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## Resolving Classes and Resolvable Spaces in Rational Homotopy Theory

Timothy L. Clark  
*Western Michigan University*

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RESOLVING CLASSES AND RESOLVABLE SPACES IN  
RATIONAL HOMOTOPY THEORY

by

Timothy L. Clark

A dissertation submitted to the Graduate College  
in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy  
Mathematics  
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Doctoral Committee:

Jeffrey Strom, Ph.D., Chair  
Michele Intermont, Ph.D.  
John Martino, Ph.D.  
Jay Wood, Ph.D.

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RATIONAL HOMOTOPY THEORY

Timothy L. Clark, Ph.D.

Western Michigan University, 2016

A class of topological spaces is called a resolving class if it is closed under weak equivalences and homotopy limits. Letting  $\mathcal{R}(A)$  denote the smallest resolving class containing a space  $A$ , we say  $X$  is  $A$ -resolvable if  $X$  is in  $\mathcal{R}(A)$ , which induces a partial order on spaces. These concepts are dual to the well-studied notions of closed class and cellular space, where the induced partial order is known as the Dror Farjoun Cellular Lattice. Progress has been made toward illuminating the structure of the Cellular Lattice. For example: Chachólski, Parent, and Stanley have shown that it is a complete lattice, while Hess and Parent have shown that the sublattice of rational spaces admits an embedding of a quotient of the Witt group.

Using the current theory surrounding closed classes and cellular spaces as a framework, this investigation is an attempt to further develop the theory of resolving classes and resolvable spaces. In particular, topological and algebraic criteria for determining if  $X$  is  $A$ -resolvable when  $X$  and  $A$  are rational spaces are found. Beyond a characterization of the rational case, these criteria are used to uncover some of the structure inherent to the resolvability relation on spaces, including the emergence of the same quotient of the Witt group discovered by Hess and Parent in the cellular lattice. Comparisons (and perhaps more interestingly, contrasts) are drawn between the theory of cellular spaces with that of resolvable spaces, and potential directions for future inquiry are discussed.

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Timothy L. Clark

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# Chapter 1

## Preliminaries

### 1.1 Overview: Introduction and Organization

#### Introduction

Many mathematical inquiries deal, either directly or indirectly, with the following question: how do certain objects<sup>1</sup> *relate* to one another? A particularly effective method of approach to this general question is to study the ways in which such objects are *put together* in terms of simpler objects. In homotopy theory, there are two basic processes by which spaces are constructed: via (homotopy) colimits or via (homotopy) limits.

The former procedure (homotopy colimits) is related to the study of closed classes and cellular spaces, a collection of ideas in homotopy theory that began to surface in the early 1990's with roots in Bousfield's work on localization ([3],[6]). A **closed class** is a class of spaces closed under weak equivalence and homotopy colimits. One of our primary examples of such a class is  $\mathcal{C}(A)$ , the smallest closed class containing a space  $A$ . The spaces in  $\mathcal{C}(A)$  are said to be  **$A$ -cellular**. We think of  $A$ -cellular spaces as being built from copies of  $A$ , reminiscent to the manner in which a CW complex is built from spheres; indeed  $\mathcal{C}(S^n)$  is the class of all  $(n-1)$ -connected spaces. If  $X$  is  $A$ -cellular and  $A$  is  $X$ -cellular, we say  $X$  is **cellularly equivalent** to  $A$ . Letting  $A \ll X$  denote  $X \in \mathcal{C}(A)$ , we induce a partial order on the collection of (cellular equivalence classes of ) spaces, known as the Dror Farjoun Cellular Lattice [6]. The complexity of this lattice has proven to be an interesting area of investigation, with various efforts being made to illuminate its structure. Hess and Parent, for example, have shown the sublattice of rational spaces to admit an embedding of a certain quotient of the Witt group [15], while

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<sup>1</sup>This word is taken loosely in this context; think "object of a category."



Chachólski, Parent, and Stanley have shown that this lattice is actually complete [4]. There are even density results concerning the sublattice of rational spaces, exhibited by Félix and Parent in [11].

The latter procedure (homotopy limits) is related to the concepts of resolving classes and resolvable spaces, appearing in the literature during the early 2000's in Strom's work on Miller spaces [25]. We say a class of spaces is a **resolving class** provided it is closed under weak equivalence and homotopy limits. Again, one of our primary examples of a resolving class is  $\mathcal{R}(A)$ , the smallest resolving class containing a space  $A$ . We say  $X \in \mathcal{R}(A)$  is  **$A$ -resolvable**, and write  $A \preceq X$ , inducing a partial order on spaces in a similar manner to the cellular case. We think of  $A$ -resolvable spaces as being formed from a product of copies of  $A$ , comparable to the way a (path-connected) space can be thought of as a "twisted" product of Eilenberg-MacLane spaces via its Postnikov tower. Resolving classes appear as a focal point in Strom's homotopy-theoretical proof of Miller's theorem [26], and have found application in Schwass's work on phantom maps [24].

Though the ideas surrounding resolving classes and resolvable spaces lie close to the heart of topological study (providing descriptions of how certain spaces relate to one another), they remain considerably less studied than their dual counterparts. The overall goal of this work is to investigate these ideas in greater detail, and (hopefully) to exhibit a complexity that warrants further interest and investigation.

## Organization

We now briefly describe the organization of the development to follow, and highlight some of the main results we will encounter in our expedition.

### Chapter 1

Section 1.2 lays out some basic terminology and notation. Section 1.3 is an overview of (Moore-)Postnikov towers. Section 1.4 involves a cursory tour through abstract homotopy theory and the related model category theory. Section 1.5 is a summary of the concepts in rational homotopy theory that will be of interest to us. Finally, Section 1.6 is a quick introduction to (systems of) rational quadratic forms and rational homogeneous polynomials in preparation for Chapter 3.

Of course, the possibility exists that some of the results concerning quadratic forms and homogeneous polynomials that appear could be useful to those studying such things; thus, the attempt was made to be reasonably self-contained, so as to be approachable to mathematicians who may not be experts in homotopy theory.

That said, the author assumes a familiarity with the basic notions of (classical) algebraic topology and category theory - references include [13] and [16].

## Chapter 2

Chapter 2 concerns resolving classes, especially in the context of rational homotopy theory. Resolving classes, resolvable spaces, the resolvability relation  $\preceq$ , and resolvable equivalence are defined. The main theorems of this section involve characterizing when  $X$  is  $A$ -resolvable in the rational case. Specifically, we have:

**Theorem 2.10.** *For (simply connected) rational spaces  $X$  and  $A$ :*

- *if the anticonnectivity of  $X$  is strictly less than that of  $A$ , then  $X \in \mathcal{R}(A)$ ;*
- *if  $X \in \mathcal{R}(A)$ , then the anticonnectivity of  $X$  is less than or equal to that of  $A$ ;*
- *$X$  is a member of the smallest strong resolving class containing  $A$  if and only if the anticonnectivity of  $X$  is less than or equal to that of  $A$ .*

**Theorem 2.12.** *Let  $X$  and  $A$  be rational spaces of anticonnectivity  $n + 1$ . Then  $A \preceq X$  (i.e.  $X \in \mathcal{R}(A)$ ) if and only if there is a homotopy class*

$$f : X \rightarrow \prod_{i \in \mathcal{I}} A$$

*for some index set  $\mathcal{I}$ , with  $\pi_n(f)$  injective.*

Taken together, these theorems completely describe when  $A \preceq X$  in the case that both spaces are rational. If their anticonnectivities are unequal, Theorem 2.10 offers us a direct answer based on how their anticonnectivities compare. In the case of equal anticonnectivity, Theorem 2.12 offers a criterion dual to the Chachólski-Parent-Stanley criterion (Theorem 1 from [15]). We note that, to the best of the author's knowledge, a proof of the Chachólski-Parent-Stanley criterion does not exist in the literature.

Theorem 2.12 has the following algebraic translation which adds to its utility:

**Theorem 2.15.** *Let  $X$  and  $A$  be rational spaces (of finite type) with anticonnectivity  $n + 1$ . Then  $A \preceq X$  (i.e.,  $X \in \mathcal{R}(A)$ ) if and only if there is a CDGA morphism*

$$\phi : \prod_{i \in \mathcal{I}} \mathcal{M}(A) \rightarrow \mathcal{M}(X)$$

*for some index set  $\mathcal{I}$ , with the linear part of  $\phi$  surjective in degree  $n$ .*

This result is the basic work-horse for the bulk of the work done in Chapter 3, and Chapter 2 ends with a discussion using Theorem 2.15 to characterize those spaces resolvably equivalent to either an Eilenberg-MacLane space, sphere, complex projective space, or quaternionic projective space (Proposition 2.17, Theorem 2.19, and Corollary 2.20).

### Chapter 3

Chapter 3 is an investigation into some of the structures present in the resolvability relation on spaces. In particular, we note  $\preceq$  on the collection of resolvable equivalence classes of spaces is a complete lower semilattice (Proposition 3.1). We also exhibit a “non-density” result:

**Proposition 3.2.** *For any simply connected rational space  $Z$  with  $S^2 \preceq Z \preceq K(\mathbb{Q}, 3)$ , we must have  $Z$  resolvably equivalent to either  $S^2$  or  $K(\mathbb{Q}, 3)$ .*

This is a notable distinction between the partial order defined by  $\preceq$  (the resolvability relation) and that by  $\ll$  (the cellular lattice).

We then move on to using Theorem 2.15 to show, amongst other things, the resolvability relation admits an embedding of the similarity classes of quadratic forms,  $\overline{W(\mathbb{Q})}$  (Corollary 3.5). This is the same quotient of the Witt group that Hess and Parent embed in the cellular lattice [15]. In light of the distinction made between the resolvability relation and the cellular lattice by our non-density result, this can be seen as a surprising turn of events. This result is shown by defining certain commutative differential graded algebras which encode, up to resolvable equivalence, information in their differential about the similarity class of a (system of) rational homogeneous polynomials (Theorem 3.3). The questions remain: is there a more conceptual explanation as to why the same copy of the Witt group appears in both the resolvability relation and the cellular lattice? Just how similar (or how different) are these two structures?

Chapter 3 ends with a description of the spaces whose minimal Sullivan model is a commutative differential graded algebra that appropriately models a homogeneous polynomial in the sense mentioned above (Proposition 3.11), and exhibits an action of quadratic forms on a collection of such spaces (Proposition 3.12).

### Chapter 4

Chapter 4 is a list of open questions and possible directions for further research related to the presented line of inquiry.

## 1.2 Basic Terminology and Notation

Unless otherwise specified, the word **space** refers to a (well-)pointed compactly generated weakly Hausdorff topological space, and the category of such spaces with continuous, base-point preserving functions between them is denoted  $\mathcal{T}_*$ . The morphisms of  $\mathcal{T}_*$  are referred to as **maps**. We write  $X \sim Y$  to denote two spaces  $X$  and  $Y$  being weakly homotopy equivalent.

Our main subject of study involves **classes of spaces**, which are defined as full subcategories of  $\mathcal{T}_*$ . As is standard, for a class of spaces  $\mathcal{C}$ ,  $X \in \text{Objects}(\mathcal{C})$  is abbreviated  $X \in \mathcal{C}$ . A functor  $F : \mathcal{I} \rightarrow \mathcal{T}_*$  is said to **take its values in  $\mathcal{C}$**  if  $F$  factors as  $\mathcal{I} \rightarrow \mathcal{C} \hookrightarrow \mathcal{T}_*$ , where  $\mathcal{C} \hookrightarrow \mathcal{T}_*$  is the inclusion functor. Given a category  $\mathcal{C}$ , its opposite category is denoted  $\mathcal{C}^{op}$ . For  $X, Y \in \mathcal{C}$ , the set of morphisms in  $\mathcal{C}$  from  $X$  to  $Y$  is written  $\mathcal{C}(X, Y)$ . We take  $\mathcal{C} \simeq \mathcal{D}$  to mean  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent as categories, while  $\mathcal{C} \cong \mathcal{D}$  means  $\mathcal{C}$  and  $\mathcal{D}$  are isomorphic as categories [16].

We employ the Quillen model structure on  $\mathcal{T}_*$ . Additionally, the notion(s) of pointed homotopy (co)limits (particularly homotopy pullbacks) play a crucial role in the development to follow. We briefly recall these concepts in Section 1.4, and we refer the reader to [1], [6], or [8] for more information as needed. Since our spaces are pointed, all (co)limits and homotopy (co)limits are assumed to be pointed. As such, we write “holim” for “holim $_*$ ” and “lim” for “lim $_*$ ”.

## 1.3 Moore-Postnikov Towers

In several of the proofs to follow, it is beneficial to investigate the Postnikov tower of a path-connected space  $X$  and the Moore-Postnikov tower of a map  $f : X \rightarrow Y$ . We will use this section to briefly recall some basic results and establish notational and indexing conventions. See [13] or [17] for more detailed expositions if desired.

Given a path-connected space  $X$ , its **Postnikov tower** is a commutative diagram

$$\begin{array}{c}
 \vdots \\
 \downarrow \\
 X_3 \\
 \downarrow \\
 X_2 \\
 \downarrow \\
 X \longrightarrow X_1
 \end{array}$$

in which:

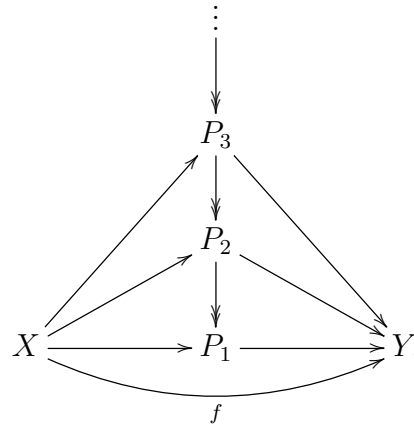
- each map  $X \rightarrow X_n$  induces an isomorphism on  $\pi_k$  for all  $k \leq n$ ;
- $\pi_k(X_n) \cong 0$  for  $k \geq n + 1$ ;
- each map  $X_n \rightarrow X_{n-1}$  is a fibration.

$X_n$  is called the  **$n$ -th Postnikov section** of  $X$ . It is easily checked that the fiber of  $X_n \rightarrow X_{n-1}$  is weakly equivalent to the Eilenberg-MacLane space  $K(\pi_n X, n)$ . If  $X$  is simply connected (more generally, if  $\pi_1 X$  acts trivially on  $\pi_n X$  for all  $n > 1$ ), then these fibrations are principal, and we have fiber sequences

$$K(\pi_n X, n) \rightarrow X_n \rightarrow X_{n-1} \rightarrow K(\pi_n X, n + 1).$$

As  $X$  is the homotopy limit of its Postnikov tower, Postnikov sections are in some sense dual to the skeleta of a space. See [18] for a more in-depth discussion on the dual relationship between Postnikov sections and skeleta.

More generally, given a map of spaces  $f : X \rightarrow Y$ , we can factor  $f$  via a commutative diagram called a **Moore-Postnikov tower**



where:

- each map  $X \rightarrow P_n$  induces an isomorphism on  $\pi_k$  for  $k < n$  and a surjection for  $k = n$ ;
- each map  $P_n \rightarrow Y$  induces an isomorphism on  $\pi_k$  for  $k > n$  and an injection for  $k = n$ ;
- each map  $P_{n+1} \rightarrow P_n$  is a fibration, with fiber  $K(\pi_n F, n)$  where  $F$  is the homotopy fiber of  $f : X \rightarrow Y$ .

Letting  $Y = *$  and setting  $X_n = P_{n+1}$  (discarding the weakly contractible  $P_1$ ) recovers the Postnikov tower of  $X$  described above. Similar to the case for Postnikov towers, if  $X$  is simply connected then the fibrations  $P_{n+1} \rightarrow P_n$  are principal, and we have fiber sequences

$$K(\pi_n F, n) \rightarrow P_{n+1} \rightarrow P_n \rightarrow K(\pi_n F, n+1).$$

## 1.4 Homotopy Theory

Classical homotopy theory is the study of topological spaces with respect to (weak) homotopy equivalence. In the late 1960's, efforts were made to axiomatize the general approach to homotopy theory, leading to the idea of a model category. This generalization acts as a simplification to the classical theory, as well as a method of importing some of the techniques and intuitions from classical homotopy theory to other (potentially surprising) contexts in abstract homotopy theory. In this section, we will review portions of (abstract) homotopy theory relevant to our later interests. The discussion in this section is far from complete; see [8] for a reference.

### Model Categories

Let  $\mathcal{M}$  be a category, and suppose  $f : A \rightarrow B$  and  $g : C \rightarrow D$  are morphisms in  $\mathcal{M}$ . We say  $f$  has the **left lifting property (LLP)** with respect to  $g$  provided:

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ f \downarrow & \dashrightarrow \exists \lambda & \downarrow g \\ B & \xrightarrow{j} & D. \end{array}$$

That is to say, if for any  $i : A \rightarrow C$  and  $j : B \rightarrow D$  in  $\mathcal{M}$  with  $g \circ i = j \circ f$ , there exists (not necessarily unique)  $\lambda : B \rightarrow C$  in  $\mathcal{M}$  such that  $\lambda \circ f = i$  and  $g \circ \lambda = j$ .

A **weak factorization system** [22] in a category  $\mathcal{M}$  is a pair of distinguished classes of morphisms  $(\mathcal{F}, \mathcal{G})$  of  $\mathcal{M}$  such that:

- every morphism  $i : A \rightarrow B$  in  $\mathcal{M}$  can be factored

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ & f \searrow & \nearrow g \\ & & X \end{array}$$

with  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$ ;

- for any  $f \in \mathcal{F}$  and  $g \in \mathcal{G}$ ,  $f$  has the LLP with respect to  $g$ ;
- $\mathcal{F}$  and  $\mathcal{G}$  are **closed under retracts**: if  $k \in \mathcal{F}$  (respectively,  $k \in \mathcal{G}$ ) and we have a diagram

$$\begin{array}{ccccc}
 & & id_A & & \\
 & & \curvearrowright & & \\
 A & \longrightarrow & C & \longrightarrow & A \\
 \downarrow h & & \downarrow k & & \downarrow h \\
 B & \longrightarrow & D & \longrightarrow & B \\
 & & \curvearrowleft id_B & & 
 \end{array}$$

then  $h \in \mathcal{F}$  (respectively,  $h \in \mathcal{G}$ ).

A **model category** is a complete and cocomplete category  $\mathcal{M}$  with three distinguished classes of morphisms  $\mathcal{C}$ ,  $\mathcal{F}$ , and  $\mathcal{W}$  (the **cofibrations**, **fibrations**, and **weak equivalences**, respectively) satisfying:

- $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  is a weak factorization system in  $\mathcal{M}$ ;
- $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$  is a weak factorization system in  $\mathcal{M}$ ;
- $\mathcal{W}$  is closed under retracts and has the **two-out-of-three property**: if, in a diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \searrow g & & \nearrow h \\
 & X & 
 \end{array}$$

any two of  $f$ ,  $g$ , and  $h$  are in  $\mathcal{W}$ , then so is the third.

We say the classes of morphisms  $\mathcal{C}$ ,  $\mathcal{F}$ , and  $\mathcal{W}$  constitute a **model structure** on  $\mathcal{M}$  in this case. The arrows  $\hookrightarrow$ ,  $\twoheadrightarrow$ , and  $\xrightarrow{\sim}$  are used to denote cofibrations, fibrations, and weak equivalences respectively. Two objects  $X, Y \in \mathcal{M}$  are said to be **weakly equivalent** provided there is a zig-zag of weak equivalences in  $\mathcal{M}$  from  $X$  to  $Y$ :

$$X \xrightarrow{\sim} A_0 \xleftarrow{\sim} A_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} A_n \xrightarrow{\sim} Y.$$

Model categories are intended to act as a generalized setting in which one can “do homotopy theory,” analogous to classical homotopy theory.

**Example.** Let  $\mathcal{T}$  be the category of *unpointed* spaces. In  $\mathcal{T}$ , if we take  $\mathcal{C}$  to be the class of retracts of maps of the form  $X \rightarrow X'$  where  $X'$  is constructed

by attaching disks to  $X$ ,  $\mathcal{F}$  to be the Serre fibrations, and  $\mathcal{W}$  to be the weak homotopy equivalences, we get the **Quillen model structure** on topological spaces. Furthermore, since  $\mathcal{T}_* = * \downarrow \mathcal{T}$  (the category whose objects are morphisms  $* \rightarrow X$  for  $X \in \mathcal{T}$ ), our category of pointed spaces inherits a model structure from this model structure on  $\mathcal{T}$ .

**Example.**  $\mathcal{T}$  has another model structure with  $\mathcal{C}$  the class of maps with the homotopy extension property,  $\mathcal{F}$  the class of maps with the homotopy lifting property, and  $\mathcal{W}$  the homotopy equivalences. This is called the **Hurewicz or Strøm model structure** on topological spaces.

**Example.** Let  $sSet$  be the category of simplicial sets [12]. Take  $\mathcal{C}$  to be the class of monomorphisms (i.e., level-wise injections),  $\mathcal{F}$  the class of Kan fibrations, and  $\mathcal{W}$  the set of morphisms whose geometric realization is a weak homotopy equivalence. This is the classical **Quillen model structure** on simplicial sets.

**Example.** For  $R$  a ring, let  $Chain_R$  be the category of non-negatively graded chain complexes of  $R$ -modules.  $Chain_R$  carries a model structure with  $\mathcal{C}$  the degree-wise monomorphisms with projective cokernel,  $\mathcal{F}$  the degree-wise epimorphisms (for degree greater than or equal to 1), and  $\mathcal{W}$  the morphisms that induce isomorphisms in homology.

Every model category  $\mathcal{M}$  has an initial object  $\emptyset$ , and a terminal object  $*$  (since it is (co)complete). So, for every object  $X \in \mathcal{M}$  we have unique morphisms  $\emptyset \rightarrow X$  and  $X \rightarrow *$ . We may then factor these morphisms as:

$$\emptyset \hookrightarrow X^C \xrightarrow{\sim} X \quad \text{and} \quad X \xrightarrow{\sim} X^F \twoheadrightarrow *.$$

An object  $X \in \mathcal{M}$  is **cofibrant** if  $\emptyset \hookrightarrow X$  is a cofibration and **fibrant** if  $X \twoheadrightarrow *$  is a fibration.  $X^C$  is called the **cofibrant replacement** of  $X$ , while  $X^F$  is called the **fibrant replacement** of  $X$ .

**Example.** In the Hurewicz model structure on  $\mathcal{T}$ , every space is both cofibrant and fibrant, so we can take  $X^F = X = X^C$ .

**Example.** In the Quillen model structure on  $\mathcal{T}$ , every space is fibrant, so we can take  $X^F = X$ . The cofibrant objects are retracts of generalized CW complexes, so we may take the cofibrant replacement of  $X$  to be its CW complex replacement.

**Example.** In the Quillen model structure on  $sSet$ , every object is cofibrant, so we can take  $K^C = K$ . The fibrant replacement of  $K$  can be taken to be its Kan complex replacement.



In general, for  $X$  in a model category  $\mathcal{M}$ , by taking a fibrant replacement of  $X^C$  (or equivalently, the cofibrant replacement of  $X^F$ ), we obtain  $(X^C)^F \sim X$ , which is the **fibrant-cofibrant replacement** of  $X$ . The fibrant-cofibrant replacement of  $X$  is denoted  $X^{CF}$ , and given a map  $f : X \rightarrow Y$  there is an induced map  $f^{CF} : X^{CF} \rightarrow Y^{CF}$ .

## The Homotopy Category of a Model Category

Let  $\mathcal{M}$  be a model category. A notion of homotopy equivalence between morphisms in  $\mathcal{M}$  can be defined in a manner that generalizes the standard notion of homotopy equivalence of continuous functions between topological spaces. This is done by first defining what it means for two morphisms  $f, g : X \rightarrow Y$  in  $\mathcal{M}$  to be **left homotopic** and **right homotopic**. Left homotopy is defined via what are called **cylinder objects**, while right homotopy is defined via **path objects**. It turns out left homotopy is an equivalence relation on  $\mathcal{M}(X, Y)$  if  $X$  is cofibrant, and right homotopy is an equivalence relation on  $\mathcal{M}(X, Y)$  if  $Y$  is fibrant.

In general, the notions of left homotopy and right homotopy are different. However, if  $X$  is cofibrant and  $Y$  is fibrant, then for any  $f, g \in \mathcal{M}(X, Y)$ ,  $f$  is left homotopic to  $g$  if and only if  $f$  is right homotopic to  $g$ . In this case, we say  $f$  is **homotopic** to  $g$ , and write  $f \simeq g$ . Denote the equivalence class of  $f$  with respect to  $\simeq$  by  $[f]$ . Now, given any model category  $\mathcal{M}$ , we can form its **homotopy category**,  $\text{Ho}(\mathcal{M})$ , as follows:

- the objects of  $\text{Ho}(\mathcal{M})$  are exactly the objects of  $\mathcal{M}$ ;
- for  $X, Y \in \mathcal{M}$  we define  $\text{Ho}(\mathcal{M})(X, Y) = \mathcal{M}(X^{CF}, Y^{CF}) / \simeq$ .

**Example.** The Quillen and Hurewicz model structures on  $\mathcal{T}$  both yield homotopy categories that are equivalent to the classical homotopy category of topological spaces.

There is a functor  $\text{Ho} : \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$  that is the identity on objects and sends  $f \in \mathcal{M}(X, Y)$  to  $[f^{CF}] \in \text{Ho}(\mathcal{M})(X, Y)$ . This is an example of what is known as a localization of  $\mathcal{M}$ .

