The Effects of Writing Assignments on Second-Semester Calculus Students' Understanding of the Limit Concept

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THE EFFECTS OF WRITING ASSIGNMENTS ON SECOND-SEMESTER CALCULUS STUDENTS' UNDERSTANDING OF THE LIMIT CONCEPT

by

Melanie A. Wahlberg

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
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The purpose of this study was to determine the impact of writing assignments on second-semester calculus students' understanding of the limit concept, using both qualitative and quantitative techniques. The study involved two sections of second-semester calculus (n = 37, 34) at Western Michigan University during the fall semester of 1997.

The treatment group completed six writing tasks focusing on the mathematical concept of limit. These conceptually-oriented writing assignments replaced some of the problem sets that the instructor would have normally assigned. The control group did not engage in writing tasks but handed in problem sets more frequently than the treatment group. Both sections used the same textbook and department syllabus. Prior to the course, the investigator met with the two instructors to confirm that both followed a traditional lecture/discussion format, and further, she monitored their classrooms intermittently during the semester.

The investigator used the writing assignments, as well as transcripts from a series of interviews of a subset (n = 5) of the treatment group, to assess cognitive
growth. This qualitative assessment was based on action-process-object-schema
(APOS) theory. Quantitative analysis of student performance was based on a
comparison of three computational problems common to the two sections' final
examination. These problems focused on the limit concept and provided for a two-
sample t-test to compare mean scores.

The students that performed the writing tasks increased measurably in their
conceptual understanding of the limit concept. Some were able to achieve elements of
object-level understanding, a level never before reported in the literature. The
interviews showed that the subjects generally rose at least one APOS level. An
exception was the one subject who did not engage in the assignments. Furthermore,
all differences in the final examination problem scores favored the treatment students.
These differences were significant in two of the three limit problems (p < .05). Finally,
students reported they saw the benefits of writing, recommending its continued use.

Writing assignments focusing on the limit concept appear to provide the
necessary cognitive conflict and resolution for students both to outgrow their
misconceptions and to improve their mathematical performance.
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CHAPTER I

INTRODUCTION

Impetus For Reform

During the past 11 years, the mathematics and mathematics education communities have fostered and witnessed a national revolution in the way calculus is taught, particularly at the college level. This revolution had its genesis in Tulane, Louisiana, at a small conference preceding the American Mathematical Society and Mathematical Association of America Joint Meetings in 1986. The conference, hosted by Ronald Douglas with funding from the Sloan Foundation, initiated a grass roots movement to reform calculus instruction on a national level. Among its 25 participants were representatives from two-year colleges, four-year colleges and advanced degree-granting research institutions.

The impetus for this reform movement was multi-layered. For many mathematicians, the disparity between the calculus they were teaching and the calculus they were taught was unsettling. In particular, participants at the "Sloan Conference" stated that both they and their colleagues were "... frustrated at the need to work so hard to help poorly prepared, poorly motivated students learn material that was a shadow of the calculus they had learned" (Assessing Calculus Reform Efforts, 1995, p13). To remedy this, there was a call among them for a reduction in "...
memorization, mimicry, templates . . . ", and things "meaningless" or "artificial" (Calculus, The Dynamics of Change, 1996, p10). At the same time, developments in technology made hand-held calculators both immensely powerful and affordable exploration tools. A change in the way the calculus was taught was a natural outcome of this progress. A third factor, contributing more to the efficacy and longevity of reform than its impetus, was the support of the National Science Foundation to several calculus reform initiatives over the past 10 years. This support was twofold; the National Science Foundation gave of its financial resources as well as giving intellectual, academic backing to the projects.

Reaction to Reform

Once several reformed curricula were being developed, the next step was implementation. This took place on a department-to-department basis. It was discovered that the easiest way to initiate more or less uniformly reformed instruction was to adopt a reformed calculus textbook department wide. This step was taken by the mathematics department at Western Michigan University in the fall semester of 1996. In the winter semester of 1996, several textbooks had been reviewed by a search committee towards selecting a calculus book appropriate for the department's undergraduate program. During this process, an issue was raised concerning the level of rigor in definitions, theorems, and proofs. In particular, it was noted that several of the reformed textbooks de-emphasized mathematical rigor in favor of applications,
heuristical arguments, and reliance on intuition. The book ultimately chosen by the committee, *Calculus, from Graphical, Numerical, and Symbolic Points of View*, by Arnold Ostebee and Paul Zorn, reflects this emphasis. Based on the feeling that "... concepts, not techniques, are truly fundamental to the course" (Ostebee and Zorn, 1997, p vii), the authors wrote a calculus book that exemplifies this differing emphasis. One can see this shift from the traditional approach in the textbook’s treatment of the limit concept. Students are provided an intuitive, informal introduction to the concept of limit and don't see the formal definition of the concept until much later. By this time, the students have used the idea not only in numerous problem situations, but in defining other related concepts as well. Clearly this approach is supported by reformers. “A student’s first understanding of calculus should be intuitive; rigor can come later.” (Calculus, The Dynamics of Change, p 4)

This approach of spending much time heuristically developing a concept before providing its definition was met with both opposition and support by members of the search committee and raised several questions regarding reformed calculus and mathematical rigor. These concerns covered all mathematical topics studied through a reformed approach, but were particularly acute with the limit concept, as it is a fundamental notion in the study of the calculus. Because of the lengthy deliberations made among faculty members (experts in the mathematical field), it was clear that students' understanding of the limit concept after experiencing instruction based upon a reformed curriculum merited disciplined inquiry and study. Research to date has
shown (Ferrini-Mundy and Graham, 1994, Monaghan, 1991, Orton, 1983, Tall, 1992, Williams, 1991) the limit concept to be particularly difficult for students to grasp. In fact, although several pedagogical strategies have been attempted (Nussbaum and Novick, 1982, Williams, 1991, Cottrill, et al., 1996) few have been able to penetrate the seemingly impermeable wall of confusion and misconceptions surrounding the concept. It seems particularly difficult to dislodge misconceptions that have formed in the minds of students of calculus. In fact, researchers (Sierpinska, 1987, Tall & Schwartzzenberger, 1978) have provided detailed accounts of having their subjects describe their current understanding, then putting them into cognitive conflict by providing an example that shows the original understanding to be erroneous and incomplete. Some at this point (Williams, 1991) even gently suggested means of resolution. None of these techniques has had a lasting effect on students’ comprehension of the limit concept. Students revert to their incorrect but comfortable original perspectives. Indeed, this phenomenon has been a primary source of frustration among researchers and practitioners alike.

Building a Framework for Undergraduate Mathematics Education Research

Interestingly, although much time, effort, and funding has been put into developing reformed calculus curricula, relatively few studies exist detailing the mathematical understanding of the undergraduate mathematics student of curricula, reformed or otherwise. Most of mathematics education research has focused on
pedagogy, student understanding, and curriculum at the kindergarten to 12th-grade levels, leaving unexplored terrain in the body of knowledge concerning mathematics students' understandings and abilities at the undergraduate levels.

In the summer of 1995, a group of mathematicians interested in beginning to fill this gap assembled at Purdue University. At a Cooperative Learning in Undergraduate Mathematics Education workshop, a subset of the participants gathered in the hopes of forming a community of researchers committed to studying the undergraduate learner of mathematics. From this gathering, the Research in Undergraduate Mathematics Education Community (RUMEC) emerged, and with it, a companion methodology for data gathering and analysis. Within this framework, the concept of understanding any mathematical idea is contingent upon moving among the action, process and object levels of facility. That is,

... understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. (Asiala, et al. 1996, p 10)

Action, process, and object are briefly defined below. These levels, together with the concept, schema, will be developed further in the Literature Review.

A subject operating at the lowest functioning level of facility is said to have an action concept of the given mathematical topic. This individual relies on mnemonics and reacts to external cues in determining how to proceed. However, after repeating and reflecting upon the action, it may be transformed in the individual's mind to a
process. The same mathematical concept is now viewed as being "... under one's control, rather than something one does in response to external cues" (ibid., p 12). Once a subject has enough familiarity with a process to consider it as a totality, the process becomes an object that may be manipulated or acted upon. In theory, then, reflective abstraction is necessary on the subject's part in order for him or her to move from one level to another. That is to say, one must pause and reflect upon one's current mathematical practice in order to broaden one's current mathematical perspective.

It has been shown (Cottrill, et al., 1996) that APOS theory is an appropriate lens through which to analyze students’ understanding of the limit concept. More specifically, APOS theory has been used to analyze interview transcripts that resulted from students’ working limit problems and discussing their thought processes while working. However, Cottrill and his colleagues demonstrated that few first-semester calculus students could be expected to be at an object-level conception of the limit notion. Thus, to see a wider spread of student understanding of the limit concept in light of the APOS structure, the understanding of students in at least their second semester of calculus would have to be examined.

Writing in the Mathematics Classroom

Working under the theoretical framework that reflective abstraction is necessary for progress along the action - process - object continuum, and that progress
along this continuum is desirable, classroom modalities that initiate stages of reflective
abstraction should be implemented. Perhaps also seeing the need for reflective
abstraction, the proponents of reform called for judiciously chosen writing assignments
in the calculus classroom.

The clear definitions and crisp standards of reasoning, shorn of all hyperbole,
make mathematics an ideal place to learn to write, but beyond that, it is
increasingly recognized that writing facilitates learning. (Calculus, The
Dynamics of Change, p 3)

It was hoped that the activity of writing would force students to pause and ponder
their current understanding or lack of understanding of a given topic. This in turn
would foster reflection regarding mathematical content and concepts. Of course, it is
entirely possible to complete a writing assignment without any reflective abstraction,
particularly if the assignment is not carefully designed. For example, if an instructor
wanted his or her students to learn about arc length from a conceptual standpoint,
having a student explicate the procedure for evaluating an arc’s length (finding the
derivative, squaring the result, adding one, taking the square root, and integrating over
the desired interval) would require little reflection and therefore contribute minimally,
if at all, to the students’ conceptual understanding. Thus, to further a student’s level
of conception in terms of APOS theory, tasks that encourage a student to study and
reflect upon the concept in question as an object should be assigned.

Several authors and researchers (Britton, 1975, Emig, 1977, Luria, 1971.)
have suggested writing on a given topic as a means of deepening one’s own subject
knowledge. More recently, others (Birken, 1989, Connolly, 1989, Havens, 1989,
Pearce and Davison, 1988, Shepard, 1993) have heralded writing specifically as a means of improving one's understanding of mathematical ideas and making robust originally fragile knowledge. Through a case study, Powell and Lopez (1989) showed that the completion of daily writing assignments enabled a developmental mathematics student to adopt a more sophisticated mathematical perspective as well as to develop more ownership of the material at hand. However, this is one of a very small number of studies examining the impact of writing assignments on students’ understanding. More studies are needed in different content areas to determine whether the anecdotal evidence teachers and researchers have gathered suggesting the benefits of writing hold true under disciplined inquiry. A natural question to ask, then, is how do writing assignments affect calculus students’ understanding of the limit concept? Furthermore, how could one ascertain and measure such effects?

Statement and Context of the Problem

This study was an inquiry into the second-semester calculus students’ understanding of the concept of limit. To a lesser extent, the concomitant concept of infinity was examined. The principal concern of this study was to ascertain the potential effects of student writing tasks on their assimilation of this concept.

The setting of this study was two sections of the 15-week-long, second-semester calculus course taught at Western Michigan University. The textbook used was Calculus from Graphical, Numerical, and Symbolic Points of View, by Arnold
Ostebee and Paul Zorn. Most students had studied the first five chapters of this text (covering functions, the derivative, applications of the derivative, and the integral) in their first-semester calculus course.

Research Questions

This systematic inquiry adopted the theoretical perspective that students, in their journey from novice understanding towards expert understanding, necessarily progress through graduated levels of sophistication in their comprehension of a topic or concept, namely action, process, and object (Asiala, et al, 1996). The specific research questions that were addressed follow.

1. What conception of limit in terms of the action-process-object-schema (APOS) theory are students reaching as a result of completing a course in second-semester calculus in which writing assignments are included?

2. Do writing assignments make a difference in students' achievement and understanding in limit-related concepts in second-semester calculus?

These questions will be posed and studied in the context of several topics, where success depends on the student's attaining a certain level of understanding of limits and their applications. Specifically, these are: the assimilation of the formal definition of limit; the definite integral defined as a limit of a Riemann sum in the context of arc length; improper integrals and convergence; and infinite sequences and series.
Brief Overview of Research Method

Two second-semester calculus sections were studied; one chosen as the treatment group, the other used for control. The treated group was given a battery of six writing assignments over the course of the semester that replaced some of the problem sets the control group worked. These writing assignments were specifically developed to put students into situations that would require them to reconcile their current understanding of limits with the rigorous mathematics they had learned in class and were expected to be able to use in more procedurally-oriented problems.

Selection of two sections allowed for quantitative comparison between the two classes as well as qualitative study of individual members of the treated group.

As one measure of limit understanding, three limit problems common to the two final exams of each of the participating sections (control and writing) were developed by the researcher and the two participating instructors. These were then assessed by the researcher to determine mathematical performance. The mean scores of the two classes (with problems twice scored for inter-rater reliability) were compared for each of the three problems. Analysis of examination performance on these three mathematical items constituted the quantitative portion of the research. This quantitative analysis, though valuable, failed to address the growth an individual learner underwent during the semester while completing the writing assignments. Thus, five students were chosen from the treatment group to participate in a series of interviews probing their changing limit concept. These students were selected based
on their responses to a questionnaire developed by Williams (1990) to ascertain a subject’s perspective of the limit concept. Once a heterogeneous group was assembled (based on Williams’ instrument), three interviews of each of the five students took place intermittently over the semester. The interviews were then transcribed and analyzed according to APOS theory. Another mathematics educator examined the transcripts and analyzed them to establish reliability of analysis.

Conclusion

With the multitude of pedagogical innovations developed within the calculus reform movement, it is natural to ask why one would focus on writing as a pedagogical device towards helping students to foster a more robust concept of limit. The researcher specifically sought a medium in which students would discern, confront, and work through inconsistencies in their concept of “how limits work.” Writing is a logical choice for such intellectual wrestling. Further, there has been a call in the literature (Powell and Lopez, 1989, Shepard, 1993) for studies documenting the supposed benefits of writing in a mathematical context. Finally, the researcher has anecdotal evidence suggesting writing’s positive impact on students’ understanding and ownership of the mathematical material at hand. It is conjectured that writing assignments will complement the problem sets the students work in such a way that conceptual understanding will improve measurably.
CHAPTER II

LITERATURE REVIEW

Introduction

This chapter will review five categories of literature: a constructivist-based theoretical framework for data analysis; APOS theory; calculus reform; college-level students' understanding of the concept of limit and infinity; and writing in the mathematics classroom as a pedagogical tool.

The section concerning the theoretical framework provides both an overview of the several different strains of constructivism and a rationale for the one chosen for this study. This choice motivates the theoretical framework and methodology for data gathering and analysis in this study. The second section details the structure of APOS theory as proposed by the members of RUMEC. The calculus reform section outlines the goals of educational reformation, discusses the difficulties in assessing its efficacy, and describes the reformed textbook used in this study. Next, the literature reflecting mathematics students' seemingly inevitable struggle with the limit concept and the germane ideas of infinity is considered. The final section explores writing as a means of learning in general and of learning mathematics, and concludes with a discussion of the practices of mathematical writing. There is a clear call heard several times in the
literature for more studies of the effects of writing on the learning of mathematics. The present study contributes to that area of research.

Theoretical Framework

Constructivism

The theoretical tradition of constructivism, developed by Piaget (1980) and further explored by countless other scholars (Skemp, 1987, Wadsworth, 1989, Bidell and Fischer, 1992, Pozo and Carretero, 1992) has had a tremendous impact on the education community. Furthermore, as is evidenced by the writings of Resnick and Ford (1981), Goldin, von Glasersfeld, Ernest, Herscovics, Janvier, Kaput and others (Steffe, et al., 1996), constructivism has been extensively developed and applied specifically within the mathematics education community. In fact, Ernest (1996, p 335) maintains that, "Following the seminal influence of Jean Piaget, constructivism is emerging as perhaps the major research paradigm in mathematics education." In particular, Piaget's work with the notion of reflective abstraction and his view of the individual as the constructor of knowledge serve to shape both the thoughts of radical constructivists and the members of RUMEC. Piaget (1980) identified four components of reflective abstraction in the learning processes of children, namely interiorization, coordination, encapsulation, and generalization. Although the majority of Piaget's work concentrated on the mathematical development of young children, Dubinsky (1991, p 95) suggests that "... this same approach can be extended to more
advanced topics going into undergraduate mathematics and beyond.” Thus, the RUMEC team has extrapolated from these four stages a model for the progression of mathematical growth of undergraduate mathematics students.

Rationale for the adoption of radical constructivism as a theoretical framework must now be included, for there exist many learning theories in mathematics education, only one of which is constructivism. In fact, constructivism is even considered by some to be not a theory at all, but rather an epistemological stance. Constructivism was chosen because, as many educational theorists agree, "...learning will always be achieved in a constructivist manner whatever you do" (Janvier, 1996, p 452). Thus, the principles of constructivism do not advocate specific action. That is, the theory and exposition of constructivism do not tell pedagogues what to do to hasten or improve the learning process, but rather attempt to explain what happens in the learning process, regardless of instructional approach.

Just as there are several prevalent learning theories, there are several strains of constructivism. In particular, radical, social and weak constructivism will be analyzed and contrasted. Finally, the rationale for choosing radical constructivism will be provided.

Von Glasersfeld (1989, p 182) puts forward two principles of constructivism, the first of which is, "Knowledge is not passively received but actively built up by the cognizing subject." This tenet is common to all three types of constructivism enumerated above, but is a sufficient theoretical basis only for weak constructivism.
The metaphor for the mind of a cognizing subject under weak constructivism (a mathematics student in this case) is an ideal "soft" computer that processes self-constructed data. A potential flaw in the epistemology of weak constructivism is its irreconcilable duality. On one hand, all individual knowledge is assumed to be constructed. However, regarding a realm of objective knowledge such as "knowable mathematics", a platonist stance is simultaneously maintained. Thus the dichotomy: how can such knowledge be obtained by a learner, if construction of the knowledge is personal and idiosyncratic? It must be, then, that knowledge is constructed to match an "outside world", rather than recursively built up internally, as the one underlying principle dictates.

Instead of a soft computer, social constructivism chooses "persons in conversation" as its metaphor for the mind (Ernest, 1996). Unlike either weak or radical constructivism, it regards the learner and the social context in which the learner participates as indissolubly linked. The model of the world is the antithesis of the platonist's: a socially constructed, shared domain. Drawing much support from the writings of Vygotsky, social constructivism proceeds from the contention that all instances of higher thinking are realized social contexts. A potential criticism stems from an observation by Voigt (1996), "The classroom culture seems to be pre-given, the students' unusual and unexpected actions could be evaluated as mere deviations, and differences among individuals' developments are rather unexplained." (p 26) Thus
it could be maintained that when the social context becomes the focus, it is difficult to reserve attention for any one specific learner.

Radical constructivism is closely based on the writings of Piaget, and has been revitalized in the last fifteen years largely by the writings of von Glasersfeld. Rather than studying how students learn, radical constructivists primarily seek to answer the question, "What does it mean to know?" Thus, radical constructivism is an epistemological stance. Its proponents embrace both von Glasersfeld's first principle (given above) and his second, "The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (von Glasersfeld, 1989). Thus, radical constructivism is neutral in its ontology, not denying the existence of an "ultimate truth," but rather making clear its inherent unknowability.

Unlike social constructivism, radical constructivism focuses strongly on personal, idiosyncratic development of meaning. As Ernest (1996) explains, "... at its heart lies sensitivity to individual construction." For this reason, this study will adopt the theoretical framework developed by radical constructivists. In particular, this supports the use of both the RUMEC framework and APOS theory, as each is based on the writings of Piaget, from whose work the basis of radical constructivism is drawn.
APOS Theory

The action-process-object-schema theoretical perspective implies a method for the analysis of qualitative data, such as interview transcripts (Asiala, et al., 1996). This method allows the researcher to interpret the level of mathematical understanding of a subject in a systematic and reliable (trustworthy) way. APOS theory has been used to analyze students’ understanding of permutations and symmetries (Asiala, Brown, Kleiman and Mathews, in press), binary operations, groups and subgroups, (Brown, Devries, Dubinsky and Thomas, in press), and other mathematical topics. The following section expands on the earlier discussion of the components of APOS theory and their interactions.

Each of the constructs, action, process, and object, is a mental perspective an individual may hold regarding a given mathematical idea. Schema refers to the mental framework within which is housed the actions, processes and objects one associates with a mathematical concept. It is quite possible, and in fact probable, that one could simultaneously maintain distinct perspectives of different topics. For instance, one could have an action perspective of derivative and an object perspective of function. The constructs increase in sophistication level, but there are times when each is needed; once a learner ascends to the next level, he or she does not abandon the previous perspective, but rather enhances it.

When a student views a mathematical concept as an action, the concept is perceived as an externally prompted command to perform. Each step is triggered by
the previous one, rather than by an overall conceptual plan. Students at this level are characterized by not knowing how to proceed if superficialities like notation or language within the problem are changed. Working with many research subjects over long periods of time has suggested to the RUMEC team that the meaning of the term action must be broad enough to include all students below the process level (Cottrill, et al., 1996).

It is helpful to examine an action-level conception within a specific mathematical context. Consider the mathematical construct \( f(x) = x^2 - 3 \). A student with an action-level conception of function would not be able to "do anything" with this construct until provided a specific value for \( x \). Then he or she could plug in that specific value for \( x \), thereby obeying the perceived command to evaluate.

Once an individual performs an action repeatedly and is put into a context where he or she ponders this action, it may then be interiorized to a process. Such interiorization requires deliberate reflection, for the process is under the control of the individual, whereas the action is not. An individual with a process conception is able to visualize and even to describe the steps of the transformation without actually needing to perform them. Such a command of the given concept enables the individual to detect properties of the mathematical idea. New processes may be formed from old by coordinating two or more existing ones, or by reversing a process.

If a student has a process-level conception of mathematics, the aforementioned construct \( f(x) = x^2 - 3 \) is seen more as an "input-output" machine. This student could
recognize properties of the function, such as all outputs are at least -3. There is still a perceived need to perform operations, but these operations may be performed mentally and several-at-a-time, allowing a greater perspective.

If an individual can conceive of a process as a totality, realizes that transformations can act on it, and can construct such transformations, the individual has achieved an object level conception of the idea. Once this happens, we say the process has been encapsulated to an object. At this level, the subject can manipulate the mathematical idea or apply other processes to it. Furthermore, this object can be "unpacked" (de-encapsulated) to reveal the underlying process, should this form be more useful in the given context.

A subject with an object-level conception of \( f(x) = x^2 - 3 \) can think of this function as, say a point in a vector space, or a polynomial to be differentiated. It is a construct upon which mathematical operations may be performed, rather than a command to do something.

Finally, a schema for a given mathematical object is a collection of one's actions, processes, and objects organized into a coherent and accessible framework. A different article than the first three described, schema is an umbrella under which actions, processes, and objects exist and interact. Thus, schema are not formed until one realizes at least one connection between two related mathematical topics. One's schema becomes more and more well-developed as relationships among actions, processes and objects are recognized and reflected upon. One's function schema
would include all one knows about functions, including the relations among the actions, processes and objects one associates with functions.

Upon establishing the roles of action, process, object and schema for learners of general mathematical topics, it is possible to examine this hierarchy within the very specific context of limits. After a preliminary conjecture followed by revisions based on student data, Cottrill, et al. (1996) use the terminology of APOS theory to create a description of how the limit concept might be constructed by a student of calculus. This description, which they term a *genetic decomposition*, is shown below. Notice the use of the terms action, process, object, and schema.

1. The action of evaluating the function \( f \) at a single point \( x \) that is considered to be close, or even equal to, \( a \).
2. The action of evaluating the function \( f \) at a few points, each successive point closer to \( a \) than was the previous point.
3. Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which \( x \) approaches \( a \).
   (b) Construction of a range process in which \( y \) approaches \( L \).
   (c) Coordination of (a),(b) via \( f \). That is, the function \( f \) is applied to the process of \( x \) approaching \( a \) to obtain the process of \( f(x) \) approaching \( L \).
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, \( 0 < |x - a| < \delta \) and \( |f(x) - L| < \varepsilon \).
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit.
7. A completed \( \varepsilon - \delta \) conception applied to specific situations. (p 177-178)
This genetic decomposition enables the researcher (teacher) to identify the conception-level at which the subject (student) must operate in order to complete a given task regardless of its accompanying notation or context. The tasks comprising this study varied in level of sophistication so as to require action, process and object conception levels, allowing the researcher to discern between and among the different levels of performance. The overarching schema that the subjects held was not probed.

Calculus Reform

The spirit and goals of calculus reform help to guide this study, providing both direction and a means for following that direction. The beginning of the theoretical reform and the subsequent implementation of its practice are explored.

Genesis

The Sloan Conference/Workshop on Calculus Instruction, January 2 - 6, 1986, (led by Ronald Douglas and often referred to as the "Tulane Conference") officially began the national movement reforming the way calculus is taught (Research in Collegiate Mathematics Education, 1994). Among the 25 participants and, later, their interested colleagues within the mathematics community, two overarching goals of this reformation took shape. These were: "...to enhance the development of conceptual understanding, and to engage students as active participants" (Calculus, The Dynamics of Change, 1996, p 8). In attempting to realize these goals, many themes

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common to different reformers' efforts emerged, including multiple representation of concepts, discerning sources of authority other than the textbook or the instructor, and writing about mathematics. These themes are developed to varying degrees within the context of the reformed calculus curriculum (Ostebee and Zorn, 1997) upon which this study is based.

**Examples of Calculus Reform Initiatives**

Before discussing the effectiveness of calculus reform, it is helpful to introduce and describe a few of the reformed curriculum initiatives. *The Calculus Consortium* based at Harvard University is a project committed to building a calculus textbook (Hallett, et al., 1994) based on the “Rule of Three.” This approach supports the belief that the three aspects of calculus -- graphical, analytical, and numerical -- are all worthy of emphasis throughout the curriculum. It presents problems that are not necessarily solvable by using traditional analytical methods, thereby forcing students to try graphical or numerical approaches. Further, *The Calculus Consortium* demonstrates the benefits of using more than one approach per problem and consequently learning how the different representations interact and complement one another. Another significant reform evident in the textbook is its authors’ willingness to streamline the curriculum by actually eliminating some traditionally cherished topics. This is done in order to allow time for more heuristical arguments and a more thorough conceptual development of the ideas introduced.
The Oregon State Calculus Connections Project (Dick and Patton, 1992) relinquishes fewer topics from traditional calculus textbooks than does The Calculus Consortium, although the authors do de-emphasize some techniques of integration. A main focus of these authors is to make intelligent use of technology, both in its implementation and in the analysis of its reports. Because of this, the development of visualization and approximation skills among the students is a secondary goal. (Assessing Calculus Reform Efforts, 1995, p 58). Numerical approaches to problems are abundant in this text.

