ldots v_{m-2}, v_{m-1}$ is a properly colored path of length m. Thus, we may assume that xv_i is color c_i for each i, $1 \leq i \leq m-2$, inductively.

Thus, xv_{m-2} and $v_{m-2}v_{m-1}$ are both color c_{m-2}. If xv_{m-1} is color c_{m-2} then we have a monochromatic triangle. If not, then $v_1, v_2, \ldots v_{m-2}, v_{m-1}, x$ is a properly colored path on m vertices.

We have shown that $CR(C_3, P_m) \leq m$. The graph K_{m-1} may be colored with every edge a different color so that it contains neither a monochromatic C_3 nor a properly colored P_m. Thus, $CR(C_3, P_m) = m$. □

However, the rainbow ramsey number grows at least exponentially for any odd cycles, including C_3.

Theorem 26. *For any integer $m \geq 2$ and any odd integer $n \geq 3$, $RR(C_n, P_m) \geq$*

$2^{m-2} + 1$.

Proof. We will define a coloring on a complete graph with 2^i vertices inductively. If $i = 1$, color K_2 with color 1.

Once the coloring on the complete graph with 2^{i-1} vertices is defined, take two identical copies of this graph, with the same colors, and color every edge between the two copies with a new color. Thus, the graph induced by the edges in this new color is a complete bipartite graph.

Since the graph induced by any particular color is bipartite, there are no monochromatic odd cycles. And since exactly i colors are used in the graph on 2^i vertices, the graph on 2^{m-2} vertices cannot contain any rainbow subgraph isomorphic to P_m. □

The existence theorem, Theorem 9, yields a rough upper bound on $RR(C_3, P_m)$. From case 2, with $n = 3$, we have

$$RR(C_3, K_{1,m-1}) \leq \frac{(m-2)^m - 1}{m-3}.$$

Using this number for N in case 3, we have

$$RR(C_3, P_m) \leq RR(C_3, P_{m-1}) \left(\frac{(m-2)^m - 1}{m-3}\right),$$

or, as observed in the discussion following the proof,

$$\begin{aligned} RR(C_3, P_m) &\leq RR(C_3, P_{m-1}) + (m-1)\left(\frac{(m-2)^m - 1}{m-3} - 1\right) \\ &= RR(C_3, P_{m-1}) + (m-1)\left(\frac{(m-2)^m - (m-2)}{m-3}\right) \\ &\leq RR(C_3, P_{m-1}) + 2(m-1)((m-2)^{m-1} - 1) \end{aligned}$$

Since $RR(C_3, P_3) = 3$, we have

$$\begin{aligned}
RR(C_3, P_m) &\leq 3 + 6(2^3 - 1) + 8(3^4 - 1) + \ldots + 2(m-1)((m-2)^{m-1} - 1) \\
&= 3 + \sum_{k=4}^{m} \left[2(k-1)((k-2)^{k-1} - 1)\right] \\
&\leq 3 + 2(m-3)(m-1)\left[(m-2)^{m-1} - 1\right]
\end{aligned}$$

A simple induction argument yields an upper bound which is only slightly better. Trivially, we have $RR(C_3, P_2) = 2$ and $RR(C_3, P_3) = 3$. For any $m \geq 3$, we claim that

$$RR(C_3, P_m) \leq m - 1 + (m-2)(RR(C_3, P_{m-1}) - 1) + 1. \tag{10}$$

Let $N = m-1+(m-2)(RR(C_3, P_{m-1})-1)+1$. By induction, we may assume that K_N contains a rainbow copy of P_{m-1}, using $m-2$ colors. Let v be an endvertex of this path. Then v has $(m-2)(RR(C_3, P_{m-1}) - 1) + 1$ neighbors not on the path. Thus, either v is incident with an edge in some new color, so that the path can be extended to a rainbow P_m, or v is incident with $RR(C_3, P_{m-1})$ edges all in the same color, say color c. Let M be the set of endpoints of these edges, excluding v. Without loss of generality, we may assume that there is a rainbow P_{m-1} on the subgraph induced by M. If color c appears on this path, then the endpoints of the edge in color c and the vertex v induce a monochromatic C_3. Otherwise, vertex v may be added to the end of this path to produce a rainbow P_m. If we solve the induction in equation 10, we have

$$\begin{aligned}
RR(C_3, P_m) &\leq m + \sum_{i=2}^{m-2} \frac{i(m-2)!}{(i-1)!} \\
&= m + \sum_{i=2}^{m-2} i(m-2)^{(m-i-1)} \\
&\leq m + 2(m-3)(m-2)!
\end{aligned}$$

Thus, $RR(C_3, P_m)$ is bounded asymptotically between 2^{m-2} and approximately $2(m-3)(m-2)!$.

We can extend the idea behind Theorem 25 to obtain upper and lower bounds on the chromatic ramsey number of a 4-cycle versus a path.

Theorem 27. *For any integer $m \geq 3$, $m+1 \leq CR(C_4, P_m) \leq 2m-2$.*

Proof. First we will color K_m so that it contains no monochromatic C_4 and no properly colored P_m. Color a triangle on vertices v_{-1}, v_0, and v_1 with color 1. For each color i, $2 \leq i \leq m-2$, add a new vertex v_i and color every edge v_iv_j for $-1 \leq j < i$ with color i. In the resulting K_m, every monochromatic subgraph is a star or a triangle, and not C_4. On any properly colored path, at most one of the vertices adjacent to v_i can have an index less than i, for $2 \leq i \leq m-2$. Thus, at most two of the vertices v_{-1}, v_0, v_1 can appear on the path.

Next, we must show that an arbitrary coloring of the edges of K_{2m-2} results in either a monochromatic C_4 or a properly colored P_m. We will proceed by induction on m. Since any P_2 is properly colored, $CR(C_4, P_2) = 2$.

For any $m \geq 3$, suppose $CR(C_4, P_{m-1}) \leq 2m-4$. Color the edges of K_{2m-2} arbitrarily. We may assume that the resulting graph contains a properly colored path P_{m-1}, say on vertices $v_1, v_2, \ldots v_{m-1}$, where edge v_iv_{i+1} is color c_i for $1 \leq i \leq m-2$. Let $M = V(K_{2m-2}) - (P_{m-1})$, so $|M| = m-1$.

Suppose there is no properly colored path of length m. We claim that for each i, at least $m-i$ of the vertices of M are joined to v_i by an edge of color c_i. If for some $x \in M$, v_1x is not color c_1, then $x, v_1, v_2, \ldots v_{m-1}$ is a properly colored path, so we may assume $m-1$ vertices are joined to v_1 with edges of color c_1.

Assume that at least $m-i$ of the vertices of M are joined to v_i by an edge of color c_i. If any one of these same vertices, say x, is joined to v_{i+1} by an edge of some color other than c_i or c_{i+1}, then $v_1, v_2, \ldots v_i, x, v_{i+1}, v_{i+2}, \ldots v_{m-1}$ is a properly colored path on m vertices. However, if more than one of these vertices, say x and y, are joined to v_{i+1} by an edge of color c_i, then v_i, x, v_{i+1}, and y form a monochromatic C_4. Thus, we may assume that at least $m-(i+1)$ of these edges are color c_{i+1}.

Thus, v_{m-2} is joined to at least 2 vertices, say x and y, in M by edges of color c_{m-2}. If any edge from v_{m-1} to M is not color c_{m-2}, then we have a properly colored path of length m. However, if $v_{m-1}x$ and $v_{m-1}y$ are both color $m-2$, then we have a monochromatic C_4. Therefore, $CR(C_4, P_m) \leq 2m-2$. □

3.3 Bounds for Stars and Paths

We can quickly obtain upper and lower bounds for the edge-chromatic and ramsey numbers of a monochromatic star and a rainbow path. These bounds suggest that $CR(K_{1,n}, P_m)$ grows roughly like the sum $n+m$, while $RR(K_{1,n}, P_m)$ grows like the product nm.

First, we establish the upper and lower bounds for $CR(K_{1,n}, P_m)$. Since $CR(K_{1,1}, P_m) = CR(K_{1,n}, P_2) = 2$, $CR(K_{1,n}, P_3) = n+1$ for $n \geq 2$, and $CR(K_{1,2}, P_m) = m$ for $m \geq 3$, we assume that $n \geq 3$ and $m \geq 4$. The upper bound requires a lemma.

Lemma 5. *For any integer $n \geq 3$, $CR(K_{1,n}, P_4) = n+1$.*

Proof. The lower bound results from coloring the edges of K_n with a single color. Suppose the edges of K_{n+1} are colored so that there is no monochromatic $K_{1,n}$. Thus, there must be two adjacent edges uv and vw which are different colors, say colors 1 and 2, respectively. There is at least one other vertex x in the graph. For any x not contained in $\{u, v, w\}$, if ux is not color 1 or uw is not color 2, then

x, u, v, w or u, v, w, x is a properly colored path of length 4. Assume that ux is color 1 and wx is color 2 for any such vertex x. Consider edge uw. If this edge is color 1, then u is the central vertex of a monochromatic $K_{1,n}$. If it is color 2, then there is a $K_{1,n}$ in color 2 with center $\{w\}$. We may assume that uw is some new color, but then v, u, w, x is a properly colored P_4. □

The general upper bound results from applying the same approach inductively.