Explicitly, a **localization** of  $\mathcal{M}$  with respect to a class of morphisms  $\mathcal{F}$  is a functor  $L : \mathcal{M} \rightarrow \mathcal{D}$  such that

- $L(f)$  is an isomorphism in  $\mathcal{D}$  for every  $f \in \mathcal{F}$ ;
- if  $R : \mathcal{M} \rightarrow \mathcal{E}$  is another functor such that  $R(f)$  is an isomorphism in  $\mathcal{E}$  for

every  $f \in \mathcal{F}$ , then

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{L} & \mathcal{D} \\ & \searrow R & \nearrow \exists! T \\ & \mathcal{E} & \end{array}$$

Localizations do not necessary exist, but note that this definition implies any two localizations of  $\mathcal{M}$  with respect to  $\mathcal{F}$  must be isomorphic if they do exist. We denote the (isomorphism class of the) category  $\mathcal{M}$  localized with respect to the class of morphisms  $\mathcal{F}$  by  $\mathcal{M}[\mathcal{F}]^{-1}$ .

It is shown in [8] that  $\text{Ho} : \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$  is a localization of  $\mathcal{M}$  with respect to the class of weak equivalences,  $\mathcal{W}$ . In other words, we have  $\text{Ho}(\mathcal{M}) \cong \mathcal{M}[\mathcal{W}]^{-1}$ . This is a conceptually powerful interpretation of the homotopy category of a model category  $\mathcal{M}$ , and we regularly identify  $\text{Ho}(\mathcal{M})$  with  $\mathcal{M}[\mathcal{W}]^{-1}$  without explicit reference. In particular, the notion of **homotopy class** can equivalently be taken to mean a morphism in  $\mathcal{M}[\mathcal{W}]^{-1}$  or  $\text{Ho}(\mathcal{M})$ .

## Quillen Equivalences

It sometimes happens that two model categories produce equivalent homotopy categories. We think of such categories as having “the same homotopy theory.” In many instances, this comes about through what is called a Quillen equivalence, which we now define.

Let  $\mathcal{M}$  and  $\mathcal{N}$  be model categories with  $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$  an adjoint pair of functors between them. We say  $(L, R)$  is a **Quillen equivalence** provided:

- $L$  preserves cofibrations and  $R$  preserves fibrations;
- for every cofibrant  $X \in \mathcal{M}$  and every fibrant  $Y \in \mathcal{N}$ ,  $X \xrightarrow{\sim} R(Y)$  is a weak equivalence if and only if its adjoint  $L(X) \xrightarrow{\sim} Y$  is as well.

There are plenty of equivalent definitions of Quillen equivalence, but the upshot of all of them is this: a Quillen equivalence  $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$  induces an equivalence of homotopy categories,  $\text{Ho}(\mathcal{M}) \simeq \text{Ho}(\mathcal{N})$ .

**Example.**  $\mathcal{T}$  with the Quillen model structure is Quillen equivalent to  $sSet$  with its Quillen model structure. The adjunction is given by

$$S_{\bullet} : \mathcal{T} \rightleftarrows sSet : | - |,$$

where  $S_{\bullet}$  is the singular simplices functor, and  $| - |$  is the geometric realization functor [12].

*Remark.* It is reasonable to ask whether every equivalence  $\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{Ho}(\mathcal{N})$  arises from a Quillen equivalence  $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ . The answer turns out to be negative, with a counterexample exhibited in [9].

## Homotopy Limits

For a small category  $\mathcal{I}$ , let  $\mathcal{T}_*^{\mathcal{I}}$  be the category whose objects are functors  $F : \mathcal{I} \rightarrow \mathcal{T}_*$  (thought of as diagrams of shape  $\mathcal{I}$  in  $\mathcal{T}_*$ ) and whose morphisms are natural transformations.

Let  $F, G \in \mathcal{T}_*^{\mathcal{I}}$ . We say a natural transformation  $F \rightarrow G$  is a **point-wise cofibration** (respectively: **point-wise fibration**, **point-wise weak equivalence**) if  $F(i) \rightarrow G(i)$  is a cofibration (respectively: fibration, weak equivalence) for every  $i \in \mathcal{I}$ .

$F \in \mathcal{T}_*^{\mathcal{I}}$  is **fibrant** if, for any  $G \rightarrow H$  in  $\mathcal{T}_*^{\mathcal{I}}$  that is a point-wise cofibration and point-wise weak equivalence, we have:

$$\begin{array}{ccc} G & \longrightarrow & F \\ \downarrow & & \nearrow \\ H & \dashrightarrow & F \end{array} \quad \exists$$

for any natural transformation  $G \rightarrow F$ .

A point-wise weak equivalence  $F \rightarrow \bar{F}$  is a **fibrant replacement** if  $\bar{F}$  is fibrant. Fibrant replacements exist (for all the functors we will work with, anyway), and the natural transformation  $F \rightarrow \bar{F}$  induces a unique map  $\lim F \rightarrow \lim \bar{F}$ . We define  $\mathrm{holim} F = \lim \bar{F}$ , and note that this is well-defined up to weak equivalence in  $\mathcal{T}_*$  since there is a point-wise weak equivalence between any two fibrant replacements of  $F$ . Homotopy limits can be thought of as the best homotopy coherent approximation to the functor  $\lim : \mathcal{T}_*^{\mathcal{I}} \rightarrow \mathcal{T}_*$ .

An important example of a homotopy limit is the homotopy limit of a pre-pullback diagram:  $\mathrm{holim}(X \rightarrow Y \leftarrow Z)$ . In this case, a diagram  $X \rightarrow Y \leftarrow Z$  is fibrant if at least one of the maps  $X \rightarrow Y$  or  $Y \leftarrow Z$  is a fibration. A fibrant replacement of a pre-pullback diagram can then be obtained by converting one, or both, of the maps in said diagram into a fibration. The homotopy limit is found by then taking the (categorical) pullback of the fibrant replacement.

**Example.** Consider  $* \rightarrow X \leftarrow *$ . We can factor  $* \rightarrow X$  as  $* \xrightarrow{\sim} P(X) \twoheadrightarrow X$ , where  $P(X)$  is the space of paths in  $X$  and  $P(X) \twoheadrightarrow X$  the path space fibration.

We then have

$$\begin{array}{ccccc}
 * & \longrightarrow & X & \longleftarrow & * \\
 \downarrow & & \downarrow id_X & & \downarrow \sim \\
 * & \longrightarrow & X & \longleftarrow & P(X)
 \end{array}$$

whence  $\text{holim}(* \rightarrow X \leftarrow *) \sim \text{lim}(* \rightarrow X \leftarrow P(X)) \cong \Omega X$ .

It is not necessarily the case that point-wise weakly equivalent diagrams have weakly equivalent limits. In fact, our example of a homotopy pullback above acts as a counterexample. However, there is a condition under which two towers have weakly equivalent limits. Suppose we have towers  $F, G : \mathbb{N}^{op} \rightarrow \mathcal{T}_*$  given by

$$\begin{aligned}
 F : \quad & \cdots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0; \\
 G : \quad & \cdots \xrightarrow{g_2} Y_2 \xrightarrow{g_1} Y_1 \xrightarrow{g_0} Y_0.
 \end{aligned}$$

Suppose further we have a natural transformation  $\eta : F \rightarrow G$ . This induces  $\pi_k \circ \eta : \pi_k \circ F \rightarrow \pi_k \circ G$  via  $(\pi_k \circ \eta)_n = \pi_k(\eta_n) : \pi_k(X_n) \rightarrow \pi_k(Y_n)$ . We say  $F$  and  $G$  are **weak pro-homotopy equivalent** [1] provided there is a natural transformation  $\eta : F \rightarrow G$  such that, for every  $k, n \in \mathbb{N}$ , there exists  $N \geq n$  and a homomorphism  $\pi_k(Y_N) \rightarrow \pi_k(X_n)$  where

$$\begin{array}{ccc}
 \pi_k(X_N) & \xrightarrow{\pi_k(\eta_N)} & \pi_k(Y_N) \\
 \downarrow & \swarrow & \downarrow \\
 \pi_k(X_n) & \xrightarrow{\pi_k(\eta_n)} & \pi_k(Y_n)
 \end{array}$$

commutes. It is shown in [1] (Ch. III, Section 3) that a weak pro-homotopy equivalence of towers  $\eta : F \rightarrow G$  induces a weak equivalence  $\text{lim } F \xrightarrow{\sim} \text{lim } G$ .

## 1.5 Rational Homotopy Theory

Loosely speaking, rational homotopy theory allows us to study spaces modulo any torsion in their homotopy groups. There are deep connections between rational homotopy theory and differential graded algebra, aiding in our computational ability. We review here some aspects of rational homotopy theory and the related algebra of which we will make use. See [14] for a more exhaustive overview, and [10] for a more complete reference including proofs.

## Commutative Differential Graded Algebra

A **rational cochain complex** consists of an integer graded  $\mathbb{Q}$ -algebra  $A^*$  and boundary maps  $d_n : A^n \rightarrow A^{n+1}$  such that  $d_{n+1} \circ d_n = 0$ . An element  $a \in A^n$  is said to have **degree**  $n$ , denoted  $|a| = n$ . It is standard to write the boundary maps as simply  $d : A^n \rightarrow A^{n+1}$ , and write the condition  $d_{n+1} \circ d_n = 0$  as  $d^2 = 0$ . It is also a common abuse of notation to use  $d$  for the boundary map of different cochain algebras, unless emphasis of distinction is needed.

The tensor product of two rational cochain algebras  $(A^*, d)$  and  $(B^*, d)$  is again a rational cochain algebra:  $(A^*, d) \otimes (B^*, d)$  is defined by  $(A^* \otimes B^*, d)$  where  $d(a \otimes b) = d(a) \otimes b + (-1)^{|a|} a \otimes d(b)$ . A morphism of cochain complexes (i.e., **cochain morphism**)  $\phi : (A^*, d) \rightarrow (B^*, d)$  is a collection of algebra morphisms  $\phi_n : A^n \rightarrow B^n$  such that

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^n & \xrightarrow{d_n} & A^{n+1} & \longrightarrow & \dots \\ & & \phi_n \downarrow & & \downarrow \phi_{n+1} & & \\ \dots & \longrightarrow & B^n & \xrightarrow{d_n} & B^{n+1} & \longrightarrow & \dots \end{array}$$

commutes for each  $n \in \mathbb{Z}$ . Nongraded  $\mathbb{Q}$ -algebras are considered cochain complexes concentrated in degree 0.

A **commutative differential graded  $\mathbb{Q}$ -algebra** (henceforth: **CDGA**) is a commutative monoid in the category of non-negatively graded, rational cochain complexes [14]. That is to say, a *CDGA* consists of a rational cochain complex  $(A^*, d)$  along with a cochain morphism  $\mu : (A^*, d) \otimes (A^*, d) \rightarrow (A^*, d)$  (the **multiplication**, written  $\mu(a \otimes b) = ab$ ) and cochain morphism  $\eta : \mathbb{Q} \rightarrow (A^*, d)$  (the **unit**) satisfying, for all  $a, b, c \in A^*$ :

- Associativity:  $(ab)c = a(bc)$ ;
- Graded commutativity:  $ab = (-1)^{|a||b|}ba$ ;
- Unit element:  $\eta(1)a = a = a\eta(1)$ .

We note, since  $\mu$  is a cochain morphism, we have:

$$\begin{aligned} d(ab) &= d(\mu(a \otimes b)) \\ &= \mu(d(a \otimes b)) \\ &= \mu(d(a) \otimes b + (-1)^{|a|} a \otimes d(b)) \\ &= d(a)b + (-1)^{|a|} ad(b). \end{aligned}$$

Hence, we call  $d$  a **differential**.

Given  $n \in \mathbb{N}$ , a CDGA  $A$  is said to be  $n$ -**connected** if  $A^0 = \mathbb{Q}$  and  $A^k = 0$  for all  $0 < k < n + 1$ . A 1-connected CDGA is called **simply connected**. The category of CDGA's with cochain morphisms between them will be denoted  $CDGA_{\mathbb{Q}}$ . A morphism in  $CDGA_{\mathbb{Q}}$  that induces an isomorphism in cohomology is called a **quasi-isomorphism** and denoted  $\xrightarrow{\sim}$ .

For a positively graded vector space  $V$ , the **tensor algebra** on  $V$  is  $TV = \bigoplus_{k=0}^{\infty} V^{\otimes k}$  where  $V^{\otimes k}$  is the  $k$ -fold tensor product of  $V$  with itself:

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \dots \otimes V}_{k \text{ factors}}$$

The free graded commutative algebra on  $V$ ,  $\Lambda V$ , is then defined as

$$\Lambda V = TV / (v \otimes w - (-1)^{|v||w|} w \otimes v).$$

Observe, if  $v \in \Lambda V$  is of odd degree, then the graded commutativity of  $\Lambda V$  forces  $v^2 = 0$ . Given another positively graded vector space  $W$ , there is a natural isomorphism  $\Lambda V \otimes \Lambda W \cong \Lambda(V \oplus W)$ .

Letting  $\Lambda^k V$  represent the linear span of elements of word length  $k$  in  $\Lambda V$ , we may write  $\Lambda V \cong \bigoplus_{k \in \mathbb{N}} \Lambda^k V$ . The differential  $d : \Lambda V \rightarrow \Lambda V$  can then be decomposed as  $d = d_{(1)} + d_{(2)} + \dots + d_{(k)} + \dots$ , where we see  $d_{(k)}$  increases word length by  $k - 1$ . It is standard to call  $d_{(2)}$  the **quadratic part** of the differential.

## A Quillen Equivalence in Rational Homotopy Theory

A space  $X$  is called **rational** if  $\pi_* X$  is a (graded)  $\mathbb{Q}$ -vector space. A map  $f : X \rightarrow Y$  is a **rational homotopy equivalence** provided  $\pi_*(f) \otimes \mathbb{Q}$  is an isomorphism. A rational homotopy equivalence  $f : X \rightarrow Y$  in which  $Y$  is a rational space is called a **rationalization** of  $X$ . Every simply connected space can be rationalized; in fact:

**Theorem 1.1.** [14] *Let  $X$  be a simply connected space. Then there is a rationalization  $q : X \rightarrow X_{\mathbb{Q}}$  such that, for any map  $f : X \rightarrow Y$  where  $Y$  is a simply connected rational space, there exists a  $g : X_{\mathbb{Q}} \rightarrow Y$  (unique up to homotopy) making*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow q & \nearrow g \\ & & X_{\mathbb{Q}} \end{array}$$

*commute.*

The **rational homotopy type** of a simply connected space  $X$  is the (weak) homotopy type of its rationalization,  $X_{\mathbb{Q}}$ . A graded vector space is of **finite type** if it is finite dimensional in each degree; a rational space  $X$  is said to be of **finite type** if  $\pi_*X$  is a graded vector space of finite type.

A fundamental result in rational homotopy theory is that there is a model structure on  $CDGA_{\mathbb{Q}}$  such that the homotopy theory of simply connected rational spaces of finite type is equivalent to the homotopy theory of simply connected CDGA's of finite type. To avoid superfluous use of adjectives, the connectedness and finiteness conditions in the previous statement will often tacitly be assumed of all rational spaces and CDGA's from here on; that is to say, we assume all rational spaces and CDGA's are simply connected and of finite type, sometimes without reference.

Letting  $sSet$  represent the category of simplicial sets, in [2], Bousfield and Gugenheim proved there is a Quillen equivalence:

$$\mathcal{A}^* : sSet \rightleftarrows CDGA_{\mathbb{Q}}^{op} : \mathcal{K}_{\bullet}.$$

If we call  $RHE$  the class of rational homotopy equivalences and  $QI$  the class of quasi-isomorphism, this Quillen equivalence produces an equivalence of localized categories:

$$sSet[RHE]^{-1} \simeq CDGA_{\mathbb{Q}}^{op}[QI]^{-1}.$$

As noted previously, the geometric realization functor  $|-| : sSet \rightarrow \mathcal{T}$  and its right adjoint, the singular simplices functor  $S_{\bullet} : \mathcal{T} \rightarrow sSet$ , also constitute a Quillen equivalence, so we may replace  $sSet$  with  $\mathcal{T}$  in the above statements. The composite functor  $\mathcal{A}^* \circ S_{\bullet}$  is denoted  $\mathcal{A}_{PL} : \mathcal{T} \rightarrow CDGA_{\mathbb{Q}}^{op}$ , and  $\mathcal{A}_{PL}(X)$  is called the **piecewise-linear de Rham forms** on  $X$ . The composite functor  $|-| \circ \mathcal{K}_{\bullet}$  is denoted  $\langle - \rangle : CDGA_{\mathbb{Q}}^{op} \rightarrow \mathcal{T}$  and called **spatial realization**.

## Minimal Sullivan Models

The cofibrant objects of  $CDGA_{\mathbb{Q}}$  are called **Sullivan algebras** and happen to have the form  $(\Lambda V, d)$  (note that not all CDGA's with free underlying algebras are Sullivan algebras; see Example 1.12 in [14]). A CDGA is called **minimal** if its differential  $d$  satisfies  $d_{(1)} = 0$ .

Given a simply connected space  $X$ , the cofibrant replacement of  $\mathcal{A}_{PL}(X)$  is a quasi-isomorphism  $(\Lambda V, d) \xrightarrow{\simeq} \mathcal{A}_{PL}(X)$  called the **minimal Sullivan model** of  $X$ . The minimal Sullivan model of a simply connected space is unique up to isomorphism, and we denote the (isomorphism class of the) minimal Sullivan model of  $X$  by  $\mathcal{M}(X)$ . Moreover, we have:

**Theorem 1.2.** [14] *Let  $X$  be a simply connected space with minimal Sullivan model  $\mathcal{M}(X) = (\Lambda V, d)$ . This model satisfies:*

- $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$ ;
- $V \cong \text{hom}_{\mathbb{Q}}(\pi_* X, \mathbb{Q})$ ;
- $d_{(1)} = 0$  and  $d_{(2)}$  is the algebraic dual of the Whitehead product;

The isomorphism  $V \cong \text{hom}_{\mathbb{Q}}(\pi_* X, \mathbb{Q})$  arises via a nondegenerate bilinear form

$$\langle -; - \rangle : V \times \pi_*(X) \rightarrow \mathbb{Q}.$$

It is shown in [10] that, for a rational space  $X$  (of finite type), the assignment  $v \mapsto \langle v; - \rangle$  constitutes a linear isomorphism  $V \rightarrow \text{hom}_{\mathbb{Q}}(\pi_*(X), \mathbb{Q})$ . We may also define a triple bracket

$$\langle -; -, - \rangle : \Lambda^2 V \times \pi_*(X) \times \pi_*(X) \rightarrow \mathbb{Q}$$

by  $\langle vw; \alpha, \beta \rangle = \langle v; \beta \rangle \langle w; \alpha \rangle + (-1)^{|w| \cdot |\alpha|} \langle v; \alpha \rangle \langle w; \beta \rangle$ . By saying  $d_{(2)}$  is algebraically dual to the Whitehead product (see [19] for more on the Whitehead product) we mean:

**Theorem 1.3.** [10] *Let  $X$  be a rational space of finite type with  $\mathcal{M}(X) = \Lambda(V, d)$ . For  $v \in V$ ,  $\alpha \in \pi_k(X)$ ,  $\beta \in \pi_n(X)$  and  $[\alpha, \beta] \in \pi_{k+n-1}(X)$ , we have:*

$$\langle d_{(2)}v; \alpha, \beta \rangle = (-1)^{k+n-1} \langle v; [\alpha, \beta] \rangle.$$

Given a morphism  $\phi : (\Lambda V, d) \rightarrow (\Lambda W, d)$  of Sullivan algebras, the **linear part** [10] of  $\phi$  is the linear map  $\bar{\phi} : V \rightarrow W$  corresponding to the morphism  $\Lambda^{\geq 1} V / \Lambda^{\geq 2} V \rightarrow \Lambda^{\geq 1} W / \Lambda^{\geq 2} W$  induced by  $\phi$  (where we note  $\Lambda^{\geq 1} V / \Lambda^{\geq 2} V \cong V$  and similarly for  $W$ ).

**Example.** As a simple example, suppose we have  $(\Lambda(x, y), dy = x^2)$  with  $|x| = 4$  and  $|y| = 7$ , and  $(\Lambda(a, b, c, z), dz = (b+c)^2)$  with  $|a| = 3$ ,  $|b| = 4 = |c|$ , and  $|z| = 7$ . Define  $\phi : (\Lambda(x, y), d) \rightarrow (\Lambda(a, b, c, z), d)$  by extending  $x \mapsto b + c$ ,  $y \mapsto z + ab$ . We notice  $\bar{\phi}_4 = \phi_4$  since everything in degree 4 of  $\Lambda(a, b, c, z)$  is of word length 1, while  $\bar{\phi}_7$  is defined by extending  $y \mapsto z$ .

Letting  $X$  and  $Y$  be (simply connected) spaces with  $\mathcal{M}(X) = (\Lambda W, d)$  and  $\mathcal{M}(Y) = (\Lambda V, d)$  then a map  $f : X \rightarrow Y$  corresponds to a morphism  $\phi : (\Lambda V, d) \rightarrow (\Lambda W, d)$ . There are potentially many choices for  $\phi$  corresponding to the homotopy class of  $f$ ; however, they all must have the same *linear part*. This linear morphism is denoted  $Q(f) : V \rightarrow W$ .



Identifying  $W \cong \text{hom}_{\mathbb{Q}}(\pi_*X, \mathbb{Q})$  and  $V \cong \text{hom}_{\mathbb{Q}}(\pi_*Y, \mathbb{Q})$ , then, in degree  $n$ ,  $Q(f)$  corresponds to the algebraic dual of  $\pi_n(f)$  (see Section 17 of [10]). Specifically:

**Theorem 1.4.** [10] *Let  $f : X \rightarrow Y$  and  $Q(f)$  be as described above. Then, for  $v \in V$  and  $\alpha \in \pi_n(X)$ :*

$$\langle Q(f)(v); \alpha \rangle = \langle v; \pi_n(f)(\alpha) \rangle.$$

In other words, we have a commutative diagram:

$$\begin{array}{ccc} V^n & \xrightarrow{Q(f)} & W^n \\ \cong \downarrow & & \downarrow \cong \\ \text{hom}_{\mathbb{Q}}(\pi_*Y, \mathbb{Q}) & \xrightarrow{\pi_n(f)^*} & \text{hom}_{\mathbb{Q}}(\pi_*X, \mathbb{Q}). \end{array}$$

We note that algebraic dualization constitutes a contravariant faithful functor; as such, it sends split monomorphisms to split epimorphisms and vice versa [21] (see [7] for a discussion on the finite-dimensional case). In particular, this implies:  $Q(f)$  is surjective in degree  $n$  if and only if  $\pi_n(f)$  is injective. This fact proves useful in the proof of Theorem 2.15 to come.

Below are some examples of minimal Sullivan models for future reference. See [10] for explicit verifications.

**Example.** For  $|x| = n$  and  $|y| = 2n - 1$ :

$$\mathcal{M}(S^n) \cong \begin{cases} (\Lambda(x, y), dy = x^2) & n \text{ even} \\ (\Lambda(x), 0) & n \text{ odd.} \end{cases}$$

This, of course, captures the rational cohomology of  $S^n$ :

$$H^k(S^n; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k \in \{0, n\} \\ 0 & k \notin \{0, n\}. \end{cases}$$

We see a nonzero differential is not needed in the odd dimensional case since  $x^2 = 0 \in \Lambda(x)$ . The presence of a nontrivial differential in the even dimensional case can be considered a basic building block in the development to follow.

**Example.**  $\mathcal{M}(K(\mathbb{Q}, n)) \cong (\Lambda(x), 0)$  where  $|x| = n$ . This is another way of stating the classical result that the rational cohomology of an Eilenberg-MacLane space is an exterior algebra if  $n$  is odd and a polynomial algebra if  $n$  is even.

Considering the previous example, we see  $S^n$  and  $K(\mathbb{Q}, n)$  have the same rational homotopy type when  $n$  is odd. More generally, it can actually be shown that  $\mathcal{M}(X) \cong (\Lambda V, 0)$  for any  $H$ -space  $X$  [10].

**Example.**  $\mathcal{M}(\mathbb{C}P^n) \cong (\Lambda(x, y), dy = x^{n+1})$  for  $|x| = 2$  and  $|y| = 2n + 1$ . We see this coincides with:

$$H^*(\mathbb{C}P^n; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1}).$$

**Example.** The model of a product is the pushout of the models. So, the model for  $X \times Y$  sits in the pushout square:

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathcal{M}(X) \\ \downarrow & & \downarrow \\ \mathcal{M}(Y) & \longrightarrow & \mathcal{M}(X \times Y). \end{array}$$

Thus,  $\mathcal{M}(X \times Y) \cong \mathcal{M}(X) \otimes \mathcal{M}(Y)$ . If  $\mathcal{M}(X) = (\Lambda V, d)$  and  $\mathcal{M}(Y) = (\Lambda W, d)$ , we find

$$\begin{aligned} \mathcal{M}(X \times Y) &\cong (\Lambda V, d) \otimes (\Lambda W, d) \\ &\cong (\Lambda(V \oplus W), d). \end{aligned}$$

**Example.** Modeling the wedge of two spaces  $X$  and  $Y$  is slightly less straightforward. The pushout square

$$\begin{array}{ccc} \star & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee Y \end{array}$$

induces a pullback square in  $CDGA_{\mathbb{Q}}$ :

$$\begin{array}{ccc} \mathcal{M}(X) \times_{\mathbb{Q}} \mathcal{M}(Y) & \longrightarrow & \mathcal{M}(X) \\ \downarrow & & \downarrow \\ \mathcal{M}(Y) & \longrightarrow & \mathbb{Q}. \end{array}$$

The fibred product may not be minimal, however. So,  $\mathcal{M}(X \vee Y)$  will be a minimal model quasi-isomorphic to  $\mathcal{M}(X) \times_{\mathbb{Q}} \mathcal{M}(Y)$ .

## 1.6 Polynomial Forms

We present here some definitions and results about quadratic forms and homogeneous polynomials related to our discussion in Section 3. See [23] for a reference including proofs of the results recounted below regarding quadratic forms, and [5] for a general reference on homogeneous polynomials.

