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Chromatic Connectivity of Graphs

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by

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CHROMATIC CONNECTIVITY OF GRAPHS

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Let G be an edge-colored connected graph. A path P is a *proper path* in G if no two adjacent edges of P are colored the same. If P is a proper $u-v$ path of length $d(u,v)$, then P is a *proper* $u-v$ *geodesic*. An edge coloring c is a proper-path coloring of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G and c is a strong proper coloring if every two vertices u and v are connected by a proper $u-v$ geodesic in G. The minimum number of colors required in a proper-path coloring (strong proper coloring) of G is called the *proper connection number* $\mathrm{pc}(G)$ (*strong proper connection number* $\mathrm{spc}(G)$) of G. These concepts are inspired by the well-known and well-studied concepts of rainbow coloring, rainbow connection number $\mathrm{rc}(G)$, strong rainbow coloring and strong connection number $\mathrm{src}(G)$ of a connected graph G.

We investigate the relationship among these four edge colorings as well as proper edge colorings in graphs, the best known edge colorings. Several realization results are established for the five edge coloring parameters $\mathrm{pc}(G)$, $\mathrm{spc}(G)$, $\mathrm{rc}(G)$, $\mathrm{src}(G)$ and chromatic index $\chi'(G)$ of a connected graph G (the minimum number of colors required in a proper edge coloring of G). Furthermore, the exact values of $\mathrm{pc}(G)$ and $\mathrm{spc}(G)$ are determined for several well-known classes of graphs G. If G is a nontrivial connected graph of size m, then $\mathrm{pc}(G) \le \mathrm{spc}(G) \le m$ and (i) $\mathrm{pc}(G) = 1$ or $\mathrm{spc}(G) = 1$ if and only of G is the complete graph K_n of order n and (ii) $\mathrm{pc}(G) = m$ or $\mathrm{spc}(G) = m$ if and only if G is the star of size m. Several well-known classes of connected graphs G are shown to have $\mathrm{pc}(G) = 2$. Connected graphs G of size m for which $\mathrm{pc}(G)$ or $\mathrm{spc}(G)$ is $m-1, m-2$ or $m-3$ are characterized.

We study proper-path colorings in those graphs obtained from some well-known graph operations, namely joins of graphs and Cartesian products of graphs, permutations graphs, line graphs, powers of graphs, coronas of graphs, vertex or edge deletions as well as the well-known class of unicyclic graphs. In fact, proper connection numbers are determined for all joins and Cartesian products of two nontrivial connected graphs, for all iterated line graphs and powers of a given connected graph. For a connected graph G, sharp lower and upper bounds are established for

the proper connection number of (i) the k-iterated corona of G in terms of $\mathrm{pc}(G)$ and k and (ii) the vertex or edge deletion graphs $G - v$ and $G - e$ where v is a non-cut-vertex of G and e is a non-bridge of G in terms of $\mathrm{pc}(G)$ and the degree of v. Since it is in general very challenging to determine $\mathrm{pc}(G)$ and $\mathrm{spc}(G)$ for any given connected graph G, we establish several sharp bounds for $\mathrm{pc}(G)$ and $\mathrm{spc}(G)$ in terms of the order, size or maximum degree of the graph G.

We extend the study of proper connection (also called chromatic connection) in connected graphs to the study of the chromatic connectivity of ℓ-connected graphs for some integer $\ell \geq 2$. Suppose that G is an ℓ-connected graph for some positive integer ℓ. It then follows from a well-known theorem of Whitney that for every integer k with $1 \leq k \leq \ell$ and every two distinct vertices u and v of G, the graph G contains k internally disjoint $u - v$ paths. An edge coloring of a connected graph G is called a *proper k-path coloring* of G for some positive integer k if for every two distinct vertices u and v of G, there exist at least k internally disjoint proper $u-v$ paths. The minimum number of the colors required in a proper k-path coloring of G is the *proper k-connectivity* $\mathrm{pc}_k(G)$ of G. Thus, $\mathrm{pc}_1(G) = \mathrm{pc}(G)$ is the proper connection number of G. We investigate the proper k-connectivity of highly connected graphs. In particular, we establish sharp lower bounds for the proper k-connectivity of certain complete bipartite graphs $K_{r,s}$ of order $r+s$ where $2 \leq r \leq s$, which are improvements of some known results. Furthermore, the proper 2-connectivity of $K_{r,s}$ has been determined for all integers r and s with $2 \leq r \leq s$. The exact values of $\mathrm{pc}_k(K_{r,s})$ have been determined for some specific values of k and for several other classes of complete bipartite graphs. Open questions are also presented on this parameter of complete bipartite graphs.

A connected graph G of diameter at least 2 is *ℓ-geodesic connected* for some positive integer ℓ if for every two *nonadjacent* vertices u and v of G, there exist at least ℓ internally disjoint $u - v$ geodesics in G. Thus, every two nonadjacent vertices u and v are connected by ℓ internally disjoint $u - v$ paths of length $d(u, v)$. An edge coloring of a graph G is called a *proper k-geodesic coloring* (or a *strong proper k-path coloring*) of G for some positive integer k if for every two nonadjacent vertices u and v of G, there exist at least k internally disjoint proper $u-v$ geodesics in G. The minimum number of colors required in a proper k-geodesic coloring of G is the *strong proper k-connectivity* $\mathrm{spc}_k(G)$ of G. In particular, $\mathrm{spc}_1(G) = \mathrm{spc}(G)$ is the *strong proper connection number* of a connected graph G. We investigate the strong proper k-connectivity of complete bipartite graphs, establish lower bounds for the strong proper k-connectivity of certain complete bipartite graphs and determine exact values of $\mathrm{spc}_k(K_{r,s})$ for some specific values of k and for some class of complete bipartite graphs. Conjectures and open questions are also presented in this area of research. Furthermore, we introduce several related topics for further study.

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Elliot Laforge

TABLE OF CONTENTS

Table of Contents - continued

List of Figures

Chapter 1

Introduction

1.1 Background and Motivation

One of the most fundamental properties that a graph G can possess is that of being connected. In a connected graph G, there is at least one path connecting every two vertices. Often one is interested in paths possessing some prescribed property, such as a path of minimum length or one of maximum length connecting each pair of vertices in G. Should the graph G be edge-colored (that is, every edge is assigned a color from some prescribed set of colors), then there are other properties of interest that have been studied or could be studied, such as coloring requirements of the edges of a path connecting each pair of vertices.

There have been several research studies dealing with certain subgraphs H of an edge-colored graph G. Examples of these are colorings of G resulting in a monochromatic H (in which every two edges of H are colored the same), a rainbow H (in which no two edges of H are colored the same) or a properly colored H (in which no two adjacent edges of H are colored the same).

If G itself is a monochromatic graph, then every two vertices of G are connected by at least one monochromatic path since G is connected. Similar statements can be made if G is a rainbow graph or if G is a properly colored graph. The primary research studies have occurred, however, when the edges of G are colored arbitrarily and a desired edge-colored subgraph (or subgraphs) H must result, or whether an edge coloring of G exists that results in such an edge-colored subgraph (or subgraphs). For example, if our major interest is that of requiring every two vertices of G to be connected by at least one rainbow path, then it is typically unnecessary for the edges of G to be colored with distinct colors. That is, if G has size m, it is quite likely that there exists an edge coloring of G using fewer than m colors and having the property that every two vertices are connected by at least one rainbow path. An edge-colored graph with this property has been called rainbow-connected. Such a coloring is called a rainbow coloring of G.

Formally, a *rainbow coloring* of a connected graph G is an edge coloring c of G with the property that for every two vertices u and v of G, there exists a $u-v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is *rainbow-connected* (*with respect to* c). The minimum number of colors needed for a rainbow coloring of G is referred to as the *rainbow connection number* of G and is denoted by $\mathrm{rc}(G)$. There is a related concept concerning rainbow colorings and distance. Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a *rainbow* $u-v$ *geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v (the length of a shortest $u-v$ path in G). The graph G is called *strongly rainbow-connected* if G contains a rainbow $u-v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum number of colors needed for a strong rainbow coloring of G is referred to as the *strong rainbow connection number* $\mathrm{src}(G)$ of G. A strong rainbow coloring of G using $\mathrm{src}(G)$ colors is called a *minimum strong rainbow coloring* of G. Thus, $\mathrm{rc}(G) \leq \mathrm{src}(G)$ for every connected graph G. These concepts were introduced and studied by Chartrand, Johns, McKeon and Zhang in 2006 where the paper [7] was published in 2008. In recent years, this topic has been studied by many (see [8, 12, 26] for example); in fact, a book [28] on rainbow colorings was published in 2012.

While this concept was introduced for the purpose of studying connected graphs by means of rainbow paths in edge-colored graphs, additional motivation for the study of this concept has occurred (see [7]). As Ericksen discussed in [16], the Department of Homeland Security in the United States was created in 2003 in response to weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. In [16] Ericksen made the following observation:

> *An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels from radio systems to databases. The technologies utilized were separate entities and prohibited*

> *shared access, meaning there was no way for officers and agents to cross check information between various organizations.*

While information related to national security needs to be protected, there must be procedures in place that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises:

> *What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?*

As described in [7], this situation can be represented (modeled) by a graph and studied by means of rainbow colorings of the graph.

1.2 Proper Colorings and Connectivity

The vertex colorings of a graph G that have received the most attention over the years are proper colorings. A *proper vertex coloring* of a graph G is a function $c : V(G) \to S$ such that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G. In our case, $S = [k] = \{1, 2, \ldots, k\}$ for some positive integer k. Since $|S| = k$ here, the coloring c is called a *k-vertex coloring* (or, more often, simply a *k-coloring*) of G. The minimum positive integer k for which G has a k-vertex coloring is called the *chromatic number* of G, denoted by $\chi(G)$. For graphs of order $n \geq 3$, it is immediate which graphs of order n have chromatic number 1, n or 2. A graph is *empty* if it has no edges; thus, a *nonempty graph* has one or more edges.

Observation 1.2.1 *If G is a graph of order $n \geq 3$, then*

(a) $\chi(G) = 1$ *if and only if G is empty;*

(b) $\chi(G) = n$ *if and only if $G = K_n$;*

(c) $\chi(G) = 2$ *if and only if G is a nonempty bipartite graph.*

An immediate consequence of Observation 1.2.1(c) is that $\chi(G) \geq 3$ if and only if G contains an odd cycle. One of the most useful lower bounds for the chromatic number of a graph is stated next.

Proposition 1.2.2 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. The following result is therefore a consequence of Proposition 1.2.2.

Corollary 1.2.3 *For every graph G, $\omega(G) \leq \chi(G)$.*

By Corollary 1.2.3 (or, in fact, by Observation 1.2.1(c)), if a graph G contains a triangle, then $\chi(G) \geq 3$. There are graphs G for which $\chi(G)$ and $\omega(G)$ may differ significantly. As far as upper bounds for the chromatic number of a graph G are concerned, the following result gives such a bound in terms of the maximum degree $\Delta(G)$ of the graph G.

Theorem 1.2.4 *For every graph G, $\chi(G) \leq \Delta(G) + 1$.*

For each positive integer n, $\chi(K_n) = n = \Delta(K_n) + 1$ and for each odd integer $n \geq 3$, $\chi(C_n) = 3 = \Delta(C_n) + 1$. The British mathematician Rowland Leonard Brooks [5] proved that these two classes of graphs are the only connected graphs with this property.

Theorem 1.2.5 (Brooks' Theorem) *If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

Now, we turn to the graph coloring that is of primary interest us. A *proper edge coloring* c of a nonempty graph G is a function $c : E(G) \to \{1, 2, \ldots, k\}$ for some positive integer k with the property that $c(e) \neq c(f)$ for every two adjacent edges e and f of G. If the colors are chosen from a set of k colors, then c is called a *k-edge coloring* of G. The minimum positive integer k for which G has a k-edge coloring is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is immediate for every nonempty graph G that $\chi'(G) \geq \Delta(G)$. The best known and most useful theorem dealing with chromatic index is one obtained by the Russian mathematician Vadim Vizing [36].

Theorem 1.2.6 (Vizing's Theorem) *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

As a result of Vizing's theorem, the chromatic index of every nonempty graph G is one of two numbers, namely $\Delta(G)$ or $\Delta(G)+1$. A graph G with $\chi'(G) = \Delta(G)$ is called a *class one graph* while a graph G with $\chi'(G) = \Delta(G)+1$ is called a *class two graph.* The chromatic index of complete graphs is given in the following result.

Theorem 1.2.7 *For each integer $n \geq 2$,*

$$\chi'(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even} \\ n & \textit{if } n \textit{ is odd.} \end{cases}$$

Therefore, K_n is a class one graph if n is even and a class two graph if n is odd. The fact that K_n is a class one graph if and only if n is even is also a consequence of the following. A 1-regular spanning subgraph of G is called a 1-*factor* of G. A graph G is 1-*factorable* if G contains 1-factors $F_1, F_2, \ldots, F_r$ such that $\{E(F_1), E(F_2), \ldots, E(F_r)\}$ is a partition of $E(G)$. In this case, $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ is called a 1-*factorization* of G. Necessarily then, every 1-factorable graph is an r-regular graph of even order for some positive integer r.

Theorem 1.2.8 *A regular graph G is a class one graph if and only if G is 1-factorable.*

An immediate consequence of this result is stated next.

Corollary 1.2.9 *Every regular graph of odd order is a class two graph.*

The next two results describe classes of graphs that are class one graphs. The first theorem is due to Denés König [25].

Theorem 1.2.10 (König's Theorem) *Every bipartite graph is a class one graph.*

If a graph G of odd order has sufficiently many edges, then G must be a class two graph. A graph G of order n and size m is called *overfull* if $m > \Delta(G)\lfloor n/2 \rfloor$. If G has even order n, then $m \leq \Delta(G)\lfloor n/2 \rfloor$ and so G is not overfull. Therefore, only graphs of odd order can be overfull.

Theorem 1.2.11 *Every overfull graph is a class two graph.*

It is also useful to review some concepts and results dealing with connectivity of graphs. A *vertex-cut* of a graph G is a set S of vertices of G such that $G - S$ is disconnected. A vertex-cut of minimum cardinality in G is called a *minimum vertex-cut* of G and this cardinality is called the *vertex-connectivity* (or the *connectivity*) of G (when G is not complete) and is denoted by $\kappa(G)$. Complete graphs do not contain vertex-cuts and so the connectivity of the complete graph of order n is defined as $n - 1$, that is, $\kappa(K_n) = n - 1$. In general, the *connectivity* $\kappa(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. Therefore, for every graph G of order n,

$$0 \leq \kappa(G) \leq n - 1.$$

Thus, a graph G has connectivity 0 if and only if either $G = K_1$ or G is disconnected; a graph G has connectivity 1 if and only if $G = K_2$ or G is a connected graph with cut-vertices; and a graph G has connectivity 2 or more if and only if G is a nonseparable graph (connected and no cut-vertex) of order 3 or more.

A graph G is *k-connected* for some positive integer k if $\kappa(G) \geq k$. That is, G is k-connected if the removal of fewer than k vertices from G results in neither a disconnected nor a trivial graph. The 1-connected graphs are then the nontrivial connected graphs, while the 2-connected graphs are the nonseparable graphs of order 3 or more. A useful characterization of k-connected graphs is due to Hassler Whitney [37].

Theorem 1.2.12 (Whitney's Theorem) *A nontrivial graph G is k-connected for some positive integer k if and only if for each pair u, v of distinct vertices of G, there are at least k internally disjoint $u - v$ paths in G.*

An *edge-cut* of a graph G is a subset X of $E(G)$ such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a *minimum edge-cut* and this cardinality is the *edge-connectivity* of G, which is denoted by $\lambda(G)$. The trivial graph K_1 does not contain an edge-cut but its edge-connectivity is defined by setting $\lambda(K_1) = 0$. Therefore, $\lambda(G)$ is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Since the set of edges incident with any vertex of a graph G of order n is an edge-cut of G, it follows that

$$0 \leq \lambda(G) \leq \delta(G) \leq n - 1.$$

A graph G is *k-edge-connected* for some positive integer k if $\lambda(G) \geq k$; namely, G is k-edge-connected if the removal of fewer than k edges from G results in neither a disconnected graph nor a trivial graph. Thus, a 1-edge-connected graph is a nontrivial connected graph and a 2-edge-connected graph is a nontrivial connected bridgeless graph. There is an edge analogue of Whitney's Theorem.

Theorem 1.2.13 *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G.*

The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G. The *diameter* $\text{diam}(G)$ of G is the greatest eccentricity among the vertices of G, while the *radius* $\text{rad}(G)$ is the smallest eccentricity among the vertices of G. The diameter of G is also the greatest distance between any two vertices of G. A vertex v with $e(v) = \text{rad}(G)$ is called a *central vertex* of G and a vertex v with $e(v) = \text{diam}(G)$ is called a *peripheral vertex* of G. Two vertices u and v of G with $d(u, v) = \text{diam}(G)$ are *antipodal vertices* of G.

1.3 Proper-Path Colorings

We now describe a major concept discussed in this dissertation. One property that a connected graph G with a proper edge coloring has is that for every two vertices u and v of G, there is a properly colored $u - v$ path in G. In fact, every $u - v$ path in such a graph G is properly colored. However, if our primary interest concerns edge colorings of G with the property that for every two vertices u and v of G, there exists a properly colored $u - v$ path in G, then this may very well be possible using fewer than $\chi'(G)$ colors.

Inspired by rainbow colorings and proper colorings of graphs, Chartrand [1] and Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza [4] independently introduced the concept of proper-path colorings of graphs. This concept has been studied further by Li, Y. F. Sun and Y. Zhao in [29] and Li, Wei and Yue in [30]. In fact, there is a dynamic survey on this topic due to Li and Magnant [27].

We now describe those concepts that we will be considering in this dissertation. Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G

if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G. If k colors are used, then c is referred to as a *proper-path k-coloring.* The minimum k for which G has a proper-path k-coloring is called the *proper connection number* $\mathrm{pc}(G)$ of G. A proper-path coloring using $\mathrm{pc}(G)$ colors is referred to as a *minimum proper-path coloring.*

To illustrate this concept, we consider the 3-regular graph G of Figure 1.1 and a proper-path 3-coloring of G using the colors 1, 2, 3 where the uncolored edges can be colored with any of these three colors. Since the three bridges in G must be assigned distinct colors, this coloring is minimum and so $\mathrm{pc}(G) = 3$. Note that this 3-regular graph G is not 1-factorable and so $\chi'(G) = 4$. That is, G has a proper-path coloring using fewer than $\chi'(G)$ colors. Indeed, if we were to replace the block of order 5 in the graph G of Figure 1.1 by a complete graph K_n of arbitrarily large odd order n, say, then the resulting graph H has the property that $\chi'(H) = n$ but $\mathrm{pc}(H) = 3$ and so $\chi'(H) - \mathrm{pc}(H)$ can be arbitrarily large. In fact, $\mathrm{pc}(K_n) = 1$ and $\chi'(K_n) = n$ if $n \geq 3$ is odd.

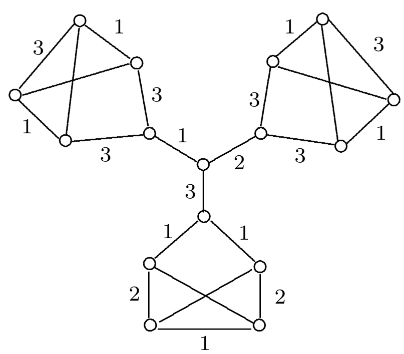

Figure 1.1: A proper-path 3-coloring of a graph G

Let G be a nontrivial connected graph of order n and size m. Since every rainbow coloring is a proper-path coloring, it follows that $\mathrm{pc}(G)$ exists and

$$1 \leq \mathrm{pc}(G) \leq \min\{\chi'(G), \mathrm{rc}(G)\} \leq m. \tag{1.1}$$

Furthermore, not only is $\mathrm{pc}(K_n) = 1$ but $\mathrm{pc}(G) = 1$ if and only if $G = K_n$. Also, $\mathrm{pc}(G) = m$ if and only if $G = K_{1,m}$ is a star of size m (see [1]).

While these concepts were introduced to parallel concepts dealing with rainbow colorings for the purpose of studying connected graphs by means of properly colored paths in edge-colored graphs, there is also corresponding motivation to that introduced for rainbow colorings of graphs. With regard to the national security discussion mentioned earlier, we are also interested in the answer to the following question:

> *What is the minimum number of passwords or firewalls that allow one or more secure paths between every two agencies where as we progress from one step to another along such a path, we are required to change passwords?*

Furthermore, Li and Magnant made the following observations in [27]:

> *When building a communication network between wireless signal towers, one fundamental requirement is that the network is connected. If there cannot be a direct connection between two towers A and B, say for example if there is a mountain in between, there must be a route through other towers to get from A to B. As a wireless transmission passes through a signal tower, to avoid interference, it would help if the incoming signal and the outgoing signal do not share the same frequency. Suppose we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and assign a color to each edge based on the assigned frequency used for the communication. Then the number of frequencies needed to assign the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph. ...*

In [14], other motivation has been given to study proper-path colorings of graphs. The Knight's Tour Problem is a famous problem that asks whether it's possible for a knight to tour an 8×8 chessboard where each square of the chessboard is visited exactly once (except that the final square visited is the initial square of the tour) and each step along the tour is a single legal move of a knight. It is well known that such a tour is, in fact, possible (see [2], for example). As described in [14], since a single move of a knight moves it from a square of one color to a square of the other color, it follows that if two knights were placed on any two squares of a chessboard, then there exists a path of legal moves from one knight to the other on a chessboard, using legal knight moves, such that the squares visited along the path alternate in color. Because the Knight's Tour Problem is equivalent to that of finding a particular type of Hamiltonian cycle on the grid $P_8 \square P_8$ (the Cartesian product of P_8 and P_8), this brings up two items, which are described in [14].

(1) Assign colors to the edges of a connected graph G so that for every two vertices u and v of G, there exists a $u - v$ path P in G such that every two adjacent edges on P have distinct colors.

(2) Assign colors to the vertices of a connected graph G so that for every two vertices u and v of G, there exists a $u - v$ path P in G such that every two adjacent vertices on P have distinct colors.

It is item (1), of course, that leads us once again to proper-path colorings of connected graphs (see [14]) .

1.4 Strong Proper-Path Colorings

Just as there is a "strong" version of rainbow colorings, there is a "strong" version of proper-path colorings. Let c be a proper-path coloring of a nontrivial connected graph G. For two vertices u and v of G, a *proper* $u - v$ *geodesic* in G is a proper $u - v$ path of length $d(u, v)$. If there is a proper $u - v$ geodesic for every two vertices u and v of G, then c is called a *strong proper-path coloring* of G or a *strong proper-path* k*-coloring* if k colors are used. The minimum number of colors needed to produce a strong proper-path coloring of G is called the *strong proper connection number* (or simply the *strong connection number*) $\text{spc}(G)$ of G. A strong proper-path coloring using $\text{spc}(G)$ colors is a *minimum strong proper-path coloring*. The concept of strong proper-path colorings of graphs was introduced by Chartrand and first studied in [1]. In general, if G is a nontrivial connected graph, then

$$1 \leq \text{pc}(G) \leq \text{spc}(G) \leq \chi'(G).$$

Since every strong rainbow coloring of G is a strong proper-path coloring of G, it follows that $\text{spc}(G) \leq \text{src}(G)$. Therefore, if G is a nontrivial connected graph of order n and size m, then

$$1 \leq \text{spc}(G) \leq \min\{\chi'(G), \text{src}(G)\} \leq m. \tag{1.2}$$

Similarly, $\text{spc}(G) = 1$ if and only if $G = K_n$ and $\text{spc}(G) = m$ if and only if $G = K_{1,m}$ is the star of size m (see [1]).

To illustrate these concepts, consider the two proper-path colorings of the 5-cycle C_5 and the proper-path coloring of the 3-regular graph G shown in Figure 1.2, where the graph G is the same graph in Figure 1.1. The coloring in Figure 1.2(a) is a minimum proper-path coloring of C_5 and so $\text{pc}(C_5) = 2$. The coloring in Figure 1.2(b) is a minimum strong proper-path coloring of C_5 and so $\text{spc}(C_5) = 3$. The coloring in Figure 1.2(c) is both a minimum proper-path coloring and a minimum strong proper-path coloring of G and so $\text{pc}(G) = \text{spc}(G) = 3$. As we saw, this 3-regular graph G is not 1-factorable and so $\chi'(G) = 4$. Notice that the coloring of the graph G in Figure 1.1 is a minimum proper-path coloring but whether it's a minimum strong proper-path coloring of G depends on how the six uncolored edges are colored. For example, if all six edges are colored 1, then the resulting coloring is not a minimum strong proper-path coloring of G.

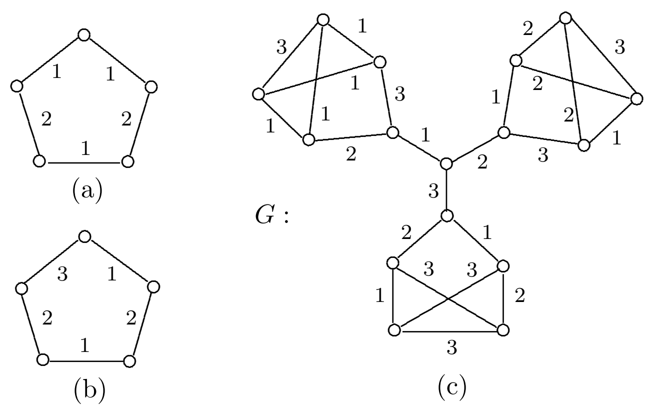

Figure 1.2: Illustrating proper-path colorings and strong proper-path colorings

We refer to the books [9, 13] for graph theory notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

Chapter 2

Preliminary Results

First, let's review some of the definitions and notation concerning proper-path colorings in graphs that were introduced in Chapter 1. Let G be an edge-colored connected graph. A path P is a *proper path* in G if no two adjacent edges of P are colored the same. If P is a proper $u - v$ path of length $d(u, v)$, then P is a *proper $u - v$ geodesic*. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u - v$ path in G and c is a *strong proper-path coloring* (or simply *strong proper coloring*) if every two vertices u and v are connected by a proper $u - v$ geodesic in G. The minimum number of colors required for a proper-path coloring and strong proper coloring of G are called the *proper connection number* $\mathrm{pc}(G)$ and *strong proper connection number* $\mathrm{spc}(G)$, respectively. As mentioned in Chapter 1, these concepts are inspired by the concepts of rainbow coloring, rainbow connection number $\mathrm{rc}(G)$, strong rainbow coloring and strong connection number $\mathrm{src}(G)$ of a connected graph G, which were introduced and first studied in [7].

In this chapter, we present some preliminary observations and results dealing with proper-path colorings and strong proper-path colorings of graphs. In this chapter, we determine $\mathrm{pc}(G)$ and $\mathrm{spc}(G)$ for several well-known classes of graphs G and investigate the relationship among these four edge colorings as well as the well-studied proper edge colorings in graphs. Furthermore, we establish several realization theorems for the five edge coloring parameters $\mathrm{pc}(G), \mathrm{spc}(G), \mathrm{rc}(G), \mathrm{src}(G)$ and the chromatic index $\chi'(G)$ of a connected graph G.

2.1 Proper Connection Numbers

Let G be a nontrivial connected graph of order n and size m. Recall, since every rainbow coloring of G is a proper-path coloring of G, that $\mathrm{pc}(G)$ exists and

$$1 \leq \mathrm{pc}(G) \leq \min\{\chi'(G), \mathrm{rc}(G)\} \leq m. \tag{2.1}$$

The graph shown in Figure 2.1 (also shown in Figure 1.1 with the same edge coloring) illustrates the following two useful facts of path connection numbers of graphs, where the uncolored edges can be colored with any one of the colors 1, 2 and 3.

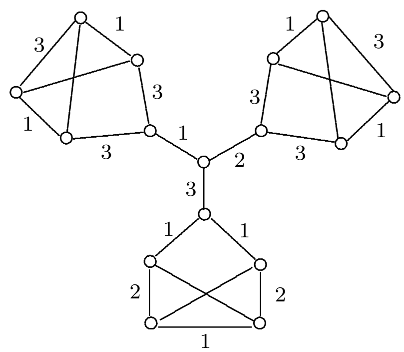

Figure 2.1: A proper-path 3-coloring of a graph

Although the following result is straightforward and easy to verify, it is very useful.

Proposition 2.1.1 *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then* $\mathrm{pc}(G) \leq \mathrm{pc}(H)$. *In particular,* $\mathrm{pc}(G) \leq \mathrm{pc}(T)$ *for every spanning tree T of G.*

Proof. Let H be a spanning subgraph of G and c_H a minimum proper-path coloring of H. Define a coloring c of G by $c(e) = c_H(e)$ if $e \in E(H)$ and $c(e) = 1$ for the remaining edges of G. Then c is a proper-path coloring of G using $\mathrm{pc}(H)$ colors and so $\mathrm{pc}(G) \leq \mathrm{pc}(H)$. ■

While the preceding result provides upper bounds for the path connection number of a graph G, the following result gives a lower bound for $\mathrm{pc}(G)$ under certain circumstances.

Proposition 2.1.2 *Let G be a nontrivial connected graph that contains bridges. If b is the maximum number of bridges incident with a single vertex in G, then* $\text{pc}(G) \geq b$.

Proof. Since $\text{pc}(G) \geq 1$, the result is trivial when $b = 1$. Thus, we may assume that $b \geq 2$. Let v be a vertex of G that is incident with b bridges and let vw_1 and vw_2 be two bridges incident with v. Since (w_1, v, w_2) is the only $w_1 - w_2$ path in G, it follows that every proper-path coloring of G must assign distinct colors to vw_1 and vw_2. Hence all b bridges incident with v are colored differently and so $\text{pc}(G) \geq b$. ■

Since every edge is a bridge in a nontrivial tree T, it follows that $\text{pc}(T) \geq \Delta(T)$ by Proposition 4.3.1. By König's theorem [25], $\chi'(G) = \Delta(G)$ for every nonempty bipartite graph G. Hence, the following is a consequence of (2.1), Proposition 4.3.1 and König's theorem.

Proposition 2.1.3 *If T is a nontrivial tree, then* $\text{pc}(T) = \chi'(T) = \Delta(T)$.

Propositions 2.1.1 and 4.3.2 provide an upper bound for the proper connection number of a graph.

Proposition 2.1.4 *For a nontrivial connected graph G,*

$$\text{pc}(G) \leq \min\{\Delta(T) : \ T \textit{ is a spanning tree of } G\}.$$

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph.* The following is an immediate consequence of Proposition 4.2.1.

Corollary 2.1.5 *If G is a traceable graph that is not complete, then*

$$\text{pc}(G) = 2.$$

We will see in the following section that the proper connection number of every complete bipartite graph $K_{s,t}$ where $s, t \geq 2$ is 2. If T is a spanning tree of $K_{2,t}$ for an arbitrarily large integer t, then $\Delta(T) \geq \lceil (t + 1)/2 \rceil$. Thus, the upper bound for $\text{pc}(G)$ of a nontrivial connected graph G in Proposition 4.2.1 can exceed $\text{pc}(G)$ by a significant amount.

We saw in (2.1) that if G is a nontrivial connected graph that is not complete such that $\text{pc}(G) = a$ and $\text{rc}(G) = b$, then $2 \leq a \leq b$. In fact, this is the only restriction on these two parameters. It is known [7] that if T is a tree, then $\mathrm{rc}(T)$ is the size of T.

Proposition 2.1.6 *For every pair a, b of positive integers with $2 \le a \le b$, there is a connected graph G such that $\mathrm{pc}(G) = a$ and $\mathrm{rc}(G) = b$.*

Proof. By Proposition 4.3.2, if T is a tree of order at least 3, then $\mathrm{pc}(T) = \Delta(T) \geq 2$. Since $\mathrm{rc}(T)$ is the size of T, it follows that for integers a and b with $2 \le a \le b$, a tree T of size b and $\Delta(T) = a$ has $\mathrm{pc}(T) = a$ and $\mathrm{rc}(T) = b$. ■

By (2.1), if G is a nontrivial connected graph that is not complete such that $\mathrm{pc}(G) = a$ and $\chi'(G) = b$, then $2 \le a \le b$. This restriction is all that is required for these two parameters.

Proposition 2.1.7 *For every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G such that $\mathrm{pc}(G) = a$ and $\chi'(G) = b$.*

Proof. If $b = 2$, then $a = b = 2$ and any path of order at least 3 has the desired property by Corollary 4.2.2. Thus, we may assume that $b \geq 3$. Let G be the graph obtained from the path $(x_1, x_2, x_3, x_4, x_5)$ of order 5 by (i) adding the $b-2$ new vertices $v_1, v_2, \ldots, v_{b-2}$ and joining each v_i $(1 \le i \le b-2)$ to both x_2 and x_4 and (ii) adding the $a-1$ new vertices $w_1, w_2, \ldots, w_{a-1}$ and joining each w_i $(1 \le i \le a-1)$ to x_5. Since G is a bipartite graph and $\Delta(G) = b$, it follows that $\chi'(G) = b$. It remains to show that $\mathrm{pc}(G) = a$. Define an edge coloring c by assigning (1) the color 1 to each of x_1x_2, x_2v_i $(1 \le i \le b-2)$, x_3x_4 and x_5w_1, (2) the color 2 to each of x_2x_3, x_4v_i $(1 \le i \le b-2)$ and x_5w_2 (3) the color i to x_5w_i $(3 \le i \le a-1$ if $a \geq 4)$ and (4) the color a to x_4x_5. Then every two vertices u and v are connected by a proper $u-v$ path. For example, v_1 and w_1 are connected by the proper path $(v_1, x_2, x_3, x_4, x_5, w_1)$. Hence, c is a proper-path coloring of G using a colors and so $\mathrm{pc}(G) \le a$. Assume, to the contrary, that $\mathrm{pc}(G) \le a-1$. Let c^* be a minimum proper-path coloring of G. Since $\deg x_5 = a$ and at most $a-1$ colors are used by c^*, there are two edges e and f incident with x_5 that are colored the same, say $e = ux_5$ and $f = x_5v$. However then, there is no proper $u-v$ path in G, which is impossible. Thus, $\mathrm{pc}(G) \geq a$ and so $\mathrm{pc}(G) = a$. ■

2.2 On Graphs with Proper Connection Number 2

By Proposition 2.1.1, if G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $\mathrm{pc}(G) \leq \mathrm{pc}(H)$. Therefore, if G is not complete and there exists a connected spanning subgraph H of G having proper connection number 2, then $\mathrm{pc}(G) = 2$. This provides added importance to study those graphs having proper connection number 2.

We saw that a nontrivial connected graph G has proper connection number 1 if and only if $G = K_n$. Also, if G contains a Hamiltonian path and G is not complete, then $\mathrm{pc}(G) = 2$ by Corollary 4.2.2. However, there are connected graphs G without a Hamiltonian path for which $\mathrm{pc}(G) = 2$. Indeed, we have mentioned that $\mathrm{pc}(K_{s,t}) = 2$ for $s, t \geq 2$ and this will be discussed in more detail soon. In fact, Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza in [4] obtained the following two results.

Theorem 2.2.1 *If G is a 3-connected graph that is not complete, then $\mathrm{pc}(G) = 2$.*

Theorem 2.2.2 *If G is a connected non-complete graph of order $n \geq 68$ and $\delta(G) \geq n/4$, then $\mathrm{pc}(G) = 2$.*

In addition, the following result was obtained by Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza in [4].

Theorem 2.2.3 *If G is a 2-connected graph that is not complete, then $\mathrm{pc}(G) \leq 3$.*

This theorem cannot be improved since the graph G of Figure 2.2 (obtained in [4]) has connectivity 2 and proper connection number 3. By Proposition 2.1.1, this graph contains no connected spanning subgraph with proper connection number 2. A proper-path 3-coloring of G is also shown Figure 2.2.

We now describe several classes of connected graphs having proper connection number 2 and present independent verifications of these facts.

The corona $\mathrm{cor}(K_n)$ of the complete graph K_n of order $n \geq 2$ is such an example. In fact, the coloring that assigns the color 1 to each edge that belongs to the subgraph K_n in $\mathrm{cor}(K_n)$ and the color 2 to each pendant edge in $\mathrm{cor}(K_n)$ is a proper-path coloring of $\mathrm{cor}(K_n)$. This example suggests that we can construct a new graph from each graph such that the resulting graph has proper connection number 2. For a

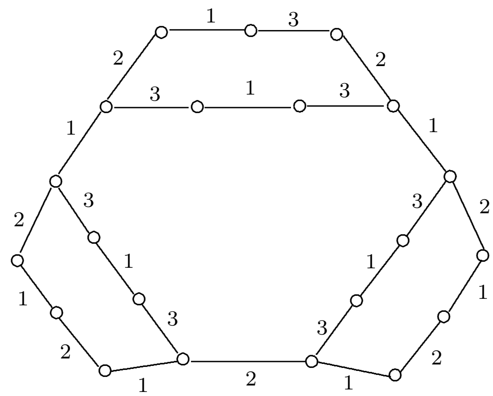

Figure 2.2: A graph G with connectivity 2
and proper connection number 3

given graph G of order n, let $\text{com}(G)$ be the graph obtained from G by replacing the n vertices of G by n mutually disjoint complete graphs, where $v \in V(G)$ is replaced by $K(v)$ of order $\deg_G v$, such that (i) one vertex in $K(v)$ is adjacent to one vertex in $K(u)$ if and only if $uv \in E(G)$ and (ii) a vertex x in $\text{com}(G)$ has degree k if and only if $x \in V(K(v))$ for some $v \in V(G)$ for which $\deg_G v = k$. This is illustrated in Figure 2.3. Thus, if G contains three or more end-vertices, then $\text{com}(G)$ contains no Hamiltonian path. In particular, if $G = K_{1,n}$ is a star, then $\text{com}(G) = \text{cor}(K_n)$. The coloring of $\text{com}(G)$ that assigns the color 1 to each edge in the complete graph $K(v)$ of order $\deg_G v \geq 2$ for every vertex v of G and the color 2 to the remaining edges of $\text{com}(G)$ is a proper-path coloring. This gives rise to the following observation.

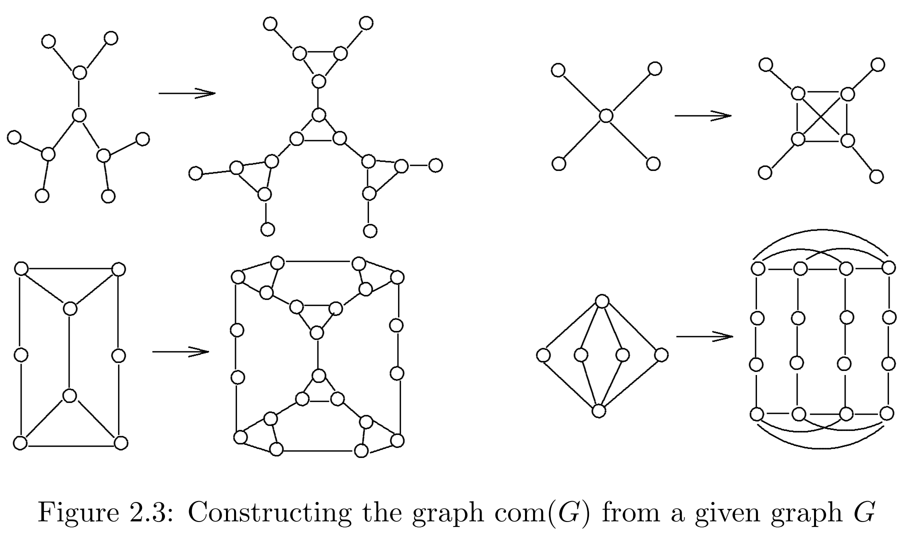

Figure 2.3: Constructing the graph $\text{com}(G)$ from a given graph G

Observation 2.2.4 *If G is a nontrivial connected graph, then*

$$\mathrm{pc}(\mathrm{com}(G)) = 2.$$

Recall that the *diameter* $\mathrm{diam}(G)$ of a connected graph G is the largest distance between two vertices of G. By an appropriate choice of a graph G, we can have $\mathrm{pc}(\mathrm{com}(G)) = 2$, $\mathrm{diam}(\mathrm{com}(G))$ arbitrarily large and the order of $\mathrm{com}(G)$ can exceed the length of a longest path in $\mathrm{com}(G)$ by an arbitrarily large number.

Next, we show that every complete multipartite graph that is neither a complete graph nor a star has proper connection number 2. In order to do this, we first present two lemmas.

Lemma 2.2.5 *Let H be the graph obtained from the cycle $C_4 = (u_1, v_1, u_2, v_2, u_1)$ of order 4 and two empty graphs $\overline{K}_r$ and $\overline{K}_s$ of order r and s, respectively, by joining each of u_1 and u_2 to every vertex in $\overline{K}_s$ and joining each of v_1 and v_2 to every vertex in $\overline{K}_r$. Then $\mathrm{pc}(H) = 2$.*

Proof. Since H is not complete, it suffices to show that H has a proper-path 2-coloring. Define a coloring c by assigning the color 1 to (i) the edge u_iv_i for $i = 1, 2$, (ii) the edge u_1x for each vertex x of $\overline{K}_s$ and (iii) the edge v_1y for each vertex y of $\overline{K}_r$ and assigning the color 2 to the remaining edges of H. We show that c is a proper-path 2-coloring of H. Let $u, v \in V(H)$. Since every two vertices on C_4 are connected by a proper path, we may assume that at least one of u and v does not belong to C_4, say v is not in C_4.

First, suppose that u is a vertex of C_4. By symmetry, we may assume that $u = u_1$ or $u = u_2$. Assume first that $u = u_1$. If v is a vertex of $\overline{K}_r$, then (u, v_1, u_2, v_2, v) is a proper $u - v$ path, while if v is a vertex of $\overline{K}_s$, then (u, v) is a proper $u - v$ path. Next, assume that $u = u_2$. If v is a vertex of $\overline{K}_r$, then (u, v_2, v) is a proper $u - v$ path, while if v is a vertex of $\overline{K}_s$, then (u, v) is a proper $u - v$ path. Next, suppose that u is not a vertex of C_4. By symmetry, we may assume u is a vertex of $\overline{K}_r$. If v is a vertex of $\overline{K}_r$, then (u, v_1, u_2, v_2, v) is a proper $u - v$ path, while if v is a vertex of $\overline{K}_s$, then (u, v_2, u_2, v) is a proper $u - v$ path. Thus, c is a proper-path 2-coloring of H and so $\mathrm{pc}(H) = 2$. ■

The proof of Lemma 2.2.5 provides the following lemma.

Lemma 2.2.6 *Let F be the graph obtained from the cycle $(v_1, v_2, v_3, v_4, v_1)$ of order 4 and an empty graphs $\overline{K}_r$ of order r by joining each of v_1 and v_3 to every vertex in $\overline{K}_r$. Then* $\mathrm{pc}(F) = 2$.

Theorem 2.2.7 *If G is a complete multipartite graph that is neither a complete graph nor a star, then* $\mathrm{pc}(G) = 2$.

Proof. Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph that is not complete, where $k \geq 2$ and $n = n_1 + n_2 + \cdots + n_k$ is the order of G. Denote the partite sets of G by $V_1, V_2, \ldots, V_k$ where $|V_i| = n_i$ for $1 \leq i \leq k$ and $|V_1| \geq |V_2| \geq \cdots \geq |V_k|$ and $|V_1| \geq 2$.

First, suppose that $k = 2$. Since G is not a star, $|V_2| \geq 2$. By Corollary 4.2.2, we may assume that $G \neq C_4$ and so $n \geq 5$. Thus, either G contains the graph H in Lemma 2.2.5 as a spanning subgraph or contains the graph F in Lemma 2.2.6 as a spanning subgraph. It then follows by Proposition 2.1.1 and Lemmas 2.2.5 and 2.2.6 that $\mathrm{pc}(G) \leq 2$. Hence, $\mathrm{pc}(G) = 2$.