Theorem 28. *For any integers $n \geq 3$ and $m \geq 4$, the edge-chromatic ramsey number $CR(K_{1,n}, P_m) \leq m + n - 3$.*

Proof. We will proceed by induction on m. The base step is handled in Lemma 5. Suppose the edges of K_{m+n-3} are colored so that there is no monochromatic $K_{1,n}$. By the inductive hypothesis, we may assume that there is a properly colored path on $m-1$ vertices, say $v_1, v_2, \ldots v_{m-1}$. Suppose edge v_1v_2 is color 1 and edge $v_{m-2}v_{m-1}$ is color 2, where colors 1 and 2 are not necessarily distinct. For any vertex x of the $n-2 \geq 1$ vertices not on this path, we may assume that v_1x is color 1 and $v_{m-1}x$ is color 2, or we have a properly colored path of length m. Consider edge v_1v_{m-1}. If this edge is color 1, then v_1 is the central vertex of a $K_{1,n}$ in color 1. Similarly, if it is color 2, there is a $K_{1,n}$ in color 2. Suppose it is some other color 3, distinct from colors 1 and 2. Let x be some vertex not on the path. Then $x, v_1, v_{m-1}, v_{m-2}, \ldots v_2$ is a properly colored path on m vertices. □

For the corresponding lower bound, we will require a special coloring. Let $v_1, v_2, \ldots v_k$ be k vertices in a complete graph K_N where $N \geq k$. A *kth order pathtrap coloring* on the edges incident with these vertices is a coloring such that every edge v_ix, where x is not in $\{v_1, v_2, \ldots v_k\}$, is color i and every edge v_iv_j is

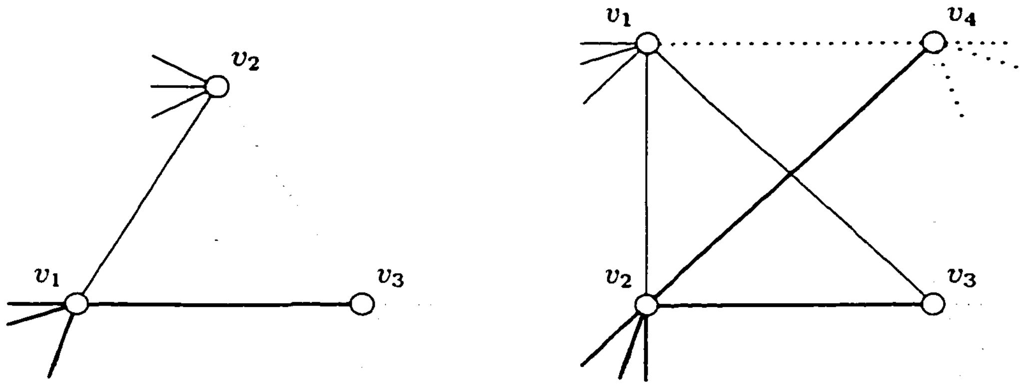

Figure 8. Examples of Pathtrap Colorings With 3 and 4 Vertices.

either color i or color j. Figure 8 shows one possible 3rd order pathtrap coloring and a possible 4th order pathtrap coloring. The set of vertices $\{v_1, v_2, \ldots v_k\}$ will be referred to as a *pathtrap*. Notice that for each k, a kth order pathtrap coloring exists in which for each i, $1 \leq i \leq k$, at most $\lceil (k-1)/2 \rceil$ of the edges $v_i v_j$ are color i. In other words, there is a kth order pathtrap coloring in which each color appears at most $\lceil (k-1)/2 \rceil$ times within the pathtrap.

Suppose a graph K_N is colored with a kth order pathtrap coloring with pathtrap $\{v_1, v_2, \ldots v_k\}$, and suppose P is a properly colored path in K_N. If P enters the pathtrap, then P cannot leave it, in the following sense. Suppose $x \in P$ where x is not in the pathtrap and x is followed on the path by v_i. Then xv_i is color i, and every edge $v_i y$, where y is not in the pathtrap, is also color i. Thus, the next vertex on the path must be some v_j, where v_j is in the pathtrap and $v_i v_j$ is color j. Every edge incident with v_j which is not color j again lies in the pathtrap. Continuing in this fashion, we see that every vertex on P after x must be a vertex of the pathtrap.

Theorem 29. *For any integers $n \geq 3$ and $m \geq 6$, the edge-chromatic ramsey number*

$$CR(K_{1,n}, P_m) \geq n + \left\lceil \frac{m+1}{2} \right\rceil - 2.$$

Proof. Let $N = n + \lceil \frac{m+1}{2} \rceil - 3$. We may assume that $N \geq m - 1$, since a rainbow colored K_{m-1} contains neither a monochromatic $K_{1,n}$ nor a rainbow P_m.

Color the edges of K_N as follows. Form an $(m-3)$rd order pathtrap coloring on vertices $v_1, v_2, \ldots v_{m-3}$ so that each color i appears at most $\lceil (m-4)/2 \rceil$ times within the pathtrap. Color the remaining edges with a new color, color $m - 2$. For each color i in the pathtrap, there are at most

$$\left\lceil \frac{m-4}{2} \right\rceil + n + \left\lceil \frac{m+1}{2} \right\rceil - 3 - (m-3) = n - 1$$

edges incident with vertex v_i in color i. Every other color appears at most once at vertex v_i. Color $m - 2$ appears incident with at most

$$n + \left\lceil \frac{m+1}{2} \right\rceil - 3 - (m-3) < n$$

vertices, and each other color appears once at each vertex outside the pathtrap. Thus, there are no monochromatic copies of $K_{1,n}$.

Since a properly colored path cannot enter and then leave the pathtrap, any vertices on the path and not in the pathtrap must appear consecutively on the path. Therefore, any such path can contain at most two vertices not in the pathtrap; thus, there is no properly colored path on more than $(m-3)+2 = m-1$ vertices. □

The upper bound for the rainbow ramsey number results from a more general upper bound for stars and trees.

Theorem 30. *For any integers $n \geq 2$ and $m \geq 3$ and any tree T of order m, the rainbow ramsey number*

$$RR(K_{1,n}, T) \leq m - 1 + (m-2)(n-1).$$

Proof. We proceed by induction on m. The only tree of order $m = 3$ can be thought of as $K_{1,2}$. If no rainbow $K_{1,2}$ appears in K_N, then the entire graph must be monochromatic. Thus, $RR(K_{1,n}, K_{1,2}) = n + 1 = 3 - 1 + (3-2)(n-1)$.

Suppose $RR(K_{1,n}, T') \leq m - 2 + (m-3)(n-1)$ for any tree T' of order $m - 1$. Let T be a tree of order m with endvertex v adjacent to a vertex u, and let $K_{m-1+(m-2)(n-1)}$ be edge-colored with any number of colors. We may assume that $K_{m-1+(m-2)(n-1)}$ contains either a monochromatic copy of $K_{1,n}$ or a rainbow copy of $T - v$. Suppose it contains a rainbow copy of $T - v$. If we remove these $m - 1$ vertices, there are $(m-2)(n-1)$ vertices remaining. Consider the edges between u and these vertices. If $m - 1$ or more colors appear on these edges, then some edge is in a color not yet appearing in $T - v$. We may add this edge to $T - v$ to obtain a rainbow copy of T. Thus, we may assume that at most $m - 2$ colors appear and that these are the same colors which appear in $T - v$. If any color appears more than $n - 1$ times, then there is a monochromatic copy of $K_{1,n}$; so we may assume that each color appears exactly $n - 1$ times. But now consider a color which appears on some edge incident with u other than uv. This color appears incident with u at least n times, so we have a monochromatic copy of $K_{1,n}$. □

Notice that if T contains an endvertex v adjacent to a vertex u with $deg_T u = k$, then the above bound can be improved to

$$RR(K_{1,n}, T) \leq m - 1 + (m-2)(n-1) - (k-2).$$

As we have seen previously, a complete graph $K_{(m-2)(n-1)+1}$ can be factored into 1-factors when $(m-2)(n-1)+1$ is even. Then $n-1$ 1-factors may be colored with each color to produce a graph with no monochromatic $K_{1,n}$ and with too few colors to contain any rainbow tree of order m. Similarly, if $(m-2)(n-1)+1$ is odd and $n-1$ is even, then $K_{(m-2)(n-1)+1}$ can be factored into hamiltonian cycles and $(n-1)/2$ of these cycles can be colored with each color. If n and m are both even, then $K_{(m-2)(n-1)}$ can be factored into 1-factors, and $n-1$ or $n-2$ of these 1-factors colored with each color so that only $m-2$ colors are used.

Combining these observations with Theorem 30, we have the following theorem and corollary.

