Comparisons of Several Medians in a Lognormal K-Sample Context Where Some Data May be Left-Censored

Stavros Costa Pouloukas
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COMPARISONS OF SEVERAL MEDIANS IN A LOGNORMAL K-SAMPLE CONTEXT WHERE SOME DATA MAY BE LEFT-CENSORED

by

Stavros Costa Pouloukas

A Dissertation
Submitted to the
Faculty of The Graduate College
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COMPARISONS OF SEVERAL MEDIANS IN A LOGNORMAL K-SAMPLE CONTEXT WHERE SOME DATA MAY BE LEFT-CENSORED

Stavros Costa Pouloukas, Ph.D.
Western Michigan University, 1994

Assume that \( k \) \((k > 2)\) independent samples of sizes \( n_i \) \((i = 1, \ldots, k)\) are available from \( k \) lognormal distributions. Four hypothesis cases \((H_1 - H_4)\) are defined. Under \( H_1 \), all \( k \) median parameters as well as all skewness parameters are equal; under \( H_2 \), all \( k \) skewness parameters are equal but not all \( k \) median parameters are equal; under \( H_3 \), all \( k \) median parameters are equal but not all \( k \) skewness parameters are equal; under \( H_4 \), neither the \( k \) median parameters nor the \( k \) skewness parameters are equal.

The Expectation Maximization (EM) algorithm is used to obtain the maximum likelihood (ML) estimates of the lognormal parameters in each of these four hypothesis cases. A \((2k - 1)\)-degree polynomial is solved at each step of the EM algorithm for the \( H_3 \) case. A three-stage procedure for testing the equality of the \( k \) medians either under skewness homogeneity or under skewness heterogeneity is also proposed and discussed. Specific details are given for the special case \( k = 3 \).

A simulation study was performed for the case \( k = 3 \) with two sample size cases (small and large) and four censoring levels \((0\%, 10\%, 25\% \text{ and } 50\%)\). The EM algorithm converged in all simulations except for a few trials in the 50% censoring case under \( H_3 \). The performance of the three-stage procedure was evaluated with respect to the frequency of trials at which it selected the hypothesis case conforming to the data simulated. This varied from 89.4% of the trials under
$H_1$ to 100% of the trials under $H_4$ except in the 50% censoring case under $H_3$ where a frequency of 84.1% was reported. The estimated Type I errors of the test for the equality of the three medians were close to the alpha level (0.05) except in case $H_3$ at 50% censoring where a Type I error rate of 0.1488 was observed.

The estimated simulated mean square errors of the ML estimates of the lognormal parameters increased as the level of censoring increased and as the sample sizes decreased. The ML estimates maintained a low estimated bias at all levels of censoring with the exception of the censored cases in hypothesis case $H_3$ where the bias of the common median parameter estimate was elevated.
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Pouloukas, Stavros Costa, Ph.D.
Western Michigan University, 1994

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Stavros Costa Pouloukas
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CHAPTER I

INTRODUCTION

Stoline (1993) proposed a procedure for comparing medians when data come from two independent lognormal distributions where some data may be left-censored. The purpose of this dissertation is to extend the work done by Stoline (1993) and adapt it to include more than two independent lognormal distributions. Stoline (1993) extended results first suggested by Harris (1989). Both of these studies were developed for data collected in environmental contexts. Both are concerned of making two-sample comparisons of environmental data in the presence of less-than-detect data.

Harris (1989) considered two parametric procedures for testing the equality of medians when the observations come from two independent lognormal distributions where some data may be left-censored. Harris (1989) performed both parametric tests on the natural logarithm of the data which follow the normal distribution when data are lognormally distributed. He used the log transformed data to test the equality of the two means under the assumption of equal variances. This is equivalent to using the original lognormal data to test the equality of the two medians under the assumption of equal skewness.

Unlike Harris (1989), Stoline did not make the assumption of skewness equality. Instead he incorporated a test of skewness homogeneity in a preliminary test as part of a three-stage procedure for testing the equality of medians. Stoline (1993) proposed two procedures for testing the hypothesis of equal medians. One method is valid under skewness homogeneity, and the other under skewness
heterogeneity. The choice of method is made by the result of the preliminary test.

Let us define the median parameters of the two lognormal distributions as \( \mu_1 \) and \( \mu_2 \) and the skewness parameters as \( \sigma_1 \) and \( \sigma_2 \). Using this notation Stoline (1993) defined four hypotheses:

1. \( H_1: \mu_1 = \mu_2 = \mu; \sigma_1 = \sigma_2 = \sigma \)
2. \( H_2: \mu_1 \neq \mu_2; \sigma_1 = \sigma_2 = \sigma \)
3. \( H_3: \mu_1 = \mu_2 = \mu; \sigma_1 \neq \sigma_2 \)
4. \( H_4: \mu_1 \neq \mu_2; \sigma_1 \neq \sigma_2 \).

Hypothesis \( H_1 \) is one of overall homogeneity of the two lognormal populations, while \( H_4 \) is a hypothesis of overall heterogeneity of the two populations. In \( H_2 \), skewness homogeneity exists, but the two medians are unequal. In \( H_3 \), skewness heterogeneity exists, but the two medians are equal. Stoline (1993) estimated each of the \( \mu_i \) and the \( \sigma_i \) parameters in cases \( H_1 \) - \( H_4 \) using the Expectation Maximization (EM) algorithm (Dempster, Laird and Rubin, 1977). The EM algorithm is used to find maximum likelihood (ML) estimates of the lognormal parameters iteratively. It is well known that the ML estimates are the simultaneous solutions of the equations which are created by equating the partial derivatives of the likelihood function to zero. The EM algorithm is used here because there are no closed form solutions for these equations. At each step of the EM algorithm the ML estimates of the lognormal parameters are updated. This process continues until convergence is achieved. In case \( H_3 \) a third degree polynomial has to be solved at each iterative step of the EM algorithm in order to obtain a ML estimate of the common value of \( \mu \) in this case.

Stoline (1993) proposed a three-stage procedure to test the equality of the median parameters of the two lognormal populations. This three-stage procedure involved the four hypothesis tests described in (1.2a) - (1.2d):
(1.2a) Test 1: $H_1$ versus $H_4$

(1.2b) Test 2: $H_2$ versus $H_4$

(1.2c) Test 3: $H_1$ versus $H_2$

(1.2d) Test 4: $H_3$ versus $H_4$.

At Stage 1 of the three-stage procedure, Test 1 is used to determine overall homogeneity. This is a test for the equality of both medians and both skewness parameters. Test 2 is performed at Stage 2. This is the preliminary test of skewness homogeneity versus skewness heterogeneity. If Test 2 is not significant (skewness homogeneity exists) then Test 3 is used to test the equality of the medians. Otherwise, if Test 2 is significant (skewness homogeneity does not exist) then Test 4 is used to test the equality of the medians. Test 3 or Test 4 comprise the Stage 3 of this three-stage procedure.

This dissertation develops comparable methods to the three-stage procedure introduced by Stoline (1993) to test the hypothesis of equal medians when data are sampled from three or more independent lognormal distributions and where some data may be left-censored.

In this study we assume independent random samples from $k$ ($k > 2$) lognormal distributions, where sample $i$ has sample size $n_i$ and parameters $\mu_i$ and $\sigma_i$ for $i = 1, \ldots, k$. In the $i^{th}$ sample there are $r_i \leq n_i$ uncensored observations and $n_i - r_i$ left-censored observations.

Four hypothesis cases, $H_1 - H_4$, are defined for the $k$ sample case, which are similar to those defined by Stoline (1993) for the two sample case. These are described in (1.3a) - (1.3d):

(1.3a) $H_1$: $\mu_1 = \ldots = \mu_k = \mu; \sigma_1 = \ldots = \sigma_k = \sigma$,

(1.3b) $H_2$: $\mu_1 \neq \ldots \neq \mu_k; \sigma_1 = \ldots = \sigma_k = \sigma$,

(1.3c) $H_3$: $\mu_1 = \ldots = \mu_k = \mu; \sigma_1 \neq \ldots \neq \sigma_k$, and
(1.3d) \( H_4: \mu_1 \neq \ldots \neq \mu_k; \sigma_1 \neq \ldots \neq \sigma_k. \)

\( H_1 \) is a hypothesis of overall homogeneity, where all \( k \) median parameters are equal and all \( k \) skewness parameters are equal. \( H_4 \) is a hypothesis of overall heterogeneity, where the \( k \) medians as well as the \( k \) skewness parameters are unequal. In hypothesis \( H_2 \) there is skewness homogeneity, but the \( k \) medians are unequal. In hypothesis \( H_3 \) there is skewness heterogeneity, but all \( k \) medians are equal.

The EM algorithm will be used to obtain the maximum likelihood (ML) estimates of the parameters in each of the four cases (1.3a) - (1.3d).

A three-stage procedure for comparing the \( k \) medians of the lognormal distributions is introduced. This procedure is an extension of the one proposed by Stoline (1993). It involves an overall homogeneity test, a preliminary test of skewness homogeneity and the final test for the equality of the \( k \) medians, depending on the result of the preliminary test. The three-stage procedure includes the following four tests:

1. Test 1: \( H_1 \) versus \( H_4 \)
2. Test 2: \( H_2 \) versus \( H_4 \)
3. Test 3: \( H_1 \) versus \( H_2 \)
4. Test 4: \( H_3 \) versus \( H_4 \).

The purpose of this three-stage procedure is to arrive at one of the following two conclusions: either all \( k \) medians are equal or at least one pair of medians is different.

The sequence of the three-stage procedure is as follows:

Stage 1: Use Test 1 to test the overall homogeneity of the \( k \) lognormal distributions at level of significance \( \alpha_1 \) with \( p \)-value \( p_1 \).

Stage 2: Use Test 2 as a preliminary test of skewness homogeneity of the \( k \)
lognormal distributions at level of significance $\alpha_2$ with $p$-value $p_2$. Note that Test 2 makes no assumption about the equality of the $k$ medians.

Stage 3: The appropriate procedure to test the equality of the $k$ medians at this final stage depends on the outcome of the preliminary test in Stage 2. If Test 2 is not significant (i.e. $p_2 \geq \alpha_2$), then use Test 3 to test the equality of the $k$ medians under the assumption of skewness homogeneity. Otherwise, if Test 2 is significant (i.e. $p_2 < \alpha_2$), then use Test 4 to test the equality of the $k$ medians under the assumption of skewness heterogeneity.

In Chapter II of this dissertation the properties of the lognormal distribution and the properties of environmental data are discussed. The use of the EM algorithm for the estimation of the lognormal parameters in the one-sample case, where there exists left-censored data, is also described. Some previous research for estimating the lognormal parameters is reviewed as well.

In Chapter III the maximum likelihood estimates of the lognormal parameters are evaluated for each case $H_1 - H_4$ via the EM algorithm for the general $k$-sample lognormal problem. A special problem arises in the case $H_3$ where a polynomial of degree $2k - 1$ in $\mu$ has to be solved at each iterative step of the EM algorithm.

In Chapter IV the three-stage procedure for testing the equality of the $k$ medians for the general $k$-sample lognormal problem is described in detail. Asymptotic chi-square tests are used to test the hypotheses at each stage of the three-stage procedure. These chi-square tests are computed by evaluating the log likelihood functions for each hypothesis $H_1 - H_4$.

In Chapter V the ML estimates of the lognormal parameters for the special case $k = 3$ are found via the EM algorithm for each of the cases $H_1 - H_4$. The procedure for the solution of a $5^{th}$ degree polynomial is explained thoroughly for
the special case $H_3$. The three-stage procedure for testing the equality of the three medians is described for the three-sample lognormal problem.

In Chapter VI a simulation study is performed to test the performance of the EM algorithm in computing ML estimates of lognormal parameters where some data may be left-censored. A Fortran program is written for this purpose. This program first simulates data from three independent lognormal populations, and then computes the ML estimates of the lognormal parameters. Finally the program utilizes the three-stage procedure by evaluating the chi-square tests and calculating the $p$-values at each stage. The performance of the three-stage procedure is evaluated for several lognormal samples of sizes $n_1 = n_2 = n_3 = 20$ and $n_1 = n_2 = n_3 = 100$ chosen from typical Case $H_1 - H_4$ situations. The estimated simulated mean squared errors and the estimated simulated biases of the ML estimates of the lognormal parameters are also calculated and compared.
CHAPTER II

BACKGROUND

2.1 Properties of the Lognormal Distribution

The random variable $X$ follows a lognormal distribution with two parameters $\mu$ and $\sigma$ if $Y = \log(X)$ is $N(\mu, \sigma)$, normally distributed with mean $\mu$ and standard deviation $\sigma$. The lognormal distribution of $X$ is conveniently denoted by $LN(\mu, \sigma)$. The probability density function of $X$ is:

$$f(x; \mu, \sigma) = \begin{cases} \frac{1}{x \sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sigma^2} (\ln x - \mu)^2 \right\}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

The moments and other important parameters of the lognormal distribution can be expressed as functions of the lognormal parameters $\mu$ and $\sigma$. The median, mean, variance and skewness of the $LN(\mu, \sigma)$ random variable $X$ are conveniently given in (2.2) - (2.5), respectively:

(2.2) median of $X = e^\mu$,

(2.3) mean of $X = e^{\mu+\sigma^2/2}$,

(2.4) variance of $X = \gamma(\gamma-1)e^{2\mu}$, and

(2.5) skewness of $X = (\gamma+2)\sqrt{\gamma-1}$, where $\gamma = e^{\sigma^2}$.

Note that the median is a function only of $\mu$ and the skewness is a function only of $\sigma$. However, the mean and the variance are functions of both $\mu$ and $\sigma$. Also note that the skewness of a lognormal variable is always positive.
2.2 Properties of Environmental Data

The analysis of environmental data has recently gained much attention from scientists, including many statisticians. Many environmental data sets tend to have a few extreme large (and sometimes toxic) observations. Many environmental data sets are characterized by a small number of high concentrations and a large number of low concentrations and are often right-skewed (Shumway, Azari, and Johnson, 1989). The lognormal distribution is positively skewed and hence can incorporate the few unusually high measurements of such environmental data in its long right-hand tail. For this reason the lognormal distribution is often applied to environmental data (Gilbert, 1987).

Concentrations of chemistry values of substances like iron, cadmium, etc. in soil, air and water media are usually measured by laboratory instruments. Small trace amounts of these substances often can not be detected by the chemistry laboratory measurement procedures. When the concentration of these substances in a sample cannot be quantified, its value is reported as nondetect. Lambert, Peterson, and Terpenning (1991) give four reasons to explain why a measurement may not be able to be quantified. First, the signal produced by the pollutant may be too small to be discriminated from background noise. Second, the instrumentation may register a low signal similar to the one registered for the "unpolluted" environmental sample. Third, a signal may be registered by the instrumentation, but certain criteria which identify the compound are not met. Finally, a threshold may be set by the client or the laboratory and any measurement below it is reported as nondetect. The lowest concentration of substance which a laboratory instrument can detect is called the detection limit. This is not a scientific definition but it will be accepted here because this dissertation is not concerned with
the determination of detection limits.

An example of environmental data are the measurements of the concentrations of chemistry parameters in groundwater samples. Sometimes a number of these measurements are below the detection limit of the laboratory instrument used to measure them. All measurements below the instrument’s detection limit are defined as censored data, and are reported as "less than the detection limit" rather than as numerical values (Gilliom and Helsel, 1986). It is assumed that the laboratory instrument is able to measure all concentrations above its detection limit. Thus there is only censoring of the observations which are less than the detection limit. This is referred to as censoring on the left, and for this reason the data are called left-censored data.

There are two types of censoring. Type I censoring occurs when all the data below a fixed value are unobservable. In our example of the measurement of the concentrations of substances in water samples, all data below the instrument’s detection limit are unobservable. The detection limit of the laboratory instrument is also called the censoring level. It is fixed in advance and all measurements below the censoring level are referred to simply as less than the detection limit (Millard and Deverell, 1988). In Type I censoring the number of left-censored data values as well as the environmental measurements themselves are random variables (Gleit, 1985).

Type II censoring occurs when a fixed percentage of the data are purposefully censored. Type II censoring is more common in life testing where, for example, a total of n items is placed on test and the data collection is terminated at the time the rth unit has failed. The value of r is decided before the data are collected (Lawless, 1982). Environmental data, and more specifically left-censored data from laboratory processes measurements typically satisfy Type I censoring.
Environmental data sets involving detection limits almost always fall into the category of Type I left-censoring (Millard and Deverell, 1988).

Another distinction needs to be made between singly censored data and multiply censored data. A sample of \( n \) observations is singly censored on the left, if \( 1 \leq r < n \) of these observations are below the single detection limit \( L \), while the remaining \( n - r \) observations are reported above the detection limit \( L \) (Millard and Deverell, 1988). A sample of \( n \) observations is multiply censored with \( m \) censoring levels, if \( r_1 < n \) observations are censored at censoring level \( L_1 \), \( r_2 < n \) observations are censored at censoring level \( L_2 \), \ldots, \( r_m < n \) observations are censored at censoring level \( L_m \), and \( n - (r_1 + \ldots + r_m) \) are uncensored observations (Millard and Deverell, 1988). Multiple detection limits can occur when the data come from several laboratories which use different detection limits.

Millard and Deverell (1988) give three additional reasons to explain why multiple detection limits may be used. First, the detection limit may depend on the method used to measure the concentration of the pollutant. Different methods may be optimal at different ranges of concentration of the pollutant and each of the methods may induce a different detection limit. A second cause for multiple detection limits may be the amount of dilution. Time constraints may induce a maximum amount of dilutions for a single sample and the detection limit depends on the number of dilutions. Therefore, several detection limits may exist in these circumstances. Finally a cause for multiple detection limits being used can arise from the improvement on the measurement techniques over a period of time. In these cases the detection limits may become smaller and smaller over time.

In this study, Type I multiple censoring on the left will be assumed.
2.3 Estimation of the Lognormal Parameters in Environmental Context via the EM Algorithm

The Expectation Maximization (EM) Algorithm is an iterative method due to Dempster et al. (1977). It is used to compute the maximum likelihood estimates of statistical parameters iteratively when incomplete data are available and a closed form solution of the likelihood equations is not possible to obtain.

Each iteration of the EM algorithm has two steps: the expectation step and the maximization step. At the expectation step the expected values of the censored data, conditional on the uncensored data and the detection limit, are calculated. The maximization step maximizes the likelihood function. This is accomplished by substituting the expected values, found at the estimation step, in place of the censored observations in the estimated likelihood function. When the complete data come from an exponential family (for example the normal and the lognormal distributions) whose maximum likelihood estimates are easy to compute, then the maximization steps of the EM algorithm are also easy to compute (Dempster et al., 1977).

Stoline (1991) gives a brief description of how the EM algorithm is used in the one sample lognormal case. He assumes \( n \) observations: \( x_1, \ldots, x_n \) from a lognormal distribution with parameters \( \mu \) and \( \sigma \). Assume there are \( r \) uncensored observations, \( x_1, \ldots, x_r \), and \( n - r \) non-detect observations with left-censored values: \( x_{r+1} = L_{r+1}, \ldots, x_n = L_n \). The log transformed values are obtained:

\[
(2.8a) \quad y_i = \log(x_i); \quad \text{if} \quad i = 1, \ldots, r
\]

\[
(2.8b) \quad y_i = \log(L_i); \quad \text{if} \quad i = r + 1, \ldots, n.
\]

The log transformed observations \( y_i \) are assumed to be \( N(\mu, \sigma) \). The likelihood function is:
where $\phi(z)$ and $\Phi(z)$ are the pdf and the cdf, respectively, of the standard unit normal variable. That is:

\begin{align*}
(2.10) \quad & \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right), \\
(2.11) \quad & \Phi(t) = \Pr[z \leq t]
\end{align*}

where $z$ is $N(0,1)$.