Next, suppose that $k \geq 3$. If $n = 4$, then G contains a 4-cycle as a spanning subgraph and so $\mathrm{pc}(G) = 2$ by Corollary 4.2.2 and Proposition 2.1.1. Thus, we may assume that $n \geq 5$. First, suppose that $n_1 \geq 2$ and $n_i = 1$ for $2 \leq i \leq k$. Define a coloring c of G by assigning the color 1 to each edge that is incident with a vertex in V_1 and the color 2 to the remaining edges of G. Let u and v be two nonadjacent vertices of G. Then $u, v \in V_1$. Let $x \in V_2$ and $y \in V_3$. Then (u, x, y, v) is a proper $u - v$ path and so c is a proper-path 2-coloring of G. Next, suppose that $n_1 \geq 2$ and $n_2 \geq 2$. Let C_4 be a cycle of order 4 in the subgraph K_{n_1,n_2} of G where the partite sets of K_{n_1,n_2} are V_1 and V_2. Then either G contains the graph H in Lemma 2.2.5 as a spanning subgraph or contains the graph F in Lemma 2.2.6 as a spanning subgraph. By Proposition 2.1.1 and Lemmas 2.2.5 and 2.2.6, $\mathrm{pc}(G) = 2$ in either case. ■

The *join* $G \vee H$ of two graphs G and H has vertex set $V(G) \cup V(H)$ and its edge set consists of $E(G) \cup E(H)$ and the set $\{uv : u \in V(G) \text{ and } v \in V(H)\}$. With the aid of Lemmas 2.2.5 and 2.2.6, the following result can be proved.

Theorem 2.2.8 *If G and H are connected graphs not both of which are complete, then* $\mathrm{pc}(G \vee H) = 2$.

Proof. Necessarily, $G \vee H$ contains either the graph in Lemma 2.2.5 as a spanning subgraph or the graph in Lemma 2.2.6 as a spanning subgraph. By Proposition 2.1.1 and Lemmas 2.2.5 and 2.2.6, it follows that $\mathrm{pc}(G \vee H) = 2$ in either case. Thus, we may assume that G is a nontrivial connected graph of order at least 3 that is not complete and $H = K_1$ where $V(H) = \{w\}$. Since $G \vee K_1$ is not complete, it follows that $\mathrm{pc}(G \vee K_1) \geq 2$ and so it remains to show that $\mathrm{pc}(G \vee K_1) \leq 2$. Let T be a spanning tree of G. By Proposition 2.1.1, it suffices to show that $\mathrm{pc}(T \vee K_1) \leq 2$. For a vertex v of T, let $e_T(v)$ denote the eccentricity of v in T. For each integer i with $1 \leq i \leq e_T(v)$, let $V_i = \{u : d(v,u) = i\}$. Hence $V_0 = \{v\}$. Define a 2-coloring c of $T \vee K_1$ by

$$c(wx) = \begin{cases} 1 & \text{if } x \in V_i,\ i \text{ is odd and } 1 \leq i \leq e_T(v) \\ 2 & \text{if } x \in V_i,\ i \text{ is even and } 0 \leq i \leq e_T(v); \end{cases}$$

$$c(xy) = \begin{cases} 1 & \text{if } x \in V_i,\ y \in V_{i+1},\ i \text{ is even and } 0 \leq i \leq e_T(v) - 1 \\ 2 & \text{if } x \in V_i,\ y \in V_{i+1},\ i \text{ is odd and } 1 \leq i \leq e_T(v) - 1. \end{cases}$$

Let x and y be two vertices of $T \vee K_1$. Since w is adjacent to every vertex in T, we may assume $x \neq w$ and $y \neq w$ and so $x, y \in V(T)$. First, suppose that $x \in V_i$ and $y \in V_j$, where $0 \leq i < j$. If i and j are of opposite parity, then (x, w, y) is a proper $x - y$ path in $T \vee K_1$. Thus, we may assume that i and j are of the same parity and so $j - i \geq 2$. Let $z \in V_{j-1}$ such that yz is an edge of T. Then (x, w, z, y) is a proper $x - y$ path in $T \vee K_1$. Next, suppose that $x, y \in V_i$ for some i with $1 \leq i \leq e_T(v)$. Let $z \in V_{i-1}$ such that xz is an edge of T. Then (x, z, w, y) is a proper $x - y$ path in $T \vee K_1$. Hence, c is a proper-path 2-coloring of $T \vee K_1$ and so $\mathrm{pc}(T \vee K_1) = 2$. Therefore, $\mathrm{pc}(G \vee K_1) = 2$. ■

The *Cartesian product* $G \,\square\, H$ of two graphs G and H has vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and two distinct vertices (u,v) and (x,y) of $G \,\square\, H$ are adjacent if either (1) $ux \in E(G)$ and $v = y$ or (2) $vy \in E(H)$ and $u = x$. The Cartesian product $G \,\square\, K_2$ of a graph G and K_2 is a special case of a more general class of graphs. We partition the edge set $G \,\square\, H$ into two sets E_1 and E_2 such that E_1 is the set of edges $(u,v)(x,y)$ in $G \,\square\, H$ such that $ux \in E(G)$ and $v = y$ and E_2 is the set of edges $(u,v)(x,y)$ in $G \,\square\, H$ such that $u = x$ and $vy \in E(H)$. We show that $\mathrm{pc}(G \,\square\, H) = 2$ for every two nontrivial connected graphs G and H. In order to do this, we first present a lemma dealing withe proper-path colorings of grids $P_s \,\square\, P_t$.

Lemma 2.2.9 *For integers s and t with $s \geq t \geq 2$, let $P_s = (u_1, u_2, \ldots, u_s)$ be a path of order s and $P_t = (v_1, v_2, \ldots, v_t)$ be a path of order t. Define a coloring of the grid $P_s \,\square\, P_t$ by assigning the color 1 to all edges in E_1 and the color 2 to all edges in E_2. Then there is a proper path from (u_1, v_1) to (u_s, v_t) in $P_s \,\square\, P_t$. Furthermore, if $s = t$, then there are two proper paths (u_1, v_1) to (u_s, v_s), one of which has its initial edge colored 1 and the other one has its initial edge colored 2.*

Proof. We consider three cases, according to whether $t = 2$, $s = t$ or $s > t \geq 3$.

Case 1. $t = 2$. If s is even, then $P = ((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_{s-1}, v_1), (u_s, v_1), (u_s, v_2))$ is a proper (u_1, v_1)-(u_s, v_2) path the colors of whose edges alternate 1 and 2. If s is odd, then $P' = ((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_{s-1}, v_2), (u_s, v_2))$ is a proper (u_1, v_1)-(u_s, v_2) path the colors of whose edges alternate 1 and 2.

Case 2. $s = t$. Observe that $P' = ((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_3), \ldots, (u_{s-1}, v_{s-1}), (u_s, v_{s-1}), (u_s, v_s))$ is a proper (u_1, v_1)-(u_s, v_s) path whose initial edge is colored 1. Furthermore, $P'' = ((u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3), (u_3, v_3), \ldots, (u_{s-1}, v_{s-1}), (u_{s-1}, v_s), (u_s, v_s))$ is a proper (u_1, v_1)-(u_s, v_s) path whose initial edge is colored 2.

Case 3. $s > t \geq 3$. Then $s = t + p$ for some positive integer p. By the same argument used in Case 1, we consider the subgraph $P_{p+1} \,\square\, P_2$ of $P_s \,\square\, P_t$, where $P_{p+1} = (u_1, u_2, \ldots, u_{p+1})$ and $P_2 = (v_1, v_2)$. If $p + 1$ is even, then $((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_p, v_1), (u_{p+1}, v_1))$ is a proper (u_1, v_1)-(u_{p+1}, v_1) path the colors of whose edges alternate 1 and 2. If $p + 1$ is odd, then $((u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_1), \ldots, (u_p, v_2), (u_{p+1}, v_2), (u_{p+1}, v_1))$ is a proper (u_1, v_1)-(u_{p+1}, v_1) path the colors of whose edges alternate 1 and 2. By Case 2, there are two proper (u_{p+1}, v_1)-(u_{p+t}, v_t) paths P' and P'' such that the initial edge of P' is colored 1 and the initial edge of P'' is colored 2. If the terminal edge of P is colored 1, then P followed by P'' is a proper path from (u_1, v_1) to (u_s, v_t); while if the terminal edge of P is colored 2, then P followed by P' is a proper path from (u_1, v_1) to (u_s, v_t). ∎

Theorem 2.2.10 *If G and H are nontrivial connected graphs, then*

$$\mathrm{pc}(G \,\square\, H) = 2.$$

Proof. As we saw, it suffices to show that $G \,\Box\, H$ has a proper-path 2-coloring. Let $V(G) = \{u_1, u_2, \ldots, u_s\}$ and $V(H) = \{v_1, v_2, \ldots, v_t\}$ where $s, t \geq 2$. Define a coloring c of $G \,\Box\, H$ by assigning the color 1 to all edges in E_1 and the color 2 to all edges in E_2. We show that c is a proper-path 2-coloring of $G \,\Box\, H$. Let (u_i, v_p) and (u_j, v_q) be two vertices of $G \,\Box\, H$, where $i, j \in \{1, 2, \ldots, s\}$ and $p, q \in \{1, 2, \ldots, t\}$.

First, suppose that either $u_i \neq u_j$ and $v_p = v_q$ or $u_i = u_j$ and $v_p \neq v_q$. We may assume, without loss of generality, that $u_i \neq u_j$. Furthermore, we can assume that v_p is adjacent to v_{p+1} in H. Since G is connected, there is a $u_i - u_j$ path P in G. Note that $P \,\Box\, (v_p, v_{p+1})$ is a subgraph of $G \,\Box\, H$ and by the proof of Case 3 of Lemma 2.2.9, there exists a proper path from (u_i, v_p) to $(u_j, v_p) = (u_j, v_q)$. Next, suppose that $u_i \neq u_j$ and $v_p \neq v_q$. Since G is connected, there is a $u_i - u_j$ path P_G in G. Similarly, since H is connected, there is a $v_p - v_q$ path P_H in H. Thus, $P_G \,\Box\, P_H$ is a subgraph of $G \,\Box\, H$ and by Lemma 2.2.9, there is a proper path from (u_i, v_p) to (u_j, v_q). ■

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let α be a permutation of the set $S = \{1, 2, \ldots, n\}$. The *permutation graph* $P_\alpha(G)$ of a graph G is the graph of order $2n$ obtained from two copies of G, where the second copy of G is denoted by G' and the vertex v_i in G is denoted by u_i in G' and v_i is joined to the vertex $u_{\alpha(i)}$ in G'. The edges $v_i u_{\alpha(i)}$ are called the *permutation edges* of $P_\alpha(G)$. This concept was first introduced by Chartrand and Harary [6]. Therefore, if α is the identity permutation on S, then $P_\alpha(G) = G \,\Box\, K_2$ is the Cartesian product of a graph G and K_2. We show that every permutation graph of a traceable graph has proper connection number 2. In order to do this, we first present a lemma. A connected graph of order 3 or more is *unicyclic* if it contains exactly one cycle.

Lemma 2.2.11 *If H is a bipartite unicyclic graph with maximum degree 3 such that H contains exactly two vertices of degree 3 each of which lies on the cycle in H, then* $\mathrm{pc}(H) = 2$.

Proof. Let $C_p = (v_1,\, v_2,\, \ldots, v_p,\, v_{p+1} = v_1)$ be the unique cycle in H, where then p is even. We may assume that $\deg_H v_1 = \deg_H v_i = 3$ for some integer i with $2 \leq i \leq p$. Suppose that $P = (v_1,\, u_1, \ldots,\, u_s)$ and $P' = (v_i,\, w_1, \ldots,\, w_t)$ are paths in H, where $s, t \geq 1$, such that $E(P) \cap E(C_p) = \emptyset$ and $E(P') \cap E(C_p) = \emptyset$. Define a 2-coloring c of G (using the colors 1 and 2) by first giving a proper coloring to C_p and then a proper coloring to P and P' such that $c(v_1u_1) = c(v_pv_1)$ and $c(v_iw_1) = c(v_iv_{i+1})$. Each of the paths $(P, v_2, \ldots, v_{i-1}, P')$, $(P, v_2, \ldots, v_p)$ and $(P', v_{i-1}, \ldots, v_1, v_p, \ldots, v_{i+1})$ is a proper path. Since every two vertices of H lie on one of these three proper paths, it follows that c is a proper-path 2-coloring and so $\mathrm{pc}(H) = 2$. ■

Theorem 2.2.12 *If G is a nontrivial traceable graph of order n, then*

$$\mathrm{pc}(P_\alpha(G)) = 2$$

for each permutation α of the set $\{1, 2, \ldots, n\}$. In particular, $\mathrm{pc}(P_\alpha(G)) = 2$ *if G is Hamiltonian.*

Proof. Let G be a nontrivial traceable graph of order n, let $(v_1, v_2, \ldots, v_n)$ be a Hamiltonian path in G and let $(v'_1, v'_2, \ldots, v'_n)$ be the corresponding Hamiltonian path in the second copy G' of G. Since $P_\alpha(G)$ is not complete for each permutation α of $\{1, 2, \ldots, n\}$, it remains to show that $\mathrm{pc}(P_\alpha(G)) \leq 2$. We consider two cases.

Case 1. $\{\alpha(1), \alpha(n)\} \cap \{1, n\} \neq \emptyset$. We may assume that $\alpha(1) = 1$ or $\alpha(n) = 1$. If $\alpha(1) = 1$, then $(v_n, v_{n-1}, \ldots, v_1, v'_1, v'_2, \ldots, v'_n)$ is a Hamiltonian path of $P_\alpha(G)$; while if $\alpha(n) = 1$, then $(v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n)$ is a Hamiltonian path of $P_\alpha(G)$. It then follows by Corollary 4.2.2 that $\mathrm{pc}(P_\alpha(G)) = 2$.

Case 2. $\{\alpha(1), \alpha(n)\} \cap \{1, n\} = \emptyset$. Suppose $\alpha(1) = i$ and $\alpha(n) = j$ where $2 \leq i \neq j \leq n-1$. We will only consider the case when $i < j$ since the argument for the case when $i > j$ is similar and we use the path $(v'_n, v'_{n-1}, \ldots, v'_2, v'_1)$ in the proof. Furthermore, assume that $\alpha(k) = 1$ for some k with $2 \leq k \leq n-1$. We consider three cases, depending on the parities of two of the integers $k-1$, $i-1$ and $n-j$.

Subcase 2.1. *$k-1$ and $i-1$ are of the same parity.* Let H be the subgraph of $P_\alpha(G)$ consisting of the even cycle $(v_1, v_2, \ldots, v_k, v'_1, v'_2, \ldots, v'_i, v_1)$ and the two paths $(v_k, v_{k+1}, \ldots, v_n)$ and $(v'_i, v'_{i+1}, \ldots, v'_n)$. Then H is a spanning subgraph of $P_\alpha(G)$. By Lemma 2.2.11, $\mathrm{pc}(H) = 2$. It then follows by Proposition 2.1.1 that $\mathrm{pc}(P_\alpha(G)) \leq 2$ and so $\mathrm{pc}(P_\alpha(G)) = 2$.

Subcase 2.2. *$k-1$ and $n-j$ are of the same parity.* Let H be the subgraph of $P_\alpha(G)$ consisting of the even cycle $(v_k, v_{k+1}, \ldots, v_n, v'_j, v'_{j-1}, \ldots, v'_1, v_k)$ and the two paths $(v_1, v_2, \ldots, v_k)$ and $(v'_j, v'_{j+1}, \ldots, v'_n)$. Then H is a spanning subgraph of $P_\alpha(G)$. By Lemma 2.2.11, $\mathrm{pc}(H) = 2$. It then follows by Proposition 2.1.1 that $\mathrm{pc}(P_\alpha(G)) \leq 2$ and so $\mathrm{pc}(P_\alpha(G)) = 2$.

Subcase 2.3. *$i-1$ and $n-j$ are of the same parity.* Let H be the subgraph of $P_\alpha(G)$ consisting of the even cycle $(v_1, v_2, \ldots, v_n, v'_j, v'_{j-1}, \ldots, v'_i, v_1)$ and the two paths $(v'_1, v'_2, \ldots, v'_i)$ and $(v'_j, v'_{j+1}, \ldots, v'_n)$. Then H is a spanning subgraph of $P_\alpha(G)$. By Lemma 2.2.11, $\mathrm{pc}(H) = 2$. It then follows by Proposition 2.1.1 that $\mathrm{pc}(P_\alpha(G)) \leq 2$ and so $\mathrm{pc}(P_\alpha(G)) = 2$. ■

By Theorem 2.2.12, every permutation graph of a traceable graph has proper connection number 2. However, traceable graphs are not the only connected graphs with this property, as we show next.

Proposition 2.2.13 *Every permutation graph of a star of order at least* 4 *has proper connection number* 2*.*

Proof. For an integer $m \geq 3$, let $G = K_{1,m}$ be the star with vertex set

$$\{v_0, v_1, \ldots, v_m\},$$

where v_0 is the central vertex. Then there are exactly two non-isomorphic permutation graphs, namely $P_{\alpha_1}(G) = G \,\square\, K_2$ where α_1 is the identity permutation on the set $\{0, 1, \ldots, m\}$ and $P_{\alpha_2}(G)$ where α_2 interchanges 0 and 1 and fixes all other integers. By Theorem 2.2.10, $\mathrm{pc}(P_{\alpha_1}(G)) = 2$. It remains to show that $P_{\alpha_2}(G) = 2$. Let

$$\{v'_0, v'_1, \ldots, v'_m\}$$

be the corresponding vertex set in the second copy G' of G. Since $P_{\alpha_2}(G)$ is not complete, it remains to show that $\mathrm{pc}(P_{\alpha_2}(G)) \leq 2$.

Define a 2-edge coloring c by assigning the color 1 to (i) the edge $v_0v'_1$, $v_1v'_0$, $v_2v'_2$, (ii) the edge v_0v_i for each $i \geq 3$ and (iii) the edge $v'_0v'_i$ for each $i \geq 3$ and assigning the color 2 to the remaining edges of $P_{\alpha_2}(G)$. We show that c is a proper-path 2-coloring of $P_{\alpha_2}(G)$. Let $u, v \in V(P_{\alpha_2}(G))$. Since every two vertices on $C_4 = (v_0, v_1, v'_0, v'_1, v_0)$ are connected by a proper path, we may assume that at least one of u and v does not belong to C_4, say v is not in C_4.

First, suppose that u is a vertex of C_4. By symmetry, we may assume that $u = v_0$ or $u = v_1$. Assume first that $u = v_0$. If $v = v_i$ where $i \geq 2$, then (u, v) is a proper $u-v$ path; while if $v=v_i'$ where $i\geq 2$, then (u,v_i,v) is a proper $u-v$ path. Next, assume that $u=v_1$. If $v=v_2$, then (u,v_0',v_2',v) is a proper $u-v$ path; while if $v=v_2'$, then (u,v_0',v) is a proper $u-v$ path. If $v=v_i$ where $i\geq 3$, then (u,v_0,v) is a proper $u-v$ path; while if $v=v_i'$ where $i\geq 3$, then (u,v_0,v_i,v) is a proper $u-v$ path. Next, suppose that u is not a vertex of C_4. By symmetry, we may assume $u=v_2$ or $u=v_3$. We first assume that $u=v_2$. If $v=v_2'$, then u and v are adjacent and so there is a proper $u-v$ path. If $v=v_i$ where $i\geq 3$, then (u,v_0,v) is a proper $u-v$ path; while if $v=v_i'$ where $i\geq 3$, then (u,v_0,v_i,v) is a proper $u-v$ path. Now, we assume that $u=v_3$. If $v=v_2'$, then (u,v_0,v_2,v) is a proper $u-v$ path; while if $v=v_3'$, then (u,v) is a proper $u-v$ path. If $v=v_i$ where $i\geq 4$, then $(u,v_0,v_2,v_2',v_0',v_i',v)$ is a proper $u-v$ path; while if $v=v_i'$ where $i\geq 4$, then (u,v_0,v_2,v_2',v_0',v) is a proper $u-v$ path. Thus, c is a proper-path 2-coloring of $P_{\alpha_2}(G)$ and so $\mathrm{pc}(P_{\alpha_2}(G))=2$. ■

We conclude this subsection by stating the following problem.

Problem 2.2.14 *Does there exist a nontrivial connected graph G such that*

$$\mathrm{pc}(P_\alpha(G))\geq 3$$

for some permutation graph $P_\alpha(G)$ of G?

Connected graphs with relatively small proper connection numbers have also been studied by Li, Wei and Yue in [30]. They proved that if G is a connected graph with diameter 2 and minimum degree at least 2, then the proper connection number of G is 2. An upper bound $\frac{3n}{\delta+1}-1$ for $\mathrm{pc}(G)$ was established for connected graphs G of order n and minimum degree δ. In [30], upper bounds for the proper connection numbers were obtained for connected interval graphs, asteroidal triple-free graphs, circular arc graphs, threshold graphs and chain graphs, all of which have minimum degree at least 2. Furthermore, it is shown that $\mathrm{pc}(G)\leq 3$ for interval graphs G and circular arc graphs G and this bound is sharp. Moreover, the relationship between proper-path colorings and dominating sets was studied [30]. It was shown that for a connected graph G with minimum degree at least 2, its proper connection number $\mathrm{pc}(G)$ is bounded above by $\mathrm{pc}(G[D])+2$, where D is a connected two-step dominating set of G and $G[D]$ is the subgraph of G induced by D (see [30]).

Proper-path colorings in random graphs have been studied (see [19], for example). In order to include some related results on graphs having proper-connection number 2, we introduce additional definitions. For a positive integer n and a real number p with $0 < p < 1$, let $G(n, p)$ denote the *Erdös-Renyi random graph* with n vertices and edges appearing with probability p (see [17]). The following two results appear in [19].

Theorem 2.2.15 *Almost all graphs have proper connection number* 2.

Theorem 2.2.16 *For sufficiently large integer n,*

$$\text{if } p \geq \frac{\log n + \alpha(n)}{n} \text{ where } \alpha(n) \to \infty, \text{ then } \mathrm{pc}(G(n,p)) \leq 2.$$

2.3 Strong Proper Connection Numbers

In this section, we turn our attention to strong proper-path colorings in graphs. First, let's recall some of the definitions and notation involved with this topic. Let c be a rainbow coloring of a connected graph G. As we mentioned in Chapter 1, for two vertices u and v of G, a *rainbow $u - v$ geodesic* in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v (the length of a shortest $u - v$ path) in G. The graph G is *strongly rainbow-connected* if G contains a rainbow $u - v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum k for which there exists a coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G such that G is strongly rainbow-connected is the *strong rainbow connection number* $\mathrm{src}(G)$ of G. A strong rainbow coloring of G using $\mathrm{src}(G)$ colors is a *minimum strong rainbow coloring* of G. Thus, $\mathrm{rc}(G) \leq \mathrm{src}(G)$ for every connected graph G.

Inspired by this concept, we consider an analogous concept for proper-path colorings. Let c be a proper-path coloring of a nontrivial connected graph G. For two vertices u and v of G, a *proper $u - v$ geodesic* in G is a proper $u - v$ path of length $d(u, v)$. If there is a proper $u - v$ geodesic for every two vertices u and v of G, then c is called a *strong proper coloring* of G or a *strong proper k-coloring* if k colors are used. The minimum number of colors needed to produce a strong proper coloring of G is called the *strong proper connection number* $\mathrm{spc}(G)$ of G. A strong proper coloring using $\mathrm{spc}(G)$ colors is referred to as a *minimum strong proper coloring*. In general, if G is a nontrivial connected graph, then $1 \leq \mathrm{pc}(G) \leq \mathrm{spc}(G) \leq \chi'(G)$.

Since every strong rainbow coloring of G is a strong proper coloring of G, it follows that $\text{spc}(G)$ exists for every nontrivial connected graph G and $\text{spc}(G) \leq \text{src}(G)$. Therefore, if G is nontrivial connected graph, then

$$1 \leq \text{pc}(G) \leq \text{spc}(G) \leq \min\{\chi'(G), \text{src}(G)\}. \tag{2.2}$$

Before proceeding further, we present some observations concerning strong proper connection numbers of graphs.

Observation 2.3.1 *Let G be a nontrivial connected graph of order n and size m.*

(1) $\text{spc}(G) = \text{pc}(G) = 1$ *if and only if* $G = K_n$.

(2) $\text{spc}(G) = \text{pc}(G) = m$ *if and only if* $G = K_{1,m}$.

(3) *If G is a tree, then* $\text{spc}(G) = \text{pc}(G) = \Delta(G)$.

(4) *If G is a connected graph with* $\text{diam}(G) = 2$, *then* $\text{spc}(G) = \text{src}(G)$.

(5) *If b is the maximum number of bridges incident with a vertex in G, then* $\text{spc}(G) \geq b$ *and* $\text{pc}(G) \geq b$.

By Observation 2.3.1 and (5.1), if G is a connected graph of order n that is not complete such that $\text{pc}(G) = a$ and $\text{spc}(G) = b$, then $2 \leq a \leq b$. In [7], it was shown that

$$\text{src}(K_{s,t}) = \lceil \sqrt[s]{t}\, \rceil \text{ for all integers } s \text{ and } t \text{ with } 2 \leq s \leq t.$$

Since $\text{diam}(K_{s,t}) = 2$, it follows by Observation 2.3.1 that

$$\text{spc}(K_{s,t}) = \text{src}(K_{s,t}) = \lceil \sqrt[s]{t}\, \rceil \text{ for } 2 \leq s \leq t.$$

Moreover, we saw in Theorem 2.2.7 that $\text{pc}(K_{s,t}) = 2$ for $2 \leq s \leq t$. Thus, if $s = 2$ and $t = b^2$ where $b \geq 2$, then $\text{spc}(K_{s,t}) - \text{pc}(K_{s,t}) = b - 2$ which can be arbitrarily large. In fact, more can be said. Next, we show that every pair a, b of integers where $2 \leq a \leq b$ is realizable as the strong proper connection number and proper connection number, respectively, of some connected graph. For the purpose of doing this, we first present a lemma.

Lemma 2.3.2 *For each integer $t \geq 2$, let $G = K_{2,t^2}$ be the complete bipartite graph of order $2 + t^2$ with partite sets U and W, where $|U| = 2$ and $|W| = t^2$. If c is a strong proper t-coloring of G using the colors $1, 2, \ldots, t$, then for each vertex $u \in U$,*

$$\{c(uw) : w \in W\} = \{1, 2, \dots, t\}.$$

Proof. Let $U = \{u_1, u_2\}$. Since $\mathrm{spc}(G) = t$, every strong proper coloring of G uses at least t colors. Assume, to the contrary, that there is a strong proper t-coloring c of G using the colors $1, 2, \dots, t$ such that

$$\{c(uw) : w \in W\} \neq \{1, 2, \dots, t\}$$

for some $u \in U$, say $t \notin \{c(u_1w) : w \in W\}$. For each vertex $w \in W$, we can associate an ordered pair $\mathrm{code}(w) = (a_1(w), a_2(w))$ called the color code of w, where $a_i(w) = c(u_iw)$ for $i = 1, 2$. Since $1 \leq a_1(w) \leq t-1$ for each $w \in W$, the number of distinct color codes of the vertices of W is at most $(t-1)t$. However, because $t^2 > (t-1)t$, there exists at least two distinct vertices w' and w'' of W such that $\mathrm{code}(w') = \mathrm{code}(w'')$. Since $c(u_iw') = c(u_iw'')$ for $i = 1, 2$, it follows that G contains no proper $w' - w''$ geodesic in G, contradicting our assumption that c is a strong proper t-coloring of G. ■

Theorem 2.3.3 *For every pair a, b of integers where $2 \leq a \leq b$, there exists a connected graph G such that $\mathrm{pc}(G) = a$ and $\mathrm{spc}(G) = b$.*

Proof. If $a = b \geq 2$, then let G be a tree with maximum degree a. Thus, $\mathrm{spc}(G) = \mathrm{pc}(G) = a$. We may assume that $2 \leq a < b$, where then $b \geq 3$. Let $H = K_{2,(b-1)^2}$ be the complete bipartite graph with partite sets

$$U = \{u_1, u_2\} \text{ and } W = \{w_1, w_2, \dots, w_{(b-1)^2}\}$$

and let $F = K_{1,a-1}$ with $V(F) = \{v, v_1, v_2, \dots, v_{a-1}\}$ where v is the central vertex of F. Now let G be the graph obtained from H and F by adding the edge u_2v.

First, we show that $\mathrm{pc}(G) = a$. Since the vertex v is incident with a bridges in G, it follows by Observation 2.3.1 that $\mathrm{pc}(G) \geq a$. Next, define an edge coloring c of G by assigning

(1) the color 1 to each of the edges u_1w_i $(2 \leq i \leq (b-1)^2)$, w_1u_2 and vv_1,

(2) the color 2 to each of u_1w_1, u_2w_i $(2 \leq i \leq (b-1)^2)$ and vv_2

(3) the color i to vv_i $(3 \leq i \leq a-1$ if $a \geq 4)$ and

(4) the color a to u_2v.

Then every two vertices x and y are connected by a proper $x-y$ path. For example, $(v_1, v, u_2, w_1, u_1, w_2)$ is a proper $v_1 - w_2$ path in G. Hence, c is a proper-path coloring of G using a colors and so $\mathrm{pc}(G) \leq a$. Thus, $\mathrm{pc}(G) = a$.

Next, we show that $\mathrm{spc}(G) = b$. First, we show that $\mathrm{spc}(G) \leq b$. Since $\mathrm{spc}(K_{2,(b-1)^2}) = b-1$, there is a strong proper $(b-1)$-coloring c_0 of the subgraph H of G using colors $1, 2, \ldots, b-1$. Define an edge coloring c_1 of G by assigning

(1) the color $c_0(e)$ to each edge e of H,

(2) the color b to the edge $u_2 v$ and

(3) the color i to vv_i for $1 \leq i \leq a-1$.

Since c_1 is a strong proper b-coloring of G, it follows that $\mathrm{spc}(G) \leq b$. Next, we show that $\mathrm{spc}(G) \geq b$. Let c be a strong proper k-coloring of G. For every two vertices x and y in the subgraph H of G, each $x-y$ geodesic lies completely in H. Hence, the restriction c_H of c to H is a strong proper coloring of H and so $k \geq b-1$. Assume, to the contrary, that $k = b-1$. By Lemma 2.3.2,

$$\{c(u_i w) : w \in W\} = \{1, 2, \ldots, b-1\}$$

for $i = 1, 2$. Since $c(u_2 v) \in \{1, 2, \ldots, b-1\}$, there exists $w \in W$ such that $c(u_2 w) = c(u_2 v)$. However then, G contains no proper $w - v$ geodesic in G, contradicting our assumption that c is a strong proper coloring of G. ■

By Observation 2.3.1 and (5.1), if G is a connected graph of order n that is not complete such that $\mathrm{spc}(G) = a$ and $\chi'(G) = b$, then $2 \leq a \leq b < n$. In fact, this is the only restriction on these three parameters.

Theorem 2.3.4 *For every triple a, b, n of integers where $2 \leq a \leq b < n$, there exists a connected graph G of order n such that $\mathrm{spc}(G) = a$ and $\chi'(G) = b$.*

Proof. First suppose that $2 \leq a = b < n$. Let G be the graph obtained from the star $K_{1,a}$ of order $a+1$ and the path P_{n-a-1} of order $n-a-1$ by adding an edge joining an end-vertex of $K_{1,a}$ and an end-vertex of P_{n-a-1}. Then G is a tree of order n with $\Delta(G) = a$ and so $\mathrm{spc}(G) = \chi'(G) = a$ by Observation 2.3.1.

Next, suppose that $2 \leq a < b < n$. We begin with a graph H constructed from the complete graph K_{b-a+2} by adding $a-1$ pendant edges at a vertex v of K_{b-a+2}. Then the graph G is obtained from H and P_{n-b-1} by adding an edge joining an end-vertex u of H and an end-vertex w of P_{n-b-1}. The graph G has order n and $\Delta(G) = b$.

We show that $\chi'(G) = \Delta(G) = b$. It suffices to provide a b-edge coloring of G. First, suppose that $b - a + 2 \geq 4$ is even. Since $\chi'(K_{b-a+2}) = b - a + 1$, there is a proper edge coloring c_1 of K_{b-a+2} using the colors $1, 2, \ldots, b - a + 1$. Since $T = G - E(K_{b-a+2})$ is a tree of maximum degree $a-1$, there is a proper edge coloring c_2 of T using the $a-1$ colors $b-a+2, b-a+3, \ldots, b$. Then the coloring c of G defined by $c(e) = c_1(e)$ if $e \in E(K_{b-a+2})$ and $c(e) = c_2(e)$ if $e \in E(T)$ is a proper b-coloring of G. Next, suppose that $b - a + 2 \geq 3$ is odd. Since $\chi'(K_{b-a+2}) = b - a + 2$, there is a proper edge coloring c_1 of K_{b-a+2} using the colors $1, 2, \ldots, b - a + 2$. Since v is incident with $b - a + 1$ edges in K_{b-a+2}, there is a color $i \in \{1, 2, \ldots, b - a + 2\}$ that is not used to color any edge incident with v in K_{b-a+2}, say $i = b - a + 2$. Now, the subgraph $T = G - E(K_{b-a+2})$ is a tree of maximum degree $a - 1$. Hence, there is a proper edge coloring c_2 of T using the $a - 1$ colors $b - a + 2, b - a + 3, b - a + 4, \ldots, b$. Then the coloring c of G defined by $c(e) = c_1(e)$ if $e \in E(K_{b-a+2})$ and $c(e) = c_2(e)$ if $e \in E(T)$ is a proper b-coloring of G. In either case, $\chi'(G) = \Delta(G) = b$.

Next, we show that $\text{spc}(G) = a$. First, define the coloring c by (i) assigning the color 1 to each edge of K_{b-a+2} and distinct colors from the set $\{2, 3, \ldots, a\}$ of colors to the $a-1$ pendant edges of H and (ii) assigning the color 1 to the edge uw and the colors 1 and 2 properly to the edges of P_{n-b-1} such that the initial edge of P_{n-b-1} (that is adjacent to uw) is colored 2. Since c is a strong proper a-coloring of G, it follows that $\text{spc}(G) \leq a$. Next, we show that $\text{spc}(G) \geq a$. Since there are $a - 1$ pendant edges incident with v, it follows by Observation 2.3.1 that $\text{spc}(G) \geq a - 1$. Suppose that $\text{spc}(G) = a - 1$. Let c' be a strong proper $(a - 1)$-coloring of G. Thus, c' must assign $a - 1$ distinct colors to the $a - 1$ edges incident with v that do not belong to K_{b-a+2}. This implies that there are edges e and f incident with v in G such that e is an edge of K_{b-a+2} and f is not an edge of K_{b-a+2} for which $c'(e) = c'(f)$. Let $e = uv$ and $f = vw$. However then, there is no proper $u - w$ geodesic in G, which is not possible. Thus, $\text{spc}(G) = a$. ■

By Observation 2.3.1, if G is a tree, then $\text{spc}(G) = \chi'(G) = \Delta(G)$. However, there are many connected graphs G that are not trees for which $\text{spc}(G) = \chi'(G)$.

For example, for the cycle C_n of order $n \geq 4$, it can be shown that

$$\mathrm{spc}(C_n) = \chi'(G) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

The following result provides a sufficient condition for a connected graph G that is not a tree such that $\mathrm{spc}(G) = \chi'(G)$. The *girth* $g(G)$ of a graph G having cycles is the length of a smallest cycle in G. In particular, $g(C_n) = n$ for each $n \geq 3$.

Proposition 2.3.5 *If G is a connected graph having $g(G) \geq 5$, then*

$$\mathrm{spc}(G) = \chi'(G).$$

Proof. Since $\mathrm{spc}(G) \leq \chi'(G)$, it suffices to show that $\chi'(G) \leq \mathrm{spc}(G)$. Let c be a minimum strong proper coloring of G using $\mathrm{spc}(G)$ colors. We claim that c is proper; for otherwise, there are adjacent edges e and f such that $c(e) = c(f)$. Let $e = uv$ and $f = vw$. Since $g(G) \geq 5$, it follows that (u, v, w) is the only $u - w$ geodesic in G. However then, there is no proper $u - w$ geodesic in G, which is a contradiction. Thus, $\chi'(G) \leq \mathrm{spc}(G)$. ■

Since the girth of the Petersen graph P is 5, it follows by Proposition 2.3.5 that $\mathrm{spc}(P) = \chi'(P) = 4$. The lower bound 5 for the girth of a graph is best possible. For example, the complete bipartite graph $G = K_{2,t^2}$ (where $t \geq 2$) has $g(G) = 4$, $\mathrm{spc}(G) = t$ and $\chi'(G) = t^2$. Also, notice that the converse of Proposition 2.3.5 is not true. For example, the 4-cycle C_4 has girth 4 and $\mathrm{spc}(C_4) = \chi'(C_4) = 2$.

We saw that (1) $\mathrm{spc}(G) \leq \mathrm{src}(G)$ for every connected graph G and (2) if G is a connected graph with $\mathrm{diam}(G) = 2$, then $\mathrm{spc}(G) = \mathrm{src}(G)$. Also, as may be expected, $\mathrm{src}(G) - \mathrm{spc}(G)$ can be arbitrarily large. In fact, more can be said. First, we present a result from [7] on the rainbow connection number and strong rainbow connection number of a graph.

Proposition 2.3.6 *Let G be a nontrivial connected graph of size m. Then*

(1) $\mathrm{src}(G) = 1$ *if and only if G is a complete graph;*

(2) $\mathrm{rc}(G) = m$ *if and only if G is a tree.*

By Observation 2.3.1 and Proposition 2.3.6, it follows that if T is a tree of size $b \geq 2$ and maximum degree a, then $\mathrm{spc}(T) = a$ and $\mathrm{src}(T) = b$. If we replace an end-vertex of such a tree T by a complete graph, the resulting graph G has $\mathrm{spc}(G) = a$ and $\mathrm{src}(G) = b$ as well. This observation gives rise to the following result.

Theorem 2.3.7 *For every triple a, b, n of integers $2 \leq a \leq b < n$, there exists a connected graph G of order n such that $\mathrm{spc}(G) = a$ and $\mathrm{src}(G) = b$.*

Chapter 3

Graphs With Large Connection Numbers

We have seen that if G is a nontrivial connected graph of size m, then $\mathrm{rc}(G) \leq m$ and $\mathrm{rc}(G) = m$ if and only if G is a tree. Furthermore, it was observed in [29] that if G is a nontrivial connected graph of size m that is not a tree, then $\mathrm{rc}(G) \leq m - 2$. Thus, there is no nontrivial connected graph G of size m for which $\mathrm{rc}(G) = m - 1$. Graphs having large rainbow connection numbers in terms of their size have been studied extensively (see [28], for example). In particular, Li, Sun and Zhao [29] presented characterizations of all connected graphs of size m having rainbow connection numbers $m - 2$ and $m - 3$. We have also seen that if G is a nontrivial connected graph of size m, then $\mathrm{pc}(G) \leq \mathrm{spc}(G) \leq m$ and $\mathrm{pc}(G) = m$ (and so $\mathrm{spc}(G) = m$ as well) if and only if G is the star of size m. In this chapter, we present characterizations of those connected graphs of size m having proper connection number or strong proper connection number $m - 1$, $m - 2$ or $m - 3$.

3.1 Introduction

It will be useful to recall the following preliminary results presented in Chapter 2.

Proposition 3.1.1 *Let G be a nontrivial connected graph containing bridges. If b is the maximum number of bridges incident with a vertex in G, then*

$$\mathrm{pc}(G) \geq b \text{ and } \mathrm{spc}(G) \geq b.$$

Proposition 3.1.2 *If T is a nontrivial tree, then*

$$\mathrm{pc}(T) = \mathrm{spc}(T) = \chi'(T) = \Delta(T).$$

Proposition 3.1.3 *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then* $\mathrm{pc}(G) \le \mathrm{pc}(H)$*. Furthermore,*

$$\mathrm{pc}(G) \le \min\{\Delta(T) : T \text{ is a spanning tree of } G\}.$$

As mentioned in Chapter 2, the following corollary is an immediate consequence of Proposition 4.2.1.

Corollary 3.1.4 *If G is a traceable graph that is not complete, then* $\mathrm{pc}(G) = 2$*.*

The following lemma provides upper bounds for the proper connection number and strong proper connection number of a connected graph in terms of the sizes of the graph and a proper connected subgraph and the proper connection numbers of the subgraph (not just *spanning* subgraphs as in Proposition 4.2.1).

Lemma 3.1.5 *If G is a connected graph of size m_G and H is a proper connected subgraph of size m_H in G, then*

$$\mathrm{pc}(G) \le m_G - m_H + \mathrm{pc}(H) \text{ and } \mathrm{spc}(G) \le m_G - m_H + \mathrm{spc}(H).$$

Proof. Let $\mathrm{pc}(H) = a$ and $b = m_G - m_H + a$. Suppose that c_H is a minimum proper-path coloring of H using the colors $1, 2, \ldots, a$. Then c_H can be extended to a proper-path b-coloring c of G by assigning the $m_G - m_H$ colors

$$a+1, a+2, \ldots, b = m_G - m_H + a$$

to the $m_G - m_H$ edges in $E(G) - E(H)$, which implies that

$$\mathrm{pc}(G) \le m_G - m_H + \mathrm{pc}(H).$$

A similar argument shows that $\mathrm{spc}(G) \le m_G - m_H + \mathrm{spc}(H)$. ∎

3.2 Graphs With Large Proper Connection Numbers

We have seen that if G is a connected graph of size m, then $\mathrm{pc}(G) \le m$ and $\mathrm{pc}(G) = m$ if and only if $G = K_{1,m}$. We now characterize those connected graphs G of size m for which $\mathrm{pc}(G) \in \{m-1, m-2, m-3\}$, beginning with those connected graphs G of size $m \ge 3$ with $\mathrm{pc}(G) = m - 1$. Recall that a *double star* is a tree of diameter 3. Thus, each double star has two *central vertices*. If the central vertices of a double star have degrees a and b, respectively, then this double star is denoted by $S_{a,b}$, where the order of $S_{a,b}$ is therefore $a+b$. By Proposition 4.3.2, if T is a nontrivial tree, then $\mathrm{pc}(T) = \Delta(T)$ and so

$$\mathrm{pc}(S_{a,b}) = \max\{a, b\}.$$

Proposition 3.2.1 *Let G be a connected graph of size $m \geq 3$. Then*

$$\mathrm{pc}(G) = m - 1 \textit{ if and only if } G = S_{2,m-1}.$$

Proof. Let $G = S_{2,m-1}$. Since $\mathrm{pc}(G) = \Delta(G) = m - 1$ by Proposition 4.3.2, it remains only to verify the converse. Let G be a connected graph of size $m \geq 3$ such that $\mathrm{pc}(G) = m - 1$. We claim that G is a tree. If this is not the case, then G contains a cycle $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $\ell \geq 3$. If $\ell = 3$, then the coloring that assigns the color 1 to each edge of C and distinct colors from the set $\{2, 3, \ldots, m-2\}$ to the remaining $m-3$ edges is a proper-path coloring of G and so $\mathrm{pc}(G) \leq m-2$, which is a contradiction. If $\ell \geq 4$, then the coloring that assigns (i) the color 1 to $v_\ell v_1$ and v_2v_3, (ii) the color 2 to v_1v_2 and v_3v_4 and (iii) distinct colors from the set $\{3, 4, \ldots, m-2\}$ to the remaining $m-4$ edges is a proper-path coloring of G and so $\mathrm{pc}(G) \leq m-2$, which again is impossible. Thus, G is a tree and so $G = S_{2,m-1}$ by Proposition 4.3.2. ■

In order to characterize all those connected graphs G of size m having ether $\mathrm{pc}(G) = m-2$ or $\mathrm{pc}(G) = m-3$, we first present two lemmas. For a nontrivial graph G for which $G + uv \cong G + xy$ for every two pairs $\{u, v\}$, $\{x, y\}$ of nonadjacent vertices of G, the graph $G+e$ is obtained from G by joining two nonadjacent vertices of G by an edge e.

Lemma 3.2.2 *If $G = K_{1,m-1} + e$ where $m \geq 4$, then*

$$\mathrm{pc}(G) = \begin{cases} m-2 & \textit{if } m = 4, 5 \\ m-3 & \textit{if } m \geq 6. \end{cases}$$

Proof. Since $K_{1,3} + e$ is a traceable graph that is not complete, it follows by Corollary 4.2.2 that $\mathrm{pc}(K_{1,3} + e) = 2$ and so $\mathrm{pc}(K_{1,3} + e) = m - 2$ when $m = 4$. Next, we show that $\mathrm{pc}(K_{1,4} + e) = 3$. By assigning the colors 1 and 2 to the two bridges of $K_{1,4} + e$ and the color 3 to the remaining edges of $K_{1,4} + e$, we obtain a proper-path 3-coloring of $K_{1,4}+e$ and so $\mathrm{pc}(K_{1,4}+e) \le 3$. Assume, to the contrary, that $K_{1,4}+e$ has a proper-path 2-coloring using the colors 1 and 2. Necessarily, the two bridges uv and uw must be colored differently, say 1 and 2, respectively. Then some vertex x of degree 2 is incident with edges of the same color, say 1. Let $e = xy$. In order for $K_{1,4}+e$ to have a properly colored $x-v$ path, uy must be colored 2. However then, $K_{1,4}+e$ contains no properly colored $y-w$ path. Therefore, $\mathrm{pc}(K_{1,4}+e) = 3 = m-2$.

