8-1981

Detector Geometry for Cascade Photon Experiments

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DETECTOR GEOMETRY FOR CASCADE PHOTON EXPERIMENTS

by

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A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Master of Arts
Department of Physics

Western Michigan University
Kalamazoo, Michigan
August 1981

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Western Michigan University, 1981

An observation of cascading optical photons can be used to determine an absolute excitation cross section for the intermediate state without the need of standard lamp calibration. In order to reduce the data an angular correlation function and an angular distribution function must be known. These functions depend on the alignment of the upper excited state and the polarization of the light emitted in the lower part of the cascade. However, at certain angles of observation the values of the required functions are known regardless of the alignment or polarization. A computer search is done to find these angles for cascade $J = 1 \rightarrow 1 \rightarrow 0$, $J = 2 \rightarrow 1 \rightarrow 0$, and $J = 3 \rightarrow 2 \rightarrow 1$ when detectors with large solid angles are used. Possible errors due to misalignment of the photon detectors are considered.

ACKNOWLEDGEMENTS

I wish to thank Dr. Dean Kaul for the hard work and guidance which he gave during the course of this project.

I would also like to thank Dr. Michitoshi Soga and Dr. Allen Dotson for their assistance in making this research project possible.

Roger Minoru Munechika

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MUNECHIKA, ROGER MINORU
DETECTOR GEOMETRY FOR CASCADE PHOTON EXPERIMENTS.

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University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106

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CHAPTER I

Introduction

A common type of experiment to measure excitation cross section is the bombardment of a gas with a monoenergetic electron beam, and the measurement of the optical radiation with a photon detector.

When measuring an absolute cross section for electron excitation, one must determine N, the total number of atoms produced in some particular excited state by electron collisions. To relate N to the number of photons recorded by the photon detector, one must determine the absolute detection efficiency of the photon detector. A standard lamp whose luminance is known is usually used for this calibration. However, the required calibration procedure is difficult, especially in the ultraviolet region. To avoid this difficulty, Kaul[1] used self-calibrating apparatus, where the absolute detection efficiency of the photon detector is obtained without the use of a standard lamp or reference to any other cross section measurement.

The apparatus observed photons emitted in an atomic cascade. The singles counts from both the upper and the lower portion of the cascade were recorded along with the number of observed true coincidences between the upper and the lower cascade events.

It can be shown that the product of the number of singles counts in the two detectors divided by the number of observed true coincidences can be used to determine N, provided certain angular distribution and correlation factors can be taken into account.

The absolute excitation cross section is then determined from N and other parameters of the bombardment apparatus.

Thus, all necessary values except angular distribution and correlation factors are directly obtained in the experiment. In this paper, the method to determine these factors is discussed, and it is shown that a separate determination of angular distribution and correlation factors can be avoided by means of appropriate detector geometry.

CHAPTER II

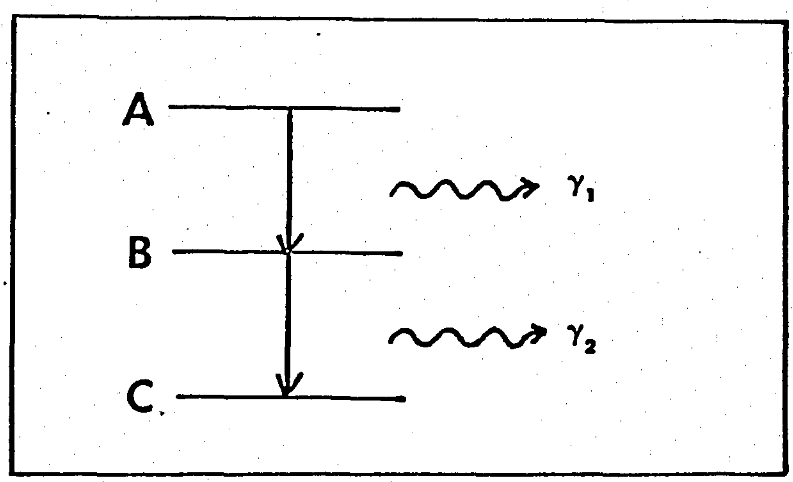
Theory

Excitation Cross Section

Consider the measurement of two radiations as shown in Figure I. The excitation cross section of state B is to be measured.

Figure I

Energy-level Diagram Showing Two Cascades



In general, state A can either be excited directly from the ground state by electron bombardment, or by optical cascade from higher levels. State A may either decay to state B or to some other state. State B may either be excited directly, by cascade from state A, or by cascade from some other state. State B may either decay to state C or to some other state.

Consider two wide-angle, single-photon detectors. Detector 1 observes the $A \rightarrow B$ transition where a photon (γ_1) of wavelength λ_1 is emitted. Let the number of photons detected by Detector 1 be N_1. Detector 2 observes the $B \rightarrow C$ transition where a photon (γ_2) of wavelength λ_2 is emitted. Let the number of photons detected by Detector 2 be N_2.

Let N be the total number of atoms excited to the intermediate state B, whether directly or by cascade. Let ξ_2 be the fraction of atoms excited to the intermediate state B that decay to the lower state C with emission of a γ_2 photon. Let e_2 be the overall detection efficiency of Detector 2 for γ_2 photons, which includes such effects as the solid angle of the lens, transmission losses in the lens and interference filter, the conversion efficiency of the photocathode, discriminator losses, etc. In addition, the fact that emission of γ_2 may be anisotropic must be included as a factor $\overline{D}_2$, the averaged angular distribution factor for dipole radiation.[2] Then

$$N_2 = e_2 \, \xi_2 \, N \, \overline{D}_2 \, . \quad (1)$$

A true cascade event detected implies that a γ_1 photon from the upper part of the cascade has been detected by Detector 1, that the atom created in the intermediate state B has decayed to the lower state C, and that the γ_2 photon emitted has been detected by Detector 2. In addition, the angular-correlation of photons γ_1 and γ_2 must be included as a factor $\overline{W}$, the averaged angular-correlation function.[3,4] Then N_t, the number of true cascade events detected is

$$N_t = e_2 \xi_2 N_1 \overline{W} . \qquad (2)$$

Combining Equation (1) and Equation (2) one obtains

$$N = \left(\frac{N_1 N_2}{N_t}\right) \left(\frac{\overline{W}}{\overline{D}_2}\right). \qquad (3)$$

Equation (3) is valid regardless of cascading, branching or quenching collisions. Note that N_1, N_2 and N_t are directly recorded by the apparatus, whereas $\overline{W}$ and $\overline{D}_2$ must be independently determined.

N from Equation (3) can be used to calculate the excitations cross section σ for the intermediate state in the observed cascade. Here, σ is the sum of a direct excitation cross section and cascade excitation cross sections.

If i is the electron bombardment current, T is the time of data accumulation, e is the magnitude of the charge on the electron, ν is the number density of atoms in the interaction region, and l is the length of the interaction region observed by Detector 1 and 2, then[5]

$$\sigma = \left(\frac{e}{iT}\right)\left(\frac{N}{\nu\, l}\right).$$

Angular Distribution Factor

The averaged angular distribution factor, $\bar{D}_2$ is given by

$$\bar{D}_2 = \frac{3\left(1 - P\,\overline{\cos^2\theta_2}\right)}{3 - P}, \quad (4)$$

where θ_2 is the angle between the direction of the electron bombardment beam and γ_2 emission, $\overline{\cos^2\theta_2}$ means averageing over Detector 2 solid angle, and P is the polarization of γ_2 radiation. P is defined as $P \equiv (I_{\parallel} - I_{\perp}) / (I_{\parallel} + I_{\perp})$ where $I_{\parallel}$ and $I_{\perp}$ are the intensities of γ_2 emission polarized parallel and perpendicular to the electron bombardment beam respectively. Both $I_{\parallel}$ and $I_{\perp}$ are measured at $\theta_2 = 90°$.

Instead of taking a measurement to determine P, certain Detector geometry can be used as done by Fite et al.[6]

Consider the angle

$$\theta_2 = \cos^{-1} \frac{1}{\sqrt{3}} . \qquad (5)$$

At this angle,

$$\frac{3 \left(1 - P \cos^2 \theta_2 \right)}{3 - P} = 1 .$$

Provided Detector 2 is circular with its center at this angle, averaging over Detector 2 solid angle still gives the same result, i.e.,

$$\bar{D}_2 = 1 .$$

Although this result is not obvious, it has been proven by Soga.[7]

Therefore, if Detector geometry can be used with θ_2 equal to that given by Equation (5), then, $\bar{D}_2$ is equal to one and one need not measure the polarization of γ_2 radiation.