Theorem 31. *For any integers $n \geq 2$ and $m \geq 3$ and any tree T of order m, the rainbow ramsey number*

$$(m-2)(n-1)+1 \leq RR(K_{1,n}, T) \leq (m-2)(n-1)+(m-1),$$

where the lower bound can be improved to $(m-2)(n-1)+2$ if n and m are not both even.

Corollary 3. *For any $n \geq 2$ and $m \geq 3$,*

$$(m-2)(n-1)+1 \leq RR(K_{1,n}, P_m) \leq (m-2)(n-1)+(m-1).$$

3.4 Bounds for Paths and Paths

We will next obtain upper bounds on the edge-chromatic and rainbow ramsey numbers of paths. First we will need a couple of lemmas for the edge-chromatic ramsey numbers.

Lemma 6. *For any integer $n \geq 3$, $CR(P_n, P_4) = n+1$.*

Proof. Suppose the edges of K_{n+1} are colored with any number of colors. If every edge is the same color, then there is certainly a monochromatic subgraph isomorphic to P_n. We may assume that there are vertices u, v and w such that uv is color 1 and vw is color 2. Let $M = V(K_{n+1}) - \{u, v, w\}$. Every edge from u to a vertex in M is color 1 or we have a properly colored P_4. Similarly, every edge from w to M must be color 2. If any edge in M is a new color, say color 3, then there is a properly colored P_4 using this edge and vertices u and w. If there are two adjacent edges in M such that one is color 1 and the other is color 2, then we may attach vertex u to the appropriate end of this path to obtain a properly colored path of order 4. Thus, we may assume that all of the edges within M are a single color, either color 1 or color 2, say color 1. We may take a path of order $n-2$ in color 1 in M and add the vertices u and v to form a monochromatic path of order n.

For the lower bound when $n \geq 4$, color K_n as follows. Fix a vertex x. Color every edge not incident with x with color 1, and color the edges incident with x with color 2. No properly colored path in this graph can contain more than one edge in color 2, since all of the edges in color 2 are adjacent; necessarily, any such edge must appear at the beginning or end of the path. Thus, this graph contains no properly colored P_4. Any path of order n must contain the vertex x and, since $n \geq 4$, at least three other vertices. But then the path must contain at least one edge incident with x and at least one edge not incident with x, so it cannot be monochromatic. If $n = 3$, then color K_3 with rainbow colors to avoid a monochromatic P_3 and a proper P_4. □

Lemma 7. *For any integers $n \geq 5$ and $m \geq 5$, the edge-chromatic ramsey numbers* $CR(P_n, P_m) \leq CR(P_n, P_{m-1}) + (m-1)(\lceil \frac{n}{4} \rceil - 1)$.

Proof. Let $N = CR(P_n, P_{m-1}) + (m-1)(\lceil \frac{n}{4} \rceil - 1)$. Assume that the edges of K_N are colored so that there is no monochromatic P_n. We may assume that there is a properly colored subgraph isomorphic to P_{m-1}. Remove these $m-1$ vertices. Since $CR(P_n, P_{m-1}) + (m-1)(\lceil \frac{n}{4} \rceil - 2)$ vertices remain, we may assume that there is another properly colored P_{m-1}. Continuing in this fashion, we may assume that K_N contains $\lceil \frac{n}{4} \rceil$ disjoint properly colored paths of order $m-1$.

Consider any two of these paths P and Q. Suppose the vertices of path P are $v_1, v_2, \ldots v_{m-1}$ and the vertices of path Q are $u_1, u_2, \ldots u_{m-1}$. If any edge v_1u_i is a color other than the color of v_1v_2, then we can extend the path P to a properly colored path of length m. Similarly, every edge $v_{m-1}u_i$ must be the same color as $v_{m-2}v_{m-1}$, every edge v_iu_1 must be the same color as the edge u_1u_2, and every edge v_iu_{m-1} must be the same color as the edge $u_{m-2}u_{m-1}$. Thus, these colors must all be the same, so $u_1, u_2, v_1, v_2, u_{m-1}, v_{m-2}, v_{m-1}, u_{m-2}$ is a monochromatic path of order 8.

Suppose we have a monochromatic path of order $4i$ beginning at u_1 and ending at u_{m-2} for some edge-chromatically colored path $u_1, u_2, \ldots u_{m-1}$, where edges u_1u_2 and $u_{m-2}u_{m-1}$ are the same color. Let R be some other edge-chromatically colored path with vertices $w_1, w_2, w_3, \ldots w_{m-1}$. Then again we may assume that edges u_1u_2, u_1w_1, u_1w_2, and w_1w_2 are all the same color, so that w_1, w_2, u_1 may be attached to the beginning of the monochromatic path. Similarly, $w_{m-1}w_{m-2}$ and $w_{m-1}u_{m-2}$ must be the same color as $u_{m-2}u_{m-1}$, so $u_{m-2}, w_{m-1}, w_{m-2}$ may be attached to the end of the monochromatic path. Proceeding by induction, we have a monochromatic path on $4\lceil \frac{n}{4} \rceil$ vertices. □

By combining these two lemmas, we have the upper bound

$$\begin{aligned} CR(P_n, P_m) &\leq (n+1) + 4\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) + 5\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) + \ldots + (m-1)\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) \\ &= n + 1 + (4 + 5 + \ldots + (m-1))\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) \\ &= n + 1 + \left(\frac{m(m-1)}{2} - 6\right)\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) \\ &= n + 1 + \frac{(m-4)(m+3)}{2}\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) \end{aligned}$$

which is summarized in the next theorem.

Theorem 32. *For any integers $n \geq 5$ and $m \geq 4$, the edge chromatic ramsey number*

$$CR(P_n, P_m) \leq \frac{(m+3)(m-4)}{2}\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) + n + 1.$$

Consider a $(m-3)$rd order pathtrap coloring on K_{m+n-4}, with the remaining edges colored with one new color. Any properly colored path contains at most 2 vertices outside the pathtrap, for a total of at most $m-1$ vertices. Each color which appears within the pathtrap induces a star, and the color outside the pathtrap induces a complete graph on $n-1$ vertices, so there is no monochromatic subgraph isomorphic to P_n for $n \geq 4$. Thus, we have the following lower bound.

Theorem 33. *For any integers $n \geq 4$ and $m \geq 6$, $CR(P_n, P_m) \geq m + n - 4$.*

For the rainbow ramsey number, start with a edge-chromatic or proper coloring of K_{m-2} using at most $m-2$ colors, labelled $1, 2, \ldots m-2$. Replace each vertex with a set of $\lfloor \frac{n-1}{2} \rfloor$ vertices, so that all of the edges between two sets are colored with the same color as the edge between the original two vertices. Color the edge between any pair of vertices in the same set with color 1. Since only $m-2$ colors appear, this graph cannot contain any rainbow subgraph isomorphic to P_m. Any monochromatic path can contain vertices from at most two sets, for a total of $2\left\lfloor\frac{n-1}{2}\right\rfloor \leq n-1$ vertices. Thus, we have a lower bound.

Theorem 34. *For any integers $m \geq 3$ and $n \geq 3$, the rainbow ramsey number $RR(P_n, P_m) \geq (m-2)\left\lfloor\frac{n-1}{2}\right\rfloor + 1$.*

3.5 The $\mathcal{F}$-free Ramsey Number

Let $\mathcal{F}$ be a family of graphs. Define an *$\mathcal{F}$-free edge coloring* of a graph G to be an edge coloring so that G does not contain any monochromatic subgraph isomorphic to any graph in $\mathcal{F}$. Thus, if $\mathcal{F} = \{2K_2, K_{1,2}\}$, then an $\mathcal{F}$-free coloring is a rainbow coloring. Similarly, if $\mathcal{F} = \{K_{1,2}\}$, then an $\mathcal{F}$-free coloring is an edge-chromatic coloring.

For a nonempty set $\mathcal{F}$ of graphs, where each graph has size at least 2, we define the *$\mathcal{F}$-free ramsey number* $R_{\mathcal{F}}(G_1, G_2)$ of two graphs G_1 and G_2 to be the minimum integer N such that any coloring of the edges of K_N, with any number of colors, must contain either a monochromatic subgraph isomorphic to G_1 or an $\mathcal{F}$-free subgraph isomorphic to G_2. The Erdös-Rado Theorem is a useful tool for determining the existence of these numbers.

Theorem 35. *Assume that $\mathcal{F}$ is a nonempty set of graphs, where each graph has size at least 2. If $\mathcal{F}$ does not contain any stars, then $R_{\mathcal{F}}(G_1, G_2)$ exists for any graphs G_1 and G_2. Otherwise, let $K_{1,n}$ be the smallest star contained in $\mathcal{F}$. Then $R_{\mathcal{F}}(G_1, G_2)$ exists if and only if G_1 is a star or G_2 does not contain any induced subgraph with minimum degree at least n.*

Proof. According to the Erdös-Rado Theorem, for any integer k, there is an integer N such that any edge-coloring of K_N contains a canonically colored K_k. Recall that for a finite graph, there are only three canonical colorings: monochromatic, rainbow, and the minimum coloring, where the color of edge ij is $min(i, j)$ for each i and j. A sufficiently large monochromatic complete graph would certainly contain a monochromatic subgraph isomorphic to G_1, and a large rainbow complete graph would contain an $\mathcal{F}$-free subgraph isomorphic to G_2. If $\mathcal{F}$ does not contain any stars, then a minimum coloring would also be $\mathcal{F}$-free. Thus, a large complete graph with a minimum coloring would contain an $\mathcal{F}$-free copy of G_2.

