Perturbed Hamiltonian System of Two Parameters with Several Turning Points

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PERTURBED HAMILTONIAN SYSTEM OF TWO PARAMETERS
WITH SEVERAL TURNING POINTS

by

Myeong Joon Ann

A Dissertation
submitted to the
Faculty of The Graduate College
in partial fulfillment of the
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PERTURBED HAMILTONIAN SYSTEM OF TWO PARAMETERS WITH SEVERAL TURNING POINTS

Myeong Joon Ann. Ph.D.

Western Michigan University. 1998

Consider a generalized Hamiltonian system with two parameters

\[ i\epsilon V' = [H_0(x) + \epsilon H_1(x, \epsilon, \eta)]V, \quad i = \sqrt{-1}. \]

for \( x \in I, \epsilon \in S_c \) and \( \eta \in N_\delta \) with \( I = [a, b], S_c = (0, c], N_\delta = (-\delta, \delta) \). where \( V, H_0(x) \) and \( H_1(x, \epsilon, \eta) \) are \( n \times n \) matrices. \( H_0(x) \) is Hermitian and analytic on \( I \) and \( H_1(x, \epsilon, \eta) \) is in the class \( C^1(I \times S_c \times N_\delta) \). Here \( a \) may be \(-\infty\) and \( b \) may be \(+\infty\). \( c \) is a positive constant, and \( \delta \) is a fixed small positive number. Any two eigenvalues of \( H_0(x) \) are allowed to coalesce finitely many times in the interval \( I \), which are called turning points of the system.

A global asymptotic solution on \( I \) of this system will be studied. In particular, two methods to compute the solution of the system will be discussed. This result can be applied to the study of the adiabatic invariance property in quantum mechanics.

This study generalizes the results of H. Gingold and P. F. Hsieh of 1987 and 1989.

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Myeong Joon Ann
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CHAPTER I

INTRODUCTION

A system of differential equations depending on a parameter $\epsilon$ is called a *singularly perturbed system* if the dimension of the system of differential equations is reduced when $\epsilon = 0$. For example, the $n$-dimensional system

\begin{equation}
\epsilon y' = A(x)y
\end{equation}

is a singularly perturbed linear system, as it is not a differential equation when $\epsilon = 0$. Here $y$ is an $n$-dimensional column vector and $A(x)$ is an $n \times n$ matrix.

For (1.1) with $A(x)$ analytic in an interval $I = [a, b]$, the diagonalization of coefficient matrix $A(x)$ or simplification process by an analytic transformation is an important step to study this system. The existence of such an analytic transformation depends on the analyticity of the eigenvalues and eigenvectors of $A(x)$.

A point $x = x_0$ on $I$ is called a *turning point* or *transition point* of (1.1) if two or more eigenvalues of $A(x)$ coalesce at this point. Many experts devoted to the study of (1.1) with a turning point in the last three quarters of century (e.g. see [1, 3, 9, 10, 18, 19, 20, 21]), and most results are restricted to an immediate neighborhood of a turning point. In order to find the global behavior of solutions of (1.1), it is desirable to find a global analytic simplification of $A(x)$. 

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Moreover, if there are several turning points on $I$, the problem becomes much more complicated. There were only a few papers in the literature could be found ten years ago, devoted to the study of a system of equations with several turning points (e.g. N. Bleisten [1] and F. Ursell [18]).

In the last ten years, H. Gingold and P. F. Hsieh [6, 7, 8, 9, 10] have several papers devoted to the global study of a special type of (1.1) with several turning points. They devoted their studies to a generalized Hamiltonian system

$$i\epsilon y' = [H_0(x) + \epsilon H_1(x, \epsilon)]y, \quad (i = \sqrt{-1}).$$

where $H_0(x)$ is an $n \times n$ Hermitian matrix, analytic on $I = [a, b]$, $(a$ may be $-\infty$ and/or $b$ may be $+\infty$), and $H_1(x, \epsilon)$ is an $n \times n$ matrix in the class $C^1(I \times \bar{S}_c)$ with $S_c = (0, c]$. In their studies, the eigenvalues of $H_0(x)$ possibly coalesce at several points of $[a, b]$; that is, there may be several turning points present in this interval. Moreover, the asymptotic solutions are obtained for entire interval $I$. Such study is important for the adiabatic invariance property in quantum mechanics. The adiabatic invariance property about crossing of energy levels (i.e., eigenvalues) was pointed out by M. Born and V. Fock in 1928 [2], and later studied by T. Kato [14] and W. Wasow [19], (see also R.L. Liboff [11]). but a rigorous study for a general case with several crossing of energy levels was not given until 1987 by H. Gingold and P. F. Hsieh [9]. The turning point problem is also important in the study of semiclassical asymptotics for simple scattering.

In this dissertation, we will generalize the results of H. Gingold and P. F. Hsieh [9, 10] to a system of equations similar to (1.2) with an additional parameter in its coefficient matrix, namely, to study a regularly perturbed problem of (1.2). The Hamiltonian system of equations with two parameters appear in quantum mechanics (cf. R.L. Riboff [6, p. 576] and H. Gingold [7]).

The main theorem will be stated in Chapter II with the assumptions described. The preliminary reductions will be carried out also in this chapter. Owing to these reductions, the proof of the main theorem is reduced to a fundamental lemma to be established in Chapter III. The proof of the fundamental lemma in different cases is done as separate lemmas in this chapter.

Two methods of computation are presented in Chapter IV with examples given. Chapter V gives problems for further studies.
CHAPTER II

THE PROBLEMS AND REDUCTIONS

2.1 The Problems and Assumptions

When we are dealing with an $n$-dimensional singularly perturbed linear system $\epsilon y' = A(x)y$ with coefficients analytic on an interval $I = [a, b]$, even when all eigenvalues of $A(x)$ are distinct on $I$, asymptotic solutions are guaranteed only for subintervals as the eigenvalues and eigenvectors of $A(x)$ may not be analytic on the entire interval. H. Gingold and P. F. Hsieh [9] obtained global asymptotic expressions for the solutions of certain singular differential systems, namely Hamiltonian systems, with several turning points on the entire interval $I$. In this paper, this result is generalized to a Hamiltonian system with an additional parameter.

Consider a singularly perturbed Hamiltonian system with two parameters:

\begin{equation}
\tag{2.1}
i \epsilon V' = H(x, \epsilon, \eta)V,
\end{equation}

where $V$ is an $n \times n$ unknown matrix. $H(x, \epsilon, \eta)$ is an $n \times n$ matrix, analytic in $(x, \epsilon, \eta)$ for $x \in I$, $\epsilon \in S_c$, and $\eta \in N_\delta$ with $I = [a, b]$, $S_c = (0, c]$, and $N_\delta = (-\delta, \delta)$. Here $a$ may be $-\infty$ and $b$ may be $+\infty$. $c$ is a positive constant.
and \( \delta \) is a fixed small positive number. Assume that (2.1) satisfies the following assumption:

(A1) The coefficient \( H(x, \epsilon, \eta) \) can be expressed as

\[
H(x, \epsilon, \eta) = H_0(x) + \epsilon H_1(x, \epsilon, \eta).
\]

where \( H_0(x) \) is an \( n \times n \) Hermitian matrix analytic on \( I \) and \( H_1(x, \epsilon, \eta) \) is an \( n \times n \) matrix in the class \( C^1(I \times \bar{S}_c \times N) \).

Let \( \{ \lambda_j(x) | j = 1, 2, ..., n \} \) be eigenvalues of \( H_0(x) \). A point \( x = x_0 \) on \( I \) such that \( \lambda_j(x_0) = \lambda_k(x_0) \) for some indices \( j \) and \( k \) \( (j \neq k) \) is called a turning point of (2.1). The order of the zero of \( \lambda_j(x) - \lambda_k(x) \) at such point \( x = x_0 \) is called the order of this turning point with respect to the pair of eigenvalues \( (\lambda_j, \lambda_k) \). In this study of (2.1), there may be several turning points with various orders for this system on \( I \).

By the results of H. Gingold and P. F. Hsieh [11] and F. Rellich [17], there exists an \( n \times n \) unitary matrix \( U(x) \) analytic and nonsingular on \( I \) such that

\[
U^{-1}(x)H_0(x)U(x) = D_0(x)
\]

\[
= \text{diag}\{\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)\}.
\]

Moreover, every eigenvalue \( \lambda_j(x) \) for \( j = 1, ..., n \) is real and analytic on \( I \). Let

\[
Y = U^{-1}(x)V.
\]
From (2.1), (2.2) and (2.3), the $n \times n$ matrix $Y$ satisfies a differential equation

\begin{equation}
 ieY' = [D_0(x) + \epsilon R_0(x, \epsilon, \eta)]Y.
\end{equation}

where

\begin{equation}
 R_0(x, \epsilon, \eta) = U^{-1}(x)H_1(x, \epsilon, \eta)U(x) - iU^{-1}(x)U'(x).
\end{equation}

Let

\begin{equation}
 D(x, \epsilon, \eta) = D_0(x) + \epsilon \text{diag}\{R_0(x, \epsilon, \eta)\}.
\end{equation}

\begin{equation}
 R(x, \epsilon, \eta) = R_0(x, \epsilon, \eta) - \text{diag}\{R_0(x, \epsilon, \eta)\}.
\end{equation}

Then the equation (2.5) can be written as:

\begin{equation}
 ieY' = [D(x, \epsilon, \eta) + \epsilon R(x, \epsilon, \eta)]Y.
\end{equation}

where $D(x, \epsilon, \eta)$ is a diagonal matrix and $R(x, \epsilon, \eta)$ is an off-diagonal matrix.

Assume further that (2.1) with (2.2) satisfies:

\begin{enumerate}
  \item[(A2)] $\lambda_j(x) - \lambda_k(x) \neq 0$ for $x \in I$, $j \neq k$.
  \item[(A3)] $\|H_1(x, \epsilon, \eta)\| \leq k_1$, for $x \in I$, $\epsilon \in S_c$, $\eta \in N_\delta$ with $k_1$ a positive constant.
  \item[(A4)] $\int_a^b \|H_1(x, \epsilon, \eta)\| dx$ is uniformly bounded for $\epsilon \in S_c$, $\eta \in N_\delta$.
  \item[(A5)] $\int_a^b \|H_1'(x, \epsilon, \eta)\| dx$ is uniformly bounded for $\epsilon \in S_c$, $\eta \in N_\delta$.
\end{enumerate}

Here $\| \|$ denotes a suitable norm of a matrix.

\section*{2.2 Statement of the Main Theorem}

We will establish the following main theorem:
Theorem 1. Under the assumptions (A1) - (A5), the fundamental matrix of
the equation (2.9) can be expressed as:

\begin{equation}
Y = Z(x, \alpha, \epsilon, \eta)(I_n + P(x, \epsilon, \eta)).
\end{equation}

where $Z(x, \alpha, \epsilon, \eta)$ is a fundamental matrix of

\begin{equation}
\begin{aligned}
\epsilon Z' &= D(x, \epsilon, \eta)Z, \\
Z(\alpha, \alpha, \epsilon, \eta) &= I_n,
\end{aligned}
\end{equation}

with $\alpha \in I$, and $P(x, \epsilon, \eta)$ is an $n \times n$ matrix in the class $C^1(I \times S_\epsilon \times N_\delta)$. 

$(0 < \epsilon < c)$. \|P(x, \epsilon, \eta)\| = O(\epsilon^d)$, with $d > 0$. uniformly on $I \times N_\delta$ as $\epsilon \to 0^-$. satisfying $P(\alpha, \epsilon, \eta) = 0$. Here $I_n$ is the $n \times n$ identity matrix.

By Theorem 1, the system (2.1) has a fundamental matrix solution

\begin{equation}
V(x, \epsilon, \eta) = U(x) \exp \left\{ -\epsilon^{-1} \int_{\alpha}^{x} D(t, \epsilon, \eta) dt \right\} (I_n + P(x, \epsilon, \eta)).
\end{equation}

which is uniformly valid on $I \times N_\delta$.

