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Semi-Strongly Regular Graphs and Generalized Cages

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SEMI-STRONGLY REGULAR GRAPHS AND GENERALIZED CAGES

by

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Two well-known classes of graphs, strongly regular graphs and cages, have been studied extensively by many researchers for a long period of time. In this dissertation, we mainly deal with semi-strongly regular graphs, a class of graphs including all strongly regular graphs, and (r, g, t)-cages, a generalization of the usual cage concept.

Chapter I introduces the two new concepts: semi-strongly regular graphs and generalized (r, g, t)-cages, gives necessary conditions for the existence of semi-strongly regular graphs and some interesting properties regarding common neighbors of pairs of vertices, and shows connections between these two new concepts and the old ones as well as connections between the two new concepts themselves.

In Chapter II, we study the existence problem of clique-disjoint semi-strongly regular graphs. We give lower bounds for the order of clique-disjoint semi-strongly regular graphs in Section 1. Then in Section 2 we prove that the necessary conditions given in Chapter I are also sufficient for a clique-disjoint semi-strongly regular graph of order n to exist when n is relatively large. And in Section 3, we show that certain clique-disjoint semi-strongly regular graphs can not exist when their orders n are too close to the lower bound.

Chapter III is devoted to generalized (r, g, t)-cages. We prove that an (r, g, t)-cage always exists in Section 1. Then we study the lower bounds for the order of (r, g, t)-cages in Section 2 and list some known (r, g, t)-cages in the last section.

Maximal graphs without C_4 on 31 vertices, which is the smallest unsettled case to a conjecture of Erdös, are studied in Chapter IV and some open questions are mentioned in Chapter V.

**To my parents
and my son Andrew**

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CHAPTER I

INTRODUCTION

1.1 Strongly Regular Graphs and Semi-Strongly Regular Graphs

In 1966, Erdös, Rényi, and Sós [7] proved the following result.

Theorem 1.1 (Friendship Theorem). Let G be a graph such that any two vertices of G have a unique common neighbor. Then $G = K_1 + mK_2$, i.e., a number of triangles with a common vertex (see Figure 1.1).

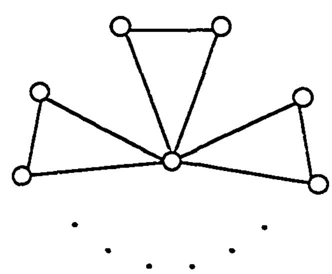

Figure 1.1. $G = K_1 + mK_2$.

Another concept concerning common neighbors of pairs of vertices, so called strongly regular graphs, was introduced by Bose [1] in 1963. An r-regular graph G of order n is said to be *[n, r, λ, μ]-strongly regular* if every two adjacent vertices of G have λ common neighbors and every two non-adjacent vertices have μ common neighbors. For example, complete graphs and regular complete bipartite graphs are strongly regular, the Petersen graph is a strongly regular graph with parameters n = 10, r = 3,

$\lambda = 0$, and $\mu = 1$, the line graph of the complete graph K_n is $[\frac{1}{2}n(n-1),$ $2(n-2), n-2, 4]$-strongly regular, and the line graph of the complete bipartite graph $K_{n,n}$ is $[n^2, 2(n-1), n-2, 2]$-strongly regular. The existence problem of strongly regular graphs have been studied extensively by many researchers. Some necessary conditions for an $[n, r, \lambda, \mu]$-strongly regular graph to exist have been developed. A very strong necessary condition is the following integrality condition. For more information about strongly regular graphs, see [3].

Theorem 1.2 (Integrality Condition). If there is a $[n, r, \lambda, \mu]$-strongly regular graph, then the numbers

$$f_{1,2} = \frac{1}{2}\left\{ n-1 \pm \frac{(n-1)(\mu-\lambda)-2r}{\sqrt{(\mu-\lambda)^2+4(r-\mu)}} \right\}$$

are non-negative integers.

We note in the following two propositions that, in certain cases, the regularity condition in the definition of a strongly regular graph is implied by the other two conditions. Proposition 1.1 can also be viewed, to some extent, as an extension of the Friendship Theorem.

Proposition 1.1. Suppose G is a graph such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have one common neighbor. If G is not regular, then $G = K_1 + mK_{\lambda+1}$.

Proof. We first show that every two non-adjacent vertices of G must have the same degree. Let v and w be two non-adjacent vertices of G. Then there is a unique vertex u adjacent to both v and w. Since every two adjacent vertices have λ common neighbors, v and u must be in a complete subgraph $H_1 = K_{\lambda+2}$, u and w must be in a complete subgraph $H_2 = K_{\lambda+2}$, and u is the only common vertex of H_1 and H_2. Let A be the set of vertices adjacent to v but not in H_1 and let B be the set of vertices adjacent to w but not in H_2. Then $A \cap B = \emptyset$. Note that A induces a union of $K_{\lambda+1}$'s, so does B. For every vertex x in A, there is a unique vertex y in B adjacent to x. A similar argument holds with A and B interchanged. So $|A| = |B|$, that is $\deg(v) = \deg(w)$.

Since G is not regular, there are two vertices u and v with different degrees. Then u and v must be adjacent. A complete subgraph $H = K_{\lambda+2}$ containing u and v is forced. If x is any vertex not in H, then x has to be adjacent to exactly one vertex in H. In fact, x must be adjacent to either u or v for otherwise, we would have $\deg(u) = \deg(v)$. Without loss of generality, assume x is adjacent to u. Then all the vertices in H other than u have the same degree and all the vertices not in H are adjacent to u. It follows that G consists of a number of $K_{\lambda+2}$'s with a common vertex u. □

A *clique* of a graph G is a maximal complete subgraph of G. We say that a graph G is *clique-disjoint* if every pair of cliques of G are edge-disjoint.

Proposition 1.2. Suppose G is a graph such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have $\mu \geq 2$ common neighbors. If G is clique-disjoint, then G must be regular.

Proof. Let u and v be two adjacent vertices of G. Then u and v have λ common neighbors, say they are $w_1, w_2, ..., w_\lambda$. Since all cliques of G are edge-disjoint, the vertices $u, v, w_1, w_2, ..., w_\lambda$ must induce a complete subgraph $K_{\lambda+2}$ in G. Let A be the set of the vertices adjacent to u but not in $\{v, w_1, w_2, ..., w_\lambda\}$ and let B be the set of the vertices adjacent to v but not in $\{u, w_1, w_2, ..., w_\lambda\}$. Since every two non-adjacent vertices have $\mu \geq 2$ common neighbors, every vertex in A has $\mu - 1$ common neighbors with the vertex v. Then every vertex in A is adjacent to $\mu - 1$ vertices in B. Similarly, every vertex in B is adjacent to $\mu - 1$ vertices in A. This implies $|A| = |B|$. Thus, deg(u) = deg(v). It follows that G is regular. □

Note that if every two adjacent vertices have exactly one common neighbor in a graph G, then G must be clique-disjoint. Proposition 1.2 has the following immediate consequence.

Corollary 1.1. Suppose G is a graph such that every two adjacent vertices have exactly one common neighbor and every two non-adjacent vertices have $\mu \geq 2$ common neighbors. Then G is regular and so it is strongly regular.

Next, we introduce a weaker concept by dropping off one of the two restrictions on common neighbors of pairs of vertices in a strongly regular graph.

Definition 1.1. An r-regular graph of order n is *[n, r, λ]-semi-strongly regular* if every two adjacent vertices of G have λ common neighbors.

For example, the graph shown in Figure 1.2 is a [12, 4, 1]-semi-strongly regular graph which is not strongly regular. From the definition, we see that if G is a triangle free r-regular graph, then the line graph L(G) of G is semi-strongly regular. The graph in Figure 1.2 is the line graph of the cube $L(Q_3)$. Also, we note that G is an [n, r, λ]-semi-strongly regular graph if and only if its complement $\bar{G}$ is a regular graph in which any two nonadjacent pair of vertices have a constant $n - 2r + \lambda$ common neighbors. In this view, one may have an equivalent definition for a semi-strongly regular graph by dropping off the restriction that every two adjacent vertices have λ common neighbors in a strongly regular graph.

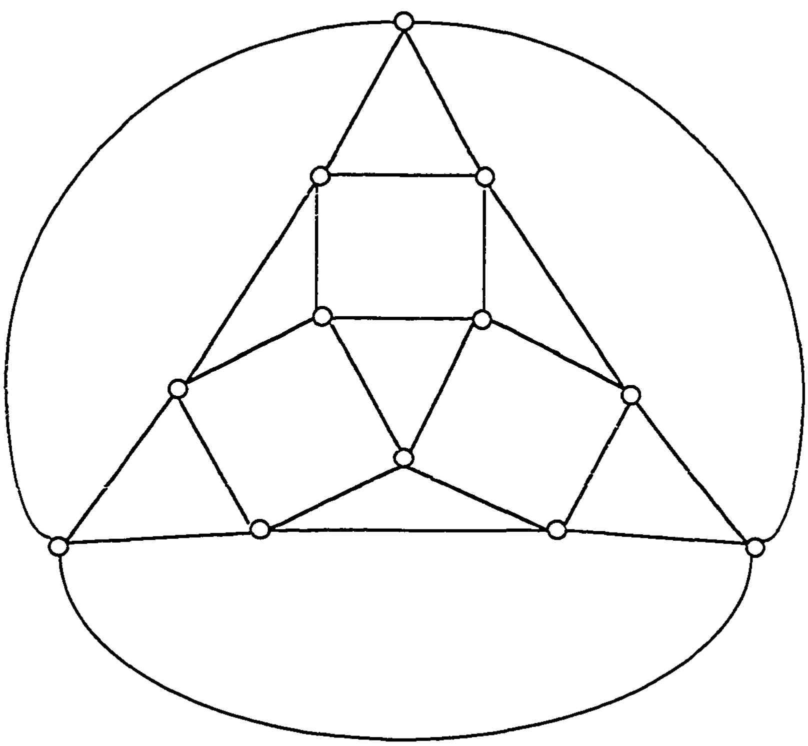

Figure 1.2. A [12, 4, 1]-semi-strongly Regular Graph.

The following proposition gives necessary conditions for an $[n, r, \lambda]$-semi-strongly regular graph to exist.

Proposition 1.3. If G is an $[n, r, \lambda]$-semi-strongly regular graph, then

(1) $6 \mid nr\lambda$,

(2) $n \geq 2r - \lambda$.

Proof. Suppose G is an $[n, r, \lambda]$-semi-strongly regular graph. For every edge e of G, let $T(e)$ be the number of triangles in G containing e. Clearly, the total number of triangles in G is $T = \frac{1}{3}\left(\sum_{e \in E(G)} T(e)\right)$. Since every edge of G is in λ triangles, it follows that $T = \frac{1}{6} nr\lambda$ and so $6 \mid nr\lambda$. By a direct calculation, we obtain (2). □

Clearly, for a clique-disjoint semi-strongly regular graph, we have the following necessary conditions.

Proposition 1.4. If G is a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph, then

(1) all cliques of G have the same order $\lambda + 2$,

(2) $(\lambda + 1)(\lambda + 2) \mid nr$,

(3) $(\lambda + 1) \mid r$ and each vertex is in $\frac{r}{\lambda + 1}$ edge-disjoint cliques.

Later, we will pay most of our attention to clique-disjoint semi-strongly regular graphs. In the next chapter, we will show by construction that the above necessary conditions are also sufficient for a clique-disjoint semi-strongly regular graph to exist when n is relatively large.

Now, we give the following connections between strongly regular graphs and semi-strongly regular graphs.

Theorem 1.3. If both a graph G and its complement $\overline{G}$ are semi-strongly regular, then G is strongly regular.

Proof. Let G be an [n, r, λ]-semi-strongly regular graph whose complement $\overline{G}$ is also semi-strongly regular. For each nonadjacent pair of vertices u and v in G, let μ_{uv} be the number of common neighbors of u and v in G. Then in $\overline{G}$, u and v have $\overline{\lambda} = n - 2r - 2 + \mu_{uv}$ common neighbors. Since $\overline{G}$ is semi-strongly regular, $\overline{\lambda}$ is a constant which implies that μ_{uv} is a constant over all nonadjacent pairs u and v. Thus, G is strongly regular. □

Corollary 1.2. If G is a self-complementary semi-strongly regular graph, then G is strongly regular.

A *clique graph* of a graph G is a new graph whose vertices are the cliques of G and two vertices are adjacent if the corresponding two cliques intersect. We state the next proposition without proof as it is an easy consequence from the definitions and Proposition 1.4.

Proposition 1.5. The clique graph of a clique-disjoint strongly regular graph is semi-strongly regular.

1.2 Cages and Generalized Cages

For $r \geq 2$ and $g \geq 3$, an *(r, g)-cage* is an r-regular graph of minimum order $n = f(r, g)$ which has girth g. That an (r, g)-cage always exists has been proved by Erdös and Sachs [8]. For a good survey on cages, see [14].

Note that in an r-regular graph with girth $g \geq 4$, every clique is a K_2, each vertex is in r cliques, and the girth g is the same as the minimum length of a cycle with edges from distinct cliques. An (r, g)-cage is simply such a graph of minimum order. This observation leads us to give the following generalization of the cage concept.

Definition 1.2. For $t \geq 2, r = k(t-1)$, and $g \geq 4$, we define an *(r, g, t)-cage* to be a clique-disjoint r-regular graph of minimum order f(r, g, t) such that every clique has t vertices, every vertex is in k cliques, and the minimum length of a cycle with edges from distinct cliques is g.

For example, the line graph L(P) of the Petersen graph P is a (4, 5, 3)-cage (See Figure 1.3).

Given a clique-disjoint graph G, we call the minimum length of a cycle in G with edges from distinct cliques the *clique-girth* of G. In fact, an (r, g, t)-cage is a clique-disjoint [n, r, t − 2]-semi-strongly regular graph of minimum order which has clique-girth g.