### Rational Quadratic Forms and the Witt Group

An  $n$ -dimensional quadratic form over a field  $\mathbb{F}$  is a polynomial  $q$  of the form

$$q(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j, \text{ for } a_{ij} \in \mathbb{F}.$$

We will be taking  $\mathbb{F}$  to be  $\mathbb{Q}$  henceforth. Writing

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

and  $A_q = (a_{ij})$  we can more conveniently express this information as  $q(x) = x^T A_q x$ . We call  $A_q$  the **coefficient matrix** of  $q$  in this case.

There are two equivalence relations that we can employ on  $n$ -dimensional quadratic forms. Two  $n$ -dimensional quadratic forms  $q$  and  $q'$  are **isometric**, denoted  $q \simeq q'$ , if there exists a  $C \in GL(n, \mathbb{Q})$  such that  $A_q = C A_{q'} C^T$ . They are **similar**, denoted  $q \sim q'$ , if there exists a nonzero  $\alpha \in \mathbb{Q}$  such that  $\alpha q \simeq q'$ . A quadratic form  $q$  is **isotropic** if there exists a nonzero vector  $v$  such that  $q(v) = 0$ , otherwise it is **anisotropic**. The empty quadratic form of dimension 0 is considered anisotropic.

Any  $n$ -dimensional quadratic form over  $\mathbb{Q}$  is isometric to a form whose coefficient matrix is diagonal (c.f. [23]). We'll use the notation  $\langle \alpha_1, \dots, \alpha_n \rangle$  to represent the  $n$ -dimensional quadratic form over  $\mathbb{Q}$  whose coefficient matrix is  $\text{Diag}(\alpha_1, \dots, \alpha_n)$ . So, given a quadratic form  $q$ , we know  $q \simeq \langle \alpha_1, \dots, \alpha_n \rangle$  for some diagonal form.

If we have two quadratic forms  $p$  and  $q$ , we can define a sum and product on their isometry classes as follows. Let  $p \simeq \langle \alpha_1, \dots, \alpha_n \rangle$  and  $q \simeq \langle \beta_1, \dots, \beta_m \rangle$ . The **sum** of (the isometry classes of)  $p$  and  $q$  is denoted  $p \perp q$ , and defined to be (isometric to)  $\langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \rangle$ . The **product** of (the isometry classes of)  $p$  and  $q$  is denoted  $p \otimes q$ , and defined to be (isometric to)  $\perp_{i=1}^n \langle \alpha_i \beta_1, \dots, \alpha_i \beta_n \rangle$ .

Notice:  $A_{p \perp q} = \text{Diag}(\alpha_1, \dots, \alpha_n) \oplus \text{Diag}(\beta_1, \dots, \beta_m)$  and  $A_{p \otimes q} = \text{Diag}(\alpha_1, \dots, \alpha_n) \otimes \text{Diag}(\beta_1, \dots, \beta_m)$  (the standard direct sum and Kronecker product of the matrices  $\text{Diag}(\alpha_1, \dots, \alpha_n)$  and  $\text{Diag}(\beta_1, \dots, \beta_m)$  [7]).

Call the set of isometry classes of anisotropic forms  $W(\mathbb{Q})$ . As described in [15], the sum and product of two anisotropic forms is not necessarily anisotropic. However, the Witt Decomposition Theorem (Corollary 5.11 [23]) states that any form  $q$  is isometric to  $\langle 1, -1 \rangle \perp \langle 1, -1 \rangle \perp \dots \perp \langle 1, -1 \rangle \perp q_o$ , where  $q_o$  is anisotropic. The form  $q_o$  is determined uniquely up to isometry and called the **core form** of  $q$ ; sending  $q \mapsto q_o$  is called the **reduction** of  $q$ .

Equipped with  $\perp$  and  $\otimes$  (under the appropriate reduction),  $W(\mathbb{Q})$  is known as the **Witt ring**. The sum  $\perp$  is not defined up to similarity, but the set of similarity classes of anisotropic forms, denoted  $\overline{W(\mathbb{Q})}$ , forms a semigroup under  $\otimes$  with a unit given by the similarity class of  $\langle 1 \rangle$ . Furthermore, there is a natural semigroup epimorphism  $W(\mathbb{Q}) \rightarrow \overline{W(\mathbb{Q})}$ . In [15], Hess and Parent identify an embedding of  $\overline{W(\mathbb{Q})}$  into the Dror Farjoun Cellular Lattice of rational spaces. In Chapter 3, we show an analogous result for the resolvability relation defined in Chapter 2.

## Higher Degree Forms and General Polynomials

We briefly generalize some of the concepts introduced for quadratic forms to homogeneous polynomials. We can extend the definition of similarity to homogeneous polynomials, and in Section 3 we identify embeddings of collections of similarity classes of degree  $m$  homogeneous polynomials into our resolvability relation.

Let  $P_n^m \subset \mathbb{Q}[x_1, \dots, x_n]$  be the subset of rational polynomials consisting of homogeneous polynomials of degree  $m$ . We call the elements of  $P_n^m$   **$n$ -dimensional  $m$ -forms**. The previous section dealt with  $q(x_1, \dots, x_n) \in P_n^2$ ; i.e., 2-forms are precisely quadratic forms. Just as for quadratic forms, write  $x = (x_1, \dots, x_n)^T$ , and  $p(x)$  for  $p(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ . Let  $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$  and denote  $x^\delta = \prod_{i=1}^n x_i^{\delta_i}$ . In this notation, we can write  $p(x) \in P_n^m$  as

$$p(x) = \sum_{\|\delta\|=m} \alpha_\delta x^\delta,$$

where  $\|\delta\| = \sum_{i=1}^n \delta_i$  and each  $\alpha_\delta \in \mathbb{Q}$ . An  $m$ -form  $p$  is **isotropic** if there exists a nonzero vector  $v$  such that  $p(v) = 0$ , otherwise it is **anisotropic**. In the case  $n = 0$ , the empty  $m$ -form is considered anisotropic.

There are two equivalence relations we may define on  $P_n^m$ , analogous to those we saw for quadratic forms:

**Definition.** We say  $p(x), q(x) \in P_n^m$  are **isometric**, written  $p \simeq q$ , provided there exists a  $C \in GL(n, \mathbb{Q})$  such that  $p(C^T x) = q(x)$ . They are **similar**, written  $p \sim q$ , provided there exists a nonzero  $\alpha \in \mathbb{Q}$  such that  $p \simeq \alpha q$ .

So,  $\sim$  identifies the  $m$ -form  $p$  with the  $m$ -form  $q$  if  $q$  can be obtained from  $p$  via a linear change of variables and a nonzero scalar multiplier. We note that these definitions indeed offer a generalization of the isometry and similarity relations defined in the case  $m = 2$  since  $q(C^T x) = x^T C A_q C^T x$  for any quadratic form  $q$ . Unlike the degree 2 case, however, it is not possible to diagonalize an  $m$ -form up to isometry in general. This hinders the creation of higher degree analogues to the Witt group and its quotients.

Now, we see  $\mathbb{Q}[x_1, \dots, x_n] = \bigoplus_{m=0}^{\infty} P_n^m$ . Thus, given  $p(x) \in \mathbb{Q}[x_1, \dots, x_n]$  of degree  $m$ , we can write  $p(x) = \sum_{l=0}^m p_l(x)$  where  $p_l \in P_n^l$  and  $p_m \neq 0$ . We may then study a general polynomial by investigating its homogeneous parts. In particular, we have the following definitions:

**Definition.** Let  $p(x), q(x) \in \mathbb{Q}[x_1, \dots, x_n]$  be two polynomials of degree  $m$ . We say they are **similar**, written  $p \sim q$ , if there exists  $C \in GL(n, \mathbb{Q})$  and  $0 \neq \alpha_l \in \mathbb{Q}$  such that  $p_l(C^T x) = \alpha_l q_l(x)$  for all  $1 \leq l \leq m$ . A polynomial  $p(x)$  is said to be **anisotropic** if each  $p_l(x)$  for  $1 \leq l \leq m$  is anisotropic as an  $l$ -form.

*Remark.* In the above definition, there is only *one* matrix  $C \in GL(n, \mathbb{Q})$ . The same matrix must satisfy all of the equations  $p_l(C^T x) = \alpha_l q_l(x)$  (for  $1 \leq l \leq m$ ) simultaneously.

## Systems of Rational Polynomial Forms

We will eventually also discuss spaces that represent certain *collections* of  $m$ -forms. It is useful, then, to establish some terminology and notation.

Given a collection  $\{q_1, \dots, q_r\}$  of  $n$ -dimensional  $m$ -forms, we call

$$\bar{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_r \end{bmatrix},$$

an  **$r$ -fold system of ( $n$ -dimensional)  $m$ -forms**. Given a vector  $v$ , write  $\bar{q}(v)$  for  $(q_1(v), \dots, q_r(v))^T$ . We can define a notion of similarity on systems of  $m$ -forms; the motivation here is topological, being explicated in Chapter 3.

**Definition.** Given two  $r$ -fold systems of  $n$ -dimensional  $m$ -forms,  $\bar{p}$  and  $\bar{q}$ , we say  $\bar{p}$  is **similar** to  $\bar{q}$ , written  $\bar{p} \sim \bar{q}$ , if there exist  $C \in GL(n, \mathbb{Q})$  and  $D \in GL(r, \mathbb{Q})$  such that  $\bar{p}(C^T x) = D^T \bar{q}(x)$  for all  $x$ .

In other words, this definition declares the system  $\bar{p}$  similar to  $\bar{q}$  if, under a linear change of variables (represented by  $C$ ), the components of  $\bar{p}$  correspond to linear combinations of the components of  $\bar{q}$  (represented by  $D$ ), i.e.:

$$p_j(C^T x) = \sum_{i=1}^r d_{ij} q_i(x)$$

for all  $1 \leq j \leq r$ , where  $D = (d_{ij})$ . Of course, we can consider a single  $m$ -form as a 1-fold system, and in this case the two definitions of similarity coincide: if  $r = 1$ , then  $D \in GL(r, \mathbb{Q})$  is simply a nonzero element of  $\mathbb{Q}$ .

**Example.** As an example, consider the 4-fold systems of 2-dimensional 3-forms

$$\bar{p}(x, y) = \begin{bmatrix} x^3 \\ x^2y - y^3 \\ y^3 \\ xy^2 + x^2y \end{bmatrix} \quad \text{and} \quad \bar{q}(x, y) = \begin{bmatrix} x^3 + 3xy^2 \\ x^2y \\ xy^2 \\ 3xy^2 + y^3 \end{bmatrix}.$$

We will show  $\bar{p} \sim \bar{q}$  by exhibiting the necessary matrices  $C$  and  $D$  as in the definition. Let  $C \in GL(2, \mathbb{Q})$  be

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and define  $D \in GL(4, \mathbb{Q})$  as

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -3 & -3 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Then we find

$$\begin{aligned} C^T \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x + y \\ y \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned}
\bar{p}\left(C^T \begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} (x+y)^3 \\ (x+y)^2y - y^3 \\ y^3 \\ (x+y)y^2 + (x+y)^2y \end{bmatrix} \\
&= \begin{bmatrix} x^3 + 3x^2y + 3xy^2 + y^3 \\ x^2y + 2xy^2 \\ y^3 \\ x^2y + 3xy^2 + 2y^3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x^3 + 3xy^2 \\ x^2y \\ xy^2 \\ 3xy^2 + y^3 \end{bmatrix} \\
&= D^T \bar{q}(x, y).
\end{aligned}$$

Given two anisotropic one dimensional  $m$ -forms, say  $p(x) = \alpha x^m$  and  $q(x) = \beta x^m$  for nonzero  $\alpha, \beta \in \mathbb{Q}$ , we find  $p \sim q$  since they are both similar to  $x^m$ . This is a consequence of the following more general result.

**Proposition 1.5.** *Let  $\bar{p}$  and  $\bar{q}$  be  $r$ -fold systems of  $n$ -dimensional  $m$ -forms such that  $\{p_j\}_{j=1}^r$  and  $\{q_j\}_{j=1}^r$  are linearly independent sets over  $\mathbb{Q}$ . If*

$$r = \binom{m+n-1}{m},$$

then  $\bar{p} \sim \bar{q}$ .

*Proof.* Suppose

$$r = \binom{m+n-1}{m}.$$

We note this binomial coefficient counts the maximum number of nonzero terms in an  $n$ -dimensional  $m$ -form. So,  $P_n^m$  (including 0) forms an  $r$ -dimensional vector space over  $\mathbb{Q}$ . Notice, then,  $\{p_j\}_{j=1}^r$  and  $\{q_j\}_{j=1}^r$  both form a basis for this vector space, whence there must exist a  $D \in GL(r, \mathbb{Q})$  such that  $\bar{p}(x) = D^T \bar{q}(x)$ .  $\square$

*Remark.* Proposition 1.5 offers another way to see  $\bar{p} \sim \bar{q}$  in the previous example. In particular, they are both similar to

$$\begin{bmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}.$$

There is no hope, in general, to appropriately diagonalize a system of quadratic forms [20], let alone a system of  $m$ -forms. However, we may, at the very least, define an  $r$ -fold system of  $m$ -forms  $\bar{q} = (q_1, \dots, q_r)$  to be **anisotropic** if each  $q_i$  is anisotropic for  $1 \leq i \leq r$ . It is important to note that this is not the standard definition of an anisotropic system; rather, this definition is topologically inspired.

Let  $\mathbb{W}$  be the collection of all anisotropic  $r$ -fold systems of  $n$ -dimensional  $m$ -forms, for  $r, n, m \in \mathbb{N}$ . Denote the collection of similarity classes of elements of  $\mathbb{W}$  by  $\overline{\mathbb{W}}$ , and for a fixed  $r \in \mathbb{N}$ , call the subset of similarity classes of  $r$ -fold systems  $\overline{\mathbb{W}}^r \subset \overline{\mathbb{W}}$ . Section 3.2 deals with  $\overline{\mathbb{W}}$  (in particular  $\overline{\mathbb{W}}^1$ ) in the context of the resolvability relation discussed in the sequel.



# Chapter 2

## Resolving Classes

Resolving classes were introduced by Strom in [25], serving as a dual to the closed classes studied by Dror Farjoun, Chachólski, et al. The reader is referred to [6] and [26] for more information on closed classes and resolving classes respectively. It should be noted that we work with *pointed* spaces; this is not always the case in the literature concerning closed classes. We present here some results about resolving classes, with special attention paid to the rational case.

### 2.1 Definition, Characterization, and Examples

We begin with a definition:

**Definition.** A (nonempty) class of spaces  $\mathcal{R}$  is called a **resolving class** if it is closed under weak equivalence and pointed homotopy limits. That is to say,  $\mathcal{R}$  is a resolving class provided:

- if  $X \in \mathcal{R}$  and  $X \sim Y$  then  $Y \in \mathcal{R}$ ; and
- any functor  $F : \mathcal{I} \rightarrow \mathcal{T}_*$  that takes its values in  $\mathcal{R}$  must have  $\text{holim}F \in \mathcal{R}$ .

If a class  $\mathcal{R}$  has the property that, for any fiber sequence  $F \rightarrow X \rightarrow Y$  with  $F, Y \in \mathcal{R}$  we have  $X \in \mathcal{R}$ , we say  $\mathcal{R}$  is closed under **extensions by fibrations**. A resolving class that is closed under extensions by fibrations is called a **strong resolving class**.

**Example.**  $\{X \mid X \sim *\}$  is a strong resolving class. In fact,  $\{Y \mid \mathcal{T}_*(X, Y) \sim *\}$  is a strong resolving class for any space  $X$ .

More generally, given a collection of spaces  $\mathcal{C}$ , define

$$\Phi(\mathcal{C}) = \{Y \mid \mathcal{T}_*(X, Y) \sim * \text{ for all } X \in \mathcal{C}\}.$$

Then  $\Phi(\mathcal{C})$  is a strong resolving class. Resolving classes of this form are called **resolving kernels**.

Interestingly, if we define

$$\Theta(\mathcal{C}) = \{X \mid \mathcal{T}_*(X, Y) \sim * \text{ for all } Y \in \mathcal{C}\},$$

then  $\Theta(\mathcal{C})$  is a (strong) closed class. In fact,  $\Theta(\Phi(\{X\}))$  is actually the Bousfield class of  $X$ . Furthermore,  $\Theta$  and  $\Phi$  (as functions from the collection of all classes of spaces to itself) form a Galois connection [26].

**Example.** The smallest resolving class containing a given space  $A$  is denoted  $\mathcal{R}(A)$ , while the smallest strong resolving class containing  $A$  is denoted  $\overline{\mathcal{R}}(A)$ . Classes of this form are our primary examples of resolving classes.

The spaces in  $\mathcal{R}(A)$  are called  **$A$ -resolvable** spaces, and we say “ $A$  forms  $X$ ”, written  $A \preceq X$ . If  $A \preceq X$  and  $X \preceq A$ , we say  $X$  and  $A$  are **resolvably equivalent**, denoted  $A \sim_r X$ . We refer to  $\preceq$  as the **resolvability relation** on spaces. This notation and terminology is purposefully chosen to be analogous to the  $A$ -cellular spaces and “building” relation developed in reference to closed classes.

Our first proposition grants some insight into how weak equivalence and resolvable equivalence compare; specifically, we see weakly equivalent spaces must be resolvably equivalent.

**Proposition 2.1.** *Let  $X$  and  $A$  be spaces.*

1.  $A \sim_r X$  if and only if  $\mathcal{R}(A) = \mathcal{R}(X)$ ;
2. if  $A \sim X$ , then  $A \sim_r X$ , i.e.  $\mathcal{R}(A) = \mathcal{R}(X)$ .

*Proof.* (1)  $A \sim_r X$  if and only if  $X \in \mathcal{R}(A)$  and  $A \in \mathcal{R}(X)$ , which is true if and only if  $\mathcal{R}(X) \subseteq \mathcal{R}(A)$  and  $\mathcal{R}(A) \subseteq \mathcal{R}(X)$ .

(2) If  $A \sim X$  then  $X \in \mathcal{R}(A)$  and  $A \in \mathcal{R}(A)$ , implying the rest of the result.  $\square$

Next, we have a simple proposition that assures us resolving classes are closed under the formation of homotopy fibers and loop spaces.

**Proposition 2.2.** *If  $\mathcal{R}$  is a resolving class, then  $*$   $\in \mathcal{R}$ . Therefore, if  $F \rightarrow X \rightarrow Y$  is a fiber sequence and  $X, Y \in \mathcal{R}$ , then  $F \in \mathcal{R}$ , and, in particular, if  $X \in \mathcal{R}$  then  $\Omega X \in \mathcal{R}$ .*

*Proof.* We see  $*$   $\sim \text{holim}(\dots \xrightarrow{*} X \xrightarrow{*} X)$  for any  $X \in \mathcal{R}$  (recall resolving classes are assumed to be nonempty). Since  $F \sim \text{holim}(* \rightarrow Y \leftarrow X)$ , and  $\Omega X \sim \text{holim}(* \rightarrow X \leftarrow *)$ , we conclude  $F, \Omega X \in \mathcal{R}$ .  $\square$

It is a result of Chachólski [3] that a class is a closed class if and only if it is closed under arbitrary (set-indexed) wedges and homotopy pushouts. As may be expected, the dual result holds: we can characterize resolving classes using only closure under homotopy pullbacks and (set-indexed) products.

**Proposition 2.3.** *A class of spaces  $\mathcal{R}$  is a resolving class if and only if  $\mathcal{R}$  is closed under weak equivalences, arbitrary products, and homotopy pullbacks.*

*Proof.* The definition immediately implies that any resolving class must be closed under weak equivalences, arbitrary products, and homotopy pullbacks. Explicitly, if  $\mathcal{R}$  is a resolving class, then for any index set  $\mathcal{I}$  with  $X_i \in \mathcal{R}$ , we have  $\prod_{i \in \mathcal{I}} X_i \sim \text{holim}\{X_i \rightarrow *\}_{i \in \mathcal{I}} \in \mathcal{R}$ ; furthermore, if  $X, Y, Z \in \mathcal{R}$  and

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

is a homotopy pullback square, then  $P \sim \text{holim}(Z \rightarrow Y \leftarrow X) \in \mathcal{R}$ .

For the converse, suppose  $\mathcal{R}$  is closed under weak equivalences, arbitrary products, and homotopy pullbacks. We are left to check that  $\mathcal{R}$  is closed under homotopy limits. Suppose  $F : \mathcal{I} \rightarrow \mathcal{R}$  is a diagram in  $\mathcal{R}$ , and call  $L = \lim F$ . As seen in [16],  $L$  is the equalizer of a pair of natural morphisms

$$f, g : \prod_{i \in \mathcal{I}} F(i) \rightarrow \prod_{\alpha: j \rightarrow k} F(k),$$

where  $\alpha$  is taken to vary over all morphisms in  $\mathcal{I}$ . Calling  $P$  the pullback of

$$\prod_{i \in \mathcal{I}} F(i) \xrightarrow{f} \prod_{\alpha: j \rightarrow k} F(k) \xleftarrow{g} \prod_{i \in \mathcal{I}} F(i),$$

we can recognize  $L$  from the pullback square

$$\begin{array}{ccc} L & \longrightarrow & P \\ \downarrow & & \downarrow \\ \prod_{i \in \mathcal{I}} F(i) & \xrightarrow{(id, id)} & \left( \prod_{i \in \mathcal{I}} F(i) \right) \times \left( \prod_{i \in \mathcal{I}} F(i) \right) \end{array}$$

Since  $\text{holim} F$  is (weakly equivalent to)  $\lim \overline{F}$  for some fibrant replacement  $F \rightarrow \overline{F}$  (in particular, for a fibrant replacement  $\overline{F}$  in which all morphisms are fibrations) and  $\mathcal{R}$  is closed under weak equivalences, the result follows.  $\square$

Dror Farjoun mentions with brief justification in [6] that closed classes are closed under retracts (though the details don't appear to be in print). We prove the dual result here.<sup>1</sup>

**Proposition 2.4.** *Let  $X \in \mathcal{R}$  where  $\mathcal{R}$  is a resolving class. Suppose  $Y$  is a retract of  $X$ , i.e. we have  $Y \xrightarrow{i} X \xrightarrow{r} Y$  with  $r \circ i = id_Y$ . Then  $Y \in \mathcal{R}$ .*

*Proof.* Call  $F : \mathbb{N}^{op} \rightarrow \mathcal{T}_*$  the diagram

$$\dots \xrightarrow{i \circ r} X \xrightarrow{i \circ r} X \xrightarrow{i \circ r} X.$$

Let  $e : F \rightarrow \overline{F}$  be a fibrant replacement for  $F$ ; in particular, the components of  $e$  are weak equivalences, and we choose  $\overline{F}$  to have the form

$$\dots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

where each  $f_n$  is a fibration. By definition, we have  $\text{holim} F \sim \lim \overline{F}$ .

Now let  $\Delta_Y$  be the constant tower at  $Y$ :  $\dots \xrightarrow{id_Y} Y \xrightarrow{id_Y} Y \xrightarrow{id_Y} Y$ . Then  $i : Y \rightarrow X$  induces a natural transformation  $\Delta_i : \Delta_Y \rightarrow F$ , which we compose with  $e$  to yield  $e \circ \Delta_i : \Delta_Y \rightarrow \overline{F}$ . In particular, for every  $n \in \mathbb{N}$ , we have a commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{i} & X & \xrightarrow[e_{n+1}]{\sim} & X_{n+1} \\ id_Y \downarrow & & i \circ r \downarrow & & \downarrow f_n \\ Y & \xrightarrow{i} & X & \xrightarrow[e_n]{\sim} & X_n. \end{array}$$

We claim  $e \circ \Delta_i$  is a weak pro-homotopy equivalence (see [1] or Section 1.4).

To verify this, let  $k \in \mathbb{N}$  and consider the commutative square:

$$\begin{array}{ccc} \pi_k Y & \xrightarrow{\pi_k(e_{n+1} \circ i)} & \pi_k X_{n+1} \\ \pi_k(id_Y) \downarrow & \swarrow \text{---} & \downarrow \pi_k(f_n) \\ \pi_k Y & \xrightarrow{\pi_k(e_n \circ i)} & \pi_k X_n. \end{array}$$

We wish to find a homomorphism  $\pi_k X_{n+1} \rightarrow \pi_k Y$  to fit in the diagram.

---

<sup>1</sup>Thanks are due to Kathryn Hess who, upon reading an early draft of this material, suggested utilizing a weak pro-homotopy equivalence in place of the lengthier elementary approach previously employed.

Define  $\phi : \pi_k X_{n+1} \rightarrow \pi_k Y$  to be the composite:

$$\pi_k X_{n+1} \xrightarrow{\pi_k(f_n)} \pi_k X_n \xrightarrow{\pi_k(e_n)^{-1}} \pi_k X \xrightarrow{\pi_k(r)} \pi_k Y .$$

Then we see

$$\begin{aligned} \phi \circ \pi_k(e_{n+1} \circ i) &= \pi_k(r) \circ \pi_k(e_n)^{-1} \circ \pi_k(f_n) \circ \pi_k(e_{n+1}) \circ \pi_k(i) \\ &= \pi_k(r) \circ \pi_k(e_n)^{-1} \circ \pi_k(e_n) \circ \pi_k(i \circ r) \circ \pi_k(i) \\ &= \pi_k(id_Y) . \end{aligned}$$

Furthermore,

$$\begin{aligned} \pi_k(e_n \circ i) \circ \phi &= \pi_k(e_n) \circ \pi_k(i) \circ \pi_k(r) \circ \pi_k(e_n)^{-1} \circ \pi_k(f_n) \\ &= \pi_k(e_n) \circ \pi_k(i \circ r) \circ \pi_k(i \circ r) \circ \pi_k(e_{n+1})^{-1} \\ &= \pi_k(e_n) \circ \pi_k(i \circ r) \circ \pi_k(e_{n+1})^{-1} \\ &= \pi_k(f_n) \circ \pi_k(e_{n+1}) \circ \pi_k(e_{n+1})^{-1} \\ &= \pi_k(f_n) . \end{aligned}$$

So,  $\phi : \pi_k X_{n+1} \rightarrow \pi_k Y$  as defined witnesses  $e \circ \Delta_i$  as a weak pro-homotopy equivalence. Therefore,  $\lim \Delta_Y \sim \lim \bar{F}$ , implying  $Y \sim \lim \bar{F} \sim \text{holim} F \in \mathcal{R}$   $\square$

**Corollary 2.5.** *Resolving classes are closed under homotopy retracts: if  $Y \xrightarrow{i} X \xrightarrow{r} Y$  is homotopic to  $id_Y$  and  $X \in \mathcal{R}$ , then  $Y \in \mathcal{R}$ .*

*Proof.* First suppose  $i : Y \hookrightarrow X$  is a cofibration. Then we have the following square commuting up to homotopy

$$\begin{array}{ccc} Y & \xrightarrow{id_Y} & Y \\ i \downarrow & & \downarrow id_Y \\ X & \xrightarrow{r} & Y . \end{array}$$

Since  $i$  is a cofibration, we may replace  $r$  with a map  $\bar{r} : X \rightarrow Y$  such that  $\bar{r} \simeq r$  and  $\bar{r} \circ i = id_Y$ . But then we are in the case of the previous proposition, so  $Y \in \mathcal{R}$ .

