Probability Polynomials for Cubic Graphs in the Framework of Random Topological Graph Theory

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PROBABILITY POLYNOMIALS FOR CUBIC GRAPHS IN THE FRAMEWORK OF RANDOM TOPOLOGICAL GRAPH THEORY

by

Esther Joy Tesar

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
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Topological graph theorists study the imbeddings of graphs on surfaces (spheres with handles). Some interesting questions in the field are on what surfaces can a graph be 2-cell imbedded and how many such imbeddings are there on each surface. The study of these and related questions is called Enumerative Topological Graph Theory. Random Topological Graph Theory uses probability models to study the 2-cell imbeddings. It generalizes the results from Enumerative Topological Graph Theory (which is the uniform case, \( p = \frac{1}{2} \)) to an arbitrary probability \( p \).

We study the model where the sample space consists of all labeled, orientable 2-cell imbeddings of a fixed connected cubic graph. In this model, neighbors of a vertex are either oriented clockwise (with probability \( p \), \( 0 \leq p \leq 1 \)) or counter-clockwise (with probability \( 1-p \)). By giving a global definition of “clockwise”, we construct probability polynomials for each surface upon which a fixed graph can be imbedded and for the expected value of the genus random variable. We investigate properties of these probability polynomials and give probability polynomials for a number of small order cubic graphs as well as some families of graphs.
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Esther Joy Tesar
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CHAPTER I

INTRODUCTION

1.1 History

Random graph theory is an area of graph theory that has produced many amazing results in the last thirty years. In random graph theory, probability models are used to answer questions about "random" graphs. One of the models has a sample space consisting of all labeled graphs of order n. Between any two vertices, there is a binary choice of whether an edge is present or not, with each of the "possible edges" equally likely. Using this model, it is possible to prove results such as almost all graphs are hamiltonian, or almost all graphs are connected [8].

Drawing upon the ideas of random graph theory, Arthur White coined the term "random topological graph theory" and defined five probability models that are based on some topological properties of graphs [14]. One of the models he proposed, Model II, is like the aforementioned model for random graph theory in that a binary choice is being made. Model II has as its sample space all orientable, 2-cell imbeddings of a fixed, connected, labeled cubic graph. In this model, the vertices of the graph of order n are labeled from the set \{1,2,\ldots,n\}. Since the graph is cubic, in each imbedding the neighbors of a vertex can only be ordered in two ways (there are only 2 non-isomorphic 3-
cycles): monotone increasing or not. If this cyclic ordering of the neighbors of a vertex is monotone increasing, then the vertex is considered to be clockwise; otherwise, it is counterclockwise.

In this research, we study a generalization of Model II which has the same sample space and the same binary choice at each vertex, but we allow the graph to be labeled differently and we allow different criteria to be used to determine if a vertex in a particular imbedding is "clockwise" or not. We will study some examples, prove some general theorems and derive formulas for the expected value of the genus random variable for several classes of graphs.

1.2 Graph Theory Definitions

We begin by reviewing some graph theory definitions. This is not necessarily an exhaustive list; more details and other related results can be found in Chartrand and Lesniak [1], in White [13] and in Gross and Tucker [6].

A graph, G, is a finite nonempty set of vertices, V(G), together with a set of unordered pairs of distinct vertices called edges. The order of a graph is the cardinality of the vertex set, and the size of a graph is the cardinality of the edge set. Two vertices of a graph are said to be adjacent if there is an edge between them. The degree of vertex v, deg(v), is the number of edges incident with vertex v; in particular, a cubic graph is a graph in which all the vertices have degree three. A graph is said to be labeled if the vertices are labeled; often we take V(G) = \{1,2,...,n\}. A walk in a graph is an alternating sequence of vertices and
edges which begins and ends with a vertex and in which each edge is incident with the two vertices immediately preceding and following it. A **path** is a walk of a graph in which no vertex is repeated. A graph is **connected** if there is a path between every pair of vertices in the graph. A graph that allows multiple adjacencies and loops is called a **pseudograph**. A **directed graph** or **digraph** is a finite nonempty set of vertices together with a set of ordered pairs of distinct vertices called **arcs** or **directed edges**.

A graph is **complete** if every two vertices are adjacent. We denote the complete graph on \( p \) vertices by \( K_p \). A graph is **bipartite** if it is possible to partition the vertices of the graph into two subsets (called **partite sets**) such that every edge is adjacent to a vertex from each partite set. A **complete bipartite** graph \( G \) is a bipartite graph with the property that there is an edge between every two vertices that are from different partite sets. We denote an complete bipartite graph by \( K_{m,n} \), where \( m \) and \( n \) are the orders of the two partite sets.

Let \( \Gamma \) be a group, and \( \Delta \) a generating set for \( \Gamma \). There is a digraph associated with \( \Gamma \) and \( \Delta \) called the **Cayley color graph of \( \Gamma \) with respect to \( \Delta \)**, which we denote by \( C_{\Delta}(\Gamma) \). The vertices of \( C_{\Delta}(\Gamma) \) correspond to the elements of \( \Gamma \), and for \( g_1, g_2 \in \Gamma \), there is an arc from \( g_1 \) to \( g_2 \) labeled \( h \) (where \( h \in \Delta \)) if and only if \( g_2 = g_1h \). If \( C_{\Delta}(\Gamma) \) contains each of the arcs \((g_1, g_2)\) and \((g_2, g_1)\) both labeled \( h \) (where \( h \) has order 2), then we represent this pair of arcs by the edge \( g_1 \rightarrow g_2 \). If \( e \notin \Delta \), and if \( \delta \in \Delta \) (\( \delta^2 \neq e \)), implies \( \delta^{-1} \notin \Delta \), then when we suppress all edge directions and the labels of the edges, we get a graph called the **Cayley graph** which is
denoted by $G_{\Delta}(\Gamma)$. Let $\Delta^* = \Delta \cup \Delta^{-1}$. Then the edge set of the Cayley graph $G_{\Delta}(\Gamma)$, equals $\{(g,g\delta) \mid \delta \in \Gamma, \delta \in \Delta^*\}$. As an example, let $\Gamma = Z_6$ and $\Delta = \{2,3\}$. The Cayley color graph, $C_{\Delta}(\Gamma)$, and the Cayley graph, $G_{\Delta}(\Gamma)$, are shown in Figure 1.

![Cayley Color Graph and Cayley Graph](image1.png)

Figure 1. Cayley Color Graph and Cayley Graph for $\Gamma = Z_6$ and $\Delta = \{2,3\}$.

One way to build a graph $G$ from two other graphs $G_1$ and $G_2$ is the **Cartesian product** denoted $G = G_1 \times G_2$. For $G$, $V(G) = V(G_1) \times V(G_2)$, and $E(G) = \{(u_1,u_2),(v_1,v_2) \mid u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$. For example, let $G_1 = K_2$ and let $G_2 = K_3$. The graph $G_1 \times G_2$ is shown in Figure 2.

![Cartesian Product of $K_2$ and $K_3$](image2.png)

Figure 2. Cartesian Product of $K_2$ and $K_3$. 

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1.3 Topological Graph Theory Results

A surface is a compact, orientable 2-manifold. Surfaces can be thought of as spheres with "handles" attached. The genus of the surface is the number of handles that have been attached. The sphere with one handle is topologically equivalent to the torus, and the sphere with two handles is topologically equivalent to a pretzel, or double torus. A pseudograph is said to be imbedded on a surface if it is "drawn" on the surface so that the edges intersect only at their common vertices. If a pseudograph G is imbedded on a surface S, then the components in S of the S – (image of G) are called regions. A region is 2-cell if it is homeomorphic to 2-dimensional Euclidean space (equivalently, to an open disk). An imbedding is called a 2-cell imbedding if every region is 2-cell. From now on, whenever we refer to an imbedding of a graph, we are referring to a 2-cell imbedding.

Imbedding a graph on a sphere is equivalent to imbedding that graph in the plane. This equivalence can be seen by performing a stereographic projection. Let G be a graph imbedded on a sphere S. Let A be any point of the sphere that is in the interior of a region of G. Let P be the plane tangent to S at the point of S diametrically opposed to A. The image of the graph G can be projected onto the plane producing an imbedding of G on the plane. The inverse of this projection shows that any graph imbedded on the plane can also be imbedded on the sphere. (For more details see [13] page 45). A graph that can be imbedded in the plane (or the sphere) is called a planar graph.
In a particular 2-cell imbedding of a pseudograph, the number of vertices, edges and regions is related to the genus. This is an important result in topological graph theory and is called the Euler Identity. Let $G$ be a connected pseudograph of order $p$ and size $q$ which is 2-cell imbedded on the surface of genus $n$, with $r$ regions; then

$$p - q + r = 2 - 2n. \tag{1.1}$$

Often we want to find the genus of the surface on which $G$ has been 2-cell imbedded, so we solve for $n$ in the Euler Identity to get

$$n = 1 + \frac{1}{2} (q - p - r) \tag{1.2}$$

which is often called the Euler formula.

The genus of a pseudograph $G$, $\gamma(G)$, is the minimum genus among all surfaces in which the pseudograph can be imbedded. If a connected pseudograph is imbedded on the surface of genus $\gamma$, then we know the imbedding is a 2-cell imbedding (see [13] page 61). The maximum genus of a pseudograph, $\gamma_M$, is the maximum genus among all surfaces in which the pseudograph can be 2-cell imbedded. Duke's Interpolation Theorem tells us that a connected graph $G$ can be 2-cell imbedded on the surface of genus $n$ if and only if $\gamma(G) \leq n \leq \gamma_M(G)$ (See for example [13] page 138).

We know that

$$\gamma_M \leq \left\lfloor \frac{q + p + 1}{2} \right\rfloor. \tag{1.3}$$

(See for example [13] page 65).

A splitting tree of a connected graph $G$ is a spanning tree $T$ for $G$ such that at most one component of $G - (the\ edges\ of\ T)$ has odd size. The maximum genus, $\gamma_M = \left\lfloor \frac{q + p + 1}{2} \right\rfloor$ if and only if $G$ has a splitting tree.
A very useful tool for describing an imbedding of a graph is Edmond's Permutation Technique. This is a method that allows us to describe an imbedding of a graph algebraically. Let G be a graph of order n, and let V(G)=(1, 2, ..., n). For each i \in V(G), let N(i) be the set of vertices adjacent to i. Let \( p_i : N(i) \to N(i) \) be a cyclic permutation on N(i), of length \( |N(i)| \); \( p_i \) is called a rotation. Let \( \rho = (p_1, p_2, ..., p_n) \) which is called a rotation scheme. There is a one-to-one correspondence between rotation schemes and 2-cell imbeddings of G. At each vertex i, there are \( (\deg(i) - 1)! \) different rotations, so there are \( \prod_{i=1}^{n} (\deg(i) - 1)! \) possible rotation schemes of G. This means there are \( \prod_{i=1}^{n} (\deg(i) - 1)! \) different labeled 2-cell imbeddings of G.

Here is an example to illustrate Edmond's Permutation Technique. Figure 3 shows an imbedding of \( K_4 \) (the complete graph on 4 vertices) on the torus.

![Figure 3](image)

**Figure 3.** An Imbedding of \( K_4 \) on the Torus.

The neighbors of vertex 1, as we go around clockwise are 2, 4, 3, in that order, so \( p_1 = (243) \). Continuing to find the rotation at each vertex, we get the rotation scheme \( \rho \) as follows:
\[ p_1: (243) \]
\[ p_2: (143) \]
\[ p_3: (142) \]
\[ p_4: (132) \]

Once we have the algebraic description of the imbedding, it is possible to determine the region boundaries from the algebraic description. The algorithm in general is as follows: replace each edge in G with a pair of symmetric arcs giving a digraph D with edge set \( E(D) \). Let \( (v_i, v_j) \in E(D) \). Let \( \pi: E(D) \rightarrow E(D) \) be such that \( \pi(v_i, v_j) = (v_j, v_{p_i(i)}) \).

Each orbit of \( \rho \) is a region boundary of the imbedding, traced out in a counterclockwise direction (see [13] page 70 or [1] page 125). So for this example, 2 precedes 4 in the cyclic ordering of \( p_1 \). This means that at vertex 1, the edge from vertex 2 to vertex 1 is followed by the edge from vertex 1 to vertex 4. We write 2-1-4. Now in \( p_4 \), 1 proceeds 3, so at vertex 4, the edge from vertex 1 to 4 is followed by the edge from 4 to 3. We add the edge 4-3 to what we had before to get 2-1-4-3. This process is continued until we encounter the edge 2-1 again, closing off the region. Then we do the same process for a new region until we have described all the regions of the imbedding. The regions in our imbedding above would be

2-1-4-3-2
and 4-1-3-4-2-3-1-2-4.

We have the four sided region with the edges 2-1, 1-4, 4-3, 3-2 and the eight sided region with edges 4-1, 1-3, 3-4, 4-2, 2-3, 3-1, 1-2, 2-4. We can
plug in $p=4$, $q=6$ and $r=2$ into the Euler formula (1.2) to see that $n = 1 + \frac{1}{2}(6-4-2) = 1$, which is correct since $K_4$ was imbedded on the torus.

An automorphism of a graph $G$ is an isomorphism of $G$ with itself. The set of all automorphisms of $G$ form a permutation group denoted $\text{Aut } G$. A map $M$ is a pair $(G, \rho)$, where $G$ is a connected graph and $\rho$ is a rotation scheme for $G$. An automorphism of a map $M$ is a graph automorphism $a \in \text{Aut } G$ such that $a(\rho) = \rho$. This means that $\rho_{a(v)} = a_\rho a^{-1}$, for all $v \in V(G)$. The set of all map automorphisms of $M$ form a group denoted $\text{Aut } M$. This brings us to an important theorem.

**Theorem 1.1** Let $G$ be graph and let $M = (G, \rho)$. The number of different labeled imbeddings of $M = \frac{|\text{Aut } G|}{|\text{Aut } M|}$. (See [12]).
CHAPTER II

MODEL II AND LABELING THE UNDERLYING GRAPH

2.1 The Original Model

In his paper, *An Introduction to Random Topological Graph Theory*, [14], Arthur White proposed several models for Random Topological Graph Theory. In this study, we are interested in the one he called Model II, which is given below ([14] page 555).

**Model II.** For each connected labeled cubic graph $G$ of order $n$ ($|V(G)| = n$) and $0 < p < 1$, the sample space $\Omega$ consists of all orientable 2-cell imbeddings, $(G,\rho)$, of $G$. For $i \in V(G)$, let $N(i) = \{j,k,l\}$, where $j < k < l$. If $\rho_j = (j,k,l)$, we say that the rotation $\rho_j$ is clockwise. If $\rho_j = (j,l,k)$, we say that $\rho_j$ is counterclockwise. Then $\rho = \{\rho_j\}_{j=1}^n$; let $\rho$ have exactly $c$ clockwise rotations $\rho_j$, $0 \leq c \leq n$. Then we define

$$P(G,\rho) = p^c (1-p)^{n-c}.$$  

To illustrate this model, we will look at the planar imbedding of $K_4$ shown in Figure 4. The rotation scheme, $\rho$, is as follows:

$$\begin{align*}
\rho_1 &: (234) \\
\rho_2 &: (143) \\
\rho_3 &: (124) \\
\rho_4 &: (132)
\end{align*}$$

We see that vertices 1 and 3 are clockwise (as emphasized by the solid vertices) and vertices 2 and 4 are counterclockwise (as emphasized by the white vertices); from this we get that $P(G,\rho) = p^2 (1-p)^2$. When we
look at the mirror image of an imbedding, every vertex that was
clockwise in the original imbedding becomes counterclockwise, and vice
versa. So the mirror image of $p$ also has probability $p^2(1-p)^2$.

Figure 4. An Imbedding of $K_4$ on the Plane.

Let $E_i$ be the event that a graph $G_i$ is imbedded on $S_i$, the surface of
genus $i$. Denote $P(E_i)$ by $f_i(G,p)$, where $f_i(G,p)$ is a polynomial in $p$. So
$f_i(G,p)$ is the probability that a given imbedding is imbedded on the
surface of genus $i$. We call $f_i(G,p)$ the probability polynomial of $G$ for
genus $i$. When the graph $G$ is understood, we will simply write $f_i(p)$. We
will call the set \( \{ f_{y(G)}(G,p), \ldots, f_{Y_M(G)}(G,p) \} \) the family of probability polynomials for $G$.

In the above example of $K_4$, the only planar imbeddings are $p$ and
its mirror image. So
\[
(2.1) \quad f_0(K_4,p) = 2p^2(1-p)^2.
\]
There are 14 imbeddings of $K_4$ on the torus and
\[
(2.2) \quad f_1(K_4,p) = (1-p)^4 + 4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p) + p^4.
\]
This can be checked by keeping track of the number of clockwise vertices
in each of the 14 imbeddings. The polynomials $f_0(G,p)$ and $f_1(G,p)$ make
up the family of probability polynomials for $K_4$. 

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Once we have a family of probability polynomials for a graph G, we can also get a probability polynomial for the expected value of the genus random variable, \( E(\gamma(G)) \). With \( E(\gamma(G)) \), we have the probability distribution where the genus of the imbedding is the random variable and so \( E(\gamma(G)) \) is the expected value of this probability distribution. Thus, \( E(\gamma(G)) = \sum_{i \geq 0} i \cdot f_i(p) \).

So for the previous example, \( E(\gamma(K_4)) = 0 \cdot f_0(p) + 1 \cdot f_1(p) = f_1(p) \).

Since the number of imbedding for a cubic graph of order \( n \) is \( 2^n \), calculating these probability polynomials becomes tedious. Because of this, a computer program was developed in PASCAL. The input for this program is the clockwise ordering of the neighbors of every vertex where the vertices are labeled from the set \{1,2,...,n\}. The computer uses the algorithm for finding the regions given the cyclic ordering of the neighbors of a vertex (algorithm was discussed in Chapter I) to calculate the regions for the imbedding where all the vertices are clockwise. After finding the regions for that imbedding, it finds the regions for another imbedding, and continues until it has found the regions for all of the imbeddings in order to find the probability polynomials for each genus and the probability polynomial for the expected value of the genus random variable. The program will output the regions in each imbedding as well as the probability polynomial, or just the polynomials. See Appendix B for the code for this program.

In Model II, the family of polynomials for a fixed connected cubic graph is not necessarily unique but can depend on the labeling of the graph used. Figure 5a shows a planar imbedding of \( G_1 = K_2 \times C_3 \). In
this imbedding there are 3 vertices clockwise and 3 vertices counterclockwise. This imbedding and its mirror image are the only planar imbeddings of $G_1$. So,

\[(2.3) \quad f_0(G_1,p) = 2p^3 (1-p)^3.\]

Figure 5b also shows a planar imbedding of the graph $G_2 = K_2 \times C_3$ but with a different labeling. This time there are 4 vertices clockwise and 2 vertices counterclockwise giving

\[(2.4) \quad f_0(G_2,p) = p^4 (1-p)^2 + p^2 (1-p)^4.\]

Since $f_0(G_1,p) \neq f_0(G_2,p)$, they must come from different families of polynomials.

\[
\begin{align*}
\text{a)} & \quad \begin{array}{c}
\text{5} \\
\text{4} & \text{3} & \text{2} \\
\text{1} & \text{6}
\end{array} \\
\text{b)} & \quad \begin{array}{c}
\text{1} \\
\text{3} & \text{6} & \text{4} \\
\text{2} & 5
\end{array}
\end{align*}
\]

Figure 5. Two Imbeddings of $K_2 \times C_3$ With Different Families of Probability Polynomials.

So the family of probability polynomials for $K_2 \times C_3$ is not independent of the labeling selected. Arthur White reported (but did not publish the proof) that the family of probability polynomials for $K_{3,3}$ is independent of the labeling used [14]. The natural question that then arises is: when are these families of probability polynomials independent of the labeling chosen?
2.2 How the Labeling Affects the Model

The smallest possible connected cubic graph is $K_4$. The graph $K_4$ only has one labeling, so using Model II, there can only be one possible family of probability polynomials for $K_4$.

There are only two non-isomorphic connected cubic graphs of order six, namely $K_2 \times C_3$ and $K_{3,3}$. In the previous section, we saw that the family of polynomials for $K_2 \times C_3$ depends on the labeling of the graph. We will prove that there is only one family of polynomials for $K_{3,3}$ but first we will need the following lemma.

**Lemma 2.1** Imbed $K_{3,3}$ hexagonally on the torus. The vertices in a fixed partite set must be all clockwise or all counterclockwise.

**Proof:** Label the vertices of $K_{3,3}$ from the set $\{a,b,c,d,e,f\}$ as shown in Figure 6. Vertex $a$ could be assigned any of the numbers from the set $\{1,..6\}$. Vertex $b$ could be assigned any of the remaining numbers, and so on. These letters then represent any possible labeling of the vertices of $K_{3,3}$. We get

\[
\begin{align*}
  p_a & : (bdf) \\
  p_b & : (aec) \\
  p_c & : (bdf) \\
  p_d & : (aec) \\
  p_e & : (bdf) \\
  p_f & : (aec)
\end{align*}
\]

So, if vertex $a$ is clockwise (counterclockwise) then vertices $e$ and $c$ are also clockwise (counterclockwise) since they have the same ordering of
neighbors. Similarly, b, d, and f are either all clockwise or all counterclockwise. □

Figure 6. Labeling of Hexagonal Imbedding of $K_{3,3}$ on the Torus.

**Theorem 2.1.** There is only one family of probability polynomials for $K_{3,3}$.

**Proof:** Call one of the partite sets of $K_{3,3}$ A and the other B. By Theorem 1.1, there are only 4 hexagonal imbeddings of $K_{3,3}$ (on the torus), since $|\text{Aut } K_{3,3}| = 72$ and $|\text{Aut } M| = 18$ where M is one of the hexagonal imbeddings. These four hexagonal imbeddings correspond exactly to the 4 following possibilities:

1. Vertices of A and B are clockwise.

2. Vertices of A are clockwise and vertices of B are counterclockwise.

3. Vertices of A are counterclockwise and vertices of B are clockwise.

4. Vertices of A and B are counterclockwise.

For any labeling of $K_{3,3}$, each one of the four possibilities above occurs in exactly one imbedding. So for any particular labeling, we can find the hexagonal imbedding that has all the vertices of both partite sets counterclockwise. From this imbedding we can follow a set algorithm.
(like the computer program given in the Appendix) to find the probability polynomials. Since in every case, we started with all vertices counterclockwise and followed the same procedure to go from one imbedding to another, the probability polynomials are identical no matter what the labeling. □

In Model II, $K_{3,3}$ may have only one family of probability polynomials, but for every $n \geq 3$, we can find a graph that has more than one probability polynomial, as the following theorem will show.

**Theorem 2.2** In model II, $K_2 \times C_n$ has more than one family of probability polynomials for all $n \geq 3$.

**Proof:** Case I, $n \geq 5$

Let $G = K_2 \times C_n$. $G$ is called an $n$-prism. By Theorem 1.1, there are only 2 labeled planar imbeddings of $K_2 \times C_n$ since $|\text{Aut } G| = 4n$ and $|\text{Aut } M| = 2n$ (where $M$ is one of the planar imbeddings). So in order to show that there is more than one family of probability polynomials for $K_2 \times C_n$, it suffices to show one labeling of $K_2 \times C_n$ with a planar imbedding that has $m$ clockwise vertices and another labeling of $K_2 \times C_n$ that has a planar imbedding with $b$ (b$m$) clockwise vertices where $m + b \neq 2n$. If this happens, $f_0(p)$ must differ for the two labelings.