2.3 Preliminary Reductions

From the differential equation (2.9), by (2.10) and (2.11).

\[ i\epsilon Y' = (D(x, \epsilon, \eta) + \epsilon R(x, \epsilon, \eta))Z(x, \alpha, \epsilon, \eta)(I_n + P). \]

\[ = D(x, \epsilon, \eta)Z(x, \alpha, \epsilon, \eta)(I_n + P) + \epsilon R(x, \epsilon, \eta)Z(x, \alpha, \epsilon, \eta)(I_n + P). \]
On the other hand, by (2.10).

\[ ieY' = ie(Z'(x, \alpha, \epsilon, \eta)(I_n + P) + Z(x, \alpha, \epsilon, \eta)P') \]

\[ = D(x, \epsilon, \eta)Z(x, \alpha, \epsilon, \eta)(I_n + P) + ieZ(x, \alpha, \epsilon, \eta)P'. \]

Equating these two expressions, we have

\[ (2.13) \quad iZ(x, \alpha, \epsilon, \eta)P' = R(x, \epsilon, \eta)Z(x, \alpha, \epsilon, \eta)(I_n + P). \]

If we multiply \( Z^{-1}(x, \alpha, \epsilon, \eta) \) on both sides of (2.13), then by (2.9), (2.10) and (2.11), the \( n \times n \) matrix \( P \) satisfies:

\[ (2.14) \quad iP' = Z^{-1}(x, \alpha, \epsilon, \eta)R(x, \epsilon, \eta)Z(x, \alpha, \epsilon, \eta)(I_n + P). \quad P(\alpha, \epsilon, \eta) = 0. \]

Thus, (2.14) can be expressed as:

\[ (2.15) \quad P = -i \int_{\alpha}^{x} Z^{-1}(t, \alpha, \epsilon, \eta)R(t, \epsilon, \eta)Z(t, \alpha, \epsilon, \eta)(I_n + P(t, \alpha, \epsilon, \eta)) \, dt. \]

Define an integral operator \( L \) as

\[ (2.16) \quad LP = -i \int_{\alpha}^{x} Z^{-1}(t, \alpha, \epsilon, \eta)R(t, \epsilon, \eta)Z(t, \alpha, \epsilon, \eta)P(t, \alpha, \epsilon, \eta) \, dt. \]

Then the integral equation (2.15) can be written as:

\[ (2.17) \quad P = LI_n + LP. \]

For a better estimate of the kernel of this equation, we take the second iteration.
that is.

\[ P = LI_n + L(LI_n + LP) \]

\[ = LI_n + L^2I_n + L^2P. \]

where

\[ L^2P = \int_{\alpha}^{\xi} Z^{-1}(s, \alpha, \epsilon, \eta)R(x, \epsilon, \eta)Z(s, \alpha, \epsilon, \eta) \]

\[ \left\{ \int_{\alpha}^{\xi} Z^{-1}(t, \alpha, \epsilon, \eta)R(t, \epsilon, \eta)Z(t, \alpha, \epsilon, \eta)P(t, \epsilon, \eta) dt \right\} ds. \]

or, by the change of order of integration.

\[ L^2P = \int_{\alpha}^{\xi} \left\{ \int_{t}^{\xi} Z^{-1}(s, \alpha, \epsilon, \eta)R(s, \epsilon, \eta)Z(s, \alpha, \epsilon, \eta) ds \right\} \]

\[ Z^{-1}(t, \alpha, \epsilon, \eta)R(t, \epsilon, \eta)Z(t, \alpha, \epsilon, \eta)P(t, \epsilon, \eta) dt. \]

To examine \( Z^{-1}(x, \alpha, \epsilon, \eta)R(x, \epsilon, \eta)Z(x, \alpha, \epsilon, \eta) \), we need some notations.

Let

\[ R_0(x, \epsilon, \eta) = [r_{jk}(x, \epsilon, \eta)]_{j,k=1}^{n}. \]

and

\[ d_j(x, \epsilon, \eta) = \lambda_j(x) + \epsilon r_{jj}(x, \epsilon, \eta). \quad (j = 1, 2, \ldots, n). \]

Note that \( d_j(x, \epsilon, \eta) \) are diagonal elements of \( D(x, \epsilon, \eta) \). Since \( D(x, \epsilon, \eta) \) is diagonal and by solving the differential equation (2.11) directly with the diagonal elements for \( D(x, \epsilon, \eta) \), \( Z \) can be written as:

\[ Z(x, \alpha, \epsilon, \eta) = \text{diag}\{z_1(x, \alpha, \epsilon, \eta), z_2(x, \alpha, \epsilon, \eta), \ldots, z_n(x, \alpha, \epsilon, \eta)\} \]
where

\[(2.24)\quad z_j(x, \alpha, \epsilon, \eta) = \exp \left\{ -i\epsilon^{-1} \int_\alpha^x d_j(t, \epsilon, \eta) \, dt \right\}.\]

Thus,

\[(2.25)\quad [Z^{-1}(x, \alpha, \epsilon, \eta) R(x, \epsilon, \eta) Z(x, \alpha, \epsilon, \eta)]_{jk}\]

\[= \begin{cases} 
  r_{jk} \exp \left\{ i\epsilon^{-1} \int_\alpha^x (d_j(t, \epsilon, \eta) - d_k(t, \epsilon, \eta)) \, dt \right\}, & \text{if } j \neq k, \\
  0, & \text{if } j = k.
\end{cases}\]

Let

\[(2.26)\quad L^2 P = [A_{jk}]_{j,k=1}^n.\]

and

\[(2.27)\quad P(x, \epsilon, \eta) = [p_{jk}(x, \epsilon, \eta)]_{j,k=1}^n.\]

Then by (2.20) and (2.25), \(A_{jk}\) can be expressed as:

\[(2.28)\quad A_{jk} = - \int_\alpha^x \sum_{l=1}^n \left[ \sum_{h=1}^n \left( \int_t^x [Z^{-1}(s, \alpha, \epsilon, \eta) R(s, \epsilon, \eta) Z(s, \alpha, \epsilon, \eta)]_{jh} \, ds \right) \right. \\
[ Z^{-1}(t, \alpha, \epsilon, \eta) R(t, \epsilon, \eta) Z(t, \alpha, \epsilon, \eta)]_{hl} \left] p_{lk}(t, \epsilon, \eta) \, dt \\
- \int_\alpha^x \sum_{l=1}^n \left[ \sum_{h=1}^n \left( \int_t^x r_{jh}(s, \epsilon, \eta) \exp \left\{ i\epsilon^{-1} \int_s^t (d_j(u, \epsilon, \eta) - d_h(u, \epsilon, \eta)) \, du \right\} \, ds \right) \\
r_{hl}(t, \epsilon, \eta) \exp \left\{ \epsilon^{-1} \int_\alpha^t (d_h(u, \epsilon, \eta) - d_l(u, \epsilon, \eta)) \, du \right\} \right] \\
p_{lk}(t, \epsilon, \eta) \, dt.\]
In order to prove the main theorem, we want to establish that

\[(2.29) \quad |||L^2 P||| \leq L(\epsilon) |||P|||. \text{ for } (x, \epsilon, \eta) \text{ in } (I \times S_\epsilon \times N_\delta). (0 < \epsilon \leq c)\]

for a suitable norm $||| \cdot |||$, where $L(\epsilon)$ is a quantity which depends only on $\epsilon$ and tends to 0 as $\epsilon \to 0^+$. 

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CHAPTER III

PROOF OF THE MAIN THEOREM

3.1 Statement of Lemmas

In order for the integral equation (2.18) to define a contraction mapping for \( x \) on \( I \), \( \epsilon \) in \( S_c \), and \( \eta \) in \( N_\delta \) with sufficiently small \( c \), we should estimate the magnitude for each entry of the kernel in the integral. The same method used in H. Gingold and P. F. Hsieh [9] will be applied in these integrals. First, we establish a fundamental lemma similar to one given in H. Gingold and P. F. Hsieh [9].

**Lemma 1.** Consider an integral expression

\[
J(a, b, \alpha, \epsilon, \eta) = \int_a^b r(s, \epsilon, \eta) \exp \left\{ i\epsilon^{-1} \int_a^s p(u) \, du \right\} ds. \quad a \leq \alpha \leq b.
\]

where \( r(x, \epsilon, \eta) \) is in the class \( C^1(I \times \bar{S}_c \times N_\delta) \) and \( p(x) \) is real analytic on \( I \).

Furthermore, assume the followings:

(a) \( p(x) \) may vanish at finitely many points of \( I \), but it is not identically zero in \( I \).

(b) \( r(x, \epsilon, \eta) \) is uniformly bounded on \( (I \times \bar{S}_c \times N_\delta) \).

(c) Both \( r(x, \epsilon, \eta) \) and \( r'(x, \epsilon, \eta) \) are absolutely integrable over \( I \) for \( \epsilon \in S_c \).

\[ \eta \in N_\delta. \]
Then, there exist positive constants $K, d$ and $c_0$ ($0 < d < 1, 0 < c_0 < c$) such that

\begin{equation}
|J(a, b, \alpha, \epsilon, \eta)| \leq K\epsilon^d \quad \text{for} \quad \epsilon \in S_{c_0}, \eta \in \mathbb{N}_\delta.
\end{equation}

Remarks

1. $a$ may be $-\infty$ and $b$ may be $+\infty$.

2. A zero point of $p(x)$ on $I$ is called a turning point of the integral expression $J(a, b, \alpha, \epsilon, \eta)$.

3. By integration by parts, (3.1) can be expressed as

\begin{equation}
J(a, b, \alpha, \epsilon, \eta) = \left[ -i\epsilon r(s, \epsilon, \eta) \frac{1}{p(s)} \exp\left\{ i\epsilon^{-1} \int_a^s p(u) \, du \right\} \right]^b_a
\end{equation}

\begin{equation}
+ i\epsilon \int_a^b \frac{r'(s, \epsilon, \eta)p(s) - r(s, \epsilon, \eta)p'(s)}{|p(s)|^2} \exp\left\{ i\epsilon^{-1} \int_a^s p(u) \, du \right\} ds.
\end{equation}

4. Since $p(x)$ is analytic on $I$, there exists a positive constant $g_1$ such that

\begin{equation}
|p'(x)| \leq g_1 \quad \text{for} \quad x \in I.
\end{equation}

5. From the conditions (b) and (c) there exist positive constants $m_1$ and $m_2$ such that for $x \in I, \epsilon \in \tilde{S}_c, \eta \in \mathbb{N}_\delta$

\begin{equation}
|r(x, \epsilon, \eta)| \leq m_1
\end{equation}

and
\[
(3.6) \ 
\int_a^b |r(s, \epsilon, \eta)| \, ds \leq m_2, \ 
\int_a^b |r'(s, \epsilon, \eta)| \, ds \leq m_2.
\]

6. For \( \alpha, \beta, \tau \in [a, b] \), \( J(a, b, \alpha, \epsilon, \eta) \) satisfies

\[
(3.7) \ 
J(a, b, \alpha, \epsilon, \eta) = J(a, t, \alpha, \epsilon, \eta) + J(t, b, \beta, \epsilon, \eta) \exp \left\{ i \epsilon^{-1} \int_a^\beta p(u) \, du \right\}.
\]

Consequently, \( |J(a, b, \alpha, \epsilon, \eta)| \) is independent of \( \alpha \) and

\[
(3.8) \ 
|J(a, b, \alpha, \epsilon, \eta)| \leq |J(a, t, \alpha, \epsilon, \eta)| + |J(t, b, \beta, \epsilon, \eta)|
\]

for all \( \alpha, \beta, \tau \in [a, b] \).

The expression (3.3) will be used to estimate the integral \( J(a, b, \alpha, \epsilon, \eta) \) as \( \epsilon \to 0^+ \). The estimate of \( J(a, b, \alpha, \epsilon, \eta) \) of each different case of \( p(x) \) will be proved as a lemma. Namely, the proof of Lemma 1 is given in different cases of \( p(x) \) stated as lemmas. We assume that all the conditions of Lemma 1 hold for following Lemmas.

The following two lemmas are the cases when the point \( x = a \) is a zero point of \( p(x) \).

**Lemma 2.** Suppose that \( a \) is finite and \( p(x) \) is expressed as

\[
(3.9) \ 
p(x) = (x - a)^{v_\alpha} \hat{p}(x)
\]

where \( v_\alpha \) is a positive integer. \( \hat{p}(x) \) is real analytic on \( I \) and satisfying

\[
(3.10) \ 
0 < g_2 \leq |\hat{p}(x)| \quad \text{for } x \in I.
\]
Then, there exists positive constants $K_a$, $l_a$ and $c_a$ satisfying $0 < v_a l_a < 1$.

$0 < c_a \leq c$ such that

\begin{equation}
| J(a + \epsilon l_a, t, \alpha, \epsilon, \eta) | \leq K_a \epsilon^{1-v_a l_a} 
\end{equation}

for $a + \epsilon l_a \leq t, \alpha \leq b, \epsilon \in S_{c_a}, \eta \in N_{\delta}$.

**Lemma 3.** Suppose that $a = -\infty$ and $p(x)$ is expressed as

\begin{equation}
p(x) = x^{-v_a} \hat{p}(x) \end{equation}

where $v_a$ is a positive integer. $\hat{p}(x)$ is real analytic on $I$ and satisfying (3.10). In addition, assume that \( r(x, \epsilon, \eta) \) can be expressed as

\begin{equation}
r(x, \epsilon, \eta) = x^{-(1+\rho)} \tilde{r}(x, \epsilon, \eta), \text{ with } | \tilde{r}(x, \epsilon, \eta) | \leq m_3
\end{equation}

for $x \in (-\infty, -q), \epsilon \in S_e$ and $\eta \in N_{\delta}$ where $\rho, m_1$ and $q$ are positive constants.

Then, there exists positive constants $K_a$, $d_a$ and $c_a$ satisfying $0 < d_a < 1$.

$0 < c_a \leq c$ such that

\begin{equation}
| J(-\infty, t, \alpha, \epsilon, \eta) | \leq K_a \epsilon^{d_a}
\end{equation}

for $-\infty < t, \alpha \leq b, \epsilon \in S_{c_a}, \eta \in N_{\delta}$.

The following two lemmas are the cases when the point $x = b$ is a zero point of $p(x)$.
Lemma 4. Suppose that $b$ is finite and $p(x)$ is expressed as

\begin{equation}
(3.15) \quad p(x) = (x - b)^{v_b} \hat{p}(x)
\end{equation}

where $v_b$ is a positive integer. $\hat{p}(x)$ is real analytic on $I$ and satisfying (3.10). Then, there exists positive constants $K_b$, $l_b$ and $c_b$ satisfying $0 < v_b l_b < 1$, $0 < c_b \leq c$ such that

\begin{equation}
(3.16) \quad |J(t, e, \alpha, \epsilon, \eta)| \leq K_b \epsilon^{1-v_b l_b}
\end{equation}

for $a < t$, $b - e l_b < b$, $\epsilon \in S_{c_b}$, $\eta \in N_\delta$.