The ML estimates of $\mu$ and $\sigma$ are those values of $\mu$ and $\sigma$ which maximize the likelihood function $L(\mu, \sigma)$ in (2.9). It is well known that the same values of $\mu$ and $\sigma$ which maximize the likelihood function $L(\mu, \sigma)$ also maximize the log of the likelihood function. The log likelihood function $\log L(\mu, \sigma)$ is:

\begin{align*}
(2.12) \quad & \log L(\mu, \sigma) = -\frac{r}{2} \log(2\pi) - r \log(\sigma) - \frac{1}{2} \sum_{i=1}^{r} \left(\frac{y_i - \mu}{\sigma}\right)^2 \\
& \quad + \sum_{i=r+1}^{n} \log \Phi\left(\frac{y_i - \mu}{\sigma}\right).
\end{align*}

In order to find the values of $\mu$ and $\sigma$ which maximize the log likelihood function in (2.12), the partial derivatives of this function with respect to $\mu$ and $\sigma$ are set equal to zero:

\begin{align*}
(2.13a) \quad & \frac{\partial \log L(\mu, \sigma)}{\partial \mu} = \sum_{i=1}^{r} \left(\frac{y_i - \mu}{\sigma}\right) - \sum_{i=r+1}^{n} W\left(\frac{y_i - \mu}{\sigma}\right) = 0, \\
(2.13b) \quad & \frac{\partial \log L(\mu, \sigma)}{\partial \sigma} = -r + \sum_{i=1}^{r} \left(\frac{y_i - \mu}{\sigma}\right)^2 - \sum_{i=r+1}^{n} \left(\frac{y_i - \mu}{\sigma}\right) W\left(\frac{y_i - \mu}{\sigma}\right) = 0
\end{align*}

where

\begin{equation}
(2.14) \quad W(x) = \frac{\phi(x)}{\Phi(x)}.
\end{equation}

The ML estimates of $\mu$ and $\sigma$ are the simultaneous solutions of (2.13a) and
(2.13b). The simultaneous solutions of these two equations do not exist in closed form. The use of the EM algorithm to compute the maximum likelihood estimates of \( \mu \) and \( \sigma \) is now described.

Each censored datum \( y_i (i = r+1, ..., n) \) is replaced by an estimate obtained from the following conditional expectation:

\[
E[Y_i | Y_j \leq y_i] = \mu - \sigma \, W\left(\frac{y_i - \mu}{\sigma}\right) \quad \text{(Wolynetz, 1979)}
\]

where it is assumed that \( Y_i \) is \( N(\mu, \sigma) \), for \( i = r+1, ..., n \) and where \( y_i = \log(L_i) \) for \( i = r+1, ..., n \).

A complete set of data are now available since we have the values of the \( r \) uncensored observations and an estimate for each of the \( n-r \) censored observations can be obtained using (2.15).

Stoline (1991) defines \( u_i \) for \( i = 1, ..., n \) as follows:

\[
\begin{align*}
(2.16a) \quad u_i &= y_i; \quad \text{if } i = 1, ..., r \\
(2.16b) \quad u_i &= \mu - \sigma \, W\left(\frac{y_i - \mu}{\sigma}\right); \quad \text{if } i = r+1, ..., n.
\end{align*}
\]

The following equation can be derived from (2.16b):

\[
(2.17) \quad \frac{u_i - \mu}{\sigma} = -W\left(\frac{y_i - \mu}{\sigma}\right); \quad \text{for } j = r+1, ..., n.
\]

Equations (2.13a) and (2.13b) can be written as (2.18a) and (2.18b), respectively, using the new \( u_i \) variables:

\[
(2.18a) \quad \sum_{i=1}^{r} \left(\frac{u_i - \mu}{\sigma}\right) + \sum_{i=r+1}^{n} \left(\frac{u_i - \mu}{\sigma}\right) - \sum_{i=r+1}^{n} \left(\frac{u_i - \mu}{\sigma}\right) - \sum_{i=r+1}^{n} W\left(\frac{y_i - \mu}{\sigma}\right) = 0
\]
(2.18b) \[-r + \sum_{i=1}^{r} \left( \frac{u_i - \mu}{\sigma} \right)^2 + \sum_{i=r+1}^{n} \left( \frac{u_i - \mu}{\sigma} \right)^2 \]
\[ - \sum_{i=r+1}^{n} \left( \frac{y_i - \mu}{\sigma} \right)^2 - \sum_{i=r+1}^{n} \left( \frac{y_i - \mu}{\sigma} \right) W \left( \frac{y_i - \mu}{\sigma} \right) = 0. \]

By using (2.17), equations (2.18a) and (2.18b) can be written as (2.19a) and (2.19b), respectively:

(2.19a) \[\sum_{i=1}^{n} \left( \frac{u_i - \mu}{\sigma} \right) + \sum_{i=r+1}^{n} W \left( \frac{y_i - \mu}{\sigma} \right) - \sum_{i=r+1}^{n} W \left( \frac{y_i - \mu}{\sigma} \right) = 0\]

(2.19b) \[-r + \sum_{i=1}^{n} \left( \frac{u_i - \mu}{\sigma} \right)^2 - \sum_{i=r+1}^{n} \left( W \left( \frac{y_i - \mu}{\sigma} \right) \right)^2 \]
\[ - \sum_{i=r+1}^{n} \left( \frac{y_i - \mu}{\sigma} \right) W \left( \frac{y_i - \mu}{\sigma} \right) = 0. \]

Finally equations (2.19a) and (2.19b) can be simplified as (2.20a) and (2.20b), respectively:

(2.20a) \[\sum_{i=1}^{n} \left( \frac{u_i - \mu}{\sigma} \right) = 0\]

(2.20b) \[-r + \sum_{i=1}^{n} \left( \frac{u_i - \mu}{\sigma} \right)^2 - \sum_{i=r+1}^{n} \lambda \left( \frac{y_i - \mu}{\sigma} \right) = 0\]

where the function \( \lambda(x) \) is defined as

(2.21) \[\lambda(x) = -W(x) [W(x) + x].\]

At step 0 of the EM algorithm all the uncensored observations are used to calculate the initial estimates of \( \mu \) and \( \sigma \) as follows:

(2.22a) \[\hat{\mu}_0 = \frac{\sum_{i=1}^{r} y_i}{r}\]

(2.22b) \[\hat{\sigma}_0 = \sqrt{\frac{\sum_{i=1}^{r} (y_i - \hat{\mu}_0)^2}{r}}.\]
At step 1, \( u_{i1} \) for \( i = 1, \ldots, n \) are defined by referring to (2.16a) and (2.16b).

Thus,

\[
(2.23a) \quad u_{i1} = y_i; \quad \text{if } i = 1, \ldots, r
\]

\[
(2.23b) \quad u_{i1} = \hat{\mu}_0 - \hat{\sigma}_0 W \left( \frac{y_i - \hat{\mu}_0}{\hat{\sigma}_0} \right); \quad \text{if } i = r + 1, \ldots, n.
\]

At step 1 equations (2.24a) and (2.24b) can be written by adapting (2.20a) and (2.20b), respectively:

\[
(2.24a) \quad \sum_{i=1}^{n} \left( \frac{u_{i1} - \hat{\mu}_1}{\hat{\sigma}_1} \right) = 0
\]

\[
(2.24b) \quad -r + \sum_{i=1}^{n} \left( \frac{u_{i1} - \hat{\mu}_1}{\hat{\sigma}_1} \right)^2 - \sum_{i=r+1}^{n} \lambda \left( \frac{y_i - \hat{\mu}_0}{\hat{\sigma}_0} \right) = 0
\]

where \( \hat{\mu}_1 \) and \( \hat{\sigma}_1 \) are the ML estimates of \( \mu \) and \( \sigma \) respectively at step 1.

From (2.24a) the updated ML estimate of \( \mu \) at step 1 can be calculated. That is,

\[
(2.25a) \quad \hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} u_{i1}.
\]

The estimate \( \hat{\mu}_1 \) can now be substituted in (2.24b). Then the updated ML estimate of \( \sigma^2 \) at step 1 can be calculated:

\[
(2.25b) \quad \hat{\sigma}_1^2 = \frac{\sum_{i=1}^{n} (u_{i1} - \hat{\mu}_1)^2}{r - \sum_{i=r+1}^{n} \lambda \left( \frac{y_i - \hat{\mu}_0}{\hat{\sigma}_0} \right)}
\]

The square root of this estimate, \( \hat{\sigma}_1 \), is the ML estimate of \( \sigma \), at step 1.

This procedure is continued until convergence is met. That is, let \( \hat{\mu}_s \) and \( \hat{\sigma}_s \) be the ML estimates of \( \mu \) and \( \sigma \) respectively at step \( s \). At step \( s + 1 \), \( u_{i(s+1)} \) for \( i = 1, \ldots, n \) are updated as follows:
\begin{align*}
(2.26a) & \quad u_{i(s+1)} = y_i; \quad \text{if } i = 1, \ldots, r \\
(2.26b) & \quad u_{i(s+1)} = \mu_s - \sigma_s W \left( \frac{y_i - \mu_s}{\sigma_s} \right); \quad \text{if } i = r + 1, \ldots, n.
\end{align*}

The ML estimates of $\mu$ and $\sigma$ at step $s + 1$ can then be found from the following two equations:

\begin{align*}
(2.27a) & \quad \sum_{i=1}^{n} \left( \frac{u_{i(s+1)} - \hat{\mu}_{s+1}}{\hat{\sigma}_{s+1}} \right) = 0 \\
(2.27b) & \quad -r + \sum_{i=1}^{n} \left( \frac{u_{i(s+1)} - \hat{\mu}_{s+1}}{\hat{\sigma}_{s+1}} \right)^2 - \sum_{i=r+1}^{n} \lambda \left( \frac{y_i - \hat{\mu}_s}{\hat{\sigma}_s} \right) = 0
\end{align*}

From (2.27a) the updated ML estimate of $\mu$ at step $s + 1$ can be calculated. That is,

\begin{equation}
(2.28a) \quad \hat{\mu}_{s+1} = \frac{\sum_{i=1}^{n} u_{i(s+1)}}{n}.
\end{equation}

The estimate $\hat{\mu}_{s+1}$ can now be substituted in (2.27b). Then the updated ML estimate of $\sigma^2$ at step $s + 1$ can be calculated:

\begin{equation}
(2.28b) \quad \hat{\sigma}_{s+1}^2 = \frac{\sum_{i=1}^{n} \left( u_{i(s+1)} - \hat{\mu}_{s+1} \right)^2}{r - \sum_{i=r+1}^{n} \lambda \left( \frac{y_i - \hat{\mu}_s}{\hat{\sigma}_s} \right)}.
\end{equation}

The square root of this estimate, $\hat{\sigma}_{s+1}$, is the ML estimate of $\sigma$, at step $s + 1$.

At each step $s + 1$, $|\hat{\mu}_s - \hat{\mu}_{s+1}|$ and $|\hat{\sigma}_s - \hat{\sigma}_{s+1}|$ are calculated. The convergence criterion is met when both $|\hat{\mu}_s - \hat{\mu}_{s+1}|$ and $|\hat{\sigma}_s - \hat{\sigma}_{s+1}|$ are less than a preassigned tolerance value, say $10^{-6}$. Once convergence is achieved the ML estimates of $\mu$ and $\sigma$ are reported as $\hat{\mu}_s$ and $\hat{\sigma}_s$ respectively.

In this study, the EM algorithm will be used to find the ML estimates of
the lognormal parameters when data come from \( k (k > 2) \) independent lognormal distributions and some data are left censored.

2.4 Previous Research in the Estimation of the Lognormal Parameters

The estimation of the lognormal parameters in the one sample case where some data may be left-censored has been examined by Gleit (1985), Gilliom and Helsel (1986), El Shaarawi (1989), Shumway et al. (1989) and Stoline (1991).

Gleit (1985) studied several methods for estimating the lognormal parameters \( \mu \) and \( \sigma \), when the sample sizes are small and the data are cut off below by detection limits. He simulated small data sets (\( n \leq 15 \)) from underlying normal distributions where some data were singly left-censored with Type I censoring. Gleit fixed the detection limit at \( L \). Gleit (1985) used these simulated data to compare the several methods of estimation of the normal parameters \( \mu \) and \( \sigma \). He then proposed that this is equivalent to estimating the lognormal parameters \( \mu \) and \( \sigma \), simply by assuming the log transformation on the initial data. This is so because if a random variable \( X \) is lognormal then \( Y = \log(X) \) is normal. Gleit (1985) concluded that the "Fill In With Expected Value" was the best method for estimating the values of the Type I left-censored data. This method is performed iteratively and is characterized by the estimation of the left-censored data conditional on being less than the detection limit. Gleit proposed an algorithm for this method, with the assumption that the detection limit is \( L \), and there are \( p \) left censored data entries. It consists of four steps as follows: at step 1 select any convenient guesses for \( \mu \) and \( \sigma \). At step 2 calculate the expected values of the first \( p \) order statistics conditional on being less than \( L \), by using the values of \( \mu \) and \( \sigma \) from step 1. At step 3 calculate the new values of \( \mu \) and \( \sigma \) by using the expected values of the censored data and the uncensored data. At step 4
calculate the absolute difference between the current and the updated versions of \( \mu \) and \( \sigma \). If they are less than the tolerance level then stop; otherwise go back to step 2. Continue this process until the tolerance level is met. Gleit reports that this algorithm converges rapidly when the tolerance level is \( 10^{-6} \). This method is the same as the Expectation Maximization (EM) algorithm.

Gilliom and Helsel (1986) performed comparative simulation studies to examine different techniques for the estimation of distributional parameters where some data are left-censored. They generated sixteen parent distributions from the lognormal distribution, the delta lognormal distribution, the contaminated lognormal distribution and the gamma distribution. The contaminated lognormal distribution used in this study consisted of a mixture of a predominant \( LN(\mu_1, \sigma_1) \) which described 80% of the population and a contaminant \( LN(\mu_2, \sigma_2) \) which described 20% of the overall population. The delta lognormal distribution is a mixture of a \( LN(\mu_1, \sigma_1) \) and a portion \( (p) \) of zeros. In their study the portion of zeros was 5%. Gilliom and Helsel (1986) used Type I censoring at four different levels (detection limits): the 20\(^{th}\), 40\(^{th}\), 60\(^{th}\) and 80\(^{th}\) percentile of the parent distribution. Then they used eight techniques to estimate the distributional parameters. They concluded that the best method for estimating the mean and the standard deviation was the log probability regression method. This method is briefly described here. First, the data were log-transformed. Then the normal scores were computed for each uncensored observation. A least squares regression of concentration on normal scores for all data above the detection limit was extrapolated to estimate the censored observations. Any estimated values falling below zero were set equal to zero.

El Shaarawi (1989) claimed that the modified maximum likelihood method performed better than the log regression method under the assumptions that the
distribution of the data are lognormal and that Type I censoring is used. This is due to the fact that the log regression method proposed by Gilliom and Helsel (1986) ignores valuable information available from the fixed value of the detection limit. The modified maximum likelihood method proposed by El Shaarawi (1989) is similar to the log regression method except that the maximum likelihood estimates of $\mu$ and $\sigma$ were used to estimate the slope of the regression line. The modified maximum likelihood method proposed by El Shaarawi (1989) is similar to "Fill In With Expected Value" method proposed by Gleit (1985).

Harris (1989) considered two parametric tests for testing the equality of the medians when the observations come from two independent lognormal distributions where some data are left-censored. The first parametric test that Harris (1989) considered was the two-sample $t$-test. The censored data were included simply replacing all censored observations by a value equal to one-half the laboratory detection limit (DL). This method for estimating the model parameters was called the DL/2 method.

The second test utilizes the linear model: $y_{ij} = \beta_0 + \beta_1 i + e_{ij}$, where $i = 0, 1, j = 1, \ldots, n_i$ and where $n_i$ is the number of observations in each group $i$. The $e_{ij}$ are assumed to be independent and identically normally distributed with mean 0 and variance $\sigma^2$. To test the equality of means the model parameter $\beta_1$ is tested to see if it is significantly different from zero using an $F$-test. The maximum likelihood estimates of the model parameters were computed by using a version of the Expectation Maximization (EM) algorithm due to Wolynetz (1979).

In his simulation study Harris (1989) used Type I censoring at two different levels (detection limits): the 20th and 60th percentile. The estimates of the model parameters produced by the EM algorithm procedure were found to be less biased than the estimates produced by the DL/2 method at both censoring levels (Harris,
At 20% censoring the mean squared error of the estimates were about equal for the two methods while at 60% censoring the mean squared error of the estimates produced by the EM algorithm procedure were higher. This is due to the fact that the DL/2 method assumes fixed values for the uncensored observations while the EM algorithm assumes that the uncensored observations are random variables. Overall the EM algorithm seems to be superior to the DL/2 method because it produces less bias in the ML estimates and because it treats the measurements as random variables. The EM algorithm will be used to estimate the lognormal parameters in this dissertation because of these findings. This approach is in agreement with Gleit (1985).
CHAPTER III

MAXIMUM LIKELIHOOD ESTIMATES OF THE LOGNORMAL PARAMETERS VIA THE EM ALGORITHM

3.1 Introduction

It is assumed that \( k \) \((k > 2)\) independent samples are available from \( k \) lognormal distributions where the \( i^{th} \) sample has parameters \( \mu_i \) and \( \sigma_i \), for \( i = 1, \ldots, k \). Assume there are \( n_i \) observations in the \( i^{th} \) sample. It is convenient to let \( x_{ij} \) denote the \( j^{th} \) measurement on the \( i^{th} \) sample where \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \). It is assumed that these data are all measured in a laboratory where the values of the detection limits are known for each observation \( x_{ij} \). For each observation \( x_{ij} \) let the known detection limit be denoted by \( L_{ij} \). If an observation \( x_{ij} \) is above its laboratory instrument’s detection limit \( L_{ij} \), then its exact value is known. In this case the measurement is reported as \( x_{ij} \). If \( x_{ij} \) is below \( L_{ij} \), then it is declared as left-censored. A value of \( L_{ij} \) is reported in this case.

It is further assumed that for the \( i^{th} \) sample \( n_i - r_i \) observations have left-censored measurements below the detection limit levels. The values of the remaining \( r_i \) observations in the \( i^{th} \) sample are assumed quantified at values above their detection limits. Therefore for each sample \( i \), there are \( r_i \) uncensored observations and \( n_i - r_i \) left-censored observations.

For simplicity, assume that the first \( r_i \) observations in the \( i^{th} \) sample are uncensored and that the last \( n_i - r_i \) observations are left-censored. Therefore, for the \( i^{th} \) sample, the data are \( x_{i1}, \ldots, x_{i(r_i)}, L_{i(r_i+1)}, \ldots, L_{i(n_i)} \). Thus, the data for all \( k \) samples are as follows:
In the attempt to find maximum likelihood (ML) estimates for $\mu_i$ and $\sigma_i$, it is convenient to log transform all the data by defining $y_{ij}$ for $i = 1, \ldots, k$ as follows:

\begin{align*}
(3.1a) & \quad y_{ij} = \log(x_{ij}); \text{ for } j = 1, \ldots, r_i \\
(3.1b) & \quad y_{ij} = \log(L_{ij}); \text{ for } j = r_i + 1, \ldots, n_i.
\end{align*}

The likelihood functions of each of the four cases expressed in (1.3a) - (1.3d) in Section 1.1 can now be determined in terms of the log transformed data $y_{ij}$. The EM algorithm will then be used to calculate the maximum likelihood (ML) estimates of $\mu_i$ and $\sigma_i$ for each of the four cases.

3.2 Case $H_1$: $\mu_1 = \ldots = \mu_k = \mu; \sigma_1 = \ldots = \sigma_k = \sigma$

3.2.1 The ML Estimates Under $H_1$

The case $H_1$ is expressed in (1.3a) in Section 1.1. In this case all $k$ population medians are equal and all $k$ skewness parameters are equal. There exists an overall homogeneity among the $k$ populations for case $H_1$ situations. All the $y_{ij}$ are assumed to be distributed as $N(\mu, \sigma)$.