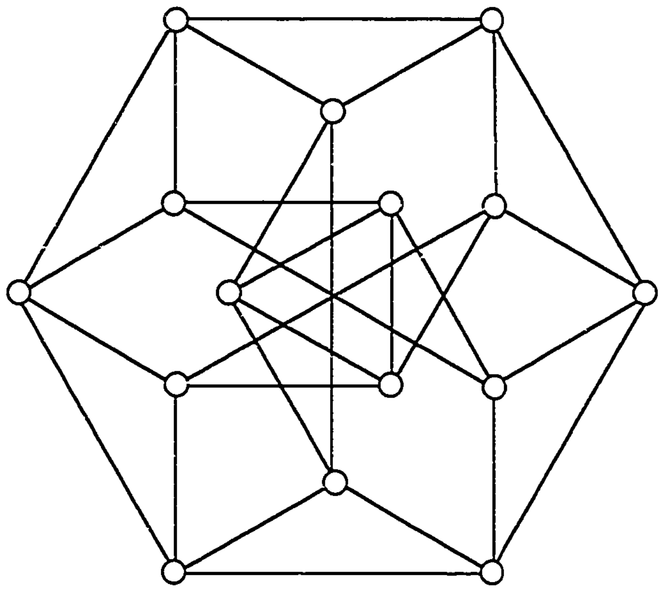

Figure 1.3. The Line Graph of Petersen Graph.

In Chapter III, we will prove that for $g \geq 4$, an (r, g, t)-cage always exists. Then, we examine lower bounds for the number f(r, g, t) and show some connections of generalized cages with generalized quadrangles, strongly regular graphs, and Moore geometries, from which many generalized cages are obtained.

CHAPTER II

SEMI-STRONGLY REGULAR GRAPHS

2.1 Lower Bounds for the Order of Clique-Disjoint Semi-Strongly Regular Graphs

We begin this section by giving a lower bound on the order of a clique-disjoint semi-strongly regular graph.

Proposition 2.1. If G is a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph, then $n \geq (\lambda + 2)(r - \lambda)$.

Proof. By Proposition 1.2, all cliques of G are of order $\lambda + 2$. Let H be a clique of G and let $u_1, u_2, \ldots, u_{\lambda+2}$ be the vertices of H. Then for $i \neq j$, u_i and u_j have no common neighbors not in H. Thus, it follows that

$$n \geq (\lambda + 2)\left[r - (\lambda + 1)\right] + \lambda + 2 = (\lambda + 2)(r - \lambda). \quad \square$$

When $\lambda = 1$, every semi-strongly regular graph is clique-disjoint. Thus, Propositions 1.4 and 2.1 give the following necessary conditions for an $[n, r, 1]$-semi-strongly regular graph.

Corollary 2.1. If G is an $[n, r, 1]$-semi-strongly regular graph, then

(1) $n \geq 3r - 3$,

(2) $6 \mid nr$,

(3) r is even.

Next, we discuss what are those clique-disjoint semi-strongly regular graphs with order equal to the lower bound given in Proposition 2.1. Before getting into it, we first introduce the following concepts (see [3] for a reference).

A *partial geometry* with parameters (k, t, h), where $k > 1$ and $t > 1$, consists of a set of points and a set of lines such that (1) every point is in k lines and every line has t points; (2) two points are in at most one common line; (3) if a point p is not in a line L, then p is collinear with exactly h points of L. A *generalized quadrangle* GQ (k, t) is a special partial geometry with $h = 1$. The *point graph* of a partial geometry is the graph whose vertices are the points of the geometry, and whose edges correspond to collinear point-pairs.

We also need Theorem 6.3 in Cameron [3] which is stated as follows.

Theorem 2.1. The parameters (k, t, h) of a partial geometry satisfy the inequality

$$(k-1)(t-2h) \leq (t-h)^2 (t-2).$$

Now, we show that a clique-disjoint semi-strongly regular graph which attains the lower bound in Proposition 2.1 must be strongly regular.

Theorem 2.2. Let G be a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph of order $n = (\lambda + 2)(r - \lambda)$. Then G must be $[n, r, \lambda, k]$-strongly regular with $k = \frac{r}{\lambda + 1}$. Moreover, G is the point graph of a generalized quadrangle $GQ(k, \lambda + 2)$.

Proof. Let G be a clique-disjoint [n, r, λ]-semi-strongly regular graph of order $n = (\lambda + 2)(r - \lambda)$. Since G is clique-disjoint and every vertex is in k edge-disjoint cliques of order $\lambda + 2$, it follows that every pair of non-adjacent vertices have at most k common neighbors. We claim that every two non-adjacent vertices have exactly k common neighbors in G. We will verify the claim by showing that for a vertex v, every vertex not adjacent to v has k common neighbors with v. Let N(v) denote the set of vertices adjacent to v. Let $w_1, w_2, \ldots, w_{n-r-1}$ be the vertices not adjacent to v and let v and w_j have μ_j common neighbors for $1 \le j \le n - r - 1$. Then all $\mu_j \le k$ as every pair of non-adjacent vertices have at most k common neighbors. We count the number of edges between N(v) and $W = \{w_1, w_2, \ldots, w_{n-r-1}\}$ in two ways. For any vertex u in N(v), there are $r - \lambda - 1$ edges joining u with vertices in W, whereas for each w_i in W, there are μ_i edges joining w_i with vertices in N(v). Thus, we have

$$r(r - \lambda - 1) = \sum_{i=1}^{n-r-1} \mu_i \le k(n - r - 1)$$

$$= k[(\lambda + 2)(r - \lambda) - r - 1] = r(r - \lambda - 1)$$

which implies that $\mu_i = k$ for all i, $1 \le i \le n - r - 1$. Thus, G is a [n, r, λ, k]-strongly regular graph. And it is easy to see that G is the point graph of a generalized quadrangle $GQ(k, \lambda + 2)$. □

Corollary 2.2. A [3r − 3, r, 1]-semi-strongly regular graph is [3r − 3, r, 1, $\frac{r}{2}$]-strongly regular. Moreover, r can only be 2, 4, 6, and 10, where K_3, $K_3 \times K_3$, and $\overline{L(K_6)}$ are the unique strongly regular graphs corresponding to $r = 2, 4, 6$, respectively, and the point graph of $GQ(5, 3)$ is $[27, 10, 1, 5]$-strongly regular.

Proof. Since every $[3r-3, r, 1]$-semi-strongly regular graph G is clique-disjoint, it follows from Theorem 2.2 that G is $[3r-3, r, 1, \frac{r}{2}]$-strongly regular and G is the point graph of a generalized quadrangle $GQ(\frac{r}{2}, 3)$. By Theorem 2.1 with $h = 1$, we have $\frac{r}{2} - 1 \leq 4$, and so $r \leq 10$. As r is even, $r = 2, 4, 6, 8,$ or 10. By Theorem 1.2 (the Integrality Condition), $r \neq 8$. Since the strongly regular graphs of order at most 15 are unique, the corollary follows. □

2.2 The Existence of Clique-Disjoint Semi-Strongly Regular Graphs

We know from Proposition 1.4 that if G is a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph, then r is a multiple of $\lambda + 1$ and $(\lambda + 1)(\lambda + 2) \mid nr$. In this section, we will show that these two necessary conditions are also sufficient for a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph to exist when n is relatively large.

Throughout this section, we let $r = (\lambda + 1)k$. We first consider the case where n is a multiple of $\lambda + 2$. Assume

$$n = (\lambda + 2)m$$

with

$$m \geq k^{\lambda}(k-1) + 1.$$

Define the graph $G_{\lambda,k,m}$ as a graph of order n with the vertex set

$$V(G_{\lambda,k,m}) = \{ (i,j) \mid 0 \leq i \leq \lambda + 1 \text{ and } 0 \leq j \leq m-1 \}$$

such that $G_{\lambda,k,m}$ consists of k factors of degree $\lambda + 1$, say $F_0, F_1, ..., F_{k-1}$ with each $(\lambda+1)$-factor F_i being a union of m complete graphs of $\lambda + 2$ vertices, that is,

$$F_i = Q_{i,0} \cup Q_{i,1} \cup ... \cup Q_{i,m-1},$$

where each $Q_{i,j}$ is a $K_{\lambda+2}$ with vertex set

$$V_{i,j} = \{(0,j)\} \cup \left\{(x, j + i k^{x-1}) \mid 1 \le x \le \lambda + 1\right\}$$

and the sum in above expression is taken modulo m.

We give the graph $G_{1,3,8}$ as an illustration of how graph $G_{\lambda,k,m}$ is constructed. $G_{1,3,8}$ consists of 3 factors of degree 2, call them F_0, F_1, and F_2, where each F_i is obtained by horizontally translating $Q_{i,0}$ shown in Figure 2.1 along the vertices in the top row. Note that the first index is computed mod 3 and the second mod 8.

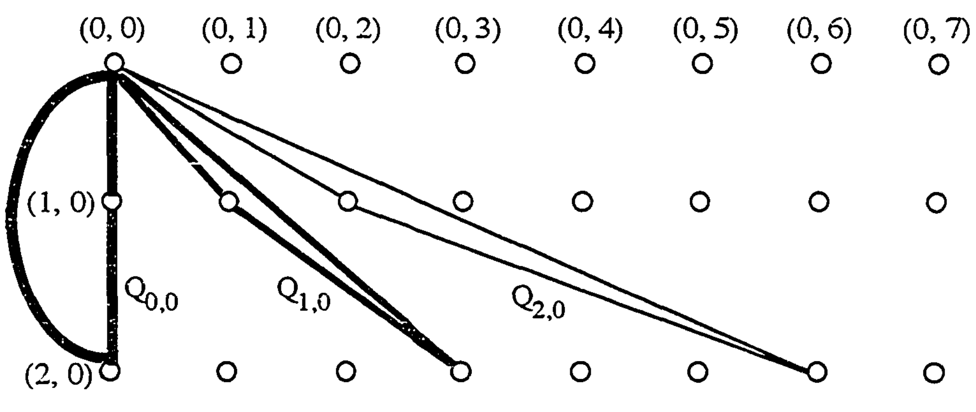

Figure 2.1. The Triangles Through the Vertex (0, 0) in $G_{1,3,8}$.

Notice that $G_{1,3,8}$ is regular of degree 6 and of order 24. For convenience, later we will call the set of vertices $\{(i,j) \mid 0 \le j \le m-1\}$ the i-th row and the set of vertices $\{(i,j) \mid 0 \le i \le \lambda + 1\}$ the j-th column.

Theorem 2.3. For positive integers λ, r, and n with $r = (\lambda+1)k$ and

$$n = (\lambda+2)m \geq (\lambda+2)\big((k-1)k^{\lambda}+1\big),$$

$G_{\lambda,k,m}$ is a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph.

Proof. Assume $k \geq 2$. We first show that $G_{\lambda,k,m}$ is r-regular by showing that $F_0, F_1, \ldots, F_{k-1}$ are edge-disjoint. Suppose, to the contrary, that F_a and F_b are not edge-disjoint for some $a \neq b$. Let edge $e \in E(Q_{a,x}) \cap E(Q_{b,y})$ for some x and y. Then there exist two integers p and q with $0 \leq p < q$ such that if $p > 0$, then the edge e is

$$\begin{aligned} e &= (p, x + ak^{p-1})(q, x + ak^{q-1}) \\ &= (p, y + bk^{p-1})(q, y + bk^{q-1}) \end{aligned}$$

and otherwise

$$\begin{aligned} e &= (0, x)(q, x + ak^{q-1}) \\ &= (0, y)(q, y + bk^{q-1}). \end{aligned}$$

This implies that $x = y$ or

$$x + ak^{p-1} \equiv y + bk^{p-1} \pmod{m}$$

and

$$x + ak^{q-1} \equiv y + bk^{q-1} \pmod{m}.$$

It follows that

$$ak^{q-1} \equiv bk^{q-1} \pmod{m}$$

or

$$(a-b)\left(k^{p-1} - k^{q-1}\right) \equiv 0 \pmod{m}$$

which implies that $a = b$ or $p = q$, contradicting the assumption that $a \neq b$ and $p \neq q$. Thus, $F_0, F_1, \ldots, F_{k-1}$ are edge-disjoint and $G_{\lambda,k,m}$ is r-regular. To see that $G_{\lambda,k,m}$ is a clique-disjoint semi-strongly regular graph, it suffices to prove that $G_{\lambda,k,m}$ has no triangle with edges from different F_i's. Let T be a triangle on the vertices $(a, x), (b, y)$ and (c, z). Then a, b and c are all distinct. Without loss of generality, assume $a < b < c$. And assume that edges

$$e_1 = (a, x)(b, y) \in F_\alpha,$$

$$e_2 = (b, y)(c, z) \in F_\beta,$$

and

$$e_3 = (a, x)(c, z) \in F_\gamma.$$

Then there exist three integers h, i and j such that

$$e_1 \in Q_{\alpha,h},$$

$$e_2 \in Q_{\beta,i},$$

and

$$e_3 \in Q_{\gamma,j}.$$

First we consider the case where $a > 0$. We have

$$x = h + \alpha k^{a-1} \equiv j + \gamma k^{a-1} \pmod{m}$$

$$y = i + \beta k^{b-1} \equiv h + \alpha k^{b-1} \pmod{m}$$

$$z = j + \gamma k^{c-1} \equiv i + \beta k^{c-1} \pmod{m}.$$

Thus,

$$\alpha k^{a-1} + \beta k^{b-1} + \gamma k^{c-1} \equiv \gamma k^{a-1} + \alpha k^{b-1} + \beta k^{c-1} \pmod{m}.$$

That is

$$(\gamma - \beta)(k^{c-1} - k^{a-1}) \equiv (\alpha - \beta)(k^{b-1} - k^{a-1}) \pmod{m}.$$

Since $m \geq (k-1)k^{\lambda}+1$, $0 \leq \alpha, \beta, \gamma \leq k-1$ and $a<b<c \leq \lambda+1$,

$$\left| (\gamma-\beta)\left(k^{c-1}-k^{a-1}\right) \right| < m$$

and

$$\left| (\alpha-\beta)\left(k^{b-1}-k^{a-1}\right) \right| < m.$$

Now if any two of α, β and γ are equal, then we can obtain $\alpha = \beta = \gamma$. Thus, we assume α, β and γ are all distinct. If $\gamma-\beta$ and $\alpha-\beta$ are both positive or both negative, then we have

$$(\gamma-\beta)\left(k^{c-1}-k^{a-1}\right) = (\alpha-\beta)\left(k^{b-1}-k^{a-1}\right),$$

i.e.,

$$|\gamma-\beta|\, k^{c-1} = \left| (\alpha-\beta)k^{b-1}+(\gamma-\alpha)k^{a-1} \right|$$

$$\leq (k-1)k^{b-1}+(k-1)k^{a-1} < k^{c-1}$$

which implies $\beta=\gamma$, a contradiction. If one of $\gamma-\beta$ and $\alpha-\beta$ is positive and the other is negative, then we have

$$|\gamma-\beta| \leq k-2$$

and

$$(\gamma-\beta)\left(k^{c-1}-k^{a-1}\right) = (\alpha-\beta)\left(k^{b-1}-k^{a-1}\right) \pm m.$$

It follows that

$$\begin{aligned} m &\leq |\gamma-\beta|\, k^{c-1} + |\alpha-\beta|\, k^{b-1} + |\gamma-\alpha|\, k^{a-1} \\ &\leq (k-2)k^{c-1}+(k-1)k^{b-1}+(k-1)k^{a-1} \\ &\leq (k-2)k^{c-1}+(k-1)k^{c-2}+(k-1)k^{c-3} \\ &\leq (k-1)k^{c-1} \leq (k-1)k^{\lambda}, \end{aligned}$$

contradicting the assumption $m \geq (k-1)k^{\lambda}+1$. If $a=0$, then we have

$$x = h = j$$

$$y = i + \beta k^{b-1} \equiv h + \alpha k^{b-1} \pmod{m}$$

$$z = j + \gamma k^{c-1} \equiv i + \beta k^{c-1} \pmod{m}$$

which gives

$$(\gamma - \beta)k^{c-1} \equiv (\alpha - \beta)k^{b-1} \pmod{m}.$$

By similar argument we have $\alpha = \beta = \gamma$. Therefore, all three edges of the triangle T are in F_α. This completes the proof. □

We now construct $[n, r, \lambda]$-semi-strongly regular graphs for r being a multiple of $(\lambda + 1)(\lambda + 2)$. The technique of constructing this kind of graphs will be modified to construct general semi-strongly regular graphs. First we introduce some terminology which will be used in the constructions.