If  $i : Y \rightarrow X$  is not a cofibration, there is an  $\bar{X} \sim X$  such that  $Y \hookrightarrow \bar{X} \xrightarrow{\sim} X$  is  $i$ . Notice that the composition  $Y \hookrightarrow \bar{X} \xrightarrow{\sim} X \rightarrow Y$  is  $r \circ i \simeq id_Y$ . So,  $Y$  is a homotopy retract of  $\bar{X}$  as well. Since  $X \in \mathcal{R}$  if and only if  $\bar{X} \in \mathcal{R}$ , we find  $Y \in \mathcal{R}$  by the previously considered case.  $\square$

## 2.2 Resolving Classes and Anticonnectivity

Taking a cue from Whitehead in [27], a space  $X$  is said to be  $n$ -**anticonnected** if  $\pi_k(X) = 0$  for all  $k \geq n$ . We say a space has **anticonnectivity**  $n + 1$ , denoted  $\text{aconn}(X) = n + 1$ , if  $X$  is  $(n + 1)$ -anticonnected but not  $n$ -anticonnected. In other words,  $X$  has anticonnectivity  $n + 1$  provided  $\pi_n(X) \neq 0$  but  $\pi_k(X) = 0$  for all  $k \geq n + 1$ . If there does not exist an  $N \in \mathbb{N}$  such that  $\pi_k(X) = 0$  for all  $k \geq N$ , we say  $X$  is infinitely anticonnected and write  $\text{aconn}(X) = \infty$ .

**Example.** Given a path-connected space  $X$ , the  $n$ -th Postnikov section of  $X$  will have  $\text{aconn}(X_n) \leq n + 1$ , and  $\text{aconn}(\Omega X) = \text{aconn}(X) - 1$ .

Furthermore, for such an  $X$ ,  $X$  is  $(n + 1)$ -anticonnected if and only if  $[K, X] = *$  for every  $n$ -connected CW complex  $K$ . To see this, suppose  $X$  is  $(n + 1)$ -anticonnected and define

$$\mathcal{C} = \{K \in \mathcal{T}_* \mid [K, X] = *\}.$$

The collection of spaces  $\mathcal{C}$  is a closed class, and since it clearly contains  $S^{n+1}$ , we have  $\mathcal{C}(S^{n+1}) \subseteq \mathcal{C}$ . We note  $\mathcal{C}(S^{n+1})$  consists of all  $n$ -connected spaces, implying the desired result. The reverse implication is immediate, considering  $S^k$  is an  $n$ -connected CW complex for all  $k \geq n + 1$ .

While closed classes encode information about connectivity, resolving classes encode information about anticonnectivity. Indeed, given  $n \in \mathbb{N}$ , the class of all spaces  $X$  with  $\text{aconn}(X) \leq n$  is a strong resolving class. This fact not only provides a nice example of a (strong) resolving class, but also proves itself useful in Section 2.4. We establish it as a corollary to Proposition 2.3, but state it as a lemma considering its application in the proof of Theorem 2.10.

**Lemma 2.6.** *Given  $n \in \mathbb{N}$ , let  $\mathcal{D}_n$  be the class of all spaces  $X$  such that  $\text{aconn}(X) \leq n$ . Then  $\mathcal{D}_n$  is a strong resolving class.*

*Proof.* The fact that  $\mathcal{D}_n$  is closed under weak equivalences is obvious. By Proposition 2.3, we are left to show that  $\mathcal{D}_n$  is closed under products, homotopy pullbacks, and extensions by fibrations.

Let  $\mathcal{I}$  be an index set and suppose  $X_i \in \mathcal{D}_n$  for all  $i \in \mathcal{I}$ . Observe  $\pi_k(\prod_i X_i) \cong \prod_i \pi_k(X_i)$ , so  $\pi_k(\prod_i X_i) \cong 0$  for all  $k \geq n$  as this is true of each  $\pi_k(X_i)$ . Therefore, the anticonnectivity of  $\prod_i X_i$  is less than or equal to  $n$ .

Now suppose that  $F \rightarrow X \rightarrow Y$  is a fiber sequence with  $F, Y \in \mathcal{D}_n$ . Apply  $\pi_k$  to yield an exact sequence  $\pi_k(F) \rightarrow \pi_k(X) \rightarrow \pi_k(Y)$ . So, if  $k \geq n$ , then  $\pi_k(X) \cong 0$  and  $X \in \mathcal{D}_n$ . Thus,  $\mathcal{D}_n$  is closed under extensions by fibrations.

Finally, suppose  $X, Y, Z \in \mathcal{D}_n$  and let  $P$  be the homotopy pullback of  $X \rightarrow Y \leftarrow Z$ . We can form the Mayer-Vietoris fiber sequence  $\Omega Y \rightarrow P \rightarrow X \times Z$ . We know  $\Omega Y, X \times Z \in \mathcal{D}_n$ , and since  $\mathcal{D}_n$  is closed under extensions by fibrations,  $P \in \mathcal{D}_n$ . With this, we are assured that  $\mathcal{D}_n$  is a strong resolving class.  $\square$

*Remark.* The proof given above is presented for the sake of explicitness. We can alternately prove  $\mathcal{D}_n$  is the resolving kernel  $\Phi(\{S^n\})$ :

$$\begin{aligned} \Phi(\{S^n\}) &= \{X \mid \mathcal{T}_*(S^n, X) \sim *\} \\ &= \{X \mid [S^k, \mathcal{T}_*(S^n, X)] = * \text{ for all } k \geq 0\} \\ &= \{X \mid [S^k \wedge S^n, X] = * \text{ for all } k \geq 0\} \\ &= \{X \mid \pi_{k+n}(X) = * \text{ for all } k \geq 0\} \\ &= \mathcal{D}_n. \end{aligned}$$

The following is another corollary of Proposition 2.3 that turns out to be valuable in Section 2.4, and is thus stated as a lemma. Recall, given a space  $X$ , we write  $X^C$  for its cofibrant replacement.

**Lemma 2.7.** *Let  $A$  be a space with  $\text{aconn}(A) = n + 1$ ,  $K$  be an  $(n - 1)$ -connected CW complex ( $n \geq 1$ ), and let  $\mathcal{D}^K$  be the class of spaces  $X$  such that*

- $\text{aconn}(X) \leq n + 1$ ;
- *there is a map  $f : X^C \rightarrow \prod_{i \in \mathcal{I}} A$  for some index set  $\mathcal{I}$  with  $[K, f]$  injective.*

*Then  $\mathcal{D}^K$  is a resolving class.*

*Proof.* The condition  $\text{aconn}(X) \leq n + 1$  is preserved under weak equivalence, products, and homotopy pullbacks as shown in Lemma 2.6. Accordingly, we will focus on the second condition.

We first show  $\mathcal{D}^K$  is closed under weak equivalence. Consider  $X \xrightarrow{\sim} Y$ . If  $Y \in \mathcal{D}^K$ , then we have a map  $Y^C \rightarrow \prod_i A$  inducing an injection on  $[K, -]$ . Of course  $X \xrightarrow{\sim} Y$  induces  $X^C \xrightarrow{\sim} Y^C$ , and composing  $X^C \xrightarrow{\sim} Y^C \rightarrow \prod_i A$  yields  $X \in \mathcal{D}^K$ . If  $X \in \mathcal{D}^K$ , then we have a map  $X^C \rightarrow \prod_i A$  injective on  $[K, -]$ . Since  $X^C$  and  $Y^C$  are cofibrant, the weak equivalence  $X^C \xrightarrow{\sim} Y^C$  has a homotopy inverse  $Y^C \xrightarrow{\sim} X^C$ . Composing  $Y^C \xrightarrow{\sim} X^C \rightarrow \prod_i A$  grants  $Y \in \mathcal{D}^K$ . Therefore,  $X \in \mathcal{D}^K$  if and only if  $Y \in \mathcal{D}^K$ , implying  $\mathcal{D}^K$  is closed under weak equivalence.

We now show  $\mathcal{D}^K$  is closed under products. Let  $X_j \in \mathcal{D}^K$  for all  $j \in \mathcal{J}$ . Then each  $X_j$  has a map  $f_j : (X_j)^C \rightarrow \prod_i A$  with  $[K, f_j]$  injective. There is then a map

$$f = \prod_j f_j : \prod_j (X_j)^C \rightarrow \prod_{j,i} A,$$

and since  $[K, f_j]$  is injective for each  $j \in J$ , it follows  $[K, f]$  is injective. Thus  $\mathcal{D}^K$  is closed under products.

Finally, suppose  $X, Y, Z \in \mathcal{D}^K$  and let  $P$  be the homotopy pullback of  $X \rightarrow Y \leftarrow Z$ . Form the Mayer-Vietoris fiber sequence  $\Omega Y \rightarrow P \rightarrow X \times Z$ , and the induced fiber sequence of cofibrant replacements  $(\Omega Y)^C \rightarrow P^C \rightarrow (X \times Z)^C$ . We know  $X \times Z \in \mathcal{D}^K$ , so we have  $(X \times Z)^C \rightarrow \prod_i A$  inducing an injection on  $[K, -]$ . Applying  $[K, -]$  yields a sequence

$$[K, (\Omega Y)^C] \rightarrow [K, P^C] \rightarrow [K, (X \times Z)^C] \rightarrow \left[ K, \prod_i A \right],$$

the first three terms of which are exact. Since  $\text{aconn}((\Omega Y)^C) = \text{aconn}(\Omega Y) \leq n$ , we know  $[K, (\Omega Y)^C] = *$ . Thus, we find  $[K, P^C] \rightarrow [K, (X \times Z)^C]$  injective, implying the composition  $P^C \rightarrow (X \times Z)^C \rightarrow \prod_i A$  has the desired property, allowing us to conclude  $P \in \mathcal{D}^K$ . By Proposition 2.3,  $\mathcal{D}^K$  is a resolving class as claimed.  $\square$

## 2.3 On Resolving Classes Containing Eilenberg-MacLane Spaces

In this short section, we consider  $K(V, n)$ , where  $n \in \mathbb{N}$  and  $V$  is a nonzero vector space over a field  $\mathbb{F}$  (with  $\mathbb{F} = \mathbb{Q}$  being the case of primary interest). For a resolving class  $\mathcal{R}$  containing  $K(V, n)$ , its closure under products, retractions, and formation of loop spaces can be used to form  $K(W, m)$  for any  $\mathbb{F}$ -vector space  $W$  and  $m \leq n$ . This property of resolving classes is the basic principle facilitating our understanding of  $\mathcal{R}(A)$  when  $A$  is a rational space. We have:

**Theorem 2.8.** *Let  $m, n \in \mathbb{N}$  and suppose  $\mathcal{R}$  is a resolving class. The following are equivalent:*

- $K(V, n) \in \mathcal{R}$  for some nonzero  $\mathbb{F}$ -vector space  $V$ ;
- $K(W, m) \in \mathcal{R}$  for all  $\mathbb{F}$ -vector spaces  $W$  and all  $m \leq n$ .

We first prove a lemma:

**Lemma 2.9.** *Suppose  $W$  and  $V$  are  $\mathbb{F}$ -vector spaces with  $W \subseteq V$ . Then there is a homotopy retraction of Eilenberg-MacLane Spaces*

$$K(W, n) \rightarrow K(V, n) \rightarrow K(W, n).$$



*Proof.* We have an algebraic retraction  $W \hookrightarrow V \rightarrow W$ , given by the inclusion of  $W$  into  $V$ ,  $W \hookrightarrow V$ , and projection of  $V$  onto  $W$ ,  $V \rightarrow W$ . This induces the desired homotopy retraction  $K(W, n) \rightarrow K(V, n) \rightarrow K(W, n)$ .  $\square$

*Proof of Theorem 2.8.* We prove the nontrivial assertion. Suppose  $V \neq 0$  is a vector space over  $\mathbb{F}$  with  $K(V, n) \in \mathcal{R}$ , and let  $W$  be an arbitrary  $\mathbb{F}$ -vector space. By Lemma 2.9 we have a homotopy retraction of Eilenberg-MacLane Spaces  $K(\mathbb{F}, n) \rightarrow K(V, n) \rightarrow K(\mathbb{F}, n)$ . Thus,  $K(\mathbb{F}, n) \in \mathcal{R}$  by Corollary 2.5.

Suppose  $W \cong \prod_{i \in \mathcal{I}} \mathbb{F}$  for some index set  $\mathcal{I}$ . Notice

$$K(W, n) = K\left(\prod_{i \in \mathcal{I}} \mathbb{F}, n\right) \sim \prod_{i \in \mathcal{I}} K(\mathbb{F}, n).$$

Since  $\mathcal{R}$  is closed under products and weak equivalence, it follows  $K(W, n) \in \mathcal{R}$ .

Now suppose  $W \cong \bigoplus_{i \in \mathcal{I}} \mathbb{F}$  for some index set  $\mathcal{I}$ . Note  $W \subseteq \prod_{i \in \mathcal{I}} \mathbb{F}$ , so according to Lemma 2.9, there is a homotopy retraction of Eilenberg-MacLane spaces  $K(W, n) \rightarrow K(\prod_{i \in \mathcal{I}} \mathbb{F}, n) \rightarrow K(W, n)$ . By what was argued above,  $K(\prod_{i \in \mathcal{I}} \mathbb{F}, n) \in \mathcal{R}$ , so by Corollary 2.5 we have  $K(W, n) \in \mathcal{R}$ .

We are left to show  $K(W, m) \in \mathcal{R}$  for all  $m \leq n$ . Using Proposition 2.2, one need only recognize  $K(W, m) \sim \Omega^{n-m} K(W, n) \in \mathcal{R}$  (for  $m < n$ ) to complete the proof.  $\square$

## 2.4 Rational Spaces in Resolving Classes

Since rational spaces have homotopy groups that are rational vector spaces, the preceding section yields some nice tools for discovering if  $X \in \mathcal{R}(A)$  when  $X$  and  $A$  are rational. In this section, we show that this question is completely determined if the anticonnectivity of  $X$  does not equal the anticonnectivity of  $A$ . In the case of equal anticonnectivity, we develop a characterization dual to the Chachólski-Parent-Stanley criterion (Theorem 1 from [15]). We continue by constructing an algebraic translation of this characterization. Recall we are assuming all rational spaces and CDGA's to be of finite type and simply connected.

For rational spaces  $X$  and  $A$  with  $\text{aconn}(X) \neq \text{aconn}(A)$ , we find:

**Theorem 2.10.** *For (simply connected) rational spaces  $X$  and  $A$ :*

- if  $\text{aconn}(X) < \text{aconn}(A)$ , then  $X \in \mathcal{R}(A)$ ;
- if  $X \in \mathcal{R}(A)$ , then  $\text{aconn}(X) \leq \text{aconn}(A)$ ;
- $X \in \overline{\mathcal{R}}(A)$  if and only if  $\text{aconn}(X) \leq \text{aconn}(A)$ .

*Remark.* We note, as a consequence of Theorem 2.10, a rational space of infinite anticonnectivity forms all rational spaces: if  $\text{aconn}(A) = \infty$ , then  $K(\pi_k X, n) \in \mathcal{R}(A)$  for all  $k, n \in \mathbb{N}$ . We can then form  $X$  as the homotopy limit of its Postnikov tower to conclude  $X \in \mathcal{R}(A)$ .

To aid in the poof of Theorem 2.10, we prove

**Lemma 2.11.** *Let  $V$  be a rational vector space and suppose  $A$  is a (simply connected) rational space with  $\text{aconn}(A) = n + 1$ . Then  $K(V, m) \in \mathcal{R}(A)$  for all  $m \leq n$ .*

*Proof.* We first show the Postnikov section  $A_{n-2} \in \mathcal{R}(A)$ . Observe that it is sufficient to consider  $n \geq 4$ : it must be the case that  $n \geq 1$  since  $\text{aconn}(A_{n-2}) \leq n - 1$ , and for  $n \in \{1, 2, 3\}$  we conclude  $A_{n-2} \sim * \in \mathcal{R}$  (since  $A$  is simply connected). So then, we continue under the hypothesis that  $n \geq 4$ . Since  $A$  is rational, we know  $\Omega A \sim \prod_j K(\pi_{j+1}(A), j)$  (c.f. Section 16 of [10]). Therefore,  $K(\pi_n A, n - 1) \in \mathcal{R}(A)$  (thus  $K(V, m) \in \mathcal{R}(A)$  for  $m \leq n - 1$  for any  $\mathbb{Q}$ -vector space  $V$  by Theorem 2.8). In the Postnikov tower for  $A$  we have fiber sequences

$$K(\pi_i A, i) \rightarrow A_i \rightarrow A_{i-1} \rightarrow K(\pi_i A, i + 1),$$

and  $A_1 \sim * \in \mathcal{R}(A)$ . With  $K(\pi_2 A, 3) \in \mathcal{R}(A)$ , we know  $A_2 \in \mathcal{R}(A)$  as it is the fiber of  $A_1 \rightarrow K(\pi_2 A, 3)$ . Inducting up the tower, we find  $A_{n-2} \in \mathcal{R}(A)$ .

Now, consider the following diagram of homotopy pullback squares:

$$\begin{array}{ccccc} K(\pi_n A, n) & \longrightarrow & F & \longrightarrow & A_n \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & K(\pi_{n-1} A, n-1) & \longrightarrow & A_{n-1} \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & A_{n-2} \end{array}$$

From what was argued above, we know  $A_{n-2}, K(\pi_{n-1} A, n - 1) \in \mathcal{R}(A)$ . Clearly  $A_n \sim A \in \mathcal{R}(A)$ . Thus  $F \in \mathcal{R}(A)$  since we have a fiber sequence  $F \rightarrow A_n \rightarrow A_{n-2}$ . But then we must also have  $K(\pi_n A, n) \in \mathcal{R}(A)$  since we have a fiber sequence  $K(\pi_n A, n) \rightarrow F \rightarrow K(\pi_{n-1} A, n - 1)$ . Theorem 2.8 then grants us the full result.  $\square$

*Proof of Theorem 2.10.* Suppose  $\text{aconn}(A) = n + 1$  and  $\text{aconn}(X) \leq n$ . With  $K(V, m) \in \mathcal{R}(A)$  for all  $m \leq n$  and any  $\mathbb{Q}$ -vector space  $V$ , in addition to  $X_1 \sim * \in \mathcal{R}(A)$ , we can induct up the Postnikov tower for  $X$  to find  $X_{n-2} \in \mathcal{R}(A)$ .

Considering  $\text{aconn}(X) \leq n$ , we have  $X \sim X_{n-1}$ , and since  $X_{n-1}$  is the fiber of  $X_{n-2} \rightarrow K(\pi_{n-1}X, n)$ , we have  $X \in \mathcal{R}(A)$ . This shows that  $\text{aconn}(X) < \text{aconn}(A)$  implies  $X \in \mathcal{R}(A)$ .

Continuing, let  $\mathcal{D}_{n+1}$  be the class described in Lemma 2.6. We know  $\mathcal{D}_{n+1}$  is a strong resolving class and clearly  $A \in \mathcal{D}_{n+1}$ , so we have both  $\overline{\mathcal{R}}(A) \subseteq \mathcal{D}_{n+1}$  and  $\mathcal{R}(A) \subseteq \mathcal{D}_{n+1}$ . Therefore, if either  $X \in \mathcal{R}(A)$  or  $X \in \overline{\mathcal{R}}(A)$ , we know  $\text{aconn}(X) \leq \text{aconn}(A)$ .

We are left to show that  $\text{aconn}(X) \leq \text{aconn}(A)$  implies  $X \in \overline{\mathcal{R}}(A)$ . In the Postnikov tower of  $X$  we have a fiber sequence  $K(\pi_n X, n) \rightarrow X_n \rightarrow X_{n-1}$ . Since  $X \sim X_n$ , it is sufficient to show  $X_{n-1} \in \overline{\mathcal{R}}(A)$ . But  $\text{aconn}(X_{n-1}) \leq n$ , so  $X_{n-1} \in \mathcal{R}(A)$  by what has already been shown. Therefore  $X_{n-1} \in \overline{\mathcal{R}}(A)$ , thus  $X_n$  is in  $\overline{\mathcal{R}}(A)$  as well. The result follows.  $\square$

We now consider the case of rational spaces  $X$  and  $A$  with the same anticonnectivity. The following characterization is dual to the Chachólski-Parent-Stanley criterion.

**Theorem 2.12.** *Let  $X$  and  $A$  be rational spaces of anticonnectivity  $n + 1$ . Then  $A \preceq X$  (i.e.  $X \in \mathcal{R}(A)$ ) if and only if there is a homotopy class*

$$f : X \rightarrow \prod_{i \in \mathcal{I}} A$$

for some index set  $\mathcal{I}$ , with  $\pi_n(f)$  injective.

*Proof.* Considering Proposition 2.1, it suffices to consider the case of  $X$  and  $A$  both cofibrant, in which case every homotopy class between  $X$  and  $\prod_i A$  is represented by a map between  $X$  and  $\prod_i A$ . Suppose, then, we have such a map  $f : X \rightarrow \prod_i A$ . Consider its Moore-Postnikov tower

$$\begin{array}{ccccccc} X & \xrightarrow{\sim} & P_{n+1} & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & \prod_i A \\ & & & & \downarrow & & & & \downarrow & & \\ & & & & K(\pi_n F, n+1) & & & & K(\pi_1 F, 2) & & \end{array}$$

where  $F$  is the homotopy fiber of  $f : X \rightarrow \prod_i A$ , and  $P_{k+1} \rightarrow P_k \rightarrow K(\pi_k F, k+1)$  are all fiber sequences. Similar to the proofs of Theorem 2.10 and Lemma 2.11, we can induct up the tower to see  $P_k \in \mathcal{R}(A)$  for  $k \leq n$ . Furthermore, the fiber sequence  $F \rightarrow X \rightarrow \prod_i A$  induces an exact sequence

$$\pi_{n+1}(\prod_i A) \longrightarrow \pi_n F \longrightarrow \pi_n X \xrightarrow{\pi_n(f)} \pi_n(\prod_i A).$$

Note  $\pi_{n+1}(\prod_i A) \cong 0$ , so  $\pi_n F \cong \ker(\pi_n(f))$ ; but  $\pi_n(f)$  is injective, so  $\ker(\pi_n(f)) = 0$ . In particular, this implies  $\pi_n F \cong 0$  and  $K(\pi_n F, n+1) \sim * \in \mathcal{R}(A)$ . Since  $P_{n+1} \rightarrow P_n \rightarrow K(\pi_n F, n+1)$  is a fiber sequence,  $P_{n+1} \in \mathcal{R}(A)$ , which is to say  $X \in \mathcal{R}(A)$ .

For the converse, consider the class  $\mathcal{D}$  of all spaces  $X$  with  $\text{aconn}(X) \leq n+1$  that have a homotopy class  $X \rightarrow \prod_i A$  inducing an injection on  $\pi_n$ . Notice,  $\mathcal{D}$  coincides with the class of spaces  $X$  such that  $\text{aconn}(X) \leq n+1$  and there exists a map  $X^C \rightarrow \prod_i A$  injective on  $\pi_n$ . This is a resolving class by Lemma 2.7 in the case  $K = S^n$ . Since  $A \in \mathcal{D}$  clearly, it follows  $\mathcal{R}(A) \subseteq \mathcal{D}$ , which forces every space in  $\mathcal{R}(A)$  to have the desired property.  $\square$

**Corollary 2.13.** *Let  $X$  and  $A$  be simply connected spaces of anticonnectivity  $n+1$ . If  $A \preceq X$ , then there is a homotopy class*

$$f : X \rightarrow \prod_{i \in \mathcal{I}} A$$

for some index set  $\mathcal{I}$ , with  $\pi_n(f)$  injective.

*Proof.* The proof of Theorem 2.12 does not rely on  $X$  and  $A$  being rational for this implication.  $\square$

*Remark.* The word ‘‘homotopy class’’ in Theorem 2.12 and its corollary can be replaced with the word ‘‘map’’ under the additional hypothesis that  $X$  is a CW complex (more generally: if  $X$  is cofibrant). A homotopy class  $X \rightarrow \prod_i A$  corresponds to a map  $X^C \rightarrow (\prod_i A)^C$ , which we can compose with  $(\prod_i A)^C \xrightarrow{\sim} \prod_i A$  for a map  $X^C \rightarrow \prod_i A$  injective on  $\pi_n$ . If  $X$  is a CW complex, then it is cofibrant, so we may take  $X^C = X$ . Likewise, if  $X$  is a CW complex and there is a map  $f : X \rightarrow \prod_i A$  injective on  $\pi_n$ , certainly there is a homotopy class  $X \rightarrow \prod_i A$  injective on  $\pi_n$ .