The likelihood function for this case is:
(3.2) \[ L_1(\mu, \sigma) = \prod_{i=1}^{k} \left( \prod_{j=1}^{r_i} \frac{1}{\sigma} \phi \left( \frac{y_{ij} - \mu}{\sigma} \right) \prod_{j=r_i+1}^{n_i} \Phi \left( \frac{y_{ij} - \mu}{\sigma} \right) \right) \]

where the functions \( \phi(z) \) and \( \Phi(t) \) were defined in (2.10) and (2.11) respectively.

The ML estimates of \( \mu \) and \( \sigma \) are those values of \( \mu \) and \( \sigma \) which maximize the likelihood function in (3.2). However, it is well known that the same values of \( \mu \) and \( \sigma \) which maximize the likelihood function, also maximize the log of the likelihood function, which is found by taking the natural logarithm on both sides of (3.2). The log likelihood function is:

(3.3) \[ \log L_1(\mu, \sigma) = \sum_{i=1}^{k} \left[ -\frac{r_i}{2} \log(2\pi) - r_i \log(\sigma) - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \mu}{\sigma} \right) \right] \]

In order to maximize the log likelihood function in (3.3) the partial derivatives of this function with respect to \( \mu \) and \( \sigma \) are set equal to zero in (3.4a) and (3.4b) respectively:

(3.4a) \[ \frac{\partial \log L_1(\mu, \sigma)}{\partial \mu} = \sum_{i=1}^{k} \left[ \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu}{\sigma} \right) \right] = 0 \]

(3.4b) \[ \frac{\partial \log L_1(\mu, \sigma)}{\partial \sigma} = \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu}{\sigma} \right) W \left( \frac{y_{ij} - \mu}{\sigma} \right) \right] = 0, \]

where \( W(x) \) is given in (2.14).

The ML estimates of \( \mu \) and \( \sigma \) are the simultaneous solutions of (3.4a) and (3.4b). There are no closed form solutions to these two simultaneous equations therefore the EM algorithm will be used to find the ML estimates of \( \mu \) and \( \sigma \) in
an iterative fashion.

3.2.2 The EM Algorithm Under $H_1$

The EM algorithm procedure used here is comparable to the procedure used by Stoline (1991) for the one sample estimation which was described in Section 2.3.

Each left-censored data value $y_{ij}$ ($i = 1, ..., k; j = r_i + 1, ..., n_i$) is replaced by an estimate obtained from the following conditional expectation:

$$(3.5) \quad E [Y | Y \leq y_{ij}] = \mu - \sigma W \left( \frac{y_{ij} - \mu}{\sigma} \right)$$

where it is assumed that $Y$ is $N(\mu, \sigma)$ and where $y_{ij} = \log(L_{ij})$ for $i = 1, ..., k$ and $j = r_i + 1, ..., n_i$.

Values of $u_{ij}$ for $i = 1, ..., k$ are defined as follows:

$$(3.6a) \quad u_{ij} = y_{ij}; \quad \text{if } j = 1, ..., r_i$$

$$(3.6b) \quad u_{ij} = \mu - \sigma W \left( \frac{y_{ij} - \mu}{\sigma} \right); \quad \text{if } j = r_i + 1, ..., n_i.$$  

The following equation can be derived from (3.6b):

$$(3.7) \quad \frac{u_{ij} - \mu}{\sigma} = -W \left( \frac{y_{ij} - \mu}{\sigma} \right); \quad \text{for } j = r_i + 1, ..., n_i.$$  

Equations (3.4a) and (3.4b) can be written as (3.8a) and (3.8b), respectively, using the new $u_{ij}$ variables:

$$(3.8a) \quad \frac{k}{2} \left[ \sum_{i=1}^{k} \left( \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu}{\sigma} \right) + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right) \right) \right] = 0$$  

$$- \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu}{\sigma} \right)$$
By using (3.7), equations (3.8a) and (3.8b) can be written as (3.9a) and (3.9b), respectively:

\[(3.9a)\]  
\[\sum_{i=1}^{k} \left[ \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right)^2 + \sum_{j=r_{i+1}}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right)^2 \right] - \sum_{j=r_{i+1}}^{n_i} \left( \frac{y_{ij} - \mu}{\sigma} \right)^2 \sum_{j=r_{i+1}}^{n_i} W \left( \frac{y_{ij} - \mu}{\sigma} \right) = 0.\]

\[(3.9b)\]  
\[\sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right)^2 - \sum_{j=r_{i+1}}^{n_i} \left( \frac{y_{ij} - \mu}{\sigma} \right)^2 \right] - \sum_{j=r_{i+1}}^{n_i} \left( \frac{y_{ij} - \mu}{\sigma} \right)^2 W \left( \frac{y_{ij} - \mu}{\sigma} \right) = 0.\]

Finally, equations (3.9a) and (3.9b) can be simplified as (3.10a) and (3.10b), respectively:

\[(3.10a)\]  
\[\sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right) = 0.\]

\[(3.10b)\]  
\[\sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma} \right)^2 - \sum_{j=r_{i+1}}^{n_i} \lambda \left( \frac{y_{ij} - \mu}{\sigma} \right) \right] = 0.\]

where the function \(\lambda(x)\) is defined in (2.21).

At step 0 of the EM algorithm all the uncensored observations are used to calculate the initial estimates of \(\mu\) and \(\sigma\) as follows:

\[(3.11a)\]  
\[\hat{\mu}_0 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{r_i} y_{ij}}{\sum_{i=1}^{k} r_i}.\]
(3.11b) \[ \hat{\sigma}_0 = \sqrt{\frac{\sum_{i=1}^{k} \sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_0)^2}{\sum_{i=1}^{k} r_i}}. \]

At step 1, \( u_{ij1} \) for \( i = 1, ..., k \) and \( j = 1, ..., n_i \) are defined by referring to (3.6a) and (3.6b). Thus,

(3.12a) \[ u_{ij1} = y_{ij} \text{ if } j = 1, ..., r_i \]

(3.12b) \[ u_{ij1} = \hat{\mu}_0 - \sigma_0 W \left( \frac{y_{ij} - \hat{\mu}_0}{\sigma_0} \right) \text{ if } j = r_i + 1, ..., n_i. \]

At step 1 equations (3.13a) and (3.13b) can be written by adapting (3.10a) and (3.10b), respectively:

(3.13a) \[ \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( u_{ij1} - \hat{\mu}_1 \right) = 0 \]

(3.13b) \[ \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( u_{ij1} - \hat{\mu}_1 \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_0}{\sigma_0} \right) \right] = 0 \]

where \( \hat{\mu}_1 \) and \( \hat{\sigma}_1 \) are the ML estimates of \( \mu \) and \( \sigma \) respectively at step 1.

From (3.13a) the updated ML estimate of \( \mu \) at step 1 can be calculated:

(3.14a) \[ \hat{\mu}_1 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} u_{ij1}}{\sum_{i=1}^{k} n_i}. \]

The estimate \( \hat{\mu}_1 \) can now be substituted in (3.13b). Then the updated ML estimate of \( \sigma^2 \) at step 1 can be calculated:

(3.14b) \[ \hat{\sigma}_1^2 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} (u_{ij1} - \hat{\mu}_1)^2}{\sum_{i=1}^{k} \left[ r_i - \sum_{j=1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_0}{\sigma_0} \right) \right]}. \]
The square root of this estimate, \( \hat{\sigma}_s \), is the ML estimate of \( \sigma \), at step 1.

This procedure is continued until convergence is met. That is, let \( \hat{\mu}_s \) and \( \hat{\sigma}_s \) be the ML estimates of \( \mu \) and \( \sigma \) respectively at step \( s \). At step \( s + 1 \) \( u_{ij(s+1)} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \) are updated as follows:

\[
(3.15a) \quad u_{ij(s+1)} = y_{ij}; \text{ if } j = 1, \ldots, r_i
\]

\[
(3.15b) \quad u_{ij(s+1)} = \hat{\mu}_s - \hat{\sigma}_s W \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right); \text{ if } j = r_i + 1, \ldots, n_i.
\]

The ML estimates of \( \mu \) and \( \sigma \) at step \( s + 1 \) can then be found from the following two equations:

\[
(3.16a) \quad \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \frac{u_{ij(s+1)} - \hat{\mu}_{s+1}}{\hat{\sigma}_{s+1}} \right) = 0
\]

\[
(3.16b) \quad \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij(s+1)} - \hat{\mu}_{s+1}}{\hat{\sigma}_{s+1}} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right) \right] = 0.
\]

From (3.16a) the ML estimate of \( \mu \) at step \( s + 1 \) can be calculated:

\[
(3.17a) \quad \hat{\mu}_{s+1} = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} u_{ij(s+1)}}{\sum_{i=1}^{k} n_i}.
\]

The estimate \( \hat{\mu}_{s+1} \) can now be substituted in (3.16b). Then the updated ML estimate of \( \sigma^2 \) at step \( s + 1 \) can be calculated:

\[
(3.17b) \quad \hat{\sigma}_{s+1}^2 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} (u_{ij(s+1)} - \hat{\mu}_{s+1})^2}{\sum_{i=1}^{k} \left[ r_i - \sum_{j=1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right) \right]}.
\]

The square root of this estimate, \( \hat{\sigma}_{s+1} \), is the ML estimate of \( \sigma \) at step \( s + 1 \).

At each step \( s + 1 \), \( |\hat{\mu}_s - \hat{\mu}_{s+1}| \) and \( |\hat{\sigma}_s - \hat{\sigma}_{s+1}| \) are calculated. The
convergence criterion is met when both \( |\hat{\mu}_s - \mu_{s+1}| \) and \( |\hat{\sigma}_s - \sigma_{s+1}| \) are less than a preassigned tolerance value, say \( 10^{-6} \). Once convergence is achieved the ML estimates of \( \mu \) and \( \sigma \) are reported as \( \hat{\mu}_s \) and \( \hat{\sigma}_s \) respectively.

3.3 Case \( H_2 \): \( \mu_1 \neq \ldots \neq \mu_k; \sigma_1 = \ldots = \sigma_k = \sigma \)

3.3.1 The ML Estimates Under \( H_2 \)

The \( H_2 \) case is described in (1.3b). Under this case all \( k \) skewness parameters are equal while the \( k \) medians are not all equal. Therefore, there is skewness homogeneity among the \( k \) populations, which differ only in location. In this case the \( k + 1 \) lognormal parameters to be estimated are: \( \mu_1, \ldots, \mu_k \) and \( \sigma \).

The likelihood function for this case is:

\[
L_2(\mu_1, \ldots, \mu_k, \sigma) = \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{\sigma} \phi \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \prod_{j=r_i+1}^{n} \Phi \left( \frac{y_{ij} - \mu_i}{\sigma} \right)
\]

where \( \phi(t) \) and \( \Phi(t) \) were defined in (2.10) and (2.11), respectively.

Similar to case \( H_1 \), the ML estimates of \( \mu_i \), for \( i = 1, \ldots, k \) and \( \sigma \) are those values which maximize the log likelihood function:

\[
\log L_2(\mu_1, \ldots, \mu_k, \sigma) = \sum_{i=1}^{k} \left[ \left( -\frac{r_i}{2} \right) \log(2\pi) - r_i \log(\sigma) \right.
- \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right)^2 + \sum_{j=r_i+1}^{n} \log \Phi \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right].
\]

The partial derivatives of the log likelihood function (3.19) with respect to \( \mu_i \) for \( i = 1, \ldots, k \) and \( \sigma \) are presented in (3.20a) and (3.20b), respectively:

\[
\frac{\partial \log L_2(\mu_1, \ldots, \mu_k, \sigma)}{\partial \mu_i} = \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right)
\]

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\[- \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) = 0 \text{ for } i = 1, \ldots, k \]

\[ \frac{\partial \log L_2(\mu_1, \ldots, \mu_k, \sigma)}{\partial \sigma} = \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right)^2 \right. \]

\left. - \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right) W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right] = 0 \]

where \( W(x) \) was defined in (2.14).

The simultaneous solutions of these \( k+1 \) equations yield the ML estimates of \( \mu_i \) for \( i = 1, \ldots, k \) and \( \sigma \). There are no closed form solutions to these \( k+1 \) simultaneous equations therefore the EM algorithm procedure will be used.

3.3.2 The EM Algorithm Under \( H_2 \)

The EM algorithm will be used to calculate the ML estimates of \( \mu_i \) for \( i = 1, \ldots, k \) and \( \sigma \) in a similar manner as was used in case \( H_1 \) in Section 3.2.2. The procedure is comparable to the procedure used by Stoline (1991) for the one sample estimation which was described in Section 2.3.

Each left-censored data value \( y_{ij} \) \((i = 1, \ldots, k; j = r_i + 1, \ldots, n_i)\) is replaced by an estimate obtained from the following conditional expectation:

\[(3.21) \ E \left[ Y \mid Y \leq y_{ij} \right] = \mu_i - \sigma W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \]

where it is assumed that \( Y \) is \( N(\mu_i, \sigma) \) and where \( y_{ij} = \log(L_{ij}) \) for \( i = 1, \ldots, k \) and \( j = r_i + 1, \ldots, n_i \).

The values of \( u_{ij} \) for \( i = 1, \ldots, k \) are then defined as follows:

\[(3.22a) \ u_{ij} = y_{ij}; \text{ if } j = 1, \ldots, r_i \]

\[(3.22b) \ u_{ij} = \mu_i - \sigma W \left( \frac{y_{ij} - \mu_i}{\sigma} \right); \text{ if } j = r_i + 1, \ldots, n_i. \]
The following equation can be derived from (3.26b):

\[
(3.23) \quad \frac{u_{ij} - \mu_i}{\sigma} = -W \left( \frac{y_{ij} - \mu_i}{\sigma} \right); \quad \text{for } j = r_i + 1, \ldots, n_i.
\]

Equations (3.20a) and (3.20b) can be written as (3.24a) and (3.24b), respectively using the new \( u_{ij} \) variables.

\[
(3.24a) \quad \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right) + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) = 0 \quad \text{for } i = 1, \ldots, k
\]

\[
(3.24b) \quad \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right)^2 + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right)^2 \right] - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) = 0.
\]

By using (3.23), equations (3.24a) and (3.24b) can be written as (3.25a) and (3.25b), respectively:

\[
(3.25a) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right) + \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) = 0, \quad \text{for } i = 1, \ldots, k
\]

\[
(3.25b) \quad \sum_{i=1}^{k} \left[ \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right)^2 \right] - \sum_{j=r_i+1}^{n_i} \left( W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right) = 0.
\]

Finally equations (3.25a) and (3.25b) can be simplified as (3.26a) and (3.26b), respectively:
(3.26a) \[
\sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right) = 0, \text{ for } i = 1, \ldots, k
\]

(3.26b) \[
\sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right] = 0
\]

where the function \( \lambda(x) \) was defined in (2.21).

At step 0 of the EM algorithm all the uncensored observations are used to calculate the initial estimates of \( \mu \) and \( \sigma \) as follows:

(3.27a) \[
\hat{\mu}_{i0} = \frac{\sum_{j=1}^{r_i} y_{ij}}{r_i}, \text{ for } i = 1, \ldots, k
\]

(3.27b) \[
\hat{\sigma}_0 = \sqrt{\frac{\sum_{i=1}^{k} \sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_{i0})^2}{\sum_{i=1}^{k} r_i}}.
\]

At step 1, \( u_{ij1} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \) are defined by referring to (3.22a) and (3.22b). Thus,

(3.28a) \( u_{ij1} = y_{ij}; \) if \( j = 1, \ldots, r_i \)

(3.28b) \( u_{ij1} = \hat{\mu}_{i0} - \hat{\sigma}_0 W \left( \frac{y_{ij} - \hat{\mu}_{i0}}{\hat{\sigma}_0} \right); \) if \( j = r_i + 1, \ldots, n_i \).

At step 1 equations (3.29a) and (3.29b) can be written by adapting (3.26a) and (3.26b), respectively:

(3.29a) \[
\sum_{j=1}^{n_i} \left( \frac{u_{ij1} - \hat{\mu}_{i1}}{\hat{\sigma}_1} \right) = 0 \text{ for } i = 1, \ldots, k
\]

(3.29b) \[
\sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij1} - \hat{\mu}_{i1}}{\hat{\sigma}_1} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{i0}}{\hat{\sigma}_0} \right) \right] = 0
\]

where \( \hat{\mu}_{i1} \) for \( i = 1, \ldots, k \) and \( \hat{\sigma}_1 \) are the ML estimates of \( \mu_i \) for \( i = 1, \ldots, k \) and \( \sigma \), respectively at step 1.
From (3.29a) the updated ML estimate of $\mu_i$ for $i = 1, \ldots, k$ at step 1 can be calculated. That is,

$$
(3.30a) \quad \hat{\mu}_{i1} = \frac{\sum_{j=1}^{n_i} u_{ij1}}{n_i}, \quad \text{for } i = 1, \ldots, k.
$$

The estimates $\hat{\mu}_{i1}$ for $i = 1, \ldots, k$ can now be substituted in (3.29b). Then the updated ML estimate of $\sigma^2$ at step 1 can be calculated:

$$
(3.30b) \quad \hat{\sigma}_1^2 = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} (u_{ij1} - \hat{\mu}_{i1})^2}{\sum_{i=1}^{k} \left[ r_i - \sum_{j=1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{i0}}{\hat{\sigma}_0} \right) \right]}
$$

The square root of this estimate, $\hat{\sigma}_1$, is the ML estimate of $\sigma$ at step 1.

This procedure is continued until convergence is met. That is, let $\hat{\mu}_{is}$ for $i = 1, \ldots, k$ and $\hat{\sigma}_s$ be the ML estimates of $\mu_i$ for $i = 1, \ldots, k$ and $\sigma$, respectively at step $s$. At step $s+1$, $u_{ij(s+1)}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ are updated as follows:

$$
(3.31a) \quad u_{ij(s+1)} = y_{ij}; \quad \text{if } j = 1, \ldots, r_i
$$

$$
(3.31b) \quad u_{ij(s+1)} = \hat{\mu}_{is} - \hat{\sigma}_s W \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_s} \right); \quad \text{if } j = r_i + 1, \ldots, n_i.
$$

The ML estimates of $\mu_i$ for $i = 1, \ldots, k$ and $\sigma$ at step $s+1$ can then be found from the following $k+1$ equations:

$$
(3.32a) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij(s+1)} - \hat{\mu}_{i(s+1)}}{\hat{\sigma}_{s+1}} \right) = 0, \quad \text{for } i = 1, \ldots, k
$$

$$
(3.32b) \quad \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij(s+1)} - \hat{\mu}_{i(s+1)}}{\hat{\sigma}_{s+1}} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_s} \right) \right] = 0.
$$

From (3.32a) the ML estimates of $\mu_i$ for $i = 1, \ldots, k$ at step $s+1$ can be
calculated. That is,
\begin{equation}
(3.33a) \quad \hat{\mu}_{i(s+1)} = \frac{\sum_{j=1}^{n_i} u_{ij(s+1)}}{n_i}, \text{ for } i = 1, \ldots, k.
\end{equation}

The estimates \( \hat{\mu}_{i(s+1)} \) can now be substituted in (3.32b). Then the updated ML estimate of \( \sigma^2 \) at step \( s + 1 \) can be calculated:
\begin{equation}
(3.33b) \quad \hat{\sigma}^2_{s+1} = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( u_{ij(s+1)} - \hat{\mu}_{i(s+1)} \right)^2}{\sum_{i=1}^{k} \left[ r_i - \sum_{j=1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_s} \right) \right]}
\end{equation}

The square root of this estimate, \( \hat{\sigma}_{s+1} \), is the ML estimate of \( \sigma \), at step \( s + 1 \).

At each step \( s + 1 \), \( |\hat{\mu}_{is} - \hat{\mu}_{i(s+1)}| \), for \( i = 1, \ldots, k \), and \( |\hat{\sigma}_s - \hat{\sigma}_{s+1}| \) are evaluated. The convergence criterion is met when all these \( k + 1 \) absolute differences are less than a preassigned tolerance value. Once convergence is achieved the ML estimates of \( \mu_i \) and \( \sigma \) are reported as \( \hat{\mu}_{is} \) and \( \hat{\sigma}_s \) respectively.