Lemma 5. Suppose that $b = +\infty$ and $p(x)$ is expressed as

\begin{equation}
(3.17) \quad p(x) = x^{-v_b} \hat{p}(x)
\end{equation}

where $v_b$ is a positive integer. $\hat{p}(x)$ is real analytic on $I$ and satisfying (3.10). In addition, assume that $r(x, \epsilon, \eta)$ can be expressed as in (3.13) for $x \in (q, +\infty)$, $\epsilon \in S_c$ and $\eta \in N_\delta$ with $q$ a positive constant. Then, there exists positive constants $K_b$, $d_b$ and $c_b$ satisfying $0 < d_b < 1$, $0 < c_b \leq c$ such that

\begin{equation}
(3.18) \quad |J(t, +\infty, \alpha, \epsilon, \eta)| \leq K_b \epsilon^{d_b}
\end{equation}

for $a < t$, $\alpha < +\infty$, $\epsilon \in S_{c_b}$, $\eta \in N_\delta$.

When both $x = a$ and $x = b$ are zero points of $p(x)$, we will establish:
Lemma 6. Suppose that $a$ and $b$ are finite and $p(x)$ is expressed as

\begin{equation}
(3.19) \quad p(x) = (x - a)^{v_a}(x - b)^{v_b} \hat{p}(x)
\end{equation}

where $v_a$ and $v_b$ are positive integers and $\hat{p}(x)$ is real analytic on $I$ satisfying (3.10). Then, there exist positive constants $K_{ab}$, $l_a$, $l_b$, $d_{ab}$ and $c_{ab}$ satisfying

$0 < v_a l_a < 1$, $0 < v_b l_b < 1$, $0 < d_{ab} < 1$ and $0 < c_{ab} \leq c$ such that

\begin{equation}
(3.20) \quad |J(a + \varepsilon l_a, b - \varepsilon l_b, \alpha, \varepsilon, \eta)| \leq K_{ab} \varepsilon^{d_{ab}}
\end{equation}

for $a \leq \alpha \leq b$, $\varepsilon \in S_{c_{ab}}$, $\eta \in N_\delta$.

When $p(x)$ has several zero points on $I$, we will establish:

Lemma 7. Suppose that $a$ and $b$ are finite, $p(x)$ is expressed as

\begin{equation}
(3.21) \quad p(x) = \left[ \prod_{j=1}^{m} (x - \alpha_j)^{v_j} \right] \hat{p}(x)
\end{equation}

where $a \leq \alpha_1 < \alpha_2 < \cdots \alpha_{m-1} < \alpha_m \leq b$ and $v_j (1 \leq j \leq m)$ are positive constants and $\hat{p}(x)$ is real analytic on $I$ satisfying (3.10). Then, there exists positive constants $K_1$, $d_1$, $c_1$ and $\delta_1$ with $(0 < d_1 < 1$, $0 < c_1 \leq c)$ such that

\begin{equation}
(3.22) \quad |J(a, b, \alpha, \varepsilon, \eta)| \leq K_1 \varepsilon^{d_1}
\end{equation}

for $a \leq \alpha \leq b$, $\varepsilon \in S_{c_1}$, $\eta \in N_\delta$. 

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3.2 Proof of Lemmas

**Proof of Lemma 2**  First, assume that $b$ is finite. From (3.3), we have

\begin{equation}
|J(a + \epsilon \alpha, t, \alpha, \epsilon, \eta)| \leq \left[ \frac{\epsilon r(s, \epsilon, \eta)}{p(s)} \right]_a ^t + \epsilon \int_a ^t \frac{|r'(s, \epsilon, \eta)p(s) - r(s, \epsilon, \eta)p'(s)|}{|p(s)|^2} ds.
\end{equation}

To find an estimate of the righthand side of (3.23), note first that for $a + \epsilon \alpha \leq x \leq b$, by the expression (3.9) for $p(x)$ with Remarks 4 and 5, we have

\begin{equation}
|p(x)| |(x - a)v''p(x)| \leq \epsilon^{-\nu \alpha \epsilon a} \frac{g_2}{g_2}.
\end{equation}

Thus,

\begin{equation}
\left[ \frac{\epsilon r(s, \epsilon, \eta)}{p(s)} \right]_a ^t \leq \epsilon r(t, \epsilon, \eta) + \epsilon r(a + \epsilon \alpha, \epsilon, \eta) \leq \frac{2m_1}{g_2} \epsilon^{1-\nu \alpha \epsilon a}.
\end{equation}

Now, the integral on righthand side of (3.23) is bounded by

\begin{equation}
\epsilon \int_a ^t \frac{|r'(s, \epsilon, \eta)p(s)|}{|p(s)|} ds + \epsilon \int_a ^t \frac{|r'(s, \epsilon, \eta)p'(s)|}{|p(s)|^2} ds.
\end{equation}

By Remark 5 and (3.24) we have

\begin{equation}
\epsilon \int_a ^t \frac{|r'(s, \epsilon, \eta)p(s)|}{p(s)} ds \leq \frac{m_2}{g_2} \epsilon^{1-\nu \alpha \epsilon a}.
\end{equation}

Note that since $\hat{p}(x)$ is analytic on $I$, there exists a constant $g_3 > 0$ such that

\begin{equation}
|\hat{p}'(x)| \leq g_3 \quad \text{for } x \in I.
\end{equation}
Since \( p'(x) = v_a(x - a)^{v_a-1} \hat{p}(x) + (x - a)^{v_a} \hat{p}'(x) \), by Remarks 5 and 6, the integrand of the second term of (3.26) satisfies:

\[
(3.29) \quad \frac{|r(s, \epsilon, \eta)p'(x)|}{|p(x)|^2} \leq \left| \frac{r(s, \epsilon, \eta)v_a(s - a)^{v_a-1} \hat{p}(s)}{(s - a)^{2v_a} |\hat{p}(s)|^2} \right| + \left| \frac{r(s, \epsilon, \eta)(s - a)^{v_a} \hat{p}'(s)}{(s - a)^{2v_a} |\hat{p}(s)|^2} \right|
\]

\[
\leq m_1 \left| \frac{v_a}{(s - a)^{v_a+1}} \right| + m_1 \left| \frac{\hat{p}'(s)}{(s - a)^{v_a} |\hat{p}(s)|^2} \right|
\]

\[
\leq \frac{m_1 v_a}{g_2} \left| \frac{1}{|s - a|^{v_a+1}} \right| + \frac{m_1 g_3}{g_2^2} \left| \frac{1}{|s - a|^{v_a}} \right|
\]

**Case 1.** If \( v_a > 1 \), then for \( \epsilon \in S_{c_a}, \eta \in N_\delta \).

\[
(3.30) \quad \int_{a+\epsilon t_a}^t \frac{|r(s, \epsilon, \eta)p'(s)|}{|p(s)|^2} ds \leq \frac{m_1 v_a}{g_2} \int_{a+\epsilon t_a}^t \frac{1}{(s - a)^{v_a+1}} ds
\]

\[
+ \frac{m_1 g_3}{g_2^2} \int_{a+\epsilon t_a}^t \frac{1}{(s - a)^{v_a}} ds
\]

\[
\leq \frac{m_1 v_a}{g_2} \left( \frac{1}{\epsilon v_a t_a} - \frac{1}{(t - a)^{v_a}} \right)
\]

\[
+ \frac{m_1 g_3}{g_2^2} \left( \frac{1}{(v_a - 1)\epsilon(v_a-1)t_a} - \frac{1}{(v_a - 1)(t - a)^{v_a-1}} \right)
\]

\[
\leq \frac{m_1 v_a}{g_2} \epsilon^{-v_a t_a} + \frac{m_1 g_3}{g_2^2} \frac{c_{t_a}^l}{(v_a - 1)\epsilon^{-v_a t_a}}
\]

\[
\leq K_1 \epsilon^{-v_a t_a}.
\]

where \( K_1 = (m_1 v_a)/g_2 + (m_1 g_3/g_2^2)(c_{t_a}^l/(v_a - 1)) \) with \( c_a \) to be specified.
Case 2. If \( \nu_a = 1 \), then for \( \epsilon < \min \{ 1, (b - a)^{1/l_a}, (b - a)^{-1/l_a} \} \)

\[
\int_{a+\epsilon l_a}^{t} \left| \frac{r(s, \epsilon, \eta) p'(s)}{[p(s)]^2} \right| ds \leq \frac{m_1}{g_2} \int_{a+\epsilon l_a}^{t} \frac{ds}{(s-a)^2} + \frac{m_1 g_3}{g_2^2} \int_{a+\epsilon l_a}^{t} \frac{ds}{(s-a)}
\]

\[
\leq \frac{m_1}{g_2} \left( \frac{1}{\epsilon l_a} + \frac{1}{(t-a)} \right)
\]

\[
+ \frac{m_1 g_3}{g_2^2} \left( |\log(t-a)| - |\log \epsilon l_a| \right)
\]

(3.31)

\[
\leq \left( \frac{m_1}{g_2} + \frac{2m_1 g_3}{g_2^2} \epsilon l_a |\log \epsilon l_a| \right) \epsilon ^{-l_a}
\]

\[
\leq \left( \frac{m_1}{g_2} + \frac{2m_1 g_3}{g_2^2} \right) \epsilon ^{-l_a}
\]

\[
\leq K_2 \epsilon ^{-\nu_a l_a}.
\]

where \( K_2 = (m_1/g_2) + (2m_1 g_3/(eg_2^2)) \) and we used the fact that \( x \log x \geq -\frac{1}{e} \) or \( x|\log x| \leq \frac{1}{e} \) for \( 0 < x < 1 \). Thus. \( c_a \) can be chosen as

(3.32) \[ c_a = \min \{ c. 1. (b - a)^{1/l_a}, (b - a)^{-1/l_a} \} . \]

Let \( K_3 = \max \{ K_1, K_2 \} \). By (3.23), (3.25), (3.26), (3.27), (3.29), (3.30) and (3.31) we have, for \( \epsilon \in S_{c_a} \) and \( \eta \in N_\delta \).

\[
|J(a + \epsilon l_a, t, \alpha, \eta)| \leq \frac{2m_1}{g_2} \epsilon ^{1-\nu_a l_a} + \frac{m_2}{g_2} \epsilon ^{1-\nu_a l_a} + K_3 \epsilon ^{1-\nu_a l_a}
\]

(3.33)

\[
\leq K_a \epsilon ^{1-\nu_a l_a}.
\]

where \( K_a = (2m_1 + m_2)/g_2 + K_3 \).

If \( b = \infty \), we use (3.23)-(3.29) and let \( t \to \infty \). For an estimation of the
second term of (3.26). we have

\[
\left| \int_{a+\epsilon l_a}^{\infty} \frac{r(s, \epsilon, \eta) p'(s)}{[p(s)]^2} ds \right| \leq m_1 \left| \int_{a+\epsilon l_a}^{\infty} \frac{p'(s)}{[p(s)]^2} ds \right|
\]

(3.34)

\[
= m_1 \left| \left[ \frac{1}{p(s)} \right]_{a+\epsilon l_a}^{\infty} \right|
\]

\[
\leq \frac{m_1}{g_2} e^{-v_a l_a}
\]

for \( \epsilon \in S_{c_a}, \eta \in N_\delta, c_a = \min\{c, 1\} \). Here we used the fact that \( p(\infty) = \infty \) from (3.9).

Similarly, Lemma 4 is proved for both finite \( a \) and \( a = -\infty \).

**Proof of Lemma 3 and 5** First, assume that \( b \) is finite. Let \( l_a \) be a positive constant satisfying \( 0 < v_a l_a < 1 \).

**Case 1.** Assume that \( b \) is negative. By Remark 6. we have

(3.35) \[ |J(-\infty, b, \alpha, \epsilon, \eta)| \leq |J(-\infty, -\epsilon^{-l_a}, \alpha, \epsilon, \eta)| + |J(-\epsilon^{-l_a}, b, \beta, \epsilon, \eta)| \]

for \( -\infty < \alpha, \beta \leq b \).

Let

(3.36) \[ s = \frac{1}{w} \]

and

(3.37) \[ c_a = \min\{1, c, q^{-1/l_a}, (-b)^{-1/l_a}\} \]
Then, for the first term of right-hand side of (3.35), from an integral expression (3.1) with (3.13) we have

\begin{equation}
J(-\infty, -\epsilon^{-l_a}, \alpha, \varepsilon, \eta) = \int_{-\infty}^{-\epsilon^{-l_a}} \hat{r}(s, \varepsilon, \eta) s^{-(1+\rho)} \exp \left\{ i\epsilon^{-1} \int_{\alpha}^{s} p(u) \, du \right\} \, ds = \int_{0}^{-\epsilon^{-l_a}} \hat{r} \left( \frac{1}{w}, \varepsilon, \eta \right) w^{(1+\rho)} \exp \left\{ i\epsilon^{-1} \int_{\alpha}^{w} \frac{1}{t} \left( -\frac{dt}{t^2} \right) \right\} \left( -\frac{dw}{w^2} \right),
\end{equation}

where \(-\epsilon^{-l_a} < \alpha^{-1} < 0\). Thus, by (3.13), we have

\begin{equation}
|J(-\infty, -\epsilon^{-l_a}, \alpha, \varepsilon, \eta)| \leq m_3 \int_{0}^{-\epsilon^{-l_a}} w^{-1+\rho} \, dw \leq \frac{m_3}{\rho} \epsilon^{\rho l_a}
\end{equation}

for \(\epsilon \in S_{\alpha}, \eta \in N_\delta\).