**Labeling 1** Label a planar imbedding of $K_2 \times C_n$ as shown in Figure 7. By observation, we see that vertices 1 and $n$ are counterclockwise and that vertices $n+1$ and $2n$ are clockwise. Let $x \in \{2,3,4,\ldots,n-1\}$. Look at the neighbors of vertex $x$ (see Figure 8). So vertex $x$ must be clockwise. Let $y \in \{n+2, n+3,\ldots, 2n-1\}$. Look at the neighbors
of vertex $y$ (see Figure 9). So vertex $y$ must be counterclockwise and with this labeling there are exactly $n$ vertices clockwise in this imbedding.

\[
\begin{align*}
&n - 1 \\
&2n - 1
\end{align*}
\]

Figure 7. A Labeling of a Planar Imbedding of $K_2 \times C_n$.

![Figure 7. A Labeling of a Planar Imbedding of $K_2 \times C_n$.](image)

Figure 8. The Neighbors of Vertex $x$ in the Planar Imbedding of $K_2 \times C_n$.

![Figure 8. The Neighbors of Vertex $x$ in the Planar Imbedding of $K_2 \times C_n$.](image)

Figure 9. The Neighbors of Vertex $y$ in the Planar Imbedding of $K_2 \times C_n$.

![Figure 9. The Neighbors of Vertex $y$ in the Planar Imbedding of $K_2 \times C_n$.](image)

**Labeling 2.** Label a planar imbedding of $K_2 \times C_n$ as shown in Figure 10. By observation of Figure 10, we see that vertices 1 and 2 are clockwise (shaded black) while vertices 3, 2n, 2n-1 and 4 are counterclockwise (shaded gray in this picture so they aren't confused.
with the vertices whose orientation is yet to be determined. This deviates from our normal convention of leaving counterclockwise vertices white.) If you add the labels of adjacent inside and outside vertices (except 1 and 2) you get $2n + 3$. Let $x \in \{5,7,9,...,2n-3\}$. See Figure 11.

![Figure 10](image)

**Figure 10.** A Labeling of a Planar Imbedding of $K_2 \times C_n$ That Gives Rise to a Different Family of Probability Polynomials.

![Figure 11](image)

**Figure 11.** The Neighbors of Vertex x in the Second Planar Imbedding of $K_2 \times C_n$.

The only way vertex x could be clockwise is if

$$x - 2 < 2n + 3 - x < x + 2$$

or

$$2x - 5 < 2n < 2x - 1$$

or

$$x - \frac{5}{2} < n < x - \frac{1}{2}.$$
For a fixed $n$, there is only one odd number that satisfies the inequalities, so besides vertex 1, there is only one other odd vertex that is clockwise.

If vertex $x$ is clockwise, then we know

$$2x - 5 < 2n < 2x - 1$$

so

$$x - 5 < 2n - x < x - 1$$

and

$$1 - x < x - 2n < 5 - x$$

so

$$2n + 1 - x < x < 2n + 5 - x.$$  

The neighbors of vertex $2n + 3 - x$ as you go around clockwise are $2n + 1 - x$, $x$, $2n + 5 - x$, so vertex $2n + 3 - x$ is also clockwise.

A similar argument would show that there is only one vertex from the set $\{6, ..., 2n - 2\}$ that can be clockwise. So with this labeling, there are only four clockwise vertices.

Since for $n \geq 5$, $n + 4 \neq 2n$, we have shown for this case that more than one family of probability polynomials exists.

**Case II, $n = 3$.**

In Section 2.1, we showed that there was more than one family of probability polynomials for $K_2 \times C_3$.

**Case III, $n = 4$.**

The graph $K_2 \times C_4$ is the 3-cube. Label the 3-cube as shown in Figure 12a. Look at the imbedding where all the vertices are clockwise.

$p_1$: (248)

$p_2$: (137)

$p_3$: (246)
If we find the regions of this imbedding, we see that the region boundaries are

2-1-4-3-6-5-8-7-2
4-1-8-5-4
8-1-2-3-4-5-6-7-8
and 3-2-7-6-3.

Using the Euler Formula, we see that this imbedding is on the torus and the terms $p^8$ and $(1-p)^8$ are part of the genus one polynomial.

![Figure 12](image.png)

**Figure 12.** Labelings of Planar Imbeddings of the 3-cube That Give Rise to Different Family of Probability Polynomials.

Now consider the labeling shown in Figure 12b. Look at the imbedding where all the vertices are clockwise.

$p_1: (247)$
$p_2: (136)$
$p_3: (248)$
If we find the regions of this imbedding, we see that the region boundaries are 2-1-4-3-8-5-4-1-7-5-8-6-2 and 7-1-2-3-4-5-7-6-8-3-2-6-7.

Using the Euler Formula, we see that the imbedding is on the double torus and the terms $p^8$ and $(1-p)^8$ are part of the genus two polynomial.

Thus the two labelings must produce different families of probability polynomials. □

We have shown that for any even number $m$, $m \geq 6$, there exists a connected cubic graph of order $m$ for which the family of probability polynomials generated under Model II is not unique. We would like to have a way to eliminate this dependence on labeling. With this idea in mind, we will define a generalized version of Model II that we will use in the rest of this study.

### 2.3 A Generalized Model

First of all, in generalizing the model, we need some sort of criterion to determine which of the two cyclic orderings of the neighbors of a vertex will be considered to be clockwise. This criterion will give a global definition of "clockwise". This definition has to satisfy the following properties:
1. For a particular labeled imbedding, the definition has to determine whether a particular vertex is clockwise or counterclockwise.

2. A fixed vertex must be clockwise in exactly $\frac{1}{2}$ of the imbeddings.

2.3.1 Model II Generalized

For each connected labeled cubic graph $G$ of order $n$, and a global definition of clockwise, and $0 \leq p \leq 1$, the sample space $\Omega$ consists of all orientable 2-cell imbeddings, $(G,\rho)$, of $G$. Then $\rho = \{p_i\}^n_1$; let $\rho$ have exactly $c$ clockwise rotations $p_i$, $0 \leq c \leq n$. Then we define

$$P(G,\rho) = p^c (1-p)^{n-c}.$$ 

There are three natural classes of "global definitions of clockwise". The first class is where the global definition of clockwise is determined by the labeling of the vertices. The original Model II is an example of this class.

A second class is where the global definition of clockwise comes from the labeling of the edges. One example in this class would be a Cayley graph. Let $\Delta^*$ be the set of generators and their inverses. Since we are concerned with cubic graphs, we want $|\Delta^*| = 3$. Call one of the permutations of $\Delta^*$ clockwise. Then consider the edges adjacent to a vertex. If, as you go around clockwise in a particular imbedding, the edges of the graph are labeled with the elements from $\Delta^*$ in the same order as the chosen permutation, this vertex would be clockwise. With a
given group and a given generating set, the probability polynomials
generated using this definition of clockwise would be unique.

A third class of definitions of clockwise would be based on a
particular imbedding of the graph. Here one picks a certain imbedding
of the graph, \( \rho \), and assigns each of the vertices to be clockwise or
counterclockwise. In a different imbedding of the graph, if a vertex has
the same rotation as it did in \( \rho \), then it is considered to have the same
orientation (clockwise or counterclockwise) as it did in \( \rho \), and if a vertex
does not have the same rotation as it did in \( \rho \), then it is considered to
have the opposite orientation as it did in \( \rho \).

There will be examples of each of these three classes in the
following chapters.
CHAPTER III

SMALL ORDER EXAMPLES AND SOME GENERAL RESULTS

3.1 Families of Probability Polynomials for Small Order Cubic Graphs

In Section 2.1 we saw the family of probability polynomials for $K_4$ using the global definition of clockwise given in the original form of Model II. In the rest of this paper, we will refer to this global definition of clockwise as the j-k-l definition of clockwise. We mentioned in Section 2.2 that using j-k-l, there is only one family of probability polynomials for $K_4$. We would get that family of probability polynomials if we considered $K_4$ as the Cayley graph, $G_{\Delta}(\Gamma)$, where $\Gamma = Z_4$ and $\Delta = \{1,2\}$. However, there are other families of probability polynomials for $K_4$ if we use different global definitions of clockwise.

The j-k-l global definition of clockwise came from the labeling of the vertices. We could get a different global definition of clockwise if we used the labeling of the edges to define “clockwise” by considering $K_4$ as the Cayley graph, $G_{\Delta}(\Gamma)$, where $\Gamma = Z_2 \times Z_2$ and $\Delta = \{(0,1),(1,0),(1,1)\}$. We will say that a vertex in a particular imbedding is clockwise if the edges adjacent to the vertex are in the order $((0,1),(1,0),(1,1))$ as you go around clockwise. Figure 13 shows $K_4$ as this Cayley graph imbedded with all vertices clockwise. The family of probability polynomials for this definition of clockwise is given below.

$$f_0(K_4,p) = (1-p)^4 + p^4.$$
$$f_1(K_4,p) = 4p(1-p)^3 + 6p^2(1-p)^2 + 4p^3(1-p)$$.

Figure 13. The Graph $K_4$ as the Cayley Graph Where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $D = \{(0,1),(1,0),(1,1)\}$.

By using the computer program mentioned before (see Appendix B), we can find all the different families of probability polynomials for a particular cubic graph. The input for the computer program is the "clockwise" rotation at each vertex, so if we input all the choices for "clockwise" at different vertices we will get all the different families of probability polynomials. For $K_4$ there are three different families of probability polynomials. We've already mentioned two of them. All three families of probability polynomials are given below.

3.1.1 Families of Probability Polynomials for $K_4$

Family 1:

$$f_0(p) = 2p^2(1-p)^2.$$  
$$f_1(p) = (1-p)^4 + 4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p) + p^4.$$  

Family 2:

$$f_0(p) = (1-p)^4 + p^4.$$  
$$f_1(p) = 4p(1-p)^3 + 6p^2(1-p)^2 + 4p^3(1-p).$$
Family 3:
\[ f_3(p) = p(1-p)^3 + p^3(1-p). \]
\[ f_2(p) = (1-p)^4 + 3p(1-p)^3 + 6p^2(1-p)^2 + 3p^3(1-p) + p^4. \]

We looked at \( K_4 \), the only cubic graph of order four; so let us now look at the cubic graphs of order six. There are two of them: \( K_2 \times K_3 \) and \( K_{3,3} \). We proved in Chapter II that there was only one family of probability polynomials for \( K_{3,3} \) using the j-k-l definition of clockwise. However \( K_{3,3} \), like \( K_4 \), can have other families of polynomials if other global definitions of clockwise are used. Overall, there are three families of probability polynomials for \( K_{3,3} \); these are given below.

3.1.2 Families of Probability Polynomials for \( K_{3,3} \)

Family 1:
\[ f_1(p) = (1-p)^6 + 9p^2(1-p)^4 + 20p^3(1-p)^3 + 9p^4(1-p)^2 + p^6. \]
\[ f_2(p) = 6p(1-p)^5 + 6p^2(1-p)^4 + 6p^3(1-p)^2 + 6p^5(1-p). \]

Family 2:
\[ f_1(p) = 4p(1-p)^5 + 10p^2(1-p)^4 + 12p^3(1-p)^3 + 10p^4(1-p)^2 + 4p^6(1-p) \]
\[ f_2(p) = (1-p)^6 + 2p(1-p)^5 + 5p^2(1-p)^4 + 8p^3(1-p)^3 + 5p^4(1-p)^2 + 2p^5(1-p) + p^6. \]

Family 3:
\[ f_1(p) = (1-p)^6 + 4p(1-p)^5 + 9p^2(1-p)^4 + 12p^3(1-p)^3 \]
\[ + 9p^4(1-p)^2 + 4p^5(1-p) + p^6. \]
\[ f_2(p) = 2p(1-p)^5 + 6p^2(1-p)^4 + 8p^3(1-p)^3 + 6p^4(1-p)^2 + 2p^5(1-p). \]
The graph $K_2 \times K_3$ has six different families of probability polynomials under the j-k-l definition of clockwise, but again the j-k-l definition of clockwise doesn't give all the possible definitions of clockwise. There are a total of eight different families of probability polynomials for $K_2 \times K_3$. These are shown below:

3.3.2 Families of Probability Polynomials for $K_2 \times K_3$

Family 1:

$\begin{align*}
  f_0(p) &= 2p^3(1-p)^3 . \\
  f_1(p) &= (1-p)^6 + 6p(1-p)^5 + 9p^2(1-p)^4 + 6p^3(1-p)^3 \\
           & \quad + 9p^4(1-p)^2 + 6p^5(1-p) + p^6 . \\
  f_2(p) &= 6p^2(1-p)^4 + 12p^3(1-p)^3 + 6p^4(1-p)^2 .
\end{align*}$

Family 2:

$\begin{align*}
  f_0(p) &= 2p^3(1-p)^3 . \\
  f_1(p) &= 4p(1-p)^5 + 10p^2(1-p)^4 + 10p^3(1-p)^3 + 10p^4(1-p)^2 + 4p^5(1-p) . \\
  f_2(p) &= (1-p)^6 + 2p(1-p)^5 + 5p^2(1-p)^4 + 8p^3(1-p)^3 \\
\end{align*}$

Family 3:

$\begin{align*}
  f_0(p) &= p^2(1-p)^4 + p^4(1-p)^2 . \\
  f_1(p) &= (1-p)^6 + 4p(1-p)^5 + 8p^2(1-p)^4 + 12p^3(1-p)^3 \\
           & \quad + 8p^4(1-p)^2 + 4p^5(1-p) + p^6 . \\
  f_2(p) &= 2p(1-p)^5 + 6p^2(1-p)^4 + 8p^3(1-p)^3 + 6p^4(1-p)^2 + 2p^5(1-p) .
\end{align*}$

Family 4:

$\begin{align*}
  f_0(p) &= p^2(1-p)^4 + p^4(1-p)^2 .
\end{align*}$
\[ f_1(p) = (1-p)^6 + 2p(1-p)^5 + 8p^2(1-p)^4 + 16p^3(1-p)^3 + 8p^4(1-p)^2 + 2p^5(1-p) + p^6. \]
\[ f_2(p) = 4p(1-p)^5 + 6p^2(1-p)^4 + 4p^3(1-p)^3 + 6p^4(1-p)^2 + 4p^5(1-p). \]

**Family 5:**
\[ f_0(p) = p^2(1-p)^4 + p^4(1-p)^2. \]
\[ f_1(p) = 4p(1-p)^5 + 9p^2(1-p)^4 + 12p^3(1-p)^3 + 9p^4(1-p)^2 + 4p^5(1-p). \]
\[ f_2(p) = (1-p)^6 + 2p(1-p)^5 + 5p^2(1-p)^4 + 8p^3(1-p)^3 + 5p^4(1-p)^2 + 2p^5(1-p) + p^6. \]

**Family 6:**
\[ f_0(p) = 2p^3(1-p)^3. \]
\[ f_1(p) = (1-p)^6 + 2p(1-p)^5 + 9p^2(1-p)^4 + 14p^3(1-p)^3 + 9p^4(1-p)^2 + 2p^5(1-p) + p^6. \]
\[ f_2(p) = 4p(1-p)^5 + 6p^2(1-p)^4 + 4p^3(1-p)^3 + 6p^4(1-p)^2 + 4p^5(1-p). \]

**Family 7:**
\[ f_0(p) = p(1-p)^5 + p^5(1-p). \]
\[ f_1(p) = (1-p)^6 + 3p(1-p)^5 + 9p^2(1-p)^4 + 12p^3(1-p)^3 + 9p^4(1-p)^2 + 3p^5(1-p) + p^6. \]
\[ f_2(p) = 2p(1-p)^5 + 6p^2(1-p)^4 + 8p^3(1-p)^3 + 6p^4(1-p)^2 + 2p^5(1-p). \]

**Family 8:**
\[ f_0(p) = (1-p)^6 + p^6. \]
\[ f_1(p) = 6p(1-p)^5 + 9p^2(1-p)^4 + 8p^3(1-p)^3 + 9p^4(1-p)^2 + 6p^5(1-p). \]
\[ f_2(p) = 6p^2(1-p)^4 + 12p^3(1-p)^3 + 6p^4(1-p)^2. \]
The 3-cube is perhaps the most common cubic graph of order eight. It is Graph 4 of the cubic graphs of order 8 given in Section 3.5. There are 12 different families of probability polynomials for the 3-cube, \( Q_3 \).

### 3.1.4 Families of Probability Polynomials for 3-Cube

**Family 1:**

\[
\begin{align*}
f_0(p) &= 2p^4(1-p)^4. \\
f_1(p) &= (1-p)^8 + 4p^2(1-p)^6 + 8p^3(1-p)^5 + 28p^4(1-p)^4 + 8p^5(1-p)^3 \\
&\quad + 4p^6(1-p)^2 + p^8. \\
f_2(p) &= 8p(1-p)^7 + 24p^2(1-p)^6 + 48p^3(1-p)^5 + 40p^4(1-p)^4 + 48p^5(1-p)^3 \\
&\quad + 24p^6(1-p)^2 + 8p^7(1-p).
\end{align*}
\]

**Family 2:**

\[
\begin{align*}
f_0(p) &= 2p^4(1-p)^4. \\
f_1(p) &= 10p^2(1-p)^6 + 8p^3(1-p)^5 + 18p^4(1-p)^4 + 8p^5(1-p)^3 \\
&\quad + 10p^6(1-p)^2. \\
f_2(p) &= (1-p)^8 + 8p(1-p)^7 + 18p^2(1-p)^6 + 48p^3(1-p)^5 + 50p^4(1-p)^4 \\
&\quad + 48p^5(1-p)^3 + 18p^6(1-p)^2 + 8p^7(1-p) + p^8.
\end{align*}
\]

**Family 3:**

\[
\begin{align*}
f_0(p) &= 2p^4(1-p)^4. \\
f_1(p) &= (1-p)^8 + 8p^2(1-p)^6 + 8p^3(1-p)^5 + 20p^4(1-p)^4 + 8p^5(1-p)^3 \\
&\quad + 8p^6(1-p)^2 + p^8. \\
f_2(p) &= 8p(1-p)^7 + 20p^2(1-p)^6 + 48p^3(1-p)^5 + 48p^4(1-p)^4 + 48p^5(1-p)^3 \\
&\quad + 20p^6(1-p)^2 + 8p^7(1-p).
\end{align*}
\]
Family 4:
\[ f_0(p) = p^3(1-p)^5 + p^5(1-p)^3. \]
\[ f_1(p) = 2p(1-p)^7 + 3p^2(1-p)^6 + 17p^3(1-p)^5 + 10p^4(1-p)^4 + 17p^5(1-p)^3 + 3p^6(1-p)^2 + 2p^7(1-p). \]
\[ f_2(p) = (1-p)^8 + 6p(1-p)^7 + 25p^2(1-p)^6 + 38p^3(1-p)^5 + 60p^4(1-p)^4 + 38p^5(1-p)^3 + 25p^6(1-p)^2 + 6p^7(1-p) + p^8. \]

Family 5:
\[ f_0(p) = p^2(1-p)^6 + p^6(1-p)^2. \]
\[ f_1(p) = (1-p)^8 + 2p(1-p)^7 + 7p^2(1-p)^6 + 6p^3(1-p)^5 + 22p^4(1-p)^4 + 6p^5(1-p)^3 + 7p^6(1-p)^2 + 2p^7(1-p) + p^8. \]
\[ f_2(p) = 6p(1-p)^7 + 20p^2(1-p)^6 + 50p^3(1-p)^5 + 48p^4(1-p)^4 + 50p^5(1-p)^3 + 20p^6(1-p)^2 + 6p^7(1-p). \]

Family 6:
\[ f_0(p) = 2p^4(1-p)^4. \]
\[ f_1(p) = 8p^2(1-p)^6 + 8p^3(1-p)^5 + 22p^4(1-p)^4 + 8p^5(1-p)^3 + 8p^6(1-p)^2. \]
\[ f_2(p) = (1-p)^8 + 8p(1-p)^7 + 20p^2(1-p)^6 + 48p^3(1-p)^5 + 46p^4(1-p)^4 + 48p^5(1-p)^3 + 20p^6(1-p)^2 + 8p^7(1-p) + p^8. \]

Family 7:
\[ f_0(p) = p^3(1-p)^5 + p^5(1-p)^3. \]
\[ f_1(p) = 3p(1-p)^7 + 3p^2(1-p)^6 + 16p^3(1-p)^5 + 10p^4(1-p)^4 + 16p^5(1-p)^3 + 3p^6(1-p)^2 + 3p^7(1-p). \]
\[ f_2(p) = (1-p)^8 + 5p(1-p)^7 + 25p^2(1-p)^6 + 39p^3(1-p)^5 + 60p^4(1-p)^4 + 39p^5(1-p)^3 + 25p^6(1-p)^2 + 5p^7(1-p) + p^8. \]
Family 8:
\[ f_0(p) = p^2(1-p)^6 + p^6 (1-p)^2. \]
\[ f_1(p) = 2p(1-p)^7 +9p^2(1-p)^6 +6p^3(1-p)^5 +20p^4 (1-p)^4 +6p^5(1-p)^3 \]
\[ +9p^6(1-p)^2 +2p^7(1-p). \]
\[ f_2(p) = (1-p)^8 +6p(1-p)^7 +18p^2(1-p)^6+50p^3(1-p)^5 +50p^4 (1-p)^4 \]
\[ +50p^5(1-p)^3 +18p^6(1-p)^2 +6p^7(1-p)+p^8. \]

Family 9:
\[ f_0(p) = p(1-p)^7 + p^7 (1-p). \]
\[ f_1(p) = (1-p)^8 +3p(1-p)^7 +7p^2(1-p)^6 +16p^3(1-p)^5 +16p^5(1-p)^3 \]
\[ +7p^6(1-p)^2 +3p^7(1-p)+p^8. \]
\[ f_2(p) = 4p(1-p)^7 +21p^2(1-p)^6 +40p^3(1-p)^5 +70p^4 (1-p)^4 +40p^5(1-p)^3 \]
\[ +21p^6(1-p)^2 +4p^7(1-p). \]

Family 10:
\[ f_0(p) = (1-p)^8 + p^8. \]
\[ f_1(p) = 8p(1-p)^7 +12p^2(1-p)^6 +14p^4 (1-p)^4 +12p^6(1-p)^2 +8p^7(1-p). \]
\[ f_2(p) = 16p^2(1-p)^6+56p^3(1-p)^5 +56p^4 (1-p)^4 +56p^5(1-p)^3 +16p^6(1-p)^2. \]

Family 11:
\[ f_0(p) = p^3(1-p)^5 + p^5 (1-p)^3. \]
\[ f_1(p) = p(1-p)^7 +3p^2(1-p)^6 +18p^3(1-p)^5 +10p^4 (1-p)^4 +18p^5(1-p)^3 \]
\[ +3p^6(1-p)^2 +p^7(1-p). \]
\[ f_2(p) = (1-p)^8 +7p(1-p)^7 +25p^2(1-p)^6+37p^3(1-p)^5 +60p^4 (1-p)^4 \]
\[ +37p^5(1-p)^3 +25p^6(1-p)^2 +7p^7(1-p)+p^8. \]
Family 12:

\[ f_0(p) = 2p^4(1-p)^4. \]

\[ f_1(p) = (1-p)^8 + 8p(1-p)^5 + 36p^2(1-p)^4 + 8p^5(1-p)^3 + p^8. \]

\[ f_1(p) = 8p(1-p)^7 + 28p^2(1-p)^6 + 48p^3(1-p)^5 + 32p^4(1-p)^4 + 48p^5(1-p)^3 \]
\[ + 28p^6(1-p)^2 + 8p^7(1-p). \]

Each of the graphs that we have looked at so far has had more than one family of probability polynomials. It appears that the number of these families of probability polynomials grows quite rapidly. Because of this ambiguity, we would like some method of specifying a unique family of probability polynomials for a particular cubic graph. In the next few sections, we will explore further some ways of doing just that. But, before we go on to look at some ways to find a unique family of probability polynomials for a given graph, let us examine some of the information we can get from these families of probability polynomials.