\section*{3.4 Case \( H_3 \): \( \mu_1 = \ldots = \mu_k = \mu; \sigma_1 \neq \ldots \neq \sigma_k \)}

\subsection*{3.4.1 The ML Estimates Under \( H_3 \)}

This case was defined in (1.3c). This is a case of equality in location and heterogeneity in skewness among the \( k \) populations. All \( k \) medians are equal, while the skewness parameters are not all equal to each other. This implies that all populations have the same \( \mu \) but a different \( \sigma_i \), for \( i = 1, \ldots, k \). The \( k + 1 \) lognormal parameters to be estimated in this case are: \( \mu, \sigma_1, \ldots, \sigma_k \). The likelihood function for this case is:
The log likelihood function is:

\[ \log L_3(\mu, \sigma_1, \ldots, \sigma_k) = \sum_{i=1}^{k} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\sigma_i) - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \mu}{\sigma_i} \right) \right] \]

The partial derivatives of the log likelihood function (3.35) with respect to \( \mu \) and \( \sigma_i \) for \( i = 1, \ldots, k \) are presented in (3.36a) and (3.36b), respectively:

\[
\begin{align*}
\frac{\partial \log L_3(\mu, \sigma_1, \ldots, \sigma_k)}{\partial \mu} &= \sum_{i=1}^{k} \frac{1}{\sigma_i} \left[ \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma_i} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu}{\sigma_i} \right) \right] = 0 \\
\frac{\partial \log L_3(\mu, \sigma_1, \ldots, \sigma_k)}{\partial \sigma_i} &= -r_i + \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu}{\sigma_i} \right) W \left( \frac{y_{ij} - \mu}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k
\end{align*}
\]

where \( W(x) \) was defined in (2.14).

The simultaneous solutions of these \( k+1 \) equations yield the ML estimates of \( \mu \) and \( \sigma_i \) for \( i = 1, \ldots, k \). There are no closed form solutions to these \( k+1 \) simultaneous equations. Therefore the EM algorithm procedure will be used to iteratively solve for the ML estimates of the lognormal parameters.

### 3.4.2 The EM Algorithm Under \( H_3 \)

The EM algorithm is used iteratively to evaluate the ML estimates of \( \mu \) and \( \sigma_i \) for \( i = 1, \ldots, k \). However, this case differs from the previous two cases in an important detail. This is similar to the case \( H_3 \) in Stoline (1993) defined in (1.1c), where a cubic equation in \( \mu \) needed to be solved at each step of the EM
algorithm. In the \( k \)-sample \( H_3 \) case, a \((2k - 1)\)-degree polynomial in \( \mu \) will be required to be solved at each step of the EM algorithm.

Each left-censored data value \( y_{ij} \) (\( i = 1, ..., k; j = r_i + 1, ..., n_i \)) is replaced by an estimate obtained from the following conditional expectation:

\[
(3.37) \quad E[Y|Y \leq y_{ij}] = \mu - \sigma_i W\left(\frac{y_{ij} - \mu}{\sigma_i}\right)
\]

where it is assumed that \( Y \) is \( N(\mu, \sigma_i) \) and where \( y_{ij} = \log(L_{ij}) \) for \( i = 1, ..., k \) and \( j = r_i + 1, ..., n_i \).

The values of \( u_{ij} \) for \( i = 1, ..., k \) are then defined as follows:

\[(3.38a) \quad u_{ij} = y_{ij}; \text{ if } j = 1, ..., r_i \]

\[(3.38b) \quad u_{ij} = \mu - \sigma_i W\left(\frac{y_{ij} - \mu}{\sigma_i}\right); \text{ if } j = r_i + 1, ..., n_i.\]

The following equation can be derived from (3.38b):

\[(3.39) \quad \frac{u_{ij} - \mu}{\sigma_i} = -W\left(\frac{y_{ij} - \mu}{\sigma_i}\right); \text{ for } j = r_i + 1, ..., n_i.\]

Equations (3.36a) and (3.36b) can be written as (3.40a) and (3.40b), respectively, using the new \( u_{ij} \) variables:

\[(3.40a) \quad \sum_{i=1}^{k} \frac{1}{\sigma_i} \left[ \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right) + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right) \right] - \sum_{j=r_i+1}^{n_i} W\left(\frac{y_{ij} - \mu}{\sigma_i}\right) = 0
\]

\[(3.40b) \quad -r_i + \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right)^2 W\left(\frac{y_{ij} - \mu}{\sigma_i}\right) = 0, \text{ for } i = 1, ..., k.
\]

By using (3.39), equations (3.40a) and (3.40b) can be written as (3.41a)
and (3.41b), respectively:

\[(3.41a) \quad \sum_{i=1}^{k} \frac{1}{\sigma_i} \left[ \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right) + \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu}{\sigma_i} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu}{\sigma_i} \right) \right] = 0 \]

\[(3.41b) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( W \left( \frac{y_{ij} - \mu}{\sigma_i} \right) \right)^2
- \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu}{\sigma_i} \right) W \left( \frac{y_{ij} - \mu}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k. \]

Finally equations (3.41a) and (3.41b) can be simplified as (3.42a) and (3.42b), respectively:

\[(3.42a) \quad \sum_{i=1}^{k} \frac{1}{\sigma_i} \left[ \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right) \right] = 0 \]

\[(3.42b) \quad -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \mu}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k \]

where the function \( \lambda(x) \) was defined in (2.21).

At step 0 of the EM algorithm all the uncensored observations are used to calculate the initial estimates of \( \mu \) and \( \sigma_i \) for \( i = 1, \ldots, k \) as follows:

\[(3.43a) \quad \hat{\mu} = \frac{\sum_{i=1}^{k} \sum_{j=1}^{r_i} y_{ij}}{\sum_{i=1}^{k} r_i} \]

\[(3.43b) \quad \hat{\sigma}_i = \sqrt{\frac{\sum_{j=1}^{r_i} (y_{ij} - \hat{\mu})^2}{r_i}}, \text{ for } i = 1, \ldots, k. \]

At step 1, \( u_{ij1} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \) are defined by referring to (3.38a) and (3.38b). Thus,

\[(3.44a) \quad u_{ij1} = y_{ij}; \text{ if } j = 1, \ldots, r_i \]
(3.44b) \( u_{ij1} = \mu_0 - \hat{\sigma}_i W \left( \frac{y_{ij} - \hat{\mu}_0}{\hat{\sigma}_0} \right) \); if \( j = r_i + 1, \ldots, n_i \).

At step 1 equations (3.45a) and (3.45b) can be written by adapting (3.42a) and (3.42b), respectively:

(3.45a) \[ \sum_{i=1}^{k} \frac{1}{\hat{\sigma}_{i1}} \sum_{j=1}^{n_i} \left( \frac{u_{ij1} - \hat{\mu}_1}{\hat{\sigma}_{i1}} \right) = 0 \]

(3.45b) \[ -\frac{r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij1} - \hat{\mu}_1}{\hat{\sigma}_{i1}} \right)^2}{n_i} - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_0}{\hat{\sigma}_0} \right) = 0 \quad \text{for} \ i = 1, \ldots, k \]

where \( \hat{\mu}_1 \) and \( \hat{\sigma}_{i1} \) for \( i = 1, \ldots, k \) are the ML estimates of \( \mu \) and \( \sigma_i \) for \( i = 1, \ldots, k \) respectively at step 1.

If the same procedure that was utilized in the previous cases were to be followed, then the ML estimate of \( \mu \) would be found from (3.45a). Then this estimate, say \( \hat{\mu}_1 \), would be substituted into (3.45b) to obtain the ML estimates of \( \sigma_i \) for \( i = 1, \ldots, k \). However, the solution of (3.45a) for \( \hat{\mu}_1 \) involves \( \hat{\sigma}_{i1} \) for \( i = 1, \ldots, k \). Therefore, the same procedure that was followed in cases \( H_1 \) and \( H_2 \) in solving for the ML estimates of the lognormal parameters at step 1, can not be applied at this point for case \( H_3 \).

Instead, the following procedure will be utilized. First, new notation is introduced as follows:

(3.46b) \[ \hat{\sigma}_{i1}^2 = \frac{1}{r_i} \sum_{j=1}^{n_i} \left( u_{ij1} - \hat{\mu}_1 \right)^2 \quad \text{for} \ i = 1, \ldots, k. \]

The expressions for \( \hat{\sigma}_{i1}^2 \) for \( i = 1, \ldots, k \) all involve \( \hat{\mu}_1 \) and are substituted in (3.45a) to give an expression (3.46a) involving \( \hat{\mu}_1 \) as the only unknown:
To simplify the expression in (3.46a) the following notation is introduced in (3.47), (3.48) and (3.49):

\begin{equation}
(3.47) \quad v_{i1} = \sum_{j=1}^{n_i} u_{ij1},
\end{equation}

\begin{equation}
(3.48) \quad w_{i1} = \sum_{j=1}^{n_i} u_{ij1}^2,
\end{equation}

\begin{equation}
(3.49) \quad \xi_{i0} = \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_0}{\hat{\sigma}_0} \right), \quad \text{for } i = 1, \ldots, k.
\end{equation}

By using (3.47), (3.48) and (3.49), the expression in (3.46a) becomes:

\begin{equation}
(3.50) \quad \sum_{i=1}^{k} \frac{(\xi_{i0} - r_i)(\hat{\mu}_1 - v_{i1}) \prod_{j \neq i}^{k} (w_{i1} - 2 v_{i1} \hat{\mu}_1 + \hat{\mu}_1^2)}{\prod_{i=1}^{k} (w_{i1} - 2 v_{i1} \hat{\mu}_1 + \hat{\mu}_1^2)} = 0.
\end{equation}

Using (3.50) the estimate of \( \mu \) at step 1 is the solution of the \((2k-1)\)-degree polynomial in \( \hat{\mu}_1 \):

\begin{equation}
(3.51) \quad \sum_{i=1}^{k} \left[ (\xi_{i0} - r_i)(\hat{\mu}_1 - v_{i1}) \prod_{j \neq i}^{k} (w_{i1} - 2 v_{i1} \hat{\mu}_1 + \hat{\mu}_1^2) \right] = 0
\end{equation}

provided that

\begin{equation}
(3.52) \quad \prod_{i=1}^{k} (w_{i1} - 2 v_{i1} \hat{\mu}_1 + \hat{\mu}_1^2) \neq 0.
\end{equation}

A numerical solution of (3.51) will give at least one real root because \(2k-1\)
is an odd integer and imaginary roots come in pairs. Let \( t \) be the number of real roots of (3.51). Then there are \( 2k - 1 - t \) imaginary roots of (3.51). Clearly \( t \geq 1 \).

One and only one of these \( t \) real roots should be chosen as the ML estimate of \( \mu \) at step 1. If there is only one real root, then that number is used to evaluate \( \hat{\mu}_1 \) provided that it satisfies inequality (3.52). Otherwise, the real root which maximizes the log likelihood function in (3.35) and satisfies inequality (3.52) is used as the estimate \( \hat{\mu}_1 \) in (3.51).

Assume at step 1 the real solutions of (3.51) are \( \rho_1, \ldots, \rho_t \), where \( 1 \leq t \leq 2k - 1 \), and \( t \) is odd. Also assume that all \( \rho_i \) for \( i = 1, \ldots, t \) satisfy inequality (3.52). Let the corresponding estimates for \( \mu \) at step 1 be:

\[
\hat{\mu}_i^{(\rho_j)} = \rho_j; \quad \text{for} \ j = 1, \ldots, t.
\]

Next, substitute each of these \( t \) values in (3.46b) to obtain \( \hat{\sigma}_{r1}^{2(\rho_j)} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, t \). Therefore, for \( j = 1, \ldots, t \),

\[
\hat{\sigma}_{r1}^{2(\rho_j)} = \frac{\sum_{j=1}^{n_1} (u_{ij} - \hat{\mu}_i^{(\rho_j)})^2}{r_i - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_0}{\hat{\sigma}_{10}} \right)} \quad \text{for} \ i = 1, \ldots, k.
\]

Let \( \hat{\sigma}_1^{(\rho_j)} \) be the corresponding square roots of \( \hat{\sigma}_{r1}^{2(\rho_j)} \) for \( i = 1, \ldots, k \). This will yield \( t \) competing sets of \( k + 1 \) ML estimates of \( \mu \) and \( \sigma_i \ (i = 1, \ldots, k) \) each.

In order to determine the proper ML parameter estimates at step 1 we substitute each of the \( \hat{\sigma}_1^{(\rho_j)} \) \((i = 1, \ldots, k)\) values along with the corresponding \( \hat{\mu}_i^{(\rho_j)} \) values into equation (3.54) which is an adaptation of (3.35):

\[
\log L_3(\hat{\mu}_1^{(\rho_j)}, \hat{\sigma}_1^{(\rho_j)}, \ldots, \hat{\sigma}_k^{(\rho_j)}) = \sum_{i=1}^{k} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\hat{\sigma}_1^{(\rho_j)}) \right.
\]

\[-\frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_i^{(\rho_j)}}{\hat{\sigma}_1^{(\rho_j)}} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_i^{(\rho_j)}}{\hat{\sigma}_1^{(\rho_j)}} \right) \right]; \quad \text{for} \ j = 1, \ldots, t.
\]
Let \((\log L_3)_1^{(\rho_i)}\) be the largest of the \(t\) evaluations in (3.54). That is:

\[
(3.55) \quad (\log L_3)_1^{(\rho_i)} = \max_{1 \leq j \leq t} \{\log L_3(\hat{\mu}_1^{(\rho_i)}, \hat{\sigma}_{11}^{(\rho_i)}, ..., \hat{\sigma}_{k1}^{(\rho_i)})\}.
\]

The updated ML estimates of \(\mu\) and \(\sigma_i\) \((i = 1, ..., k)\) at step 1 are reported as \(\hat{\mu}_i = \hat{\mu}_1^{(\rho_i)}\) and \(\hat{\sigma}_{ii} = \hat{\sigma}_{11}^{(\rho_i)}\) for \(i = 1, ..., k\).

This procedure is continued until convergence is met. That is, let \(\hat{\mu}_s\) and \(\hat{\sigma}_{is}\) for \(i = 1, ..., k\) be the ML estimates of \(\mu\) and \(\sigma_i\) for \(i = 1, ..., k\) respectively at step \(s\). At step \(s + 1\), \(u_{ij(s+1)}\) for \(i = 1, ..., k\) and \(j = 1, ..., n_i\) are updated as follows:

\[
(3.56a) \quad u_{ij(s+1)} = y_{ij}; \text{ if } j = 1, ..., r_i
\]

\[
(3.56b) \quad u_{ij(s+1)} = \hat{\mu}_s - \hat{\sigma}_{is} W \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_{is}} \right); \text{ if } j = r_i + 1, ..., n_i.
\]

The ML estimates of \(\mu\) and \(\sigma_i\) for \(i = 1, ..., k\) at step \(s + 1\) are then the simultaneous solutions of the following \(k + 1\) equations:

\[
(3.57a) \quad \sum_{i=1}^{k} \frac{1}{\hat{\sigma}_{i(s+1)}} \sum_{j=1}^{n_i} \left( \frac{u_{ij(s+1)} - \hat{\mu}_{s+1}}{\hat{\sigma}_{i(s+1)}} \right) = 0
\]

\[
(3.57b) \quad -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij(s+1)} - \hat{\mu}_{s+1}}{\hat{\sigma}_{i(s+1)}} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_{is}} \right) = 0 \text{ for } i = 1, ..., k.
\]

Like in step 1 of this procedure, the solution of (3.57a) in terms of \(\hat{\mu}_{s+1}\) involves all of the \(\hat{\sigma}_{i(s+1)}\) for \(i = 1, ..., k\). Therefore a \((2k - 1)\)-degree polynomial in \(\hat{\mu}_{s+1}\) has to be solved at step \(s + 1\) in a similar manner in which it was solved at step 1.

The notation \(\hat{\sigma}_{i(s+1)}^2\) is introduced as follows:
The expressions for \( \hat{\sigma}^2_{i(s+1)} \) for \( i = 1, \ldots, k \), all of which involve \( \hat{\mu}_{s+1} \), are substituted into (3.57a) to give an expression involving \( \hat{\mu}_{s+1} \) as the only unknown:

\[
(3.58a) \quad \sum_{i=1}^{k} \left( r_i - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{s+1}}{\hat{\sigma}_{is}} \right) \right) \left[ \sum_{j=1}^{n_i} \left( u_{ij(s+1)} - \hat{\mu}_{s+1} \right) \right] = 0.
\]

To simplify the expression in (3.58a) the following notation is introduced in (3.59), (3.60) and (3.61):

\[
(3.59) \quad v_{i(s+1)} = \frac{\sum_{j=1}^{n_i} u_{ij(s+1)}}{n_i},
\]

\[
(3.60) \quad w_{i(s+1)} = \frac{\sum_{j=1}^{n_i} u_{ij(s+1)}^2}{n_i},
\]

\[
(3.61) \quad \xi_is = \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{s+1}}{\hat{\sigma}_{is}} \right), \quad \text{for } i = 1, \ldots, k.
\]

By using (3.59), (3.60) and (3.61), the expression in (3.58a) becomes:

\[
(3.62) \quad \sum_{i=1}^{k} \left( \xi_is - r_i \right) \left( \hat{\mu}_{s+1} - v_{i(s+1)} \right) \prod_{j=1 \atop j \neq i}^{k} \left( w_{i(s+1)} - 2v_{i(s+1)} \hat{\mu}_{s+1} + \hat{\mu}^2_{s+1} \right) = 0.
\]

Therefore, the estimate of \( \mu \) at step \( s + 1 \) is the solution of the \((2k - 1)\)-degree polynomial in \( \hat{\mu}_{s+1} \):
provided that

\[ (3.64) \prod_{i=1}^{k} \left( w_{i(s+1)} - 2v_{i(s+1)} \hat{\mu}_{s+1} + \hat{\mu}^2_{s+1} \right) \neq 0. \]

Like in step 1, a numerical solution of (3.63) will yield at least one real root, because \(2k - 1\) is an odd integer. Again, let \(t\), where \(1 \leq t \leq 2k - 1\) be the number of real roots. Then, there are \(2k - 1 - t\) imaginary roots, an even number.

As in step 1, one and only one of these real roots should be chosen as the ML estimate of \(\mu\) at step 1. The imaginary roots are discarded and the real root which maximizes the log likelihood function in (3.35) and also satisfies inequality (3.64) is used to evaluate \(\hat{\mu}_{s+1}\).