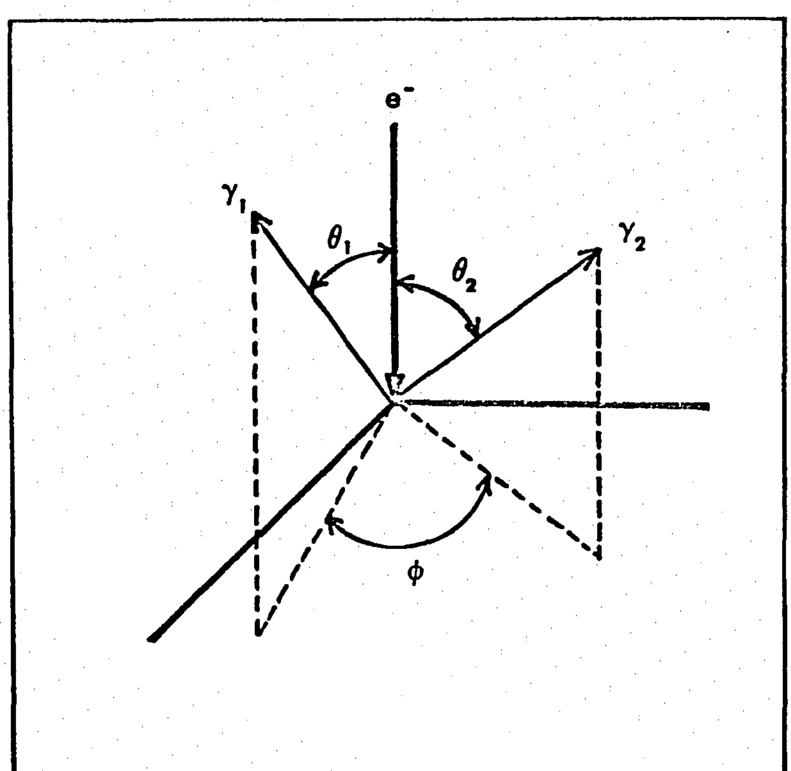
<u>Angular-Correlation Function</u>

In general, the angular-correlation function depends on the relative directions of the electron bombardment beam, γ_1 emission, and γ_2 emission as well as the angular momenta of the cascading states and the relative populations of the upper state of the cascade.[8]

Let θ_1, θ_2, ϕ be angles defined in Figure II.

Figure II

Definition of Angles θ_1, θ_2, and ϕ



Then the angular-correlation function is given by[8]

$$W(\theta_1, \theta_2, \phi) = 2 \sum_{KMN} a_{KM}^{N} \cdot \bar{P}_K^N(\cos\theta_1) \cdot \bar{P}_M^N(\cos\theta_2) \cdot \cos(N\phi)$$

with $$a_{KM}^{N} = \sum_{m} p(m) \cdot C_{KM}^{N}(m) .$$

The indices K and M can take on only the values 0, 2, and 4. The index N is limited to positive integral values (and zero) such that the absolute value of N does not exceed either K or M. The index m can take on the values 0, 1, 2, . . . , J for the integral spin J, and the values 1/2, 3/2, . . . , J for the half-integral J. The coefficients $C_{KM}^{N}(m)$ are tabulated[8], m is the magnetic quantum number of the upper cascading state referring to the beam direction as quantization axis. For that axis, it can be shown that the populations of the magnetic substates m and -m are equal. p(m) represents the population of m and -m taken together, which satisfies

$$\sum_{m=o}^{J} p(m) = 1 .$$

Here, the integral spin J is considered. However, all discussions can be applied to the half integral J. For instance, the above equation would be

$$\sum_{m=1/2}^{J} p(m) = 1 ,$$

for the half-integral J.

$\bar{P}_K^N$ (cos θ) is given by

$$\bar{P}_K^N (\cos \theta) = \sqrt{\frac{2K+1}{2}} \sqrt{\frac{(K-N)!}{(K+N)!}} P_K^N (\cos \theta),$$

where P_K^N (cos θ) is the associated Legendre function.

By substituting $a_K^N M$ and changing the order of the summation, one obtains

$$W(\theta_1, \theta_2, \phi) = \sum_m p(m) \sum_{KMN} 2\, C_{KM}^N (m)\, \bar{P}_K^N (\cos \theta_1)\, \bar{P}_M^N (\cos \theta_2) \cos (N \phi)$$

or, by denoting

$$A(\theta_1, \theta_2, \phi ; m) = \sum_{KMN} 2\, C_{KM}^N (m)\, \bar{P}_K^N (\cos \theta_1)\, \bar{P}_M^N (\cos \theta_2) \cos (N \phi),$$

$$W(\theta_1, \theta_2, \phi) = \sum_m p(m)\, A(\theta_1, \theta_2, \phi ; m). \qquad (6)$$

Instead of making a measurement to determine the alignment of the upper state of the cascade, certain detector geometry can be used so that the value of the angular-correlation function is known regardless of the alignment of the upper cascading state. If the angles θ_1, θ_2 and ϕ are chosen so that $A(\theta_1, \theta_2, \phi; m)$ is independent of m, i.e.,

$$A(\theta_1, \theta_2, \phi; o) = A(\theta_1, \theta_2, \phi; 1) =$$

$$\ldots = A(\theta_1, \theta_2, \phi; J) = A,$$

then the value of the angular-correlation function at these particular angles is

$$W = A \sum_m p(m) = A.$$

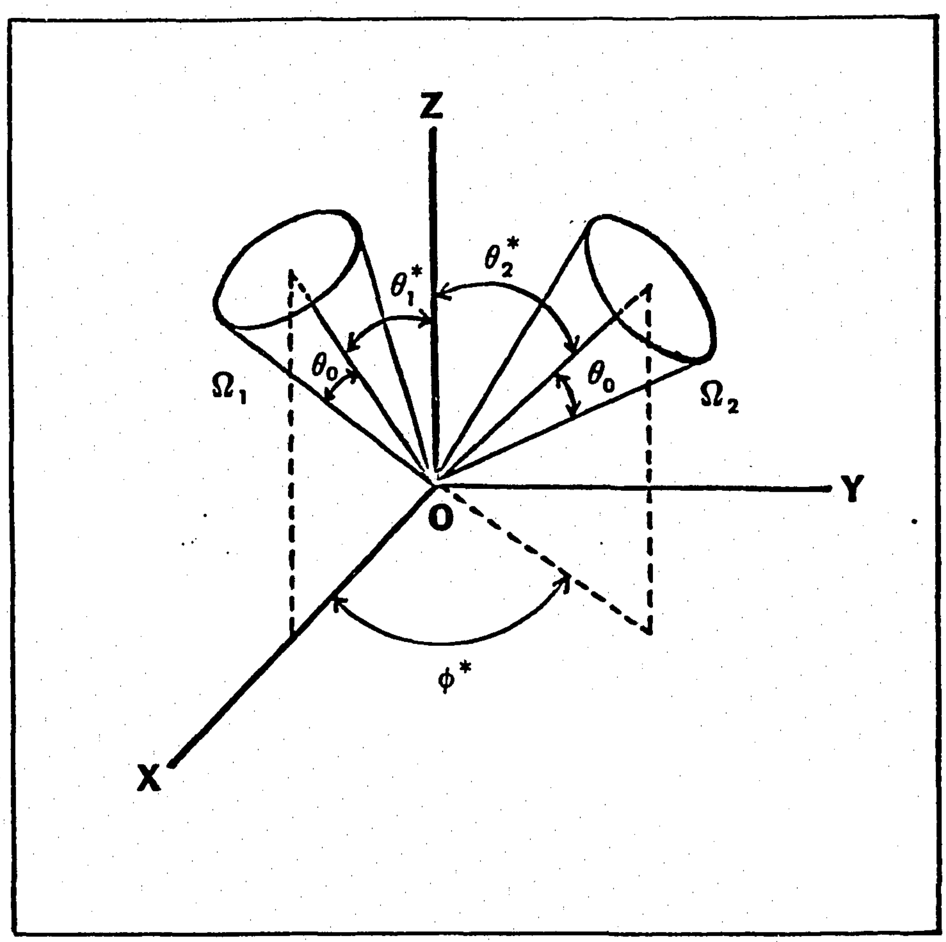
In an actual experiment, the large solid angle of the detectors would require that $W(\theta_1, \theta_2, \phi)$ be averaged over solid angle. Then,

$$\overline{W} = \int_{\Omega_1} \int_{\Omega_2} W(\theta_1, \theta_2, \phi)\, d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1} \int_{\Omega_2} d\Omega_1 d\Omega_2 ,$$

where $d\Omega_1$ is an element of solid angle for Detector 1 and $d\Omega_2$ is an element of solid angle for Detector 2. Solid angle Ω_1 is determined by the angle $\theta_1{}^*$ pointing at the center of Detector 1 and the half cone angle θ_o, and Ω_2 is determined by the angle $\theta_2{}^*$ pointing at the center of Detector 2, the half cone angle θ_o, and the azimuth angle ϕ * as shown in Figure III.

Figure III

Definition of Angles $\theta_1{}^*$, $\theta_2{}^*$, ϕ * and θ_o



In this case, if the angles $\theta_1{}^*$, $\theta_2{}^*$ and ϕ * are chosen so that

$$\int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;0)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1\, d\Omega_2$$

$$= \int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;1)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1\, d\Omega_2$$

$$= \cdots = \int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;J)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1\, d\Omega_2$$

$$= \bar{A}$$

then the value of the averaged angular-correlation function at these particular angles is

$$\overline{W} = \bar{A} \sum_m p(m) = \bar{A} \quad .$$

Therefore, if the angles $\theta_1{}^*$, $\theta_2{}^*$ and $\phi\,{}^*$ are chosen to satisfy Equation (7), then the value of $\overline{W}$ is known and one need not determine the alignment of the upper cascading state. Note that this method is applicable up to J equal to three, because there are three independent parameters θ_1, θ_2, and ϕ to satisfy Equation (7). If one chose $\theta_2{}^*$ equal to that given by Equation (5) so that $\overline{D}_2$ is equal to one, then there are only two parameters θ_1 and ϕ, and this method is applicable up to J equal to two.