First of all, let us examine Family 1 for \( K_4 \). Using Maple, we can find that the probability polynomial for the expected value of the genus random variable (which in this case is the same as \( f_1(p) \)) has critical points at \( p = 0, 1, \frac{1}{2} \) with a maximum when \( p = 0, 1 \) and a minimum when \( p = \frac{1}{2} \). This polynomial has points of infection at \( \frac{1}{2} \pm \frac{1}{6}\sqrt{3} \) (approximately .2113248654 and .7886751346). Figure 14 shows a plot of the expected value of the genus random variable for Family 1. So if we looked at \( K_4 \) as the Cayley map (p=0 or 1) where \( \Gamma=Z_4 \) and \( \Delta = \{1,2\} \), we would be maximizing the expected value of the genus random variable. The probability polynomial for genus zero, \( f_0(p) \), is maximized at \( p=\frac{1}{2} \), and is zero when \( p=0 \) or \( p=1 \).
Figure 14. Plot of the Expected Value of the Genus Random Variable for Family 1 of $K_4$.

Now, if we look at Family 2 for $K_4$, we find that the probability polynomial for the expected value random variable (which is the same as $f_1(p)$) also has critical points at $p = 0, 1, \frac{1}{2}$; but in this case, it has a maximum when $p = \frac{1}{2}$ and a minimum when $p = 0, 1$. This polynomial has no points of infection. So, looking at the graph as a Cayley map with $\Gamma = Z_2 \times Z_2$ and $\Delta = \{(0,1),(1,0),(1,1)\}$ would minimize the expected value of the genus random variable (the expected value for the genus would be 0). The probability polynomial for genus zero, $f_0(p)$, is minimized at $p = \frac{1}{2}$, and is maximized when $p = 0$ or $p = 1$. Figure 15 shows a plot of the expected value of the genus random variable for Family 2.

If we look at Family 3, the probability polynomial for the expected value random variable (which in this case again is the same as $f_1(p)$) has a maximum when $p = 0, 1$ and a minimum when $p = \frac{1}{2}$. This
polynomial has no points of infection. Letting \( p = 0 \) or 1 would maximize the expected value of the genus random variable. The probability polynomial for genus zero, \( f_0(p) \), is maximized at \( p = \frac{1}{2} \), and is zero when \( p = 0, 1 \). Figure 16 shows a plot of the expected value of the genus random variable for Family 3.

![Figure 15. Plot of the Expected Value of the Genus Random Variable for Family 2 of \( K_4 \).](image1)

![Figure 16. Plot of the Expected Value of the Genus Random Variable for Family 3 of \( K_4 \).](image2)
Tables 1 through 3 will summarize some of the information (such as where the maximum and minimum occur) that we can find out for the families of probability polynomials for the graphs of $K_4$, $K_2 \times K_3$, and $K_{3,3}$. Exact data can be calculated for the 3-cube, but it is too lengthy to put in a table, so we have not included a table that deals with the families of probability polynomials for the 3-cube.

3.2 Small Order Cubic Cayley Graphs

There is a unique family of probability polynomials for a Cayley Graph with a given group and a given generating set (since we are dealing with cubic graphs, we are interested in the generating sets where $|\Delta^*| = 3$). The global definition of clockwise is determined by the labeling of the arcs in the Cayley Color Graph (see [13] page 24 for more information on Cayley Color Graphs). The edges adjacent to a fixed vertex are labeled with precisely the elements from $\Delta^*$. We will chose one of the cyclic permutations of $\Delta^*$ to be "clockwise". If, as we encounter the neighbors of a vertex, the cyclic ordering of the neighbors is the "clockwise" permutation, we will say that the given vertex is clockwise in that imbedding. By doing this, we will be able to derive probability polynomials. We will give the family of probability polynomials for some small order groups using all possible generating sets such that $|\Delta^*| = 3$.

We will present the family of probability polynomials for these Cayley Graphs in several different ways. First, we will organize them by different groups; then we will organize them by some classes of
groups with specific generating sets; then we will organize the probability polynomials by graph; and finally we will show some interesting examples.

3.2.1 Organized by Group

3.2.1.1 Groups of Order 4

The only groups of order four are $Z_4$ and $Z_2 \times Z_2$.

3.2.1.1.1 The Group $Z_4$. For the group $Z_4$, with $\Delta = \{1,2\}$, $G_\Delta(\Gamma) = K_4$. The family of probability polynomials is Family 1 given in Section 1 for $K_4$.

The only other set of generators for $Z_4$ that form a cubic graph is $\Delta = \{2,3\}$, but $\Delta^* = \{1,2,3\}$, so this is equivalent to $\Delta = \{1,2\}$.

3.2.1.1.2 The Group $Z_2 \times Z_2$. For the group $Z_2 \times Z_2$, with $\Delta = \{(0,1),(1,0),(1,1)\}$, $G_\Delta(\Gamma) = K_4$. The family of probability polynomials is Family 2 given for $K_4$.

3.2.1.2 Groups of Order 6

The only groups of order six are $Z_6$ and $S_3$.

3.2.1.2.1 The Group $Z_6$. For the group $Z_6$, with $\Delta = \{1,3\}$, $G_\Delta(\Gamma) = K_{3,3}$. The family of probability polynomials is Family 1 given for $K_{3,3}$.

For the group $Z_6$, with $\Delta = \{2,3\}$, $G_\Delta(\Gamma) = K_2 \times K_3$. The family of probability polynomials is Family 1 given for $K_2 \times K_3$. 

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All other sets of generators that produce cubic graphs for $Z_6$ are equivalent to the ones we've mentioned.

3.2.1.2.2 The Group $S_3$. For the group $S_3$, with $A = \{(12),(123)\}$, $G_\Delta(\Gamma) = K_2 \times K_3$. The family of probability polynomials is Family 8 given for $K_2 \times K_3$.

For the group $S_3$, with $A = \{(12),(13),(23)\}$, $G_\Delta(\Gamma) = K_{3,3}$. The family of probability polynomials is Family 1 given for $K_2 \times K_3$.

All other sets of generators for $S_3$ that produce cubic graphs are equivalent to the ones we've mentioned.

3.2.1.3 Groups of Order 8

The groups of order eight are $Z_8$, $Z_4 \times Z_2$, $Z_2 \times Z_2 \times Z_2$, $D_4$, and the quaternions.

3.2.1.3.1 The Group $Z_2$. For the group $Z_8$, with $A = \{1,4\}$, $G_\Delta(\Gamma)$ is Graph 5 (called Möbius Ladder by McGeoch in his dissertation [7]) of the cubic graphs of order eight given in Section 3.5. The family of probability polynomials is given in Table 4 ($n = 4$).

For the group $Z_8$, with $A = \{3,4\}$, $G_\Delta(\Gamma)$ gives the same graph and family of probability polynomials as $Z_8$, $A = \{1,4\}$ above.

All other sets of generators that produce cubic graphs for $Z_8$ are equivalent to the ones we've mentioned.

3.2.1.3.2 The Group $Z_4 \times Z_2$. For the group $Z_4 \times Z_2$, with $A = \{(0,1),(1,0)\}$, $G_\Delta(\Gamma)$ is the 3-cube. The family of probability polynomials is Family 3 given for 3-cube.
All other sets of generators that produce cubic graphs for $Z_4 \times Z_2$ are equivalent to the ones we've mentioned.

3.2.1.3.3 The Group $Z_2 \times Z_2 \times Z_2$. For the group $Z_2 \times Z_2 \times Z_2$, with $\Delta = \{(0,0,1),(0,1,0),(1,0,0)\}$, $G_\Delta(\Gamma)$ is the 3-cube. The family of probability polynomials is Family 12 given for 3-cube.

All other sets of generators that produce cubic graphs for $Z_2 \times Z_2 \times Z_2$ are equivalent to the ones we've mentioned.

3.2.1.3.4 The Group $D_4$. For the group $D_4$, with $\Delta = \{s^2, t, ts \mid s^4 = 1, t^2 = 1, tst = s^{-1}\}$, $G_\Delta(\Gamma)$ is the 3-cube. The family of probability polynomials is Family 10 given for 3-cube.

For the group $D_4$, with $\Delta = \{s^2, t, ts \mid s^4 = 1, t^2 = 1, tst = s^{-1}\}$, $G_\Delta(\Gamma)$ is Graph 5 (Möbius Ladder) of the cubic graphs of order eight given in Section 3.5. The family of probability polynomials is the same as $Z_8$, $\Delta = \{1,4\}$ which are given in Table 4.

For the group $D_4$, with $\Delta = \{ts^2, t, ts \mid s^4 = 1, t^2 = 1, tst = s^{-1}\}$, $G_\Delta(\Gamma)$ is the 3-cube. The family of probability polynomials is Family 3 given for 3-cube.

All other sets of generators that produce cubic graphs for $D_4$ are equivalent to the ones we've mentioned.

3.2.1.3.5 The Group $Q$. There are no cubic graphs with the group $Q$ (quaternions). $Q$ only has one element of order 2, call this element $a$. If $\Delta = \{a, b\}$ where $b^4 = 1$, then $b^2 = a$, and we only get a group of order 4.
3.2.1.4 Groups of Order 10

The groups of order ten are $Z_{10}$, and $D_5$.

3.2.1.4.1 The Group $Z_{10}$. For the group $Z_{10}$, with $\Delta = \{1,5\}$. This graph is Graph 17 (Möbius Ladder) of the cubic graphs of order 10 given in Section 3.5. The family of probability polynomials is given in Table 4 (n=5).

For the group $Z_{10}$, with $\Delta = \{2,5\}$, $G_{\Delta}(\Gamma)$ is Graph 15 (prism graph) of the cubic graphs of order 10 given in Section 3.5. The family of probability polynomials is given in Table 6 for $Z_n \times Z_2$ (n=5).

All other sets of generators that produce cubic graphs for $Z_{10}$ are equivalent to the ones we've mentioned.

3.2.1.4.2 The Group $D_5$. For the group $D_5$, with $\Delta = \{s,t, | s^5=1, t^2=1=(st)^2\}$, $G_{\Delta}(\Gamma)$ is Graph 15 (prism graph) of the cubic graphs of order 10 given in Section 3.5. The family of probability polynomials is given in Table 6 (n = 5).

For the group $D_5$, with $\Delta = \{ts^2,t,ts | s^5=1, t^2=1=(st)^2\}$, $G_{\Delta}(\Gamma)$ is Graph 17 (Möbius Ladder) of the cubic graphs of order 10 given in Section 3.5. The family of probability polynomials is given below.

$$f_1(p) = (1-p)^{10} + 5p^2(1-p)^8 + 20p^4(1-p)^6 + 20p^5(1-p)^5 + 20p^6(1-p)^4 + 5p^8(1-p)^2 + p^{10},$$

$$f_2(p) = 10p(1-p)^9 + 20p^2(1-p)^8 + 60p^3(1-p)^7 + 130p^4(1-p)^6 + 192p^5(1-p)^5 + 130p^6(1-p)^4 + 60p^7(1-p)^3 + 20p^8(1-p)^2 + 10p^9(1-p).$$

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\( f(p) = 20p^2(1-p)^8 + 60p^3(1-p)^7 + 60p^4(1-p)^6 \\
+ 40p^5(1-p)^5 + 60p^6(1-p)^4 + 60p^7(1-p)^3 + 20p^8(1-p)^2. \\
+ 40p^5(1-p)^5 + 60p^6(1-p)^4 + 60p^7(1-p)^3 + 20p^8(1-p)^2. \\

All other sets of generators that produce cubic graphs for \( D_5 \) are equivalent to the ones we've mentioned.

3.2.2 Organized by Classes of Groups With a Specific Generating Set

Here we will look at several classes of groups with a specific generating set and see what the family of probability polynomials are for the different Cayley Graphs. Table 4 gives the family of probability polynomials for \( Z_{2n} \) with \( \Delta = \{1,n\} \). The graph \( G_\Delta(\Gamma) \) is what McGeoch calls a Möbius Ladder [7]. In Table 5, we give the expected value of the genus random variable for the Cayley Graph \( G_\Delta(\Gamma) \) where \( \Gamma = Z_{2n} \) with \( \Delta = \{1,n\} \). Figure 17 shows a Maple plot of the expected value of the genus random variable.

\[ \text{Figure 17. Expected Value of Genus Random Variable for } G_\Delta(\Gamma), \Gamma = Z_{2n} \text{ With } \Delta = \{1,n\}, 2 \leq n \leq 7. \]
Table 6 shows the family of probability polynomials for $Z_n \times Z_2$, $3 \leq n \leq 8$, with $\Delta = \{(1,0),(0,1)\}$. Figure 18 shows a Maple plot of the expected value of the genus random variable with $2 \leq n \leq 7$. Table 7 shows the expected value of the genus random variable for the Cayley graphs where $\Gamma = Z_n \times Z_2$ with $\Delta = \{(1,0),(0,1)\}$; $G_\Delta(\Gamma)$ is the prism graph.

![Figure 18. Expected Value of Genus Random Variable for $G_\Delta(\Gamma)$, $\Gamma = Z_n \times Z_2$ With $\Delta = \{(1,0),(0,1)\}$; $3 \leq n \leq 8$.]

Table 8 shows the family of probability polynomials for the Cayley Graph $G_\Delta(\Gamma)$ where $\Gamma = D_n$, $3 \leq n \leq 7$, with $\Delta = \{s,t | s^n = 1, t^2 = 1, tst = s^{-1}\}$. Figure 19 shows a Maple plot of the expected value of the genus random variable. Table 9 shows the expected value of the genus random variable for $D_n$, $3 \leq n \leq 7$, with $\Delta = \{s,t | s^n = 1, t^2 = 1, tst = s^{-1}\}$; again $G_\Delta(\Gamma)$ is the prism graph.
3.2.3 Organized by Graph

We will give some small order cubic graphs and tell which probability polynomials are the result of considering the graph as a Cayley Graph.

![Figure 19. Expected Value of Genus Random Variable for $G_{\Delta}(\Gamma)$, $\Gamma = D_n$ With $\Delta = \{s,t | s^n = 1, t^2 = 1, tst = s^{-1}\}; 3 \leq n \leq 7.$](image)

3.2.3.1 The Graph $K_4$

Family 1 is the family of probability polynomials for $G_{\Delta}(\Gamma)$ for $\Gamma = Z_4$ with $\Delta = \{1,2\}$.

Family 2 is the family of probability polynomials for $G_{\Delta}(\Gamma)$ for $\Gamma = Z_2 \times Z_2$ with $\Delta = \{01,10,11\}$.

3.2.3.2 The Graph $K_{2,3}$

Family 1 is the family of probability polynomials for $G_{\Delta}(\Gamma)$ for $\Gamma = Z_6$ with $\Delta = \{1,3\}$.
Family 1 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = S_3$ with $\Delta = \{(12),(13),(23)\}$.

### 3.2.3.3 The Graph $K_2 \times K_3$

Family 1 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = Z_6$ with $\Delta = \{2,3\}$.

Family 8 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = S_3$ with $\Delta = \{(12),(123)\}$.

### 3.2.3.4 The Graph 3-Cube

Family 3 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = Z_4 \times Z_2$ with $\Delta = \{(0,1),(1,0)\}$

Family 12 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = Z_2 \times Z_2 \times Z_2$ with $\Delta = \{(0,0,1),(0,1,0),(1,0,0)\}$.

Family 10 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = D_4$ with $\Delta = \{s,t|s^4=1, t^2=1, tst=s^{-1}\}$.

Family 1 is the family of probability polynomials for $G_A(\Gamma)$ for $\Gamma = D_4$ with $\Delta = \{t,ts,ts^2|s^4=1, t^2=1, tst=s^{-1}\}$.

### 3.2.4 Special Examples

Even if we have a Cayley Graph, not every family of probability polynomials can be obtained using the Cayley definitions. An example of this is Family 3 for $K_4$.

We can find graphs that are the Cayley Graphs of different groups but still have the same family of probability polynomials. An example of
this is $\Gamma = Z_6$ with $\Delta = \{1,3\}$ and $\Gamma = S_3$ with $\Delta = \{(12),(13),(23)\}$. In both cases $G_\Delta(\Gamma)$ is $K_{3,3}$ and they both give Family 1 of the probability polynomials.

We can find examples where a graph is a Cayley Graph for different groups, and the different groups give different families of probability polynomials. An example of this is $K_4$. This is the Cayley Graph of $\Gamma = Z_4$ with $\Delta = \{1,2\}$ as well as $\Gamma = Z_2 \times Z_2$ with $\Delta = \{01,10\}$. These have probability polynomials Family 1 and Family 2, respectively.

We can find examples of a Cayley Graph which is the Cayley Graph for the same group with different generating sets and different families of probability polynomials result. An example of this is the 3-cube. The 3-cube is the Cayley Graph of $\Gamma = D_4$ with $\Delta = \{s,t | s^4 = 1, t^2 = 1, tst = s^{-1}\}$ as well as the Cayley Graph of $\Gamma = D_4$ with $\Delta = \{t, ts, ts^2 | s^4 = 1, t^2 = 1, tst = s^{-1}\}$. The family of probability polynomials are Family 10 and Family 3, respectively.

**3.3 Some Special Cubic Graphs**

It is helpful to have a unique family of probability polynomials for a fixed graph with a specific global definition of clockwise, so there is no confusion over which family of probability polynomials is being discussed. In the previous section, we used Cayley graphs to specify a unique family of probability polynomials. However, in this section, we are going to give families of probability polynomials for some special graphs. These families of probability polynomials are not unique, but
because the graphs are special cubic graphs, it is worthwhile to find families of probability polynomials for them.

We will consider the Petersen Graph as the graph $O_5$. The vertices will be given as all two-element subsets of the set $\{1,2,3,4,5\}$, where two adjacent vertices have no elements in common. See Figure 20. An edge will be labeled with the element that does not appear in either of the sets of the labels of the two vertices to which it is adjacent. In a particular imbedding, a vertex will be considered clockwise if the clockwise ordering of the edges forms a strictly increasing cyclic permutation. The family of probability polynomials using this definition is given Table 10.

![Figure 20. A Labeling of the Petersen Graph as the Graph $O_5$.](image)

The genus distribution for the Petersen Graph is the value of the uniform case of this family of probability polynomials. And the genus
distribution is given in the right-hand column of Table 11. A Maple plot of the expected value of the genus random variable for the Petersen Graph is given in Figure 21. The average genus for the Petersen Graph is 2.273, with a variance of 0.277.

![Expected Value of the genus Ran. Var. for the Petersen Graph](image)

**Figure 21.** Expected Value of the Genus Random Variable for the Petersen Graph as the Graph $O_5$. 

The expected value of the genus random variable has a maximum when $p = \frac{1}{2}$, and a minimum when $p = 0$ or $1$. The points of inflection occur at approximately $0.1770166878$ and $0.822983312$.

If we take a planar graph of the dodecahedron and identify the antipodal vertices, we get the Petersen graph imbedded on the projective plane. So if we labeled the vertices and edges down below as the odd graph, $O_5$, the labeling down below would give rise to a unique labeling up above. Figure 22 shows the Petersen Graph on the projective plane and the dodecahedron as the double cover of the Petersen Graph.
The labeling of the edges of the graph of the dodecahedron is forced by the labeling of the edges of the Petersen Graph, so we can use the same global definition of clockwise that we used for the Petersen graph. The family of probability polynomials for the dodecahedron is given in Table 11. The genus distribution is given in the right-hand column of this table. The average genus for the dodecahedron is 4.509456636. A Maple plot of the expected value of the genus random variable for the dodecahedron is given in Figure 23. The expected value of the genus random variable has a maximum at $p = \frac{1}{2}$ and a minimum at approximately 0.0194686691 and 0.9805313344 (Maple only can find exact values when the polynomial is of degree 8 or less).

3.4 General Results for Cubic Graphs

There are some general results that are true for all the families of probability polynomials for a fixed connected, labeled cubic graph. Gross and Furst mention a number of invariants for imbeddings of graphs [4]. Many of these can be found as a special case of a family of probability polynomials. Each value of the genus distribution is precisely the value of the numerator of the uniform case of a probability polynomial for that surface (i.e. $2^{2n_f_1} \frac{1}{2}$). So, no matter how we globally define clockwise, we can get the genus distribution from any probability polynomial. We can find the average genus by looking at the uniform case of the probability polynomial for the expected value of the genus distribution. The genus range as well as the minimum genus...
Figure 22. Dodecahedron as Antipodal Double Cover of the Petersen Graph.
and maximum genus can also be found from each family of probability polynomials for a graph.

Below are several theorems that relate to the probability polynomials of each surface ($f_i(p)$) and to the probability polynomials for the expected value of the genus random variable ($E(\gamma(p))$). We will let $f(p)$ stand for both $f_i(p)$ and $E(\gamma(p))$.

**Theorem 3.4.1** The polynomial $f(p)$ always has a relative maximum or minimum at $p = \frac{1}{2}$ and $f(p)$ is symmetrical about the line $p = \frac{1}{2}$.

**Proof:** Let $(G, \rho)$ be an orientable 2-cell imbedding of a cubic graph of order $2m$. Under a given global definition of clockwise, if $(G, \rho)$ has $s$ clockwise vertices and is imbedded on surface of genus $i$, then the mirror image of $(G, \rho)$ is also imbedded on the surface of genus $i$, but has...
2m-s clockwise vertices. Thus if \( f(p) \) has a term of the form \( p^s(1-p)^{2m-s} \),
then \( f(p) \) also has a term of the form \( p^{2m-s}(1-p)^s \), so \( f(p) \) is symmetrical
about the line \( p = \frac{1}{2} \).

We can write \( f(p) = \sum_{i=0}^{m-1} a_i[p^i(1-p)^{2m-i} + p^{2m-i}(1-p)^i] + a_m p^m(1-p)^m, \)
\( a_i \in \mathbb{R} \).

Thus \( f'(p) = \sum_{i=0}^{m-1} a_i[\frac{1}{2}(1-p)^{2m-i} - (2m-i)p^{2m-i-1} + \)
(\(2m-i)p^{2m-i-1}(1-p)^i - ip^{2m-i}(1-p)^i-1 \] ) +
\( ma_m p^{m-1}(1-p)^m - ma_m p^m(1-p)^{m-1} \).

So,
\[
f'(\frac{1}{2}) = \sum_{i=0}^{m-1} a_i\left[\left(\frac{1}{2}\right)^{2m-1} - (2m-i)\left(\frac{1}{2}\right)^{2m-1} + (2m-i)\left(\frac{1}{2}\right)^{2m-1} - i\left(\frac{1}{2}\right)^{2m-1}\right] +
ma_m\left(\frac{1}{2}\right)^{2m-1} - ma_m\left(\frac{1}{2}\right)^{2m-1} = 0
\]

Now we can look at two cases.

**Case 1** If \( f(p) \) is constant on an interval \((\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)\), for some \( \epsilon > 0 \), then \( f \)
has both a local maximum and a local minimum at \( p = \frac{1}{2} \).

**Case 2** The polynomial \( f(p) \) is not constant on an interval \((\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)\), for
all \( \epsilon > 0 \). So there must be an interval \((\frac{1}{2} - \epsilon, \frac{1}{2})\) on which \( f \) is either strictly
increasing or strictly decreasing. If \( f \) is strictly increasing on \((\frac{1}{2} - \epsilon, \frac{1}{2})\),
then since \( f \) is symmetrical about the line \( p = \frac{1}{2} \), \( f \) is strictly decreasing on
\((\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)\) and \( f \) has a local maximum at \( p = \frac{1}{2} \).

