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Bandwidth, Edgesum and Profile of Graphs

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BANDWIDTH, EDGESUM AND PROFILE OF GRAPHS

by

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For a graph $G = (V, E)$, each 1-1 mapping $f : V \longrightarrow \{1, 2, \ldots, |V|\}$ is called a *proper numbering* of G. The *bandwidth of graph* G is $\min \max |f(u) - f(v)|$, where the maximum is taken over each edge $uv \in E(G)$, and the minimum is over all proper numberings f. For graphs in general it is well known that the decision problem associated with finding bandwidth is NP-complete.

The *edgesum of* G is the number $\min \sum_{uv \in E} |f(u) - f(v)|$, where the minimum is taken over all proper numberings f. Determination of the edgesum for arbitrary graphs is known to be NP-complete.

For a proper numbering f, the *profile width* $w_f(v)$ *of a vertex* v in a graph G is the number $w_f(v) = \max_{x \in N[v]}(f(v) - f(x))$ where $N[v] = \{x \in V : x = v \text{ or } xv \in E\}$ is the closed neighborhood of v. The *profile of graph* G is $\min \sum_{v \in V(G)} w_f$ where the minimum is taken over all proper numberings f. It is known that determination of the profile for arbitrary graphs is NP-complete.

The graph parameters bandwidth, edgesum and profile are examined in detail. The results of an extensive survey as to the solved bandwidth, edgesum and profile problems for classes of graphs are presented.

Graphs are appropriate models for many computer applications. Several

application areas are discussed.

The exact values of the profile of the composition of a path with other graphs, a cycle with other graphs, a complete graph with other graphs and a complete bipartite graph with other graphs are given. The exact value of the bandwidth of a butterfly is established. A polynomial time approximation algorithm to find the edgesum and profile of a butterfly is presented. An approximation algorithm to find the profile of a hypercube is presented. Several tight bounds on the profile of the corona of two graphs are developed and the exact value of the profile of the tensor product of a path with a complete bipartite graph is provided.

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For Jung-Chih and Alice

whose love, patience and understanding make this possible.

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Finally, I would like to dedicate this dissertation to our heavenly Father for His guidance and help in my life.

Yung-Ling Lai

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CHAPTER I

INTRODUCTION

1.1 Definitions and Examples

For a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G.

Let $G = (V, E)$ be a graph on n vertices. A 1-1 mapping $f : V \longrightarrow \{1, 2, \ldots, n\}$ is called a *proper numbering* of G. The *bandwidth* $B_f(G)$ *of a proper numbering* f *of* G is the number

$$B_f(G) = max\{|f(u) - f(v)| : uv \in E\},$$

and the *bandwidth B(G) of G* is the number

$$B(G) = min\{B_f(G) : f \text{ is a proper numbering of } G\}.$$

A proper numbering f is called a *bandwidth numbering of* G if $B_f(G) = B(G)$.

For example, Figure 1 shows a bandwidth numbering for the graphs $P_4, C_5, K_{1,4}$ and $K_{2,3}$. In general, $B(P_n) = 1$, $B(C_n) = 2$, $B(K_{1,n}) = \lceil n/2 \rceil$ and $B(K_{m,n}) = m + \lceil n/2 \rceil - 1$ for $m \leq n$.

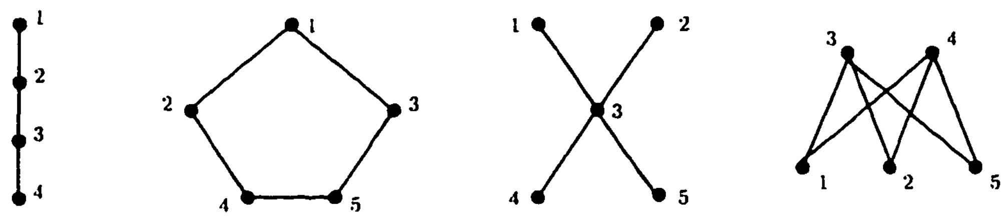


Figure 1. A Bandwidth Numbering for $P_4, C_5, K_{1,4}$ and $K_{2,3}$.

For a proper numbering f, the *edgesum* $s_f(G)$ *produced by* f is given by

$$s_f(G) = \sum_{uv \in E(G)} |f(u) - f(v)|.$$

The *edgesum* $s(G)$ of a graph G is defined by

$$s(G) = min\{s_f(G) : f \text{ is a proper numbering of } G\}.$$

A proper numbering that achieves the edgesum of a graph is called an *edgesum numbering*. The term here called *edgesum* means exactly the same as the terms *bandwidth sum* or *minimum sum* which have been used by a number of previous investigators.

For example,Figure 2 shows an edgesum numbering for the graphs $P_4, C_5, K_{1,4}$ and $K_{2,3}$. In general, $s(P_n) = n - 1$ and $s(C_n) = 2(n - 1)$. Yao and Wang[86]

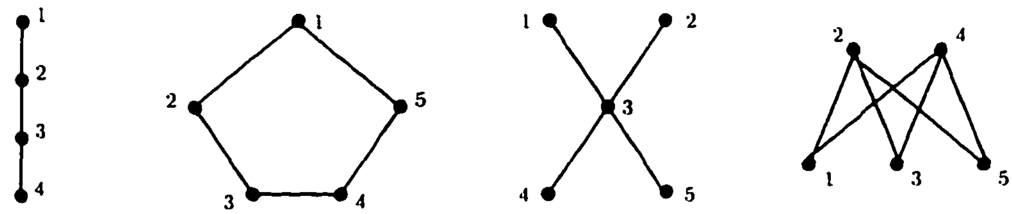


Figure 2. An Edgesum Numbering for $P_4, C_5, K_{1,4}$ and $K_{2,3}$.

showed that

$$s(K_{1,n}) = \begin{cases} \dfrac{n(n+2)}{4} & \text{if } n \equiv 0 \pmod 2 \\ \\ (\dfrac{n+1}{2})^2 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

and for $m \leq n$, Williams[82] verified that

$$s(K_{m,n}) = \begin{cases} \dfrac{3n^2m - m^3 + 6m^2n + 4m}{12} & \text{if } n - m \equiv 0 (\text{mod} 2) \\ \dfrac{3n^2m - m^3 + 6m^2n + m}{12} & \text{if } n - m \equiv 1 (\text{mod} 2). \end{cases}$$

Also, for a proper numbering f, the *profile width $w_f(v)$ of a vertex v* in a graph G is the number

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x))$$

where $N[v] = \{x \in V : x = v \text{ or } xv \in E\}$ is the closed neighborhood of v. The

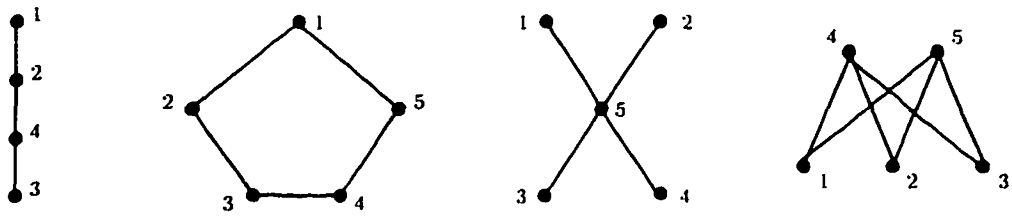


Figure 3. Profile Numberings for $P_4, C_5, K_{1,4}$ and $K_{2,3}$.

profile $P_f(G)$ *of a proper numbering* f *of* G is defined by

$$P_f(G) = \sum_{v \in V} w_f(v)$$

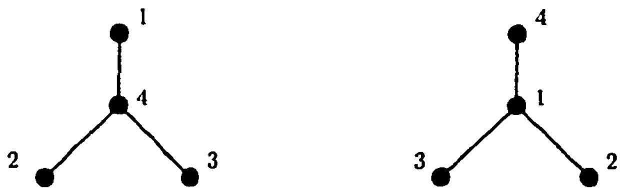
and the *profile* $P(G)$ *of* G is the number

$$P(G) = \min\{P_f(G) : f \text{ is a proper numbering of } G\}.$$

A proper numbering f is called a *profile numbering of* G if $P_f(G) = P(G)$.

For example,Figure 3 shows a profile numbering for the graphs P_4, C_5, $K_{1,4}$ and $K_{2,3}$. In general, $P(P_n) = n-1$, $P(C_n) = 2n-3$, $P(K_{1,n}) = n$ and for $m \leq n$, $P(K_{m,n}) = mn + m(m-1)/2$ (see Lin and Yuan[53]).

For a proper numbering f on a graph G of order n, the *complementary numbering* f' on G is defined by $f'(v) = n + 1 - f(v)$ for each vertex v in G. Then $B_{f'}(G) = B_f(G)$ and $s_{f'}(G) = s_f(G)$. Thus the complementary numbering of any bandwidth (or edgesum) numbering is also a bandwidth (or edgesum) numbering. However this relation does not hold for profile numberings, as is



$P_f(G) = 3$ $P_{f'}(G) = 6$

Figure 4. Complementary Numberings, Different Profile Results.

illustrated inFigure 4.

The decision problem corresponding to finding the bandwidth of an arbitrary graph was shown to be NP-complete by Papadimitriou[68]. Garey, Graham, Johnson and Knuth[24] showed that the problem is NP-complete even for trees of maximum degree 3. The decision problem associated with determining the edgesum for an arbitrary graph (sometimes call linear layout or the linear arrangement problem) was shown to be NP-complete by Garey, Johnson and Stockmeyer[26]; but the edgesum for trees is polynomal. In fact, Chung[14] provided the most efficient algorithm known to date to achieve an edgesum numbering for arbitrary trees. Also, Chung[15] found the edgesum for the complete binary trees. Lin and Yuan[53] indicated that the profile minimization problem of an arbitrary graph is equivalent to the interval graph completion problem, which was shown to be NP-complete by Garey and Johnson[25]. Kuo and Chang[42] provides a polynomial algorithm to achieve a profile numbering for an arbitrary tree of order n.

1.2 A Survey of Known Results

A large number of approximation algorithms for the bandwidths of a variety of graphs have been given. Approximations have been developed for general graphs (see Cheng[9], Cuthill and McKee[18], Gibbs, Poole, Jr. and Stockmeyer[27], GowriSankaran and Opatrny[30], Jeff[39], King[41], Luo[60], Quoc and O'Leary[69], Smyth[74], Wiegers and Monien[81]) and also for trees or caterpillars (see GowriSankaran, Miller and Opatrny[29], Odlyzko and Wilf[67], Haralambides, Makedon and Monien[31], and Smithline[73]).

For the profile of general graphs, Everstine[21] gives a comparison of three profile approximation algorithms given by Cuthill and McKee[18], Gibbs, Poole, Jr. and Stockmeyer[27] (their algorithms are called GPS) and Levy[51]. Among those three, GPS is exceptionally fast and the best able to reduce profile. Koo and Lee[43], Luo[60], Quoc and O'LearySnay[69], Snay[75], and Wiegers and Monien[81] also give approximation algorithms to reduce the profile of general graphs.

A number of upper and lower bounds are known which relate bandwidth to various graph invariants (see Bascuñán and Ruiz[2], Bascuñán, Ruiz and Slater[3], Brualdi and McDougal[5], Chinn, Chvátalová, Dewdney and Gibbs[10], de la Véga[19], Erdös, Hell and Winkler[22], Jeffs[39], Leung, Vornberger and Witthoff[50], Lin[52], and Miller[64]). Some upper and lower bounds are known for edge-sum problems (see Yao and Wang[86]); however, the exact values of the bandwidth, edgesum and profile have only been discovered for a few classes of graphs. These classes include paths, cycles, complete graphs, complements of complete graphs, and stars. The exact value of the bandwidth of some planar graphs (see Hochberg[36]), bandwidth of triangulated cycles (see Hochberg, McDiarmid and Saks[37]), bandwidth of the k-th power of paths (see Lee, Saba and Sun[49]) and certain graphs built from other graphs have also been determined. Surveys by Chung[13] and Chinn, Chvátalová, Dewdney and Gibbs[10] contain a number of results pertaining to solved bandwidth problems.

Tables 1 through 4 summarize the known exact results for bandwidth, edgesum, and profile on graphs built from other graphs.

The *corona* of graphs G_1 and G_2, on n_1 and n_2 vertices respectively, is denoted by $G_1 \wedge G_2$ and contains one copy of G_1 and n_1 copies of G_2. Each distinct vertex of G_1 is joined to every vertex of the corresponding copy of G_2.

The *Cartesian product* of two graphs G and H, denoted $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G and $v_1 = v_2$ or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

The *sum* $G_1 + G_2 + \ldots + G_k$ (also known as *join*) of k pairwise disjoint graphs for some $k \geq 2$ is the graph with vertex set $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k)$ and edge set $E(G) = \cup_{i=1}^{k} E(G_i) \cup \{(u, v) : u \in V(G_i), v \in V(G_j) \text{ and } i \neq j\}$.

The *composition* of two graphs G and H, denoted $G[H]$, is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

The *tensor product* of graphs G_1 and G_2, denoted $G_1(T_p)G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, v_1) is adjacent to (u_2, v_2) if $(u_1, u_2) \in E(G_1)$ and $(v_1, v_2) \in E(G_2)$.

The *strong product* of graphs G_1 and G_2, denoted $G_1(S_P)G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, v_1) is adjacent to (u_2, v_2) if one of the following holds: (a) $(x_1, x_2) \in E(G_1)$ and $(y_1, y_2) \in E(G_2)$, (b) $x_1 = x_2$ and $(y_1, y_2) \in E(G_2)$, or (c) $y_1 = y_2$ and $(x_1, x_2) \in E(G_1)$.

A graph G is *bipartite* if it is possible to partition $V(G)$ into two subsets V_1 and V_2 such that every element of $E(G)$ has one endvertex in V_1 and another endvertex in V_2. A *complete bipartite graph* $G = (V, E) = K_{m,n}$ is a bipartite graph with partite sets V_1, V_2 where $|V_1| = m$, $|V_2| = n$, $V_1 \cup V_2 = V$ and $E = \{(uv) : u \in V_1 \text{ and } v \in V_2\}$.

The d-dimensional hypercube has $n = 2^d$ vertices and $d2^{d-1}$ edges. Each vertex corresponds to a d-bit binary number, and two vertices are adjacent if and only if their binary number differs in only one bit.

A *tree* is an acyclic connected graph.

Table 1

Corona and Cartesian Product

	Corona	Cartesian Product
Bandwidth	Give bounds for two graphs, solved for complete graph with complete graph, cycle and path with K_1 and cycle with i copies of K_1 [11]	Solved for path with path and path with cycle [16], [17], [35]
		Solved for cycle with cycle [35], [48]
		Solved for complete graph with path, cycle and complete graph [35]
		Give bounds for two graphs and k graphs [16], [17]
Edgesum	Give bounds for two graphs, solved for path with path and complete graphs with K_1 [85]	Solved for path with path and cycle with cycle [65]
		Solved for m complete graphs [55]
Profile		Solved for path with path and path with cycle [54]
		Solved for path with complete graph, cycle with complete graph and cycle with cycle [61]

Table 2

Sum and Composition

	Sum	Composition
Bandwidth	Solved for two graphs [57], [87]	Solved for complete graph with other graph [35], [58]
		Solved for certain graph powers [12]
	Solved for k graphs [45]	Solved for star with other graph [58]
		Solved for path and cycle with other graph [35]
Edgesum	Solved for k graphs [47]	Solved for complete graph with path and complete graph with cycle [58]
	Solved for certain graph sums [82]	Solved for path with path and path with cycle [56]
Profile	Solved for two graphs [53]	

Table 3

Tensor Product and Strong Product

	Tensor Product	Strong Product
Bandwidth	Solved for paths and cycles with complete graphs [83]	Solved for path with path [35], [48], [88]
	Solved for paths with complete bipartite graphs [84]	Solved for complete graph with path and complete graph with cycle [35]
	Solved for path with path, path with cycle and cycle with cycle [46]	Solved for cycle with cycle and path with cycle [48], [88]
		Give upper bound for two graphs [88]
Edgesum	Solved for paths with complete bipartite graphs [84]	

Table 4

Complete Bipartite Graph, Hypercube and Tree

	Complete Bipartite Graph	Hypercube	Tree
Bandwidth	Solved [16]	Solved [33]	NP-complete [24]
Edgesum	Solved [82], [86]	Solved [34]	Solved [14], [28], [71]
Profile	Solved [53]		Solved [42]

CHAPTER II

APPLICATION AREAS

Bandwidth, edgesum and profile are useful parameters for many applications. In this chapter, we discuss some of the better known application areas.