As was previously mentioned, the subjects in this particular study used a reformed calculus text, namely, Calculus, from Graphical, Symbolic, and Numerical Points of View (Ostebee & Zorn, 1997). The treatment of limits and infinity epitomize both the spirit of this book and the cautious side of reform. For example, students work problems requiring the use of limit notation before seeing its rigorous definition. In fact, the derivative of a function at a point, an idea heavily dependent on the limit concept, is formally defined (p 144), using the term limit, before limit is defined (p 155). However, the fact that the authors ultimately do provide formal definitions for mathematical terms puts Ostebee and Zorn’s textbook into a slightly less radical category than some other widely-used reformed calculus texts.
Assessment of Calculus Reform's Efficacy

Since its inception over 11 years ago, substantial effort has been made to assess both the scope and the efficacy of calculus reform, particularly within institutions of higher learning. This task is daunting though, due to the large-scale and highly irregular implementation of a wide variety of curricula (Assessing Calculus Reform Efforts, 1995). However, this could be overcome with enough data collection and perseverance of research teams. What remains an obstacle in assessing the success of the new teaching methods is that with changing curriculum, pedagogy, and technology comes a concomitant change in teaching goals. No longer are mathematical skills developed to such a degree that little time for conceptual development is left. Instead, the conceptual understanding is what is emphasized and demanded. Thus, instruments measuring "head-to-head" performances of traditional and reformed curricula can have severely compromised validity. The studies that have been done (Bookman & Friedman, 1994, Heid, 1984) generally show that students from reformed curricula tend to do better on conceptually-oriented questions than their "traditional" counterparts. Similarly, students from the traditional curricula tend to do well on procedurally-oriented questions, despite a lack of conceptual understanding (Selden, Selden & Mason, 1994, Ferrini-Mundy & Graham, 1994). Johnson (1995) found that students from traditional calculus courses tended to outperform their reformed counterparts in subsequent mathematics courses. Other general observations are that retention rate (both during the course and in taking
subsequent courses) is higher among reform classes, and there tend to be both fewer A's and fewer failures in reformed classrooms (Assessing Calculus Reform Efforts, 1995).

Limits

Students’ understanding of the mathematical concept of limit merits scholarly inquiry due to the fundamental role the limit concept plays in calculus and mathematical analysis. Failure to learn the limit concept correctly or completely has debilitating consequences in subsequent mathematics courses.

Cognitive Obstacles

A number of researchers (Cornu, 1991, Tall and Schwartzengerger, 1978, Williams, 1991, Monaghan, 1991), primarily studying mathematics students at the late high school or early college level, have determined that students face both epistemological and cognitive obstacles in their study of limits. In his study of students' concepts of limit, Bernard Cornu (1991) coined the term spontaneous conception to refer to ideas formed prior to formal teaching. The spontaneous conceptions tend not to enhance the ideas rigorously introduced in class, but rather lead to conflicting notions held simultaneously in the mind of an individual. Similarly, Tall and Schwarzenberger (1978), in questioning first year university students in mathematics whether .99999... is equal to 1, discovered the majority not only
answered incorrectly, but exhibited a wide range of incorrect reasoning to support their claims. According to the authors, these incorrect responses stem from both conscious and subconscious conflicts held by the students. Seeds for conscious conflict are sown when a subtle, high-level concept like limit is brought down to the level of the students in such a way that precision is compromised, making ambiguous the things that were originally clear. For example, when instructors, under the guise of simplifying the limit concept for their students, replace rigorous epsilon-delta arguments with vague notions of "closeness", students are confronted with a new concept with little means for coming to terms with it. Unconscious conflict has a more nebulous genesis, and seems to occur simply due to the complications inherent in limits and infinity.

**Difficulties With Language**

When administering his own questionnaire, Williams (1991) found that "... an overwhelming majority of students ... said on the initial questionnaire that a limit was unreachable" (p 225). Ferrini-Mundy and Graham (1994) found similar results in profiling a student named Sandy, who struggled with the question of whether .999... is equal to 1. Sandy argues that, "You can get as close as you can which would be .999..., but it wouldn't be quite 1. For the work we do in limits, .999... would be close enough to solve the problems that we needed to solve" (p 38). Ferrini-Mundy and Graham go on to remark, "It appears that the traditional language used to explain the
limit notion to students may be helping Sandy to shape a very fuzzy concept of what is meant by finding the limit of a function at a particular point” (p 38). Such a remark suggests that students may have been experiencing cognitive difficulties due to the everyday English meaning of the word limit, as was discussed by Monaghan (1991). Monaghan found, for example, that when students were asked to use the word “limit” in a sentence using any context except speed limit, almost all used a case of physical limit, such as the height one can jump. Frid (1994) also discovered that “…students using limit notation and terminology ascribed meanings to limits by interpreting ‘limit’ as an everyday language term, making reference to such things as barriers, swimming endurance and borders” (p 89). Such a limit is highly unlikely to be passed, and thereby further cements a student’s perception of a limit as something unsurpassable, if not unreachable. Similar difficulties were found with the terms “converges”, “tends to”, and “approaches.”

Theoretical Pedagogical Strategies

Cornu suggests the use of tasks that engage the students in reflective abstraction and concept image development (that is, development of the immediate mental picture evoked when the word “limit” is mentioned) as a means of helping students to acquire the elusive ideas. He recognizes the difficulty students must have in grasping the notion of limit, for it was an elusive idea that historically plagued mathematicians developing its use. He then enumerates four epistemological obstacles
that historically inhibited mathematicians as early as 430 B.C. from developing a robust understanding of limit. They are: (1) the failure to link geometry with numbers, (2) the notion of the infinitely large and the infinitely small, (3) the metaphysical aspect of limit, and (4) whether or not the limit is obtained. In discussing these obstacles, Norman and Pritchard (1994) add,

We wonder why, if it took mathematicians such a long time to formalize the notion of limit, we should expect students to understand adequately the rather unmotivated formalized version presented in calculus courses -- and in one class period at that! (p 74)

Perhaps not surprisingly, the last three of these obstacles historically plaguing the mathematicians were also found by Williams (1991), in his study of second-semester calculus students. He discovered that even the most deliberately applied, focused treatments were largely unsuccessful in dislodging students' misconceptions about limits. For example, no student out of the ten interviewed was completely convinced that "... plugging a finite number of points into a function does not always give a correct idea about the limit" (ibid., p229). This is likely because such a strategy works with all the problems students have been assigned for homework. Another more metaphysical conflict was faced when Williams probed his subjects' understanding of whether or not the limit is reached. Several previous studies (Cornu, 1991, Tall and Schwartzzenberger, 1978, Monaghan, 1991), had shown students to be steeped in the misconception that a function can never achieve its limit value. At times when this happens, students are willing even to distort reality in order to preserve their current perspective. Williams also found this phenomenon among his students, and
poignantly evident in one subject in particular. He asked her to consider the following problem:

Define a function \( f(x) \) by letting \( f(x) \) be the distance from a certain train to the station at time \( x \), where \( x \) is measured in hours after 12:00 noon on March 1, 1989. At exactly 2:00 p.m. on that day, the train arrives and comes to a complete stop at the station. Discuss the limit of \( f(x) \) as \( x \) approaches 2.

It was clear to her that the train most certainly would reach the station, giving both a limit of 0 and a function value of 0. This, however, conflicted with her belief that a function cannot reach its limit. Thus, she found a way to distort reality to accommodate her misconception:

I thought about the train example [laughter]. This is going to get really philosophical, maybe, but like the train comes to a stop at like a certain time, right, but it’s ... what is stopped? I mean, you can question the words, “no movement.” Does anything ever stop moving? I mean, is there such a thing as no motion? And you can say, slows down enough for passengers to come in and get on it, but what is exactly stopping, unless there is like, I suppose at a certain temperature level, there’s absolutely no motion, otherwise, how do you define stopping? (p 227)

This subject was also confronted with a different counterexample to her misconception that a function cannot reach its limit. She was asked to consider a constant function, a mathematical object with which she was well-acquainted. She quickly recognized that the limit of this function was the function’s value, but then just as quickly wondered, “Would you really call this a function, where the \( x \)-value is totally irrelevant to the \( y \)-value? See, I wouldn’t even consider that a function.” (p 227) Again, the subject changed her perception of reality to coincide with her beliefs. As Williams notes,
"Because she didn't consider the constant function to be a function, the problem of its reaching its limit was effectively mitigated." (p 227)

Towards countering these seemingly immovable cognitive obstacles, Williams details a pedagogical strategy designed by Nussbaum and Novick (1982) to force students into a state of mental disequilibrium, in which changing previous notions about the limit concept is both natural and desirable. The strategy is to arrange a three-part instructional sequence for students to pass through, involving an *exposing event*, a *discrepant event*, and a period of *resolution*. The exposing event requires a student to disclose his or her current thoughts about a concept through answering questions (by writing or being interviewed). The instructor can then respond to this disclosure with a discrepant event, one that points out an example of a case the student may not have thought of, that was either left out of or misunderstood in the exposing event. The intended result of these two stages is resolution: the learner reconciles the given "new case" with his or her current understanding by updating the previous understanding. It is important here to note that, unlike the first two stages of exposing event and discrepant event, the stage resolution is almost impervious to the actions of the instructor. That is, the first two stages are planned out and orchestrated by the instructor, who must then wait hopefully for the stage of resolution stage to take place. However, it has been discovered (Ferrini-Mundy and Graham, 1994, Williams, 1991, Sierpinska, 1987) that students are quite willing to live with the mathematical inconsistencies that instructors go to such lengths to point out. Some students have
been found (Ferrini-Mundy and Graham, 1994) to have a seemingly incompatible faith and mistrust in mathematics. A research subject Sandy remarked that, "You just have to practice the problems and then you understand", and yet, "There's always, like, one exception or two exceptions to ... every rule that's supposed to happen all the time."

As Ferrini-Mundy and Graham caution,

...if students are comfortable with the inconsistencies, contradictions, and competing meanings that emerge as a result, then the challenge of helping them reach a workable means of connecting these representations is very complex. (p 44)

Similarly, Williams found that even after going through this three-step process, many still clung to misconceptions they maintained throughout the treatment by simply ignoring or viewing as pathological any conflicting cases.

This rather disappointing finding by Williams may be redeemed by the research of Anna Sierpinska (1987) who, in an ethnographic study of a small group of seventeen-year-old humanities students, also found that epistemological obstacles abounded in the topic areas of limits. Through interviews, organized pedagogical sessions, and qualitative analysis, she determined four notions that were the main sources of these obstacles: scientific knowledge, infinity, function, and real number. These contextual areas are much broader than the ones enumerated by Cornu, but underscore some of the same problems, such as discerning the difference between "arbitrarily small, positive number", and zero.

Using infinite series as a mathematical context, she discovered a valuable fact: counterexamples alone, even when validated by mathematical proofs, will not upset
the conceptions of students unless their attitudes towards the validity, reliability, and accessibility of mathematical knowledge change. Validity of a mathematical statement refers to its perceived truthfulness. Students might be willing to accept a seemingly questionable conjecture on faith, simply because they do not realize the value and basis of mathematical proof. Further, a student who doubts the reliability of mathematical truth in general may feel there is room for "mathematical opinion." Thus it would be difficult for this student to fully acquire a mathematical concept. Finally, if a student finds mathematical results to be generally inaccessible to them (because of lack of confidence or a weak background, for example), a counterexample to a student-offered conjecture will have little, if any, impact.

If students feel no ownership towards a mathematical statement, even the reading of one with which they disagree will not provide the tension necessary for Novick and Nussbaum's discrepant event to be realized. This implies, however, that there is hope in the three-step pedagogy employed by Williams, if student attitudes are carefully regarded. Sierpinska, for example, discovered that students' attitudes towards the result \(0.99999... = 1\) were largely determined by their attitudes towards mathematical knowledge and the infinite. As a result, she subsequently developed eight classes (with three subclasses each) into which her students might fall, depending on their attitudes towards infinity.

From the preceding discussion, it is clear that some type of pedagogical treatment is necessary for the development of mathematical attitudes that are more
consonant with the thoughtful accepting or wrestling with a mathematical statement.

Along these lines, Fischbein, Tirosh, and Hess (1979) studied students from fifth grade
to ninth grade to determine their current understanding of the concept of infinity.

Before analyzing the data, they hypothesized, based on classroom experience, that
correctness of students' responses to questions posed on a written questionnaire would
remain fairly stable across age and grades. In other words, current teaching practice
does not significantly affect students' understanding of infinity. What they found
partially confirmed this hypothesis. They delineated the ideas of "intuition of infinity"
and "concept of infinity" as follows: intuition is what we really feel as being true or
self-evident, while concept is what we accept as being true as a consequence of a
logical explicit analysis. Once this idea was separated into two components, they were
able to conclude that the concept of infinity may develop itself through formal training,
while intuition of infinity remains unchanged. This leads to the question posed earlier,
"What can help students develop their intuition of infinity? Are there pedagogical
treatments that facilitate this development?"

At this point, the traditional mode of teaching collegiate mathematics via
lecturing and giving periodic timed, in-class exams, has been challenged (Calculus, The
Dynamics of Change, 1996), and several alternatives have been suggested. These
include, among many others, cooperative groupwork (Reynolds, et al., 1995), frequent
visits to the computer laboratory (Asiala, et al. 1996), judicious use of hand-held
calculators in the classroom and on exams (Shoaf-Grubbs, 1994), and writing
assignments (Using Writing to Teach Mathematics, 1991, Havens, 1989, Keith, 1988, Mett, 1988, Price, 1989). Because of growing interest among mathematics educators in writing as a vehicle for learning, the call for writing assignments by the participants at the Sloan Conference, and the need to select one pedagogical variable in this study, writing was chosen as a focus. Thus the previous question can be narrowed to, "Are there writing tasks that facilitate the development of one's intuition of infinity, improving one's chances of understanding limits?"

Admittedly, the literature is not particularly hopeful in the existence of any pedagogical treatments positively affecting students' understanding of limits. As was mentioned previously, students seem to have inevitable cognitive conflict with limits and the related ideas of continuity, differentiability, and infinity (Orton, 1983; Tall, 1992). And, as say Cottrill, et al., (1996, p 172), "We have not ... found any reports of success in helping students overcome these difficulties." Similar results were experienced by Williams (1991), who found that even though he was able, through diligent interviewing efforts, to place his students in cognitive conflict, he still remarks (p 229), "The stage was set for cognitive conflict, and in fact, some conflict did occur. What did not occur was real cognitive change." At this point it can be noted that both Williams and Cottrill and his colleagues used student interviews as their medium for initiating cognitive conflict, and found that this was not sufficient to result in resolution on the part of their students. Perhaps a medium that was more likely to force resolution was needed. If students must complete a writing assignment having them
disclose their current conceptions and reconcile them with what they are learning in class, they must at least face the disparities (if not resolve them) before handing in the assignment. Thus, this study seeks to examine a pedagogical treatment that could be a much needed catalyst for more permanent cognitive change.

Writing

Researchers and scholars in various fields have long attributed intellectual benefits to the practice of writing (Sanders and Littlefield, 1975, Emig, 1977, Luria, 1971). In a very powerful paragraph supporting writing as a means for developing necessary thought processes, cognitive psychologist Luria states,

Written speech ... assumes a much slower, repeated mediating process of analysis and synthesis, which makes it possible not only to develop the required thought, but even to revert to its earlier stages, thus transforming the sequential chain of connections in a simultaneous, self-reviewing structure. Written speech thus represents a new and powerful instrument of thought. (p 118)

Luria later points out that this slower pace of written speech compared to audible speech actually encourages the moving among past, present and even future stages of understanding. Emig (1977) similarly supports the use of writing as a vehicle for successful learning, remarking, “The medium of written verbal language requires the establishment of systematic connections and relationships” (p 126).
Writing in Mathematics

Within the specific discipline of mathematics, the use of writing as an instructional tool has been receiving a great deal of support in mathematics education literature (Havens, 1989, Keith, 1988, Mett, 1988, Price, 1989, Petersen and Nahrgang, 1986). Several benefits have been proposed, including a better understanding of conceptual relationships (Pearce and Davison, 1988, Birken, 1989) and a facilitation of “personal ownership” of knowledge (Connolly, 1989, Mett, 1989). In fact, as was mentioned earlier, the participants at the Sloan Conference/Workshop on Calculus Instruction, meeting in 1986, called for regular writing assignments in calculus courses. This came in an attempt to help students develop ownership of the mathematical ideas they were studying. However, despite various claims and expectations of the beneficial effects of writing on learning mathematics, little research evidence has tangibly demonstrated these effects. As Powell and Lopez (1989) remark,

A number of mathematics educators have asserted that writing facilitates mathematics learning; however, little evidence of students’ conceptual development or increased mathematical maturity has been proffered to support the reasonableness of this assertion.  (p 160)

In a similar vein, Birken (1989) comments,

Although I am confident that my students learn more about mathematics when they write and my students confirm repeatedly, both formally through questionnaires and informally in discussion, that they have a much deeper level of understanding of the concepts in a course and the connections between these concepts and between related courses when they write, it is time to set up research studies and publish the results. (p 35)
In attempting to do just this, Powell and Lopez perform a case study in which a student of Developmental Mathematics submits a series of journal entries and a series of freewritings created over the course of a semester. The writings are analyzed in reference to a model developed by Britton and his colleagues (Britton et al., 1975) in which are defined the terms expressive writing and transactional writing. According to Britton, expressive writing is “thinking aloud on paper” (p 89). It reveals the writer and verbalizes his or her consciousness. On the other hand, the more sophisticated transactional writing speaks with greater authority as it advises, persuades or instructs. According to Shepard (1993), “With respect to conceptual development, it is the transactional type of writing which may best facilitate cognitive changes” (p 289). As Powell and Lopez analyzed the written submissions of the subject through this lens, tangible results were found.

He constructed and reconstructed meaning. He wrote and revised his reflections. As Jose began to express his ideas with greater clarity and confidence and selected language that more accurately described his perceptions and actions, his writing shifted from expressive to transactional. (p 173-74)

... over time [Jose’s writing] shifted inward and began to include reflections that claimed that patterns were being noticed and that described his feelings in relation to assignments. (p 168)

Notice the similarity of Jose’s shift toward inner source of verification and the theoretical cognitive growth necessary for progressing from the externally-motivated action-level conception a subject might possess to the internally-motivated process.
Thus, this case study supports the claim that writing can have measurable benefits on mathematical learning.

In a study similar to the present study, Schurle (1991) compared the performances of two sections of a college course on differential equations. In the treated section, writing assignments were substituted for a portion of the traditional homework assignments. Schurle discovered that writing assignments did not have the effect of improving test scores. However, survey results indicated that students felt the writing assignments improved their comprehension. In the context of an elementary algebra course, Hirsch and King (1983) compared two sections; one section was given 15 assignments covering mathematics problems while the other was given 15 assignments requiring written responses to conceptual questions. The researchers went to great lengths to insure neither teacher knew whether his or her students were completing the writing assignments or the problem sets. An independent grader scored all assignments. No significant difference was found between groups on either pre- or post-performance measures. Hirsch and King concluded that when writing assignments are employed without teacher engagement, they are no more effective than traditional assignments.

In light of the relative paucity of research, as well as the varied and somewhat weak results, many more studies are needed, using different types of writing assignments, different mathematical contexts, and different research frameworks, to substantiate the beliefs of the mathematics education community at large.
When one actually attempts to develop writing tasks to facilitate mathematical learning among students, more precise learning outcomes must be defined and subsequently matched with appropriate writing activities (Shepard, 1993).

Recognizing this, Azzolino (1990) lists thirteen pedagogical goals associated with writing, among them: to help the student to summarize, organize, relate, and associate ideas; to provide an opportunity for a student to define, discuss, or describe an idea or concept; to permit the student to experiment with, create, or discover mathematics independently; and to assist in the translating or decoding of mathematical notation.

Clearly, writing assignments must be specifically crafted so as to realize these goals. Such crafting would require a more holistic perspective of the concept in question than would choosing a set of problems.

**Mathematical Writing in Practice**

To motivate students to improve their writing skills within the mathematics classroom and to help them reach that goal, Crannell (1990) has developed *A Guide to Writing in Mathematics Classes*, to be used by her own students, as well as by professors in other classes for their students. She develops strategies for students to employ towards strengthening their written communication and provides words and phrases proven to be helpful in getting mathematical ideas across to a lay audience.

She remarks, "... one of the simplest reasons for writing in a math class is that writing
helps you to learn the mathematics better. By explaining a difficult concept to other
people, you end up explaining it to yourself” (p 3).

In response to the call from the Sloan workshop participants, Smith and his
colleagues at Duke University have collaborated with members of the English
department in a "Writing Across the Curriculum Program" (Gopen & Smith, 1990).
Through extensive implementation and study of the program, the following discoveries
were made:

1. Thought and expression of thought are so inextricably intertwined for
students that improving one improves the other, and

2. Writing assignments in mathematics courses will improve student
comprehension.

Anecdotal evidence from others (Brosnan & Ralley, 1995, Rishel, 1994) corroborate
these discoveries, with claims that there is no substitute for writing in a mathematics
class, as it gets the concepts across differently than a problem set can. In fact, Tucker
(‘Calculus, The Dynamics of Change, 1996, p19) remarks that, "Other disciplines have
long known that to write is to think. Ideas only take shape when they are put into
works, sentences, and paragraphs." Besides being a natural thinking and learning
process, writing also provides a way for a researcher to see into the thought processes
of a subject. As Smith (ibid., p32) states, "The cleanest window we have is student
writing."
At this point, little research has been performed investigating the impact of writing tasks on undergraduate mathematics students' understanding of a given concept. This study contributes in that area by examining the way in which student writing assignments affect student comprehension of limits. There is good reason to believe that writing is a natural way for students to improve their understanding of the calculus. As Frid (1994) concluded, “Calculus instruction might be more successful for students if enabled students were more personally involved in the construction of their calculus conceptualizations” (p 93). Emig (1977) described writing as “...active, engaged, personal – notably self-rhythmed” (p 128). Thus one may conclude that a way to increase the success of calculus instruction is to involve students more personally in their own learning processes by having them write.
CHAPTER III

METHODOLOGY

This study addresses the following research questions using a combination of qualitative and quantitative methods:

1. What conception of limit (in terms of action-process-object-schema theory) are students reaching as a result of completing a course in second-semester calculus in which writing assignments are included?, and

2. Do writing assignments make a difference in students' achievement and understanding in limit-related concepts in second-semester calculus?

This chapter is a detailed account of the specific methodologies and procedures used in the present study. In the overview, each data-gathering and analytical phase of the study is very briefly mentioned, pausing only long enough to provide the reader with a sense of the structure of the study. The phase is then fully described in subsequent sections.

Overview

In the Fall semester of 1997, at Western Michigan University, the Department of Mathematics and Statistics offered six sections of second-semester calculus. In
order to administer a treatment to one of the sections and have a different section established as a control, two sections were needed to participate in the present study.

**Selection of Participating Sections**

Prior to the beginning of the fall semester, I distributed a memo to each of the six instructors scheduled to teach second-semester calculus, asking for volunteers to participate in the study. I received three positive responses. Based on the desire of minimizing both variables of “class meeting time” and “instructor style”, I chose the two instructors whose classes met in the middle of the day (11:00 a.m. and 12:00 noon). Because the instructor teaching at 11:00 a.m. had demonstrated professional interest in pedagogical issues, and in particular, alternative assessments, his section was designated as the treatment group. Because of the random nature of the enrollment process for two sections of the same class meeting at the same general time of day, the two sections were assumed to be equivalent in relation to mathematical ability and prior content understanding.

A survey measuring students’ beginning understanding of the limit concept (Appendix C) was administered to both the treatment group and the control group. The results of this survey revealed not only the participants’ initial individual understanding, but also provided data for comparison of the two sections. These data will be detailed in Chapter IV, but for now it should be noted that the two sections
were quite comparable in limit-understanding as measured by this instrument, with slightly fewer misconceptions demonstrated by the control group.

In the treatment section, writing assignments (to be discussed in further detail below) replaced some of the mathematical problem sets the instructor normally would have assigned. In the control section, no writing tasks were given, but rather, problem sets were more frequently assigned and collected. In order to isolate the variable being studied, an effort was made to eliminate or at least to minimize other classroom variables between the control group and the treatment group. The researcher met with the two participating instructors to establish that a common general teaching philosophy existed between the two. The instructors were asked to describe their general classroom procedures. Each spent the first few minutes addressing questions about previous homework assignments, then lectured on the new material. Further, both gave periodic in-class, timed examinations as their main form of assessment, and also collected and graded problem sets from problems out of the common textbook, *Calculus from Graphical, Numerical, and Symbolic Points of View* (Ostebee and Zorn, 1996). Finally, the two instructors followed the same department-created syllabus. Thus, the major difference in student assessment was the use of writing assignments in the treatment section. The control group instructor assured me that he would not assign any type of writing assignments like those to be used in the study. In order to verify the two sections maintained a comparable learning environment, the
two classrooms were observed by the researcher. This observation is detailed later in
the chapter.

The Treatment

Recall that this study was conducted based on the conjecture that tasks that
foster and initiate reflective abstraction will promote growth along the APOS
continuum of conceptual understanding. Thus, an effective treatment would require
such reflection. Based on a review of the literature concerning writing and on a pilot
study conducted in the summer of 1997, I concluded that writing tasks with certain
structures were conducive to eliciting reflective abstraction among students. In
particular, the pilot showed that unless a well-composed rubric was provided, students
were able to fulfill the letter of the directions without engaging in the reflective
abstraction I sought to initiate. For instance, I discovered that simply asking for an
example led to students parroting an example given in the book. However, when I
asked for an example and a non-example, with explanation (and further made sure this
information was not available in the textbook), much more thoughtful responses were
obtained.

The first writing assignment in the current study asked the students to reconcile
the formal definition of the limit (involving epsilon and delta) with a picture showing
the function graphed in an appropriate epsilon-delta window. This set the stage for
working with the limit, and made students consider it outside the trappings of a
specific mathematical context. The next four assignments all dealt with applications of
the limit process: arc length, improper integrals, L'Hopital's rule, and sequences and
series. The sixth and final writing assignment required students to give a holistic
perspective of the topics they had written about over the course of the semester. Here
students were asked to write about the thread of the limit concept running through the
previous five writing assignments.