CHAPTER III

Program

A computer program is used to find the values of θ_1^*, θ_2^* and ϕ^* which satisfy Equation (7) in each case. An example for $J = 3 \rightarrow 2 \rightarrow 1$ case is given in Appendix C.

Coordinate Transformation

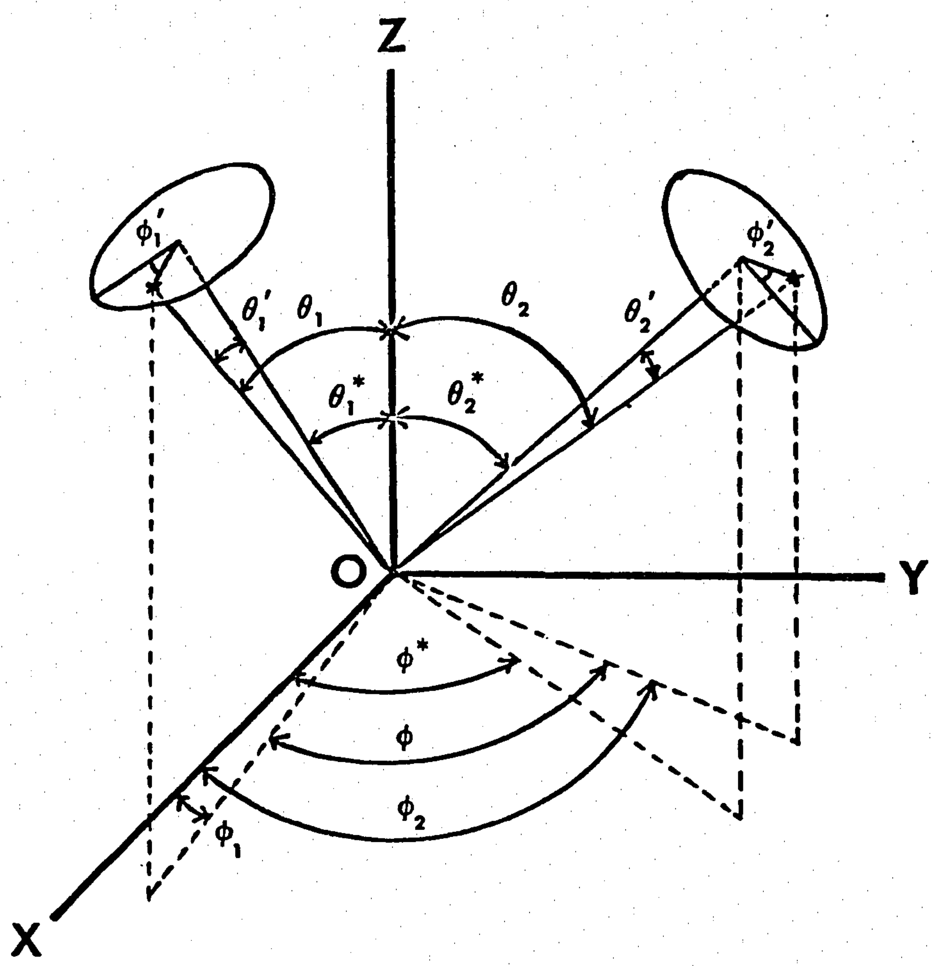
In general, in order to perform averaging $W(\theta_1, \theta_2, \phi)$ over given solid angle θ_o, the coordinate transformation from $(\theta_1^*, \theta_2^*, \phi^*, \theta_1', \theta_2', \phi_1', \phi_2')$ to $(\theta_1, \theta_2, \phi_1, \phi_2)$ is required. (See Figure IV)

It can be shown that the required transformations are[9]

$$\text{for Detector 1} \quad \left\{ \begin{aligned} \sin\theta_1\cos\phi_1 &= \cos\theta_1^*\sin\theta_1'\cos\phi_1' + \sin\theta_1^*\cos\theta_1' \\ \sin\theta_1\sin\phi_1 &= \sin\theta_1'\sin\phi_1' \\ \cos\theta_1 &= -\sin\theta_1^*\sin\theta_1'\cos\phi_1' + \cos\theta_1^*\cos\theta_1' \end{aligned} \right.$$

Figure IV

Coordinate Transformation



and for Detector 2

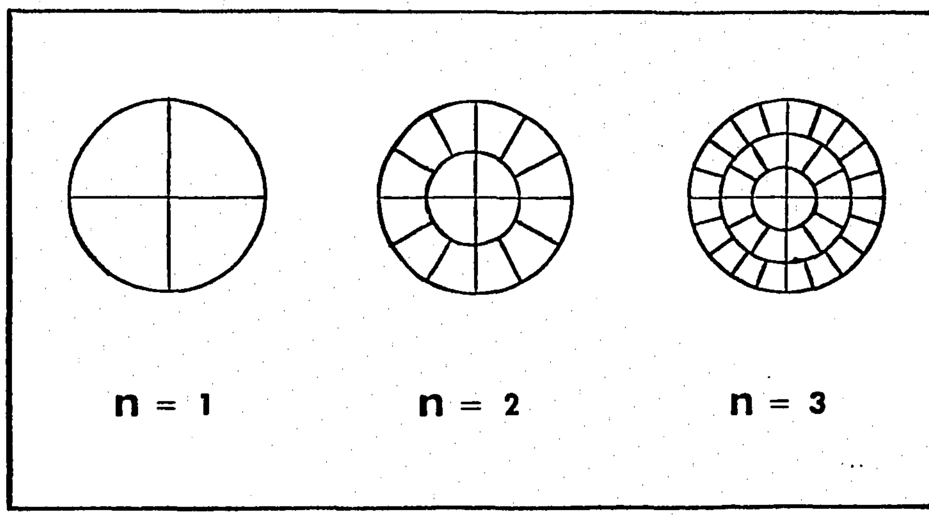
$$
\begin{cases}
\sin\theta_2\cos\phi_2 = \cos\theta_2{*}\cos\phi{*}\sin\theta_2'\cos\phi_2' - \\
\qquad \sin\phi{*}\sin\theta_2'\sin\phi_2' + \\
\qquad \sin\theta_2{*}\cos\phi{*}\cos\theta_2' \\
\sin\theta_2\sin\phi_2 = \cos\theta_2{*}\sin\phi{*}\sin\theta_2'\cos\phi_2' + \\
\qquad \cos\phi{*}\sin\theta_2'\sin\phi_2' + \\
\qquad \sin\theta_2{*}\sin\phi{*}\cos\theta_2' \\
\cos\theta_2 = -\sin\theta_2{*}\sin\theta_2'\cos\phi_2' + \\
\qquad \cos\theta_2{*}\cos\theta_2'
\end{cases}
$$

Averaging Over Solid Angle

The integration in Equation (7) must be performed numerically. As shown in Figure V, each detector surface is divided into small sectors of an annulus subtending equal solid angle, and the value of W at the center of each sector is calculated.

Figure V

Examples of Dividing Detector Surface



The center of each sector is determined by the set of coordinates (θ_1', ϕ_1') or (θ_2', ϕ_2'), where θ_1' and ϕ_1' are given by

$$\cos \theta_1' = 1 - \left(\frac{m_1^2 - m_1 + 1/2}{n^2} \right) (1 - \cos \theta_o) ,$$

$$m_1 = 1, 2, \ldots, n,$$

$$\phi_1' = \frac{360^\circ (K_1 - 1/2)}{8m_1 - 4} ,$$

$$K_1 = 1, 2, \ldots, 8m_1 - 4,$$

and θ_2' and ϕ_2' are given by the same equations except for the subscript being changed to 2. The total number of segments determined by n is equal to $4n^2$.

In all cases, $n = 5$, $4n^2 = 100$ sectors, is used because of the computer calculation time except $\theta_o = 0^o$, where $n = 1$ is used. For example, to calculate W in $J = 3 \rightarrow 2 \rightarrow 1$ case, $n = 10$ requires approximately 16 times more CPU time than $n = 5$, while it makes less than 0.001% difference between obtained values of W.

Search Program

In order to find the values of θ_1^*, θ_2^* and ϕ^* which satisfy Equation (7), the following procedure is used.