Assume that at step \(s\) the real solutions of \(\hat{\mu}_{s+1}\) in (3.63) are \(\rho_1, \ldots, \rho_t\), where \(1 \leq t \leq 2k - 1\), and \(t\) is odd. Also assume that all \(\rho_i\) for \(i = 1, \ldots, t\) satisfy inequality (3.64). Let the corresponding estimates for \(\mu\), at step \(s + 1\), be:

\[ (3.65a) \quad \hat{\mu}_{s+1}^{(\rho_j)} = \rho_j; \quad \text{for } j = 1, \ldots, t. \]

Next, substitute each of these \(t\) values in (3.58b) to get \(\hat{\sigma}_{i(s+1)}^{2(\rho_j)}\) for \(i = 1, \ldots, k\) and \(j = 1, \ldots, t\). Therefore, for \(j = 1, \ldots, t\),

\[ (3.65b) \quad \hat{\sigma}_{i(s+1)}^{2(\rho_j)} = \left( \sum_{j=1}^{n_i} \left( u_{ij(s+1)} - \hat{\rho}_{s+1}^{(\rho_j)} \right) \right)^2 \]

\[ r_i - \sum_{j=r_{i+1}}^{n_i} \lambda \left( y_{ij} - \hat{\mu}_z \right) \hat{\sigma}_{i(s+1)} \]

Let \(\hat{\sigma}_{i(s+1)}^{(\rho_j)}\) for \(i = 1, \ldots, k\) be the corresponding square roots. This will yield \(t\) competing sets of \(k + 1\) ML estimates of \(\mu\) and \(\sigma_i\) \((i = 1, \ldots, k)\) each. In order to find the right set of ML estimates at step \(s + 1\) substitute each of the \(\hat{\sigma}_{i(s+1)}^{(\rho_j)}\)
values for $i = 1, \ldots, k$ along with the corresponding $\mu^{(p_i)}_{z+1}$ values into equation (3.66), which is an adaptation of (3.35). Therefore, for $j = 1, \ldots, t$,

$$
(3.66) \quad \log L_3(\mu^{(p_j)}_{z+1}; \sigma_{i(s+1)}^{(p_j)}, \ldots, \sigma_{k(s+1)}^{(p_j)}) = \sum_{i=1}^{k} \left[ -\frac{r_i}{2} \log(2\pi) - r_i \log(\sigma_{i(s+1)}^{(p_j)}) \right] + \frac{-1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu^{(p_j)}_{z+1}}{\sigma_{i(s+1)}^{(p_j)}} \right)^2 + \sum_{j=r_i+1}^{n} \log \Phi \left( \frac{y_{ij} - \mu^{(p_j)}_{z+1}}{\sigma_{i(s+1)}^{(p_j)}} \right); \quad \text{for } i = 1, \ldots, k.
$$

Let, $(\log L_3)^{(p_i)}_{z+1}$ be the largest of the $t$ evaluations in (3.66). That is,

$$
(3.67) \quad (\log L_3)^{(p_i)}_{z+1} = \max_{1 \leq s \leq t} \{ \log L_3(\mu^{(p_j)}_{z+1}; \sigma_{i(s+1)}^{(p_j)}, \ldots, \sigma_{k(s+1)}^{(p_j)}) \}.
$$

Then, the updated ML estimates of $\mu$ and $\sigma_i$ ($i = 1, \ldots, k$), that is $\hat{\mu}_{z+1}$ and $\hat{\sigma}_{i(s+1)}$ ($i = 1, \ldots, k$) are $\hat{\mu}^{(p_i)}_{z+1}$ and $\hat{\sigma}_{i(s+1)}^{(p_i)}$ ($i = 1, \ldots, k$) respectively.

At each step $s + 1$, $|\hat{\mu}_s - \hat{\mu}_{z+1}|$, for $i = 1, \ldots, k$, and $|\hat{\sigma}_s - \hat{\sigma}_{i(s+1)}|$ are evaluated. The convergence criterion is met when all these $k + 1$ absolute differences are less than a preassigned tolerance value. Once convergence is achieved the ML estimates of $\mu_i$ and $\sigma$ are reported as $\hat{\mu}_s$ and $\hat{\sigma}_i$ respectively.

3.5 Case $H_4$: $\mu_1 \neq \ldots \neq \mu_k; \sigma_1 \neq \ldots \neq \sigma_k$

3.5.1 The ML Estimates Under Case $H_4$

This case is defined in (1.3d). It is a case of overall heterogeneity. That is, there exists skewness heterogeneity and at least one pair of the $k$ lognormal distributions differ in location. In this case the $2k$ parameters to be estimated are: $\mu_i$ and $\sigma_i$ for $i = 1, \ldots, k$. The likelihood function for this case is:

$$
(3.68) \quad L_4(\mu_i, \sigma_i; i = 1, \ldots, k) = \prod_{i=1}^{k} \left[ \prod_{j=1}^{r_i} \frac{1}{\sigma_i} \phi \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \prod_{j=r_i+1}^{n} \Phi \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \right]
$$

where $\phi(t)$ and $\Phi(t)$ were defined in (2.10) and (2.11), respectively.
The log likelihood function is:

\[(3.69) \quad \log L_i(\mu_i, \sigma_i; i = 1, \ldots, k) = \sum_{i=1}^{k} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\sigma_i) \right. \]
\[\left. - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \right].\]

The partial derivatives of the log likelihood function (3.52) with respect to $\mu_i$ and $\sigma_i$, for $i = 1, \ldots, k$ are:

\[(3.70a) \quad \frac{\partial \log L_i(\mu_i, \sigma_i)}{\partial \mu_i} = \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) = 0\]

\[(3.70b) \quad \frac{\partial \log L_i(\mu_i, \sigma_i)}{\partial \sigma_i} = -r_i + \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right)^2 \]
\[\quad - \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) = 0\]

where $W(x)$ was defined in (2.14).

The simultaneous solutions of these $2k$ equations give the ML estimates of $\mu_i$ and $\sigma_i$, for $i = 1, \ldots, k$. There are no closed form solutions to these $2k$ simultaneous equations. Therefore, the EM algorithm procedure will be used.

3.5.2 The EM Algorithm Under $H_4$

The EM algorithm will be used to calculate the ML estimates of $\mu_i$ and $\sigma_i$ for $i = 1, \ldots, k$ in a similar manner that it was used in cases $H_1$ and $H_2$. This is comparable to the procedure used by Stoline (1991), which was described in Section 2.3.

Each left-censored data value $y_{ij}$ ($i = 1, \ldots, k; j = r_i + 1, \ldots, n_i$) is replaced by an estimate obtained from the following conditional expectation:
(3.71) \[ E [Y | Y \leq y_{ij}] = \mu_i - \sigma_i \cdot W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \]

where it is assumed that \( Y \) is \( N(\mu_i, \sigma_i) \) and where \( y_{ij} = \log(L_{ij}) \) for \( i = 1, \ldots, k \) and \( j = r_i + 1, \ldots, n_i \).

Values of \( u_{ij} \) for \( i = 1, \ldots, k \) are defined as follows:

(3.72a) \[ u_{ij} = y_{ij}; \text{ if } j = 1, \ldots, r_i \]

(3.72b) \[ u_{ij} = \mu_i - \sigma_i \cdot W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right); \text{ if } j = r_i + 1, \ldots, n_i. \]

The following equation can be derived from (3.76b):

(3.73) \[ \frac{u_{ij} - \mu_i}{\sigma_i} = -\log(L_{ij}) \cdot \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right); \text{ for } j = r_i + 1, \ldots, n_i, \]

Equations (3.70a) and (3.70b) can be written as (3.74a) and (3.74b), respectively using the new \( u_{ij} \) variables.

(3.74a) \[ \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right) + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right) \]

\[ - \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right) - \sum_{j=r_i+1}^{n_i} \log(L_{ij}) \cdot \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k \]

(3.74b) \[ -r_i + \sum_{j=1}^{r_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right)^2 \]

\[ - \sum_{j=r_i+1}^{n_i} \log(L_{ij}) \cdot \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k. \]

By using (3.73), equations (3.74a) and (3.74b) can be written as (3.75a) and (3.75b), respectively:
\[(3.75a) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right) + \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) - \sum_{j=r_i+1}^{n_i} W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k \]

\[(3.75b) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \left( W \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \right)^2 - \sum_{j=r_i+1}^{n_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right) W \left( \frac{y_{ij} - \mu_i}{\sigma} \right) = 0, \text{ for } i = 1, \ldots, k. \]

Finally equations (3.75a) and (3.75b) can be simplified as (3.76a) and (3.76b), respectively:

\[(3.76a) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right) = 0, \text{ for } i = 1, \ldots, k \]

\[(3.76b) \quad \sum_{i=1}^{k} \left[ -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij} - \mu_i}{\sigma_i} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \right] = 0 \]

where the function \(\lambda(x)\) was defined in (2.21).

At step 0 of the EM algorithm all the uncensored observations are used to calculate the initial estimates of \(\mu_i\) and \(\sigma_i\) for \(i = 1, \ldots, k\) as follows:

\[(3.77a) \quad \hat{\mu}_{i0} = \frac{\sum_{j=1}^{r_i} y_{ij}}{r_i}, \text{ for } i = 1, \ldots, k \]

\[(3.77b) \quad \hat{\sigma}_{i0} = \sqrt{\frac{\sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_{i0})^2}{r_i}}, \text{ for } i = 1, \ldots, k. \]

At step 1, \(u_{ij1}\) for \(i = 1, \ldots, k\) and \(j = 1, \ldots, n_i\) are defined by referring to (3.72a) and (3.72b). Thus,

\[(3.78a) \quad u_{ij1} = y_{ij}; \text{ if } j = 1, \ldots, r_i \]
At step 1 equations (3.79a) and (3.79b) can be written by adapting (3.76a) and (3.76b), respectively:

\begin{align}
(3.79a) \quad \sum_{j=1}^{n_i} \left( \frac{u_{ij1} - \hat{\mu}_{i1}}{\hat{\sigma}_{i1}} \right) &= 0 \quad \text{for } i = 1, \ldots, k \\
(3.79b) \quad -r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij1} - \hat{\mu}_{i1}}{\hat{\sigma}_{i1}} \right)^2 - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{i0}}{\hat{\sigma}_{i0}} \right) &= 0, \quad \text{for } i = 1, \ldots, k
\end{align}

where \( \hat{\mu}_{i1} \) and \( \hat{\sigma}_{i1} \) for \( i = 1, \ldots, k \) are the ML estimates of \( \mu_i \) and \( \sigma_i \) for \( i = 1, \ldots, k \) respectively at step 1.

From (3.79a) the updated ML estimate of \( \mu_i \) for \( i = 1, \ldots, k \) at step 1 can be calculated. That is,

\begin{equation}
(3.80a) \quad \hat{\mu}_{i1} = \frac{\sum_{j=1}^{n_i} u_{ij1}}{n_i}, \quad \text{for } i = 1, \ldots, k.
\end{equation}

The estimates \( \hat{\mu}_{i1} \) for \( i = 1, \ldots, k \) can now be substituted in (3.79b). Then the updated ML estimate of \( \sigma_i^2 \) for \( i = 1, \ldots, k \) at step 1 can be calculated.

\begin{equation}
(3.80b) \quad \hat{\sigma}_{i1}^2 = \frac{\sum_{j=1}^{n_i} (u_{ij1} - \hat{\mu}_{i1})^2}{r_i - \sum_{j=1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{i0}}{\hat{\sigma}_{i0}} \right)}, \quad \text{for } i = 1, \ldots, k.
\end{equation}

The square root of these estimates, \( \hat{\sigma}_{i1} \) for \( i = 1, \ldots, k \), are the ML estimates of \( \sigma_i \) for \( i = 1, \ldots, k \) at step 1.

This procedure is continued until convergence is met. That is, let \( \hat{\mu}_{is} \) and \( \hat{\sigma}_{is} \) for \( i = 1, \ldots, k \) be the ML estimates of \( \mu_i \) and \( \sigma_i \) for \( i = 1, \ldots, k \) respectively at step \( s \). At step \( s + 1 \), \( u_{ij(s+1)} \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, n_i \) are updated as follows:
The ML estimates of $\mu_i$ and $\sigma_i$ for $i = 1, ..., k$ at step $s+1$ can then be found from the following $2k$ equations:

\begin{equation}
\sum_{j=1}^{n_i} \frac{u_{ij}(s+1) - \hat{\mu}_i(s+1)}{\hat{\sigma}_i(s+1)} = 0, \text{ for } i = 1, ..., k.
\end{equation}

\begin{equation}
-r_i + \sum_{j=1}^{n_i} \left( \frac{u_{ij}(s+1) - \hat{\mu}_i(s+1)}{\hat{\sigma}_i(s+1)} \right)^2 - \sum_{j=r_i+1}^{n} \lambda \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}_i} \right) = 0, \text{ for } i = 1, ..., k.
\end{equation}

From (3.82a) the ML estimates of $\mu_i$ for $i = 1, ..., k$ at step $s+1$ can be calculated. That is,

\begin{equation}
\hat{\mu}_i(s+1) = \frac{\sum_{j=1}^{n_i} u_{ij}(s+1)}{n_i}, \text{ for } i = 1, ..., k.
\end{equation}

The estimates $\hat{\mu}_i(s+1)$ can now be substituted in (3.82b). Then the updated ML estimate of $\sigma_i^2$ for $i = 1, ..., k$ at step $s+1$ can be calculated:

\begin{equation}
\hat{\sigma}_i^2(s+1) = \frac{\sum_{j=1}^{n_i} \left( u_{ij}(s+1) - \hat{\mu}_i(s+1) \right)^2}{r_i - \sum_{j=1}^{n} \lambda \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}_i} \right)}, \text{ for } i = 1, ..., k.
\end{equation}

The square roots of these estimates, $\hat{\sigma}_i(s+1)$ for $i = 1, ..., k$, are the ML estimates of $\sigma_i$ for $i = 1, ..., k$, at step $s+1$.

At each step $s+1$, $|\hat{\mu}_i - \hat{\mu}_i(s+1)|$ for $i = 1, ..., k$ and $|\hat{\sigma}_i - \hat{\sigma}_i(s+1)|$ for $i = 1, ..., k$ are evaluated. The convergence criterion is met when all these $2k$ absolute differences are less than a preassigned tolerance value. Once convergence is achieved the ML estimates of $\mu_i$ and $\sigma_i$ for $i = 1, ..., k$ are reported as $\hat{\mu}_i$ and $\hat{\sigma}_i$ respectively.
CHAPTER IV

HYPOTHESIS TESTS FOR THE EQUALITY OF THE K MEDIANS

4.1 Introduction

In this chapter the three stage procedure described in Chapter I will be utilized to test the equality of the \( k \) medians. It will be assumed that the data are as described in Section 3.1.

The eventual goal of the three-stage procedure is to determine whether the \( k \) medians of the \( k \) independent lognormal distributions are equal. Therefore Test 1 through Test 4 defined in (1.4a) through (1.4d), respectively, will be conducted in a sequence according to the three-stage procedure. This procedure was described in Chapter I. These tests incorporate Case \( H_1 \) through Case \( H_4 \) defined in (1.3a) through (1.3d) respectively. Asymptotic \( \alpha \)-level chi-square tests will be used throughout this test procedure. These chi-square tests will be calculated using the estimated log likelihood functions which will be defined in Section 4.2.

4.2 The Estimated Log Likelihood Functions for Cases \( H_1 - H_4 \)

It has been shown in Chapter III how the EM algorithm can be used to find the ML estimates of the lognormal parameters for each of the four cases defined in (1.3a) through (1.3d). Once the ML estimates are obtained, the corresponding log likelihood functions can be evaluated. The estimated log likelihood functions are now defined for each of the four cases.

For Case \( H_1 \), equation (3.3) can be estimated to yield the estimated log

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likelihood function as follows:

\[(4.1) \quad LL_1 = \log L_1(\hat{\mu}, \hat{\sigma}) = \sum_{i=1}^{k} \left[ -\frac{r_i}{2} \log(2\pi) - r_i \log(\hat{\sigma}) \right. \\
- \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}}{\hat{\sigma}} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}}{\hat{\sigma}} \right) \]  

using estimates $\hat{\mu}$ and $\hat{\sigma}$ which are the estimates of $\mu$ and $\sigma$ respectively at step $s$ of the EM algorithm in Section 3.2.2.

For Case $H_2$, equation (3.19) can be estimated to yield the estimated log likelihood function as follows:

\[(4.2) \quad LL_2 = \log L_2(\hat{\mu}_1, \ldots, \hat{\mu}_k, \hat{\sigma}) = \sum_{i=1}^{k} \left[ \left( -\frac{r_i}{2} \right) \log(2\pi) - r_i \log(\hat{\sigma}) \right. \\
- \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}} \right) \]  

using estimates $\hat{\mu}_i$ ($i = 1, \ldots, k$) and $\hat{\sigma}$ which are the estimates of $\mu_i$ ($i = 1, \ldots, k$) and $\sigma$, respectively, at step $s$ of the EM algorithm in Section 3.3.2.

For Case $H_3$, equation (3.35) can be estimated to yield the estimated log likelihood function as follows:

\[(4.3) \quad LL_3 = \log L_3(\hat{\mu}, \hat{\sigma}_1, \ldots, \hat{\sigma}_k) = \sum_{i=1}^{k} \left[ \left( -\frac{r_i}{2} \right) \log(2\pi) - r_i \log(\hat{\sigma}_i) \right. \\
- \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}}{\hat{\sigma}_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}}{\hat{\sigma}_i} \right) \]  

using estimates $\hat{\mu}$ and $\hat{\sigma}_i$ ($i = 1, \ldots, k$) which are the estimates of $\mu$ and $\sigma_i$ ($i = 1, \ldots, k$), respectively, at step $s$ of the EM algorithm in Section 3.4.2.

For Case $H_4$, equation (3.69) can be estimated to yield the estimated log likelihood function as follows:
\[ LL_4 = \log L_4(\hat{\mu}_i, \hat{\sigma}_i; i = 1, 2, ..., k) = \sum_{i=1}^{k} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\hat{\sigma}_i) \right. \]
\[ - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}_i} \right) \]

using estimates \( \hat{\mu}_i \) and \( \hat{\sigma}_i \) (\( i = 1, ..., k \)) which are the estimates of \( \mu_i \) and \( \sigma_i \) (\( i = 1, ..., k \)), respectively, at step \( s \) of the EM algorithm in Section 3.5.2.

4.3 The Three-Stage Procedure for Testing the Equality of the \( k \) Medians

Four hypothesis cases are defined for the \( k \) sample case in (4.5a) - (4.5d) as follows:

(4.5a) \( H_1: \mu_1 = ... = \mu_k = \mu; \sigma_1 = ... = \sigma_k = \sigma, \)

(4.5b) \( H_2: \mu_1 \neq ... \neq \mu_k; \sigma_1 = ... = \sigma_k = \sigma, \)

(4.5c) \( H_3: \mu_1 = ... = \mu_k = \mu; \sigma_1 \neq ... \neq \sigma_k, \) and

(4.5d) \( H_4: \mu_1 \neq ... \neq \mu_k; \sigma_1 \neq ... \neq \sigma_k. \)

The three-stage procedure for testing the equality of the \( k \) medians includes the following four tests:

(4.6a) Test 1: \( H_1 \) versus \( H_4 \)

(4.6b) Test 2: \( H_2 \) versus \( H_4 \)

(4.6c) Test 3: \( H_1 \) versus \( H_2 \)

(4.6d) Test 4: \( H_3 \) versus \( H_4 \).

The three stages which comprise the three-stage procedure are now described for the \( k \)-sample case.

4.3.1 Stage 1

At Stage 1 a test of overall homogeneity is conducted. Overall homogeneity is expressed in Case \( H_1 \) defined in (4.5a). To test the overall homogeneity,
hypothesis $H_1$ will be tested versus hypothesis $H_4$ at level of significance $\alpha_1$ and $p$-value $p_1$. This is Test 1 defined in (4.6a). The chi-square test at Stage 1 is:

(4.7) $\chi^2_1 = -2(LL_1 - LL_4)$

where $LL_1$ and $LL_4$ were defined in (4.1) and (4.4) respectively. A decision against $H_1$ and in favor of $H_4$ is concluded if:

(4.8) $\chi^2_1 > \chi^2(\alpha_1, 2k - 2)$.

Let:

(4.9) $p_1 = Pr[\chi^2(2k - 2) > \chi^2_1]$.

Thus, if $p_1 \geq \alpha_1$, then $H_1$ can not be rejected and overall homogeneity is concluded; if $p_1 < \alpha_1$, $H_1$ can be rejected in favor of $H_4$ and overall heterogeneity is concluded at a significance level of $\alpha_1$.

4.3.2 Stage 2

At Stage 2 a test of skewness homogeneity is performed. Hypothesis $H_2$, defined in (4.5b), will be tested against hypothesis $H_4$, defined in (4.5d). This is Test 2 defined in (4.6b). It will be performed at Stage 2 at level of significance $\alpha_2$ and $p$-value $p_2$.

The chi-square test at this stage is:

(4.10) $\chi^2_2 = -2(LL_2 - LL_4)$

where $LL_2$ and $LL_4$ were defined in (4.2) and (4.4), respectively. A decision against $H_2$ and in favor of $H_4$ is concluded if:

(4.11) $\chi^2_2 > \chi^2(\alpha_2, k - 1)$.