2.1 Solving Linear Equations

The analysis of network systems with digital computers requires the solution of a large number of linear equations. The use of the finite element method also involves solving a large set of linear algebraic equations of the form $[A][X] = [B]$ where $[A]$ is a large symmetric sparse matrix. The matrix $[A]$ is said to be *sparse* if the number of nonzero entries is small in comparison with its total number of entries. Such matrices also appear very often as coefficient matrices of systems of differential equations in numerical analysis and physics.

Although storage availability and internal speed has been greatly increased in recent years, the solution of equations still takes a huge amount of space and time. To store such an $n \times n$ matrix would requre storage of all its n^2 entries, and most of them are 0s. When performing operations on these matrices, a large number of the computations only involve multiplying or adding 0s. If we can somehow focus on the nonzero entries, discarding the zero entries, that will save a large portion of the storage space and it may also save some computation time.

The focus on the nonzero entries of a matrix $[A]$ is easier if all these entries are grouped close to the main diagonals of $[A]$ and are said to be *banded.* A small band of $[A]$ allows the use of relatively little memory for storing $[A]$ since one can keep track of each nonzero entry by simply recording the row to which it belongs and the distance from the diagonal in the row. Among the solution methods of linear equations, the standard Gauss algorithm is well known and most other methods are only modifications of this basic algorithm to reduce the number of calculations. The small band of $[A]$ makes it possible to carry out only a small fraction of all the computations involved and still get the desired result since the remaining computations involve all zero entries and thus have predictable results.

Of course, the small band does not always occur in the matrices we deal with. When the matrix $[A]$ does not have a small band, we may permute the rows and columns of $[A]$ in order to obtain a matrix that has a smaller band. In other words, we may rearrange the columns and rows of $[A]$ to translate the original system of equations $[A][X] = [B]$ into an equivalent system $[A'][X'] = [B']$ where $[A']$ is a symmetric matix with a smaller band than $[A]$. Then we may work with $[A']$ rather than with $[A]$.

We know that there is a direct one-to-one relationship between symmetric matrices and graphs. The position of the nonzero entries of an $n \times n$ symmetric matrix can define an adjacency matrix of a graph G on n vertices (See Jeffs[39]).

That is, each row or column of $[A]$ can be represented by a vertex of G and the edge $uv \in E(G)$ if and only if the entry in column u, row v of the matrix $[A]$ is nonzero. For an $n \times n$ matrix $[A]$, define the *column height* P_j of column j $(1 \leq j \leq n)$ as

$$P_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \text{ for } 1 \leq i \leq j \\ j - i & i \text{ is the smallest integer such that } a_{ij} \neq 0 \end{cases}$$

Then the bandwidth of the matrix A denoted $B(A)$ is $\max_j P_j$ and the profile of the matrix denoted $P(A)$ is $\sum_{j=1}^{n} P_j$. This definition is equivalent to the bandwidth and profile definition on the corresponding graph.

The smallest possible maximum column height achievable (over all row and column permutations) for a given matrix $[A]$ is equivalent to the bandwidth of the corresponding graph G. And the execution time arising from solving the system of linear equations is proportional to the sum of the squares of the column heights. See King[41]. For storage, the bandwidth represents the maximum length of a column which must be stored, and the profile represents the total amount of storage needed. A number of papers address this application area, including Cheng[9], Everstine[21] and Jennings[40], King[41], Koo and Lee[43], Lin and Yuan[54], Luo[60], Miller[63], Quoc and O'Leary[69], Snay[75] and Veldhorst[79].

2.2 VLSI Layout

Given a set of modules, the VLSI layout problem consists of placing the modules on a two-dimensional grid in a non-overlapping manner and then wiring together the terminals on the different modules according to a given wiring specification in such a way that the wires do not interfere with each other. Thus, there are two stages in VLSI layout: placing the modules on a board, called *the placement problem*, and then after the modules are situated, wiring together the terminals of different modules that should be connected. This is called *the routing problem.*

The first formal model for VLSI layout was developed by Thompson [76], [77]. The model is consistent with the VLSI design rules established by Mead and Conway [62] and is also similar to the widely used Manhattan wiring model. There is a natural one-to-one correspondence between VLSI circuits and graphs. Assume that the graphs are of bounded degrees and that vertices require only a constant area of silicon. We can then model a VLSI circuit in a graph, with the vertices representing the modules and the edges of the graph representing the wires. The graph provides a simplified model of the circuit which can help us to obtain better solutions for the real-world model. No deterministic exact algorithms are known for placement and routing problems in the real world and the techniques used are based on more or less efficient heuristic algorithms.

The goal of the placement problem is to place the modules in such a way that the total wire-length is minimized. The wiring on a VLSI chip is restricted to following along grid tracks and is not allowed to overlap on the same track although vertical path segment may cross a horizontal path segment. In other words, routing is done by a grid of vertical and horizontal tracks or channels (see Nanan and Kurtzberg[66], Shing and Hu[72]). Let $d_{ij} = |x_i - x_j| + |y_i - y_j|$ be the distance function that measures the real wire-length between two pins (x_i, y_i) and (x_j, y_j), and let c_{ij} be the number of wires between the corresponding modules. Then we want to find the placement of the modules in such a way that it minimizes the function $\sum_{i \neq j} c_{ij} d_{ij}$. This problem is equivalent to finding a numbering on the corresponding graph G which minimizes $\sum_{uv \in E} |f(u) - f(v)|$ which is the edgesum of the graph G. This problem is also known as *Optimal Linear Ordering* (see Adolphson and Hu[1]).

The bandwidth measures the maximum distance between modules. Signals do not propagate instantaneously across wires, and the longer the wire, the longer the propagation delay. In pipelined or systolic systems, the effect of propagation delays is even more dramatic. The maximum delay determines the clockperiod, and hence the throughput, of the system. Therefore, minimizing the bandwidth is equivalent to minimizing the delay communication between modules which is an important parameter when solving the routing problem of VLSI layout. Several papers discuss this application areas including Adolphson and Hu[1], Bhatt and Leighton[4], Diaz[20], Miller[63], Shing and Hu[72] and Ullman[78].

2.3 Interconnection Networks

We can use a graph $G = (V, E)$ to represent an *interconnection network* (sometimes called a *parallel computation network*). Each vertex of G represents a different computer and $uv \in E$ if and only if there is a direct link between computers u and v. In the network, each computer receives part or all of the original input data and the master program will control all the computers and specify what computation need to be performed on each computer. At every time unit each computer can pass the results of the computation to one of its neighboring computers (the ones joined to it by an edge) and these neighbors will use those results as inputs for their own computations later.

If we have a problem P which needs to be solved on an interconnection network G when G is not available, but there is another interconnection network H available, we might want to "simulate" the program for G by a program for H which solves the same problem. The simulation of G by H is a way of describing how to use the program P as a guide so that H can accomplish the same task as G by assigning its computers the tasks assigned to those of G.

We seek a one-to-one mapping (embedding) f from G to H. Each computation of the program P at a computer x in G will now be replaced by the same computation at the corresponding computer $f(x)$ in H. Also, each communication of P between connected computers x_i and x_j will be replaced by $f(x_i)$ and $f(x_j)$ in H. The efficiency of the map f is measured by the time delay factor (dilation)

d since whereas x_i and x_j could communicate in some unit time t because they are neighbors, the vertices $f(x_i)$ and $f(x_j)$ require communication time dt where d is the distance between $f(x_i)$ and $f(x_j)$ in H. The bandwidth of G represents the worst possible delay (dilation) of the embedding from an interconnection network G to a linear array (path).

It is also natural to consider the *average* time delay caused by this embedding. Although the dilation might be large, the embedding might still be a good map if it has a small time delay on a large fraction of all the edges of G. We would calculate the average by summing the individual delays and dividing by the total number of edges in G. The edgesum of G represents the sum of each of the individual delays. Taking the ratio of the edgesum to the size of G, one might ask for the smallest possible average time delay over all possible mappings. This would be provided by an edgesum numbering. (See Miller[63].)

2.4 Constraint Satisfaction Problem

Graph bandwidth also has some applications to a class of search problems known as *constraint satisfaction problems* (CSPs) from the field of artificial intelligence. The discussion of CSPs in this section closely follows the problem description given in Zabih[89]. CSPs have an associated constraint graph. In the graph the vertices represent the variables of the search problem, and there is an edge between two vertices if there is a (nontrivial) constraint between those variables. Graph bandwidth provides a link between the syntactic structure of a constraint satisfaction problem and the complexity of the underlying search task.

A CSP has a set of variables and a domain of values. Every variable must be assigned a value. A CSP also consists of some constraints describing which assignments are compatible. Most interesting problems are binary CSPs where the constraints involve pairs of variables. Such a constraint consists of the simultaneously permitted assignments. The constraint graph associated with the binary CSP has edge set $E = \{(v_i, v_j)$: there is a contraint between v_i and $v_j\}$. The constraint graph hides a great deal of the information about the search problem, particularly the tightness of the constraints.

There are several reasons to believe that the bandwidth of the constraint graph of a CSP reflects the locality of the search problem. If the CSP has limited bandwidth, each vertex in the constraint graph can have no more than a fixed number of neighbors. This suggests that graphs with small bandwidth should be solvable by only dealing with a small subset of the variables at any instant. A stronger argument can be made on the basis of the claim that nearly decomposable search problems should be easy to solve. In particular, it should be possible to solve them efficiently by solving their subparts more or less independently and then by using divide and conquer (or dynamic programming) to put them together into a solution. The bandwidth of the constraint graph of a CSP serves as a measure of its decomposability. It turns out that any CSP of limited bandwidth can be solved in polynomial time by dynamic programming. The basic strategy is to find an ordering with minimal bandwidth, and then to use this ordering to solve the CSP.

When using backtrack search, the decision actually responsible for a failure can occur significantly before the failure itself is detected. The greater the number of intervening decisions, the worse the backtracking performs. Our goal is to prune the search tree as much as possible (see [70], [6]). The search trees resulting from small bandwidth orderings are significantly smaller than those resulting from orderings with greater bandwidth. If a CSP is searched in a fixed order using intelligent backtracking, the bandwidth of that ordering provides a bound on the amount of the search tree that intelligent backtracking will prune. It is possible to obtain restrictions on the nodes in the search tree, where these exceptions can occur in terms of the k-consistency (see [23]) of the original CSP. So the bandwidth of a search ordering can provide a measure of its quality. If small bandwidth orderings are really useful, then the large body of heuristics that has been developed by numerical analysts for finding such orderings may prove to be useful for solving CSPs. For more detail about this application, see Zabih[89].

2.5 The Visual Stimuli Application

There are some other applications of edgesum problems such as representing two-dimensional arrays on a sequential file in a computer. If one wishes to perform local calculations around a point in the array, as in the case of evaluating a differential operator, then $|f(u) - f(v)|$ measures the distance that must be traversed between u and v which is a local operation in the file. The edgesum measures the total cost of the operation. The following description of the problem is from Mitchison and Durbin[65], page 571.

> An analogous problem in computer hardware occurs when placing components of a multi-dimensional array processor on a lower dimensional chassis. Originally interest in this problem was raised by a biological question. The cortex of the brain of higher mammals can be regarded as a sheet of nerve cells. In the part of vortex devoted to vision, cells respond to certain visual stimuli, such as oriented bars of light against a dark background. A major discovery in recent years is that variables used to describe these stimuli, such as the location of an edge in space or its orientation, are mapped in a systematic manner on the cortex (see [38]). This mapping of more than two variables onto a two-dimensional sheet is accomplished by cycling through the values of variables to give striped patterns. This suggests that the nervous system may be trying to achieve as much continuity as possible in mapping these variables onto the cortex. The numbering of an array represents the most simplified mathematical model of this problem.

CHAPTER III

SOME CORRECTIONS OF PREVIOUS RESULTS

3.1 Correction of Previous Result on $P_m \times C_n$

Definition 1 Given two graphs G and H, the *cartesian product* of G and H denoted by $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if (a) u_1 is adjacent to u_2 in G and $v_1 = v_2$ or (b) $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

In [54], Lin and Yuan described the profile of $P_m \times P_n$ of $P_{m,n}$ (defined in [54]) and of $P_m \times C_n$. The result they provide for $P_m \times C_n$ with $2m \geq n$ is

$$P(P_m \times C_n) = \begin{cases} mn^2 - \dfrac{n}{12}(2n^2 - 3n + 16), & n \equiv 0 \pmod 2 \\[2ex] mn^2 - \dfrac{1}{12}(2n^3 - 3n^2 + 16n - 3), & n \equiv 1 \pmod 2. \end{cases}$$

The numbering pattern they provide to achieve this result is given in Figure 5. However, this numbering pattern is in error and is not actually optimal. A corrected pattern is shown in Figure 6. The proper numbering f of Figure 5 as given in [54] is $P_f(P_4 \times C_6) = 111$. However, the correct profile numbering as illustrated in Figure 6 gives the correct profile, which is $P(P_4 \times C_6) = 109$.

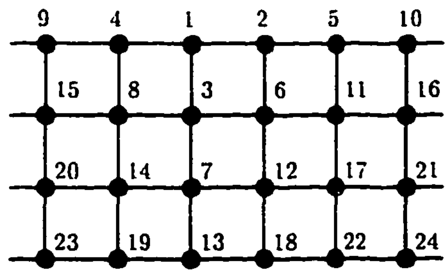


Figure 5. Lin and Yuan's [71] Numbering for Profile of $P_m \times C_n$.

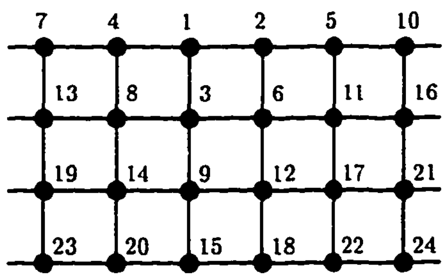


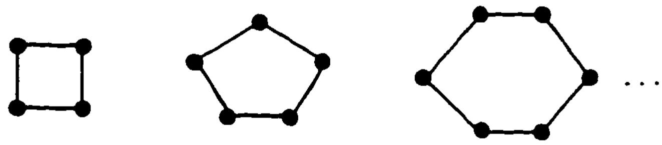
Figure 6. Corrected Numbering for Profile of $P_m \times C_n$ When $2m \geq n$.

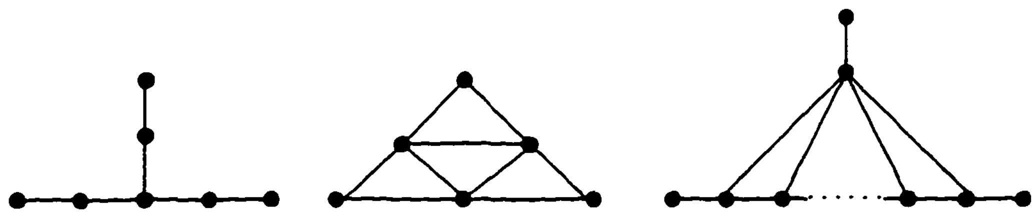
3.2 Interval Graphs and Profile

Definition 2 Let $J_1, J_2, \ldots, J_n$ be intervals on a line. An *interval graph* $G = (V, E)$ is a graph with vertex set $V = \{J_1, J_2, \ldots, J_n\}$ and $(J_i, J_k) \in E$ if and only if intervals J_i and J_k have a point in common.

Proposition 1 *(From Lovasz[59]) A graph G is an interval graph if and only if G does not contain any of the graphs shown in Figure 7 as an induced subgraph.*

The following three results are all from Lin and Yuan[53]. The numbers identifying these results are taken directly from [53] and do not represent identifying numbers in this dissertation.