Consider $J = 3 \rightarrow 2 \rightarrow 1$ case, which is the most complicated case, since there are three parameters to vary. Equation (7) in this case is

$$\int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;0)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

$$= \int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;1)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

$$= \int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;2)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

$$= \int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;3)\,d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

Let D_1 be the difference between the first expression and the second, i.e.,

$$D_1 = \int_{\Omega_1}\int_{\Omega_2} A(\theta_1, \theta_2, \phi ; 1)\, d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

$$- \int_{\Omega_1}\int_{\Omega_2} A(\theta_1, \theta_2, \phi ; 0)\, d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

D_2 be the difference between the second and the third, and D_3 be the difference between the third and the fourth. D_1, D_2 and D_3 are functions of three variables θ_1, θ_2 and ϕ,

$$D_1 = D_1 (\theta_1, \theta_2, \phi),$$

$$D_2 = D_2 (\theta_1, \theta_2, \phi),$$

$$D_3 = D_3 (\theta_1, \theta_2, \phi).$$

The search is initiated by substituting a trial guess for the values of θ_1^*, θ_2^* and ϕ^* into D's. If the substitution makes D's equal to zero, then these trial guesses are the required values of θ_1^*, θ_2^* and ϕ^*, and the search is not necessary. Otherwise, an iteration procedure must be developed to improve initial guesses. Suppose substitution yields non-zero values of D's,

Δ_1, Δ_2, and Δ_3. Corrections in θ_1, θ_2 and ϕ are wanted to counteract Δ_1, Δ_2 and Δ_3 so that the D's vanish. Changes in D's, ΔD_1, ΔD_2 and ΔD_3 are related to these corrections in angles, $\Delta\theta_1$, $\Delta\theta_2$ and $\Delta\phi$ as

$$\Delta D_1 = \frac{\partial D_1}{\partial \theta_1}\Delta\theta_1 + \frac{\partial D_1}{\partial \theta_2}\Delta\theta_2 + \frac{\partial D_1}{\partial \phi}\Delta\phi ,$$

$$\Delta D_2 = \frac{\partial D_2}{\partial \theta_1}\Delta\theta_1 + \frac{\partial D_2}{\partial \theta_2}\Delta\theta_2 + \frac{\partial D_2}{\partial \phi}\Delta\phi ,$$

$$\Delta D_3 = \frac{\partial D_3}{\partial \theta_1}\Delta\theta_1 + \frac{\partial D_3}{\partial \theta_2}\Delta\theta_2 + \frac{\partial D_3}{\partial \phi}\Delta\phi .$$

Therefore, desired corrections can be found by solving following linear equations.

$$\frac{\partial D_1}{\partial \theta_1}\Delta\theta_1 + \frac{\partial D_1}{\partial \theta_2}\Delta\theta_2 + \frac{\partial D_1}{\partial \phi}\Delta\phi = -\Delta_1 ,$$

$$\frac{\partial D_2}{\partial \theta_1}\Delta\theta_1 + \frac{\partial D_2}{\partial \theta_2}\Delta\theta_2 + \frac{\partial D_2}{\partial \phi}\Delta\phi = -\Delta_2 ,$$

$$\frac{\partial D_3}{\partial \theta_1}\Delta\theta_1 + \frac{\partial D_3}{\partial \theta_2}\Delta\theta_2 + \frac{\partial D_3}{\partial \phi}\Delta\phi = -\Delta_3 .$$

Partial derivatives $\frac{\partial D_1}{\partial \theta_1}, \frac{\partial D_1}{\partial \theta_2}, \ldots, \frac{\partial D_3}{\partial \phi}$ can be evaluated analytically, although it would not be simple. For simplicity, these derivatives are calculated numerically as

$$\frac{\partial D_1}{\partial \theta_1} = \frac{D_1(\theta_1 + \xi, \theta_2, \phi) - D_1(\theta_1, \theta_2, \phi)}{\xi},$$

$$\frac{\partial D_1}{\partial \theta_2} = \frac{D_1(\theta_1, \theta_2 + \xi, \phi) - D_1(\theta_1, \theta_2, \phi)}{\xi},$$

and so on, where ξ represents a small change in angle.

The solution of linear equations is well known. In this program, subroutines SGECO and SGESL are called from the LINPAK package.[10] After $\Delta\theta_1$, $\Delta\theta_2$ and $\Delta\phi$ are found, new guesses are given by

$$\theta_1(\text{new}) = \theta_1(\text{old}) + \Delta\theta_1,$$

$$\theta_2(\text{new}) = \theta_2(\text{old}) + \Delta\theta_2,$$

$$\phi(\text{new}) = \phi(\text{old}) + \Delta\phi,$$

and the same procedure is repeated until all D's become sufficiently small. In our case, D's become less than 10^{-5} after 3 iterations, where the initial ξ is 1 degree, and ξ is reduced by a factor of 10 in each iteration.

CHAPTER IV

Results

Possible detector geometries are searched to reduce $\overline{D}$ and $\overline{W}$ in the following cases.

$$J = 1 \rightarrow 1 \rightarrow 0$$
$$J = 2 \rightarrow 1 \rightarrow 0$$
$$J = 3 \rightarrow 2 \rightarrow 1$$

$\underline{J = 1 \rightarrow 1 \rightarrow 0}$

In this case, the angular-correlation function given by Equation (6) is (See Appendix A)

$$W_{1 \rightarrow 1 \rightarrow 0} (\theta_1, \theta_2, \phi) = p(o) A (\theta_1, \theta_2, \phi ; o) + p(1) A (\theta_1, \theta_2, \phi ; 1),$$

where

$$A (\theta_1, \theta_2, \phi ; o) = \frac{9}{16} (1+\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta_1 \sin^2 \theta_2 \cos 2 \phi),$$

$$A (\theta_1, \theta_2, \phi ; 1) = \frac{9}{8} (1 - \cos^2 \theta_1 \cos^2 \theta_2 - \sin \theta_1 \cos \theta_1 \sin \theta_2 \cos \theta_2 \cos \phi).$$

To make $\overline{W}$ equal to a constant regardless of P(o) and p(1), Equation (7) must be satisfied, which in this case is

$$\frac{\int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;o)\,d\Omega_1\,d\Omega_2}{\int_{\Omega_1}\int_{\Omega_2} d\Omega_1\,d\Omega_2} = \frac{\int_{\Omega_1}\int_{\Omega_2} A(\theta_1,\theta_2,\phi\,;1)\,d\Omega_1\,d\Omega_2}{\int_{\Omega_1}\int_{\Omega_2} d\Omega_1\,d\Omega_2}$$

Since there are three parameters to vary, two of them can be fixed.

Three different geometries are studied here. These are chosen to be applicable to actual experiment settings.

In the first geometry, $\theta_1{}^*$ is set equal to $\theta_2{}^*$, and ϕ^* is fixed at 90°. The values of $\theta_1{}^*$ and $\theta_2{}^*$ which satisfy Equation (7) for detector half cone angles 0°, 10°, 20° and 30° were searched. Results are shown in Table I with corresponding constant $\overline{W}$ values.

TABLE I

Detector Geometry For

$J = 1 \rightarrow 1 \rightarrow 0$, $\theta_1{}^* = \theta_2{}^*$ and $\phi^* = 90°$

θ_0	$\theta_1{}^* = \theta_2{}^*$	$\overline{W}$
0°	90.00°	1.125
10°	73.88°	1.117
20°	68.20°	1.098
30°	64.40°	1.073

In the second geometry, θ_2^* is set equal to $\cos^{-1}(1/\sqrt{3})$ so that $\bar{D}_2$ is equal to one, θ_1^* is set equal to θ_2^*, and the value of ϕ^* which satisfies Equation (7) is searched. Results are shown in Table II. Note that in this case, even though the detector half cone angle θ_0 varies, the value of ϕ^* remains the same.

TABLE II

Detector Geometry For

$J = 1 \rightarrow 1 \rightarrow 0, \quad \theta_1^* = \theta_2^* = \cos^{-1}(1/\sqrt{3})$

θ_0	ϕ^*	$\bar{W}$
0°	120.0°	1.125
10°	120.0°	1.119
20°	120.0°	1.104
30°	120.0°	1.082

In the third geometry, θ_2^* is set equal to $\cos^{-1}(1/\sqrt{3})$, and ϕ^* is fixed at 90°.

Results are shown in Table III.

TABLE III

Detector Geometry For

$J = 1 \rightarrow 1 \rightarrow 0, \quad \theta_2^* = \cos^{-1}(1/\sqrt{3}), \quad \phi^* = 90°$

θ_0	θ_1^*	$\overline{W}$
0°	90.00°	1.125
10°	83.42°	1.118
20°	78.11°	1.099
30°	72.78°	1.074

$J = 2 \rightarrow 1 \rightarrow 0$

In this case, the angular-correlation function is

$$W_{2 \rightarrow 1 \rightarrow 0}(\theta_1, \theta_2, \phi) = p(0)\, A(\theta_1, \theta_2, \phi; 0) + p(1)\, A(\theta_1, \theta_2, \phi; 1) + p(2)\, A(\theta_1, \theta_2, \phi; 2)$$

where

$$A(\theta_1, \theta_2, \phi; 0) = \frac{3}{16}(9 - 7\cos^2\theta_1 - 7\cos^2\theta_2 + 9\cos^2\theta_1 \cos^2\theta_2 - 8\sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \cos\phi + \sin^2\theta_1 \sin^2\theta_2 \cos 2\phi),$$

$$A(\theta_1, \theta_2, \phi; 1) = \frac{9}{8}(1 - \cos^2\theta_1 \cos^2\theta_2 + \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \cos\phi),$$

$$A(\theta_1, \theta_2, \phi; 2) = \frac{9}{16}(1 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_1 \cos^2\theta_2).$$

Since one parameter can be fixed, $\theta_2{}^*$ is set equal to $\cos^{-1}(1/\sqrt{3})$ so that $\bar{D}_2$ is equal to one, and the values of $\theta_1{}^*$ and ϕ^* which satisfy Equation (7) are searched. Results are shown in Table IV.

TABLE IV

Detector Geometry For

$J = 2 \rightarrow 1 \rightarrow 0, \quad \theta_2{}^* = \cos^{-1}(1/\sqrt{3})$.