Suppose $\mathcal{F}$ does contain a star, and let $K_{1,n}$ be the smallest such star. If G_1 is a star, then a sufficiently large minimum coloring would contain a monochromatic G_1. Suppose every induced subgraph of G_2 has minimum degree strictly less than n. Then we claim that a complete graph of order $|V(G_2)|$ with the minimum coloring has an $\mathcal{F}$-free subgraph isomorphic to G_2. Let v_1 be a vertex of G_2 with degree strictly less than n. Then let v_2 be a vertex of $G_2 - v_1$ so that its degree in the graph induced by $V(G_2) - v_1$ is less than n. Continuing in this fashion, we can label the vertices of G_2 so that in a minimum coloring, no color appears more than $n - 1$ times at any vertex. Since $\mathcal{F}$ contains no stars smaller than $K_{1,n}$, this is an $\mathcal{F}$-free coloring of G_2.

Now, suppose that G_1 is not a star, so G_1 is not a monochromatic subgraph of any complete graph with the minimum coloring, and suppose that G_2 has an induced subgraph H with minimum degree at least n. If G_2 is a subgraph of some complete graph with the minimum coloring, then let v be the vertex in H with the minimum index. Now, $\deg_H(v) \geq n$, and every edge from v to any other vertex in H is the same color, so G_2 must contain a monochromatic subgraph isomorphic to $K_{1,n}$. Thus, we may color any complete graph with the minimum coloring to avoid both a monochromatic G_1 and an $\mathcal{F}$-free G_2. □

Notice that Theorem 35 generalizes Theorem 9, the existence theorem for the rainbow ramsey numbers.

The following observations are immediate, but useful. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then any $\mathcal{F}_2$-free coloring is necessarily $\mathcal{F}_1$-free.

Observation 1. *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then for any graphs G_1 and G_2 for which both numbers are defined,*

$$R_{\mathcal{F}_1}(G_1, G_2) \leq R_{\mathcal{F}_2}(G_1, G_2)$$

Thus, for example, the rainbow ramsey number is always an upper bound on the edge-chromatic ramsey number. In fact, since a rainbow coloring is $\mathcal{F}$-free for any set $\mathcal{F}$ of graphs such that each graph has size at least two, $R_{\mathcal{F}}(G_1, G_2) \leq RR(G_1, G_2)$.

The second observation follows from the fact that any coloring of G_2 (or of $K_{|V(G_2)|}$) either contains a monochromatic G_1 or it does not.

Observation 2. *If $\mathcal{F} = \{G_1\}$, where G_1 is a graph of size at least 2, then for any graph G_2,*

$$R_{\mathcal{F}}(G_1, G_2) = |V(G_2)|$$

Finally, notice that if no graph in $\mathcal{F}$ is a subgraph of G_2, then *any* coloring of G_2 is $\mathcal{F}$-free.

Observation 3. *If no graph in the set $\mathcal{F}$ is contained in the graph G_2, and G_1 is a graph with size at least 2, then*

$$R_{\mathcal{F}}(G_1, G_2) = |V(G_2)| \qquad .$$

We will concentrate on the cases $\mathcal{F} = \{K_{1,2}, 2K_2\}$, $\mathcal{F} = \{K_{1,2}\}$, and $\mathcal{F} = \{2K_2\}$. In the first two cases, the $\mathcal{F}$-free ramsey number is precisely the rainbow ramsey number and the edge-chromatic ramsey number, respectively. As noted in Observation 1,

$$CR(G_1, G_2) \leq RR(G_1, G_2)$$

$$R_{2K_2}(G_1, G_2) \leq RR(G_1, G_2)$$

for any graphs G_1 and G_2 for which these numbers are defined. The other two numbers cannot be placed in a consistent linear order, however. For example,

$$4 = CR(K_{1,2}, P_4) < R_{2K_2}(K_{1,2}, P_4) = 5$$

but

$$4 = R_{2K_2}(2K_2, P_4) < CR(2K_2, P_4) = 5.$$

See Figure 9 for colorings of K_4 containing no monochromatic $K_{1,2}$ and no $\{2K_2\}$-free P_4 or no monochromatic $2K_2$ and no $\{K_{1,2}\}$-free P_4, respectively. However, a brief argument shows that

$$CR(P_4, P_4) = 5 = R_{2K_2}(P_4, P_4).$$

Thus, any ordering of these two ramsey numbers is possible.

3.6 The $2K_2$-free Ramsey Number

The $2K_2$-free ramsey number $R_{2K_2}(G_1, G_2)$ of two graphs G_1 and G_2 is the smallest integer N such that any edge coloring of K_N contains either a monochromatic copy of G_1 or a copy of G_2 in which no two nonadjacent edges are the same color. Thus, each color in G_2 must induce either a star or a triangle. This

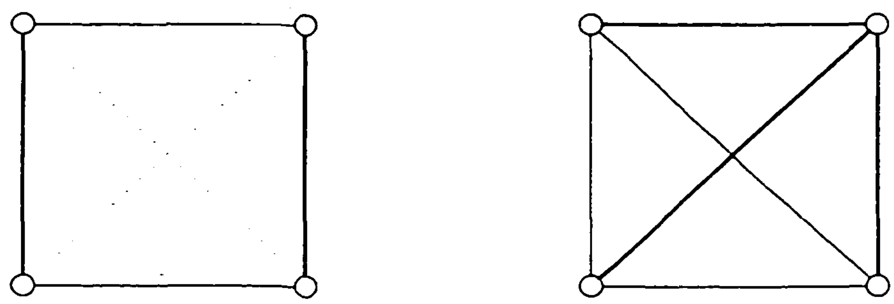

Figure 9. Colorings of K_4 Showing $R_{2K_2}(K_{1,2}, P_4) \geq 5$ and $CR(2K_2, P_4) \geq 5$.

particular $\mathcal{F}$-free ramsey number exists for any graphs G_1 and G_2. According to Theorem 35, $R_{2K_2}(K_n, K_m)$ is defined. The next theorem gives upper and lower bounds on this number.

Theorem 36. *For any positive integers $n \geq 3$ and $m \geq 3$,*

$$(n-1)^{m-3} + 1 \leq R_{2K_2}(K_n, K_m) \leq$$
$$m + \sum_{i=1}^{(n-2)(m-2)-1} \left[\frac{(m-1)^{i+1}(m-2)^i(m-3)^i}{2^i}\right] + \left[\frac{(m-1)(m-2)(m-3)}{2}\right]^{(n-2)(m-2)}$$

Proof. For the lower bound, we require a coloring of the edges of K_N, where $N = (n-1)^{m-3}$, with neither a monochromatic K_n nor a $\{2K_2\}$-free K_m. There are two ways to view the relevant coloring. We may start with a copy of K_{n-1} in color 1 and proceed inductively. For each color, take $n-1$ copies of the previous graph and color the edges between these copies with the new color. Continue until $m-3$ colors are used. Alternately, we may label the edges of K_N with the $(n-1)^{m-3}$ different $(m-3)$-tuples of the numbers $\{1, 2, \ldots, n-1\}$. Color an edge between two vertices with the index of the first entry in which their $(m-3)$-tuples differ. Either of these descriptions yields the same coloring.

Consider any n vertices $v_1, v_2, \ldots v_n$. Suppose each edge incident with v_1 is the same color, say i. Then the ith entry in the tuple for v_1 differs from the ith entry in each of the tuples for the other vertices. Thus, there are only $n-2$ choices for the ith entry in these $n-1$ tuples. The tuples for some pair of vertices, say v_2 and v_3, must have the same ith entry. Thus, edge v_2v_3 is not color i; there cannot be any monochromatic copy of K_n.

Next, consider any set of m vertices. Since at most $m-3$ colors are used to color their edges, we may apply Corollary 1 to Cockayne and Lorimer's theorem with $c = m-3$ and $n = 2$. We have

$$r(2K_2, 2K_2, \ldots 2K_2) = m$$

Thus, the subgraph induced by any m vertices must contain a copy of $2K_2$ in some color, so there is no $\{2K_2\}$-free copy of K_m.