Now suppose that $m \ge 6$ and $G = K_{1,m-1}+e$, where $V(G) = \{v, v_1, v_2, \ldots, v_{m-1}\}$, v is the central vertex of $K_{1,m-1}$ and $e = v_{m-2}v_{m-1}$. By Proposition 4.3.1, $\mathrm{pc}(G) \ge m-3$. Define the coloring $c : E(G) \to \{1, 2, \ldots, m-3\}$ by $c(vv_i) = i$ for $1 \le i \le m-3$ and $\{c(vv_{m-1}), c(vv_{m-2}), c(v_{m-2}v_{m-1})\} = \{1, 2, 3\}$. Since c is a proper-path $(m-3)$-coloring of G, it follows that $\mathrm{pc}(G) \le m-3$ and so $\mathrm{pc}(G) = m-3$. ■

Recall that a graph is *unicyclic* if it is connected and contains exactly one cycle. Thus, the graph $K_{1,m-1}+e$ considered in Lemma 3.2.2 is unicyclic, whose lone cycle is a triangle. For an integer $m \ge 5$, let $S_{3,m-3}$ be the double star whose central vertices have degrees 3 and $m-3$. The unicyclic graph U_m of size m is the graph obtained from $S_{3,m-3}$ by adding an edge joining the two neighboring end-vertices of the central vertex of degree 3 in $S_{3,m-3}$ (see Figure 3.1).

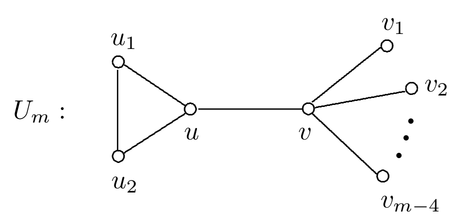

Figure 3.1: The unicyclic graph U_m of size $m \ge 5$

Lemma 3.2.3 *For each integer $m \ge 5$, $\mathrm{pc}(U_m) = m-3$.*

Proof. Let $H = S_{3,m-3}$ be the double star in U_m. Suppose that u and v are the two central vertices of H with $\deg_H u = 3$ and $\deg_H v = m-3$, where u_1 and u_2 are the two end-vertices of H adjacent to u and $v_1, v_2, \ldots, v_{m-4}$ are the $m-4$ end-vertices adjacent to v (see Figure 3.1). Since each of the $m-3$ edges incident with v must be assigned different colors in a proper-path coloring of U_m, it follows that $\mathrm{pc}(U_m) \ge m-3$. On the other hand, the coloring that assigns the color 1 to each edge in $\{uu_1, uu_2, u_1u_2\}$, the color i to the edge vv_i for $1 \le i \le m-4$ and the color

$m-3$ to the edge uv is a proper-path $(m-3)$-coloring of U_m and so $\mathrm{pc}(U_m) \le m-3$. Therefore, $\mathrm{pc}(U_m) = m-3$. ■

We now present a characterization of those connected graphs of size $m \ge 4$ having proper connection number $m-2$.

Theorem 3.2.4 *Let G be a connected graph of size $m \ge 4$. Then $\mathrm{pc}(G) = m-2$ if and only if G is a tree with $\Delta(G) = m-2$ or $G \in \{C_4, K_{1,3}+e, K_{1,4}+e\}$.*

Proof. If G is a tree with $\Delta(G) = m-2$, then $\mathrm{pc}(G) = \Delta(G)$ by Proposition 4.3.2. By Corollary 4.2.2 and Lemma 3.2.2,

$$\mathrm{pc}(G) = m-2 \text{ if } G \in \{C_4, K_{1,3}+e, K_{1,4}+e\}.$$

Thus, it remains to verify the converse. Let G be a connected graph of size $m \ge 4$ such that $\mathrm{pc}(G) = m-2$. Assume, to the contrary, that

$$G \text{ is not a tree and } G \notin \{C_4, K_{1,3}+e, K_{1,4}+e\}.$$

Since G is not a tree, G contains a cycle. By Corollary 4.2.2, G is not a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G. First, suppose that $\ell \ge 5$. Since $\mathrm{pc}(C) = 2$ by Corollary 4.2.2, it follows by Lemma 3.1.5 that

$$\mathrm{pc}(G) \le m - \ell + 2 \le m-3,$$

which is impossible. Thus, $\ell = 4$ or $\ell = 3$. If $\ell = 4$, then either G contains $C_4 + e$ as a subgraph or there is a vertex x of G such that xv_i is an edge in G where $1 \le i \le 4$. In either case, G contains a subgraph H of size 5 with $\mathrm{pc}(H) = 2$ by Corollary 4.2.2. It then follows by Lemma 3.1.5 that $\mathrm{pc}(G) \le m-3$, which is a contradiction. Finally, if $\ell = 3$, then G contains a subgraph that is isomorphic to one of the three graphs H_1, H_2, H_3 in Figure 3.2. [Note that G may also contain the graphs G_5 and G_6 shown in Figure 3.3 as subgraphs. Because H_2 is a subgraph of G_5 and H_1 is a subgraph of G_6, we need only consider the three graphs H_1, H_2, H_3.] Since $\mathrm{pc}(H_i) = m(H_i) - 3$ for $i = 1, 2, 3$, where $m(H_i)$ is the size of H_i, it follows that $\mathrm{pc}(G) \le m-3$, a contradiction. ■

Next, we characterize those connected graphs G of size $m \ge 5$ with $\mathrm{pc}(G) = m-3$.

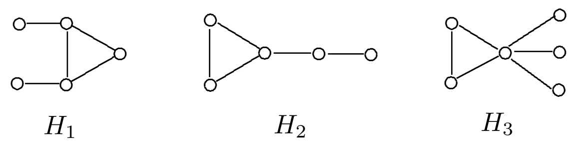

Figure 3.2: Subgraphs H_1, H_2, H_3 in the proof of Theorem 3.2.4

Theorem 3.2.5 *Let G be a connected graph of size $m \geq 5$. Then* $\mathrm{pc}(G) = m - 3$ *if and only if*

(i) G is a tree with $\Delta(G) = m - 3$,

(ii) $G = K_{1,m-1} + e$, where $m \geq 6$,

(iii) $G = U_m$ or

(iv) G is one of the graphs in Figure 3.3.

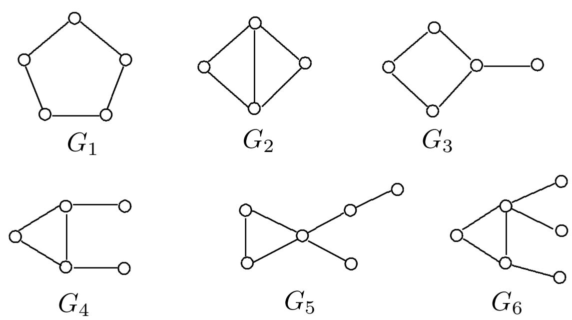

Figure 3.3: The six graphs in Theorem 3.2.5

Proof. By Proposition 4.3.2 and Lemmas 3.2.2 and 3.2.3, if G is a tree with $\Delta(G) = m - 3$ or $G = K_{1,m-1} + e$, where $m \geq 6$ or $G = U_m$ for $m \geq 5$, then $\mathrm{pc}(G) = m - 3$. If G is one of the traceable graphs G_i ($1 \leq i \leq 4$) of size 5 in the Figure 3.3, then $\mathrm{pc}(G) = 2 = m - 3$ by Corollary 4.2.2. If $G = G_i$ for $i = 5, 6$, then G is obtained from $K_{1,4} + e$ by adding a pendant edge $f = xv$ at a vertex v of $K_{1,4} + e$. Observe that for every two vertices u and w of $G - x$, each $u - w$ path of G completely lies in $G - x$. This implies that the restriction of a proper-path coloring of G to the subgraph $G - x$ is also a proper-path coloring of $G - x$. Since $\mathrm{pc}(K_{1,4} + e) = 3$ by Lemma 3.2.2, it follows that $\mathrm{pc}(G) \geq 3$. Since there is a proper-path 3-coloring of $G = G_i$ for $i = 5, 6$ (as shown in Figure 3.4) and G has size 6, it follows that $\mathrm{pc}(G) \leq 3$ and so $\mathrm{pc}(G) = 3 = m - 3$.

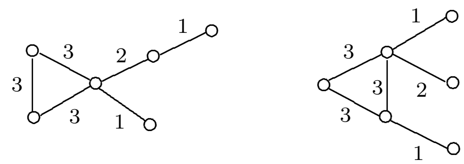

Figure 3.4: Proper-path 3-colorings of G_5 and G_6 in Figure 3.3

For the converse, let G be a connected graph of size $m \geq 5$ such that $\mathrm{pc}(G) = m-3$. Assume, to the contrary, that G is not a tree, $G \neq K_{1,m-1}+e$, where $m \geq 6$, $G \neq U_m$ and G is not any of the graphs shown in Figure 3.3. Since G is not a tree, it follows that G contains a cycle. By Corollary 4.2.2, $G \neq C_n$ for $n \geq 6$. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G. First, suppose that $\ell \geq 6$. Since $\mathrm{pc}(C) = 2$ by Corollary 4.2.2, it follows by Lemma 3.1.5 that $\mathrm{pc}(G) \leq m-4$, which is impossible. Thus, $\ell \in \{3, 4, 5\}$. If $\ell = 5$, then either G contains C_5+e as a subgraph or G contains the subgraph obtained from C_5 by adding exactly one pendant edge at a vertex of C_5. In either case, G contains a traceable subgraph H of size 6 with $\mathrm{pc}(H) = 2$ by Corollary 4.2.2. It then follows by Lemma 3.1.5 that $\mathrm{pc}(G) \leq m-4$, which is a contradiction. If $\ell = 4$, then G contains K_4 as a subgraph or one of the graphs of size 6 in Figure 3.5 as a subgraph. Since $\mathrm{pc}(K_4) = 1$ and each of these graphs has proper connection number 2 (where a proper-path 2-coloring of each graph is also shown in Figure 3.5), it follows by Lemma 3.1.5 that $\mathrm{pc}(G) \leq m-4$, which is a contradiction.

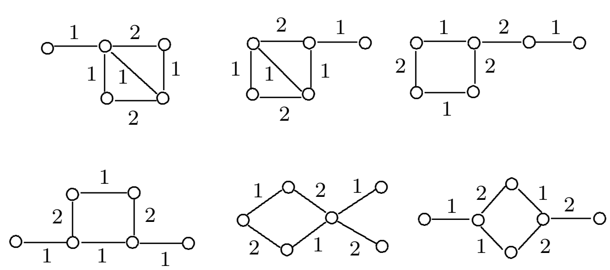

Figure 3.5: Subgraphs having proper connection number 2
in the proof of Theorem 3.2.5

Finally, if $\ell = 3$, then G contains a subgraph that is isomorphic to one of the seven graphs $H_1, H_2, \ldots, H_7$ in Figure 3.6. For $i = 1, 2, 3, 4$, the graph H_i has size 6 and $\mathrm{pc}(H_i) = 2$; while for $i = 5, 6, 7$, the graph H_i has size 7 and $\mathrm{pc}(H_i) = 3$. A minimum proper-path coloring of each graph H_i ($1 \leq i \leq 7$) is also shown in

Figure 3.6. Hence, $\mathrm{pc}(H_i) = m(H_i) - 4$ for $1 \le i \le 7$ and so $\mathrm{pc}(G) \le m - 4$ by Lemma 3.1.5, which is a contradiction. ∎

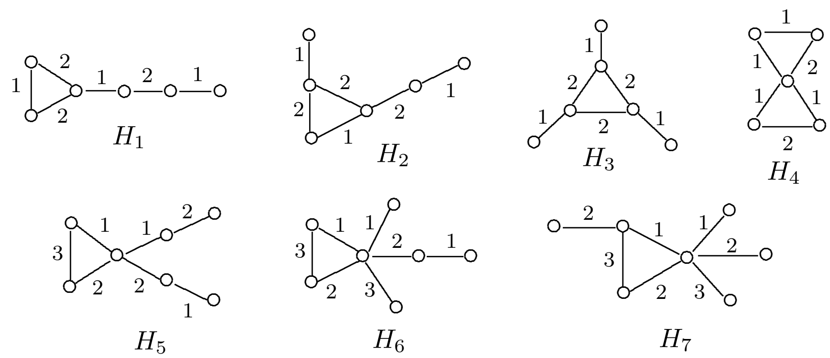

Figure 3.6: Subgraphs $H_1, H_2, \ldots, H_7$ in the proof of Theorem 3.2.5

3.3 Graphs with Large Strong Connection Numbers

By Proposition 3.2.1, the double star $S_{2,m-1}$ of size $m \ge 3$ is the only connected graph of size m with proper connection number $m - 1$. Employing an argument similar to the one used in the proof of Proposition 3.2.1, we now show that this is also true for the strong connection numbers.

Proposition 3.3.1 *Let G be a connected graph of size $m \ge 3$. Then*

$$\mathrm{spc}(G) = m - 1 \text{ if and only if } G = S_{2,m-1}.$$

Proof. If $G = S_{2,m-1}$, then $\mathrm{spc}(G) = \Delta(S_{2,m-1}) = m-1$ by Proposition 4.3.2. For the converse, let G be a connected graph of size $m \ge 3$ such that $\mathrm{spc}(G) = m - 1$. We claim that G is a tree. If this is not the case, then G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $\ell \ge 3$, be a cycle of G. If $\ell = 3$, then the coloring that assigns the color 1 to each edge of C and distinct colors from the set $\{2, 3, \ldots, m-2\}$ to the remaining $m-3$ edges is a strong proper-path coloring of G and so $\mathrm{spc}(G) \le m - 2$, which is a contradiction. If $\ell \ge 4$, then the coloring that assigns

(i) the color 1 to $v_\ell v_1$ and $v_2 v_3$,

(ii) the color 2 to $v_1 v_2$ and $v_3 v_4$ and

(iii) distinct colors from the set $\{3, 4, \ldots, m - 2\}$ to the remaining $m - 4$ edges is a strong proper-path coloring of G and so $\mathrm{spc}(G) \le m-2$, which is again impossible. Thus, as claimed, G is a tree and so $G = S_{2,m-1}$. ■

In order to characterize all connected graphs G of size m having $\mathrm{spc}(G) = m-2$, we first present two useful lemmas, the first of which was observed in Chapter 2.

Lemma 3.3.2 *For an integer $n \ge 4$,*

$$\mathrm{spc}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3.3.3 *For an integer $m \ge 4$, $\mathrm{spc}(K_{1,m-1} + e) = m-2$.*

Proof. Let $G = K_{1,m-1} + e$ with $V(G) = \{v, v_1, v_2, \ldots, v_{m-1}\}$, where v is the central vertex of $K_{1,m-1}$ and $e = v_{m-2}v_{m-1}$. Define the coloring $c : E(G) \to \{1, 2, \ldots, m-2\}$ by $c(vv_i) = i$ for $1 \le i \le m-3$ and $c(vv_{m-1}) = c(vv_{m-2}) = c(v_{m-2}v_{m-1}) = m-2$. Since c is a strong proper $(m-2)$-coloring of G, it follows that $\mathrm{spc}(G) \le m-2$. By Proposition 4.3.1, $\mathrm{spc}(G) \ge m-3$. If G has a strong proper $(m-3)$-coloring c', then c' must assign the same color to an edge $\{vv_i : 1 \le i \le m-3\}$ and an edge in $\{vv_{m-1}, vv_{m-2}\}$, say $c'(vv_1) = c'(vv_{m-1})$. However then, there is no proper $v_1 - v_{m-1}$ geodesic in G, which is impossible. Hence, $\mathrm{spc}(K_{1,m-1} + e) = m-2$. ■

Theorem 3.3.4 *Let G be a connected graph of size $m \ge 4$. Then $\mathrm{spc}(G) = m-2$ if and only if G is a tree with $\Delta(G) = m-2$ or $G \in \{C_4, C_5, K_{1,m-1} + e\}$.*

Proof. If G is a tree with $\Delta(G) = m-2$ or $G \in \{C_4, C_5, K_{1,m-1} + e\}$, then $\mathrm{spc}(G) = m-2$ by Proposition 4.3.2 and Lemmas 3.3.2 and 3.3.3. For the converse, let G be a connected graph of size $m \ge 4$ such that $\mathrm{spc}(G) = m-2$. Assume, to the contrary, that G is not a tree and $G \notin \{C_4, C_5, K_{1,m-1} + e\}$. Since G is not a tree, it follows that G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G and $\ell \ge 3$. By Lemma 3.3.2, $G \ne C_n$ for $n \ge 6$. If $\ell \ge 6$, then

$$\mathrm{spc}(G) \le m - \ell + 3 \le m-3$$

by Lemmas 3.3.2 and 3.1.5, which is impossible. Therefore, $\ell = 5, 4, 3$. We consider these three possibilities. If $\ell = 5$, then either G contains $H_1 = C_5 + e$ as a subgraph or G contains the subgraph H_2 obtained from C_5 by adding a pendant edge. Since

H_1 has size 6 with $\text{spc}(H_1) = 2$ and H_2 has size 6 with $\text{spc}(H_2) = 3$, it then follows by Lemma 3.1.5 that $\text{spc}(G) \le m - 3$. If $\ell = 4$, then either G contains $F_1 = C_4 + e$ as a subgraph or G contains the subgraph F_2 obtained from C_4 by adding a pendant edge. In each case, the size of F_i is 5 and $\text{spc}(F_i) = 2$ for $i = 1, 2$. It then follows by Lemma 3.1.5 that $\text{spc}(G) \le m - 3$, which is impossible.

Finally, assume that $\ell = 3$. Since $G \ne K_{1,m-1} + e$ where $m - 1 \ge 3$, there are two vertices x and y of G that do not lie on C such that either (1) x is adjacent to a vertex v_1, say, of C and $xy \in E(G)$ or (2) x and y are adjacent to different vertices of C, say $xv_1, yv_2 \in E(G)$. First, suppose that (1) occurs. If $yv_1 \notin E(G)$, then the coloring that assigns

(i) the color 1 to xy, v_1v_2 and v_1v_3,

(ii) the color 2 to xv_1 and v_2v_3 and

(iii) distinct colors from the set $\{3, 4, \ldots, m - 3\}$ to the remaining $m - 5$ edges

is a strong proper-path coloring of G and so $\text{spc}(G) \le m - 3$, which is impossible. If $yv_1 \in E(G)$, then the coloring that assigns

(i) the color 1 to each edge of C,

(ii) the color 2 to xy, xv_1 and yv_1 and

(iii) distinct colors from the set $\{3, 4, \ldots, m - 4\}$ to the remaining $m - 6$ edges

is a strong proper-path coloring of G and so $\text{spc}(G) \le m - 4$, which is impossible. Next suppose that (2) occurs. Then the coloring that assigns

(i) the color 1 to each edge of C,

(ii) the color 2 to xv_1 and yv_2 and

(iii) distinct colors from the set $\{3, 4, \ldots, m - 3\}$ to the remaining $m - 5$ edges

is a strong proper-path coloring of G and so $\text{spc}(G) \le m - 3$, which is impossible. ■

In order to determine all connected graphs of size $m \ge 5$ with strong connection number $m-3$, we first describe three classes of such graphs of size m. For an integer $m \ge 5$, let H_m be the graph of size m obtained from a 4-cycle C_4 by adding $m - 4$ pendant edges at a vertex of C_4 and let F_m be the graph of size m obtained from $K_{1,m-2}+e$ by adding a pendant edge at an end-vertex of $K_{1,m-2}+e$ (see Figure 3.7).

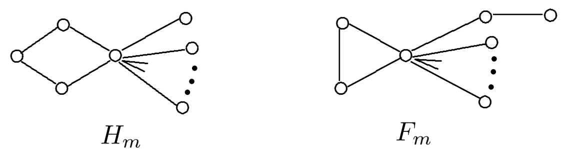

Figure 3.7: The graphs H_m and F_m of size $m \geq 5$

Lemma 3.3.5 *For an integer $m \geq 5$,* $\text{spc}(H_m) = \text{spc}(F_m) = m - 3$.

Proof. Let $V(H_m) = \{u, v, w, x, v_1, v_2, \ldots, v_{m-4}\}$ where (u, v, w, x, u) is the 4-cycle in H_m and vv_i is an edge of H_m for $1 \leq i \leq m-4$. Define the coloring $c : E(H_m) \to \{1, 2, \ldots, m-3\}$ by $c(vv_i) = i$ for $1 \leq i \leq m-4$ and $c(uv) = c(vw) = m-3$, $c(ux) = 1$ and $c(xw) = 2$. Since c is a strong proper $(m-3)$-coloring of H_m, it follows that $\text{spc}(H_m) \leq m-3$. By Proposition 4.3.1, $\text{spc}(H_m) \geq m-4$. If H_m has a strong proper $(m-4)$-coloring c', then c' must assign the same color to an edge $\{vv_i : 1 \leq i \leq m-4\}$ and an edge in $\{vu, vw\}$, say $c'(vu) = c(vv_1)$. However then, there is no proper $u - v_1$ geodesic in H_m, which is impossible. Hence, $\text{spc}(H_m) = m-3$.

A similar argument shows that $\text{spc}(F_m) = m-3$ for $m \geq 5$. ■

Recall that U_m denotes the unicyclic graph of size $m \geq 5$ shown in Figure 3.1. An argument similar to the one used in the proof of Lemma 3.2.3 yields the following lemma.

Lemma 3.3.6 *For each integer $m \geq 5$,* $\text{spc}(U_m) = m - 3$.

Theorem 3.3.7 *Let G be a connected graph of size $m \geq 5$. Then* $\text{spc}(G) = m - 3$ *if and only if*

(i) G is a tree with $\Delta(G) = m - 3$,

(ii) $G \in \{H_m, F_m, U_m\}$ or

(iii) G is one of the four graphs in Figure 3.8.

Proof. By Proposition 4.3.2 and Lemmas 3.3.5 and 3.3.6, if G is a tree with $\Delta(G) = m-3$ or $G \in \{H_m, F_m, U_m\}$, then $\text{pc}(G) = m-3$. Also, it is easy to see that $\text{pc}(G_i) = m-3$ for $1 \leq i \leq 4$ for each graph G_i in Figure 3.8. A minimum strong proper-path coloring of G_i $(1 \leq i \leq 4)$ is shown in Figure 3.9.

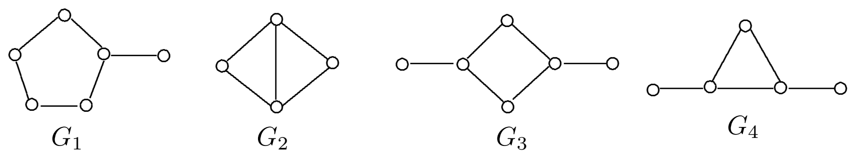

Figure 3.8: The graphs in Theorem 3.3.7

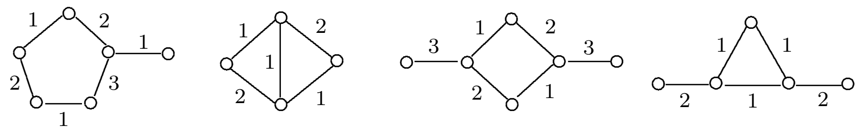

Figure 3.9: Minimum strong proper-path colorings of the graphs in Figure 3.8

It remains to verify the converse. Let G be a connected graph of size $m \geq 5$ such that $\text{spc}(G) = m - 3$. Assume, to the contrary, that G is not a tree, $G \notin \{H_m, F_m, U_m\}$ and G is not any of the graphs shown in Figure 3.8. Since G is not a tree, it follows that G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G. First, suppose that $\ell \geq 6$. Since $\text{spc}(C) = 2$ if ℓ is even and $\text{spc}(C) = 3$ if ℓ is odd by Lemma 3.3.2, it follows that $\text{spc}(C) \leq \ell - 4$. Hence, $\text{spc}(G) \leq m - 4$ by Lemma 3.1.5, which is impossible. Thus, $\ell \in \{3, 4, 5\}$. First, suppose that $\ell = 5$. Since $G \neq C_5$, it follows that G contains a subgraph isomorphic to one of the graphs R_1, R_2, R_3, R_4 in Figure 3.10, where a minimum strong proper-path coloring is also shown for each graph. Thus, $\text{spc}(R_1) = 2$ and $\text{spc}(R_i) = 3$ for $i = 2, 3, 4$. Since $\text{spc}(R_i) \leq m(R_i) - 4$ where $m(R_i)$ is the size of R_i for $i = 1, 2, 3, 4$, it follows that $\text{spc}(G) \leq m - 4$ by Lemma 3.1.5, which is a contradiction.

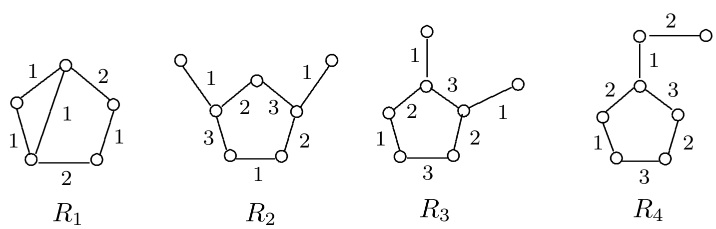

Figure 3.10: The subgraphs R_1, R_2, R_3, R_4 in the proof of Theorem 3.3.7

Next, suppose that $\ell = 4$. Then G contains a subgraph that is isomorphic to K_4 or to one of the three graphs of size 6 in Figure 3.11(a). Since $\text{spc}(K_4) = 1$ and the strong connection number of each of the graphs in Figure 3.11(a) is 2, (a minimum strong proper-path coloring for each graph is shown the figure as well), it follows by Lemma 3.1.5 that $\mathrm{pc}(G) \leq m - 4$, a contradiction.

Finally, suppose that $\ell = 3$. Then G contains a subgraph isomorphic to one of the three graphs of size 6 in Figure 3.11(b), where a minimum strong proper-path coloring is also shown for each graph. Since the strong connection number of each of these graphs is 2, $\mathrm{spc}(G) \leq m - 4$ by Lemma 3.1.5, which is a contradiction. ■

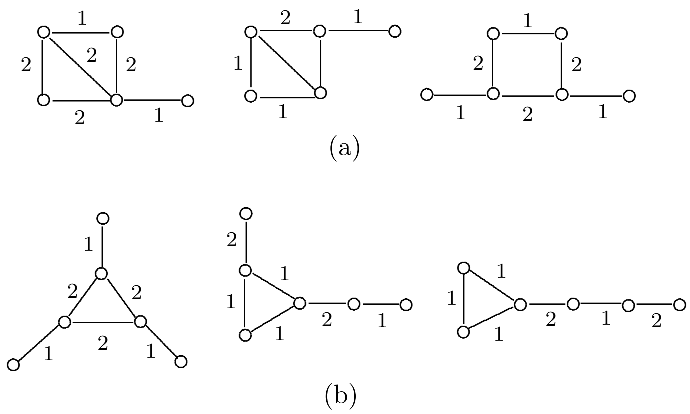

Figure 3.11: The subgraphs of G in the proof of Theorem 3.3.7

Chapter 4

Special Classes of Graphs

In this chapter, we study proper-path colorings in those graphs obtained through certain well-known graph operations, including line graphs, powers of graphs, coronas and vertex or edge deletions. Specifically, we determine the proper connection numbers of the line graph and powers of a given connected graph. For a given nontrivial connected graph G, we also establish sharp lower and upper bounds for the proper connection number of (i) the k-iterated corona of G in terms of $\mathrm{pc}(G)$ and k and (ii) the vertex or edge deletion graphs $G-v$ and $G-e$ for a non-cut-vertex v of G and a non-bridge e of G in terms of $\mathrm{pc}(G)$ and the degree of v. We also present some related results and open questions in this chapter.

4.1 Line Graphs

The most familiar graph operation of a graph is probably that of the line graph. The *line graph* $L(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. We determine the proper connection number of $L(G)$ for every connected graph G of order at least 3. In order to do this, we first present an additional definition.

For a connected graph G and two sets X and Y of vertices of G, the *distance* $d(X,Y)$ *between* X *and* Y is defined as

$$d(X,Y) = \min\{d(x,y) :\ x \in X \text{ and } y \in Y\}.$$

Thus, $d(X,Y) = 0$ if and only if $X \cap Y \neq \emptyset$. In particular, if $X = \{x\}$ and $Y = \{y\}$, then $d(X,Y) = d(x,y)$.

Since the line graph of a connected graph G of order 3 or more is complete if and only if G is either a star or K_3, it follows that $\mathrm{pc}(L(T)) \geq 2$ if G is neither of these graphs.

Theorem 4.1.1 *For each connected graph G of order at least 3 that is neither a star nor K_3,*

$$\mathrm{pc}(L(G)) = 2.$$

Proof. Let G be a connected graph of order $n \geq 3$ with $V(G)= \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and T a rooted spanning tree of G where the root of T is v_0. For each integer d with $0 \leq d \leq e_T(v_0)$, let $L_d = \{v \in V(G) : d_T(v_0, v) = d\}$. Thus, $L_0 = \{v_0\}$. First, we define a vertex coloring $c : V(G) \to \{1, 2\}$ of G as follows. For $0 \leq k \leq n-1$ and $0 \leq d \leq e_T(v_0)$, let

$$c(v_k) = \begin{cases} 1 & \text{if } v_k \in L_d \text{ and } d \text{ is odd} \\ 2 & \text{if } v_k \in L_d \text{ and } d \text{ is even.} \end{cases}$$

Observe that each vertex of $L(G)$ can be represented as v_iv_j for some i and j where $0 \leq i \neq j \leq n-1$ since it corresponds to an edge v_iv_j in G. The two vertices v_iv_j and v_kv_ℓ of $L(G)$ are adjacent if and only if (exactly) one of these four conditions occurs: $i = k$, $i = \ell$, $j = k$ or $j = \ell$.

Next, we define an edge coloring $c_L : E(L(G)) \to \{1, 2\}$ of $L(G)$ by $c_L(e) = c(v_j)$ where, without loss of generality, $e = ab$ in which $a = v_iv_j$ and $b = v_jv_k$ are edges of G. It remains to show that c_L is a proper-path 2-coloring of $L(G)$.

For $x, y \in V(L(G))$, we show that there is a properly colored $x-y$ path in $L(G)$. We may assume x and y are nonadjacent. Let $x = v_iv_j$ and $y = v_kv_\ell$, where then v_i, v_j, v_k and v_ℓ are distinct vertices of G. Without loss of generality, for $X = \{v_i, v_j\}$ and $Y = \{v_k, v_\ell\}$ we let $d_T(X, Y) = d_T(v_j, v_k) \geq 1$. Then there is a unique $v_j - v_k$ path P' in T that does not contain v_i and v_ℓ.

Let P be a $v_i - v_\ell$ path of G obtained from the path P' in T by joining v_i to v_j and joining v_ℓ to v_k. Thus, $P = (v_i = w_1, v_j = w_2, w_3, \ldots, w_{q-1} = v_k, w_q = v_\ell)$ for some integer $q \geq 4$. Moreover, we may assume for $2 \leq t \leq q-1$ that

$$c(w_t) = \begin{cases} 1 & \text{if } t \text{ is odd} \\ 2 & \text{if } t \text{ is even.} \end{cases} \tag{4.1}$$

Then $L(P) = (e_1, e_2, \ldots, e_{q-1})$ is a path of order $q-1$ in $L(G)$, where $e_i = w_i w_{i+1}$ $(1 \le i \le q-1)$. In particular, $e_1 = x$ and $e_{q-1} = y$. Observe that e_1 and e_2 are both incident with w_2 and so the edge e_1e_2 in $L(G)$ is colored 2 by c_L; that is $c_L(e_1e_2) = c(w_2) = 2$ by (4.1). Next, e_2 and e_3 are both incident with w_3 and so $c_L(e_2e_3) = c(w_3) = 1$ again by (4.1). This is illustrated in Figure 4.1 for $q = 8$, where the edges in $L(G)$ are indicated by dash lines. In general,

$$c_L(e_ie_{i+1}) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Since the edges of $L(P)$ are colored alternatively by 1 and 2, it follows that $L(P)$ is a properly colored $x-y$ path in $L(G)$. Therefore, $\mathrm{pc}(L(G)) \le 2$. ∎

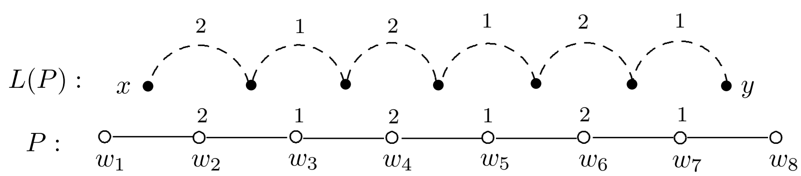

Figure 4.1: Illustrating the proper-path coloring c_L in the proof of Theorem 4.1.1

Proof #2 of Theorem 4.1.1: Let G be a connected graph of order at least 3 that is neither a star nor K_3 and let T be a spanning tree of G. Let V_1 and V_2 be the partite sets of T. Define a vertex coloring $c : V(G) \to \{1,2\}$ of G by $c(v) = i$ if $v \in V_i$ $(i = 1,2)$. We now define an edge coloring $c_L : E(L(G)) \to \{1,2\}$ of $L(G)$. Let $e' = xy$ be an edge of $L(G)$. Then x and y correspond to adjacent edges pq and qr in G, where then $p, q, r \in V(G)$. We then define $c_L(e') = c(q)$. In particular, this implies that if e_1 and e_2 are any two adjacent edges of T, then $\{c_L(e_1), c_L(e_2)\} = \{1,2\}$. We now show that c_L is a proper-path coloring of $L(G)$.

Let u and v be two distinct vertices of $L(G)$. We show that there is a proper $u-v$ path in $L(G)$. If u and v are adjacent, then (u,v) is a proper path. Hence, we may assume that u and v are not adjacent. Let u and v correspond to the edges e and f of G, where $e = tw$ and $f = yz$. Then $d_T(\{t,w\},\{y,z\}) \ge 1$. Suppose that $d_T(\{t,w\},\{y,z\}) = d_T(w,y)$. Let $P' = (w = v, v_1, \ldots, v_k = y)$ be the unique $w-y$ path in T. The path P' gives rise to a $u-v$ path P in $L(G)$ whose edges have the colors $c(v_0), c(v_1), \ldots, c(v_k)$. Since these colors alternate between 1 and 2, the path P is a proper $u-v$ path in $L(G)$. Therefore, c_L is a proper-path coloring of $L(G)$. ∎

For a connected graph G, the line graph $L(G)$ of G is complete if and only if G is a star or G is a triangle. Since the complete graph of order $n \geq 2$ is the only connected graph of order n with proper connection number 1, the following is a consequence of Theorem 4.1.1.

Corollary 4.1.2 *If G is a connected graph of order $n \geq 3$, then*

$$\mathrm{pc}(L(G)) = \begin{cases} 1 & \textit{if } G \in \{K_3, K_{1,n-1}\} \\ 2 & \textit{otherwise.} \end{cases}$$

There is a more general concept in line graphs. For a nonempty graph G, we write $L^0(G)$ to denote G and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the *k-iterated line graph* $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. In order to determine the proper connection number of each iterated line graph of a graph, we first present some useful information in line graphs. A graph H is called a *line graph* if there exists a graph G such that $H = L(G)$. A natural question to ask is whether a given graph H is a line graph. Several characterizations of line graphs have been obtained, perhaps the best known of which is a 1970 forbidden subgraph characterization due to Beineke.

Theorem 4.1.3 [3] *A graph H is a line graph if and only if none of the graphs of Figure 4.2 is isomorphic to an induced subgraph of H.*

By Theorem 4.1.3, the star $K_{1,n-1}$ of order $n \geq 4$ is not a line graph. Furthermore, $L(K_{1,n-1}) = K_{n-1}$ and $L^k(K_3) = K_3$ for each positive integer k. It then follows by Corollary 4.1.2 that if $G \neq K_{1,3}$, then $L^k(G) = 2$ for each integer $k \geq 2$. Therefore, the following corollary is a consequence of Theorem 4.1.3 and Corollary 4.1.2, which provides the exact value of $\mathrm{pc}(L^k(G))$ for every connected graph G and each $k \geq 1$.

Corollary 4.1.4 *If G is a connected graph of order $n \geq 3$ and k is a positive integer, then*

$$\mathrm{pc}(L^k(G)) = \begin{cases} 1 & \textit{if either (i) } G \in \{K_3, K_{1,3}\} \textit{ and } k \geq 1 \\ & \textit{or (ii) } G = K_{1,n-1} \textit{ where } n \neq 4 \textit{ and } k = 1 \\ 2 & \textit{otherwise.} \end{cases}$$

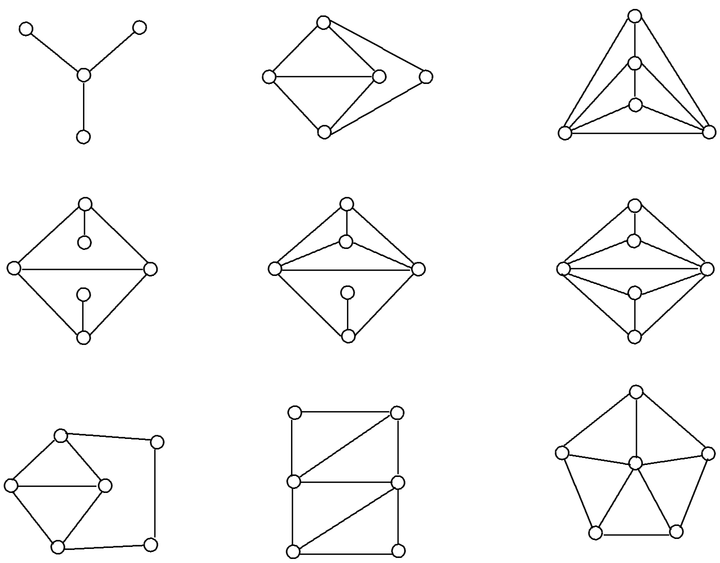

Figure 4.2: The induced subgraphs not contained in any line graph

4.2 Powers of Graphs

For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \le d_G(u,v) \le k$. The graph G^2 is called the *square* of G and G^3 is the *cube* of G. In order to determine the proper connection numbers of powers of graphs, we first recall the following two results presented in Chapter 2 (and repeated in Chapter 3). A fundamental property of the chromatic number or chromatic index is that if H is a subgraph of a graph G, then $\chi(H) \le \chi(G)$ and $\chi'(H) \le \chi'(G)$. For the proper connection number, the situation is different, as we mentioned before.

Proposition 4.2.1 *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then* $\mathrm{pc}(G) \le \mathrm{pc}(H)$. *Furthermore,*

$$\mathrm{pc}(G) \le \min\{\Delta(T) : \ T \textit{ is a spanning tree of } G\}.$$

Again, as we mentioned before, the following corollary is an immediate consequence of Proposition 4.2.1.

Corollary 4.2.2 *If G is a traceable graph that is not complete, then* $\mathrm{pc}(G) = 2$.

We now determine the proper connection number of the square of connected graphs of order at least 3. If G is such a graph of diameter at most 2, then G^2 is complete and so $\mathrm{pc}(G^2) = 1$. Hence, it suffices to consider only connected graphs of diameter at least 3.

Theorem 4.2.3 *For each connected graph G of diameter at least* 3*,*

$$\mathrm{pc}(G^2) = 2.$$

Proof. Let G be a connected graph of diameter at least 3 and T a spanning tree of G. Thus, T is a spanning tree of G^2 as well. Moreover, T^2 is a spanning subgraph of G^2. Define an edge coloring $c : E(G^2) \to \{1, 2\}$ of G^2 by

$$c(e) = \begin{cases} 1 & \text{if } e \in E(T) \\ 2 & \text{if } e \in E(G^2) - E(T). \end{cases}$$

Let $x, y \in V(G^2) = V(G)$. We show that there is a properly colored $x - y$ path in T^2 and therefore in G^2 as well. Let $P = (x = v_0, v_1, v_2, \ldots, y = v_d)$ be the unique $x - y$ path in T. Thus, P is also a path in T^2. We may assume that x and y are not adjacent in T^2 and so $d \geq 3$. We claim that there is a properly colored $x - y$ path P' in T^2. We consider three cases, according to whether d is congruent to 0, 1 or 2 modulo 3.

Case 1. $d \equiv 0 \pmod 3$. Then $d = 3t$ for some positive integer t. Thus,

$$P' = (x = v_0, v_1, v_3, v_4, v_6, \ldots, v_{3(t-1)}, v_{3t-2}, v_{3t} = y)$$

is an $x - y$ path in T^2 whose edges are alternately colored 1 and 2.

Case 2. $d \equiv 1 \pmod 3$. Then $d = 3t + 1$ for some positive integer t. Thus,

$$P' = (x = v_0, v_1, v_3, v_4, v_6, v_7, \ldots, v_{3(t-1)}, v_{3t-2}, v_{3t}, v_{3t+1} = y)$$

is an $x - y$ path in T^2 whose edges are alternately colored 1 and 2.

Case 3. $d \equiv 2 \pmod 3$. Then $d = 3t + 2$ for some positive integer t. Thus,

$$P' = (x = v_0, v_2, v_3, v_5, v_6, v_8, \ldots, v_{3(t-1)}, v_{3t-1}, v_{3t}, v_{3t+2} = y)$$

is an $x - y$ path in T^2 whose edges are alternately colored 1 and 2.

In any case, P' is a properly colored $x - y$ path in G^2. Hence, $\mathrm{pc}(G^2) = 2$. ■

In 1960 Sekanina [35] proved that the cube of every connected graph G is Hamiltonian-connected and, consequently, G^3 is Hamiltonian if its order is at least 3.

Furthermore, the kth power G^k of a connected graph G is complete if and only if $\mathrm{diam}(G) \leq k$. Thus, the following is a consequence of Corollary 4.2.2 and Theorem 4.2.3.

Corollary 4.2.4 *Let $k \geq 2$ be an integer. If G is a connected graph of order at least 3, then $\mathrm{pc}(G^k) \leq 2$. Furthermore, $\mathrm{pc}(G^k) = 1$ if and only if $\mathrm{diam}(G) \leq k$.*

4.3 Iterated Corona Graphs

For a given graph G, the *corona* $\mathrm{cor}(G)$ of G is obtained from G by adding a pendant edge to each vertex of G. Thus, if the order of G is n and its size is m, then the order of $\mathrm{cor}(G)$ is $2n$ and its size is $m+n$. For a nonempty graph G, we write $\mathrm{cor}^0(G)$ to denote G and $\mathrm{cor}^1(G)$ to denote $\mathrm{cor}(G)$. For an integer $k \geq 2$, the *k-iterated corona graph* $\mathrm{cor}^k(G)$ is defined as $\mathrm{cor}(\mathrm{cor}^{k-1}(G))$. Recall the following useful two results.

Proposition 4.3.1 *Let G be a nontrivial connected graph containing bridges. If the maximum number of bridges incident with a single vertex in G is b, then $\mathrm{pc}(G) \geq b$.*

Proposition 4.3.2 *If T is a nontrivial tree, then $\mathrm{pc}(T) = \chi'(T) = \Delta(T)$.*

With the aid of Propositions 4.3.1 and 4.3.2, we will see that iterated corona graphs have relatively large proper connection numbers. First, we introduce an additional definition. For a proper-path coloring c of a graph G, the *restriction* c_H of c to a subgraph H of G is the coloring defined by $c_H(e) = c(e)$ for every edge e of H.

Theorem 4.3.3 *If G is a nontrivial connected graph and k is a positive integer, then*

$$\max\{\mathrm{pc}(G), k\} \leq \mathrm{pc}(\mathrm{cor}^k(G)) \leq \mathrm{pc}(G) + k. \tag{4.2}$$

Proof. Let G be a connected graph of order $n \geq 2$. In the graph $\mathrm{cor}^k(G)$, each vertex of G is incident with at least k bridges. It then follows by Proposition 4.3.1 that $\mathrm{pc}(\mathrm{cor}^k(G)) \geq k$. Furthermore, for every two vertices of u and v in $\mathrm{cor}^k(G)$ such that $u, v \in V(G)$, each $u - v$ path lies completely in G. This implies that the restriction of a proper-path coloring of $\mathrm{cor}^k(G)$ to G must be a proper-path coloring of G as well. Hence, $\mathrm{pc}(\mathrm{cor}^k(G)) \geq \mathrm{pc}(G)$. Therefore, the lower bound in (4.2) holds.