For the second term of right-hand side of (3.35), from the integral expression (3.3) with (3.12) we have

\begin{equation}
\left| J(-\epsilon^{-l_a}, b, \beta, \varepsilon, \eta) \right| \leq \epsilon \left| \left[ r \left( \frac{1}{w}, \varepsilon, \eta \right) \frac{1}{w^{v^*} p \left( \frac{1}{w} \right)} \right]^{b^{-1}} \right| \left. + \epsilon \int_{b^{-1}}^{\epsilon^{-l_a}} \left. \left| \frac{r' \left( \frac{1}{w}, \varepsilon, \eta \right) p \left( \frac{1}{w} \right) - r \left( \frac{1}{w}, \varepsilon, \eta \right) p' \left( \frac{1}{w} \right) \right| \frac{dw}{[p \left( \frac{1}{w} \right)]^2} \right. \right. 
\end{equation}

where \(r' = dr/dw, p' = dp/dw\). By (3.5) and (3.10)

\begin{equation}
\left| \left[ r \left( \frac{1}{w}, \varepsilon, \eta \right) \frac{1}{w^{v^*} p \left( \frac{1}{w} \right)} \right]^{b^{-1}} \right| \leq \left| \frac{r(b, \varepsilon, \eta) b^{v^*}}{\hat{p}(b)} \right| + \left| \frac{r(-\epsilon^{-l_a}, \varepsilon, \eta)}{-\epsilon^{l_a} v^* \hat{p}(-\epsilon^{-l_a})} \right| 
\end{equation}

\begin{equation}
\leq \frac{m_1}{g_2} (-b)^{v^*} + \frac{m_1}{g_2} \epsilon^{-v^* l_a} \leq \frac{2m_1}{g_2} \epsilon^{-v^* l_a}.
\end{equation}
since \( \epsilon \leq (-b)^{-1/l} \) for \( \epsilon \in S_{ca} \). By (3.6) and (3.10)

\[
(3.42) \quad \int_{b^{-1}}^{-\epsilon^{l_a}} \left| \frac{r'(s, \epsilon, \eta)p\left(\frac{1}{w}\right)}{[p\left(\frac{1}{w}\right)]^2} \right| dw = \int_{b^{-1}}^{-\epsilon^{l_a}} \left| \frac{r'(s, \epsilon, \eta)}{w^{v_a} \tilde{p}\left(\frac{1}{w}\right)} \right| dw \leq \frac{m_2}{g_2} \epsilon^{-v_a l_a}.
\]

Similar to (3.30) there exists a positive constant \( K_3 \) such that

\[
(3.43) \quad \int_{b^{-1}}^{-\epsilon^{l_a}} \left| \frac{r(s, \epsilon, \eta)p\left(\frac{1}{w}\right)}{[p\left(\frac{1}{w}\right)]^2} \right| dw \leq K_3 \epsilon^{-v_a l_a}
\]

for \( \epsilon \in S_{ca} \), \( \eta \in N_\delta \). Thus, by (3.40) - (3.43), we have

\[
(3.44) \quad \left| J\left(-\epsilon^{-l_a}, b, \beta, \epsilon, \eta\right) \right| \leq K_4 \epsilon^{1-v_a l_a}
\]

for a suitable positive constant \( K_4 \), and for \( \epsilon \in S_{ca} \). Hence, by (3.39), (3.44) and (3.35),

\[
(3.45) \quad \left| J\left(-\infty, b, \alpha, \epsilon, \eta\right) \right| \leq K_5 \epsilon^{d_a}
\]

for a suitable positive constant \( K_5 \), and

\[
(3.46) \quad d_a = \min\{p l_a, 1 - v_a l_a\}.
\]

**Case 2.** Assume that \( b \) is non-negative. By Remark 6, we have

\[
(3.47) \quad \left| J\left(-\infty, b, \alpha, \epsilon, \eta\right) \right| \leq \left| J\left(-\infty, -\epsilon^{-l_a}, \alpha, \epsilon, \eta\right) \right| + \left| J\left(-\epsilon^{-l_a}, 0, \beta, \epsilon, \eta\right) \right| + \left| J\left(0, b, \gamma, \epsilon, \eta\right) \right|
\]

for \( -\infty < \alpha, \beta, \gamma \leq b \).

Note that (3.39) holds for \( \epsilon \in S_{ca}, \eta \in N_\delta \). (3.44) holds also for \( b = 0 \) and \( \epsilon \in S_{ca}, \eta \in N_\delta \). For the third term of (3.47), observe that \( p(s) \) is analytic.
nonzero for $0 \leq s \leq b$ so that $1/|p(s)|$ is bounded in that interval. So, from the expression (3.3) and by (3.4), (3.5) and (3.6)

\begin{equation}
|J(0, b, \gamma, \epsilon, \eta)| \leq \epsilon \left[ \frac{r(s, \epsilon, \eta)}{|p(s)|} \right]_0^b + \epsilon \int_0^b \frac{|r'(s, \epsilon, \eta)p(s) - r(s, \epsilon, \eta)p'(s)|}{|p(s)|^2} ds
\end{equation}

\leq K_6 \epsilon

for a suitable positive constant $K_6$, and for $\epsilon \in S_{c_{\alpha}} \cdot \eta \in N_\delta$. Therefore.

\begin{equation}
|J(-\infty, b, \alpha, \epsilon, \eta)| \leq K_7 \epsilon^a
\end{equation}

for a suitable positive constant $K_7$.

If $b = +\infty$, then by Remark 4. we can apply

\begin{equation}
|J(-\infty, \infty, \alpha, \epsilon, \eta)| \leq |J(-\infty, \hat{\alpha}, \alpha, \epsilon, \eta)| + |J(\hat{\alpha}, \hat{\beta}, \beta, \epsilon, \eta)| + |J(\hat{\beta}, \infty, \gamma, \epsilon, \eta)|
\end{equation}

for suitable finite constants $\alpha, \beta, \gamma, \hat{\alpha}$ and $\hat{\beta}$ with $\alpha < \hat{\beta}$.

The first term can be estimated as shown above. The estimates of the second and the last terms can be obtained by applying Lemma 2. or repeatedly using Lemma 2. if necessary. This proves Lemma 3.

Similarly, Lemma 5 is proved for both finite $a$ and $a = -\infty$.

**Proof of Lemma 6** For $p(x)$ in the form of (3.19) and $\epsilon \in S_{c_{ab}}, \eta \in N_\delta$ with $c_{ab}$ to be specified, by Lemma 4 and Lemma 2 there exist positive constants
$K_{1a}$ and $K_{1b}$ such that

$$
J(a + \epsilon^a, \frac{a + b}{2}, \alpha, \epsilon, \eta) \leq K_{1a} \epsilon^{1 - v_a l_a}.
$$

(3.51)

$$
J(\frac{a + b}{2}, b - \epsilon^b, \beta, \epsilon, \eta) \leq K_{1b} \epsilon^{1 - v_b l_b}
$$

for $\alpha, \beta \in [a, b]$. $0 < v_a l_a < 1.0 < v_b l_b < 1$. Since $\epsilon$ should satisfy $a + \epsilon^a \leq \frac{a + b}{2}$ and $\frac{a + b}{2} \leq b - \epsilon^b$. $c_{ab}$ can be chosen as

$$
c_{ab} = \min\left\{ 1, \left(\frac{b - a}{4}\right)^{1/l_a}, \left(\frac{b - a}{4}\right)^{1/l_b}, \left(\frac{b - a}{4}\right)^{-1/l_a}, \left(\frac{b - a}{4}\right)^{-1/l_b} \right\}.
$$

(3.52)

Then, by (3.8) there exists

$$
J(a + \epsilon^a, b - \epsilon^b, \alpha, \epsilon, \eta) \leq J(a + \epsilon^a, \frac{a + b}{2}, \alpha, \epsilon, \eta) + J(\frac{a + b}{2}, b - \epsilon^b, \beta, \epsilon, \eta)
$$

$$
\leq K_{1a} \epsilon^{1 - v_a l_a} + K_{1b} \epsilon^{1 - v_b l_b} \leq K_{ab} \epsilon^{d_{ab}}
$$

for $\alpha, \beta \in [a, b]$, where $K_{ab} = k_{1a} + k_{1b}$ and

$$
d_{ab} = \min\{1 - v_a l_a, 1 - v_b l_b\}.
$$

(3.54)

Proof of Lemma 7 First, assume that $a < \alpha_1$ and $\alpha_m < b$. Choose $l_1, l_2, \cdots, l_m$ such that $0 < v_j l_j < 1$ for $j = 1, 2, \cdots, m$. Let

$$
c_1 = \min\left\{ 1, \left(\frac{\alpha_1 - a}{2}\right)^{1/l_1}, \left(\frac{\alpha_1 - a}{2}\right)^{-1/l_1}, \left(\frac{\alpha_j + 1 - \alpha_j}{4}\right)^{1/l_j}, \left(\frac{\alpha_j + 1 - \alpha_j}{4}\right)^{-1/l_j}, \left(\frac{b - \alpha_m}{2}\right)^{1/l_m}, \left(\frac{b - \alpha_m}{2}\right)^{-1/l_m} \right\}, \quad 1 \leq j \leq m - 1.
$$

(3.55)

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For $\epsilon \in S_{c_1}$ let

$$I_1 = \bigcup_{j=1}^{m} [\alpha_j - \epsilon^{l_j}, \alpha_j + \epsilon^{l_j}].$$

(3.56)

$$I_2 = [a, b] - \text{Int}(I_1).$$

Also let $J_1$ be the sum of the integrals over each subinterval of $I_1$ and $J_2$ be the sum of those over each subinterval of $I_2$.

For the estimation of $J_1$. by using (3.5) we have

$$|J_1| \leq \int_{I_1} |r(s, \epsilon, \eta)| ds \leq 2m_1 \sum_{j=1}^{m} \epsilon^{l_j} \leq \hat{K}_1 \epsilon^{\hat{d}}$$

(3.57)

for suitable positive constants $\hat{K}_1$ and $\hat{d}$, and $\epsilon \in S_{c_1}, \eta \in N_\delta$.

Note that $J_2$ is the sum

$$J_2 = J(a, \alpha_1 - \epsilon^{l_1}, \beta_0, \epsilon, \eta) + \sum_{j=1}^{m-1} J(\alpha_j + \epsilon^{l_j}, \alpha_{j+1} - \epsilon^{l_{j+1}}, \beta_j, \epsilon, \eta)$$

$$+ J(\alpha_m + \epsilon^{l_m}, b, \beta_m, \epsilon, \eta)$$

(3.58)

for $a < \beta_0 < \alpha_1 - \epsilon^{l_1}, \alpha_j + \epsilon^{l_j} < \beta_j < \alpha_{j+1} - \epsilon^{l_{j+1}}$ ($j = 1, 2, \ldots, m - 1$), and $\alpha_m + \epsilon^{l_m} < \beta_m < b$. In order to apply Lemma 6, instead of using (3.10) we need

$$\hat{g}_j = \left( \inf_{\alpha_j \leq x \leq \alpha_{j+1}} \prod_{k \neq j, j+1}^\infty \frac{|x - \alpha_k|^{v_k}}{k^{j+1}} \right) g_2, \quad j = 1, 2, \ldots, m - 1.$$  

(3.59)

Let

$$\hat{d}_j = \min\{1 - v_j l_j, 1 - v_{j+1} l_{j+1}\}, \quad j = 1, 2, \ldots, m - 1.$$  

(3.60)
Then, by Lemma 6. for suitable $K_j$ we have

$$\left| J(\alpha_j + \epsilon_j, \alpha_{j+1} - \epsilon_{j+1}, \beta_j, \epsilon, \eta) \right| \leq K_j \epsilon^{d_j}, \quad j = 1, 2, \ldots, m - 1.$$  \hspace{2cm} (3.61)

for $\epsilon \in S_{c_1}$. For the estimation of $J(a, \alpha_1 - \epsilon^1, \beta_0, \epsilon, \eta)$ and $J(\alpha_m + \epsilon^m, b, \beta_m, \epsilon, \eta)$, we apply Lemma 4 and Lemma 2, respectively. Again, instead of using (3.10) we need

$$\tilde{g}_0 = \left( \inf_{a \leq x \leq \alpha_1} \prod_{j=1}^{m} |x - \alpha_j|^\nu_j \right) g_2,$$  \hspace{2cm} (3.62)

and

$$\tilde{g}_m = \left( \inf_{\alpha_m \leq x \leq b} \prod_{j=1}^{m-1} |x - \alpha_j|^\nu_j \right) g_2.$$  \hspace{2cm} (3.63)

Let

$$\tilde{d}_0 = 1 - \nu_1 l_1, \quad \tilde{d}_m = 1 - \nu_m l_m.$$  \hspace{2cm} (3.64)

Then, by Lemmas 4 and 2, for suitable $K_0$ and $K_m$ we have

$$\left| J(a, \alpha_1 - \epsilon^1, \beta_0, \epsilon, \eta) \right| \leq K_0 \epsilon^{\tilde{d}_0}, \quad \text{for } \epsilon \in S_{c_1},$$  \hspace{2cm} (3.65)

and

$$\left| J(\alpha_m + \epsilon^m, b, \beta_m, \epsilon, \eta) \right| \leq K_m \epsilon^{\tilde{d}_m}, \quad \text{for } \epsilon \in S_{c_1}.$$  \hspace{2cm} (3.66)

By combining above arguments, there exist positive constants $K_1$, $d_1$
such that

\[ |J(a, b, \alpha, \epsilon, \eta)| \leq |J_1| + |J_2| \leq \hat{K}_1 \epsilon^d + \sum_{j=0}^{m} \hat{K}_j \epsilon^{d_j} \leq \bar{K}_1 \epsilon^d \]

for \( \epsilon \in S_{\varepsilon_1}, \eta \in N_\delta. \)

For the case \( a = \alpha_1, \) Lemma 7 is proved by combining the above argument with Lemmas 2 and 3. Similarly, Lemma 7 is proved for the case \( \alpha_m = b \) by using Lemma 5. Thus, Lemma 7 is proved.