For the upper bound, recall that a minimum coloring is a coloring in which edge ij is color $min(i,j)$ for each pair of vertices i and j. For our present purposes, we will allow colors to be repeated, so that for instance, color 1 might be the same as color 3. We claim that any complete graph on $(n-2)(m-2)+2$ vertices with the minimum coloring must contain either a monochromatic nK_2 or a $\{2K_2\}$-free K_m. Label the vertices of such a graph $1, 2, \ldots (n-2)(m-2)+2$. The graph is colored with colors $1, 2, \ldots (n-2)(m-2)+1$, which are not necessarily distinct. If any $n-1$ of these colors are identical, say color c_1 = color c_2 = ... = color c_{n-1}, then the vertices $c_1, c_2, \ldots c_{n-1}$ and $(n-2)(m-2)+2$ form a monochromatic subgraph K_n. Otherwise, there must be at least $m-1$ different colors. Suppose color c_1, color c_2, ..., color c_{m-2}, and color c_{m-1} are all distinct. In this case, the vertices $c_1, c_2, \ldots c_{m-1}$ and $(n-2)(m-2)+2$ induce a $\{2K_2\}$-free copy of K_m.

Now, let $a(n, m, N)$ be the smallest integer M such that any edge-coloring of K_M contains a minimum-colored copy of K_N, a monochromatic copy of K_n, or a $2K_2$-free copy of K_m. From the discussion above, we know that

$$R_{2K_2}(K_n, K_m) \leq a(n, m, (n-2)(m-2)+2).$$

Since any graph on 1 or 2 vertices is minimum-colored, $a(n, m, 1) = 1$ and $a(n, m, 2) \leq 2$. Any graph on 3 or fewer vertices is necessarily $\{2K_2\}$-free, so $a(n, 2, N) \leq 2$ and $a(n, 3, N) \leq 3$. Next, we claim that $a(n, m, N)$ is bounded above by

$$max[a(n, m-1, N), m + \frac{1}{2}(m-1)(m-2)(m-3)(a(n, m, N-1)-1)] \quad (11)$$

for $n \geq 3$, $m \geq 4$ and $N \geq 3$.

Let $L = max[a(n, m-1, N), m + \frac{1}{2}(m-1)(m-2)(m-3)(a(n, m, N-1)-1)]$. Color the edges of K_L with any number of colors. Since $L \geq a(n, m-1, N)$, we may assume without loss of generality that there is a $\{2K_2\}$-free copy of K_{m-1} in K_L. Label the vertices of K_{m-1} by $v_1, v_2, \ldots v_{m-1}$. Let H be the set of vertices $V(K_L) - V(K_{m-1})$. Define $m-1$ subsets $H_1, H_2, \ldots H_{m-1}$ of H by

$$H_i = \{u \in H | uv_i \text{ is the same color as } v_j v_k \text{ for some } j, k \neq i, 1 \leq j < k \leq m-1\}.$$

If any vertex $u \in H$ is not in H_i for any i, then $K_{m-1} + u$ is a $\{2K_2\}$-free K_m. Of course, some vertices may be in H_i for more than one value of i. Thus, we may assume that

$$\sum_{i=1}^{m-1} |H_i| \geq \frac{1}{2}(m-1)(m-2)(m-3)(a(n, m, N-1)-1)+1.$$

Assume without loss of generality that $|H_1| \geq |H_i|$ for $2 \leq i \leq m-1$. Then $|H_1| \geq \frac{1}{2}(m-2)(m-3)(a(n, m, N-1)-1)+1$. Now, there are at most $\binom{m-2}{2} =$

$\frac{1}{2}(m-2)(m-3)$ colors used in K_{m-1} on edges not incident with vertex v_1. Divide the set H_1 into at most $\frac{1}{2}(m-2)(m-3)$ subsets $L_1, L_2, \ldots$ by defining

$$L_j = \{u \in H_1 | uv_1 \text{ is color } j\}.$$

There are at most $\frac{1}{2}(m-2)(m-3)$ such subsets, and their union is H_1. Thus, there must be some subset, say L_1, such that

$$|L_1| \geq \left\lceil \frac{|H_1|}{\frac{1}{2}(m-2)(m-3)} \right\rceil \geq a(n, m, N-1).$$

We may assume that the subgraph induced by L_1 contains a copy of K_{N-1} with the minimum coloring. Since every edge from v_1 to L_1 is the same color, this K_{N-1} along with the vertex v_1 yields a K_N with the minimum coloring.

Now, solving the recursion in equation 11 with respect to N shows that for $N \geq 3$,

$$a(n,m,N) \leq m + \sum_{i=1}^{N-3} \left[\frac{1}{2^i}(m-1)^{i+1}(m-2)^i(m-3)^i\right] + \left[\frac{1}{2}(m-1)(m-2)(m-3)\right]^{N-2}.$$

If we set $N = (n-2)(m-2)+2$, we have the desired upper bound. □

3.7 Bounds for Stars and Cycles

We will consider one more set of bounds for a class of graphs. The proofs of the following bounds help illustrate the relationships between the various $\mathcal{F}$-free ramsey numbers. Since a rainbow coloring is $\mathcal{F}$-free for any set $\mathcal{F}$ of graphs with size at least two, the previous upper bounds for rainbow ramsey numbers are useful for any $\mathcal{F}$-free ramsey number. In the proof of the following bound, we actually force a stricter coloring than necessary in order to simplify the proof.

Theorem 37. *For any integers $n \geq 3$ and $m \geq 3$,*

$$R_{2K_2}(K_{1,n}, C_m) \leq (2n-1)(m-2) \quad .$$

Proof. Let $N = (2n-1)(m-2)$. Suppose K_N is edge-colored with no monochromatic $K_{1,n}$. According to Corollary 3, $RR(K_{1,n}, P_{m-1}) \le (m-3)(n-1)+(m-2)$. Since $(m-3)(n-1)+(m-2) < (2n-1)(m-2)$, we may assume without loss of generality that there is a rainbow P_{m-1} in K_N. Let M be the set of vertices in K_N which are not on this path. Then

$$\begin{aligned} |M| &= (2n-1)(m-2)-(m-1) \\ &= 2(n-1)(m-2)-1. \end{aligned}$$

Let v and w be the end vertices of the path. Since there is no monochromatic $K_{1,n}$, each of the $m-2$ colors which appear on the path can be used at most $n-1$ times on the edges between v and M. The color of the edge incident with v on the path can be used at most $n-2$ times. Thus, at most $(n-1)(m-2)-1$ of the edges from v to M are colored in colors which appear on the path. Similarly, at most $(n-1)(m-2)-1$ of the edges from w to M can be colored in colors which appear on the path. Since $|M| > 2[(n-1)(m-2)-1]$, there must be some vertex $u \in M$ such that neither uv nor uw are colored with any of the colors appearing on the path P_{m-1}. Notice that uv and uw could be the same color; regardless, the path P_{m-1} and these two edges form a $\{2K_2\}$-free copy of C_m. □

Essentially the same idea can be used to prove the following upper bound for the rainbow ramsey number, although a little greater care is needed in the final step to ensure that the last two edges of the cycle are not the same color.

Theorem 38. *For any integers $n \ge 2$ and $m \ge 3$,*

$$RR(K_{1,n}, C_m) \le n^3 + n^2(m-5) - n(m-5) + (m-2)$$

Proof. Let $N = n^3+n^2(m-5)-n(m-5)+(m-2)$. Suppose the edges of K_N are colored so that there is no monochromatic subgraph isomorphic to $K_{1,n}$. For $n \ge 2$

and $m \geq 3$, we have $n^3+n^2(m-5)-n(m-5)+(m-2) \geq (m-4)(n-1)+(m-3)$. Since there is no monochromatic $K_{1,n}$, corollary 3 guarantees a rainbow P_{m-2}. Let v and w be the endpoints of this path. Since $m-3$ colors are used on the path, one of which is incident with v, at most $(m-3)(n-1)-1$ of the edges from v to vertices not on the path are the same color as edges on the path. Each new color incident with v can appear at most $n-1$ times. Since

$$(n-1)(m-3)-1+(n-1)(n-2) < n^3+n^2(m-5)-n(m-5),$$

there must be at least $n-1$ new colors appearing on edges incident with v. Let $u_1, u_2, \ldots u_{n-1}$ be vertices not on the path such that the edges vu_1, vu_2, ... vu_{n-1} are all colored with distinct new colors.

Let M be the set of remaining vertices, that is, $M = V(K_N) - V(P_{m-2}) - \{u_1, u_2, \ldots u_{n-1}\}$. Then $|M| = n^3+n^2(m-5)-n(m-5)-(n-1) = n^3+n^2(m-5)-n(m-4)+1$. At this point, we have used $m+n-4$ colors. Since there is no monochromatic $K_{1,n}$, for each i, $1 \leq i \leq n-1$, at most $(n-1)(m+n-4)$ of the edges from u_i to M are colored with colors already used. Similarly, at most $(n-1)(m+n-4)$ of the edges from w to M are colored with colors already used. Since

$$\begin{aligned} n(n-1)(m+n-4) &= n^3+n^2(m-5)-n(m-4) \\ &< |M|, \end{aligned}$$

there must be a vertex $x \in M$ such that each of the edges $xu_1, xu_2, \ldots xu_{n-1}$ and xw are colored with colors not previously used. If these n edges are all the same color, then we have a monochromatic copy of $K_{1,n}$. Otherwise, there is some i such that xu_i is not the same color as xw. Thus, if we add the edges vu_i, u_ix and xw to the path P_{m-2}, we obtain a rainbow copy of C_m. □

The proof of the upper bound for the edge-chromatic ramsey number is very similar, except that we can apply Theorem 28 instead of corollary 3.