To show that $\text{pc}(\text{cor}^k(G)) \leq \text{pc}(G)+k$, let c_G be a minimum proper-path coloring of G using the colors $1, 2, \ldots, \text{pc}(G)$ and $F = \text{cor}^k(G) - E(G)$. Then F is a forest consisting of n components $T_1, T_2, \ldots, T_n$. The maximum degree of each component T_i is k. By Proposition 4.3.2, $\text{pc}(T_i) = k$. Let c_{T_i} be a minimum proper-path coloring of T_i using the colors $\text{pc}(G)+1, \text{pc}(G)+2, \ldots, \text{pc}(G)+k$. Now, define the coloring c of $\text{cor}^k(G)$ by

$$c(e) = \begin{cases} c_G(e) & \text{if } e \in E(G) \\ c_{T_i}(e) & \text{if } e \in E(T_i) \text{ for } 1 \leq i \leq n. \end{cases}$$

Since c is a proper-path coloring of $\text{cor}^k(G)$ using exactly $\text{pc}(G)+k$ colors, it follows that $\text{pc}(\text{cor}^k(G)) \leq \text{pc}(G)+k$. ■

Both upper and lower bounds in Theorem 4.3.3 are sharp. For example, if G is a tree, then $\text{cor}^k(G)$ is a tree with $\Delta(\text{cor}^k(G)) = \Delta(G)+k$. Hence, $\text{pc}(\text{cor}^k(G)) = \text{pc}(G)+k$ by Proposition 4.3.2 and so the upper bound in Theorem 4.3.3 is sharp. Furthermore, there are connected graphs G that are not trees for which $\text{pc}(\text{cor}^k(G)) = \text{pc}(G)+k$ for each integer $k \geq 1$. For example, the 3-regular graph G of Figure 4.3 has $\text{pc}(G) = 3$ (a proper-path 3-coloring is shown in the figure) and $\text{pc}(\text{cor}^k(G)) = 3+k$ for each positive integer k.

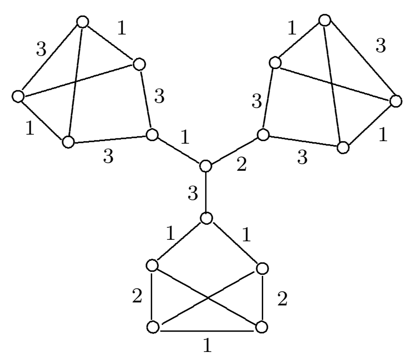

Figure 4.3: A 3-regular graph G with $\text{pc}(G) = 3$

To show that the lower bound in Theorem 4.3.3 is sharp, we determine $\text{pc}(\text{cor}^k(K_n))$ for each complete graph K_n of order $n \geq 3$ and each positive integer k.

Theorem 4.3.4 *For integers $k \geq 1$ and $n \geq 3$,*

$$\text{pc}(\text{cor}^k(K_n)) = \begin{cases} k+1 & \text{if either } k=1 \text{ or } k=2 \text{ and } n=3 \\ k & \text{if either } k=2 \text{ and } n \geq 4 \text{ or } k \geq 3. \end{cases} \tag{4.3}$$

Proof. In the graph $\mathrm{cor}^k(K_n)$, let $V(K_n) = \{u_1, u_2, \dots, u_n\}$ where $k \ge 1$ and $n \ge 3$. Let G be the k-corona of K_n; that is, G is obtained from K_n by adding exactly k pendant edges at each vertex of of K_n. Since $\mathrm{pc}(T) = \Delta(T)$ for each tree T, every proper-path coloring of G can be extended to a proper-path coloring of $\mathrm{cor}^k(K_n)$ with the same set of colors and so $\mathrm{pc}(G) \ge \mathrm{pc}(\mathrm{cor}^k(K_n))$. On the other hand, if x and y are vertices of $\mathrm{cor}^k(K_n)$ such that $x, y \in V(G)$, then every $x - y$ path lies completely in G. Hence, the restriction of a proper-path coloring of $\mathrm{cor}^k(K_n)$ to G is a proper-path coloring of G and so $\mathrm{pc}(G) \le \mathrm{pc}(\mathrm{cor}^k(K_n))$ and so $\mathrm{pc}(G) = \mathrm{pc}(\mathrm{cor}^k(K_n))$. Thus, it suffices to show that $\mathrm{pc}(G)$ satisfies the formula given in (4.3). For $1 \le i \le n$, let $v_{i,1}, v_{i,2}, \dots, v_{i,k}$ be the k end-vertices adjacent to u_i in G. Thus, every proper-path coloring of G must assign distinct k colors to these k pendant edges at each vertex of K_n. We consider three cases, according to whether $k = 1$, $k = 2$ or $k \ge 3$.

Case 1. $k = 1$. Since $\mathrm{pc}(G) \ge 2$, it suffices to find a proper-path 2-coloring of G. The coloring that assigns the color 1 to each pendant edge of G and the color 2 to the remaining edges of G is a proper-path 2-coloring of G and so $\mathrm{pc}(G) = 2$.

Case 2. $k = 2$. First, suppose that $n = 3$. The coloring that assigns the colors 1 and 2 to the two pendant edges at each vertex of K_3 and the color 3 to each edge of K_3 is a proper-path 3-coloring of G and so $\mathrm{pc}(G) \le 3$. If there were a proper-path 2-coloring c of G, then c must assign two edges of K_3 the same color, say $c(u_1u_2) = c(u_2u_3) = 1$. Furthermore, we may assume that $c(u_2v_{2,1}) = 1$. However then, there is neither a proper $v_{2,1} - u_1$ path nor a proper $v_{2,1} - u_3$ path in G. Thus, $\mathrm{pc}(G) = 3$.

Next, suppose that $n \ge 4$. Since $\mathrm{pc}(G) \ge 2$, it suffices to find a proper-path 2-coloring of G. Let $C = (u_1, u_2, \cdots, u_n, u_1)$ be a Hamiltonian cycle of K_n. Define a coloring c by assigning (1) the colors 1 and 2 to the two pendant edges at each vertex of K_n, (2) the color 1 to each edge of C and (3) the color 2 to the remaining edges of K_n. Since c is a proper-path 2-coloring of G, it follows that $\mathrm{pc}(G) = 2$.

Case 3. $k \ge 3$. Since $\mathrm{pc}(G) \ge k$ by Proposition 4.3.1, it suffices to find a proper-path k-coloring of G. The coloring defined on Case 2 can be extended to a proper-path k-coloring of G; that is, we assign (1) the colors $1, 2 \dots, k$ to the k pendant edges at each vertex of K_n, (2) the color 1 to each edge of a Hamiltonian cycle of K_n and (3) the color 2 to the remaining edges of K_n. Therefore, $\mathrm{pc}(G) = k$. ■

Every connected graph G that we considered so far has the property that either $\mathrm{pc}(\mathrm{cor}^k(G)) = \max\{\mathrm{pc}(G), k\}$ or $\mathrm{pc}(\mathrm{cor}^k(G)) = \mathrm{pc}(G) + k$. Furthermore, we know of no connected graphs G for which $\max\{\mathrm{pc}(G), k\} < \mathrm{pc}(\mathrm{cor}^k(G)) < \mathrm{pc}(G) + k$. Thus, we are left with the following question.

Problem 4.3.5 *Are there connected graphs G and integers $k \geq 3$ for which*

$$\max\{\mathrm{pc}(G), k\} < \mathrm{pc}(\mathrm{cor}^k(G)) < \mathrm{pc}(G) + k?$$

4.4 Vertex or Edge Deletions

Let G be a connected graph of order at least 3. For each vertex v of G and each edge e of G, it is known that

$$\chi(G) - 1 \leq \chi(G - v) \leq \chi(G) \text{ and } \chi(G) - 1 \leq \chi(G - e) \leq \chi(G).$$

However, this is not the case for the proper connection number of a graph in general. In order to show this, we first present a useful observation.

Observation 4.4.1 *If T is a nontrivial tree with maximum degree Δ and having n_1 end-vertices, then $\Delta \leq n_1$.*

Theorem 4.4.2 *Let G be a connected graph of order at least 3. If v is a non-cut-vertex of G, then*

$$\mathrm{pc}(G) - 1 \leq \mathrm{pc}(G - v) \leq \mathrm{pc}(G) + \deg v. \tag{4.4}$$

Proof. Suppose that $\mathrm{pc}(G - v) = a$ and $\deg v = d$. First, observe that if c is a proper-path coloring of $G - v$ using the colors $1, 2, \ldots, a$, then c can be extended to a proper-path coloring of G by assigning the color $a + 1$ to each edge incident with v in G. Thus, $\mathrm{pc}(G) \leq \mathrm{pc}(G - v) + 1$, establishing the lower bound.

To verify the upper bound, let $c_G : E(G) \to \{1, 2, \ldots, k\}$ be a minimum proper-path coloring of G and let $N(v)$ be the neighborhood of v, where then $|N(v)| = d$. Since $G - v$ is connected, there is a tree T of minimum order in $G - v$ such that $N(v) \subseteq V(T)$. Necessarily, each end-vertex of T belongs to $N(v)$. Thus, if the number of end-vertices of T is n_1, then $n_1 \leq d$. Now, let $\Delta = \Delta(T)$ be the maximum degree of T. By Proposition 4.3.2 and Observation 4.4.1, it follows that $\chi'(T) = \mathrm{pc}(T) = \Delta \leq n_1 \leq d$. Let $c_T : E(T) \to \{k+1, k+2, \ldots, k+\Delta\}$ be a proper edge coloring of T. Define an edge coloring

$$c : E(G - v) \to \{1, 2, \ldots, k + \Delta\}$$

of $G - v$ by

$$c(e) = \begin{cases} c_G(e) & \text{if } e \in E(G - v) - E(T) \\ c_T(e) & \text{if } e \in E(T). \end{cases} \tag{4.5}$$

It remains to show that c is a proper-path coloring of $G - v$. Let x and y be two nonadjacent vertices of $G - v$. We show that there is a properly colored $x - y$ path in $G - v$. Since c_G is a proper-path coloring of G, there is an $x - y$ path in G that is properly colored by the edge coloring c_G of G. We consider two cases.

Case 1. *There is an $x - y$ path P in G that does not contain v and is properly colored by c_G.* If $E(P) \cap E(T) = \emptyset$, then P is an $x - y$ path in $G - v$ that is properly colored by c. Thus, we may assume that $E(P) \cap E(T) \neq \emptyset$. We now divide the path P into a finite number of blocks $A_1, B_1, A_2, B_2, \ldots$ for which

$$P = (A_1, B_1, A_2, B_2, \ldots)$$

where each block is a subpath of P such that $E(A_i) \subseteq E(G - v) - E(T)$ for each $i \geq 1$ and $E(B_j) \subseteq E(T)$ for each $j \geq 1$ (or $E(A_i) \subseteq E(T)$ for each $i \geq 1$ and $E(B_j) \subseteq E(G - v) - E(T)$ for each $j \geq 1$). Since P is properly colored by c_G and T is properly colored by c_T, it follows by the definition of c in (4.5) that each of the blocks A_i and B_j is properly colored by c in $G - v$. Furthermore, the colors of edges in A_i belong to $\{1, 2, \ldots, k\}$ and the colors of edges in B_j belong to $\{k+1, k+2, \ldots, k+\Delta\}$. Therefore, P is properly colored by c and so P is a proper $x - y$ path in $G - v$.

Case 2. *Every $x - y$ path in G that is properly colored by c_G contains the vertex v.* Let Q be an $x - y$ path in G that is properly colored by c_G. Thus, Q contains a subpath (u, v, w) where $u, w \in N(v)$. Let W be the $x - y$ walk in $G - v$ obtained from Q by replacing the subpath (u, v, w) by the $u - w$ path R in T. Let $Q_{x,u}$ be the $x - u$ subpath of Q and $Q_{w,y}$ be the $w - y$ subpath of Q. Furthermore, let u' be the first vertex (from x to u) that belongs to $V(Q_{x,u}) \cap V(R)$ and let w' be the last vertex (from w to y) that belong to $V(Q_{w,y}) \cap V(R)$. Then $u' \neq w'$ where it is possible that $u = u'$ or $w = w'$. Now let $Q_{x,u'}$ be the $x - u'$ subpath of Q, let $Q_{w',y}$ be the $w' - y$ subpath of Q and let $R_{u',w'}$ be the $u' - w'$ subpath of R. Now the path $P = (Q_{x,u'}, R_{u',w'}, Q_{w',y})$ is an $x - y$ path in $G - v$. Since the colors of edges in $Q_{x,u'}$ and $Q_{w',y}$ belong to $\{1, 2, \ldots, k\}$ and the colors of edges in $R_{u',w'}$ belong to $\{k+1, k+2, \ldots, k+\Delta\}$, it follows that P is properly colored by c and so P is a proper $x - y$ path in $G - v$.

Therefore, the edge coloring $c : E(G - v) \to \{1, 2, \ldots, k + \Delta\}$ defined in (4.5) is a proper-path coloring of $G - v$ and so $\mathrm{pc}(G - v) \le k + \Delta \le \mathrm{pc}(G) + \deg v$. ■

Both lower and upper bounds in (4.4) are sharp. For example, let $G = K_{1,t}$ be the star of order $t + 1 \ge 3$ and let v be an end-vertex of G. Since $\mathrm{pc}(G) = t$ and $\mathrm{pc}(G - v) = t - 1 = \mathrm{pc}(G) - 1$, it follows that the lower bound is sharp. For the upper bound in (4.4), we start with the complete bipartite graph $K_{2,t}$ of order $2 + t \ge 4$ where u and v are the vertices of degree t in $K_{2,t}$. It was shown in [1] that if G is a complete multipartite graph that is neither a complete graph nor a tree, then $\mathrm{pc}(G) = 2$. Thus, $\mathrm{pc}(K_{2,t}) = 2$. The graph H is then obtained from $K_{2,t}$ by adding two pendant edges at the vertex u of degree t in $K_{2,t}$. It can be shown that $\mathrm{pc}(H) = 2$. In fact, a proper-path 2-coloring of H can be obtained from a proper-path 2-coloring of $K_{2,t}$ (using the colors 1 and 2) by assigning the colors 1 and 2 to the two pendant edges incident with the vertex u in H. Then $H - v = K_{1,t+2}$. Since $\deg_H v = t$ and $\mathrm{pc}(H - v) = t + 2$, it follows that

$$\mathrm{pc}(H - v) = \mathrm{pc}(H) + \deg_H v.$$

Therefore, the upper bound in (4.4) is sharp. Furthermore, strict equalities are also possible in (4.4). For example, let $F = K_{2,t}$ where $t \ge 3$ and so $\mathrm{pc}(F) = 2$. Now let v be a vertex of degree t in F. Then $F - v = K_{1,t}$ and so $\mathrm{pc}(F - v) = t$. Therefore,

$$\mathrm{pc}(F) < \mathrm{pc}(F - v) = t = \mathrm{pc}(F) + \deg v - 2 < \mathrm{pc}(F) + \deg v.$$

Theorem 4.4.3 *Let G be a connected graph of order at least 3. If e is an edge of G that is not a bridge, then*

$$\mathrm{pc}(G) \le \mathrm{pc}(G - e) \le \mathrm{pc}(G) + 2. \tag{4.6}$$

Proof. Since $G - e$ is a spanning subgraph of G for each nonbridge e of G, it follows by Proposition 4.2.1 that $\mathrm{pc}(G) \le \mathrm{pc}(G - e)$, establishing the lower bound. It remains to verify the upper bound. Suppose that $\mathrm{pc}(G) = k$ and $e = uv$ where $u, v \in V(G)$. Let $c_G : E(G) \to \{1, 2, \ldots, k\}$ be a minimum proper-path coloring of G. Since e is not a bridge, there is a $u - v$ path P in $G - e$. Let $c_P : E(P) \to \{k + 1, k + 2\}$ be a proper edge coloring of P. Define an edge coloring $c : E(G-e) \to \{1, 2, \ldots, k+2\}$ of $G - e$ by

$$c(e) = \begin{cases} c_G(e) & \text{if } e \in E(G-e) - E(P) \\ c_P(e) & \text{if } e \in E(P). \end{cases}$$

Applying an argument similar to one used in the proof of Theorem 4.4.2 where the tree T is replaced by the $u - v$ path P, it can be shown that c is a proper-path coloring of $G - e$ and so $\mathrm{pc}(G - e) \leq k + 2 = \mathrm{pc}(G) + 2$. ∎

Both lower and upper bounds in (4.6) are sharp. For example,

$$\mathrm{pc}(C_n) = \mathrm{pc}(C_n - e) = 2$$

for $n \geq 4$ and any edge e in C_n. Furthermore, if $G = K_{1,t} + e$ where $t \geq 5$, then $\mathrm{pc}(G) = t - 2$ and so $\mathrm{pc}(G - e) = t = \mathrm{pc}(G) + 2$.

4.5 Unicyclic Graphs

We saw that the proper connection numbers of trees and cycles have been determined, namely if T is a nontrivial tree, then

$$\mathrm{pc}(T) = \mathrm{spc}(T) = \chi'(T) = \Delta(T).$$

For each integer $n \geq 4$, $\mathrm{pc}(C_n) = 2$ and

$$\mathrm{spc}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, if G is a cycle, then the iterated corona graph $\mathrm{cor}^k(G)$ of G is a connected graph with exactly one cycle; that is, $\mathrm{cor}^k(G)$ is a unicyclic graph. In this section, we study proper-path colorings of unicyclic graphs in general. First, we establish bounds for the proper connection number of a graph in terms of its maximum degree.

Proposition 4.5.1 *If G is a unicyclic graph, then*

$$\Delta(G) - 2 \leq \mathrm{pc}(G) \leq \Delta(G). \tag{4.7}$$

Proof. Let C be the unique cycle in a unicyclic graph G and let e be an edge of C such that $\Delta(G-e) = \Delta(G)$. Since $G - e$ is a spanning tree of G, it follows by Proposition 4.2.1 that $\mathrm{pc}(G) \leq \Delta(G-e) = \Delta(G)$. Furthermore,

$$\mathrm{pc}(G-e) \leq \mathrm{pc}(G) + 2$$

by Theorem 4.4.3 and so $\mathrm{pc}(G) \geq \mathrm{pc}(G-e) - 2 = \Delta(G) - 2$. ■

By Proposition 4.5.1, the proper connection number of a unicyclic graph G is one of the three numbers $\Delta(G), \Delta(G)-1$ and $\Delta(G)-2$. Next, we show that there is an infinite class of unicyclic graphs G for which $\mathrm{pc}(G) = \Delta(G) - i$ for each $i \in \{0,1,2\}$, beginning with $i = 0$. For example, let $G_0 = C_n$ for some even integer $n \geq 4$, let G_1 be the unicycle graph obtained from G_0 by adding a new vertex and joining this new vertex to a vertex of G_0 and let G_2 be the unicycle graph obtained from G_0 by adding two new vertices and joining both of these two new vertices to a vertex of G_0. Then $\Delta(G_0) = 2$, $\delta(G_1) = 3$ and $\Delta(G_2) = 4$. Now color the edges of G_0 properly with the colors 1 and 2, color the pendant of G_1 by color 1 and color the two pendant edges of G_2 by colors 1 and 2, producing a proper-path 2-coloring of G_i for $i = 0, 1, 2$. Therefore, $\mathrm{pc}(G_i) = 2$ for $i = 0, 1, 2$. In fact, these examples can be extended to infinite classes of unicyclic graphs G of arbitrarily large maximum degree for which $\mathrm{pc}(G) = \Delta(G) - i$ for each $i \in \{0,1,2\}$. To see this, let k and n be integers with $k \geq 1$ and $n \geq 4$ and consider the k-iterated corona graph $\mathrm{cor}^k(C_n)$ of the n-cycle C_n, which is a unicyclic graph with $\Delta(\mathrm{cor}^k(C_n)) = k + 2$ for integers $k \geq 1$ and $n \geq 4$.

Proposition 4.5.2 *For integers $k \geq 1$ and $n \geq 4$,*

$$\mathrm{pc}(\mathrm{cor}^k(C_n)) = \Delta(\mathrm{cor}^k(C_n)) = \mathrm{pc}(C_n) + k.$$

Proof. In the graph $\mathrm{cor}^k(C_n)$ where $k \geq 1$, let

$$C_n = (u_1,\, u_2,\, \ldots,\, u_n,\, u_{n+1} = u_1)$$

be a cycle of order $n \geq 4$ and for each integer i with $1 \leq i \leq n$, let $e_i = u_i u_{i+1}$ be an edge of the cycle. Let $H_{n,k}$ be the k-corona of C_n; that is, $H_{n,k}$ is obtained from C_n by adding exactly k pendant edges at each vertex of C_n and for each $1 \leq i \leq n$, let $v_{i,1}, v_{i,2}, \ldots, v_{i,k}$ be the k end-vertices adjacent to u_i in $H_{n,k}$. Since $\mathrm{pc}(T) = \Delta(T)$ for each tree T, every proper-path coloring of $H_{n,k}$ can be extended to a proper-path coloring of $\text{cor}^k(C_n)$ with the same set of colors and so $\text{pc}(H_{n,k}) \geq \text{pc}(\text{cor}^k(C_n))$. Furthermore, for vertices $x, y \in \text{cor}^k(C_n)$, if $x, y \in H_{n,k}$ then every $x - y$ path lies completely in $H_{n,k}$ and therefore the restriction of a proper-path coloring of $\text{cor}^k(C_n)$ to its subgraph $H_{n,k}$ is a proper-path coloring of $H_{n,k}$. Hence, $\text{pc}(H_{n,k}) \leq \text{pc}(\text{cor}^k(C_n))$ and so $\text{pc}(H_{n,k}) = \text{pc}(\text{cor}^k(C_n))$. Thus, it suffices to show that

$$\text{pc}(H_{n,k}) = \text{pc}(C_n) + k = 2 + k.$$

Observe that $H_{n,k} - e_1$ is a spanning tree of $H_{n,k}$ and $\Delta(H_{n,k} - e_1) = k + 2$, it follows by Proposition 4.2.1 that $\text{pc}(H_{n,k}) \leq k + 2$. It remains to show that $\text{pc}(H_{n,k}) \geq k + 2$. Assume, to the contrary, that there is a proper-path $(k+1)$-coloring c of $H_{n,k}$ using the colors $1, 2, \ldots, k+1$. Thus, c must assign distinct k colors to the k pendant edges at each vertex of C_n. Let u be a vertex of C_n. We may assume $u = u_1$ and $c(u_1v_{1,j}) = j$ for $1 \leq j \leq k$. Observe that it is impossible that $c(e_1) = c(e_n) \in \{1, 2, \ldots, k\}$. Thus, either $\{c(e_1), c(e_n)\} = \{k+1, i\}$ for some i with $1 \leq i \leq k+1$ or $\{c(e_1), c(e_n)\} = \{i, j\}$ where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$. Hence, we may assume, without loss of generality, that $\{c(e_1), c(e_n)\}$ is one of the three sets $\{k+1, 1\}$, $\{k+1\}$ and $\{1, 2\}$. We consider these three cases.

Case 1. *$\{c(e_1), c(e_n)\} = \{k+1, 1\}$, say $c(e_1) = k+1$ and $c(e_n) = 1$.* Since $(v_{1,1}, u_1, u_n)$ is not a proper path, the path $(v_{1,1}, u_1, u_2, u_3, \ldots, u_n)$ is properly colored. Thus, $c(e_2) \neq k+1$ and for each $1 \leq j \leq k$, $c(u_2v_{2,j}) \neq k+1$; for otherwise, there is no proper $v_{1,1} - u_3$ path and no proper $v_{1,1} - v_{2,j}$ path for some j, respectively, which is a contradiction. It then follows that there is a vertex v adjacent to u_2 such that $c(u_2v) = c(e_2) = t \in \{1, 2, \ldots, k\}$. Moreover, $c(e_{n-1}) \neq 1$ and for each $1 \leq j \leq k$, $c(u_nv_{n,j}) \neq 1$; for otherwise there is no proper $v - u_{n-1}$ path and no proper $v - v_{n,j}$ path for some j, respectively, which is impossible. So there is a vertex w adjacent to u_n such that $c(u_nw) = c(e_{n-1}) = \ell \in \{2, 3, \ldots, k+1\}$. However then, there is no proper $w - v_{1,1}$ path, which is a contradiction.

Case 2. *$c(e_1) = k+1 = c(e_n)$.* Since (u_2, u_1, u_n) is not a proper path, the path $(u_2, u_3, u_4, \ldots, u_n)$ is properly colored. Let $c(e_2) = t \in \{1, 2, \ldots, k, k+1\}$ then $c(e_3) \neq t$ and for each $1 \leq j \leq k$, $c(u_3v_{3,j}) \neq t$; for otherwise, there is no proper $u_2 - u_4$ path and no proper $u_2 - v_{3,j}$ path for some j, respectively, which is a contradiction. This implies that there is a vertex v adjacent to u_3 such that $c(u_3v) = c(e_3) \in \{1, 2, \ldots, k, k+1\} - \{t\}$. However then, there is no proper $v - u_4$ path, which is a contradiction.

Case 3. $\{c(e_1), c(e_n)\} = \{1, 2\}$, *say* $c(e_1) = 2$ *and* $c(e_n) = 1$. Then $c(e_2) \neq 2$ and $c(u_2v_{2,j}) \neq 2$ for each j with $1 \leq j \leq k$; for otherwise, there is no proper $v_{1,1} - u_3$ path and no proper $v_{1,1} - v_{2,j}$ path for some j, respectively, which is a contradiction. Hence, there is a vertex v adjacent to u_2 such that $c(u_2v) = c(e_2) = \ell \in \{1, 2, 3, \ldots, k+1\} - \{2\}$. Similarly, $c(e_{n-1}) \neq 1$ and for each $1 \leq j \leq k$, $c(u_nv_{n,j}) \neq 1$; for otherwise, there is no proper $v_{1,2} - u_{n-1}$ path and no proper $v_{1,2} - v_{n,j}$ path for some j, respectively, which is impossible. So there is a vertex w adjacent to u_n such that $c(u_nw) = c(e_{n-1}) = t \in \{2, 3, \ldots, k+1\}$. This means that there is no proper $v_{1,1} - w$ path, which is impossible. Therefore, $\mathrm{pc}(H_{n,k}) \geq k+2$ and so $\mathrm{pc}(H_{n,k}) = k+2$. ■

In fact, the proof of Proposition 4.5.2 provides the following more general result.

Corollary 4.5.3 *Let G be a unicyclic graph that is not a cycle such that the unique cycle in G is not a triangle. If the cycle in G contains three or more consecutive vertices of degree $\Delta(G)$, then $\mathrm{pc}(G) = \Delta(G)$.*

For integers $k \geq 1$ and $n \geq 4$, let $G_{n,k}$ be the unicyclic graph obtained from the cycle C_n of order n by adding exactly k pendant edges at one vertex of C_n. Then $\Delta(G_{n,k}) = k+2$. Next, we show that $\mathrm{pc}(G_{n,k}) = \Delta(G_{n,k}) - 2$ for each integer $k \geq 3$.

Proposition 4.5.4 *For integers $k \geq 1$ and $n \geq 4$,*

$$\mathrm{pc}(G_{n,k}) = \begin{cases} \Delta(G_{n,k}) - 1 = k+1 & \text{if either } (i)\ n \text{ is even and } k = 1 \\ & \quad \text{or } (ii)\ n \text{ is odd and } k = 1, 2 \\ & \\ \Delta(G_{n,k}) - 2 = k & \text{if either } (i)\ n \text{ is even and } k \geq 2 \\ & \quad \text{or } (ii)\ n \text{ is odd and } k \geq 3. \end{cases}$$

Proof. In the graph $G = G_{n,k}$, let $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ be the cycle of order $n \geq 4$ in G. Suppose that $v_1, v_2, \ldots, v_k$ are the k end-vertices of G that are adjacent to $u = u_1$ on C_n. Thus, $\mathrm{pc}(G) \geq k$ by Proposition 4.3.1. Moreover, since $G - \{u_1u_2\}$ is a spanning tree of G and $\Delta(G - \{u_1u_2\}) = k+1$, it follows by Proposition 4.2.1 that $\mathrm{pc}(G) \leq k+1$. Therefore, $\mathrm{pc}(G) = k$ or $\mathrm{pc}(G) = k+1$. If $k = 1$ then G is traceable. Since G is not complete, $\mathrm{pc}(G) = 2 = k+1$ by Corollary 4.2.2 when $k = 1$. Thus, we may assume $k \geq 2$.

First, suppose that n is even. It suffices to show that there is a proper-path k-coloring of G. If we assign the distinct colors 1, 2, …, k to the k pendant edges incident with u and color the n-cycle C_n properly by the colors 1 and k, then we obtain a proper-path k-coloring of G and so $\mathrm{pc}(G) = k$. Next, suppose that n is odd. We consider two cases.

Case 1. $k = 2$. We show that $\mathrm{pc}(G) = 3$. Assume, to the contrary, that there is a proper-path 2-coloring c of G using the colors 1 and 2. Without loss of generality, we assume $c(uv_1) = 1$ and $c(uv_2) = c(uu_n) = 2$. Since (v_2, u, u_n) is not a proper path, the path $(v_2, u, u_2, u_3, \ldots, u_n)$ is properly colored. Thus, for $1 \le i \le n-1$, $c(u_iu_{i+1}) = 1$ if i is odd and $c(u_iu_{i+1}) = 2$ if i is even. However then, there is no proper $v_1 - u_{n-1}$ path, which is a contradiction. Therefore, $\mathrm{pc}(G) = 3 = k + 1$.

Case 2. $k \ge 3$. Define an edge coloring c of G by assigning (1) the distinct colors 1, 2, …, k to the k pendant edges incident with u, (2) the colors 1 and 2 properly to the edges of C_n except uu_n and (3) the color k to the edge uu_n. Since c is a proper-path k-coloring of G, it follows that $\mathrm{pc}(G) = k$. ■

For integers $n \ge 4$ and $k \ge 1$, let $F_{n,k}$ be the unicyclic graph obtained from the n-cycle C_n by adding exactly k pendant edges at two consecutive vertices of C_n. Thus, $\Delta(F_{n,k}) = k + 2$.

Proposition 4.5.5 *For integers $k \ge 1$ and $n \ge 4$,*

$$\mathrm{pc}(F_{n,k}) = \Delta(F_{n,k}) - 1 = k + 1.$$

Proof. In the graph $F = F_{n,k}$, let $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ be the cycle of order $n \ge 4$ in F. Let $u = u_1$ and $w = u_2$ be the two consecutive vertices of C_n such that $\deg u = \deg w = k + 2$. Suppose that $v_1, v_2, \ldots, v_k$ are the k end-vertices of F that are adjacent to u and $w_1, w_2, \ldots, w_k$ are the k end-vertices of F that are adjacent to w on C_n. Thus, $\mathrm{pc}(F) \ge k$ by Proposition 4.3.1. Moreover, since $F - \{uw\}$ is a spanning tree of F and $\Delta(F - \{uw\}) = k + 1$, it follows by Proposition 4.2.1 that $\mathrm{pc}(F) \le k + 1$. Therefore, $\mathrm{pc}(F) = k$ or $\mathrm{pc}(F) = k + 1$. If $k = 1$, then F is traceable and non-complete and so $\mathrm{pc}(F) = 2 = k + 1$ by Corollary 4.2.2. Now assume that $k \ge 2$. Suppose, to the contrary, that there is a proper-path k-coloring c of F using the colors 1, 2, …, k. Assume, without loss of generality, that $c(uv_i) = i$ for $1 \le i \le k$, Observe that it is impossible that

$c(uw) = c(uu_n) \in \{1, 2, \ldots, k\}$. Thus, $\{c(uw), c(uu_n)\} = \{i, j\}$ where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$, say $c(uw) = 1$ and $c(uu_n) = 2$. Since $(v_2, u, u_n, u_{n-1}, \ldots, w, w_i)$ is not a proper path, it follows that (v_2, u, w, w_i) is a properly colored path. Thus, $c(ww_i) \neq 1$ for $1 \leq i \leq k$; for otherwise, there is no proper $v_2 - w_i$ path for some i. Since the k pendant edges ww_i $(1 \leq i \leq k)$ must be assigned distinct colors in $\{2, 3, \ldots, k\}$, this is impossible. Therefore, $\mathrm{pc}(F) = k + 1$. ■

Chapter 5

Strong Proper-Path Colorings

In this chapter, we study strong proper-path colorings of three well-known classes of graphs, namely unicyclic graphs, line graphs and powers of graphs.

5.1 Unicyclic Graphs

We begin by investigating a class of unicyclic graphs G with $\mathrm{spc}(G) = \Delta(G)$. Recall once again that if G is nontrivial connected graph, then

$$1 \le \mathrm{pc}(G) \le \mathrm{spc}(G) \le \min\{\chi'(G), \mathrm{src}(G)\}. \tag{5.1}$$

Proposition 5.1.1 *If G is a unicyclic graph with $\Delta(G) \ge 3$ whose unique cycle has order at least 4, then $\mathrm{spc}(G) = \Delta(G)$.*

Proof. Let $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ be the unique cycle of order $n \ge 4$ in G and let $\Delta(G) = \Delta$. For each integer i with $1 \le i \le n$, let $e_i = u_i u_{i+1}$ and let u be a vertex of G such that $\deg u = \Delta \ge 3$. In any strong proper-path coloring of G, if $v, w \in N(u)$ then uv and uw are assigned distinct colors. Hence, $\mathrm{spc}(G) \ge \Delta$. It remains to show that $\mathrm{spc}(G) \le \Delta$.

First, suppose that n is even. Then G is a bipartite graph and so $\chi'(G) = \Delta$. It then follows by (5.1) that $\mathrm{spc}(G) \le \Delta$ and so $\mathrm{spc}(G) = \Delta$. Next, suppose that $n \ge 5$ is odd. Let $F = G - E(C_n)$ be the spanning forest of G consisting of n components $F_1, F_2, \ldots, F_n$ such that F_i contains u_i for $1 \le i \le n$ (where F_i may be a trivial tree). Since $\Delta(F_i) \le \Delta$ for $1 \le i \le n$, it follows that $\mathrm{spc}(F_i) \le \Delta$. For each integer i with $1 \le i \le n$ such that F_i is nontrivial, let $c_i : E(F_i) \to \{1, 2, \ldots, \Delta\}$ be a proper coloring of F_i such that the color of each edge e incident with u_i in F_i satisfies the following conditions:

- if $i = 1$, then $c(e) \in \{2, 3, \ldots, \Delta - 1\}$,
- if $2 \le i \le n-1$, then $c(e) \in \{1, 2, \ldots, \Delta - 2\}$ and
- if $i = n$, then $c(e) \in \{2, 3, \ldots, \Delta - 2, \Delta\}$.

Define an edge coloring $c : E(G) \to \{1, 2, \ldots, \Delta\}$ by

$$c(e) = \begin{cases} \Delta & \text{if } e = e_i \text{ where } i \text{ is odd and } 1 \le i \le n-2 \\ \Delta - 1 & \text{if } e = e_i \text{ where } i \text{ is even and } 2 \le i \le n-1 \\ 1 & \text{if } e = e_n \\ c_i(e) & \text{if } e \in E(F_i) \text{ for } 1 \le i \le n. \end{cases}$$

This is illustrated in Figure 5.1, where $[\Delta - 2] = \{1, 2, \ldots, \Delta - 2\}$. Since c is a strong proper-path Δ-coloring of G, it follows that $\mathrm{spc}(G) \le \Delta(G)$ and so $\mathrm{spc}(G) = \Delta(G)$ when n is odd. ■

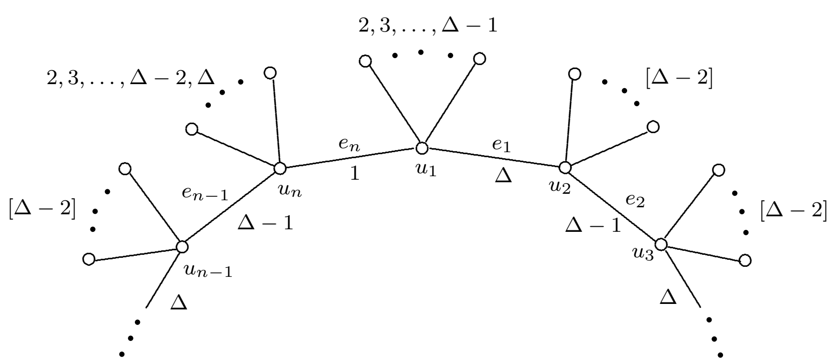

Figure 5.1: Illustrating a portion of the coloring c in the proof of Theorem 5.1.1 when n is odd

The next result shows that if G is a unicyclic graph containing a triangle, then there are only two possible values for $\mathrm{spc}(G)$.

Proposition 5.1.2 *If G is a unicyclic graph containing a triangle, then*

$$\Delta(G) - 1 \le \mathrm{spc}(G) \le \Delta(G).$$

Proof. Let $C_3 = (u_1, u_2, u_3, u_4 = u_1)$ be the unique cycle in G and let $\Delta(G) = \Delta$. First, suppose that there is a vertex u in $V(G) - V(C_3)$ such that $\deg u = \Delta$. Since all incident edges of u must be assigned distinct colors by any strong proper-path coloring of G, it follows that $\text{spc}(G) \geq \Delta$. The coloring c described in the proof of Theorem 5.1.1 (when n is odd) is also a strong proper-path Δ-coloring of G in this case. Hence, $\text{spc}(G) \leq \Delta$ and so $\text{spc}(G) = \Delta$.

Next, suppose that $\deg u \leq \Delta - 1$ for all $u \in V(G) - V(C_3)$. Thus, $\deg u_i = \Delta$ for some $i = 1, 2, 3$, say $\deg u_1 = \Delta$. Let $F = G - E(C_3)$ be the spanning forest of G consisting of three components F_1, F_2, F_3 such that F_i contains u_i for $i = 1, 2, 3$ (where F_i may be a trivial tree). Hence, $\Delta(F_i) \leq \Delta - 1$ for $i = 1, 2, 3$ and so $\text{spc}(F_i) \leq \Delta - 1$. Let $e_i = u_i u_{i+1}$ for $i = 1, 2, 3$. In any strong proper-path coloring of G, if $f \in E(F_1)$ such that f and e_1 are adjacent, then f and e_1 are assigned distinct colors. Similarly, f and e_3 are assigned distinct colors. Thus, $\text{spc}(G) \geq \Delta - 1$. It remains to show that G has a strong proper-path $(\Delta - 1)$-coloring. For each $i = 1, 2, 3$ such that F_i is nontrivial, let

$$c_i : E(F_i) \to \{1, 2, \ldots, \Delta - 1\}$$

be a proper edge coloring of F_i such that $c_i(e) \in \{1, 2, \ldots, \Delta - 2\}$ for each edge e incident with u_i in F_i. Define an edge coloring

$$c : E(G) \to \{1, 2, \ldots, \Delta - 1\}$$

by

$$c(e) = \begin{cases} \Delta(G) - 1 & \text{if } e \in E(C_3) \\ c_i(e) & \text{if } e \in E(F_i) \text{ where } 1 \leq i \leq 3. \end{cases}$$

Since c is a strong proper-path $(\Delta - 1)$-coloring of G, it follows that

$$\text{spc}(G) \leq \Delta(G) - 1$$

and so $\text{spc}(G) = \Delta(G) - 1$. ∎

Recall that for each integer $n \geq 4$, $\text{pc}(C_n) = 2$ and

$$\text{spc}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

As a consequence of this result and Propositions 5.1.1 and 5.1.2, we have the following result which provides the strong proper connection number of every unicyclic graph.

Theorem 5.1.3 *If G is a unicyclic graph of order at least 3, then*

$$\Delta(G) - 1 \leq \text{spc}(G) \leq \Delta(G) + 1.$$

Furthermore, if C is the unique cycle in G, then

(1) $\text{spc}(G) = \Delta(G) + 1$ *if and only if $G = C$ is an odd cycle of order at least 5,*

(2) $\text{spc}(G) = \Delta(G)$ *if and only if*

  (i) $G = C$ *is an even cycle,*

  (ii) $\Delta(G) \geq 3$ *and C is a cycle of order at least 4 or*

  (iii) $C = C_3$ *such that* $\deg_G u = \Delta(G)$ *for some* $u \in V(G) - V(C)$ *and*

(3) $\text{spc}(G) = \Delta(G) - 1$ *if and only if* $C = C_3$ *and* $\deg_G u \leq \Delta(G) - 1$ *for all* $u \in V(G) - V(C)$.

5.2 Line Graphs

While it is challenging, in general, to determine the strong proper connection number $\text{spc}(L(G))$ of the line graph of a connected graph G, we determine in this section the strong proper connection number $\text{spc}(L(T))$ of every tree T of order at least 3.

Theorem 5.2.1 *If T is a tree of order at least 3, then*

$$\text{spc}(L(T)) \leq 2.$$

Proof. Let T be a tree of order at least 3. Since $\chi(T) = 2$, there is a proper vertex coloring c of T using the colors 1 and 2. We define an edge coloring $c_L : E(L(T)) \to \{1, 2\}$ of $L(T)$ as follows. For each $e = xy \in E(L(T))$, where $x = uv$ and $y = vw$ are edges of T, let $c_L(e) = c(v)$. It remains to show that c_L is a strong proper-path 2-coloring of $L(T)$. For $x, y \in V(L(T))$, we show that there is a properly colored $x - y$ geodesic in $L(T)$. We may assume that x and y are nonadjacent. Let $x = uu'$ and $y = w'w$, where then the four vertices u, u', w', w are distinct in T. Let P be the unique $x - y$ path in T joining the edges x and y, say $P = (u = w_1, u' = w_2, w_3, \ldots, w_{q-1} = w', w_q = w)$, where $w_1w_2 = x$ and $w_{q-1}w_q = y$. We may assume for $1 \leq t \leq q$ that

$$c(w_t) = \begin{cases} 1 & \text{if } t \text{ is odd} \\ 2 & \text{if } t \text{ is even.} \end{cases}$$

Then $L(P) = (e_1, e_2, \ldots, e_{q-1})$ is a path of order $q-1$ in $L(T)$, where $e_i = w_i w_{i+1}$ ($1 \le i \le q-1$). In particular, $e_1 = x$ and $e_{q-1} = y$. Since the edges of $L(P)$ are colored alternatively by 1 and 2, it follows that $L(P)$ is a properly colored $x-y$ path in $L(T)$. It remains to show that $L(P)$ is an $x-y$ geodesic in $L(T)$. Assume, to the contrary, that $L(P)$ is not an $x-y$ geodesic in $L(T)$. Let Q be an $x-y$ geodesic in $L(T)$. Necessarily, the length of Q is strictly less than the length of $L(P)$. This would imply there is a path P^* in T joining the edges x and y whose length is smaller than the length of P, where then $L(P^*) = Q$ and $P^* \ne P$. Since P is the unique $x-y$ path in T, this is impossible. Therefore, $L(P)$ is an $x-y$ geodesic and so $\text{spc}(L(T)) \le 2$. ■

If T is a tree of diameter at least 3, then there are two vertices of $L(T)$ that are not adjacent; that is, $L(T)$ is complete if and only if T is a star. Thus, the following is a consequence of Theorem 5.2.1.

Corollary 5.2.2 *If T is a nontrivial tree that is not a star, then*

$$\text{spc}(L(T)) = 2.$$

For an odd integer $n \ge 5$, it is known that $\text{spc}(C_n) = 3$. Since $L(C_n) = C_n$, it follows that $\text{spc}(L(C_n)) = 3$ for each odd integer $n \ge 5$. However, it is not known whether there is any other connected graph G for which $\text{spc}(L(G)) \ge 3$. In fact, there is a more general question.

Problem 5.2.3 *Is there a constant c such that $\text{spc}(L(G)) \le c$ for every nontrivial connected graph G?*

5.3 Squares of Graphs

In this section, we study strong proper-path colorings of squares of graphs, with special emphasis on trees. For each integer $n \ge 3$, the square $K^2_{1,n-1}$ of the star $K_{1,n-1}$ is the complete graph of order n and so we have the following observation.

Observation 5.3.1 *If T is a star of order at least 3, then $\text{spc}(T^2) = 1$.*

The following result gives an upper bound for the strong connection number of the square of a tree in terms of the maximum degree of the tree.