(C) C_k for $k \geq 4$. i.e. 



(A) (B) (D)

Figure 7. The Induced Subgraphs Not Contained in Any Interval Graph.

Lemma 1.2 A graph G is an interval graph if and only if there exists a numbering f such that if $f(x) < f(y) < f(z)$ and $xz \in E(G)$ then $yz \in E(G)$.

Theorem 1.3 For any graph G, $P(G) \geq |E(G)|$; and $P(G) = |E(G)|$ if and only if G is an interval graph.

Theorem 1.4 For any graph G, $P(G)$ is the minimum number of edges of an interval supergraph of G.

Each of the results stated from [53] will be shown to be in error in this section.

Lemma 1 *Let $G^* = (V, E)$ be as shown in Figure 8. Then G^* does not contain any of the graphs shown in Figure 7 as an induced subgraph.*

Proof: Let H_1 be the graph shown in Figure 7 (A). Note that $|V(H_1)| = 7$ and $|E(H_1)| = 6$. But $|V(G^*)| = 8$, $|E(G^*)| = 13$ and there is no vertex in G^* of degree 7, so G^* does not contain H_1 as an induced subgraph.

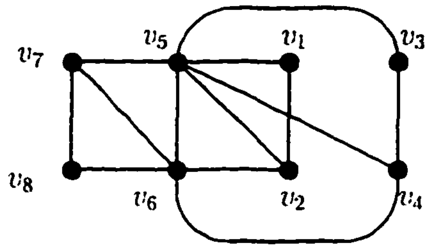


Figure 8. An Interval Graph G^*.

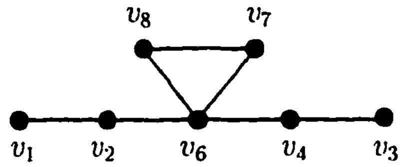


Figure 9. Subgraph Induced by Removing v_5 From G^*.

Let H_2 be the graph shown in Figure 7 (B). There are three vertices in H_2 of degree 4 but only two vertices in G^* of degree greater than 3. So G^* does not contain H_2 as an induced subgraph.

Assume there is an induced C_k for $k \geq 4$ in G. Suppose v_5 is not on C_k. Then the subgraph induced by removing v_5 from G^* is (V', E') where $V' = V - \{v_5\}$ and $E' = \{(v_1, v_2), (v_2, v_6), (v_3, v_4), (v_4, v_6), (v_6, v_7), (v_6, v_8)\}$ (as shown in Figure 9) which does not contain an induced C_k for $k \geq 4$. Suppose v_5 is on C_k. Note that v_5 has degree 6 and there are only two vertices adjacent to v_5 that can be on C_k. Since $k \geq 4$, v_8 must be on C_k. But v_8 is adjacent to v_6 and v_7, and $(v_6, v_7) \in E$. So G^* does not contain C_k for $k \geq 4$ as an induced subgraph.

In group (D) of Figure 7, we only need to show that G^* does not contain either H_3 or H_4 (shown in Figure 10) as an induced subgraph. For H_3, note that

Figure 10. H_3 and H_4.

$|V(H_3)| = 7$ and $|E(H_3)| = 8$, and v_5 is the only vertex in G with degree 6. But $\langle G - v_5\rangle \not\cong H_3$. So G^* does not contain H_3 as an induced subgraph. For H_4, note that $|V(H_4)| = 6$ and $|E(H_4)| = 6$. The maximum degree of the vertices in H_4 is 3. v_5 has degree 6 in G^*, there is no way to remove two vertices from G^* and cause v_5 to have degree less than 4. So v_5 must not be on the induced subgraph of H_4. There is only one vertex with degree greater than 2 in $\langle G^* - v_5\rangle$. So G does not contain H_4 as an induced subgraph. Hence, G^* does not contain any of the graphs shown in Figure 7 as an induced subgraph. $\square$

By Proposition 1 and Lemma 1 we have the following theorem.

Theorem 1 *The graph G^* of Figure 8 is an interval graph.*

Through exhaustive computer testing, we know that there does not exists a numbering f such that if $f(x) < f(y) < f(z)$ and $xz \in E(G)$ then $yx \in E(G)$. So Lemma 1.2 in [53] has been disproved. Also by exhaustive computer testing, we have $P(G^*) = 14$. In fact, in Figure 8, if we let $f(v_i) = i$ for $1 \leq i \leq 8$, then f is a profile numbering which achieves $P_f(G^*) = P(G^*) = 14$. Since $|E(G^*)| = 13$, we have the following theorem.

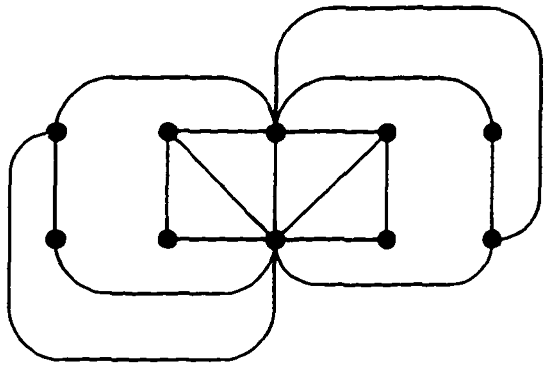


Figure 11. Another Interval Graph G'.

Theorem 2 *There exists an interval graph $G = (V, E)$ such that $P(G)$ exceeds $|E(G)|$.*

It can be shown in a similar way that the graph G' in Figure 11 is an interval graph and $|E(G')| = 17$. Through exhaustive computer testing, we know that $P(G') = 19$. The two examples G^* and G' not only disprove Theorem 1.3 in [53] but also tell us that the difference between the profile and the size of an interval graph need not be a fixed number.

Also by exhaustive computer testing, we obtain the following lemma.

Lemma 2 *If $G = K_{1,3} \times P_2$, then $P(G) = 14$.*

By Lemmas 1 and 2 we know that G^* is an interval supergraph of G and $P(G) = 14 > 13 = |E(G^*)|$. So Theorem 1.4 in [53] has been disproved.

Thus each of Lemma 1.2, Theorem 1.3 and Theorem 1.4 is now shown to be false. However, the first part of Theorem 1.3 is true and we state and prove this as our next theorem.

Theorem 3 *For any graph G, $P(G) \geq |E(G)|$.*

Proof: Assume there is a graph G with $P(G) < |E(G)|$. Let f be a profile numbering of G. Define $e_f(v) = |\{u : uv \in E(G) \text{ and } f(u) < f(v)\}|$. It is clear that for all $v \in V(G)$, $w_f(v) \geq e_f(v)$. Then we have $P(G) = P_f(G) = \sum_{v \in V(G)} w_f(v) \geq \sum_{v \in V(G)} e_f(v) = |E(G)|$ contradicting our assumption. □

The following theorem corrects the result in Lemma 1.3.

Theorem 4 *Given a graph $G = (V, E)$. Then $P(G) = |E(G)|$ if and only if there exist a numbering f such that if $f(x) < f(y) < f(z)$ and $xz \in E(G)$ then $yz \in E(G)$.*

Proof: Define $e_f(v) = |\{u : uv \in E(G) \text{ and } f(u) < f(v)\}|$. Note that $\sum_{v \in V} e_f(v) = |E(G)|$.

First, let f be a numbering such that if $f(x) < f(y) < f(z)$ and $xz \in E(G)$ then $yz \in E(G)$. We have $w_f(v) = e_f(v)$ for all $v \in V$. So $P_f(G) = \sum_{v \in V} w_f(v) = \sum_{v \in V} e_f(v) = |E(G)|$. Then we have $P(G) \leq P_f(G) = |E(G)|$. By Theorem 2 we know that $P(G) \geq |E(G)|$. Therefore $P(G) = |E(G)|$.

Now suppose $P(G) = |E(G)|$. Let f be a profile numbering. For all $v \in V(G)$, it is clear that $e_f(v) \leq w_f(v)$. Assume that there exists v such that $e_f(v) < w_f(v)$. Then $P_f(G) = \sum_{v \in V} w_f(v) > \sum_{v \in V} e_f(v) = |E(G)|$ which is a contradiction. Hence $w_f(v) = e_f(v)$ for all $v \in V$. Thus we have if $f(x) < f(y) < f(z)$ and $xz \in E(G)$ then $yz \in E(G)$. □

CHAPTER IV

PROFILE OF COMPOSITION

4.1 Definition and Examples

Definition 3 The *composition* $G[H]$ of a graph G with a graph H is the graph with vertex set $V(G) \times V(H)$ such that (u_1, v_1) is adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G or if $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

Figure 12 shows $P_3[P_4]$. The composition problems for a complete graph with a path, a complete graph with a cycle, a path with a path, and a path with a cycle have been solved for bandwidth and edgesum. In this chapter, we investigate the profile of the composition of paths, cycles, complete graphs and complete bipartite graphs with other graphs.

For a composition graph $G[H] = (V, E)$ with graphs G of order m and H of order n, we represent the vertex set as $V = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ where

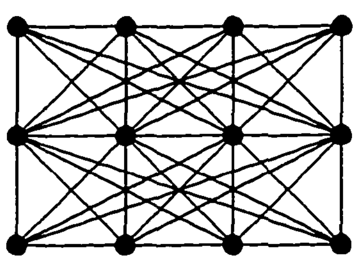


Figure 12. $P_3[P_4]$.

column j is denoted by Q_j $(1 \leq j \leq n)$, which represents a copy of G, and row i $(1 \leq i \leq m)$ is denoted by R_i, which represents a copy of H.

The following result from [54] is essential for the work done in this chapter.

Proposition 2 *(From Lin and Yuan[54]) Let G be a graph of order n. For any proper numbering f of G,*

$$P_f(G) = \sum_{i=1}^{n} |N(S_i)| \text{ where } S_t = \{v : v \in V(G) \text{ and } f(v) \leq t\}.$$

4.2 Paths With Other Graphs

In this section, we establish the profile of the composition of paths with other graphs.

Theorem 5 *Let $G = P_m[H]$ where H is a graph with n vertices. Then*

$$P(G) = \begin{cases} m - 1 & \text{for } n = 1 \\ P(H) & \text{for } m = 1 \\ P(H) + \dfrac{3n^2 - n}{2} & \text{for } m = 2 \text{ and } n > 1 \\ 2P(H) + \dfrac{mn(3n-1)}{2} - 2n^2 + n & \text{for } m \geq 3 \text{ and } n > 1. \end{cases}$$

Proof: For $n = 1$, $G = P_m$; so $P(G) = m - 1$. For $m = 1$, $G = P_1[H] = H$; so $P(G) = P(H)$. For $m = 2$, notice that every vertex in R_2 is adjacent to every vertex in R_1, once we number a vertex in both rows, all of the unnumbered vertices are in $N_f(S)$. By Proposition 2 we know that the profile numbering of G must completely number one row before numbering any vertex in the other row. This will minimize $\sum_{k=1}^{mn} |N(S_k)|$ which in turn minimizes $P_f(G)$. Without loss of generality, assume we completely number R_1 and then R_2. We want to number the vertices in R_1 in the order of a profile numbering (say f) of H. Since every vertex in R_2 is adjacent to the vertex $f^{-1}(1)$, it does not matter how we number the vertices in R_2. Hence, $P(G) = P(H) + \sum_{i=0}^{n-1}(n+i) = P(H) + (3n^2 - n)/2$.

For $m \geq 3$, we first show that $P(G) \leq 2P(H) + mn(3n-1)/2 - 2n^2 + n$. Assume that g is a profile numbering of H. Consider a numbering f such that

$$f(v_{i,j}) = \begin{cases} g(v_j) + (i-1)n & \text{for } 1 \leq i \leq m-2,\ 1 \leq j \leq n \\ g(v_j) + (m-1)n & \text{for } i = m-1,\ 1 \leq j \leq n \\ g(v_j) + (m-2)n & \text{for } i = m,\ 1 \leq j \leq n. \end{cases}$$

Then

$$\begin{aligned} P_f(G) &= P(H) + (m-3)\sum_{i=0}^{n-1}(n+i) + P(H) + \sum_{i=0}^{n-1}(2n+i) \\ &= 2P(H) + \frac{mn(3n-1)}{2} - 2n^2 + n. \end{aligned}$$

Let h be a profile numbering of G. Now, assume that $P_h(G) < 2P(H) + mn(3n-1)/2 - 2n^2 + n$. For the same reason as in the case with $m = 2$, h must completely number a row before starting another row. Furthermore, h must begin with one of the end rows (say R_1); otherwise $P_h(G) \geq P_f(G) + n$. We claim that h must number the rows in the order $R_1, R_2, \ldots, R_{m-2}, R_m, R_{m-1}$.

We prove the claim by contradiction. Assume that this pattern is not followed. So the first violation by h is at R_p to R_q, where $q = p + r$ and $r > 1$ and $p < m - 2$. Then if $q < m$,

$$\begin{aligned}\sum_{i=p+1}^{q+1} \sum_{j=1}^{n} w_h(v_{i,j}) &\geq P(H) + (5n^2 - n) + \frac{(r-2)(3n^2 - n)}{2} \\ &> \frac{(r+1)(3n^2 - n)}{2} \\ &= \sum_{i=p+1}^{q+1} \sum_{j=1}^{n} w_f(v_{i,j}).\end{aligned}$$

If $q = m$, then

$$\begin{aligned}\sum_{i=p+1}^{q} \sum_{j=1}^{n} w_h(v_{i,j}) &\geq P(H) + (5n^2 - n) + \frac{(r-3)(3n^2 - n)}{2} \\ &> P(H) + \frac{5n^2 - n}{2} + \frac{(r-2)(3n^2 - n)}{2} \\ &= \sum_{i=p+1}^{q} \sum_{j=1}^{n} w_f(v_{i,j}).\end{aligned}$$

So, the claim is proved and h must number the rows in the same order as f.

Since within each row, f numbers the same way as g, which is a profile numbering of H, it follows that

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} w_f(v_{i,j}) &= 2P(H) + \frac{mn(3n-1)}{2} - 2n^2 + n \\ &< \sum_{j=1}^{n} w_h(v_{1,j}) + \sum_{j=1}^{n} w_h(v_{m,j}) + \frac{mn(3n-1)}{2} - 2n^2 + n \\ &= P_h(G),\end{aligned}$$

which implies that $P_f(G) < P_h(G)$, which is a contradiction. □

Corollary 1 *For $m \geq 3$,*

$$P(G) = \begin{cases} \frac{mn(3n-1)}{2} - 2n^2 + 3n - 2 & \textit{for } G = P_m[P_n] \\ \frac{mn(3n-1)}{2} - 2n^2 + 5n - 6 & \textit{for } G = P_m[C_n] \\ \frac{mn(3n-1)}{2} - n^2 & \textit{for } G = P_m[K_n]. \end{cases}$$

4.3 Cycles and Complete Graphs With Other Graphs

In this section, we establish the profile of the composition of cycles with other graphs and the profile of the composition of complete graphs with other graphs.