For an $[n, r, \lambda]$-semi-strongly regular graph G with $\lambda > 0$, we say a set $S = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of G is *a good set* if

(i) all $G_1, G_2, \ldots, G_t$ are vertex-disjoint complete graphs of order $\lambda + 2$,

(ii) G has no edges incident with one vertex in G_i and the other vertex in G_j for $i \neq j$,

(iii) there exists a vertex v_i in each subgraph G_i such that v_i and v_j have no common neighbors for $i \neq j$.

We can now present the construction.

Suppose G is an $[n, r, \lambda]$-semi-strongly regular graph with $\lambda > 0$ and $r = (\lambda + 1)(\lambda + 2)x$. For a good set $S = \{G_1, G_2, \ldots, G_{(\lambda+1)x}\}$ of G, we can obtain a new graph $G(S)$ from G as follows: We remove all the edges from v_i to the other vertices in G_i for $1 \leq i \leq (\lambda + 1)x$, add a new vertex u and join u to every vertex in each G_i and add an edge between each pair of vertices in

$$X_i = \left\{ x_j \mid (i-1)(\lambda+1)+1 \le j \le i(\lambda+1) \right\}$$

for $0 \le i \le x$. Then it is clear that $G(S)$ is an r-regular graph of $n+1$ vertices and every edge is in at least one complete graph of order $\lambda + 2$.

Lemma 2.1. Let G be an $[n, r, \lambda]$-semi-strongly regular graph with $\lambda > 0$ and $r = (\lambda+1)(\lambda+2)x$ and let $S = \{G_1, G_2, ..., G_{(\lambda+1)x}\}$ be a good set of G. Then $G(S)$ is $[n+1, r, \lambda]$-semi-strongly regular. In addition, if G is clique-disjoint, then $G(S)$ is also clique-disjoint.

Proof. From the construction, we know that $G(S)$ is an r-regular graph of order $n+1$ and every two adjacent vertices of $G(S)$ have at least λ common neighbors. We now show that every two adjacent vertices of $G(S)$ have exactly λ common neighbors. Let W be the set consisting of all vertices in G_i for $1 \le i \le (\lambda+1)x$ together with the new vertex u and let v_1, v_2 be two adjacent vertices in $G(S)$. If both v_1 and v_2 are not in W, then clearly v_1 and v_2 have λ common neighbors in $G(S)$ since the original graph G is semi-strongly regular. If exactly one of v_1 and v_2 is in W, then no common neighbors of v_1 and v_2 are in W, and hence v_1 and v_2 have exactly λ common neighbors in $G(S)$. If both v_1 and v_2 are in W, then by the conditions (ii) and (iii) of S and the construction, the edge v_1v_2 is in a unique $K_{\lambda+2}$, and the subgraph induced by W has no triangles which contain two edges from different complete graph of order $\lambda + 2$. Therefore, $G(S)$ is $[n+1, r, \lambda]$-semi-strongly regular. □

Theorem 2.4. Let λ, r, n be positive integers with

$$r = (\lambda+1)\,k = (\lambda+1)(\lambda+2)\,x$$

and

$$\left\lfloor \frac{n}{\lambda+2} \right\rfloor > \begin{cases} (k-1)\,k + \left(\frac{2}{3}k - 1\right)(k+2) & \text{if } \lambda = 1\,; \\ (k-1)\,k^{\lambda} + \left(\frac{\lambda+1}{\lambda+2}k - 1\right)(k^2+1) & \text{if } \lambda \geq 2\,. \end{cases}$$

Then there exists a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph.

Proof. Let $n = (\lambda+2)\,m + y$ with $0 \leq y \leq \lambda + 1$. Then

$$m > \begin{cases} (k-1)\,k + \left(\frac{2}{3}k - 1\right)(k+2) & \text{if } \lambda = 1\,; \\ (k-1)\,k^{\lambda} + \left(\frac{\lambda+1}{\lambda+2}k - 1\right)(k^2+1) & \text{if } \lambda \geq 2\,. \end{cases}$$

By Theorem 2.3, $G_{\lambda,k,m}$ is a clique-disjoint $[n_1, r, \lambda]$-semi-strongly regular graph, where $n_1 = n - y = (\lambda+2)\,m$. We intend to add y new vertices to $G_{\lambda,k,m}$ in a proper way so that the resulting graph is a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph. By Lemma 2.1, it suffices to find y vertex-disjoint good sets $S_0, S_1, \ldots, S_{y-1}$ in $G_{\lambda,k,m}$. Let Q_i be the complete graph $K_{\lambda+2}$ on the i-th column $\{(0, i), (1, i), \ldots, (\lambda+1, i)\}$, for $0 \leq i \leq m-1$. Next let

$$S_j = \left\{ Q_{f(i,j)} \,\middle|\, 0 \leq i \leq (\lambda+1)x - 1 \right\}$$

for $0 \leq j \leq y-1$, where

$$f(i,j) = \begin{cases} i\,(k+1) + j & \text{if } \lambda = 1 \text{ and } k \text{ is even;} \\ i\,(k+2) + j & \text{if } \lambda = 1 \text{ and } k \text{ is odd;} \\ i\,(k^2+1) + j & \text{if } \lambda \geq 2. \end{cases}$$

Since $j \leq y - 1 \leq \lambda \leq k-2$ and $k = (\lambda+2)x$, we have

$$i \le (\lambda+1)x - 1 \le k-2,$$

$$i(k+2) + j \le (k-2)(k+2) + k - 2 \le (k-1)k + \left(\frac{2}{3}k - 1\right)(k+2) < m$$

and

$$i\left(k^2+1\right) + j \le \left[(\lambda+1)x - 1\right]\left(k^2+1\right) + k - 2 < m$$

which imply that $S_0, S_1, \ldots, S_{y-1}$ are vertex-disjoint. Next, we prove that each S_j is a good set in $G_{\lambda,k,m}$. Clearly, condition (i) is satisfied. To verify condition (iii), we let $x_{i,j} = (0, f(i, j))$ for $0 \le i \le (\lambda+1)x - 1$. Then we claim that $N(x_{a,j}) \cap N(x_{b,j}) = \emptyset$ for $a \ne b$. Suppose, to the contrary, that (d, h) is adjacent to both $x_{a,j}$ and $x_{b,j}$. Then by the construction of $G_{\lambda,k,m}$, there exist $0 \le p, q \le k-1$ such that

$$h = f(a,j) + pk^{d-1} \equiv f(b,j) + qk^{d-1} \pmod{m},$$

i.e.,

$$f(a,j) - f(b,j) \equiv -(p-q)k^{d-1} \pmod{m}.$$

It follows that

$$f(a,j) - f(b,j) = -(p-q)k^{d-1}$$

or

$$f(a,j) - f(b,j) = -(p-q)k^{d-1} \pm m.$$

For the former case, we claim that $d = 1$. If $d > 1$, then $k \mid [f(a,j) - f(b,j)]$. Since $\gcd(k, k+1) = 1 = \gcd(k, k^2+1)$ and $\gcd(k, k+2) = 1$ if k is odd, the definition of f and $|a-b| \le k-2$ imply that $k \mid [f(a,j) - f(b,j)]$ is impossible. Thus, $f(a,j) - f(b,j) = -(p-q)$. Since $|p-q| \le k-1$, we have $a = b$, contradicting the condition $a \ne b$. For the latter case, since

$$0 \le a, b \le (\lambda+1)x - 1 = \frac{(\lambda+1)k}{\lambda+2} - 1,$$

$$|a-b| \le \frac{(\lambda+1)k}{\lambda+2} - 1,$$

and since $d - 1 \le \lambda$, we have

$$m \le |f(a,j) - f(b,j)| + |p - q| k^{d-1} \le |f(a,j) - f(b,j)| + (k-1)k^{\lambda}$$
$$< m,$$

a contradiction. Thus the claim holds and condition (iii) for S_j is satisfied.

We now show that condition (ii) is also true for every good set S_j. Assume, to the contrary, that there exist two integers a and b with $0 \le a \ne b \le (\lambda+1)x - 1$ such that there is an edge e joining a vertex of $Q_{f(a,j)}$ to a vertex of $Q_{f(b,j)}$, say

$$e = (h, f(a,j))(d, f(b,j)),$$

where $0 \le h < d \le \lambda + 1$. Then $e \in Q_{p,q}$ for some integers p and q with $0 \le p \le k-1$ and $0 \le q \le m-1$. Then

$$f(a,j) \equiv \begin{cases} q \pmod m & \text{if } h = 0; \\ q + p\,k^{h-1} \pmod m & \text{if } h > 0 \end{cases}$$

and

$$f(b,j) \equiv q + pk^{d-1} \pmod m.$$

It follows that

1) when $h = 0$, $f(b,j) - f(a,j) \equiv pk^{d-1} \pmod m$;

2) when $h > 0$, $f(b,j) - f(a,j) \equiv p(k^{d-1} - k^{h-1}) \pmod m$.

From the discussion for condition (iii) in the last paragraph, we conclude that 1) can not happen. For 2), by similar argument, we must have $h = 1$ which gives

$$f(b,j) - f(a,j) \equiv p(k^{d-1} - 1) \pmod{m}.$$

It follows that either

$$f(b,j) - f(a,j) = p\left(k^{d-1} - 1\right) - m$$

or

$$f(b,j) - f(a,j) = p\left(k^{d-1} - 1\right).$$

If $f(b,j) - f(a,j) = p\left(k^{d-1} - 1\right) - m$, then we have

$$\begin{aligned} m &= p(k^{d-1} - 1) + |f(b,j) - f(a,j)| \\ &\le (k-1)\left(k^{\lambda} - 1\right) + |f(b,j) - f(a,j)| \\ &\le (k-1)k^{\lambda} + |f(b,j) - f(a,j)| < m, \end{aligned}$$

a contradiction. Now we only have to check the case where

$$f(b,j) - f(a,j) = p\left(k^{d-1} - 1\right).$$

We first consider when $\lambda \ge 2$. Then

$$\begin{aligned} p\left(k^{d-1} - 1\right) &= f(b,j) - f(a,j) \\ &= (b-a)\left(k^2 + 1\right) \\ &\le \left[(\lambda + 1)x - 1\right]\left(k^2 + 1\right) \\ &= \left[\frac{(\lambda+1)k}{\lambda+2} - 1\right]\left(k^2 + 1\right) \\ &< (k-1)\left(k^2 + 1\right) \end{aligned}$$

which implies that $d \le 3$ or $p = 0$, for otherwise

$$p\left(k^{d-1} - 1\right) \ge k^3 - 1 \ge (k-1)\left(k^2 + 1\right).$$

Since $b \neq a$, we must have $p > 0$ and $d = 3$. That is,

$$p\left(k^2 - 1\right) = (b - a)\left(k^2 + 1\right) = t\left(k^2 + 1\right)$$

for some positive integer t. Thus, $(p - t)k^2 = p + t < 2k$ which implies $p = t = 0$ as $k = (\lambda + 2)x \geq 2$. This gives $a = b$, contradicting the assumption $a \neq b$. Finally we consider the remaining case with $\lambda = 1$. When k is even, we have

$$p\left(k^{d-1} - 1\right) = f(b, j) - f(a, j) = (b - a)(k + 1) < (k - 1)(k + 1)$$

which implies $d = 2$ and $p(k - 1) = (b - a)(k + 1) = t(k + 1)$ where $t = b - a$. Thus, $(p - t)k = p + t < 2k$. So $k \mid (p + t)$ and $p - t \leq 1$. Since k is even, we have $p + t$ is even, and hence $p - t$ is also even. Therefore $p = t = 0$ and $a = b$, a contradiction. Else when k is odd,

$$k = (\lambda + 2)x = 3x.$$

Similarly, we have

$$p(k - 1) = (b - a)(k + 2) = t(k + 2),$$

where $t = b - a$. Then, $(p - t)k = p + 2t < 3k$. Note that k is divisible by 3. We conclude that $3 \mid (p + 2t)$, and then $3 \mid (p - t)$ so $p = t = 0$ and $a = b$, again a contradiction. Therefore, condition (ii) holds.

This completes the proof. □

From Theorems 2.3 and 2.4, the next result follows immediately.

Corollary 2.3. Let λ, r, n be integers such that $\lambda + 2$ is a prime and

$$\left\lfloor \frac{n}{\lambda+2} \right\rfloor > \begin{cases} (k-1)\,k + \left(\frac{2}{3}k - 1\right)(k+2) & \text{if } \lambda = 1; \\ (k-1)\,k^{\lambda} + \left(\frac{\lambda+1}{\lambda+2}k - 1\right)(k^2+1) & \text{if } \lambda \geq 2, \end{cases}$$

where $k = \frac{r}{\lambda+1}$. Then a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph exists if and only if $(\lambda+1) \mid r$ and $(\lambda+1)(\lambda+2) \mid nr$.