Let:

(4.12) $p_2 = Pr[\chi^2(k - 1) > \chi^2_2]$.

Thus, if $p_2 \geq \alpha_2$, then $H_2$ can not be rejected and skewness homogeneity is concluded; on the other hand, if $p_2 < \alpha_2$, $H_2$ can be rejected in favor of $H_4$ and
skewness heterogeneity is concluded at a significance level of $\alpha_2$.

4.3.3 Stage 3

This is the final stage of the procedure. The test to be conducted at this stage depends on the outcome of the preliminary test for skewness homogeneity performed at Stage 2. If skewness homogeneity is concluded in the preliminary test at Stage 2 then, at Stage 3, hypothesis $H_1$ will be tested versus hypothesis $H_2$. This is Test 3 defined in (4.6c). On the other hand, if skewness heterogeneity is concluded in the preliminary test at Stage 2 then, at Stage 3, hypothesis $H_3$ defined in (4.5c) will be tested versus hypothesis $H_4$. This is Test 4 defined in (4.6d).

If Test 3 is to be conducted at Stage 3, then the chi-square test at this stage is:

\[ \chi_3^2 = -2(LL_1 - LL_2) \]

where $LL_1$ and $LL_2$ were defined in (4.1) and (4.2), respectively. A decision against $H_1$ and in favor of $H_2$ is concluded, at level of significance $\alpha_3$ and $p$-value $p_3$ if:

\[ \chi_3^2 > \chi^2(\alpha_3, k - 1). \]

Let:

\[ p_3 = \Pr[\chi^2(k - 1) > \chi_3^2]. \]

Then the equality of the medians is concluded if $p_3 \geq \alpha_3$; if $p_3 < \alpha_3$ then at least one pair of medians is unequal.

If Test 4 is to be conducted at Stage 3, then the chi-square test at this stage is:

\[ \chi_4^2 = -2(LL_3 - LL_4) \]

where $LL_3$ and $LL_4$ were defined in (4.3) and (4.4), respectively. A decision
against $H_3$ and in favor of $H_4$ is concluded, at level of significance $\alpha_4$ and $p$-value $p_4$ if:

$$\chi^2_4 \geq \chi^2(\alpha_4, k - 1).$$

Let:

$$p_4 = \Pr \left[ \chi^2(k - 1) > \chi^2_4 \right].$$

Then the equality of the medians is concluded if $p_4 \geq \alpha_4$; if $p_4 < \alpha_4$ then at least one pair of medians is unequal.
CHAPTER V

THE ML ESTIMATES AND THE THREE-STAGE PROCEDURE FOR K=3

5.1 Introduction

In this chapter we consider the simplest case, k = 3. It is assumed that three independent samples are available from three lognormal distributions, each with parameters \( \mu_i \) and \( \sigma_i \) for \( i = 1, 2, 3 \). Assume there are \( n_i \) observations in the \( i^{th} \) sample. It is convenient to let \( x_{ij} \) denote the \( j^{th} \) measurement on the \( i^{th} \) sample where \( i = 1, 2, 3 \) and \( j = 1, ..., n_i \). It is assumed that these data are all measured in a laboratory where the values of the detection limits, denoted by \( L_{ij} \), are known for each observation \( x_{ij} \).

It is further assumed that for the \( i^{th} \) sample \( n_i - r_i \) observations have left-censored measurements below the detection limit levels while the remaining \( r_i \) observations are uncensored. For simplicity, assume that the first \( r_i \) observations in the \( i^{th} \) sample are uncensored and that the last \( n_i - r_i \) observations are left-censored. Thus, the data for all three samples are as follows:

<table>
<thead>
<tr>
<th>Sample 1</th>
<th>Sample 2</th>
<th>Sample 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{11} )</td>
<td>( x_{21} )</td>
<td>( x_{31} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( x_{1,r_1} )</td>
<td>( x_{2,r_2} )</td>
<td>( x_{3,r_3} )</td>
</tr>
<tr>
<td>( L_{1,r_1+1} )</td>
<td>( L_{2,r_2+1} )</td>
<td>( L_{3,r_3+1} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( L_{1,n_1} )</td>
<td>( L_{2,n_2} )</td>
<td>( L_{3,n_3} )</td>
</tr>
</tbody>
</table>
In the attempt to find maximum likelihood (ML) estimates for $\mu_i$ and $\sigma_i$, it is convenient to log transform the data by defining $y_{ij}$, for $i = 1, 2, 3$ as follows:

\begin{align*}
(5.1a) & \quad y_{ij} = \log(x_{ij}); \text{ for } j = 1, \ldots, r_i \\
(5.1b) & \quad y_{ij} = \log(L_{ij}); \text{ for } j = r_i + 1, \ldots, n_i.
\end{align*}

Four cases were defined for the $k$-sample in (1.3a) - (1.3d) in Section 1.1. The corresponding four cases for the case $k = 3$ are now defined in (5.2a) - (5.2d):

\begin{align*}
(5.2a) & \quad H_1: \mu_1 = \mu_2 = \mu_3 = \mu; \sigma_1 = \sigma_2 = \sigma_3 = \sigma \\
(5.2b) & \quad H_2: \mu_1 \neq \mu_2 \neq \mu_3; \sigma_1 = \sigma_2 = \sigma_3 = \sigma \\
(5.2c) & \quad H_3: \mu_1 = \mu_2 = \mu_3 = \mu; \sigma_1 \neq \sigma_2 \neq \sigma_3 \\
(5.2d) & \quad H_4: \mu_1 \neq \mu_2 \neq \mu_3; \sigma_1 \neq \sigma_2 \neq \sigma_3.
\end{align*}

The log likelihood functions of each of the four cases expressed in (5.2a) - (5.2d) can now be determined in terms of the log transformed data $y_{ij}$. The EM algorithm will then be used to derive the maximum likelihood (ML) estimates of $\mu_i$ and $\sigma_i$, $i = 1, 2, 3$, for each of the four cases. These estimates will then be used to evaluate the log likelihood functions, $LL_1 - LL_4$ defined in (4.1) - (4.4) in Section 4.1.

A computer program has been written to obtain the ML estimates of the lognormal parameters. This program is used in a simulation study described in Chapter VI. Details are provided here for the specific method used to estimate the lognormal parameters, especially for Case $H_3$ with the solution of the 5th degree polynomial.

5.2 The ML Estimates in the Case $k = 3$

5.2.1 The ML Estimates Under $H_1: \mu_1 = \mu_2 = \mu_3 = \mu, \sigma_1 = \sigma_2 = \sigma_3 = \sigma$

The log likelihood function for the $k$-sample $H_1$ case was given in (3.5) in
Section 3.2.1. This can be adapted for the case $k = 3$ as follows:

\begin{equation}
(5.3) \quad \log L_1(\mu, \sigma) = \sum_{i=1}^{3} \left[ \frac{-r_i}{2} \log(2\pi) - r_i \log(\sigma) - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma} \right)^2 \right. \\
\left. + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \mu}{\sigma} \right) \right].
\end{equation}

The EM algorithm is used to calculate the ML estimates of $\mu$ and $\sigma$ iteratively, as it was described in Section 3.2.2.

All the uncensored observations are used at step 0 to calculate the initial estimates of $\mu$ and $\sigma$ as follows:

\begin{align}
(5.4a) \quad \hat{\mu}_0 &= \frac{\sum_{i=1}^{3} \sum_{j=1}^{r_i} y_{ij}}{\sum_{i=1}^{k} r_i} \\
(5.4b) \quad \hat{\sigma}_0 &= \sqrt{\frac{\sum_{i=1}^{3} \sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_0)^2}{\sum_{i=1}^{3} r_i}}.
\end{align}

Let $\hat{\mu}_s$ and $\hat{\sigma}_s$ be the ML estimates of $\mu$ and $\sigma$ respectively, at step $s$. At step $s + 1$, the values of $u_{ij}$, for $i = 1, 2, 3$ are updated as follows:

\begin{align}
(5.5a) \quad u_{ij(s+1)} &= y_{ij}; \quad \text{if } j = 1, \ldots, r_i \\
(5.5b) \quad u_{ij(s+1)} &= \hat{\mu}_s - \hat{\sigma}_s W \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right); \quad \text{if } j = r_i + 1, \ldots, n_i.
\end{align}

The updated ML estimates of $\mu$ and $\sigma$ at step $s + 1$ can be calculated:

\begin{align}
(5.6a) \quad \hat{\mu}_{s+1} &= \frac{\sum_{i=1}^{3} \sum_{j=1}^{n_i} u_{ij(s+1)}}{n_i}
\end{align}
\[ (5.6b) \quad \hat{\sigma}_{s+1} = \sqrt{\frac{3}{\sum_{i=1}^{3} \sum_{j=1}^{n_i} \left( u_{ij(s+1)} - \hat{\mu}_{s+1} \right)^2}{\sum_{i=1}^{3} \left[ r_i - \sum_{j=1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right) \right]}}. \]

At each step \( s + 1 \), \( |\hat{\mu}_s - \hat{\mu}_{s+1}| \) and \( |\hat{\sigma}_s - \hat{\sigma}_{s+1}| \) are calculated. The convergence criterion is met when both \( |\hat{\mu}_s - \hat{\mu}_{s+1}| \) and \( |\hat{\sigma}_s - \hat{\sigma}_{s+1}| \) are less than a preassigned tolerance value, say \( 10^{-6} \). Once convergence is achieved the ML estimates of \( \mu \) and \( \sigma \) are reported as \( \hat{\mu}_s \) and \( \hat{\sigma}_s \) respectively.

The log likelihood function of the ML estimates can now be found by adapting (5.3) as:

\[ (5.7) \quad LL_1(\hat{\mu}_s, \hat{\sigma}_s) = \sum_{i=1}^{3} \left[ -\frac{r_i}{2} \log(2\pi) - r_i \log(\hat{\sigma}_s) - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right)^2 \right] \]

\[ + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_s} \right). \]

5.2.2 The ML Estimates Under \( H_2: \mu_1 \neq \mu_2 \neq \mu_3, \sigma_1 = \sigma_2 = \sigma_3 = \sigma \)

The log likelihood function for the \( k \)-sample \( H_2 \) case was given in (3.23). This can be adapted for the case \( k = 3 \) as follows:

\[ (5.8) \quad \log L_2(\mu_1, \mu_2, \mu_3, \sigma) = \sum_{i=1}^{3} \left[ -\frac{r_i}{2} \log(2\pi) - r_i \log(\sigma) \right. \]

\[ \left. -\frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu_i}{\sigma} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \mu_i}{\sigma} \right) \right]. \]

The EM algorithm is used to calculate the ML estimates of \( \mu_1, \mu_2, \mu_3 \) and \( \sigma \) iteratively as it was described in Section 3.3.2.

All the uncensored observations are used to calculate the initial estimates of \( \mu_1, \mu_2, \mu_3 \) and \( \sigma \) at step 0 as follows:
\begin{align}
(5.9a) \quad \hat{\mu}_{i0} &= \frac{\sum_{j=1}^{r_i} y_{ij}}{r_i}, \quad \text{for } i = 1, 2, 3 \\
(5.9b) \quad \hat{\sigma}_0 &= \sqrt{\frac{\sum_{i=1}^{3} \sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_{i0})^2}{\sum_{i=1}^{3} r_i}}. \\
\end{align}

Let \( \hat{\mu}_{1s}, \hat{\mu}_{2s}, \hat{\mu}_{3s}, \) and \( \hat{\sigma}_s \) be the ML estimates of \( \mu_1, \mu_2, \mu_3 \) and \( \sigma \) respectively, at step \( s \). At step \( s + 1 \) the values of \( u_{ij} \), for \( i = 1, 2, 3 \), are updated as follows:

\begin{align}
(5.10a) \quad u_{ij(s+1)} &= y_{ij}; \quad \text{if } j = 1, \ldots, r_i \\
(5.10b) \quad u_{ij(s+1)} &= \hat{\mu}_{is} - \hat{\sigma}_s W \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_s} \right); \quad \text{if } j = r_i + 1, \ldots, n_i. \\
\end{align}

The updated ML estimates of \( \mu_1, \mu_2 \) and \( \mu_3 \) at step \( s + 1 \) can be calculated:

\begin{align}
(5.11a) \quad \hat{\mu}_{i(s+1)} &= \frac{\sum_{j=1}^{n_i} u_{ij(s+1)}}{n_i}, \quad \text{for } i = 1, 2, 3 \\
(5.11b) \quad \hat{\sigma}_{s+1}^2 &= \sqrt{\frac{\sum_{i=1}^{3} \sum_{j=1}^{n_i} (u_{ij(s+1)} - \hat{\mu}_{i(s+1)})^2}{\sum_{i=1}^{3} r_i - \sum_{j=r_i+1}^{n_i} \frac{\lambda (y_{ij} - \hat{\mu}_{is})}{\hat{\sigma}_s}}}.
\end{align}

At each step \( s + 1 \), \( | \hat{\mu}_{is} - \hat{\mu}_{i(s+1)} | \), for \( i = 1, 2, 3 \), and \( | \hat{\sigma}_s - \hat{\sigma}_{s+1} | \) are evaluated. The convergence criterion is met when all of these four absolute differences are less than a preassigned tolerance value. Once convergence is achieved the ML estimates of \( \mu_i \) and \( \sigma \) are reported as \( \hat{\mu}_{is} \) and \( \hat{\sigma}_s \) respectively.

The log likelihood function of the ML estimates can now be evaluated by adapting (5.8) as:
(5.12) \[ LL_2(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}) = \sum_{i=1}^{3} \left[ \left(\frac{-r_i}{2}\right) \log(2\pi) - r_i \log(\hat{\sigma}) \right. \]
\[ \left. - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}} \right) \right]. \]

5.2.3 The ML Estimates Under \( H_3; \mu_1 = \mu_2 = \mu_3 = \mu, \sigma_1 \neq \sigma_2 \neq \sigma_3 \)

The log likelihood function for the \( k \)-sample \( H_3 \) case was given in (3.39). This can be adapted for the case \( k = 3 \) as follows:

(5.13) \[ \log L_3(\mu, \sigma_1, \sigma_2, \sigma_3) = \sum_{i=1}^{3} \left[ \left(\frac{-r_i}{2}\right) \log(2\pi) - r_i \log(\sigma_i) \right. \]
\[ \left. - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \mu}{\sigma_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \mu}{\sigma_i} \right) \right]. \]

The EM algorithm is used to evaluate the ML estimates of \( \mu, \sigma_1, \sigma_2 \) and \( \sigma_3 \) iteratively as it was described in Section 3.4.2. This case differs from the previous two cases in an important detail. Like in the \( k \)-sample \( H_3 \), described in section 3.4.2, a polynomial of degree 5 in \( \mu \) is solved at each step of the EM algorithm. The iterative procedure suggested in Section 3.4.2 for the \( k \)-sample case will now be adapted and used for the case \( k = 3 \).

At step 0 of the EM algorithm all the uncensored observations are used to calculate the initial estimates of \( \mu \) and \( \sigma_i \), for \( i = 1, 2, 3 \).

(5.14a) \[ \hat{\mu}_0 = \frac{3}{\sum_{i=1}^{3} \sum_{j=1}^{r_i} y_{ij}} \]
(5.14b) \[ \hat{\sigma}_{i0} = \sqrt{\frac{3}{\sum_{i=1}^{3} r_i} \frac{\sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_0)^2}{r_i}}, \quad \text{for } i = 1, 2, 3. \]
Let $\hat{\mu}_s$, $\hat{\sigma}_{1s}$, $\hat{\sigma}_{2s}$ and $\hat{\sigma}_{3s}$ be the ML estimates of $\mu$, $\sigma_1$, $\sigma_2$, and $\sigma_3$ respectively, at step $s$. At step $s+1$ the values of $u_{ij}$ for $i = 1, 2, 3$ and $j = 1, \ldots, n_i$ are updated as follows:

(5.15a) $u_{ij(s+1)} = y_{ij};$ if $j = 1, \ldots, r_i$

(5.15b) $u_{ij(s+1)} = \hat{\mu}_s - \hat{\sigma}_{is} W \left( \frac{y_{ij} - \hat{\mu}_s}{\hat{\sigma}_{is}} \right);$ if $j = r_i + 1, \ldots, n_i$

The estimate of $\hat{\mu}$ at step $s+1$ is the solution of the following fifth degree polynomial in $\hat{\mu}_{s+1}$.

(5.16) $\sum_{i=1}^{3} \left[ (\xi_{is} - r_i)(\hat{\mu}_{s+1} - v_{i(s+1)}) \prod_{j=1, j \neq i}^{3} (w_{i(s+1)} - 2v_{i(s+1)}\hat{\mu}_{s+1} + \hat{\mu}_{s+1}^2) \right] = 0$

provided that

(5.17) $\prod_{i=1}^{3} (w_{i(s+1)} - 2v_{i(s+1)}\hat{\mu}_{s+1} + \hat{\mu}_{s+1}^2) \neq 0$

where $v_{i(s+1)}$, $w_{i(s+1)}$ and $\xi_{is}$ were defined in (3.59), (3.60) and (3.61), respectively.

Equation (5.16) can be written in the form:

(5.18) $a_5\hat{\mu}_{s+1}^5 + a_4\hat{\mu}_{s+1}^4 + a_3\hat{\mu}_{s+1}^3 + a_2\hat{\mu}_{s+1}^2 + a_1\hat{\mu}_{s+1} + a_0 = 0$

where

(5.19) $a_5 = \sum_{i=1}^{3} (\xi_{is} - r_i)$

(5.20) $a_4 = \sum_{i=1}^{3} (r_i - \xi_{is}) \left( v_i + 2 \sum_{j=1, j \neq i}^{3} v_j \right)$

(5.21) $a_3 = \sum_{i=1}^{3} (\xi_{is} - r_i) \left( v_i \left( 2 \sum_{j=1, j \neq i}^{3} v_j \right) + 4 \prod_{j=1, j \neq i}^{3} v_j + \sum_{j=1, j \neq i}^{3} w_j \right)$
The polynomial in (5.18) has five roots, at least one of which is real, since imaginary roots come in pairs. A numerical solution of (5.17) will give either one, or three, or five real roots. The real root that maximizes the log likelihood function in (5.13) is used to evaluate \( \hat{\mu}_{s+1} \), provided that it also satisfies the inequality given in (5.17).

For example, assume that at step \( s + 1 \) there are five real solutions of (5.16) all of which also satisfy inequality (5.17). Let these five real roots be denoted by \( \rho_j \) for \( j = 1, 2, 3, 4, 5 \). Then the corresponding estimates for \( \mu \), at step \( s + 1 \), are:

(5.25a) \[ \hat{\mu}_{s+1}(\rho_j) = \rho_j; \quad \text{for } j = 1, 2, 3, 4, 5. \]

The corresponding ML estimates of \( \sigma_i \) for \( i = 1, 2, 3 \) at step \( s + 1 \) are:

(5.25b) \[ \hat{\sigma}_{i(s+1)}^{(\rho_j)} = \sqrt{\frac{\sum_{j=1}^{n_i} (u_{ij(s+1)} - \hat{\mu}_{s+1}(\rho_j))^2}{r_i - \sum_{j=r_i+1}^{n_i} \lambda (\hat{y}_{ij} - \hat{\mu}_s)}} \quad \text{for } j = 1, 2, 3, 4, 5. \]

By adapting the log likelihood function in (5.13) the log likelihood function of the ML estimates at step \( s + 1 \) is:
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(5.26) \[ LL_3(\hat{\mu}_{s+1}, \hat{\sigma}_1(s+1), \hat{\sigma}_2(s+1), \hat{\sigma}_3(s+1)) = \sum_{i=1}^{3} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\hat{\sigma}_i(s+1)) \right. \]

\[- \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_{s+1}}{\hat{\sigma}_i(s+1)} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_{s+1}}{\hat{\sigma}_i(s+1)} \right) \right].