Theorem 6 *Let $G = C_m[H]$ where H is a graph with n vertices. Then*

$$P(G) = P(H) + \frac{n(5mn - 7n - m + 1)}{2}.$$

Proof: Let g be a profile numbering of H. Define a numbering $f(v_{i,j}) = (i-1)n + g(v_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then using an argument similar to that in the proof of the previous theorem, we see that a profile numbering is produced, namely,

$$\begin{aligned} P_f(G) &= P(H) + \frac{(m-2)n(3n-1)}{2} + \frac{n(2mn - n - 1)}{2} \\ &= P(H) + \frac{n(5mn - 7n - m + 1)}{2}. \end{aligned}$$

□

Corollary 2 *For $m \geq 3$ and $n \geq 3$,*

$$P(G) = \begin{cases} \dfrac{n(5mn - 7n - m + 3)}{2} - 1 & \textit{for } G = C_m[P_n] \\ \dfrac{n(5mn - 7n - m + 5)}{2} - 3 & \textit{for } G = C_m[C_n] \\ \dfrac{n(5mn - 6n - m)}{2} & \textit{for } G = C_m[K_n]. \end{cases}$$

Theorem 7 *For $m \geq 1$, let $G = K_m[H]$ where H is a graph with n vertices. Then*

$$P(G) = P(H) + \frac{(mn+n-1)(mn-n)}{2}.$$

Proof: In the composition of a complete graph with another graph, every vertex in R_i is adjacent to all other vertices which are in rows other than R_i. Hence once we number a vertex 1, we know

$$\sum_{i=n+1}^{mn} w_f(v_i) = \frac{(mn+n-1)(mn-n)}{2}.$$

Since $\min \sum_{i=1}^{n} w_f(v_i) = P(H)$, we then have $P(G) \geq P(H)+(mn+n-1)(mn-n)/2$. And $P(H)+(mn+n-1)(mn-n)/2$ is achievable by numbering R_1 with $1, \ldots, n$; R_2 with $n+1, \ldots, 2n$ etc. within each row follow a profile numbering of H. So, $P(G) = P(H) + (mn+n-1)(mn-n)/2$. □

Corollary 3 *For $m \geq 1$ and $n \geq 1$,*

$$P(K_m[K_n]) = \frac{mn(mn-1)}{2}.$$

4.4 Complete Bipartite Graphs With Other Graphs

In this section, we present the profile of the composition of a complete bipartite graph with any arbitrary graph of order l.

Theorem 8 *Let $G = K_{m,n}[H]$ where $m \leq n$ and H is a graph of order l. Then*

$$P(G) = mnl^2 + \frac{ml(ml-1)}{2} + nP(H).$$

Proof: Assume the two partite sets of vertices in $K_{m,n}$ are $\{v_1, v_2, \ldots, v_n\}$ and $\{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\}$. Also assume that g is a profile numbering of H. Consider a numbering f such that $f(v_{i,j}) = g(v_j) + (i-1)l$ for $1 \leq i \leq n+m$. Then

$$\begin{aligned} P_f(G) &= nP(H) + \sum_{i=nl+1}^{(m+n)l} (i-1) \\ &= mnl^2 + \frac{ml(ml-1)}{2} + nP(H). \end{aligned}$$

Now assume that h is a profile numbering of G. For $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$, every vertex in $\bigcup_{i=1}^{n} R_i$ is adjacent to every vertex in $\bigcup_{j=n+1}^{n+m} R_j$ and for $m \leq n$, the vertex $v = h^{-1}(1)$ should be in $\bigcup_{i=1}^{n} R_i$. So

$$\sum_{i=n+1}^{m+n} \sum_{j=1}^{l} w_h(v_{i,j}) = mnl^2 + \frac{ml(ml-1)}{2}.$$

Since $\min \sum_{i=1}^{n} \sum_{j=1}^{l} w_h(v_{i,j}) = nP(H)$, we have $P_h(G) = mnl^2 + ml(ml-1)/2 + nP(H)$. □

A direct application of Theorem 8 leads to Corollary 4.

Corollary 4

$$P(G) = \begin{cases} mnl^2 + nl - n + \dfrac{ml(ml-1)}{2} & \textit{for } G = K_{m,n}[P_l] \\ mnl^2 + 2nl - 3n + \dfrac{ml(ml-1)}{2} & \textit{for } G = K_{m,n}[C_l] \\ mnl^2 + \dfrac{nl(l-1)}{2} + \dfrac{ml(ml-1)}{2} & \textit{for } G = K_{m,n}[K_l]. \end{cases}$$

The profile of the composition of a star with any other graph also follows directly from Theorem 8.

Corollary 5 *For $n \geq 1$, let H be a graph with l vertices. Then*

$$P(K_{1,n}[H]) = nl^2 + \frac{l(l-1)}{2} + nP(H).$$

Corollary 6 *For $n \geq 1$ and $l \geq 1$,*

$$P(G) = \begin{cases} nl^2 + nl - n + \dfrac{l(l-1)}{2} & \textit{for } G = K_{1,n}[P_l] \\ nl^2 + 2nl - 3n + \dfrac{l(l-1)}{2} & \textit{for } G = K_{1,n}[C_l] \\ nl^2 + \dfrac{l(l-1)(n+1)}{2} & \textit{for } G = K_{1,n}[K_l]. \end{cases}$$

CHAPTER V

NETWORK ARCHITECTURES

5.1 Introduction

Linear arrays (paths), rings (cycles), completely connected (K_n), binary trees, stars ($K_{1,n}$), 2-dimensional meshes ($P_n \times P_m$), 2-dimensional tori ($C_m \times C_n$), hypercubes and butterflies are some of the most commonly used network architectures. It is known that for the linear arrays $B(P_n) = 1$, $s(P_n) = n - 1$ and $P(P_n) = n - 1$. For rings $B(C_n) = 2$, $s(C_n) = 2n - 2$ and $P(C_n) = 2n - 3$. For completely connected graphs $B(K_n) = n - 1$, $s(K_n) = n(n-1)/2$ and $P(K_n) = n(n-1)/2$. For binary trees the bandwidth problem is NP-complete. Chung[14] gives a solution for the edgesum of a complete binary tree. There are several polynomial algorithms to find the edgesum for arbitrary trees (see [13], [28] and [71]). In fact, Chung[13] provided an $O(n^{1.6})$ algorithm, which is the most efficient known, to achieve optimal numberings for arbitrary trees. Also, an $O(n^{1.72})$ algorithm which gives the profile for arbitrary trees was provided by Chang[42]. For stars, $B(K_{1,n}) = \lceil n/2 \rceil$, $s(K_{1,n}) = (\lceil n/2 \rceil(\lceil n/2 \rceil + 1) + \lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1))/2$ and $P(K_{1,n}) = n$. For 2-dimensional meshes, with $m \leq n$, $B(P_m \times P_n) = m$. Mitchison and Durbin[65] provides a polynomial algorithm to solve the edgesum problem of a 2-dimensional mesh and Lin and Yuan[54] gives $P(P_m \times P_n) = m^2n - m(2m^2 - 3m + 7)/6$. For the 2-dimensional torus,

$$B(C_m \times C_n) = \begin{cases} 2m & \text{for } 3 \leq m \leq n \\ 2m - 1 & \text{for } 3 \leq m = n. \end{cases}$$

The same numbering given by Mitchison and Durbin[65] for the 2-dimensional mesh also achieves the edgesum of the 2-dimensional torus. Furthermore, Mai[61] provides the profile $P(C_m \times C_n) = \lceil 2m - 2n/3 + 1/2)n^2 - 16n/3 + 3 - \min\{1, m - n\}\rceil$ with $m \geq n \geq 3$. There is a polynomial algorithm for the bandwidth of a hypercube given by Harper[33] and there is a polynomial algorithm for the edgesum of a hypercube (see Harper[34]), but there is no work done for the profile of a hypercube. Also, there is no work done on butterfly architectures.

In this chapter, we investigate the bandwidth, edgesum and profile of butterflies and the profile of hypercubes in an attempt to complete the work on the most commonly used network architectures.

5.2 Butterfly Architecture

A d-dimensional butterfly has $(d + 1)2^d$ vertices and $d2^{d+1}$ edges. The vertices correspond to pairs (l, w) where l is the *level* or *dimension* of the vertex $(0 \leq l \leq d)$ and w is an d-bit binary number that denotes the position (column) of the vertex. Two vertices (l_1, w_1) and (l_2, w_2) are adjacent if and only if $l_1 = l_2 + 1$

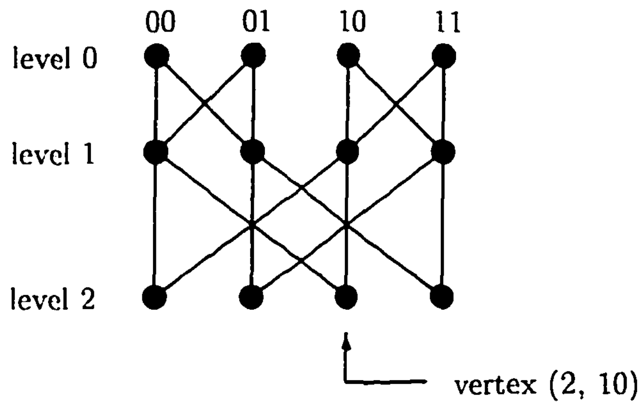


Figure 13. A 2-dimensional Butterfly.

and either (1) $w_1 = w_2$ or (2) w_1 and w_2 differ in only the l_1th bit. If $w_1 = w_2$, the edge is said to be a *straight edge*. Otherwise, the edge is called a *cross edge*. For example, Figure 13 shows a 2-dimensional butterfly.

In this section, we discuss the bandwidth, edgesum and profile of the butterfly architectures.

For $S \subseteq V(G)$, ∂S denotes the set of vertices in S adjacent to those in $V(G) - S$ and $N(S)$ denotes the set of vertices in $V(G) - S$ adjacent to those in S. Let $p(G)$ denote $|V(G)|$ and $D(G)$ be the diameter of G. For $S \subseteq V$, $\overline{S}$ denotes $V(G) - S$. For a given numbering f, let $S_t = \{v \in V(G) : f(v) \leq t\}$. The following propositions are used in the proofs that follow.

Proposition 3 *(From Harper[33]) If G is a connected graph, then*

$$B(G) \geq \max_{k} \min_{|S|=k} |\partial S|.$$

Proposition 4 *(From Lai and Williams[46]) Let f be a bandwidth numbering of*

G. Then for $t \in \{1, 2, \ldots, p(G)\}$, $|\partial_f S_t| \leq B(G)$.

Proposition 5 *(From Lai and Williams[46]) Let f be a bandwidth numbering of G. Then for $t \in \{1, 2, \ldots, p(G)\}$, $|N_f(S_t)| = |\partial_f \overline{S_t}| \leq B(G)$.*

Given a graph $G = (V, E)$, define a *cut* of G as a subset of E which when removed from E disconnects G into two connected subgraphs H_1 and H_2. Given a cut R of G, we define $t(R) = \{v \in V : v \text{ is an endvertex of an edge in } R\}$.

Lemma 3 *Let $G = (V, E)$ be a graph with n vertices. If every cut R of G that disconnects G into two subgraphs H_1 and H_2 for which $|V(H_1)| = \lceil n/2 \rceil$ and $|V(H_2)| = \lfloor n/2 \rfloor$ has the property that $|t(R)| \geq m$, then $B(G) \geq m/2$.*

Proof: Let f be a bandwidth numbering of G and let $V(H_1) = S_{\lceil n/2 \rceil}$. For every cut R of G, we have $|t(R)| \geq m$, and $|t(R)| = |N_f(S_{\lceil n/2 \rceil})| + |\partial_f(S_{\lceil n/2 \rceil})| \geq m$. So either $|N_f(S_{\lceil n/2 \rceil})| \geq m/2$ or $|\partial_f(S_{\lceil n/2 \rceil})| \geq m/2$. By Propositions 4 and 5 we have $B(G) = B_f(G) \geq m/2$. □

Theorem 9 *If G is a d-dimensional butterfly, then $B(G) = 2^d$.*

Proof: By [44] we know that the bisection width of a d-dimensional butterfly is 2^d. In fact, every cut R of a d-dimensional butterfly which disconnects G into two subgraphs of equal order has the property that $|t(R)| \geq 2^{d+1}$. Then by Lemma 3, we have $B(G) \geq 2^d$.

We number the left-most half of the graph sequentially row by row left-to-right until the last row where we number the whole row. Then we reverse the

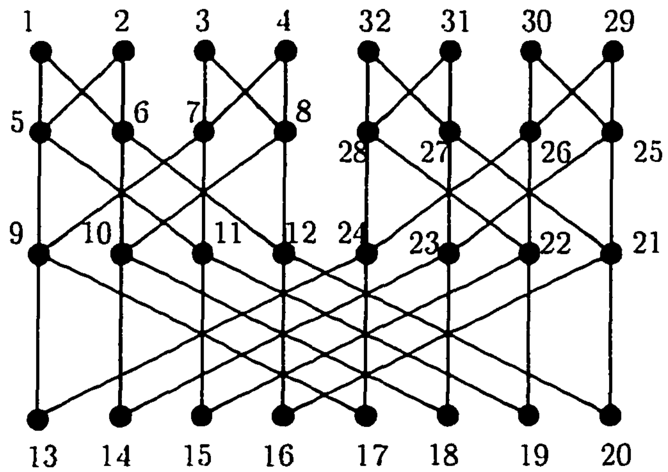


Figure 14. A Bandwidth Numbering of 3-dimensional Butterfly.

order in the second right-most half of the graph. This numbering will achieve the bandwidth 2^d (see Figure 14). So $B(G) = 2^d$. □

Proposition 6 *(From Harper[34]) If f is a proper numbering of a graph G on n vertices, then*

$$\sum_{uv \in G} |f(u) - f(v)| = \sum_{t=1}^{n} e(S_t),$$

where $e(S_t)$ is the number of edges with one endvertex in S_t and the other endvertex in $V(G) - S_t$.

The following algorithm appears to provide an edgesum numbering for butterflies of dimension d and order n.

Algorithm 1 *Edgesum of butterflies.*

Input: A d-dimensional butterfly of $n = (d+1)2^d$ vertices named $1, 2, \ldots, n$ in a consecutive order from level 0, row by row.

Output: A believed edgesum numbering of the given graph.

Variables:

V is an array which holds the numbering. The index in V indicates the name of the vertex.

NbList is an array which holds $N(S_t)$ where S_t is the set of vertices that have been numbered $1, 2, \ldots, t$.

cur is the lowest number which is not yet assigned.

Methods:

1. Initialize $V[i]$ to zero for $1 \leq i \leq n$.

2. Set $V[1] = 1$, $cur = 2$.

3. Put $N(S_{cur-1})$ into *NbList*.

4. In *NbList*, find a vertex j such that $V[j] = cur$ will give a minimum value of $e(S_{cur})$.

5. Assign *cur* to $V[j]$ and increase *cur* by one.

6. Loop from step 3 to step 5 until all the vertices have been numbered.

End of Algorithm 1

To analyze the time complexity of Algorithm 1 we note that step 1 takes $O(n)$, step 2 takes $O(1)$ and step 3 takes no more than $O(n^2)$, step 4 takes no more than $O(n^3)$, step 5 takes $O(1)$ and Step 3,4,5 loop $O(n)$ times, so the total time complexity for Algorithm 1 is no more than $O(n^4)$ which, of course, is polynomial. A version of the implementation of Algorithm 1 in C++ code is given in Appendix

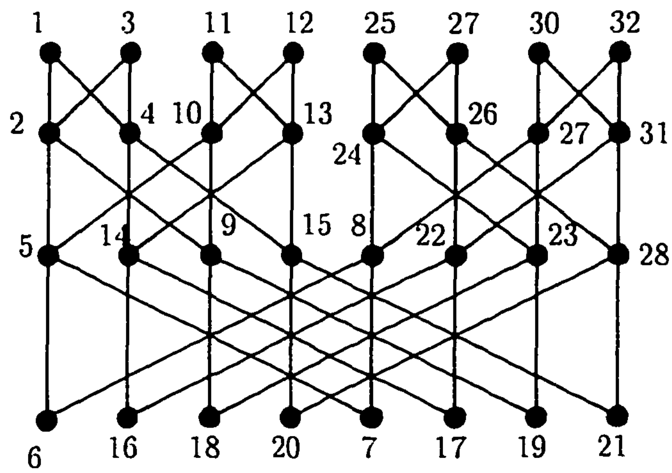


Figure 15. Algorithm 1 Applied to a 3-dimensional Butterfly, $s_f(G) = 202$.

A. Figure 15 shows the numbering of Algorithm 1 applied to a 3-dimensional butterfly.

Through exhaustive computer testing it has been established that Algorithm 1 provides an edgesum numbering for all butterflies with $n \leq 12$. We have tried, by computer testing, a number of reasonable techniques for $32 \leq n \leq 80$, and this algorithm continues to provide the best numbering. Therefore it is believed that this algorithm will provide an edgesum numbering for all butterflies.

The following algorithm appears to provide a profile numbering for butterflies of dimension d and order n.

Algorithm 2 *Profile of butterflies.*

Input: A d-dimensional butterfly of $n = (d+1)2^d$ vertices named $1, 2, \ldots, n$ in a consecutive order from level 0, row by row.