In practice, the mysterious index set  $\mathcal{I}$  and assertion of a homotopy class (as opposed to the assertion of a map) in Theorem 2.12 can be a little cumbersome. In the remark above, we saw that we can assert the existence of a map provided  $X$  is a CW complex. Luckily, there are also ways to achieve more concrete conditions on the index set:

**Corollary 2.14.** *Let  $X$  and  $A$  be simply connected rational CW complexes of anticonnectivity  $n+1$ . Then  $A \preceq X$  if and only if either of the two following conditions hold:*

1. The map  $F : X \rightarrow \prod_{[X,A]} A$  defined by

$$\begin{array}{ccc}
 & & f \\
 & \curvearrowright & \\
 X & \xrightarrow{F} & \prod_{[X,A]} A \xrightarrow{pr_f} A \\
 & & \searrow
 \end{array}$$

for  $f \in [X, A]$  has  $\pi_n(F)$  injective;

2. For each nonzero  $\alpha \in \pi_n(X)$ , there is a map  $f : X \rightarrow A$  such that  $\pi_n(f)(\alpha)$  is nonzero.

*Proof.* (1) Certainly if  $\pi_n(F)$  is injective then  $A \preceq X$  by Theorem 2.12.

Conversely, it follows from Theorem 2.12 that  $A \preceq X$  implies the existence of a map  $f : X \rightarrow \prod_{i \in \mathcal{I}} A$  for some index set  $\mathcal{I}$  with  $\pi_n(f)$  injective. But then we must have a factorization

$$\begin{array}{ccc}
 & X & \\
 F \swarrow & & \searrow f \\
 \prod_{[X,A]} A & \overset{g}{\dashrightarrow} & \prod_{i \in \mathcal{I}} A
 \end{array}$$

Setting  $pr_i \circ f = f_i$ ,  $g$  is defined by the property  $pr_i \circ g = pr_{f_i} : \prod_{[X,A]} A \rightarrow A$  for each  $i \in \mathcal{I}$ ; then  $g \circ F = f$  since, for every  $i \in \mathcal{I}$

$$\begin{aligned}
 pr_i \circ (g \circ F) &= pr_{f_i} \circ F \\
 &= f_i.
 \end{aligned}$$

Finally, we see  $\pi_n(F)$  is injective because  $\pi_n(g) \circ \pi_n(F) = \pi_n(f)$  is injective.

(2) Suppose  $A \preceq X$  and let  $0 \neq \alpha \in \pi_n(X)$ . By part (1),  $\pi_n(F)$  is injective, implying  $\pi_n(F)(\alpha) \neq 0$ , i.e.  $F \circ \alpha \neq 0$ . In particular, this means there must be some  $f \in [X, A]$  such that  $pr_f \circ (F \circ \alpha) \neq 0$ . But then  $f \circ \alpha \neq 0$ , i.e.  $\pi_n(f)(\alpha) \neq 0$ .

For the reverse implication, suppose each  $0 \neq \alpha \in \pi_n(X)$  has a map  $f_\alpha : X \rightarrow A$  such that  $\pi_n(f_\alpha)(\alpha) \neq 0$ . Define  $f : X \rightarrow \prod_{\alpha} A$  by  $pr_\alpha \circ f = f_\alpha$  where  $\alpha$  varies over all nonzero elements of  $\pi_n(X)$ . Then if  $\alpha \neq 0$ ,  $\pi_n(f)(\alpha) = f \circ \alpha \neq 0$  since  $f_\alpha \circ \alpha \neq 0$ , whence  $\pi_n(f)$  is injective as it has trivial kernel.  $\square$

By shifting our attention from a map  $X \rightarrow \prod_i A$  to maps  $X \rightarrow A$ , the element-wise characterization of condition (2) from Corollary 2.14 can be easier to handle conceptually than the original statement of Theorem 2.12. Condition (1) from Corollary 2.14 can be thought of as a *universal example* for the characterization

from Theorem 2.12, and its focus on a particular index set lends itself nicely to discussions about *obstructions* to forming  $X$  from  $A$  based on the cardinality of the collection of homotopy classes from  $X$  to  $A$ : if  $[X, A]$  is, in some sense, too small, then we cannot possibly form  $X$  from  $A$ .

**Example.** Let  $X$  and  $A$  be as in Corollary 2.14. If  $X \in \Theta(\{A\})$ , then  $X$  is not formed by  $A$  (i.e.,  $X \notin \mathcal{R}(A)$ ). Recall

$$\Theta(\{A\}) = \{X \mid \mathcal{T}_*(X, A) \sim *\},$$

so, in particular, if  $X \in \Theta(\{A\})$ , then  $[X, A] = *$ . Then it certainly cannot be the case  $X \rightarrow \prod_{[X, A]} A$  is injective on  $\pi_n$ , as this would imply  $* : X \rightarrow A$  induces an injection on  $\pi_n$  which would contradict our assumption  $\text{aconn}(X) = n + 1$ .

In [15], Hess and Parent describe an algebraic translation of the Chachólski-Parent-Stanley criterion to add to its utility. Following suit, we have:

**Theorem 2.15.** *Let  $X$  and  $A$  be rational spaces (of finite type) with anticonnectivity  $n + 1$ . Then  $A \preceq X$  (i.e.,  $X \in \mathcal{R}(A)$ ) if and only if there is a CDGA morphism*

$$\phi : \coprod_{i \in \mathcal{I}} \mathcal{M}(A) \rightarrow \mathcal{M}(X)$$

for some index set  $\mathcal{I}$ , with the linear part of  $\phi$  surjective in degree  $n$ .

*Proof.* Suppose  $X \in \mathcal{R}(A)$ . By Theorem 2.12, we have a homotopy class  $f : X \rightarrow \prod_i A$  such that  $\pi_n(f)$  is injective. This implies we have a morphism  $\phi : \coprod_i \mathcal{M}(A) \rightarrow \mathcal{M}(X)$ . Since  $\pi_n(f)$  is injective, we know  $\bar{\phi}$  is surjective in degree  $n$  as it corresponds to the algebraic dual of  $\pi_n(f)$  (see Theorem 1.4, or Section 17 of [10]).

Now suppose there is a CDGA morphism  $\phi : \coprod_i \mathcal{M}(A) \rightarrow \mathcal{M}(X)$  with  $\bar{\phi}$  surjective in degree  $n$ . Since  $X$  is of finite type, we may assume  $\mathcal{I}$  is finite. So,  $\phi$  corresponds to a homotopy class  $X \rightarrow \prod_i A$  that induces an injection on  $\pi_n$ . This implies  $X \in \mathcal{R}(A)$  by Theorem 2.12.  $\square$

Now, given two minimal Sullivan algebras,  $(\Lambda V, d)$  and  $(\Lambda W, d)$ , we have corresponding rational spaces  $X \sim \langle (\Lambda V, d) \rangle$  and  $Y \sim \langle (\Lambda W, d) \rangle$  (where we recall  $\langle - \rangle$  denotes spatial realization; see Section 1.5). Write

$$(\Lambda V, d) \preceq (\Lambda W, d) \text{ if and only if } X \preceq Y,$$

and write

$$(\Lambda V, d) \sim_r (\Lambda W, d) \text{ if and only if } X \sim_r Y.$$

Section 3.2 deals extensively with  $\preceq$  on a collection CDGA's specially chosen to model (systems of)  $m$ -forms. In the situations that we will encounter there (and elsewhere), it often happens to be the case that

$$\mathcal{M}(X) \cong (\Lambda(V \oplus \mathbb{Q} \cdot y), d),$$

where  $V$  is concentrated in degree  $k$  (for  $k \geq 2$  even) and  $|y| = 2k - 1$ . In this case, Theorem 2.15 has a simpler incarnation; we record this result as a lemma in light of its later applications.

**Lemma 2.16.** *Let  $k \geq 2$  be even,  $X$  be as above, and suppose  $A$  is a rational space of anitconnectivity  $2k$ . Then  $A \preceq X$  if and only if there is a CDGA morphism*

$$\phi : \mathcal{M}(A) \rightarrow \mathcal{M}(X)$$

with  $\phi$  nontrivial in degree  $2k - 1$ .

*Proof.* We first note that, in this case,  $\bar{\phi} = \phi$  in degrees  $k$  and  $2k - 1$ ; there are no elements of word length 2 or greater in degrees  $k$  or  $2k - 1$  (for degree reasons), so the linear part of  $\phi$  coincides with  $\bar{\phi}$  in those degrees.

Now, if there is a CDGA morphism  $\phi : \mathcal{M}(A) \rightarrow \mathcal{M}(X)$  that is nontrivial in degree  $2k - 1$ , then it is necessarily onto in degree  $2k - 1$ , considering that  $\mathcal{M}(X)$  is simply  $\mathbb{Q} \cdot y$  in that degree. Therefore, we have a CDGA morphism  $\mathcal{M}(A) \rightarrow \mathcal{M}(X)$  with surjective linear part in degree  $2k - 1$ , thus  $A \preceq X$  by Theorem 2.15.

Supposing  $X \preceq A$ , Theorem 2.15 implies the existence of a CDGA morphism

$$\phi : \coprod_{i \in \mathcal{I}} \mathcal{M}(A) \rightarrow \mathcal{M}(X),$$

with linear part surjective in degree  $2k - 1$ . In other words,  $\phi$  is nontrivial in degree  $2k - 1$ , whence one of the compositions

$$\mathcal{M}(A) \hookrightarrow \coprod_{\mathcal{I}} \mathcal{M}(A) \rightarrow \mathcal{M}(X)$$

must be nontrivial in degree  $2k - 1$ . This establishes the result.  $\square$

## 2.5 Characterizing Some Resolvable Equivalences

We end this chapter by utilizing Theorem 2.15 to discover when a rational space  $X$  is resolvably equivalent to either a sphere, complex projective space, quaternionic

projective space, or Eilenberg-MacLane Space. We determine when a rational space is resolvably equivalent to an Eilenberg-MacLane space first.

**Proposition 2.17.** *Let  $X$  be a rational space with  $\text{aconn}(X) = n+1$  and  $\mathcal{M}(X) = (\Lambda V, d)$ . Then  $X \sim_r K(\mathbb{Q}, n)$  if and only if  $d(V^n) = 0$ .*

*Proof.* Lemma 2.11 implies  $K(\mathbb{Q}, n) \in \mathcal{R}(X)$ , i.e.  $X \preceq K(\mathbb{Q}, n)$ , so we are left to show  $K(\mathbb{Q}, n) \preceq X$ . The minimal Sullivan model of  $K(\mathbb{Q}, n)$  is  $(\Lambda(\omega), 0)$  where  $|\omega| = n$ . Pick a basis  $\{x_1, \dots, x_k\}$  of  $V^n$  and define  $\phi : \prod_{i=1}^k (\Lambda(\omega_i), 0) \rightarrow (\Lambda V, d)$  by  $\omega_i \mapsto x_i$ . The condition that  $d(V^n) = 0$  ensures this is a CDGA morphism, and Theorem 2.15 then grants  $X \in \mathcal{R}(K(\mathbb{Q}, n))$ , i.e.  $K(\mathbb{Q}, n) \preceq X$ .  $\square$

Rationally,  $K(\mathbb{Q}, n) \sim S^n$  when  $n$  is odd, so Proposition 2.17 also characterizes spaces that are resolvably equivalent to odd dimensional (rational) spheres. Furthermore, Proposition 2.17 implies if  $X$  is a rational  $H$ -space with  $\text{aconn}(X) = n+1$ , then  $X \sim_r K(\mathbb{Q}, n)$ .

The rest of the spaces on our list are characterized as corollaries of our next theorem, but it seems pertinent to set up some notation and a lemma before tackling the proof. For  $k \geq 2$  even, let  $A$  be a space with

$$\mathcal{M}(A) = (\Lambda(x, y), dy = x^{n+1}),$$

where  $|x| = k$ ,  $|y| = k(n+1) - 1$ , and  $n \geq 1$ . For example, in the case  $n = 1$  we can take  $A$  to be an even dimensional sphere  $S^k$ . In the case  $k = 2$ , we can take  $A = \mathbb{C}P^n$ , and if  $k = 4$  set  $A = \mathbb{H}P^n$ . Generally, the space

$$(\Omega S^{k+1})_{nk},$$

i.e. the  $nk$ -th Postnikov section of  $\Omega S^{k+1}$ , will have the desired model.

Now, suppose  $X$  is a rational space with

$$\mathcal{M}(X) = (\Lambda(V \oplus W \oplus Z), d),$$

where:

- $V$  is concentrated in degree  $k$ ;
- $W$  is nontrivial in at most degrees  $q$  for  $2 \leq q \leq k-1$  and  $k+1 \leq q \leq k(n+1) - 2$ ; and
- $Z$  is concentrated in degree  $k(n+1) - 1$ .



Let  $X_0$  be a rational space with  $\mathcal{M}(X_0) = (\Lambda(V \oplus Z), D)$ , where  $D$  is induced by  $d$  after quotienting by  $W$ . The projection  $V \oplus W \oplus Z \rightarrow V \oplus Z$  induces a CDGA morphism  $p : \mathcal{M}(X) \rightarrow \mathcal{M}(X_0)$  defined as the identity in degrees  $k$  and  $k(n+1) - 1$  while being 0 in all other positive degrees. Of course,  $p$  is surjective in degree  $k(n+1) - 1$ , so by Theorem 2.15, we have  $X \preceq X_0$ . Thus, if  $X_0 \preceq A$  then certainly  $X \preceq A$ .

This is a useful simplification of the situation at hand: we can effectively ignore the part of  $\mathcal{M}(X)$  that is not in degree  $k$  or degree  $k(n+1) - 1$ . The next lemma shows that, in fact,  $X \preceq A$  implies  $X_0 \preceq A$  as well, and offers a characterization of precisely when such a relationship holds.

**Lemma 2.18.** *Let  $A$ ,  $X$ , and  $X_0$  be as above. The following are equivalent:*

1.  $X \preceq A$ ;
2.  $X_0 \preceq A$ ;
3.  $D \neq 0$ .

*Proof.* (1)  $\Rightarrow$  (2). Suppose  $X \preceq A$ . By Lemma 2.16, there is a CDGA morphism  $\phi : \mathcal{M}(X) \rightarrow \mathcal{M}(A)$  with  $\phi_{k(n+1)-1} \neq 0$ . Given such a  $\phi : \mathcal{M}(X) \rightarrow \mathcal{M}(A)$ , we have  $\phi(W) = 0$  for degree reasons. Thus,  $\phi$  must factor as

$$\begin{array}{ccc} \mathcal{M}(X) & \xrightarrow{\phi} & \mathcal{M}(A) \\ & \searrow p & \nearrow \psi \\ & & \mathcal{M}(X_0) \end{array}$$

where we note  $\psi_{k(n+1)-1} \neq 0$ . Therefore, it must be the case  $X_0 \preceq A$ , again by Lemma 2.16.

(2)  $\Rightarrow$  (3). Suppose  $X_0 \preceq A$ . If we further suppose  $D = 0$ , then  $X_0 \sim_r K(\mathbb{Q}, k(n+1) - 1)$  by Proposition 2.17. However,  $K(\mathbb{Q}, k(n+1) - 1)$  does not form  $A$ : there is no well-defined CDGA morphism  $(\Lambda(w), 0) \rightarrow (\Lambda(x, y), dy = x^{n+1})$  that is nontrivial in degree  $k(n+1) - 1$  since supposing otherwise would imply  $x^{n+1} = 0$ . We must then conclude  $D \neq 0$ .

(3)  $\Rightarrow$  (1). Lastly, suppose  $D \neq 0$ . From the remarks above, we know  $X \preceq X_0$ . It suffices, then, to prove  $X_0 \preceq A$ . Let  $Z$  have basis  $\{z_j\}_{j=1}^m$ . For a given basis  $\{v_i\}_{i=1}^r$  of  $V$ , we have

$$D(z_j) = \sum_{\|\delta\|=n+1} \alpha_{j,\delta} (v_1, \dots, v_r)^\delta,$$

for  $1 \leq j \leq m$ . Now, there must exist some  $z_j \in \{z_j\}_{j=1}^m$  such that not all of the coefficients  $\alpha_{j,\delta} \in \mathbb{Q}$  of  $D(z_j)$  are equal to 0 (otherwise  $D = 0$ ). Without loss of generality, suppose  $z_1$  has this property. Since  $D(z_1)$  is not the 0 polynomial [5], we can find  $(t_1, \dots, t_r) \in \mathbb{Q}^r$  such that

$$\sum_{\|\delta\|=n+1} \alpha_{1,\delta}(t_1, \dots, t_r)^\delta \neq 0.$$

For each  $1 \leq j \leq m$ , define

$$\alpha_j = \sum_{\|\delta\|=n+1} \alpha_{j,\delta}(t_1, \dots, t_r)^\delta.$$

Then we are assured  $\alpha_1 \neq 0$ .

Now, define  $\phi : \mathcal{M}(X_0) \rightarrow \mathcal{M}(A)$  by  $v_i \mapsto t_i x$  and  $z_j \mapsto \alpha_j y$  for  $1 \leq i \leq r$  and  $1 \leq j \leq m$  respectively. Notice  $\phi \circ D(v_i) = 0 = D \circ \phi(v_i)$  for every  $1 \leq i \leq r$ . Furthermore, observe:

$$\begin{aligned} \phi((v_1, \dots, v_r)^\delta) &= \phi\left(\prod_{i=1}^r v_i^{\delta_i}\right) \\ &= \prod_{i=1}^r \phi(v_i)^{\delta_i} \\ &= \prod_{i=1}^r t_i^{\delta_i} x^{\delta_i} \\ &= \left(\prod_{i=1}^r t_i^{\delta_i}\right) x^{n+1}. \end{aligned}$$

Thus, for  $1 \leq j \leq m$ , we also find:

$$\begin{aligned}
\phi \circ D(z_j) &= \phi \left( \sum_{\|\delta\|=n+1} \alpha_{j,\delta} (v_1, \dots, v_r)^\delta \right) \\
&= \sum_{\|\delta\|=n+1} \alpha_{j,\delta} \phi((v_1, \dots, v_r)^\delta) \\
&= \sum_{\|\delta\|=n+1} \alpha_{j,\delta} \left( \prod_{i=1}^r t_i^{\delta_i} \right) x^{n+1} \\
&= \sum_{\|\delta\|=n+1} \alpha_{j,\delta} (t_1, \dots, t_r)^\delta x^{n+1} \\
&= \alpha_j x^{n+1} \\
&= D \circ \phi(z_j).
\end{aligned}$$

This goes to show  $\phi$  is a CDGA morphism with  $\phi_{k(n+1)-1} \neq 0$  since  $\alpha_1 \neq 0$ . Lemma 2.16 implies  $X_0 \preccurlyeq A$ , thus we conclude  $X \preccurlyeq A$  as desired.  $\square$

We are now in a position to prove the following theorem, characterizing when a rational space  $X$  is resolvably equivalent to a space  $A$  (for  $A$  as above). It is worth noting that the characterization given by Theorem 2.19 is a bit obtuse. Roughly speaking, what this theorem tells us is, for  $X$  and  $A$  as in Theorem 2.19,  $X \sim_r A$  if and only if, after modding out by  $W$ , the reduced differential is nonzero, and  $Z$  has a basis  $\{z_j\}_{j=1}^m$  that, modulo nontrivial products, has  $\alpha_j d(z_j) \equiv v_j^{n+1}$  for  $1 \leq j \leq m$  where  $v_j \in \ker(d_k)$ .

**Theorem 2.19.** *For  $k \geq 2$  even, let  $A$  be a space with*

$$\mathcal{M}(A) = (\Lambda(x, y), dy = x^{n+1})$$

where  $|x| = k$ ,  $|y| = k(n+1) - 1$ , and  $n \geq 1$ . Let  $X$  be a rational space with

$$\mathcal{M}(X) = (\Lambda(V \oplus W \oplus Z), d),$$

and  $X_0$  be the rational space with  $\mathcal{M}(X_0) = (\Lambda(V \oplus Z), D)$  where:

- $V$  is concentrated in degree  $k$ ;
- $W$  is nontrivial in at most degrees  $q$  for  $2 \leq q \leq k-1$  and  $k+1 \leq q \leq k(n+1) - 2$ ;
- $Z$  is concentrated in degree  $k(n+1) - 1$ ; and

- $D$  is induced from  $d$  after quotienting by  $W$ .

Then the following equivalences hold:

1.  $X \preceq A$  if and only if  $D \neq 0$ .
2.  $A \preceq X$  if and only if  $Z$  has a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_k)$ ,  $0 \neq \alpha_j \in \mathbb{Q}$ , and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  such that  $v_j^{n+1} - \alpha_j d(z_j) = d(w_j)$  for all  $1 \leq j \leq m$ .

*Proof.* (1) This is immediate from Lemma 2.18.

(2) Suppose  $A \preceq X$  and let

$$\phi : \prod_{i \in \mathcal{I}} (\Lambda(x_i, y_i), dy_i = x_i^{n+1}) \rightarrow \mathcal{M}(X)$$

be a CDGA morphism with surjective linear part in degree  $k(n+1) - 1$ . In this case, for a basis  $\{z_j\}_{j=1}^m$  of  $Z$ , there exist  $y_j \in \{y_i\}_{i \in \mathcal{I}}$  such that  $\phi(y_j) = \alpha_j z_j + w_j$  where  $0 \neq \alpha_j \in \mathbb{Q}$  and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  for  $1 \leq j \leq m$ . Then  $d(\phi(y_j)) = \alpha_j d(z_j) + d(w_j)$ . Since  $d \circ \phi(y_j) = \phi \circ d(y_j) = \phi(x_j)^{n+1}$ , we have

$$\begin{aligned} \phi(x_j)^{n+1} &= \alpha_j d(z_j) + d(w_j) \\ \phi(x_j)^{n+1} - \alpha_j d(z_j) &= d(w_j). \end{aligned}$$

Letting  $v_j = \phi(x_j)$  for each  $1 \leq j \leq m$ , we find  $v_j^{n+1} - \alpha_j d(z_j) = d(w_j)$  for  $0 \neq \alpha_j \in \mathbb{Q}$ ,  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$ , and  $v_j \in \ker(d_k)$  (since  $d(v_j) = d(\phi(x_j)) = \phi(d(x_j)) = 0$ ).

Now suppose  $Z$  has such a basis  $\{z_j\}_{j=1}^m$  with  $v_j^{n+1} - \alpha_j d(z_j) = d(w_j)$  where  $v_j \in \ker(d_k)$ ,  $0 \neq \alpha_j \in \mathbb{Q}$ , and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$ . Define

$$\phi : \prod_{j=1}^m (\Lambda(x_j, y_j), dy_j = x_j^{n+1}) \rightarrow \mathcal{M}(X)$$

by  $x_j \mapsto v_j$  and  $y_j \mapsto \alpha_j z_j + w_j$ . Then, for all  $1 \leq j \leq m$ ,  $d \circ \phi(x_j) = d(v_j) = 0 = \phi \circ d(x_j)$ , and

$$\begin{aligned} d \circ \phi(y_j) &= d(\alpha_j z_j + w_j) \\ &= \alpha_j d(z_j) + d(w_j) \\ &= v_j^{n+1} \\ &= \phi \circ d(y_j). \end{aligned}$$

Thus,  $\phi$  is a CDGA morphism. Moreover,  $\bar{\phi} : y_j \mapsto \alpha_j z_j$  with  $\alpha_j \neq 0$  for each  $1 \leq j \leq m$ , so  $\phi$  has surjective linear part in degree  $k(n+1) - 1$ . Hence  $A \preceq X$ .  $\square$

As promised, this theorem has the following corollary that determines when a rational space  $X$  is resolvably equivalent to either an even dimensional sphere, complex projective space, or quaternionic projective space.

**Corollary 2.20.** *Let  $X$  be as in Theorem 2.19 (with  $k \geq 2$  even). We have:*

1.  $X \sim_r S^k$  if and only if  $D \neq 0$  and  $Z$  as a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_k)$ ,  $0 \neq \alpha_j \in \mathbb{Q}$ , and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  such that  $v_j^2 - \alpha_j d(z_j) = d(w_j)$  for all  $1 \leq j \leq m$ .
2.  $X \sim_r \mathbb{C}P^n$  if and only if  $D \neq 0$  and  $Z$  as a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_2)$ ,  $0 \neq \alpha_j \in \mathbb{Q}$ , and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  such that  $v_j^{n+1} - \alpha_j d(z_j) = d(w_j)$  for all  $1 \leq j \leq m$ .
3.  $X \sim_r \mathbb{H}P^n$  if and only if  $D \neq 0$  and  $Z$  as a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_4)$ ,  $0 \neq \alpha_j \in \mathbb{Q}$ , and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  such that  $v_j^{n+1} - \alpha_j d(z_j) = d(w_j)$  for all  $1 \leq j \leq m$ .

*Proof.* (1) This is Theorem 2.19 in the case  $A = S^k$ , i.e. restricting to  $n = 1$ .

(2) This is Theorem 2.19 when we consider  $A = \mathbb{C}P^n$ , i.e. specifying  $k = 2$ .

(3) This is Theorem 2.19 if we let  $A = \mathbb{H}P^n$ , i.e. asserting  $k = 4$ .  $\square$

If  $X$  satisfies the additional hypothesis that  $\mathcal{M}(X)$  has  $d = d_{(2)}$  (i.e.,  $d$  increases word length by exactly 1), part (1) of Corollary 2.20 has a moderately less convoluted statement:

**Corollary 2.21.** *Let  $X$  be as in Theorem 2.19 (with  $k \geq 2$  even) and suppose further  $\mathcal{M}(X)$  has  $d = d_{(2)}$ . Then the following equivalences hold:*

1.  $X \preceq S^k$  if and only if  $D \neq 0$ .
2.  $S^k \preceq X$  if and only if  $Z$  as a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_k)$  and  $0 \neq \alpha_j \in \mathbb{Q}$  such that  $v_j^2 = \alpha_j d(z_j)$  for all  $1 \leq j \leq m$ .

*In other words:*  $X \sim_r S^k$  if and only if  $D \neq 0$  and  $Z$  as a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_k)$  and  $0 \neq \alpha_j \in \mathbb{Q}$  such that  $v_j^2 = \alpha_j d(z_j)$  for all  $1 \leq j \leq m$ .