Theorem 39. *For any integers $n \geq 2$ and $m \geq 3$, the edge-chromatic ramsey number*

$$CR(K_{1,n}, C_m) \leq n(n-1) + (m-2)$$

Proof. Let $N = n(n-1) + (m-2)$. Suppose that the edges of K_N are colored so that there is no monochromatic subgraph isomorphic to $K_{1,n}$. Since $n(n-1) + (m-2) > m + n - 5$, we may apply Theorem 28. There must be an edge-chromatically colored path on $m-2$ vertices. Let w and v be the end vertices of this path, and let w' and v' be the vertices on the path which are adjacent to w and v, respectively.

There are $n(n-1)$ vertices not on the path. Since each color can appear on at most $n-1$ edges incident with v, there must be at least n different colors appearing on edges between v and the vertices not on the path, including at least $n-1$ colors different from the color of edge vv'. Let $u_1, u_2, \ldots u_{n-1}$ be vertices not on the path so that the edges $vu_1, vu_2, \ldots vu_{n-1}$ and vv' are all colored with different colors.

Let M be the set of remaining vertices, so that $M = V(K_N) - V(P_{m-2}) - \{u_1, u_2, \ldots u_{n-1}\}$. Then $|M| = (n-1)^2$. For each i, $1 \leq i \leq n-1$, at most $n-2$ of the edges between u_i and M are the same color as vu_i. Otherwise, we would have a monochromatic $K_{1,n}$ in that color. Similarly, at most $n-2$ of the edges between w and M are the same color as ww'. Since $(n-1)^2 > n(n-2)$, there must be some vertex x in the set M such that xu_i is not the same color as u_iv, for each i, and xw is not the same color as ww'. If all of the edges $xu_1, xu_2, \ldots xu_{n-1}$ and

xw are the same color, then we have a monochromatic copy of $K_{1,n}$. Otherwise, there is some i such that xu_i is not the same color as xw. Thus, if we add the edges vu_i, u_ix and xw to the edge-chromatically colored path on $m-2$ vertices, we have an edge-chromatically colored cycle on m vertices. □

DISCONNECTED GRAPHS

Suppose a graph G has components G_2 and G_3. If we know $R_{\mathcal{F}}(G_1, G_2)$ and $R_{\mathcal{F}}(G_1, G_3)$, what can we say about $R_{\mathcal{F}}(G_1, G)$? When $\mathcal{F} = \{K_{1,2}\}$, so that $R_{\mathcal{F}}$ is the edge-chromatic ramsey number, we can obtain bounds.

Theorem 40. *For any graphs G_1, G_2 and G_3 for which the following numbers are defined, the edge-chromatic ramsey number satisfies*

$$CR(G_1, G_2) \leq CR(G_1, G_2 \cup G_3) \leq max(|V(G_3)| + CR(G_1, G_2), CR(G_1, G_3))$$

Proof. The lower bound is clear. Suppose $N \geq max(|V(G_3)| + CR(G_1, G_2), CR(G_1, G_3))$. Color the edges of K_N with any number of colors. Since $N \geq CR(G_1, G_3)$, we may assume without loss of generality that there is a properly colored subgraph isomorphic to G_3. If we remove these $|V(G_3)|$ vertices, the remaining graph contains either a monochromatic copy of G_1 or a properly colored copy of G_2 which is disjoint from the copy of G_3. Thus, we have a monochromatic copy of G_1 or a properly colored copy of $G_2 \cup G_3$. □

Notice that the roles of G_2 and G_3 are interchangeable. Since $CR(G_1, G_2) \geq max(|V(G_1)|, |V(G_2)|)$ for any graphs G_1 and G_2 with size at least 2, we have the following corollary.

Corollary 4. *For any graphs G_1 and G_2 of size at least 2 for which the following numbers are defined,*

$$CR(G_1, G_2) \leq CR(G_1, 2G_2) \leq 2\ CR(G_1, G_2)$$

As an example, Theorem 40 gives the following bounds:

$$6 \leq CR(K_{1,3}, 2K_{1,3}) \leq 10$$

The actual value of this parameter is $CR(K_{1,3}, 2K_{1,3}) = 8$. The complete graph K_7 may be edge-colored rainbow to avoid both graphs. Suppose that K_8 is edge-colored so that no color appears more than twice at any vertex. Pick an edge uv colored with color 1. At least three other colors must be used on edges incident with vertex u. Suppose, then, that edges ua, ub, and uc are colors 2, 3, and 4, respectively. Let d, e and f be the remaining vertices of the graph. If vd, ve and vf are all different colors, then we have a properly colored $2K_{1,3}$. If they are all the same color, then we have a monochromatic $K_{1,3}$. We may assume without loss of generality that vd and ve are color c_1, possibly equal to 1, 2, 3, or 4, and vf is color c_2, where $c_1 \neq c_2$, but c_2 could be 1, 2, 3, or 4. No other edge incident with v is color c_1 and at most one other edge is color c_2. Thus, we may assume wlog that edges va and vb are not color c_1 or color c_2.

At most one of the edges ud and ue can be color 4. Assume wlog that ud is not color 4. If ud is color 2, then the edges ub, uc, ud, va, ve, and vf form a properly colored subgraph isomorphic to $2K_{1,3}$. If ud is not color 2, then the edges ua, uc, ud, vb, ve, and vf form the desired subgraph.

Theorem 40 can be generalized for $\mathcal{F}$-free ramsey numbers provided all of the graphs in $\mathcal{F}$ are connected.

Theorem 41. *Suppose $\mathcal{F}$ is a family of* connected *graphs. Then for any graphs G_1, G_2, and G_3 for which the numbers are defined,*

$$R_{\mathcal{F}}(G_1, G_2) \leq R_{\mathcal{F}}(G_1, G_2 \cup G_3) \leq max(|V(G_3)| + R_{\mathcal{F}}(G_1, G_2), R_{\mathcal{F}}(G_1, G_3)).$$

The proof is completely analogous to the proof of Theorem 40. We again have a corollary.

Corollary 5. *Suppose $\mathcal{F}$ is a family of* connected *graphs. Then for any graphs G_1 and G_2 of size at least 2 for which the numbers are defined,*

$$R_{\mathcal{F}}(G_1, G_2) \leq R_{\mathcal{F}}(G_1, 2G_2) \leq 2R_{\mathcal{F}}(G_1, G_2)$$

However, the condition that $\mathcal{F}$ contain only *connected* graphs is essential. For example, $RR(G_1, 2G_2)$ is not less than $2RR(G_1, G_2)$ in general. For at least two small examples, the opposite inequality holds. The rainbow ramsey number $RR(K_{1,n}, K_2) = 2$ while $RR(K_{1,n}, 2K_2) \geq n+1$. Similarly, $RR(K_{1,n}, K_{1,2}) = n+1$, while $RR(K_{1,n}, 2K_{1,2}) \geq 3n - 2$. To see this last inequality, notice that when n is odd, K_{3n-2} may be decomposed into $3(n-1)/2$ hamiltonian cycles; $(n-1)/2$ cycles may be colored with each color, so that no monochromatic $K_{1,n}$ appears and only three colors are used. When n is even, K_{3n-2} may be decomposed into $3(n-1)$ perfect matchings and $n-1$ matchings may be colored with each color.

SYMMETRY IN $\mathcal{F}$-FREE RAMSEY NUMBERS

The definition of the traditional ramsey numbers is symmetric, in the sense that $r(G_1, G_2) = r(G_2, G_1)$. If we have a coloring of K_N with no red G_1 and no blue G_2, we merely need to interchange the colors to obtain a coloring with no red G_2 and no blue G_1. For the rainbow, edge-chromatic, and other $\mathcal{F}$-free ramsey numbers, however, the definitions contain no such symmetry.

For example, there is no simple relationship between $RR(G_1, G_2)$ and $RR(G_2, G_1)$ in general. We described bounds for the number $RR(C_3, P_m)$, but $RR(P_n, C_3)$ does not exist for $n \geq 4$. In cases where both numbers exist and $G_1 \subseteq G_2$, we have seen numerous examples where $RR(G_1, G_2) \leq RR(G_2, G_1)$. Recall, for instance, that

$$RR(4K_2, 5K_2) = 17$$
$$RR(5K_2, 4K_2) = 18.$$

On the other hand, $K_{1,2} \subseteq K_m$ for $m \geq 3$. Since any coloring of K_m that is not monochromatic must contain two adjacent edges in different colors, $RR(K_m, K_{1,2}) = m$. However, $RR(K_{1,2}, K_m) \geq 2(m-2)+1$. To see this inequality, color $K_{2(m-2)}$ as follows. Color a perfect matching in one color, say color 1, and color the other edges with different colors. Thus, there are no two adjacent edges in the same color, but any set of m vertices must contain at least two edges of the perfect matching in color 1.

It is perhaps surprising, then, that there is an almost symmetrical relationship for the edge-chromatic ramsey number when one of the graphs is a star.