Interviews

Because I was looking for cognitive growth as well as mathematical
performance, I needed more than numerical data. I wanted to examine the treatment
group's individual thought processes, and to do this would require a more fluid
medium than computational mathematics problems. Thus, I interviewed a subset (n =
5) of the students who participated in the writing treatment. (Students that were
willing to participate in the interviewing process were offered an hour of free tutoring
after the interviews had been completed for the semester as an incentive for
participating in the study. Three of the five accepted this free tutoring.) These
interviews took place three times during the semester, and lasted from one-half hour to
an hour. I had students explain concepts and work through problems out loud,
communicating their thought processes as they worked. Because I was looking for the
cognitive growth of students participating in writing assignments, there was no need to
select students to interview from the control group.
Classroom Observation

As was mentioned earlier, to ascertain the classroom climate that was developed in the control section and in the treatment section, I observed the two participating instructors teach their respective sections. I developed an observation checklist (Appendix B) that enabled me more objectively to assess pedagogical constructs like "opportunities for reflective abstraction" and "APOS level of question being posed to the students."

Means of Analysis

The two research questions posed required the use of different methodologies. To assess what level of conceptual understanding the students were reaching required qualitative data, while determining if writing tasks make a difference in students mathematical achievement required a comparison of quantitative data.

Qualitative Analysis

Qualitative data on the treatment group were gathered from several sources: limit questionnaires (Appendix C), writing tasks (Appendix A), interview transcripts (with the researcher), and end-of-semester surveys (Appendix D.) They were then analyzed according to APOS theory, developed by the Research in Undergraduate Mathematics Education Community. In particular, cognitive growth along the APOS continuum as a result of participating in writing assignments was sought.
Quantitative Analysis

To assess differences in mathematical performance required a means of comparison between the treatment group and the control group. The participating instructors and I developed three limit-based calculus problems (Appendix E) to include on the two sections' final examinations. (Three was the largest number of heavily limit-oriented problems that would normally be included on a second-semester calculus examination.) I graded these and compared the mean scores of the two sections on a problem-by-problem basis using a two-tailed two-sample t-test.

Initial Limit Survey

The initial limit survey (Appendix C), adapted from Williams (1991) was comprised of a list of six statements regarding the limit concept. Students were to mark each one true or false, then choose the one that most closely described their own point of view. Each statement has a misconception embedded within it, with the exception of the third statement. The six statements are:

1. A limit describes how a function moves as \( x \) moves toward a certain point.
2. A limit is a number or point past which a function cannot go.
3. A limit is a number that the \( y \)-values of a function can be made arbitrarily close to by restricting \( x \)-values.
4. A limit is a number or point the function gets close to but never reaches.
5. A limit is an approximation that can be made as accurate as you wish.
6. A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

The questionnaire served two purposes in this study. First, it was used simply to ascertain study participants' general perceptions of the limit, prior to any classroom experience in second-semester calculus. It was thus administered to both the treatment group and the control group during the first week of the course. The second purpose was that it could be used to classify the treatment students according to limit perspective. A careful review of the literature concerning limits and mathematics students (particularly Cottrill, et al, 1996, Sierpinska, 1987, Tall and Vinner, 1981, and Williams, 1990), along with several semester's experience teaching second-semester calculus, showed that students begin their second semester of calculus with a wide variety of concept images of the limit. As was previously mentioned, the intent of this study was to focus on students’ thought processes. Thus, a subset of the participants in the treatment group were interviewed. Based on the work of the scholars above, I wished to include as many different limit perspectives as possible within the small subset of interviewees. Based upon the results of the survey, I partitioned the treatment group into “perspective classes”, according to their responses. Then I randomly chose one or two students from each perspective class, until I had seven subjects. I then invited these seven to participate in my study; five accepted. These five represented four perspectives, statements 1, 2, 4 and 6 from above, and so comprised a heterogeneous group in terms of one’s limit concept image.
Writing Tasks

The main reason for developing writing tasks was to provide the treatment students a vehicle for reflective abstraction. Besides giving the students a chance for such abstraction in several limit-oriented contexts, the writing assignments also gave students an opportunity for seeing the limit as an object. It is certain that there are problems in the calculus textbook (Ostebee and Zorn, 1996) that, if done in a thorough manner, would also encourage an object-level perception of limit. For example, one may assign a sequence of questions that requires more and more sophisticated views of the limit within the case of, say, definite integrals. However, I had previously discovered during private tutoring sessions that students did not perform the reflective abstraction necessary to pull together the steps of the sequence. Instead, they would proceed through each step as if in isolation, then move quickly onto the next problem. There was no connection made between the processes of the first few steps and the object of the final step. I felt if I could develop writing tasks that would prolong their engagement and require them to make this connection, it would be worth pursuing.

Description of the Writing Tasks

Over the course of the semester, I developed limit-based writing tasks (see Appendix A) to be assigned to the treatment group. The treatment instructor and I decided that within a fourteen-week period, six writing tasks were as many as could comfortable be assigned without overwhelming the students. This figure was also
based on the writing of Stehney (1991), who recommended that when requiring students to hand in four-to-six page mathematical essays, three or four papers a term is a reasonable expectation. Because the questions I posed required only two or three pages to answer, I raised the number of papers to six.

Initially, factoring in time for test-taking and Thanksgiving break, this implied that a writing task would be assigned approximately every two weeks. However, because of a slow start to the implementation of the writing tasks, the first one was not collected, assessed, and returned until the sixth week of class. Hence, a writing task had to be assigned almost once a week thereafter, to insure that all six would be assigned by the end of the semester. This certainly intensified the students' academic focus onto writing tasks during the second half of the course. This intensity might have been advantageous if the students were learning how to get more out of the writing from each task. They would have less time to forget what strategies seemed to work well for them. This may have had the disadvantage of "wearing students out" in terms of writing; students may have tapered off in their efforts simply due to writing fatigue.

Prior to the study, it was decided that the topics of improper integrals, L'Hopital's rule, and sequences and series would be included in the writing assignments to be completed by the treatment group. The use of other topics would emerge according to the material covered in class at the appropriate time of assigning the next writing task. As a result, the first paper was an introduction to the limit.
concept; the second dealt with arc length (an application of limits at an appropriate
time for a writing task); the third was about improper integration; the fourth required
explanations of L'Hopital's rule, the $n$th term test for divergence of a series, and the
definition of the limit of a sequence; the fifth paper had the students discern between
sequences and series, and the sixth involved reflecting on the first five writing
assignments.

When the tasks were assigned by the treatment instructor, a page of
instructions including a scoring rubric was given to each student. I required an
expository writing about two or three pages in length, that addressed all the questions
posed in the task. The students were given a week to complete the task, and each task
was designed to focus the students' thinking on the limit concept. According to my
classroom observation and informal conversations with them, students made the
natural assumption that the tasks were developed by the treatment instructor; most did
not know I was involved at all.

Although the set-up for each writing task was slightly different, the basic
premise was to get students to reconcile their concept images with their concept
definitions. The pedagogical device I used was to have students reword a concept as
though they were communicating with a student who had asked for tutoring. This was
based on a suggestion from Azzolino (1990), who states, "Rewording is a way of
demonstrating understanding." This form of writing gave students incentive for

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explaining concepts in their own words, and in some cases, seemed to make talking about mathematics a more natural part of their experience.

Development of Each Task

A large part of the work involved in this study was the development of writing tasks that lent themselves to student engagement. In addition to attempting this, I also needed to create opportunities for students to see, to achieve, and to demonstrate action-, process-, and object-level conceptions of the limit. The following is a description of the development of each writing task and how I anticipated enabling students to progress along the APOS continuum.

Development of Writing Task #1

The goal of the first writing task was to refresh students’ memory about the limit concept and, more specifically, to have them reconcile their limit concept image with the rigorous epsilon-delta definition given in the textbook (Ostebee and Zorn, 1996). The text provided a picture of a continuous function within an epsilon-delta window, and showed readers graphically that if the limit of a given function was $L$, then for every $x$-value that was chosen between $x - \delta$ and $x + \delta$, the output would be between $L - \varepsilon$ and $L + \varepsilon$. If one did not have a solid understanding of limit, however, the well-drawn picture provided by the authors could seem unconnected to the rigorous definition.
To the end of helping them to build a more solid understanding, I had students write, as if to a student they were tutoring, a reconciliation of these two representations of the limit concept. To do this, they were to choose a specific function \( f(x) \) and a specific value of \( a \), determine the limit using whatever method they wished (with explanation), choose an epsilon less than 1 and determine the resulting delta, and sketch the graph of their chosen function in the corresponding epsilon-delta window.

The successful completion of this task would require a process-level understanding of the limit concept. If students were unable to evaluate a limit by any other means than evaluating the function at the point in question (that is, if they had only an action-level concept of limit), they would not understand the relationship between epsilon and delta. This is because to visualize delta as depending on epsilon requires one to envision the process of evaluating the limit indefinitely many times without actually performing the evaluations. Furthermore, no knee-jerk response to any kind of idea developed in class would enable a student to determine delta based on the chosen epsilon and function.

**Development of Writing Task #2**

This task was developed to compel students to reconcile the formal definition (that is, the formula) of arc length with the approximation obtained by setting up a sum of lengths of line segments. In particular, I wanted them to see an application
connecting a sum and a definite integral. They were to choose a non-linear function $f(x)$ and sketch it over a chosen interval $[a, b]$; divide $[a, b]$ into four subintervals of equal length; set up a sum of lengths of four line segments accompanied by an appropriate sketch; evaluate both the sum and the definite integral $\int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx$; and determine why the sum underestimated the true answer obtained by the definite integral.

The successful completion of this writing task again required at least a process-level conception of the limit. In particular, one would not realize that the sum would always underestimate the definite integral unless one could envision finer and finer partitions of the $x$-axis, with line segments that were necessarily shorter than the curve's length over that subinterval. An action-level concept would therefore be inadequate.

Upon completing this writing task, a student would have had the chance to see the limit as an object. In this case, that object would be a number that represents the length of a curve over an interval. Here, students who initially thought that a limit could be made as accurate as desired would have the chance to see the limit as a fixed object. It is the approximations that are getting more and more accurate; the length of a curve is a fixed entity. Theoretically, this exercise would help students to encapsulate the process of finding a limit (picturing finer and finer summations) to the object represented by the definite integral.
Development of Writing Task #3

The goal of the third writing task was to have students realize that an integral with infinite limits of integration can have a finite value. This is both conceptually challenging and counter-intuitive. To enable students to wrestle with this cognitive hurdle, I asked them to evaluate three given improper integrals (using an appropriate limit technique); provide an accompanying graph with each that represents the value determined to be the answer; and write a paragraph explaining each of four possible cases of infinite or finite limits of integration and infinite or finite value of the integral.

It is clear in this setting that the action-level response of evaluating the function at the point in question will prove completely inadequate. That response would translate to evaluating the function at a singularity, just by the nature of improper integrals. Hence at least a process-level conception was required. However, to write the required paragraph characterizing limits of proper integrals as finite or infinite implicitly demanded an object-level understanding of limit, for the writer of such a paragraph must refer to a limit as a (perhaps infinite) number. At the very least, students see the limit as an object upon successfully completing this writing task.

Development of Writing Task #4

In order to help students to prepare for an upcoming exam requiring a paragraph of writing about limits, the fourth writing task required a different structure than usual. Instead of one theme, developed and maintained over two or three pages,
this task called for three short, unrelated paragraphs exploring the limit in the contexts of L'Hopital's rule, the definition of a sequence, and the \( n \)-th term test for divergence of series. Each paragraph will be discussed.

Students first had to detail under what circumstances L'Hopital's rule could be correctly used, then describe the process of its application. This required only an action-level concept of limit, because students could rely on their textbook for examples and explanations, and choose ratios of continuous functions to which to apply L'Hopital's rule. This produced limits that were truly best evaluated by "plugging in points", namely, the point \( a \) that \( x \) approached (since for continuous functions, \( \lim_{x \to a} f(x) = f(a) \)). Thus, this part of the task served mostly as a warm-up to the next two parts.

The second paragraph asked students to describe how one computes the limit of a sequence. They were to include the definition of the limit of a sequence, an example of a convergent sequence whose limit is not 0, and an example of a divergent sequence. To create a convergent sequence whose limit is not 0 (or a divergent sequence) required a process-level understanding of the limit concept. Action-level conceptions only allow a subject to react to a given mathematical object, not to create one from scratch that has certain desired properties. Further, this exercise allowed students to see the limit as an object, namely, the number to which a sequence converges.
The third paragraph required a discussion of the inverse of the $n$th-term test for divergence of a series. Specifically, students were to determine whether or not a series would converge if the limit of the terms of the related sequence was 0. This task required a process-level understanding of the limit concept, as students would have to envision adding the terms of an infinite sum.

Development of Writing Task #5

Writing Task #5 was the least structured of the six tasks. After studying sequences and series in conjunction for two weeks, students were asked to identify similarities and differences between them. The successful completion of this task required an object-level limit conception because, for the first time, students had to identify a series as a sum, that is, as a fixed mathematical object. Process-level conception was required for other parts of the task, such as examining the sequence of partial sums, but would be inadequate when trying to describe a series as a (perhaps infinite) number.

Development of Writing Task #6

The final writing task was developed to give students a tangible context for reflecting upon the previous five writing assignments. Besides choosing the one they found to be the most helpful, they also had to address three misconceptions that had emerged among the students during the study. They were: (1) the limit of a function
can be made as accurate as desired, (2) the sum of a list of decreasing, positive terms is a finite number, and (3) a shape with infinite length must have infinite area. Notice that to address any one of these three misconceptions required an object-level understanding of the limit concept, for all three show implicitly (1) or explicitly (2 and 3) that the limit is a fixed and immovable object.

**Rubrics**

A rubric is an assessment guide which highlights general performance expectations for a given task. It differs from an answer key in that it contains guidelines rather than specific responses. Rubrics may be developed exclusively for the instructor or may be shared with students to steer their work.

Rubrics for these writing tasks were developed for two reasons: to help the scoring be more objective and to help the students be thorough, on task, and engaged in reflective abstraction. The rubrics were provided right along with the directions for the task itself. I would give the students a brief set-up outlining the limit-based concept to be explored and the context in which they were to write. I then provided bullets -- short descriptions of elements they must include in their tasks in order to receive full credit -- with point allocations for each bullet. The descriptions for what must be included in the tasks became progressively more vague to allow students to develop some mathematical autonomy as they grew in their conceptual understandings.
For example, below in Figure 1 is the rubric included in the assignment of Writing Task #2, dealing with arc length.

_The purpose of this task is to explore the relationship between Riemann sums and definite integrals by considering the graph of a curve and estimating its arc length. To do this,
- choose a non-linear function \( f(x) \) and an interval \([a, b]\) and sketch \( f(x) \) over this interval
- divide \([a, b]\) into four subintervals of equal length
- set up and evaluate a Riemann sum (with four terms) that approximates the arc length of \( f(x) \) over \([a, b]\)
- draw line segments on the sketch of \( f(x) \) whose lengths represent the terms in your Riemann sum
- determine the arc length of your curve "exactly" over the given interval using the integral formula above ... you will likely have to use numerical methods._

Write up this assignment in paragraph form, word-processed, with sketches provided (hand drawn is fine.) In the concluding paragraph, address the question of why your approximation was an underestimate of the true arc length.

Each bullet is worth three points, and completing the assignment earns five points, for a total of 20 points possible.

Figure 1. Rubric for Writing Task #2.

The goal was to give the students enough structure so that they would fulfill the "spirit" of the assignment, while providing enough leeway for creativity and open-ended exploration. The former was achieved by first completing the assignment myself, then writing out instructions that would encourage the thought processes I went through. The latter was achieved by allowing students to select their own examples to explore and discuss.
Evaluation

Finally, a very important part of the treatment process was the prompt and abundant feedback students received from me in the form of detailed, personally tailored comments in the margins of their submitted papers, accompanied by a numerical score. With the exception of the first writing task, all were returned to the students by the next class meeting, filled with suggestions for improvement, gentle nudging towards more complete understanding, and counterexamples to broad and incorrect statements made within the papers. I also provided the subjects with a “model solution” (Appendix A) that I had written as though I were a second-semester calculus student, including all the elements that were requested in the scoring rubric.

Implementation

After developing the writing tasks, I gave them, one at a time, to the treatment section’s instructor to distribute in class. There was no pattern to how the writing tasks were given to students; sometimes it was at the beginning of class, sometimes it was at the end. Prior to classtime, the instructor and I would discuss what helpful oral instructions should accompany the tasks. I discovered from observation that the instructions provided were a subset of the ones we had discussed, and were usually given in a voice that could not compete with the general din of the classroom. I was struck by the apparent lack of impact of the instructions, as many students would not even stop talking while the instructor provided an explanation of the assignment.
The students were given a week to complete the tasks, after which the instructor would collect them and give them to me for grading. The instructor and I had hour-long debriefing sessions after I had assessed the writing tasks, in which I told him general trends and issues that merited more classroom discussion, as they were missed by so many students. Again a subset of the misconceptions were mentioned, either at the beginning of class, when the bustle of latecomers and talkers overwhelmed the instructor's voice, or at the end of class, when students had begun to pack up and all but ceased to listen. In either case, their perceived impact was minimal. For example, based on my classroom observation, the instructor told his students repeatedly the value of reading the model solutions distributed in class. However, during the third interview, I asked a treatment subject if he ever read the model solutions. "No," he said. "Why, are we supposed to?" (I3.3). Apparently, the admonition to read the models had not impacted him.

Interviews

As was previously mentioned, interviews probing understanding of the limit concept were conducted with five subjects from the treatment group. These five were selected on the basis of their responses to a limit perspective true-false questionnaire (Appendix C); a heterogeneous group was desired to ascertain if certain initial limit perspectives lessened or increased the effects of the writing tasks.
I used the interviewing process as a means for discerning cognitive growth. In particular, I wanted to see if a given subject's limit conception would grow in sophistication as writing tasks were performed. I looked for action-level conceptions to progress to process-level conceptions, and from process-level to object-level. For example, when a subject named Evan was asked in the first interview to describe his concept of a limit, he remarked,

Um, like if you gave me a graph, and you had like any kind of function there, and you said as $x$ approaches 2, which is there, I could say like, that value of it, whatever $f(x)$ is at 2. Now I could tell him what that is. Now, is that right? Like if you had some ... at $x = 2$, you put 2 into the $f(x)$ equation, isn't that what the limit is right there? (II, 1)

Because he clearly identifies the limit as "you put the 2 into the $f(x)$ equation," the subject had an action-level conception of the limit. However, this same subject said in the third interview,

You can evaluate the limit like at a certain $x$ and your $f(x)$ is going to be, say, 0.9, 0.99, 0.999. But your limit is still going to be one. This (probably referring to the value of $f(x)$) can change but your limit stays the same. (Evan, 13, 5)

Recall that to realize that the limit is an immovable number regardless of the input variable is a characterization of object-level conception. In discussing the concept of arc length, Evan wrote,

We used Riemann sums, breaking up a curve into small subdivisions, and we found the length of each subdivision's segment. Next we added all the segments to get an estimated arc length. Summing all of these lengths turned out to be an integral problem. If we took the limit of a Riemann sum, we got a definite integral that was the length of the arc over the interval. (WT6)
Again, Evan described a limit as a fixed number, in this case, the length of a curve over a given interval. In a different context, he is exhibiting a sign of object-level understanding. Thus, in the case of this subject, I was able to discern cognitive growth over the course of the treatment.

**Implementation**

A total of fifteen interviews took place during the data-gathering phase; three interviews each for five research subjects. The interviewing format I chose was based on the *standardized open-ended interview*, defined by Patton (1980) as,

> ... a set of questions carefully worded and arranged with the intention of taking each respondent through the same sequence and asking each respondent the same questions with essentially the same words. Flexibility in probing is more or less limited, depending on the nature of the interview ... (p 198)

The probing of the understanding level of subjects in this study was flexible; I began the interviews with a well-organized list of open-ended questions, but allowed myself to prolong the investigation of certain topics by continuing to ask germane questions, based on the subject's original response. Also, when a question of mine drew more than thirty seconds of silence from a subject, I would rephrase the question or provide a gentle prompt. The strengths of this type of interview, as enumerated by Patton (1980), include that all the respondents answer the same questions, thus increasing comparability of responses, and that the organization and analysis of data are facilitated by the initial structure of the interview. These strengths motivated my choice of interview type. However, weaknesses were mentioned as well: there is little
flexibility in relating the interview to particular individuals, and the standardized wording of questions may constrain and limit the naturalness and relevance of questions and answers. I was able to combat these weaknesses by not restricting myself to the questions previously developed for the interviews. Rather, they were used as a guide (all were eventually asked) but were not adhered to in a restricting manner. Thus I was able to relate the interview to the individual, but still maintain comparability of responses from different respondents.

Veteran interviewer and research scholar Douglas (1985), in discussing the interviewing process, writes,

Creative interviewing ... involves the use of many strategies and tactics of interaction, largely based on an understanding of friendly feelings and intimacy, to optimize cooperative, mutual disclosure and a creative search for mutual understanding. (p. 25)

Based on the desire to create friendly feelings, I scheduled interviews at a time mutually agreed upon by the subject and myself, and we met in a room relatively free of passerby traffic. Although the material we discussed was not sensitive, I discovered during the pilot study that students were inhibited if they felt as though others, particularly mathematics professors or students, could hear their responses. Thus every effort was made to strike a balance between too little privacy (where subjects felt embarrassed or reticent to speak) and too much privacy (where subjects felt intimidated by what seemed like an intense interviewing process.) Meeting in an out-of-the-way conference room with the door open seemed to strike that balance.
An interview’s length was a function of the subject’s length of responses and overall engagement with the interview. Typically, one lasted from a half-hour to an hour. I discovered during the pilot study that most mathematical conversation after an hour lacked the focus necessary to learn more about the subject’s thought processes.

Interview Protocol

Interview questions were based on Interview Guides developed by a subset of the Research in Undergraduate Mathematics Education Community who also had studied students’ understanding of the limit concept in terms of APOS theory. The Guides provided me a structure from which to begin. I also developed questions of my own to more specifically address certain topics. The pilot study also provided me with the opportunity to reshape and improve questions that were unfruitful. For example, some of the limit problems I had originally developed were too demanding technically. In one pilot case, I had asked a subject to evaluate the improper integral

$$\int_{0}^{3} \frac{1}{x^2 - x} \, dx$$

The function had too many singularities within the interval of integration for the subjects to deal with. In most cases, the subject would work so hard on the computational aspects of the problem that any generality about limits that one might learn was lost in the technical details. I was able to eliminate much, but not all, of this phenomenon in the research study, by revising my questions.
I sought to achieve a balance between prompting enough to get the subject to reveal his or her understanding but not so much that the interview became a tutoring session. Indeed, I guarded most carefully against the latter situation to protect my study from contamination. I anticipated questions regarding the material might arise, so I warned the students ahead of time that I would not be able to answer their limit-oriented questions until after the study was over. The very few times the subjects asked me questions about limits, I reminded them of our agreement that I would not answer questions until after the study was completed. The full set of interview questions from each of the three sessions may be found in Appendix F.

Classroom Observation

Recall that, in order best to isolate the writing variable, every effort was made to eliminate or at least to minimize other differences between the two sections. As was mentioned, the textbook and syllabus used were identical. The variable with the most impact then, that couldn’t be eliminated due to scheduling constraints, was the pedagogical climate each instructor effected in the classroom. To assess this, I observed the instructors, eight times apiece, as they taught their classes. (The observations were roughly once every two weeks, but became more frequent as the semester progressed.) Because the writing tasks were a vehicle for reflective abstraction, I was particularly concerned about a discrepancy between the two classrooms in opportunities for such abstraction. Based on APOS theory, I developed
a checklist (Appendix B) that enabled me to record instances of such opportunities. I recorded phenomena such as length of the instructor's wait time after posing a question, APOS level of question being posed by the instructor, and engagement of the students in the lecture.

End-of-Semester Surveys

To learn about the students' attitudes towards the writing assignments they had completed, an end-of-semester survey (Appendix D) was administered. The students were asked, among other things, which writing assignment was the most helpful, if they worked alone or with a classmate, and if they had further comments about the writing assignments. These data were not specifically analyzed, but will be informally reported in Chapters IV and V.

Analysis of Data

Two broad types of data were collected and analyzed towards answering my research questions, qualitative and quantitative. The main emphasis was the qualitative data, which came from several sources. The quantitative data were used to verify and support the qualitative.
Qualitative Data

As explained in the overview, qualitative data were gathered from student responses to the initial limit perspective questionnaires, interview transcripts, the writing tasks, and exit surveys. These were elicited for the purpose of assessing a research subject's level of understanding according to APOS theory. I first studied the individual as a unit, profiling five subjects to discern some of the causes of their cognitive growth during the semester.

By design, I had several sources of qualitative data for each individual, namely, the limit surveys, the three interview transcripts, the writing tasks, and the end-of-semester survey. In my proposal, I had originally planned to triangulate the data (that is, to verify a theoretical position from three different data sources) by listening to the students in their interviews, reading what the students wrote in their papers, and observing how the students spoke and behaved in class. However, it soon became clear from classroom observation that due to the teacher-centered atmosphere, little in the way of student behavior or language could be observed. Hence, I had the two main data sources of interviews and writing tasks to cross-verify one another. In determining APOS levels of the five students profiled, I corroborated theoretical statements from two data sources.

After learning some specific trends among the five, I then studied the treatment class as a unit. In particular, I searched the data for a collective level of understanding
among the subjects, and common trends in their cognitive growth, to see if the class exhibited some of the same characteristics as the individuals.

**Individual as a Unit**

For a more in-depth look at students' conceptual understanding of the limit process, I looked at five subjects in the treatment group. Besides the initial limit questionnaire, the writing tasks, and the end-of-semester survey, I also had three interview transcripts per subject to analyze. This provided ample opportunity to establish an initial limit-conception level, as well as to ascertain growth along the APOS continuum. By again using the genetic decomposition, I was able to identify action-, process- and some object-level thinking among the subjects. The combined findings of the two methods of analysis enabled me to answer my first research question. "What conception of limit (in terms of action-process-object-schema theory) are students reaching as a result of passing through a course in second-semester calculus in which writing activities are included?"

**Class as a Unit**

My main means of assessing the class as a unit was through the writing tasks they submitted. I gathered them into six files and read them for evidence of understanding of the limit concept. The writing tasks served largely to present subjects with an object view of the limit within a specific mathematical context. Upon
discovering this, I began searching for evidence that subjects had, in fact, been able to take advantage of this example of "limit-as-object," rather than becoming too entrenched in the details to benefit in this way.

In determining the APOS level of understanding the class had collectively reached, I focused on one student at a time, then aggregated my findings. To consistently assess APOS level across writing tasks and among subjects, I relied heavily on the genetic decomposition developed by RUMEC, provided in Chapter II and in Appendix G. I had to extrapolate from the general limit context in which RUMEC worked to a more specific one, but the extrapolation was a natural extension of their work, and enabled me to maintain a higher level of objectivity as I analyzed the data.