On the other hand, if \( f \) is strictly decreasing on \((\frac{1}{2} - \epsilon, \frac{1}{2})\), then since
\( f \) is symmetrical about the line \( p = \frac{1}{2} \), \( f \) is strictly increasing on \((\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)\)
and \( f \) has a local minimum at \( p = \frac{1}{2} \).
Theorem 3.4.2 Let \( \sum_{i=0}^{n} b_i p^{i(1-p)^n} \) be the expected value polynomial of the genus random variable for some family of probability polynomials for a graph \( G \) of order \( n=2m \); then \( \sum_{i=0}^{n} b_i \) is constant over all families of probability polynomials.

Proof: Let \( a_{ij} \) be the coefficient of \( p^{i(1-p)^n} \) in \( f_j(p) \). Then,
\[
\sum_{i=0}^{n} b_i = \sum_{i=0}^{n} \sum_{j} j \cdot a_{ij} = \sum_{j} j \sum_{i=0}^{n} a_{ij}.
\]
But \( \sum_{i=0}^{n} a_{ij} \) is the number of imbeddings \( G \) has on surface \( j \), which is constant. So, \( \sum_{j} j \sum_{i=0}^{n} a_{ij} \) is constant no matter which family of probability polynomials is used. \( \square \)

Theorem 3.4.3 Let \( E(p) = \sum_{i=0}^{n} c_i p^i \) be the expected value polynomial (in expanded form) of the genus random variable for some family of probability polynomials for a graph \( G \) of order \( n \). Let \( g \) be the genus of the surface on which the imbedding of \( G \) with all the vertices clockwise (counterclockwise) is imbedded. Then \( g = \sum_{i=0}^{n} c_i = c_0 \).

Proof: It is immediate that \( g = E(0) = E(1) \). But \( E(0) = c_0 \) and
\[
E(1) = \sum_{i=0}^{n} c_i p^i = \sum_{i=0}^{n} c_i. \quad \square
\]
Theorem 3.4.4 Let $f_j(p) = \sum_{i=0}^{n} a_{ij}p^i(1-p)^{n-i}$. Then $\sum_{j} \sum_{i=0}^{n} a_{ij} = 2^n$.

**Proof:** We know that $\sum a_{ij} = \binom{n}{1}$, so

$$\sum_{j} \sum_{i=0}^{n} a_{ij} = \sum_{i=0}^{n} \sum_{j} a_{ij} = \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$ \square

3.5 All Cubic Graphs of Order Eight and Ten

There are 5 cubic graphs of order 8 and 19 cubic graphs of order 10. The cubic graphs of order 8 are shown in Figure 24 and the cubic graphs of order 10 are shown in Figure 25. Table 12 lists the genus distribution for all the cubic graphs of order 8 and Table 13 lists the genus distribution for all the cubic graphs of order 10.

For each of the cubic graphs of order 8 and order 10, we have given the genus distribution and average genus (see Tables 12 and 13). There may be many different family of probability polynomials for these graphs; however, as we mentioned in Section 3.4, the uniform case of any one of these family of probability polynomials gives the genus distribution for the graph. So, we just found one family of probability polynomials and used it get the genus distribution.

Some of the cubic graphs of orders 8 and 10 are special graphs and we will call special attention to them. First of all, for the graphs of order 8, Graph 2 is the Ringel ladder $R_3$ which we will discuss in Chapter 5. Graph 4 is the 3-cube, $Q_3$ or the graph of a cube drawn in the plane, it is sometimes called the prism graph of the square. For graphs of order 10, Graph 7 is the Ringel ladder $R_4$. Graph 15 is the prism graph of the
pentagon. Graph 17 is the Möbius Ladder, and Graph 19 is the Petersen Graph, which we discussed in Section 3.3.

Figure 24. Cubic Graphs of Order Eight.
Figure 25. Cubic Graphs of Order Ten.
Figure 25--Continued

Graph 9

Graph 10

Graph 11

Graph 12

Graph 13

Graph 14

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Figure 25--Continued

Graph 15  

Graph 16

Graph 17  

Graph 18

Graph 19
CHAPTER IV

CLOSED-END LADDERS

4.1 Probability Polynomials for Closed-End Ladders

The genus distributions for closed-end ladders were given by Furst, Gross, and Statman [2]. They used combinatorial techniques to count how many imbeddings of a closed-end ladder there were on each surface. This number of imbeddings on each surface is precisely the value of the numerator of the uniform case of a probability polynomial for that surface (i.e. $2^{2n_1} (\frac{1}{2})$). While using their basic techniques, we also keep track of how many vertices are clockwise in each imbedding and thus develop probability polynomials for closed-end ladders.

Closed-end ladders, $L_n$, are the pseudographs formed by taking $P_n \times K_2$ and doubling both of the ends. Figure 26 shows the closed-end ladder $L_4$. Closed-end ladders are similar to the ladders Ringel used in his proof of the Heawood Map Coloring Theorem (see [13] page 125, or [10]); we will look at Ringel Ladders in more detail in Chapter V. We will, as Ringel did, color the vertices that are clockwise black and will leave the counterclockwise vertices white. ([13] page 134)

Our global definition of clockwise will be based on a fixed planar imbedding of $L_n$ (as extending Figure 26). All the vertices in this planar imbedding will be “clockwise”. When we are discussing an imbedding of $L_n$, we will not actually describe the imbedding on the surface, but will
consider whether a vertex is clockwise or not in the imbedding. Using this method, we can picture an imbedding of $L_n$ by simply drawing a planar imbedding and coloring the vertices of this imbedding black or white. To count the number of regions in this imbedding, we count the number of edge-orbits. Figure 27 shows the three edge-orbits for a particular imbedding of $L_3$. (See [6] page 113 for more details on edge-orbits). Then using the Euler formula, we can figure out the genus of the surface on which the closed-end ladder is imbedded.

![Figure 26. The Closed-End Ladder $L_4$.](image)

![Figure 27. Edge-Orbits for an Imbedding of $L_3$.](image)

The rungs on the ends are called **end-rungs**, while the ones in the middle are called **mid-rungs**. A mid-rung is called **matched** if it is the same color at both ends, otherwise it is called **unmatched**. If a rung is either an end-rung or a matched mid-rung, it is called an **m/e rung**. Two m/e rungs are **consecutive m/e rungs** if there are no unmatched rungs between them.
Furst, Gross, and Statman use the expression $b(p, q, r)$ to stand for the number of ways to put $p$ identical balls into $q$ distinct boxes, so that exactly $r$ boxes have an even number of balls [2]. They let $s(n, m, k)$ stand for the number of rotation systems for the ladder $L_n$ that have $m$ matched mid-rungs of which $k$ pairs of the m/e rungs are evenly separated. The expressions for $b(p, q, r)$ and $s(n, m, k)$ are given below:

$$b(p, q, r) = \begin{cases} 0 & \text{if } p-q+r \text{ is odd} \\ \binom{q}{r} \binom{\frac{p+q+r}{2} + q-1}{q-1} & \text{otherwise} \end{cases}$$

and

$$s(n, m, k) = 2^n b(n-m, m+1, k) = 2^n \binom{m+1}{k} \binom{\frac{n+k-1}{2}}{m}.$$ 

Note, since $b(p, q, k) = 0$ if $p-q+k$ is odd, $s(n, m, k) = 0$ if $(n-m)-(m+1) + k = n-2m+k-1$ is odd.

Let $s(n, m, k, c)$ be the number of rotation systems for $L_n$ that have $m$ matched mid-rungs of which $k$ pairs of m/e rungs are evenly separated such that $c$ vertices are clockwise. Since there are $n-m$ unmatched mid-rungs and each of these has exactly one clockwise vertex, $s(n, m, k, c) = 0$, if $c < n-m$. Also, every matched mid-rung has either zero or two clockwise vertices so $s(n, m, k, c) = 0$ if $c \not\equiv (n-m) \mod 2$.

Thus,

$$s(n, m, k, c) = \begin{cases} 0 & \text{if } c < n-m \text{ or } c \not\equiv (n-m) \mod 2 \\ \frac{m}{c(n-m)} 2^{n-m} b(n-m, m+1, k) & \text{if } (n-m) \leq c \leq n \\ \text{and } c \equiv (n-m) \mod 2 \end{cases}$$
If we sum over all values for $c$, we get $s(n,m,k)$ as seen below.

\[
\sum_{c=0}^{2n} s(n,m,k,c) = \frac{c-(n-m)}{2} \sum_{a=0}^{m} \binom{m}{a} 2^{n-m} b(n-m,m+1,k) \\
= 2^{n-m} b(n-m,m+1,k) \sum_{a=0}^{m} \binom{m}{a} \\
= 2^{n-m} b(n-m,m+1,k) 2^m \\
= 2^n b(n-m,m+1,k) = s(n,m,k)
\]

Let $f(n,k)$ denote the number of imbeddings of $L_n$ that have $k$ regions. Furst, Gross, and Statman found that the number of regions was one more than the number of evenly separated pairs of m/e rungs, so we get:

\[
f(n,k) = \sum_{m=0}^{n} s(n,m,k-1)
\]

Define $f(n,k,c)$ to be the number of imbeddings of $L_n$ that have $k$ faces with $c$ clockwise vertices. Then

\[
f(n,k,c) = \sum_{m=0}^{n} s(n,m,k-1,c).
\]

If we sum over all values for $c$, we get $f(n,k)$, as seen below;

\[
\sum_{c=0}^{2n} f(n,k,c) = \sum_{c=0}^{2n} \sum_{m=0}^{n} s(n,m,k-1,c) \\
= \sum_{m=0}^{n} \sum_{c=0}^{2n} s(n,m,k-1,c)
\]
\[
\sum_{m=0}^{n} s(n,m,k-1) = f(n,k).
\]

Let \(g_i(L_n)\) be the number of imbeddings of \(L_n\) on surface \(S_i\), and let \(g_{i,c}(L_n)\) be the number of imbeddings of \(L_n\) on surface \(S_i\) that have \(c\) vertices clockwise. Using the Euler identity, we get \(k=n+2-2i\). So,

\[
g_{i,c}(L_n) = f(n,n+2-2i,c)
= \sum_{m=0}^{n} s(n,m,n+1-2i,c).
\]

Thus, for genus \(i\), the probability polynomial \(f_i(p)\) would be

\[
f_i(p) = \sum_{c=0}^{2n} \sum_{m=0}^{n} s(n,m,n+1-2i,c)p^c(1-p)^{2n-c}
= \sum_{m=0}^{n} \sum_{c=0}^{2n} s(n,m,n+1-2i,c)p^c(1-p)^{2n-c}
\]

(Again, \(a=\frac{c-(n-m)}{2}\), so \(c= n-m+2a\). Also, the terms are all zero when \(c< n-m\).) So,

\[
f_i(p) = \sum_{m=0}^{n} \sum_{a=0}^{m} s(n,m,n+1-2i,n-m+2a)p^{n-m+2a}(1-p)^{n+m-2a}
= \sum_{m=0}^{n} \sum_{a=0}^{m} \binom{m}{a} 2^{n-m} b(n-m,m+1,n+1-2i)p^{n-m+2a}(1-p)^{n+m-2a}
= \sum_{m=0}^{n} \sum_{a=0}^{m} \binom{m}{a} 2^{n-m} \left( \binom{m+1}{n+1-2i} \left( \frac{n+1-2i}{2} \right) \right) p^{n-m+2a}(1-p)^{n+m-2a}
= \sum_{m=0}^{n} \sum_{a=0}^{m} \binom{m}{a} 2^{n-m} \left( \binom{m+1}{n+1-2i} \left( \frac{n-i}{m} \right) \right) p^{n-m+2a}(1-p)^{n+m-2a}.
\]

The probability polynomial for the expected value of the genus random variable for \(L_n\) would be
\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} f_i(p)
\]

\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} \sum_{m=0}^{n} \sum_{a=0}^{m} \binom{m}{a} 2^{n-m} \binom{m+1}{n+1-i} \binom{n-i}{m} p^{n-m+2a(1-p)^{n+m}-2a}
\]

\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} \sum_{m=0}^{n} 2^{n-m} \binom{m+1}{n+1-i} \binom{n-i}{m} p^{n-m(1-p)^{n+m}} \sum_{a=0}^{m} \binom{m}{a} p^{2a(1-p)^{2a}}
\]

\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} \sum_{m=0}^{n} 2^{n-m} \binom{m+1}{n+1-i} \binom{n-i}{m} p^{n-m(1-p)^{n+m}} \sum_{a=0}^{m} \binom{m}{a} \left( \frac{p^2}{(1-p)^2} \right)^a
\]

\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} \sum_{m=0}^{n} 2^{n-m} \binom{m+1}{n+1-i} \binom{n-i}{m} p^{n-m(1-p)^{n+m}} \left( 1 + \frac{p^2}{(1-p)^2} \right)^m
\]

\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} \sum_{m=0}^{n} 2^{n-m} \binom{m+1}{n+1-i} \binom{n-i}{m} p^n(1-p)^n \left( \frac{1-p}{p} + \frac{p}{1-p} \right)^m
\]

\[
\mathbb{E}(\gamma(L_n)) = \sum_{i=0}^{\lfloor n+1/2 \rfloor} \sum_{m=0}^{n} 2^{n-m} \binom{m+1}{n+1-i} \binom{n-i}{m} p^n(1-p)^n \left( \frac{1-2p+2p^2}{p(1-p)} \right)^m
\]
4.2 Illustrating the Probability Polynomials

When \( p = \frac{1}{2} \), the numerator of the probability polynomials \( f_i(p) \), with denominator \( 2^{2n} \), gives us the genus distribution of the graph. So, in the case of closed-end ladders \( 2^{2n} f_i \left( \frac{1}{2} \right) \) should give the number of imbeddings on surface of genus \( i \) (\( g_i \)) as given by Furst, Gross, and Statman.

\[
2^{2n} f_i \left( \frac{1}{2} \right) = 2^{2n} \sum_{m=0}^{n} \sum_{a=0}^{m} \binom{m}{a} 2^{n-m} \left( \binom{m+1}{n+1-2i} \right) \left( \binom{n-i}{m} \right) \left( \frac{1}{2} \right)^{n-m+2a} \left( \frac{1}{2} \right)^{n+m-2a}
\]

\[
= 2^{2n} \left( \frac{1}{2} \right) 2^n \sum_{m=0}^{n} \left[ 2^{n-m} \left( \binom{m+1}{n+1-2i} \right) \left( \binom{n-i}{m} \right) \sum_{a=0}^{m} \binom{m}{a} \right]
\]

\[
= \sum_{m=0}^{n} 2^{n-m} \left( \binom{m+1}{n+1-2i} \right) \left( \binom{n-i}{m} \right) 2^m
\]

\[
= 2^n \sum_{m=0}^{n} \left( \binom{m+1}{n+1-2i} \right) \left( \binom{n-i}{m} \right)
\]

Let \( k = n+2-2i \)

\[
= 2^n \sum_{m=0}^{n} \left( \binom{m+1}{k-1} \right) \left( n - \left\lfloor \frac{k+n+2}{2} \right\rfloor \right)
\]

\[
= 2^n \sum_{m=0}^{n} \left( \binom{m+1}{k-1} \right) \left( \frac{n+k-2}{2} \right)
\]

\[
= f(n,k)
\]

\[
f(n,n+2-2i)
\]

\[
g_i(L_n)
\]

\[
= \left\{ \begin{array}{ll}
2^{n-1+i} \left( \frac{n+i-1}{i} \right) & \text{if } i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \\
0 & \text{otherwise}
\end{array} \right.
\]
Table 14 gives the probability polynomials for each surface on which the closed-end ladders $L_1 - L_5$ can be imbedded. In the uniform case ($p=\frac{1}{2}$), the genus distribution and the average genus agree with results given previously.([2] page 30;[14] page 550). When we let $p=\frac{1}{2}$ in the probability polynomials for the expected value of the genus random variable, we get the average genus of the closed-end ladder as given in [14]. Table 15 shows the probability polynomials for the expected value of the genus as well as the average genus for $L_1 - L_5$. Figure 28 shows a plot done on Maple of the probability polynomials of the expected value of the genus random variable for $L_1 - L_5$.

![Maple Plot for Probability Polynomials of the Expected Value of the Genus Random Variable for $L_1 - L_5$.](image)

Figure 28.
CHAPTER V

RINGEL LADDERS

5.1 Genus Distribution of Ringel Ladders

As we mentioned in Chapter IV, Ringel ladders are the graphs that were used by Ringel to solve the Haywood Map Coloring Theorem (see [10], or for example [13], page 125). We will find the genus distribution and the average genus for Ringel ladders (the genus distribution for Ringel ladders has been found independently by Klein, [3]). To do this, we will use some of the combinatorial techniques that Furst, Gross, and Statman used for closed-end ladders as well as some definitions and concepts used by McGeoch in his Ph.D. dissertation [7].

Ringel ladders, $R_n$, are the graphs formed by subdividing the end-rungs of the closed-end ladders and adding an edge between these two new vertices. Figure 29 shows the Ringel ladder $R_4$. Ringel ladders are the graphs Ringel used in his proof of the Heawood Map Coloring Theorem. We will, as Ringel did, color the vertices that are clockwise black and will leave the counterclockwise vertices white. (See for example [13] page 134).

As we did with closed-end ladders (Chapter IV), our global definition of clockwise will be based on a fixed planar imbedding of $R_n$, as in Figure 29. All the vertices in this planar imbedding will be "clockwise". When we are discussing an imbedding of $R_n$, we will not
actually describe the imbedding on the surface, but will consider whether a vertex is clockwise or not in the imbedding. Using this method, we can picture an imbedding of $R_n$ by simply drawing a planar imbedding and coloring the vertices of this imbedding black or white. To count the number of regions in this imbedding, we count the number of edge-orbits. Then using the Euler formula, we can figure out the genus of the surface on which the closed-end ladder is imbedded.

Figure 29. The Ringel Ladder $R_4$.

We will find the genus distribution for the Ringel ladders by taking imbeddings of closed-end ladders, subdividing each of the end-rungs, and adding an edge between each of these end-rungs. We will then use a special case of the Ringel-White Edge Adding Lemma ([13], page 135) to find the genus of that particular Ringel ladder. In the following we will be referring to closed-end ladders unless we specifically mention otherwise. The special case of the Ringel-White Edge Adding Lemma that we will need will be proved after we discuss the notation we will use for closed-end ladders (this notation will be very similar to the notation used in Chapter IV for closed-end ladders, but there will be a few differences).
The rungs on the ends are called **end-rungs**, while the ones in the middle are called **mid-rungs**. A mid-rung is called **matched** if it is the same color at both ends, otherwise it is called **unmatched**. If a rung is either an end-rung or a matched mid-rung, it is called an **m/e rung**. Two m/e rungs are **consecutive m/e rungs** if there are no unmatched rungs between them. Two consecutive matched mid-rungs form an **even gap (odd gap)** if the number of unmatched rungs between them is even (odd). NOTE: This differs from what Furst, Gross, and Statman used for closed-end ladders [2]. They counted m/e rungs when they were considering even gaps. We don't want to include the end-rungs in the even (odd) gaps. An end-rung and its consecutive matched rung form an **even-end gap (odd-end gap)** if there is an even (odd) number of unmatched rungs between them. Two consecutive unmatched mid-rungs are called **similar** if they are colored the same at the top and bottom. Figure 30 shows two similar unmatched rungs.

![Figure 30. Two Similar Unmatched Rungs.](image)

Some notation that we will use is as follows:

- **n** = number of mid-rungs
- **m** = number of matched mid-rungs
- **k** = number of even gaps
- **e** = number of even-end gaps

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5.1.1 The Special Case of the Ringelstein-White Edge Adding Lemma

Take a closed-end ladder, $L_n$, and subdivide the two end-rungs, getting $L_n'$. Call the two new vertices created by the subdivision $v$ and $u$. Let $R_n = L_n' + uv$. Corresponding to each imbedding of $L_n$, there are four imbeddings of $R_n$. Suppose $L_n$ can be 2-cell imbedded on the surface $S_h$ with $r$ regions. In this imbedding, $v$ and $u$ each lie on two regions (not necessarily distinct). The four imbeddings of $R_n$ which correspond to this imbedding of $L_n$ can be determined from the imbedding of $L_n$ by the following five cases.

**Case 1** None of the regions that border $v$ are the same as any of the regions that border $u$.

In this case, all **four** of the imbeddings of $R_n$ are on $S_{h+1}$ with $r-1$ regions.

**Case 2** There are two distinct regions bordering each vertex. One of the regions bordering $v$ also borders $u$.

In this case, there is **one** imbedding of $R_n$ on $S_h$ with $r+1$ regions and **three** imbeddings of $R_n$ on $S_{h+1}$ with $r-1$ regions.

**Case 3** Without loss of generality, there are two distinct regions bordering $v$ and one distinct region bordering $u$. One of the regions bordering $v$ is the same as the region bordering $u$.

In this case, there are **two** imbeddings of $R_n$ on $S_h$ with $r+1$ regions and **two** imbeddings of $R_n$ on $S_{h+1}$ with $r-1$ regions.
Case 4 All four of the regions are the same region.

In this case, all four imbeddings of $R_n$ on $S_h$ with $r+1$ regions.

Case 5 There are two distinct regions bordering $v$ and two distinct regions bordering $u$. Each of the regions bordering $v$ is the same as one of the regions bordering $u$.

In this case, there are two imbeddings of $R_n$ on $S_h$ with $r+1$ regions and two imbeddings of $R_n$ on $S_{h+1}$ with $r-1$ regions.

5.1.2 Ringel Ladders Using Ringel-White Edge Adding Lemma

An edge-orbit is said to reverse direction at a rung if it was going from left to right and is now going from right to left or vice versa. In Figure 32, two of the edge-orbits reverse direction, and two do not. In Figure 31, none of the edge-orbits reverse direction. At an unmatched rung, none of the edge-orbits reverse direction. Unmatched rungs fall into two categories. One (with its edge-orbits) can be seen in Figure 31. The other is its mirror image. We know that when we take the mirror image of a map, the vertices that were clockwise become counterclockwise and vice versa. Taking the mirror image of a map doesn't have any effect on the edge-orbits, so henceforth, we will not mention the mirror image, although it also may be needed to illustrate the point. At a matched mid-rung, two of the edge-orbits reverse direction and two do not as seen in Figure 32.
Case 5 happens if and only if \( n \) is even and there are no matched mid-rungs \((m=0)\).

**Proof:** In order to be in Case 5, neither of the edge-orbits can reverse direction except at the end-rungs, since both of the end-rungs have to lie on both of the edge-orbits. Because of this, we don’t want to have any matched rungs.

By Furst, Gross, and Statman’s Lemma 2.1, ([2] page 26), any odd gap can be replaced with a gap of length one without having any effect on the edge-orbits. Any even gap can be replaced by either no rungs or two dissimilar unmatched rungs. So, essentially the only possibility for an odd gap when \( m=0 \) is shown in Figure 33a. Ringel ladders of this type are in Case 4. If \( m=0 \), essentially the only possibility for an even gaps is shown in Figure 33b. These fall into case 5.
So Case 5 happens precisely when \( n \) is even and \( m=0 \).

In the above proof, we also showed that if \( m=0 \), and \( n \) is odd, then we are in Case 4. So, now we will look at what happens when \( m \geq 1 \).

**Case 1** We need all of the edge-orbits that border the end-rungs to reverse direction before they reach the other end-rung. An odd gap allows two of the edge-orbits to pass through, see Figure 34, so the odd gaps do not affect whether the imbedding is in Case 1.