Theorem 42. *For any graph G and any positive integer n,*

$$CR(K_{1,n}, G) \leq CR(G, K_{1,n+1})$$

Proof. Suppose $CR(K_{1,n}, G) = N + 1$ for some integer N. Color the edges of K_N so that there are no monochromatic copies of $K_{1,n}$ and no edge-chromatically colored copies of G.

Now, define a new coloring of K_N as follows. For each color in the original coloring, recolor the edges of the subgraph induced by that color with an edge-chromatic coloring in colors $1, 2, \ldots k$, using as few colors as possible. According to Vizing's Theorem, the edge-chromatic number of any graph is at most one more than its maximum degree. Since the maximum degree of the subgraph induced by any color class in the original coloring is at most $n - 1$, at most n colors are used in this new coloring. Thus, there can be no edge-chromatically colored $K_{1,n+1}$ in the new coloring.

Suppose a monochromatic copy of G appears in the new coloring. If e and f are any two adjacent edges of G in the new coloring, notice that they must have been different colors in the original coloring. Thus, this copy of G was edge-chromatically colored in the original coloring, which is a contradiction.

Thus, we have an edge-coloring of K_N with no monochromatic G and no edge-chromatic $K_{1,n+1}$, so $CR(G, K_{1,n+1}) \geq N + 1$. □

DIGRAPH RAINBOW AND EDGE-CHROMATIC RAMSEY NUMBERS

Determining rainbow or edge-chromatic ramsey numbers for paths is difficult, in part, because the paths could move through the vertices of the complete graph in any order. If we order the directions of both the paths and the complete graphs, this difficulty is eliminated. This idea leads naturally to the definition of rainbow ramsey and edge-chromatic ramsey numbers for acyclic digraphs.

Let D_1 and D_2 be any two acyclic digraphs. We define the ***digraph rainbow ramsey number*** $DRR(D_1, D_2)$ as the minimum integer N such that any arc-coloring of the complete acyclic digraph D_N must contain either a monochromatic subdigraph isomorphic to D_1 or a rainbow subdigraph isomorphic to D_2. In what follows, an *outstar* is a star $K_{1,n}$ in which every edge is directed away from the central vertex, and an *instar* is a star in which every edge is directed towards the central vertex.

For which digraphs D_1 and D_2 do these numbers exist? For any integer N, label the vertices of D_N with the integers $1, 2, \ldots N$ so that the edge from vertex i to vertex j, where $i < j$, is directed from i to j. If this graph is colored with the minimum coloring, so that the arc ij is color $min(i, j)$ for each i and j, then the only monochromatic subdigraphs are outstars and no rainbow subdigraph can contain a vertex with outdegree greater than 1. If this same graph is colored with the maximum coloring, so that arc ij is colored with color $max(i, j)$ for each i and j, then the only monochromatic subdigraphs are instars and no rainbow subdigraph contains a vertex with indegree more than 1. In the next theorem, we will show that if the digraph rainbow ramsey number of any pair of digraphs D_1 and D_2 exists for these two colorings, then it exists for any coloring of K_N.

Theorem 43. *Let D_1 and D_2 be two nontrivial acyclic digraphs, so that D_1 has at least 2 arcs. Then the digraph rainbow ramsey number $DRR(D_1, D_2)$ exists if and only if* one *of the following holds:*

1. *D_1 is an outstar and D_2 has no vertex with indegree greater than 1*
2. *D_1 is an instar and D_2 has no vertex with outdegree greater than 1,* or
3. *D_2 is a union of directed paths, that is, D_2 has no vertex with outdegree greater than 1 and no vertex with indegree greater than 1.*

Proof. The examples given above show that $DRR(D_1, D_2)$ does not exist unless one of these three requirements is satisfied. If D_1 is an outstar, then an acyclic digraph with the maximum coloring contains no monochromatic D_1 and no rainbow subdigraph with indegree greater than 1. If D_1 is an instar, then an acyclic digraph with the minimum coloring contains no monochromatic D_1 and no rainbow subdigraph with outdegree greater than 1. Finally, if D_1 is neither an instar nor an outstar, then neither the minimum nor maximum colorings contain a monochromatic D_1. The only rainbow subdigraphs contained in both colorings are those digraphs with no outdegree greater than 1 and no indegree greater than 1.

Suppose D_1 is an outstar with underlying graph $K_{1,n}$ and suppose D_2 is an acyclic digraph of order m such that *indeg* $v \leq 1$ for all $v \in V(D_2)$. We will show that $DRR(D_1, D_2)$ exists by induction on m. If $m = 2$, then $DRR(D_1, D_2) = 2$ trivially.

Since D_2 has no directed cycles and no vertex with indegree greater than 1, there cannot be any cycles in its underlying graph. Thus, the underlying graph is a tree or a union of trees, each component of which must have at least two end-vertices. Let u and v be two end-vertices in the same component of D_2. If both u and v have positive outdegree, then there must be a vertex on the path between u and v with indegree at least 2. This is a contradiction; we may assume without loss of generality that u has outdegree 0 and indegree 1.

By induction, we know that $DRR(D_1, D_2-u)$ exists. Let $N = DRR(D_1, D_2-u) + (n-1)(m-2) + 1$. Consider any coloring of the arcs of D_N. On the $DRR(D_1, D_2-u)$ vertices with highest outdegree, there must be either a monochromatic copy of D_1 or a rainbow copy of $D_2 - u$. Suppose there is a rainbow copy of D_2-u. Let w be the vertex adjacent to u in D_2. There are at least $(n-1)(m-2)+1$ vertices in D_N which are not in $D_2 - u$ and which are adjacent from w. If any n of the arcs from w to these vertices are the same color, we have a monochromatic copy of D_1. Otherwise, there must be arcs in at least $m-1$ different colors. Since the underlying graph of $D_2 - u$ is acyclic and has order $m-1$, $D_2 - u$ contains at most $m-2$ arcs. Thus, at least one of these colors must be a new color. We may add this arc to $D_2 - u$ to obtain a rainbow copy of D_2.

Thus, $DRR(D_1, D_2)$ exists, where D_1 is an outstar with underlying graph $K_{1,n}$ and D_2 is an acyclic digraph with no vertex with indegree greater than 1. Solving the recursive bound

$$\begin{aligned} DRR(D_1, D_2) &= 2 && \text{when } D_2 \text{ has order } 2 \\ DRR(D_1, D_2) &\le DRR(D_1, D_2 - u) + (n-1)(m-2) + 1 && \text{when } D_2 \text{ has order } m \end{aligned}$$

yields the upper bound

$$DRR(D_1, D_2) \le \frac{1}{2}(m-2)(m-1)(n-1) + m. \tag{12}$$

A very similar argument shows that if D_1 is an instar with underlying graph $K_{1,n}$ and D_2 is an acyclic digraph with no vertex with outdegree greater than 1, then $DRR(D_1, D_2)$ exists and

$$DRR(D_1, D_2) \leq \frac{1}{2}(m-2)(m-1)(n-1) + m.$$

Suppose D_2 is a digraph such that each vertex has outdegree at most one and indegree at most one. We will need some additional definitions and a lemma to show the existence of $DRR(D_1, D_2)$ in this case. We will say that a complete acyclic digraph D, along with a coloring of its arcs, is a *type-A digraph* or has a *type-A coloring* if any two arcs incident from the same vertex in D are colored with the same color. Similarly, we will say that D is a *type-B digraph* or has a *type-B coloring* if any two arcs incident from the same vertex are different colors. Thus, type-A is a generalization of the minimum coloring, while type-B includes both the maximum and the rainbow colorings. Let $AB(k, j)$ be the minimum positive integer N such that any coloring of the complete acyclic digraph D_N contains either a type-A complete digraph on k vertices or a type-B complete digraph on j vertices.

Lemma 8. *For any positive integers k and j, the number $AB(k, j)$ exists.*

We will prove the lemma by induction on k and j. If $k = 1$ or 2 or $j = 1$ or 2, then the number exists trivially; any coloring of any complete digraph on 1 or 2 vertices is both type-A and type-B. Suppose both $AB(k-1, j)$ and $AB(k, j-1)$ exist. Let $N = (AB(k-1, j) - 1)(AB(k, j-1) - 1) + 2$. Color the arcs of D_N arbitrarily. Let v be the vertex in D_N with maximum outdegree. Suppose there are $AB(k-1, j)$ arcs incident from v in the same color. In this case, the vertices incident from these arcs induce a digraph containing either a type-B digraph of order j or a type-A digraph of order $k-1$. The vertex v could be added to a type-A digraph of order $k-1$ to produce a type-A digraph of order k. Otherwise, there must be $AB(k,j-1)$ arcs incident from v such that each arc is a different color. The digraph induced by the vertices incident from these arcs must contain either a type-A digraph of order k or a type-B digraph of order $j-1$. The vertex v could be added to a type-B digraph of order $j-1$ to produce a type-B digraph of order j. Thus, $AB(k,j)$ exists and $AB(k,j) \leq (AB(k-1,j)-1)(AB(k,j-1)-1)+2$ for $k \geq 3$ and $j \geq 3$.