Now, we are ready to construct a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph for any n large enough if

$$(\lambda+1) \mid r \text{ and } (\lambda+1)(\lambda+2) \mid nr.$$

Given positive integers λ, $r = (\lambda+1)k$ and $n = (\lambda+2)m + y$ with $0 \leq y \leq \lambda+1$ such that $(\lambda+1)(\lambda+2) \mid nr$. Then we have

$$(\lambda+2) \mid ky.$$

Let

$$ky = (\lambda+2)p.$$

By Theorem 2.3, $G_{\lambda,k,m}$ is a clique-disjoint $[(\lambda+2)\,m, r, \lambda]$-semi-strongly regular graph. We want to modify the graph $G_{\lambda,k,m}$ and add y vertices together with certain edges to obtain a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph. Suppose in graph $G_{\lambda,k,m}$ there exist a set $S = \{G_1, G_2, \ldots, G_{ky}\}$ of disjoint cliques of order $\lambda+2$ and a set $X = \{x_1, x_2, \ldots, x_{ky}\}$ of independent vertices with $x_i \in V(G_i)$. We define a new graph $G[S, X]$ on n vertices as follows: First, remove all the edges from x_i to the vertices in $V_i = V(G_i) - \{x_i\}$ for $1 \leq i \leq ky$, and add y new vertices $u_1, u_2, \ldots, u_y$.

For $1 \le j \le y$, join vertex u_j to every vertex in V_i for $(j-1)k+1 \le i \le jk$; then, we divide the set X into p subsets $X_1, X_2, ..., X_p$ with $\lambda + 2$ vertices in each subset, i.e.,

$$X_i = \left\{ x_j \mid (i-1)(\lambda+2)+1 \le j \le i(\lambda+2) \right\},$$

and add an edge between each pair of vertices in X_i to form a $K_{\lambda+2}$ for $1 \le i \le p$. It is clear that $G[S, X]$ is r-regular. Similarly to Lemma 2.1, we can derive the following lemma.

Lemma 2.2. Let $S = \{ G_1, G_2, ..., G_{ky} \}$ be a set of disjoint cliques of order $\lambda+2$ in $G_{\lambda,k,m}$ such that each $S_j = \{ G_{(j-1)k+1}, G_{(j-1)k+2}, ..., G_{jk} \}$ is a good set and $X = \{ x_1, x_2, ..., x_{ky} \}$ with $x_i \in V(G_i)$ is an independent set of vertices satisfying $N(x_i) \cap N(x_j) = \emptyset$ for $i \ne j$, then $G[S, X]$ is a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph.

Now, we can present our general existence theorem.

Theorem 2.5. Let $\lambda \ge 2$, n and r be integers and let $n = (\lambda + 2)m + y$ with $0 \le y \le \lambda+1$ and

$$m > (k-1)k^{\lambda} + (ky-1)(k^t+1)$$

where $k = \dfrac{r}{\lambda+1} \ge 2$ and t is an integer such that

$$k^{t-2} < \lambda + 1 \le k^{t-1}.$$

Then a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph exists if and only if $(\lambda+1) \mid r$ and $(\lambda+1)(\lambda+2) \mid nr$.

Proof. Clearly, $t \ge 2$. The necessary condition follows from Proposition 1.4.

Conversely, assume $(\lambda + 1) \mid r$ and $(\lambda + 1)(\lambda + 2) \mid nr$. We know that these two conditions imply $(\lambda + 2) \mid ky$. Assume

$$ky = (\lambda + 2)p.$$

Let G be the graph G[S, X] constructed above by adding y vertices to the graph $G_{\lambda,k,m}$. By Lemma 2.2, it suffices to show that $G_{\lambda,k,m}$ has a set $S = \{G_1, G_2, ..., G_{ky}\}$ of disjoint cliques of order $\lambda + 2$ such that each subset $S_j = \{G_{(j-1)k+1}, G_{(j-1)k+2}, ..., G_{jk}\}$ is a good set and there exists a set $X = \{x_1, x_2, ..., x_{ky}\}$ of independent vertices with $x_i \in V(G_i)$ satisfying $N(x_i) \cap N(x_j) = \emptyset$ for $i \neq j$. In fact, we take G_i to be the clique on the $(i-1)(k^t + 1)$-th column and x_i to be the top vertex of that column, i.e.,

$$x_i = (0, (i-1)(k^t + 1)) \quad \text{for} \quad 1 \le i \le ky.$$

Let

$$S = \{G_1, G_2, ..., G_{ky}\},$$
$$X = \{x_1, x_2, ..., x_{ky}\}$$

and

$$S_j = \{G_{(j-1)k+1}, G_{(j-1)k+2}, ..., G_{jk}\}.$$

It is clear that X is an independent set. We will show that each S_j is a good set. First, we prove S_j satisfies the condition (ii) of a good set. Suppose, to the contrary, that there exist two integers a and b with $(j-1)k + 1 \le a \neq b \le jk$ such that there is an edge from a vertex of G_a to a vertex of G_b, say

$$e = \big(h, (a-1)(k^t + 1)\big)\big(d, (b-1)(k^t + 1)\big)$$

with $0 \le h < d \le \lambda + 1$. Then $e \in Q_{p,q}$ for some $0 \le p \le k-1$ and $0 \le q \le m-1$. This implies that

$$(a-1)(k^t+1) \equiv \begin{cases} q \pmod{m} & \text{if } h=0; \\ q+pk^{h-1} \pmod{m} & \text{if } h>0 \end{cases}$$

and

$$(b-1)(k^t+1) \equiv q+pk^{d-1} \pmod{m}.$$

It follows that

1) when $h=0$, $(b-a)(k^t+1) \equiv pk^{d-1} \pmod{m}$;

2) when $h>0$, $(b-a)(k^t+1) \equiv p(k^{d-1}-k^{h-1}) \pmod{m}$.

Since $|b-a| \le k-1$, we have

$$|(b-a)(k^t+1)| \le (k-1)(k^t+1) < m.$$

Also,

$$p(k^{d-1}-k^{h-1}) \le pk^{d-1} \le (k-1)k^{\lambda} < m.$$

If 2) is true, that is,

$$(b-a)(k^t+1) \equiv p(k^{d-1}-k^{h-1}) \pmod{m} \text{ with } h>0,$$

then we must have

$$(b-a)(k^t+1) = p(k^{d-1}-k^{h-1}), \tag{2.1}$$

for otherwise,

$$(b-a)(k^t+1)+m = p(k^{d-1}-k^{h-1})$$

which implies

$$m \le (k-1)k^{\lambda}+(ky-1)(k^t+1),$$

contradicting the definition of m. Since $0 \le b-a \le k-1$ and

$$\gcd(k,\ k^t+1)=1,$$

equation (2.1) implies $h=1$ which gives

$$(b-a)(k^t+1) \equiv p(k^{d-1}-1),$$

or equivalently

$$0 \le pk^{d-1}-(b-a)k^t = p+(b-a) \le 2k-2.$$

Recall that $t \ge 2$ which implies $d-1 \le 1$. Thus,

$$pk^{d-1}-(b-a)k^t = p+(b-a) \le pk-(b-a)k^t$$

which leads to $k^2+1 \le (b-a)(k^t+1) \le p(k-1) \le (k-1)^2$ as $b-a > 0$, a contradiction that completes the discussion of 2). Similarly, for 1), we must have $d = 1$. Then

$$k^2+1 \le (b-a)(k^t+1) = p \le k-1$$

which is impossible. Thus, each S_j satisfies the condition (ii).

To see that $N(x_i) \cap N(x_j) = \emptyset$ for $i \ne j$, we suppose, to the contrary, that there exists a vertex (d, h) adjacent to both x_i and x_j for some $i \ne j$. Then by the construction of $G_{\lambda,k,m}$, there exist $0 \le p, q \le k-1$ such that

$$h \equiv (i-1)(k^t+1)+ pk^{d-1} \equiv (j-1)(k^t+1)+qk^{d-1} \pmod m,$$

that is,

$$(i-j)(k^t+1) \equiv (q-p)k^{d-1} \pmod m.$$

Notice that

$$|(i-j)(k^t+1)| \le (ky-1)(k^t+1) < m$$

and

$$|(q-p)k^{d-1}| \le (k-1)k^{\lambda} < m.$$

Then either

$$(i-j)(k^t+1) = (q-p)k^{d-1} \pm m,$$

which leads to

$$m \le (k-1)\,k^{\lambda} + (ky-1)\,(k^t+1),$$

again contradicting the definition of m;

or

$$(i-j)\,(k^t+1) = (q-p)k^{d-1}.$$

For this second case, if $d-1 \ge t$, then $k^t \mid (i-j)\,(k^t+1)$. That is the same as $k^t \mid (i-j)$. But

$$|\,i-j\,| \le ky-1 \le (\lambda+1)k-1 \le k^t-1$$

which is impossible. Thus, $d-1 < t$. We have

$$k^{d-1} \mid (i-j)$$

and

$$q-p = a(k^t+1)$$

for some integer a. But $a(k^2+1) \le a(k^t+1) = q-p \le k-1$ implies $t = 0$, and so $i = j$, contradicting the assumption $i \ne j$. Therefore the independent set $X = \{x_1, x_2, \ldots, x_{ky}\}$ satisfies conditon $N(x_i) \cap N(x_j) = \emptyset$ for $i \ne j$ and each $S_j = \{G_{(j-1)k+1}, G_{(j-1)k+2}, \ldots, G_{jk}\}$ is a good set. This completes the proof. □

2.3 The Nonexistence of Some Clique-Disjoint Semi-Strongly Regular Graphs

Recall that a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph must satisfy the necessary conditions given in Proposition 1.4 as well as the lower bound $n \ge (\lambda+2)\,(r-\lambda)$ given in Proposition 2.1. Then, in Section 2.2, we showed that when n is large enough, such as n is bigger than the lower bound given in Theorem 2.5, the necessary conditions in Proposition 1.4 are also sufficient for a clique-disjoint $[n, r, \lambda]$-semi-strongly regular graph to exist. Now, an immediate question is: Do there exist $[n, r, \lambda]$-semi-strongly regular graphs for those n between the two bounds mentioned above? In this section we will give further study to the existence problem of $[n, r, \lambda]$-semi-strongly regular graphs and prove the nonexistence of certain such graphs for some remaining values of n with $\lambda = 1$.

In Corollary 2.2 we observed that the lower bound $n = 3r - 3$ is attained for certain $[n, r, 1]$-semi-strongly regular graphs which then also become strongly regular. We now wish to show that other $[n, r, 1]$-semi-strongly regular graphs can not exist if n is too close to the bound $3r - 3$. Specifically:

Theorem 2.6. For $r \neq 6, 12$, there is no $[n, r, 1]$-semi-strongly regular graph for $n = 3r-2$ or $3r-1$.

Proof. For $r \not\equiv 0 \pmod 6$, by Corollary 2.1, n is divisible by 3, so the result follows. Assume $r = 6k$ with $k \geq 3$. Suppose, to the contrary, that there exists an $[n, r, 1]$-semi-strongly regular graph for $n = 3r - 2$ or $3r-1$. Clearly, every vertex of G lies in $3k$ triangles. Let v_1, v_2, v_3 be the vertices of a triangle in G. For $1 \leq i \leq 3$ and for $0 \leq t \leq 3k - 2$, let v_i, $x_{i,2t+1}, x_{i,2t+2}$ be the vertices of the other $3k-1$ triangles containing v_i (for $k = 3$ see Figure 2.2) and let

$$X_i = \left\{ x_{i,j} \mid 1 \leq j \leq 6k-2 \right\}.$$

Since each edge is in a unique triangle, $X_i \cap X_j = \emptyset$ for $i \neq j$ and each X_i induces a matching with $3k-1$ edges $x_{i,2t+1}x_{i,2t+2}$ for $0 \leq t \leq 3k-2$. Moreover, for each $v \neq v_i$, at most one of the two vertices $x_{i,2t+1}$ and

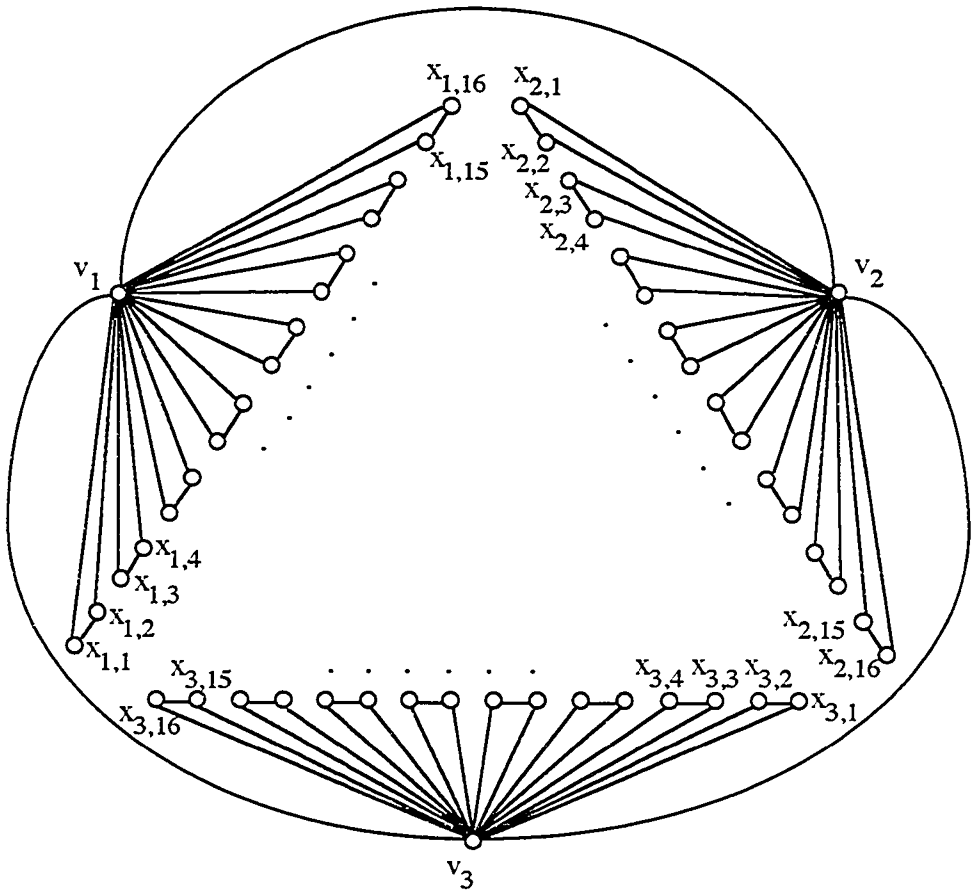

Figure 2.2. A Subgraph of G for k = 3.