![Figure 34. Edge-Orbits Passing Through Odd Gaps.](image)

Even-end (Figure 35a) and odd-end gaps (Figure 35b) also allow two of the edge-orbits to pass through, and thus, also do not affect whether or not the imbedding is in Case 1.
So only even gaps can have any bearing on whether a particular imbedding is in Case 1 or not. Shown below in Figure 36 are essentially all the possibilities for even gaps (not including mirror images).

Only the even gaps on the right hand side of Figure 36 reverse all four edge-orbits that come into the gap. Call even gaps that are the type on the right hand side of Figure 36 turnabouts. So an imbedding is in Case 1 if, and only if, it has at least one turnabout.

Case 2 happens if, and only if, both of the end gaps are even-end gaps and there are no turnabouts.
Proof: If one of the end gaps is odd, the end rung bordering that end gap lies on only one region, see Figure 35b, so the imbedding is not in Case 2.

Suppose both of the end gaps are even-end gaps and that there are no turnabouts. In Figure 35a, we saw that one of the regions bordering the end-rung turns around and closes itself off, while the other continues past the matched rung. If there are no turnabouts, the region that continues past the matched rung will continue onto the other end-rung, so we are in Case 2.

Case 3 happens if, and only if, exactly one of the end gaps is an even-end gap and there are no turnabouts.

Proof: If there is one odd-end gap and one even-end gap, one end-rung will lie on only one region (Figure 35b) and the other end-rung will lie on two regions (Figure 35a), only one of which continues past the matched rung. Again, if there are no turnabouts, the region that continues past the matched rung will continue until the other end.

Case 4 happens if, and only if, both end gaps are odd-end gaps.

Proof: Similar to Cases 2 and 3.

5.1.3 Counting the Number of Imbeddings in Each Case

We will use a combinatorial expression to help us determine how many imbeddings of \( L_n \) (with \( m \) matched rungs) are in each of the above cases. Let \( b(p,q,k,e) \) equal the number of ways to put \( p \) identical balls into \( q \) distinct boxes (the boxes are in a row) so that \( e \) of the end
boxes have an even number of balls ($e = 0, 1, 2$) and so $k$ non-end boxes also have an even number of balls (this is similar to the expression $b(p, q, r)$ used in Chapter IV, page 3).

We will end up with $k + e$ boxes with an even number of balls, and $q - (k + e)$ boxes with an odd number of balls. Of the boxes with an odd number of balls, $2 - e$ of them will be end boxes. This means there will be $[q - (k - e) - (2 - e)] = q - k - 2$ boxes that contain an odd number of balls and are not end boxes.

First, one ball is placed in each of the odd boxes. Then we will put the remaining balls by pairs into the $q$ boxes. We have 2 end boxes and need to choose $2 - e$ of these to be odd. Then consider the $q - 2$ non-end boxes; we need to put one ball in the $q - 2 - k$ of these that are odd. After doing this, there are $p - (2 - e) - (q - 2 - k) = p - q + k + e$ balls left. These balls are distributed in pairs into the $q$ boxes. In order to do this, we use the combinatorial formula, $\binom{r + n - 1}{r} = \binom{r + n - 1}{n - 1}$ which counts the number of ways to put $r$ identical objects into $n$ different boxes. (See for example, [11] page 194). In this case, the $r$ identical objects are the $\frac{p - q + k + e}{2}$ pairs of balls; and $q$ is the number of boxes. So,

$$b(p, q, k, e) = \begin{cases} 
0 & \text{if } p - q + k + e \text{ is odd} \\
\frac{2}{2-e} \binom{q-2}{q-2-k} \left( \frac{p-q+k+e}{2} + q-1 \right) & \text{otherwise}
\end{cases}$$

$$= \begin{cases} 
0 & \text{if } p - q + k + e \text{ is odd} \\
\frac{2}{2-e} (q-2) \left( \frac{p-q+k+e}{2} + q-1 \right) & \text{otherwise}.
\end{cases}$$
Let \( s(n,m,k,e) = \) the number of rotation systems for \( L_n \) that have \( m \) matched mid-rungs of which \( k \) pairs are evenly separated and which have \( e \) even-end rungs. (This is similar to \( s(n,m,k) \) used in Chapter IV, page 3). We have \( n-m \) unmatched mid-rungs that are to be inserted into the \( m+1 \) distinct boxes formed by the \( m \) matched rungs. Thus,

\[
s(n,m,k,e) = 2^n b(n-m,m+1,k,e)
\]

\[
= 2^n \binom{2}{e} \left( \frac{(m+1)-2}{k} \right) \left[ \frac{\frac{(n-m)-(m+1)+k+e}{2}}{(m+1)-1} \right]
\]

\[
= 2^n \binom{2}{e} \binom{m-1}{k} \left( \frac{n+k+e-1}{2} \right)
\]

Note, since \( b(p,q,k,e) = 0 \) if \( p-q+k+e \) is odd, \( s(n,m,k,e) = 0 \) if \( (n-m)-(m+1)+k+e = n + k -2m +e -1 \) is odd. So, \( s(n,m,k,e) = 0 \) if \( n + k +e -1 \) is odd.

Let \( NT(s,m,k,e) = \) the number of imbeddings of \( L_n \) which have \( m \) matched mid-rungs of which \( k \) pairs are evenly separated and which have \( e \) even-end rungs which also don't contain any turnabouts. Therefore,

\[
NT(s,m,k,e) = 2^{n-k} b(n-m,m+1,k,e)
\]

\[
= 2^{n-k} \binom{2}{e} \binom{m-1}{k} \left( \frac{n+k+e-1}{2} \right)
\]

5.1.3.1 \( m=0 \)

5.1.3.1.1 The Number \( n \) is Even. This happens when we are in Case 5. There are \( 2^n \) imbeddings of \( L_n \) with two regions, so there are
$2^{n+1}$ imbeddings of $R_n$ with three regions and $2^{n+1}$ imbeddings of $R_n$ with one region.

5.1.3.1.2 The Number $n$ is Odd. We are in Case 5. There are $2^n$ imbeddings of $L_n$ with two regions, so there are $2^{n+1}$ imbeddings of $R_n$ with three regions and $2^{n+1}$ imbeddings of $R_n$ with one region.

5.1.3.2 $m \geq 1$

Case 1. There are $\sum_{e=0}^{2} [ s(n,m,k,e) - NT(n,m,k,e) ]$ imbeddings of $L_n$ with $k+e+1$ regions. So there are $4 \sum_{e=0}^{2} [ s(n,m,k,e) - NT(n,m,k,e) ]$ imbeddings of $R_n$ with $k+e$ regions.

Case 2. There are $NT(n,m,k,2)$ imbeddings of $L_n$ with $k+3$ regions, so there are $NT(n,m,k,2)$ imbeddings of $R_n$ with $k+4$ regions and $3NT(n,m,k,2)$ imbeddings of $R_n$ with $k+2$ regions.

Case 3. There are $2NT(n,m,k,1)$ imbeddings of $L_n$ with $k+2$ regions, so there are $2NT(n,m,k,1)$ imbeddings of $R_n$ with $k+3$ regions and $2NT(n,m,k,1)$ imbeddings of $R_n$ with $k+1$ regions.

Case 4. There are $NT(n,m,k,0)$ imbeddings of $L_n$ with $k+1$ regions, so there are $4NT(n,m,k,0)$ imbeddings of $R_n$ with $k+2$ regions.

Let $f(n,m,r)$ be the number of imbeddings of $L_n$ that have $m$ matched rungs and $r$ regions. See Chapter IV, page 4. For $m \geq 1$,
\[ f(n, m, k) = 4 \sum_{e=0}^{2} \left[ s(n, m, k-e, e) - NT(n, m, k-e, e) \right] \]

\[ + \ NT(n, m, k-4, 2) + 3NT(n, m, k-2, 2) + 2NT(n, m, k-3, 1) \]

\[ + 2NT(n, m, k-1, 1) + 4NT(n, m, k-2, 0) \]

\[ = 4 \sum_{e=0}^{2} \left[ \binom{2n - 2n-(k-e)}{2} \binom{m-1}{k-e} \left( \frac{n+(k-e)+e-1}{m} \right) \right] + \]

\[ 2n-(k-4) \binom{2}{2} \binom{m-1}{k-4} \left( \frac{n+(k-4)+2-1}{m} \right) + \]

\[ 3 \cdot 2n-(k-2) \binom{2}{2} \binom{m-1}{k-2} \left( \frac{n+(k-2)+2-1}{m} \right) + \]

\[ 2 \cdot 2n-(k-3) \binom{2}{1} \binom{m-1}{k-3} \left( \frac{n+(k-3)+2-1}{m} \right) + \]

\[ 2 \cdot 2n-(k-1) \binom{2}{1} \binom{m-1}{k-1} \left( \frac{n+(k-1)+2-1}{m} \right) + \]

\[ 4 \cdot 2n-(k-2) \binom{2}{0} \binom{m-1}{k-2} \left( \frac{n+(k-2)+2-1}{m} \right) + \]

\[ = \sum_{e=0}^{2} \left[ \binom{2n+2 - 2n-k+e+2}{2} \binom{m-1}{k-e} \left( \frac{n+k-1}{m} \right) \right] + \]

\[ 2n-k+4 \binom{m-1}{k-4} \left( \frac{n+k-3}{m} \right) + 3 \cdot 2n-k+2 \binom{m-1}{k-2} \left( \frac{n+k-1}{m} \right) + \]
Let \( f(n, k) \) = the number of imbeddings of \( R_n \) that have \( k \) regions (see [2] page 28). Then

\[
f(n, k) = f(n, 0, k) + \sum_{m=1}^{n} f(n, m, k)
\]

\[
\begin{align*}
&\begin{cases}
2n+1 & \text{if } k = 1 \\
2n+2 & \text{if } k = 2 \\
2n + 1 & \text{if } k = 3 \\
0 & \text{otherwise}
\end{cases} + \\
&\sum_{m=1}^{n} \left( \sum_{e=0}^{2} \binom{2n+2 - 2n-k+e+2}{e} \binom{m-1}{k-e} \left( \frac{n+k-1}{2m} \right) \right) + \\
&2n-k+4 \binom{m-1}{k-4} \left( \frac{n+k-3}{2m} \right) + 3 \cdot 2n-k+2 \binom{m-1}{k-2} \left( \frac{n+k-1}{2m} \right) + \\
&2n-k+5 \binom{m-1}{k-3} \left( \frac{n+k-3}{2m} \right) + 2n-k+3 \binom{m-1}{k-1} \left( \frac{n+k-1}{2m} \right) + \\
&+ 2n-k+4 \binom{m-1}{k-2} \left( \frac{n+k-3}{2m} \right)
\end{align*}
\]

Let \( i \) be the genus of the surface on which \( R_n \) is imbedded. Using the Euler Identity, we get

\[
r = 2 - 2i - (2n+2) + \left( \frac{3}{2} (2n+2) \right) = n - 2i + 3.
\]
Let \( g_i(R_n) \) denote the number of imbeddings of \( R_n \) in the surface \( S_i \).

Then

\[
g_i(R_n) = f(n, n-2i+3)
\]

\[
= \begin{cases} 
2n+1 & \text{if } n-2i = -2 \\
2n+2 & \text{if } n-2i = -1 \\
2n+1 & \text{if } n-2i = 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\sum_{m=1}^{n} \left( \sum_{e=0}^{2} \left( 2n+2 - 2(n-2i+3)-e \right) \left( \begin{array}{c} m-1 \\ 2 \end{array} \right) \left( \frac{n+(n-2i+3)-1}{m} \right) \right) +
\]

\[
2n-(n-2i+3)+4 \left( \begin{array}{c} m-1 \\ (n-2i+3)-4 \end{array} \right) \left( \frac{n+(n-2i+3)-3}{m} \right) +
\]

\[
3 \cdot 2n-(n-2i+3)+2 \left( \begin{array}{c} m-1 \\ (n-2i+3)-2 \end{array} \right) \left( \frac{n+(n-2i+3)-1}{m} \right) +
\]

\[
2n-(n-2i+3)+5 \left( \begin{array}{c} m-1 \\ (n-2i+3)-3 \end{array} \right) \left( \frac{n+(n-2i+3)-3}{m} \right) +
\]

\[
2n-(n-2i+3)+3 \left( \begin{array}{c} m-1 \\ (n-2i+3)-1 \end{array} \right) \left( \frac{n+(n-2i+3)-1}{m} \right) +
\]

\[
2n-(n-2i+3)+4 \left( \begin{array}{c} m-1 \\ (n-2i+3)-2 \end{array} \right) \left( \frac{n+(n-2i+3)-3}{m} \right) +
\]

\[
= \begin{cases} 
2n+1 & \text{if } n-2i = -2 \\
2n+2 & \text{if } n-2i = -1 \\
2n+1 & \text{if } n-2i = 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\sum_{m=1}^{n} \left( \sum_{e=0}^{2} \left( 2n+2 - 2(n-2i+3)-e \right) \left( \begin{array}{c} m-1 \\ n-2i+3-e \end{array} \right) \left( \frac{n-i+1}{m} \right) \right) +
\]

\[
2^{2i+1} \left( \begin{array}{c} m-1 \\ n-2i-1 \end{array} \right) \left( \frac{n-i}{m} \right) + 3 \cdot 2^{2i-1} \left( \begin{array}{c} m-1 \\ n-2i+1 \end{array} \right) \left( \frac{n-i+1}{m} \right) +
\]

\[
2^{2i+2} \left( \begin{array}{c} m-1 \\ n-2i \end{array} \right) \left( \frac{n-i}{m} \right) + 2^{2i} \left( \begin{array}{c} m-1 \\ n-2i+2 \end{array} \right) \left( \frac{n-i+1}{m} \right)
\]
We will simplify this expression using the following combinatorial identities (see for example, [2] page 28):

1. \( \binom{m+1}{k-1} = \binom{m}{k-1} + \binom{m}{k-2} \)

2. \( \sum_{q=0}^{p} \binom{q}{r} \binom{p}{q} = \binom{p}{r} \cdot 2^{p-r} \).

So, by writing out each term \( e=0,1,2 \), we get

\[
g_i(R_n) = \begin{cases} 
2^n+1 & \text{if } n-2i = -2 \\
2^n+2 & \text{if } n-2i = -1 \\
2^n+1 & \text{if } n-2i = 0 \\
0 & \text{otherwise}
\end{cases} + \sum_{m=1}^{n} \left( (2n+2-2^{2i-1})\binom{m-1}{n-2i+3}\binom{n-i+1}{m} + (2n+3-2^{2i+1})\binom{m-1}{n-2i+2}\binom{n-i+1}{m} \right. \\
+ (2n+2-2^{2i+1})\binom{m-1}{n-2i+1}\binom{n-i+1}{m} + 2^{2i+1}\binom{m-1}{n-2i-1}\binom{n-i}{m} + \\
3\cdot2^{2i-1}\binom{m-1}{n-2i+1}\binom{n-i+1}{m} + 2^{2i+2}\binom{m-1}{n-2i}\binom{n-i}{m} + \\
2^{2i}\binom{m-1}{n-2i+2}\binom{n-i+1}{m} + 2^{2i+1}\binom{m-1}{n-2i+1}\binom{n-i}{m} \right)
\]

Then, by grouping similar combinations, we get

\[
g_i(R_n) = \begin{cases} 
2^n+1 & \text{if } n-2i = -2 \\
2^n+2 & \text{if } n-2i = -1 \\
2^n+1 & \text{if } n-2i = 0 \\
0 & \text{otherwise}
\end{cases} + \sum_{m=1}^{n} \left( (2n+2-2^{2i-1})\binom{m-1}{n-2i+3} \right. + \\
(2n+3-2^{2i+1} + 2^{2i})\binom{m-1}{n-2i+2} + (2n+2-2^{2i+1} + 3\cdot2^{2i-1})\binom{m-1}{n-2i+1}\binom{n-i+1}{m} \right)
\]
\[
\left[ 2^{2i+1} \binom{m-1}{n-2i+1} + 2^{2i+2} \binom{m-1}{n-2i} + 2^{2i+1} \binom{m-1}{n-2i+1} \right] \binom{n-i}{m}.
\]

Let \( L = \left\{ \begin{array}{ll}
2^{n+1} & \text{if } n-2i = -2 \\
2^{n+2} & \text{if } n-2i = -1 \\
2^{n+1} & \text{if } n-2i = 0 \\
0 & \text{otherwise}
\end{array} \right. \)

and let \( B = \sum_{m=1}^{n} \left( \left( 2^{n+2} \cdot 2^{2i-1} \right) \binom{m-1}{n-2i+3} + \left( 2^{n+3} \cdot 2^{2i+1} + 2^{2i} \right) \binom{m-1}{n-2i+2} + \left( 2^{n+2} \cdot 2^{2i+1} + 3 \cdot 2^{2i-1} \right) \binom{m-1}{n-2i+1} \right) \binom{n-i+1}{m} \)

Then
\[
g_i(R_n) = L + B + \sum_{m=1}^{n} \binom{m-1}{n-2i+1} \left[ \binom{m-1}{n-2i-1} + 2\binom{m-1}{n-2i} + \binom{m-1}{n-2i+1} \right] \binom{n-i}{m}.
\]

Using identity (1) twice, we get
\[
= L + B + \sum_{m=1}^{n} 2^{2i+1} \left[ \binom{m}{n-2i} + \binom{m}{n-2i+1} \right] \binom{n-i}{m}
\]
\[
= L + B + 2^{2i+1} \sum_{m=1}^{n} \binom{m}{n-2i} \binom{n-i}{m} + 2^{2i+1} \sum_{m=1}^{n} \binom{m}{n-2i+1} \binom{n-i}{m}
\]
\[
= L + B + 2^{2i+1} \sum_{m=0}^{n} \binom{m}{n-2i} \binom{n-i}{m} + 2^{2i+1} \sum_{m=0}^{n} \binom{m}{n-2i+1} \binom{n-i}{m} - 2^{2i+1} \binom{0}{n-2i} \binom{n-i}{0} - 2^{2i+1} \binom{0}{n-2i+1} \binom{n-i}{0}.
\]

Using identity (2), we get
\begin{align*}
\mathsf{g}_i(R_n) &= L + B + 2^{2i+1} \binom{n-i}{n-2i} 2^{n-i-(n-2i)} + 2^{2i+1} \binom{n-i}{n-2i+1} 2^{n-i-(n-2i+1)} - \\
&\quad \begin{cases}
2^{2i+1} & \text{if } n-2i = 0 \\
2^{2i+1} & \text{if } n-2i = -1 \\
0 & \text{otherwise}
\end{cases} \\
&= L + B + 2^{3i+1} \binom{n-i}{n-2i} + 2^{3i} \binom{n-i}{n-2i+1} - \\
&\quad \begin{cases}
2^{2i+1} & \text{if } n-2i = 0 \\
2^{2i+1} & \text{if } n-2i = -1 \\
0 & \text{otherwise}
\end{cases}
\end{align*}

Let \( K = \begin{cases}
2^{n+1} & \text{if } n-2i = -2 \\
0 & \text{otherwise}
\end{cases} \). Let \( A = 2^{3i+1} \binom{n-i}{n-2i} + 2^{3i} \binom{n-i}{n-2i+1} \).

Then
\begin{align*}
\mathsf{g}_i(R_n) &= K + A + B \\
&= K + A + \sum_{m=1}^{n} \left( \left[ (2n+2-2^{2i-1}) \left( \binom{m-1}{n-2i+3} \right) \\
(2n+3-2^{2i+1}+2^{2i}) \left( \binom{m-1}{n-2i+2} \right) + (2n+2-2^{2i+1}+3-2^{2i-1}) \left( \binom{m-1}{n-2i+1} \right) \right] \left[ \binom{n-i+1}{m} \right] \\
&= K + A + 2^{2i+1}(2n-2i+3-1) \sum_{m=1}^{n} \left[ \binom{m-1}{n-2i+3} + 2 \binom{m-1}{n-2i+2} + \binom{m-1}{n-2i+1} \right] \left[ \binom{n-i+1}{m} \right]
\end{align*}

Using identity (1) twice, we get
\begin{align*}
\mathsf{g}_i(R_n) &= K + A + 2^{2i-1}(2n-2i+3-1) \sum_{m=1}^{n} \left( \left[ \binom{m}{n-2i+3} + \binom{m}{n-2i+2} \right] \left[ \binom{n-i+1}{m} \right]
\right.
&= K + A + (2n+2-2^{2i-1}) \sum_{m=1}^{n} \left( \binom{m}{n-2i+3} \binom{n-i+1}{m} \right) +
\end{align*}
Claim $n-2i \geq -2$. Ringel ladders have a splitting tree, so they are upper imbeddable.

So $\gamma_M = \left\lfloor \frac{q-p+1}{2} \right\rfloor = \gamma_M \leq \left\lfloor \frac{n+2}{2} \right\rfloor$. So $i \leq \frac{n+2}{2}$. Thus, $n-2i \geq n - 2 \left( \frac{n+2}{2} \right) \geq -2$.

So,

$$g_i (R_n) = K + A + (2^{n+2} \cdot 2^{2i-1}) \sum_{m=0}^{n} \left( \begin{array}{c} m \\ n-2i+2 \end{array} \right) \left( \begin{array}{c} n-i+1 \\ m \end{array} \right)$$

Using identity (2) twice, we get

$$g_i (R_n) = K + A + (2^{n+2} \cdot 2^{2i-1}) \left( \begin{array}{c} n-2i+2 \\ m \end{array} \right) \left( \begin{array}{c} n-i+1 \\ m \end{array} \right) - \left( \begin{array}{c} 0 \\ m \end{array} \right) \left( \begin{array}{c} n-i+1 \\ 0 \end{array} \right)$$

Canceling with $K$, we get

$$g_i (R_n) = 23i+1 \left( \begin{array}{c} n-1 \\ n-2i \end{array} \right) + 23i \left( \begin{array}{c} n-i \\ n-2i+1 \end{array} \right) + \left( 2n+i - 23i-3 \right) \left( \begin{array}{c} n-i+1 \\ i-2 \end{array} \right) + \left( 2n+i+1 - 23i-2 \right) \left( \begin{array}{c} n-i+1 \\ i-1 \end{array} \right)$$

We will give some examples of the genus distributions of Ringel Ladders in the next section.
5.2 Illustrating the Genus Distribution

Values of the genus distribution for the Ringel ladders $R_1 - R_5$ are shown in Table 16. We have mentioned that Ringel and Youngs used these ladders in their proof of the Haywood Map Coloring Theorem. They needed to find a maximum genus imbedding of the ladder with one region. They then used this imbedding as a current graph to find $\gamma(K_n)$. (The one region gives the rotation for a Cayley Map for $K_n$). Below we show the probability of getting such an imbedding. We see that these imbeddings with one region are very rare and hard to find if you just constructed an imbedding at random. This shows how clever Ringel and Youngs were to find their index-one current graphs.