θ_0	$\theta_1{}^*$	ϕ^*	$\bar{W}$
0°	56.46	97.41	0.9790
10°	56.41	97.39	0.9800
20°	56.30	97.36	0.9826
30°	56.11	97.32	0.9864

$\underline{J = 3 \rightarrow 2 \rightarrow 1}$

In this case, the angular-correlation function is

$$W_{3\to 2\to 1}(\theta_1, \theta_2, \phi) = p(0)\, A(\theta_1, \theta_2, \phi; 0) + p(1)\, A(\theta_1, \theta_2, \phi; 1) + p(2)\, A(\theta_1, \theta_2, \phi; 2) + p(3)\, A(\theta_1, \theta_2, \phi; 3),$$

where

$$A(\theta_1, \theta_2, \phi; 0) = \frac{9}{80}(13 - 7\cos^2\theta_1 - 7\cos^2\theta_2 + 5\cos^2\theta_1\cos^2\theta_2 - 4\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\cos\phi + \sin^2\theta_1\sin^2\theta_2\cos 2\phi),$$

$$A(\theta_1, \theta_2, \phi; 1) = \frac{3}{80}(35 - 13\cos^2\theta_1 - 13\cos^2\theta_2 + 3\cos^2\theta_1\cos^2\theta_2 + 2\sin^2\theta_1\sin^2\theta_2\cos 2\phi),$$

$$A(\theta_1, \theta_2, \phi; 2) = \frac{3}{16}(5 + \cos^2\theta_1 + \cos^2\theta_2 - 3\cos^2\theta_1\cos^2\theta_2 + 4\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\cos\phi),$$

$$A(\theta_1, \theta_2, \phi; 3) = \frac{9}{16}(1 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_1\cos^2\theta_2)$$

Here, all parameters must be varied to satisfy Equation (7). Since $\theta_2{}^*$ can no longer be set equal to $\cos^{-1}(1/\sqrt{3})$,

$\bar{D}_2$ must be determined separately. The values of $\theta_1{}^*$, $\theta_2{}^*$ and ϕ^* are searched within a practical range, and results are shown in Table V.

TABLE V

Detector Geometry For

$J = 3 \rightarrow 2 \rightarrow 1$

θ_0	$\theta_1{}^*$	$\theta_2{}^*$	ϕ^*	$\bar{W}$
0°	60.93	50.39	99.59	0.9779
10°	61.42	49.89	100.04	0.9787
20°	62.29	48.98	100.95	0.9812
30°	63.04	48.15	101.86	0.9850

$J = 1 \rightarrow 1 \rightarrow 0$ and $J = 2 \rightarrow 1 \rightarrow 0$

Although a separate polarization measurement is necessary to determine $\bar{D}_2$, it is possible to find values of $\theta_1{}^*$, $\theta_2{}^*$ and ϕ^* which satisfy Equation (7) in both $J = 1 \rightarrow 1 \rightarrow 0$ case and $J = 2 \rightarrow 1 \rightarrow 0$ case simultaneously. One example is shown in Table VI to show this possibility.

TABLE VI

Detector Geometry For

$J = 1 \rightarrow 1 \rightarrow 0$ and $J = 2 \rightarrow 1 \rightarrow 0$

θ_0	$\theta_1{}^*$	$\theta_2{}^*$	ϕ^*	$\bar{W}$ $1 \rightarrow 1 \rightarrow 0$	$\bar{W}$ $2 \rightarrow 1 \rightarrow 0$
0°	69.51	42.47	114.09	1.125	0.9750

Symmetry

If θ_1^* and θ_2^* are interchanged, Equation (7) is still satisfied and $\overline{W}$ remains the same. This is obvious from each functional form of $A(\theta_1, \theta_2, \phi; m)$.

In addition, it is found that changing any two of three parameters to their supplement keeps results unchanged, i.e., Equation (7) is still satisfied by new parameters and $\overline{W}$ remains the same. This is valid in all cases, and for any detector solid angle.

Examples are shown in Tables VII and VIII.

TABLE VII

Possible Variations For

$J = 1 \rightarrow 1 \rightarrow 0$

θ_0	θ_1^*	θ_2^*	ϕ^*	$\overline{W}$	Variation
20°	78.1°	54.74°	90.0°	1.099	θ_1^*, θ_2^*, ϕ^*
20°	101.9°	125.26°	90.0°	1.099	$\overline{\theta_1}^*$, $\overline{\theta_2}^*$, ϕ^*
20°	101.9°	54.74°	90.0°	1.099	$\overline{\theta_1}^*$, θ_2^*, $\overline{\phi}^*$
20°	78.1°	125.26°	90.0°	1.099	θ_1^*, $\overline{\theta_2}^*$, $\overline{\phi}^*$

Note. In this case, the supplement of ϕ^* is ϕ^* itself, $\overline{\phi}^* = \phi^*$. $\cos^{-1}(1/\sqrt{3}) = 54.74°$. $\overline{\theta}1^*$ means the supplement of θ_1^*.

TABLE VIII

Possible Variations For

$J = 3 \rightarrow 2 \rightarrow 1$.

θ_0	$\theta_1{}^*$	$\theta_2{}^*$	ϕ^*	$\overline{W}$	Variation
0°	60.93	50.39	99.59	0.9779	$\theta_1{}^*$, $\theta_2{}^*$, ϕ^*
0°	119.07	129.61	99.59	0.9779	$\overline{\theta_1}{}^*$, $\overline{\theta_2}{}^*$, ϕ^*
0°	119.07	50.39	80.41	0.9779	$\overline{\theta_1}{}^*$, $\theta_2{}^*$, $\overline{\phi}^*$
0°	60.93	129.61	80.41	0.9779	$\theta_1{}^*$, $\overline{\theta_2}{}^*$, $\overline{\phi}^*$
0°	50.39	60.93	99.59	0.9779	$\theta_1{}^* \leftrightarrow \theta_2{}^*$, ϕ^*
0°	129.61	119.07	99.59	0.9779	$\overline{\theta_1}{}^* \leftrightarrow \overline{\theta_2}{}^*$, ϕ^*
0°	50.39	119.07	80.41	0.9779	$\overline{\theta_1}{}^* \leftrightarrow \theta_2{}^*$, ϕ^*
0°	129.61	60.93	80.41	0.9779	$\theta_1{}^* \leftrightarrow \overline{\theta_2}{}^*$, $\overline{\phi}^*$,

Note that in the geometry such as shown in Table VII, where $\theta_2{}^*$ is set equal to $\cos^{-1}(1/\sqrt{3})$ so that $\overline{D}_2$ is equal to one, by using the supplement of $\theta_2{}^*$, one still obtains $\overline{D}_2 = 1$. This comes about because in Equation (4), $\cos^2\theta_2$ appears rather than $\cos\theta_2$, so that although $\cos(\pi - \theta_2) = -\cos\theta_2$, the final result is not affected. On the other hand, interchanging $\theta_1{}^*$ and $\theta_2{}^*$ destroys $\overline{D}_2$ being equal to one, although it still keeps $\overline{W}$ unchanged.

In Table VIII, where $\overline{D}_2$ can no longer be set equal to one, all possible variations are shown.

Chosing their supplements rather than θ_1* and θ_2* themselves may be favorable if the apparatus has a design similar to that used by Kaul [1], where a truncated cone-shaped input collimator and truncated cone-shaped electrode define each end of the interaction region.

CHAPTER V

Experimental Errors

In an actual experiment it is impossible to set up the detectors at the desired angles θ_1^*, θ_2^*, ϕ^* with absolute precision. In addition the effective half angle of the detectors (θ_0) may not be the same as that determined geometrically. Then, the calculated values of $\overline{W}$ that have been tabulated would be in error. In addition, the value used for $\overline{D}_2$ may be in error. The question addressed in this section is how large these errors might be.

The averaged angular-correlation function $\overline{W}$ is given by

$$\overline{W} = \sum_m p(m)\, \overline{A}\,(m),$$

where

$$\overline{A}\,(m) = \int_{\Omega_1}\int_{\Omega_2} A(\theta_1, \theta_2, \phi\,;\, m)\, d\Omega_1 d\Omega_2 \Big/ \int_{\Omega_1}\int_{\Omega_2} d\Omega_1 d\Omega_2$$

Thus, $\overline{A}(m)$ is a function of θ_0 as well as of θ_1^*, θ_2^* and ϕ^*. For small changes in angle

$$\Delta\overline{W} = \Big(\sum_m p(m) \frac{\partial A(m)}{\partial \theta_0} \Big) \Delta\theta_0 + \Big(\sum_m p(m) \frac{\partial A(m)}{\partial \theta_1^*} \Big) \Delta\theta_1^*$$

$$+ \Big(\sum_m p(m) \frac{\partial A(m)}{\partial \theta_2^*} \Big) \Delta\theta_2^* + \Big(\sum_m p(m) \frac{\partial A(m)}{\partial \phi^*} \Big) \Delta\phi^* .$$

A sample of the required partial derivatives is given in Tables IX through XI. These partial derivatives are estimated as

$$\frac{\partial \bar{A}(o)}{\partial \theta_1} = \frac{\bar{A}(\theta_1^* + 1°, \theta_2^*, \phi^*; o) - \bar{A}(\theta_1^*, \theta_2^*, \phi^*; o)}{1°},$$

etc. Since they involve averaging over solid angle, the derivatives had to be evaluated numerically.

As an example of the magnitude of the error that might arise, a $J = 2 \rightarrow 1 \rightarrow 0$ cascade is considered here. It is assumed that the experimenter believes one has detectors with half angle $\theta_0 = 30°$ and has optimum detector geometry as given in Table IV, i.e., $\theta_1^* = 56.11°$, $\theta_2^* = 54.74°$, $\phi^* = 97.32°$. Table IV gives the value $\bar{W} = 0.9864$ for the averaged angular-correlation function.