*Proof.* (1)  $X \preceq S^k$  if and only if  $D \neq 0$  by the same argument presented in Theorem 2.19 and Corollary 2.20 (1).

(2) Suppose  $S^k \preceq X$ . By Theorem 2.19,  $Z$  has a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_k)$ ,  $0 \neq \alpha_j \in \mathbb{Q}$ , and  $w_j \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  such that  $v_j^2 - \alpha_j d(z_j) = d(w_j)$  for all  $1 \leq j \leq m$ . Thus, for each  $1 \leq j \leq m$ ,

$$v_j^2 - \alpha_j d(z_j) = d(w_j) \in \Lambda^{\geq 3}(V \oplus W \oplus Z).$$

However, if  $d = d_{(2)}$ , then  $d(z_j) \in \Lambda^2(V \oplus W \oplus Z)$ . Hence,

$$v_j^2 - \alpha_j d(z_j) \in \Lambda^2(V \oplus W \oplus Z) \cap \Lambda^{\geq 3}(V \oplus W \oplus Z) = \{0\},$$

i.e.,  $v_j^2 = \alpha_j d(z_j)$ .

If we assume  $Z$  has a basis  $\{z_j\}_{j=1}^m$  with  $v_j \in \ker(d_k)$  and  $0 \neq \alpha_j \in \mathbb{Q}$  such that  $v_j^2 = \alpha_j d(z_j)$  for all  $1 \leq j \leq m$ , then setting  $w_j = 0 \in \Lambda^{\geq 2}(V \oplus W \oplus Z)$  for every  $1 \leq j \leq m$  yields  $v_j^2 - \alpha_j d(z_j) = d(w_j)$ . This implies  $S^k \preceq X$  by Theorem 2.19.

Therefore,  $X \sim_r S^k$  if and only if both of the conditions from (1) and (2) hold, as claimed.  $\square$

# Chapter 3

## Investigating the Resolvability Relation

This section is dedicated to using the results we've developed (specifically, the algebraic criterion of Theorem 2.15) to further elucidate some of the structure inherent to the resolvability relation. In particular, we uncover the same quotient of the Witt group discovered within the cellular lattice of rational spaces in [15], along with results concerning higher degree forms. In the final section, we bring things back to the realm of topology by studying some of the spaces corresponding to the algebraic undertakings in which we will indulge.

### 3.1 Some Basic Properties

We call the collection of all spaces resolvably equivalent to a given space  $X$  the **resolvable equivalence class** of  $X$ . We make it a habit to denote a resolvable equivalence class with a particular representative. We immediately have our first observation:

**Proposition 3.1.** *Let  $\mathcal{S}$  be the collection of resolvable equivalence classes of spaces. Then  $(\mathcal{S}, \preceq)$  is a complete lower semilattice.*

*Proof.* The fact that  $\preceq$  is a partial order on  $\mathcal{S}$  is clear. To say the partial order  $(\mathcal{S}, \preceq)$  is a complete lower semilattice is to say that any subset  $S \subseteq \mathcal{S}$  has a greatest lower bound with respect to  $\preceq$ . In other words, we must find an  $A \in \mathcal{S}$  such that:

- $A \preceq X$  for every  $X \in S$ , and
- if  $Z \preceq X$  for every  $X \in S$ , then  $Z \preceq A$ .

Observe  $A = \prod_{X \in \mathcal{S}} X$  satisfies these conditions: the first following from the closure of  $\mathcal{R}(\prod_{X \in \mathcal{S}} X)$  under retracts, and the second following from the closure of  $\mathcal{R}(Z)$  under products.  $\square$

*Remark.* It is a nontrivial result that the cellular lattice is complete [4]. At the time of this writing, the question of whether or not  $(\mathcal{S}, \preccurlyeq)$  is a complete lattice remains open.

In the case that  $X$  forms  $Y$  but  $Y$  does not form  $X$ , we say this forming relation is **strict**, and write  $X \overset{s}{\preccurlyeq} Y$  (there is analogous notation for a strict building relation, namely  $\overset{s}{\ll}$ ). Theorem 2.10 immediately yields an infinite sequence of strict forming relations:

$$\dots \overset{s}{\preccurlyeq} K(\mathbb{Q}, n) \overset{s}{\preccurlyeq} K(\mathbb{Q}, n-1) \overset{s}{\preccurlyeq} \dots \overset{s}{\preccurlyeq} *.$$

So, in particular, we see  $(\mathcal{S}, \preccurlyeq)$  is at least countably infinite. Moreover, Theorem 2.10 implies, for any space  $X$  with  $\text{aconn}(X) = n + 1$ :

$$K(\mathbb{Q}, n+1) \overset{s}{\preccurlyeq} X \overset{s}{\preccurlyeq} K(\mathbb{Q}, n-1).$$

Also, if  $\text{aconn}(X) = n + 1$  and  $\text{aconn}(Y) = n - 2$ , then we have:

$$X \overset{s}{\preccurlyeq} K(\mathbb{Q}, n-1) \overset{s}{\preccurlyeq} Y.$$

In [11], Félix and Parent show that the rational cellular lattice satisfies a much stronger density condition:

**Theorem.** (*Rational Density [11]*) *If  $X$  and  $Y$  are cohomologically finite, simply connected rational spaces such that  $X \overset{s}{\ll} Y$ , then there exist infinitely many non-comparable cohomologically finite, simply connected rational spaces  $Z_n$  (for  $n \geq 1$ ) such that  $X \overset{s}{\ll} Z_n \overset{s}{\ll} Y$ .*

In a scathing act of defiance,  $(\mathcal{S}, \preccurlyeq)$  violates this notion of density outright:

**Proposition 3.2.**  $S^2 \overset{s}{\preccurlyeq} K(\mathbb{Q}, 3)$ , but there does not exist a (simply connected) rational space  $Z$  such that  $S^2 \overset{s}{\preccurlyeq} Z \overset{s}{\preccurlyeq} K(\mathbb{Q}, 3)$ .

*Proof.* Suppose we have a space  $Z$  such that  $S^2 \preccurlyeq Z \preccurlyeq K(\mathbb{Q}, 3)$ . Then, it must be the case that  $Z$  is simply connected and 4 anticonnected; therefore  $\mathcal{M}(Z) = (\Lambda V, d)$  with  $V$  nontrivial in at most degrees 2 and 3. Consider the differential,  $d$ .



If  $d = 0$ , then  $Z \sim_r K(\mathbb{Q}, 3)$  by Proposition 2.17. If  $d \neq 0$ , then  $Z \sim_r S^2$  by part (1) of Corollary 2.21. In either case, one of the forming relations is not strict.  $\square$

The preceding result is simple, but its implications about the relationship between the cellular lattice and  $(\mathcal{S}, \preceq)$  are important. It is tempting to reason that the theory surrounding  $(\mathcal{S}, \preceq)$  should be dual to that surrounding the cellular lattice, but Proposition 3.2 implies they are very much different structures! This makes the next section all the more interesting, since the same copy of the Witt group can be found in both. The extent to which the theories *are* strictly dual to one another is currently unclear.

## 3.2 Polynomial Forms in the Resolvability Relation

Using our algebraic criterion, we can identify several embeddings of purely algebraic structures into  $(\mathcal{S}, \preceq)$ . As one might expect, this is done via an intermediate journey through the category of CDGA's. We start in the general case by considering systems of homogeneous polynomial forms, before moving on to specific examples, the most important of which being the case of quadratic forms.

### From Systems of Rational Polynomial Forms to Spaces

Recall (from Section 1.6), two  $r$ -fold systems of  $n$ -dimensional  $m$ -forms  $\bar{q}$  and  $\bar{p}$  are said to be similar, written  $\bar{p} \sim \bar{q}$ , provided there exist  $C \in GL(n, \mathbb{Q})$  and  $D \in GL(r, \mathbb{Q})$  such that  $\bar{p}(C^T x) = D^T \bar{q}(x)$  for all  $x = (x_1, \dots, x_n)^T$ . We have denoted the collection of all anisotropic  $r$ -fold systems of  $n$ -dimensional  $m$ -forms, for  $r, n, m \in \mathbb{N}$ , by  $\mathbb{W}$ . Furthermore, the collection of similarity class of elements of  $\mathbb{W}$  is denoted  $\overline{\mathbb{W}}$ , and for a fixed  $r \in \mathbb{N}$ , the subset of similarity classes of  $r$ -fold systems is called  $\overline{\mathbb{W}}^r \subset \overline{\mathbb{W}}$ . The goal of this section is to prove:

**Theorem 3.3.** *For every even  $k \geq 2$ , there is a well-defined function*

$$e_k : \overline{\mathbb{W}} \rightarrow \mathcal{S}$$

*that is injective when restricted to  $\overline{\mathbb{W}}^1$ .*

We start by defining the types of models we will end up identifying with the similarity classes of systems of forms:

**Definition.** For  $k \geq 2$  even,  $n \in \mathbb{N}$ , and  $\bar{q} = (q_1, \dots, q_r)^T$  an anisotropic  $r$ -fold system of  $n$ -dimensional  $m$ -forms, define

$$L^k(n, \bar{q}) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_r), dy_j = q_j(x))$$

where  $|x_i| = k$  for all  $1 \leq i \leq n$ , and  $|y_j| = mk - 1$  for  $1 \leq j \leq r$ .

We proceed by disposing of a technical lemma regarding the collection of  $L^k(n, \bar{q})$ .

**Lemma 3.4.** *Suppose  $\bar{p}$  and  $\bar{q}$  are anisotropic  $r$ -fold systems of  $n$ -dimensional  $m$ -forms. Let  $\phi : L^k(n, \bar{p}) \rightarrow L^k(n, \bar{q})$  be a degree-preserving function where*

$$\begin{aligned} L^k(n, \bar{p}) &= (\Lambda(x_1, \dots, x_n, y_1, \dots, y_r), dy_j = p_j(x)) \\ L^k(n, \bar{q}) &= (\Lambda(x_1, \dots, x_n, y'_1, \dots, y'_r), dy'_j = q_j(x)) \end{aligned}$$

with  $|x_i| = k$  for all  $i$ , and  $|y_j| = mk - 1 = |y'_j|$  for all  $j$ .

Denote the matrix corresponding to the linear map  $\phi_k$  as  $C_{n \times n}$  (i.e.,  $\phi_k : v \mapsto Cv$ ), and the matrix corresponding to  $\phi_{mk-1}$  as  $D_{r \times r}$  (i.e.,  $\phi_{mk-1} : v \mapsto Dv$ ). Then:

1.  $\phi$  is a CDGA morphism if and only if  $\bar{p}(C^T x) = D^T \bar{q}(x)$ ;
2. In the case  $r = 1$ :  $L^k(n, p) \cong L^k(n, q)$  if and only if there is a nontrivial CDGA morphism  $L^k(n, p) \rightarrow L^k(n, q)$ .

*Remark.* Before we start the proof, it seems prudent to attempt to elucidate the potentially peculiar appearance of  $C^T$  and  $D^T$  in part (1). Writing  $C_i$  for the  $i$ -th column of  $C$ , we notice:

$$\begin{aligned} C^T x &= \begin{bmatrix} C_1^T x \\ \vdots \\ C_n^T x \end{bmatrix} \\ &= \begin{bmatrix} \phi(x_1) \\ \vdots \\ \phi(x_n) \end{bmatrix}. \end{aligned}$$

It is precisely this last vector at which we will evaluate  $p_j$  ( $1 \leq j \leq r$ ) in the calculations to follow.

Similarly, writing  $D_j$  for the  $j$ -th column of  $D$ , we have (for  $1 \leq j \leq r$ )

$$\phi(y_j) = D_j^T \begin{bmatrix} y'_1 \\ \vdots \\ y'_r \end{bmatrix},$$

which, again, will be vectors of particular interest in the forthcoming proof.

*Proof of Lemma 3.4.* (1) Write  $D_j$  for the  $j$ -th column of  $D$  as in the remark above. To say  $\bar{p}(C^T x) = D^T \bar{q}(x)$  is to say:

$$\begin{aligned} \begin{bmatrix} p_1(C^T x) \\ \vdots \\ p_r(C^T x) \end{bmatrix} &= D^T \begin{bmatrix} q_1(x) \\ \vdots \\ q_r(x) \end{bmatrix} \\ &= \begin{bmatrix} D_1^T \bar{q}(x) \\ \vdots \\ D_r^T \bar{q}(x) \end{bmatrix} \end{aligned}$$

In other words,  $\bar{p}(C^T x) = D^T \bar{q}(x)$  means  $p_j(C^T x) = D_j^T \bar{q}(x)$  for each  $1 \leq j \leq r$ .

Now, let  $D = (a_{lj})$  and consider, for each  $1 \leq j \leq r$ ,  $d \circ \phi(y_j)$ :

$$\begin{aligned} d \circ \phi(y_j) &= d \left( D_j^T \begin{bmatrix} y'_1 \\ \vdots \\ y'_r \end{bmatrix} \right) \\ &= d \left( \sum_{l=1}^r a_{lj} y'_l \right) \\ &= \sum_{l=1}^r a_{lj} d(y'_l) \\ &= \sum_{l=1}^r a_{lj} q_l(x) \\ &= D_j^T \begin{bmatrix} q_1(x) \\ \vdots \\ q_r(x) \end{bmatrix} \\ &= D_j^T \bar{q}(x). \end{aligned}$$

Next, let  $j$  be in  $\{1, \dots, r\}$ , and write  $p_j(x) = \sum_{\|\delta\|=m} \alpha_\delta(x_1, \dots, x_n)^\delta$ . We investigate  $\phi \circ d(y_j)$ :

$$\begin{aligned}
\phi \circ d(y_j) &= \phi(p_j(x)) \\
&= \phi \left( \sum_{\|\delta\|=m} \alpha_\delta(x_1, \dots, x_n)^\delta \right) \\
&= \sum_{\|\delta\|=m} \alpha_\delta \phi(x_1, \dots, x_n)^\delta \\
&= \sum_{\|\delta\|=m} \alpha_\delta (\phi(x_1), \dots, \phi(x_n))^\delta \\
&= p_j(C^T x).
\end{aligned}$$

What we have found is, for a degree-preserving function  $\phi : L^k(n, \bar{p}) \rightarrow L^k(n, \bar{q})$ ,

$$\phi \circ d(y_j) = p_j(C^T x) \text{ and } D_j^T \bar{q}(x) = d \circ \phi(y_j)$$

for every  $1 \leq j \leq r$ . If  $\phi$  is a CDGA morphism, we must then have  $\bar{p}(C^T x) = D^T \bar{q}(x)$ , and if  $\bar{p}(C^T x) = D^T \bar{q}(x)$  then  $\phi \circ d = d \circ \phi$  (since  $\phi_k \circ d = 0 = d \circ \phi_k$  for degree reasons).

(2) Suppose  $r = 1$ , with  $p$  and  $q$  both anisotropic  $n$ -dimensional  $m$ -forms. In light of Lemma 2.16, we know  $L^k(n, p) \preceq L^k(n, q)$  if and only if there is a CDGA morphism  $\phi : L^k(n, p) \rightarrow L^k(n, q)$  that is nontrivial in degree  $2k - 1$ . In particular, if  $L^k(n, p) \preceq L^k(n, q)$ , then there is a nontrivial CDGA morphism  $L^k(n, p) \rightarrow L^k(n, q)$ .

Now suppose there exists a nontrivial CDGA morphism  $\phi : L^k(n, p) \rightarrow L^k(n, q)$ . In this case our  $r \times r$  matrix  $D$  is simply a scalar  $\alpha \in \mathbb{Q}$ . To say  $\phi$  is nontrivial is to say either  $\alpha \neq 0$  or  $C \neq 0$ . If  $\alpha \neq 0$ , then  $\phi_{mk-1} \neq 0$  and  $L^k(n, p) \preceq L^k(n, q)$  by Lemma 2.16. So, assume  $C \neq 0$ . By part (1), we must have  $p(C^T x) = \alpha q(x)$ . Then  $\alpha$  cannot be 0, otherwise  $p$  would be isotropic.  $\square$

We can now commence with the proof of Theorem 3.3 forthwith:

*Proof of Theorem 3.3.* Fix  $k \geq 2$  even. We claim defining  $e_k : \overline{\mathbb{W}} \rightarrow \mathcal{S}$  by

$$[\bar{q}] \mapsto L^k(n, \bar{q}) \mapsto \langle L^k(n, \bar{q}) \rangle$$

where  $[\bar{q}]$  denotes the similarity class of  $\bar{q}$  and  $n$  is the dimension of the components of  $\bar{q}$  satisfies the conditions stated in the theorem.

We first verify  $e_k$  is well-defined. Let  $\bar{p}$  and  $\bar{q}$  be similar anisotropic  $r$ -fold

systems of  $n$ -dimensional  $m$ -forms. Then, by definition, there exist  $C \in GL(n, \mathbb{Q})$  and  $D \in GL(r, \mathbb{Q})$  such that  $\bar{p}(C^T x) = D^T \bar{q}(x)$ . By part (1) of Lemma 3.4,

$$\phi : L^k(n, \bar{p}) \rightarrow L^k(n, \bar{q})$$

defined by  $\phi_k : v \mapsto Cv$  and  $\phi_{mk-1} : v \mapsto Dv$  is a CDGA morphism. Furthermore, since  $D \in GL(r, \mathbb{Q})$ ,  $\phi_{mk-1}$  is surjective, thus  $L^k(n, \bar{p}) \preceq L^k(n, \bar{q})$ . Similarly, since  $(D^{-1})^T \bar{p}(x) = \bar{q}((C^{-1})^T x)$ , part (1) of Lemma 3.4 implies

$$\psi : L^k(n, \bar{q}) \rightarrow L^k(n, \bar{p})$$

defined by  $\psi_k : v \mapsto C^{-1}v$  and  $\psi_{mk-1} : v \mapsto D^{-1}v$  is a CDGA morphism with  $\psi_{mk-1}$  surjective; hence,  $L^k(n, \bar{q}) \preceq L^k(n, \bar{p})$ . This goes to show:  $\bar{p} \sim \bar{q}$  implies  $L^k(n, \bar{p}) \sim_r L^k(n, \bar{q})$ , thus the assignment  $[\bar{q}] \mapsto L^k(n, \bar{q})$  is a well-defined, implying  $e_k : \overline{\mathbb{W}} \rightarrow \mathcal{S}$  is well-defined.

Now consider the case of  $r = 1$ , where we wish to show  $e_k$  is also injective. Let  $p$  and  $q$  be two anisotropic  $n$ -dimensional  $m$ -forms. There is a unique empty  $m$ -form of dimension 0, so assume  $n \geq 1$  for the remainder of the proof. We need to show  $L^k(n, p) \sim_r L^k(n, q)$  implies  $p \sim q$ . Recall, two  $n$ -dimensional  $m$ -forms  $p$  and  $q$  are similar provided there exist  $C \in GL(n, \mathbb{Q})$  and  $0 \neq \alpha \in \mathbb{Q}$  such that  $p(C^T x) = \alpha q(x)$ . If  $L^k(n, p) \sim_r L^k(n, q)$  then, in particular,  $L^k(n, p) \preceq L^k(n, q)$ . By Lemma 3.4 (2), this means there is a nontrivial CDGA morphism  $\phi : L^k(n, p) \rightarrow L^k(n, q)$ , so there exists an  $n \times n$  matrix  $C$  and  $\alpha \in \mathbb{Q}$  such that  $p(C^T x) = \alpha q(x)$ . Not only is  $\phi$  nontrivial, but neither  $\alpha$  nor  $C$  can be 0: if  $\alpha = 0$  then  $C$  must also be 0 to guarantee  $\phi$  is a CDGA morphism, but then  $\phi$  would be trivial; if  $C = 0$ , then  $\alpha$  must be 0 to avoid contradicting the fact that  $q$  is anisotropic. It remains to verify  $C \in GL(n, \mathbb{Q})$ . If not, then there is a  $v \in \ker(C^T)$ , thus  $q(v) = (1/\alpha)p(C^T v) = 0$ , implying  $q$  is isotropic. Therefore,  $\alpha \neq 0$  and  $C \in GL(n, \mathbb{Q})$ , so we are assured  $p \sim q$ . It follows that  $e_k : \overline{\mathbb{W}}^1 \rightarrow \mathcal{S}$  is injective, as claimed.  $\square$

## Uncovering the Witt Group

Just as Hess and Parent were able to uncover  $\overline{W(\mathbb{Q})}$  in the cellular lattice [15], the same can be done for  $(\mathcal{S}, \preceq)$  as a consequence of Theorem 3.3:

**Corollary 3.5.** *For every even  $k \geq 2$ , there is a subcollection of  $\mathcal{S}$  that is in bijective correspondence with  $\overline{W(\mathbb{Q})}$ .*

*Proof.* Let  $[q] \in \overline{W(\mathbb{Q})}$  denote the similarity class of  $q \in W(\mathbb{Q})$ . By Theorem 3.3 in the case  $r = 1$  and  $m = 2$ , the function  $e_k : \overline{W(\mathbb{Q})} \rightarrow \mathcal{S}$  defined by  $[q] \mapsto L^k(n, q) \mapsto \langle L(n, q) \rangle$  is injective, thus  $\overline{W(\mathbb{Q})}$  is in bijective correspondence with the image  $e_k$ .  $\square$

Of course, Theorem 3.3 implies a result for anisotropic systems of quadratic forms as well. Let  $\overline{\mathbb{W}}_2$  denote the collection of such systems. We have:

**Corollary 3.6.** *For every even  $k \geq 2$ , there is a well-defined function*

$$e_k : \overline{\mathbb{W}}_2 \rightarrow \mathcal{S}.$$

*Proof.* Theorem 3.3 in the case  $m = 2$  defines a function  $e_k : \overline{\mathbb{W}}_2 \rightarrow \mathcal{S}$ , implying the stated result.  $\square$

Fix an even  $k \geq 2$ , and let  $\mathbb{L}^r$  denote the image of the function given by  $[\bar{q}] \mapsto L^k(n, \bar{q})$  where  $\bar{q} \in \overline{\mathbb{W}}_2$ ; so,  $\mathbb{L}^r$  is the collection of resolvable equivalence classes of CDGA's of the form  $L^k(n, \bar{q})$  for  $n \in \mathbb{N}$  and  $\bar{q}$  an  $r$ -fold anisotropic system of  $n$ -dimensional quadratic forms. In particular,  $\mathbb{L}^1$  is the collection of resolvable equivalence classes of CDGA's modeling anisotropic quadratic forms.

Given the bijection between  $\overline{W(\mathbb{Q})}$  and  $\mathbb{L}^1$ , we can formally define the (unreduced) product of  $L^k(n, q)$  and  $L^k(m, q')$  by:

$$L^k(n, q) \odot L^k(m, q') = L^k(nm, q \otimes q'),$$

where we use  $\odot$  to avoid confusion with the standard tensor product of CDGA's. We can then reduce this product to have it align with  $\otimes$  in  $\overline{W(\mathbb{Q})}$ ; that is, define the reduced product of (the resolvable equivalence classes of)  $L^k(n, q)$  and  $L^k(m, q')$  by:

$$L^k(n, q) \widetilde{\odot} L^k(m, q') = L^k(l, (q \otimes q')_o),$$

where, in general,  $l \leq nm$  (corresponding to the dimension of the reduction  $(q \otimes q')_o$ ; see Section 1.6). In this way, we can consider  $(\mathbb{L}^1, \widetilde{\odot})$  and  $(\overline{W(\mathbb{Q})}, \otimes)$  to be isomorphic as semigroups (note  $\mathcal{M}(S^k) = (\Lambda(x, y), dy = x^2)$  corresponds to the unit element  $\langle 1 \rangle \in \overline{W(\mathbb{Q})}$ ).

Recall the sum  $\perp$  on  $W(\mathbb{Q})$  is not defined up to similarity, so a sum will not be well-defined as a binary operation on  $\mathbb{L}^1$ . Nonetheless, we can still formally define the reduced and unreduced sum of models  $L^k(n, q)$  and  $L^k(m, q')$  by

$$L^k(n, q) \perp L^k(m, q') = L^k(n + m, q \perp q')$$

and

$$L^k(n, q) \widetilde{\perp} L^k(m, q') = L^k(l, (q \perp q')_o),$$

where, again,  $l \leq n + m$  corresponds to the dimension of  $(q \perp q')_o$ .

Likewise, though resolvable equivalence of CDGA's does not correspond to similarity on *isotropic* quadratic forms in general, it is still the case that  $q \sim q'$

implies  $L^k(n, q) \sim_r L^k(n, q')$  even when  $q$  and  $q'$  are isotropic. Therefore, the assignment  $[q] \mapsto L^k(n, q)$  is well-defined for *all* quadratic forms.

The following proposition collects a few observations about this product and sum on CDGA's:

**Proposition 3.7.** *For  $L^k(n, q)$ ,  $L^k(m, q')$ ,  $\odot$ , and  $\perp$ , as above:*

1.  $\odot$  and  $\perp$  are commutative;
2.  $L^k(n, q) \perp L^k(m, q') \preceq L^k(n, q)$  and  $L^k(n, q) \perp L^k(m, q') \preceq L^k(m, q')$ ;
3.  $L^k(n, q) \odot L^k(m, q') \preceq L^k(n, q)$  and  $L^k(n, q) \odot L^k(m, q') \preceq L^k(m, q')$ ;

*Proof.* (1) This follows directly from the corresponding facts on quadratic forms.

(2) Without loss of generality, we may assume  $q$  and  $q'$  are diagonal forms. Write  $q = \langle \alpha_1, \dots, \alpha_n \rangle$  and  $q' = \langle \beta_1, \dots, \beta_m \rangle$ . Then  $L(n, q) \perp L(m, q')$  is

$$(\Lambda(x_1, \dots, x_n, w_1, \dots, w_m, z), dz = \langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \rangle).$$

Define  $\phi : L^k(n, q) \perp L^k(m, q') \rightarrow L^k(n, q)$  by  $w_j \mapsto 0$  for all  $1 \leq j \leq m$ ,  $x_i \mapsto x_i$  for all  $1 \leq i \leq n$ , and  $z \mapsto y$ . Then  $\phi$  is CDGA morphism with surjective linear part in degree  $2k - 1$ , so  $L^k(n, q) \perp L^k(m, q') \preceq L^k(n, q)$ .  $L^k(n, q) \perp L^k(m, q') \preceq L^k(m, q')$  follows from the commutativity of  $\perp$ .