We have seen that for Ringel ladders
\[
\gamma_M = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad \sqrt{\frac{n+2}{2}}
\]
So, we have one region precisely when $n$ is even. Let $n=2t$. Thus $\gamma_M = t + 1$.

\begin{align*}
g_{t+1}(R_{2t}) &= 2^{3(t+1)+1} \binom{2t}{t+1} \left( \frac{(2t)-(t+1)}{(t+1)} \right) + 2^{3(t+1)} \binom{(2t)-(t+1)}{(t+1)-1} + \\
&\quad (2(2t)+(t+1)\cdot 2^{3(t+1)-3}) \binom{(2t)-(t+1)+1}{(t+1)-2} + \\
&\quad (2(2t)+(t+1)+1\cdot 2^{3(t+1)-2}) \binom{(2t)-(t+1)+1}{(t+1)-1} \\
&= 2^{3t+4} \binom{t+1}{t+1} + 2^{3t+3} \binom{t+1}{t} + (2^{3t+1} \cdot 2^{3t}) \binom{t}{t+1} + (2^{3t} \cdot 2^{3t+1}) \binom{t}{t} \\
&= (2^{3t+1} \cdot 2^{3t}) t + (2^{3t+3} \cdot 2^{3t+1}) = 2^{3t} (t+2).
\end{align*}

So, the probability of finding at random an imbedding with one region is
\[
\frac{2^{3t} (t+2)}{2^{2(2t)+2}} = \frac{t+2}{2^{t+2}}, \quad \text{and} \quad \lim_{t \to \infty} \frac{t+2}{2^{t+2}} = 0.
\]
Ringel and Youngs needed an imbedding of \( R_n \) with one region, but for completeness, we will also find a formula for the number of maximum genus imbeddings when \( n \) is odd. Let \( n=2t+1 \). Thus \( \gamma_M = t + 1 \).

\[
g_{t+1}(R_{2t+1}) = 2^3(t+1)+1 \left( \frac{(2t+1)-(t+1)}{(t+1)} \right) + 2^3(t+1) \left( \frac{(2t+1)-(t+1)}{(t+1)} \right) +
\frac{(2(2t+1)+(t+1) - 2^3(t+1)-3)}{(2^3(t+1)-(t+1)+1)} \frac{(2t+1)-(t+1)+1}{(t+1)-2}
\]

\[
= 2^{3t+4} \left( \frac{t}{t+1} \right) + 2^{3t+3} \left( \frac{t}{t+1} \right) + (2^{3t+2} - 2^{3t}) \left( \frac{t+1}{t+2} \right) + (2^{3t+1} - 2^{3t+1}) \left( \frac{t+1}{t} \right)
\]

\[
= 2^{3t+3} + (2^{3t+3} - 2^{3t+1})(t+1)
\]

\[
= 2^{3t+1} [16 + (4-1)t^2 + (4-1)t + (16-4)t + (16-4)]
\]

\[
= 2^{3t+1} (3t^2 + 15t + 28)
\]

So, the probability of finding at random an imbedding with two regions is as follows:

\[
\frac{2^{3t+1} (3t^2 + 15t + 28)}{2^{2(2t+1)+2}} = \frac{3t^2 + 15t + 28}{2^{2t+5}}, \text{ and } \lim_{t \to \infty} \frac{3t^2 + 15t + 28}{2^{2t+5}} = 0.
\]

We can compute \( E(\gamma(R_n)) = \sum_{i \geq 0} i \frac{g_i}{2^{2n+2}} \). Let

Let \( f_m(x) = \sum_{i \geq 0} \binom{m-i}{i} x^i; \) then

\[
f_m(x) = \frac{(1+\alpha)^{m+1} - (1-\alpha)^{m+1}}{2m+1\alpha}, \text{ where } \alpha = \sqrt{1+4x}. \text{ See [9], page 76.}
\]

Thus, \( f'_m(x) = \sum_{i \geq 0} i \binom{m-i}{i} x^{i-1}; \) also

\[
f'_m(x) = \frac{\alpha(m+1)(1+\alpha)^m + (1-\alpha)^m - (1+\alpha)^{m+1} - (1-\alpha)^{m+1}}{2m\alpha^3}.
\]
\[
2^{n+2} E(\gamma(R_n)) = \sum_{i=0}^{2^n+1} i2^{2i+1} + \sum_{i=0}^{2^n-1} i2^{2i} + \sum_{i=0}^{2^n-1} i2^{2n+i+1} - \sum_{i=0}^{2^{n-1}} i2^{2i+3(i-1)} + \sum_{i=0}^{2^{n-1}} i2^{2n+i+1} - \sum_{i=0}^{2^{n-2}} i2^{2i-2}.
\]

\[
= 16\sum_{i=0}^{2^n} i2^3(i-1)(n_i - 1) + 64\sum_{i=0}^{2^n} (i+1) 2^3i-3(n_i - 1) +
\]

\[
2^n+3\sum_{i=0}^{2^n+1} (i+2) 2^i-1(n_i - 1) - 64\sum_{i=0}^{2^n+1} 2^3i-3(n_i - 1) +
\]

\[
2^n+3\sum_{i=0}^{2^n+1} (i+1) 2^i-1(n_i - 1) - 16\sum_{i=0}^{2^n+1} 2^3i-3(n_i - 1)
\]

\[
= 16f_n(8) + (64f_{n-1}(8) + 8f_n(8)) + (2^n+3f_{n-1}(2) + 2^n+3f_n(2)) -
\]

\[
(64f_{n-1}(8) + 16f_n(8)) + (2^n+3f_n(2) + 2^n+2f_n(2)) -
\]

\[
(16f_n(8) + 2f_n(8))
\]

\[
= -8f_{n-1}(8) - 2f_n(8) + 2^n+3f_{n-1}(2) + 2^n+2f_n(2) + 2^n+3f_n(2) + 2^n+3f_n(2).
\]

Now,

\[
f_{n-1}(2) = \frac{4n-(-2)^n}{3\cdot2n} = \frac{2^n + (-1)n+1}{3},
\]

\[
f_n(2) = \frac{4n+1-(-2)^n+1}{3\cdot2n+1} = \frac{2^n+1 - (-1)n}{3},
\]

\[
f_{n-1}(2) = \frac{(3n)[4n-1 - (-2)^n+1] - [4n - (-2)^n]}{27\cdot2n+1}
\]

\[
= \frac{3n2n+3n(-1)n - 2n+1 + 2(-1)n}{27}
\]

\[
f_n(2) = \frac{3n2n + 3\cdot2n+3n(-1)n + 3(-1)n - 2n+2 + 2(-1)n+1}{27},
\]

and

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\[ f_n(8) = \frac{(1+\sqrt{33})^{n+1} - (1-\sqrt{33})^{n+1}}{\sqrt{33} \cdot 2^{n+1}}. \]

Using the binomial theorem, we get
\[
f_n(8) = \frac{1}{\sqrt{33} \cdot 2^{n+1}} \sum_{k=0}^{n} \binom{n+1}{k} \left( (\sqrt{33})^k \cdot 1^{n+1-k} \cdot (-1)^{n+1-k} \cdot (\sqrt{33})^{1 \cdot 0} \cdot 1^{n+1-k} \right)
\]
\[
= \frac{1}{2^n} \left[ \sum_{k=0}^{n} \binom{n+1}{2k+1} (33)^k \right]
\]
which tells us that \( f_n(8) \) is a rational number. Also, we see that
\[
f_{n-1}(8) = \frac{1}{2^{n-1}} \left[ \sum_{k=0}^{n} \binom{n}{2k+1} (33)^k \right].
\]

So \( \mathbb{E}(\gamma(R_n)) = \frac{1}{2^{2n+2}} \left[ 2^{n+3} \left( \frac{2^{n+1} + (-1)^{n+1}}{3} \right) + 2^{n+2} \left( \frac{2^{n+1} + (-1)^n}{3} \right) \right] +
\]
\[
2^{n+3} \left( \frac{3n2^{n-1} + 3(-1)^{n-1} - 2^{n+1} + 2(-1)^n}{27} \right) +
\]
\[
2^{n+3} \left( \frac{3n2^n + 3 \cdot 2^n + 3(-1)^n - 2^{n+2} + 2(-1)^{n+1}}{27} \right) -
\]
\[
\frac{1}{2^{n-4}} \left[ \sum_{k=0}^{n} \binom{n}{2k+1} (33)^k \right] - \frac{1}{2^{n-1}} \left[ \sum_{k=0}^{n+1} \binom{n+1}{2k+1} (33)^k \right]
\]
\[
= \frac{1}{2^{2n+2}} \left[ 2^{n+2} \left( \frac{2^{n+2} + (-1)^{n+1}}{3} \right) + 2^{n+3} \left( \frac{2^{n-1}(9n-6) + 3(-1)^n}{27} \right) \right] -
\]
\[
\frac{1}{2^{n-4}} \left[ \sum_{k=0}^{n} \binom{n}{2k+1} (33)^k \right] - \frac{1}{2^{n-1}} \left[ \sum_{k=0}^{n+1} \binom{n+1}{2k+1} (33)^k \right]
\]
\[
= \frac{1}{2^{2n+2}} \left[ 9 \left( 3 \cdot 2^{n+2} + 3(-1)^{n+1} + (3n-2)2^n + 2(-1)^n \right) \right] -
\]
\[
\frac{1}{2^{n-4}} \left[ \sum_{k=0}^{n} \binom{n}{2k+1} (33)^k \right] - \frac{1}{2^{n-1}} \left[ \sum_{k=0}^{n+1} \binom{n+1}{2k+1} (33)^k \right]
\]
\[
\frac{1}{2^{n+2}} \frac{2^{n+2}}{9} \left( 2^n(3n+10)+(-1)^n+1 \right) - \frac{1}{2^{n-4}} \left( \sum_{k=0}^{n} \frac{n+1}{2k+1} (33)^k \right) - \frac{1}{2^{n-1}} \left( \sum_{k=0}^{n+1} \frac{n+1}{2k+1} (33)^k \right)
\]

Let \( h_m(x) = \sum_{k=0}^{n} \frac{n+1}{2k+1} x^k \).

(Note, it is interesting that \( \frac{1}{2^n-1} h_n(5) \) is the \( n \)th Fibonnaci number.)

So,

\[
E(\gamma(R_n)) = \frac{1}{2^{n+2}} \frac{2^{n+2}}{9} \left( 2^n(3n+10)+(-1)^n+1 \right) - \frac{1}{2^{n-1}} \left( 8h_n(33) + h_{n+1}(33) \right)
\]

So,

\[
E(\gamma(R_n)) \leq \frac{1}{2^{n+2}} \frac{2^{n+2}}{9} \left( 2^n(3n+10)+(-1)^n+1 \right) \leq \frac{3n+2+1}{9} = \frac{n+1}{3}.
\]

Since \( \gamma_M = \left\lfloor \frac{n+2}{2} \right\rfloor \), we know that the average genus of the Ringel Ladders is not asymptotic to the maximum genus.

Moreover, we know that the average genus of a graph is at least as large as the average genus of any of its subgraphs [5]. So in particular, we know the average genus of \( R_n \) is at least as large as the average genus of the subgraph of \( R_n \) obtained by deleting the edge that runs connects both ends of \( R_n \). However, this gives us precisely the closed-end ladder \( L_n \) with both of the end-rungs subdivided. But this subgraph has the same genus distribution as \( L_n \). From [14] we know

\[
E(\gamma(L_n)) = \frac{3n+1+(-1)^n+1/2^n}{9} \geq \frac{3n}{9} = \frac{n}{3}.
\]

So,

\[
\frac{n}{3} \leq E(\gamma(R_n)) \leq \frac{n+1}{3}.
\]

Thus \( E(\gamma(R_n)) \sim \frac{n}{3} \).
CHAPTER VI

CONCLUSIONS

In this final chapter, we would like to summarize what we have done and mention some open problems for the future. There are four major concepts that we have explored. First of all, we looked at Model II for Random Topological Graph Theory, and saw that probabilities from this model often depended on the labeling of the graph. As a result of this, we proposed a generalized model. Secondly, after proposing this generalized model, and developing some general results, we used it to study a number of small order cubic graphs. These small order graphs include some Cayley graphs, the Petersen graph, the graph of the dodecahedron and all cubic graphs of orders eight and ten. The third major concept we investigated was probability polynomials for closed-end ladders. We used the genus distribution (which was known) to develop the probability polynomials. The fourth area that we studied was Ringel ladders. We found formulas for the genus distribution and average genus of Ringel ladders.

The field of Random Topological Graph Theory is so open there are numerous problems and areas that can be studied. Here I will mention only some of the problems that are most pertinent to things discussed in this work.

1. We have found the genus distribution for Ringel ladders, but have not yet found probability polynomials for Ringel Ladders.
2. McGeoch found the genus distributions for circular ladders (prism graphs) and Möbius ladders [7]. Probability polynomials for these graphs have not been found. Also, we do not have the average genus for these graphs in a compact form (i.e. a formula for calculating the average genus without calculating the genus distribution first).

3. We can ask questions such as: what kinds of graphs achieve the maximum (minimum) of the probability polynomial for the genus (maximum genus) surface when $p=0$ or 1, or when $p=\frac{1}{2}$?

4. What can these probability polynomials tell us about properties of the graph?

5. Also, we could consider a second random variable of interest, the number of map automorphisms.
Appendix A

Tables
Table 1  
Data for All the Families of Probability Polynomials for $K_4$

**Family 1:**  
$E(\gamma(K_4)) = f_1(p) = (1-p)^4 + 4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p) + p^4$.  

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p =$</th>
<th>Min at $p =$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0, 1</td>
<td>$\frac{1}{2}$ $\pm \frac{1}{6}\sqrt{3}$</td>
</tr>
<tr>
<td>$f_1(p)$</td>
<td>0, 1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$ $\pm \frac{1}{6}\sqrt{3}$</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>0, 1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$ $\pm \frac{1}{6}\sqrt{3}$</td>
</tr>
</tbody>
</table>

**Family 2:**  
$E(\gamma(K_4)) = f_1(p) = 4p(1-p)^3 + 6p^2(1-p)^2 + 4p^3(1-p)$.  

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p =$</th>
<th>Min at $p =$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(p)$</td>
<td>0, 1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
<tr>
<td>$f_1(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0, 1</td>
<td>none</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>$\frac{1}{2}$</td>
<td>0, 1</td>
<td>none</td>
</tr>
</tbody>
</table>

**Family 3:**  
$E(\gamma(K_4)) = f_1(p) = (1-p)^4 + 3p(1-p)^3 + 6p^2(1-p)^2 + 3p^3(1-p) + p^4$.  

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p =$</th>
<th>Min at $p =$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0, 1</td>
<td>none</td>
</tr>
<tr>
<td>$f_1(p)$</td>
<td>0, 1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>0, 1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
</tbody>
</table>
Data for All the Families of Probability Polynomials for $K_{3,3}$


<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p =$</th>
<th>Min at $p =$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(p)$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{6} \sqrt{3}$</td>
<td>$\frac{1}{3}, \frac{2}{3}$</td>
</tr>
<tr>
<td>$f_2(p)$</td>
<td>$\frac{1}{2} \pm \frac{1}{6} \sqrt{3}$</td>
<td>0,1</td>
<td>$\frac{1}{2}, \frac{2}{3}$</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>$\frac{1}{2} \pm \frac{\sqrt{3}}{6}$</td>
<td>0,1</td>
<td>$\frac{1}{2}, \frac{2}{3}$</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p =$</th>
<th>Min at $p =$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>$f_2(p)$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{6} \sqrt{-33 + 6\sqrt{31}}$</td>
<td>$\frac{1}{2} \pm \frac{1}{30} \sqrt{-495 + 6\sqrt{69}}$</td>
</tr>
<tr>
<td>$E(\gamma(K_{3,3}))$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{6} \sqrt{-24 + 3\sqrt{73}}$</td>
<td>$\frac{1}{2} \pm \frac{1}{10} \sqrt{-40 + 5\sqrt{69}}$</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p =$</th>
<th>Min at $p =$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(p)$</td>
<td>0,1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
<tr>
<td>$f_2(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>0,1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
</tbody>
</table>
Table 3
Data for All the Families of Probability Polynomials for $K_2 \times K_3$

### Family 1: $E(\gamma(K_2 \times K_3)) = -(1-p)^6 + 6p(1-p)^5 + 21p^2(1-p)^4 + 30p^3(1-p)^3 + 21p^4(1-p)^2 + 6p^5(1-p) + p^6$

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p=$</th>
<th>Min at $p=$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{10} \sqrt{5}$</td>
</tr>
<tr>
<td>$f_1(p)$</td>
<td>0,1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2} \pm \frac{1}{10} \sqrt{75-30\sqrt{5}}$</td>
</tr>
<tr>
<td>$f_2(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{10} \sqrt{3}$</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>$\frac{1}{2} \pm \frac{1}{10} \sqrt{-57+6\sqrt{97}}$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{10} \sqrt{-95+10\sqrt{94}}$</td>
</tr>
</tbody>
</table>

### Family 2: $E(\gamma(K_2 \times K_3)) = 3(1-p)^6 + 10p(1-p)^5 + 19p^2(1-p)^4 + 22p^3(1-p)^3 + 19p^4(1-p)^2 + 10p^5(1-p) + 3p^6$

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p=$</th>
<th>Min at $p=$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{30} \sqrt{45-30\sqrt{6}}$</td>
</tr>
<tr>
<td>$f_1(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>$f_2(p)$</td>
<td>0,1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>0,1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
</tbody>
</table>

### Family 3: $E(\gamma(K_2 \times K_3)) = -(1-p)^6 + 8p(1-p)^5 + 20p^2(1-p)^4 + 28p^3(1-p)^3 + 20p^4(1-p)^2 + 8p^5(1-p) + p^6$.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at $p=$</th>
<th>Min at $p=$</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>$f_1(p)$</td>
<td>0,1</td>
<td>$\frac{1}{2}$</td>
<td>none</td>
</tr>
<tr>
<td>$f_2(p)$</td>
<td>$\frac{1}{2}$</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>$E(\gamma(K_4))$</td>
<td>$\frac{1}{2} \pm \frac{1}{6} \sqrt{69-12\sqrt{31}}$</td>
<td>0,1</td>
<td>$\frac{1}{2} \pm \frac{1}{30} \sqrt{1035-30\sqrt{1149}}$</td>
</tr>
</tbody>
</table>
Table 3 -- Continued

**Family 4:**  
\[ E(\gamma(K_2 \times K_3)) = (1-p)^6 + 10p(1-p)^5 + 20p^2(1-p)^4 + 24p^3(1-p)^3 + 20p^4(1-p)^2 + 10p^5(1-p) + p^6. \]

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at ( p = )</th>
<th>Min at ( p = )</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>( f_1(p) )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{20} \sqrt{636-6\sqrt{105}} )</td>
<td>( \frac{1}{2} \pm \frac{1}{30} \sqrt{945-30\sqrt{969}} )</td>
</tr>
<tr>
<td>( f_2(p) )</td>
<td>( \frac{1}{2} \pm \frac{1}{10} \sqrt{5} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{30} \sqrt{15} )</td>
</tr>
<tr>
<td>( E(\gamma(K_4)) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{10} \sqrt{-45+10\sqrt{29}} )</td>
</tr>
</tbody>
</table>

**Family 5:**  
\[ E(\gamma(K_2 \times K_3)) = (1-p)^6 + 8p(1-p)^5 + 19p^2(1-p)^4 + 28p^3(1-p)^3 + 19p^4(1-p)^2 + 8p^5(1-p) + 2p^6. \]

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at ( p = )</th>
<th>Min at ( p = )</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>( f_1(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>( f_2(p) )</td>
<td>0,1</td>
<td>( \frac{1}{2} )</td>
<td>none</td>
</tr>
<tr>
<td>( E(\gamma(K_4)) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{10} \sqrt{-45+10\sqrt{29}} )</td>
</tr>
</tbody>
</table>

**Family 6:**  
\[ E(\gamma(K_2 \times K_3)) = (1-p)^6 + 10p(1-p)^5 + 21p^2(1-p)^4 + 22p^3(1-p)^3 + 21p^4(1-p)^2 + 10p^5(1-p) + p^6. \]

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at ( p = )</th>
<th>Min at ( p = )</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{10} \sqrt{5} )</td>
</tr>
<tr>
<td>( f_1(p) )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{20} \sqrt{-51+12\sqrt{19}} )</td>
<td>( \frac{1}{2} \pm \frac{1}{30} \sqrt{-765+30\sqrt{669}} )</td>
</tr>
<tr>
<td>( f_2(p) )</td>
<td>( \frac{1}{2} \pm \frac{1}{10} \sqrt{5} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{30} \sqrt{15} )</td>
</tr>
<tr>
<td>( E(\gamma(K_4)) )</td>
<td>( \frac{1}{2} \pm \frac{1}{5} \sqrt{69-12\sqrt{31}} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \pm \frac{1}{30} \sqrt{1035-10\sqrt{1149}} )</td>
</tr>
</tbody>
</table>
Table 3 -- Continued

**Family 7:** \( E(\gamma(K_2xK_3)) = (1-p)^6 + 7p(1-p)^5 + 21p^2(1-p)^4 + 28p^3(1-p)^3 + 21p^4(1-p)^2 + 7p^5(1-p) + p^6. \)

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at ( p = )</th>
<th>Min at ( p = )</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \frac{1}{10}\sqrt{5} )</td>
</tr>
<tr>
<td>( f_1(p) )</td>
<td>0,1</td>
<td>( \frac{1}{2} )</td>
<td>none</td>
</tr>
<tr>
<td>( f_2(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>( E(\gamma(K_4)) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \frac{1}{10}\sqrt{-45+10\sqrt{29}} )</td>
</tr>
</tbody>
</table>

**Family 8:** \( E(\gamma(K_2xK_3)) = 6p(1-p)^5 + 21p^2(1-p)^4 + 32p^3(1-p)^3 + 21p^4(1-p)^2 + 6p^5(1-p). \)

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Max at ( p = )</th>
<th>Min at ( p = )</th>
<th>Inflection pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_0(p) )</td>
<td>0,1</td>
<td>( \frac{1}{2} )</td>
<td>none</td>
</tr>
<tr>
<td>( f_1(p) )</td>
<td>( \frac{1}{2} \frac{1}{2}\sqrt{-9+2\sqrt{21}} )</td>
<td>0,1</td>
<td>( \frac{1}{2} \frac{1}{10}\sqrt{-135+10\sqrt{186}} )</td>
</tr>
<tr>
<td>( f_2(p) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>none</td>
</tr>
<tr>
<td>( E(\gamma(K_4)) )</td>
<td>( \frac{1}{2} )</td>
<td>0,1</td>
<td>none</td>
</tr>
</tbody>
</table>
Table 4

Families of Probability Polynomials for $G_\Delta(\Gamma)$ With $\Gamma = Z_{2n}$ and $\Delta = \{1, n\}$

<table>
<thead>
<tr>
<th>n</th>
<th>Family of Probability Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$f_0(p) = 2p^2(1-p)^2$.</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = (1-p)^4 + 4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p) + p^4$.</td>
</tr>
<tr>
<td></td>
<td>(Family 1 for $K_4$)</td>
</tr>
<tr>
<td>3</td>
<td>$f_1(p) = (1-p)^6 + 9p^2(1-p)^4 + 20p^3(1-p)^3 + 9p^4(1-p)^2 + p^6$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 6p^2(1-p)^4 + 6p(1-p)^5 + 6p^4(1-p)^2 + 6p^5(1-p)$.</td>
</tr>
<tr>
<td>4</td>
<td>$f_1(p) = 4p^2(1-p)^6 + 8p^3(1-p)^5 + 32p^4(1-p)^4 + 8p^5(1-p)^3 + 4p^6(1-p)^2$.</td>
</tr>
<tr>
<td>5</td>
<td>$f_1(p) = 10p^4(1-p)^6 + 52p^5(1-p)^5 + 10p^6(1-p)^4$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = (1-p)^{10} + 25p^2(1-p)^8 + 100p^3(1-p)^7 + 140p^4(1-p)^6 + 100p^5(1-p)^5 + 140p^6(1-p)^4 + 100p^7(1-p)^3 + 25p^8(1-p)^2 + p^{10}$.</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = 10p(1-p)^9 + 20p^2(1-p)^8 + 20p^3(1-p)^7 + 60p^4(1-p)^6 + 100p^5(1-p)^5 + 60p^6(1-p)^4 + 20p^7(1-p)^3 + 20p^8(1-p)^2 + 10p^9(1-p)$.</td>
</tr>
<tr>
<td>6</td>
<td>$f_1(p) = 12p^5(1-p)^7 + 88p^6(1-p)^6 + 12p^7(1-p)^5$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 6p^2(1-p)^{10} + 36p^3(1-p)^9 + 204p^4(1-p)^8 + 288p^5(1-p)^7 + 260p^6(1-p)^6 + 288p^7(1-p)^5 + 204p^8(1-p)^4 + 36p^9(1-p)^3 + 6p^{10}(1-p)^2$.</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = (1-p)^{12} + 12p(1-p)^{11} + 60p^2(1-p)^{10} + 184p^3(1-p)^9 + 291p^4(1-p)^8 + 492p^5(1-p)^7 + 576p^6(1-p)^6 + 492p^7(1-p)^5 + 291p^8(1-p)^4 + 184p^9(1-p)^3 + 60p^{10}(1-p)^2 + 12p^{11}(1-p) + (1-p)^{12}$.</td>
</tr>
</tbody>
</table>
Table 4 -- Continued