$x_{i,2t+2}$ is adjacent to v for $0 \leq t \leq 3k-2$. Thus, each vertex v not in $\{v_i\} \cup X_i$ can have at most $3k-1$ neighbors in X_i. Let

$$X = X_1 \cup X_2 \cup X_3$$

and

$$A = \{v_1, v_2, v_3\} \cup X.$$

Then $|A| = 3r-3$. Observe that a triangle whose vertices are in X must contain exactly one vertex from each X_i. We call such a triangle an inner triangle. For $x \in X$, let $m(x)$ be the number of inner triangles containing x. We claim that for $1 \le i \le 3$, if G has a vertex $w \in V(G)-A$ such that

$$d = \sum_{j \ne i} d_j \ge 4k,$$

where $d_j = |N(w) \cap X_j|$, then there is at most one vertex y in X_i adjacent to w with $m(y) = 3k-2$. For otherwise, suppose that there are two vertices y_1 and y_2 in X_i which are adjacent to w such that $m(y_1) = m(y_2) = 3k-2$. Since every inner triangle containing y_1 or y_2 contains no vertices in $N(w)$, it follows that y_1 and y_2 have at least $d_j - 2$ common neighbors in X_j for $j \ne i$. For the set N of common neighbors of y_1 and y_2 in X, $|N| \ge \sum_{j \ne i} d_j - 4$ and N must be an independent set. Since y_1 is in $m(y_1) = 3k-2$ inner triangles, we have

$$2(3k-2) - |N| \ge |N|,$$

i.e.,

$$3k-2 \ge |N| \ge d-4 \ge 4k-4$$

which yields $k \le 2$, contradicting the assumption that $k \ge 3$. Thus, the claim holds. Now, we consider the cases when $n = 3r-2$ and $n = 3r-1$ seperately.

Case 1. $n = 3r-2$. Then $|V(G)-A| = 1$. Let u be the additional vertex. Let

$$d_i = |N(u) \cap X_i|$$

for $1 \le i \le 3$. It is obvious that each $d_i \le 3k-1$ and $d_1 + d_2 + d_3 = r = 6k$. Without loss of generality, assume $d_1 \ge d_2 \ge d_3$. Then $d_1 + d_2 \ge 4k$ and $d_3 \ge 2$. Since $m(x) = 3k-2$ for every x adjacent to u, from the claim we have $d_3 \le 1$, a contradiction.

Case 2. $n = 3r-1$. Then $|V(G) - A| = 2$. Let u and v be the two additional vertices and let

$$d_i = |N(u) \cap X_i|$$

and

$$t_i = |N(v) \cap X_i|$$

for $1 \le i \le 3$. It is easy to see that $d_i \le 3k-1$ and $t_i \le 3k-1$. Without loss of generality, assume $d_1 \ge d_2 \ge d_3$. Then $d_1 + d_2 \ge \frac{2}{3}(6k-1)$ which implies that $d_1 + d_2 \ge 4k$ and $d_3 \le 2k$. Let n_i be the number of triangles containing exactly one vertex in X_i. Then $n_1 = n_2 = n_3 = (6k-2)(3k-1)$. This implies that if $uv \in E(G)$, then

$$\begin{cases} d_i + t_i = 4k-1 & \text{if } w \notin X_i; \\ d_i + t_i = 4k & \text{if } w \in X_i, \end{cases}$$

where w is the unique common neighbor of u and v; if $uv \notin E(G)$, then

$$d_i + t_i = 4k$$

for all $1 \le i \le 3$. For the former case, u and v have the unique common neighbor w. Then we must have $m(x) = 3k-2$ for every x adjacent to u in X. By the claim, $d_3 \le 1$. However, $k \ge 3$, $t_3 \le 3k-1$, and $d_3 + t_3 \ge 4k-1$ imply $d_3 \ge k \ge 3$, a contradiction. Thus, we assume $uv \notin E(G)$. It is clear that

$$d_1 + d_2 + d_3 = t_1 + t_2 + t_3 = 6k.$$

Since $d_i + t_i = 4k$ for all $1 \le i \le 3$ and $d_1 \ge d_2 \ge d_3$, we have $t_1 \le t_2 \le t_3$ and so $t_2 + t_3 \ge 4k$ and $t_1 \le 2k$. Notice that for each

$$x \in [N(u) \cup N(v) - N(u) \cap N(v)] \cap X,$$

$m(x) = 3k - 2$. By the claim, it follows that u and v have $a \ge t_1 - 1$ common neighbors in X_1 and $b \ge d_3 - 1$ common neighbors in X_3. It is obvious that $d_3 \ge 2$ and $t_1 \ge 2$. Without loss of generality, let $x_{3,1}$ and $x_{3,3}$ be two vertices adjacent to u in X_3 with $m(x_{3,1}) = 3k - 3$. Then $x_{3,1}$ and $x_{3,3}$ have $p \ge d_1 - 4$ common neighbors in X_1 and $q \ge d_2 - 4$ common neighbors in X_2. Since $x_{3,1}$ is in $3k - 3$ inner triangles, it follows that

$$3k - 3 \ge p + q \ge d_1 + d_2 - 8.$$

As $k \ge 3$, $d_1 + d_2 \le 4k + 2$ which implies that $d_3 \ge 2k - 2$. By symmetry, we also have $t_1 \ge 2k - 2$. Thus, $d_1 \le 2k + 2$ and $t_3 \le 2k + 2$, and so $d_2 + d_3 \ge 4k - 2$ and $t_1 + t_2 \ge 4k - 2$. Since $d_3 \ge 2k - 2$, $t_1 \ge 2k - 2$, and $d_i + t_i = 4k$ for $1 \le i \le 3$, we have either $t_1 + t_3 \ge 4k$ or $d_1 + d_3 \ge 4k$. By the claim, u and v have $c \ge \min\{d_2 - 1, t_2 - 1\}$ common neighbors in X_2. It follows that u and v have

$$\begin{aligned} a + b + c &\ge \min\{t_1 - 1 + d_3 - 1 + d_2 - 1,\ t_1 - 1 + d_3 - 1 + t_2 - 1\} \\ &\ge (2k - 2) + (4k - 2) - 3 = 6k - 7 \end{aligned}$$

common neighbors. This implies that $3k \ge 6k - 7$, i.e., $k \le 2$, a contradiction. Therefore, the theorem holds. □

Next, we make several remarks which cover the cases $r = 6$ and $r = 12$.

Remark 2.1. There is no $[3r - 2, r, 1]$-semi-strongly regular graph for $r = 6$ or 12.

Proof. Suppose, to the contrary, that there exists a $[3r-2, r, 1]$-semi-strongly regular graph G for $r = 6k$ with $k = 1$ or 2. Let v_1, v_2, v_3 be the vertices of a triangle in G. For $1 \le i \le 3$, let $v_i, x_{i,2t+1}, x_{i,2t+2}$, for $0 \le t \le 3k-2$, be the vertices of the other $3k-1$ triangles containing v_i and let

$$X_i = \left\{ x_{i,j} \mid 1 \le j \le 6k-2 \right\}.$$

Since G has $3r-2$ vertices, we let u be the only additional vertex of G. Clearly, each inner triangle (i.e., a triangle not containing v_1, v_2, v_3 and u) of G must contain one vertex from each X_i, this implies that u is adjacent to exactly $2k$ vertices in each X_i, say $x_{i,2j+1}$ for $0 \le j \le 2k-1$ in X_i. Without loss of generality, let $3k$ triangles containing u be $ux_{1,2j-1}x_{2,2j+2k-1}, ux_{2,2j-1}x_{3,2j+2k-1}, ux_{3,2j-1}x_{1,2j+2k-1}$ for $1 \le j \le k$ (see Figure 2.3 for $k = 1$). We consider the following two cases.

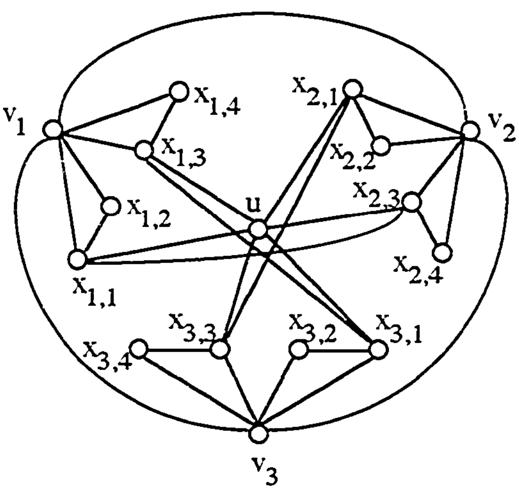

Figure 2.3. A Spanning Subgraph of G for k = 1.

Case 1. $k = 1$. In this case, since $x_{1,1}$ is in an inner triangle, $x_{1,1}$ must be adjacent to $x_{2,2}$. Similarly, $x_{1,3}$ is adjacent to $x_{3,4}$, $x_{2,1}$ is adjacent to $x_{3,2}$, and $x_{3,3}$ is adjacent to $x_{2,4}$. Then $x_{1,1}$ and $x_{2,2}$ must have the common neighbor $x_{3,4}$. Now, $x_{1,3}$ and $x_{3,4}$ can not have any common neighbor, contradicting the condition that every edge of G is in a triangle.

Case 2. k = 2. Then each vertex in $X = \{ x_{3,1}, x_{3,3}, x_{3,5}, x_{3,7} \}$ is in four additional inner triangles. This implies that every pair of vertices in X have at least two common neighbors in each of X_1 and X_2. Moreover, if $x_{3,1}$ and $x_{3,3}$ have exactly two common neighbors in X_1, then $x_{3,5}$ and one of $x_{3,1}$ and $x_{3,3}$ must have at least three common neighbors in X_1. It follows that there are two vertices x and y in X which have at least five common neighbors in $X_1 \cup X_2$. Then those five common neighbors must be independent which implies that neither x nor y is in four inner triangles, a contradiction. □

Finally, we note that a [3r − 1, r, 1]-semi-strongly regular graph for r = 6 exists (see the graph shown in Figure 2.4).

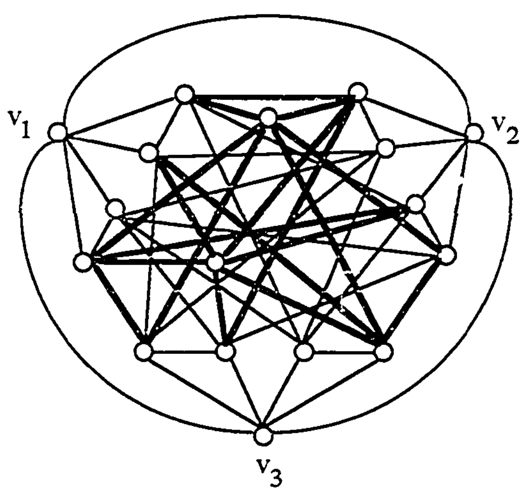

Figure 2.4. A [17, 6, 1]-semi-strongly Regular Graph.

CHAPTER III

GENERALIZED CAGES

3.1 The Existence of Generalized Cages

In Section 1.3 we generalized the usual cage concept and defined an (r, g, t)-cage to be an r-regular graph of minimum order $n = f(r, g, t)$ such that every clique has t vertices, every vertex is in $\frac{r}{(t-1)}$ cliques, every edge is in one clique, and the minimum length of a cycle with edges from distinct cliques is g. Clearly, an (r, g)-cage is the same as an (r, g, 2)-cage. It was proved by Erdös and Sachs [8] that an (r, g)-cage always exists. Our purpose in this section is to establish the existence of generalized cages.

We begin with the following lemma.

Lemma 3.1. For $g \geq 5$, if an (r, g, t)-cage of order $f(r, g, t)$ exists, then there also exists an (r, g − 1, t)-cage of order $f(r, g-1, t)$, and $f(r, g-1, t) \leq f(r, g, t)$.

Proof. For $t = 2$, $f(r, g, 2)$ is the same as the order $f(r, g)$ of an (r, g)-cage. It is known that $f(r, g-1) \leq f(r, g)$ (see [14]). Thus, we assume $t \geq 3$. Let G be an (r, g, t)-cage and let $C = u_1 u_2 \dots u_g u_1$ be a cycle of length g in G such that edge $u_i u_{i+1}$ is in clique F_i for $1 \leq i \leq g$, where $u_{g+1} = u_1$ and $F_i \neq F_j$ for $i \neq j$. Let $V(F_1) = \{u_1, u_2, v_1, v_2, \dots, v_{t-2}\}$ and $V(F_2) = \{u_2, u_3, w_1, w_2, \dots, w_{t-2}\}$. Suppose that G^* is the graph obtained from G by replacing two cliques F_1 and F_2 with two new cliques F_1' and F_2' on the vertex sets $(V(F_1) - \{u_1\}) \cup \{w_1\}$ and $(V(F_2) - \{w_1\}) \cup \{u_1\}$, respectively. Then it is clear that every vertex is in k cliques and G^* has a cycle $C' = u_1u_3u_4 \dots u_gu_1$ with edges from different cliques. Thus the clique girth of G^* is at most $g-1$. We now claim that the clique girth of G^* is $g-1$. Suppose, to the contrary, that G^* has a cycle $C^* = a_1a_2 \dots a_ha_1$ with $h \le g-2$ and with edges from different cliques. Then C^* must contain at least one new edge, i.e., an edge in $E_1 = \{u_1u_3, u_1w_j \mid 2 \le j \le t-2\}$ or $E_2 = \{w_1v_j \mid 1 \le j \le t-2\}$. Since E_1 is contained in the clique F_2' and E_2 is contained in the clique F_1', C^* contains at most one edge from each E_i. If C^* contains exactly one new edge a_1a_2, then either $a_1a_2 = u_1a_2$, where $a_2 = u_3$ or $a_1a_2 = w_1v_j$, for some j. Then G must has a cycle $a_1u_2a_2a_3 \dots a_ha_1$ of length $h + 1 \le g-1$, which implies that the clique girth is at most $g-1$, a contradiction. If C^* contains two new edges, say $a_1a_2 = u_1a_2$ where $a_2 = u_3$ or w_i, and $a_ja_{j+1} = w_1v_s$, for some j $(3 \le j \le h-1)$, then G has a cycle $a_2a_3 \dots a_ja_2$ of length $j-1 \le g-4$ and a cycle $a_{j+1}a_{j+2} \dots a_ha_1a_{j+1}$ of length $h-(j-1) \le h-2 \le g-4$, both imply that the clique girth is less than g, a contradiction. Thus, an $(r, g-1, t)$-cage exists and $f(r, g-1, t) \le f(r, g, t)$. □

We define the distance $d(H_1, H_2)$ between two subgraphs of a graph G to be the shortest distance from any vertex $v_1 \in H_1$ to any vertex $v_2 \in H_2$. For convenience, we shall invent some names for certain quantities that arise repeatedly, specifically let

$$\varphi(k, g, t) = 1 + (k-1)(t-1) + (k-1)^2(t-1)^2 + \dots + (k-1)^{g-2}(t-1)^{g-2}$$

and

$$h(k, g, t) = t\,(t-1)\,[\,\varphi(k, g, t) + (k-1)^{g-2}\,(t-1)^{g-1}\,] + 1.$$

For each vertex v of a graph G, let $c(v)$ be the number of cliques in G containing v. Suppose that $G = (V, E)$ is a clique-disjoint graph with t vertices in each clique such that $k-1 \le c(x) \le k$ for all $x \in V$. Then we have the following immediate inequalities:

(1) If $p \in V$ and $c(p) = k-1$, then

$$|\{x \in V \mid d(x, p) \le g-2\}| \le \varphi(k, g, t).$$

(2) Let K_t be a clique in G. Then

$$|\{x \in V \mid d(x, K_t) \le g-2\}| \le t\,\varphi(k, g, t).$$

Let $G(k, g, t) = \{\, G \mid G$ is a clique-disjoint semi-strongly regular graph with $\lambda = t-2$, $c(x) = k$ for all $x \in V(G)$ and clique-girth at least $g\,\}$.