Now the following question is asked. Suppose the experimenter has overestimated the effective solid angle of one's detector and that the true value is $\theta_0 = 20°$. Suppose in addition that the experiment has made a $\pm 3°$ error in θ_1^*, θ_2^* and ϕ^* in such a way that will lead to maximal error in $\bar{W}$. The errors in experimental set up mentioned above are larger than a careful experimenter should expect to make. Now consider what percentage error in $\bar{W}$ will result.

Using the derivatives in Table X one obtains

$$\Delta\overline{W} = (.000604\ p(o) + .00504\ p(1) + .000203\ p(2))\ \Delta\theta_0$$
$$+ (.008047\ p(o) + .005162\ p(1) - .009717\ p(2))\ \Delta\theta_1^*$$
$$+ (.008624\ p(o) + .004948\ p(1) - .009772\ p(2))\ \Delta\theta_2^*$$
$$+ (.004222\ p(o) - .002770\ p(1))\ \Delta\phi^*,$$

where $\Delta\theta_0$, $\Delta\theta_1^*$, $\Delta\theta_2^*$, $\Delta\phi^*$ are given in degrees. Using the relation $p_2 = 1 - p_0 - p_1$, dividing by $\overline{W}$, and multiplying by 100, one obtains the percentage change in $\overline{W}$

$$\%\text{ change in } \overline{W} = 100 \cdot \frac{\Delta\overline{W}}{\overline{W}}$$
$$= (.041\ p(o) + .031\ p(1) + .021)\ \Delta\theta_0$$
$$+ (1.801\ p(o) + 1.508\ p\ (1) - .985)\ \Delta\theta_1^*$$
$$+ (1.865\ p(o) + 1.492\ p(1) - .991)\ \Delta\theta_2^*$$
$$+ (.428\ p(o) - .281\ p(1))\ \Delta\phi^*$$

With no knowledge of the relative populations of the sublevels of the upper state, the limits on p(o) and P(1) are

$$0 \leqq p(o) \leqq 1, \quad 0 \leqq p(1) \leqq 1,$$
$$p(o) + p(1) \leqq 1.$$

It has been postulated that $\Delta\theta_0 = -10°$ and $\pm 3°$ for $\Delta\theta_1^*$, $\Delta\theta_2^*$ and $\Delta\phi^*$. Some contemplation and sample calculations show that maximal error will occur with p(o) = 1, p(1) = 0, and $\Delta\theta_1^* = \Delta\theta_2^* = \Delta\phi^* = -3°$. The maximum percentage change predicted for $\overline{W}$ is then - 7.0%. An exact calculation of $\overline{W}$ for $\theta_0 = 20°$, $\theta_1^* = 53.11°$, $\theta_2^* = 51.74°$, $\phi^* = 94.32°$ with p(o) = 1 and p(1) = p(2) = 0 gives $\overline{W}$ = .9104 which is 7.7% below the value .9864 for $\overline{W}$ that the experimenter would be using in the reduction of one's data. An alternate view is that the experimenter would be using a value of $\overline{W}$ that was 8.3% too high in the reduction of one's data. It is seen here that the derivative in Tables IX, X and XI are useful in estimating possible errors in $\overline{W}$.

TABLE IX

Partial Derivatives For

$J = 1 \rightarrow 1 \rightarrow 0$ in $(\text{deg})^{-1}$

	$\partial\overline{A}(o)$	$\partial\overline{A}(1)$
$\partial\theta_0$	-.002422	-.002422
$\partial\theta_1$	-.007905	.003898
$\partial\theta_2$	-.007905	.003897
$\partial\phi$	-.004981	.002454

$\theta_0 = 30°$, $\theta_1^* = 54.74°$, $\theta_2^* = 54.74°$

$\phi^* = 120.0°$, $\overline{W} = 1.082$

TABLE X

Partial Derivatives For

$J = 2 \rightarrow 1 \rightarrow 0$ in $(\deg)^{-1}$

	$\partial \bar{A}(o)$	$\partial \bar{A}(1)$	$\partial \bar{A}(2)$
$\partial\theta_0$	.000604	.000504	.000203
$\partial\theta_1$	.008047	.005162	-.009717
$\partial\theta_2$	.008624	.004948	-.009772
$\partial\phi$	.004222	-.002770	0

$\theta_0 = 30°$, $\theta_1{}^* = 56.11°$, $\theta_2{}^* = 54.74°$

$\phi^* = 97.32°$, $\overline{W} = 0.9864$

TABLE XI

Partial Derivatives For

$J = 3 \rightarrow 2 \rightarrow 1$ in $(\deg)^{-1}$

	$\partial \bar{A}(o)$	$\partial \bar{A}(1)$	$\partial \bar{A}(2)$	$\partial \bar{A}(3)$
$\partial\theta_0$	.000588	.000531	.000397	.000301
$\partial\theta_1$	.005378	.004601	.001039	-.009007
$\partial\theta_2$	.008224	.005839	-.000721	-.009676
$\partial\phi$	.001478	.000315	-.001676	0

$\theta_0 = 30°$, $\theta_1{}^* = 63.04$, $\theta_2{}^* = 48.15$,

$\phi^* = 101.86$, $\overline{W} = 0.9850$

The angular distribution factor $\bar{D}_2$ is also needed in the data reduction. The factor $\bar{D}_2$ is given by

$$\bar{D}_2 = \frac{3\,(1-P\,\overline{\cos^2\theta_2})}{3-P}$$

where P is the polarization of the light emitted at $\theta_2 = 90°$ in the intermediate state to lower state decay. In general

$$\Delta\bar{D}_2 = \frac{-3P}{3-P}\left(\frac{\partial\,\overline{\cos^2\theta_2}}{\partial\theta_0}\,\Delta\theta_0 + \frac{\partial\,\overline{\cos^2\theta_2}}{\partial\theta_{2^*}}\,\Delta\theta_{2^*}\right).$$

Then

$$\frac{\Delta\bar{D}_2}{\bar{D}_2} = \frac{-P}{1-P\,\overline{\cos^2\theta_2}}\left(\frac{\partial\,\overline{\cos^2\theta_2}}{\partial\theta_0}\,\Delta\theta_0 + \frac{\partial\,\overline{\cos^2\theta_2}}{\partial\theta_{2^*}}\,\Delta\theta_{2^*}\right).$$

It must be remembered that it is the ratio $\frac{\bar{W}}{\bar{D}_2}$ that is used in data reduction. Errors in experimental set-up that maximize the error in $\bar{W}$ do not necessarily maximize the error in $\frac{\bar{W}}{\bar{D}_2}$. The fractional change in the ratio $\frac{\overline{W}}{\overline{D}_2}$ is given by

$$\frac{\Delta\left(\frac{\overline{W}}{\overline{D}_2}\right)}{\frac{\overline{W}}{\overline{D}_2}} = \frac{\Delta \overline{W}}{\overline{W}} - \frac{\Delta \overline{D}_2}{\overline{D}_2}$$

In the particular case considered here, i.e., a $J = 2 \rightarrow 1 \rightarrow 0$ cascade, $\theta_2{}^*$ has been chosen as 54.74° so that $\overline{D}_2 = 1$ and $\overline{\cos^2\theta_2}$ is independent of θ_0. Then

$$\frac{\Delta \overline{D}_2}{\overline{D}_2} = \frac{-3P}{3-P} \frac{\partial\, \overline{\cos^2\theta_2}}{\partial \theta_2{}^*} \Delta\theta_2{}^*$$

substituting the appropriate derivative from Table XII and multiplying by 100 one obtains the percentage change in $\overline{D}_2$

$$\%\text{ change in } \overline{D}_2 = 100 \cdot \frac{\Delta \overline{D}_2}{\overline{D}_2} = \frac{3P}{3-P} \cdot 1.321 \cdot \Delta\theta_2{}^*$$

Combining the above expression with Equation (8) one obtains

$$\%\text{ change in } \frac{\overline{W}}{\overline{D}_2} = 100 \cdot \frac{\Delta\left(\frac{\overline{W}}{\overline{D}_2}\right)}{\frac{\overline{W}}{\overline{D}_2}} = 100 \cdot \frac{\Delta \overline{W}}{\overline{W}} - 100 \cdot \frac{\Delta \overline{D}_2}{\overline{D}_2}$$

$$\begin{aligned}
&= (.041\ p(o) + .031\ p(1) + .021)\ \Delta\theta_0 \\
&+ (1.801\ p(o) + 1.508\ p(1) - .985)\ \Delta\theta_1{}^* \\
&+ (1.865\ p(o) + 1.492\ p(1) - .991 - \tfrac{3P}{3-P} \cdot 1.321)\ \Delta\theta_2{}^* \\
&+ (.428\ p(o) - .281\ p(1)\ )\ \Delta\phi^*.
\end{aligned}$$

TABLE XII

Partial Derivatives At

$\theta_2^* = 54.74°$ in $(\text{deg})^{-1}$

θ_0	$\dfrac{\partial\, \overline{(\cos^2 \theta_2)}}{\partial\, \theta_2^*}$
0°	-0.01635
10°	-0.01598
20°	-0.01490
30°	-0.01321

TABLE XIII

Relation Between P' and the Populations

of the Sublevels of the Upper State.