<table>
<thead>
<tr>
<th>n</th>
<th>Family of Probability Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$f_1(p) = 14p^6(1-p)^8 + 156p^7(1-p)^7 + 14p^8(1-p)^6$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 56p^4(1-p)^{10} + 392p^5 (1-p)^9$</td>
</tr>
<tr>
<td></td>
<td>$+ 602p^6(1-p)^8 + 532p^7(1-p)^7 + 602p^8(1-p)^7 + 392p^9 (1-p)^5 + 56p^{10} (1-p)^4$.</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = (1-p)^{14} + 49p^2(1-p)^{12} + 280p^3(1-p)^{11} + 679p^4 (1-p)^{10} + 1064p^5 (1-p)^9$</td>
</tr>
<tr>
<td></td>
<td>$+ 1799p^6(1-p)^8 + 2240p^7(1-p)^7 + 1799p^8(1-p)^6 + 1064p^9 (1-p)^5$</td>
</tr>
<tr>
<td></td>
<td>$+ 679p^{10} (1-p)^4 + 280p^{11} (1-p)^3 + 49p^{12} (1-p)^2 + (1-p)^{14}$.</td>
</tr>
<tr>
<td></td>
<td>$f_4(p) = 14p(1-p)^{11} + 42p^2(1-p)^{12} + 84p^3(1-p)^{11} + 266p^4 (1-p)^{10} + 546p^5 (1-p)^9$</td>
</tr>
<tr>
<td></td>
<td>$+ 588p^6(1-p)^8 + 504p^7(1-p)^7 + 588p^8(1-p)^6 + 546p^9 (1-p)^5$</td>
</tr>
<tr>
<td></td>
<td>$+ 266p^{10} (1-p)^4 + 84p^{11} (1-p)^3 + 42p^{10} (1-p)^2 + 14p^{11} (1-p)^2$.</td>
</tr>
<tr>
<td>8</td>
<td>$f_1(p) = 16p^7(1-p)^{10} + 288p^8(1-p)^8 + 16p^9(1-p)^7$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 80p^5(1-p)^{11} + 752p^6 (1-p)^{10} + 1248p^7(1-p)^9 + 1152p^8(1-p)^8$</td>
</tr>
<tr>
<td></td>
<td>$+ 1248p^9(1-p)^7 + 752p^{10} (1-p)^6 + 80p^{11} (1-p)^5$.</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = 8p^2(1-p)^{14} + 96p^3(1-p)^{13} + 704p^4 (1-p)^{12} + 1808p^5(1-p)^{11}$</td>
</tr>
<tr>
<td></td>
<td>$+ 3144p^6(1-p)^{10} + 4944p^7(1-p)^9 + 6112p^8(1-p)^8 + 4944p^9 (1-p)^7$</td>
</tr>
<tr>
<td></td>
<td>$+ 3144p^{10} (1-p)^6 + 1808p^{11} (1-p)^5 + 704p^{12} (1-p)^4 + 96p^{13} (1-p)^3 + 8p^{14} (1-p)^2$.</td>
</tr>
<tr>
<td></td>
<td>$f_4(p) = (1-p)^{16} + 16p(1-p)^{15} + 112p^2(1-p)^{14} + 464p^3(1-p)^{13} + 1116p^4 (1-p)^{12}$</td>
</tr>
<tr>
<td></td>
<td>$+ 2480p^5(1-p)^{11} + 4112p^6(1-p)^{10} + 5232p^7(1-p)^9 + 5318p^8 (1-p)^8$</td>
</tr>
<tr>
<td></td>
<td>$+ 5232p^9(1-p)^7 + 4112p^{10} (1-p)^6 + 2480p^{11} (1-p)^5 + 1116p^{12} (1-p)^4$</td>
</tr>
<tr>
<td></td>
<td>$+ 464p^{13} (1-p)^3 + 112p^{14} (1-p)^2 + 16p^{15} (1-p)^1 + (1-p)^{16}$.</td>
</tr>
</tbody>
</table>
Table 5

Probability Polynomials for the Expected Value of the Genus Random Variable of $G_{\Delta}(\Gamma)$ With $Z_{2n}$ and $\Delta = \{1,n\}$

<table>
<thead>
<tr>
<th>n</th>
<th>Expected Value of the Genus Random Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$E(\gamma(p)) = (1-p)^4 + 4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p) + p^4$.</td>
</tr>
<tr>
<td>4</td>
<td>$E(\gamma(p)) = 2(1-p)^8 + 16p(1-p)^7 + 52p^2(1-p)^6 + 104p^3(1-p)^5 + 108p^4(1-p)^4 + 104p^5(1-p)^3 + 52p^6(1-p)^2 + 16p^7(1-p) + p^8$.</td>
</tr>
<tr>
<td>6</td>
<td>$E(\gamma(p)) = 3(1-p)^12 + 36p(1-p)^{11} + 192p^2(1-p)^{10} + 624p^3(1-p)^9 + 1281p^4(1-p)^8 + 2064p^5(1-p)^7 + 2336p^6(1-p)^6 + 2064p^7(1-p)^5 + 1281p^8(1-p)^4 + 624p^9(1-p)^3 + 192p^{10}(1-p)^2 + 36p^{11}(1-p) + 3(1-p)^{12}$.</td>
</tr>
<tr>
<td>7</td>
<td>$E(\gamma(p)) = 3(1-p)^14 + 56p(1-p)^{13} + 315p^2(1-p)^{12} + 1176p^3(1-p)^{11} + 3213p^4(1-p)^{10} + 6160p^5(1-p)^9 + 8967p^6(1-p)^8 + 8956p^7(1-p)^7 + 8967p^8(1-p)^6 + 6160p^9(1-p)^5 + 3213p^{10}(1-p)^4 + 1176p^{11}(1-p)^3 + 315p^{12}(1-p)^2 + 56p^{13}(1-p) + 3(1-p)^{14}$.</td>
</tr>
<tr>
<td>8</td>
<td>$E(\gamma(p)) = 4(1-p)^{14} + 64p(1-p)^{13} + 472p^2(1-p)^{12} + 2144p^3(1-p)^{11} + 6576p^4(1-p)^{10} + 15504p^5(1-p)^9 + 27384p^6(1-p)^8 + 38272p^7(1-p)^7 + 42200p^8(1-p)^6 + 38272p^9(1-p)^5 + 27384p^{10}(1-p)^4 + 15504p^{11}(1-p)^3 + 6576p^{12}(1-p)^2 + 2144p^{13}(1-p)^3 + 472p^{14}(1-p)^2 + 64p^{15}(1-p) + 4(1-p)^{16}$.</td>
</tr>
</tbody>
</table>
Table 6
Families of Probability Polynomials for $G_{\Delta}(\Gamma)$ With $Z_n \times Z_2$
and $\Delta = \{(1,0),(0,1)\}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Family of Probability Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$f_0(p) = 2p^3(1-p)^3$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 6p^2(1-p)^4 + 12p^3(1-p)^3 + 6p^4(1-p)^2$.</td>
</tr>
<tr>
<td>4</td>
<td>$f_0(p) = 2p^4(1-p)^4$.</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = (1-p)^8 + 8p^2(1-p)^6 + 8p^3(1-p)^5 + 20p^4(1-p)^4$ + $8p^5(1-p)^3 + 8p^6(1-p)^2 + p^8$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 8p(1-p)^7 + 20p^2(1-p)^6 + 48p^3(1-p)^5 + 48p^4(1-p)^4$ + $48p^5(1-p)^3 + 20p^6(1-p)^2 + 8p^7(1-p)$.</td>
</tr>
<tr>
<td>5</td>
<td>$f_0(p) = 2p^5(1-p)^5$.</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = 10p^3(1-p)^7 + 10p^4(1-p)^6 + 30p^5(1-p)^5 + 10p^6(1-p)^4 + 10p^7(1-p)^3$.</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = 20p^2(1-p)^6 + 60p^3(1-p)^5 + 60p^4(1-p)^4 + 40p^5(1-p)^5$ + $60p^6(1-p)^4 + 60p^7(1-p)^3 + 20p^8(1-p)^2$.</td>
</tr>
<tr>
<td>6</td>
<td>$f_0(p) = 2p^6(1-p)^6$.</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = (1-p)^{12} + 18p^2(1-p)^{10} + 36p^3(1-p)^9 + 135p^4(1-p)^8 + 288p^5(1-p)^7$ + $372p^6(1-p)^6 + 288p^7(1-p)^5 + 135p^8(1-p)^4 + 36p^9(1-p)^3$ + $18p^{10}(1-p)^2 + (1-p)^{12}$.</td>
</tr>
</tbody>
</table>
Table 6 -- Continued

<table>
<thead>
<tr>
<th>n</th>
<th>Family of Probability Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>$f_0(p) = 2p^7(1-p)^7$</td>
<td></td>
</tr>
<tr>
<td>$f_1(p) = 14p^5(1-p)^9 + 14p^6(1-p)^8 + 126p^7(1-p)^7 + 14p^8(1-p)^6 + 14p^9(1-p)^5$</td>
<td></td>
</tr>
<tr>
<td>$f_2(p) = 28p^3(1-p)^{11} + 56p^4(1-p)^{10} + 266p^5(1-p)^9 + 602p^6(1-p)^8 + 728p^7(1-p)^7$</td>
<td></td>
</tr>
<tr>
<td>$+ 602p^8(1-p)^6 + 266p^9(1-p)^5 + 56p^{10}(1-p)^4 + 28p^{11}(1-p)^3$</td>
<td></td>
</tr>
<tr>
<td>$f_3(p) = (1-p)^{14} + 14p^3(1-p)^{12} + 168p^3(1-p)^{11} + 679p^4(1-p)^{10}$</td>
<td></td>
</tr>
<tr>
<td>$+ 1386p^5(1-p)^9 + 1799p^6(1-p)^8 + 1799p^7(1-p)^7 + 1799p^8(1-p)^6 + 1386p^9(1-p)^5$</td>
<td></td>
</tr>
<tr>
<td>$+ 679p^{10}(1-p)^4 + 168p^{11}(1-p)^3 + 49p^{12}(1-p)^2 + 14p^{13}(1-p) + (1-p)^{14}$</td>
<td></td>
</tr>
<tr>
<td>$f_4(p) = 42p^2(1-p)^{12} + 168p^3(1-p)^{11} + 266p^4(1-p)^{10} + 336p^5(1-p)^9 + 588p^6(1-p)^8$</td>
<td></td>
</tr>
<tr>
<td>$+ 784p^7(1-p)^7 + 588p^8(1-p)^6 + 336p^9(1-p)^5 + 266p^{10}(1-p)^4 + 168p^{11}(1-p)^3$</td>
<td></td>
</tr>
<tr>
<td>$+ 42p^{12}(1-p)^2$</td>
<td></td>
</tr>
</tbody>
</table>

| 8  |                                  |
| $f_0(p) = 2p^8(1-p)^8$ |
| $f_1(p) = 16p^6(1-p)^{10} + 16p^7(1-p)^9 + 254p^8(1-p)^8 + 16p^9(1-p)^7 + 16p^{10}(1-p)^6$ |
| $f_2(p) = 40p^4(1-p)^{12} + 80p^5(1-p)^{11} + 576p^6(1-p)^{10} + 1248p^7(1-p)^9 + 1424p^8(1-p)^8$ |
| $+ 1248p^9(1-p)^7 + 576p^{10}(1-p)^6 + 80p^{11}(1-p)^5 + 40p^{12}(1-p)^4$ |
| $f_3(p) = (1-p)^{16} + 32p^2(1-p)^{14} + 96p^3(1-p)^{13} + 500p^4(1-p)^{12} + 1808p^5(1-p)^{11}$ |
| $+ 3728p^6(1-p)^{10} + 4944p^7(1-p)^9 + 5302p^8(1-p)^8 + 4944p^9(1-p)^7$ |
| $+ 3728p^{10}(1-p)^6 + 1808p^{11}(1-p)^5 + 500p^{12}(1-p)^4 + 96p^{13} + 32p^{14}(1-p)^2 + (1-p)^{16}$ |
| $f_4(p) = 16p(1-p)^{15} + 88p^2(1-p)^{14} + 464p^3(1-p)^{13} + 1280p^4(1-p)^{12}$ |
| $+ 2480p^5(1-p)^{11} + 3688p^6(1-p)^{10} + 5232p^7(1-p)^9 + 5888p^8(1-p)^8$ |
| $+ 5232p^9(1-p)^7 + 3688p^{10}(1-p)^6 + 2480p^{11}(1-p)^5 + 1280p^{12}(1-p)^4$ |
| $+ 464p^{13} + 88p^{14}(1-p)^2 + 16p^{15}(1-p)$ |

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Table 7
Probability Polynomials for the Expected Value of the Genus Random Variable of $G_{\Delta}(\Gamma)$ With $\Gamma = Z_n \times Z_2$ and $\Delta = ((1,0),(0,1))$

<table>
<thead>
<tr>
<th>n</th>
<th>Expected Value of Genus Random Variable</th>
</tr>
</thead>
</table>
| 3   | $E(\gamma(p)) = (1-p)^6 + 6p(1-p)^5 + 21p^2(1-p)^4 + 30p^3(1-p)^3$  
| 4   | $E(\gamma(p)) = (1-p)^8 + 16p(1-p)^7 + 48p^2(1-p)^6 + 104p^3(1-p)^5 + 116p^4(1-p)^4$  
| 5   | $E(\gamma(p)) = 2(1-p)^{10} + 20p(1-p)^9 + 110p^2(1-p)^8 + 290p^3(1-p)^7$  
      | $+ 470p^4(1-p)^6 + 510p^5(1-p)^5 + 470p^6(1-p)^4 + 290p^7(1-p)^3$  
      | $+ 110p^8(1-p)^2 + 20p^9(1-p) + 2p^{10}$ |
| 6   | $E(\gamma(p)) = 2(1-p)^{12} + 36p(1-p)^{11} + 180p^2(1-p)^{10} + 624p^3(1-p)^9 + 1326p^4(1-p)^8$  
      | $+ 2064p^5(1-p)^7 + 2270p^6(1-p)^6 + 2064p^7(1-p)^5 + 1326p^8(1-p)^4$  
      | $+ 624p^9(1-p)^3 + 180p^{10}(1-p)^2 + 36p^{11}(1-p) + 2(1-p)^{12}$ |
| 7   | $E(\gamma(p)) = 3(1-p)^{14} + 42p(1-p)^{13} + 315p^2(1-p)^{12} + 1232p^3(1-p)^{11}$  
      | $+ 3213p^4(1-p)^{10} + 6048p^5(1-p)^9 + 8967p^6(1-p)^8$  
      | $+ 10094p^7(1-p)^7 + 8967p^8(1-p)^6 + 6048p^9(1-p)^5$  
      | $+ 3213p^{10}(1-p)^4 + 1232p^{11}(1-p)^3 + 315p^{12}(1-p)^2 + 42p^{13}(1-p)^1 + 3(1-p)^{14}$ |
| 8   | $E(\gamma(p)) = 3(1-p)^{16} + 64p(1-p)^{15} + 448p^2(1-p)^{14} + 2144p^3(1-p)^{13}$  
      | $+ 6700p^4(1-p)^{12} + 15504p^5(1-p)^{11} + 27104p^6(1-p)^{10}$  
      | $+ 38272p^7(1-p)^9 + 42560p^8(1-p)^8 + 38272p^9(1-p)^7$  
      | $+ 27104p^{10}(1-p)^6 + 15504p^{11}(1-p)^5 + 6700p^{12}(1-p)^4 +$  
      | $+ 2144p^{13}(1-p)^3 + 448p^{14}(1-p)^2 + 64p^{15}(1-p)^1 + 3(1-p)^{16}$ |
### Table 8

**Families of Probability Polynomials for $G_\Delta(\Gamma)$ With $\Gamma = D_n$ and $\Delta = \{s, t | s^2 = 1, t^2 = 1, tst = s^{-1}\}$**

<table>
<thead>
<tr>
<th>$n$</th>
<th>Family of Probability Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$f_0(p) = (1-p)^6 + p^6$</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = 6p(1-p)^5 + 9p^2(1-p)^4 + 8p^3(1-p)^3 + 9p^4(1-p)^2 + 6p^5(1-p)$</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 6p^2(1-p)^4 + 12p^3(1-p)^3 + 6p^4(1-p)^2$</td>
</tr>
<tr>
<td>4</td>
<td>$f_0(p) = (1-p)^8 + p^8$</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = 8p(1-p)^7 + 12p^2(1-p)^6 + 14p^4(1-p)^4 + 12p^6(1-p)^2 + 8p^7(1-p)$</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 16p^2(1-p)^6 + 56p^3(1-p)^5 + 56p^4(1-p)^4 + 56p^5(1-p)^3 + 16p^6(1-p)^2$</td>
</tr>
<tr>
<td>5</td>
<td>$f_0(p) = (1-p)^{10} + p^{10}$</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = 10p(1-p)^9 + 15p^2(1-p)^8 + 10p^4(1-p)^6 + 10p^6(1-p)^4 + 15p^8(1-p)^2$</td>
</tr>
<tr>
<td></td>
<td>$+10p^9(1-p)$</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 30p^2(1-p)^8 + 100p^3(1-p)^7 + 120p^4(1-p)^6 + 132p^5(1-p)^5$</td>
</tr>
<tr>
<td></td>
<td>$+120p^6(1-p)^4 + 100p^7(1-p)^3 + 30p^8(1-p)^2$</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = 20p^3(1-p)^7 + 80p^4(1-p)^6 + 120p^5(1-p)^5$</td>
</tr>
<tr>
<td></td>
<td>$+80p^6(1-p)^4 + 20p^7(1-p)p^3$</td>
</tr>
<tr>
<td>6</td>
<td>$f_0(p) = (1-p)^{12} + (p)^{12}$</td>
</tr>
<tr>
<td></td>
<td>$f_1(p) = 12p(1-p)^{11} + 18p^2(1-p)^{10} + 15p^4(1-p)^8 + 20p^6(1-p)^6 + 15p^8(1-p)^4$</td>
</tr>
<tr>
<td></td>
<td>$+18p^{10}(1-p)^2 + 12p^{11}(1-p)$</td>
</tr>
<tr>
<td></td>
<td>$f_2(p) = 48p^2(1-p)^{10} + 156p^3(1-p)^9 + 180p^4(1-p)^8 + 168p^5(1-p)^7$</td>
</tr>
<tr>
<td></td>
<td>$+224p^6(1-p)^6 + 168p^7(1-p)^5 + 180p^8(1-p)^4 + 156p^9(1-p)^3$</td>
</tr>
<tr>
<td></td>
<td>$+48p^{10}(1-p)^2 +$</td>
</tr>
<tr>
<td></td>
<td>$f_3(p) = 64p^3(1-p)^9 + 300p^4(1-p)^8 + 624p^5(1-p)^7$</td>
</tr>
<tr>
<td></td>
<td>$+680p^6(1-p)^6 + 624p^7(1-p)^5 + 300p^8(1-p)^4 + 64p^9(1-p)^3$</td>
</tr>
<tr>
<td>n</td>
<td>Family of Probability Polynomials</td>
</tr>
<tr>
<td>----</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>7</td>
<td>( f_0(p) = (1-p)^{14} + p^{14} )</td>
</tr>
<tr>
<td></td>
<td>( f_1(p) = 14p(1-p)^{13} + 21p^2(1-p)^{12} + 21p^4(1-p)^{10} + 35p^6(1-p)^8 + 35p^8(1-p)^6 )</td>
</tr>
<tr>
<td></td>
<td>( + 21p^{10}(1-p)^4 + 21p^{12}(1-p)^2 + 14p^{14}(1-p) )</td>
</tr>
<tr>
<td></td>
<td>( f_2(p) = 70p^2(1-p)^{12} + 224p^3(1-p)^{11} + 252p^4(1-p)^{10} + 266p^5(1-p)^9 )</td>
</tr>
<tr>
<td></td>
<td>( + 336p^6(1-p)^8 + 336p^7(1-p)^7 + 336p^8(1-p)^6 + 266p^9(1-p)^5 )</td>
</tr>
<tr>
<td></td>
<td>( + 252p^{10}(1-p)^4 + 224p^{11}(1-p)^3 + 70p^{12}(1-p)^2 )</td>
</tr>
<tr>
<td></td>
<td>( f_3(p) = 140p^3(1-p)^{11} + 672p^4(1-p)^{10} + 1440p^5(1-p)^9 )</td>
</tr>
<tr>
<td></td>
<td>( + 1792p^6(1-p)^8 + 1976p^7(1-p)^7 + 1792p^8(1-p)^6 + 1440p^9(1-p)^5 )</td>
</tr>
<tr>
<td></td>
<td>( + 672p^{10}(1-p)^4 + 140p^{11}(1-p)^3 )</td>
</tr>
<tr>
<td></td>
<td>( f_4(p) = 56p^4(1-p)^{10} + 336p^5(1-p)^9 + 840p^6(1-p)^8 + 1120p^7(1-p)^7 + 840p^8(1-p)^6 )</td>
</tr>
<tr>
<td></td>
<td>( + 336p^9(1-p)^5 + 56p^{10}(1-p)^4 )</td>
</tr>
<tr>
<td>8</td>
<td>( f_0(p) = (1-p)^{16} + (1-p)^{16} )</td>
</tr>
<tr>
<td></td>
<td>( f_1(p) = 16p(1-p)^{15} + 24p^2(1-p)^{14} + 28p^4(1-p)^{12} + 56p^6(1-p)^{10} + 70p^8(1-p)^8 )</td>
</tr>
<tr>
<td></td>
<td>( + 56p^{10}(1-p)^6 + 28p^{12}(1-p)^4 + 24p^{14}(1-p)^2 + 16p^{15}(1-p) )</td>
</tr>
<tr>
<td></td>
<td>( f_2(p) = 96p^2(1-p)^{14} + 304p^3(1-p)^{13} + 336p^4(1-p)^{12} + 400p^5(1-p)^{11} )</td>
</tr>
<tr>
<td></td>
<td>( + 544p^6(1-p)^{10} + 624p^7(1-p)^9 + 704p^8(1-p)^8 + 624p^9(1-p)^7 )</td>
</tr>
<tr>
<td></td>
<td>( + 544p^{10}(1-p)^6 + 400p^{11}(1-p)^5 + 336p^{12}(1-p)^4 + )</td>
</tr>
<tr>
<td></td>
<td>( + 304p^{13}(1-p)^3 + 96p^{14}(1-p)^2 )</td>
</tr>
<tr>
<td></td>
<td>( f_3(p) = 256p^3(1-p)^{13} + 1232p^4(1-p)^{12} + 2560p^5(1-p)^{11} )</td>
</tr>
<tr>
<td></td>
<td>( + 3440p^6(1-p)^{10} + 4032p^7(1-p)^9 + 4480p^8(1-p)^8 + 4032p^9(1-p)^7 )</td>
</tr>
<tr>
<td></td>
<td>( + 3440p^{10}(1-p)^6 + 2560p^{11}(1-p)^5 + 1232p^{12}(1-p)^4 + 256p^{13}(1-p)^3 )</td>
</tr>
<tr>
<td></td>
<td>( f_4(p) = 224p^4(1-p)^{12} + 1408p^5(1-p)^{11} )</td>
</tr>
<tr>
<td></td>
<td>( + 3968p^6(1-p)^{10} + 6784p^7(1-p)^9 + 7616p^8(1-p)^8 + 6784p^9(1-p)^7 )</td>
</tr>
<tr>
<td></td>
<td>( + 3968p^{10}(1-p)^6 + 1408p^{11}(1-p)^5 + 224^{12}(1-p)^4 )</td>
</tr>
</tbody>
</table>
Table 9