For example, according to APOS theory, a subject is working at the process level of limit understanding if the subject can envision evaluating the function at infinitely many points as $x$ approaches a given value, without having to perform these evaluations. For this study, it was necessary to know what this meant in the specific context of each limit topic the subjects explored. If a subject was working within the context of arc length, for example, a process-level conception would translate as the ability to envision finer and finer partitions of the $x$-axis, yielding to better and better estimates of the arc's true length without having to perform the computations necessary to calculate these estimates. Such a subject would also be able to determine that the resulting estimates of the arc's true length would always be underestimates.
because the process of estimation could be envisioned, allowing the subject to picture line segments that would be shorter than curves.

**Quantitative Data**

The numerical data gathered for this study were in the form of scores on three limit-related final examination problems. The final examinations taken by treatment and control students, were identical in the first three problems. The mean scores for each problem and section were then computed and compared. The specific hypotheses I attended to were:

\[ H_0: \text{Students performing writing tasks intermittently throughout the semester will have the same final examination scores on the three limit-related problems as the non-writing students will, and,} \]

\[ H_a: \text{Students performing writing tasks intermittently throughout the semester will have significantly higher final examination scores on the three limit-related problems as the non-writing students will.} \]

In order to most objectively grade the final examination scores, I developed a detailed rubric, with input from two mathematics educators and two mathematicians (Appendix H.) Further, to verify inter-rater reliability, I provided another researcher with five randomly chosen final examinations to score and a copy of my rubric. Similar scores on fifteen problems would convince me of the reliability of the scoring rubric. The scores, reported in Table 1, were related by a correlation coefficient of \( r = .913 \). A high correlation between the two scorers was achieved because the rubric was further refined as I scored, taking into account almost every idiosyncrasy of the
performance of the seventy-one students participating in this study. After the scores
from the common final examination problems for both the treatment group and the
control group were determined, a two-tailed two-sample t-test was performed; the
variance was not pooled. A significance level of \( \alpha = .05 \) was fixed, and significant
differences were sought. The results from the qualitative data were then reconsidered
in light of the quantitative findings.

Table 1

Scores Awarded by the Two Raters on Three Final Examination
Problems Over Five Research Subjects

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<th>Subject</th>
<th>Problem Number</th>
<th>Rater 1 Score</th>
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CHAPTER IV

ANALYSIS OF DATA

This chapter begins with a qualitative analysis of data gathered from the treatment group. Materials from each of the five interviewed subjects are organized into profiles and analyzed utilizing the APOS framework. Conceptual growth is determined on a student-by-student basis by aggregating all data (initial limit survey, the three interview transcripts, the six writing tasks, and the exit survey) into a file for each of the five interviewed subjects, and looking for movement along the APOS continuum in the subject’s perception of the limit concept. Finally, the treatment group’s change in understanding of the limit concept is measured by considering the entire class as the object of study, and examining their collective writing efforts task-by-task.

Next is provided a quantitative comparison of students’ performance on three calculus problems common to both final examinations. Explanations for differences in performance between the treatment group and the control group are given. Before beginning this discussion, however, the stage will be set with the observed teaching climates developed in the control and the treatment sections as well as the results of the initial limit survey of the two groups.
Results From Teaching Observation

Recall that teaching observation checklists were developed to allow for systematic observation (and subsequent comparison) of the two participating instructors. The following is a distillation of the results of this observation.

Occasionally, about once or twice a class period, the treatment instructor would pose a question whose answer would require reflective abstraction. However, insufficient wait time was allowed (silence becomes uncomfortable quickly). Even if the students had had time for reflective abstraction, they had little incentive to engage in it, for the answer was soon revealed. The vast majority of questions that were answered only demanded action-level thought processes -- reactions to notation or mathematical context. These were procedurally-oriented. For example, the professor would be in the middle of evaluating a definite integral, and would then ask, “Now what do I do?” There was nothing in the atmosphere that seemed to inhibit students from answering the questions requiring an action-level answer (the professor did not have an intimidating demeanor, for example), but higher-level questions either were not posed or were not given sufficient think time to elicit answers.

In the control class, there was more explicit encouragement for students to think. More specifically, the control professor was comfortable with longer periods of silence while students pondered a question. He was quite willing to take the students’ perhaps misleading suggestions and try them up at the board. Whether it was intended or not, these temporarily fruitless endeavors showed the students the potential struggle

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behind many second-semester calculus problems, such as choosing a workable
equation for use in the substitution method of integration. In the control section,
most questions required action-level to process-level answers.

On the whole, the two classes maintained atmospheres that were relaxed and
open for questions. They were instructor-centered with almost exclusive use of the
lecture method. There was more evidence of opportunities for reflective abstraction
during classtime in the control section. (Based on the results of this study, however, it
did not appear that these opportunities were capitalized upon.)

Both instructors assigned practice problems after almost every lecture.
Exceptions were the day before an examination or the times when they sensed students
needed more time on the previous assignment. In addition, both instructors collected
problem sets. The timing of this collection was based on how much material each
professor had covered at the time, and so did not correspond exactly. However, the
same number of assignments were collected by each instructor: the control instructor
collected ten problem sets and the treatment instructor collected four problem sets and
the six writing assignments.

Results of the Initial Limit Survey

The primary reason for collecting the initial limit survey (Appendix C) in the
two sections was to assess the participants collective limit understanding prior to any
treatment. However, it can also be used to further establish the comparability of the
two sections used in the study. As can be seen from the bar charts on the next page, (Figures 2 and 3) both sections favor the misconceptions of choices 1 and 4. (Refer to page 48 for the limit perspectives.) However, not as many participants in the control were taken in by misconception 4. Recall that the only correct statement out of the six was statement 3; five students in the control group identified that as the statement that most closely matched their own, while only two in the treatment group did.

Qualitative Findings

For an in-depth study of a subject’s growth along the APOS continuum, student profiles, based on initial limit perspective surveys, three interviews, the six writing tasks, and the end-of-semester surveys, proved to be a rich and revealing source. In particular, amassing all data relevant to one subject allowed me to detect thought patterns that appeared in both speaking and writing. To analyze these data, I used the APOS framework described in Chapter III.

Student Profiles

The Profile of Amy

A conscientious willing-to-work student, Amy described herself as one that had always been “good at math” but was for the first time really struggling to grasp even basic second-semester calculus concepts. She told me of many nights helping girls in her dormitory with their lower-level algebra, precalculus, and calculus, only to
Figure 2. The Collective Limit Perspectives of the Treatment Group.

Figure 3. The Collective Limit Perspectives of the Control Group.
return to her own room and have "... no one that understands or is taking the mathematics that I am supposed to do." She often sought to understand why a mathematical procedure was true, saying things like, "I've practiced this technique he gave us in class enough to be able to do it, but I would remember it better if I knew why I was doing it." (casual conversation, pre-interview 1).

Despite this desire to know why, Amy also displayed a reverence for predetermined or set formulas. In discussing the role of technology in her calculus course, she remarked, "That's the thing with the book we're using that's really making me ... it's so much visualization, and I was always taught with just direct formulas, not a lot of graphical visualization and things like that." (11, 6) Later in an assignment discussing the possibility of using the limit concept to get a value for arc length, Amy demonstrated her trust in the reliability of formulas by stating, "The true arc length was determined using a set formula, which then produced a very accurate result." (WT2)

Based on her arc length comment, it is perhaps not surprising that, when pressed for an answer (in the initial questionnaire, in the writing assignments, and during the interview sessions), Amy often tried to rely on isolated strategies and "cases." She frequently used language like, "I can't recall any specific instances like this one," or "I will try to do what we did last time, if I can remember," or, "Oh, this must be a different type of problem, because what I am trying is not working this time." However, especially in the interviewing sessions, when I was able to press her
to think about the context and let go of the desire to find a template for proceeding, she would make conceptual connections. Evidence of this will be seen later in this profile.

Amy began to reveal specifics about her understanding of the limit concept when filling out the questionnaire given to all subjects at the beginning of the semester. (See Appendix C.) She identified the following statement as the truest of six possible representatives of her own understanding of the limit concept: “A limit describes how a function moves as $x$ moves towards a certain point.” From this it can be inferred that for Amy, the limit concept is bound up with the concept of motion. This is corroborated by the fact that she characterized the statement, “A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached” as true. Several times she referred to plugging in points as part of taking a limit. This indicates an at most process-level conception of limit, as limit is never referred to as a number, but rather as an operation built from the repetition of actions.

A student perceiving the limit concept as “motion-bound” will have difficulty viewing the limit as a fixed number. Thus it is not surprising that Amy denied that “A limit is a number that the $y$-values of a function can be made arbitrarily close to by restricting the $x$-values”, even though this is a correct definition. In fact, she tried to validate her denial by explaining, “When I have dealt with limits, they were always as $\lim_{x \to \text{a number}}$ (expression)” (initial limit survey), as though such an object could not itself be a number.
Finally, the last sentence of her questionnaire, combined with the previous data, made it plain that to Amy, the limit is more an action or process than an object. She wrote, "I understand a limit to be a way of figuring out solutions to problems when direct substitution does not work" (emphasis mine.) In other words, the limit is a means to an end upon which one relies, in her words, "... to get an answer out by not ordinary means", rather than an end in itself.

Amy's preference for direct substitution was maintained throughout much of the course. In her first writing assignment, she explored the mathematics of

$$\lim_{x \to a} \frac{x^2 + 3x - 2}{x - 2}.$$  

She remarked, "The simplest way to tackle a problem such as this is by substituting the specific value of $a$, which is 6, into the equation for the variable $x$" (WT1). She subsequently correctly determined this limit value to be 7. Recall, however, that Amy had previously expressed the opinion that a limit is a method for finding solutions to problems when direct substitution does not work. Here, Amy is using substitution to evaluate a limit. One might infer that Amy primarily relied on substitution as a means for "tackling" limit problems, but simultaneously felt she thus avoided the work that would have been involved had she "really" been forced to take a limit.

This inference was confirmed in the first interview during a lengthy discussion of strategies Amy used for evaluating limits. She seems to struggle when the limit cannot be found by evaluating the limit at a point.
First of all, I guess if it's really direct you can just substitute in, but I do remember that if it's not, ... there's other ways of solving it. You could try factoring ... you can take the highest variable in the denominator and divide every character, variable, number, whatever ... usually when I do something like that I'll mess around with it to see if I could get it to do much. Try a bunch of things -- look back in an old math book! (II, 1).

To evaluate the function at one point is her preferred method, which is natural as it is the simplest. However, she appeared not to realize that substitution is valid only when a function is continuous. Rather, she relied on the form of the expression whose limit she is finding.

Later in her first interview, when discussing the meaning of \( \lim_{x \to a} f(x) = L \), Amy remarked, "... it means that, pretty much, as we put \( a \) into the function, the function's gonna equal \( L \). I don't know how to elaborate on it much more, I mean ... just put \( a \) in for \( x \) into the function" (II, 1). Not only did she espouse this viewpoint theoretically, she very frequently put it into practice. In determining limits of functions that were discontinuous at their point of evaluation (as in \( \lim_{x \to 2} |x| \)), Amy would arrive at an incorrect answer because her method of direct substitution worked only for continuous functions. Such procedurally-oriented practice also led to confusion later when she examined \( \lim_{x \to 0} \cos(1/x) \). She reasoned, "Ok, um, well, since you can't substitute it, and you can't simplify it much more, I'm gonna graph it and look at it first" (II, 5). Of course, this is not a bad strategy, by any means. However, she later concluded, "Even though it's doing some crazy stuff in there and all that, um, it
actually appears to at least come together to the same point from both sides” (ibid.). This led her to believe (erroneously) that this limit did indeed exist.

Although one goal of the writing assignments and the interviews was to put the students into cognitive conflict, thereby forcing students to reconcile their intuition with the mathematics they were performing, Amy effectively resisted in this case by not allowing herself to become personally involved with the material. In an attempt to confront students with the surprising phenomenon of an improper integral whose interval of integration is infinite and whose value is finite, Writing Task #3 asked students to evaluate the two improper integrals $\int_1^\infty \frac{1}{\sqrt{x}} \, dx$ and $\int_0^\infty e^{-x} \, dx$ and then to draw and appropriately shade graphs representing their respective areas. In each case, Amy correctly evaluated the integral (obtaining infinity and 1, respectively), then drew the graph of the integrand only, disregarding both the interval over which she was integrating and the need to shade underneath the graph of the integrand to accurately represent area. This problem was specifically designed to illustrate the seeming dichotomy of infinite intervals of integration yielding in one case infinite area and in the other finite, even tiny, area. However, Amy’s potential tension was effectively mitigated by her not seeing the similarity of the two pictures (because of her incomplete drawing) and the enormous disparity in their area values.
Later this topic was substantially revisited in an interview. Amy was exploring
\[ \int_{1}^{\infty} \frac{1}{x^2} \, dx \] to determine whether this value was finite or infinite. With prompting from me, she noted that the interval of integration was infinite (recall she had ignored this in her writing assignment) and had performed the process of calculating the integral’s value as 1. She failed to reconcile the two objects (\( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) and 1) and notice the possible incongruity. Thus I pressed her to explore their relationship. “Could such an integral have a finite value, even though the limits of integration are infinite?” From her lengthy pause and very hesitant response, I could tell this was the first time she had considered such a question. Amy slowly answered,

Seems like ... see, when it ... I’m trying to think. Because in one respect I would say no because it’s never is really going to hit 0. But it’s going to tend to get really close to it. So, I mean, that’s what happened when we had 1 over \( x \) here. I mean, in actuality it’s still going to be 1 over infinity, it just tends so close to it that we call it 0. So when it is like all the way out here, if we just called it 0, then you would have like a definite area. Even though it was infinite. (my emphasis, I2, 4)

Although there is much confusion and half-formed ideas bound up in this paragraph, Amy is making progress because she is becoming involved in the material and actively trying to reconcile her intuition with what she is discovering “mathematically.” I asked again, “So would that area be finite or infinite?” She replied,

Area would be infinite ... but that contradicts what I just said. It confuses me because graphically I can see it’s like infinite value because there’s no stopping point on it. But whenever you had like a problem like numerically ... you can plug in these numbers and it would come out to an actual numerical value. (I2, 5)
Later she seems to come to some resolution by erroneously thinking of the limit as an approximation. Once deciding this, she is noticeably relieved. Still discussing the finiteness or infiniteness of \( \int_{1}^{r} \frac{1}{x^2} \, dx \), she says,

I'm just not sure, because when I look at it graphically it would make sense that it would go on forever, but when I look at it this way (numerically) it has a definite value for an answer. ... once you get down this far, it's going to be such a small area that they just stop computing it. You'll go on forever and it won't add up. *So, I guess this is more of an approximation.* I mean if you wanted to get really technical about it and carry it out forever and ever, it would be probably a little off. But, I mean, after you get out so far, it's going to be such a small amount that it's not even worth noting (my emphasis, 12, 6.)

Amy shows this almost disparaging view of "limit as a technicality" later in this interview, while discussing the difference between .999... and 1. She says, "Technically, .999... would be less than 1 because it still is. I mean, like I said, it's going to be like .000 forever 1, less than 1. But in problems, just to save ourselves, I guess, any sort of trouble or grief for me, use 1." (12, 7). Again by thinking of the limit as a process involving estimation or compromise of accuracy, Amy allowed herself to let go of any tensions she experienced in trying to reconcile her intuition with the rigors of mathematics.

One misconception that Amy did not ignore but rather worked through was her confusion between sequences and series. In the fourth writing assignment she gave what she thought to be an example of a convergent sequence whose limit is not 0. In fact, the mathematical object she provided was \( \sum_{k=1}^{\infty} \frac{1}{k^2 + 3} \). There were two
problems with this response. First of all, \( \left\{ \frac{1}{k^2 + 3} \right\}_{k=1}^{\infty} \) is a convergent sequence, but the limit is 0. Secondly, Amy provided a series when she meant to be discussing a sequence. Both of these misconceptions would prevent much progress in developing an understanding of sequences and series. On the next writing assignment, Amy defined a series by writing, “A series involves summing an infinite amount of values substituted in for one equation” (WT5). Again it seems as though her idea of series, like her idea of limit, involves the concept of motion. It appears as though she is interpreting the notation of series as a command to begin summing. This is a context-specific symptom of an action-level understanding of the limit concept in general. To nudge her away from this idea, I commented on her paper, “Talk more about how divergence and convergence of sequences and series are related. You can come right out and say, ‘A series is a sum.’ Then talk about how the terms of the sequence are the addends of the sum.” Later, in an end-of-semester exit survey, Amy detailed a turning point in her sequence and series understanding. She wrote,

I had a few basic ideas about series and sequences, some of the ideas were on the right track and some were misconceptions. The comments written on my paper explaining that the sequence is the list of numbers and the series is the sum of the numbers really clarified things for me and the remaining material about sequences and series fell into place.

Amy was actually told that a sequence is a list and a series is a sum in class (I heard it!) but it seemed to make little impact on her until she really struggled with the material on her own in the writing assignments.
What is interesting and necessary to analyze now is Amy’s concept of limit after progressing through the six writing assignments. Recall that in her first writing task, she wrote that the best way to tackle limit type problems was to substitute the value that \( x \) was approaching into the given expression. She also maintained this view in the first interview and the initial limit questionnaire. Thus Amy began the class with an action-concept of limit firmly in her mind. This notion was challenged during the course, particularly in the writing assignments. However, as the assignments progressed from topic to topic, Amy could display an action-concept of the topic at hand (say arc length) and still be progressing in sophistication of her limit viewpoint. For Amy then, the writing tasks were less a showcase for well-formed conceptions than a means for uncovering her misconceptions that I could then point out.

According to her exit survey, she “… usually had to spend about an hour just thinking about the question and what was meant by the question.” Several times, both in casual conversations and during interviews, Amy remarked that it was the comments on her writing assignments that were extremely beneficial in guiding her away from her erroneous thinking.

A tangible benefit Amy gained from her work with limits was evidenced in the third interview. Again speaking of plugging in points as a means of evaluating a limit, Amy says,

Well, that wouldn’t always be true … when you have a simple, really straightforward limit that is going to equal, um, a set number, then you can plug in, um, points, and get the limit that you want. But whenever you’re going to have limits that go to zero or that can go to positive or negative infinity, you can run into some complications (13, 7).
In order to answer this way, a subject would need a larger perspective of the limit than as a command to plug in a point, or even a series of points. Thus Amy has progressed in her understanding of the limit concept, and can be characterized as having elements of a process-level understanding. However, since she never speaks of the limit without some implicit or explicit reference to motion, she is certainly not higher than process-level. In fact, in her final writing assignment, she still is displaying misconceptions that are characteristic of a subject at the action level. She writes,

There are three common misconceptions regarding limits and infinity. The first one is that "the limit of the function can be made as accurate as desired." If the student correctly completed writing assignment #3, this misconception would be clarified. The correct way of thinking about this involves the first example of the improper integral assignment. The area under a curve that goes from one to infinity is going to equal infinity. There is no way of increasing the accuracy of this. (WT6)

I would have liked to see Amy drop this notion that the area under a curve of infinite length must be infinite. Thinking that any improper integral with infinity as a limit of a integration must have an infinite value is a knee-jerk reaction to notation. This behavior characterizes the action-level subject. Another point of this citation is that she does not understand why the limit of a function does not vary in its "accuracy."

Thus, she still associates motion with the limit.

**Summary**

Amy began the semester with a motion-bound, pre-action to action-level concept of limit. By the end of the term, she demonstrated progress towards process-
level understanding in writing assignments and interviews. The improvement she made in her limit concept, based on observation and on her own admission, seems largely due to the proactive response she made to the comments I wrote on her writing tasks. They impacted her notably more than the same comments made in class, probably because they were one-on-one in nature and specifically addressed matters she had struggled with.

The Profile of Brett

Brett was a student that was particularly willing to get personally involved with the material. This was evidenced by the substantial amount of time he put into the writing assignments (reported in the third interview) as well as the vehemence with which he would defend his mathematical viewpoints during the interviews. He also read the model writing task write-ups that were passed out with the graded student papers, unlike several others interviewed. Within the writing tasks, his writing style was fluid and comfortable, even casual, from which I inferred that he was not intimidated by the assignments themselves. There was an authenticity in his papers that stood out from the formality of other subjects' writing. That is, in other interviewees papers, I found a writing style quite incongruous with how the subject spoke. They would write things that I couldn't imagine they would say. I did not find this disparity in Brett's work. To exemplify this phenomenon requires citation of
Brett’s work as well as the work of his cohorts, so differences may be seen. The following excerpts are taken from Writing Task #5.

An example of a sequence follows: \( a_k = \frac{k - 1}{k} \). First make a list of the numbers by substituting positive integers in for \( k \). \( 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{10}{11}, \ldots \). From this set of numbers you should study them carefully and look at their pattern. You should be saying to yourself, ‘Hey!, if this sequence goes on for awhile and I keep plugging in greater positive numbers, then this will converge to 1. That is exactly what you need to do to take the limit of a sequence. There is no magic as it may seem, but all you need to do is make a list of the sequence and watch where the numbers are getting closer to. (Brett)

This is very accessible language, and sounds almost like spoken conversation.

Regardless of the correctness of his thought processes, there is a continuity between the way he spoke in interviews and the way he wrote in papers. On the other hand, consider the language used by some classmates of Brett:

The main difference between sequences and series is that sequences make a function tend to a certain value and series functions are obtained by summing the inputted values of the function. (S31)

Specific limits (if one exists within the domain of whole numbers) can be evaluated as more sets of numbers are generated from a particular function. (S15)

A sequence is a continuing list of infinite numbers, meaning numbers going to infinity. (S20)

The words these students put together to describe sequences and series are awkward and unnatural, and do not sound like spoken language.

Perhaps as a result of his engagement with the material, Brett got quite frustrated with what he initially perceived to be “arbitrary rules” to be followed in
working limits. When he came face-to-face with the fact that some integrals over an infinite interval yield finite values while other such integrals yield infinite values, he was visibly uncomfortable, saying, "It's the fault of whoever came up with integrals ... why is it like that?" (I2, 10). When I reminded him of the definition of improper integrals as a limit, he said, disappointedly, "Just by the definition?" This feeling of frustration because his intuition conflicted with his mathematical understanding was a catalyst for change -- dramatic change -- that took place over the semester. This change will be detailed in the section describing Brett's engagement with the writing assignments.

Brett's initial survey results reveal a motion-bound concept of limit. In reacting to the statement, "A limit describes how a function moves as $x$ moves towards a certain point," Brett says "true" because, "It gives the motion." Other evidence of a motion-bound limit concept is provided in our first interview. Brett began telling me about the relationship between $\lim_{x \to a} f(x)$ and $f(a)$. I asked him, "What happens to $f(x)$ as $x$ gets close to $a$?" Brett declared, "The limit is more precise" (I2, 1). Probing further, I asked, "Does the limit move?" Brett answered yes. At this point, he clearly did not think of the limit as a fixed number.

The statement that Brett felt most characterized his view of limit is, "A limit is a number or point the function gets close to but never reaches," saying, "It can be a hole in the line." This feeling of the limit being associated with a hole is prevalent in much of the early data from Brett. In fact, for the first several weeks the best way to
describe his thinking towards the entire limit concept is “suspicious.” For example, in
describing what it means to say that the limit of a function \( f \) as \( x \) approaches \( s \) is some
number \( L \), Brett wrote, “As \( f(x) \) gets closer to \( s \), it really isn’t there, but I’m not sure
what it’s used for” (my emphasis, initial limit survey.) Later in the second interview,
when discussing \( \lim_{x\to0} \frac{1}{x} \), Brett cried, “It’s undefined! It becomes a ‘null’, a zero, it’s
just space, it’s -- I don’t know, it’s not there ... it’s just gone!” (I1, 4).

Following his initial limit survey, Brett began the course with a very
procedurally-oriented (aberrant) first writing assignment. He missed the opportunity
to get deeply involved with the limit notion by explaining it in terms of a continuous
function, thus failing to mentally discern between the mathematical objects \( \lim_{x\to a} f(x) \)
and \( f(a) \). In his example, he evaluated the limit by simply plugging in the value for \( a \)
into the function. Although such a procedure does work here because Brett chose a
continuous function, there is a tacit assumption that this method will always work.
Such an assumption suggested Brett was at the action-level conception of the limit, as
it corresponds to an act precipitated by notation and context. This level became more
clearly evident in the first interview. I asked Brett to explain the difference between
\( \lim_{x\to a} f(x) \) and \( f(a) \). He said, “I would say that there is no difference between them
and that they are the same all the time” (I1, 2). He held to this misconception very
tightly at the time. Even when I showed him a graphical example of when this clearly
would not hold, Brett wouldn’t budge from his intuition. He simply said “something
strange and odd” happens in the case that I drew. This confirms the elements of action-level thinking that were lurking in the limit schema of Brett.

Perhaps more than all other subjects in the study, Brett showed dramatic improvement over the semester, finishing with a much clearer concept of limit than the one with which he began. Recall that in the first interview, he claimed that the limit moves, or has a changeable value, and further claimed that \( \lim_{x \to a} f(x) \) and \( f(a) \) were the same. In his last writing assignment, Brett wrote,

A common misconception that many students have is with the statement that “the limit of a function can be made as accurate as desired.” A correct way to state that would be like: “a function of a limit can be made as accurate as desired to that function’s limit.” After all, the function is an approximation of the limit ... the limit ... is not going to get any more accurate (WT6).

In the paragraph above, Brett is actually displaying elements of an object-level conception of limit. This is tremendous, measurable growth from his first writing assignment. He also addresses a topic that gave him problems in the second interview. He writes,

Yet another misconception is that a shape with infinite length must have infinite area. When you first hear that statement, it makes sense that since something is going to infinity that its area should be infinite too, but that is not the case. The answer or the proof as to why the statement is wrong is hidden inside the limit. When you take the limit, ... you apply your improper integral techniques and you find that the area is indeed a finite number (WT6).

Brett further supplied an example of an integral whose upper limit of integration is infinite, but whose value is finite, along with a correctly done proof. He had let go of
the misconception that an infinite interval of integration necessarily implies an infinite value for that integral.

A third previous misconception of his own that Brett addressed during the last few weeks of the semester was his thought that one could always evaluate a limit simply by plugging in a finite number of points near $a$ and evaluating the function $f$ at those points. Brett supplied a step function to show why plugging in points to evaluate a limit at the point of discontinuity of the step function could be misleading.

Perhaps the reason the writing assignments had such an impact on Brett is that he got quite involved with them. On his end-of-semester survey, he wrote, “In my opinion, the sequence and series paper was the most helpful for me since I had great confusion over the ideas. The paper forced me to jump into the concepts. Don’t change a thing about [the writing tasks.] They helped a lot.” Also in the last interview, Brett commented, “[Writing] made me ... It helped me to understand what was going on. I wouldn’t have paid as much attention to the concepts if I wouldn’t have written it down. So that helped a lot.” (13, 1).