Output: A believed profile numbering of the given graph.

Variables:

V is an array which holds the numbering. The index in V indicates the name of the vertex.

cur is the lowest number which is not yet assigned.

Methods:

1. Initialize $V[i]$ to zero for $1 \leq i \leq n$.

2. Set $V[1] = 1$, $cur = 2$.

3. Among all vertices which have not been numbered, find a vertex j such that setting $V[j] = cur$ will result in a minimum value of $|N(S_{cur})|$.

4. Assign *cur* to $V[j]$ and increase *cur* by 1.

5. Loop from step 3 to step 4 until all the vertices have been numbered.

End of Algorithm 2

The time complexity analysis of Algorithm 2 shows that step 1 takes $O(n)$ time, step 2 takes $O(1)$, step 3 takes no more than $O(n^3)$, and step 4 takes $O(1)$. steps 3 and 4 loop $O(n)$ times, so the total time complexity for Algorithm 1 is no more than $O(n^4)$ which is polynomial. A version of the implementation of Algorithm 2 in C++ code is in Appendex B. Figure 16 shows the numbering of Algorithm 2 applied to a 3-dimensional butterfly.

Through exhaustive computer testing it has been established that Algorithm 2 provides a profile numbering for all butterflies with $n \leq 12$. Again, we have tested, by computer, a number of reasonable techniques for $32 \leq n \leq 80$, and

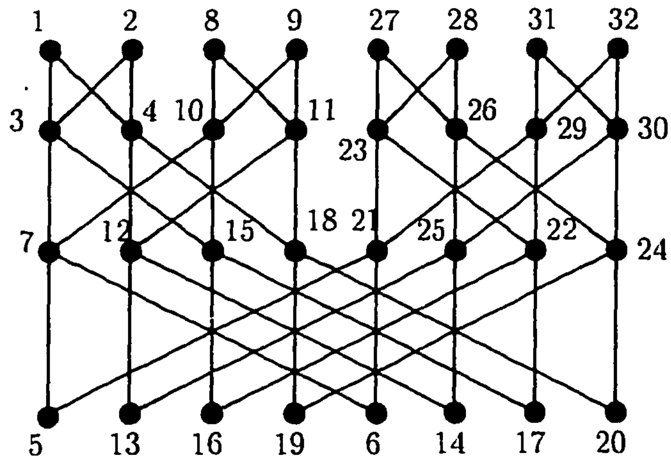


Figure 16. Algorithm 2 Applied to a 3-dimensional Butterfly, $P_f(G) = 125$.

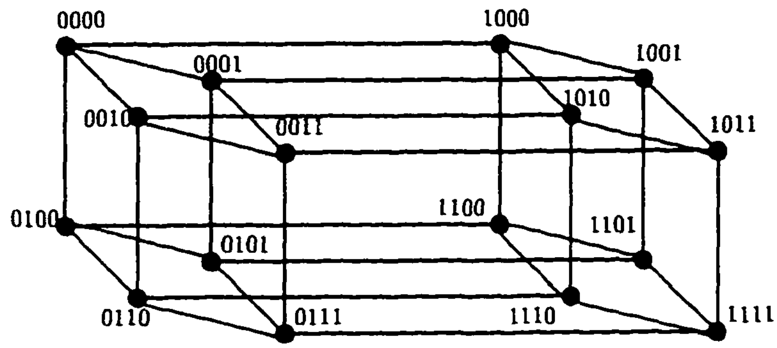


Figure 17. A 4-dimensional Hypercube.

this algorithm still provides the best numbering. It is believed that this algorithm will provide a profile numbering for all butterflies.

5.3 Profiles of Hypercubes

Recall that a d-dimensional hypercube has $n = 2^d$ vertices and $d2^{d-1}$ edges. Each vertex corresponds to a d-bit binary number, and two vertices are adjacent if and only if their binary number differs in only one bit. Figure 17 show a 4-dimensional hypercube.

Polynomial numbering algorithms to provide the bandwidth and edgesum of the hypercube are known. For the profile of a hypercube, the following algorithm provides a numbering for a hypercube of order n which has dimension $d = \log n$. Through exhaustive computer testing, we have established that this algorithm provides a profile numbering for all hypercubes with $d \leq 4$. It is believed that it will provide a profile numbering for all hypercubes.

Algorithm 3 *Profile of hypercubes.*

Input: A hypercube with n vertices named $0, 1, 2, \ldots, n-1$ according to the natural numbering which uses the decimal equivalent of the binary number.

Output: A numbering which provides a low profile.

Variables:

V is an array which holds the numbering. The index in V indicates the name of the vertex.

cur is the lowest number which is not yet assigned.

pre is the lowest number of a vertex whose neighbors are not all assigned a number yet.

Methods:

1. Initialize $V[i]$ to zero for $1 \leq i \leq n$.
2. Set $V[0] = 1$ and $cur = 2$.
3. Set $pre = 0$.
4. Number all unnumbered vertices which are adjacent to *pre* as *cur* and

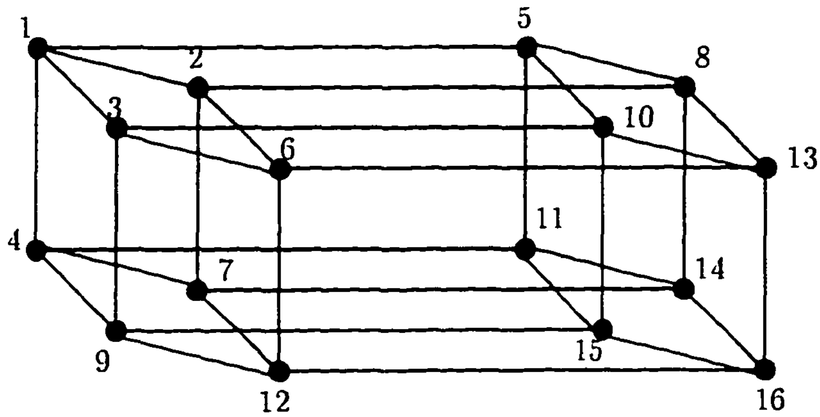


Figure 18. Algorithm 3 Applied to a 4-dimensional Hypercube, $P_f(G) = 75$.

increase *cur* by 1 after numbering each vertex.

5. Increase *pre* by 1.

6. Repeat step 4 and 5 until all the vertices are numbered.

End of Algorithm 3

The time complexity analysis of Algorithm 3 shows that step 1 takes $O(n)$, step 2 to step 3 takes $O(1)$ time, step 4 takes $O(n)$ and step 5 takes $O(1)$ and steps 4 and 5 loop $O(n)$ times, so the total time complexity for Algorithm 3 is $O(n^2)$ which is polynomial. Figure 18 shows the numbering of Algorithm 3 applied to a 4-dimensional hypercube.

Since none of Algorithms 1, 2 and 3 have been mathematically proved optimal except for graphs of small order, they must be regarded as providing approximation numberings. Also, their result provides an upper bound on the exact edgesum and profile.

CHAPTER VI

PROFILE OF CORONA

6.1 Definition and Examples

The corona of two graphs was first defined by Harary [32].

Definition 4 Given graphs G_1 and G_2 on n_1 and n_2 vertices respectively, the *corona* $G = G_1 \wedge G_2$ *of* G_1 *with respect to* G_2 has $V(G) = V(G_1) \cup \{n_1$ distinct copies of $V(G_2)$ denoted $V(G_2)_1, V(G_2)_2, \ldots, V(G_2)_{n_1}\}$, and $E(G) = E(G_1) \cup \{n_1$ distinct copies of $E(G_2)\} \cup \{(u_i, v) : u_i \in V(G_1),\ v \in V(G_2)_i\}$.

Figure 19 shows $P_3 \wedge P_4$. Chinn, Lin and Yuan[11] determined a tight upper bound for the bandwidth of the corona of two graphs and gave solutions for special cases. Williams[85] established a tight upper bound and a tight lower bound for the edgesum of the corona of two graphs and gave solutions for some special cases. There is no work done for the profile of the corona of two graphs. This chapter determine a tight upper bound and a tight lower bound for the profile of the corona of two graphs. Also, the exact values are determined for the profile

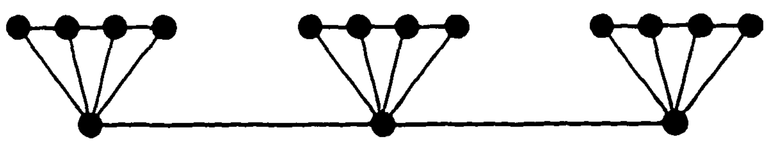


Figure 19. $P_3 \wedge P_4$.

of the corona of several families of graphs.

6.2 Tight Bounds for General Cases

In this section we give an upper bound and a lower bound for the profile of the corona of any two graphs and show some special cases to achieve the upper bound and the lower bound.

Recall by Proposition 2 that for a graph G of order n and proper numbering f of G, $P_f(G) = \sum_{i=1}^{n} |N_f(S_i)|$, where $S_i = \{v : f(v) \leq i\}$.

Lemma 4 *Given a graph $G = (V(G), E(G))$, let $H = (V(H), E(H))$ such that $V(H) \cap V(G) = \emptyset$ and $u \in V(G)$. Let $G' = (V', E')$ where $V' = V(G) \cup V(H)$ and $E' = E(G) \cup E(H) \cup \{uv : v \in V(H)\}$. Then there exists a profile numbering f of G' such that for each $v \in V(H)$, $f(v) < f(u)$.*

Proof: Let g be a profile numbering of G'. Suppose that there is $v \in V(H)$ such that $g(v) > g(u)$. Let $a = g(u)$ and $b = \max\{g(v) : v \in H\}$. Define a proper numbering f of G as follows:

$$f(x) = \begin{cases} b & x = u \\ g(x) - 1 & a < g(x) \leq b \\ g(x) & g(x) < a \text{ or } g(x) > b. \end{cases}$$

Then for $t < a$ or $t > b$, $|N_f(S_t)| = |N_g(S_t)|$, $\sum_{t=a}^{b} |N_f(S_t) \cap V(G)| \leq \sum_{t=a}^{b} |N_g(S_t) \cap V(G)| + c$ where $c = |\{x : x \in V(G), g(x) < b \text{ and } xv \notin E(G)\}|$ and $\sum_{t=a}^{b} |N_f(S_t) \cap$

$V(H)| < \sum_{t=a}^{b} |N_g(S_t) \cap V(H)| - cc'$ where $c' = |\{v : v \in V(H) \text{ and } v \notin N_g(S_{a-1})\}|$. Since g is a profile numbering, if $c > 0$ then $c' > 0$. So $\sum_{t=1}^{n(m+1)} |N_f(S_t)| \leq \sum_{t=1}^{n(m+1)} |N_g(S_t)|$. Thus, f must be a profile numbering of G' and for each $v \in V(H)$, $f(v) < f(u)$. □

Lemma 5 *Given a graph $G = (V(G), E(G))$, let $H = (V(H), E(H))$ such that $V(H) \cap V(G) = \emptyset$ and $u \in V(G)$. Let $G' = (V', E')$ where $V' = V(G) \cup V(H)$ and $E' = E(G) \cup E(H) \cup \{uv : v \in V(H)\}$. Then $P(G') \geq P(G) + P(H) + |V(H)|$.*

Proof: Let f be a profile numbering of G' which satisfies Lemma 4. Define a proper numbering h of G as follows:

$$h(x) = \begin{cases} f(x) & f(x) < \min\{f(v) : v \in V(H)\} \\ f(x) - k & k = |\{v : f(x) > f(v) \text{ and } v \in V(H)\}|. \end{cases}$$

Since $f(u) > f(v)$ for each $v \in V(H)$, we know that $\sum_{v \in V(H)} w_f(v) \geq P(H)$ and $w_f(u) \geq w_h(u) + |V(H)|$. So, $\sum_{x \in V(G')} w_f(x) = \sum_{x \in V(G)} w_f(x) + \sum_{x \in V(H)} w_f(x) \geq \sum_{x \in V(G)} w_h(x) + |V(H)| + P(H) \geq P(G) + P(H) + |V(H)|$ which implies that $P(G') \geq P(G) + P(H) + |V(H)|$. □

Applying Lemma 5 n_1 times, we have Theorem 10.

Theorem 10 *Let G_1 and G_2 be two graphs of orders n_1 and n_2 respectively. Then*

$$P(G_1 \wedge G_2) \geq P(G_1) + n_1 P(G_2) + n_1 n_2.$$

The bound in Theorem 10 is tight as illustrated later in Theorem 11. Lemma 6 is immediate since G is connected.

Lemma 6 *If G is a connected graph of order n and f is a proper numbering on G, then $|N_f(S_t)| \geq 1$ for $1 \leq t < n$.*

Theorem 11 *If G is a graph of order m, then $P(P_n \wedge G) = nP(G) + mn + n - 1$.*

Proof: Let the vertices in P_n be named sequentially, from one end to the other, v_i for $1 \leq i \leq n$. Let f be a profile numbering of G. Define a proper numbering g of $P_n \wedge G$ as follows:

$$g(x) = \begin{cases} (m+1)i & x = v_i \text{ and } 1 \leq i \leq n \\ (i-1)(m+1) + f(x) & x \in V(G_i) \text{ and } 1 \leq i \leq n. \end{cases}$$

Then $P_g(P_n \wedge G) = nP(G) + (m+1)(n-1) + m = nP(G) + mn + n - 1$. So, $P(P_n \wedge G) \leq nP(G) + mn + n - 1$.

Suppose that $P(P_n \wedge G) < nP(G) + mn + n - 1$. Let h be a profile numbering of $P_n \wedge G$. We note that

$$|N_g(S_t)| = \begin{cases} 0 & t = (m+1)n \\ 1 & t \equiv 0 \pmod{m+1} \text{ and } t \neq (m+1)n \\ |N_f(S_t)| + 1 & \textit{otherwise} \end{cases}$$

By Proposition 2, there exists an integer t with $1 \leq t < (m+1)n$ such that $|N_h(S_t)| < |N_g(S_t)|$. By Lemma 6, $t \not\equiv 0 \pmod{m+1}$.

For either $h^{-1}(t) \in V(G_i)$ or $h^{-1}(t) \in V(P_n)$, we have $|N_h(S_t)| \geq |N_f(S_t)| + 1 = |N_g(S_t)|$ which produces a contradiction. Thus, $P(P_n \wedge G) = nP(G) + mn + n - 1$. □

Theorem 12 *If G_1 and G_2 are two graphs of orders n_1 and n_2 respectively, then*

$$P(G_1 \wedge G_2) \leq n_1 P(G_2) + (n_2 + 1)P(G_1) + n_1 n_2.$$

Proof: Let f_1 be a profile numbering of G_1 and let v_i denote the vertex in G_1 with $f_1(v_i) = i$ for $1 \leq i \leq n_1$. Let f_2 be a profile numbering of G_2 and let u_j denote the vertex in G_2 with $f_2(u_j) = j$ for $1 \leq j \leq n_2$. Define a proper numbering g of $H = G_1 \wedge G_2$ as follows:

$$g(x) = \begin{cases} (n_2+1)f_1(x) & x \in V(G_1) \\ (n_2+1)(f_1(v_i) - 1) + f_2(x) & x \in V(G_2) \text{ and } (x, v_i) \in E(H) \end{cases}$$

Then $P_g(G_1 \wedge G_2) = n_1 P(G_2) + (n_2 + 1)P(G_1) + n_1 n_2$. Thus, $P(G_1 \wedge G_2) \leq n_1 P(G_2) + (n_2 + 1)P(G_1) + n_1 n_2$. □

The bound in Theorem 12 is tight as illustrated later in Theorem 13.

Definition 5 The *complement* of a graph $G = (V, E)$ is denoted as $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{uv : uv \notin E\}$.