Probability Polynomials for the Expected Value of Genus Random Variable of $G_\Delta(\Gamma)$ With $\Gamma = D_n$ and $\Delta = \{s,t \mid s^n = 1, t^2 = 1, \text{tst}=s^{-1}\}$

<table>
<thead>
<tr>
<th>n</th>
<th>Expected Value of the Genus Random Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$E(y(p)) = 6p(1-p)^5 + 21p^2(1-p)^4 + 32p^3(1-p)^3 + 21p^4(1-p)^2 + 6p^5(1-p)$</td>
</tr>
<tr>
<td>4</td>
<td>$E(y(p)) = 8p(1-p)^7 + 44p^2(1-p)^6 + 112p^3(1-p)^5 + 126p^4(1-p)^4 + 112p^5(1-p)^3 + 44p^6(1-p)^2 + 8p^7(1-p)$</td>
</tr>
<tr>
<td>5</td>
<td>$E(y(p)) = 10p(1-p)^9 + 75p^2(1-p)^8 + 260p^3(1-p)^7 + 490p^4(1-p)^6 + 624p^5(1-p)^5 + 490p^6(1-p)^4 + 260p^7(1-p)^3 + 75p^8(1-p)^2 + 10p^9(1-p)$</td>
</tr>
<tr>
<td>6</td>
<td>$E(y(p)) = 12p(1-p)^11 + 114p^2(1-p)^10 + 504p^3(1-p)^9 + 1275p^4(1-p)^8 + 2208p^5(1-p)^7 + 2508p^6(1-p)^6 + 2208p^7(1-p)^5 + 1275p^8(1-p)^4 + 504p^9(1-p)^3 + 114p^{10}(1-p)^2 + 12p^{11}(1-p)$</td>
</tr>
<tr>
<td>8</td>
<td>$E(y(p)) = 16p(1-p)^15 + 216p^2(1-p)^14 + 1376p^3(1-p)^13 + 5292p^4(1-p)^12 + 14112p^5(1-p)^11 + 27336p^6(1-p)^10 + 40480p^7(1-p)^9 + 45382p^8(1-p)^8 + 40480p^9(1-p)^7 + 27336p^{10}(1-p)^6 + 14112p^{11}(1-p)^5 + 5292p^{12}(1-p)^4 + 1376p^{13}(1-p)^3 + 216p^{14}(1-p)^2 + 16p^{15}(1-p)^1$</td>
</tr>
</tbody>
</table>
### Table 10

**Family of Probability Polynomials for the Peterson Graph Considered as the Graph $O_5$**

<table>
<thead>
<tr>
<th>Genus</th>
<th>$f_i(p)$</th>
<th># of imbeddings</th>
</tr>
</thead>
</table>
| 1     | $5p^2(1-p)^8 + 5p^3(1-p)^7 + 5p^4(1-p)^6 + 10p^5(1-p)^5$
        | $+ 5p^6(1-p)^4 + 5p^7(1-p)^3 + 5p^8(1-p)^2$ | 40 |
| 2     | $(1-p)^{10} + 10p(1-p)^9 + 30p^2(1-p)^8 + 75p^3(1-p)^7 + 135p^4(1-p)^6$
        | $+ 162p^5(1-p)^5 + 135p^6(1-p)^4 + 75p^7(1-p)^3 + 30p^8(1-p)^2$
        | $+ 10p^9(1-p) + p^{10}$ | 664 |
| 3     | $10p^2(1-p)^8 + 40p^3(1-p)^7 + 70p^4(1-p)^6 + 80p^5(1-p)^5 + 70p^6(1-p)^4$
        | $+ 40p^7(1-p)^3 + 10p^8(1-p)^2$ | 320 |
| $7$   | $2(1-p)^{10} + 20p(1-p)^9 + 95p^2(1-p)^8 + 275p^3(1-p)^7 + 485p^4(1-p)^6$
        | $+ 574p^5(1-p)^5 + 485p^6(1-p)^4 + 275p^7(1-p)^3 + 95p^8(1-p)^2$
        | $+ 20p^9(1-p)^2 + 2p^{10}$ | 2,273 |
Table 11

Family of Probability Polynomials for the Dodecahedron
Considered as the 2-fold Antipodal Covering of the Peterson Graph

<table>
<thead>
<tr>
<th>Genus</th>
<th>Probability Polynomials</th>
<th># of imbeddings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2p^{10}(1-p)^{10}$</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>$20p^8(1-p)^{12}+20p^9(1-p)^{11}+20p^{10}(1-p)^{10}+20p^{11}(1-p)^9$ $+20p^{12}(1-p)^8$</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>$25p^4(1-p)^{16}+12p^5(1-p)^{15}+100p^6(1-p)^{14}+180p^7(1-p)^{13}$ $+305p^8(1-p)^{12}+380p^9(1-p)^{11}+490p^{10}(1-p)^{10}$ $+380p^{11}(1-p)^9+305p^{12}(1-p)^8+180p^{13}(1-p)^7$ $+100p^{14}(1-p)^6+12p^{15}(1-p)^5+25p^{16}(1-p)^4$</td>
<td>2494</td>
</tr>
<tr>
<td>3</td>
<td>$35p^2(1-p)^{18}+200p^3(1-p)^{17}+520p^4(1-p)^{16}+1220p^5(1-p)^{15}$ $+2200p^6(1-p)^{14}+3660p^7(1-p)^{13}+5260p^8(1-p)^{12}$ $+6920p^9(1-p)^{11}+7510p^{10}(1-p)^{10}+6920p^{11}(1-p)^9$ $+5260p^{12}(1-p)^8+3660p^{13}(1-p)^7+2200p^{14}(1-p)^6$ $+1220p^{15}(1-p)^5+520p^{16}(1-p)^4+200p^{17}(1-p)^3$ $+35p^{18}(1-p)^2$</td>
<td>47,540</td>
</tr>
<tr>
<td>4</td>
<td>$(1-p)^{20}+20p(1-p)^{19}+125p^2(1-p)^{18}+620p^3(1-p)^{17}$ $+2590p^4(1-p)^{16}+7632p^5(1-p)^{15}+17160p^6(1-p)^{14}$ $+31440p^7(1-p)^{13}+48935p^8(1-p)^{12}+62960p^9(1-p)^{11}$ $+68434p^{10}(1-p)^{10}+62960p^{11}(1-p)^9+48935p^{12}(1-p)^8$ $+31440p^{13}(1-p)^7+17160p^{14}(1-p)^6+7632p^{15}(1-p)^5$ $+2590p^{16}(1-p)^4+620p^{17}(1-p)^3+125p^{18}(1-p)^2$ $+20p^{19}(1-p)+p^{20}$</td>
<td>411,400</td>
</tr>
</tbody>
</table>

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### Table 11—Continued

<table>
<thead>
<tr>
<th>Genus</th>
<th>Probability Polynomials</th>
<th># of imbeddings</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$30p^2(1-p)^{18} + 320p^3(1-p)^{17} + 1710p^4(1-p)^{16} + 6640p^5(1-p)^{15}$</td>
<td>587,040</td>
</tr>
<tr>
<td></td>
<td>$+ 19300p^6(1-p)^{14} + 42240p^7(1-p)^{13} + 71450p^8(1-p)^{12}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 97680p^9(1-p)^{11} + 108300p^{10}(1-p)^{10} + 97680p^{11}(1-p)^9$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 71450p^{12}(1-p)^9 + 42240p^{13}(1-p)^7 + 19300p^{14}(1-p)^6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 6640p^{15}(1-p)^5 + 1710p^{16}(1-p)^4 + 320p^{17}(1-p)^3 + 30p^{18} (1-p)^2$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$4(1-p)^{20} + 80p(1-p)^{19} + 755p^2(1-p)^{18} + 4680p^3(1-p)^{17}$</td>
<td>4.50945663</td>
</tr>
<tr>
<td></td>
<td>$+ 20520p^4(1-p)^{16} + 67412p^5(1-p)^{15} + 171940p^6(1-p)^{14}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 348300p^7(1-p)^{13} + 569400p^8(1-p)^{12} + 761780p^9(1-p)^{11}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 838766p^{10}(1-p)^{10} + 761780p^{11}(1-p)^9$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 569400p^{12}(1-p)^8 + 348300p^{13}(1-p)^7 + 171940p^{14}(1-p)^6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 67412p^{15}(1-p)^5 + 20520p^{16}(1-p)^4 + 4680p^{17}(1-p)^3 + 755p^{18} (1-p)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ 80p^{19}(1-p) + 4p^{20}$</td>
<td></td>
</tr>
</tbody>
</table>
Table 12
Genus Distribution and Average Genus for the Cubic Graphs of Order 8

<table>
<thead>
<tr>
<th>Graph</th>
<th>Genus 0</th>
<th>Genus 1</th>
<th>Genus 2</th>
<th>Average Genus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>92</td>
<td>160</td>
<td>1.609375</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>70</td>
<td>184</td>
<td>1.7109375</td>
</tr>
<tr>
<td>3</td>
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<td>64</td>
<td>192</td>
<td>1.75</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>54</td>
<td>200</td>
<td>1.7734375</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>56</td>
<td>200</td>
<td>1.78125</td>
</tr>
</tbody>
</table>
Table 13
Genus Distribution and Average Genus for Cubic Graphs of Order 10

<table>
<thead>
<tr>
<th>Graph</th>
<th>Genus 0</th>
<th>Genus 1</th>
<th>Genus 2</th>
<th>Genus 3</th>
<th>Average Genus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>48</td>
<td>656</td>
<td>320</td>
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<td>2.265625</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>720</td>
<td>256</td>
<td></td>
<td>2.203125</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>160</td>
<td>720</td>
<td>128</td>
<td>1.9453125</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>78</td>
<td>656</td>
<td>288</td>
<td>2.201171875</td>
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<td>72</td>
<td>664</td>
<td>288</td>
<td></td>
<td>2.2109375</td>
</tr>
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<td>L</td>
<td>Probability Polynomials</td>
<td>(2^{2n} f_1^{1/2}(n))</td>
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<tr>
<td>L1</td>
<td>(f_0(p)= (1-p)^2 + p^2)</td>
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<td>(f_1(p)=2p(1-p))</td>
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<td>L2</td>
<td>(f_0(p)= (1-p)^4 + 2p^2(1-p)^2 + p^4)</td>
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<td>(f_1(p)=4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p))</td>
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<td>L3</td>
<td>(f_0(p)= (1-p)^6 + 3p^2(1-p)^4 + 3p^4(1-p)^2 + p^6)</td>
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<td>(f_1(p)=6p(1-p)^5 + 8p^2(1-p)^4 + 12p^3(1-p)^3 + 8p^4(1-p)^2 + 6p^5(1-p))</td>
<td>40</td>
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<td>(f_2(p)= 4p^2(1-p)^4 + 8p^3(1-p)^3 + 4p^4(1-p)^2)</td>
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<td>L4</td>
<td>(f_0(p)= (1-p)^8 + 4p^2(1-p)^6 + 6p^4(1-p)^4 + 8p^6(1-p)^2 + p^8)</td>
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<td>(f_1(p)= 8p(1-p)^7 + 12p^2(1-p)^6 + 24p^3(1-p)^5 + 24p^4(1-p)^4 + 24p^5(1-p)^3 + 12p^6(1-p)^2 + 8p^7(1-p))</td>
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<td>(f_2(p)= 12p^2(1-p)^6 + 32p^3(1-p)^5 + 40p^4(1-p)^4 + 32p^5(1-p)^3)</td>
<td>128</td>
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<tr>
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<td>+ 12p^6(1-p)^2)</td>
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<tr>
<td>L5</td>
<td>(f_0(p)= (1-p)^{10} + 5p^2(1-p)^8 + 10p^4(1-p)^6 + 10p^6(1-p)^4 + 5p^8(1-p)^2 + p^{10})</td>
<td>32</td>
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<tr>
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<td>(f_1(p)= 10p(1-p)^9 + 16p^2(1-p)^8 + 40p^3(1-p)^7 + 48p^4(1-p)^6 + 60p^5(1-p)^5 + 48p^6(1-p)^4 + 40p^7(1-p)^3 + 16p^8(1-p)^2 + 10p^9(1-p)^)</td>
<td>288</td>
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<td>(f_2(p)= 24p^2(1-p)^7 + 72p^3(1-p)^6 + 120p^4(1-p)^5 + 144p^5(1-p)^4 + 120p^6(1-p)^3 + 72p^7(1-p)^2 + 24p^8(1-p)^2)</td>
<td>576</td>
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<td>(f_3(p)= 8p^3(1-p)^7 + 32p^4(1-p)^6 + 48p^5(1-p)^5 + 32p^6(1-p)^4 + 8p^7(1-p)^3)</td>
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<tr>
<td>L_1</td>
<td>$E(\gamma(L_1)) = 2p(1-p)$</td>
<td>$\frac{1}{2}$</td>
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<tr>
<td>L_2</td>
<td>$E(\gamma(L_2)) = 4p(1-p)^3 + 4p^2(1-p)^2 + 4p^3(1-p)$</td>
<td>$\frac{3}{4}$</td>
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<tr>
<td>L_3</td>
<td>$E(\gamma(L_2)) = 6p(1-p)^5 + 16p^2(1-p)^4 + 28p^3(1-p)^3 + 16p^4(1-p)^2 + 6p^5(1-p)$</td>
<td>$\frac{9}{8}$</td>
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<tr>
<td>L_4</td>
<td>$E(\gamma(L_2)) = 8p(1-p)^7 + 36p^2(1-p)^6 + 88p^3(1-p)^5 + 104p^4(1-p)^4 + 88p^5(1-p)^3 + 36p^6(1-p)^2 + 8p^7(1-p)$</td>
<td>$\frac{23}{16}$</td>
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<tr>
<td>L_5</td>
<td>$E(\gamma(L_2)) = 10p(1-p)^9 + 64p^2(1-p)^8 + 208p^3(1-p)^7 + 384p^4(1-p)^6 + 492p^5(1-p)^5 + 384p^6(1-p)^4 + 208p^7(1-p)^3 + 64p^8(1-p)^2 + 10p^9(1-p)$</td>
<td>$\frac{57}{36}$</td>
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Table 15

Expected Value of the Genus Random Variable
for Closed-End Ladders

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<table>
<thead>
<tr>
<th>Genus 0</th>
<th>Genus 1</th>
<th>Genus 2</th>
<th>Genus 3</th>
<th>Average Genus</th>
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<td></td>
<td>14/16</td>
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<td>R₂</td>
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<td>24</td>
<td>38/24</td>
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<td>R₃</td>
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<td>R₅</td>
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<td>198</td>
<td>1656</td>
<td>10230/4096</td>
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</table>
Appendix B

Computer Program (PASCAL) That Calculates Probability Polynomials
program graph (input, output, vfile);

const max = 60;  (* max size of array *)
type
  matrix = array [0..max,1..3] of integer;
  matrix2 = array [0..max,0..max] of integer;
  matrix3 = array [0..max,0..3*max] of integer;
  list = array [0..max] of integer;

var
  vfile: text;
  filename: packed array [1..12] of char;
  a: matrix;  (* stores the neighbors of each vertex in incr. order *)
  s: matrix2;  (* stores the successor [in vertex i, of vertex j] *)
  n: integer;  (* number of vertices in graph *)
  i,j,k,m: integer;
  r: list;  (* list of vertices on a particular region *)
  done: boolean;  (* tells if every edge used in finding regions *)
  region: boolean;  (* tells if region is completed *)
  cw: list;  (* list of whether vertices are clockwise or counter-cw
          cw[i] = 0 if i is ccw, cw[i] = 1 if i is cw *)
  poss: integer;  (* used to calculate clockwise and ccw *)
  remain: integer;  (* used *)
  two: list;  (* lists powers of 2 *)
  numreg: list;  (* lists the number of regions of each size *)
  used: matrix2;  (* matrix telling which edges have been used - 1 =
          used *)
  totalreg: integer;  (* total number of regions in this rotation *)
  genus: integer;  (* genus of this rotation *)
  print_out: boolean;  (* whether or not a print out of all region is
          wanted *)
  test: char;  (* response to whether or no a print out is wanted *)
  poly: matrix3;  (* stores info for polynomial *)
  num_cw: integer;  (* number of clockwise rotations *)
  c: list;  (* stores the perm. of a particular labeling *)
  exp_val: list;  (* stores the poly of the expected value *)

procedure findnext;
  (* finds the successor [in vertex i, of vertex j] *)
begin
  for i := 1 to n do
    for j := 1 to n do s[i,j] := 0;
  for i := 1 to n do
  begin
    if cw[i] = 1 then
    begin
      s[i,a[i,1]] := a[i,2];
      s[i,a[i,2]] := a[i,3];
    end;
  end;
end;
procedure getrotations;
    {* determines whether each vertex is clockwise or counterclockwise *>}
begin
    remain := k;
    for i := n downto 1 do
    begin
        cw[i] := remain div two[i-1];
        remain := remain - cw[i] * two[i-1];
        end; {* end for *} 
    num_cw := 0;
    for i := 1 to n do
    num_cw := num_cw + cw[i];
    end; {* end procedure getrotations *}

procedure findunused(var x,y:integer);
    {* finds the next unused edge *}

label 10;

begin
    x:= 0;
    y:= 0;
    for i := 1 to n do
    begin
        for j := 1 to n do
        begin
            if used[i,j] = 0 then
            begin
                x := i;
                y := j;
                used[i,j] := 1;
                goto 10;
            end;
        end; {* end for j *}
    end; {* end for i *}. 

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10: if \( x = 0 \) then done := true;
end;  

procedure printrotation;
  {* prints the orientation for this rotation  *}  

begin
  writeln;
  writeln ('=============================================');
  writeln ('\textit{Rotation}\, (k+1):1);  
  for i := 1 to n do  
    begin
      if cw[i] = 1 then  
        begin
          write ('vertex \, i:1,','
          write (',a[i,1]:1,';'  
          write (a[i,2]:1,');  
          write (a[i,3]:1,')');  
          writeln (\textit{\footnotesize Clockwise});  
        end
      else
        begin
          write ('vertex \, i:1,','
          write (',a[i,1]:1,';  
          write (a[i,3]:1,')');  
          writeln (\textit{\footnotesize Counter-clockwise});  
        end;
    end;  
  writeln;
  writeln ('Regions are: ');  
end;  

procedure printregion(next:integer);
  {* prints the vertices of a particular region  *}  

begin
  write (r[1]:1);  
  for i:= 2 to next do  write (',r[i]:1);  
  writeln;
end;  

procedure getregions;
{ * calculates the regions for a given rotation * }

var x,y,next:integer;
k,l:integer;
begin
for i:= 1 to n do
  for j:= 1 to n do used [i,j] := 1;
for i:= 1 to n do
  for j:= 1 to 3 do used [i, a[i,j]] := 0;
done := false;
for i := 3 to (3*n) do num_reg[i] := 0;
findunused(x,y);
total_reg := 0;
while not done do
  begin
    region := false;
    for i:= 1 to (3*n) do r[i] := 0;
    r[l] := x;
    r[2] := y;
    next := 3;
    while not region do
      begin
        r[next] := s[r[next-1],r[next-2]];
        used [r[next-1],r[next]] := 1;
        if (r[next-1] = r[1]) and (r[next] = r[2]) then
          begin
            region := true;
            if print_out then print_region(next);
            end; /* end if */
        next := next + 1;
      end; /* end while not region */
    findunused(x,y);
  end; /* end while not done */
for i:= 3 to (3*n) do total_reg := total_reg + num_reg[i];
genus := trunc(1.0 - 0.5*(total_reg-0.5*n));
if print_out then
  begin
    writeln;
    writeln ('number of regions');
    for i:= 3 to (3*n) do
      begin
        if num_reg[i] > 0 then write (num_reg[i]:1,'-r','i:1,' ');
      end; /* end for */
    writeln ('total number of regions ', total_reg:1);
    writeln ('genus ',genus:1);
    end; /* end if */
  poly[genus,num_cw]:=poly[genus,num_cw] + 1;
end; /* end procedure getregions *
procedure print_poly;
  (* prints the prob. polynomial for this labeling *)

  var i,j:integer;

begin
  writeln;
  for i:= 0 to 3*n do exp_val[i]:=0;
  for i:= 0 to n do 
  begin
    write ('genus ',i:1,',' );
    for j:= 0 to (3*n) do 
    begin
      if poly[i,j] > 0 then 
        write (poly[i,j]:1,'pA ',j:1,',' );
      exp_val[j] := exp_val[j] + i*poly[i,j];
    end;
    writeln;
  end;
  writeln;
  write ('Expected value ');
  for j:= 0 to 3*n do if exp_val[j] > 0 then 
    write (exp_val[j]:1,'pA ',j:1,',' );
  writeln;
end; (* ends procedure print_poly *)

begin (* main *)
  writeln ('Enter the filename for the data you would like to run');
  readln (filename);
  open (vfile,filename,old);
  reset (vfile);
  writeln ('Do you want all the regions printed out?');
  readln (test);
  print_out := true;
  if (test = 'n') or (test = 'N') then print_out := false;
  while not eof(vfile) do
  begin
    read(vfile,n);
    for i:= 1 to n do read(vfile,c[i]);
    for i:= 0 to n do 
    for j:= 0 to (3*n) do poly[i,j] := 0;
    for i:= 1 to n do 
    for j:= 1 to 3 do read(vfile,a[i,j]);
    writeln;
    if print_out then
begin
writeln;
writeln;
write('***************************************************************************');
writeln('***************************************************************************');
write('***************************************************************************');
writeln('***************************************************************************');
writeln('This graph has ,n:1, vertices');
for i:= 1 to n do
begin
   write('vertex ',i,1,' has neighbors ');
   writeln(a[i,1]:1,'',a[i,2]:1,'',a[i,3]:1);
end;
end
else
begin
   write('--- ');
   for i:= 1 to n do write(c[i]:1);
   write(' ');
   for i:= 1 to n do
   begin
      for j:= 1 to 3 do write(a[i,j]:1,'');
      write(' ');
      writeln;
   end;
   poss := 1;
two[0]:= 1;
for i := 1 to n do
begin
   poss := poss * 2;
two[i] := poss;
end;
for k:= 0 to (poss-1) do
begin
   for m := 0 to n do cw[m] := 0;
   getrotations;
   findnext;
   if print_out then printrotation;
   getregions;
   end; /* end for */
print_poly;
end; /* end while */
close(vfile);
end.

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REFERENCES