Even though Brett seemed to get more out of the actual completing of the writing assignments than other classmates did, he also benefited noticeably from the comments that were written on the graded paper. In discussing the paper on arc length, he remarked, “Though I got a whole bunch wrong on it, it helped me to understand after, when I got it back. I looked it over. Well, remembered that ‘oh yea,
this makes sense now" (13, 1). Based on this evidence, it was clear that writing had an impact on his understanding of the limit concept.

Summary

Brett began the course with an action-level concept and finished with a concept between process- and object-level. This is substantial and noteworthy progress. Because Brett seemed to respond particularly well to the battery of writing tasks, I searched his writing submissions and interview transcripts for reasons why this might be so. The word that I found to describe Brett’s approach is authentic. As I commented earlier, his writing voice was quite similar to his speaking voice. He did not try to fill his submissions with esoteric vocabulary and vague abstractions -- he simply revealed as much as he knew. This had the major benefit of allowing me to pinpoint his misconceptions or incomplete areas of understanding and give specific help. Brett was then proactive in reading the comments I provided on his paper, discovering his weak spots.

The Profile of Charles

Out of the five students I interviewed, Charles was the least willing to involve himself in the material or tasks at hand. This lack of involvement manifested itself in many different ways. In his end-of-semester survey, Charles wrote that he appreciated the writing tasks but would have preferred to have fewer assigned, because he became
"burned out" on having to write so many. He entirely neglected to do one of the
tasks, and did another in such a short time that he did not word-process it. The
writing task that many students characterized as the most helpful (assignment #5) was
only four sentences long for Charles, containing a most superficial treatment of the
two subjects. He revealed much more annoyance at having to complete the
assignments than other subjects. For example, he said, "I got fed up by this time on
doing writing assignments so I just wanted a break and ... so I didn't even put any time
into it ... And then I just got fed up and I just said forget it." (13, 2) Finally, he
admitted he had never read the model solutions passed out in class, and, even after
being told why reading them would help, continued to ignore them upon receipt.

Another more subtle way that Charles avoided involvement with the material
took place during the interviews. He ended nearly every answer with a request to go
on to the following topic. "Ok ... next question?" was a frequent response. This rush
to the next question had the effect of nipping in the bud any cognitive conflict or
means of resolution. It was difficult for me to steer Charles back into a question he
had just abandoned, because in his mind we were "done with that one."

Charles revealed a rather immature limit concept in his initial limit survey. He
was the only subject to write "I don't know" or "I have no idea" as responses to some
of the true/false statements about limits. The statement he most closely identified with
was "A limit is a number or point past which a function cannot go." He agreed,
commenting, "Once the limit is reached, it can go no further unless the limit is
infinity.” (Presumably Charles means the function by the word “it.”) He also perhaps inadvertently disclosed a misconception that resurfaced frequently within the writing tasks and the interviews. Namely, in being asked to paraphrase the statement, “... the limit of a function \( f \) as \( x \to s \) is some number \( L \),” Charles wrote, “As \( x \) moves towards \( s \), then \( f(s) \) will equal \( L \).” This statement reveals an action-level understanding, as it indicates Charles equates \( f(s) \) and \( L \).

The following experience with Charles highlights a subtlety of APOS theory, namely, how does one discern between a series of actions and a process? The answer lies in the subject’s motivation: is he reacting to notation, and the previous step in a procedure, or is he responding to a problem’s context, with a vision towards the problem’s solution. Interestingly, two subjects might perform the same series of acts, with one at the process level and the other at the action level. Thus, the researcher must rely on clues from the subject about from where his motivation stems.

Charles was quite confident with the calculator’s ability to provide a graph from which the answer to a limit problem may be read. In the second interview, he told me that he had two strategies for evaluating a limit. He would either plug in the one point that \( x \) was approaching (call it \( a \)), demonstrating an action-level conception of limit, or he would graph the function and look at the behavior of the curve at the point \( x = a \). When using technology to evaluate the limit, it is difficult to tell if he is working at the process-level ( picturing the behavior of the graph using technology) or at action-level (reacting to the notation of the limit problem.) In fact, he claims,
... if I want to find a limit, I'm usually given an equation, and you can tell from
the equation where it usually, like where it can't be 0 ... It just seems easier to
graph it. I mean, I've never had anything that wouldn't, I guess, work for
graphing it (II,3).

However, just a few moments later, I presented Charles with the task of evaluating
\[ \lim_{x \to 0} \cos\left(\frac{1}{x}\right) \]. After grabbing his calculator, he zoomed in closer and closer to the
point where \( x = 0 \), and found that his calculator would give him no useful information.

This reaction, punching commands into his calculator based on the results of the
previous screen, lends credibility to identifying Charles as having an action-level
conception of limit in this context. A student with more intuition about the behavior
of the function \( 1/x \) near \( x = 0 \) would have probably extrapolated from the crazy-
looking curve given on the calculator screen that perhaps this limit is not defined.

Instead, Charles seemed to give up, and called himself a moron. Only after extensive
discussion (with too much talking done by me) was he able to see that since
\[ \lim_{x \to 0^+} \left(\frac{1}{x}\right) = \infty \], it must be that \( \lim_{x \to 0^+} \cos\left(\frac{1}{x}\right) \) is undefined.

Like several other interviewees, Charles clung to his misconceptions even
when faced with clear, understandable counterexamples. For example, when asked if
the terms of a sequence ever reach the limit of that sequence, he flatly said no (II, 6).

Later he worked with a sequence whose odd terms were \( \frac{1}{2n} \) and whose even terms
were 0. Thus the limit of the sequence, as well as every other term of the sequence,
was 0. Charles came to this conclusion himself and felt quite comfortable with it, until
I pointed out to him that this conflicted with his previous declaration that the terms of

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a sequence never reached their limit. He said rather dubiously, “But my thinking is
that you can never reach your limit. I mean, I don't know” (I2, 6) Even when the
counterexample was unquestionably understood by Charles, he still was reluctant to
change his belief.

As was previously discussed, Charles put minimal time and effort into the
writing assignments. However, he admitted that he initially hadn’t realized there
would be more than one writing task, and so threw himself wholeheartedly into the
first one (I3, 1). The result was a fairly well-done, thorough explanation of limit.
Nonetheless, although he understood that delta must depend on epsilon in the
definition of limit of a function, he failed to make this relationship plain. This failure
allowed for ambiguity in his own mind.

Similarly, his second attempt at writing, based on the notion of arc length,
produced a good, straightforward paper explaining how to use a Riemann sum to
estimate the arc length of a given curve. He then described how to find it exactly, by
using a formula he cited from the book containing a definite integral. However, he
never attempted to reconcile the sum with the integral, thereby never realizing that the
limit of a Riemann sum is a definite integral. In fact, in describing the paper in the final
writing task, he claimed that the idea of the limit was not present in the second paper,
except that “… trying to find the arc length of an infinite long arc could definitely be
hard if not impossible” (WT6). As was mentioned before, Charles did the third writing
assignment quickly and without a word processor, the fourth he skipped entirely, and
the fifth was only four sentences. Thus it would be surprising to find much evidence of cognitive growth among the writing tasks and interviews.

Summary

As was noted, Charles began the semester with an action-level understanding of the limit concept. Occasionally but unpredictably, elements of process-level understanding surfaced. For example, even at the beginning of the semester, he realized that delta is somehow connected to epsilon in the formal definition of limit, (he wrote, "The whole reasons we needed to have an ε is so we could pick our δ") thereby demonstrating that he knew restricting values \( x \) can take on will bring \( y \) closer to the limit. However, such understanding was sporadic. At one point, subjects had the opportunity to show their limit-conception in the final interview where they were to evaluate a limit that did not lend itself to the plug and chug method. I was curious to see if perhaps Charles had made some progress, despite his limited participation in the writing tasks. However, the third interview had Charles using many of the same incorrect strategies (such as equating \( f(a) \) with \( L \) and relying on his calculator when doing so was not helpful) that he used in earlier settings. From this I could see that his thinking remained at a level at which he began the course -- between action and process.
The Profile of DeMahli

From the way she spoke in the first interview, I could see that DeMahli began the course with a firm trust in her procedural knowledge. Thus, the first writing assignment, dealing with a fairly deep conceptual topic, understandably shook her up.

In the first interview, she herself brought up the topic of the first writing assignment, saying,

...to tell you the truth, I really did not understand much about that because I'm a transfer student, and we used a different book, a whole different author and everything ... I was kind of confused as to where [the epsilon-delta] came from, because I understood this portion, the limit and the function and everything, but I didn't understand where the rest of it came from exactly. I mean, I knew what I was doing, but I didn't know how it originated or anything like that (11,1).

At least at the beginning of the semester, DeMahli equated procedural facility (ability to carry out actions) with conceptual understanding. Despite her self-described successes in first-semester calculus with evaluating limits, interview probing and analysis of assignments showed severe weaknesses in her limit schema. These weaknesses included confusing domain and range, the feeling that a function cannot reach its limit, a viewpoint of the limit as a technicality, and a discomfort with the notion of infinity, discussed in subsequent paragraphs.

DeMahli was in agreement with five out of the six possible viewpoints of limit (Appendix C). She revealed an action-level tendency by identifying the statement, “A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached” as her favorite. She further demonstrated an
association of "undefined point" with "limit of a function" by responding, "Once the function is undefined it has been reached." She marked the statement, "A limit is a number or point past which a function cannot go" as true, stating, "Otherwise the function would not work outside the given numbers." In the final writing assignment, after shaking other misconceptions, she still writes, "You should gather that if a function converges to a limit you will get closer and closer to the limit, however, never actually reaching it" (WT6). Earlier, in investigating a sequence whose odd terms are 0 and whose even terms are \( \frac{1}{2n} \), she struggled to determine the limit. She said, "The evens are always 0 ... the odds will never actually get to 0, but they'll approach 0. And since the evens are already 0, it'll be like they're meeting. With the evens being 0, I don't know if 0 could be a limit." (I3, 13) This feeling that the limit cannot be reached was apparently never eradicated from her limit schema.

Despite these several misconceptions with the notion of limit, she did note in the initial survey that the statement, "A limit is an approximation that can be made as accurate as you wish," is false, because, "A limit is exact." However, based on the preponderance of other comments she made, it seems as though this answer is a guess, rather than a response that indicates true understanding.

One source of confusion that initially arrested DeMahli’s development of a more complete limit understanding was her tendency to confuse the value that \( x \) was approaching (which was commonly referred to as \( a \) with the value that \( f(x) \) was approaching (the limit!) In fact, in her first interview, she said, "Well, my
understanding of it is, um, for any function, the value as \( x \) approaches um, a certain point, um limit, the whole function gets closer and closer to a certain, in this case \( L \), it'll get closer and closer to a certain value for \( L \)." (II, 1) Notice the only time she used the word limit was in referring to the value that the domain variable is approaching. Later in this same interviewing session, she states, "Ok, so um, well then the function \( f(a) \) cannot get any larger then, 'cause as \( x \) is approaching \( a \), \( a \) is like the absolute ... limit I guess, like the smallest er, the absolute largest that a number can get?" (her emphasis, II, 4) She also reveals another misconception that takes the form of an over-generalization. She holds (perhaps subconsciously) the idea that all functions are increasing. In discussing the difference between \( \lim_{x \to a} f(x) \) and \( f(a) \), she says,

And um, \( f(x) \) could just be anything in between, like inside the limit? So I guess \( f(a) \) would be like the absolute limit itself? And then \( f(x) \) would be ... it could be the limit, but ... it may also be something inside of the bounds on the limit. I guess I see it more clearly now, like \( f(x) \) could be any number inside the function (II, 4-5).

Further probing during the interview showed that in referring to \( f(x) \) as being inside the limit, she meant \( f(x) < f(a) \). This implies that \( x \) must increase as it approaches \( a \), and \( f(x) \) must increase (as \( x \) increases) up to \( f(a) \), "the absolute limit itself." This preconceived narrow view of limit would lessen her ability to visualize a more general case, such as the oscillating sequence of 0 and \( \frac{1}{2n} \). DeMahli seems to display a discomfort with limits and especially with infinity. One way she shows this is...
to use the term "technical" in an almost disparaging way in reference to limit
consequences. This term came up in several contexts, the first of which is improper
integrals. I was pressing her to determine, first intuitively, then rigorously, whether
\[ \int_{1}^{r} \frac{1}{x} \, dx \] was finite or infinite. She notes,

It seems like it would be finite because it gets so small that it doesn’t really
matter anymore, but technically it never really ends. I mean, there’s always
going to be another one that’s a little, little bit smaller. So I guess it has to be
infinite (12, 4).

It almost seemed as though she knew the answer I wanted to hear was infinite, but she
truly felt that the area underneath that curve was finite. After hearing DeMahli and
several other subjects use the word “technical” when describing a limit problem, I
came to learn that it almost always indicated discomfort and perhaps suspicion. In this
case, after a lengthy debate (mostly with herself) on the value of \[ \int_{1}^{r} \frac{1}{x} \, dx \], she ultimately
grinds out the limit and discovers \[ \int_{1}^{r} \frac{1}{x} \, dx = \infty \]. She seems to feel cheated, and remarks,

“Okay, I see. So the area under the ... Okay. So it is like in ... I mean, for all it’s
worth, it stops, but technically it does go on and on forever” (12, 7). The word
“technical” peppered her interview transcripts, always in places where she exhibited
feelings of vague discomfort or doubt.

According to DeMahli, she previously had success with her procedural
methods. Now she seems to be wrestling with concepts for the first time. Because
she faced situations in the writing assignments that forced her to work with her concept image along with formal mathematics, she naturally became uncomfortable. Actually, because of her initial procedural orientation, and the conceptual nature of the tasks, she was well-poised to benefit from them.

Perhaps more so than any other interviewee, DeMahli demonstrated a good understanding of the goal of each writing assignment. She consistently began each task with an introductory paragraph putting the points of her paper into a larger framework. I worked and worked to get the entire class into the habit of doing this, for I felt it encouraged both reflective abstraction (the main point of the writing tasks) and the development of ownership of the material. This practice seemed to come naturally to DeMahli.

In assessing her progress over the semester towards gaining a more complete view of the limit concept, it is helpful first to consider what level limit-conception she revealed in her first interview. During this dialogue, I asked her to describe how she normally goes about determining a limit. She replied,

Um, I usually just ... sometimes I pick numbers and just plug 'em in and see as it's approaching the number, a lot of times I just guess, or, y'know, just pick 'cause I'm like, well, I'm not really sure where this comes from or what I should be doing, so, a lot of times it's just ... a lot of plugging and seeing what happens (11,2).

Characteristically, when determining the limit of a step function, DeMahli "just took the value for \( a \) and plugged it into the function." This led to an incorrect answer due to the function's discontinuity. During this interview, she displayed signs of a pre-
action-level understanding of the limit concept -- responding to the limit notation as a command to evaluate the function at the point $a$.

During our final interview, I was curious to see how reliable DeMahli felt the algorithm of “plugging and seeing what happens” felt to her after completing most of the course and all of the writing assignments. She remarked,

Um ... a function can diverge, or else it doesn’t necessarily have to have like one certain point. A pattern can vary. So, um, by plugging in ... just plugging in numbers doesn’t really do you a lot of good, if you don’t really know what is going on (I3, 7-8).

She at least sees the fallibility of her favorite method.

Recall also that DeMahli believed that a function could not reach its limit point. In her final interview she finally declares, “There are cases where it does and there are cases where it doesn’t.” She then goes on to point out (astutely) that many of the problems she worked in class were the type where the function increases up to the limit as $x$ approaches infinity, suggesting that the limit is never reached. This demonstrated conceptual growth on her part, because the action-level conception of limit was seen to be inadequate by her.

She had tremendous enthusiasm for the writing tasks themselves, and told me in her last interview that,

...all calculus classes should have to do writing assignments because it really makes you get into your sections and understand them and try to explain it to somebody else in simpler terms. And in the process of doing that, I learned a lot more myself than I normally would have. I would have just skipped over all that stuff, but I was forced to sit down and think about it and it helped a lot in my understanding of the limit and the different concepts in calculus (I3, 9).
Note that the activity of breaking down concepts into simpler terms can actually move one along the APOS continuum, because it fosters internalization of the concept, making one respond to a context rather than react to notation.

Summary

It appears at least from the two instances of changed perception (realizing the fallibility of the “plugging in points” method and discovering that a function can reach its limit) that DeMahli progressed over the semester in her limit concept. The writing assignments had presented her with situations whose resolution forced her to outgrow the pre-action- to action-level conception with which she began class. Some comments indicate an ability to imagine evaluating a function infinitely many times without actually performing any evaluations, which suggests at least a process-level understanding. For example, in her final interview while discussing \( \lim_{x \to 0} \cos \left( \frac{1}{x} \right) \), she says, “... if you just guessed bigger and bigger numbers, then you wouldn’t have it converging to any specific number. It would just grow and grow and grow, so it wouldn’t ... you wouldn’t be able to find a limit if there wasn’t one.” (I3, 5) (I assumed that “it” always refers to the output of the function \( f(x) = \cos \left( \frac{1}{x} \right) \).) However, when provided the opportunity, she showed no indications of an object-level understanding. For example, in the final writing task, she attempts to address the misconception that a limit of a function can be made as accurate as desired. The most basic reason this is false is that the limit is a fixed object. No amount of motion of the input variable will
change the value of the limit. However, DeMahli does not identify the limit as an object as other students were able to. Instead she writes, "... you should gather that if a function converges to a limit you will get closer and closer to the limit, however, never actually reaching getting there" (WT6). She substitutes another misconception to account for the first one. Since she did not use this opportunity to consider the limit as a fixed number whose accuracy does not change, but did recognize the limit as a process, we can infer that over the course of the semester, DeMahli progressed from pre-action-/action-level to process-level understanding.

**The Profile of Evan**

Evan began his first interview by telling me that he knew nothing about limits, that the whole topic was a mystery to him. His only strategy for dealing with limits was plugging $a$ into the function $f$, the hallmark of pre-action-level understanding. Even the language used to describe this action made him uncomfortable — he saw no reason to pursue an abstraction even as superficial as calling a real number by a letter (as in letting $a$ denote whatever number $x$ is approaching.) He said, "I mean, see, I don’t work with like, $x = a$, I have to have a number there" (11,1).

Evan was rather skeptical about most of the statements on the initial limit survey. He marked the first four false, choosing only to identify as true the statements, "A limit is an approximation that can be made as accurate as you wish," (his favorite) and "A limit is determined by plugging in numbers closer and closer to a given number
until the limit is reached." Each statement demonstrates both an action-level concept of limit and an idea of the limit as a motion-bound procedure rather than a mathematical object.

This idea of plugging in numbers was one to which Evan clung tightly and brought up frequently. When I first questioned him in very general terms what he thought about the limit, he replied,

Um, like if you gave me a graph, and you had like any kind of function there, and you said as $x$ approaches 2, which is there, I could say like, that value of it, whatever $f(x)$ is at 2. Now I could tell you what that is. Now, is that right? Like if you had some ... at $x = 2$, you put 2 into the $f(x)$ equation, isn't that what the limit is right there? (11,1)

It was apparent from later discussions that although Evan seemed to believe that the best way to determine the limit of a function $f$ at a point $x = a$ was to evaluate $f(a)$, this belief was somewhat subconscious. Later when I asked him point blank for the difference between $\lim_{x \to a} f(x)$ and $f(a)$, he said, "Isn't that ... they are like the same thing, aren't they? 'Cause, this is like, means that it's going to get really near $a$, and this is actually $f(a)$, so they'd be approximately the same (11,1)." I then asked,

"Approximately the same, or the same?" Evan cried, "Does it matter? Like on a test I'd say they are the same. Like, as $x$ tends to $a$. "Why do they even have the arrow in the first place? Why don't they just have $a$? What's the point of that?" (11,1). It was clear that although Evan seemed to hold at least a subconscious belief that the limit of a function at a point is simply the value of the function at that point, it was not something he had been made to articulate. The above articulation process forced him
to see the potential waste of notation that would occur if \( \lim_{x \to a} f(x) \) meant nothing more than \( f(a) \). At this point I felt I had a window of opportunity to reach Evan and erase this misconception. I quickly thought of an example of a function and an input value for which \( \lim_{x \to a} f(x) \) did not equal \( f(a) \). I drew a concave down parabola with a hollow circle at its vertex (the \( x \)-value was 2 and the \( y \)-value was 4) and a filled in circle at the point (2, 1). Thus the value of \( a \) was 2, and \( \lim_{x \to a} f(x) \) equaled 4 while \( f(a) \) was 1. However, it took Evan so long mathematically to process the circumstances under which we were working that no cognitive conflict was worked through and resolved. His final comment was, "I don't know, I really don't know what you do with that" (11,2). Thus, he was frustrated, but was theoretically well-positioned to benefit from the cognitive struggle brought on by the writing tasks.

Two features stood out to me as I analyzed the collection of writing tasks Evan submitted: his consistency of effort and the way he brought together some very important ideas in the last writing task. Several times during the course of the semester, he would hand in a thoroughly-done writing task that revealed an action-level limit conception or an incomplete understanding. By the time he wrote the final task, however, he expressed many once-hazy concepts articulately and at a higher APOS level. For example, in the second writing task, Evan wrote about the procedure of finding a reasonable estimate for arc length (using a finite sum) very competently. He then dictated how one could use the integral formula given in class to accomplish...
an exact answer. However, he never discussed how the estimate and the exact answer were related. The fact that the limit of a finite sum is a definite integral completely escaped his scrutiny, and the point of the assignment was missed. I wrote on his paper to further consider the relationship between the estimate and the exact answer.

It seemed as though Evan had never been exposed to the idea that a definite integral is by definition the limit of a sequence of Riemann sums. Classroom observation, however, revealed that his professor had stated in class that the limit of the expression for arc length (namely a sum of line segments' lengths) was a definite integral. Evan had heard this in class, but it apparently made no impact on him. Thus it was surprising and gratifying to read in his final writing assignment,

> We used Riemann sums, breaking up a curve into small subdivisions, and we found the length of each subdivision's segment. Next we added all the segments to get an estimated arc length. Summing all of these lengths turned out to be an integral problem. If we took the limit of a Riemann sum, we got a definite integral that was the length of the arc over the interval. (WT6)

Here the connection between Evan's progress and the writing assignments appears quite strong. Being able to view a definite integral as the result of taking infinitely many sums demonstrates a process-level understanding of the limit. Evan can see the result of investigating a more and more finely minced sum, but certainly does not have to compute all the summations in order to appreciate the value of the definite integral. Thus it appears that he can envision the process of taking a limit. This ability was not evident in the first writing tasks.
Another concept that Evan resolved during the course of the semester was that of the fixed nature of a limit. Recall that Evan began the course with an action-level concept of limit, quite dependent on the “plugging in numbers” strategy. In fact, the statement that most closely reflected his own understanding of the limit concept was, “A limit is an approximation that can be made as accurate as you wish.” This has the vague association of changeability within the limit, and prevents an object-level understanding. In his final paper, however, he writes,

The limit of a function cannot be made as accurate as desired. This was a misconception of mine for awhile. It's easier to explain this problem by example. For instance, a function that has a limit at 1, gets closer to 1 by 0.9, 0.99, 0.999, ... This is getting increasingly closer to 1. However the limit of this sequence is 1. *It doesn't change.* (my emphasis.)

At least in this context, Evan is associating the limit with a fixed number rather than a motion-oriented concept or a procedure for finding a number. This is substantial progress and marks the beginning of an object-level understanding. Evan himself was able to note his progress, stating in the end of semester exit survey that, “This writing assignment (#6) was the best because I tied in all of the topics together.”

Summary

The profile of Evan is a success story. It begins with a student at a pre-action to action-level understanding of the limit, and a willingness to work on the writing tasks. It ends with a student who has developed a strong process-level understanding with elements of object-level evidenced in his writing. This obvious progress,
combined with Evan's deliberate attempts at the writing tasks, suggests a correlation
between one's level-of-participation in the tasks and the intellectual benefits one gains
from completing the tasks.

Having completed the final profile, we now may gain a more general picture of
the class' limit perspective by investigating the writing tasks one at a time and
examining overall class performance on each one.

**Writing Tasks and APOS Theory**

The goal of the writing assignments (Appendix A) was to initiate and foster
reflective abstraction on the part of the subjects. In particular, I sought to put the
students into cognitive conflict by having them reconcile their concept images with
their concept definitions (Tall and Vinner, 1981). The writing tasks were to be a
vehicle for the resolution of this conflict, because their successful completion required
subjects to work through their misconceptions well enough to explain the topic at
hand in their own words, but with reference to rigorous definitions and formulas. The
results from these assignments will be part of the data that attend to the question of
whether the students grew in their conceptual understanding of limit.

Another goal of the writing assignments was to guide the subjects into
mathematical contexts where the limit would emerge as an object (such as the length
of an arc, or the area under a curve.) The subjects who were able to complete the
application would have the experience of viewing the limit as an object, at least within
that narrow context. This by no means suggests that this subject will maintain an object-level understanding of the limit concept in general, but rather that the application provides him or her a means for seeing the limit as an object in a way that a problem set having students evaluate the arc lengths of curves would not encourage. The final writing task contained several questions that were almost litmus tests for action, process, or object-level understanding (there is no such thing as a true litmus test for APOS understanding, as a student could be temporarily using, for example, a process-level understanding because of the problem context, while truly having mental access to an object-level understanding.) Because of this, some of the analysis of the treatment group's collective understanding-level of limit is postponed until the discussion of the final writing task. The first five are discussed in the context of: what subjects would have to achieve to complete the task; what each APOS level would require; how subjects performed on the task; and which APOS levels were at least temporarily demonstrated.

Writing Task #1

The goal of this assignment was to have students reconcile the formal definition of limit with a graph showing epsilon and delta pictorially. The students had to come up with both their own function and a specific value for \( x \) to approach. To demonstrate an action-level understanding, subjects would have to at least evaluate their chosen function at the point that the domain variable was approaching. A
process-level understanding would require a comprehension of the relationship between delta and epsilon, and how restricting the domain of the function would ultimately allow a better and better approximation of the limit of the function. Finally, to achieve an object-level understanding would require a subject to apply a completed delta-epsilon conception to specific situations. Because this task did not demand the application of the limit concept to a specific situation, but rather asked subjects to evaluate a limit with reference to the delta-epsilon connection, an object-level limit conception was not required.