Theorem 13 *Let $G = \overline{K_n} \wedge K_m$. Then $P(G) = nm(m+1)/2$.*

Proof: Note that $P(\overline{K_n}) = 0$ and $P(K_m) = m(m-1)/2$. By Theorem 12, $P(G) \leq nP(K_m) + mP(\overline{K_n}) + mn = nm(m-1)/2 + mn = nm(m+1)/2$. Since $\overline{K_n} \wedge K_m$ is n separate copies of K_{m+1}, $P(G) = nm(m+1)/2$. □

6.3 Tight Bounds on the Corona of Graphs With $\overline{K_m}$

A tight upper bound and a tight lower bound is given in this section for the corona of a connected graph with $\overline{K_m}$.

Given $H = \overline{K_m}$ and applying Lemma 5 n times, we have Theorem 14.

Theorem 14 *If G is a graph of order n, then $P(G \wedge \overline{K_m}) \geq P(G) + mn$.*

Theorem 15 shows that the bound in Theorem 14 is tight.

Theorem 15 *If $G = P_n \wedge \overline{K_m}$, then $P(G) = mn + n - 1$.*

Proof: Let the vertices in P_n be named sequentially, from one end to the other, v_i for $1 \leq i \leq n$. Also, u_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq m$ will be a vertex in $\overline{K_m}$ which is joined to v_i. Let $f(v_i) = (m+1)i$ and $f(u_{ij}) = (m+1)i - j$. Then $P_f(G) = (m+1)(n-1) + m = mn + n - 1$. So, $P(G) \leq mn + n - 1$. By Theorem 10, we have $P(G) \geq P(P_n) + mn = (n-1) + mn = mn + n - 1$. Hence, $P(G) = mn + n - 1$. □

Corollary 7 *If $G = P_n \wedge K_1$, then $P(G) = 2n - 1$.*

Lemma 7 *For any connected graph G of order $n \geq 3$, there exists a profile numbering f of G such that at least two vertices of G, say u and v, have $w_f(u) > 0$*

and $w_f(v) > 0$.

Proof: Let g be a profile numbering of G. Let the vertices of G be named $v_1, v_2, \ldots, v_n$ where $g(v_i) = i$. Then we know that $w_g(v_n) > 0$. If there exists a vertex v_i for $1 \leq i < n$ such that $w_g(v_i) > 0$, then the lemma follows. So we assume that $w_g(v_i) = 0$ for $1 \leq i < n$. Then both of v_1 and v_{n-1} must only be adjacent to v_n. Define f as follows:

$$f(v_i) = \begin{cases} i & 1 \leq i \leq n-2 \\ n & i = n-1 \\ n-1 & i = n \end{cases}$$

Then $\sum_{i=1}^{n} w_f(v_i) = \sum_{i=1}^{n} w_g(v_i)$. So f is a profile numbering of G and $w_f(v_{n-1}) = w_g(v_n) - 1 > 0$ and $w_f(v_n) = 1 > 0$. □

Theorem 16 *If* $G = (V, E)$ *is a connected graph of order* n *for* $n \geq 3$, *then* $P(G \wedge \overline{K_m}) \leq (m+1)P(G) + m(n-2)$.

Proof: For each profile numbering h of G, define $W_h = \{v : w_h(v) > 0\}$. Let f be the profile numbering such that $|W_f| = \max |W_h|$ where the maximum is taken over all profile numberings of G. Let the vertices of G be named $v_1, v_2, \ldots, v_n$ where $f(v_i) = i$. Let the vertices of the ith copy of $\overline{K_m}$ in $G \wedge \overline{K_m}$ be denoted by u_{ij} for $1 \leq j \leq m$. Then define a proper numbering g of $H = G \wedge \overline{K_m}$ as $g(v_i) = (m+1)i$ and $g(u_{ij}) = (m+1)(i-1) + j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Then $P_g(G \wedge \overline{K_m}) = (m+1)P(G) + m(n - |W_f|)$. By Lemma 7, we know that $|W_f| \geq 2$ so the theorem follows. □

Corollary 8 *If G is a connected graph of order n for $n \geq 3$, then $P(G \wedge K_1) \leq 2P(G) + n - 2$.*

The bound of Theorem 16 is tight as is illustrated in Theorem 17.

Theorem 17 *If $G = K_{1,n} \wedge \overline{K_m}$, then $P(G) = 2mn - m + n$.*

Proof: Note that the order of $K_{1,n}$ is $n+1$ and $P(K_{1,n}) = n$. By Theorem 16, $P(G) \leq (m+1)P(K_{1,n}) + m((n+1)-2) = (m+1)n + m(n+1-2) = 2mn - m + n$. Let the vertices in $K_{1,n}$ be denoted as $v_1, v_2, \ldots, v_n$ and v_{n+1} where $deg(v_i) = 1$ for $1 \leq i \leq n$. Name the vertices in $\overline{K_m}$ joined to v_i as u_{ij} for $1 \leq i \leq n+1$ and $1 \leq j \leq m$. Define f as follows:

$$f(v_i) = \begin{cases} (m+1)i & 1 \leq i \leq n-1 \\ (m+1)(n+1) & i = n \\ (m+1)n & i = n+1 \end{cases}$$

and $f(u_{ij}) = f(v_i) - j$ for $1 \leq j \leq m$ and $1 \leq i \leq n+1$. Then $P_f(G) = m(n-1) + m + 1 + (n-1)(m+1) = 2mn - m + n$.

Now suppose that $P(G) < 2mn - m + n$. Let g be a profile numbering of G. By Proposition 2, there exists an integer t with $1 \leq t < 2n+2$ such that $|N_g(S_t)| < |N_f(S_t)|$. Note that

$$|N(S_i)| = \begin{cases} 0 & i = mn + n \\ 1 & 1 \leq i \leq m \\ 1 & i > (n-1)(m+1) - 1 \\ 1 & i \equiv 0 \pmod{m+1} \text{ and } i \not\equiv mn + n \\ 2 & i \not\equiv 0 \pmod{m+1} \text{ and } m < i < (n-1)(m+1). \end{cases}$$

By Lemma 6, t must satisfy following conditions: $t \not\equiv 0 \pmod{m+1}$, $m < t < (n-1)(m+1)$ and $|N_g(S_t)| = 1$. However, for any $|N_g(S_t)| < |N_f(S_t)|$, we must have $|N_g(S_{t+i})| > |N_f(S_{t+i})|$ for some i such that $1 \leq i \leq m$. Since we have $t \not\equiv 0 \pmod{m+1}$ and $m < t < (n-1)(m+1)$, then $g^{-1}(t) = u_{(n+1)j}$ for some integer j such that $1 \leq j \leq m$ and $N_g(S_t) = \{v_{n+1}\}$. Consider the set X of vertices such that for all $x \in X$, we have $t < g(x) \leq t + m$. If all such elements of X are in copies of $\overline{K_m}$, then there is at least one of them, say y, such that $|N_g(S_{g(y)})| = 2$. If there is a vertex v_i such that $t + 1 \leq g(v_i) \leq t + m$, then $|N_g(S_{g(v_i)})| \geq 2$. So $\sum_{t=1}^{(m+1)(n+1)} |N_g(S_t)| \geq \sum_{t=1}^{(m+1)(n+1)} |N_f(S_t)|$. Therefore, $P_g(G) \geq P_f(G)$ producing a contradiction. Thus, $P(G) = 2mn - m + n$. $\square$

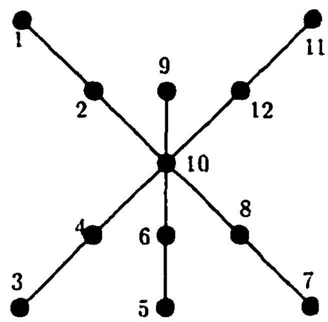


Figure 20. A Profile Numbering of $G = K_{1,5} \wedge K_1$.

6.4 Special Cases

In this section, we determine the exact values of the profile of $K_n \wedge K_m$ and $C_n \wedge G$.

Corollary 9 is a direct consequence of Theorem 17.

Corollary 9 *If $G = K_{1,n} \wedge K_1$, then $P(G) = 3n - 1$.*

A profile numbering of $G = K_{1,5} \wedge K_1$ is shown in Figure 20.

Lemma 8 *Let G be a graph of order m and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ denote the vertex set of K_n. Then there exists a profile numbering f of $K_n \wedge G$ such that for each $u \in V(G_i)$, $f(u) < f(v_i)$ for $1 \leq i \leq n$.*

Proof: Let g be a profile numbering of G. We denote the vertex set of G_i by $V(G_i)$. Suppose that there is an integer i with $1 \leq i \leq n$ such that $g(v_i) < g(u)$ for some $u \in V(G_i)$. Let $a = g(v_i)$ and $b = \max\{g(u) : u \in G_i\}$. Define f as follows:

$$f(v) = \begin{cases} b & v = v_i \\ g(v) - 1 & a < g(v) \le b \\ g(v) & g(v) < a \text{ or } g(v) > b. \end{cases}$$

Then for $t < a$ or $t > b$, $|N_f(S_t)| = |N_g(S_t)|$. For $a \le t \le b$, $|N_f(S_t) \cap V(K_n)| \le |N_g(S_t) \cap V(K_n)|$, $|N_f(S_t) \cap V(G_i)| \le |N_g(S_t) \cap V(G_i)|$ and $|N_f(S_t) \cap V(G_j)| = |N_g(S_t) \cap V(G_j)|$ for $j \neq i$. So $\sum_{t=1}^{n(m+1)} |N_f(S_t)| \le \sum_{t=1}^{n(m+1)} |N_g(S_t)|$. Thus, f must be a profile numbering of $K_n \wedge G$ and for each $u \in V(G_i)$, $f(u) < f(v_i)$ for $1 \le i \le n$. □

Theorem 18 *If $G = K_n \wedge K_m$ for positive integers n and m, then*

$$P(G) = \frac{nm(m-1)}{2} + \frac{m+1}{2}(\lceil\frac{n}{2}\rceil^2 + \lfloor\frac{n}{2}\rfloor^2) + \frac{m-1}{2}(\lceil\frac{n}{2}\rceil + \lfloor\frac{n}{2}\rfloor) + \lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor.$$

Proof: Let the vertices in K_n be denoted as $v_1, v_2, \ldots, v_n$ and the vertices in K_m that are joined to v_i be $u_{i,j}$ for $1 \le i \le n$, $1 \le j \le m$. Let $V = \{v_i : 1 \le i \le n\}$ and $U_i = \{u_{i,j} : 1 \le i \le n \text{ and } 1 \le j \le m\}$. Now we define a proper numbering f of G as follows:

$$f(u_{i,j}) = \begin{cases} (i-1)m + j & 1 \le i \le \lceil\frac{n}{2}\rceil \text{ and } 1 \le j \le m \\ (i-1)(m+1) + j & \lceil\frac{n}{2}\rceil < i \le n \text{ and } 1 \le j \le m, \end{cases}$$

and

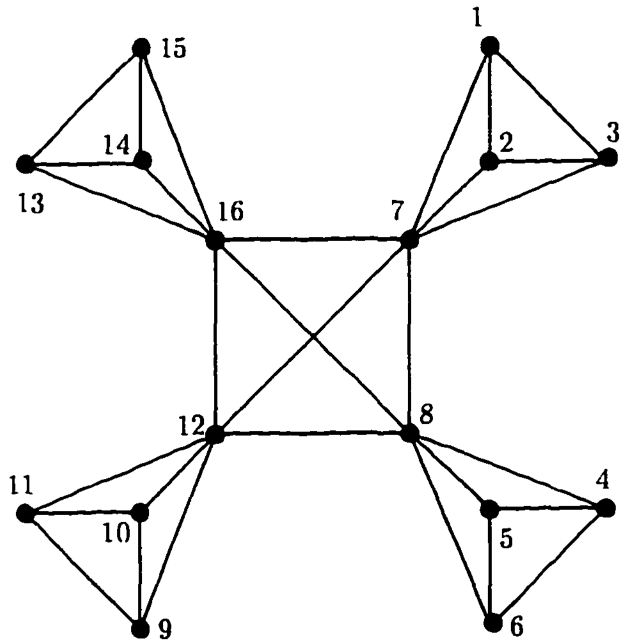


Figure 21. A Profile Numbering of $K_4 \wedge K_3$.

$$f(v_i) = \begin{cases} m\lceil \frac{n}{2} \rceil + i & 1 \le i \le \lceil \frac{n}{2} \rceil \\ (m+1)i & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

Then

$$\begin{aligned} P_f(G) &= \frac{nm(m-1)}{2} + \sum_{i=0}^{\lceil n/2 \rceil - 1} (m(\lceil \frac{n}{2} \rceil - i) + i) + \sum_{i=\lceil n/2 \rceil + 1}^{n} (m(i - \lceil \frac{n}{2} \rceil) + i - 1) \\ &= \frac{nm(m-1)}{2} + \frac{m+1}{2}(\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2) + \frac{m-1}{2}(\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor) + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor. \end{aligned}$$

Figure 21 shows f applied to $K_4 \wedge K_3$.

Then the following is true.

$$|N(S_i)| = \begin{cases} k+km-i & (k-1)m+1 \le i \le km \text{ and} \\ & 1 \le k \le \lceil \frac{n}{2} \rceil \\ n-(i-m\lceil \frac{n}{2} \rceil) & m\lceil \frac{n}{2} \rceil < i \le (m+1)\lceil \frac{n}{2} \rceil \\ k(m+1)-i+n-k & (k-1)(m+1)+1 \le i \le k(m+1) \text{ and} \\ & \lceil \frac{n}{2} \rceil < k \le n. \end{cases}$$

Suppose, to the contrary, that $P(G) < nm(m-1)/2 + (m+1)(\lceil n/2 \rceil^2 + \lfloor n/2 \rfloor^2)/2 + (m-1)(\lceil n/2 \rceil + \lfloor n/2 \rfloor)/2 + \lceil n/2 \rceil \lfloor n/2 \rfloor$. Let g be a profile numbering of G which satisfies Lemma 8. By Proposition 2 there exists an integer t with $1 \le t \le n(m+1)$ such that $|N_g(S_t)| < |N_f(S_t)|$. The following cases are possible for t.

Case 1: $(k-1)m+1 \le t \le km$ and $1 \le k \le \lceil n/2 \rceil$. Then $S_f(t) \cap V = \emptyset$. If $|S_g(t) \cap V| = l$ for $0 < l \le t$, then $|N_g(S_t)| \ge n - l + km - t + l > k + km - t = |N_f(S_t)|$. Since f completely numbers one copy of K_m before numbering another copy of K_m, f minimizes $|N_f(S_t) \cap (\cup_{i=1}^{n} U_i)|$. If $|S_g(t) \cap V| = 0$, then $|N_g(S_t)| = |N_g(S_t) \cap (\cup_{i=1}^{n} U_i)| \ge |N_f(S_t) \cap (\cup_{i=1}^{n} U_i)| = |N_f(S_t)|$.

Case 2: $m\lceil n/2 \rceil < t \le (m+1)\lceil n/2 \rceil$. In this case $|N_f(S_t) \cap U_i| = 0$ for $1 \le i \le n$. If $|S_g(t) \cap V| - |S_f(t) \cap V| = l$ for $0 < l \le n$, $|N_g(S_t) \cap V| = |N_f(S_t) \cap V| - l$ and $|N_g(S_t) \cap (\cup_{i=1}^{n} U_i)| \ge |N_f(S_t) \cap (\cup_{i=1}^{n} U_i)| + l$. So, $|N_g(S_t)| \ge |N_f(S_t)|$. If $|S_g(t) \cap V| = |S_f(t) \cap V|$ $|N_g(S_t) \cap V| = |N_f(S_t) \cap V|$ and $|N_g(S_t) \cap (\cup_{i=1}^{n} U_i)| \ge$

$|N_f(S_t) \cap (\cup_{i=1}^n U_i)|$. So, $|N_g(S_t)| \geq |N_f(S_t)|$. If $|S_f(t) \cap V| - |S_g(t) \cap V| = l$ for $0 < l \leq \lceil n/2 \rceil$, $|N_g(S_t) \cap V| = |N_f(S_t) \cap V| + l$ and $|N_g(S_t) \cap (\cup_{i=1}^n U_i)| \geq |N_f(S_t) \cap (\cup_{i=1}^n U_i)|$. So $|N_g(S_t)| > |N_f(S_t)|$.