These quantities happen to lead to the proof of Theorem 3.1, but the reason they have been chosen will not be evident until we reach the end of the proof. The proof is motivated by a method in [12].

Theorem 3.1. For $t \ge 2$, $g \ge 4$, $k \ge 1$, $n \ge h(k, g, t)$ and $n \equiv 0 \pmod t$, there exists a graph $G \in G(k, g, t)$ so that $|V(G)| = n$.

Proof. (By induction on k.) The theorem obviously holds for $k = 1$. Assume the result is true for $k-1$, where $k > 1$. Since $n \ge h(k, g, t) > h(k-1, g, t)$, there exists a graph $G_0 \in G(k-1, g, t)$ of order n. Let

$N = \{\, H \mid H$ is a clique-disjoint graph of order n with t vertices in each clique, $k-1 \le c(x) \le k$ for any $x \in V(H)$, and clique-girth at least $g\,\}$.

Then $N \neq \emptyset$ since $G_0 \in N$. Let $c(G)$ denote the number of cliques in graph G. Assume $G \in N$ such that $c(G) \geq c(H)$ for every $H \in N$. We claim that G is the desired graph (i.e., $G \in G(k, g, t)$).

To prove the claim, it suffices to show that $c(x) = k$, for every x in $V(G)$. Suppose, to the contrary, that $V' = \{ x \mid x \in V(G), c(x) = k - 1 \} \neq \emptyset\}$. The number of distinct pairs (x, K_t) with x in K_t is $t \cdot c(G) = nk - |V'|$. Since $n \equiv 0 \pmod t$, $|V'|$ is a multiple of t and so $|V'| \geq t$. Let A be a subset of V' and $|A| = t$. Next we will show that there are $t-1$ distinct cliques of t vertices $Q_1, Q_2, \ldots, Q_{t-1}$ such that

$$d(Q_i, A) \geq g - 1 \qquad (3.1)$$

and

$$d(Q_i, Q_j) \geq g - 1 \text{ for } i \neq j. \qquad (3.2)$$

Suppose that there are at most m cliques $Q_1, Q_2, \ldots, Q_m$ satisfying (3.1) and (3.2) and $0 \leq m \leq t - 2$. Let

$$B = A \cup V(Q_1) \cup V(Q_2) \cup \ldots \cup V(Q_m)$$

and let

$$C = \{ x \in V(G) \mid d(x, B) \leq g - 2 \}.$$

Then

$$|C| \leq (m + 1)\, t\, \varphi(k, g, t) \leq t\,(t - 1)\, \varphi(k, g, t).$$

Define $D = V(G) - C$, then

$$|D| = n - |C| > t\,(t - 1)\,(k - 1)^{g-2}\,(t - 1)^{g-1}.$$

The set D contains no K_t by the maximality condition on m. Let

$$E = \{ v \in V(G) \mid d(v, B) = g - 2 \}.$$

Then $|E| \le (m+1)\, t\, (k-1)^{g-2} (t-1)^{g-2} \le t\, (k-1)^{g-2} (t-1)^{g-1}$. Let

$$D' = \{ K_t \mid V(K_t) \cap D \neq \emptyset \}.$$

Since D contains no complete K_t and the vertices of D are at distance at least $g-1$ from B, it follows that if $K_t \in D'$, then $K_t \cap E \neq \emptyset$ and $V(K_t) \subseteq E \cup D$. Since each vertex in E is in at most $k-1$ cliques K_t in D', it follows that

$$|D'| \le (k-1)\, |E|.$$

Also since each vertex of D is in at least $k-1$ cliques K_t in D' and every K_t in D' contains at most $t-1$ vertices in D, we have

$$(k-1)\, |D| \le \sum_{v \in D} c(v) \le (t-1)\, |D'|.$$

We now have the contradiction

$$t\,(t-1)\,(k-1)^{g-2}(t-1)^{g-1} < |D| \le \frac{t-1}{k-1}\, |D'|$$

$$\le (t-1)\, |E| \le t(t-1)\,(k-1)^{g-2}(t-1)^{g-1}.$$

Thus, there are $Q_1, Q_2, \ldots, Q_{t-1}$ satisfying (3.1) and (3.2).

Since A, $V(Q_1)$, $V(Q_2)$, ..., $V(Q_{t-1})$ are t disjoint sets each with t elements, there are disjoint sets $X_1, X_2, \ldots, X_t$ such that $|X_i| = t$ and

$$|X_i \cap A| = 1, \quad |X_i \cap V(Q_j)| = 1 \text{ for } 1 \le i \le t,\ 1 \le j \le t-1.$$

Now, let $G_1 = G - E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_{t-1})$ and G_2 be the graph obtained from G_1 by joining all possible pairs of vertices in each X_i and denote the new cliques on each X_i by Z_i. Clearly, G_2 is a clique-disjoint graph with t vertices in every clique and $c(x) = k-1$ or k for all $x \in V(G_2)$. We now show that G_2 has clique-girth at least g. Suppose, to the contrary, that G_2 contains a cycle $e_1e_2...e_be_1$ with length $b < g$ such that each edge e_i is from a different clique F_i. Since the clique-girth of G is at least g, one of the cliques F_j must be one of the Z_i, say $F_1 = Z_1$. Let $p \in F_1 \cap F_2$ and $q \in F_1 \cap F_b$, then $p \neq q$ and by the definition of Z_1 we may assume $p \notin A$. Since $Z_1, Z_2, ..., Z_t$ are mutually disjoint, $F_2 \notin \{ Z_1, Z_2, ..., Z_t \}$, and hence, there is some s with $2 \leq s \leq b$ such that F_i is in G_1 for $1 \leq i \leq s$ and $F_{s+1} \in \{ Z_1, Z_2, ..., Z_t \}$. This implies that there is a path of length at most $g-2$ joining p and a vertex $w \in A \cup V(Q_1) \cup V(Q_2) \cup ... \cup V(Q_{t-1})$, where w and p are not in the same Q_j, which contradicts (3.1) and (3.2). Therefore, the clique-girth of G_2 is at least g and $G_2 \in N$. But $c(G_2) = c(G) + 1$, contradicting the maximality of c(G). Hence $G \in G(k, g, t)$. □

By Lemma 3.1 and Theorem 3.1, we have the following existence theorem for an (r, g, t)-cage.

Theorem 3.2. For $g \geq 4$ and $r = k(t-1)$, there exists an (r, g, t)-cage and

$$f(r, g, t) \leq t(t-1)\left(\sum_{i=1}^{g-2} (k-1)^i (t-1)^i + (k-1)^{g-2} (t-1)^{g-1} \right) + t.$$

3.2 Lower Bounds for the Order of (r, g, t)-Cages

For $g \geq 4$ and $r = k(t-1)$, let

$$f_0(r, g, t) = \begin{cases} 1 + \sum_{i=1}^{d} k (k-1)^{i-1} (t-1)^i & \text{if } g = 2d+1; \\ \sum_{i=0}^{d} t (k-1)^i (t-1)^i & \text{if } g = 2d+2. \end{cases}$$

Then it is easy to see that $f(r, g, t) \geq f_0(r, g, t)$. For a clique-disjoint $[n, r, t-2]$-semi-strongly regular graph G with clique-girth g, define $e = n - f_0(r, g, t)$ to be *the excess of G*. Recall that any (r, g, t)-cage must be a clique-disjoint $[n, r, t-2]$-semi-strongly regular graph. The following statement is equivalent to Theorem 2.2.

Proposition 3.1. For $g = 4$ and $r = k(t-1)$, an (r, 4, t)-cage G of order $n = f_0(r, 4, t)$ must be $[n, r, t-2, k]$-strongly regular and it is the point graph of a generalized quadrangle GQ(k, t). Conversely, the point graph of a generalized quadrangle GQ(k, t) is an (r, 4, t)-cage with $r = (t-1)k$.

Next, we derive that any (r, 5, t)-cage with excess 0 must also be strongly regular.

Proposition 3.2. For $g = 5$ and $r = k(t-1)$, any (r, 5, t)-cage G of order $n = f_0(r, 5, t)$ is $[n, r, t-2, 1]$-strongly regular.

Proof. Let G be a (r, 5, t)-cage of order $n = f_0(r, 5, t)$, where $r = k(t-1)$, and

$$f_0(r, 5, t) = 1 + k(t-1) + k(k-1)(t-1)^2.$$

We only need to verify the condition that every two non-adjacent vertices have exactly one common neighbor. Since the clique-girth of G is 5, every pair of non-adjacent vertices have at most one common neighbor. Let u and v be two non-adjacent vertices. The set N(u) of neighboring vertices of u contains r elements. Then v is in the set W of remaining vertices. Since G is of minimum order, every vertex of W is adjacent to some vertex of N(u). Therefore, u and v have exactly one common neighbor, and G is strongly regular. □

Note that for $g = 2d + 1$, an (r, g, t)-cage of order $n = f_0(r, g, t)$ is a Moore geometry of diameter d (see [9] for a definition). It is known that there exists no nontrivial Moore geometry of diameter greater than 2 (see [6], [9] and [10]). Thus, we have the following proposition.

Proposition 3.3. For $g = 2d + 1 \geq 7$ and $r = k(t-1) > 2$,

$$f(r, g, t) > f_0(r, g, t).$$

Theorem 3.3. There is no clique-disjoint $[n, r, t-2]$-semi-strongly regular graph with clique-girth 5 and excess one.

Proof. Suppose, to the contrary, that there is an $[n, r, t-2]$-semi-strongly regular graph G with clique-girth 5 and excess one. Then

$$n = 2 + k(t-1) + k(k-1)(t-1)^2$$

and $r = (t-1)k$ for some positive integer k. For convinence, let $\lambda = t - 2$. Then $r = (\lambda + 1)k$ and $n = r^2 - \lambda r + 2$. Without loss of generality, we assume $r \geq 3$. Let A be the adjacency matrix of G. Then we obtain

$$A^2 - (\lambda - 1) A - (r-1) I = J - B$$

where J is the n×n matrix all of whose entries are 1 and B is an adjacent matrix of a perfect matching, i.e., a direct sum of the adjacent matrices of K_2's with a suitable relabelling of G. It follows that n is even and $J - B - I$ is the adjacency matrix of K(2, 2,..., 2). Since the eigenvalues of K(2, 2,..., 2) are $n-2, 0, -2$ with multiplicities $1, \frac{n}{2}, \frac{n}{2} - 1$, respectively, $J-B$ has eigenvalues $n-1, 1, -1$ with multiplicities $1, \frac{n}{2}$, $\frac{n}{2}-1$, respectively. Any eigenvector of A having eigenvalue x must also be an eigenvector of $J-B$ with eigenvalue $x^2-(\lambda-1)x-(r-1)$. As A is real and symmetric, it must have one eigenvalue satisfying

$$x^2-(\lambda-1)x-(r-1)=n-1$$

which gives one eigenvalue $x=r$, $\frac{n}{2}$ eigenvalues satisfying

$$x^2-(\lambda-1)x-(r-1)=1,$$

i.e.,

$$x_{1,2}=\frac{(\lambda-1)\pm s}{2} \quad \text{where} \quad s=\sqrt{(\lambda-1)^2+4r},$$

and $\frac{n}{2}-1$ eigenvalues satisfying

$$x^2-(\lambda-1)\,x-(r-1)=-1,$$

i.e.,

$$x_{3,4}=\frac{(\lambda-1)\pm t}{2} \quad \text{where} \quad t=\sqrt{(\lambda-1)^2+4(r-2)}.$$

Suppose that the multiplicities of the distinct eigenvalues x_1, x_2, x_3, x_4 for adjacency matrix A are a, b, c, d, respectively. Then

$$a+b=\frac{n}{2} \quad \text{and} \quad c+d=\frac{n}{2}-1.$$

Since the trace of A is zero, we obtain the following identity:

$$r+\frac{\lambda-1}{2}(n-1)+\frac{s}{2}(a-b)+\frac{t}{2}(c-d)=0. \tag{3.3}$$

Considering A^3, the number in each (i,i)-entry gives the number of (i,i) walks of length 3. Since each triangle containing vertex i gives two such walks, the (i,i)-entry of A^3 is equal to $\frac{r}{(\lambda+1)}\binom{\lambda+1}{2}2 = \lambda r$. Thus,

$$tr(A^3) = n\lambda r.$$

On the other hand,

$$tr(A^3) = r^3 + x_1^3\, a + x_2^3\, b + x_3^3\, c + x_4^3\, d$$

which leads to the following identity:

$$n\lambda r = r^3 + \frac{(\lambda-1)^3 + 3\,(\lambda-1)\,s^2}{8}\,\frac{n}{2} + \frac{3\,(\lambda-1)^2\,s + s^3}{8}(a-b)$$
$$+\frac{(\lambda-1)^3 + 3\,(\lambda-1)\,t^2}{8}\left(\frac{n}{2}-1\right) + \frac{3\,(\lambda-1)^2\,t + t^3}{8}(c-d).$$

By substituting s^2 and t^2 in the the above identity we obtain

$$n\lambda r = r^3 + \frac{(\lambda-1)^3 + 3\,(\lambda-1)\,r}{2}\,\frac{n}{2} + \left[(\lambda-1)^2 + r\right]\frac{s}{2}(a-b)$$
$$+\frac{(\lambda-1)^3 + 3\,(\lambda-1)\,(r-2)}{2}\left(\frac{n}{2}-1\right) + \left[(\lambda-1)^2 + (r-2)\right]\frac{t}{2}(c-d). \qquad (3.4)$$

Now we consider the following four cases:

Case 1. Both s and t are rational, hence both integral. Since $s^2 - t^2 = 8$, we must have $s = 3$ and $t = 1$. Thus $(\lambda-1)^2 + 4r = 9$ which implies $r \le 2$, contradicting the assumption $r \ge 3$.