**Proof of Lemma 1** As Lemmas 2 through 7 cover all cases of \( p(x) \), including the case when \( p(x) \neq 0 \) (as in Lemma 2), thus Lemma 1 is proved.

3.3 Proof of the Main Theorem

Let

\[ \alpha_{jk}(x) = \lambda_j(x) - \lambda_k(x). \]

(3.68)

\[ \beta_{jk}(x, \epsilon, \eta) = r_{jj}(x, \epsilon, \eta) - r_{kk}(x, \epsilon, \eta). \]

and

(3.69)

\[ \Delta_{jk}(x, \epsilon, \eta) = d_j(x, \epsilon, \eta) - d_k(x, \epsilon, \eta). \]

Then, by (2.22) and the notation (3.68)

\[ \Delta_{jk}(x, \epsilon, \eta) = \lambda_j(x) - \lambda_k(x) + \epsilon(r_{jj}(x, \epsilon, \eta) - r_{kk}(x, \epsilon, \eta)) \]

(3.70)

\[ = \alpha_{jk}(x) + \epsilon \beta_{jk}(x, \epsilon, \eta). \]
Let

\[ J_{jk}(a, b, \alpha, \epsilon, \eta) = \int_{a}^{b} r_{jk}(s, \epsilon, \eta) \exp \left\{ i\epsilon^{-1} \int_{\alpha}^{s} \Delta_{jk}(u, \epsilon, \eta) du \right\} ds \]

(3.71)

\[ = \int_{a}^{b} r_{jk}(s, \epsilon, \eta) \exp \left\{ i\epsilon^{-1} \int_{\alpha}^{s} \left( \alpha_{jk}(u) + \epsilon \beta_{jk}(u, \epsilon, \eta) \right) du \right\} ds \]

\[ = \int_{a}^{b} \tilde{r}_{jk}(s, \alpha, \epsilon, \eta) \exp \left\{ i\epsilon^{-1} \int_{\alpha}^{s} \alpha_{jk}(u) du \right\} ds \]

where

(3.72) \[ \tilde{r}_{jk}(x, \alpha, \epsilon, \eta) = r_{jk}(x, \epsilon, \eta) \exp \left\{ i \int_{\alpha}^{x} \beta_{jk}(u, \epsilon, \eta) du \right\} . \]

By the fact that eigenvalues \( \lambda_j(x) \) are real and analytic on \( I \) and the assumption of (A2), the functions \( \alpha_{jk}(x) \) are real and analytic on \( I \) and satisfy the first condition of Lemma 1. In order to show that \( \tilde{r}_{jk}(x, \alpha, \epsilon, \eta) \) satisfies conditions (b) and (c) of Lemma 1, note that the unitary matrix \( U(x) \) in (2.6) is analytic on \( I \). By (2.21) and Assumptions (A2)-(A5), the elements \( r_{jk}(x, \epsilon, \eta) \) of \( R_0(x, \epsilon, \eta) \) given by (2.6) and their derivatives are uniformly bounded and absolutely integrable over \( I \) for \( \epsilon \in S_c, \eta \in N_\delta \). Thus, \( \tilde{r}_{jk}(x, \alpha, \epsilon, \eta) \) satisfy conditions (b) and (c) of Lemma 1. Therefore, for each pair \( (j, k) \), \( (j, k = 1, 2, \ldots, n) \), there exists a function \( G_{jk}(\epsilon) \) and \( \epsilon_{jk} \) \( (0 < \epsilon_{jk} \leq c) \), such that

(3.73)

\[ |J_{jk}(a, t, \alpha, \epsilon, \eta)| \leq G_{jk}(\epsilon) = O(\epsilon^{d_{jk}}) \quad \text{for} \ t \in I, \ \eta \in N_\delta, \ \text{and} \ 0 < \epsilon < \epsilon_{jk} \]

where \( d_{jk} \) are positive constants given by the results of Lemma 1.
Define a matrix norm $\| A \|$ as

$$\tag{3.74} \| A \| = \max_{j,k} \{|a_{jk}|\}. \quad (3.74)$$

where $a_{jk}$ is the $(j,k)$ element of $A$. and let

$$\tag{3.75} \| | P | | = \sup_{x \in I, \eta \in N_\delta} \| P(x, \epsilon, \eta) \|. \quad (3.75)$$

Then, with the expression (2.28) of $A_{jk}$

$$|A_{jk}| \leq \left| \int_{\alpha}^{x} \sum_{l=1}^{n} \left[ \sum_{h=1}^{n} J_{jh}(a, t, a, t, \epsilon, \eta) \right] r_{hl}(t, \epsilon, \eta) \exp \left\{ i \epsilon^{-1} \int_{\alpha}^{t} \Delta_{hl}(u, \epsilon, \eta) \, du \right\} p_{lk}(t, \epsilon, \eta) \, dt \right| \quad (3.76)$$

for $x \in I$, $0 < \epsilon < \epsilon_{jk}$, $\eta \in N_\delta$ where $\hat{G}_{hl}$ are suitable positive constants.

Here $J_{jk}(a, t, a, t, \alpha, \epsilon, \eta)$ are functions in the form of (3.1) and $G_{jk}(\epsilon)$ are their estimates obtained by Lemma 1. Moreover $G_{jk}(\epsilon)$ ($j, k = 1, 2, \ldots, n$) tend to $0$ as $\epsilon \to 0^+$.

Now, choose $c_1$ sufficiently small such that

$$\tag{3.77} c_1 \leq \min \{ \epsilon_{jk} \mid j, k = 1, 2, \ldots, n \}. \quad (3.77)$$
For $\epsilon < c_1$, let

$$
L(\epsilon) = \max_{1 \leq j \leq n} \left\{ \sum_{l,h=1}^{n} G_{jh}(\epsilon) \hat{G}_{hl} \right\}.
$$

Then, we have

$$
|||L^2 P||| \leq ||L^2 P|| = \max_{1 \leq j,k \leq n} \{|A_{jk}|\}
$$

(3.79)

\[ \leq L(\epsilon)|||P|||. \]

where $L(\epsilon)$ tends to 0 as $\epsilon \to 0^+$. 

Now choose $c_2$ such that $0 < L(\epsilon) < 1$ for $\epsilon \in S_{c_2}$. Then, we can define a sequence of successive approximations of the solutions of (2.18) as follows:

$$
P_0(x, \epsilon, \eta) = LI_n + L^2 I_n
$$

(3.80)

\[
P_{m+1}(x, \epsilon, \eta) = P_0(x, \epsilon, \eta) + L^2 P_m. \quad m = 0, 1, 2, \ldots.
\]

Since

$$
\lim_{m \to \infty} P_m(x, \epsilon, \eta) = P_0(x, \epsilon, \eta) + \sum_{j=1}^{\infty} L^{2j} P_0(x, \epsilon, \eta)
$$

(3.81)

and

$$
||| \lim_{m \to \infty} P_m(x, \epsilon, \eta)||| \leq \{1 + \sum_{j=1}^{\infty} L(\epsilon)^j \} |||P_0|||
$$

(3.82)

\[ = \frac{1}{1 - L(\epsilon)} |||P_0|||. \]

Thus, \{\(P_m(x, \epsilon, \eta)\)|\(m = 0, 1, 2, \ldots\} is a sequence convergent uniformly in \(I \times S_\hat{c} \times N_\delta\). Denote

$$
\hat{P}(x, \epsilon, \eta) = \lim_{m \to \infty} P_m(x, \epsilon, \eta) = P_0(x, \epsilon, \eta) + \sum_{j=1}^{\infty} L^{2j} P_0(x, \epsilon, \eta).
$$

(3.83)
Then, \( \dot{P}(x, \epsilon, \eta) \) satisfies (2.18) since

\[
(3.84) \quad P_0 + L^2 \dot{P} = P_0 + L^2(P_0 + \sum_{j=1}^{\infty} L^{2j} P_0) = P_0 + \sum_{k=1}^{\infty} L^{2k} P_0 = \dot{P}.
\]

Therefore, a solution \( \dot{P}(x, \epsilon, \eta) \) of (2.18) exists, and which is in class \( C^1(I \times S_\varepsilon \times N_\delta) \) where \( \varepsilon = \min\{\epsilon_1, \epsilon_2\} \). Furthermore, by (3.73), let

\[
(3.85) \quad \dot{L}(\epsilon) = \|[G_{jk}(\epsilon)]_{j,k=1}^{n}||.
\]

Then, we have

\[
(3.86) \quad \|[L_1]| \leq \dot{L}(\epsilon) = O(\epsilon^d). \quad \text{in } I \times S_\varepsilon \times N_\delta.
\]

where \( d = \min_{1 \leq j, k \leq n} \{d_{jk}\} \). Similarly, \( L(\epsilon) \) in (3.78) satisfies

\[
(3.87) \quad \dot{L}(\epsilon) = O(\epsilon^d). \quad \text{in } S_\varepsilon.
\]

Thus,

\[
(3.88) \quad \|[P_0(x, \epsilon, \eta)|| = O(\epsilon^d). \quad \text{in } I \times S_\varepsilon \times N_\delta.
\]

By (3.82), we have

\[
(3.89) \quad \|[P(x, \epsilon, \eta)|| = O(\epsilon^d). \quad \text{in } I \times S_\varepsilon \times N_\delta.
\]

Moreover, since \( P_0(\alpha, \epsilon, \eta) = 0 \). \( P_m(\alpha, \epsilon, \eta) = 0 \) for \( m = 1, 2, 3, \ldots \).

Hence, \( \dot{P}(\alpha, \epsilon, \eta) = 0 \). Thus, Theorem 1 is proved.
CHAPTER IV

COMPUTATION

4.1 Introduction

In this chapter, we will discuss how to calculate the matrix $P(x, \epsilon, \eta)$ by two methods depending on dimension $n$. In Section 4.2, $P(x, \epsilon, \eta)$ will be calculated in the form of decoupled integral equations for $n = 2$. In Section 4.3, the successive approximations method will be used to find approximations for $P(x, \epsilon, \eta)$ of any dimension $n$. The error estimate formula for such approximations is given also in Section 4.3. In Section 4.4, two examples will be given for $n = 2$ by direct calculation and for $n = 3$ by the successive approximations method.

4.2 Direct Calculation for $n = 2$

For a two dimensional system (2.1), by the preliminary reductions in Section 2.3, the matrix $Y$ satisfies the differential equation (2.9):

$$i\epsilon Y' = [D(x, \epsilon, \eta) + \epsilon R(x, \epsilon, \eta)] Y \quad (4.1)$$

where $D(x, \epsilon, \eta)$ is a square diagonal matrix and $R(x, \epsilon, \eta)$ is a square off-diagonal matrix and they are given by (2.7) and (2.8).
From Theorem 1, the fundamental matrix of (4.1) can be expressed as:

\[(4.2)\quad Y = Z(x,\alpha,\epsilon,\eta)(I_2 + P(x,\epsilon,\eta)).\]

where \(Z(x,\alpha,\epsilon,\eta)\) is a fundamental matrix of

\[(4.3)\quad i\epsilon Z' = D(x,\epsilon,\eta)Z, \quad Z(\alpha,\alpha,\epsilon,\eta) = I_2.\]

with \(I_2\) a 2 \times 2 identity matrix and the 2 \times 2 matrix \(P(x,\epsilon,\eta)\) tends to 0 as \(\epsilon \to 0^+\).

From the discussion in Section 2.3, \(P\) satisfies the following integral equation:

\[(4.4)\quad P = LI_2 + L^2I_2 + L^2P.\]

where

\[(4.5)\quad L^2P = -\int_{\alpha}^{\infty}\left\{\int_{t}^{x} Z^{-1}(s,\alpha,\epsilon,\eta)R(s,\epsilon,\eta)Z(s,\alpha,\epsilon,\eta)ds\right\} \cdot Z^{-1}(t,\alpha,\epsilon,\eta)R(t,\epsilon,\eta)Z(t,\alpha,\epsilon,\eta)P(t,\epsilon,\eta)dt.\]

Put

\[(4.6)\quad Z = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},\]

and

\[(4.7)\quad R = \begin{bmatrix} 0 & r_{12} \\ r_{21} & 0 \end{bmatrix}.\]
By the notations (4.6) and (4.7), we have

\[ Z^{-1} R Z = \begin{bmatrix} 0 & z_1^{-1} r_{12} z_2 \\ z_2^{-1} r_{21} z_1 & 0 \end{bmatrix}. \tag{4.8} \]

\[ Z^{-1} R Z P = \begin{bmatrix} z_1^{-1} r_{12} z_2 p_{21} & z_1^{-1} r_{12} z_2 p_{22} \\ z_2^{-1} r_{21} z_1 p_{11} & z_2^{-1} r_{21} z_1 p_{12} \end{bmatrix}. \]