Theorem 5.3.2 *If T is a tree with maximum degree Δ, then*

$$\text{spc}(T^2) \le (\Delta - 1)^2 + 1.$$

Proof. We proceed by mathematical induction on the order $n \ge 3$ of a tree. Since $\text{spc}(P_3^2) = \text{spc}(K_3) = 1$, the statement is true for the tree of order 3. Assume that the statement holds for all trees of order n for some integer $n \ge 3$. Let T be a tree of order $n+1 \ge 4$ with $\Delta(T) = \Delta$. Since $\text{spc}(K_{1,n}^2) = \text{spc}(K_{n+1}) = 1$, we may assume that T is not a star and so T has an end-vertex v such that $T_1 = T - v$ is a tree of order n and $\Delta(T_1) = \Delta$. Let $k = (\Delta - 1)^2 + 1$. By induction hypothesis, T_1^2 has a strong proper-path k-coloring c_1 using colors in the set $[k] = \{1, 2, \ldots, k\}$. Let u be the vertex in T_1 such that $uv \in E(T)$ and let $N_{T_1}(u) = \{w_1, w_2, \ldots, w_d\}$ where then $\deg_{T_1} u = d \le \Delta - 1$. To extend the coloring c_1 to a strong proper-path k-coloring c of T^2, we show that there are colors $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_d \in [k]$ such that $c(uv) = \alpha$ and $c(vw_i) = \alpha_i$ for $1 \le i \le d$ and the resulting coloring c is a strong proper-path k-coloring of T^2.

For each integer i with $1 \le i \le d$, let X_i be the set of neighbors of w_i that are end-vertices of T and let $Y_i = N_T(w_i) - (X_i \cup \{u\})$ (where then $N_T(w_i) - \{u\} = X_i \cup Y_i$). Let $|X_i| = p_i \ge 0$, $|Y_i| = q_i \ge 0$ and $p_i + q_i = \deg_T w_i - 1 \le \Delta - 1$ for $1 \le i \le d$. If $q_i \ge 1$, then let $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\}$. For each pair i, j of integers with $1 \le i \le d$ and $1 \le j \le q_i$, let $Z_{i,j} = N_T(y_{i,j}) - \{w_i\}$ and so $0 \le |Z_{i,j}| \le \Delta - 1$. For each pair i, j with $1 \le i \le d$ and $1 \le j \le q_i$, let

$$\begin{aligned} c_1[u, N_T(w_i) - \{u\}] &= \{c_1(uw) : \ w \in N_T(w_i) - \{u\}\} \\ c_1[w_i, Z_{i,j}] &= \{c_1(w_i z) : \ z \in Z_{i,j}\}. \end{aligned}$$

Then

$$0 \le |c_1[u, N_T(w_i) - \{u\}]| \le \Delta - 1 \text{ and } 0 \le |c_1[w_i, Z_{i,j}]| \le \Delta - 1.$$

For $1 \le i \le d$, let

$$\begin{aligned} c_1(W) &= \bigcup_{i=1}^{d} c_1[u, N_T(w_i) - \{u\}] \\ c_1(W_i) &= \bigcup_{j=1}^{q_i} c_1[w_i, Z_{i,j}]. \end{aligned}$$

This is illustrated in Figure 5.2, where the solid edges belong to T and the dashed edges belong to $E(T^2) - E(T)$.

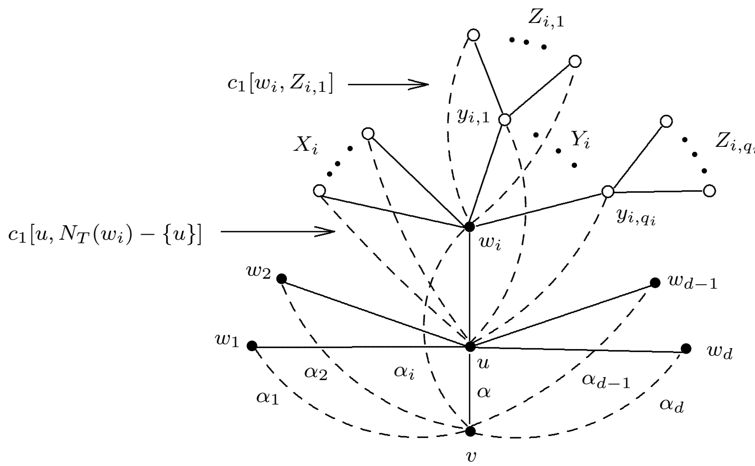

Figure 5.2: A step in the proof of Theorem 5.3.2

Since $d \leq \Delta - 1$ and $q_i \leq \Delta - 1$ for $1 \leq i \leq d$, it follows that

$$
\begin{aligned}
|c_1(W)| &\leq d(\Delta - 1) \leq (\Delta - 1)^2 \\
|c_1(W_i)| &\leq q_i(\Delta - 1) \leq (\Delta - 1)^2.
\end{aligned}
$$

Let $\alpha \in [k] - c_1(W)$ and let $\alpha_i \in [k] - c_1(W_i)$ for $1 \leq i \leq d$. Define an edge coloring $c : E(T^2) \to [k]$ by

$$
c(e) = \begin{cases} c_1(e) & \text{if } e \in E(T_1^2) \\ \alpha & \text{if } e = uv \\ \alpha_i & \text{if } e = vw_i \text{ where } 1 \leq i \leq d. \end{cases}
$$

It remains to show that c is a strong proper-path k-coloring of T^2. Let $x, y \in V(T^2)$ such that $xy \notin E(T^2)$. We show that there is a properly colored $x - y$ geodesic in T^2. If $x, y \in V(T_1)$, then there is a properly colored $x - y$ geodesic in T_1^2 (and so in T^2). Thus, we may assume that $x = v$. If $d_T(v, y) = 3$, then (v, u, y) is a properly colored $v - y$ geodesic in T^2; while $d_T(v, y) = 4$, then there exists $i \in \{1, 2, \ldots, d\}$ such that (v, w_i, y) is a properly colored $v - y$ geodesic in T^2. Thus, we may assume that $d_T(v, y) \geq 5$. If $d_T(v, y) \geq 5$ is odd, then let y^* be the vertex on the $v - y$ path in T such that $d_T(v, y*) = 3$; while if $d_T(v, y) \geq 6$ is even, then let y^* be the vertex on the $v-y$ path in T such that $d_T(v, y*) = 4$. In either case, $d_T(y^*, y)$ is even. Let P be a properly colored $y^* - y$ geodesic in T_1^2. If $d_T(v, y) \geq 5$ is odd, then the path (v, u, y^*) followed by P is a properly colored $v-y$ geodesic in T^2; while if $d_T(v, y) \geq 6$ is even, then the path (v, w_i, y^*) followed by P is a properly colored $v-y$ geodesic in T^2. Thus, c is a strong proper-path k-coloring of T^2 and so $\mathrm{spc}(T^2) \leq k = (\Delta - 1)^2 + 1$. ■

There is reason to believe that if T is not a star, then $\mathrm{spc}(T^2)$ is bounded below by 2 and above by $\Delta(T)$, resulting in the following conjecture.

Conjecture 5.3.3 *For each tree T that is not a star,*

$$2 \leq \mathrm{spc}(T^2) \leq \Delta(T).$$

On the other hand, if Conjecture 5.3.3 is true, then it cannot be improved. More precisely, we show that for each integer $\Delta \geq 2$, there is an infinite class of trees T having $\Delta(T) = \Delta$ such that $\mathrm{spc}(T^2) = \Delta$. We begin with the case when $\Delta = 2$.

Proposition 5.3.4 *For each integer $n \geq 4$, $\mathrm{spc}(P_n^2) = 2$.*

Proof. Let $P_n = (u_1, u_2, \ldots, u_n)$ be a path of order $n \geq 4$. Define a 2-edge coloring c of P_n^2 by

$$c(e) = \begin{cases} 1 & \text{if } e = u_iu_{i+1} \text{ where } i \equiv 1, 2 \pmod 4 \text{ or} \\ & \text{if } e = u_iu_{i+2} \text{ where } i \equiv 0, 1 \pmod 4 \\ 2 & \text{if } e = u_iu_{i+1} \text{ where } i \equiv 0, 3 \pmod 4 \text{ or} \\ & \text{if } e = u_iu_{i+2} \text{ where } i \equiv 2, 3 \pmod 4 \end{cases}$$

This coloring is illustrated in Figure 5.3 for P_8^2 and P_9^2, where each solid edge is colored 1 and each dashed edge is colored 2. Let u_i, u_j be two distinct vertices of P_n^2 where $u_iu_j \notin E(P_n^2)$. Without loss of generality, we may assume that $j = i + a$ for some integer $a \geq 3$. If a is even then $d(u_i, u_j) = a/2$. The path $P = (u_i, u_{i+2}, \ldots, u_{i+a})$ is a properly colored path of length $a/2$. If a is odd then $d(u_i, u_j) = \lceil a/2 \rceil$. The path $P = (u_i, u_{i+1}, u_{i+3}, \ldots, u_{i+a})$ is a properly colored path of length $\lceil a/2 \rceil$. Thus, c is a strong proper-path coloring of P_n^2. Since P_n^2 is not complete when $n \geq 4$, it follows that $\mathrm{spc}(P_n^2) = 2$. ■

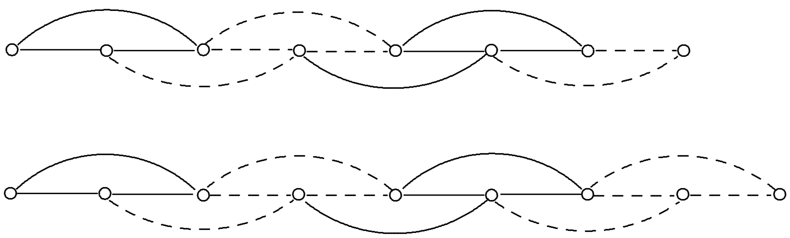

Figure 5.3: Strong proper-path 2-colorings of P_8^2 and P_9^2

For each integer $\Delta \geq 3$, let $K_{1,\Delta}$ be a star with $V(K_{1,\Delta}) = \{u, u_1, u_2, \ldots, u_\Delta\}$ where u is the central vertex of the star. Let $\mathcal{S}(K_{1,\Delta})$ denote the set of all subdivisions H of $K_{1,\Delta}$ such that each edge uu_i is subdivided at least once for $1 \leq i \leq \Delta$. Hence, every vertex u_i is adjacent to a vertex of degree 2 in H and $\deg_H u_i = 1$ for each i with $1 \leq i \leq \Delta$. Thus, u is the only vertex of H such that $\deg_H u = \Delta(H)$ and u is also referred to as the *central vertex* of H. Let $[\Delta] = \{1, 2, \ldots, \Delta\}$.

Proposition 5.3.5 *If $H \in \mathcal{S}(K_{1,\Delta})$ where $\Delta \geq 3$, then*

$$\mathrm{spc}(H^2) = \Delta(K_{1,\Delta}) = \Delta.$$

Proof. Let u be the central vertex of H, where then $\deg_H u = \Delta(H) = \Delta$, and let $N_H(u) = \{w_1, w_2, \ldots, w_\Delta\}$. Thus, $H^2[N_H(u)] = K_\Delta$. First, we show that $\mathrm{spc}(H^2) \geq \Delta$. For each integer i with $1 \leq i \leq \Delta$, let $N_H(w_i) = \{u, v_i\}$. If $i, j \in [\Delta]$ with $i \neq j$, then $d_H(v_i, v_j) = 4$ and so $d_{H^2}(v_i, v_j) = 2$. Because (v_i, u, v_j) is the only $v_i - v_j$ geodesic in H^2, every strong proper-path coloring of H^2 must assign distinct colors to uv_i and uv_j in H^2, which implies that $\mathrm{spc}(H^2) \geq \Delta$.

To show that $\mathrm{spc}(H^2) \leq \Delta$, we construct a strong proper-path coloring $c : E(H^2) \to [\Delta]$ of H^2 as follows. For each integer i with $1 \leq i \leq \Delta$, let q_i be an end vertex of H and let $Q_i = (u = u_{i,1},\ w_i = u_{i,2},\ \ldots,\ q_i = u_{i,n_i})$ be the $u - q_i$ path in H, where then $n_i \geq 3$. Using the coloring of P_n^2 described in the proof of Proposition 5.3.4, we first define a strong proper-path 2-coloring $c_i : E(Q_i^2) \to \{i, i+1\}$ of Q_i^2 for each i with $1 \leq i \leq \Delta$ (where each integer i is expressed as an integer modulo Δ). For a fixed integer $i \in [\Delta]$ and each integer p with $1 \leq p \leq n_i$, the coloring c_i is defined by

$$c_i(e) = \begin{cases} i & \text{if } e = u_{i,p}u_{i,p+1} \text{ where } p \equiv 1,2 \pmod 4 \text{ or} \\ & \text{if } e = u_{i,p}u_{i,p+2} \text{ where } p \equiv 0,1 \pmod 4 \\ i+1 & \text{if } e = u_{i,p}u_{i,p+1} \text{ where } p \equiv 0,3 \pmod 4 \text{ or} \\ & \text{if } e = u_{i,p}u_{i,p+2} \text{ where } p \equiv 2,3 \pmod 4). \end{cases}$$

This is illustrated in Figure 5.4, where each solid edge belongs to H while each dashed edge belongs to $E(H^2) - E(H)$.

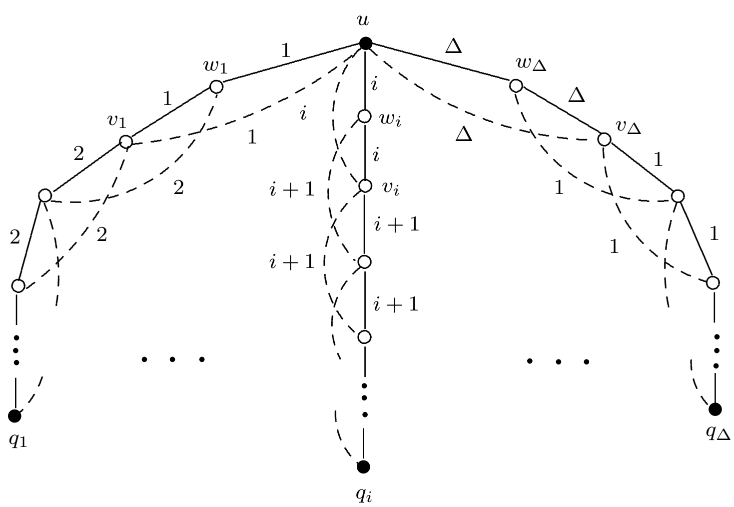

Figure 5.4: A step in the proof of Proposition 5.3.5

Next, we extend the colorings c_i of Q_i^2 for $1 \le i \le \Delta$ to a strong proper-path Δ-coloring c of H^2. For each pair i, j of integers with $1 \le i, j \le \Delta$ and $i \ne j$, let $\ell_{i,j} \in [\Delta] - \{i+1, j+1\}$ (where recall that each integer i is expressed as an integer modulo Δ). Define the coloring $c : E(H^2) \to [\Delta]$ by

$$c(e) = \begin{cases} c_i(e) & \text{if } e \in E(Q_i^2) \text{ for } 1 \le i \le \Delta \\ \ell_{i,j} & \text{if } e = w_i w_j \text{ for } 1 \le i \ne j \le \Delta. \end{cases}$$

It remains to show that c is a strong proper-path Δ-coloring of H^2, that is, every two vertices of H^2 are connected by a properly colored geodesic. Let $x, y \in V(H^2)$ such that $xy \notin E(H^2)$. If $x, y \in V(Q_i^2)$ for some i with $1 \le i \le \Delta$, then, as we have seen in Proposition 5.3.4, there is a properly colored $x - y$ geodesic in Q_i^2 (and so in H^2). Thus, we may assume that $x \in V(Q_i^2)$ and $y \in V(Q_j^2)$ where $1 \leq i, j \leq \Delta$ and $i \neq j$. Observe that $d_H(x,y) = d_H(x,u) + d_H(u,y) \geq 3$.

- If $d_H(x,u)$ and $d_H(u,y)$ are both even, then the properly colored $x-u$ geodesic followed by the properly colored $u-y$ geodesic results in a properly colored $x-y$ geodesic.

- If $d_H(x,u)$ and $d_H(u,y)$ are both odd, then the properly colored $x-w_i$ geodesic followed by $P = (w_i, w_j)$ and then followed by the properly colored $w_j - y$ geodesic produces a properly colored $x-y$ geodesic in H^2.

- If $d_H(x,u)$ is odd and $d_H(u,y)$ is even, then the properly colored $x - w_i$ geodesic followed by $P = (w_i, u)$ and then followed by the properly colored $u-y$ geodesic produces a properly colored $x-y$ geodesic in H^2.

- If $d_H(x,u)$ is even and $d_H(u,y)$ is odd, then the properly colored $x-u$ geodesic followed by $P = (u, w_j)$ and then followed by the properly colored $w_j - y$ geodesic produces a properly colored $x-y$ geodesic in H^2.

Thus, c is a strong proper-path Δ-coloring of H^2 and so $\text{spc}(H^2) = \Delta$. ■

Next, we establish an upper bound for the strong proper connection number of a connected graph G in terms of its maximum degree. For two disjoint subsets A and B of $V(G)$, let $[A, B]$ denote the set of edges joining a vertex in A and a vertex in B. If $A = \{v\}$ consists of a single vertex, then we write $[A, B] = [v, B]$. For an edge coloring c of G, let

$$c[A, B] = \{c(ab) : a \in A \text{ and } b \in B\}.$$

Theorem 5.3.6 *If G is a connected graph with maximum degree Δ, then*

$$\text{spc}(G^2) \leq \Delta(\Delta - 1) + 1. \tag{5.2}$$

Proof. We proceed by mathematical induction on the order $n \geq 3$ of a connected graph. Since $\text{spc}(P_3^2) = \text{spc}(K_3^2) = \text{spc}(K_3) = 1$, the statement is true for connected graphs of order 3. Assume that the statement holds for all connected graphs of order n for an integer $n \geq 3$. Let G be a connected graph of order $n+1 \geq 4$ with $\Delta(G) = \Delta$. If $\text{diam}(G) \leq 2$, then G^2 is complete and so $\text{spc}(G^2) = 1$. Thus, we may assume that $\text{diam}(G) \geq 3$. Let u and v be antipodal vertices of G, that is, $d(u, v) = \text{diam}(G)$.

Then u and v are not cut-vertices of G and so $G-u$ and $G-v$ are connected graphs of order n. Furthermore, either $\Delta(G-u)=\Delta(G)$ or $\Delta(G-v)=\Delta(G)$. Suppose, without loss of generality, that $\Delta(G-v)=\Delta(G)$. Let

$$G_1 = G - v \text{ and } k = \Delta(\Delta-1)+1.$$

By the induction hypothesis, G_1^2 has a strong proper path k-coloring c_1 using colors in the set $[k]=\{1,2,\ldots,k\}$. We next extend the coloring c_1 to a strong proper path k-coloring c of G^2 by assigning colors to the edges in $E(G^2)-E(G_1^2)$.

For each integer t with $1 \le t \le e(v) = \mathrm{diam}(G)$, let

$$V_t = \{x : d(v,x)=t\} \text{ for } t=1,2,\ldots,e(v) \text{ where } e(v)\ge 3. \tag{5.3}$$

In particular, $V_1 = N_G(v) = \{u_1,u_2,\ldots,u_d\}$ where $d=\deg_G v \le \Delta$. For each integer i with $1\le i\le d$, let

$$U_i = N_G(u_i)\cap V_2 = X_i \cup Y_i \subseteq V_2, \tag{5.4}$$

where each vertex in X_i is an end-vertex of G and each vertex in Y_i has degree at least 2 in G. Furthermore, let

$$r_i = |U_i| \le \deg_G u_i - 1 \le \Delta - 1.$$

Let $|X_i| = p_i \ge 0$ and $|Y_i| = q_i \ge 0$, where then $p_i+q_i=r_i$ for $1\le i\le d$. If $q_i\ge 1$, let

$$Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\} \subseteq V_2. \tag{5.5}$$

First, we choose the d colors $\alpha_1,\alpha_2,\ldots,\alpha_d \in [k]$ that can be assigned to the edges in the set $[v,N_G(v)] = \{vu_i : 1\le i\le d\}$. For each pair i,j of integers with $1\le i\le d$ and $1\le j\le q_i$, let

$$Z_{i,j} = N_G(y_{i,j})\cap V_3 \subseteq V_3 \tag{5.6}$$

and so $0\le |Z_{i,j}| \le \Delta-1$ (see Figure 5.5). For each integer i with $1\le i\le d$, let

$$W_i = c_1[u_i,U_i] \cup \left(\bigcup_{j=1}^{q_i} c_1[u_i, Z_{i,j}] \right), \tag{5.7}$$

where $[u_i,U_i]\subseteq E(G)$ and $[u_i,Z_{i,j}] \subseteq E(G^2)-E(G)$. Since $q_i\le \Delta-1$ and $|U_i|\le \Delta-1$, it follows that

$$|W_i| \leq (\Delta - 1) + q_i(\Delta - 1) \leq (\Delta - 1) + (\Delta - 1)^2 = \Delta(\Delta - 1).$$

Because $k = \Delta(\Delta - 1) + 1$ for $1 \leq i \leq d$, it follows that

$$\text{there is } \alpha_i \in [k] - W_i \text{ and let } c(vu_i) = \alpha_i \text{ for } 1 \leq i \leq d. \tag{5.8}$$

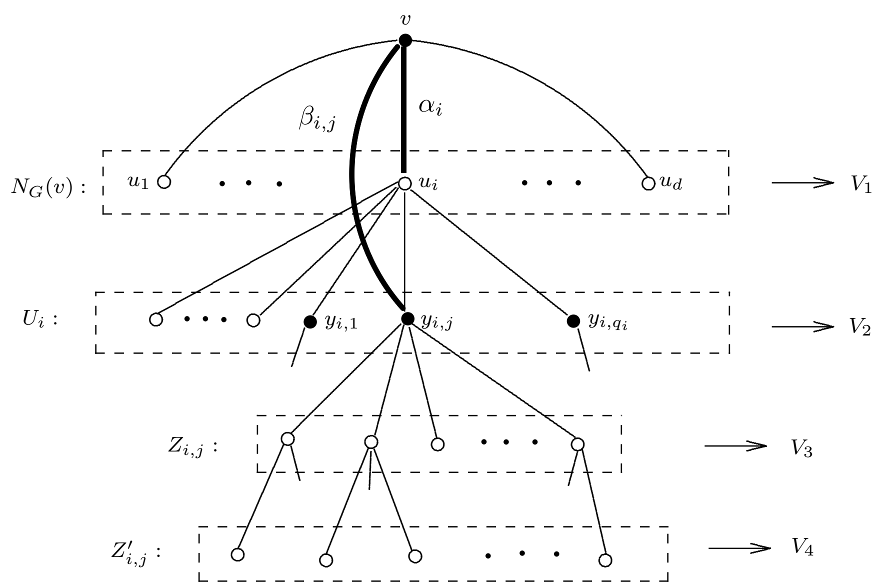

Figure 5.5: A step in the proof of Theorem 5.3.6

Next, for a fixed integer i with $1 \leq i \leq d$, we choose the r_i colors $\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,r_i}$ belonging to the set $[k]$ that can be assigned to the r_i edges in $[v, U_i] \subseteq E(G^2) - E(G)$ by c. If $e \in [v, X_i]$, then, since every vertex in X_i is an end-vertex of G, we can color e arbitrarily with a color in $[k]$. Thus, it remains to consider the colors for the edges in $[v, Y_i]$ when $|Y_i| = q_i \geq 1$. For each $y_{i,j} \in Y_i$ in (5.5), let $Z_{i,j}$ be defined by (5.6) and let

$$Z'_{i,j} = \{x \in V_4 : d(y_{i,j}, x) = 2\} \subseteq V_4.$$

Because each vertex $y_{i,j} \in Y_i \subseteq V_2$ is adjacent to at most $\Delta - 1$ vertices in V_3 and each of these $\Delta - 1$ vertices in V_3 is adjacent to at most $\Delta - 1$ vertices in V_4, it follows that $Z'_{i,j} \leq (\Delta - 1)^2$. Since

$$|Z_{i,j} \cup Z'_{i,j}| \leq (\Delta - 1) + (\Delta - 1)^2 = \Delta(\Delta - 1),$$

there is

$$\beta_{i,j} \in [k] - c_1[y_{i,j}, Z_{i,j} \cup Z'_{i,j}] \tag{5.9}$$

that is available for $vy_{i,j}$ for all i, j with $1 \le i \le d$ and $1 \le j \le q_i$.

In summary, the edge coloring $c : E(G^2) \to [k]$ of G^2 is defined by

$$c(e) = \begin{cases} c_1(e) & \text{if } e \in E(G_1^2) \\ \alpha_i & \text{if } e = vu_i \\ \beta_{i,j} & \text{if } e = vy_{i,j} \text{ where } 1 \le i \le d \text{ and } 1 \le j \le q_i. \end{cases}$$

It remains to show that c is a strong proper-path k-coloring of G^2. Let $x, y \in V(G^2)$ such that $xy \notin E(G^2)$. We show that there is a properly colored $x - y$ geodesic in G^2. If $x, y \in V(G_1)$, then there is a properly colored $x - y$ geodesic in G_1^2 (and so in G^2). Thus we may assume that $x = v$.

- ⋆ If $d_G(v, y) = 3$, then $P_1 = (v, u_i, y)$ is a properly colored $v - y$ geodesic in G^2. Here, note that $c(vu_i) = \alpha_i \in [k] - W_i$ and $c(u_i y) \in W_i$ where W_i is defined in (5.7).

- ⋆ If $d_G(v, y) = 4$, then there exists $i \in \{1, 2, \ldots, d\}$ such that $P_2 = (v, y_{i,j}, y)$ is a properly colored $v - y$ geodesic in G^2 by (5.9).

Thus, we may assume that $d_G(v, y) \ge 5$ and so $y \in V_t$, described in (5.3), for some $t \ge 5$.

- ⋆ If $d_G(v, y) \ge 5$ is odd, then let y^* be the vertex on a $v - y$ geodesic in G such that $d_G(v, y*) = 3$.

- ⋆ If $d_G(v, y) \ge 6$ is even, then let y^* be the vertex on a $v - y$ geodesic in G such that $d_G(v, y*) = 4$.

In either case, $d_G(y^*, y)$ is even. Let P be a properly colored $y^* - y$ geodesic in G_1^2.

- ⋆ If $d_G(v, y) \ge 5$ is odd, then the path $P_1 = (v, u_i, y^*)$ is proper by (5.8) and P_1 followed by P is a properly colored $v - y$ geodesic in G^2.

- ⋆ If $d_G(v, y) \ge 6$ is even, then the path $P_2 = (v, y_{i,j}, y^*)$ is proper by (5.9) and P_2 followed by P is a properly colored $v - y$ geodesic in G^2.

Thus, c is a strong proper-path k-coloring of G^2 and so $\text{spc}(G^2) \le k = \Delta(\Delta - 1) + 1$. ■

Next, we show that Theorem 5.3.6 is attainable when $\Delta = 2$. More precisely, if $n \ge 9$ and $n \equiv 1, 2, 3 \pmod 4$, then $\text{spc}(C_n^2) = 3 = \Delta(\Delta - 1) + 1$ for $\Delta = 2$.

Theorem 5.3.7 *For each integer $n \geq 3$,*

$$\mathrm{spc}(C_n^2) = \begin{cases} 1 & \text{if } n = 3,4,5 \\ 2 & \text{if (i) } n = 6,7 \text{ or (ii) } n \geq 8 \text{ and } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \geq 9 \text{ and } n \equiv 1,2,3 \pmod 4. \end{cases}$$

Proof. For each integer $n \geq 3$, let $G = C_n^2$, where

$$C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1).$$

If $n = 3,4,5$, then $G = K_n$ and so $\mathrm{spc}(G) = 1$. If $n \geq 6$, then G is not complete and so $\mathrm{spc}(G) \geq 2$. For $n = 6$, let $c : E(G) \to \{1,2\}$ be the edge coloring of G defined by $c(e) = 1$ if $e \in \{v_1v_2, v_1v_3, v_1v_5, v_2v_3\}$ and $c(e) = 2$ otherwise. For $n = 7$, let $c : E(G) \to \{1,2\}$ be defined by assigning the color 1 to each edge in the two triangles (v_1, v_2, v_3, v_1) and (v_5, v_6, v_7, v_5) and the color 2 to the remaining edges of G. These two colorings are illustrated in Figure 5.6, where each solid edge is colored 1, while each dashed edge is colored 2. In each case, c is a strong proper-path 2-coloring of G and so $\mathrm{spc}(G) = 2$.

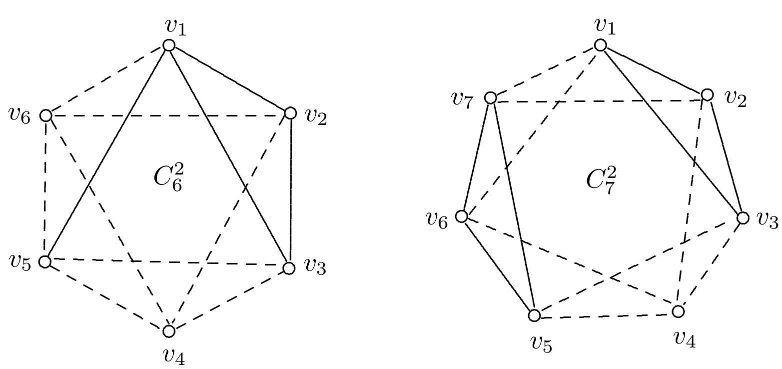

Figure 5.6: Strong proper-path 2-colorings of C_6^2 and C_7^2

For $n \geq 8$, we consider four cases, according to whether $n \equiv r \pmod 4$ for $r = 0,1,2,3$. In each of these cases, the subscript of every vertex is expressed as an integer modulo n.

Case 0. $n \equiv 0 \pmod 4$ *and* $n \geq 8$. For each odd positive integer i with $1 \leq i \leq n-1$, let $T_i = (v_i, v_{i+1}, v_{i+2}, v_i)$ be the triangle in G. Define the edge coloring $c : E(G) \to \{1, 2\}$ of G by

$$c(e) = \begin{cases} 1 & \text{if } e \in E(T_i) \text{ and } i \equiv 1 \pmod 4 \text{ or} \\ & e = v_i v_{i+2} \text{ and } i \equiv 0 \pmod 4 \\ 2 & \text{if } e \in E(T_i) \text{ and } i \equiv 3 \pmod 4 \text{ or} \\ & e = v_i v_{i+2} \text{ and } i \equiv 2 \pmod 4). \end{cases}$$

This is illustrated in Figure 5.7 for $n = 8$, where each solid edge is colored 1, while each dashed edge is colored 2. Since c is a strong proper-path 2-coloring of G, it follows that $\text{spc}(G) = 2$.

Case 1. *$n \equiv 1 \pmod 4$ and $n \geq 9$.* Let

$$C = (v_1, v_3, v_5, \ldots, v_n, v_2, v_4, \ldots, v_{n-1}, v_1)$$

be a Hamiltonian cycle in G, where the distance between every two consecutive vertices is 2 in C_n. Since every strong proper-path coloring of G must be a proper coloring of C and $\chi'(C) = 3$, it follows that $\text{spc}(G) \geq 3$. It remains to show that there exists a strong proper-path 3-coloring of G. Define the edge coloring $c : E(G) \to \{1, 2, 3\}$ of G by

$$c(e) = \begin{cases} 1 & \text{if } e = v_i v_{i+2} \text{ and } i \equiv 0, 1 \pmod 4 \text{ and } i \neq n-1 \\ 2 & \text{if } e = v_i v_{i+2} \text{ and } i \equiv 2, 3 \pmod 4 \\ 3 & \text{if } e = v_1 v_{n-1} \text{ or } e \in E(C_n). \end{cases}$$

This is illustrated in Figure 5.7 for $n = 9$, where each solid edge is colored 1, each dashed edge is colored 2 and each bold edge is colored 3. Since c is a strong proper-path 3-coloring of G, it follows that $\text{spc}(G) = 3$.

Case 2. *$n \equiv 2 \pmod 4$ and $n \geq 10$.* Let $n = 4k + 2$ for some integer $k \geq 2$. Let $C' = (v_1, v_3, v_5, \ldots, v_{4k+1}, v_1)$ and $C'' = (v_2, v_4, v_6, \ldots, v_{4k+2}, v_2)$ be the two edge-disjoint cycles of order $2k+1$ in G. Since every strong proper-path coloring of G must be a proper coloring of C' (and C''), it follows that $\text{spc}(G) \geq 3$. It remains to find a strong proper-path 3-coloring of G. Define the edge coloring $c : E(G) \to \{1, 2, 3\}$

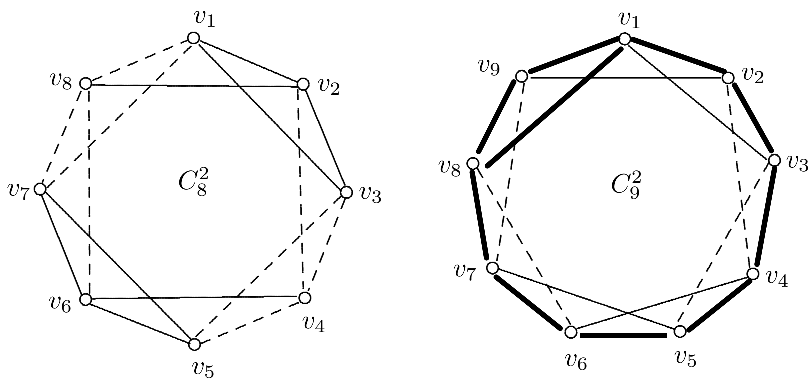

Figure 5.7: A strong proper-path 2-coloring of C_8^2 and a strong proper-path 3-coloring of C_9^2

of G by

$$c(e) = \begin{cases} 1 & \text{if } e = v_iv_{i+2} \text{ and } i \equiv 0, 1 \pmod 4 \text{ and } i \neq n-1 \text{ or} \\ & e \in \{v_1v_2, v_{n-1}v_n\} \\ 2 & \text{if } e = v_iv_{i+2} \text{ and } i \equiv 2, 3 \pmod 4 \text{ and } i \neq n \\ 3 & \text{otherwise.} \end{cases}$$

This is illustrated in Figure 5.8 for $n = 10$, where, as expected, each solid edge is colored 1, each dashed edge is colored 2 and each bold edge is colored 3. Since c is a strong proper-path 3-coloring of G, it follows that $\text{spc}(G) = 3$.

Case 3. *$n \equiv 3 \pmod 4$ and $n \geq 11$.* Let

$$C = (v_1, v_3, v_5, \ldots, v_n, v_2, v_4, \ldots, v_{n-1}, v_1)$$

be a Hamiltonian cycle in G, where the distance between every two consecutive vertices is 2 in C_n. Since every strong proper-path coloring of G must be a proper coloring of C and $\chi'(C) = 3$, it follows that $\text{spc}(G) \geq 3$. It remains to show the existence of a strong proper-path 3-coloring of G. Define the edge coloring $c : E(G) \to \{1, 2, 3\}$ of G by

$$c(e) = \begin{cases} 1 & \text{if } e = v_iv_{i+2} \text{ and } i \equiv 1, 2 \pmod 4 \text{ and } i \neq n-1 \\ 2 & \text{if } e = v_iv_{i+2} \text{ and } i \equiv 0, 3 \pmod 4 \\ 3 & \text{if } e = v_1v_{n-1} \text{ or } e \in E(C_n). \end{cases}$$

This is illustrated in Figure 5.8 for $n = 11$. Since c is a strong proper-path 3-coloring of G, it follows that $\text{spc}(G) = 3$. ∎

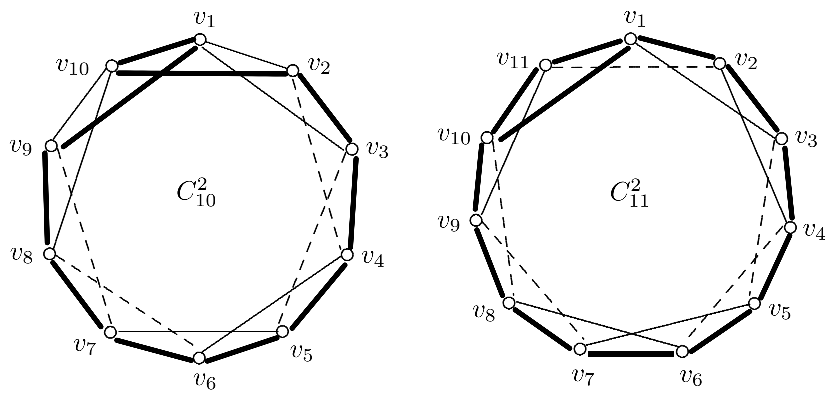

Figure 5.8: Strong proper-path 3-colorings of C_{10}^2 and C_{11}^2

Among other problems which may be of interest to study are the following.

Problem 5.3.8 *For connected graphs G and H, what is* $\text{spc}(G \vee H)$*?*

Problem 5.3.9 *For connected graphs G and H, what is* $\text{spc}(G \,\square\, H)$*?*

Problem 5.3.10 *For a connected graph G of order n and a permutation α of the set $[n]$, what is* $\text{spc}(P_\alpha(G))$*?*

Problem 5.3.11 *For each non-cut-vertex v of G and each nonbridge e of G, how are* $\text{spc}(G - v)$ *and* $\text{spc}(G - e)$ *for $v \in V(G)$ and $e \in E(G)$ related to* $\text{spc}(G)$*?*

Chapter 6

Proper Connectivity

A *vertex-cut* of a connected graph G is a set S of vertices of G such that $G - S$ is disconnected. A vertex-cut of minimum cardinality in G is called a *minimum vertex-cut* of G and this cardinality is the *vertex-connectivity* (or the *connectivity*) of G (when G is not complete) and is denoted by $\kappa(G)$. Complete graphs of order n do not contain vertex-cuts and in this case its connectivity is defined as $n - 1$, that is, $\kappa(K_n) = n - 1$. Therefore, the *connectivity* $\kappa(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. The connectivity is a common measure of connectedness for a graph. A graph G is *ℓ-connected* for some positive integer ℓ if $\kappa(G) \geq \ell$. That is, G is ℓ-connected if the removal of fewer than ℓ vertices from G results in neither a disconnected nor a trivial graph. Suppose that G is an ℓ-connected graph for some positive integer ℓ. It then follows from a well-known theorem of Whitney [37] that for every integer k with $1 \leq k \leq \ell$ and every two distinct vertices u and v of G, the graph G contains k internally disjoint $u - v$ paths.

6.1 Proper k-Path Colorings

Let G be a graph with connectivity $k \geq 1$. The *chromatic connectivity* $\kappa_\chi(G)$ of G is the minimum number of colors needed in an edge coloring of G such that every two distinct vertices u and v of G are connected by k internally disjoint proper $u-v$ paths. For a graph G with $\kappa_\chi(G) \geq 3$, there are *intermediate concepts* between the proper connection number $\text{pc}(G)$ and the chromatic connectivity $\kappa_\chi(G)$ of the graph G. This also leads to a more general concept.

An edge coloring of a connected graph G is called a *proper k-path coloring* of G

for some positive integer k if for every two distinct vertices u and v of G, there exist at least k internally disjoint proper $u - v$ paths. The minimum number of colors required for a proper k-path coloring of G is the *proper k-connectivity* $\mathrm{pc}_k(G)$ of G. Thus, $\mathrm{pc}_1(G) = \mathrm{pc}(G)$ is the proper connection number of G. If $\kappa(G) = \kappa$, then $\mathrm{pc}_\kappa(G) = \kappa_\chi(G)$ is the chromatic connectivity of G. The concept $\mathrm{pc}_k(G)$ has been studied in [4]. If G is an ℓ-connected graph and the edges of G are properly colored, then every two vertices of G are connected by at least ℓ internally disjoint proper paths by Whitney's theorem. Therefore, we have the following observation.

Observation 6.1.1 *If G is an ℓ-connected graph for some $\ell \geq 1$ and k is an integer with $1 \leq k \leq \ell$, then $\mathrm{pc}_k(G)$ exists and*

$$\mathrm{pc}_k(G) \leq \chi'(G). \tag{6.1}$$

Furthermore, if k and k' are two integers with $1 \leq k \leq k' \leq \ell$, then

$$\mathrm{pc}_k(G) \leq \mathrm{pc}_{k'}(G).$$

Another useful observation is stated next.

Observation 6.1.2 *Let G be a nontrivial connected graph. If H is an ℓ-connected spanning subgraph of G for some positive integer ℓ and k is an integer with $1 \leq k \leq \ell$, then*

$$\mathrm{pc}_k(G) \leq \mathrm{pc}_k(H). \tag{6.2}$$

That equality in (6.1) and in (6.2) are both possible is shown next.

Proposition 6.1.3 *If G is a Hamiltonian graph of order $n \geq 3$, then*

$$\mathrm{pc}_2(G) \leq \chi'(C_n).$$

In particular, if $n \geq 4$ is even, then $\mathrm{pc}_2(G) = 2$.

Proof. Since G is Hamiltonian, there is a Hamiltonian cycle C_n of order n. Define an edge coloring of G such that C_n is properly colored with $\chi'(C_n)$ colors. Thus, for every two distinct vertices u and v of G, there exist two internally disjoint proper $u - v$ paths on C_n. Hence, $\mathrm{pc}_2(G) \leq \chi'(C_n)$. If $n \geq 4$ is even, then since $\mathrm{pc}_2(G) \geq 2$ and $\chi'(C_n) = 2$, it follows that $\mathrm{pc}_2(G) = 2$. ■

The following results by Dirac [15] and Ore [33] are well known.

Theorem 6.1.4 (Dirac) *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 6.1.5 (Ore) *If G is a graph of order $n \geq 3$ such that*

$$\deg u + \deg v \geq n$$

for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u - v$ path for every pair u, v of distinct vertices of G. Ore [34] also proved the following two theorems in 1963.

Theorem 6.1.6 (Ore) *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

Theorem 6.1.7 (Ore) *If G is a graph of order $n \geq 4$ such that*

$$\deg u + \deg v \geq n + 1$$

for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian-connected.

The following two theorems are due to Huang, Li and Wang [20], where the results for n even are immediate consequences of Theorems 6.1.4 and 6.1.5 and Proposition 6.1.3. We present an independent proof of Theorem 6.1.8.

Theorem 6.1.8 *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then $\mathrm{pc}_2(G) = 2$.*

Proof. By Proposition 6.1.3, we may assume that $n \geq 5$ is odd. Since $\delta(G) \geq n/2$ and n is odd, it follows that $\delta(G) \geq (n+1)/2$. Thus, G is Hamiltonian-connected by Theorem 6.1.6. Let $H = G - v$ for some vertex v of G. Then

$$\delta(H) \geq (n+1)/2 - 1 = (n-1)/2.$$

Since the order of H is $n-1$, it follows by Theorem 6.1.4 that H is Hamiltonian. Let $C = (v_1, v_2, \ldots, v_{n-1}, v_1)$ be a Hamiltonian cycle of H. Because $\deg_G v \geq (n+1)/2$, the vertex v is adjacent to two consecutive vertices v_1 and v_2 of C. Define an edge coloring $c : E(G) \to \{1, 2\}$ such that $c(vv_1) = c(vv_2) = c(v_1v_2) = 1$ and the edges of

C are colored properly. It remains to show that c is a proper 2-path coloring of G. Let $x, y \in V(G)$. If $x, y \in V(C)$, then there are two internally disjoint proper $x-y$ paths on C. Thus, we may assume that $x = v$ and $y = v_i$ for some integer i with $1 \le i \le n-1$. Then $(x, v_1, v_{n-1}, v_{n-2}, \ldots, v_i = y)$ and $(x, v_2, v_3, \ldots, v_i = y)$ are two internally disjoint proper $x-y$ paths in G. Therefore, $\mathrm{pc}_2(G) = 2$. ■

Theorem 6.1.9 *If G is a graph of order $n \ge 3$ such that $\deg u + \deg v \ge n$ for each pair u, v of nonadjacent vertices of G, then $\mathrm{pc}_2(G) = 2$.*

In this chapter, we investigate the proper k-connectivity $\mathrm{pc}_k(G)$ where $k \ge 2$ for complete bipartite graphs G that are not stars.

6.2 Proper r-Connectivity of $K_{r,s}$

For integers r and s with $2 \le r \le s$, let $K_{r,s}$ be the complete bipartite graph with partite sets U and V where $|U| = r$ and $|V| = s$. Since $\kappa(K_{r,s}) = r$, it follows that $K_{r,s}$ is k-connected for each integer k with $2 \le k \le r$. Furthermore, by Whitney's theorem [37], every two distinct vertices of $K_{r,s}$ are connected by k internally disjoint paths for $2 \le k \le r$. Therefore, we consider the proper k-connectivity $\mathrm{pc}_k(K_{r,s})$ of $K_{r,s}$ for $k = 2, 3, \ldots, r$, beginning with the proper r-connectivity $\mathrm{pc}_r(K_{r,s})$ of $K_{r,s}$. First, we determine $\mathrm{pc}_r(K_{r,s})$ when $s = r$.