Case 2. Both s and t are irrational. First suppose that s and t are linearly dependent over the rationals. Then s^2 and t^2 must have the same square-free part a, which must divide their difference 8. Then a can only be 2. Let $s = u\sqrt{2}$ and $t = v\sqrt{2}$. We have $s^2 - t^2 = 2(u^2 - v^2) = 8$ which gives $u^2 - v^2 = 4$. But there are no two positive integers u and v with the difference of the squares equal to 4. Thus s and t must be linearly independent over rationals. This implies that the eigenvalues x_1 and x_2 occur in pairs; so also do the eigenvalues x_3 and x_4. Since one of $\frac{n}{2}$ and $\frac{n}{2} - 1$ is odd, this is impossible.

Case 3. s is irrational and t is rational. Then the eigenvalues x_1 and x_2 must occur in pairs which implies that $\frac{n}{2}$ is even and $a = b$. Since

$$n = r^2 - \lambda r + 2 = (\lambda + 1)^2 k - \lambda(\lambda + 1)k + 2, \quad \text{where } r = (\lambda + 1)k,$$

we must have $\lambda + 1$ even and k odd. Moreover, $\lambda + 1$ can not be divisible by 4. It follows that $\lambda + 1 \equiv 2 \pmod 4$, i.e., $\lambda \equiv 1 \pmod 4$. As $a = b$, (3.3) and (3.4) give

$$r + \frac{\lambda - 1}{2}(n-1) + \frac{t}{2}(c - d) = 0$$

and

$$n\lambda r = r^3 + \frac{(\lambda-1)^3 + 3(\lambda-1)r}{2}\frac{n}{2} + \frac{(\lambda-1)^3 + 3(\lambda-1)(r-2)}{2}\left(\frac{n}{2}-1\right)$$
$$+ [(\lambda-1)^2 + (r-2)]\frac{t}{2}(c-d).$$

After simplifying the second identity above we have

$$2n\lambda r - 2r^2 + 2(\lambda-1)^2 r + 2r(r-2) - 3(\lambda-1)nr + (\lambda-1)(n+2)r$$
$$= (\lambda-1)(-n+4) \tag{3.5}$$

which implies that $r \mid (\lambda - 1)(n-4) = (\lambda-1)(r^2 - \lambda r + 2)$, i.e., $r \mid 2(\lambda - 1)$. Since $r = (\lambda + 1)k$, we have $(\lambda + 1) \mid 2(\lambda - 1)$, i.e., $(\lambda + 1) \mid 4$. Then $\lambda + 1 = 1, 2,$ or 4. Recall that $\lambda \equiv 1 \pmod 4$. Thus $\lambda = 1$, which eliminates several terms in (3.5) and leaving $2nr - 2r^2 + 2r(r-2) = 0$. That is $nr = 2r$, a contradiction.

Case 4. s is rational and t is irrational. Then the eigenvalues x_3 and x_4 appear in pairs. That is, $c = d$. Similar to Case 3, by combining (3.3) and (3.4) and simplifying the resulting identities, we obtain

$$3(\lambda - 1)r + 2r - 2\lambda r = \lambda^2 - \lambda - 4 \qquad (3.6)$$

which implies that $r \mid \lambda^2 - \lambda - 4$. Since $r = (\lambda + 1)k$, $(\lambda + 1) \mid \lambda^2 - \lambda - 4$, i.e., $(\lambda + 1) \mid 2$. Thus, $\lambda + 1 = 1$ or 2, that is $\lambda = 0$ or 1. For $\lambda = 0$, (3.6) yields that $r = 4$. But the (4, 5)-cage has 19 vertices which is greater than $n = 18$, a contradiction. For $\lambda = 1$, it follows from (3.6) that $0 = -4$, also a contradiction.

Therefore, the theorem follows. □

By letting $\lambda = 0$, we obtain Brown's result [2] as the following corollary.

Corollary 3.1. There are no r-regular graphs with excess one and girth 5.

3.3 The Known (r, g, t)-Cages

We begin this section with the following observation that shows a connection between (r, g)-cages and generalized (2(r – 1), g, r)-cages.

Proposition 3.4. For $g \geq 4$, a graph G is an (r, g)-cage if and only if the line graph L(G) of G is a (2(r – 1), g, r)-cage.

Proof. Assume that $g \geq 4$. Let H be a $(2(r-1), g, r)$-cage. Then, by a characterization for a graph to be a line graph of some graph (see [13]), it follows that H must be a line graph of another graph G. Furthermore, G must be r-regular and has girth g. Since $H = L(G)$ is of minimum order, G is an (r, g)-cage. Conversely, let G be an (r, g)-cage. Then it is easy to see that the line graph L(G) of G is a $2(r-1)$-regular graph with every clique of the same order r and clique-girth g. This implies that L(G) is a $(2(r-1), g, r)$-cage for otherwise a $2(r-1)$-regular graph of smaller order with all cliques of the same order r and the clique-girth g would give an r-regular graph with girth g which has order less than the order of G, contradicting the choice of G. □

So far, (r, g)-cages with $g \geq 4$ have been found for the pairs (see [14] for detail information)

$(r, g) = (r, 4)$: order $n = 2r$, $K(r, r)$ with $r \geq 2$;

$(3, 5)$: order $n = 10$, the Petersen graph;

$(4, 5)$: order $n = 19$, the Robertson graph;

$(5, 5)$: order $n = 30$, three known cages;

$(6, 5)$: order $n = 40$;

$(7, 5)$: order $n = 50$, the Hoffman-Singleton graph;

$(3, 6)$: order $n = 14$, the Heawood graph;

$(7, 6)$: order $n = 90$;

$(r, 6)$: order $n = f_0(r, 6, 2)$ with $r-1$ a prime power;

$(3, 7)$: order $n = 24$;

$(3, 8)$: order $n = 30$;

$(r, 8)$: order $n = f_0(r, 8, 2)$ with $r-1$ a prime power;

$(3, 10)$: order $n = 70$, three known cages;

$(r, 12)$: order $n = f_0(r, 12, 2)$ with $r-1$ a prime power.

Thus, by Proposition 3.4 we have a corresponding $(2(r-1), g, r)$-cage for each pair (r, g) listed above.

The above generalized cages are special cases where each vertex is in two cliques. In the following, we present some generalized (r, g, t)-cages with each vertex in at least three cliques.

We first consider (r, g, t)-cages with $g = 4$. By Proposition 3.1, any $(r, 4, t)$-cage of order $n = f_0(r, 4, t) = t\,[1 + (k-1)(t-1)]$ is $[n, r, t-2, k]$-strongly regular and is the point graph of a generalized quadrangle $GQ(k, t)$, where $k = \frac{r}{t-1}$; and conversely. It is known (see [3]) that a generalized quadrangle $GQ(k, t)$ exists for the pairs $(k, t) = (2, q+1)$, $(q+1, 2)$, $(q+1, q+1)$, $(q^2+1, q+1)$, $(q+1, q^2+1)$, (q^2+1, q^3+1), and (q^3+1, q^2+1), where q is a prime power. Thus, we have $(r, 4, t)$-cages for

$$
\begin{aligned}
(r, 4, t) = {} & (2q, 4, q+1), \\
& ((q+1)q, 4, q+1), \\
& ((q^2+1)q, 4, q+1), \\
& ((q+1)q^2, 4, q^2+1), \\
& ((q^2+1)q^3, 4, q^3+1), \\
& ((q^3+1)q^2, 4, q^2+1),
\end{aligned}
$$

where q is a prime power.

Recall that each (r, g, t)-cage of order n corresponds to a clique-disjoint [n, r, t − 2]-semi-strongly regular graph of minimum order with clique-girth g. In particular, for g = 4 and t = 3, it follows from Corollary 2.1 that $n \geq 3r - 3 = f_0(r, 4, 3)$. By Corollary 2.2, $n = 3r - 3$ only for r = 2, 4, 6, and 10. Since r = 2 produces K_3, the other three give three (r, 4, 3)-cages:

(4, 4, 3)-cage, (6, 4, 3)-cage, (10, 4, 3)-cage

which can also be obtained by taking q = 2 in the first three triples of the

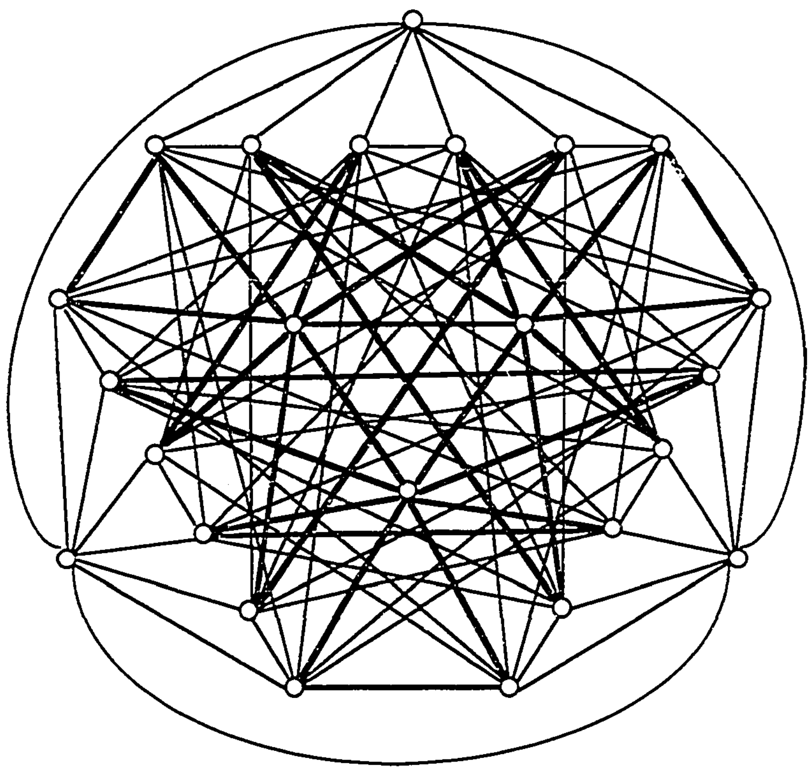

Figure 3.1. A (8, 4, 3)-cage.

list in the preceding paragraph. Thus, any other (r, 4, 3)-cages must have order $n > 3r - 3$. As all cliques K_3 in an (r, 4, 3)-cage are edge-disjoint, 3 must divide nr which implies that a (8, 4, 3)-cage must have order $n \geq 3r = 24$. The graph shown in Figure 3.1 is a (8, 4, 3)-cage which has 24 vertices. The smallest unknown (r, 4, 3)-cage is (12, 4, 3)-cage.

For (r, g, t)-cages with $g = 5$, by Proposition 3.2, any (r, 5, t)-cage of order $n = f_0(r, 5, t)$ is [n, r, t − 2, 1]-strongly regular. In particular:

(a) When $t = 3$, $(t-1) \mid r$ means r must be even. By the Friendship Theorem (Theorem 1.1), there is no [n, r, 1, 1]-strongly regular graph other than C_3. It follows that any (r, 5, 3)-cage must have order $n > f_0(r, 5, 3)$. And then Theorem 3.3 implies that any (r, 5, 3)-cage must have order $n \geq f_0(r, 5, 3) + 2$. The line graph L(P) (see Figure 1.3) of the unique (3, 5)-cage, the Petersen graph P, is the unique (4, 5, 3)-cage of order $15 = f_0(4, 5, 3) + 2$.

(b) For $t = 4$, $(t-1) \mid r$ implies that r is a multiple of 3. It can be shown by the Integrality Condition (Theorem 1.2) that an [n, r, 2, 1]-strongly regular graph can possibly exist only for $r = 21$. It is not known whether a [400, 21, 2, 1]-strongly regular graph exists. Such a graph is necessarily a (21, 5, 4)-cage. Any (r, 5, 4)-cage with $r \neq 21$ will have order $n > f_0(r, 5, t)$, and then Theorem 3.3 implies that $n \geq f_0(r, 5, t) + 2$.

For odd clique-girth $g = 2d + 1 \geq 7$, by Proposition 3.3, any (r, g, t)-cage must have order $n \geq f_0(r, g, t) + 1$.

CHAPTER IV

MAXIMAL GRAPHS WITHOUT C_4

4.1 Introduction of the Problem

Recall that an (r, 5, 3)-cage is an r-regular graph of minimum order such that every clique is a C_3, all cliques are edge-disjoint, and clique-girth is 5. Such a graph can not contain a C_4. If we are just interested in the restriction of without C_4, then a different interesting question could be: What is the maximum size $f(n)$ among all graphs of order n which have no C_4? This is an old problem posed by Erdős in 1938 (see [11] for a reference). It is well-known that $f(n) \sim \frac{1}{2} n \sqrt{n}$. It is also well-known that

$$f(n) \le \frac{1}{4} n (1 + \sqrt{4n-3}).$$

For $n = q^2 + q + 1$, the above inequality gives

$$f(q^2 + q + 1) \le \frac{1}{2}(q^2 + q + 1)(q + 1).$$

Next, we define the graphs $ER(q^2 + q + 1)$ due to Erdös, Rényi, and Sós [7].