(3) Again, without loss of generality, we may assume  $q = \langle \alpha_1, \dots, \alpha_n \rangle$  and  $q' = \langle \beta_1, \dots, \beta_m \rangle$ . Moreover, since  $\odot$  and  $\otimes$  are defined up to similarity, by dividing through by  $\alpha_1$  and  $\beta_1$ , we may assume  $\alpha_1 = 1 = \beta_1$ . Then

$$\begin{aligned} q \otimes q' &= \perp_{i=1}^n \langle \alpha_i \beta_1, \dots, \alpha_i \beta_n \rangle \\ &= q' \perp (\perp_{i=2}^n \langle \alpha_i \beta_1, \dots, \alpha_i \beta_n \rangle). \end{aligned}$$

Thus  $L^k(n, q) \odot L^k(m, q') \preceq L^k(m, q')$  by part (2). By the commutativity of  $\odot$ , we also find  $L^k(n, q) \odot L^k(m, q') \preceq L^k(n, q)$ .  $\square$

The reduced sum and product do not share the forming relations of their unreduced counterparts described in Proposition 3.7:

**Example.** Consider  $\langle 1, 1 \rangle \perp \langle -1, -1 \rangle = \langle 1, 1, -1, -1 \rangle$ . This form reduces to the empty form of dimension 0, thus it cannot be the case  $L^k(2, \langle 1, 1 \rangle) \perp L^k(2, \langle -1, -1 \rangle)$  forms either  $L^k(2, \langle 1, 1 \rangle)$  or  $L^k(2, \langle -1, -1 \rangle)$ .

Similarly,  $(\langle 1, 1 \rangle \otimes \langle -1, 2 \rangle)_o$  is the empty form. So, there is no hope to have  $L^k(2, \langle 1, 1 \rangle) \odot L^k(2, \langle -1, 2 \rangle)$  form either of its factors.

With a bijection between  $\mathbb{L}^1$  and  $\overline{W(\mathbb{Q})}$  in hand, we can also define a partial order  $\preceq$  on  $\overline{W(\mathbb{Q})}$  by  $q \preceq q'$  if and only if  $L(n, q) \preceq L(m, q')$ . Considering Lemma 3.4, we can equivalently state  $q \preceq q'$  to mean there exists an  $m \times n$  matrix  $C$  (not necessarily invertible) and  $0 \neq \alpha \in \mathbb{Q}$  such that  $CA_qC^T = \alpha A_{q'}$ . By the above example, we cannot say anything in general about how  $q \otimes q'$  relates to  $q$  or  $q'$ , but if  $q \otimes q'$  is anisotropic then  $q \otimes q' \preceq q$  and  $q \otimes q' \preceq q'$ . A similar partial order can be put on  $\mathbb{W}^r$  for  $r > 1$ . The extent to which these partial orders can actually be of use to the study of quadratic forms is yet to be established.

## The Case of Homogeneous Polynomials

As one might expect, Theorem 3.3 has an immediate consequence in regards to homogeneous polynomials.

Recall, we have denoted the collection of degree  $m$  homogeneous polynomials in  $n$  variables as  $P_n^m \subset \mathbb{Q}[x_1, \dots, x_n]$ . For  $p, q \in P_n^m$ , we have defined  $p$  to be similar to  $q$ , written  $p \sim q$ , to mean there exists a  $C \in GL(n, \mathbb{Q})$  and  $0 \neq \alpha \in \mathbb{Q}$  such that  $p(C^T x) = \alpha q(x)$ . Write  $\overline{P_n^m}$  for the collection of similarity classes of anisotropic  $n$ -dimensional  $m$ -forms. We have:

**Corollary 3.8.** *For every even  $k \geq 2$ , there is a subcollection of  $\mathcal{S}$  that is in bijective correspondence with  $\overline{P_n^m}$ .*

*Proof.* The assignment  $e_k : [p] \mapsto L^k(n, p) \mapsto \langle L^k(n, p) \rangle$  defines an injective function  $e_k : \overline{P_n^m} \rightarrow \mathcal{S}$  via Theorem 3.3 in the case  $r = 1$ . Thus  $\overline{P_n^m}$  is in bijective correspondence with the image of  $e_k$ .  $\square$

There are not higher degree analogues to the Witt group, but one can generalize the discussion at the end of the previous section to construct a partial order on  $\overline{P_n^m}$  using the bijection from Corollary 3.8. The applications of this partial order to the study of homogeneous polynomials (and related topics) is open for consideration.

## The Case of Nonhomogeneous Polynomials

We end this section by considering general rational polynomials, unveiling a certain collection of their similarity classes to be embedded in  $(\mathcal{S}, \preceq)$ .

Given a polynomial  $p(x) \in \mathbb{Q}[x_1, \dots, x_n]$  of degree  $m$ , we can represent it as

$$p(x) = \sum_{l=0}^m p_l(x)$$



where  $p_l \in P_n^l$ , and we've defined a notion of similarity on the elements of  $\mathbb{Q}[x_1, \dots, x_n]$ . Namely,  $p \sim q$  if there is a  $C \in GL(n, \mathbb{Q})$  and  $0 \neq \alpha_l \in \mathbb{Q}$  such that  $p_l(C^T x) = \alpha_l q_l(x)$  for all  $1 \leq l \leq m$ . Let  $\overline{\mathbb{Q}^{\geq 2}[x_1, \dots, x_n]}$  represent the collection of similarity classes of anisotropic polynomials in  $\mathbb{Q}[x_1, \dots, x_n]$  of degree greater than or equal to 2. For an even  $k \geq 2$ , we assign to the similarity class of such an anisotropic  $p$  the CDGA

$$L^k(n, p) = (\Lambda(x_1, \dots, x_n, y_2, \dots, y_m), dy_l = p_l(x)),$$

where  $|x_i| = k$  for each  $1 \leq i \leq n$  and  $|y_l| = lk - 1$  for each  $2 \leq l \leq m$ .

We begin with a lemma characterizing an alternate description of  $L^k(n, p)$ .

**Lemma 3.9.** *The CDGA  $L^k(n, p) = (\Lambda(x_1, \dots, x_n, y_2, \dots, y_m), dy_l = p_l(x))$  is isomorphic to the colimit of  $\{(\Lambda(x_1, \dots, x_n, 0) \hookrightarrow L^k(n, p_l))\}_{l=2}^m$ :*

$$L^k(n, p) \cong L^k(n, p_2) \otimes_{(\Lambda(\{x_i\}), 0)} L^k(n, p_3) \otimes_{(\Lambda(\{x_i\}), 0)} \dots \otimes_{(\Lambda(\{x_i\}), 0)} L^k(n, p_m).$$

*Proof.* We verify our assertion by showing  $L^k(n, p)$  satisfies the universal property of  $\text{colim}\{(\Lambda(x_1, \dots, x_n, 0) \hookrightarrow L^k(n, p_l))\}_{l=2}^m$ .

For each  $2 \leq l \leq m$ , define  $\iota_l : L^k(n, p_l) \rightarrow L^k(n, p)$  by  $x_i \mapsto x_i$  for all  $1 \leq i \leq n$  in degree  $k$ ,  $y_l \mapsto y_l$  in degree  $lk - 1$ , and 0 in all other positive degrees. This is certainly a CDGA morphism. Moreover, each  $\iota_l$  restricts to the same morphism

$$(\Lambda(x_1, \dots, x_n), 0) \hookrightarrow L^k(n, p) \xrightarrow{\iota_l} L^k(n, p)$$

defined by  $x_i \mapsto x_i$  for all  $1 \leq i \leq n$ ; call this morphism  $\iota : (\Lambda(x_1, \dots, x_n), 0) \rightarrow L^k(n, p)$ .

Now, suppose we have another CDGA  $(A, d)$  with CDGA morphisms  $\epsilon_l : L^k(n, p) \rightarrow (A, d)$  for each  $2 \leq l \leq m$  for which all the compositions

$$(\Lambda(x_1, \dots, x_n), 0) \hookrightarrow L^k(n, p) \xrightarrow{\epsilon_l} (A, d)$$

are equal; call this composite morphism  $\epsilon : (\Lambda(x_1, \dots, x_n), 0) \rightarrow (A, d)$ . We wish to show there exists a unique CDGA morphism  $\phi : L^k(n, p) \rightarrow (A, d)$  such that  $\phi \circ \iota_l = \epsilon_l$  for each  $2 \leq l \leq m$ , as in the diagram:

$$\begin{array}{ccccc}
 & & \iota & & \\
 & & \curvearrowright & & \\
 (\Lambda(x_i), 0) & \longrightarrow & L^k(n, p_l) & \xrightarrow{\iota_l} & L^k(n, p) \\
 & \searrow \epsilon & \downarrow \epsilon_l & \swarrow \exists! \phi & \\
 & & (A, d) & & 
 \end{array}$$

Define  $\phi : L^k(n, p) \rightarrow (A, d)$  by:

- $\phi_k : x_i \mapsto \epsilon(x_i)$  for  $1 \leq i \leq n$ ;
- $\phi_{lk-1} : y_l \mapsto \epsilon_l(y_l)$  for  $2 \leq l \leq m$ .

We first verify  $\phi$  is a CDGA morphism. Consider, for each  $1 \leq i \leq n$ , since  $\epsilon$  is a CDGA morphism:

$$\begin{aligned}
 d \circ \phi(x_i) &= d \circ \epsilon(x_i) \\
 &= \epsilon \circ d(x_i) \\
 &= \epsilon(0) \\
 &= 0 \\
 &= \phi(0) \\
 &= \phi \circ d(x_i).
 \end{aligned}$$

Also, since  $\epsilon_l$  and  $\iota_l$  are CDGA morphisms for each  $2 \leq l \leq m$  we have:

$$\begin{aligned}
 d \circ \phi(y_l) &= d \circ \epsilon_l(y_l) \\
 &= \epsilon_l \circ d(y_l) \\
 &= \phi \circ \iota_l \circ d(y_l) \\
 &= \phi \circ d \circ \iota_l(y_l) \\
 &= \phi \circ d(y_l).
 \end{aligned}$$

Thus,  $\phi$  is a CDGA morphism.

Note that the definition of  $\phi$  is all but forced by the equation  $\phi \circ \iota_l = \epsilon_l$ . Suppose  $\psi : L^k(n, p) \rightarrow (A, d)$  is a CDGA morphism that also satisfies  $\psi \circ \iota_l = \epsilon_l$  for each  $2 \leq l \leq m$ . Then

$$\begin{aligned}
 \psi(x_i) &= \psi \circ \iota_l(x_i) \\
 &= \epsilon_l(x_i) \\
 &= \epsilon(x_i) \\
 &= \phi(x_i),
 \end{aligned}$$

and

$$\begin{aligned}
 \psi(y_l) &= \psi \circ \iota_l(y_l) \\
 &= \epsilon_l(y_l) \\
 &= \phi(y_l).
 \end{aligned}$$

Therefore,  $\phi$  is the unique CDGA morphism satisfying  $\phi \circ \iota_l = \epsilon_l$ , and with our universal property satisfied, we are left to conclude:

$$\begin{aligned} L^k(n, p) &\cong \operatorname{colim}\{(\Lambda(x_1, \dots, x_n, 0) \hookrightarrow L^k(n, p_l))\}_{l=2}^m \\ &\cong L^k(n, p_2) \otimes_{(\Lambda(\{x_i\}), 0)} L^k(n, p_3) \otimes_{(\Lambda(\{x_i\}), 0)} \dots \otimes_{(\Lambda(\{x_i\}), 0)} L^k(n, p_m). \end{aligned}$$

□

We are now ready to prove:

**Theorem 3.10.** *For every even  $k \geq 2$ , there is a subcollection of  $\mathcal{S}$  that is in bijective correspondence with  $\overline{\mathbb{Q}^{\geq 2}[x_1, \dots, x_n]}$ .*

*Proof.* Fix an even  $k \geq 2$ . We show the assignment

$$[p] \mapsto L^k(n, p) \mapsto \langle L^k(n, p) \rangle$$

defines an injective function  $\overline{\mathbb{Q}^{\geq 2}[x_1, \dots, x_n]} \rightarrow \mathcal{S}$ , and so  $\overline{\mathbb{Q}^{\geq 2}[x_1, \dots, x_n]}$  is in bijective correspondence with the image of this function.

By Lemma 3.9, we can alternately represent  $L(n, p)$  as

$$L^k(n, p_2) \otimes_{(\Lambda(\{x_i\}), 0)} L^k(n, p_3) \otimes_{(\Lambda(\{x_i\}), 0)} \dots \otimes_{(\Lambda(\{x_i\}), 0)} L^k(n, p_m).$$

In other words,  $L^k(n, p) \cong \operatorname{colim}\{(\Lambda(x_1, \dots, x_n), 0) \hookrightarrow L^k(n, p_d)\}_{d=2}^m$ . In particular, for every  $l$  we have CDGA morphisms  $\iota_l : L^k(n, p_l) \rightarrow L^k(n, p)$  with the property  $(\Lambda(x_1, \dots, x_n), 0) \hookrightarrow L^k(n, p_l) \rightarrow L^k(n, p)$  is the inclusion  $(\Lambda(x_1, \dots, x_n), 0) \hookrightarrow L^k(n, p)$ .

Suppose  $L^k(n, p) \sim_r L^k(n, q)$  for anisotropic  $p, q \in \overline{\mathbb{Q}^{\geq 2}[x_1, \dots, x_n]}$  of degree  $m$ . In particular, we must have  $L^k(n, p) \preceq L^k(n, q)$  and thus a CDGA morphism

$$\Phi : L^k(n, p) \rightarrow L^k(n, q)$$

with  $\Phi_{mk-1} \neq 0$ . Let  $C$  be the matrix representation of  $\Phi_k$ , and suppose  $\Phi_{lk-1} : y_l \mapsto \alpha_l y'_l$  with  $\alpha_l \in \mathbb{Q}$  for every  $2 \leq l \leq m$ . The restriction of  $\Phi$  to  $L^k(n, p_l)$  factors through  $L^k(n, q_l)$  as in the following diagram:

$$\begin{array}{ccc} L^k(n, p_l) & \xrightarrow{\phi} & L^k(n, q_l) \\ \iota_l \downarrow & & \downarrow \iota_l \\ L^k(n, p) & \xrightarrow{\Phi} & L^k(n, q). \end{array}$$

We see  $\phi$  is defined by  $\phi_k = \Phi_k$ ,  $\phi_{dk-1} = \Phi_{lk-1}$ , while being 0 in all other positive degrees. By Lemma 3.4, we must have  $p_l(C^T x) = \alpha_l q(x)$  for every  $l$ ; in particular,

$\alpha_l \neq 0$  since supposing otherwise would imply  $p_l$  is isotropic. To show  $p \sim q$ , we are left to show that  $C$  is invertible, but this follows from the fact that  $q_m$  is anisotropic (as in the proof of Lemma 3.4). Therefore,  $p \sim q$  by definition.

Now suppose  $p \sim q$ . Then there exists  $C \in GL(n, \mathbb{Q})$  and  $0 \neq \alpha_l \in \mathbb{Q}$  such that  $p_l(C^T x) = \alpha_l q_l(x)$  for every  $l$ . So, for each  $l$ , define  $\phi : L^k(n, p_l) \rightarrow L^k(n, q_l)$  by  $\phi_k : v \mapsto Cv$  and  $\phi_{lk-1} : y_l \mapsto \alpha_l y'_l$ . These will all be CDGA morphisms by Lemma 3.4. Moreover, they all restrict to the same morphism  $(\Lambda(x_1, \dots, x_n), 0) \rightarrow (\Lambda(x_1, \dots, x_n), 0)$  in degree  $k$  given by  $v \mapsto Cv$ . Considering the collection of CDGA morphisms  $u_l \circ \phi : L^k(n, p_l) \rightarrow L^k(n, q)$ , there must exist a unique CDGA morphism  $\Phi : L^k(n, p) \rightarrow L^k(n, q)$  such that  $\Phi \circ u_l = u_l \circ \phi$ . Since  $\alpha_m \neq 0$ ,  $\Phi_{mk-1}$  is nontrivial, whence  $L^k(n, p) \preceq L^k(n, q)$ . To show  $L^k(n, q) \preceq L^k(n, p)$ , start by defining  $\psi : L^k(n, q_l) \rightarrow L^k(n, p_l)$  by  $\psi_k : v \mapsto C^{-1}v$  and  $\psi_{lk-1} : y'_l \mapsto (1/\alpha_l)y_l$ . Similar to the previous situation, the collection of  $u_l \circ \psi : L^k(n, q_l) \rightarrow L^k(n, p)$  will induce a CDGA morphism  $\Psi : L^k(n, q) \rightarrow L^k(n, p)$  with  $y'_m \mapsto (1/\alpha_m)y_m$  (i.e.,  $\Psi_{mk-1}$  nontrivial). So,  $L^k(n, p) \sim_r L^k(n, q)$  if and only if  $p \sim q$ , implying the desired result.  $\square$

### 3.3 Topological Perspectives

In this section, we look at some of the geometric interpretations of the algebraic structures we've so far been studying. In particular, we'll exhibit examples of spaces whose models are of the form  $L^k(n, q)$  for a diagonal  $n$ -dimensional  $m$ -form  $q$ , as well as exhibit an action of the set of quadratic forms on a certain collection of rational spaces.

#### Spatial Models of Homogeneous Polynomial Forms

Let  $q(x) = \sum_{i=1}^m \alpha_i x_i^m$  be a diagonal  $n$ -dimensional  $m$ -form, and let  $A$  be a space with

$$\mathcal{M}(A) = (\Lambda(x, y), dy = x^m)$$

where  $|x| = k$  and  $|y| = mk - 1$  (where  $k \geq 2$  is even, and  $n, m \in \mathbb{N}$ ). As in the discussion preceding Lemma 2.18, in the case  $m = 2$  we could take  $A = S^k$  and in the case  $k = 2$  we can take  $A = \mathbb{C}P^{m-1}$ . We use  $A$  as our building block to find a space  $X$  with  $\mathcal{M}(X) \cong L^k(n, q)$ . We note that every quadratic form is isometric (hence similar) to a diagonal form, so our impending discussion is enough to model all quadratic forms up to similarity. This is not the case for  $m \geq 3$ , though the diagonal homogeneous forms do play a particularly important role in algebraic geometry [5].

Consider  $\prod_{i=1}^n A$ , whose model has the form

$$\begin{aligned} \mathcal{M} \left( \prod_{i=1}^n A \right) &\cong \prod_{i=1}^n \mathcal{M}(A) \\ &\cong (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n), dy_i = x_i^m) \end{aligned}$$

where  $|x_i| = k$  and  $|y_i| = mk - 1$  for  $1 \leq i \leq n$ . Let  $v_1 = \sum_{i=1}^n \alpha_i y_i$  and extend this to a basis  $\{v_1, \dots, v_n\}$  of  $\Lambda^1(x_1, \dots, x_n, y_1, \dots, y_n)$  in degree  $2k - 1$ . With this, we may write

$$\mathcal{M} \left( \prod_{i=1}^n S^k \right) \cong (\Lambda(x_1, \dots, x_n, v_1, \dots, v_n), d).$$

Notice, then,  $dv_1 = \sum_{i=1}^n \alpha_i x_i^m$ , which is precisely the diagonal form we wish to model; so, we are left to eliminate the other elements in degree  $2k - 1$ .

Each  $v_i$  is dual to a basis element  $f_i \in \pi_{mk-1}(\prod_{i=1}^n A)$ , in the sense of Theorem 1.7. In other words, we have  $\pi_{mk-1}(\prod_{i=1}^n A) \cong \bigoplus_{i=1}^n \mathbb{Q} \cdot f_i$  where  $\langle v_i, f_i \rangle = 1$  while  $\langle v_i, f_j \rangle = 0$  for any  $i \neq j$  (i.e.,  $\langle v_i, - \rangle^* = f_i$ ). Let

$$in_i : S^{mk-1} \hookrightarrow \bigvee_{i=2}^n S^{mk-1}$$

be the inclusion of the  $i$ -th summand, and call

$$f : \bigvee_{i=2}^n S^{mk-1} \rightarrow \prod_{i=1}^n A$$

the unique map such that  $f \circ in_i = f_i$  for each  $2 \leq i \leq n$ . Construct the pushout

$$\begin{array}{ccc} \bigvee_{i=2}^n S^{mk-1} & \xrightarrow{f} & \prod_{i=1}^n A \\ \downarrow & & \downarrow h \\ \bigvee_{i=2}^n D^{mk} & \longrightarrow & \left( \prod_{i=1}^n A \right) \cup_f \left( \bigvee_{i=2}^n D^{mk} \right). \end{array}$$

Call the resulting adjunction space  $X_{(0)}$ . We claim  $\pi_{mk-1}(X_{(0)}) \cong \mathbb{Q} \cdot h_*(f_1)$ .

To see this, consider the cofiber sequence

$$\bigvee_{i=2}^n S^{mk-1} \xrightarrow{f} \prod_{i=1}^n A \xrightarrow{h} X_{(0)}.$$

Now,  $S^{mk-1}$  is  $(mk-2)$ -connected and  $X_{(0)}$  is  $(k-1)$ -connected, so the Blakers-Massey Theorem implies the sequence

$$\pi_{mk-1} \left( \bigvee_{i=2}^n S^{mk-1} \right) \xrightarrow{f_*} \pi_{mk-1} \left( \prod_{i=1}^n A \right) \xrightarrow{h_*} \pi_{mk-1}(X_{(0)}) \rightarrow \pi_{mk-2} \left( \bigvee_{i=2}^n S^{mk-1} \right)$$

is exact (since, for  $k \geq 2$ , we have  $mk-1 \leq (mk-2) + (k-1)$ ). But  $\pi_{mk-2}(\bigvee_{i=2}^n S^{mk-1}) = 0$ , so

$$\begin{aligned} \pi_{mk-1}(X_{(0)}) &\cong \pi_{mk-1} \left( \prod_{i=1}^n A \right) / \ker(h_*) \\ &\cong \pi_{mk-1} \left( \prod_{i=1}^n A \right) / \text{im}(f_*) \\ &\cong \left( \bigoplus_{i=1}^n \mathbb{Q} \cdot f_i \right) / \text{im}(f_*). \end{aligned}$$

It remains to verify what  $\text{im}(f_*)$  is. We notice

$$\pi_{mk-1} \left( \bigvee_{i=2}^n S^{mk-1} \right) \cong \bigoplus_{i=2}^n \mathbb{Q} \cdot in_i,$$

so, on the  $i$ -th summand,  $f_* : in_i \mapsto f \circ in_i = f_i \in \pi_{mk-1}(\prod_{i=1}^n A)$ . In other words,  $\text{im}(f_*) \cong \bigoplus_{i=2}^n \mathbb{Q} \cdot f_i$ , and we conclude  $\pi_{mk-1}(X_{(0)}) \cong \mathbb{Q} \cdot h_*(f_1)$ .

Take the  $mk$ -th Postnikov section of  $X_{(0)}$ , denoted  $(X_{(0)})_{mk}$ , and we have

$$\mathcal{M}((X_{(0)})_{mk}) = \left( \Lambda(x_1, \dots, x_n, v_1), dv_1 = \sum_{i=1}^n \alpha_i x_i^m \right).$$

All of this goes to show:

**Proposition 3.11.** *Suppose  $A$  is a space with  $\mathcal{M}(A) = (\Lambda(x, y), dy = x^m)$  as above, and let  $X$  be a space with  $\mathcal{M}(X) = L^k(n, q)$  where  $q(x) = \sum_{i=1}^n \alpha_i x_i^m$  is a diagonal  $m$ -form. There is a cone decomposition of  $X$  given by*

$$\begin{array}{ccccccc}
\bigvee_{i=2}^n S^{mk-1} & & \bigvee & S^{mk} & & \bigvee & S^{mk+n} & & \bigvee & S^{mk+n+1} \\
\downarrow & & \downarrow & \pi_{mk}(X_{(0)}) & & \downarrow & \pi_{mk+n}(X_{(n)}) & & \downarrow & \pi_{mk+n+1}(X_{(n+1)}) \\
\prod_{i=1}^n A & \longrightarrow & X_{(0)} & \longrightarrow & \cdots & \longrightarrow & X_{(n)} & \longrightarrow & X_{(n+1)} & \longrightarrow & \cdots
\end{array}$$

**Example.** As an example, let us construct a space whose model is  $L^2(2, \langle \alpha, \beta \rangle)$  for nonzero  $\alpha, \beta \in \mathbb{Q}$ . Start with  $S^2 \times S^2$  which has

$$\mathcal{M}(S^2 \times S^2) = (\Lambda(x_1, x_2, y_1, y_2), dy_i = x_i^2).$$

Let  $v_1 = \alpha y_1 + \beta y_2$  and  $v_2 = \alpha y_1 - \beta y_2$ . Then

$$\mathcal{M}(S^2 \times S^2) \cong (\Lambda(x_1, x_2, v_1, v_2), dv_1 = \alpha x_1^2 + \beta x_2^2, dv_2 = \alpha x_1^2 - \beta x_2^2).$$

Denote  $\pi_2(S^2 \times S^2) \cong \mathbb{Q} \cdot \iota_1 \oplus \mathbb{Q} \cdot \iota_2$  and define

$$f = \left(-\frac{1}{4\alpha}\right) [\iota_1, \iota_1] + \left(\frac{1}{4\beta}\right) [\iota_2, \iota_2] \in \pi_3(S^2 \times S^2),$$

where we recall  $[-, -]$  denotes the Whitehead product [19].