Proposition 6.2.1 *For an integer $r \ge 2$, let c be an edge coloring of the complete regular bipartite graph $K_{r,r}$. Then c is a proper r-path coloring if and only if c is a proper edge coloring of $K_{r,r}$. Hence, $\mathrm{pc}_r(K_{r,r}) = \chi'(K_{r,r}) = r$.*

Proof. Since a proper edge coloring of $K_{r,r}$ is a proper r-path coloring, it remains to verify the converse. Assume, to the contrary, that c is a proper r-path coloring but there are two adjacent edges xy and yz such that $c(xy) = c(yz)$. However then, there are at most $r-1$ internally disjoint $x-z$ paths in $K_{r,r}$, which is impossible. Then it follows by (6.1) that $\mathrm{pc}_r(K_{r,r}) = \chi'(K_{r,r}) = r$. ■

We saw in Proposition 6.2.1 that $\mathrm{pc}_r(K_{r,r}) = r$ for each integer $r \ge 2$. In fact, this is a special case of a more general result.

Proposition 6.2.2 *If r and s are integers with $2 \le r \le s$, then*

$$\mathrm{pc}_r(K_{r,s}) = s.$$

Proof. Since $\chi'(K_{r,s}) = \Delta(K_{r,s}) = s$, it follows that $\mathrm{pc}_r(K_{r,s}) \leq s$ by (6.1). Next, we show that $\mathrm{pc}_r(K_{r,s}) \geq s$. Let U and V be the partite sets of $K_{r,s}$, where $|U| = r$ and $|V| = s$. Let c be a proper r-path coloring of $K_{r,s}$. We claim that all edges incident with each vertex $u \in U$ must be assigned different colors by c; for suppose that $c(uv_1) = c(uv_2) = 1$, where $v_1, v_2 \in V$. Then there are at most $r-1$ internally disjoint $v_1 - v_2$ paths in $K_{r,s}$, which is impossible. Since $\deg u = s$ for each $u \in U$, it follows that c must use at least s colors and so $\mathrm{pc}_r(K_{r,s}) \geq s$. Therefore, $\mathrm{pc}_r(K_{r,s}) = s$. ■

We now make some observations concerning proper r-path colorings of $K_{r,s}$. By Proposition 6.2.1, if an edge coloring c of $K_{r,r}$ is a proper r-path coloring, then c is a proper edge coloring of $K_{r,r}$. This is, however, not the case for $K_{r,s}$ when $r < s$. We saw that if c is a proper r-path coloring of $K_{r,s}$, then all edges incident with a vertex of degree s in $K_{r,s}$ must be assigned different colors. On the other hand, it is not necessary that all edges incident with a vertex of degree r be assigned different colors. For example, the edge coloring c of $K_{3,4}$ of Figure 6.1 is a proper 3-path coloring but it is not a proper edge coloring of $K_{3,4}$ as $c(u_1v_1) = c(v_1u_2) = 1$.

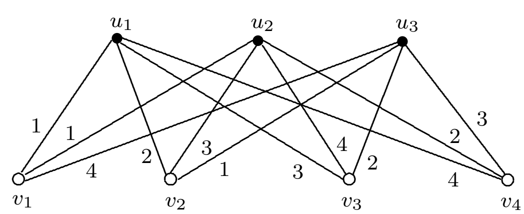

Figure 6.1: A proper 3-path coloring of $K_{3,4}$

If $G = K_{2,s}$ where $s \geq 3$, then there is *only one* edge coloring c_0 (up to permutations of colors) of G that assigns distinct colors to all edges incident with each of the two vertices of degree s in G but is not a proper 2-path coloring. Hence, all other edge colorings that assign distinct colors to all edges incident with each of the two vertices of degree s in G are proper 2-path colorings of G. To see this, let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of G. Define $c_0(u_1v_i) = c_0(u_2v_i) = i$ for $i = 1, 2, \ldots, s$. Since G does not have two internally disjoint proper $u_1 - u_2$ paths, it follows that c_0 is not a proper 2-path coloring. On the other hand, if c is an edge coloring of G such that

$$\{c(u_1v_i) : 1 \leq i \leq s\} = \{c(u_2v_i) : 1 \leq i \leq s\} = [s]$$

where $c(u_1v_a) \neq c(u_2v_a)$ and $c(u_1v_b) \neq c(u_2v_b)$ for some $a, b \in [s]$ with $a \neq b$, then (u_1, v_a, u_2) and (u_1, v_b, u_2) are two internally disjoint proper $u_1 - u_2$ paths in G. Thus, c is a proper 2-path coloring of G.

6.3 Lower Bounds for $\mathrm{pc}_k(K_{r,s})$

In [4], Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza obtained a lower bound for the proper k-connectivity of certain complete bipartite graphs.

Theorem 6.3.1 [4] *If r, s, k are positive integers where $r = 2k - 1$ and $s > 2^r$, then*

$$\mathrm{pc}_k(K_{r,s}) > 2.$$

Next, we establish an improved lower bound for the proper k-connectivity of certain complete bipartite graphs. In order to do this, we first provide some preliminary results beginning with a useful fact about the graphs $K_{r,s}$ for $2 \leq r \leq s$. The following lemma is an immediate consequence of a theorem of Whitney [37].

Lemma 6.3.2 *For integers r and s with $2 \leq r \leq s$, let $G = K_{r,s}$ with partite sets U and V where $|U| = r$ and $|V| = s$ and let k be a positive integer such that $k \leq r < 2k$. For every two distinct vertices $v, v' \in V$, there are k internally disjoint $v - v'$ paths, each of length at least 2.*

Lemma 6.3.3 *For integers r, s and k with $2 \leq r \leq s$ and $k \leq r < 2k$, let $G = K_{r,s}$ with partite sets U and V where $|U| = r$ and $|V| = s$ and let there be given a proper k-path coloring of G. For every two distinct vertices $v, v' \in V$ and every set $\mathcal{S}$ of k internally disjoint properly colored $v - v'$ paths in G, at least $2k - r$ of the paths in $\mathcal{S}$ have length 2.*

Proof. Let c be a proper k-path coloring of G and let $v, v' \in V$. Since c is a proper k-path coloring of G, there are k internally disjoint $v - v'$ paths $P_1, P_2, \ldots, P_k$ in G. We claim that at least $2k - r$ of these k paths have length 2. Assume, without loss of generality, that $P_1, P_2, \ldots, P_t$ have length 2 for some integer t with $0 \leq t \leq k$ and $P_{t+1}, P_{t+k}, \ldots, P_k$ have length 3 or more. Since (i) $P_1, P_2, \ldots, P_k$ are internally disjoint, (ii) each path P_i $(1 \leq i \leq t)$ contains exactly one vertex of U and (iii) each path P_j $(t + 1 \leq j \leq k)$ contains at least two vertices of U, it follows that $P_1, P_2, \ldots, P_k$ contain at least $t + 2(k-t)$ distinct vertices of U. Hence, $t + 2(k-t) \leq |U| = r$ and so $t \geq 2k - r$. ∎

For integers r and s with $2 \leq r \leq s$, let $G = K_{r,s}$ with partite sets

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } V = \{v_1, v_2, \ldots, v_s\}.$$

For each integer i with $1 \leq i \leq s$, let $G_i = G[U, v_i] = K_{1,r}$ be the subgraph induced by the set $[U, v_i]$ of all edges incident with v_i in G. For an edge coloring $c : E(G) \to [n] = \{1, 2, \ldots, n\}$ of G and each integer i with $1 \leq i \leq s$, there is an edge coloring $c_i : E(G_i) \to [n]$ obtained by restricting the coloring c to G_i. Each edge coloring c_i $(1 \leq i \leq s)$ can be considered as an integer-valued function $c_i : [r] \to [n]$ of G_i defined by

$$c_i(p) = c(u_p v_i) \text{ for each } p \in [r]. \tag{6.3}$$

Lemma 6.3.4 *For integers r and s with $2 \leq r \leq s$, let $G = K_{r,s}$ with partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$. If k is a positive integer such that $k \leq r < 2k$ and $c : E(G) \to [n]$ is a proper k-path coloring, then the s integer-valued functions $c_1, c_2, \ldots, c_s$ defined in (7.6) are distinct.*

Proof. We show that $c_i \neq c_j$ for each pair i, j of distinct integers in $\{1, 2, \ldots, s\}$. Let v_i and v_j be two vertices of V. Since $c : E(G) \to [n]$ is a proper k-path coloring of G, there is a set S of k internally disjoint proper $v_i - v_j$ paths in G. It then follows by Lemma 6.3.3 that S contains at least $2k - r \geq 1$ proper $v_i - v_j$ paths Q of length 2, say $Q = (v_i, u_p, v_j)$ for some $u_p \in U$. Since $c(u_p v_i) \neq c(u_p v_j)$, it follows that $c_i(p) \neq c_j(p)$ and so $c_i \neq c_j$. Therefore, $c_1, c_2, \ldots, c_s$ are distinct functions. ∎

We are now prepared to present a lower bound for $\mathrm{pc}_k(K_{r,s})$ in terms of r and s when $k \leq r < 2k$ or, equivalently, $r/2 < k \leq r$.

Theorem 6.3.5 *For positive integers r, s and k with $r \leq s$ and $r/2 < k \leq r$,*

$$\mathrm{pc}_k(K_{r,s}) \geq \lceil \sqrt[r]{s}\, \rceil.$$

Proof. Let $G = K_{r,s}$ and suppose that $\mathrm{pc}_k(K_{r,s}) = n$. Assume, to the contrary, that $n < \lceil \sqrt[r]{s}\, \rceil$. Let $c : E(G) \to [n]$ be a proper k-path coloring of G. Consider the s integer-valued functions $c_1, c_2, \ldots, c_s$, where $c_i : [r] \to [n]$ is defined from c as described in (7.6) for $1 \leq i \leq s$. Let $\mathcal{F}(r, n)$ denote the set of all functions from $[r]$ to

$[n]$. Since $|\mathcal{F}(r,n)| = n^r$ and $s > n^r$, it follows that $c_1, c_2, \ldots, c_s$ are not distinct. It then follows by Lemma 6.3.4 that c is not a proper k-path coloring, a contradiction. Therefore, $\mathrm{pc}_k(K_{r,s}) = n \geq \lceil \sqrt[r]{s}\, \rceil$. ■

Consider the graph $K_{3,28}$ and its proper 2-connectivity. Since $k = 2, r = 3 = 2k-1$ and $s = 28 > 2^3 = 2^r$, it follows from Theorem 6.3.1 that $\mathrm{pc}_2(K_{3,28}) > 2$ or, equivalently, that $\mathrm{pc}_2(K_{3,28}) \geq 3$. On the other hand, since $r/2 = 1.5 < 2 = k \leq 6 = 2r$, it follows by Theorem 6.3.5 that $\mathrm{pc}_2(K_{3,28}) \geq \lceil \sqrt[3]{28}\, \rceil = 4$. Therefore, Theorem 6.3.5 gives an improved lower bound for $\mathrm{pc}_2(K_{3,28})$ over that given by Theorem 6.3.1. [We will visit this graph in some detail in Section 6.4 and show that $\mathrm{pc}_2(K_{3,28}) = 4$.] Furthermore, we saw in Theorem 6.3.1 that if r, s, k are positive integers with $r = 2k-1$ and $s > 2^r$, then $\mathrm{pc}_k(K_{r,s}) > 2$. If $s > 2^r$, then $\lceil \sqrt[r]{s}\, \rceil > \lceil \sqrt[r]{2^r}\, \rceil = 2$. Consequently, Theorem 6.3.1 is a special case of Theorem 6.3.5. As we will soon, the lower bound for $\mathrm{pc}_k(K_{r,s})$ presented in Theorem 6.3.5 is best possible when $k = 2$ and $r = 3$ for all $s \geq r$. More generally, this lower bound is best possible for all integers k, r and s with $k \geq 2$, $r = 2k-1$ and $s \geq 3^r$, as we show next.

Theorem 6.3.6 *If k, r and s are integers with $k \geq 2$, $r = 2k-1$ and $s \geq 3^r$, then*

$$\mathrm{pc}_k(K_{r,s}) = \lceil \sqrt[r]{s}\, \rceil.$$

Proof. Let $G = K_{r,s}$ and $N = \lceil \sqrt[r]{s}\, \rceil$, where $r = 2k-1 \geq 3$ and $s \geq 3^r$. Since $\mathrm{pc}_k(G) \geq N$ by Theorem 6.3.5, it remains to show that $\mathrm{pc}_k(G) \leq N$; that is, G has a proper k-path coloring using colors from the set $[N]$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of G and let $\mathcal{F}(r,N)$ be the set of all functions $f : [r] \to [N]$. Then $|\mathcal{F}(r,N)| = N^r = (\lceil \sqrt[r]{s}\, \rceil)^r \geq s$. Now, let

$$\mathcal{F}(r,N) = \{c_1, c_2, \ldots, c_{N^r}\}$$

such that for $1 \leq i \leq 2^r$, the range of each c_i is a subset of $\{1,2\}$ and for $2^r + 1 \leq i \leq 3^r$, the range of each c_i is a subset of $\{1,2,3\}$. We now define an edge coloring $c : E(G) \to [N]$ by using the s integer-valued functions $c_1, c_2, \ldots, c_s \in \mathcal{F}(r,N)$ as follows. For each integer i with $1 \leq i \leq s$, let $c(u_p v_i) = c_i(p)$ for each $p \in [r]$. Next, we show that c is a proper k-path coloring of G; that is, we show that every two distinct vertices x and y of G are connected by k internally disjoint properly colored $x - y$ paths in G. We consider three cases.

Case 1. $\{x, y\} = \{u_i, u_j\}$ *where* $1 \le i < j \le r$. For each fixed pair i, j of integers with $1 \le i < j \le r$, there are 2^{r-1} functions $f \in \mathcal{F}(r, N)$ whose range in [2] satisfies $f(i) \ne f(j)$. Thus, there is a subset $V' \subseteq V$ with $|V'| = 2^{r-1}$ such that $c(u_i v) \ne c(u_j v)$ for each $v \in V'$. Thus, (u_i, v, u_j) is a proper $x - y$ path of length 2 in G for each $v \in V'$. Since $2^{r-1} \ge \frac{r+1}{2} = k$ for each integer $r \ge 3$, there are at least k internally disjoint properly colored $x - y$ paths in G when $\{x, y\} = \{u_i, u_j\}$.

Case 2. $\{x, y\} = \{v_i, v_j\}$ *where* $1 \le i < j \le s$. Since $c_i \ne c_j$, there is $p \in [r]$ such that $c_i(p) \ne c_j(p)$ and so $c(u_p v_i) \ne c(u_p v_j)$. Thus, $P = (v_i, u_p, v_j)$ is a proper $x - y$ path of length 2 in G. We may assume, without loss of generality, that $p = 1$ and so $P = (v_i, u_1, v_j)$. Next, we show that there are $k - 1$ internally disjoint properly colored $x - y$ paths of length 4 in G such that u_1 does not belong to any of these $k - 1$ paths. In fact, we show that there are $k - 1$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ of V such that the following $k - 1$ paths of length 4 are proper $v_i - v_j$ paths in G:

$$\begin{aligned}
P_1 &= (v_i, u_2, v_{\ell_1}, u_3, v_j)\\
P_2 &= (v_i, u_4, v_{\ell_2}, u_5, v_j)\\
&\cdots\\
P_t &= (v_i, u_{2t}, v_{\ell_t}, u_{2t+1}, v_j)\\
&\cdots\\
P_{k-1} &= (v_i, u_{r-1}, v_{\ell_{k-1}}, u_r, v_j)
\end{aligned}$$

It remains to show that these $k - 1$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ exist in V.

In order for $P_t = (v_i, u_{2t}, v_{\ell_t}, u_{2t+1}, v_j)$ to be proper for each integer t with $1 \le t \le k - 1$, we must have

$$c(v_i u_{2t}) \ne c(u_{2t} v_{\ell_t}),\ c(u_{2t} v_{\ell_t}) \ne c(v_{\ell_t} u_{2t+1}) \text{ and } c(v_{\ell_t} u_{2t+1}) \ne c(u_{2t+1} v_j).$$

Let $a_t \in [3] - \{c(v_i u_{2t})\}$ and $b_t \in [3] - \{a_t, c(u_{2t+1} v_j)\}$. Thus, $a_t, b_t \in \{1, 2, 3\}$ and $a_t \ne b_t$. Now, let $f \in \{c_1, c_2, \ldots, c_{3^r}\}$ (where $f(p) \in [3]$ for each $p \in [r]$) such that $f(2t) = a_t$ and $f(2t + 1) = b_t$. Suppose that $f = c_{\ell_t}$ for some integer ℓ_t with $1 \le \ell_t \le 3^r$. Then we choose $v_{\ell_t} \in V$ and so $P_t = (v_i, u_{2t}, v_{\ell_t}, u_{2t+1}, v_j)$ is a proper $v_i - v_j$ path. For each fixed integer t with $1 \le t \le k - 1$, the remaining $r - 2$ values of $c_{\ell_t}(p)$, where $p \in [r] - \{2t, 2t+1\}$, are elements in [3]. Thus, there are $2 \cdot 3^{r-2}$ possible choices for these $k - 1$ elements $c_{\ell_1}, c_{\ell_2}, \ldots, c_{\ell_{k-1}}$ in $\mathcal{F}(r, N)$ (or these $k - 1$ vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ in V) to construct the $k - 1$ proper $x - y$ paths $P_1, P_2, \ldots, P_{k-1}$.

Since $2 \cdot 3^{r-2} > \frac{r-1}{2} = k - 1$ for each integer $r \geq 3$, these $k-1$ internally disjoint properly colored $x - y$ paths $P_1, P_2, \ldots, P_{k-1}$ exist in G.

Case 3. $\{x, y\} = \{u_i, v_j\}$ *where* $1 \leq i \leq r$ *and* $1 \leq j \leq s$. We may assume that $u_i = u_1$ and so (u_1, v_j) is a properly colored $x - y$ path of length 1. Next, we show that there are $k-1$ internally disjoint properly colored $x-y$ paths of length 3 in G. In fact, we show that there are $k-1$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ of V such that the following $k-1$ paths of length 3 are proper $u_1 - v_j$ paths in G:

$$\begin{array}{rcl} P_1 & = & (u_1, v_{\ell_1}, u_2, v_j) \\ P_2 & = & (u_1, v_{\ell_2}, u_3, v_j) \\ & \cdots & \\ P_t & = & (u_1, v_{\ell_t}, u_{t+1}, v_j) \\ & \cdots & \\ P_{k-1} & = & (u_1, v_{\ell_{k-1}}, u_k, v_j) \end{array}$$

It remains to show that these $k-1$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ exist in V. For each integer t with $1 \leq t \leq k-1$, in order for $P_t = (u_1, v_{\ell_t}, u_{t+1}, v_j)$ to be proper, we must have

$$c(u_1 v_{\ell_t}) \neq c(v_{\ell_t} u_{t+1}) \text{ and } c(v_{\ell_t} u_{t+1}) \neq c(u_{t+1} v_j).$$

Let $a_t \in [3] - \{c(u_1 v_{\ell_t}), c(u_{t+1} v_j)\}$ and let $f \in \{c_1, c_2, \ldots, c_{3^r}\}$ such that $f(t) = a_t$. Suppose that $f = c_{\ell_t}$ for some integer ℓ_t with $1 \leq \ell_t \leq 3^r$. Then we choose $v_{\ell_t} \in V$ and so $P_t = (u_1, v_{\ell_t}, u_{t+1}, v_j)$ is a proper $u_1 - v_j$ path. For each fixed integer t with $1 \leq t \leq k-1$, the remaining $r-1$ values of $c_{\ell_t}(p)$, where $p \in [r] - \{t\}$, are elements in [3]. Thus, there are 3^{r-1} possible choices for these $k-1$ elements $c_{\ell_1}, c_{\ell_2}, \ldots, c_{\ell_{k-1}}$ in $\mathcal{F}(r, N)$ (or the $k-1$ vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ in V) to construct these $k-1$ proper paths $P_1, P_2, \ldots, P_{k-1}$. Since $3^{r-1} > \frac{r-1}{2} = k-1$ for each integer $r \geq 3$, these $k-1$ internally disjoint properly colored $x - y$ paths $P_1, P_2, \ldots, P_{k-1}$ exist in G.

Thus, $c : E(G) \to [N]$ is a proper k-path coloring of G and so $\text{pc}_k(G) \leq N$. Therefore, $\text{pc}_k(K_{r,s}) = N = \lceil \sqrt[r]{s}\ \rceil$. ■

Next, we present another lower bound for $\text{pc}_k(K_{r,s})$ for certain values of r, s and k. In order to do this, we introduce some additional definitions. Let $\mathcal{F}(r, n)$ denote the set of all functions $f : [r] \to [n]$. For a positive integer k with $k \le r$, a subset $S \subseteq \mathcal{F}(r,n)$ is called a *k-distinct subset* if for each pair f, g of functions in S, there are at least k distinct elements $p \in [r]$ such that $f(p) \neq g(p)$ (or at least k elements in the domain $[r]$ have distinct images).

Next, let $M(r,n,k)$ denote the maximum size of a k-distinct subset of $\mathcal{F}(r,n)$; that is,

$$M(r,n,k) = \max\{|S| : S \text{ is a } k\text{-distinct subset of } \mathcal{F}(r,n)\}.$$

With the aid of Lemma 6.3.3, we present a lower bound for $\mathrm{pc}_k(K_{r,s})$ for certain values of r, s and k.

Theorem 6.3.7 *Let r, s and k be positive integers with $2 \le r \le s$ and $k \le r < 2k$. If N is the smallest positive integer such that $M(r, N, 2k-r) \ge s$, then*

$$\mathrm{pc}_k(K_{r,s}) \ge N.$$

Proof. For integers r and s with $2 \le r \le s$, let $G = K_{r,s}$ with partite sets

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } V = \{v_1, v_2, \ldots, v_s\}.$$

Suppose that $\mathrm{pc}_k(G) = n$ and let $c : E(G) \to [n]$ be a proper k-path coloring of G using n colors. For each integer i with $1 \le i \le s$, let c_i be the integer-valued function defined in (7.6). Now, let $S = \{c_1, c_2, \ldots, c_s\}$. We claim that S is a $(2k-r)$-distinct subset of $\mathcal{F}(r,n)$; that is, if $c_i, c_j \in S$ where $i \neq j$, then there are at least $2k-r$ distinct elements $p \in [r]$ such that $c_i(p) \neq c_j(p)$. For $v_i, v_j \in V$ where $i \neq j$, let $Q_1, Q_2, \ldots, Q_k$ be k internally disjoint proper $v_i - v_j$ geodesics in G. By Lemma 6.3.3, there are at least $2k-r$ of $Q_1, Q_2, \ldots, Q_k$ having length 2. We may assume, without loss of generality, that $Q_p = (v_i, u_p, v_j)$ for $p = 1, 2, \ldots, 2k-r$. Since Q_p is properly colored, it follows that $c(v_iu_p) \neq c(u_pv_j)$ and so $c_i(p) \neq c_j(p)$ for $p = 1, 2, \ldots, 2k-r$. Therefore, as claimed, S is a $(2k-r)$-distinct subset of $\mathcal{F}(r,n)$. Since $M(r,n,2k-r)$ is the maximum size of a $(2k-r)$-distinct subset of $\mathcal{F}(r,n)$, it follows that $M(r,n,2k-r) \ge |S| = s$. Also, since N is the smallest positive integer such that $M(r,N,2k-r) \ge s$, it follows that c must use at least N colors and so $n \ge N$. Therefore, $\mathrm{pc}_k(G) = n \ge N$. ■

It should be mentioned that the concepts of k-distinct subsets of the set $\mathcal{F}(r,n)$ of all functions $f : [r] \to [n]$ and the maximum size $M(r,n,k)$ of a k-distinct subset of $\mathcal{F}(r,n)$ can be expressed under the context of *error-correcting codes* in coding theory (see [10, pp.123-126] or [18, pp.477, 683]). In terms of error-correcting codes, the set $\mathcal{F}(r,n)$ is the set of all r-tuples where each coordinate is an element of $[n]$. A collection C of r-tuples is called a *code* and each element in C is called a *code word of length* r. For two code words x and y in a code C, the *distance* $d(x,y)$ between x and y is the number of coordinates at which x and y differ. This distance is referred to as the *Hamming distance* between x and y. For a collection C of code words of length r, the *distance of* C is defined as

$$d(C) = \min\{d(x,y) : x, y \in C\}.$$

Thus, a k-distinct subset of the set $\mathcal{F}(r,n)$ is, in fact, a set C of code words of length r (each of whose coordinates is an element of $[n]$) such that $d(C) \geq k$ and $M(r,n,k)$ is the maximum size of a code C of code words of length r (each of whose coordinates is an element of $[n]$) distance is at least k. Although the following result is known [24], we present an independent proof expressed in our notation and terminology.

Proposition 6.3.8 *For integers r, k and n with $2 \leq k \leq r$ and $n \geq 2$,*

$$M(r,n,k) \leq n^{r-k+1}.$$

Proof. Let S be a k-distinct subset of $\mathcal{F}(r,n)$ such that $|S| = M(r,n,k)$. Since $k \geq 2$, it follows that $1 \leq r-k+1 \leq r$. For each function $f \in S$, let

$$f' : [r-k+1] \to [n]$$

be the restriction of f to the set $[r-k+1] \subseteq [r]$. Now, let

$$S' = \{f' : \ f \in S\}.$$

Then S' is a subset of $\mathcal{F}(r-k+1,n)$. We claim that $|S'| = |S|$. Let $f, g \in S$ and $f \neq g$. Since S is a k-distinct subset, there are at least k distinct elements $p \in [r]$ such that $f(p) \neq g(p)$. Thus, there exists $p \in [r-k+1]$ such that $f(p) \neq g(p)$ and so $f' \neq g'$ in $\mathcal{F}(r-k+1,n)$. Hence, $|S'| = |S|$, as claimed. Therefore,

$$M(r,n,k) = |S| = |S'| \leq |\mathcal{F}(r-k+1,n)| = n^{r-k+1},$$

as desired. ■

As we will soon see, the upper bound in Proposition 7.2 is sharp for *some* values of r, k and n.

By Theorem 6.3.5, if r, s and k are positive integers with $r \leq s$ and $r/2 < k \leq r$ (or, equivalently, $k \leq r < 2k$), then

$$\mathrm{pc}_k(K_{r,s}) \geq \lceil \sqrt[r]{s}\, \rceil.$$

With the aid of Theorem 6.3.7 and Proposition 7.2, we are able to establish a lower bound for $\mathrm{pc}_k(K_{r,s})$ for certain values of r, s and k that is an improvement of the lower bound in Theorem 6.3.5.

Theorem 6.3.9 *If r, s and k are integers with $2 \leq r \leq s$ and $2 \leq k \leq r < 2k$, then*

$$\mathrm{pc}_k(K_{r,s}) \geq \lceil \sqrt[2r-2k+1]{s}\, \rceil. \tag{6.4}$$

Proof. Let N be the smallest positive integer such that $M(r, N, 2k - r) \geq s$. It then follows by Theorem 6.3.7 that $\mathrm{pc}_k(K_{r,s}) \geq N$. Since

$$M(r, N, 2k - r) \leq N^{2r-2k+1}$$

by Proposition 7.2, it follows that

$$s \leq M(r, N, 2k - r) \leq N^{2r-2k+1}$$

and so $N \geq \lceil \sqrt[2r-2k+1]{s}\, \rceil$. Therefore, $\mathrm{pc}_k(K_{r,s}) \geq N \geq \lceil \sqrt[2r-2k+1]{s}\, \rceil$, as desired. ■

Since $k \leq r < 2k$, it follows that $r \geq 2r - 2k + 1$ and so $\lceil \sqrt[2r-2k+1]{s}\, \rceil \geq \lceil \sqrt[r]{s}\, \rceil$. Hence, the lower bound in Theorem 6.3.9 is an improvement of the lower bound in Theorem 6.3.5. Furthermore, if $r = 2k - 1$, then these two lower bounds are equal; while if $r = k$, then equality in (6.4) holds by Proposition 6.2.2. We will soon see that equality in (6.4) also holds when $k = 2$. If $k = 3$, then

$$\mathrm{pc}_3(K_{r,s}) \geq \lceil \sqrt[2r-5]{s}\, \rceil \text{ for } 3 \leq r < 6. \tag{6.5}$$

6.4 Two Examples

In this section, we present two example to illustrate Theorems 6.3.5, 6.3.7 and 6.3.9, namely, we show that $\mathrm{pc}_3(K_{4,27}) = 3$ and $\mathrm{pc}_3(K_{4,28}) = 4$, beginning with $\mathrm{pc}_3(K_{4,27}) =$

3. By Theorem 6.3.5, $\mathrm{pc}_3(K_{4,27}) \geq \lceil \sqrt[4]{27}\, \rceil = 3$. To apply Theorem 6.3.7, observe that $k = 3$, $2k - r = 2$, $r = 4$ and $s = 27$. Since the smallest positive integer N such that $M(r, N, 2k - r) = M(4, N, 2) \geq 27$ is 3, it follows by Theorem 6.3.7 that $\mathrm{pc}_3(K_{4,27}) \geq \lceil \sqrt[4]{27}\, \rceil = 3$. Furthermore, by Theorem 6.3.9 or by (6.5), it follows that $\mathrm{pc}_3(K_{4,27}) \geq \lceil \sqrt[3]{27}\, \rceil = 3$. In fact, $\mathrm{pc}_3(K_{4,27}) = 3$, as we show next.

Proposition 6.4.1 $\mathrm{pc}_3(K_{4,27}) = 3$.

Proof. Let $G = K_{4,27}$ with partite sets

$$U = \{u_1, u_2, u_3, u_4\} \text{ and } V = \{v_1, v_2, \ldots, v_{27}\}.$$

As we mentioned, $\mathrm{pc}_3(G) \geq 3$ by Theorem 6.3.5 or by Theorem 6.3.7. Hence, it remains to show that there is a proper 3-path coloring $c : E(G) \to [3]$ of G. Let $\mathcal{F}(4, 3)$ denote the set of all functions $f : [4] \to [3]$. To simplify notation, if $f \in \mathcal{F}(4, 3)$ such that $f(1) = a$, $f(2) = b$, $f(3) = c$ and $f(4) = d$, where $a, b, c, d \in [3]$, then we write $f = (abcd)$.

First, we select a 27-element subset $S = \{f_1, f_2, \ldots, f_{27}\} \subseteq \mathcal{F}(4, 3)$, namely

$$\begin{array}{lll}
f_1 = (1111) & f_2 = (2222) & f_3 = (3333) \\
f_4 = (2211) & f_5 = (3322) & f_6 = (1133) \\
f_7 = (3311) & f_8 = (1122) & f_9 = (2233) \\
\\
f_{10} = (1212) & f_{11} = (2323) & f_{12} = (3131) \\
f_{13} = (2312) & f_{14} = (3123) & f_{15} = (1231) \\
f_{16} = (3112) & f_{17} = (1223) & f_{18} = (2331) \\
\\
f_{19} = (2113) & f_{20} = (3221) & f_{21} = (1332) \\
f_{22} = (3213) & f_{23} = (1321) & f_{24} = (2132) \\
f_{25} = (1313) & f_{26} = (2121) & f_{27} = (3232).
\end{array}$$

Since for each pair f_i, f_j of functions in S where $1 \leq i < j \leq 27$, there are *at least* two distinct elements $p \in [4]$ such that $f_i(p) \neq f_j(p)$, it follows that S is a 2-distinct subset of $\mathcal{F}(4, 3)$. We make two useful observations:

> **Observation 1.** For each pair i, t of integers where $i \in [4]$ and $t \in [3]$, there are *exactly nine* functions $f \in S$ such that $f(i) = t$. For example, the nine functions f in S for which $f(1) = 2$ are $f_2, f_4, f_9, f_{11}, f_{13}, f_{18}, f_{19}, f_{24}, f_{26}$.

Observation 2. For each pair i, j of two distinct elements of $[4]$, there are *exactly nine* functions $f \in S$ such that $f(i) = f(j)$. For example, the nine functions in S for which $f(1) = f(2)$ are f_i for $1 \leq i \leq 9$ and the nine functions f in S for which $f(1) = f(4)$ are $f_1, f_2, f_3, f_{13}, f_{14}, f_{15}, f_{22}, f_{23}, f_{24}$.

We now construct an edge coloring $c : E(G) \to [3] = \{1, 2, 3\}$ of G using the 27 functions in S. For each integer i with $1 \leq i \leq 27$, if $f_i = (a_i\ b_i\ c_i\ d_i)$, then define the colors of the four edges incident with v_i by

$$c(u_1v_i) = a_i,\ c(u_2v_i) = b_i,\ c(u_3v_i) = c_i \text{ and } c(u_4v_i) = d_i. \tag{6.6}$$

For example, for

$$f_1 = (1111),\ f_4 = (2211),\ f_7 = (3311),\ f_{10} = (1212) \text{ and } f_{13} = (2312),$$

the edges incident with the vertices $v_1, v_4, v_7, v_{10}, v_{13}$ are shown in Figure 6.2, where each thin regular edge is colored 1, each dashed edge is colored 2 and each bold edge is colored 3.

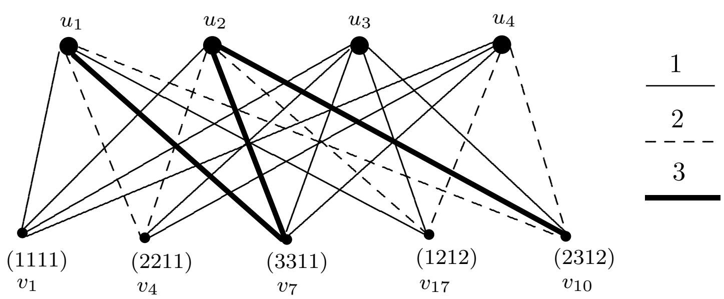

Figure 6.2: Illustrating a potion of the coloring c of $K_{4,27}$

For each integer i with $1 \leq i \leq 27$, let $G_i = G[U, v_i] = K_{1,4}$ be the subgraph induced by the set $[U, v_i]$ of the four edges incident with v_i in G. For this edge coloring c of G described above and each integer i with $1 \leq i \leq 27$, there is an edge coloring $c_i : E(G_i) \to [3]$ obtained by restricting the coloring c to G_i. Each edge coloring c_i ($1 \leq i \leq 27$) is in fact the integer-valued function $c_i : [4] \to [3]$ of G_i such that

$$c_i(p) = c(u_pv_i) \text{ for each } p \in [4]. \tag{6.7}$$

By the definition of the coloring c described in (6.6), it follows that

$$c_i = f_i \text{ for } 1 \leq i \leq 27. \tag{6.8}$$

Thus, in what follows, we use c_i for f_i for each function $f_i \in S$ where $1 \leq i \leq 27$.

Next, we show that c is a proper 3-path coloring of G; that is, we show that every two distinct vertices of G are connected by at least three internally disjoint proper paths in G.

First, consider two vertices v_i and v_j in V where $1 \leq i < j \leq 27$. If $c_i(p) \neq c_j(p)$ for at least three elements $p \in \{1, 2, 3, 4\}$, say p_1, p_2, p_3, then there are three internally disjoint proper $v_i - v_j$ paths of length 2, namely (v_i, u_{p_1}, v_j), (v_i, u_{p_2}, v_j) and (v_i, u_{p_3}, v_j). Since $c_i(p) \neq c_j(p)$ for at least two elements $p \in \{1, 2, 3, 4\}$, we may assume that $c_i(p) \neq c_j(p)$ for *exactly* two elements $p \in \{1, 2, 3, 4\}$, say p_1 and p_2. Thus, there are two internally disjoint proper $v_i - v_j$ paths $P_1 = (v_i, u_{p_1}, v_j)$ and $P_2 = (v_i, u_{p_2}, v_j)$ of length 2 in G. It remains to show that there is a proper $v_i - v_j$ path P_3 that is internally disjoint from P_1 and P_2. Since there are exactly two elements $x, y \in [4] - \{p_1, p_2\}$ such that $c_i(x) = c_j(x)$ and $c_i(y) = c_j(y)$, a proper $v_i - v_j$ path P_3 must have length 4 and so P_3 has the form

$$P_3 = (v_i, u_x, v_\alpha, u_y, v_j), \tag{6.9}$$

where $\alpha \in [27] - \{i, j\}$. In order for P_3 to be proper, we must have

$$\text{(i) } c(v_i u_x) \neq c(u_x v_\alpha), \text{ (ii) } c(u_x v_\alpha) \neq c(v_\alpha u_y), \text{ (iii) } c(v_\alpha u_y) \neq c(u_y v_j).$$

In terms of the functions in S, the function $c_\alpha \in S$ must satisfy the following conditions:

$$\text{(i) } \ c_i(x) \neq c_\alpha(x), \text{ (ii) } \ c_\alpha(x) \neq c_\alpha(y), \text{ (iii) } \ c_\alpha(y) \neq c_j(y).$$

We claim that there is at least one vertex $v_\alpha \in V$ (or at least one function $c_\alpha \in S$) that satisfies (i), (ii) and (iii).

By Observation 1, constraint (i) eliminates nine choices for $v_\alpha \in V$ (or $c_\alpha \in S$). Similarly, constraint (iii) eliminates nine choices for v_α. By Observation 2, constraint (ii) eliminates nine choices for v_α. Thus, the three constraints (i), (ii) and (iii) give rise to three subsets S_1, S_2, S_3 of S, each of which consists of nine forbidden elements for v_α. However, these three sets are not disjoint. Hence, $S - (S_1 \cup S_2 \cup S_3) \neq \emptyset$ and so there is at least one choice for v_α. For example, let $v_i = v_1$ and $v_j = v_8$. Since $c_1 = (1111)$ and $c_8 = (1122)$, it follows that $c(v_1u_1) = (v_1u_2) = (v_1u_3) = (v_1u_4) = 1$, $c(v_8u_1) = (v_8u_2) = 1$ and $(v_8u_3) = (v_8u_4) = 2$. In this case, $c_1(1) = c_8(1) = 1$ and $c_1(2) = c_8(2) = 1$, $1 = c_1(3) \neq c_8(3) = 2$ and $1 = c_1(4) \neq c_8(4) = 2$. Thus, $p_1 = 3$, $p_2 = 4$, $x = 1$ and $y = 2$. Therefore,

$$P_1 = (v_1, u_3, v_8),\ P_2 = (v_1, u_4, v_8)$$

and the path P_3 described in (6.9) has the form

$$P_3 = (v_1, u_1, v_\alpha, u_2, v_8),$$

where v_α satisfies

$$\text{(i)}\ \ c_\alpha(1) \neq c_1(1) = 1,\ \text{(ii)}\ c_\alpha(1) \neq c_\alpha(2) \text{ and (iii)}\ \ c_\alpha(2) \neq c_8(2) = 1.$$

Therefore, if $c_\alpha = (abcd)$, then $a, b \neq 1$ and $a \neq b$. Observe that for each integer $\alpha \in \{11, 13, 18, 20, 22, 27\}$, the function $c_\alpha \in S$ has these properties. Also, observe that the three sets S_1, S_2, S_3 of nine forbidden elements for the function $c_\alpha \in S$ are the following:

$$\begin{aligned}
S_1 &= \{c_i : c_i(1) = c_1(1) = 1\} = \{c_1, c_6, c_8, c_{10}, c_{15}, c_{17}, c_{21}, c_{23}, c_{25}\} \\
S_2 &= \{c_i : c_i(1) = c_i(2)\} = \{c_1, c_2, \ldots, c_9\} \\
S_2 &= \{c_i : c_i(2) = c_8(2) = 1\} = \{c_1, c_6, c_8, c_{12}, c_{14}, c_{16}, c_{19}, c_{24}, c_{26}\}.
\end{aligned}$$

Note that S_1, S_2 and S_3 are not disjoint and in fact

$$S - (S_1 \cup S_2 \cup S_3) = \{c_{11}, c_{13}, c_{18}, c_{20}, c_{22}, c_{27}\}.$$

Once again, as we observed earlier, there are six choices for v_α, namely, each integer

$$\alpha \in \{11, 13, 18, 20, 22, 27\}$$

gives rise to a choice for v_α. For example, suppose that we choose $v_\alpha = v_{11}$. Since $c_{11} = (2323)$, it follows that $c(v_{11}u_1) = 2$, $(v_{11}u_2) = 3$, $(v_{11}u_3) = 2$ and $(v_{11}u_4) = 3$. Then the three internally disjoint proper $v_1 - v_8$ paths P_1, P_2 and P_3 are

$$\begin{aligned}
P_1 &= (v_1, u_3, v_8) \\
P_2 &= (v_1, u_4, v_8) \\
P_3 &= (v_1, u_1, v_{11}, u_2, v_8).
\end{aligned}$$

This is illustrated in Figure 6.3.

Next, consider two vertices u_i and u_j in U where $1 \leq i < j \leq 4$. A proper $u_i - u_j$ path of length 2 has the form (u_i, v_α, u_j) for some $\alpha \in [27]$ such that $c(u_i v_\alpha) \neq c(v_\alpha u_j)$. Thus, the function c_α satisfies $c_\alpha(i) \neq c_\alpha(j)$. By Observation 2, there are 18 choices for v_α and so there are at least three internally disjoint proper

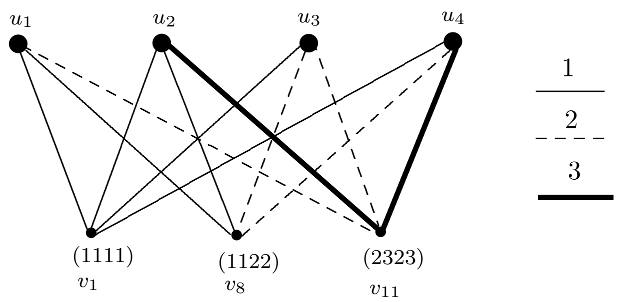

Figure 6.3: Three internally disjoint proper $v_1 - v_8$ paths in $K_{4,27}$

$u_i - u_j$ paths of length 2. (In fact, there are 18 internally disjoint proper $u_i - u_j$ paths of length 2.)

Finally, we consider two vertices $u_i \in U$ and $v_j \in V$ where $1 \leq i \leq 4$ and $1 \leq j \leq 27$. In this case, each $u_i - v_j$ path has odd length. Since (u_i, v_j) is a proper $u_i - v_j$ path, we only need to determine two internally disjoint proper $u_i - v_j$ paths P_1 and P_2 of length at least 3. In fact, we show next that there are two internally disjoint proper $u_i - v_j$ paths P_1 and P_2 of length 3. Suppose that

$$\begin{aligned} P_1 &= (u_i, v_\alpha, u_x, v_j) \\ P_2 &= (u_i, v_\beta, u_y, v_j), \end{aligned}$$

where $v_\alpha, v_\beta \in V$ and $u_x, u_y \in U$ such that $\{v_\alpha, u_x\} \cap \{v_\beta, u_y\} = \emptyset$.