The nature of this writing assignment was different from the next five because it was the only one that did not nestle the limit into a mathematical application. Rather, the focus was solely on the notion of limit itself. Thus, the first assignment served as what Novick and Nussbaum (1982) refer to as an exposing event. The subjects exposed their understandings to me and I provided written feedback to point out discrepancies in their reasoning. This feedback, along with model solutions I created that were distributed in class with the evaluated writings, was to nudge the students towards resolution. Because these were the first data I collected specifically that were associated with the treatment, I could assess current limit conceptions of students (the vast majority of which were at action-level) but could not ascertain cognitive growth. In completing the task, all students chose a continuous function, thus making the limit quite easy to determine. The selection of a continuous function made it impossible to determine whether students realized that it was the special
property of continuity that made the limit the same as the function’s value at that point. Some claimed such a “plug and chug” method would always work. Others plugged in a few points near the value of \(a\) they had chosen, then extrapolated cavalierly, saying it was clear that the limit of their function, as \(x\) approached \(a\), was \(f(a)\). These two approaches, that all but a handful of the students used, corroborated the findings of the first interview of the five selected students, which showed over-reliance on “plugging in points” to be common. The few students that did not use this approach in evaluating the limit instead used the capacity of their graphing calculators to symbolically evaluate the limit for them, effectively mitigating the dilemma.

Another problem that was consistently notable was the lack of algebraic connection students would make between their chosen epsilon and the subsequent delta. After following the directions to choose an epsilon less than 1, some simply guessed at a delta, then checked to see that it provided a small enough domain to keep the outputs within epsilon of \(x\). Even this primitive approach showed more understanding than subjects who failed to notice any connection whatsoever between epsilon and delta. Some chose delta arbitrarily (and often incorrectly) without checking its fit to the limit context. Finally, some never mentioned delta and epsilon again after stating the definition. The comment I wrote the most frequently on writing task #1 was, “How does delta depend on epsilon?”

Thus, analysis based on APOS theory showed the majority of students to be at action-level understanding, with a very small number (\(n = 3\)) at a process-level understanding. The task served well as an exposing event, but showed the class was
weak in its collective limit understanding.

**Writing Task #2**

The goal of the second writing task, Arc Length (see Appendix), was to expose the students to a mathematical application of the limit. Recall that the previous writing assignment required only a process-level understanding of limit. This task required students to make a connection between a limit of a sequence of Riemann sums, and the length of a function’s curve over an interval. Since students would see the *fixed length* of a curve as the limit of a sequence of estimates of a curve’s length, this connection should encourage an object-level concept of limit. Subjects that could make this connection would see the limit as a fixed and immovable number. Thus they would achieve a moment of object-level understanding of the limit concept. This would expose them to a circumstance in which it was clear that the value of the *limit* would not change, regardless of the number of subdivisions one used in computing an estimate.

A subject with an action-level understanding of limit could evaluate the definite integral given in the formula for the length of arc $f(x)$ over the interval $[a,b]$, namely,

$$\int_a^b \sqrt{1 + [f'(x)]^2} \, dx.$$  

In fact, the hallmark "symptom" of action-level understanding is plugging the point that $x$ is approaching, say $a$, into the function. The context-specific analogue of this theory-based behavior is to disregard any possible sequence of sums.
whose limit is a definite integral and proceed rather blindly to the integral itself, plug in numbers, and evaluate.

A subject with a process-level understanding of the limit concept would see the connection between the sums that estimate the arc’s length and the definite integral that gives the exact length (namely, that the integral is the limit of the sequence of sums.) One of the bulleted questions of Writing Task #2 asked why the sum that the subject computed was necessarily an underestimate of the number obtained when correctly evaluating the definite integral of the arc length formula. The reason is that the addends of the sum are lengths of line segments. Since the shortest distance between any two points in the plane is the length of the segment determined by the points, the sum will underestimate the length of the curve. One would need to see the curve’s length as being obtained from the sequence of sums in order to realize why an underestimate is always obtained. This would require a process-level understanding of limit.

Finally, a subject with object-level understanding of limit could simultaneously recognize arc length as a limit of a sequence of sums and use arc length to, say, compare two functions. Such a subject would realize, without computation, that the arc length of the function $f(x) = \sin x$ will be the same over any interval of length $2\pi$.

Over half of the students (22 of 37) were unable to connect the arc’s true length over the interval $[a, b]$ with the estimate they had computed, because they used inappropriate means for finding the estimate. They computed an estimate for the area
under the curve over the interval \([a, b]\). The formula for arc length then seemed to come from thin air. I wrote, "Where did this come from?" on the papers of subjects that didn't attempt to reconcile their erroneous estimate with the answer they got from the definite integral. When students evaluated the wrong sum, it was clear they had little understanding of why arc length could be represented as a definite integral, or, more specifically, as a limit of a sequence of sums of line segments' lengths. Thus, these subjects were displaying an action-level understanding. The ones that correctly reconciled the estimate with true value showed at least a process-level understanding. Object-level understanding was not ostensibly required for the completion of this task.

Writing Task #3

This writing task on improper integrals was designed to disabuse students of the notion that any integral with at least one infinite limit of integration would have an infinite value, regardless of the integrand. I wanted students to have the experience of sketching some representation of a curve whose domain was unbounded, shading underneath the curve, perhaps assuming the area would be infinite, and ultimately discovering this area to be actually finite. To complete the task, students would have to realize that any combination of finite and infinite limits of integration and a finite or infinite value of the integral was possible.

Within the body of the rather lengthy directions, the students were asked, "Can an integral with an infinite upper limit of integration ever have a finite value? Can an
integral with finite limits of integration ever have an infinite value?” The subjects were then provided with three improper integrals to compute, each having a different combination of finite or infinite limits and finite or infinite value.

Again, subjects were considering an integral as a limit. This time, however, the integral was the limit of a proper integral, as one of the limits of integration approached a singularity of the integrand. A subject exhibiting an action-level concept of limit would ignore the singularity and simply evaluate the integral as if it were proper. The subject might even go through the motions of rewriting the improper integral as a limit, but would then immediately plug the singularity back into the integral and evaluate.

A subject performing at the process-level of limit understanding would rewrite the integral as a limit. After applying an appropriate method for evaluating the integral in terms of the variable substituted for the singularity (such as the Fundamental Theorem of Calculus) the subject will take the necessary limit by envisioning plugging in a sequence of points that approach the singularity.

Finally, subjects with object-level limit understanding will be able to think of an improper integral as a (perhaps infinite) number, rather than as a process one goes through. A subject must have an object-level understanding to get the message that a shape with one infinite dimension (such as the area under a curve over the interval from 1 to infinity) may have finite area. There must eventually be an immediate
association between the improper integral and a number representing area for a subject to achieve object-level understanding.

Because of how I organized the task sheet, the subjects were not specifically asked to address the questions “Can an integral with an infinite upper limit of integration ever have a finite value? Can an integral with finite limits of integration ever have an infinite value?” Thus it was easy for them to become bogged down in the computations involved in the limit problems to the point where they lost the big picture, namely that there exist improper integrals whose upper limit of integration is infinite but whose value is finite (such as \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \)). This certainly lessened the impact of the writing assignment. I noted later that when the above question “Can an integral with an infinite upper limit of integration ever have a finite value?” was asked in the interviews, three of the five subjects answered as though they had never considered such a phenomenon, even after completing this writing assignment.

Because students generally tended to perform the process of evaluating an improper integral, then neglected to encapsulate this process enough to consider an improper integral as a number, I could determine that the class collectively was at the process level of limit understanding.
Writing Task #4

This task required the students to write short paragraphs as an avenue for exploring the limit in three contexts: L'Hopital's rule, whether a sequence is convergent or divergent, and the $n$th term test for convergence of sequences. The three main goals of the task, then, were: to allow students to see that L'Hopital's rule can not be applied haphazardly; to have them give an example of a sequence that converged to something other than 0; and to rid them of the misconception that if

$$\lim_{n \to \infty} a_n = 0,$$

the series $\sum_{n=1}^\infty a_n$ must converge.

The first context asked the students under what circumstances L'Hopital's rule may be used, and what the consequences are for misapplying the rule. Because students were able to take much of this procedurally-oriented information from their notes and textbook, and because the students could choose simplistic examples to illustrate the application of the rule, no more than an action-level concept of limit was required to successfully complete this part of the task.

The second context asked students to describe a method for finding the limit of a sequence. This portion was designed to give students experience in wrestling with the concept of sequence before exploring the more complicated concept of series. To complete this part of Task #4 successfully, one needed at least a process-level understanding of the limit. The mental constructs necessary to describe a general method for evaluating a limit (envisioning performing an infinite number of
evaluations) would not be accommodated by an action-level conception. Its completion provided opportunities for uncovering misconceptions; students showed both vague, half-formed ideas and completely-formed, erroneous perspectives. Each phenomenon is seen in this student's attempt, who writes,

Consider the following sequence as converging: \( \sum_{b=0}^{\infty} \frac{b^2}{b^2 + b - 1} \). Tabulating numbers, we find that "b" approaches 1 but never reaches it. Consequently, the limit of the sequence is 1, and is converging. (S8)

He unnecessarily brings in the idea of series, and it is difficult to tell whether he even realizes he is not making a distinction between sequence and series. His discussion concerning \( b \) is quite sloppy (\( b \) is not approaching 1, as \( b \) begins at 0 and tends to infinity) and his last sentence indicates at best a process-level understanding of "the limit of a sequence." Unfortunately, this student's paper and misconceptions were typical of the group. Because many students suffered the same problem of confusing sequence and series, it was plain that another writing assignment specifically dealing with the difference between the two was desirable.

The third portion of Task #4 began by reminding students that if the series \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \). They were asked to discuss the converse, namely, if the limit of the \( n \)th term of the sequence approaches 0, must the corresponding series converge? Approximately three-fifths of class correctly responded that the converse was not true. All who answered correctly provided the harmonic series as a counterexample. There was still a sizable group, however, that
answered incorrectly, and the root of nearly all their individual problems could be traced to a confusion between sequence and series. For example, one subject wrote,

The answer is yes, the series would converge. For a series to converge, its terms must tend to zero. Here is an example of a limit that converges:

\[ \sum_{k=0}^{\infty} \frac{6^k}{k!} = \lim_{x \to \infty} \frac{6^k}{k!} = 0. \]  

(S24)

The fact that this student is confusing sequences with series is evidenced in two parts of his response. First, he refers to the “terms” of a series. A series is a number, it does not have terms. Perhaps he is thinking of the sequence of partial sums. However, even in a convergent series (with positive addends) the terms of the sequence of partial sums do not tend to 0. Thus it is more likely that he is thinking of the sequence defined by \( a_k = \frac{6^k}{k!} \). As he shows, these terms do tend to zero. This does not insure convergence of the series, however, but in fact shows the danger of exchanging sequences for series.

The second way the subject reveals he is mixing up sequences and series is in equating the sum and the limit in the first part of his mathematical argument. When examining the convergence of a sequence, it is logical at least to look at the limit of the \( n \)th term of the sequence as \( n \) approaches infinity. This is what this subject does when examining the series, an incorrect response. (The use of the equals sign is purely wrong, but for examining his APOS level of limit, contributes little to the discussion.)

Another subject responded in more general terms:

The converse of Theorem 5 in our text is true. If the limit approaches zero then the limit must converge. This is true by definition of convergence. When
any limit is found it converges to that limit. Converging basically means that the numbers are tending to, or approaching a certain value. (S5)

This person does not appear to address the sum with this process of examining the convergence or divergence of a series. It becomes a rather thin topic if all one considers is whether or not the individual terms of the sequence go to zero. Although the goal of analysis of this task was not primarily to provide a profile of the class' collective conception of series, it became clear that most were at the process level. They viewed series as an opportunity to perform the operation of finding a limit. From this and from individual responses to the second and third items on this task, it was clear students needed a chance to compare sequences and series head-to-head.

The APOS levels required for task #4 must be analyzed in terms of each context separately. For L'Hopital's rule, once subjects made the leap from $\lim_{x \to a} \frac{f(x)}{g(x)}$ to $\lim_{x \to a} \frac{f(x)}{g(x)}$, the manner in which they would evaluate the top and bottom would relate directly to the APOS level they would express taking the limit of any function. These have been discussed in both the genetic decomposition provided in Chapter 2 and in the APOS-level analysis of Task #1 at the beginning of the section on writing tasks. Below is the work of one representative subject that shows the procedural nature of the class' responses.

To use L'Hopital's rule the function must be in the indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If it isn't in this form it may not work. An example where L'Hopital's
rule can be applied is \( \lim_{n \to \infty} \frac{x^2}{2^n} \). As \( x \) approaches infinity both \( x^2 \) and \( 2^n \) go to infinity so it is in the indeterminate form \( \frac{x}{x} \). Now you need to take the derivative of the expression again so it will no longer be in indeterminate form ...

In the second context, a subject with an action-level understanding of limit would describe finding the limit of a sequence as “plugging infinity into the subscript.” In other words, the subject would try to leap to the “last term of the sequence” by evaluating the \( n \)th term when \( n = \infty \). The curly braces, subscripts, and other sequence notation would be perceived as a command to plug infinity in for \( n \) and evaluate.

A process-level understanding of the limit concept in the context of infinite sequences would require a subject to recognize the connection between the individual terms of the sequence and the limit of the sequence. The subject would realize that there is some value of \( n \) beyond which all sequence terms are within a given tolerance of the limit. This subject, in discussing finding the limit of a sequence, would thus refer to visualizing the operation of evaluating the \( n \)th term of the sequence infinitely many times as \( n \) increases without actually performing these evaluations.

Finally, a subject with object-level understanding would recognize the limit of a real-valued sequence as a number. This recognition would provide a deeper understanding of other limits of sequences and this understanding could be applied in contexts such as definite integrals or series.
Writing Task #5

The two questions students faced in this task were, “What is the difference between a sequence and a series,” and, “How are sequences and series related?” Ideally, in making the distinction between these two subjects they had studied quickly in succession, students would realize that a series is a sum -- a number! -- thereby encapsulating their process-level conception of series to an object. I soon found that while some students did discover that “a series is a sum,” it was a bit too hopeful to think that this realization would lift them permanently to a more sophisticated level of conception. The association of series with the process of finding a limit was difficult for them to temporarily “put away” in order to consider series as an object. For example, here is the response given by a subject who admits that the series is a summation, but clearly maintains a process-level understanding:

Series are a summation of terms of a sequence. So, they are unlike sequences, which were solely related to their respective terms. Series are the addition of each excessive term. In essence a series is composed of two parts. A sequence of terms, and the addition of each term. (S2)

To say that “series are the addition of each excessive term” indicates that one is focusing on the process of adding, rather than on the object of the sum itself. A classmate wrote “A series derives one number, the sum, while a sequence is composed of a bunch of terms.” (S19) In other words, a series is a process whose result is a sum. Collectively, the class was at the process-level of limit as a series.
Another way subjects associated motion with the series concept was to refer to taking the limit of a series. (One does not take the limit of a series -- a series is a (possibly infinite) number.) When students did this, I commented on their papers that in order to determine whether a series converges or diverges, we take the limit of a sequence, namely the sequence of partial sums.

Writing Task #6

The goal of the final writing task was to have the students reflect upon the previous five assignments they had completed and discern the concept that was a common thread (the limit concept). This task was designed so that students would revisit each application of the limit concept and, looking at their own submissions or at the model solutions passed out in class, see the limit as an object in several contexts. In addition, students were asked to respond to three commonly-held misconceptions that they and their classmates had revealed to me over the semester. This writing assignment was really the culmination of the qualitative data gathered from the subjects, and proved quite fruitful, both from an educator's and a researcher's perspective.

The first misconception the students addressed was “the limit of a function can be made as accurate as desired.” Many subjects addressed this adeptly, even ones who had originally agreed with this statement on the Williams (1990) questionnaire at the beginning of the term. One wrote,
[This] is false because the limit of a function is the value it approaches. The value of the function gets closer and closer to the limit but the limit doesn't change. For example, the limit of the function $1/x$ as $x$ approaches infinity equals 0. The function gets closer to 0 as $x$ increases, but the limit doesn't change, therefore the accuracy of the limit cannot change. (S3)

This was particularly gratifying to read, because at the beginning of the semester, this subject had written on his questionnaire that the statement, "A limit is an approximation that can be made as accurate as you wish" was true. In his words, "The boundary that you want to look at can be set to anything you want," as though the limit itself was moveable. Thus over the term this subject was disabused of his misconception to the point where he was able to encapsulate his limit conception to object-level understanding.

The second misconception to which students reacted was, "The sum of an infinite list of decreasing, positive terms is a finite number." Here, most students cited the harmonic series as a counterexample to this statement. Apparently, little tension was involved with this concept, and hence little resolution was necessary.

The final misconception was that, "A shape with infinite length must have infinite area." Most subjects again provided a counterexample. A typical response was an improper integral with infinity for the upper limit of integration, evaluated by taking a limit, and accompanied by the corresponding picture. For example, one subject evaluated $\int e^-x \, dx$ and correctly got 1. He then remarked,

Even though the shape under the function has been given an infinite length, an area was determined, and in this case, it was finite. Thus it would have been.
unfair to pass such a comment without consideration to the function and its limits. (S18)

This subject also went on to give an example of an improper integral whose limits of integration were finite but whose value was infinite.

In general, the sixth writing task was apparently beneficial to the students, as it made them crystallize their thinking and perhaps face a few misconceptions they themselves still held.

Conclusions

The analyses of the writing tasks demonstrate the difficulty in getting students to change their perspectives about the limit concept. However, they do reveal the treatment group’s collective conceptual growth. In fact, all but one of the profiled students (the one who participated minimally in the writing tasks) grew in mathematical sophistication according to APOS theory. This encouraging finding is particularly remarkable in light of the literature review, in which previous researchers had not been able to exploit situations of cognitive conflict to effect cognitive change. A natural question to ask now is if the writing tasks actually improved mathematical performance. We turn to the findings of the quantitative analysis.

Quantitative Findings

In order to answer the question, “Do writing assignments make a difference in students’ achievement and understanding in limit-related concepts in second-semester
calculus?”, I needed to compare a group of subjects who had participated in writing assignments to a group of subjects who had not. A quantitative study does this well. Hence, I gathered numerical scores from both treatment group and control group to compare performance on limit-based problems worked within a testing situation. Recall that, based on randomness of enrollment, similar meeting times, and classroom observations, the two sections were assumed to be equivalent.

The first three problems on each of the final examinations of the two sections were identical, and students’ performance on them provided the data for this comparison. Problems, as I assessed them, were worth 10 points each, and partial credit was awarded according to the rubric described in Chapter 3. I was looking more for evidence of conceptual understanding than for procedural knowledge; hence, my scoring emphasized contextual awareness. The three questions assessed were:

1. Given the improper integral \[ \int_{0}^{1} \frac{x}{\sqrt{1-x^2}} \, dx \]:

   (a) (3 points) Draw a picture representing the value of this integral.

   (b) (7 points) Does the integral converge or diverge? Support your response.

2. (10 points) Find the limit of the sequence \( \{a_k\} \) if \( a_k = \frac{2k}{k-3} \).

3. (10 points) Find the radius and center of convergence of \( \sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n} \).
The first problem tested a subject's ability to evaluate a definite integral after drawing the corresponding graph and shading the appropriate area being measured. The second problem simply tested if subjects understood the difference between series and sequences and furthermore, whether they could compute the limit of a straightforward sequence. The third problem demanded that students work through a potentially lengthy array of computations to determine a limit and make further use of the answer.

Using the scores the two groups separately achieved, a two-tailed, two-sample t-test, with unpooled variances, provided the following means and p-values. A significance-level of $\alpha = .05$ was used. The specific hypotheses I attended to were:

$H_0$: Students performing writing tasks intermittently throughout the semester will have the same final examination scores on the three limit-related problems as the non-writing students will, and,

$H_a$: Students performing writing tasks intermittently throughout the semester will have significantly higher final examination scores on the three limit-related problems as the non-writing students will.

The results from scoring the three problems are given in Table 2. Following the table is a problem-by-problem analysis of the performance of the students, both treatment and control, along with preliminary comments suggesting the specific impact the writing assignments had on each of the three problems. Notice that all differences favor the treatment group. However, because significance was established at the $\alpha = .05$ level, only the second and third problems show a significant difference in performance between the two groups. Following the table, an analysis of the results is provided for each of the three problems.
Table 2
Results on Common Final Examination Limit-Based Problems

<table>
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<th>Problem</th>
<th>Treatment</th>
<th>Control</th>
<th>p-value</th>
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</thead>
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<td>.229</td>
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<tr>
<td>2</td>
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<tr>
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<td>3.44</td>
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</table>

Problem #1

Although the difference in average score did favor the treatment group, there was no significant difference in performance. The students who had been assigned the writing tasks performed better, in general, on part (a) of the problem. They had been considering "area under the curve" during the semester. This was done in the context of determining if such area could be finite even if the domain for the curve were infinite. Such consideration took conceptually-oriented efforts, and perhaps showed subjects the value of a correctly-drawn graph. Thus, those in the treatment group were more likely to indicate a vertical asymptote at $x = 1$ (29 of 37 did so), because they were accustomed to wrestling with the idea of seemingly infinite area being described by a finite number. On the other hand, many students in the control group neglected to indicate a vertical asymptote (only 16 of 34 did so), which implied to me that they had not come face-to-face with this seeming paradox.
Once the students got to the more procedurally-oriented part (b), the two groups proceeded at very comparable levels of facility. Apparently, the operation for determining whether an integral converges or diverges was not impacted by the writing assignments, which explains the similar overall performance. The small difference in performance on part (a) was not enough to achieve a significant difference in performance on this problem. It is interesting and perhaps surprising that neither group performed very well, particularly on a concept so fundamental to the course. However, if a subject failed to make the substitution \( u = 1 - x^2 \), the problem was quite difficult. This is a reason for the similarly low scores. The substitution issue could have hidden differences in the two groups’ understandings of the limit concept. A more helpful test question could have been developed that would avoid such issues and better isolate the limit concept for study.

Problem #2

Interpretation of the results of problem #2 is straightforward: since no credit was awarded for interpreting the given sequence as a series (which led to an incorrect conclusion), those subjects who were unclear of the difference between sequence and series were penalized strongly. This begins to explain the significant difference. The vast majority of those who did not get full credit on problem #2 unnecessarily introduced summation notation. (Simultaneously, most subjects who interpreted the sequence correctly as a function instead of a summation were able to use one of a
variety of methods in finding the limit.) Because the treatment group had completed a
writing task about how to evaluate the limit of a sequence, they were much less likely
than the control group to mix up sequence and series. This problem indicates an
impact the writing assignments had had on the treatment group -- they saw the
difference between sequences and series. In particular, they recognized the limit of a
sequence was different from the value of the corresponding series.

Problem #3

There was a notable difference in performance between the treatment group
and the control group in the third problem. The correct solution to the problem (as
taught by both instructors) required setting up the limit of a ratio. If a subject were
unable to correctly set up this limit, the rest of the points of the problem were almost
impossible to earn, as each step of the solution built upon the previous one. Thus,
subjects that did not know to compute \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \), were at a distinct and measurable
disadvantage. Very few of those in the control group made this connection and,
interestingly, even those who did were, in general, unable to evaluate it.

Conclusions

An analysis of the final examination problem scores seems to indicate that
writing about the limit process enables subjects to perform better on even
procedurally-oriented mathematics problems in a testing situation. There were subtle differences between the treatment group and the control group, but the largest difference was the writing assignments. Thus it is likely that the difference in mathematical performance can be attributed to writing. This finding complements the findings obtained by analysis of qualitative data, namely that writing about the limit process enabled students in the treatment group to grow in their conceptual understanding as well. The next chapter provides more in-depth conclusions about the qualitative results, the quantitative results, and their interplay.
CHAPTER V

CONCLUSIONS

The goals of this study were first, to measure the level of understanding students reach as a result of participating in a semester of second-semester calculus in which writing activities are included, and second, to determine if completing carefully chosen writing assignments has a beneficial effect on the students' understanding. Analyses of both qualitative and quantitative data were performed and will be summarized to this end. Responses to these two inquiries are provided in this chapter, as well as a section on general discoveries, pedagogical implications, and recommendations for future research. It was discovered that writing assignments do seem to have a beneficial effect on the understanding of the second-semester calculus students who engage in them, allowing them to develop more sophisticated views of the limit concept (namely, outgrowing a pre-action to action-level concept in favor of seeing the limit as a process and occasionally as an object.)

Question 1

The first question formally posed in this study was, "What conception of limit, in terms of the action-process-object-schema (APOS) theory, are students reaching as a result of completing a semester of calculus in which writing activities are included?"
To answer this required extensive analysis of qualitative data, examined both by fixing
the subject and letting the writing task vary and by fixing the writing task and letting
the subject vary. That is, I profiled both a subset of the treatment subjects and the full
set of writing assignments.

Subject Profiles

Recall that five research subjects were chosen from the group participating in
writing tasks. To obtain a sample that was heterogeneous with respect to limit
perception, the subjects were chosen on the basis of their responses to the
questionnaire developed by Williams (1991). Four out of the six statements were
represented within my sample of five subjects. The following is a brief distillation of
what I learned from each subject, with particular emphasis on the individual’s growth
along the APOS continuum (in light of their initial limit perspective), accompanied by
rationale as to why their pattern of growth developed as it did.

The Profile of Amy

Amy began the course with a motion-bound idea of limit. The statement with
which she identified most strongly was, “A limit describes how a function moves as x
moves towards a certain point.” Interviews and writing assignments demonstrated
that she began the course with an action-level concept and finished with a process-
level concept. She appeared to grow largely due to her proactive response to the
comments I put on her submitted writing tasks. My written comments had more impact on her than the same comments made in class, perhaps due to their one-on-one nature, or the fact that Amy had struggled seriously with the material being commented upon.

The Profile of Brett

Out of the five subjects profiled, Brett showed the most dramatic growth in conceptual understanding of the limit concept. He originally identified with the statement, “A limit is a number or point the function gets close to but never reaches,” but overcame this misconception during the semester and grew from an action-level to a very strong process-level understanding, even demonstrating object-level understanding in certain contexts. I attribute the fruitfulness of Brett’s participation to what I called the “authentic voice” in which he wrote and spoke in trying to make sense of limits, and also to the proactive manner in which he learned from my written comments.

The Profile of Charles

Recall that Charles was the subject who all but refused to engage in the writing assignments or the interviews. He began the course with the perception that, “A limit describes how a function moves as x moves towards a certain point.” Over the course of the semester, he never lost the motion-bound concept of limit. Analysis showed
that he began and ended the treatment at a level between action and process. He was
the only subject profiled who did not improve in understanding and was also the only
one to demonstrate no conceptual growth. Thus I attribute his lack of growth to his
lack of participation.