By (4.5), we have the following decoupled equations for entries of \( P \):

\[ p_{11}(x, \epsilon, \eta) = -\int_{\alpha}^{x} \left\{ \int_{t}^{x} z_1^{-1}(s, \alpha, \epsilon, \eta) r_{12}(s, \epsilon, \eta) z_2(s, \alpha, \epsilon, \eta) \, ds \right\} z_2^{-1}(t, \alpha, \epsilon, \eta) r_{21}(t, \epsilon, \eta) z_1(t, \alpha, \epsilon, \eta) \, dt \]

\[ -\int_{\alpha}^{x} \left\{ \int_{t}^{x} z_1^{-1}(s, \alpha, \epsilon, \eta) r_{12}(s, \epsilon, \eta) z_2(s, \alpha, \epsilon, \eta) \, ds \right\} z_2^{-1}(t, \alpha, \epsilon, \eta) r_{21}(t, \epsilon, \eta) z_1(t, \alpha, \epsilon, \eta) p_{11}(t, \epsilon, \eta) \, dt \]

\[ p_{12}(x, \epsilon, \eta) = -i \int_{\alpha}^{x} \left\{ \int_{t}^{x} z_1^{-1}(s, \alpha, \epsilon, \eta) r_{12}(s, \epsilon, \eta) z_2(s, \alpha, \epsilon, \eta) \, ds \right\} \]

\[ z_2^{-1}(t, \alpha, \epsilon, \eta) r_{21}(t, \epsilon, \eta) z_1(t, \alpha, \epsilon, \eta) p_{12}(t, \epsilon, \eta) \, dt \]

\[ p_{21}(x, \epsilon, \eta) = -i \int_{\alpha}^{x} \left\{ \int_{t}^{x} z_2^{-1}(s, \alpha, \epsilon, \eta) r_{21}(s, \epsilon, \eta) z_2(s, \alpha, \epsilon, \eta) \, ds \right\} z_1^{-1}(t, \alpha, \epsilon, \eta) r_{12}(t, \epsilon, \eta) z_2(t, \alpha, \epsilon, \eta) p_{21}(t, \epsilon, \eta) \, dt \]

\[ p_{22}(x, \epsilon, \eta) = -\int_{\alpha}^{x} \left\{ \int_{t}^{x} z_2^{-1}(s, \alpha, \epsilon, \eta) r_{21}(s, \epsilon, \eta) z_2(s, \alpha, \epsilon, \eta) \, ds \right\} \]

\[ z_1^{-1}(t, \alpha, \epsilon, \eta) r_{12}(t, \epsilon, \eta) z_2(t, \alpha, \epsilon, \eta) p_{22}(t, \epsilon, \eta) \, dt \]
By Theorem 1, the solutions of these integral equations (4.9) - (4.12),

\[ p_{jk}(x, \epsilon, \eta). \ (j, k = 1, 2) \]

exist and are in the class \( C^1(I \times S_\delta \times N_{\delta}). \ \)\( (0 < \delta \leq c) \).

\[ \| p_{jk}(x, \epsilon, \eta) \| \leq \epsilon^d \]

with \( d > 0 \) uniformly on \( I \times N_\delta \) as \( \epsilon \to 0^+ \). When these integrals cannot be found explicitly, suitable numerical methods can be used to find the approximations of solutions.

### 4.3 Successive Approximations Method

In order to find the global solution (2.10) for a system (2.1) with dimension \( n \), note that from (2.18), \( P \) satisfies the integral equation:

\[ P = LI_n + L^2I_n + L^2P \tag{4.13} \]

with \( L \) and \( L^2 \) given by (2.16) and (2.20), respectively, and we have

\[ |||L^2P||| \leq L(\epsilon)|||P|||. \quad L(\epsilon) < 1 \tag{4.14} \]

for \((x, \epsilon, \eta)\) in \((I \times S_\delta \times N_\delta)\). \( (0 < \delta \leq c) \). A sequence of successive approximations of the solution of (4.13) is defined as (3.80). Then, \( P_m(x, \epsilon, \eta) \) converges uniformly to \( P(x, \epsilon, \eta) \) for \((x, \epsilon, \eta)\) in \( I \times S_\delta \times N_\delta \). When \( P(x, \epsilon, \eta) \) is approximated by \( P_m(x, \epsilon, \eta) \), since

\[ P(x, \epsilon, \eta) - P_m(x, \epsilon, \eta) = L^{m+2}(I_n + LI_n) + L^{2m+4}(I_n + LI_n) + \cdots \tag{4.15} \]

we have the error estimate:

\[ |||P(x, \epsilon, \eta) - P_m(x, \epsilon, \eta)||| \leq \{L(\epsilon)^{m+2} + L(\epsilon)^{m+4} + \cdots\}|||I_n + LI_n||| \]

\[ = \frac{L(\epsilon)^{m+2}}{1 - L(\epsilon)}|||I_n + LI_n|||. \tag{4.16} \]
4.4 Examples

**Example 1.** Consider the following two-dimensional system:

\[
(4.17) \quad i\epsilon V' = \left( \frac{1}{2} \begin{bmatrix} x + x^3 & x - x^3 \\ x - x^3 & x + x^3 \end{bmatrix} + \epsilon \begin{bmatrix} 2 & 0 \\ 0 & 2\eta x \end{bmatrix} \right) V
\]

for \( x \in I, I = [-2, 2], \epsilon \in (0, c] \) and \( \eta \in (-\delta, \delta) \) with positive constants \( c \) and \( \delta \).

Let

\[
H_0(x) = \frac{1}{2} \begin{bmatrix} x + x^3 & x - x^3 \\ x - x^3 & x + x^3 \end{bmatrix}
\]

which is a Hermitian matrix and let

\[
H_1(x, \epsilon, \eta) = \begin{bmatrix} 2 & 0 \\ 0 & 2\eta x \end{bmatrix}
\]

which satisfies Assumptions (A2)-(A5) in Section 2.2. Denote the eigenvalues of \( H_0(x) \) as:

\[
\lambda_1(x) = x, \quad \lambda_2(x) = x^3.
\]

and

\[
D_0(x) = \begin{bmatrix} x & 0 \\ 0 & x^3 \end{bmatrix}.
\]

Then,

\[
\lambda_1(x) - \lambda_2(x) = x(x + 1)(x - 1).
\]

Thus, \( x = -1, x = 0 \) and \( x = 1 \) are turning points of order 1. Note that \( U^{-1}H_0(x)U = D_0(x) \) by the unitary matrix

\[
U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]
and that

\[ R_0(x, \epsilon, \eta) = U^{-1}H_1(x, \epsilon, \eta)U \]

\[ = \begin{bmatrix} 1 + \eta x & -1 + \eta x \\ -1 + \eta x & 1 + \eta x \end{bmatrix}. \]

Thus, by the transformation \( Y = U^{-1}V \), (4.17) can be expressed as:

\[ i\epsilon Y' = \begin{bmatrix} x & 0 \\ 0 & x^3 \end{bmatrix} + \epsilon \begin{bmatrix} 1 + \eta x & -1 + \eta x \\ -1 + \eta x & 1 + \eta x \end{bmatrix}Y. \]

or

\[ i\epsilon Y' = \begin{bmatrix} x + \epsilon(1 + \eta x) & 0 \\ 0 & x^3 + \epsilon(1 + \eta x) \end{bmatrix} + \epsilon \begin{bmatrix} 0 & -1 + \eta x \\ -1 + \eta x & 0 \end{bmatrix}Y. \]

Let

\[ Z = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \]

be a fundamental matrix of

\[ i\epsilon Z' = \begin{bmatrix} x + \epsilon(1 + \eta x) & 0 \\ 0 & x^3 + \epsilon(1 + \eta x) \end{bmatrix} Z. \quad Z(x, \alpha, \epsilon, \eta) = I_2. \]

Here, we assume that \( \alpha = -2 \). Then,

\[ z_1 = \exp \left\{ -i\epsilon^{-1} \int_{-2}^{x} [t + \epsilon(1 + \eta t)] \, dt \right\} = \exp \left\{ -i\epsilon^{-1} \left[ \frac{1}{2}(x^2 - 4) + \epsilon(x + 2) + \frac{1}{2}\epsilon\eta(x^2 - 4) \right] \right\}. \]

\[ z_2 = \exp \left\{ -i\epsilon^{-1} \int_{-2}^{t} (t^3 + \epsilon(1 + \eta t)) \, dt \right\} = \exp \left\{ -i\epsilon^{-1} \left[ \frac{1}{4}(x^4 - 16) + \epsilon(x + 2) + \frac{1}{2}\epsilon\eta(x^2 - 4) \right] \right\}. \]
By Theorem 1, (4.19) has a fundamental solution

\[ Y(x, \epsilon, \eta) = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} (I_2 + P(x, \epsilon, \eta)) \]

with

\[ \| P(x, \epsilon, \eta) \| \leq K \epsilon^d \]

for suitable positive constant \( K, d \). Therefore, (4.17) has a fundamental matrix solution

\[ V(x, \epsilon, \eta) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} (I_2 + P(x, \epsilon, \eta)). \]

Put

\[ P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}. \]

and

\[ R = \begin{bmatrix} 0 & r_{12} \\ r_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 + \eta x \\ -1 + \eta x & 0 \end{bmatrix}. \]

Then, by (4.9),

\[ Z^{-1}RZ = \begin{bmatrix} 0 & z_1^{-1}r_{12}z_2 \\ z_2^{-1}r_{21}z_1 & 0 \end{bmatrix}. \]

where

\[ z_1^{-1}r_{12}z_2 = (-1 + \eta x) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} x^4 - \frac{1}{2} x^2 - 2 \right) \right\}. \]

\[ z_2^{-1}r_{21}z_1 = (-1 + \eta x) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} x^4 - \frac{1}{2} x^2 - 2 \right) \right\}. \]

By (4.4) or (4.9) - (4.12), \( p_{jk}, (j, k = 1, 2) \) satisfy following decoupled integral equations:
\[ p_{11}(x, \epsilon, \eta) = - \int_{-2}^{x} \left\{ \int_{t}^{x} (-1 + \eta s) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \right\} \]
\[ \cdot (-1 + \eta t) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} t^4 - \frac{1}{2} t^2 - 2 \right) \right\} dt \]
\[ - \int_{-2}^{x} \left\{ \int_{t}^{x} (-1 + \eta s) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \right\} \]
\[ \cdot (-1 + \eta t) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} t^4 - \frac{1}{2} t^2 - 2 \right) \right\} p_{11}(t, \epsilon, \eta) dt. \]

\[ p_{12}(x, \epsilon, \eta) = -i \int_{-2}^{x} (-1 + \eta s) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \]
\[ - \int_{-2}^{x} \left\{ \int_{t}^{x} (-1 + \eta s) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \right\} \]
\[ \cdot (-1 + \eta t) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} t^4 - \frac{1}{2} t^2 - 2 \right) \right\} p_{12}(t, \epsilon, \eta) dt. \]

\[ p_{21}(x, \epsilon, \eta) = -i \int_{-2}^{x} (-1 + \eta s) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \]
\[ - \int_{-2}^{x} \left\{ \int_{t}^{x} (-1 + \eta s) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \right\} \]
\[ \cdot (-1 + \eta t) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} t^4 - \frac{1}{2} t^2 - 2 \right) \right\} p_{21}(t, \epsilon, \eta) dt. \]

\[ p_{22}(x, \epsilon, \eta) = - \int_{-2}^{x} \left\{ \int_{t}^{x} (-1 + \eta s) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \right\} \]
\[ \cdot (-1 + \eta t) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} t^4 - \frac{1}{2} t^2 - 2 \right) \right\} dt \]
\[ - \int_{-2}^{x} \left\{ \int_{t}^{x} (-1 + \eta s) \exp \left\{ i \epsilon^{-1} \left( \frac{1}{4} s^4 - \frac{1}{2} s^2 - 2 \right) \right\} ds \right\} \]
\[ \cdot (-1 + \eta t) \exp \left\{ -i \epsilon^{-1} \left( \frac{1}{4} t^4 - \frac{1}{2} t^2 - 2 \right) \right\} p_{22}(t, \epsilon, \eta) dt. \]

The existence of solutions \( p_{jk}(x, \epsilon, \eta), \ (j, k = 1, 2) \), is assured by Theorem 1. As explicit forms of these solutions are not always obtainable, numerical approximations are desirable.
**Example 2.** Consider the following three-dimensional system:

\[
    i\epsilon V' = \left( \begin{array}{ccc}
    (x - 1)^3 & 0 & 0 \\
    0 & (x - 1)(2x - 3) & 0 \\
    0 & 0 & 3(x - 1)(2x - 5) \\
    \end{array} \right) + \epsilon \left( \begin{array}{ccc}
    0 & 0 & x \\
    x & 0 & \eta x^2 \\
    0 & \epsilon \eta x & 0 \\
    \end{array} \right) V
\]

(4.20)

for \( x \in I \), \( I = [0.5] \), \( \epsilon \in (0, c] \) and \( \eta \in (-\delta, \delta) \) with positive constants \( c \) and \( \delta \).

Let

\[
    R(x, \epsilon, \eta) = \left( \begin{array}{ccc}
    0 & 0 & x \\
    x & 0 & \eta x^2 \\
    0 & \epsilon \eta x & 0 \\
    \end{array} \right).
\]

Denote

\[
    \lambda_1(x) = (x - 1)^3, \quad \lambda_2(x) = (x - 1)(2x - 3),
\]

\[
    \lambda_3(x) = 3(x - 1)(2x - 5).
\]

and

\[
    D(x) = \text{diag}\{\lambda_1(x), \lambda_2(x), \lambda_3(x)\}.
\]

Then.

\[
    \lambda_1(x) - \lambda_2(x) = (x - 1)(x - 2)^2.
\]

\[
    \lambda_1(x) - \lambda_3(x) = (x - 1)(x - 4)^2.
\]

\[
    \lambda_2(x) - \lambda_3(x) = -4(x - 1)(x - 3).
\]

Thus, \( x = 1 \), \( x = 2 \), \( x = 3 \) and \( x = 4 \) are turning points of order 1, 2, 1 and 2, respectively.