Case 3: $(k-1)(m+1)+1 \leq t \leq k(m+1)$ and $\lceil n/2 \rceil < k \leq n$. In this case, $|N_f(S_t) \cap U_i| = 0$ for $1 \leq i \leq k-1$ and $|S_g(t) \cap V| \leq |S_f(t) \cap V|$. If $|S_g(t) \cap V| = |S_f(t) \cap V|$, then $|N_g(S_t) \cap V| = |N_f(S_t) \cap V|$ and $|N_g(S_t) \cap (\cup_{i=1}^n U_i)| \geq |N_f(S_t) \cap (\cup_{i=1}^n U_i)|$. So $|N_g(S_t)| \geq |N_f(S_t)|$. If $|S_f(t) \cap V| - |S_g(t) \cap V| = l$ for $0 < l \leq k$, then $|N_g(S_t) \cap V| = |N_f(S_t) \cap V| + l$ and $|N_g(S_t) \cap (\cup_{i=1}^n U_i)| \geq |N_f(S_t) \cap (\cup_{i=1}^n U_i)| - l$. So $|N_g(S_t)| \geq |N_f(S_t)|$.

Thus in all cases $|N_g(S_t)| \geq |N_f(S_t)|$, producing a contradiction. Therefore, we must have $P(G) = nm(m-1)/2 + (m+1)(\lceil n/2 \rceil^2 + \lfloor n/2 \rfloor^2)/2 + (m-1)(\lceil n/2 \rceil + \lfloor n/2 \rfloor)/2 + \lceil n/2 \rceil \lfloor n/2 \rfloor$. □

Corollary 10 *If $G = K_n \wedge K_1$, then*

$$P(G) = \lceil \frac{n}{2} \rceil^2 + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor^2.$$

In addition, it is believed the following conjecture to be true.

Conjecture 1 *If G is a graph of order m, then*

$$P(K_n \wedge G) = nP(G) + \frac{m+1}{2}(\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2) + \frac{m-1}{2}(\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor) + \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor.$$

Theorem 19 *If G is a graph of order m, then*

$$P(C_n \wedge G) = nP(G) + (m-1)(2n-3) + m \textit{ for } n \geq 3.$$

Proof: Let the vertices in C_n be named sequentially $v_1 \dots v_n$. Let f be a profile numbering of G. Define a proper numbering g of $C_n \wedge G$ as follows:

$$g(x) = \begin{cases} (m+1)i & x = v_i \text{ and } 1 \leq i \leq n \\ (i-1)(m+1) + f(x) & x \in V(G_i) \text{ and } 1 \leq i \leq n. \end{cases}$$

Then $P_g(C_n \wedge G) = nP(G) + (m+1)(2n-3) + m$. So, $P(C_n \wedge G) \leq nP(G) + (m+1)(2n-3) + m$. Suppose that $P(C_n \wedge G) < nP(G) + (m+1)(2n-3) + m$.

$$|N_g(S_t)| = \begin{cases} 0 & t = (m+1)n \\ 1 & t = (m+1)n - 1 \\ 2 & t \equiv 0 \pmod{m+1} \text{ and } t \neq (m+1)n \\ |N_f(S_t)| + 1 & t \leq m \\ |N_f(S_t)| + 2 & \textit{otherwise} \end{cases}$$

Let h be a profile numbering of $C_n \wedge G$. Then by Proposition 2, there exists an integer t with $1 \leq t < (m+1)n$ such that $|N_h(S_t)| < |N_g(S_t)|$.

Case 1: $1 \leq t \leq m$. For either $h^{-1}(t) \in V(G_i)$ or $h^{-1}(t) \in V(C_n)$, we have $|N_h(S_t)| \geq |N_f(S_t)| + 1 = |N_g(S_t)|$.

Case 2: $m < t < (m+1)n - 1$ and $t \equiv 0 \pmod{m+1}$. For either $h^{-1}(t) \in V(G_i)$ or $h^{-1}(t) \in V(C_n)$, we have $|N_h(S_t)| \geq 2 = |N_g(S_t)|$.

Case 3: $m < t < (m+1)n - 1$ and $t \not\equiv 0 \pmod{m+1}$. For either $h^{-1}(t) \in V(G_i)$ or $h^{-1}(t) \in V(C_n)$, we have $|N_h(S_t)| \geq |N_f(S_t)| + 2 = |N_g(S_t)|$.

Thus in all cases we have $|N_h(S_t)| \geq |N_g(S_t)|$, producing a contradiction. Thus, $P(C_n \wedge G) = nP(G) + (m+1)(2n-3) + m$. □

CHAPTER VII

PROFILE OF TENSOR PRODUCT

7.1 Definition and Examples

The tensor product of two graphs was defined in Capobianco and Molluzzo[7].

Definition 6 The *tensor product* of graphs G_1 and G_2 is defined to be $G = (V, E)$ where $V = V(G_1) \times V(G_2)$ and $((x_1, y_1), (x_2, y_2)) \in E$ whenever $(x_1, x_2) \in E(G_1)$ and $(y_1, y_2) \in E(G_2)$.

We use $G_1(T_P)G_2$ to denote G. Figure 22 shows $C_3(T_P)P_4$.

Proposition 7 *(From Weichsel[80]) If G_1 and G_2 are connected, then $G_1(T_P)G_2$ is connected if and only if G_1 or G_2 has an odd cycle.*

Proposition 8 *(From Miller[63]) If $G = G_1(T_P)G_2$ for connected graphs G_1 and G_2, then G consists of exactly two components if and only if G_1 and G_2 are both bipartite.*

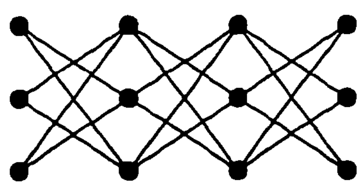


Figure 22. $C_3(T_P)P_4$.

For the tensor product of a path with a complete bipartite graph, Williams [84] provided a numbering to achieve the bandwidth and a linear algorithm to achieve edgesum. In this chapter, we provide linear algorithms to achieve the profile of paths with complete bipartite graphs.

In this chapter, when discussing $G_1(T_P)G_2$ for G_1 a path and G_2 a complete bipartite graph, we assume the vertices of G_1 and G_2 are identified in the following natural manner. In the case of a path, the vertices are identified sequentially proceeding from one end vertex of the path to the other. In the case of the complete bipartite, graph vertices are identified sequentially within each partite set in an arbitrary manner. We use (r, i) to denote the ith vertex of the r-partite and (m, i) to denote the ith vertex of the m-partite. We then use $(i, (r, j))$ to denote vertex $(v_i, v_{(r,j)}) \in V(P_n(T_P)K_{r,m})$.

By Propositions 7 and 8, for $n \geq 2$ $P_n(T_P)K_{r,m}$ always consists of two components. We define *row i*, R_i, of each component separately as $R_i = \{(i, j) : 1 \leq j \leq m \text{ and } (i, j) \in V(G)\}$. For $m \geq r$, let H_1 be the component of G containing $(1, (m, 1))$ and H_2 be the other component. Figure 23 illustrates $P_6(T_P)K_{2,5}$.

Proposition 9 *(From Lin and Yuan[54]) If G has components $G_1, G_2, \ldots, G_m$, then*

$$P(G) = \sum_{i=1}^{m} P(G_i).$$

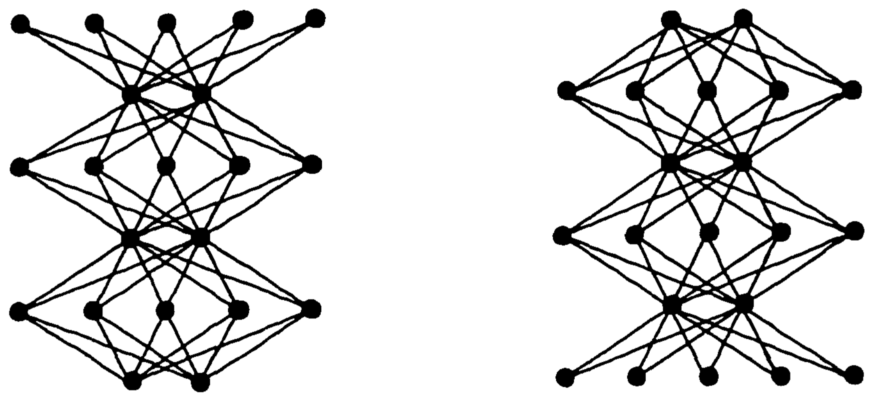


Figure 23. Two Components of $P_6(T_P)K_{2,5}$.

7.2 Profile of $P_n(T_P)K_{m,m}$

A graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency; that is, $uv \in E(G_1)$ if and only if $\phi u \phi v \in E(G_2)$. If G_1 is isomorphic to G_2, then we say G_1 and G_2 are *isomorphic* and write $G_1 \cong G_2$.

Theorem 20 *Let* $G = P_n(T_P)K_{m,m}$. *Then*

$$P(G) = \begin{cases} 0 & n = 1 \\ m(3m-1) & n = 2 \\ (3n-4)m^2 - (n-2)m & n \geq 3. \end{cases}$$

Proof: For $n = 1$, $G = \overline{K_{2m}}$; so $P(G) = 0$. For $n = 2$, $H_1 \cong H_2 \cong K_{m,m}$. By Proposition 9, $P(G) = 2P(H_1) = m(3m-1)$. For $n \geq 3$, $H_1 \cong H_2 \cong P_n[\overline{K_m}]$. By

Proposition 9 and Theorem 5,

$$P(G) = 2(2P(\overline{K_m}) + mn(3m-1)/2 - 2m^2 + m) = (3n-4)m^2 - (n-2)m.$$

□

7.3 Profile of $P_n(T_P)K_{r,m}$ for Even Values of n

Algorithm 4 *Finds a proper numbering h for component H_1 of $G = P_n(T_P)K_{r,m}$ with $r < m$ where n is even and $n \geq 4$.*

Begin

1. Number R_1 with integer $1, 2, \ldots, m$.
2. If $m^2 + m \geq 2r^2$ then

   for $i = 1$ to $(n-4)/2$

   Number R_{2i+1} then R_{2i} with next $m + r$ integers

   else for $i = 1$ to $(n-4)/2$

   Number R_{2i} then R_{2i+1} with next $r + m$ integers.
3. If $m^2 - m \geq 3r^2 - r$ then

   Number R_{n-1} then R_{n-2} then R_n sequentially

   else Number R_{n-2} then R_n then R_{n-1} sequentially.

End of Algorithm

Theorem 21 *Let $G = P_n(T_P)K_{r,m}$ with $r < m$ for even values of n with $n \geq 4$. Algorithm 4 produces a profile numbering of component H_1 in linear time.*

Proof: Let g be a profile numbering of H_1. By Proposition 1, we know that $P(H_1) = \sum_{t=1}^{(r+m)n/2} |N_g(S_t)|$. We want to show that the numbering h we get from Algorithm 4 is no worse than g for H_1 of G. Since every vertex in R_i is adjacent to every vertex in rows R_{i-1} and R_{i+1} for $1 < i < n$, g must completely number a row before starting another row.

Claim 1: For $0 \leq p \leq (n-6)/2$, let $a(p) = m + p(r+m) + 1$ and $b(p) = m + (p+1)(r+m)$, then $\sum_{t=a(p)}^{b(p)} |N(S_t)| \geq \min\{2rm + r^2 + r(r-1)/2, 2rm + r(r-1)/2 + m(m-1)/2\}$. Furthermore $\sum_{t=a(p)}^{b(p)} |N(S_t)| = \min\{2rm + r^2 + r(r-1)/2, 2rm + r(r-1)/2 + m(m-1)/2\}$ is achievable.

Proof of Claim 1: First suppose g numbers a row $|R_i| = m$ and then numbers R_j with $|R_j| = r$. Then $\sum_{t=a(p)}^{b(p)} |N(S_t)| \geq 2rm + r^2 + r(r-1)/2$. Next suppose g numbers a row R_j with $|R_j| = r$ and then numbers R_i with $|R_i| = m$. Then $\sum_{t=a(p)}^{b(p)} |N(S_t)| \geq 2rm + r(r-1)/2 + m(m-1)/2$.

Next suppose g numbers a row R_i with $|R_i| = m$ and then a row R_j with $|R_j| = m$. Then $\sum_{t=a(p)}^{b(p)} |N(S_t)| \geq 2rm + r^2 + 2r^2 > 2rm + r^2 + r(r-1)/2$. Next suppose g numbers a row R_i with $|R_i| = r$ and then a row R_j with $|R_j| = r$. Then $\sum_{t=a(p)}^{b(p)} |N(S_t)| > 3rm + r(r+1)/2 > 2rm + r^2 + r(r-1)/2$.

Therefore, in all cases, $\sum_{t=a(p)}^{b(p)} |N(S_t)| \geq \min\{2rm + r^2 + r(r-1)/2, 2rm + r(r-1)/2 + m(m-1)/2\}$. But note that $\min\{2rm + r^2 + r(r-1)/2, 2rm + r(r-

$1)/2+m(m-1)/2\}$ is achieved by Algorithm 4.

Claim 2: h is no worse than g.

Proof of Claim 2: Let $a = m + (n-4)(r+m)/2 + 1$ and $b = (m+r)n/2$. We establish Claim 2 by considering the following two cases.

Case 1: $R_1 \in S_{a-1}$. In this case, $\sum_{t=a}^{b} |N_g(S_t)| \geq \min\{2rm + r^2 + r(r-1), 2rm + (r(r-1) + m(m-1))/2\}$. This is true since if there are three or more rows of length r in $\overline{S_{a-1}}$, then $\sum_{t=a}^{b} |N_g(S_t)| \geq 3r(r-1)/2 + 3rm > 2rm + r^2 + r(r-1)$. Suppose there are two rows of length r in $\overline{S_{a-1}}$.

If g numbers both rows of length r before the row of length m, then $\sum_{t=a}^{b} |N_g(S_t)| \geq 2rm + (r(r-1) + m(m-1))/2$. Otherwise, $\sum_{t=a}^{b} |N_g(S_t)| \geq 2rm+r^2+r(r-1)$. If there is only one row of length r in $\overline{S_{a-1}}$, then $\sum_{t=a}^{b} |N_g(S_t)| \geq 2rm + (r^2 + m^2 - 1)/2 > 2rm + (r(r-1) + m(m-1))/2$.

If there are no rows of length n in $\overline{S_{a-1}}$, then $\sum_{t=a}^{b} |N_g(S_t)| \geq 2rm+m(m-1)/2 + 2r(2r-1)/2 > 2rm + (r(r-1) + m(m-1))/2$. Note that $\sum_{t=a}^{b} |N_g(S_t)| = \min\{2rm+r^2+r(r-1), 2rm+(r(r-1)+m(m-1))/2\}$ is achieved by Algorithm 4.