Definition 4.1. For any prime power q, the graph $ER(q^2 + q + 1)$ is defined as follows: the vertices are the points of the finite projective plane PQ(2, q) over a feild of order q and the points (x, y, z) and (x', y', z') are adjacent if and only if $xx' + yy' + zz' = 0$.

The graph $ER(q^2+q+1)$ has q^2+q+1 vertices and $\frac{1}{2}q(q+1)^2$ edges and it contains no C_4. This establishes the following proposition.

Proposition 4.1. If q is a prime power, then $\frac{1}{2}q(q+1)^2 \le f(q^2+q+1)$.

It was then conjectured by Erdös in 1976 that

$$f(q^2+q+1)=\frac{1}{2}q(q+1)^2.$$

In 1983, Füredi [11] proved that $f(q^2+q+1)=\frac{1}{2}q(q+1)^2$ when q is a power of 2. For $n \le 21$, the exact values of $f(n)$ have been determined in [5]. Thus, the smallest unsettled case for the Erdös's conjecture above is when $q=5$, i.e., $f(31)$. By Proposition 4.1, $f(31) \ge 90$. On the other hand, it follows from the next proposition due to Füredi [11] that $f(31) \le 93$. In this chapter we will show that $f(31) \ne 92, 93$, and so $f(31)=90$ or 91.

Proposition 4.2. Let G be a graph on q^2+q+1 vertices which has no C_4. If the maximum degree of G satisfies $\Delta(G) \ge q+2$, then $|E(G)| \le \frac{1}{2}q(q+1)^2$.

4.2 Maximal Graphs Without C_4 on 31 Vertices

We now prove the following result.

Theorem 4.1. The maximum size $f(31)$ of a graph on 31 vertices without C_4 satisfies $90 \le f(31) \le 91$.

Proof. Let G be a graph on 31 vertices with $f(31)$ edges which has no C_4. Note that $31 = 5^2+5+1$. By Proposition 4.1, $f(31) \ge 90$. If the maximum degree of G is $\Delta = \Delta(G) \geq 7$, then it follows from Proposition 4.2 that $f(31) = |E(G)| \leq 90$. Thus, we assume $\Delta = 6$ and so $f(31) = |E(G)| \leq 93$. Moreover, if every vertex of degree 6 has a neighbor of degree at most 5, then G has at least 6 vertices of degree at most 5, and so $|E(G)| \leq 90$. Hence, we can assume further that G has a vertex y of degree 6 such that all the neighbors of y have degree 6. This implies that G has a spanning subgraph as shown in Figure 4.1. For convenience, for $1 \leq i \leq 6$ we let X_i be the set of the other four neighbors of vertex y_i. We consider the following two cases.

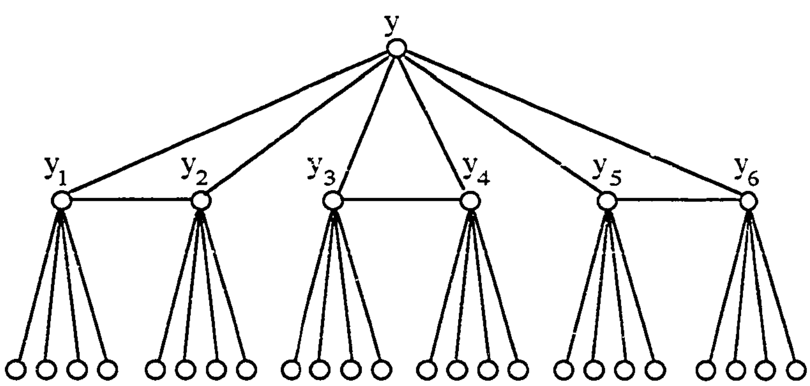

Figure 4.1. A Spanning Subgraph of G.

Case 1. $f(31) = 93$. In this case, G must be 6-regular and every edge of G is in exactly one C_3. This implies that G must be a [31, 6, 1]-semi-strongly regular graph with clique-girth 5. It follows from Proposition 3.2 that G is [31, 6, 1, 1]-strongly regular, i.e., every two vertices of G have exactly one common neighbor. By the Friendship Theorem (Theorem 1.1), no such graph exists.

Case 2. $f(31) = 92$. Since G has no C_4, for each $1 \leq i \leq 6$ the subgraph induced by X_i has maximum degree at most 1 and each vertex in X_i of degree 6 in G must be adjacent to exactly one vertex in each X_j for $j \neq i,\ i+1$ if i is odd, and $j \neq i-1, i$ if i is even. This implies that G must have exactly two vertices v_1 and v_2 of degree 5 and all other vertices of degree 6. Moreover, v_1 and v_2 can only be in either the same X_i, say X_6, or different X_i and X_j with $\{i,j\} \neq \{1,2\}, \{3,4\}$, or $\{5,6\}$, say X_4 and X_6. For the former case, by moving the vertex y_1 to the position of the vertex y in Figure 4.1, we end up with the latter case since there is no vertex other than y_6 which is adjacent to both v_1 and v_2. Thus, we assume the two vertices v_1 and v_2 of degree 5 are in X_4 and X_6, respectively. Then G must contain a spanning subgraph H shown in Figure 4.2. There are $\binom{31}{2}$ pairs of vertices and $29\binom{6}{2} + 2\binom{5}{2}$ pairs are

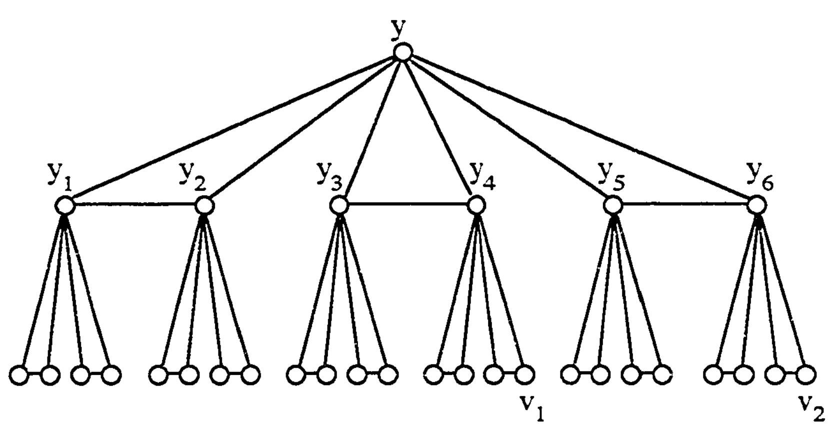

Figure 4.2. The Spanning Subgraph H of G.

covered in G. Thus, there are 10 uncovered pairs. This meas that there are at most 10 edges using uncovered pairs, that is, at least 82 edges of G are in C_3's. It follows that there are at least 28 C_3's in G. Note that

H contains 15 C_3's. Then G has at least 13 additional C_3's. We can rearrange the vertices of G as in Figure 4.3.

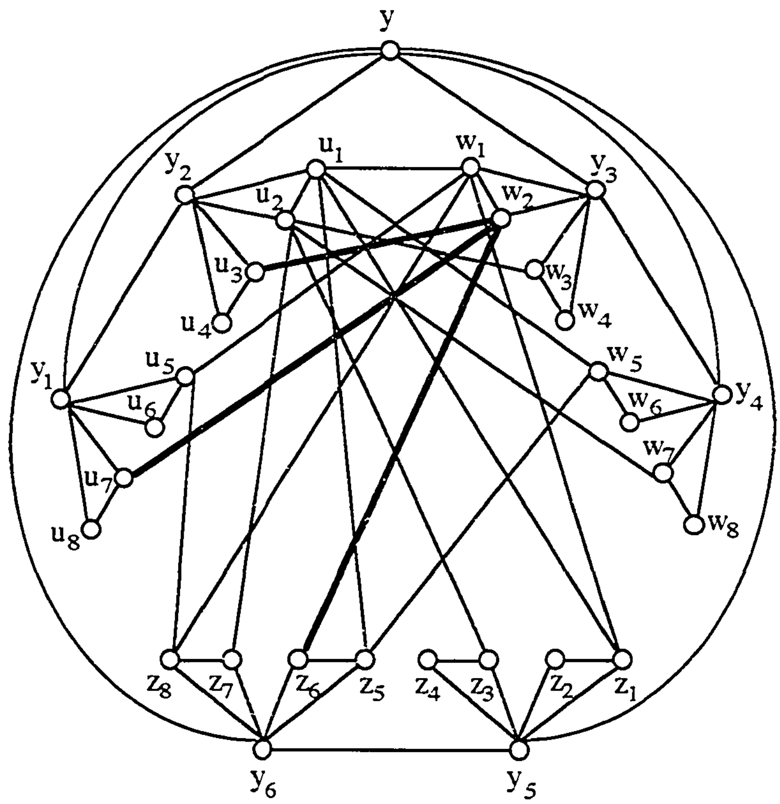

Figure 4.3. A Spanning Subgraph of G Containing H.

For convenience, we call C_3's not in H interior triangles. Thus we have at least 13 interior triangles. Let $A = X_1 \cup X_2$, $B = X_3 \cup X_4$, and $C = X_5 \cup X_6$. Then all vertices in A have degree 6, and so every vertex in A must be adjacent to exactly one vertex in X_i for $3 \le i \le 6$. Clearly, each interior triangle contains exactly one vertex from each of A, B, and C. Since there are at least 13 interior triangles, in each of A, B, and C there exist two adjacent vertices each of which is in two interior triangles.

Assume u_1 and u_2 are two such vertices in A. Without loss of generality, assume the interior triangles containing the vertex u_1 are as shown in Figure 4.3 and let the four neighbors of u_2 in B and C are w_3, w_7, z_3, and z_7. Then there are two ways to form two interior triangles containing u_2: either $u_2w_3z_3u_2$ and $u_2w_7z_7u_2$ or $u_2w_3z_7u_2$ and $u_2w_7z_3u_2$. Now, without loss of generality, let w_1 and w_2 be two adjacent vertices in B each of which is in two interior triangles. Then each of w_1 and w_2 must be adjacent to one vertex in each X_i for $i = 1, 2, 5, 6$. This forces that w_1 is adjacent to z_8 in X_6 and one of the vertices in X_1, say u_5. Thus, the second interior triangle containing w_1 is $w_1u_5z_8w_1$. Similarly, the edge w_2z_6 is forced and w_2 is adjacent to one of z_3 and z_4 in X_5, one of u_3 and u_4 in X_2, say u_3, and one of u_7 and u_8 in X_1, say u_7. Now, there are two ways to form two interior triangles containing w_2 according to whether the triangle containing z_6 is $w_2u_7z_6w_2$ or $w_2u_3z_6w_2$. For the former case, since u_3 is adjacent to one vertex in X_6, the edge u_3z_8 is forced and then $u_3z_8w_1w_2u_3$ is a cycle of length 4, a contradiction. For the latter case, since w_2 is adjacent to one of z_3 and z_4 in X_5, the second interior triangle T containing w_2 is either $w_2u_7z_3w_2$ or $w_2u_7z_4w_2$. For $T = w_2u_7z_3w_2$, since u_7 is adjacent to one vertex in X_6, the edge u_7z_7 is forced and then $u_7z_7u_2z_3u_7$ is a cycle of length 4, a contradiction. For $T = w_2u_7z_4w_2$, the edge u_3z_2 is forced, and then the edge u_4z_4 is forced. This gives a cycle $u_3u_4z_4w_2u_3$ on four vertices, again a contradiction. Therefore, the result follows. □

CHAPTER V

OPEN PROBLEMS

5.1 Problems on Semi-Strongly Regular Graphs

For clique-disjoint [n, r, λ]-semi-strongly regular graphs, Proposition 2.1 gives a lower bound on the orders $n \geq (\lambda + 2)(r - \lambda)$. We then showed in Section 2.2 that when n is large enough, such as n is bigger than the lower bound given in Theorem 2.5, a clique-disjoint [n, r, λ]-semi-strongly regular graph exists as long as the necessary conditions in Proposition 1.4 are satisfied. Now, an immediate question follows.

Problem 5.1. Do there exist clique-disjoint [n, r, λ]-semi-strongly regular graphs for those n between the two bounds mentioned above?

In a special case where $\lambda = 1$, Theorem 2.5 and Remark 2.1 indicate that an [n, r, λ]-semi-strongly regular graph can not exist when n is too close to the lower bound $3r - 3$. In general, when n is reasonably far away from the lower bound, such a clique-disjoint [n, r, λ]-semi-strongly regular graph should exist.

When $n = (\lambda + 2)(r - \lambda)$, Theorem 2.2 tells us that a clique-disjoint [n, r, λ]-semi-strongly regular graph is also strongly regular and is the point graph of a generalized quadrangle $GQ(k, \lambda + 2)$. Thus, finding clique-disjoint [n, r, λ]-semi-strongly regular graphs with order attaining the lower bound is the same as to discover unknown clique-disjoint [n, r, λ, k]-strongly regular graphs with $k = \frac{r}{\lambda + 1}$ (or equivalently, generalized quadrangles $GQ(k, \lambda + 2)$). Corollary 2.2 gives all possible [n, r, 1, k]-strongly regular graphs.

Problem 5.2. For $\lambda \geq 2$, determine all clique-disjoint [n, r, λ, k]-strongly regular graphs where $k = \frac{r}{\lambda + 1}$ (or equivalently, all generalized quadrangles $GQ(k, \lambda + 2)$).

5.2 Problems on Generalized Cages

In Chapter III, we showed the existence of generalized (r, g, t)-cages in Section 3.1 and listed all known (r, g, t)-cages in Section 3.3. A general question here is as follows.

Problem 5.3. Determine the values of $f(r, g, t)$ and find unknown (r, g, t)-cages (in particular, find a (12, 4, 3)-cage which is a smallest unknown generalized cage).

As shown in Proposition 3.2, any (r, 5, t)-cage of order n attaining the lower bound $f_0(r, 5, t)$ must be strongly regular. Again, to look for such generalized cages is the same to discover [n, r, t – 2, 1]-strongly regular graphs. It can be shown that if a [n, r, 2, 1]-strongly regular graph exists, then $r = 21$, and so $n = 400$. So far, whether a [400, 21, 2, 1]-strongly regular graph exists is still open.

Problem 5.4. Determine whether a [400, 21, 2, 1]-strongly regular graph exists (if exists, it is a (21, 5, 4)-cage).

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