Let

\[
    Z = \left[ \begin{array}{ccc}
    z_1 & 0 & 0 \\
    0 & z_2 & 0 \\
    0 & 0 & z_3 \\
    \end{array} \right]
\]
be a fundamental matrix of

\[ ieZ' = D(x)Z, \quad Z(x, 0, \epsilon, \eta) = I_3. \]

Then,

\[
\begin{align*}
 z_1 &= \exp\left\{-i\epsilon^{-1} \int_0^x (t-1)^3 \, dt \right\} = \exp\left\{-i\epsilon^{-1}(0.25x^4 - x^3 + 1.5x^2 - x) \right\}. \\
 z_2 &= \exp\left\{-i\epsilon^{-1} \int_0^t (t-1)(2t - 3) \, dt \right\} = \exp\left\{-i\epsilon^{-1}(1.5x^4 - 2.5x^2 + 3x) \right\}. \\
 z_3 &= \exp\left\{-i\epsilon^{-1} \int_0^x 3(t-1)(2t - 5) \, dt \right\} = \exp\left\{-i\epsilon^{-1}(2x^3 - 10.5x^2 + 15x) \right\}.
\end{align*}
\]

By Theorem 1, (4.20) has a fundamental solution

\[
V(x, \epsilon, \eta) = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{bmatrix} (I_3 + P(x, \epsilon, \eta))
\]

with

\[ \| P(x, \epsilon, \eta) \| \leq K\epsilon^d \]

for suitable positive constant \( K, d. \)

Since

\[
(4.21) \quad Z^{-1}RZ = \begin{bmatrix} 0 & 0 & xz_1^{-1}z_3 \\ xz_2^{-1}z_1 & 0 & \eta x^2 z_2^{-1}z_3 \\ 0 & \epsilon \eta xz_3^{-1}z_2 & 0 \end{bmatrix}.
\]

the first two terms \( P_0(x, \epsilon, \eta) \) and \( P_1(x, \epsilon, \eta) \) of successive approximations given in (4.15) can be expressed as:

\[
P_0(x, \epsilon, \eta) = I_3 + L I_3
\]

\[
= I_3 + \begin{bmatrix} 0 & 0 & -i \int_0^x t z_1^{-1}z_3 \, dt \\ -i \int_0^x t z_2^{-1}z_1 \, dt & 0 & -i \int_0^x \eta t^2 z_2^{-1}z_3 \, dt \\ 0 & -i \int_0^x \epsilon \eta t z_3^{-1}z_2 \, dt & 0 \end{bmatrix}.
\]
\[ P_1(x, \epsilon, \eta) = I_3 + LI_3 + L^2 I_3 + L^3 I_3 \]
\[ = I_3 + \begin{bmatrix}
0 & 0 & -i \int_0^x t z_1^{-1} z_3 dt \\
-i \int_0^x t z_2^{-1} z_1 dt & 0 & -i \int_0^x \eta t z_2^{-1} z_3 dt \\
0 & -i \int_0^x \epsilon t z_3^{-1} z_2 dt & 0
\end{bmatrix} + \begin{bmatrix}
w_{11} & 0 & w_{13} \\
w_{21} & 0 & w_{23} \\
w_{31} & 0 & w_{33}
\end{bmatrix}.\]

where

\[ v_{22} = -\int_0^x \eta s^2 z_2^{-1} z_3 \left\{ \int_0^s \epsilon t z_3^{-1} z_2 dt \right\} ds. \]
\[ v_{23} = -\int_0^x s z_2^{-1} z_1 \left\{ \int_0^s t z_1^{-1} z_3 dt \right\} ds. \]
\[ v_{31} = -\int_0^x \epsilon s z_3^{-1} z_2 \left\{ \int_0^s t z_2^{-1} z_1 dt \right\} ds. \]
\[ v_{33} = -\int_0^x \epsilon s z_3^{-1} z_2 \left\{ \int_0^s \eta t^2 z_2^{-1} z_3 dt \right\} ds. \]

and

\[ w_{11} = i \int_0^x s z_1^{-1} z_3 \left\{ \int_0^s \epsilon t z_3^{-1} z_2 \left\{ \int_0^t u z_2^{-1} z_1 du \right\} dt \right\} ds. \]
\[ w_{13} = i \int_0^x s z_1^{-1} z_3 \left\{ \int_0^s \epsilon t z_3^{-1} z_2 \left\{ \int_0^t \eta u^2 z_2^{-1} z_3 du \right\} dt \right\} ds. \]
\[ w_{21} = i \int_0^x \eta s^2 z_2^{-1} z_3 \left\{ \int_0^s \epsilon t z_3^{-1} z_2 \left\{ \int_0^t u z_2^{-1} z_1 du \right\} dt \right\} ds. \]
\[ w_{23} = i \int_0^x \eta s^2 z_2^{-1} z_3 \left\{ \int_0^s \epsilon t z_3^{-1} z_2 \left\{ \int_0^t \eta u^2 z_2^{-1} z_3 du \right\} dt \right\} ds. \]
\[ w_{32} = i \int_0^x \epsilon s z_3^{-1} z_2 \left\{ \int_0^s \eta t^2 z_2^{-1} z_3 \left\{ \int_0^t \epsilon u z_3^{-1} z_2 du \right\} dt \right\} ds. \]
\[ w_{33} = i \int_0^x \epsilon s z_3^{-1} z_2 \left\{ \int_0^s \epsilon t z_3^{-1} z_2 \left\{ \int_0^t u z_2^{-1} z_1 du \right\} dt \right\} ds. \]

In order to estimate the error \[|||P(x, \epsilon, \eta) - P_1(x, \epsilon, \eta)|||\]. note that \(Z^{-1} R Z\).
in (4.21) has non-zero entries at (1, 3), (2, 1), (2, 3) and (3, 2) positions. This will be carried out for a fixed, but positive small $\delta$. First let

\[(4.22) \quad J_{13}(0.5, 0, \eta) = \int_0^\infty s \exp\left\{i \epsilon^{-1} \int_0^s (t - 1)(t - 4)^2 \, dt\right\} ds\]

with turning points $\alpha_1 = 1$ and $\alpha_2 = 4$ of $J_{13}$ and their orders $v_1 = 1$ and $v_2 = 2$, respectively. We will apply Lemma 7 to estimate $J_{13}$. Choose $l_1 = \frac{1}{2}$ and $l_2 = \frac{1}{3}$ so that $0 < v_j l_j < 1$ for $j = 1, 2$. By (3.55), $c_1$ can be chosen as

\[c_1 = \min\left\{1, \left(\frac{1}{2}\right)^2, 2^2, \left(\frac{3}{4}\right)^2, \left(\frac{4}{3}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{4}{3}\right)^3, \left(\frac{1}{2}\right)^3, 2^3\right\} = \frac{1}{8}.\]

For this integral, let

\[I_1 = [1 - \epsilon^{\frac{1}{2}} \cdot 1 + \epsilon^{\frac{1}{2}}] \cup [4 - \epsilon^{\frac{1}{2}} \cdot 4 + \epsilon^{\frac{1}{2}}].\]

\[I_2 = [0.5] - \text{Int}(I_1).\]

Then, for $J_{13}^{(13)}$, the integral of $J_{13}$ over $I_1$, we have

\[(4.23) \quad |J_{13}^{(13)}| \leq \int_{I_1} s \, ds = 2\epsilon^{\frac{1}{2}} + 8\epsilon^{\frac{1}{3}} \leq 10\epsilon^{\frac{1}{3}}\]

for $\epsilon \in S_{c_1}$ and $\eta \in N_{\delta}$. Here, the integration was evaluated directly for better estimation, rather than using (3.5) or (3.57).

For the estimation of $J_{13}^{(13)}$, the integral of $J_{13}$ over $I_2$, let

\[\hat{p}_0(x) = (x - 4)^2, \quad \hat{p}_1(x) = 1, \quad \hat{p}_2(x) = x - 1.\]

Then, by (3.59), (3.62), (3.63), and the fact that $g_2 = 1$.

\[\hat{g}_0 = \inf_{0 \leq x \leq 1} |\hat{p}_0(x)| = 9, \quad \hat{g}_1 = 1, \quad \hat{g}_2 = \inf_{4 \leq x \leq 5} |\hat{p}_2(x)| = 3.\]
By (3.60) or (3.64), let
\[ d_1 = \min\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{3}. \]

Observe that by (3.5) and (3.6), \( m_1 = 5 \) and \( m_2 = \frac{25}{2} \). Then, by (3.61), (3.65) and (3.66), we have
\[
\begin{align*}
\left| J_{13}(0, 1 - \epsilon^\frac{1}{2}, \beta_0, \eta) \right| &\leq \hat{K}_0 \epsilon^\frac{1}{3}. \\
\left| J_{13}(1 + \epsilon^\frac{1}{2}, 4 - \epsilon^\frac{1}{2}, \beta_1, \eta) \right| &\leq \hat{K}_1 \epsilon^\frac{1}{3}. \\
\left| J_{13}(4 + \epsilon^\frac{1}{2}, 5, \beta_2, \eta) \right| &\leq \hat{K}_2 \epsilon^\frac{1}{3}
\end{align*}
\]
for \( 0 < \beta_0 < \beta_1 < \beta_2 < 5, \epsilon \in S_{c_1} \) and \( \eta \in N_5 \). By (3.33) and (3.30) or (3.31), and using \( \hat{g}_j \) instead of \( g_2 \) and using \( \hat{g}_{3j} \) instead of \( g_3 \) (\( j = 1, 2, 3 \))
\[ \hat{K}_0 = \frac{2m_1 + m_2}{\hat{g}_0} + \frac{m_1}{\hat{g}_0} + \frac{2m_1 \hat{g}_{30}}{\epsilon \hat{g}_0^2} \approx 3.8. \]
\[ \hat{K}_2 = \frac{2m_1 + m_2}{\hat{g}_2} + \frac{m_1 \hat{g}_2}{\hat{g}_2} + \frac{m_1 \hat{g}_{32}}{\hat{g}_2^2} \epsilon_1^2 v_2 - 1 \approx 11.1. \]
where \( |\hat{p}_0(x)| \leq \hat{g}_{30} = 16, (0 \leq x \leq 1) \) and \( |\hat{p}_2(x)| \leq \hat{g}_{32} = 1, (4 \leq x \leq 5) \). and by (3.53).
\[ \hat{K}_1 \leq \hat{K}_0 + \hat{K}_2 \approx 14.9. \]

Note that above inequality is true since the values of \( m_1 \) and \( m_2 \) are given in the whole interval rather than each sub-interval in (4.24). Thus.
\[
(4.25) \quad |J_{2}^{(13)}| \leq \hat{K}_0 \epsilon^\frac{1}{3} + \hat{K}_1 \epsilon^\frac{1}{3} + \hat{K}_2 \epsilon^\frac{1}{3} \leq 29.8 \epsilon^\frac{1}{3}.
\]

Combining (4.23) and (4.25), we have
(4.26) \[ |J_{13}(0.5.0.\eta)| \leq |J_{1}^{(13)}| + |J_{2}^{(13)}| \leq 39.8\epsilon^{\frac{1}{3}} = G_{13}(\epsilon). \]

Note that the above estimate for \( J_{13} \) may not be sharp. We could get better estimate for \( J_{13} \) if we used different \( m_1 \) and \( m_2 \) for each interval in (4.24).

Next, let

\[ J_{21}(0.5.0.\eta) = \int_{0}^{\epsilon} s \exp\left\{ i\epsilon^{-1} \int_{0}^{s} -(t - 1)(t - 2)^2 \, dt \right\} ds. \]

Here, \( \alpha_1 = 1 \) and \( \alpha_2 = 2 \) with \( v_1 = 1 \) and \( v_2 = 2 \). Choose \( l_1 = \frac{1}{2}, l_2 = \frac{1}{3} \). Let

\[ c_2 = \min \{ 1, (\frac{1}{2})^2, 2^2, (\frac{1}{4})^2, 4^2, (\frac{1}{4})^3, 4^3, (\frac{3}{2})^3, (\frac{2}{3})^3 \} = \frac{1}{64} \]

and, for this integral,

\[ I_1 = [1 - \epsilon^\frac{1}{2}, 1 + \epsilon^\frac{1}{2}] \cup [2 - \epsilon^\frac{1}{2}, 2 + \epsilon^\frac{1}{2}]. \]

\[ I_2 = [0.5] - \text{Int}(I_1). \]

Then, for \( J_{1}^{(21)} \), the integral of \( J_{21} \) over \( I_1 \), we have

(4.28) \[ |J_{1}^{(21)}| \leq \int_{I_1} s \, ds = 2\epsilon^{\frac{1}{3}} + 4\epsilon^{\frac{1}{3}} \leq 6\epsilon^{\frac{1}{3}} \]

for \( \epsilon \in S_{c_2} \) and \( \eta \in N_{\delta} \).