We claim this  $f$  is dual to  $v_2$ . Firstly, note (c.f. Theorem 1.3):

$$\begin{aligned}
\langle y_i; [\iota_j, \iota_j] \rangle &= (-1)^3 \langle dy_i; \iota_j, \iota_j \rangle \\
&= -\langle x_i^2; \iota_j, \iota_j \rangle \\
&= -(\langle x_i; \iota_j \rangle \langle x_i; \iota_j \rangle + (-1)^4 \langle x_i; \iota_j \rangle \langle x_i; \iota_j \rangle) \\
&= \begin{cases} -2 & i = j \\ 0 & i \neq j. \end{cases}
\end{aligned}$$

Now consider:

$$\begin{aligned}
\langle v_2; f \rangle &= \langle \alpha y_1 - \beta y_2; f \rangle \\
&= \alpha \langle y_1; f \rangle - \beta \langle y_2; f \rangle \\
&= \alpha \left\langle y_1; \left(-\frac{1}{4\alpha}\right) [\iota_1, \iota_1] + \left(\frac{1}{4\beta}\right) [\iota_2, \iota_2] \right\rangle \\
&\quad - \beta \left\langle y_2; \left(-\frac{1}{4\alpha}\right) [\iota_1, \iota_1] + \left(\frac{1}{4\beta}\right) [\iota_2, \iota_2] \right\rangle \\
&= \alpha \left( \left\langle y_1; \left(-\frac{1}{4\alpha}\right) [\iota_1, \iota_1] \right\rangle + \left\langle y_1; \left(\frac{1}{4\beta}\right) [\iota_2, \iota_2] \right\rangle \right) \\
&\quad - \beta \left( \left\langle y_2; \left(-\frac{1}{4\alpha}\right) [\iota_1, \iota_1] \right\rangle + \left\langle y_2; \left(\frac{1}{4\beta}\right) [\iota_2, \iota_2] \right\rangle \right) \\
&= \alpha \left( \left(-\frac{1}{4\alpha}\right) \langle y_1; [\iota_1, \iota_1] \rangle + \left(\frac{1}{4\beta}\right) \langle y_1; [\iota_2, \iota_2] \rangle \right) \\
&\quad - \beta \left( \left(-\frac{1}{4\alpha}\right) \langle y_2; [\iota_1, \iota_1] \rangle + \left(\frac{1}{4\beta}\right) \langle y_2; [\iota_2, \iota_2] \rangle \right) \\
&= \alpha \left(-\frac{1}{4\alpha}\right) (-2) - \beta \left(\frac{1}{4\beta}\right) (-2) \\
&= 1.
\end{aligned}$$

So then, as described above, we take the pushout

$$\begin{array}{ccc}
S^3 & \xrightarrow{f} & S^2 \times S^2 \\
\downarrow & & \downarrow \\
D^4 & \longrightarrow & (S^2 \times S^2) \cup_f D^4,
\end{array}$$

and then

$$((S^2 \times S^2) \cup_f D^4)_4$$

will have minimal Sullivan model  $L^2(2, \langle \alpha, \beta \rangle) = (\Lambda(x_1, x_2, v_1), dv_1 = \alpha x_1^2 + \beta x_2^2)$ .



## An Action of Quadratic Forms on Spaces

Call the collection of all quadratic forms  $Q$ . The operations  $\perp$  and  $\otimes$  are defined on arbitrary quadratic forms as corresponding to the direct sum and Kronecker product of their coefficient matrices. That is, for  $q, q' \in Q$ :

$$\begin{aligned} A_{q \perp q'} &= A_q \oplus A_{q'}; \\ A_{q \otimes q'} &= A_q \otimes A_{q'}. \end{aligned}$$

These definitions correspond with the operations on isometry classes of diagonal forms defined on  $W(\mathbb{Q})$ . However,  $(Q, \perp, \otimes)$  is not a ring: for one thing, the addition  $\perp$  is not strictly commutative on  $Q$ . Be that as it may,  $\otimes$  does distribute over  $\perp$  from the right; note, for  $p, q, q' \in Q$ :

$$\begin{aligned} A_{(q \perp q') \otimes p} &= (A_q \oplus A_{q'}) \otimes A_p \\ &= (A_q \otimes A_p) \oplus (A_{q'} \otimes A_p) \\ &= A_{(q \otimes p) \perp (q' \otimes p)}. \end{aligned}$$

For  $k \geq 2$  even, say  $L^k(m, \bar{q}) = (\Lambda(v_1, \dots, v_m, z_1, \dots, z_r), dz_l = q_l(v))$  where  $|v_j| = k$  for  $1 \leq j \leq m$ ,  $|z_l| = 2k - 1$  for  $1 \leq l \leq r$ , and  $\bar{q}$  is an  $r$ -fold system of  $n$ -dimensional quadratic forms. Let  $\Gamma^r$  be the collection of all CDGA's of the form  $L^k(n, \bar{q})$  where  $n \in \mathbb{N}$  and  $\bar{q}$  is a (not necessarily anisotropic)  $r$ -fold system of  $n$ -dimensional quadratic forms. We can extend our definition of  $\perp$  on the models of quadratic forms to  $r$ -fold systems of quadratic forms in the obvious way:

$$L^k(m, \bar{q}) \perp L^k(n, \bar{p}) = L^k(n + m, (q_1 \perp p_1, \dots, q_r \perp p_r)).$$

Then  $Q$  acts on the elements of  $\Gamma^r$  for  $r \geq 1$  in the following sense:

**Proposition 3.12.** *Let  $q \in Q$  be an  $n$ -dimensional quadratic form, and  $r \geq 1$ . Define a binary operation*

$$\cdot : Q \times \Gamma^r \rightarrow \Gamma^r$$

by  $q \cdot L^k(m, \bar{q}) = L^k(nm, (q \otimes q_1, \dots, q \otimes q_r))$ . For  $p, q \in Q$ , we have:

1.  $(p \perp q) \cdot L^k(m, \bar{q}) = p \cdot L^k(m, \bar{q}) \perp q \cdot L^k(m, \bar{q})$ ;
2.  $\langle 1 \rangle \cdot (-) : \Gamma^r \rightarrow \Gamma^r$  defined by  $L^k(m, \bar{q}) \mapsto \langle 1 \rangle \cdot L^k(m, \bar{q})$  is the identity function;
3.  $p \cdot (q \cdot L^k(m, \bar{q})) = (p \otimes q) \cdot L^k(m, \bar{q})$ .

*Proof.* (1) Observe, for  $p, q \in Q$  of dimension  $l$  and  $n$  respectively, and  $r \geq 1$ :

$$\begin{aligned} (p \perp q) \cdot L^k(m, \bar{q}) &= L^k((l+n)m, ((p \perp q) \otimes q_1, \dots, (p \perp q) \otimes q_r)) \\ &= L^k(lm + nm, ((p \otimes q_1) \perp (q \otimes q_1), \dots, (p \otimes q_r) \perp (q \otimes q_r))) \\ &= p \cdot L^k(m, \bar{q}) \perp q \cdot L^k(m, \bar{q}). \end{aligned}$$

(2) Notice,

$$\begin{aligned} \langle 1 \rangle \cdot L^k(m, \bar{q}) &= L^k(m, (\langle 1 \rangle \otimes q_1, \dots, \langle 1 \rangle \otimes q_r)) \\ &= L^k(m, (q_1, \dots, q_r)) \\ &= L^k(m, \bar{q}). \end{aligned}$$

Thus  $\langle 1 \rangle \cdot (-)$  is the identity function.

(3) We find:

$$\begin{aligned} p \cdot (q \cdot L^k(m, \bar{q})) &= p \cdot L^k(m, (q \otimes q_1, \dots, q \otimes q_r)) \\ &= L^k(m, (p \otimes (q \otimes q_1), \dots, p \otimes (q \otimes q_r))) \\ &= L^k(m, ((p \otimes q) \otimes q_1, \dots, (p \otimes q) \otimes q_r)) \\ &= (p \otimes q) \cdot L^k(m, \bar{q}). \end{aligned}$$

□

*Remark.* If we defined the action on the right instead, i.e.

$$L^k(m, \bar{q}) \cdot q = L^k(mn, (q_1 \otimes q, \dots, q_r \otimes q)),$$

in place of (2) from Proposition 3.12, we would have:

$$(L^k(m, \bar{q}) \perp L^k(n, \bar{p})) \cdot q = L^k(m, \bar{q}) \cdot q \perp L^k(n, \bar{p}) \cdot q.$$

Of course, letting  $\Gamma = \{\Gamma^r\}_{r \geq 1}$ , we can think of  $Q$  as acting on this graded structure  $\Gamma$  via the binary operation

$$\cdot : Q \times \Gamma \rightarrow \Gamma$$

defined as in Proposition 3.12 in degree  $r$ .

Furthermore, if we have a rational space  $X$  with model  $\mathcal{M}(X) = L^k(m, \bar{q})$  we can then define, for  $q \in Q$ ,

$$q \cdot X = \langle q \cdot L^k(m, \bar{q}) \rangle.$$

This gives us an action of quadratic forms on the collection of rational spaces  $X$  with minimal models corresponding to some system of quadratic forms  $\bar{q}$ . Hess and Parent describe a similar action of quadratic forms on spaces in [15], though created from their embedding of the Witt group into the rational cellular lattice. It is currently unknown what the implications of either of these actions are, as is what connections (if any) exist between them.

# Chapter 4

## Potential Directions for Future Inquiry

This section gathers an assortment of questions that cropped up throughout our journey, and, in some instances, discusses various ideas currently surrounding them. Some questions are rather concrete, while others are more open-ended. Moreover, some questions have experienced more advanced development than others. That being said, the author notes the questions appear in no particular order.

### On the Resolvability Relation and the Cellular Lattice

Clearly there are some parallels between  $(\mathcal{S}, \preceq)$  and the cellular lattice. The fundamental concepts underlying them (resolving classes and closed classes, respectively) are dual to one another, for instance; not to mention the same quotient of the Witt group appearing in both, nor the Galois connection between strong closed classes and strong resolving classes ([26]; see Section 2.1). Just how far do these comparisons go? In what ways are these structures dual to one another? Is there a conceptual reason as to why the same copy of the Witt group is embedded in both? How much of what we did here dualizes to the cellular case?

### Obstruction Theory on Resolvable Spaces

Corollary 2.14 offers a criterion for determining if  $A \preceq X$  in terms of a product indexed by  $[X, A]$ . In particular, if  $[X, A]$  is “too small”, then we are able to conclude  $X \notin \mathcal{R}(A)$ . What exactly does “too small” mean in this situation? What sort of conditions on  $[X, A]$  can obstruct a forming relation  $A \preceq X$ ?

## Resolving Rational Density

As seen previously,  $(\mathcal{S}, \preceq)$  does not satisfy the same density conditions as the cellular lattice. This was shown in Proposition 3.2 where it was argued

$$S^2 \stackrel{s}{\preceq} K(\mathbb{Q}, 3),$$

but there does not exist a simply connected rational space  $Z$  with

$$S^2 \stackrel{s}{\preceq} Z \stackrel{s}{\preceq} K(\mathbb{Q}, 3).$$

Is this, perhaps, a pathological example? In other words, is there some (considerable) subcollection of  $(\mathcal{S}, \preceq)$  satisfying density theorems analogous to those available in the cellular lattice?

A related question is the following: what happens to the spaces between  $S^2$  and  $K(\mathbb{Q}, 3)$  in  $(\mathcal{S}, \preceq)$  after we rationalize them? Which ones become  $S^2$  in  $\mathcal{S}$ ? Which ones become  $K(\mathbb{Q}, 3)$  in  $\mathcal{S}$ ? Why? Answering this may offer some more insight into what a resolvable equivalence tells us about the spaces involved.

## A Matter of Principal

We define a closed (resp. resolving) class  $\mathcal{D}$  to be **cellularly** (resp. **resolvably**) **generated by a set**  $S \subseteq \mathcal{D}$  (called **strongly generated** in [4]) if  $\mathcal{D} = \mathcal{C}(S)$ , where  $\mathcal{C}(S)$  is the smallest closed class containing the set  $S$  (resp. if  $\mathcal{D} = \mathcal{R}(S)$ , where  $\mathcal{R}(S)$  is the smallest resolving class containing  $S$ ). Given a class of spaces  $\mathcal{D}$ , say  $\mathcal{D}$  is **principally resolved** if  $\mathcal{D} = \mathcal{R}(A)$  for some  $A \in \mathcal{D}$ , and **principally cellular** provided  $\mathcal{D} = \mathcal{C}(A)$  for some  $A \in \mathcal{D}$ . Note, in the terminology of [4], to say  $\mathcal{D} = \mathcal{C}(A)$  is to say  $A$  is a **cellular generator** of  $\mathcal{D}$ , so we can analogously call  $A$  a **resolving generator** of  $\mathcal{D}$  if  $\mathcal{D} = \mathcal{R}(A)$ .

As it turns out, the class of  $(n - 1)$ -connected spaces is principally cellular, being  $\mathcal{C}(S^n)$ . What of the class of  $(n + 1)$ -anticonnected spaces,  $\mathcal{D}_{n+1}$ ? We know by Theorem 2.10

$$\mathcal{D}_{n+1} \subset \mathcal{R}(K(\mathbb{Q}, n + 1)),$$

but this containment is strict. Is it true that  $\mathcal{D}_{n+1} = \mathcal{R}(A)$  for some  $A \in \mathcal{D}_{n+1}$ ?

Answering this question could involve coming to a better understanding of when a resolving class is principally resolved. In [4], conditions are laid out under which a closed class  $\mathcal{D}$  is cellularly generated by a set  $S \subseteq \mathcal{D}$ . In this case

$$\mathcal{D} = \mathcal{C} \left( \bigvee_{X \in S} X \right),$$

where, since  $S$  is a set,  $\bigvee_{X \in S} X \in \mathcal{D}$ . If  $\mathcal{D}$  is a resolving class, can we find similar conditions under which a set  $S$  resolvably generates  $D$ ? If so, we can guarantee said  $\mathcal{D}$  would be principally resolved via  $A = \prod_{X \in S} X \in \mathcal{D}$ .

## “Lattice” Ask a Question

As discussed in Section 3.1, it has been shown that the cellular lattice is complete [4]. At the time of this writing, we know  $(\mathcal{S}, \preceq)$  is a complete lower semilattice, but it is unknown whether  $(\mathcal{S}, \preceq)$  has least upper bounds.

For a set  $S \subseteq \mathcal{S}$ , to say  $Z$  is a least upper bound of  $S$  with respect to  $\preceq$  is to say:

1.  $X \preceq Z$  for every  $X \in S$  (i.e.,  $Z \in \mathcal{R}(X)$  for every  $X \in S$ );
2. if  $X \preceq A$  for every  $X \in S$ , then  $Z \preceq A$  (i.e., if  $A \in \mathcal{R}(X)$  for every  $X \in S$ , then  $A \in \mathcal{R}(Z)$ ).

Notice then,  $Z$  is a least upper bound for  $S$  if and only if  $\bigcap_{X \in S} \mathcal{R}(X) = \mathcal{R}(Z)$ . So, the existence of least upper bounds with respect to  $\preceq$  is tied to the previous question concerning when a resolving class is principally resolved.

## Inducing Equivalence Relations on Finite Sets

In Lemma 2.18 and Theorem 2.19, we had a rational space  $X$  with

$$\mathcal{M}(X) = (\Lambda(V \oplus W \oplus Z), d)$$

where:

- $V$  is concentrated in degree  $k$  (for  $k \geq 2$  even);
- $W$  is nontrivial in at most degrees  $q$  for  $2 \leq q \leq k - 1$  and  $k + 1 \leq q \leq k(n + 1) - 2$ ; and
- $Z$  is concentrated in degree  $k(n + 1) - 1$ .

We then let  $X_0$  be a rational space with  $\mathcal{M}(X_0) = (\Lambda(V \oplus Z), D)$ , where  $D$  is induced by  $d$  after quotienting by  $W$ , and found the projection  $V \oplus W \oplus Z \rightarrow V \oplus Z$  induces a CDGA morphism  $p : \mathcal{M}(X) \rightarrow \mathcal{M}(X_0)$  that is nontrivial in degree  $k(n + 1) - 1$ . We concluded  $X \preceq X_0$ .

More generally, let  $X$  be an  $(n+1)$ -anticonnected rational space with  $\mathcal{M}(X) = (\Lambda V, d)$ , and say  $\mathcal{I} = \{2, \dots, n\}$  (i.e., the collection of  $i$  such that  $V^i$  is potentially nontrivial). For a subset  $\mathcal{J} \subseteq \mathcal{I}$  define

$$V^{\mathcal{J}} = \bigoplus_{j \in \mathcal{J}} V^j,$$

and call  $X_{\mathcal{J}}$  the space with  $\mathcal{M}(X_{\mathcal{J}}) \cong (\Lambda(V^{\mathcal{I}-\mathcal{J}}), D)$ . If we have subsets  $\mathcal{J} \subseteq \mathcal{L} \subseteq \mathcal{I}$ , then the projections

$$\bigoplus_{i \in \mathcal{I}} V^i \rightarrow \bigoplus_{i \in \mathcal{I}-\mathcal{J}} V^i \rightarrow \bigoplus_{i \in \mathcal{I}-\mathcal{L}} V^i$$

induce CDGA morphisms

$$\mathcal{M}(X) \rightarrow \mathcal{M}(X_{\mathcal{J}}) \rightarrow \mathcal{M}(X_{\mathcal{L}}).$$

As it turns out, for  $\mathcal{J} \subseteq \mathcal{L} \subseteq \mathcal{I}$ , we find  $X \approx X_{\mathcal{J}} \approx X_{\mathcal{L}}$ .

**Example.** In the case of  $X$  is as in Theorem 2.19, let  $\mathcal{I} = \{2, \dots, k(n+1) - 1\}$  and  $\mathcal{J} = \{2, \dots, k-1, k+1, \dots, k(n+1) - 2\}$ . Then what we were calling  $X_0$  can equivalently be taken as  $X_{\mathcal{J}}$ .

We can define an equivalence relation on  $2^{\mathcal{I}}$  (the set of all subsets of  $\mathcal{I}$ ) corresponding to  $\sim_r$  on spaces as follows:

**Definition.** For  $X$  and  $\mathcal{J}, \mathcal{L} \subseteq \mathcal{I}$  as above, define  $\mathcal{J} \sim_r \mathcal{L}$  provided  $X_{\mathcal{J}} \sim_r X_{\mathcal{L}}$ .

What information about the sets  $\mathcal{J}$  and  $\mathcal{L}$  does  $\sim_r$  encode, and how does the space  $X$  affect the induced equivalence relation?

**Example.** Let  $X = \prod_{i=2}^n K(\mathbb{Q}, i)$  and say  $\mathcal{I} = \{2, \dots, n\}$ . Then, for  $\mathcal{J}, \mathcal{L} \subseteq \mathcal{I}$ ,  $\mathcal{J} \sim_r \mathcal{L}$  if and only if the maximum element of  $\mathcal{I} - \mathcal{J}$  is also the maximum element of  $\mathcal{I} - \mathcal{L}$  (under the usual ordering of natural numbers). To see this, notice  $\text{aconn}(X_{\mathcal{J}}) = \max(\mathcal{I} - \mathcal{J}) + 1$  and similarly for  $X_{\mathcal{L}}$ . The claim follows since, as products of Eilenberg-MacLane spaces,  $X_{\mathcal{J}} \sim_r X_{\mathcal{L}}$  if and only if they have the same anticonnectivity.

So, for a product of Eilenberg-MacLane spaces as above, the equivalence relation on  $2^{\mathcal{I}}$  isn't terribly complicated. Spaces with more complicated minimal models may induce more complicated equivalence relations. By choosing an appropriate space  $X$ , can we recover known combinatorial relations? What sorts of equivalence relations on  $2^{\mathcal{I}}$  are induced by a space  $X$  in this manner? Can we use them to learn more about the space  $X$  involved?

## May I Take Your Order?

As discussed in Section 3.2, the relation  $\preceq$  on CDGA's of the form  $L^k(n, q)$  (for  $q$  an  $n$ -dimensional anisotropic  $m$ -form) can be used to put a partial order on  $\overline{W(\mathbb{Q})}$ , and, more generally,  $\overline{\mathbb{W}^1}$  (the collection of similarity classes of all  $m$ -forms,  $m \geq 2$ ). Can we learn anything about quadratic forms or homogeneous polynomials from these partial orders? How does our partial order on  $\overline{W(\mathbb{Q})}$  compare to the one defined by Hess and Parent in [15]?

## Taking Action

At the end of Section 3.3, we define an action of quadratic forms on the collection of rational spaces  $X$  with model  $\mathcal{M}(X) = L^k(m, \bar{q})$  (where  $\bar{q} = (q_1, \dots, q_r)^T$  is a system of  $m$ -dimensional quadratic forms). Specifically, for  $q$  an  $n$ -dimensional quadratic form, we take

$$q \cdot X = \langle q \cdot L^k(m, \bar{q}) \rangle,$$

where  $q \cdot L^k(m, \bar{q}) = L^k(nm, (q \otimes q_1, \dots, q \otimes q_r))$ .

As mentioned, Hess and Parent do something similar in [15] based on their embedding of  $\overline{W(\mathbb{Q})}$  into the cellular lattice. The questions here are: what effect does this action have topologically, and how does our action compare to the one defined by Hess and Parent?

## Geometric Version of the Unreduced Sum and Product

For a given even  $k \geq 2$ , in Section 3.2 we constructed embeddings

$$e_k : \overline{W(\mathbb{Q})} \rightarrow \mathcal{S}.$$

For the remainder of this section, fix an even  $k \geq 2$  and let  $\mathcal{S}_W$  represent the image of  $e_k$ . We previously discussed using this embedding to produce a partial order on  $\overline{W(\mathbb{Q})}$  corresponding to  $\preceq$ . We can also describe an operation on  $\mathcal{S}_W$  corresponding to  $\otimes$ .

Specifically, define  $\otimes : \mathcal{S}_W \times \mathcal{S}_W \rightarrow \mathcal{S}_W$  via the diagram:

$$\begin{array}{ccc} \overline{W(\mathbb{Q})} \times \overline{W(\mathbb{Q})} & \xrightarrow{\otimes} & \overline{W(\mathbb{Q})} \\ e_k \times e_k \downarrow & & \downarrow e_k \\ \mathcal{S}_W \times \mathcal{S}_W & \xrightarrow{\otimes} & \mathcal{S}_W. \end{array}$$



In other words, for  $X, Y \in \mathcal{S}_W$  with  $\mathcal{M}(X) = L^k(n, q)$  and  $\mathcal{M}(Y) = L^k(m, q')$ , we define

$$X \otimes Y \sim_r \langle L^k(nm, q \otimes q') \rangle.$$

Of course, this is not a particularly insightful definition. It would be nice if we could describe geometric versions of  $\otimes$  (and, for that matter,  $\perp$ ) in terms of other operations on spaces with which we are already familiar. In [10], Hess and Parent describe the smash product,  $\wedge$ , as modeling  $\otimes$  geometrically (with a “shift in generators”). As a colimit construction, the smash product seems an unlikely fit for the geometric model for  $\otimes$  on  $\mathcal{S}_W$  (though it would be an interesting development if it did). Moreover, the dual idea, i.e. taking the fiber of  $X \vee Y \rightarrow X \times Y$ , does not produce a geometric model for  $\otimes$ .

**Example.** Suppose we have a fiber sequence

$$F \rightarrow X \vee Y \rightarrow X \times Y.$$

Furthermore, let  $X, Y \in \mathcal{S}_W$  where, for simplicity, assume  $\mathcal{M}(X) \cong L^k(2, \langle 1, 1 \rangle) \cong \mathcal{M}(Y)$ . Specifically, say  $\mathcal{M}(X) = (\Lambda(x_1, x_2, y), dy = x_1^2 + x_2^2)$  and  $\mathcal{M}(Y) = (\Lambda(w_1, w_2, z), dz = w_1^2 + w_2^2)$ . We note, then:

$$\mathcal{M}(X \times Y) \cong (\Lambda(x_1, x_2, w_1, w_2, y, z), d),$$

while

$$\mathcal{M}(X \vee Y) \cong (\Lambda(x_1, x_2, w_1, w_2, y, z, v_{ij}, \dots), d),$$

where  $|v_{ij}| = 2k - 1$  for  $i, j \in \{1, 2\}$  with  $dv_{ij} = x_i w_j$  (be aware: we are only listing  $\mathcal{M}(X \vee Y)$  in low degrees; this space is infinitely anticonnected).

Observe, we have an exact sequence:

$$\pi_{n+1}(X \times Y) \rightarrow \pi_n(F) \rightarrow \pi_n(X \vee Y) \rightarrow \pi_n(X \times Y).$$

Thus  $\pi_n(F) \cong \pi_n(X \vee Y)$  for all  $n \geq 2k$ . In particular,  $F$  is infinitely anticonnected, so  $F \notin \mathcal{S}_W$ . Therefore,  $F$  does not model a quadratic form (even up to resolvable equivalence).

## Without a Trace

There is a fiber sequence

$$\Omega X * \Omega Y \rightarrow X \vee Y \rightarrow X \times Y.$$

Although it was argued in the previous section that  $\Omega X * \Omega Y$  does not model a quadratic form, by killing off elements of the homotopy groups of this space, we can obtain a space that may indeed model a quadratic form. Importantly, the elements acting as the variables for the quadratic form in this situation would be in degree  $2k - 1$ . Now,  $2k - 1$  is odd, so the quadratic form represented will necessarily be traceless! This could potentially be used to produce a new operation on similarity classes of quadratic forms taking its values in  $sl(n, \mathbb{Q})$  (the Lie algebra of the special linear group  $SL(n, \mathbb{Q})$ ).

Similarly, something to this effect (that is, using a geometric operation to induce an operation on similarity classes of  $m$ -forms) could possibly be used to create analogues to  $\overline{W}(\mathbb{Q})$  for  $m \geq 3$ . The (many) details here have yet to be investigated to any notable degree. There is a glimmer of hope, however, that liberating our topological intuition to this situation can possibly produce a work-around in describing higher degree analogues of  $\overline{W}(\mathbb{Q})$ .

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