⋆ In order for P_1 to be proper, we must have

$$c(u_i v_\alpha) \neq c(v_\alpha u_x) \text{ and } c(v_\alpha u_x) \neq c(u_x v_j).$$

Hence,

$$\text{(i)} \quad c_\alpha(i) \neq c_\alpha(x) \quad \text{and} \quad \text{(ii)} \quad c_\alpha(x) \neq c_j(x). \tag{6.10}$$

⋆ In order for P_2 to be proper, we must have

$$c(u_i v_\beta) \neq c(v_\beta u_y) \text{ and } c(v_\beta u_y) \neq c(u_y v_j).$$

Hence,

$$\text{(iii)} \quad c_\beta(i) \neq c_\beta(y) \quad \text{and} \quad \text{(iv)} \quad c_\beta(y) \neq c_j(y). \tag{6.11}$$

By Observations 1 and 2, the four constraints (i), (ii), (iii) and (iv) in (6.10) and (6.11) give rise to four subsets S_1, S_2, S_3, S_4 of S such that each of S_1 and S_2 consists of nine forbidden elements for v_α and each of S_3 and S_4 consists of nine forbidden elements for v_β. However, these four sets are not disjoint. In fact, $|S-(S_1\cup S_2\cup S_3\cup S_4)|\geq 2$ and so there are choices for v_α and v_β. For example, let $u_i=u_2$ and $v_j=v_{10}$. Since $c_{10}=(1212)$, it follows that

$$c(v_{10}u_1)=1,\ c(v_{10}u_2)=2,\ c(v_{10}u_3)=1 \text{ and } c(v_{10}u_4)=2.$$

First, we choose two distinct vertices $u_x,u_y\in U-\{u_2\}$. We may assume, without loss of generality, that $\{u_x,u_y\}=\{u_1,u_3\}$ or $\{u_x,u_y\}=\{u_1,u_4\}$. We only consider the case where $\{u_x,u_y\}=\{u_1,u_3\}$ (as the case where $\{u_x,u_y\}=\{u_1,u_4\}$ is similar). In this case, c_α and c_β must satisfy

$$\begin{array}{c}\text{(i)}\ \ c_\alpha(2)\neq c_\alpha(1)\ \ \text{and}\ \ \text{(ii)}\ \ c_\alpha(1)\neq c_{10}(1)=1;\\ \text{(iii)}\ \ c_\beta(2)\neq c_\beta(3)\ \ \text{and}\ \ \text{(iv)}\ \ c_\beta(3)\neq c_{10}(3)=1.\end{array}$$

The four subsets S_1,S_2,S_3,S_4 of nine forbidden elements for v_α or v_β are

$$\begin{array}{rcl}S_1&=&\{c_i:c_i(1)=c_i(2)\}=\{c_1,c_2,\ldots,c_9\}\\ S_2&=&\{c_i:c_i(1)=c_{10}(1)=1\}=\{c_1,c_6,c_8,c_{10},c_{15},c_{17},c_{21},c_{23},c_{25}\}\\ S_3&=&\{c_i:c_i(2)=c_i(3)\}=\{c_1,c_2,c_3,c_{16},c_{17},c_{18},c_{19},c_{20},c_{21}\}\\ S_4&=&\{c_i:c_i(3)=c_{10}(3)=1\}=\{c_1,c_4,c_7,c_{10},c_{13},c_{16},c_{19},c_{22},c_{25}\}\end{array}$$

Note that S_1, S_2, S_3 and S_4 are not disjoint. In fact,

$$S-(S_1\cup S_2\cup S_3\cup S_4)=\{c_{11},c_{12},c_{14},c_{20},c_{23},c_{24},c_{26},c_{27}\}.$$

Thus, there are eight choices for v_α and v_β, namely, every two distinct elements

$$\alpha,\beta\in\{11,12,14,20,23,24,26,27\}$$

give rise to two internally disjoint proper u_i-v_j paths P_1 and P_2 of length 3 in G. For example, suppose that we choose $v_\alpha=v_{11}$ and $v_\beta=v_{12}$. Since $c_{11}=(2323)$ and $c_{12}=(3131)$, it follows that

$$\begin{array}{c}c(v_{11}u_1)=2,\ (v_{11}u_2)=3,\ (v_{11}u_3)=2 \text{ and } (v_{11}u_4)=3,\\ c(v_{12}u_1)=3,\ (v_{12}u_2)=1,\ (v_{12}u_3)=3 \text{ and } (v_{12}u_4)=1.\end{array}$$

Then the three internally disjoint proper u_2-v_{10} paths P_1, P_2 and P_3 are

$$\begin{array}{rcl}P_1&=&(u_2,v_{11},u_1,v_{10})\\ P_2&=&(u_2,v_{12},u_3,v_{10})\\ P_3&=&(u_2,v_{10}).\end{array}$$

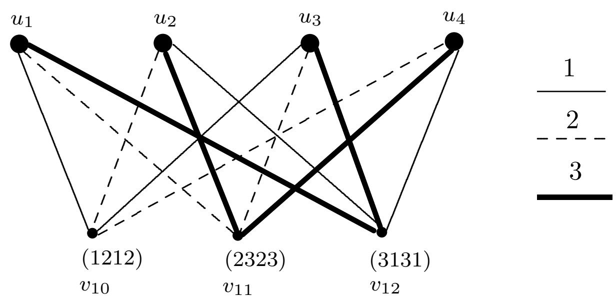

Figure 6.4: Three internally disjoint proper $u_2 - v_{10}$ paths in $K_{4,27}$

This is illustrated in Figure 6.4.

Therefore, c is a proper 3-path coloring of G and so $\text{pc}_3(K_{4,27}) = 3$. ■

Proposition 6.4.2 $\text{pc}_3(K_{4,28}) = 4$.

Proof. By Theorem 6.3.5,

$$\text{pc}_3(K_{4,28}) \geq \lceil \sqrt[4]{28}\ \rceil = 3$$

or by Theorem 6.3.9,

$$\text{pc}_3(K_{4,28}) \geq \lceil \sqrt[3]{28}\ \rceil = 4.$$

To see that $\text{pc}_3(K_{4,28}) \leq 4$, let $G = K_{4,28}$ with partite sets $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, \ldots, v_{28}\}$ and let $H = K_{4,27}$ with partite sets U and $V - \{v_{28}\}$. Next, let $c_H : E(H) \to [3]$ be the proper 3-path coloring of H described in the proof of Proposition 6.4.1. Define the edge coloring $c : E(G) \to [4]$ by $c(e) = c_H(e)$ if $e \in E(H)$ and $c(e) = 4$ if $e \notin E(H)$.

Next, we show that c is a proper 3-path coloring of G; that is, we show that every two distinct vertices x and y of G are connected by at least three internally disjoint proper paths. If $x, y \in V(H)$, then there are three internally disjoint proper $x - y$ paths in H and so in G. Thus, we may assume that $x \in V(H)$ and $y = v_{28}$.

First, suppose that $x \in U$. Then (x, y) is a proper path in G. Since c_H is a proper 3-path coloring of H, there are two internally disjoint proper $x - v_1$ paths of length at least 3 in H. In fact, as described in the proof of Proposition 6.4.1, there are two internally disjoint proper $x - v_1$ paths P_1 and P_2 of length 3 in H, say

$$\begin{aligned} P_1 &= (x, v_{i_1}, u_{j_1}, v_1) \\ P_2 &= (x, v_{i_2}, u_{j_2}, v_1) \end{aligned}$$

Replacing the edge $u_{j_1}v_1$ in P_1 by $u_{j_1}y$ and the edge $u_{j_2}v_1$ in P_2 by $u_{j_2}y$, we obtain two internally disjoint proper $x-y$ paths Q_1 and Q_2 of length 3 in G.

Next, suppose that $x \in V$. Since the edges incident with v_{28} are the only edges colored 4 in G, it follows that the four proper paths $Q_i = (x, u_i, y)$, where $i = 1, 2, 3, 4$, are four internally disjoint proper $x-y$ paths in G.

In each case, there are three internally disjoint proper $x-y$ paths in G. Therefore, c is a proper 3-path coloring of G and so $\mathrm{pc}_3(K_{4,28}) = 4$. ■

6.5 Proper 2-Connectivity of $K_{r,s}$

In this section, we determine the proper 2-connectivity of $K_{r,s}$ for all integers r and s with $2 \le r \le s$, beginning with the regular complete bipartite graphs $K_{r,r}$ for $r \ge 2$. A formula for $\mathrm{pc}_2(K_{r,s})$ was obtained by Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza in [4]; however, our proof of this formula is different from theirs and provides useful information on this topic. Consequently, we include this proof in the current section.

It is known that a complete bipartite graph of order at least 3 is Hamiltonian if and only if it is regular. The following is a consequence of Proposition 6.1.3 and the fact that $\chi'(C_n) = 2$ if $n \ge 4$ is even.

Proposition 6.5.1 *For each integer $r \ge 2$, $\mathrm{pc}_2(K_{r,r}) = 2$.*

By Proposition 6.2.2, $\mathrm{pc}_2(K_{2,s}) = s$ for $s > 2$. Next, we consider $\mathrm{pc}_2(K_{r,s})$ for integers r and s with $3 \le r < s$. First, we present an upper bound for $\mathrm{pc}_2(K_{r,s})$ in terms of r and s. As we will soon see, the next theorem (Theorem 6.5.2) is not surprising. However, the proof of this theorem provides information on constructing various proper paths in $K_{r,s}$, which may be used to study different proper paths in $K_{r,s}$ having certain prescribed properties. For this reason, we include this result here.

Theorem 6.5.2 *If r and s are integers with $3 \le r < s$, then*

$$\mathrm{pc}_2(K_{r,s}) \le \lceil s/r \rceil.$$

Proof. We show that $G = K_{r,s}$ has a proper 2-path coloring using $\lceil s/r \rceil$ colors. Write $s = pr + q$ where $p \ge 1$ and $0 \le q < r$. Let

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } V = V_1 \cup V_2 \cup \cdots \cup V_{p+1}$$

be the partite sets of G, where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,r}\}$ for $1 \le i \le p$ and

$$V_{p+1} = \{v_{p+1,1}, v_{p+1,2}, \ldots, v_{p+1,q}\}$$

if $q \neq 0$ (if $q = 0$, then $V_{p+1} = \emptyset$). For each integer i with $1 \le i \le p$, let

$$G_i = G[U \cup V_i] = K_{r,r}$$

and let $M_i = \{u_1v_{i,1}, u_2v_{i,2}, \ldots, u_rv_{i,r}\}$ be a maximum (perfect) matching of G_i. If $q \neq 0$, let $G_{p+1} = G[U \cup V_{p+1}] = K_{r,q}$ and let

$$M_{p+1} = \{u_1v_{p+1,1}, u_2v_{p+1,2}, \ldots, u_qv_{p+1,q}\}$$

be a maximum matching of G_{p+1}.

Define an edge coloring $c : E(G) \to \{1, 2, \ldots, \lceil s/r \rceil\}$ by

$$c(e) = \begin{cases} 1 & e \in M_1 \cup M_3 \cup \cdots \cup M_{p+1} \text{ or } e \in E(G_2) - M_2 \\ 2 & e \in E(G_1) - M_1 \text{ or } e \in M_2 \\ i & e \in E(G_i) - M_i \text{ for } 3 \le i \le p+1. \end{cases}$$

(If $q = 0$, then $\lceil s/r \rceil = p$ and $E(G_{p+1}) - M_{p+1} = \emptyset$.) The coloring c is illustrated in Figure 6.5 for $K_{3,8}$ using the colors 1, 2 and 3.

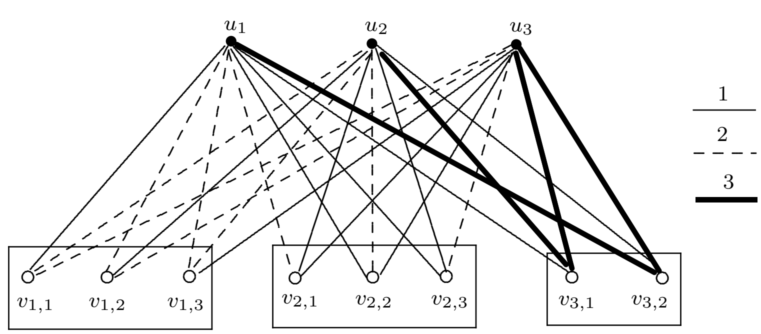

Figure 6.5: A proper 2-path coloring of $K_{3,8}$ using three colors

In general, for each i with $1 \le i \le p+1$, the colors of the edges of G_i are shown in Figure 6.6, where each edge in M_1 or M_i $(3 \le i \le p+1)$ is colored 1, each edge in M_2 is colored 2, each edge in $E(G_1) - M_1$ is colored 2, each edge in $E(G_2) - M_2$ is colored 1 and each edge in $E(G_i) - M_i$ is colored i for $3 \le i \le p+1$.

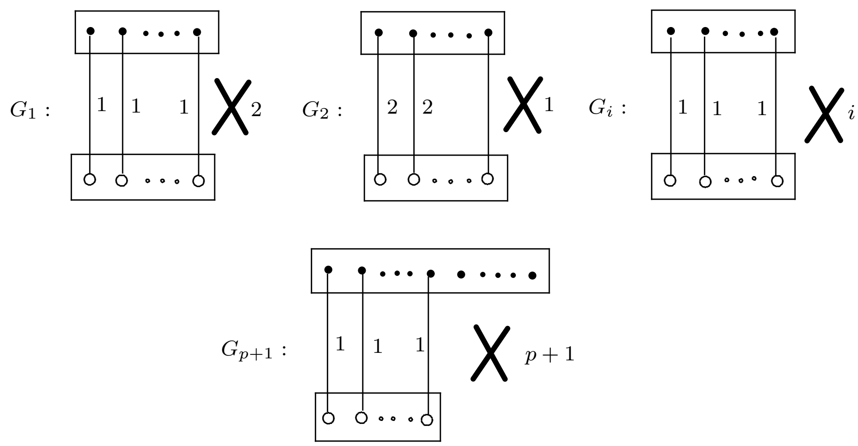

Figure 6.6: Illustrating the proper path coloring of $K_{r,s}$ which is described in the proof of Theorem 6.5.2

It remains to show that for each pair x, y of vertices of G, there are two internally disjoint proper $x - y$ paths in G. We consider two cases, according to whether $\{x, y\} \subseteq V(G_i)$ for *some* i with $1 \le i \le p+1$ or $\{x, y\} \not\subseteq V(G_i)$ for *all* i with $1 \le i \le p+1$.

Case 1. *$\{x, y\} \subseteq V(G_i)$ for some integer i with $1 \le i \le p+1$.* We consider two subcases.

Subcase 1.1. *$\{x, y\} \subseteq V(G_i)$ for some integer i with $1 \le i \le p$.* We may assume, without loss of generality, that $i = 1$ and $x, y \in \{u_1, u_2, v_{1,1}, v_{1,2}\}$. Since $c(u_1v_{1,1}) = c(u_2v_{1,2}) = 1$ and $c(u_1v_{1,2}) = c(u_2v_{1,1}) = 2$, it follows that $(u_1, v_{1,2}, u_2, v_{1,1}, u_1)$ is a properly colored 4-cycle in G that contains x and y. Hence, there are two internally disjoint proper $x - y$ paths in G.

Subcase 1.2. *$\{x, y\} \not\subseteq V(G_i)$ for all integers i with $1 \le i \le p$ and $x, y \in V(G_{p+1})$.* Thus, at least one of x and y belongs to V_{p+1} and so $|V_{p+1}| = q \ge 1$. We may assume, without loss of generality, that $x \in U \cup V_{p+1}$ and $y \in V_{p+1}$. Let $U_1 = \{u_1, u_2, \ldots, u_q\}$ and $U_2 = \{u_{q+1}, u_{q+2}, \ldots, u_r\}$. There are three subcases, according to whether $x \in U_1$, $x \in U_2$ or $x \in V_{p+1}$.

Subcase 1.2.1. *$x \in U_1$.* If $q \ge 2$, then we may assume, without loss of generality, that $x, y \in \{u_1, u_2, v_{p+1,1}, v_{p+1,2}\}$ and the argument in Subcase 1.1 shows that G has a properly colored 4-cycle that contains x and y and so there are two internally disjoint proper $x - y$ paths in G. Thus, we may assume $q = 1$ and so $x = u_1$ and $y = v_{p+1,1}$. Since $c(u_1v_{p,2}) = p$, $c(u_2v_{p,2}) = c(u_1v_{p+1,1}) = 1$ and $c(u_2v_{p+1,1}) = p+1$, it follows that $(x = u_1,\ v_{p,2},\ u_2,\ v_{p+1,1} = y,\ u_1)$ is a properly colored 4-cycle in G that contains x and y. Hence, there are two internally disjoint proper $x - y$ paths in G.

Subcase 1.2.2. *$x \in U_2$, say $x = u_{q+1}$.* First, observe that $Q_1 = (x, y)$ is a proper $x - y$ path. If $y \neq v_{p+1,2}$, then $Q_2 = (x = u_{q+1}, v_{p,2}, u_2, v_{p+1,1} = y)$ is proper $x - y$ path in G; while if $y = v_{p+1,2}$, then $Q_2 = (x = u_{q+1}, v_{p,1}, u_1, v_{p+1,1} = y)$ is proper $x - y$ path in G. In either case, Q_1 and Q_2 are two internally disjoint proper $x - y$ paths in G.

Subcase 1.2.3. *$x \in V_{p+1}$, say $x = v_{p+1,1}$ and $y = v_{p+1,2}$.* Hence, $x, y \in \{u_1,\ u_2,\ v_{p+1,1},\ v_{p+1,2}\}$. Then $(u_1,\ v_{p+1,2},\ u_2,\ v_{p+1,1},\ u_1)$ is a properly colored 4-cycle in G that contains x and y. Hence, there are two internally disjoint proper $x - y$ paths in G.

Case 2. *$\{x, y\} \nsubseteq V(G_i)$ for all i with $1 \leq i \leq p + 1$.* Thus, $x \in V_i$ and $y \in V_j$ where $1 \leq i \neq j \leq p + 1$. There are three subcases.

Subcase 2.1. *$i = 1$, say $x = v_{1,1}$.* First, suppose that $j = 2$. Suppose, without loss of generality, that $y = v_{2,1}$ or $y = v_{2,2}$.

- If $y = v_{2,1}$, then since $c(u_1v_{1,1}) = c(u_2v_{2,1}) = 1$ and $c(u_1v_{2,1}) = c(u_2v_{1,1}) = 2$, it follows that $(u_1,\ v_{1,1},\ u_2,\ v_{2,1},\ u_1)$ is a properly colored 4-cycle in G that contains x and y. Thus, there are two internally disjoint proper $x - y$ paths in G.

- If $y = v_{2,2}$, then since $c(v_{1,1}u_3) = 2$ and $c(u_3v_{2,2}) = 1$, it follows that $Q_1 = (x = v_{1,1}, u_3, v_{2,2} = y)$ is a proper $x - y$ path in G. Furthermore, since $c(v_{1,1}u_1) = 1$, $c(u_1v_{2,1}) = 2$, $c(v_{2,1}u_2) = 1$ and $c(u_2v_{2,2}) = 2$, it follows that $Q_2 = (v_{1,1}, u_1, v_{2,1}, u_2, v_{2,2})$ is another proper $x - y$ path in G. Thus, Q_1 and Q_2 are two internally disjoint proper $x - y$ paths in G.

Next, suppose that $3 \leq j \leq p + 1$. Assume, without loss of generality, that $y = v_{j,1}$ or $y = v_{j,2}$.

If $y = v_{j,1}$, then $(v_{1,1}, u_2, v_{j,1})$ and $(v_{1,1}, u_3, v_{j,1})$ are two internally disjoint proper $x - y$ paths in G.

If $y = v_{j,2}$, then $(v_{1,1}, u_1, v_{j,2})$ and $(v_{1,1}, u_2, v_{j,1})$ are two internally disjoint proper $x - y$ paths in G.

Subcase 2.2. *$i = 2$, say $x = v_{2,1}$.* Thus, $j \geq 3$. Assume, without loss of generality, that $y = v_{j,1}$ or $y = v_{j,2}$.

> If $y = v_{j,1}$, then $(v_{2,1}, u_1, v_{j,1})$ and $(v_{2,1}, u_2, v_{j,1})$ are two internally disjoint proper $x - y$ paths in G.
>
> If $y = v_{j,2}$, then $(v_{2,1}, u_1, v_{j,2})$ and $(v_{2,1}, u_3, v_{j,1})$ are two internally disjoint proper $x - y$ paths in G.

Subcase 2.3. *$i \geq 3$, say $x = v_{3,1}$* and $y \in V_j$ where $j \geq 4$. Then $(v_{3,1}, u_2, y)$ and $(v_{2,1}, u_3, y)$ are two internally disjoint proper $x - y$ paths in G. ■

By Proposition 6.5.1 and Theorem 6.5.2, $\mathrm{pc}_2(K_{3,3}) = 2 = \sqrt[3]{s}$ and $\mathrm{pc}_2(K_{3,s}) \leq \lceil s/3 \rceil$ for $s > 3$. For $4 \leq s \leq 6$, since $\lceil s/3 \rceil = 2 = \lceil \sqrt[3]{s} \, \rceil$ and $\mathrm{pc}_2(K_{3,s}) \geq 2$ for all $s \geq 3$, it follows that $\mathrm{pc}_2(K_{3,3}) = 2 = \lceil \sqrt[3]{s} \, \rceil$ for $4 \leq s \leq 6$. In fact, the exact value of $\mathrm{pc}_2(K_{3,s})$ can be determined for all $s \geq 3$, as we show next.

Theorem 6.5.3 *For each integer $s \geq 3$,*

$$\mathrm{pc}_2(K_{3,s}) = \begin{cases} 3 & \text{if } s = 7, 8 \\ \lceil \sqrt[3]{s} \, \rceil & \text{if } s \neq 7, 8. \end{cases}$$

Proof. We saw that $\mathrm{pc}_2(K_{3,s}) = \lceil \sqrt[3]{s} \, \rceil$ when $3 \leq s \leq 6$. Thus, we may assume that $s \geq 7$. First, suppose that $s \in \{7, 8\}$. By Theorem 6.5.2, $\mathrm{pc}_2(K_{3,s}) \leq \lceil s/3 \rceil = 3$ for $s \in \{7, 8\}$. Hence, it remains to show that $K_{3,s}$ has no proper 2-path coloring. Assume, to the contrary, that there is a proper 2-path coloring $c : E(K_{3,s}) \to [2]$ using colors 1 and 2 when $s \in \{7, 8\}$. Let $c_1, c_2, \ldots, c_s$ be the s distinct functions constructed from c, as described in (7.6). Since there are exactly six surjective functions from [3] to [2] and $s \geq 7$, there is at least one function $c_i : [3] \to [2]$ for some $i = 1, 2, \ldots, s$ such that c_i is not surjective. We may assume that $c_i = c_1$ and $c_1(p) = 1$ for $p \in \{1, 2, 3\}$. Thus, $c(u_1v_1) = c(u_2v_1) = c(u_2v_1) = 1$. Furthermore, there is at least one function $c_j : [3] \to [2]$ such that $c_j(p) = c_j(q) = 1$ for some $p, q \in \{1, 2, 3\}$ and $p \neq q$. We may assume $c_j = c_2$ and $c_j(1) = c_j(2) = 1$ and so $c(u_1v_2) = c(u_2v_2) = 1$. However then, there is only one proper $v_1 - v_2$ path in G, namely (v_1, u_3, v_2). This implies that c is not a proper 2-path coloring, a contradiction. Therefore, $\mathrm{pc}_2(K_{3,s}) = 3$ for $s \in \{7, 8\}$.

Next, suppose that $s \geq 9$. Since $s > 2^3 = 8$, it follows that $\mathrm{pc}_2(K_{3,s}) \geq 3$. Let $n = \lceil \sqrt[3]{s}\ \rceil$. By Theorem 6.3.5, $\mathrm{pc}_2(K_{3,s}) \geq n$. It remains to construct an n-edge coloring $c : E(K_{3,s}) \to [n]$ such that c is a proper 2-path coloring of $K_{3,s}$.

Let $G = K_{3,s}$. For each integer i with $1 \leq i \leq s$, let $G_i = G[U, v_i] = K_{1,3}$ be the subgraph of G induced by the three edges in $[U, v_i]$. Thus, $\{G_1, G_2, \ldots, G_s\}$ is a $K_{1,3}$-decomposition of G. Let $\mathcal{F}(3, s)$ be the set of n^3 functions from [3] to $[n]$. Observe that each function $f : [3] \to [n]$ gives rise to an edge coloring of $K_{1,3}$ using colors from the set $[n]$ and visa versa. To see this, let $V(K_{1,3}) = \{u_1, u_2, u_3, v\}$ where v is the central vertex of $K_{1,3}$ and assign the color $f(p)$ to the edge u_pv for $p = 1, 2, 3$. On the other hand, if c^* is an edge coloring of $K_{1,3}$ using colors from the set $[n]$, then let $f^*(p) = c^*(u_pv)$ for $p = 1, 2, 3$ and so $f^* \in \mathcal{F}(3, n)$. Since $s \leq n^3 = |\mathcal{F}(3, n)|$, there are at least s distinct functions in $\mathcal{F}(3, n)$. We select s distinct functions from $\mathcal{F}(3, n)$, denoted by $c_1, c_2, \ldots, c_s$, to construct a 2-path coloring $c : E(G) \to [n]$ of G as follows:

$\star$ For $1 \leq i \leq 6$, let c_i be the edge coloring of G_i such that

$c_1(u_1v_1) = 1,\ c_1(u_2v_1) = 2,\ c_1(u_3v_1) = 3,$

$c_2(u_1v_2) = 3,\ c_2(u_2v_2) = 2,\ c_2(u_3v_2) = 1,$

$c_3(u_1v_3) = 2,\ c_3(u_2v_3) = 3,\ c_3(u_3v_3) = 1,$

$c_4(u_1v_4) = 2,\ c_4(u_2v_4) = 1,\ c_4(u_3v_4) = 3,$

$c_5(u_1v_5) = 3,\ c_5(u_2v_5) = 1,\ c_5(u_3v_5) = 2,$

$c_6(u_1v_6) = 1,\ c_6(u_2v_6) = 3,\ c_6(u_3v_6) = 2.$

This is shown in Figure 6.7. Thus, $c_1, c_2, \ldots, c_6$ can also be considered as six functions in $\mathcal{F}(3, n)$ such that each c_i is a bijection from [3] to [3].

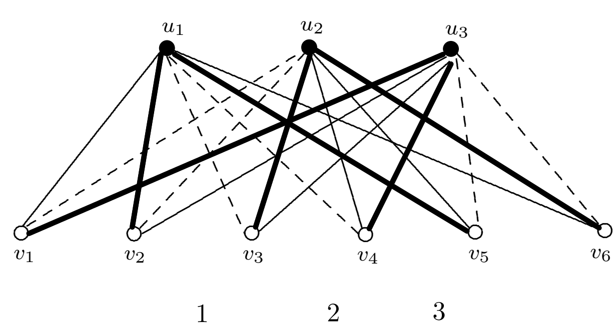

Figure 6.7: A step in the proof of Theorem 6.5.3

$\star$ Next, choose $s-6$ additional distinct functions $c_7, c_8, \ldots, c_s$ from the set $\mathcal{F}(3,n) - \{c_1, c_2, \ldots, c_6\}$. For each integer i with $7 \leq i \leq s$, the function c_i gives rise to an edge coloring of $G_i = G[U, v_i]$ using colors from $[n]$ by assigning $c_i(p)$ to the edge $u_p v_i$ of G_i for $p = 1, 2, 3$.

Now, the n-edge coloring $c : E(G) \to [n]$ is defined by

$$c(e) = c_i(e) \text{ if } e \in E(G_i) \text{ for } 1 \leq i \leq s. \tag{6.12}$$

It remains to show that c is a proper 2-path coloring of G.

Let $x, y \in V(G)$ such that $x \neq y$. We show that there are at least two internally disjoint proper $x-y$ paths in G. We consider three cases.

Case 1. $x, y \in U$, say $x = u_1$ and $y = u_2$. Then (u_1, v_1, u_2) and (u_1, v_2, u_2) are two internally disjoint proper $x-y$ paths in G.

Case 2. $x \in U$ *and* $y \in V$. Thus, (x, y) is a proper $x-y$ path. It remains to find another proper $x-y$ path in G. We may assume that $x = u_1$. First, suppose that $y = v_i$ for some $i \in \{1, 2, \ldots, 6\}$, say $y = v_1$. Then (u_1, v_2, u_3, v_1) is a proper $x-y$ path. Next, suppose that $y = v_i$ for some $i \geq 7$, say $y = v_7$. Let $c(u_2 v_7) = \alpha$ and $c(u_3 v_7) = \beta$.

$\star$ If $\alpha \neq 2$, then (u_1, v_1, u_2, v_7) is a proper $x-y$ path.

$\star$ If $\beta \neq 3$, then (u_1, v_1, u_3, v_7) is a proper $x-y$ path.

$\star$ If $\alpha = 2$ and $\beta = 3$, then (u_1, v_2, u_3, v_7) is a proper $x-y$ path.

Thus, there are two internally disjoint proper $x-y$ paths in G in this case (see Figure 6.8).

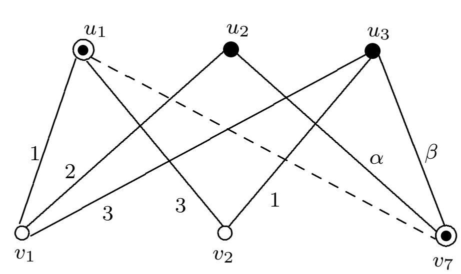

Figure 6.8: A step in the proof of Case 2 in Theorem 6.5.3

Case 3. $x, y \in V$. We consider three situations.

- If $x, y \in \{v_1, v_2, \ldots, v_6\}$, say $x = v_1$ and $y = v_2$, then (v_1, u_1, v_2) and (v_1, u_3, v_2) are two internally disjoint proper $x - y$ paths in G.

- If $x = v_i$ for $1 \leq i \leq 6$ and $y = v_j$ for some $j \geq 7$, then we may assume that $x = v_1$ and $y = v_7$. Since $c_1 \neq c_7$ in $\mathcal{F}(3, n)$, we may assume that $c_1(1) \neq c_7(1)$ and so $c(u_1v_1) \neq c(u_1v_7)$. Hence, (v_1, u_1, v_7) is a proper $x - y$ path. It remains to find another proper $x - y$ path in G. Let $c(u_2v_7) = \alpha$ and $c(u_3v_7) = \beta$.

  ⋆ If $\alpha \neq 2$, then (v_1, u_2, v_7) is a proper $x - y$ path.

  ⋆ If $\beta \neq 3$, then (v_1, u_3, v_7) is a proper $x - y$ path.

  ⋆ If $\alpha = 2$ and $\beta = 3$, then $(v_1, u_2, v_3, u_3, v_7)$ is a proper $x - y$ path.

- If $x = v_i$ and $y = v_j$ where $7 \leq i < j \leq s$, then we may assume that $x = v_7$ and $y = v_8$. Since $c_7 \neq c_8$ in $\mathcal{F}(3, n)$, we may assume that $c_7(1) \neq c_8(1)$ and so $c(u_1v_7) \neq c(u_1v_8)$. Hence, (v_7, u_1, v_8) is a proper $x - y$ path. It remains to find another proper $x - y$ path in G. Let

$$c(u_2v_7) = \alpha,\ c(u_3v_7) = \beta,\ c(u_2v_8) = \alpha' \text{ and } c(u_3v_8) = \beta'.$$

  If $\alpha \neq \alpha'$ or $\beta \neq \beta'$, then either (v_7, u_2, v_8) or (v_7, u_3, v_8) is a proper $x - y$ path. Thus, we may assume that $\alpha = \alpha'$ or $\beta = \beta'$. Then either $\alpha = \beta$ or $\alpha \neq \beta$.

  ⋆ If $\alpha = \beta$, say $\alpha = \beta \notin \{2, 3\}$, then $(v_7, u_2, v_1, u_3, v_8)$ is a proper $x - y$ path.

  ⋆ If $\alpha \neq \beta$, say $\alpha \neq 1$ and $\beta \notin \{1, 2\}$, then $(v_7, u_2, v_5, u_3, v_8)$ is a proper $x - y$ path.

Thus, there are two internally disjoint proper $x - y$ paths in G in this case (see Figure 6.9).

Hence, $c : E(G) \to [n]$ is a proper 2-path coloring of G. Therefore, $\mathrm{pc}_2(K_{3,s}) \geq n$ and $\mathrm{pc}_2(K_{3,s}) = n$ for $s \geq 9$. ■

Finally, we show that $\mathrm{pc}_2(K_{r,s}) = 2$ for $4 \leq r \leq s$.

Theorem 6.5.4 *If r and s are integers with $4 \leq r \leq s$, then* $\mathrm{pc}_2(K_{r,s}) = 2$.

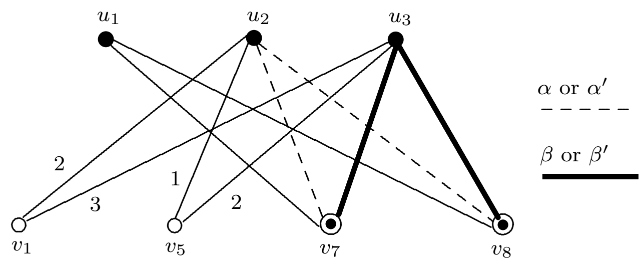

Figure 6.9: A step in the proof of Case 3 in Theorem 6.5.3

Proof. Let $G = K_{r,s}$ where $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ are the partite sets of G. We show that G has a proper 2-path coloring using colors 1 and 2. Let $M = \{v_i u_{i+1} : 1 \le i \le r\}$ be a matching of size r in G where $u_{r+1} = u_1$. If $r < s$, then let $H = K_{2,s-r}$ be the subgraph of G induced by $\{u_1, u_4\} \cup \{v_{r+1}, v_{r+2}, \ldots, v_s\}$. Define the edge coloring $c : E(G) \to \{1, 2\}$ by $c(e) = 1$ if $e \in M \cup E(H)$ and $c(e) = 2$ if $e \in E(G) - (M \cup E(H))$. This is shown in Figure 6.10 for $K_{4,6}$, where each solid edge is colored 1 and each dashed edge is colored 2. We show that c is a proper 2-path coloring of G. That is, we show that for each pair x, y of vertices of G, there are two internally disjoint proper $x - y$ paths in G.

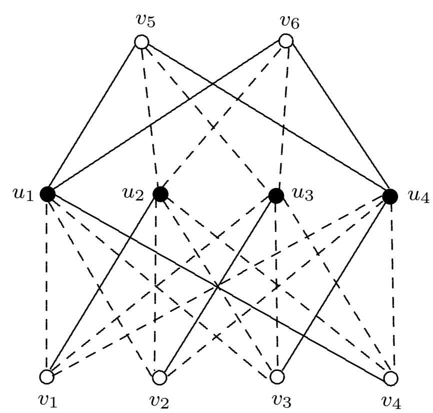

Figure 6.10: A proper 2-path coloring of $K_{4,6}$

Let $x, y \in V(G)$. First, observe that $(u_1, v_1, u_2, v_2, \ldots, u_r, v_r, u_1)$ is a properly colored cycle of order $2r$ in G. Thus, if $x = u_i$ and $y = v_j$ where $1 \le i, j \le r$, then x and y are connected by two internally disjoint proper paths in G. Thus, we may assume that $r < s$ and at least one of x and y does not belong to $U \cup \{v_1, v_2, \ldots, v_r\}$, say $y \notin U \cup \{v_1, v_2, \ldots, v_r\}$. There are two cases, according to whether $x \in U$ or $x \in V$.

Case 1. *$x \in U$, say $x = u_i$ where $1 \le i \le r$.* Since $y \in V$, it follows that $P_1 = (x, y)$ is an $x - y$ path. By the argument above, we may assume that $y = v_{r+1}$.

If $i = 2$, then let $P_2 = (x = u_2, v_1, u_1, v_{r+1} = y)$.

If $i \ne 2$, then let $P_2 = (x = u_i, v_1, u_2, v_{r+1} = y)$.

Then P_1 and P_2 are two internally disjoint proper $x - y$ paths in G.

Case 2. *$x \in V$.* Assume, without loss of generality, that $x = v_i$ and $y = v_j$ where $1 \le i < j \le s$ and $j \ge r + 1$.

- ⋆ For $1 \le i \le r$, we may assume that $j = r + 1$.

  If $i \ne 3, r$, then $P_1 = (v_i, u_1, v_{r+1})$ and $P_2 = (v_i, u_4, v_{r+1})$ are two internally disjoint proper $x - y$ paths in G.

  For $i = 3$, let $Q_1 = (v_3, u_3, v_2, u_4, v_{r+1})$. Then P_1 and Q_1 are two internally disjoint proper $x - y$ paths in G.

  For $i = r$, let $Q_2 = (v_r, u_1, v_1, u_2, v_{r+1})$. Then P_2 and Q_2 are two internally disjoint proper $x - y$ paths in G.

- ⋆ For $i \ge r + 1$, we may assume that $i = r + 1$ and $j = r + 2$. Then

  $$(v_{r+1}, u_1, v_1, u_2, v_{r+2}) \text{ and } (v_{r+1}, u_3, v_2, u_4, v_{r+2})$$

  are two internally disjoint proper $x - y$ paths in G.

In each case, x and y are connected by two internally disjoint proper paths in G and so c is a proper 2-path coloring using two colors. Therefore, $\mathrm{pc}_2(K_{r,s}) = 2$ for $4 \le r \le s$. ■

In summary, the following result provides the proper 2-connectivity of each complete bipartite graph $K_{r,s}$ for $2 \le r \le s$.

Theorem 6.5.5 *For integers r and s with $2 \le r \le s$,*

$$\mathrm{pc}_2(K_{r,s}) = \begin{cases} s & \text{if } r = 2 \\ 3 & \text{if } r = 3 \text{ and } s = 7, 8 \\ \lceil \sqrt[3]{s}\, \rceil & \text{if } r = 3 \text{ and } s \ne 7, 8 \\ 2 & \text{if } 4 \le r \le s. \end{cases}$$

In [4], Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero and Tuza showed that $\mathrm{pc}_k(K_{r,s}) = 2$ for positive integers r, s and k with $k \geq 3$ and $2k \leq r \leq s$; while for the remaining integers r, s and k, the values of $\mathrm{pc}_k(K_{r,s})$ are unknown in general.

6.6 Another Special Case

We saw in Theorem 6.3.6 that if k, r and s are integers with $r = 2k - 1 \geq 3$ and $s \geq 3^r$, then $\mathrm{pc}_k(K_{r,s}) = \lceil \sqrt[r]{s}\ \rceil$. We now consider another special case of $\mathrm{pc}_k(K_{r,s})$, namely when $r = 2k - 2 \geq 4$ and s is sufficiently large. First, recall that by Proposition 7.2, if r, k and n are integers with $2 \leq k \leq r$ and $n \geq 2$, then $M(r, n, k) \leq n^{r-k+1}$. In fact, it is known that

$$M(r, n, 2) = n^{r-1}. \tag{6.13}$$

Nevertheless, we provide an independent proof of a more general statement here.

Theorem 6.6.1 *For each pair r, n of integers with $r, n \geq 2$, $M(r, n, 2) = n^{r-1}$. Furthermore, $\mathcal{F}(r, n)$ can be partitioned into n disjoint 2-distinct sets of size n^{r-1}.*

Proof. By proposition 6.3.8 $M(r, n, 2) \leq n^{r-1}$, it only remains to construct a 2-distinct set of this size. We proceed by induction. We first show the statement is true for each $n \geq 2$ when $r = 2$ and we then show that the r, n case implies the $r + 1, n$ case.

Let G be the subgroup of the symmetric group S_n generated by the permutation $(234 \ldots n1)$. Thus G is an n-cycle and $|G| = n$. We let G act on the set $\mathcal{F}(2, n)$ by composition. This is clearly well defined since each element of G is a bijection $[n] \to [n]$. Thus the orbits of this action partition $\mathcal{F}(2, n)$. Since $|G| = n$, it follows that these orbits have size n. Since $|\mathcal{F}(2, n)| = n^2$, there must be n distinct orbits. Finally, we observe that each of these orbits is a 2-distinct set. Indeed, the orbit of an element (x, y) is (taking coordinates mod n):

$$\{(x, y), (x + 1, y + 1), (x + 2, y + 2), \ldots, (x + n - 1, y + n - 1)\}$$

which is easily seen to be 2-distinct. Thus we have a partition of $\mathcal{F}(2, n)$ into n disjoint 2-distinct sets of size $n = n^{2-1}$. Now we assume the statement is true for integers $r, n \geq 2$ and show it is true for integers $r + 1, n$. Let $A_1, A_2, \ldots, A_n$ be a partition of $\mathcal{F}(r,n)$ where each A_i is 2-distinct and $|A_i| = n^{r-1}$. For each $1 \le i \le n$ and $1 \le j \le n$ define

$$B_{i,j} = \{f \in \mathcal{F}(r+1,n) : f|_{[r]} \in A_i, f(r+1) = i+j\}.$$

We claim that for each fixed integer j, the set $\bigcup_{i=1}^{n} B_{i,j}$ is 2-distinct and has size n^r. Let $f, g \in \bigcup_{i=1}^{n} B_{i,j}$, say $f \in B_{x,j}$ and $g \in B_{y,j}$. If $x = y$, then $f|_{[r]} \in A_x$ and $g|_{[r]} \in A_x$. By assumption, A_x is 2-distinct and so f, g must differ at least twice. If $x \neq y$, then $f(r+1) \neq g(r+1)$. Furthermore, since $f|_{[r]}$ and $g|_{[r]}$ are distinct functions, it follows that they must differ at least once on $[r]$. Thus, $\bigcup_{i=1}^{n} B_{i,j}$ is 2-distinct. Next observe that $B_{x,j}$ and $B_{y,j}$ are disjoint if $x \neq y$. Indeed, if $f \in B_{x,j}$, then $f(r+1) = x + j \neq y + j$ and so $f \notin B_{y,j}$. Hence,

$$\left| \bigcup_{i=1}^{n} B_{i,j} \right| = n \cdot |A_i| = n^r.$$

Finally we claim that

$$\bigcup_{i=1,j=1}^{n} B_{i,j} = \mathcal{F}(r+1,n).$$

We show that the sets $\bigcup_{i=1}^{n} B_{i,1}$, $\bigcup_{i=1}^{n} B_{i,2}$, $\ldots$, $\bigcup_{i=1}^{n} B_{i,n}$ are pairwise disjoint. Assume, to the contrary, that there are x and y with $1 \le x \neq y \le n$ such that $\left(\bigcup_{i=1}^{n} B_{i,x}\right) \bigcap \left(\bigcup_{i=1}^{n} B_{i,y}\right) \neq \emptyset$. Let $f \in \left(\bigcup_{i=1}^{n} B_{i,x}\right) \bigcap \left(\bigcup_{i=1}^{n} B_{i,y}\right)$, where say $f \in B_{\alpha,x}$ and $f \in B_{\beta,y}$. Then $f(r+1) = \alpha + x$ and $f|_{[r]} \in A_\alpha$. Also, $f(r+1) = \beta + y$ and $f|_{[r]} \in A_\beta$. So we must have that $\alpha + x = \beta + y$, or $\alpha - \beta = y - x$. Since $y - x \neq 0$, it follows that $\alpha - \beta \neq 0$ and so $\alpha \neq \beta$. Thus A_α and A_β are disjoint, implying that f cannot exist. Hence, $\bigcup_{i=1}^{n} B_{i,1}$, $\bigcup_{i=1}^{n} B_{i,2}$, $\ldots$, $\bigcup_{i=1}^{n} B_{i,n}$ are pairwise disjoint, as claimed.

Thus, we have

$$\left|\bigcup_{i=1,j=1}^{n} B_{i,j}\right| = n \cdot |\bigcup_{i=1}^{n} B_{i,j}| = n \cdot n^r = n^{r+1} = |\mathcal{F}(r+1,n)|.$$

Therefore, $\mathcal{F}(r+1,n)$ has been partitioned into n disjoint 2-distinct sets of size n^r, as desired. ■

With the aid of Theorem 6.3.9 and (7.3), we now determine the formula for $\mathrm{pc}_k(K_{r,s})$ when $r = 2k-2$ and $s \geq 3^{r-1}$. By Theorem 6.5.5, we may assume that $r = 2k-2 \geq 4$ and so $k \geq 3$.

Theorem 6.6.2 *If k, r and s are integers with $k \geq 3$, $r = 2k-2$ and $s \geq 3^{r-1}$, then*

$$\mathrm{pc}_k(K_{r,s}) = \lceil \sqrt[r-1]{s}\ \rceil.$$

Proof. Let $G = K_{r,s}$ and $N = \lceil \sqrt[r-1]{s}\ \rceil$, where $r = 2k-2 \geq 4$ and $s \geq 3^{r-1}$. First, observe that if $r = 2k-2$, then $2r-2k+1 = r-1$. It then follows by Theorem 6.3.9 that $\mathrm{pc}_k(G) \geq \lceil \sqrt[r-1]{s}\ \rceil$. Thus, it remains to show that

$$\mathrm{pc}_k(G) \leq \lceil \sqrt[r-1]{s}\ \rceil.$$

That is, we show that G has a proper k-path coloring using colors from the set $[N]$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of G and let $\mathcal{F}(r,N)$ be the set of all functions $f : [r] \to [N]$.