For the estimation of \( J_{2}^{(21)} \), the integral of \( J_{21} \) over \( I_2 \), let

\[ \hat{p}_0(x) = -(x - 2)^2, \quad \hat{p}_1(x) = -1, \quad \hat{p}_2(x) = -(x - 1). \]

Then,

\[ \hat{g}_0 = \inf_{0 \leq x \leq 1} |\hat{p}_0(x)| = 1, \quad \hat{g}_1 = 1, \quad \hat{g}_2 = \inf_{2 \leq x \leq 5} |\hat{p}_2(x)| = 1. \]
Observe that \( m_1 = 5 \), \( m_2 = \frac{25}{2} \) and \( \hat{d}_1 = \frac{1}{3} \). Thus, by Lemma 7

\[
|J_{21}(0, 1 - \epsilon^{\frac{1}{3}}, \beta_0, \eta)| \leq \hat{K}_0 \epsilon^{\frac{1}{3}}, \quad |J_{21}(2 + \epsilon^{\frac{1}{3}}, 5, \beta_2, \eta)| \leq \hat{K}_2 \epsilon^{\frac{1}{3}}.
\]

\[
|J_{21}(1 + \epsilon^{\frac{1}{3}}, 2 - \epsilon^{\frac{1}{3}}, \beta_1, \eta)| \leq \hat{K}_1 \epsilon^{\frac{1}{3}}
\]

for \( 0 < \beta_0 < \beta_1 < \beta_2 < 5 \), \( \epsilon \in S_{c_2} \) and \( \eta \in N_5 \), and

\[
\hat{K}_0 = \frac{2m_1 + m_2}{\hat{g}_0} + \frac{m_1}{\hat{g}_0} + \frac{2m_1 \hat{g}_{30}}{\epsilon \hat{g}_0^2} \approx 42.2.
\]

\[
\hat{K}_2 = \frac{2m_1 + m_2}{\hat{g}_2} + \frac{m_1 v_2}{\hat{g}_2} + \frac{m_1 \hat{g}_{32}}{\epsilon \hat{g}_2^2} \frac{c_2^2}{v_2 - 1} \approx 34.5.
\]

where \( |\hat{p}'_0(x)| \leq \hat{g}_{30} = 4 \), \( (0 \leq x \leq 1) \) and \( |\hat{p}'_2(x)| \leq \hat{g}_{32} = 1 \), \( (2 \leq x \leq 5) \), and

\[
\hat{K}_1 \leq \hat{K}_0 + \hat{K}_2 \approx 76.7.
\]

Hence,

\[
(4.29) \quad |J_{21}(0, 5, 0, \eta)| \leq |J_{21}(0, 5, 0, \eta)| + |J_{21}(0, 5, 0, \eta)| \leq 153.4 \epsilon^{\frac{1}{3}}.
\]

Thus, combining (4.28) and (4.29), we have

\[
(4.30) \quad |J_{21}(0, 5, 0, \eta)| \leq |J_{21}(0, 5, 0, \eta)| + |J_{21}(0, 5, 0, \eta)| \leq 159.4 \epsilon^{\frac{1}{3}} = G_{21}(\epsilon).
\]

Now, let

\[
(4.31) \quad J_{23}(0, 5, 0, \eta) = \int_0^x \eta s^2 \exp \left\{ i \epsilon^{-1} \int_0^s -4(t - 1)(t - 3) dt \right\} ds.
\]
Here, $\alpha_1 = 1$ and $\alpha_2 = 3$ with $v_1 = v_2 = 1$. Choose $l_1 = l_2 = \frac{1}{2}$. Let

$$c_3 = \min\{1, \left(\frac{1}{2}\right)^2, 2^2\} = \frac{1}{4}$$

and, for this integral,

$$I_1 = [1 - \epsilon^\frac{3}{2}, 1 + \epsilon^\frac{3}{2}] \cup [3 - \epsilon^\frac{3}{2}, 3 + \epsilon^\frac{3}{2}],$$

$$I_2 = [0, 5] - \text{Int}(I_1).$$

Then, for $J_1^{(23)}$, the integral of $J_{23}$ over $I_1$, we have

$$(4.32) \quad |J_1^{(23)}| \leq \int_{I_1} \eta s^2 \, ds = \frac{\eta}{3} (60\epsilon^\frac{1}{2} + 4\epsilon^\frac{3}{2}) \leq \frac{64\delta}{3}\epsilon^\frac{1}{2}$$

for $\epsilon \in S_{c_3}$ and $\eta \in N_\delta$.

For the estimation of $J_2^{(23)}$, the integral of $J_{23}$ over $I_2$, let

$$\hat{\rho}_0(x) = -4(x - 3), \quad \hat{\rho}_1(x) = -4, \quad \hat{\rho}_2(x) = -4(x - 1).$$

Then, $\hat{g}_0 = 8, \hat{g}_1 = 4, \hat{g}_2 = 8$. Observe that $m_1 = 25\delta, m_2 = \frac{125\delta}{3}$ and $d_1 = \frac{1}{2}$.

Then, by Lemma 7

$$\left| J_{23}(0, 1 - \epsilon^\frac{1}{2}, \beta_0, \eta) \right| \leq \hat{K}_0 \epsilon^\frac{1}{2}, \quad \left| J_{23}(3 + \epsilon^\frac{1}{2}, 5, \beta_2, \eta) \right| \leq \hat{K}_2 \epsilon^\frac{1}{2},$$

$$\left| J_{23}(1 + \epsilon^\frac{3}{2}, 3 - \epsilon^\frac{1}{2}, \beta_1, \eta) \right| \leq \hat{K}_1 \epsilon^\frac{1}{2}.$$ 

for $0 < \beta_0 < \beta_1 < \beta_2 < 5, \epsilon \in S_{c_3}$ and $\eta \in N_\delta$, and

$$\hat{K}_0 = \frac{2m_1 + m_2}{\hat{g}_0} + \frac{m_2}{\hat{g}_0} + \frac{2m_1 \hat{g}_0}{\epsilon \hat{g}_0^2} \approx 15.7\delta.$$
\[
K_2 = \frac{2m_1 + m_2}{\hat{g}_2} + \frac{m_2}{\hat{g}_2} + \frac{2m_1 \hat{g}_{32}}{e \hat{g}_2^2} \approx 18.9\delta.
\]

where \(|\hat{p}_0(x)| \leq \hat{g}_{30} = 4. (0 \leq x \leq 1)\) and \(|\hat{p}_2(x)| \leq \hat{g}_{32} = 4. (3 \leq x \leq 5)\). and \(\hat{K}_1 \leq \hat{K}_0 + \hat{K}_2 \approx 34.6\delta\).

Hence,

\[(4.33) \quad |J_2^{(23)}| \leq \hat{K}_0 \epsilon^{\frac{1}{2}} + \hat{K}_1 \epsilon^{\frac{1}{2}} + \hat{K}_2 \epsilon^{\frac{1}{2}} \leq 69.2\delta \epsilon^{\frac{1}{2}}.\]

Thus, combining (4.32) and (4.33), we have

\[(4.34) \quad |J_{23}(0.5.0.\eta)| \leq |J_1^{(23)}| + |J_2^{(23)}| \leq 90.5\delta \epsilon^{\frac{1}{2}} = G_{23}(\epsilon).\]

Finally, let

\[(4.35) \quad J_{32}(0.5.0.\eta) = \int_0^x \epsilon \eta s \exp \left\{i \epsilon^{-1} \int_0^s 4(t - 1)(t - 3) dt \right\} ds.\]

Here, \(\alpha_1 = 1\) and \(\alpha_2 = 3\) with \(v_1 = v_2 = 1\). Choose \(l_1 = l_2 = \frac{1}{2}\). Let

\[c_4 = \min\{1, (\frac{1}{2})^2, 2^2\} = \frac{1}{4}\]

and, for this integral.

\[I_1 = [1 - \epsilon^{\frac{1}{2}}, 1 + \epsilon^{\frac{1}{2}}] \cup [3 - \epsilon^{\frac{1}{2}}, 3 + \epsilon^{\frac{1}{2}}],\]

\[I_2 = [0.5] - \text{Int}(I_1).\]

Then, for \(J_1^{(32)}\), the integral of \(J_{32}\) over \(I_1\), we have

\[(4.36) \quad |J_1^{(32)}| \leq \int_{I_1} \epsilon \eta s ds = \epsilon \eta (2\epsilon^{\frac{1}{2}} + 6\epsilon^{\frac{1}{2}}) \leq 8\delta \epsilon^{\frac{1}{2}}\]
for $\epsilon \in S_{c_4}$ and $\eta \in N_\delta$.

For the estimation of $J_2^{(32)}$, the integral of $J_{32}$ over $I_2$, let

$$\hat{p}_0(x) = 4(x - 3), \quad \hat{p}_1(x) = 4, \quad \hat{p}_2(x) = 4(x - 1).$$

Then, $\hat{g}_0 = 8, \hat{g}_1 = 4, \hat{g}_2 = 8$. Observe that $m_1 = 5c_4\delta = \frac{5\delta}{4}, m_2 = \frac{25c_4\delta}{2} = \frac{25\delta}{8}$ and $d_1 = \frac{1}{8}$. Then, similar to $J_{23}$.

$$|J_2^{(32)}| \leq \hat{K}_0 + \hat{K}_1 + \hat{K}_2 \approx 0.9\delta + 1.1\delta + 2.0\delta = 4\delta.$$  

Thus, combining (4.36) and (4.37), we have

$$|J_{32}(0.5, 0, \eta)| \leq |J_1| + |J_2| \leq 12\delta\epsilon^\frac{1}{2} = G_{32}(\epsilon).$$

Let

$$\hat{c}_1 = \min\{c_1, c_2, c_3, c_4\} = \frac{1}{64}.$$  

Therefore, assuming $\delta = 0.1$

$$|||I_3 + LI_3||| = \max\{G_{13}(\epsilon), G_{21}(\epsilon), G_{23}(\epsilon), G_{32}(\epsilon)\} \approx 38\epsilon^\frac{1}{2}$$

for $\epsilon < \hat{c}_1$

Now, we need to find $L(\epsilon)$ such that $|||L^2P||| \leq L(\epsilon)|||P|||$, which is given by (3.78).

$$|J_{13}(0.5, 0, \eta)| \leq \int_0^x |s| \, ds \leq \frac{25}{2} = \hat{G}_{13},$$

$$|J_{21}(0.5, 0, \eta)| \leq \int_0^x |s| \, ds \leq \frac{25}{2} = \hat{G}_{21},$$

$$|J_{23}(0.5, 0, \eta)| \leq \delta \int_0^x |s^2| \, ds \leq \frac{12.5}{3} = \hat{G}_{23},$$

$$|J_{32}(0.5, 0, \eta)| \leq \hat{c}_1\delta \int_0^x |s| \, ds \leq \frac{25}{8} = \hat{G}_{32}. $$

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Thus, by the estimates (4.26), (4.30), (4.34) and (4.38).

\[
L(\varepsilon) = \max_{1 \leq j \leq 3} \left\{ \sum_{l,h=1}^{3} G_{Jh}(\varepsilon) \hat{G}_{hl} \right\} \\
= \max \left\{ G_{13}(\varepsilon) \hat{G}_{32}, G_{21}(\varepsilon) \hat{G}_{13} + G_{23}(\varepsilon) \hat{G}_{32}, G_{32}(\varepsilon)(\hat{G}_{21} + \hat{G}_{23}) \right\} \\
= \max \left\{ 12.4\varepsilon^{\frac{1}{3}}, 1993.8\varepsilon^{\frac{1}{3}}, 20\varepsilon^{\frac{1}{3}} \right\} < 2000\varepsilon^{\frac{1}{3}} \leq 1
\]

for \( \varepsilon \in S_{c}, \hat{c} = 2000^{-\frac{1}{3}}. \) Thus, the error estimate, given by (4.21), is

\[
\|\|P(x, \varepsilon, \eta) - P_1(x, \varepsilon, \eta)\|\| \leq \frac{L(\varepsilon)^3}{1 - L(\varepsilon)} \|\|I_3 + LI_3\|\|. 
\]
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CHAPTER V

FUTURE STUDY

In this section, we present four problems related to this dissertation for future studies. For the first three problems we need suitable assumptions similar to those given in Chapter II to find the asymptotic solutions valid in the entire given interval.

First, if the first assumption (A1) is relaxed, the coefficient \( H(x, \epsilon, \eta) \) can be expressed as

\[
H(x, \epsilon, \eta) = H_0(x) + \epsilon H_1(x, \epsilon, \eta) + \eta H_2(x, \eta)
\]

where \( H_0(x) \) is an \( n \times n \) Hermitian matrix analytic on \( I \) and \( H_1(x, \epsilon, \eta) \) is an \( n \times n \) matrix in the class \( C^1(I \times \mathbb{S} \times N_s) \) and \( H_2(x, \eta) \) is an \( n \times n \) matrix in the class \( C^1(I \times N_s) \). It is believed that the method used in this dissertation cannot be applied immediately to this problem.

The second problem is the following 2 \( \times \) 2 dimensional system:

\[
i \begin{bmatrix}
\epsilon \sigma_1 y_1' \\
\epsilon \sigma_2 y_2'
\end{bmatrix} = (D_0(x) + \epsilon R_0(x, \epsilon)) \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
\]

where \( \sigma_1, \sigma_2, (\sigma_1 \neq \sigma_2) \), are positive constants, and \( D_0(x) \) is a 2 \( \times \) 2 analytic diagonal matrix and \( R_0(x, \epsilon) \) is a 2 \( \times \) 2 matrix in the class \( C_1 \) in \( (x, \epsilon) \).
The third problem is the following $2 \times 2$ dimensional system by combining the first and second problems.

$$i \begin{bmatrix} e^{\sigma_1 y'_{1}} \\ e^{\sigma_2 y'_{2}} \end{bmatrix} = (D_0(x) + R_0(x, \epsilon, \eta) + \eta Q_0(x, \eta)) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where $\sigma_1, \sigma_2$ ($\sigma_1 \neq \sigma_2$) are positive constants, and $D_0(x)$ is a $2 \times 2$ analytic diagonal matrix. $R_0(x, \epsilon, \eta)$ is a $2 \times 2$ matrix in the class $C_1$ in $(x, \epsilon, \eta)$ and $Q_0(x, \eta)$ is a $2 \times 2$ matrix in the class $C_1$ in $(x, \eta)$.

Finally, we discussed the decoupled equations for entries of $P$ in Chapter IV. In most cases, it is expected that the integrals cannot be found explicitly. Therefore, suitable numerical integrations are necessary. Because of the existence of turning points and pure imaginary exponential functions (e.g. see Example 1 and 2), we may need irregular oscillatory integrations (cf. P. Davis and P. Rabinowitz [4], and G. Evans [5]) in the neighborhood of the turning points. One may consider collocation method (cf. G. Evans [5], D. Kincaid and W. Cheney [15]) for this problem. Also, good error analysis is required.
BIBLIOGRAPHY