Now, substitute \( \hat{\sigma}_j^{(p_j)} \), and for \( j = 1, 2, 3, 4, 5 \) in (5.26). The result will be five competing evaluations of the log likelihood function, one for each of the five competing sets of ML estimates at step \( s + 1 \). That is, for \( j = 1, 2, 3, 4, 5 \):

(5.27) \[ (LL_3)^{(p_j)}_{s+1}(\hat{\mu}_{s+1}, \hat{\sigma}_1^{(p_j)}, \hat{\sigma}_2^{(p_j)}, \hat{\sigma}_3^{(p_j)}) = \sum_{i=1}^{k} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) \right. \]

\[- \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_{s+1}}{\hat{\sigma}_i^{(p_j)}(s+1)} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_{s+1}}{\hat{\sigma}_i^{(p_j)}(s+1)} \right) \right].

Let \( (LL_3)^{(p_j)}_{s+1} \) be the largest of the five competing evaluations of the log likelihood function of the ML estimates at step \( s + 1 \) expressed in (5.27). That is,

(5.28) \[ (LL_3)^{(p_j)}_{s+1} = \max_{1 \leq j \leq 5} \{ LL_3(\hat{\mu}_{s+1}^{(p_j)}, \hat{\sigma}_1^{(p_j)}, \hat{\sigma}_2^{(p_j)}, \hat{\sigma}_3^{(p_j)}) \}. \]

Then, the updated ML estimates of \( \mu, \sigma_1, \sigma_2, \) and \( \sigma_3, \) at step \( s + 1 \) are \( \hat{\mu}_{s+1}^{(p_j)}, \)

\( \hat{\sigma}_1^{(p_j)}(s+1), \hat{\sigma}_2^{(p_j)}(s+1), \) and \( \hat{\sigma}_3^{(p_j)}(s+1), \) respectively.

At each step \( s + 1 \), \( |\hat{\mu}_s - \hat{\mu}_{s+1}|, \) and \( |\hat{\sigma}_{is} - \hat{\sigma}_{is(s+1)}|, \) for \( i = 1, 2, 3 \) are evaluated. The convergence criterion is met when all of these 4 absolute differences are less than a preassigned tolerance value. Once convergence is achieved the ML estimates of \( \mu \) and \( \sigma_1 \) are reported as \( \hat{\mu}_s \) and \( \hat{\sigma}_{is}, \) respectively.

The log likelihood function of the ML estimates can now be computed by evaluating (5.27) at step \( s \). Therefore we have:
The log likelihood function for the \( k \)-sample \( H_4 \) case was given in (3.73) in Section 3.5.1. This can be adapted for the case \( k = 3 \) as follows:

\[
\log L_4(\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3) = \sum_{i=1}^{3} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\sigma_i) - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_i}{\hat{\sigma}_i} \right)^2 + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_i}{\sigma_i} \right) \right].
\]

\[\text{Section 5.2.4 The ML Estimates Under Case } H_4 : \mu_1 \neq \mu_2 \neq \mu_3, \sigma_1 \neq \sigma_2 \neq \sigma_3\]

The EM algorithm is used to calculate the ML estimates of \( H_i \) and \( \sigma_i \) for \( i = 1, 2, 3 \) iteratively as it was described in Section 3.5.2.

At step 0 all the uncensored observations are used to calculate the initial estimates of \( \mu_i \) and \( \sigma_i \) for \( i = 1, 2, 3 \) as follows:

\[
\hat{\mu}_{i0} = \frac{\sum_{j=1}^{r_i} y_{ij}}{r_i}, \quad \text{for } i = 1, 2, 3
\]

\[
\hat{\sigma}_{i0} = \sqrt{\frac{\sum_{j=1}^{r_i} (y_{ij} - \hat{\mu}_{i0})^2}{r_i}}, \quad \text{for } i = 1, 2, 3.
\]

Let \( \hat{\mu}_{is} \) and \( \hat{\sigma}_{is} \), for \( i = 1, 2, 3 \), be the ML estimates of \( \mu_i \) and \( \sigma_i \) for \( i = 1, 2, 3 \) respectively at step \( s \). At step \( s + 1 \) the values of \( u_{ij} \) for \( i = 1, 2, 3 \) and \( j = 1, ..., n_i \) are updated as follows:

\[
u_{ij(s+1)} = y_{ij} \text{; if } j = 1, ..., r_i
\]
(5.32b) \[ u_{ij(s+1)} = \hat{\mu}_{is} - \hat{\sigma}_{is} W \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_{is}} \right); \text{ if } j = r_i + 1, \ldots, n_i. \]

The updated ML estimates of \( \mu_i \) and \( \sigma_i \) for \( i = 1, 2, 3 \) at step \( s + 1 \) can be calculated:

(5.33a) \[ \hat{\mu}_{i(s+1)} = \frac{\sum_{j=1}^{n_i} u_{ij}}{n_i}, \quad \text{for } i = 1, 2, 3 \]

(5.33b) \[ \hat{\sigma}_{i(s+1)}^2 = \frac{\sum_{j=1}^{n_i} (u_{ij(s+1)} - \hat{\mu}_{i(s+1)})^2}{r_i - \sum_{j=r_i+1}^{n_i} \lambda \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_{is}} \right)}; \quad \text{for } i = 1, 2, 3. \]

At each step \( s + 1 \), \(| \hat{\mu}_{is} - \hat{\mu}_{i(s+1)} | \), for \( i = 1, 2, 3 \), and \(| \hat{\sigma}_{is} - \hat{\sigma}_{i(s+1)} | \) for \( i = 1, 2, 3 \) are evaluated. The convergence criterion is met when all six absolute differences are less than a preassigned tolerance value. Once convergence is achieved the ML estimates of \( \mu_i \) and \( \sigma_i \) are reported as \( \hat{\mu}_{is} \) and \( \hat{\sigma}_{is} \) respectively.

The log likelihood function of the ML estimates can now be evaluated by adapting (5.20) as:

(5.34) \[ LL_4(\hat{\mu}_{1s}, \hat{\mu}_{2s}, \hat{\mu}_{3s}, \hat{\sigma}_{1s}, \hat{\sigma}_{2s}, \hat{\sigma}_{3s}) = \sum_{i=1}^{3} \left[ \left( \frac{-r_i}{2} \right) \log(2\pi) - r_i \log(\hat{\sigma}_{is}) - \frac{1}{2} \sum_{j=1}^{r_i} \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_{is}} \right)^2 \right] + \sum_{j=r_i+1}^{n_i} \log \Phi \left( \frac{y_{ij} - \hat{\mu}_{is}}{\hat{\sigma}_{is}} \right). \]

5.3 Hypothesis Tests for the Equality of the Three Medians

The three stage procedure to test the equality of the \( k \) medians in the \( k \)-sample case was described in section 4.2. A comparable procedure will be used to test the equality of the 3 medians in the special case \( k = 3 \), as follows:
Stage 1

At this stage $H_1$ is tested versus $H_4$ at level of significance $\alpha_1$ and $p$-value $p_1$. This is a test of overall homogeneity, expressed by $H_1$ versus overall heterogeneity, expressed by $H_4$. The test statistic is:

$$\chi_1^2 = -2[LL_1(\mu, \sigma) - LL_4(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \sigma_1, \sigma_2, \sigma_3)]$$

where $LL_1(\mu, \sigma)$ and $LL_4(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \sigma_1, \sigma_2, \sigma_3)$ were defined in (5.7) and (5.34) respectively. A decision against $H_1$ and in favor of $H_4$ is concluded if:

$$\chi_1^2 > \chi^2(\alpha_1, 4).$$

Let:

$$p_1 = \Pr[\chi^2(4) > \chi_1^2].$$

Thus, if $p_1 \geq \alpha_1$ then $H_1$ cannot be rejected and overall homogeneity is concluded; on the other hand, if $p_1 < \alpha_1$ then $H_1$ is rejected in favor of $H_4$ and overall heterogeneity is concluded at level of significance $\alpha_1$.

Stage 2

The preliminary test for skewness homogeneity is performed at this stage. This test is performed without any assumptions on the equality of the 3 medians. $H_2$ is tested versus $H_4$ at level of significance $\alpha_2$ and $p$-value $p_2$. The test statistic is:

$$\chi_2^2 = -2[LL_2(\mu_1, \mu_2, \mu_3, \sigma) - LL_4(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \sigma_1, \sigma_2, \sigma_3)],$$

where $LL_2(\mu_1, \mu_2, \mu_3, \sigma)$ and $LL_4(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \sigma_1, \sigma_2, \sigma_3)$ were defined in (5.12) and (5.34), respectively. A decision against $H_2$ and in favor of $H_4$ is concluded if:

$$\chi_2^2 > \chi^2(\alpha_2, 2).$$
Let:

\[ p_2 = \Pr [\chi^2(2) > \chi^2_2]. \]  

Thus, if \( p_2 \geq \alpha_2 \) then \( H_2 \) cannot be rejected and skewness homogeneity is concluded; on the other hand, if \( p_2 < \alpha_2 \) then \( H_2 \) is rejected in favor of \( H_4 \) and skewness heterogeneity is concluded at a significance level of \( \alpha_2 \).

Stage 3

This is the final stage of the procedure at which the test for the equality of the medians is performed. The null hypothesis that the three medians are equal will be tested either under skewness homogeneity or under skewness heterogeneity depending on the outcome of the test at Stage 2. That is, if skewness homogeneity is concluded at Stage 2, then at Stage 3 \( H_1 \) should be tested versus \( H_2 \) at level of significance \( \alpha_3 \) and \( p \)-value \( p_3 \). This is a test for the equality of the three medians under skewness homogeneity. On the other hand, if skewness heterogeneity is concluded at Stage 2 then, at Stage 3, \( H_3 \) should be tested versus \( H_4 \) at level of significance \( \alpha_3 \) and \( p \)-value \( p_3 \). This is a test for the equality of the three medians under skewness heterogeneity.

When the test for the equality of the three medians is performed under skewness homogeneity, the test statistic is:

\[ \chi^2_3 = -2 [LL_1(\hat{\mu}, \hat{\sigma}) - LL_2(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma})], \]

where \( LL_1(\hat{\mu}, \hat{\sigma}) \) and \( LL_2(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}) \) were defined in (5.7) and (5.12), respectively. A decision against \( H_1 \) and in favor of \( H_2 \) is concluded if:

\[ \chi^2_3 > \chi^2(\alpha_3, 2). \]

Let:

\[ p_3 = \Pr [\chi^2(2) > \chi^2_3]. \]

Thus, if \( p_3 \geq \alpha_3 \) then \( H_1 \) cannot be rejected and the equality of the three
medians is concluded; on the other hand, if \( p_3 < \alpha_3 \) then \( H_1 \) is rejected in favor of \( H_2 \) and non equality of the three medians is concluded.

When the test for the equality of the three medians is performed under skewness heterogeneity the test statistic is:

\[(5.44) \quad \chi^2_4 = -2 [LL_3(\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3) - LL_4(\hat{\mu_1}, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)],\]

where \( LL_3(\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3) \) and \( LL_4(\hat{\mu_1}, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3) \) were defined in (5.29) and (5.34), respectively. A decision against \( H_3 \) and in favor of \( H_4 \) is concluded if:

\[(5.45) \quad \chi^2_3 > \chi^2(\alpha_4, 2).\]

Let:

\[(5.46) \quad p_4 = \Pr[\chi^2(2) > \chi^2_4].\]

Thus, if \( p_4 \geq \alpha_4 \), then \( H_3 \) cannot be rejected and the equality of the three medians is concluded; on the other hand, if \( p_4 < \alpha_4 \) then \( H_3 \) is rejected in favor of \( H_4 \) and non equality of the three medians is concluded.
CHAPTER VI

A SIMULATION STUDY

6.1 Simulation Methods

A simulation study was performed for the case \( k = 3 \), which was described in Chapter V. In this study, data from three independent lognormal distributions with specified median and skewness parameters are created and analyzed. A Fortran program was written for this purpose.

Two sample size cases were created. In the first case: \( n_1 = n_2 = n_3 = 20 \), and in the second case: \( n_1 = n_2 = n_3 = 100 \). These will be referred to as the small sample size case and the large sample size case, respectively. For each of these two sample size cases four parameter choices were specified, each conforming to one of the four hypothesis cases \( H_1 - H_4 \) defined in (5.2a) - (5.2d). This yielded a total of eight cases. These are referred to as simulation sets \( A_{20} \), \( B_{20} \), \( C_{20} \), and \( D_{20} \) for the small sample size case and simulation sets \( A_{100} \), \( B_{100} \), \( C_{100} \) and \( D_{100} \) for the large sample size case. The parameter choices for the eight simulation sets are presented in table form in Table 1. The parameter choices for each of these eight simulations are now described.

The lognormal parameters for simulation sets \( A_{20} \) and \( A_{100} \) are chosen to conform to hypothesis \( H_1 \): \( \mu_1 = \mu_2 = \mu_3 = \mu, \sigma_1 = \sigma_2 = \sigma_3 = \sigma \) defined in (5.2a) for the small sample size case and the large sample size case, respectively. In simulation set \( A_{20} \) three random samples of size 20 each are selected from \( LN(\mu = 0, \sigma = .1) \). In simulation set \( A_{100} \) three random samples of size 100 each
### Table 1
Parameter Choices in the Generation of Data

#### Small Sample Size Case($n_1 = n_2 = n_3 = 20$)

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Random Sample 1</th>
<th>Random Sample 2</th>
<th>Random Sample 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{20}$</td>
<td>$LN(\mu=0, \sigma=.1)$</td>
<td>$LN(\mu=0, \sigma=.1)$</td>
<td>$LN(\mu=0, \sigma=.1)$</td>
</tr>
<tr>
<td>$B_{20}$</td>
<td>$LN(\mu_1=-1, \sigma=.1)$</td>
<td>$LN(\mu_2=0, \sigma=.1)$</td>
<td>$LN(\mu_3=1, \sigma=.1)$</td>
</tr>
<tr>
<td>$C_{20}$</td>
<td>$LN(\mu=0, \sigma_1=.1)$</td>
<td>$LN(\mu=0, \sigma_2=.33)$</td>
<td>$LN(\mu=0, \sigma_3=1)$</td>
</tr>
<tr>
<td>$D_{20}$</td>
<td>$LN(\mu_1=-1, \sigma_1=.1)$</td>
<td>$LN(\mu_2=0, \sigma_2=.33)$</td>
<td>$LN(\mu_3=1, \sigma_3=1)$</td>
</tr>
</tbody>
</table>

#### Large Sample Size Case($n_1 = n_2 = n_3 = 100$)

<table>
<thead>
<tr>
<th>Simulation</th>
<th>Random Sample 1</th>
<th>Random Sample 2</th>
<th>Random Sample 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{100}$</td>
<td>$LN(\mu=0, \sigma=.1)$</td>
<td>$LN(\mu=0, \sigma=.1)$</td>
<td>$LN(\mu=0, \sigma=.1)$</td>
</tr>
<tr>
<td>$B_{100}$</td>
<td>$LN(\mu_1=-1, \sigma=.1)$</td>
<td>$LN(\mu_2=0, \sigma=.1)$</td>
<td>$LN(\mu_3=1, \sigma=.1)$</td>
</tr>
<tr>
<td>$C_{100}$</td>
<td>$LN(\mu=0, \sigma_1=.1)$</td>
<td>$LN(\mu=0, \sigma_2=.33)$</td>
<td>$LN(\mu=0, \sigma_3=1)$</td>
</tr>
<tr>
<td>$D_{100}$</td>
<td>$LN(\mu_1=-1, \sigma_1=.1)$</td>
<td>$LN(\mu_2=0, \sigma_2=.33)$</td>
<td>$LN(\mu_3=1, \sigma_3=1)$</td>
</tr>
</tbody>
</table>

are selected from $LN(\mu = 0, \sigma = .1)$. Therefore in simulations $A_{20}$ and $A_{100}$ three data sets are created from lognormal distributions with the same median and skewness parameters.

The lognormal parameters for simulation sets $B_{20}$ and $B_{100}$ are chosen to conform to hypothesis $H_2$: $\mu_1 \neq \mu_2 \neq \mu_3, \sigma_1 = \sigma_2 = \sigma_3 = \sigma$ defined in (5.2b) for the small sample size case and the large sample size case, respectively. In simulation set $B_{20}$ a random sample of size 20 is selected from $LN(\mu = -1, \sigma = .1)$,
a second random sample of size 20 is selected from $LN(\mu = 0, \sigma = .1)$, and a third random sample of size 20 is selected from $LN(\mu = 1, \sigma = .1)$. In simulation set $B_{100}$ a random sample of size 100 is selected from $LN(\mu = -1, \sigma = .1)$, a second random sample of size 100 is selected from $LN(\mu = 0, \sigma = .1)$, and a third random sample of size 100 is selected from $LN(\mu = 1, \sigma = .1)$. Therefore in simulations $B_{20}$ and $B_{100}$ three data sets are created from lognormal distributions with the same skewness parameter but different median parameters.

The lognormal parameters for simulation sets $C_{20}$ and $C_{100}$ are chosen to conform to hypothesis $H_3$: $\mu_1 = \mu_2 = \mu_3 = \mu$, $\sigma_1 \neq \sigma_2 \neq \sigma_3$ defined in (5.2c) for the small sample size case and the large sample size case, respectively. In simulation set $C_{20}$ a random sample of size 20 is selected from $LN(\mu = 0, \sigma = .1)$, a second random sample of size 20 is selected from $LN(\mu = 0, \sigma = .33)$, and a third random sample is selected from $LN(\mu = 0, \sigma = 1)$. In simulation set $C_{100}$ a random sample of size 100 is selected from $LN(\mu = 0, \sigma = .1)$, a second random sample of size 100 is selected from $LN(\mu = 0, \sigma = .33)$, and a third random sample of size 100 is selected from $LN(\mu = 0, \sigma = 1)$. Therefore in simulations $C_{20}$ and $C_{100}$ three data sets are created from lognormal distributions with the same median parameter but different skewness parameters.

The lognormal parameters for simulation sets $D_{20}$ and $D_{100}$ are chosen to conform to hypothesis $H_4$: $\mu_1 \neq \mu_2 \neq \mu_3$, $\sigma_1 \neq \sigma_2 \neq \sigma_3$ defined in (5.2d) for the small sample size case and the large sample size case, respectively. In simulation set $D_{20}$ a random sample of size 20 is selected from $LN(\mu = -1, \sigma = .1)$, a second random sample of size 20 is selected from $LN(\mu = 0, \sigma = .33)$, and a third random sample is selected from $LN(\mu = 1, \sigma = 1)$. In simulation set $D_{100}$ a random sample of size 100 is selected from $LN(\mu = -1, \sigma = .1)$, a second random sample of size 100 is selected from $LN(\mu = 0, \sigma = .33)$, and a third random sample of size
100 is selected from $LN(\mu = 1, \sigma = 1)$. Therefore in simulations $D_{20}$ and $D_{100}$ three data sets are created from lognormal distributions with different median parameters and different skewness parameters.

Type I censoring at four different levels were used for each of the eight sets of simulations. Censoring cuts were set at the $0^{th}$, $10^{th}$, $25^{th}$ and $50^{th}$ percentile of the parent distribution. These are referred to as the uncensored case, the $10\%$ censoring case, the $25\%$ censoring case and the $50\%$ censoring case, respectively. Thus, a total of 32 conditions were simulated in this study. For each condition 1000 trials were run.

In each trial the EM algorithm was used to calculate the ML estimates of the lognormal parameters under cases $H_1$ - $H_4$. Specifically, the EM algorithm was used to calculate: $\hat{\mu}$ and $\hat{\sigma}$ under case $H_1$, as was described in Section 5.2.1; $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\sigma}$ under case $H_2$, as was described in Section 5.2.2; $\hat{\mu}$, $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$ under case $H_3$, as was described in Section 5.2.3; and $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$, $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$ under case $H_4$, as was described in Section 5.2.4. After the ML estimates are obtained, the log likelihood functions are evaluated under cases $H_1$ - $H_4$. That is, $LL_1(\hat{\mu}, \hat{\sigma})$, $LL_2(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma})$, $LL_3(\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ and $LL_4(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ defined in (5.7), (5.12), (5.29) and (5.34), respectively, are evaluated. Then $\chi^2_1$, $\chi^2_2$, $\chi^2_3$ and $\chi^2_4$ defined in (5.35), (5.38), (5.41) and (5.44), respectively, are evaluated along with the corresponding $p$-values which are $p_1$, $p_2$, $p_3$ and $p_4$, defined in (5.37), (5.40), (5.43) and (5.46), respectively.