The Profile of DeMahli

DeMahli initially relied heavily on the “plug in points” algorithm for evaluating
a limit. Accordingly, the limit perspective she most closely agreed with was, “A limit
is determined by plugging in numbers closer and closer to a given number until the
limit is reached.” Over the semester, through cognitive conflict and resolution, she
overcame this and other misconceptions to grow from a pre-action to a process level
of understanding

The Profile of Evan

Recall that Evan demonstrated consistency of effort in completing the writing
tasks throughout the term. In particular, he brought together several concepts in the
final task, and addressed three previously-held misconceptions. He began the study
thinking of a limit as a motion-oriented procedure and finished the study describing the
limit as a fixed number. In terms of APOS theory, he began with a pre-action- to
action-level understanding and finished at process- to object-level understanding.
Conclusion From Profiles

Notice that, with the exception of Charles, each subject progressed at least a full APOS level; most moved one-and-a-half to two levels. I originally thought that I could attribute Charles’ lack of movement to his initial limit perspective. Perhaps there was some misconception among individuals with his perspective that was particularly difficult to overcome and hence was impervious to the treatment developed. However, Evan began with the same limit perspective as Charles, and yet advanced two APOS levels. The only difference I found between Charles and his cohorts, then, was amount of engagement with the material via writing.

This difference prompted a need for defining engagement in terms of APOS theory. To make pedagogical recommendations that encourage student-engagement in the material, requires sharing a definition of this term with the reader. Recall that one of the characteristics of an action-level understanding is that a subject perceives mathematical notation as a command to do something. Furthermore, the motions that the subject performs upon perceiving that command are externally-motivated. On the other hand, the subject with a process-level or higher understanding of the given topic is guided by the problem situation and its related mathematics and relies more on internal guidance. Thus, a subject engaged in the material responds to mathematical context rather than reacts to mathematical notation. Therefore, according to this definition, a subject willing to engage in the material gives him- or herself the chance to operate at the process level. Perhaps more relevant to this study is the
contrapositive: a subject unwilling to engage in the material does not afford him- or herself the chance to develop a process-level understanding.

The cognitive growth that was measured among those interviewed is a remarkable finding, particularly in light of what Williams (1991) declared after trying to use interviews to improve students' conceptual understanding: “The stage was set for cognitive conflict, and in fact, some conflict did occur. What did not occur was real cognitive change” (p. 229). Within my interview transcripts and writing tasks, I was able to see evidence of “real cognitive change.” Recall also that in discussing the cognitive obstacles that calculus students face in learning about limits, (many of which were specifically addressed within this study) Cottrill and his colleagues (Cottrill, et al., 1996) say, “We have not ... found any reports of success in helping students overcome these difficulties” (p. 172). The summary of the profiles from the present study provides promising results.

To attend to the research question, “What APOS level are the students reaching?”, I found that with the exception of Charles, all subjects profiled finished the treatment with at least a process-level understanding. Two displayed elements of object-level understanding.

Based on the conclusions of the profiles and summary, one can then examine the writing tasks alone and gain more of an understanding of the individual effects of each assignment on students' understanding. Also, one can assess how the treatment class as a whole performed.
**Writing Tasks**

To determine the individual effects of each writing task, I looked at how all thirty-seven subjects in the treated group performed on the task, what the pedagogical goal was, and which APOS level was minimally required to complete it. In this way I could correlate the demands of the assignment with the growth of the subjects. Such knowledge would be particularly helpful in developing further writing tasks. What I found, however, as I analyzed the impact the tasks had on the students' understanding, was that no one task had the effect of permanently raising a group of students performing at one APOS level to the next higher level. Rather, it was the sum of writing assignments, all requiring that the subject make plain the relations between formal definitions and personally constructed concept images, that enabled students to progress in their thinking as assessed by the APOS structure.

As was discussed in Chapter IV, one of the purposes of Tasks #2 - #5 was to provide students a context for exploring a mathematical concept in which the limit would emerge as an object. Specifically, these objects were: the length of a curve, the area under a positive-valued function with a vertical or horizontal asymptote, the number to which a sequence converges, and the sum of an infinite sequence of terms. Thus the first and last tasks served as exposing events, where subjects would disclose their individual limit concepts, and allow me to assess their cognitive change.
Conclusions Regarding Question 1

The findings of this study are particularly notable since Cottrill and his colleagues (Cottrill, et al. 1996) were unable to find any reports of success in helping students to grow in their conceptual understanding of limits. Similarly, other authors (Li & Tall, 1993; Monaghan et al., 1994; Tall, 1992) have reported that even technology that allows students to visualize the limit-taking process has not been successful in alleviating subjects' misconceptions in taking limits. Thus, to discover that writing assignments made a measurable difference in both the treatment students' understanding and their achievement is noteworthy.

After profiling five students individually and analyzing students' collective performance on each writing task individually, I can attend to the question, "What conception of limit, in terms of the action-process-object-schema (APOS) theory, are students reaching as a result of completing a semester of calculus in which writing activities are included?" To summarize, I discovered that students who engage in the assignments progress from their action-level construct to minimally a process level understanding of the limit concept. In this particular study, the collection of writing assignments enabled some students to see limit as an object to such an extent that they were able to incorporate elements of object-level understanding into their limit schemas.
In order to answer the question, "Do writing assignments make a difference in students' achievement and understanding in limit-related concepts in second-semester calculus?", I analyzed both qualitative and quantitative data. Because I wanted to measure a difference, I needed a comparison between a group of subjects who engaged in writing tasks and a group who did not. Thus, I performed a statistical comparison between the two groups' mean scores on three final examination problems. Conclusions from this comparison are reported below. I also asked subjects from the treatment group questions in the third interview and in an exit survey that were specifically about their writing. These qualitative data are subsequently reported.

Quantitative Analysis

Conclusions from the quantitative analysis are first drawn problem-by-problem, then summative statements are made based on the collection of problems.

Problem #1

Students were asked to work with the improper integral \( \int_0^1 \frac{x}{\sqrt{1-x^2}} dx \) by first drawing a picture representing the value of this integral, then determining whether the integral converges or diverges. Because the treatment group had spent a week writing
a paper about improper integrals and the different possibilities for convergence and divergence, it was expected that they would outperform their counterparts in the control group. In fact, their scores were only slightly higher; statistically, there was no significant difference ($p \approx 0.23$). This result, however, is fully corroborated by the qualitative data gathered in the interviews and writing assignments. Recall that in Writing Task #3, the body of prose explaining the expectations of the task contained the question, “Can an integral with infinite limits of integration be finite? Can an integral with finite limits of integration be infinite?” It would seem that conscientious attendance to such questions during the semester would improve student performance on Problem #1. However, recall that during the second interview, I discovered from three out of the five interviewees that those two questions had not even registered with them as they skimmed the directions for Writing Task #3. This explains the minimal impact of the assignment, and suggests a means of improvement of the writing task as well.

**Problem #2**

Students were asked to find the limit of the sequence $\{a_k\}$ if $a_k = \frac{2k}{\sqrt{k+3}}$. In this case, the treatment students performed significantly better than the control students. This was because a much greater proportion of control students tried to interpret the given sequence as a series. Thus, they inappropriately summed the terms
instead of finding their limit. This led them to believe that the sequence diverged, when in fact, it converges. Writing Task #5, completed by treatment students four weeks earlier, asked the difference between a sequence and a series, thus very few treatment subjects made this mistake. This time, the corresponding writing assignment had a notable and measurable impact.

**Problem #3**

Recall that the third problem asked students to find the radius and center of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$. This was the problem on which the treatment group and the control group had such remarkably different results. The students who had been writing regularly about limits, including series and power series, were much more able to set up an accurate ratio with limit notation and correctly find an appropriate limit. Many of those in the control group were unable to make any progress towards setting up the limit of a ratio, let alone evaluating it. Again, the writing assignments had a measurable impact.

**Conclusions From Final Examination Problems**

Differences in mathematical performance favored treatment students on all three problems. Two out of the three problems showed significantly better performance by the treatment group, suggesting that participating in writing tasks does
make a difference in students’ understanding of the limit-related concepts. The lack of difference between the two groups' performance on the first task might have been the poor structuring of the explanations of the requirements of the task: they were arranged in such a way that students could address all the requirements without coming face-to-face with the seeming dichotomy of infinite limits of integration and a finite value of the integral. Still, the overall disparity in performance, coupled with the literature’s declaration that no effective pedagogical treatment has yet been discovered for helping students overcome limit misconceptions, strongly suggests that the six writing tasks did make a difference in students’ understanding and achievement in limit-related concepts.

Students’ Perceptions

It is interesting to consider how the students participating in the study felt about the writing tasks they performed. In an end-of-semester survey, I asked all subjects who had performed the writing assignments for their comments on the individual tasks as well as the overall experience of writing in their calculus class. The response was overwhelmingly positive. Thirty-six out of the 37 participants highly praised the use of writing in the classroom. I was pleasantly surprised at the genuinely complimentary remarks the students consistently made. Many offered comments from which one could infer that the tasks did accomplish their goal of promoting reflective abstraction. For example: “[Writing] not only taught the methods of the topic, but
also the why of the topic” (S18); “To complete them, you kind of had to go back, do some reading, do some thinking, and relearn the material” (S23); and, “... it reinforced my vague knowledge to a better understanding of convergence and divergence” (S24).

Another response I got with surprising frequency (n = 8) was gratitude for being assigned the writing tasks. Students could see the progress they had made as a result of their writing. They wrote, “Teachers make you do something, and you might get out a whole lot more than what you thought. Thanks” (S33); and, “I think I learned much more from them than I ever would have learned with standard homework assignments. Thanks!” (anonymous). The students were not expressing gratitude for being assigned tasks that lightened their workload. Rather, they appreciated the opportunity and the demand to do assignments that improved their conceptual understanding.

It should be noted that most of the students did not realize that someone other than the instructor was grading their writing tasks. A few surmised my involvement from my presence in the classroom coupled with an unfamiliar handwriting on their graded writing tasks. Two of the interviewed subjects asked me if I were grading the writing tasks. The majority, however, based on their responses to the end-of-semester survey, attributed the comments to the treatment instructor.
Conclusions Regarding Question 2

In considering the research question, "Do writing assignments make a difference in students' achievement and understanding in limit-related concepts in second-semester calculus?", I can respond positively based on the test scores of the students who participated in the study, the writing assignments, and the written remarks of the treatment group. Further studies need to be conducted to determine if other pedagogical devices would foster this type of reflective abstraction.

General Discoveries

In getting to understand students' thinking through reading their writing assignments, interviewing them and observing their behavior in class, I became aware of two interesting phenomena that were not specifically addressed through quantitative or qualitative means. These were patterns of thought that allowed students to cope with the frustration of not understanding limits and an overwhelming desire to stop thinking so hard about an uncomfortable subject.

The first phenomenon was one I would call a difference in perspective that the students seem to hold from the typical perspective of professional mathematicians. More specifically, the concept of limit is considered by many mathematicians to be a symbol of what made mathematics rigorous -- an exact, precise science. The celebrated mathematicians Cauchy, Liebnitz, Newton, and others took great pains to create a framework in which the theory of infinitesimals would fit. Once this was
accomplished, much of the mystery and suspicion that had previously surrounded the
limit concept was eliminated. It is ironic, then, that this very symbol of mathematical
rigor among developers and users of the calculus, the limit, is the pinnacle of
ambiguity, approximation, even frustration among many students of the calculus.

The result of this difference in perspective, based on insight from this study, is
that students perceive a certain inevitability of imprecision that is companion to the
limit concept. This feeling made Amy declare, when working with improper integrals
that converged, “The area goes on forever, but they pretend it stops” (13, 4). She
“knew” the area could not possibly be finite if the horizontal dimension continued
forever, and she also knew that limit answers were protected from reality. Thus she
could explain away the answer she felt was true and the answer she knew was
considered correct by mathematical authority (the interviewer, in this case.) This same
feeling made Evan say, “Technically, .999 repeating forever is less than one, but in real
life they are the same” (12, 6) Although this phenomenon was not specifically studied,
but rather observed, its emergence has significant implications. By allowing students
to maintain this dualistic notion of what they think is true about limits and what they
think “authority” thinks is true, we lose an opportunity to expose them to the benefits
of mathematics over our own (sometimes faulty) intuition. This fosters the next
phenomenon I found to emerge.

As mathematicians, we trust the validity of mathematics enough to know that
our intuition will at times fail us. We use mathematics in these cases to correct our
intuition and adjust our concept images accordingly. This is especially practiced in the cases of limits and infinity. Early students of mathematics usually believe in their intuition much more strongly than in mathematics, thus, when they are confronted with seeming dichotomies, they further adjust their understanding of mathematics to fit their intuitions. This corroborates the findings of Fischbein, Tirosh and Hess (1979) who discovered that one's concept of infinity could grow more and more sophisticated while one's intuition of infinity remained unchanged.

The result of this adjustment is that students will contradict statements they know to be true in order to preserve the correctness of their intuitions. This is what allowed DeMahli, who believed that a function could never reach its limit, to say blandly, "The constant function is not really a function" (I2, 7) when she knew perfectly well that it was a function. Again, this phenomenon can have debilitating effects on students in that it actually supports the attribution of illogical conclusions to the perceived inconsistency of mathematics. This makes it difficult for students to develop mathematical autonomy, because if they don't believe in the validity of mathematics, they have no independent standard (irrespective of person or textbook) by which to evaluate their work. Secondly, it makes it difficult for them to identify their own mistakes, because mathematical inconsistency is seen as normative, rather than as an indication that one's argument is flawed.
Pedagogical Implications

Several of the findings from this study suggest the importance of writing assignments in the second-semester calculus classroom. First, I discovered that students are reaching at least a process-level understanding of the limit concept as a consequence of performing a battery of writing assignments. Such conceptual growth has never before been documented, even with the three-step pedagogical treatment of Nussbaum and Novick (1982) and the computer use of Tall and Li (1993). Thus, the qualitative data imply the usefulness of writing tasks.

Second, students who worked through the series of writing assignments significantly outperformed their non-writing counterparts on two of the three final examination questions (see Table 2, Chapter IV). It is inferred that writing helped them make mathematical connections that enabled them to perform limit-oriented problems with greater facility. Hence the quantitative data imply the usefulness of writing tasks as well.

Finally, the two phenomena of “rigorous versus approximate” and “the laws of mathematics versus intuition” clearly demonstrate a need for tasks that force students to reconcile their concept images with their concept definitions. These tasks must allow, then, for reflective abstraction. The writing tasks developed for this study were designed to effect and prolong reflective abstraction. Thus these two phenomena imply the use of writing tasks, too.
If one is going to recommend the use of writing tasks, it is very important to characterize the classroom climate that envelops the tasks. In other words, what type of instructor-given support should accompany the writing assignments in order to maximize their impact? Informal data gathered during this study serves to guide instruction for supporting the writing tasks.

I discovered during the first interview (as well as subsequent interviews and the exit survey) that subjects were taking the comments I wrote on their papers quite seriously. At least 14 made explicit reference to comments, either in interviews, casual conversation with the investigator, or on the end-of-semester interview. They used the comments to improve both the style and the content of future submissions, as well as deepen their understanding of the current topic. Upon learning this, I began to treat each writing assignment as a private tutoring session. Each paper a student submitted served as an exposing event. My comments, then, were tailored specifically for that student’s misconceptions. It is my recommendation that the instructor gently guide the student away from his or her incorrect or incomplete ideas and suggest means of improvement, rather than crossing out the student’s work and providing the correct answer. A specific mathematical focus, combined with personally-oriented instructions, seemed to give students incentive to uncover their mistakes and learn more of the conceptual underpinnings of the topic.

Furthermore, the fact that the writing assignments were returned in such a timely fashion -- by the next class period -- made an impact on the students. Three out
of the five students profiled specifically commented (unsolicited) that they appreciated getting the tasks back, with detailed comments, in such a short time.

Recall that the writing tasks were designed to initiate, foster and prolong reflective abstraction. Initially, I conjectured that any task that was designed to this end would improve students' conceptual understanding of the limit concept. However, there were several unanticipated benefits (such as the impact of personal written feedback, mentioned earlier) associated specifically with writing tasks that might not be found in other alternative assessments.

One final influence I noted was that the language that mathematicians and mathematics instructors use in describing limits does not seem conducive to helping students develop an object-level understanding of the concept. In particular, much of the germane vocabulary and sentence structure is motion-bound. For example, when talking about an infinite sum, a number, we ask the students, "Does this series converge or diverge?" In other words, what does the sum do, as though it moves. A pedagogical suggestion based on my findings would be occasionally to rephrase the question as, "Is the sum infinite or finite?", in order to reinforce the fixed nature of series. Of course, this extends to improper integrals, and other settings where the limit can be beneficially viewed as a process or an object, depending on the circumstance.
Recommendations for Future Research

The recommendations that follow are partitioned into two groups. First, ways in which the present study could have been more effectively executed are provided. Next, suggestions for further studies that would build on this one are discussed.

Improvements for Present Study

To increase the impact of the writing assignments on the treatment group, a closer working relationship between the researcher and the treatment instructor would be desirable. For example, in the implementation of the present treatment, the instructor would pass out the writing tasks with few comments, or with comments to which the class was not attentive. Perhaps if the researcher and the instructor more closely outlined goals of the writing assignments, the subjects would feel the influence of the instructor's beliefs, and respond by putting more time into the tasks. One way to overcome this potential obstacle would have been to teach the course I was investigating. However, this almost assuredly compromises the objectivity of the researcher and perhaps the validity of any research instruments and replicability of the study. For this reason, I felt it was better to choose instructors other than myself to carry out the study.

The writing variable would have been more strictly isolated if the two instructors had assigned the exact same problems when they each assigned a problem set. This synchronization is difficult to achieve without infringing upon one or both of
the teaching styles of the participating instructors, but it would strengthen the case for
the writing's impact.

To better isolate the specific phenomenon of infinite limits of integration on an
integral of finite value, the wording on the third writing task should be improved. In
particular, a specific question dealing with this occurrence could be put in the "bullets"
section, so students could not complete the assignment without coming face-to-face
with this issue.

Finally, to better measure conceptual growth, the questionnaire from Williams
(1991) could be administered to both sections at the beginning and at the end of the
study, so that the change in subjects' responses could be analyzed. Although there is
the problem of subjects learning from the pretest and thus artificially inflating their
posttest scores, this problem would occur in both sections, thus neutralizing its
confounding effects. Respective changes in the two sections' responses could then be
compared and analyzed.

Suggestions for Further Studies

Recall that there is a specific call in the literature for more studies detailing the
impact of writing on students' mathematical performance. Because of this, studies
that build from the results of the present study would be welcomed. In particular,
further studies are needed to see if the impact of the writing tasks is changed if certain
variables are changed. For example, do writing assignments continue to improve
conceptual understanding if they focus on a different mathematical topic, such as functions, or groups and subgroups, or vector spaces? Do different students (such as those in third-semester calculus or later classes) respond differently to the tasks? Finally, as was touched upon before, is it the writing, or is it reflective abstraction and engagement in general, that effected the changes documented in the present study?
Appendix A

Writing Tasks and Model Solutions
Making Sense of Limits
Writing Task #1

The concept of limit is one that is used frequently in second-semester calculus. To help you refamiliarize yourself with this important idea, complete the following writing assignment.

In first-semester calculus, you studied the concept of the limit of a function $f(x)$ at a point $a$. Informally, if $f(x)$ tends to a single number $L$ as $x$ approaches $a$, then

$$\lim_{x \to a} f(x) = L.$$ 

The formal definition, according to Calculus, from Graphical, Numerical and Symbolic Points of View, page 155, says:

Suppose that for every positive number $\epsilon$ (epsilon), no matter how small, there is a positive number $\delta$ (delta) so that

$$|f(x) - L| \lt \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$ 

Then,

$$\lim_{x \to a} f(x) = L.$$ 

Oliver, a student enrolled in first-semester calculus, has just read this definition for the first time, and sees no connection between the formal definition that he's seen in class and the graphical representation he saw and read in the text (p 155). He seeks you out as a tutor and asks how to make sense of the words and the pictures.

To help him reconcile these two perspectives of the limit, you

- choose a specific function $f(x)$ and a specific value $a$ (3 points)
- determine the limit at this point using whatever method you feel comfortable using, explaining your method as you work (3 points)
- choose an epsilon less than one and determine the resulting delta, with specific reference to the formal definition (4 points)
- draw the specific graph corresponding to the function you chose, like the sketch on page 155, but with epsilon and delta as concrete numbers, and your function in for the curve (5 points)

Your paper will probably be one to two pages, word-processed. You may do the graphical sketches by hand, or with software.
Hello, Oliver. I can relate to your puzzlement regarding the whole limit concept. It is not an easy idea, so let me give you a specific example to help you understand the relationship between the graphical and the symbolic.

Let's begin with the function \( f(x) = x^3 \), and consider the limit of \( f \) as \( x \) approaches 0. We'll call this limit \( L \). In symbols, this is

\[
\lim_{x \to 0} f(x) = L.
\]

Our goal is to find the numerical value of \( L \).

Consider the function \( f(x) = x^3 \). Because it is a polynomial, it is a continuous function, and therefore

\[
\lim_{x \to 0} x^3 = 0^3 = 0.
\]

In other words, \( L = 0 \). Now what does this have to do with the formal definition of limit and the picture given in the textbook? To see this better, let's choose a specific value for \( \varepsilon \), and determine a corresponding \( \delta \). Say \( \varepsilon = 1/8 \). Then, according to the definition in the textbook, since \( L = 0 \) and \( a = 0 \), there is a positive number \( \delta \) so that

\[
|f(x) - 0| < 1/8 \quad \text{whenever} \quad 0 < |x - 0| < \delta.
\]

Put more simply, we are looking for a positive number \( \delta \) so that

\[
|x^3| < 1/8 \quad \text{whenever} \quad |x| < \delta.
\]

Now this is just an algebraic inequality to solve. If \(-1/8 < x^3 < 1/8\), then \(-1/2 < x < 1/2\). Thus, \( \delta = 1/2 \). The picture below gives the graphical representation. Hope this helps you to understand better, Oliver!
Arc Length: Sums and Integrals
Writing Task #2

In class, the topic of arc length was discussed, both from a theoretical and a practical point of view. The arc length of a curve $f(x)$ over an interval $[a,b]$ was shown to be

$$\int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.$$

The purpose of this task is to explore the relationship between Riemann sums and definite integrals by considering the graph of a curve and estimating its arc length. To do this,

- choose a non-linear function $f(x)$ and an interval $[a,b]$ and sketch $f(x)$ over this interval
- divide $[a, b]$ into four subintervals of equal length
- set up and evaluate a Riemann sum (with four terms) that approximates the arc length of $f(x)$ over $[a,b]$
- draw line segments on the sketch of $f(x)$ whose lengths represent the terms in your Riemann sum
- determine the arc length of your curve "exactly" over the given interval using the integral formula above ... you will likely have to use numerical methods.

Write up this assignment in paragraph form, word-processed, with sketches provided (hand drawn is fine.) In the concluding paragraph, address the question of why your approximation was an underestimate of the true arc length.

Each bullet is worth three points, and completing the assignment earns five points, for a total of 20 points possible.
Arc Length: Sums and Integrals
Possible Response

The following is an exploration of the relationship between Riemann sums and definite integrals in the context of arc length.

Consider the function $f(x) = x^{1/2}$ over the interval $[1, 9]$. Below is a sketch of the graph, with the interval divided into four subintervals of equal length. Line segments beginning at each of the four subintervals ($x_1 = 1, x_2 = 3, x_3 = 5, x_4 = 7$) are drawn in as dotted lines.

How long is each dotted segment? If we can sum the lengths of the four dotted segments, we can estimate the length of the curve $f$ over this interval $[1, 9]$. To find the length of one segment, we use the distance formula. Recall that the distance from any two points in the plane $(x_1, y_1)$ to $(x_2, y_2)$ is given by $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$. Thus in my case, the first segment has length

$$\sqrt{(f(3) - f(1))^2 + (3 - 1)^2} = \sqrt{(\sqrt{3} - 1)^2 + 4} \approx 2.1298 \text{ units.}$$

But this is only one segment, and I must evaluate the length of all four, then sum them. In other words, I must find

$$\sum_{i=1}^{4} \sqrt{(f(x_{i+1}) - f(x_i))^2 + (x_{i+1} - x_i)^2}.$$ Broken down term by term, this is

$$2.1298 + 2.0625 + 2.0415 + 2.0311 = (approximately) 8.2649.$$

Thus the sum of the lengths of the four line segments is a little more than 8 and one-quarter units. How well does this sum compare to the integral formula, which gives an exact value for the arc length?
First of all, notice from the picture that the Riemann sum should be an underestimate of the true value, since the line segment joining two points is the shortest "curve" possible joining the two points. (See sketch on back.)

This length: \[ \text{will always be smaller than this length:} \]

What does the integral formula give us?

The arc length of \( f(x) = x^{\frac{1}{2}} \) over the interval \([1, 9]\) is equal to

\[
\int_{1}^{9} \sqrt{1 + \left(\frac{1}{2} x^{-\frac{1}{2}}\right)^2} \, dx = \int_{1}^{9} \sqrt{1 + \frac{0.5}{x^{\frac{1}{2}}}^2} \, dx = \int_{1}^{9} \sqrt{1 + 0.25x^{-1}} \, dx = 8.2681459
\]

using Simpson's rule with 100 subdivisions. Notice this true value is just a little bit (a very little bit!) larger than the estimate we got using Riemann sums.
Improper Integrals
Writing Task #3

To complete this assignment, you will study definite integrals whose limits of integration cause the integrals to be improper. Recall that in class, you were given the definition of the value of such an improper integral as follows:

\[ \int_{0}^{b} f(x) \, dx = \lim_{b \to -\infty} \int_{0}^{b} f(x) \, dx. \]

In other words, the "trouble spot" (infinity in this case) is replaced by a finite number \( b \). Why must the integral be written before it can be evaluated?

Can an integral with an infinite upper limit of integration ever have a finite value? Can an integral with finite limits of integration ever have an infinite value?

To get started in answering these questions, evaluate the three definite integrals below:

1) \( \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \)

2) \( \int_{0}^{\infty} e^{-x} \, dx \)

3) \( \int_{1}^{\infty} \frac{1}{2x} \, dx \)

To complete the writing assignment,

- write an introductory paragraph explaining what you are investigating (3 points)
- evaluate the three integrals above, showing your work, and providing an accompanying graph with each that represents the value you determine to be the answer (6 points)
- answer "why must the integral be rewritten to be evaluated?" (3 points)
- write a paragraph explaining each possible case of infinite or finite limits and infinite or finite value (there are four possible combinations.) Feel free to use the examples you worked earlier. (8 points)

This task is worth 20 points. The explanations must be word-processed, but the calculations and graphs may be done by hand.
The following is an exploration of improper integrals. In particular, the questions of whether a definite integral with finite limits can ever be infinite, or whether a definite integral with infinite limits can ever be finite will be answered. To do this, we will begin by applying the definition of improper integrals to the three given definite integrals.