Next, suppose D_1 is a complete acyclic digraph of order n and suppose D_2 is a directed path of order m. We claim that $DRR(D_1, D_2)$ exists. Let $N = AB((n-2)(m-2)+2, m(m-1)/2+1)$. Consider any coloring of the arcs of the complete acyclic digraph D_N. Either this digraph contains a complete subdigraph of type A with $(n-2)(m-2)+2$ vertices or a complete subdigraph of type B with $m(m-1)/2+1$ vertices. Suppose there is a type-A digraph on $(n-2)(m-2)+2$ vertices. Label the vertices with their outdegrees $0, 1, 2, \ldots (n-2)(m-2)+1$. The arcs out of any given vertex are all the same color, so at most $(n-2)(m-2)+1$ colors are used. If $n-1$ of these colors are the same, then these $n-1$ vertices and the vertex with outdegree 0 form a monochromatic copy of D_1. Otherwise, there are $m-1$ different colors appearing. The corresponding $m-1$ vertices and the vertex with outdegree 0, in order, produce a directed rainbow path of order m.

Suppose, instead, that there is a type-B digraph on $m(m-1)/2+1 = 1+1+2+3+\ldots+(m-1)$ vertices. Label the vertices, in order from highest to lowest outdegree, by $v_1, v_2, \ldots v_{m(m-1)/2+1}$. Start with arc v_1v_2. At most one of the arcs v_2v_3 and v_2v_4 can be the same color as v_1v_2; choose whichever one is a different color, say v_2v_4. Now, at least one of the arcs v_4v_5, v_4v_6, and v_4v_7 must be in some new color; choose this arc. Continuing in this fashion, we can choose a directed rainbow path of order at least m.

Thus, $DRR(D_1, D_2)$ exists, where D_1 is a complete acyclic digraph and D_2 is a directed path. It follows that $DRR(D_1, D_2)$ exists for any acyclic digraph D_1 and any union of directed paths D_2.

To obtain an upper bound in this case, we first need an upper bound on $AB(k, j)$. First, notice that $AB(2, j) = AB(k, 2) = 2$. For $k \geq 3$ and $j \geq 3$, we have

$$\begin{aligned} AB(k,j) &\leq (AB(k-1,j)-1)(AB(k,j-1)-1)+2 \\ &= AB(k-1,j)AB(k,j-1) - AB(k-1,j) - AB(k,j-1)+3 \\ &\leq AB(k-1,j)AB(k,j-1). \end{aligned}$$

Solving this bound recursively yields

$$AB(k,j) \leq 2^{\binom{k+j-4}{k-2}}$$

We can verify this formula by induction. If $k = 2$, then $\binom{j-2}{0} = 1$, so $AB(2, j) = 2^1$. If $j = 2$, then we have $\binom{k-2}{k-2} = 1$ so $AB(k, j) = 2^1$. Suppose the formula holds for $AB(k, j-1)$ and $AB(k-1, j)$. Then

$$\begin{aligned} AB(k,j) &\leq AB(k,j-1)AB(k-1,j) \\ &\leq 2^{\binom{k+j-5}{k-2}} 2^{\binom{k+j-5}{k-3}} \\ &= 2^{\binom{k+j-4}{k-2}}. \end{aligned}$$

Thus, when D_1 is a complete acyclic digraph of order n and D_2 is a directed path of order m, we have the bound

$$\begin{aligned} DRR(D_1, D_2) &\leq AB((n-2)(m-2)+2, m(m-1)/2+1) \\ &\quad \binom{(n-2)(m-2)+m(m-1)/2-1}{(n-2)(m-2)} \\ &\leq 2. \end{aligned}$$

□

Suppose we color the complete acyclic digraph on $m+\frac{1}{2}(n-1)(m-2)(m-3)$ vertices so that edge $v_i v_{i+j}$ is colored with color $\lceil\frac{j}{n-1}\rceil$. This digraph has no monochromatic subdigraph isomorphic to an outstar with underlying graph $K_{1,n}$. Any edge in color i must skip at least $(n-1)(i-1)+1$ indices, so a rainbow-colored directed path on m vertices must skip at least $1+(1+(n-1))+(1+2(n-1))+\ldots+(1+(m-2)(n-1))$ indices from the first vertex to the last vertex on the path. Combining this discussion with the bound in equation 12, we have the following theorem.

Theorem 44. *If D_1 is an outstar (or, similarly, if D_1 is an instar) with underlying graph $K_{1,n}$ and D_2 is a directed path on m vertices, then*

$$DRR(D_1, D_2) = m + \frac{1}{2}(n-1)(m-1)(m-2).$$

If we apply the Pigeonhole Principle to the colors of the arcs from the vertex with maximum outdegree, the next formula is immediate. The lower bound follows from a coloring of the complete acyclic digraph on $(m-1)(n-1)+1$ vertices with $m-1$ colors, so that no vertex has more than $n-1$ arcs adjacent from it in the same color.

Theorem 45. *If D_1 is an outstar with underlying graph $K_{1,n}$ and D_2 is an outstar with underlying graph $K_{1,m}$, then*

$$DRR(D_1, D_2) = (m-1)(n-1) + 2.$$

For any outstar D_1 with underlying graph $K_{1,n}$ and any acyclic digraph D_2 on m vertices containing no vertex with indegree greater than 1, the digraph rainbow ramsey number lies between $m + \frac{1}{2}(n-1)(m-1)(m-2)$ and $(m-2)(n-1) + 2$. The upper bound is included in the existence proof; the lower bound follows from the coloring for the last theorem.

POSSIBLE DIRECTIONS FOR FURTHER STUDY

Many questions remain open for study. Certainly, the rainbow and edge-chromatic ramsey numbers for other classes of graphs could be considered, and the bounds we have found may be improved. Some other generalizations seem natural. For example, we could define $RR(G_1, G_2, G_3, \ldots ; H)$ for a graph H and a sequence of graphs $G_1, G_2, G_3, \ldots$ to be the smallest integer N such that any coloring of K_N with colors $1, 2, 3, \ldots$ must contain either a monochromatic copy of G_i in color i for some i or a rainbow copy of H.

Relationships between the various parameters could be explored. For example, we have bounds on the rainbow ramsey number in terms of the generalized ramsey number. Could similar bounds be found for the edge-chromatic ramsey number? Is there any relationship between $R_{\mathcal{F}}(H_1, G_1)$, $R_{\mathcal{F}}(H_2, G_2)$, and $R_{\mathcal{F}}(H_1 \times H_2, G_1 \times G_2)$ for some family of graphs $\mathcal{F}$?

We have often found an optimal lower bound coloring for the rainbow ramsey number $RR(G_1, G_2)$ by using $m - 1$ colors, where G_2 has m edges. The lower-bound coloring for $RR(K_{1,3}, 3K_2)$ must use 3 colors, but we have seen few other examples. It would be interesting to find significant examples where this optimal coloring is forced to use m or more colors or, perhaps, to establish that such examples do not exist except in special cases such as $RR(K_{1,3}, 3K_2)$.

The traditional and generalized ramsey numbers involve only monochromatic graphs, with a maximum number of colors used to color K_N. We might define a similar rainbow number, involving only rainbow subgraphs, if we set a minimum number of colors. Is there a natural way to set a minimum number of colors to use in coloring K_N, presumably dependent on N? What relationships might we expect between such a number, the traditional ramsey number, and the rainbow ramsey number?

In another direction, the edge-chromatic number of a graph is the minimum number of colors needed to color its edges with no two adjacent edges the same color, that is, with no monochromatic subgraph isomorphic to $K_{1,2}$. We might also explore the $\mathcal{F}$-free edge-chromatic number for other families of graphs $\mathcal{F}$, defined of course to be the minimum number of colors needed to edge-color G so that there is no monochromatic subgraph isomorphic to any graph in $\mathcal{F}$. Thus, when $\mathcal{F} = \{K_{1,2}\}$, we have the usual edge-chromatic number. The rainbow or $\{2K_2, K_{1,2}\}$-free chromatic number is simply the number of edges in the graph, that is, its size.

The $\{2K_2, K_3\}$-free edge-chromatic number equals the vertex cover number $\alpha(G)$, defined [see 4, p. 243] to be the minimum number of vertices needed to cover all of the edges in the graph. If $\{v_1, v_2, \ldots, v_k\}$ is a set of vertices that covers all of the edges in G, then G could be colored $\{2K_2, K_3\}$-free using k colors, by decomposing G into k stars with centers at $v_1, v_2, \ldots, v_k$. Conversely, if G is $\{2K_2, K_3\}$-free colored, then G is decomposed into k monochromatic stars. Their centers $v_1, v_2, \ldots, v_k$ necessarily cover all of the edges of G. We might also consider the $\{2K_2\}$-free edge-chromatic number. For which graphs does it differ from the $\{2K_2, K_3\}$-free edge-chromatic number?

These and many other questions involving both ramsey theory and colorings of graphs could naturally be considered.

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