Case 2: $R_1 \in \overline{S_{a-1}}$. In this case $\sum_{t=a}^{b} |N_g(S_t)| \geq \min\{2rm + r^2 + r(r-1), 2rm + (r(r-1) + m(m-1))/2\} - rm$ and $\sum_{t=1}^{m} |N_g(S_t)| \geq 2rm$. We know that $\sum_{t=1}^{m} |N_h(S_t)| = rm$; so by the proof given above, we have $P_h(H_1) \leq P_g(H_1)$. $\square$

Theorem 22 *Let $G = P_n(T_P)K_{r,m}$ with $r < m$ and n is even. Then*

$$P(G) = \begin{cases} 2(r+m-1) & n = 2 \\ 2mr(n-1) + r^2(n-2) + \dfrac{nr(r-1)}{2} & n \geq 4 \text{ and} \\ & m^2 - m \geq 3r^2 - r \\ 2mr(n-1) + r^2(n-4) + \dfrac{r(r-1)(n-2)}{2} & n \geq 4, m^2 + m \geq 2r^2 \\ & \text{and } m^2 - m < 3r^2 - r \\ 2mr(n-1) + \dfrac{(r^2 - r + m^2 - m)(n-2)}{2} & n \geq 4 \text{ and} \\ & m^2 + m < 2r^2. \end{cases}$$

Proof: Note that when n is even, H_1 and H_2 are isomorphic. By Proposition 9, $P(G) = 2P(H_1)$. For $m = 2$, G becomes two disjoint paths of length $r + m$ so $P(G) = 2(r + m - 1)$. For $m \geq 4$, by Theorem 21, when $m^2 - m \geq 3r^2 - r$ then $P(H_1) = mr(n-1) + r^2(n-2)/2 + nr(r-1)/4$. When $m^2 + m < 2r^2$, $P(H_1) = 2rm + (n-2)(2rm + r(r-1)/2 + m(m-1)/2)/2$. Also when $m^2 - m < 3r^2 - r$ and $m^2 + m \geq 2r^2$ then $P(H_1) = 3rm + (n-4)(2rm + r^2 + r(r-1)/2)/2 + (r(r-1) + m(m-1))/2$. By Proposition 9, the result follows. □

7.4 Profile of $P_n(T_P)K_{r,m}$ for Odd Values of n

Algorithm 5 *Finds a proper numbering h for $G = P_n(T_P)K_{r,m}$ with $r < m$, n is odd and $n \geq 5$.*

Begin

1. Number R_1 of H_1 with integers $1, 2, \ldots, m$.

2. If $m^2 + m \geq 2r^2$ then

    for $i = 1$ to $(n-3)/2$

        Number R_{2i+1} then R_{2i} of H_1 with the next $m + r$ integers

    else for $i = 1$ to $(n-3)/2$

        Number R_{2i} then R_{2i+1} of H_1 with the next $r + m$ integers.

3. Number R_n then R_{n-1} of H_1 sequentially.

4. If $m^2 - m \geq 3r^2 - r$ then

    Number R_2 then R_1 of H_2 with the next $m + r$ integers

    else Number R_1 then R_2 of H_2 with the next $r + m$ integers.

5. If $m^2 + m \geq 2r^2$ then

    for $i = 2$ to $(n-3)/2$

        Number R_{2i} then R_{2i-1} in H_2 with the next $r + m$ integers,

    else for $i = 2$ to $(n-3)/2$

        Number R_{2i-1} then R_{2i} in H_2 with the next $m + r$ integers.

6. If $m^2 - m \geq 3r^2 - r$ then

    Number R_{n-1}, R_{n-2} then R_n of H_2 sequentially,

    else Number R_{n-2}, R_n then R_{n-1} of H_2 sequentially.

End of Algorithm

Theorem 23 *Let $G = P_n(T_P)K_{r,m}$ with $r < m$ where n is odd and $n \geq 5$. Then Algorithm 5 produces a profile numbering of G in linear time.*

Proof: In Algorithm 5, steps 1 to 3 number H_1 of G and steps 4 to 6 number H_2 of G. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. By an argument similar to that in the proof of Theorem 21, we see that steps 1 to 3 in Algorithm 5 give a profile numbering of H_1.

It remains to show that steps 4 to 6 of Algorithm 5 give a profile numbering of H_2. Define a proper numberig h' of H_2 as $h'(v) = h(v) - n_1$, then $P_{h'}(H_2) = P_h(H_2)$. Let g be a profile numbering of H_2. By the same reasons as seen in the proof of Theorem 21, g must completely number each row before starting to number another row. The following claims can then be shown in a similar manner.

Claim 1: $\sum_{t=1}^{m+r} |N_g(S_t)| \geq \min\{2mr + m(m-1)/2, 2mr + r^2 + r(r-1)/2\}$.

Claim 2: For $1 \leq p \leq (n-5)/2$, let $a(p) = p(r+m)+1$ and $b(p) = (p+1)(r+m)$. Then $\sum_{t=a(p)}^{b(p)} |N_g(S_t)| \geq \min\{2rm+r(r-1)/2+m(m-1)/2, 2rm+r^2+r(r-1)/2\}$.

Claim 3: Let $a = (n-3)(r+m)/2 + 1$ and $b = ((n-1)m + (n+1)n)/2$. Then $\sum_{t=a}^{b} |N_g(S_t)| \geq \min\{2rm + r(r-1)/2 + m(m-1)/2, 2rm + r^2 + r(r-1)\}$.

Note that the lower bounds in all the claims are achievable by Algorithm 5. By Proposition 9, Algorithm 5 produces a profile numbering of G in linear time. $\square$

Theorem 24 *Let $G = P_n(T_P)K_{r,m}$ with $r < m$ and n is odd. Then*

$$P(G) = \begin{cases} 0 & n = 1 \\ 4rm + r^2 + \dfrac{3r(r-1)}{2} & n = 3 \text{ and} \\ & m^2 - m \geq 4r^2 - 2r \\ 4rm + \dfrac{r(r-1) + m(m-1)}{2} & n = 3 \text{ and} \\ & m^2 - m < 4r^2 - 2r \\ 2mr(n-1) + r^2(n-2) + \dfrac{nr(r-1)}{2} & n \geq 5 \text{ and} \\ & m^2 - m \geq 3r^2 - r \\ 2mr(n-1) + r^2(n-4) + \dfrac{r(r-1)(n-2)}{2} & n \geq 5, m^2 + m \geq 2r^2 \\ & \text{and } m^2 - m < 3r^2 - r \\ 2mr(n-1) + \dfrac{(r^2 - r + m^2 - m)(n-2)}{2} & n \geq 5 \text{ and} \\ & m^2 + m < 2r^2. \end{cases}$$

Proof: When $n = 1$ G becomes a collection of isolated vertices so $P(G) = 0$. When $n = 3$, in H_1 we must number both end rows (i.e., R_1 and R_3) before numbering R_2 since every vertex in R_2 is adjacent to every vertex in R_1 and R_3. So, $P(H_1) = 2mr + r(r-1)/2$. In H_2, since we need to completely number each row before starting another, and since R_1 and R_3 are equivalent, there are only three ways to proceed. Let f number H_2 in the order of R_1, then number R_3 then number R_2. Then we have $P_f(H_2) = 2rm + m(m-1)/2$. Let g number H_2 in the order of R_1 then number R_2 then number R_3. Then we have $P_g(H_2) =$

$rm + rm + m(m-1)/2 + r(r-1)/2 = 2rm + m(m-1)/2 + r(r-1)/2$. Let h number H_2 in the order of R_2 then number R_1 then number R_3. Then we have $P_h(H_2) = 2rm + r^2 + r(r-1)/2 + r(r-1)/2 = 2rm + r^2 + r(r-1) = 2rm + 2r^2 - r$.

Since $P_g(H_2) \geq P_f(H_2)$ is always true, then if $m^2 - m \geq 4r^2 - 2r$, $P(H_2) = P_h(H_2)$ else $P(H_2) = P_f(H_2)$. By Proposition 9, when $m^2 - m \geq 4r^2 - 2r$, $P(G) = P(H_1) + P(H_2) = 4rm + r^2 + 3r(r-1)/2$ and when $m^2 - m < 4r^2 - 2r$, $P(G) = P(H_1) + P(H_2) = 4rm + (r(r-1) + m(m-1))/2$. When $n \geq 5$, by Theorem 23, for $m^2 + m \geq 2r^2$, $P(H_1) = 2mr + (n-3)(2rm + r^2 + r(r-1)/2)/2 + r(r-1)/2$; otherwise $P(H_1) = 2rm + (n-3)(2rm + r(r-1)/2 + m(m-1)/2)/2 + r(r-1)/2$. When $m^2 - m \geq 3r^2 - r$, $P(H_2) = 4rm + 2r^2 + 3r(r-1)/2 + (n-5)(2rm + r^2 + r(r-1)/2)/2$. When $m^2 - m < 3r^2 - r$ and $m^2 + m \geq 2r^2$, $P(H_2) = 4rm + m(m-1) + r(r-1)/2 + (n-5)(2rm + r^2 + r(r-1)/2)/2$. When $m^2 - m < 2r^2$, $P(H_2) = 4rm + m(m-1) + r(r-1)/2 + (n-5)(2rm + r(r-1) + m(m-1))/2$. By Proposition 9, the result follows. $\square$

CHAPTER VIII

SUMMARY

This dissertation has investigated the bandwidth, edgesum and profile of a number of classes of graphs and has provided solutions for several of them.

In Chapter II, various application areas of the graph parameters bandwidth, edgesum and profile are discussed. These include minimizing the storage and computation time for solving linear equations; solving the placement problem and the routing problem in VLSI design layout; embedding one network in another and solving constraint satisfaction problems in AI.

In Chapter III, we indicated the errors of some theorems which were given by previous researchers and gave the corrected results.

In Chapter IV, we solved the profile of the composition of paths, cycles, complete graphs and complete bipartite graphs with other graphs.

In Chapter V, the bandwidth of a d-dimension butterfly was solved. A polynomial time aproximation algorithm was presented to number a butterfly in order to minimize the edgesum and the profile. Also, a polynomial time algorithm was presented to minimize the profile of a hypercube.

In Chapter VI, a tight upper bound and a tight lower bound were given for the profile of the corona of any two graphs. A tight upper bound and a tight lower bound were provided for the profile of the corona of any connected graph with $\overline{K_m}$. The exact value of the corona of a complete graph with a complete graph and a cycle with any other graph were established.

Finally, in Chapter VII, the exact value of the profile of the tensor product of a path with a complete bipartite graph was established.

Appendix A

Code of Edgesum of Butterflies

```
#include<iostream.h>

#include<fstream.h>

#include<stdlib.h>

const int max = 80; // max number of vertices in a graph

class graph{

    int n;  // number of vertices in the graph

    int adj[max][max]; // adjancency matrix of  the graph

    int num[max]; // numbering of the graph

public:

    graph(){ for(int  i = 0;  i < max;  i++)

                num[i] = 0; }

    void readAdj(ifstream&);  // read in adjancency matrix

                              // from a file

    void doit(ifstream& fin);

    void printNum(); // print out current numbering

    int edgesum(); // return the Edgesum of a numbering

    int CountNb1(int*); // count  e(S_t)

    void mynum4(); // my way to number a graph - select a neighbor,

                   // - keep smallest e(S_t)

    int find4(int); // find the vertex to be numbered next

};
```

```
void graph::doit(ifstream& fin){
    fin >> n;
    readAdj(fin);
}

void graph::readAdj(ifstream& fin){
    for (int  i = 0;  i < n;  i ++)
        for(int j = 0; j < n; j++)
            fin >> adj[i][j];
}

void graph::printNum(){
    cout << endl;
    for(int i=0; i<n; i++) {
        if (num[i] < 10) cout << ' ';
        cout << num[i] << "  ";
        switch(n){ // for butterfly output
           case 12: if(!((i+1)%4)) cout << endl; break;
           case 32: if(!((i+1)%8)) cout << endl; break;
           case 80: if(!((i+1)%16)) cout << endl;  break;
```

```
        }
    }
    cout << endl << endl;
}

int graph::edgesum(){
    int e=0; //edgesum
    for (int i=0; i<n; i++)
        for (int j=i+1; j<n; j++)
           if (adj[i][j] == 1)
               e = e + abs(num[i]-num[j]);
    return e;
}

// count number of neighbors -  edge version
int graph::CountNb1(int* tnum){
int k=0;
    for(int i=0; i<n; i++)
        if(tnum[i])
            for(int j=0; j < n; j++)
                if (adj[i][j]&&(!tnum[j])) k++;
```

```
    return k;
}


int graph::find4(int c){
int tnum[max]; // numbering array that I am using now
    for(int i = 0; i < n; i++) // initialize tnum[]
tnum[i] = num[i];
int j, m,
    k=0,        // number of neighbors we have right now
    b,          // the vertex that I choose to assign
    nb[max];    // neighbor list
    for(i=0; i<n; i++)        // initialize neighbor list
        if(num[i])
            for(j=0; j < n; j++)
                if ((adj[i][j])&&(!tnum[j])){
                    nb[k++] = j;
                    tnum[j] = -1;
                }
    for(i=0; i<n; i++) // change the mark on the tnum[] back
       if (tnum[i] ==-1) tnum[i] = 0;
```

```
int min = n*(n-1)/2;
    for(int l=0; l<k; l++){
        tnum[nb[l]] = c;
        m = CountNb1(tnum);  // count e(S_cur)
        if (m < min){
            min = m;
            b =nb[l];
        }
        tnum[nb[l]] = 0;
    }
    return b;
}

void graph::mynum4(){ // for edgesum
int i,j;
    num[0] = 1;
    for (i=2; i <= n; i++){
        j = find4(i); // find the vertex to number
        num[j] = i;
    }
}
```

```
int main(){

ifstream finN("adj.dat");

int e;

    while (finN) {

        graph g1;

        g1.doit(finN);

        if (finN.eof()) break;

        g1.mynum4();

        cout << "*** by mynum5 *** The numbering is : ";

        g1.printNum();

        e = g1.edgesum();

        cout << "The edgesum is " << e << endl<< endl<< endl;

    }

}
```

Appendix B

Code of Profile of Butterflies

```
#include<iostream.h>

#include<fstream.h>

#include<stdlib.h>

const int max = 80; // max number of vertices in a graph

class graph{

    int n;  // number of vertices in the graph

    int adj[max][max]; // adjacency matrix of  the graph

    int num[max]; // numbering of the graph

public:

    graph(){ for(int  i = 0;  i < max;  i++)

                num[i] = 0; }

    void readAdj(ifstream&);  // read in adjacency matrix

                              // from a file

    void doit(ifstream& fin);

    void printNum(); // print out current numbering

    int profile(); // return the Profile of a numbering

    int CountNb(int*); // count  e(S_t)

    void mynum(); // my way to number a graph - select any vertex,

                  // - keep smallest |N(S)|

    int find(int); // find the vertex which should be numbered next

};
```

```
void graph::doit(ifstream& fin){
    fin >> n;
    readAdj(fin);
}

void graph::readAdj(ifstream& fin){
    for (int  i = 0;  i < n;  i ++)
        for(int j = 0; j < n; j++)
            fin >> adj[i][j];
}

void graph::printNum(){
    cout << endl;
    for(int i=0; i<n; i++) {
        if (num[i] < 10) cout << ' ';
        cout << num[i] << "  ";
        switch(n){ // for butterfly output
           case 12: if(!((i+1)%4)) cout << endl; break;
           case 32: if(!((i+1)%8)) cout << endl; break;
           case 80: if(!((i+1)%16)) cout << endl;  break;
```

```
        }
    }
    cout << endl << endl;
}


int graph::profile(){
    int p = 0; //profile
    for (int i=0; i<n; i++){
        int min = num[i];
        for (int j=0; j < n; j ++)
            if (adj[i][j] == 1 && num[j] <min)
                min = num[j];
        p = p+num[i]-min;
    }
    return p;
}


// count number of neighbors
int graph::CountNb(int* tnum){
int k=0;
    for(int i=0; i<n; i++)
```

```
        if(tnum[i]&&(tnum[i] != -1))
            for(int j=0; j < n; j++)
                if (adj[i][j]&&(!tnum[j])){
                    k++;
                    tnum[j] = -1;
                }
    for(i=0; i<n; i++)
        if (tnum[i] ==-1) tnum[i] = 0;
    return k;
}


int graph::find(int c){
int tnum[max]; // numbering array that I am using now
    for(int i = 0; i < n; i++) // initialize tnum[]
        tnum[i] = num[i];
int m,b;
int min = n*(n-1)/2;
    for(int l=0; l<n; l++)
        if(!tnum[l]) {
           tnum[l]= c;
           m = CountNb(tnum);
```

```
            if (m < min){
                min = m;
                b = l;
            }
            tnum[l] = 0;
        }
    return b;
}

void graph::mynum(){
int i,j;
    num[0] = 1;
    for (i=2; i <= n; i++){
        j = find(i); // find the vertex to number
        num[j] = i;
    }
}

int main(){
ifstream finN("adj.dat");
int p;
```

```
    while (finN) {
        graph g1;
        g1.doit(finN);
        if (finN.eof()) break;
        g1.mynum();
        cout << "*** by mynum *** The numbering is : ";
        g1.printNum();
        p = g1.profile();
        cout << "The profile is " << p << endl<< endl<< endl;
    }
}
```

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