We will first consider \( \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \). The area under the curve from 1 to infinity can be represented in the following graph:

![Graph of \( \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \)](image)

Applying the definition of improper integral, we see that

\[
\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{\sqrt{x}} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^{-\frac{1}{2}} \, dx = \lim_{b \to \infty} 2x^{\frac{1}{2}} \bigg|_{1}^{b} = \lim_{b \to \infty} 2\sqrt{b} - 2 = \infty.
\]

Thus this improper integral diverges to infinity.

2) Next we consider \( \int_{0}^{\infty} e^{-x} \, dx \). A graph representing this integral is given below.

![Graph of \( \int_{0}^{\infty} e^{-x} \, dx \)](image)

Now, \( \int_{0}^{\infty} e^{-x} \, dx = \lim_{b \to \infty} \int_{0}^{b} e^{-x} \, dx = \lim_{b \to \infty} -e^{-x} \bigg|_{0}^{b} = \lim_{b \to \infty} -e^{-b} + 1 = 1 \). Thus this improper integral converges to 1.
3) Finally, we consider the definite integral \( \int_{1}^{2} \frac{1}{2x} \, dx \). The graph below represents this integrand.

Now, \( \int_{1}^{2} \frac{1}{2x} \, dx = \int_{1}^{2} \left( \frac{1}{2x} \right) \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{2x} \, dx = \lim_{b \to \infty} \frac{1}{2} \ln|x|^b \bigg|_1^b + \lim_{a \to 0^+} \frac{1}{2} \ln|a|^b = \infty \)

Thus the integral diverges to infinity.

Through these three examples, we can see why the improper integral must be rewritten. Otherwise, the integral would have to be evaluated at points for which the function was not defined (impossible to compute) or at the value infinity (which makes no sense to do.)

Finally, we will consider the four possible cases of infinite or finite limits of integration producing infinite or finite values.

**Case 1:** Finite limits of integration producing a finite value.
This is probably the most straightforward case to discuss. An example could be \( \int_{1}^{5} x^3 \, dx = \frac{x^4}{3} \bigg|_1^5 = \frac{124}{3} \). Looking at the graph of this curve, it is clear that the area underneath between 1 and 5 is finite.

**Case 2:** Infinite limits of integration producing an infinite value.
This phenomenon is seen in the example above of \( \int_{0}^{\infty} e^{-x} \, dx \). The upper limit of integration is infinite, and yet the value of the integral is finite.

**Case 3:** Finite limits of integration producing an infinite value.
Again, we saw this type of behavior in the example of \( \int_{1}^{2} \frac{1}{2x} \, dx \). The limits of integration are finite, yet the integral that they yield is infinite.

**Case 4:** Infinite limits of integration producing an infinite value.
This concept was seen in the first example \( \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \). The upper limit of integration is infinite and the value of the integral is infinite as well.

Apparently there are no rules telling whether to expect an infinite or a finite value for a definite integral, based exclusively on the limits of integration. These types of problems must be examined case-by-case.
Explanatory Paragraphs
Writing Task #4

To help you prepare for the upcoming exam, answer the following three questions on L'Hopital's rule, limits of sequences, and the nth term test for convergence of a sequence. Each response should have the format of a paragraph written to a classmate who needs a pre-exam review of the material. A question of this type ("explain the concept in a paragraph") will be on exam #2, so this exercise will provide practice.

1) Under what circumstances may L'Hopital's rule be applied? What can happen if the rule is misapplied, that is, applied in a situation that is not appropriate? (Include an example of each of the two situations, "can be applied" and "cannot be applied.") Finally, give an example of an expression to which L'Hopital's rule initially appears not to apply, that may be manipulated to fit the desired indeterminate form.

2) How does one compute the limit of a sequence? In answering this, include the definition of the limit of a sequence, an example of a convergent sequence whose limit is not 0, and an example of a divergent sequence, all using proper sequence notation.

3) Theorem 5 in your textbook is called the nth term test for divergence. It says if \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges. Is the inverse of this statement true? That is, if \( \lim_{n \to \infty} a_n = 0 \), does that mean that \( \sum_{n=1}^{\infty} a_n \) converges? If so, explain why. If not, provide a counterexample with justification.

Each paragraph will be worth 6 points, with 2 points awarded for handing in the assignment (word-processed and complete) on time. Thus the assignment is worth 20 points total.
1) **Under what circumstances may L'Hôpital's rule be applied? What can happen if the rule is misapplied, that is, applied in a situation that is not appropriate?**

L'Hôpital’s rule may be applied to the limit of any ratio of functions that has the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ when the limit of the top and bottom are taken individually. Such a ratio of functions is in “indeterminate form”, and is suitable for application of the rule. For example, a good candidate for application of L'Hôpital’s rule is $\lim_{x \to 0} \frac{\cos 2x - 1}{\sin x}$. In this case, the ratio is of the form $\frac{0}{0}$. A place where L'Hôpital’s rule may not be applied is when the ratio is not in indeterminate form, such as $\lim_{x \to 2} \frac{x^2 + 5}{x^2}$. Because the function $\frac{x^2 + 5}{x^2}$ is continuous at $x = 2$, the limit is simply $\frac{2^2 + 5}{2^2} = \frac{9}{4}$. However, if we tried to apply L'Hôpital's rule to this ratio, we would obtain $\lim_{x \to 2} \frac{x^2 + 5}{x^2} = \lim_{x \to 2} \frac{2x}{2x} = 4$. This is “garbage.” Finally, we consider an expression that initially seems not to be in indeterminate form: $\lim_{x \to 0} (-\ln x)(x^2)$. The limit as $x$ approaches 0 is of the form $\frac{\infty \cdot 0}{\infty}$. If we rewrite this as a ratio, however, we get $\lim_{x \to 0} \frac{-\ln x}{x^2}$, which yields the desired indeterminate form of $\frac{\infty}{\infty}$. Thus the ratio’s limit may then be evaluated.

2. **How does one compute the limit of a sequence?**

To answer this, we should start with the definition of the limit of a sequence. Let $\{a_n\}$ be a sequence. We define the limit of $\{a_n\}$ to be $L$ if $a_n$ approaches $L$ to within any desired tolerance as $n$ increases without bound.

To better see how one might compute this number, let’s consider an example of a convergent sequence. For example, let $a_n = \frac{2n + 5}{n + 2}$. 

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Let's consider the first several terms of \( \{a_n\} \):

\[
\begin{align*}
  a_1 &= 7/3 \\
  a_2 &= 9/4 \\
  a_3 &= 11/5 \\
  &\vdots \\
  a_{10} &= 25/12.
\end{align*}
\]

It appears that the sequence is approaching 2.

To prove this, we take the limit of \( a_n \) as \( n \) approaches infinity.

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n + 5}{n + 2} = \lim_{n \to \infty} \frac{2}{1} = 2.
\]

(Notice that L'Hopital's rule was legitimately applied in the second to last step.)

Finally, let's look at a divergent sequence. Let \( b_k = \frac{k!}{k^2} \). Again, we will consider the first few terms, to get a feel for the sequence:

\[
\begin{align*}
  b_1 &= 1 \\
  b_2 &= 1/2 \\
  b_3 &= 6/4 \\
  b_4 &= 24/16 \\
  b_5 &= 120/25 \\
  &\vdots \\
  b_{10} &= 3628800/100
\end{align*}
\]

It appears that \( b_k \) increases without bound as \( k \) approaches infinity. Taking the limit of the sequence proves this, as \( k! \) grows faster than \( k^2 \).

3) If \( \lim_{n \to \infty} a_n = 0 \), does that mean that \( \sum_{n=1}^{\infty} a_n \) converges?

No, this statement is not true. A good counterexample to this is the harmonic series, denoted \( \sum_{k=1}^{\infty} \frac{1}{k} \). That this sequence diverges is stated on page 565 in the text. The reason people tend to believe this statement is because the converse is true: if the series converges, the terms must tend to 0. (This was discussed in class as the difference between a necessary and a sufficient condition. For a sequence to converge, it is necessary that the terms approach 0, however, this is not sufficient.)
Sequences and Series
Writing Task #5

In completing this exercise, remember the goal is to get you to reflect on your thought processes. In other words, it is expected that this will take a little while. In fact, completing the task should take you an hour, which includes reflecting, writing and proofreading.

The questions you are to answer in this task are: What is the difference between a sequence and a series? and How are they related?

Context: imagine that you are explaining this to a Math 123 classmate who has hired you as a tutor. In your paragraphs explaining these differences and relationships, you will probably find it helpful to include the definitions. However, since your classmate already has the text, do not take definitions straight from the book. Instead, explain ideas in your own words, in terms he or she can understand more easily.

Evaluation: the directions are less specific this time, as you have had several examples (writing tasks #1 - #4) of the type of writing expected. It is up to you to determine the critical elements to include (what would you want in a complete lesson on the differences and similarities between sequences and series?) The task is worth 20 points total, and will be evaluated according to the “critical elements” you include and the thoroughness and clarity of your explanations of them.
Sequences and Series

Possible Response

In this tutoring session, I will attempt to answer a couple questions for you regarding the material from Chapter 11 in the textbook. First, what is the difference between a sequence and a series? Secondly, how are sequences and series related? To answer these, we should first have the definitions of both to work with. What I'll do is give you the book's definition, then tell you what it means to me.

Definitions and Examples: Let's begin with sequences, because as you'll soon see, the idea of sequence is used in series. The textbook says that an infinite sequence is a real-valued function that is defined for positive integers. Well, ok, ... but that doesn't help me too much, because when I think of sequences, I almost always think of subscripts, and patterns, and stuff like that, so just knowing that a sequence is a function isn't too illuminating. I personally think of a sequence as an ordered, infinite collection of real numbers. Often when discussing a particular sequence, the book gives an expression for the nth term of the sequence -- in other words, what a typical term looks like. I like this kind of sequence, because I can figure out the next term easily. Here are some examples of sequences:

1) 1, 1, 4, 2, 5, 9, 3, 5, 2, ...

2) \(3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\)

3) \(\left\{\frac{n!}{n^2}\right\}_{n=1}^{\infty}\)

Each suggests an infinite list of real numbers. The three have different properties, however. The first list does nothing to suggest a pattern. I cannot determine what the "nth term" of the sequence is. However, it is a legitimate sequence because each term is a real number.

The second list suggests more of a pattern. Even though I am not explicitly told, I can tell that the first term is equal to \(\frac{3}{2^0}\). The second term is equal to \(\frac{3}{2^1}\). The third term is equal to \(\frac{3}{2^2}\). In general, the nth term is equal to \(\frac{3}{2^{n-1}}\). Even though I wasn't given an equation for the sequence terms, I can now generate as many as I want, based on the formula I derived.
Finally, in the third list, we are in the opposite situation as the second. This time, instead of being given terms, we are given a formula to "unpack." Let's generate some terms:

\[ a_1 = \frac{1}{1} = 1 \]
\[ a_2 = \frac{2 \cdot 1}{2^2} = \frac{1}{2} \]
\[ a_3 = \frac{3 \cdot 2 \cdot 1}{3^2} = \frac{6}{9} \]
\[ a_6 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6^2} = \frac{620}{36} = 20 \]

From just the "naked" terms, it would have been difficult to guess the formula. I'm glad that it was given. These examples and the definition should give you an idea of what sequences are about. Let's move on to series.

The book tells us that an **infinite series** is an infinite sum. In my words, it is the result of adding together infinitely many real numbers. Aha!! "Infinitely many real numbers" sounds like a sequence. Thus, a series is the sum of a sequence. Unfortunately, the term "series" makes it sound like "motion" is involved. But we can tell from the definition that a series is really just a number, like 5. It's just obtained in a certain way, namely by adding up infinitely many terms, a job that would take forever unless we develop some mathematical methods for summing.

Now let's take a look at some series. To make things simple, I'll just sum the terms of the sequences given as examples above.

1) \[ 1 + 1 + 4 + 2 + 5 + 9 + 3 + 5 + 2 + \ldots \]

2) \[ 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots = \sum_{n=1}^{\infty} \frac{3}{2^{n-1}} \]

3) \[ \sum_{n=1}^{\infty} \frac{n!}{n^2} = \frac{1}{1} + \frac{2}{4} + \frac{6}{9} + \frac{24}{16} + \ldots \]

As I explained before, each of these series represents a number (possibly positive or negative infinity.) This, to me, brings up a fundamental difference between sequences and series. A sequence is an infinite **collection** of real numbers, whereas a series is the **sum** of such a collection.
Convergence and Divergence: Going back to sequences, a question we often consider is, "To what value do the terms of the sequence tend (or, what is the limit?)" To answer this, we first derive a formula for the \( n \)th term of the sequence, then take the limit of the formula as \( n \) approaches infinity. If the formula approaches a certain number \( L \) as \( n \) grows without bound, then \( L \) is the limit of the sequence, and we say the sequence converges to \( L \). If the function oscillates or grows without bound, the function diverges. In our three examples of sequences, the limit of the first is impossible to determine, the limit of the second is 0, and the limit of the third is positive infinity.

We may also talk about the convergence or divergence of a series, but this refers to the series' value: is it finite, infinite, or impossible to determine because of the oscillation of the terms being summed? We may answer this actually by looking at a special sequence — the sequence of partial sums. (Here is a way that sequences and series are closely related!) If the sequence of partial sums converges, the series converges. If the sequence of partial sums diverges, the series diverges.

In our examples, the value of the first series is again impossible to determine. The second one converges to 6 (geometric series), and the third diverges to infinity.

Conclusion: In summary, the biggest difference between sequences and series is structural. A sequence represents a list, while a series represents a number. The two are closely related, however, especially when one talks about convergence and divergence. Let's take one final (often seen) example -- the harmonic series.

The sequence \( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \) pretty clearly has terms that approach 0. Thus the sequence converges to 0. However, the related series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges to infinity. Thus despite similar appearances, sequences and series are very different (but related) animals.
Looking Back
Writing Task #6

In your first five writing tasks, you explored and discussed five topics:
- the formal definition of limit
- arc length
- improper integrals
- summary paragraphs about L'Hopital's rule, $n$th term test and sequences
- the differences and similarities between sequences and series

To complete this series of writing assignments, your final writing task (worth 20 points) will require you to look back and to address the following questions.

1) What concept was a common thread running through all five topics? How was this concept utilized in each paper?

2) Three common misconceptions regarding limits and infinity are
   a. "the limit of a function can be made as accurate as desired"
   b. the sum of a list of decreasing, positive terms is a finite number
   c. a shape with infinite length must have infinite area

Choose three writing tasks (one per misstatement) whose completion would require a student to overcome their misconception. (In other words, if the student developed a reasonable response, there is no way possible the student could maintain their misconception.)

Assessment:
1) Item 1 requires a clear, specific response highlighting how the identified concept was addressed in each paper. (Probably two sentences per paper.) (5 points.)

2) Item 2 demands a response that clearly and specifically shows how each misconception would be overcome by the student carefully responding to the writing task. (5 points apiece -- 15 points for entire item.)
Part 1: One common thread that ran through all five topics was the concept of limit. The concept was introduced in the first paper, which required the formal definition of limit. Then the concept was applied in the second paper, dealing with arc length. The arc length of a curve between two points on the x-axis can be found by first approximating this length with small line segments connecting intermittent points on the function, then taking the limit of this sum of line segments as the points get infinitely close to one another. In other words, arc length is a limit of a sum of line segment lengths.

The third topic, indefinite integrals, requires the replacing of at least one of the limits of integration (one that makes the integral improper) with a variable, such as $t$, then taking the limit as $t$ approaches the original limit of integration. An improper integral is the limit of a sequence of proper integrals. The fourth topic, L'Hopital's rule, is a method for finding the limit of a quantity originally presented in indeterminate form. This rule lets us take the limit of the numerator and the denominator separately and arrive at the same result as if we had taken the limit of the entire quotient (usually a much more difficult task.) Finally, a major aspect of both sequences and series is their behavior as $n$ approaches infinity. The value of a series (a sum) is the limit of the sequence of partial sums.

Part 2: a) The misconception that “the limit of a function can be made as accurate as desired” is addressed in the first writing assignment, which required a student to first define a limit. The informal definition given in the textbook is that “If $f(x)$ tends to a single number $L$ as $x$ approaches $a$, then $\lim_{x \to a} f(x) = L$.” Thus, the limit of a function is a single fixed number, not a quantity that can change as desired.

b) The idea that “the sum of a list of decreasing, positive terms is a finite number” seems plausible, but after correctly completing either writing assignment about sequences, no student could retain that misconception. For one thing, the terms themselves must approach 0 if the series is going to converge. The terms of the sequence $\left\{ \frac{k + 1}{k} \right\}_{k=1}^n$ are decreasing as $k$ increases, yet the series diverges because it fails the $n$th term test.

c) Finally, the idea that a shape with infinite length must have infinite area would be corrected while completing the writing assignment dealing directly with improper integrals. The area under the curve $f(x) = e^{-x}$ from 0 to infinity has an infinitely long
base. However, \( \int_0^\infty e^{-x} \, dx = \lim_{b \to \infty} \int_0^b e^{-x} \, dx = \lim_{b \to \infty} \left[-e^{-x}\right]_0^b = \lim_{b \to \infty} -e^{-b} + 1 = 1 \). Thus the writing assignments, in one way or another, addressed each of these misconceptions.
Appendix B

Classroom Observation Checklists
Classroom Observation Checklist
Opportunities for Reflective Abstraction

Instructor Behavior:
1. Does the instructor provide “think time” after posing questions? Does he provide time for students to respond to their classmates’ responses?

2. How does the instructor synthesize ideas for the students during the lecture? Are students ever responsible during class to perform such synthesis?

3. What level of questions are being asked of the students? (Action, process, or object?)

4. What steps does the instructor take towards encouraging metacognition?

Student Behavior:
1. How do students respond to the questions the instructor asks? Approximately what percentage of the students engage in this question-and-answer dialogue?

2. How thoroughly do students present results when doing problems on the board for one another?
Appendix C

Limit Questionnaires
**Limit Questionnaire**

A. Please mark the following six statements about limits as being true or false:

1. T  F  A limit describes how a function moves as $x$ moves towards a certain point.

2. T  F  A limit is a number or point past which a function cannot go.

3. T  F  A limit is a number that the $y$-values of a function can be made arbitrarily close to by restricting $x$-values.

4. T  F  A limit is a number or point the function gets close to but never reaches.

5. T  F  A limit is an approximation that can be made as accurate as you wish.

6. T  F  A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle one.)

```
1  2  3  4  5  6  None
```

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function $f$ as $x \to s$ is some number $L$. 

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Appendix D

End-of-Semester Surveys
End-of-semester survey

Several of you have remarked that the writing assignments have been helpful in developing a more complete grasp of the material. So that these writing tasks can continue to be helpful, I would like your responses to the following questions. Please respond as honestly and completely as possible. You will be graded only on thoroughness of response, not on whether or not you had the “correct” opinion. (Complete this after doing the final writing task.) This survey is worth 10 points.

1. Which of the six writing assignments was the most helpful? (Remember you have a list of the first five on your final writing assignment.) In what ways (be specific) was it helpful?

2. Did you work independently, or did you consult with a classmate or with Dr. Pence before “plunging in?”

3. What further comments about the writing do you have?
Appendix E

Three Common Final Examination Problems
Each problem is worth 10 points.

1. Given the improper integral \( \int_{0}^{1} \frac{x}{\sqrt{1 - x^2}} \, dx \).
   
   a) (3 points) Draw a picture representing the value of this integral.
   
   b) (7 points) Does the integral converge or diverge? Support your response.

2. Find the limit of the sequence \( \{a_k\} \) if \( a_k = \frac{\sqrt{2k}}{k + 3} \).

3. Find the radius and center of convergence of \( \sum_{n=1}^{\infty} \frac{(x - 2)^n}{n \cdot 3^n} \).
Appendix F

Interview Questions
Interview 1

1. What does it mean for a function $f$ to have the limit $L$ at the domain point $a$? In symbols,

$$\lim_{x \to a} f(x) = L.$$ 

2. Apply your idea of limit to the case in which $f$ is given by $f(x) = \lceil x \rceil$, the greatest integer function? In other words, evaluate $\lim_{x \to a} f(x)$ for $a = 1/2$ and for $a = 1$.

3. a) Apply your idea of limit to the case in which $f$ is given by $f(x) = \cos(1/x)$, $x = 0$. In symbols, evaluate

$$\lim_{x \to 0} f(x) = \cos(1/x).$$

b) Could $f$ have a value at 0? If so, what would it be? If not, why not?

4. Apply your idea of limit to the case in which $f$ is given by $f(x) = x \cos(1/x)$, $x = 0$.

5. Consider the sequence whose $n$th term is given by

$$a_n = \frac{2n^2 - 1}{1 + 5n^2}$$

a) What is the limit $L$ of this sequence?

b) Express as rigorously as you can what it means to say that $\lim_{n \to \infty} a_n = L$. 

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Interview 2

1. In your last writing assignment, you discussed improper integrals. Can an integral with finite limits of integration ever have an infinite value?

2a) Compare the values 0.9999 and 1. How are they related?

b) Evaluate the limit of the sequence \( \frac{10^n - 1}{10^n} \).

3a) Is the limit of a sequence, as n approaches infinity, ever reached?

b) Evaluate the limit of the sequence \( \{a_n\} \) below:

\[
a_n = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1/2n & \text{if } n \text{ is odd}
\end{cases}
\]

c) Define a function \( f(x) \) by letting \( f(x) \) be the distance from a certain train to the station at time \( x \), where \( x \) is measured in hours after 12:00 noon on November 1, 1997. At exactly 2:00 p.m. that day, the train arrives and comes to a complete stop at the station. Discuss the limit of \( f(x) \) as \( x \) approaches 2.

4. A student was given a function \( F \) and asked to find the limit of \( F \) as \( x \) approached 0. He plugged in numbers on each side of 0 and made the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9</td>
</tr>
<tr>
<td>0.01</td>
<td>0.99</td>
</tr>
<tr>
<td>0.001</td>
<td>0.999</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.999</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.9</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.9</td>
</tr>
</tbody>
</table>

What can you conclude about the limit of the function \( F \) as \( x \) approaches 0?
Interview 3

1. Describe to me your “process” of completing a writing assignment. What were things you did for every one that were similar?

2. Comment individually on each of the first five writing assignments — the level of difficulty, how it helped you to understand a concept, and anything else you want to include.

   - general definition of limit
   - arc length
   - indefinite integrals
   - several short paragraphs
   - sequences vs. series

3. How did these writing assignments connect to your studying the course material?

4. Describe some benefits and some drawbacks of having writing assignments.

5. Given that \( F(x) = x + 1 + \frac{1}{10^{20} x} \), evaluate \( \lim_{x \to 0} F(x) \).

6. A classmate makes several comments about limits. What is your response to each of the following?

   a) It is always possible to find the limit of a function by plugging in a finite number of points.

   b) A limit is an approximation that can be made as accurate as you wish.

   c) A limit is a number that the \( y \)-values can be made arbitrarily close to by restricting the \( x \)-values.
Appendix G

Revised Genetic Decomposition of the Limit Concept
Genetic Decomposition

1. The action of evaluating the function $f$ at a single point $x$ that is considered to be close, or even equal to, $a$.
2. The action of evaluating the function $f$ at a few points, each successive point closer to $a$ than was the previous point.
3. Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which $x$ approaches $a$.
   (b) Construction of a range process in which $y$ approaches $L$.
   (c) Coordination of (a),(b) via $f$. That is, the function $f$ is applied to the process of $x$ approaching $a$ to obtain the process of $f(x)$ approaching $L$.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit.
7. A completed $\varepsilon - \delta$ conception applied to specific situations. (p 177-178)
Appendix H

Final Examination Scoring Rubric
Final Examination Scoring Rubric

Each problem is worth 10 points.

1. Given the improper integral \( \int_{0}^{1} \frac{x}{\sqrt{1-x^2}} \, dx \),

   a) (3 points) Draw a picture representing the value of this integral.
   - 1 point for shading area under the function.
   - 1 point for some indication of a vertical asymptote such as a dotted line or an arrowhead on the function's curve.
   - 1 point for an accurately drawn curve over the appropriate interval \([0,1]\).

   b) (7 points) Does the integral converge or diverge? Support your response.
   - 3 points for demonstrating the integral is improper and setting up a meaningful limit: \( \lim_{k \to 1} \int_{0}^{k} \frac{x}{\sqrt{1-x^2}} \, dx \).
   - 4 points for evaluating the integral and limit correctly (including a substitution such as \( u = 1 - x^2 \)).
   - No credit given if the sequence is interpreted as a series.
   - 4 points total for a meaningful but incorrect attempt (L'Hopital's rule that leads nowhere, correctly writing out first few terms of sequence and finding their decimal approximations, etc.)
   - 8 points total for graphing expression as a function of \( x \) and estimating the limit to be about 1.4
   - 10 points total for finding by taking an appropriate limit (such as \( \lim_{k \to \infty} \sqrt{\frac{2}{1 + \frac{3}{k}}} \)).

2. Find the limit of the sequence \( \{a_k\} \) if \( a_k = \frac{2k}{k + 3} \).
   - No credit given if the sequence is interpreted as a series.
   - 4 points total for a meaningful but incorrect attempt (L'Hopital's rule that leads nowhere, correctly writing out first few terms of sequence and finding their decimal approximations, etc.)
   - 8 points total for graphing expression as a function of \( x \) and estimating the limit to be about 1.4
   - 10 points total for finding \( \sqrt{2} \) by taking an appropriate limit (such as \( \lim_{k \to \infty} \sqrt{\frac{2}{1 + \frac{3}{k}}} \)).
3. Find the radius and center of convergence of \( \sum_{n=1}^{\infty} \frac{(x - 2)^n}{n \cdot 3^n} \).

3 points for setting up the correct ratio with limit notation, namely \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \).

3 points for correctly finding the limit \( \frac{|x - 2|}{3} \).

2 points for setting the limit less than 1.
1 point for identifying the radius.
1 point for identifying the center.

(per instructions from the participating professors, students may disregard endpoint behavior.)
Appendix I

Protocol Clearance From the Human Subjects Institutional Review Board
Date: 27 August 1997

To: Christine Browning, Principal Investigator
   Melanie Wahlberg, Student Investigator

From: Richard Wright, Chair

Re: HSIRB Project Number 97-08-02

This letter will serve as confirmation that your research project entitled "The Impact of Writing on the Second-Semester Calculus Student’s Understanding of the Limit Concept" has been approved under the exempt category of review by the Human Subjects Institutional Review Board. The conditions and duration of this approval are specified in the Policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note that you may only conduct this research exactly in the form it was approved. You must seek specific board approval for any changes in this project. You must also seek reapproval if the project extends beyond the termination date noted below. In addition if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the HSIRB for consultation.

The Board wishes you success in the pursuit of your research goals.

Approval Termination: 27 August 1998
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