$J = 1 \rightarrow 1$	$P' = \dfrac{-2\,p(o) + p(1)}{2\,p(o) + 3\,p(1)}$
$J = 2 \rightarrow 1$	$P' = \dfrac{6\,p(o) + 3\,p(1) - 6\,p(2)}{10\,p(o) + 9\,p(1) + 6\,p(2)}$
$J = 3 \rightarrow 2$	$P' = \dfrac{12\,p(o) + 9\,p(1) - 15\,p(3)}{24\,p(o) + 23\,p(1) + 20\,p(2) + 15\,p(3)}$

It is assumed that in addition to having no knowledge of the relative populations of the sublevels of the upper state one also has no knowledge of the polarization of the light emitted in the intermediate state to lower state decay. Then P may range from -1 to +1, i.e., $-1 \leq P \leq +1$. With $\Delta \theta_0 = -10°$, sample calculations show that maximal error will occur with $p(o) = 0$, $p(1) = 0$ (implying $p(2) = 1$), $P = +1$, and $\Delta \theta_1{}^* = \Delta \theta_2{}^* = +3°$. (With $p(o) = p(1) = 0$ the error is independent of $\Delta \phi$ *.) The maximum percentage change predicted for $\frac{\overline{W}}{\overline{D}_2}$ is then -12.1%. An exact calculation of $\frac{\overline{W}}{\overline{D}_2}$ for $\theta_0 = 20°$, $\theta_1{}^* = 59.11°$, $\theta_2{}^* = 57.74°$ with $p(o) = p(1) = 0$, $p(2) = 1$, and $P = +1$ gives $\frac{\overline{W}}{\overline{D}_2} = .8638$ which is 12.4% below the value .9864 for $\frac{\overline{W}}{\overline{D}_2}$ that the experimenter would be using in the reduction of one's data. An alternate view is that the experimenter would be using a value of $\frac{\overline{W}}{\overline{D}_2}$ that was 14.2% too high in the reduction of one's data. It is seen that the derivatives are useful in estimating possible errors in $\frac{\overline{W}}{\overline{D}_2}$.

A considerable improvement may result in our maximal error limits if the polarization of radiation (at $\theta_1 = 90°$ and $\theta_2 = 90°$) is measured for the upper and lower transitions of the cascade. For instance, if P is less than 100%, a smaller error in $\overline{D}_2$ will result than that estimated above for the $J = 2 \rightarrow 1 \rightarrow 0$ cascade. In fact, if it is found that $P = 0$ for the lower portion of the cascade ($J = 1 \rightarrow 0$), then no error in $\overline{D}_2$ will result from errors in detector geometry.

The error limit for $\overline{W}$ may be decreased if P', the polarization for the upper portion of the cascade is measured. Measurement of P' provides one with an additional constraint on the populations of the sublevels of the upper state and may reduce the maximal error for $\overline{W}$. Table XIII gives the relation between P' and the populations of the sublevels of the upper state for $J = 1 \rightarrow 1$, $J = 2 \rightarrow 1$, and $J = 3 \rightarrow 2$ transitions which are the upper transitions of the three cascades considered in this thesis. These relations are obtained from the dipole ($\overline{p}$) matrix elements.[11]

Consider a $J = 2 \rightarrow 1 \rightarrow 0$ again. The polarization P' for the $J = 2 \rightarrow 1$ transition is given by

$$P' = \frac{6\,p(o) + 3\,p(1) - 6\,p(2)}{10\,p(o) + 9\,p(1) + 6\,p(2)}$$

Using the relation $p(2) = 1 - p(o) - p(1)$ one obtains

$$P' = \frac{3\,(\,4\,p(o) + 3\,p(1) - 2)}{(\,4\,p(o) + 3\,p(1) + 6)}$$

or

$$P' = \frac{3\ (X-2)}{(X+6)}$$

where

$$X = 4\ p(o) + 3\ p(1).$$

Here, a constraining relation between p(o) and p(1) is seen. One might think that if $P' = 0$ one could guarantee equal populations of the sublevels, i.e., p(o) = 1/5, p(1) = p(2) = 2/5. This is not necessarily so although the inverse is true. One might also think that if one measured the polarizations for $J = 2 \rightarrow 2$ and $J = 2 \rightarrow 3$ transitions originating from the same upper state for which one measured P' that additional constraints would result which would uniquely determine p(0), p(1), and p(2). This is not the case. The polarizations for $J = 2 \rightarrow 2$ and $J = 2 \rightarrow 3$ depend only on X = 4 p(o) + 3 p (1). The polarizations are not independent and no additional information results. This comes about in general because the alignment of the upper excited state created by electron bombardment is given by $\langle 3\ J_z^2 - J^2 \rangle$. This can be specified by one parameter and in turn determines the polarization of radiation in any decay transition. Conversely, the measurement of the polarization for one decay transition determines the alignment uniquely even though it may not specify the relative sublevel populations completely. Subsequent measurements of polarization for other decay transitions will furnish no additional information.

To proceed further assume that a measurement of P' gives $P' = 0$. Now find the maximal error limit for $\overline{W}$. If $P' = 0$ then $4\,p(0) + 3\,p(1) = 2$ or $p(1) = (2/3) - (4/3)\,p(0)$. Substituting this relation into the previous expression for the % change in $\overline{W}$ one obtains

$$\%\text{ change in } \overline{W} = 100 \cdot \frac{\Delta \overline{W}}{\overline{W}}$$

$$\begin{aligned} &= (.041)\ \Delta\theta_0 \\ &+ (-.210\,p(0) + .020)\ \Delta\theta_1^* \\ &+ (-.125\,p(0) + .004)\ \Delta\theta_2^* \\ &+ (.802\,p(0) - .187)\ \Delta\phi^* \end{aligned}$$

With $P' = 0$ the limits on $p(0)$ are $0 \leq p(0) \leq \frac{1}{2}$. With $\Delta\theta_0 = -10°$, sample calculations show that maximal error will occur with $p(0) = \frac{1}{2}$ (implying $p(1) = 0$, $p(2) = \frac{1}{2}$), $\Delta\theta_1^* = \Delta\theta_2^* = +3°$, and $\Delta\phi^* = -3°$. The maximal percentage change predicted for $\overline{W}$ is then -1.5%. An exact calculation for $\overline{W}$ for $\theta_0 = 20°$, $\theta_1^* = 59.11°$, $\theta_2^* = 57.74°$, $\phi^* = 94.32°$ with $p(0) = \frac{1}{2}$, $p(1) = 0$, $p(2) = \frac{1}{2}$ gives $\overline{W} = .9719$ which is

1.5% below the value .9864 for $\overline{W}$ that the experimenter would be using in the reduction of one's data. An alternate view is that the experimenter would be using a value of $\overline{W}$ that was 1.5% too high in the reduction of one's data.

It is seen that a measurement of the polarizations P and P' may reduce the maximal error limits for $\overline{D}_2$ and $\overline{W}$ respectively. It may well be worth the experimenter's time to make a crude measurement of these polarizations if they are not available from previous experiments. One should recall that the P and P' used here are the polarizations for γ_2 photon and γ_1 photon measured at $\theta_2 = 90°$ and $\theta_1 = 90°$ respectively. If the polarization is measured at an angle θ other than 90° then the polarization at 90° can be obtained by

$$P(90°) = \frac{P(\theta)}{P(\theta)\cos^2\theta + \sin^2\theta} \; .$$

CHAPTER VI

Conclusions

It was shown in this work that the appropriate detector geometry can be used to avoid separate determination of angular distribution and correlation factors required in the determination of an absolute excitation cross section.

This method is applicable up to $J = 3$ for the angular correlation factor to be known, and up to $J = 2$ for both angular distribution and correlation factors to be known. Required geometries were searched for integral J. By means of this method, it is also possible to find the appropriate geometries for the half-integral J.

It was also shown that the possible errors due to the misalignment of the photon detectors may be reduced considerably if the polarization of radiation is measured for the upper and lower transitions of the cascade.

FOOTNOTES

1. Kaul, R. D. Measurement of excitation cross section of the 6 $^3P_1$ state of Hg using self-calibrating apparatus. Journal of the Optical Society of America, 1979, 69, 150-158.