Let T be a 2-distinct subset of maximum size in $\mathcal{F}(r,N)$. Then $|T| = N^{r-1} \geq s$ by (7.3). For each $f \in T$, let f' be the restriction of f to $[r-1]$, that is, $f'(x) = f(x)$ for each $x \in [r-1]$. Now let $T' = \{f' : f \in T\}$. Since T is a 2-distinct subset of $\mathcal{F}(r,N)$, it follows that if $f, g \in T$ and $f \neq g$, then there are at least two distinct elements p_1 and p_2 in $[r]$ such that $f(p_1) \neq g(p_1)$ and $f(p_2) \neq g(p_2)$. We may assume that $p_1 \neq r$. Hence, $f'(p_1) \neq g'(p_1)$ where $p_1 \in [r-1]$ and so $f' \neq g'$ in T'. Therefore, $|T'| = |T| = N^{r-1}$. Since $T' \subseteq \mathcal{F}(r-1,N)$ and $|\mathcal{F}(r-1,N)| = N^{r-1} = |T'|$, it follows that $T' = \mathcal{F}(r-1,N)$. Because $N^{r-1} \geq s \geq 3^{r-1}$, it follows that $N \geq 3$. Let $c'_1, c'_2, \ldots, c'_{3^{r-1}} \in T' = \mathcal{F}(r-1,N)$ such that the range of each c'_i $(1 \leq i \leq 3^{r-1})$ belongs to $[3]$. Since $|\mathcal{F}(r-1,3)| = 3^{r-1}$, it follows that

$$\{c'_1, c'_2, \ldots, c'_{3^{r-1}}\} = \mathcal{F}(r-1,3).$$

Furthermore, let

$$T = \{c_1, c_2, \ldots, c_{3^{r-1}}, \ldots, c_{N^{r-1}}\},$$

where c'_i is the restriction of c_i to $[r-1]$ for $1 \le i \le 3^{r-1}$.

We now define an edge coloring $c : E(G) \to [N]$ by using the *first* s integer-valued functions $c_1, c_2, \ldots, c_s \in T$ as follows. For each integer i with $1 \le i \le s$, let $c(u_p v_i) = c_i(p)$ for each $p \in [r]$. Next, show that c is a proper k-path coloring of G using colors in $[N]$. That is, we show that every two distinct vertices x and y of G are connected by k internally disjoint properly colored $x - y$ paths in G. We consider three cases.

Case 1. $\{x, y\} = \{u_i, u_j\}$ *where* $1 \le i < j \le r$. Let $x \in [r] - \{i, j\}$. For each integer i with $1 \le i \le 3^{r-1}$, let c_i^* be the restriction of c_i to $[r] - \{x\}$. Now let $A = \{c_1^*, c_2^*, \ldots, c_{3^{r-1}}^*\}$. Again, since T is a 2-distinct subset of $\mathcal{F}(r, N)$ and the range of each c_i $(1 \le i \le 3^{r-1})$ belongs to $[3]$, it follows that $|A| = 3^{r-1}$ and so $A = \mathcal{F}(r-1, 3)$ is the set of functions from $[r] - \{x\} \to [3]$. There are $6 \cdot 3^{r-3}$ elements f in $\mathcal{F}(r-1, 3)$ such that

$$f(i) \ne f(j). \tag{6.14}$$

[Note that there are 6 choices for $f(i)$ and $f(j)$ in [3] such that $f(i) \ne f(j)$ and 3^{r-3} choices for the remaining $r-3$ values of f in [3].] Since $6 \cdot 3^{r-3} \ge \frac{r+2}{2} = k$ for each integer $r \ge 4$, there are k distinct elements $c_{\ell_1}, c_{\ell_2}, \ldots, c_{\ell_k}$ in $\mathcal{F}(r, N)$, each of whose ranges belongs to [3], that satisfy the condition in (6.14). Therefore, there are the k distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_k}$ in V such that each path (u_i, v_{ℓ_t}, u_j), $1 \le t \le k$, is proper. Hence, there are k internally disjoint properly colored $x - y$ paths in G.

Case 2. $\{x, y\} = \{v_i, v_j\}$ *where* $1 \le i < j \le s$. Since $c_i \ne c_j$, there are two elements $p, q \in [r]$ such that $c_i(p) \ne c_j(p)$ and $c_i(q) \ne c_j(q)$. Hence, $c(u_p v_i) \ne c(u_p v_j)$ and $c(u_q v_i) \ne c(u_q v_j)$. Thus, $Q_1 = (v_i, u_p, v_j)$ and $Q_2 = (v_i, u_q, v_j)$ are two proper $x - y$ paths of length 2 in G. We may assume, without loss of generality, that $p = r-1$ and $q = r$. Next, we show that there are $k-2$ internally disjoint properly colored $x - y$ paths of length 4 in G such that u_{r-1} and u_r do not belong to any of these $k-2$ paths. In fact, we show that there are $k-2$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-2}}$ of V such that the following $k-2$ paths of length 4 are proper

$v_i - v_j$ paths in G:

$$\begin{aligned}
P_1 &= (v_i, u_1, v_{\ell_1}, u_2, v_j)\\
P_2 &= (v_i, u_3, v_{\ell_2}, u_4, v_j)\\
&\cdots\\
P_t &= (v_i, u_{2t-1}, v_{\ell_t}, u_{2t}, v_j)\\
&\cdots\\
P_{k-2} &= (v_i, u_{r-3}, v_{\ell_{k-2}}, u_{r-2}, v_j).
\end{aligned}$$

In order for $P_t = (v_i, u_{2t-1}, v_{\ell_t}, u_{2t}, v_j)$ to be proper for each integer t with $1 \le t \le k-1$, we must have

$$c(v_i u_{2t-1}) \ne c(u_{2t-1} v_{\ell_t}),\ c(u_{2t-1} v_{\ell_t}) \ne c(v_{\ell_t} u_{2t}) \text{ and } c(v_{\ell_t} u_{2t}) \ne c(u_{2t} v_j).$$

Let $a_t \in [3] - \{c(v_i u_{2t-1})\}$ and $b_t \in [3] - \{a_t, c(u_{2t} v_j)\}$. Thus, $a_t, b_t \in \{1,2,3\}$ and $a_t \ne b_t$. If $f \in \{c_1, c_2, \ldots, c_{3^{r-1}}\}$ (where $f(p) \in [3]$ for each $p \in [r]$) such that $f(2t-1) = a_t$ and $f(2t) = b_t$, then f gives rise to a vertex $v_{\ell_t} \in V$ that can be used to construct P_t. There are at least $2 \cdot 3^{r-3}$ elements f in $\mathcal{F}(r-1,3)$ which satisfy this condition. [Note that there are two choices for a_t, once a_t has been chosen, there is one choice for b_t and there are 3^{r-3} choices for the remaining $r-3$ values of f in [3].] Since $2 \cdot 3^{r-3} > \frac{r-2}{2} = k-2$ for each integer $r \ge 4$, there are $k-2$ distinct elements in $\mathcal{F}(r-1,3)$ that give rise to distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-2}} \in V$ that can be used to construct these $k-2$ internally disjoint properly colored $x-y$ paths $P_1, P_2, \ldots, P_{k-2}$ in G. Thus, $Q_1, Q_2, P_1, P_2, \ldots, P_{k-2}$ are k internally disjoint properly colored $x-y$ paths in G.

Case 3. *$\{x,y\} = \{u_i, v_j\}$ where $1 \le i \le r$ and $1 \le j \le s$.* We may assume that $u_i = u_1$ and so (u_1, v_j) is a properly colored $x-y$ path of length 1. Next, we show that there are $k-1$ internally disjoint properly colored $x-y$ paths of length 3 in G. In fact, we show that there are $k-1$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ of V such that the following $k-1$ paths of length 3 are proper $u_1 - v_j$ paths in G:

$$\begin{aligned} P_1 &= (u_1, v_{\ell_1}, u_2, v_j) \\ P_2 &= (u_1, v_{\ell_2}, u_3, v_j) \\ &\cdots \\ P_t &= (u_1, v_{\ell_t}, u_{t+1}, v_j) \\ &\cdots \\ P_{k-1} &= (u_1, v_{\ell_{k-1}}, u_k, v_j). \end{aligned}$$

In order for $P_t = (u_1, v_{\ell_t}, u_{t+1}, v_j)$ to be proper for each integer t with $1 \le t \le k-1$, we must have

$$c(u_1 v_{\ell_t}) \neq c(v_{\ell_t} u_{t+1}) \text{ and } c(v_{\ell_t} u_{t+1}) \neq c(u_{t+1} v_j).$$

Let $a_t \in [3] - \{c(u_{t+1}v_j)\}$ and let $b_t \in [3] - \{a_t\}$. If $f \in \{c_1, c_2, \ldots, c_{3^{r-1}}\}$ such that $f(t+1) = a_t$ and $f(1) = b_t$, then f gives rise to a vertex $v_{\ell_t} \in V$ with the desired property. There are at least $4 \cdot 3^{r-3}$ elements f of $\mathcal{F}(r-1, 3)$ which satisfy this condition. [Note that there are at least two choices for a_t and at least two choices for b_t and there are 3^{r-3} choices for the remaining $r-3$ values of f in [3].] Since $4 \cdot 3^{r-3} \ge \frac{r}{2} = k-1$, there are $k-1$ distinct vertices $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_{k-1}}$ in V that can be used to construct the $k-1$ proper paths $P_1, P_2, \ldots, P_{k-1}$. Therefore, there are k internally disjoint properly colored $x - y$ paths in G.

Thus, $c : E(G) \to [N]$ is a proper k-path coloring of G and so $\mathrm{pc}_k(G) \le N$. Therefore, $\mathrm{pc}_k(K_{r,s}) = N = \lceil \sqrt[r-1]{s}\, \rceil$. ■

Our work suggests the problems below.

Problem 6.6.3 *Determine* $\mathrm{pc}_3(K_{4,s})$ *for* $s \ge 5$.

Problem 6.6.4 *Determine* $\mathrm{pc}_{r-1}(K_{r,s})$ *for* $4 \le r < s$.

Problem 6.6.5 *Determine* $\mathrm{pc}_{\frac{r-1}{2}}(K_{r,s})$ *for* $7 \le r < s$ *and* r *is odd.*

Chapter 7

Strong Proper Connectivity & Related Topics

In this chapter, we introduce the concept of strong proper connectivity resulting from strong proper-path colorings of connected graphs. We also discuss some related topics for further study.

7.1 Proper k-Geodesic Colorings

For every nontrivial connected graph G and every two distinct vertices u and v of G, there is at least one $u - v$ path of length $d(u, v)$, namely a $u - v$ geodesic. If u and v are adjacent vertices in G, then (u, v) is the unique $u - v$ geodesic in G; while if u and v are not adjacent, then there may be more than one $u - v$ geodesic in G. For example, if $G = K_{2,s}$ with partite sets $\{u, v\}$ and $\{w_1, w_2, \ldots, w_s\}$ where $s \geq 2$, then there are s internally disjoint $u - v$ geodesics in G, namely (u, w_i, v) for $1 \leq i \leq s$ and for $1 \leq i < j \leq s$, there are two internally disjoint $w_i - w_j$ geodesics in G, namely (w_i, u, w_j) and (w_i, v, w_j). Hence, there are at least two internally disjoint geodesics connecting every two nonadjacent vertices in G. However, it is also possible that there is a unique $u - v$ geodesic connecting two nonadjacent vertices u and v in a nontrivial connected graph. For example, if P is the Petersen graph, then $d(u, v) = 2$ for every two nonadjacent vertices u and v of P. Since the girth of P is 5, there is a unique $u - v$ geodesic connecting *every* two nonadjacent vertices u and v in P.

A connected graph G of diameter at least 2 is *ℓ-geodesic connected* for some positive integer ℓ if for every two *nonadjacent* vertices u and v of G, there exist at least ℓ internally disjoint $u - v$ geodesics in G. For example, the Petersen graph is 1-geodesic connected and not ℓ-geodesic connected for any $\ell \geq 2$. For positive integers r and s with $r \leq s$, the complete bipartite $K_{r,s}$ is r-geodesic connected. In general, if $G = K_{n_1,n_2,\ldots,n_p}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_p$, is a complete p-partite graph for some integer $p \geq 2$ and $\ell = n_1 + n_2 + \cdots + n_{p-1}$, then G is ℓ-geodesic connected.

An edge coloring of a connected graph G is called a *proper k-geodesic coloring* (or a *strong proper k-path coloring*) of G for some positive integer k if for every two nonadjacent vertices u and v of G, there exist at least k internally disjoint proper $u - v$ geodesics in G. The minimum number of the colors required in a proper k-geodesic coloring of G is the *strong proper k-connectivity* $\mathrm{spc}_k(G)$ of G. In particular, $\mathrm{spc}_1(G) = \mathrm{spc}(G)$ is the *strong proper connection number* of G. If G is an ℓ-geodesic connected graph and the edges of G are properly colored, then every two nonadjacent vertices of G are connected by at least ℓ internally disjoint proper geodesics. Therefore, we have the following observation.

Observation 7.1.1 *If G is an ℓ-geodesic connected graph for some positive ℓ and k is an integer with $1 \leq k \leq \ell$, then $\mathrm{spc}_k(G)$ exists and*

$$\mathrm{spc}_k(G) \leq \chi'(G). \tag{7.1}$$

Furthermore, if k and k' are integers with $1 \leq k \leq k' \leq \ell$, then $\mathrm{spc}_k(G) \leq \mathrm{spc}_{k'}(G)$.

We investigate the following problem.

For integers k, r and s with $2 \leq k \leq r \leq s$, what is $\mathrm{spc}_k(K_{r,s})$?

Since $\mathrm{diam}(K_{r,s}) = 2$, for each pair u, v of vertices of $K_{r,s}$, each $u - v$ geodesic P is a $u - v$ path of length 1 or 2. Therefore, for every proper edge coloring of $K_{r,s}$, the path P is a rainbow path. The question here is to determine the minimum number of the colors required in a proper k-geodesic coloring of $K_{r,s}$ such that every two nonadjacent vertices of $K_{r,s}$ are connected by at least k internally disjoint rainbow geodesics in $K_{r,s}$.

It will be useful to recall some definitions from Chapter 6. For integers r and n, where $r, n \geq 2$, let $\mathcal{F}(r, n)$ denote the set of all functions $f : [r] \to [n]$. For a positive integer k with $k \leq r$, a subset $S \subseteq \mathcal{F}(r, n)$ is called a *k-distinct subset* if for each pair f, g of functions in S, there are at least k distinct elements $p \in [r]$ such that $f(p) \neq g(p)$. As we will see soon, the k-distinct subsets of $\mathcal{F}(r, n)$ play an important role in constructing proper k-geodesic colorings of $K_{r,s}$ using the colors from the set $[n]$. Let $M(r,n,k)$ denote the maximum size of a k-distinct subset of $\mathcal{F}(r,n)$; that is,

$$M(r,n,k) = \max\{|S| : \ S \text{ is a } k\text{-distinct subset of } \mathcal{F}(r,n)\}.$$

Recall the following result on $M(r,n,k)$ established in Chapter 6. For integers r, k and n with $2 \le k \le r$ and $n \ge 2$,

$$M(r,n,k) \le n^{r-k+1}. \tag{7.2}$$

Furthermore,

$$M(r,n,2) = n^{r-1}. \tag{7.3}$$

In particular,

$$M(3,2,2) = 2^2 = 4, \ M(3,3,2) = 3^2 = 9 \text{ and } M(4,3,2) = 3^3 = 27.$$

For example,

$$S = \{(112), (121), (211), (222)\} \subseteq \mathcal{F}(3,2) \tag{7.4}$$

is a 2-distinct subset of $\mathcal{F}(3,2)$. The set

$$S = \{f_1, f_2, \ldots, f_9\} \subseteq \mathcal{F}(3,3) \tag{7.5}$$

is a 2-distinct subset of $\mathcal{F}(3,3)$, where

$$\begin{array}{lll} f_1 = (111) & f_2 = (222) & f_3 = (333) \\ f_4 = (123) & f_5 = (132) & f_6 = (213) \\ f_7 = (231) & f_8 = (312) & f_9 = (321). \end{array}$$

The set $S = \{f_1, f_2, \ldots, f_{27}\} \subseteq \mathcal{F}(4,3)$ is a 2-distinct subset of $\mathcal{F}(4,3)$, where

$$\begin{array}{lll} f_1 = (1111) & f_2 = (2222) & f_3 = (3333) \\ f_4 = (2211) & f_5 = (3322) & f_6 = (1133) \\ f_7 = (3311) & f_8 = (1122) & f_9 = (2233) \\ \\ f_{10} = (1212) & f_{11} = (2323) & f_{12} = (3131) \\ f_{13} = (2312) & f_{14} = (3123) & f_{15} = (1231) \\ f_{16} = (3112) & f_{17} = (1223) & f_{18} = (2331) \\ \\ f_{19} = (2113) & f_{20} = (3221) & f_{21} = (1332) \\ f_{22} = (3213) & f_{23} = (1321) & f_{24} = (2132) \\ f_{25} = (1313) & f_{26} = (2121) & f_{27} = (3232). \end{array}$$

We mentioned that the k-distinct subsets of $\mathcal{F}(r,n)$ play an important role in constructing proper k-geodesic colorings of $K_{r,s}$ using the colors from the set $[n]$. For example, if $S = \{f_1, f_2, \ldots, f_s\}$ is a 2-distinct subset of $\mathcal{F}(r,n)$, then we can use S to construct a proper 2-geodesic coloring of $K_{r,s}$ using colors in the set $[n]$. We now illustrate this by providing a proper 2-geodesic coloring of $K_{3,4}$ and $K_{3,9}$.

We begin with the graph $K_{3,4}$. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4\}$ be the partite sets of $K_{3,4}$. Let $S = \{(112), (121), (211), (222)\}$ be the 4-element 2-distinct subset of $\mathcal{F}(3,2)$ described in (7.4). For each integer i with $1 \le i \le 4$, if $f_i = (a_i\ b_i\ c_i)$, then define the colors of the three edges incident with v_i by $c(u_1v_i) = a_i$, $c(u_2v_i) = b_i$ and $c(u_3v_i) = c_i$. This coloring c is shown in Figure 7.1. Since c is a proper 2-geodesic coloring c of $K_{3,4}$ and $\text{spc}_2(K_{3,4}) \ge 2$, it follows that $\text{spc}_2(K_{3,4}) = 2$.

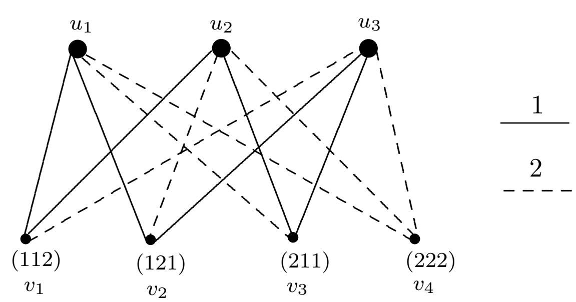

Figure 7.1: A proper 2-geodesic coloring of $K_{3,4}$

As another example, we consider the graph $K_{3,9}$ whose partite sets are $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, \ldots, v_9\}$. We now select the 9-element 2-distinct subset $S = \{f_1, f_2, \ldots, f_9\} \subseteq \mathcal{F}(3,3)$ described in (7.5); that is,

$$\begin{array}{lll} f_1 = (111) & f_2 = (222) & f_3 = (333) \\ f_4 = (123) & f_5 = (132) & f_6 = (213) \\ f_7 = (231) & f_8 = (312) & f_9 = (321). \end{array}$$

We now construct an edge coloring $c : E(G) \to [3] = \{1,2,3\}$ of G using the 9 functions in S. For each integer i with $1 \le i \le 9$, if $f_i = (a_i\ b_i\ c_i)$, then define the colors of the three edges incident with v_i by $c(u_1v_i) = a_i$, $c(u_2v_i) = b_i$ and $c(u_3v_i) = c_i$. This coloring is shown in Figure 7.2, where each fine edge is colored 1, each dashed edge is colored 2 and each bold edge is colored 3. Since $c : E(K_{3,9}) \to [3]$ is a proper 2-geodesic coloring of $K_{3,9}$, it follows that $\text{spc}_2(K_{3,9}) \le 3$. Since

$\mathrm{pc}_2(K_{3,9} = 3$, it follows by (7.1) that $\mathrm{spc}_2(K_{3,9}) = 3$.

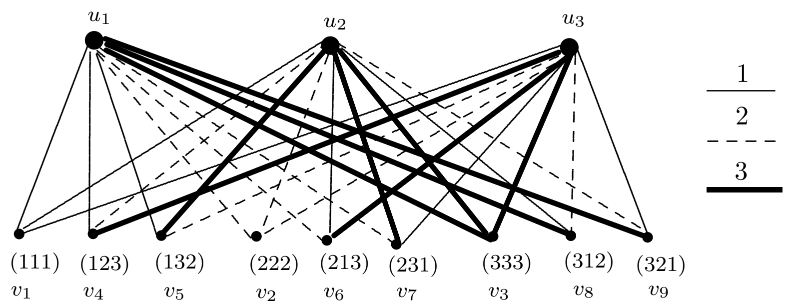

Figure 7.2: A proper 2-geodesic coloring of $K_{3,9}$

It is will be useful to recall the following definition. Let $G = K_{r,s}$ with partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ where $2 \le r \le s$. For each integer i with $1 \le i \le s$, let $G_i = G[U, v_i] \cong K_{1,r}$ be the subgraph induced by the set $[U, v_i]$ of all edges incident with v_i in G. For an edge coloring $c : E(G) \to [n] = \{1, 2, \ldots, n\}$ of G and each integer i with $1 \le i \le s$, there is an edge coloring $c_i : E(G_i) \to [n]$ obtained by restricting the coloring c to G_i. Each edge coloring c_i $(1 \le i \le s)$ can also be considered as an integer-valued function $c_i : [r] \to [n]$ of G_i defined by

$$c_i(p) = c(u_p v_i) \text{ for each } p \in [r]. \tag{7.6}$$

7.2 Lower Bounds for $\mathrm{spc}_k(K_{r,s})$

Recall that in Theorem 6.3.7, we established the following lower bound on the proper k-connectivity of a connected graph. That is, if r, s and k are positive integers such that $2 \le r \le s$ and $k \le r < 2k$ and N is the smallest positive integer such that $M(r, N, 2k - r) \ge s$, then

$$\mathrm{pc}_k(K_{r,s}) \ge N.$$

We next show that there is a similar result on $\mathrm{spc}_k(K_{r,s})$.

Theorem 7.2.1 *Let r, s, k be integers such that $2 \le k \le r \le s$. If N is the smallest positive integer such that $M(r, N, k) \ge s$, then*

$$\mathrm{spc}_k(K_{r,s}) \ge N.$$

Proof. For integers r and s with $2 \le r \le s$, let $G = K_{r,s}$ with partite sets $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$. Since $\mathrm{diam}(G) = 2$, if x and y are nonadjacent vertices of G, then each $x-y$ geodesic has length 2. Suppose that $\text{spc}_k(G)=n$ and let $c: E(G)\to[n]$ be a proper k-geodesic coloring of G using n colors. For each integer i with $1\le i\le s$, let c_i be the integer-valued function defined by (7.6). Now, let $S=\{c_1,c_2,\ldots,c_s\}$. We claim that S is a k-distinct subset of $\mathcal{F}(r,n)$; that is, if $c_i,c_j\in S$ where $i\neq j$, then there are at least k distinct elements $p\in[r]$ such that $c_i(p)\neq c_j(p)$. For $v_i,v_j\in V$ where $i\neq j$, let $Q_1,Q_2,\ldots,Q_k$ be k internally disjoint proper v_i-v_j geodesic in G. We may assume, without loss of generality, that $Q_p=(v_i,u_p,v_j)$ for $p=1,2,\ldots,k$. Since Q_p is properly colored, it follows that $c(v_iu_p)\neq c(u_pv_j)$ and so $c_i(p)\neq c_j(p)$ for $p=1,2,\ldots,k$. Therefore, as claimed, S is a k-distinct subset of $\mathcal{F}(r,n)$. If $M(r,n,k)$ is the maximum size of a k-distinct subset of $\mathcal{F}(r,n)$, then $M(r,n,k)\ge|S|=s$. Since N is the smallest positive integer such that $M(r,N,k)\ge s$, it follows that c must use at least N colors and so $n\ge N$. Therefore, $\text{spc}_k(G)=n\ge N$. ■

Recall by Theorem 6.3.9 that if r,s and k are integers with $2\le r\le s$ and $2\le k\le r<2k$, then

$$\text{pc}_k(K_{r,s})\ge\lceil \sqrt[2r-2k+1]{s}\ \rceil.$$

With the aid of Theorem 7.2.1 and (7.2), we are able to establish two additional lower bounds for $\text{spc}_k(K_{r,s})$, the first of which is similar to the one for $\text{pc}_k(K_{r,s})$.

Theorem 7.2.2 *If r,s and k are positive integers with $2\le r\le s$, then*

$$\text{spc}_k(K_{r,s})\ge\lceil \sqrt[r-k+1]{s}\ \rceil. \tag{7.7}$$

Proof. Let N be the smallest positive integer such that $M(r,N,k)\ge s$. It then follows by Theorem 7.2.1 that $\text{pc}_k(K_{r,s})\ge N$. Since $M(r,N,k)\le N^{r-k+1}$ by (7.2), it follows that

$$s\le M(r,N,k)\le N^{r-k+1}$$

and so $N\ge\lceil \sqrt[r-k+1]{s}\ \rceil$. Therefore, $\text{spc}_k(K_{r,s})\ge N\ge\lceil \sqrt[r-k+1]{s}\ \rceil$, as desired. ■

The following is an immediate consequence of Observation 7.1.1 and Theorem 7.2.2 when $r=k$.

Corollary 7.2.3 *If r and s are integers with $2\le r\le s$, then*

$$\text{spc}_r(K_{r,s})=s.$$

Proposition 7.2.4 *Let r, s, k, n be integers such that $2 \le k \le r \le s$ and $n \ge 2$.*

$$\textit{If } k > r + 1 - \log_n(s), \textit{ then } \mathrm{spc}_k(K_{r,s}) > n.$$

Proof. First, we verify the following statement:

$$\text{If } \mathrm{spc}_k(K_{r,s}) = n, \text{ then } k \le r + 1 - \log_n(s). \tag{7.8}$$

Let $G = K_{r,s}$ and let $c : E(G) \to [n]$ be a strong proper k-path coloring of G. Then $S = \{c_1, c_2, \dots, c_s\}$ is a k-distinct subset of $\mathcal{F}(r, n)$, where each c_i $(1 \le i \le s)$ is the integer-valued function $c_i : [r] \to [n]$ of the subgraph G_i as described in (7.6). Hence, $s = |S| \le M(r, n, k) \le n^{r-k+1}$ by (7.2) and so $s \le n^{r-k+1}$. Thus,

$$k \le r + 1 - \log_n(s).$$

Restating the implication in (7.8) in its contrapositive form, we obtain the following statement:

$$\text{If } k > r + 1 - \log_n(s), \text{ then } \mathrm{spc}_k(K_{r,s}) \ne n. \tag{7.9}$$

Since the statement in (7.9) holds for all integers m with $2 \le m < n$, it follows that if $k > r + 1 - \log_n(s)$, then $\mathrm{spc}_k(K_{r,s}) > n$, as desired. ∎

With the aid of Theorem 7.2.2 and (7.3), we are able to determine a formula for the strong proper 2-connectivity of $K_{r,s}$. We begin with the case when s is sufficiently large.

Theorem 7.2.5 *Let r and s be integers with $2 \le r \le s$. If $s \ge 2^{r-1}$, then*

$$\mathrm{spc}_2(K_{r,s}) = \lceil \sqrt[r-1]{s}\, \rceil.$$

Proof. By Theorem 7.2.2, $\mathrm{spc}_2(K_{r,s}) \ge \lceil \sqrt[r-1]{s}\, \rceil$. Thus, it remains to show that

$$\mathrm{spc}_2(K_{r,s}) \le \lceil \sqrt[r-1]{s}\, \rceil.$$

Let $G = K_{r,s}$. If $r = 2$, then $\lceil \sqrt[r-1]{s}\, \rceil = s = \chi'(G)$. Since $\mathrm{spc}_2(G) \le \chi'(G)$ by Observation 7.1.1, it follows that $\mathrm{spc}_2(G) \le \lceil \sqrt[r-1]{s}\, \rceil$ and so $\mathrm{spc}_2(G) = \lceil \sqrt[r-1]{s}\, \rceil$ when $r = 2$. Thus, we may assume that $r \ge 3$ and $s \ge 2^{r-1} \ge 4$. Let

$$U = \{u_1, u_2, \dots, u_r\} \text{ and } V = \{v_1, v_2, \dots, v_s\}$$

be the partite sets of G. Furthermore, let $N = \lceil \sqrt[r-1]{s}\ \rceil \geq 2$ and let A be a 2-distinct subset of maximum size in $\mathcal{F}(r, N)$. Since $s \geq 2^{r-1}$, it follows by (7.3) that $|T| = N^{r-1} \geq s$. For each $f \in A$, let f' be the restriction of f to $[r-1]$; that is, $f'(x) = f(x)$ for each $x \in [r-1]$. Now let $A' = \{f' : f \in A\}$. Since A is a 2-distinct subset of $\mathcal{F}(r, N)$, it follows that if $f \neq g \in A$, then there are two distinct elements p_1 and p_2 in $[r]$ such that $f(p_1) \neq g(p_1)$ and $f(p_2) \neq g(p_2)$. We may assume that $p_1 \neq r$. Hence, $f'(p_1) \neq g'(p_1)$ and so $f' \neq g'$ in T'. Therefore, $|A'| = |A| = N^{r-1}$. Since $A' \subseteq \mathcal{F}(r-1, N)$ and $|\mathcal{F}(r-1, N)| = N^{r-1} = |A'|$, it follows that $A' = \mathcal{F}(r-1, N)$. Let $A' = \{c'_1, c'_2, \ldots, c'_{N^{r-1}}\}$ such that the ranges of the first 2^{r-1} elements $c'_1, c'_2, \ldots, c'_{2^{r-1}}$ in A' belong to [2]. Now, let $A = \{c_1, c_2, \ldots, c_{N^{r-1}}\}$, where then the restriction of c_i to $[r-1]$ is $c'_i \in A'$ for $1 \leq i \leq N^{r-1}$.

We define an edge coloring $c : E(G) \to [N]$ by using the first s integer-valued functions $c_1, c_2, \ldots, c_s \in A$ as follows. For each integer i with $1 \leq i \leq s$, let $c(u_p v_i) = c_i(p)$ for each $p \in [r]$. It remains to show that c is a strong proper 2-path N-coloring of G. Let x and y be two nonadjacent vertices of G. First, suppose that $x = v_i$ and $y = v_j$, where $1 \leq i < j \leq s$. Since A is a 2-distinct subset of $\mathcal{F}(r, N)$, there are $p, q \in [r]$ such that $c_i(p) \neq c_j(p)$ and $c_i(q) \neq c_j(q)$. Thus, (v_i, u_p, v_j) and (v_i, u_q, v_j) are two internally disjoint properly colored $v_i - v_j$ geodesics in G. Next, suppose that $x = u_i$ and $y = u_j$, where $1 \leq i < j \leq r$. Since $r \geq 3$, it follows that $[r] - \{i, j\} \neq \emptyset$. Let $x \in [r] - \{i, j\}$. For each i with $1 \leq i \leq 2^{r-1}$, let c^*_i be the restriction of c_i to $[r] - \{x\}$. Now let $B' = \{c^*_1, c^*_2, \ldots, c^*_{2^{r-1}}\}$. Since A is a 2-distinct subset of $\mathcal{F}(r, N)$, it follows that $|B'| = 2^{r-1}$. Since the range of each c_i belongs to [2] for $1 \leq i \leq 2^{r-1}$, it follows that $B' \subseteq \mathcal{F}(r, 2)$. Again, $|\mathcal{F}(r, 2)| = 2^{r-1}$ and so $B' = \mathcal{F}(r, 2)$. Let $B = \{c_1, c_2, \ldots, c_{2^{r-1}}\}$ where the restriction of c_i to $[r] - \{x\}$ is c^*_i for $1 \leq i \leq 2^{r-1}$. Since B is a 2-distinct subset of $\mathcal{F}(r, N)$, we saw that $|B| = |B'| = 2^{r-1}$. There are at least $2^{r-2} \geq 2$ elements f in B such that $f(i) \neq f(j)$, say $c_1(i) \neq c_1(j)$ and $c_2(i) \neq c_2(j)$. Then (u_i, v_1, u_j) and (u_i, v_2, u_j) are two internally disjoint properly colored $u_i - u_j$ geodesics in G. In each case, c is a strong proper 2-path N-coloring of G and so $\text{spc}_2(G) \leq N$. Therefore,

$$\text{spc}_2(K_{r,s}) = N = \lceil \sqrt[r-1]{s}\ \rceil$$

when $s \geq 2^{r-1} \geq 2$. ■

Next, we show that the strong proper 2-connectivity of $K_{r,s}$ is 2 for all integers r and s with $2 \leq r \leq s \leq 2^{r-1}$.

Theorem 7.2.6 *Let r and s be integers with $2 \le r \le s \le 2^{r-1}$. Then*

$$\mathrm{spc}_2(K_{r,s}) = 2.$$

Proof. Let $G = K_{r,s}$. It suffices to show that G has a strong proper 2-path coloring using the colors 1 and 2. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the partite sets of G. We consider two cases, according to whether $r = s$ or $r < s$.

Case 1. $r = s$. Define an edge coloring $c : E(G) \to \{1, 2\}$ by

$$c(u_iv_j) = \begin{cases} 1 & \text{if } i \neq j \\ 2 & \text{if } i = j. \end{cases} \tag{7.10}$$

We show that c is a strong proper 2-path coloring of G. Let x and y be two nonadjacent vertices of G. First, assume that $x = u_i$ and $y = u_j$ where $1 \le i < j \le r$. Then (u_i, v_i, u_j) and (u_i, v_j, u_j) are two internally disjoint properly colored $u_i - u_j$ geodesics in G. Next, assume that $x = v_i$ and $y = v_j$ where $1 \le i < j \le s$. By symmetry, there are two internally disjoint properly colored $v_i - v_j$ geodesics in G. However, we provide a different argument which will be useful for the general case, namely Case 2.

For the edge coloring $c : E(G) \to \{1, 2\}$ defined in (7.10), consider the induced color functions $c_i : [r] \to [2]$ for each integer i with $1 \le i \le r$, where $c_i(x) = c(v_iu_x)$ for each $x \in [r]$. Thus,

$$c_i(x) = \begin{cases} 1 & \text{if } i \neq x \\ 2 & \text{if } i = x. \end{cases} \tag{7.11}$$

Let $A = \{c_i :\ 1 \le i \le r\}$. We show that A is a 2-distinct subset of $\mathcal{F}(r, 2)$. Let $c_i, c_j \in A$, where $1 \le i \neq j \le r$. Since $c_i(i) = 2$ and $c_i(j) = 1$ and $c_j(i) = 1$ and $c_j(j) = 2$, it follows that $c_i(i) \neq c_j(i)$ and $c_i(j) \neq c_j(j)$. Furthermore, $c_i(x) = c_j(x)$ for each $x \in [r] - \{i, j\}$. Hence, (v_i, u_i, v_j) and (v_i, u_j, v_j) are two internally disjoint properly colored $v_i - v_j$ geodesics in G. Thus, c is a strong proper 2-path coloring of G and so $\mathrm{spc}_2(G) = 2$ in this case.

Case 2. $r < s \le 2^{r-1}$. First, we show that there exists 2-distinct subset T of $\mathcal{F}(r, 2)$ such that $|T| = 2^{r-1}$ and $A \subseteq T$. For each $f \in \mathcal{F}(r, 2)$, let

$$w(f) = |\{x \in [r] : f(x) = 2\}|$$

(which is referred to as the *Hamming weight of* f in coding theory). Thus, if $f \in A$, then $w(f) = 1$. We claim that if $f, g \in \mathcal{F}(r, 2)$ such that $w(f)$ and $w(g)$ are of the same parity, then $\{f, g\}$ is a 2-distinct subset of $\mathcal{F}(r, 2)$. If this were not the case, then we may assume, without loss of generality, that $f(1) \neq g(1)$ and $f(x) = g(x)$ for all $x \in [r] - \{1\}$. Since $f(1), g(1) \in [2]$ and $f(1) \neq g(1)$, we may assume that $f(1) = 1$ and $g(1) = 2$. Because $f(x) = g(x)$ for each $x \in [r] - \{1\}$, it follows that $w(g) = w(f) + 1$ and so $w(f)$ and $w(g)$ are of opposite parity, which is a contradiction. Now, let

$$T = \{f \in \mathcal{F}(r, 2) : \ w(f) \text{ is odd}\}.$$

Since $w(f) = 1$ for each $f \in A$, we have $A \subseteq T$. By the argument above, T is a 2-distinct subset of $\mathcal{F}(r, 2)$. Furthermore,

$$|T| = \frac{1}{2}|\mathcal{F}(r, 2)| = \frac{2^r}{2} = 2^{r-1}.$$

Since $s \leq 2^{r-1}$, we can choose a s-element subset B of T such that $A \subseteq B$.

Next, suppose that $B = \{c_1, c_2, \ldots, c_s\}$ where c_i is defined in (7.11) for $1 \leq i \leq r$. We now define an edge coloring $c : E(G) \to [2]$ by using the s integer-valued functions $c_1, c_2, \ldots, c_s \in B$ as follows. For each integer i with $1 \leq i \leq s$, let $c(u_p v_i) = c_i(p)$ for each $p \in [r]$. It remains to show that c is a strong proper 2-path 2-coloring of G. Let x and y be two nonadjacent vertices of G. If $x = u_i$ and $y = u_j$, where $1 \leq i < j \leq r$, then the argument in Case 1 shows that there are 2 internally disjoint properly colored $u_i - u_j$ geodesics in G. Next, suppose that $x = v_i$ and $y = v_j$, where $1 \leq i < j \leq s$. There are $p, q \in [r]$ where $p \neq q$ such that $c_i(p) \neq c_j(p)$ and $c_i(q) \neq c_j(q)$. Hence, (v_i, u_p, v_j) and (v_i, u_q, v_j) are 2 internally disjoint properly colored $v_i - v_j$ geodesics in G. Therefore, c is a strong proper 2-path 2-coloring of G and so $\mathrm{spc}_2(G) = 2$. ■

If r and s are integers with $2 \leq r \leq s \leq 2^{r-1}$, then $\sqrt[r-1]{s} \leq 2$ and so $\lceil \sqrt[r-1]{s} \, \rceil = 2$. Consequently, Theorems 7.2.5 and 7.2.6 give rise to a formula of $\mathrm{spc}_2(K_{r,s})$ for all r and s with $2 \leq r \leq s$.

Theorem 7.2.7 *Let r and s be integers with $2 \leq r \leq s$. Then*

$$\mathrm{spc}_2(K_{r,s}) = \lceil \sqrt[r-1]{s} \, \rceil.$$

We plan to continue investigating $\mathrm{spc}_k(K_{r,s})$ for $k \geq 3$.

7.3 Topics for Further Study

We plan to continue investigating open questions on proper-connection and strong-connection numbers as well as on proper and strong connectivities in graphs. Furthermore, we also plan to explore related concepts and problems in this area of research. We present some of these in this section.

7.3.1 Edge Analogue of Proper Connectivity

Recall that an *edge-cut* of a graph G is a subset X of $E(G)$ such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a *minimum edge-cut* and this cardinality is the *edge-connectivity* of G, which is denoted by $\lambda(G)$. The trivial graph K_1 does not contain an edge-cut but we define $\lambda(K_1) = 0$. Therefore, $\lambda(G)$ is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Since the set of edges incident with any vertex of a graph G of order n is an edge-cut of G, it follows that $0 \leq \lambda(G) \leq \delta(G) \leq n-1$. A graph G is *k-edge-connected* for some positive integer k, if $\lambda(G) \geq k$, namely, G is k-edge-connected if the removal of fewer than k edges from G results in neither a disconnected graph nor a trivial graph. As with the connectivity of a graph, the edge connectivity of a graph is also a common measures of connectedness of the graph. As we mentioned in Chapter 1, there is an edge analogue of Whitney's Theorem; namely, a nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G.

Let G be a graph with edge connectivity $k \geq 1$. The *edge chromatic connectivity* $\lambda_\chi(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by k edge-disjoint proper $u - v$ paths. As expected, for a graph G with $\lambda_\chi(G) \geq 3$, there are *intermediate concepts* between the proper connection number $\mathrm{pc}(G)$ and the edge chromatic connectivity $\lambda_\chi(G)$ of the graph G. This is an analogue of $\mathrm{pc}_k(G)$ in terms of edge-disjoint proper paths. We plan to explore this new concept.

7.3.2 Proper-Tree Colorings

Let G be a nontrivial connected graph of order n. In a proper-path coloring of G, there is, for every two vertices u and v, a properly colored $u - v$ path in G.

For each fixed integer s with $3 \leq s \leq n$, an edge coloring c is an *s-proper-tree* *coloring* (or simply an *s-tree coloring*) of G if for every set S of s vertices of G, there is a properly colored tree in G that contains S. If k colors are used, then c is referred to as an *s-proper-tree k-coloring* or (or simply an *s-tree k-coloring*). The minimum k for which G has an s-tree k-coloring is called the *proper s-tree connection number* (or simply the *s-tree connection number*) $\mathrm{tc}_s(G)$ of G. An s-tree coloring using $\mathrm{tc}_s(G)$ colors is referred to as a *minimum s-tree coloring* of G. This concept was suggested by Chartrand in 2013 as an analogue to rainbow-tree colorings first studied in [12, 22, 32].

Problem 7.3.1 *Study 3-tree colorings of graphs.*

7.3.3 Proper Steiner Tree Colorings

Let G be a nontrivial connected graph of order n. In a strong proper-path coloring of G, there is, for every two vertices u and v, a properly colored $u-v$ geodesic in G. Once again, recall that *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G and that a $u-v$ path of length $d(u,v)$ is called a *$u-v$ geodesic*. Generalizations of distance and geodesics were introduced in [11]. For a nonempty set S of vertices in a connected graph G, the *Steiner distance* $d(S)$ of S is the minimum size of a connected subgraph of G containing S. Necessarily, each such subgraph is a tree and is called a *Steiner tree with respect to S* or a *Steiner S-tree* (see [23]). If $S=\{u,v\}$, then $d(S)=d(u,v)$ and a Steiner tree of S is a $u-v$ geodesic. If G has order n and $|S|=n$ (so $S=V(G)$), then $d(S)=n-1$ and every spanning tree of G is a Steiner tree for S. For example, let $S=\{u,v,x\}$ in the graph G of Figure 7.3. Here $d(S)=4$. There are several trees of size 4 containing S, one of which is the tree T of Figure 7.3.

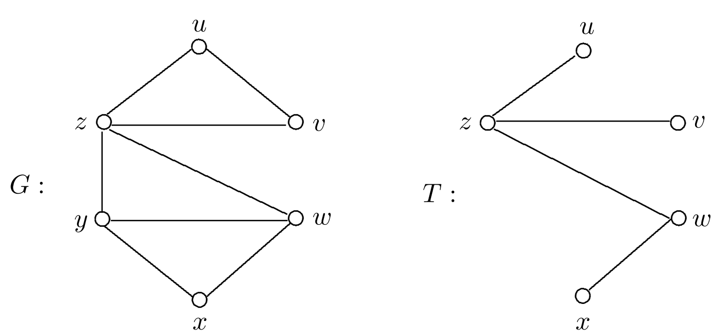

Figure 7.3: A graph G and a Steiner tree T

For a connected graph G of order $n \geq 2$ and a fixed integer s with $3 \leq s \leq n$, an edge coloring c is an *s-proper Steiner tree coloring* (or simply an *s-Steiner coloring*) of G if every set S of s vertices of G are connected by a proper Steiner tree in G. If k colors are used, then c is referred to as an *s-proper Steiner tree k-coloring* or (or simply an *s-Steiner k-coloring*). The minimum number of colors needed to produce an s-Steiner coloring of G is called the *proper s-Steiner tree connection number* (or simply the *s-Steiner connection number*) $\mathrm{sc}_s(G)$ of G. An s-Steiner coloring using $\mathrm{sc}_s(G)$ colors is referred to as a *minimum s-Steiner coloring* of G. This concept was suggested by Chartrand in 2013 as an analogue to the concept of tree-connectivity first studied in [12, 22, 32].

Problem 7.3.2 *Study s-Steiner colorings of graphs.*

7.3.4 Chromatic-Distance Problem

Let c be a proper-path coloring of a connected graph G. Then c may not be a strong proper-path coloring. However, for every two vertices u and v, there is a proper $u-v$ path of minimum length. For two vertices u and v, let $\mathrm{pd}(u, v)$ denote the length of a shortest properly colored $u - v$ path in G, which could be called the *chromatic distance between u and v*. Thus, $\mathrm{pd}(u, v) \geq d(u, v)$. This suggests looking at the proper eccentricity, proper radius and proper diameter of G.

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