2. Condon, E. U., & Shortley, G. H. The theory of atomic spectra. Cambridge, England: Cambridge University, 1963, 90.

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APPENDICES

APPENDIX A

Sample Calculations to Obtain W

In general, the angular-correlation function is given by [see Equation (6)].

$$W(\theta_1, \theta_2, \phi) = \sum_m p(m)\, A(\theta_1, \theta_2, \phi; m),$$

where

$$A(\theta_1, \theta_2, \phi; m) = \sum_{KMN} 2\, C^N_{KM}(m)\, \bar{P}^N_K(\cos\theta_1)\, \bar{P}^N_M(\cos\theta_2) \cos(N\phi).$$

Consider $J = 2 \rightarrow 1 \rightarrow 0$ case, and consider $m = 0$. From the table of coefficients[1], required C^N_{KM} $(m = 0)$ is as follows:

TABLE A

Coefficients for the Electric Dipole Radiation

K	M	N	C^N_{KM} (m = 0)
0	0	0	+1.00000
0	2	0	-0.17889
2	0	0	-0.17889
2	2	0	+0.05000
2	2	1	-0.06000
2	2	2	+0.06000

Here, the coefficients for the electric dipole radiation with $L_1 = L_1' = L_2 = L_2' = 1$ must be used. $A(\theta_1, \theta_2, \phi; 0)$ will then have six terms,

$$
\begin{aligned}
A(\theta_1, \theta_2, \phi; 0) = \; & C^0_{00}\, \bar{P}_0{}^0(\cos\theta_1)\, \bar{P}_0{}^0(\cos\theta_2)\cos(0) \\
+ \; & C^0_{02}\, \bar{P}_0{}^0(\cos\theta_1)\, \bar{P}_2{}^0(\cos\theta_2)\cos(0) \\
+ \; & C^0_{20}\, \bar{P}_2{}^0(\cos\theta_1)\, \bar{P}_0{}^0(\cos\theta_2)\cos(0) \\
+ \; & C^0_{22}\, \bar{P}_2{}^0(\cos\theta_1)\, \bar{P}_2{}^0(\cos\theta_2)\cos(0) \\
+ \; & C^1_{22}\, \bar{P}_2{}^1(\cos\theta_1)\, \bar{P}_2{}^1(\cos\theta_2)\cos(\phi) \\
+ \; & C^2_{22}\, \bar{P}_2{}^2(\cos\theta_1)\, \bar{P}_2{}^2(\cos\theta_2)\cos(2\phi).
\end{aligned}
$$

$\bar{P}_K{}^N(\cos\theta)$ is given by

$$\bar{P}_K{}^N(\cos\theta) = \sqrt{\frac{2K+1}{2}}\sqrt{\frac{(K-N)!}{(K+N)!}}\; P_K{}^N(\cos\theta),$$

where $P_K{}^N(\cos\theta)$ is the associated Legendre function.[2]

Some $\bar{P}_K{}^N(\cos\theta)$'s needed here are

$$\bar{P}_0{}^0(\cos\theta) = \sqrt{\frac{1}{2}},$$

$$\bar{P}_2{}^0(\cos\theta) = \sqrt{\frac{5}{2}}\left(\frac{3\cos^2\theta - 1}{2}\right),$$

$$\overline{P}_2^{\,1}(\cos\theta) = \sqrt{\frac{15}{2}}\,\sin\theta\,\cos\theta\,,$$

$$\overline{P}_2^{\,2}(\cos\theta) = \sqrt{\frac{15}{4}}\,\sin^2\theta\,.$$

After substituting these functions and C^N_{KM}'s, $A(\theta_1, \theta_2, \phi; 0)$ can be reduced to

$$A(\theta_1, \theta_2, \phi; 0) = \frac{9}{16}\left(1 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2\cos 2\phi\right).$$

1. Smith, P. B. Nuclear reactions II, Amsterdam: North-Holland, 1962, 248.

2. Morse, P. & Feshbach, H. Methods of mathematical physics, New York: McGraw-Hill, 1953, 1325.

APPENDIX B

Hyperfine Correction

Perturbations of the intermediate state can change the photon angular correlation.

One such perturbation could be caused by an external magnetic field, such as the earth's magnetic field.[1] Another unavoidable perturbation arises from the hyperfine interaction. Perturbations of angular correlations are caused in the odd isotopes, lead to a rapid precession of the angular-correlation function. However, the angular correlation theory proves that the anisotropy does not vanish but is reduced to the "hard-core-value."[2] Thus, the effect is to attenuate the angular-correlation function for the odd isotopes.

1. Popp, M., Schafer, G., & Bodenstedt, E. Investigation of photon-photon correlations in atomic mercury. Z. Physik. 240, 1970, 71-92.

2. Frauenfelder, H., & Steffen, R. M. In K. Siegbahn, (Ed.) Alpha-beta, and gamma-ray spectroscopy. Amsterdam: North-Holland, 1965, 997.

APPENDIX C

Search Program for J = 3 → 2 → 1

```
C SEARCH PROGRAM FOR J=3>2>1

        REAL A(3,4)
        REAL Z(3)
        INTEGER IPVT(3)
        REAL T(3)
        REAL D(3)
        REAL B(3)

        DEL=1.

1000    READ (5,200) (T(I),I=1,3)
200     FORMAT(3F)
        IF(T(1).LT.0.) STOP

        CALL F1234(T,D)
        DO 10 I=1,3
                A(I,4)=-D(I)
                DO 10 J=1,3
                        A(I,J)=D(I)
10      CONTINUE

        DO 15 N=1,3
                DO 20 I=1,3
                        B(I)=T(I)
20              CONTINUE

                B(N)=B(N)+DEL
                CALL F1234(B,D)
                DO 30 J=1,3
                        A(J,N)=(D(J)-A(J,N))/DEL
30              CONTINUE

15      CONTINUE

        WRITE(5,300) ((A(I,J),J=1,4),I=1,3)
300     FORMAT(1H-,(4F15.8/))

        LDA=3
        JOB=0
        N=3
```

```
      CALL SGECO(A,LDA,N,IPVT,RCOND,Z)
      COND=1./RCOND
      WRITE(5,130) RCOND,COND
130   FORMAT (1H-,2G15.8)
      IF(1.+RCOND .EQ.1.) GO TO 500
      CALL SGESL(A,LDA,N,IPVT,A(1,4),JOB)

      DO 40 I=1,3
            T(I)=T(I)+A(I,4)
40    CONTINUE

      WRITE(5,140) (T(I),I=1,3)
140   FORMAT(1H-,3F15.8/'OCONTINUE')

      DEL=DEL/10.

      GO TO 1000

500   WRITE(5,150)
150   FORMAT('ONO SOLUTION')

      STOP

      END

      SUBROUTINE F1234(T,D)

      REAL T(3)
      REAL D(3)

      DATA PI/3.141592654/

      N=5
      TOD=30.

      T1SD=T(1)
      T2SD=T(2)
      PSD=T(3)

      TO=TOD*PI/180.
            COSTO=COS(TO)

      T2S=T2SD*PI/180.
      T1S=T1SD*PI/180.
            SINT1S=SIN(T1S)
            COST1S=COS(T1S)
            SINT2S=SIN(T2S)
            COST2S=COS(T2S)
      PS=PSD*PI/180.
            SINPS=SIN(PS)
            COSPS=COS(PS)
```

```
      SUM1=0.
      SUM2=0.
      SUM3=0.
      SUM4=0.

      DO 10 M1=1,N
      COST1P=1.-(FLOAT(M1**2-M1)+.5)*(1.-COST0)/FLOAT(N**2)
      SINT1P=SQRT(1.-COST1P**2)
      L1=8*M1-4

      DO 10 K1=1,L1
      P1P=2.*PI*(FLOAT(K1)-.5)/FLOAT(L1)
              COSP1P=COS(P1P)
              SINP1P=SIN(P1P)
      COST1=-SINT1S*SINT1P*COSP1P+COST1S*COST1P
      SINT1=SQRT(1.-COST1**2)
      SINP1=SINT1P*SINP1P/SINT1
      P1=ASIN(SINP1)

      DO 10 M2=1,N
      COST2P=1.-(FLOAT(M2**2-M2)+.5)*(1.-COST0)/FLOAT(N**2)
      SINT2P=SQRT(1.-COST2P**2)
      L2=8*M2-4

      DO 10 K2=1,L2
      P2P=2.*PI*(FLOAT(K2)-.5)/FLOAT(L2)
              COSP2P=COS(P2P)
              SINP2P=SIN(P2P)
      COST2=-SINT2S*SINT2P*COSP2P+COST2S*COST2P
      SINT2=SQRT(1.-COST2**2)
      COSP2=(COST2S*COSPS*SINT2P*COSP2P-SINPS*SINT2P*SINP2P
   1          +SINT2S*COSPS*COST2P)/SINT2
      P2=ACOS(COSP2)
      T1=ASIN(SINT1)
      T2=ASIN(SINT2)
      P=P2-P1

      F1=(37./80.)-(63./80.)*COS(T1)**2
      1       -(63./80.)*COS(T2)**2
      2       +(9./16.)*COS(T1)**2*COS(T2)**2
      3       -(9./20.)*SIN(T1)*COS(T1)*SIN(T2)*COS(T2)*COS(P)
      4       +(9./80.)*SIN(T1)**2*SIN(T2)**2*COS(2*P)

      F2=(5./16.)-(39./80.)*COS(T1)**2
      1       -(39./80.)*COS(T2)**2
      2       +(9./80.)*COS(T1)**2*COS(T2)**2
      3       +(3./40.)*SIN(T1)**2*SIN(T2)**2*COS(2*P)
```

```
      F3=-(1./16.)+(3./16.)*COS(T1)**2
      1       +(3./16.)*COS(T2)**2
      2       -(9./16.)*COS(T1)**2*COS(T2)**2
      3       +(3./4.)*SIN(T1)*COS(T1)*SIN(T2)*COS(T2)*COS(P)

      F4=-(7./16.)+(9./16.)*COS(T1)**2
      1       +(9./16.)*COS(T2)**2
      2       +(9./16.)*COS(T1)**2*COS(T2)**2

      SUM1=SUM1+F1
      SUM2=SUM2+F2
      SUM3=SUM3+F3
      SUM4=SUM4+F4

10    CONTINUE

      DD=FLOAT((4*N**2)**2)

      AVG1=SUM1/DD
      AVG2=SUM2/DD
      AVG3=SUM3/DD
      AVG4=SUM4/DD

      D(1)=AVG2-AVG1
      D(2)=AVG3-AVG2
      D(3)=AVG4-AVG3

      WRITE(5,100) T,D
100   FORMAT(1H-,3(F15.8))

      RETURN
      END
```

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