The $p$-value from Stage 2 ($p_2$) together with the $p$-value from Stage 3 (either $p_3$ or $p_4$) are used to classify the simulated data sets into one of the four cases $H_1$ - $H_4$ defined in (5.2a) - (5.2d). It is noted here that the $p$-value from Stage 1 was not utilized in this classification procedure because at Stage 1 a test of overall homogeneity was performed. All hypothesis tests were performed at the 5% level.
of significance. That is, the alpha levels were set at: $\alpha_2 = \alpha_3 = \alpha_4 = .05$. The classification procedure is now described:

- if $p_2 \geq .05$ and $p_3 \geq .05$ then the data conform to case $H_1$;
- if $p_2 \geq .05$ and $p_3 < .05$ then the data conform to case $H_2$;
- if $p_2 < .05$ and $p_4 \geq .05$ then the data conform to case $H_3$;
- if $p_2 < .05$ and $p_4 < .05$ then the data conform to case $H_4$.

Tables 3 through 10 summarize the results of the use of this classification procedure for the eight simulation sets described in Table 1. Table 3 contains the results of the classification procedure for simulation set $A_{20}$ for the uncensored case, the 10% censoring case, the 25% censoring case and the 50% censoring case. For each of the four censoring cases in simulation set $A_{20}$ the frequencies of conformity to cases $H_1 - H_4$ are given along with the frequency of inconclusive trials. Tables 4, 5, 6, 7, 8, 9 and 10 contain the same results for simulation sets $A_{100}$, $B_{20}$, $B_{100}$, $C_{20}$, $C_{100}$, $D_{20}$ and $D_{100}$, respectively.

In some trials the ML estimates of the lognormal parameters under hypothesis case $H_3$ were not able to be computed due to non-convergence of the EM algorithm. As a result the value of $LL_3(\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ defined in (5.35) and hence the value of $\chi^2_4$ defined in (5.44) and the value of $p_4$ defined in (5.45) could not be computed. The value of $p_4$ is needed in this classification procedure if $p_2 < .05$. Therefore, if in a specific trial the ML estimates under $H_3$ were not able to be computed due to non-convergence of the EM algorithm and $p_2 < .05$ then the classification procedure described above is declared as inconclusive. The frequency of inconclusive trials is reported in Tables 3 through 10 under the column INC.

Table 2 contains the frequencies of non-convergence in the EM algorithm for case $H_3$ in each of the eight simulation sets. It is noted that the EM algorithm under cases $H_1$, $H_2$ and $H_4$ converged in all trials of this simulation study. The
problem of non-convergence of the EM algorithm under case \( H_3 \) is probably due to the non-convergence in the root-finding procedure of the 5\(^{th}\) degree polynomial.

As was mentioned in Chapter I the eventual goal of the three-stage procedure is to arrive to one of the following two conclusions: either the three medians are equal or the three medians are unequal. Let \( H_0: \mu_1 = \mu_2 = \mu_3 = \mu \) and \( H_a: \mu_1 \neq \mu_2 \neq \mu_3 \). At Stage 3 of the three-stage procedure \( H_0 \) is tested either under skewness homogeneity or skewness heterogeneity at \( \alpha = .05 \). The median parameters are equal \((\mu_1 = \mu_2 = \mu_3 = 0)\) in simulations \( A_20, A_{100}, C_20 \) and \( C_{100} \). The median parameters are unequal \((\mu_1 \neq \mu_2 \neq \mu_3)\) in simulations \( B_20, B_{100}, D_20 \) and \( D_{100} \). In a hypothesis test of \( H_0: \mu_1 = \mu_2 = \mu_3 \) versus \( H_a: \mu_1 \neq \mu_2 \neq \mu_3 \) in simulations \( A_20, A_{100}, C_20 \) and \( C_{100} \) the Type I error is estimated as follows:

(6.1) Estimated Type I Error = \( \frac{FC_H_2 + FC_H_4}{1000 - INC} \)

where

\( FC_H_2 = \) Number of trials conforming to case \( H_2 \),

\( FC_H_4 = \) Number of trials conforming to case \( H_4 \), and

\( INC = \) Frequency of Inconclusive Trials.

In simulations \( B_20, B_{100}, D_20 \) and \( D_{100} \) the power of the hypothesis test of \( H_0: \mu_1 = \mu_2 = \mu_3 \) versus \( H_a: \mu_1 \neq \mu_2 \neq \mu_3 \) is estimated as follows:

(6.2) Estimated Power = \( \frac{FC_H_2 + FC_H_4}{1000 - INC} \)

The estimated simulated Type I error rates for simulations \( A_20, A_{100}, C_20 \) and \( C_{100} \) are presented in Table 11. The estimated simulated power rates for simulations \( B_20, B_{100}, D_20 \) and \( D_{100} \) are presented in Table 12.

For each of the 32 conditions the ML estimates of the lognormal parameters computed via the EM algorithm were evaluated with respect to their estimated
simulated mean square error and with respect to their estimated simulated bias. These will be referred to as the MSE and the ESB, respectively. The MSE was calculated using the following equation:

\[ \text{MSE}(\hat{\theta}) = \frac{1}{c} \sum_{i=1}^{c} \left( \hat{\theta}_i - \bar{\theta} \right)^2, \]

where 
- \( \hat{\theta}_i \) = the ML estimate of the lognormal parameter \( \theta \) for the \( i^{th} \) trial,
- \( c \) = the number of trials,
- \( \bar{\theta} = \frac{1}{c} \sum_{i=1}^{c} \hat{\theta}_i \) = the sample mean of the \( c \) ML estimates of parameter \( \theta \).

Tables 13, 14, 17, 18, 21, 22, 25 and 26 contain the estimated simulated mean square errors (MSE) of the ML estimates of the lognormal parameters in each of the four censoring cases for simulation sets \( A_{20}, A_{100}, B_{20}, B_{100}, C_{20}, C_{100}, D_{20}, \) and \( D_{100} \), respectively.

The ESB of the ML estimates of the lognormal parameters was computed using the following equation:

\[ \text{ESB} = \hat{\theta} - \theta. \]

Tables 15, 16, 19, 20, 23, 24, 27 and 28 contain the estimated simulated biases (ESB) of the ML estimates of the lognormal parameters in each of the four censoring cases for simulation sets \( A_{20}, A_{100}, B_{20}, B_{100}, C_{20}, C_{100}, D_{20}, \) and \( D_{100} \), respectively.

In Tables 3 through 28 C.L. stands for Censoring Level while UNC, 10%, 25% and 50% stand for the uncensored case, the 10% censoring case, the 25% censoring case and the 50% censoring case, respectively.
6.2 Simulation Results

6.2.1 Non-convergence Results

As was mentioned in Section 6.1 there was non-convergence of the EM algorithm under case $H_3$ for some trials. From Table 2 it can be seen that the problem of non-convergence in the EM algorithm under case $H_3$ is present only in the 50% censoring case. In the 50% censoring case of simulation set $A_{20}$ the EM algorithm failed to converge under $H_3$ in 108 trials while in simulation set $A_{100}$ the EM algorithm always converged. In simulation set $B_{20}$ non-convergence occurred in 75 trials while in simulation set $B_{100}$ non-convergence occurred in just one trial. In simulation set $C_{20}$ the EM algorithm failed to converge in 133 trials while in simulation set $C_{100}$ non-convergence occurred in only 8 trials. In simulation set $D_{20}$ the EM algorithm failed to converge in 102 trials while in simulation set $D_{100}$ non-convergence occurred in 8 trials. We conclude that the number of trials for which the EM algorithm did not converge under case $H_3$ in the 50% censoring case decreases as the sample sizes increase.

6.2.2 $H_1$ Case Simulation Results

Under case $H_1$: $\mu_1 = \mu_2 = \mu_3 = \mu, \sigma_1 = \sigma_2 = \sigma_3 = \sigma$ the median parameters are equal and the skewness parameters are equal. In simulations $A_{20}$ and $A_{100}$ the three-stage procedure results in the correct conclusion when $p_2 \geq .05$ and $p_3 \geq .05$. The percentage of trials for which the three-stage procedure resulted in the correct conclusion in simulations $A_{20}$ and $A_{100}$ is computed as follows:

\[
FCC = \frac{FC \cdot H_1}{1000}
\]

where
Table 2
Number of Trials For Which the EM Algorithm Failed to Converge When Computing the ML Estimates Under Case $H_3$

<table>
<thead>
<tr>
<th>Simulation</th>
<th>UNC.</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{20}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>108</td>
</tr>
<tr>
<td>$B_{20}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>75</td>
</tr>
<tr>
<td>$C_{20}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>133</td>
</tr>
<tr>
<td>$D_{20}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>102</td>
</tr>
<tr>
<td>$A_{100}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$B_{100}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$C_{100}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$D_{100}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

$FCC = \text{Percent Frequency of Correct Conclusion,}$

$FC\ H_1 = \text{Number of Trials Conforming to Case } H_1,$

The percent frequency of correct conclusions in simulation $A_{20}$ is reported in Table 3 under FCC. It can be seen from Table 3 that in simulation $A_{20}$ the three-stage procedure has a frequency of correct conclusion close to 90% of the trials at all levels of censoring. Specifically, the frequency of correct conclusions in simulation $A_{20}$ is 89.4% of the trials in the uncensored case, 89.8% of the trials in the 10% censoring case, 91.3% of the trials in the 25% censoring case and 89.8% of the trials in the 50% censoring case.

The results for simulation $A_{100}$ are summarized in Table 4. The frequency
Table 3
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $A_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>INC.</th>
<th>FCC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>894</td>
<td>48</td>
<td>54</td>
<td>4</td>
<td>0</td>
<td>89.4</td>
</tr>
<tr>
<td>10%</td>
<td>898</td>
<td>45</td>
<td>50</td>
<td>7</td>
<td>0</td>
<td>89.8</td>
</tr>
<tr>
<td>25%</td>
<td>913</td>
<td>41</td>
<td>42</td>
<td>4</td>
<td>0</td>
<td>91.3</td>
</tr>
<tr>
<td>50%</td>
<td>898</td>
<td>50</td>
<td>32</td>
<td>15</td>
<td>5</td>
<td>89.8</td>
</tr>
</tbody>
</table>

Table 4
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $A_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>INC.</th>
<th>FCC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>904</td>
<td>46</td>
<td>48</td>
<td>2</td>
<td>0</td>
<td>90.4</td>
</tr>
<tr>
<td>10%</td>
<td>903</td>
<td>42</td>
<td>53</td>
<td>2</td>
<td>0</td>
<td>90.3</td>
</tr>
<tr>
<td>25%</td>
<td>907</td>
<td>42</td>
<td>50</td>
<td>1</td>
<td>0</td>
<td>90.7</td>
</tr>
<tr>
<td>50%</td>
<td>921</td>
<td>43</td>
<td>29</td>
<td>7</td>
<td>5</td>
<td>92.1</td>
</tr>
</tbody>
</table>

of correct conclusions in the three-stage procedure is close to 90% of the trials for the uncensored case, the 25% censoring case and the 50% censoring case at 90.4%, 90.3% and 90.7%, respectively. The frequency of correct conclusions for
the 50% censoring case is 92.1% of the trials, which is somewhat higher than the other three censoring cases.

6.2.3 \( H_2 \) Case Simulation Results

Under case \( H_2 \): \( \mu_1 \neq \mu_2 \neq \mu_3 \), \( \sigma_1 = \sigma_2 = \sigma_3 = \sigma \) the skewness parameters are equal but the median parameters are not equal. In simulations \( B_{20} \) and \( B_{100} \) the three-stage procedure results in the correct conclusion when \( p_2 \geq .05 \) and \( p_3 < .05 \). The percentage of trials for which the three-stage procedure resulted in the correct conclusion in simulations \( B_{20} \) and \( B_{100} \) is computed as follows:

\[
(6.6) \quad PFCC = \frac{FC_{H_2}}{1000}
\]

where

\( FCC = \) Percent Frequency of Correct Conclusion,
\( FC_{H_2} = \) Number of Trials Conforming to Case \( H_2 \),

The percent frequency of correct conclusions in simulation \( B_{20} \) is reported in Table 5 under FCC. It can be seen from Table 5 that in simulation \( B_{20} \) the three-stage procedure has a frequency of correct conclusion close to 95% of the trials at all levels of censoring. Specifically, the frequency of correct conclusions in simulation \( B_{20} \) is 94.2% of the trials in the uncensored case, 94.3% of the trials in the 10% censoring case, 95.4% of the trials in the 25% censoring case and 94.8% of the trials in the 50% censoring case.

The results for simulation \( B_{100} \) are summarized in Table 6. The frequency of correct conclusions in the three-stage procedure is close to 95% of the trials for the uncensored case, the 10% censoring case and the 25% censoring case, where levels of 95.0%, 94.5% and 94.9% are reported, respectively. The frequency of correct conclusions is 96.4% of the trials for the 50% censoring case, which is
Table 5
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $B_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>INC.</th>
<th>FCC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>0</td>
<td>942</td>
<td>0</td>
<td>58</td>
<td>0</td>
<td>94.2</td>
</tr>
<tr>
<td>10%</td>
<td>0</td>
<td>943</td>
<td>0</td>
<td>57</td>
<td>0</td>
<td>94.3</td>
</tr>
<tr>
<td>25%</td>
<td>0</td>
<td>954</td>
<td>0</td>
<td>46</td>
<td>0</td>
<td>95.4</td>
</tr>
<tr>
<td>50%</td>
<td>0</td>
<td>948</td>
<td>0</td>
<td>48</td>
<td>4</td>
<td>94.8</td>
</tr>
</tbody>
</table>

Table 6
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $B_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>INC.</th>
<th>FCC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>0</td>
<td>950</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>95.0</td>
</tr>
<tr>
<td>10%</td>
<td>0</td>
<td>945</td>
<td>0</td>
<td>55</td>
<td>0</td>
<td>94.5</td>
</tr>
<tr>
<td>25%</td>
<td>0</td>
<td>949</td>
<td>0</td>
<td>51</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>50%</td>
<td>0</td>
<td>964</td>
<td>0</td>
<td>36</td>
<td>0</td>
<td>96.4</td>
</tr>
</tbody>
</table>

somewhat higher.
6.2.4 $H_3$ Case Simulation Results

Under case $H_3$: $\mu_1 = \mu_2 = \mu_3 = \mu, \sigma_1 \neq \sigma_2 \neq \sigma_3$ the median parameters are equal, but the skewness parameters are not equal. In simulations $C_{20}$ and $C_{100}$ the three-stage procedure results in the correct conclusion when $p_2 < .05$ and $p_4 \geq .05$. The percentage of trials for which the three-stage procedure resulted in the correct conclusion in simulations $C_{20}$ and $C_{100}$ is computed as follows:

$$ (6.7) \quad FCC = \frac{FC H_3}{1000 - INC} $$

where

$FCC = \text{Percent Frequency of Correct Conclusion,}$

$FC H_3 = \text{Number of Trials Conforming to Case } H_3,$

$INC = \text{Frequency of Inconclusive Trials.}$

The percent frequency of correct conclusions in simulation $C_{20}$ is reported in Table 7 under $FCC$. It can be seen from Table 7 that in simulation $C_{20}$ the three-stage procedure has a frequency of correct conclusions close to 94% of the trials for the uncensored case, the 10% censoring case and the 25% censoring case with observed levels of 94.5%, 94.4% and 94.1%, respectively. The observed frequency of correct conclusions is much lower for the 50% censoring case with 85.1% of the trials classified correctly into case $H_3$.

The results for simulation $C_{100}$ are summarized in Table 8. The frequency of correct conclusions in the three-stage procedure is close to 95% of the trials for the uncensored case, the 10% censoring case and the 25% censoring case with observed levels of 95.2%, 95.5% and 95.2%, respectively. The observed frequency of correct conclusions is slightly lower for the 50% censoring case with 94.6% of the trials classified correctly into case $H_3$. 

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Table 7
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $C_{20}$

<table>
<thead>
<tr>
<th>Frequency Conforming to Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.L.</td>
</tr>
<tr>
<td>UNCP</td>
</tr>
<tr>
<td>10%</td>
</tr>
<tr>
<td>25%</td>
</tr>
<tr>
<td>50%</td>
</tr>
</tbody>
</table>

Table 8
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $C_{100}$

<table>
<thead>
<tr>
<th>Frequency Conforming to Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.L.</td>
</tr>
<tr>
<td>UNCP</td>
</tr>
<tr>
<td>10%</td>
</tr>
<tr>
<td>25%</td>
</tr>
<tr>
<td>50%</td>
</tr>
</tbody>
</table>

6.2.5 $H_4$ Case Simulation Results

Under case $H_4$: $\mu_1 \neq \mu_2 \neq \mu_3 \neq \mu$, $\sigma_1 \neq \sigma_2 \neq \sigma_3$ the median parameters
are unequal and the skewness parameters are unequal. In simulations $D_{20}$ and $D_{100}$ the three-stage procedure results in the correct conclusion when $p_2 < .05$ and $p_4 < .05$. The percentage of trials for which the three-stage procedure resulted in the correct conclusion in simulations $D_{20}$ and $D_{100}$ is computed as follows:

\[
(6.8) \quad FCC = \frac{FC_{H_4}}{1000 - INC}
\]

where

- $FCC = \text{Percent Frequency of Correct Conclusion},$
- $FC_{H_4} = \text{Number of Trials Conforming to Case } H_4,$
- $INC = \text{Frequency of Inconclusive Trials}.$

The percent frequencies of correct conclusions in simulations $D_{20}$ and $D_{100}$ are reported in Tables 9 and 10, respectively, under FCC. It can be seen from these tables that in simulations $D_{20}$ and $D_{100}$ the three-stage procedure has a frequency of correct conclusion equal to 100\% of the trials at all levels of censoring.

### 6.2.6 Summary and Conclusions on Simulation Results for Cases $H_1$-$H_4$

The frequency of correct conclusions about medians in the classification procedure in all 32 conditions is bigger than 85\% of the trials. The lowest frequency is 85.1\% in the 50\% censoring case of simulation $C_{20}$ and the highest is 100.0\% observed in all of the censoring cases of simulations $D_{20}$ and $D_{100}$. The frequency of correct conclusions remains steady as the censoring level increases in the simulations conforming to cases $H_1$, $H_2$ and $H_4$ in both sample size cases. Specifically the frequency is close to 90\% of the trials in simulations $A_{20}$ and $A_{100}$, which conform to case $H_1$, close to 95\% of the trials in simulations $B_{20}$ and $B_{100}$, which conform to case $H_2$, and equal to 100.0\% of the trials in simulations $D_{20}$ and $D_{100}$, which conform to case $H_4$. In simulation $C_{100}$, which is the large sample
Table 9
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $D_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>INC.</th>
<th>FCC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>100.0</td>
</tr>
<tr>
<td>10%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>100.0</td>
</tr>
<tr>
<td>25%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>100.0</td>
</tr>
<tr>
<td>50%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>898</td>
<td>102</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Table 10
Results of the Classification Procedure for Evaluating the Three-Stage Procedure in Simulation $D_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>INC.</th>
<th>FCC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>100.0</td>
</tr>
<tr>
<td>10%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>100.0</td>
</tr>
<tr>
<td>25%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>100.0</td>
</tr>
<tr>
<td>50%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>992</td>
<td>8</td>
<td>100.0</td>
</tr>
</tbody>
</table>

size case conforming to case $H_3$, the frequency of correct conclusions about medians in the classification procedure is observed to be about 95% of the trials, at all levels of censoring. In simulation $C_{20}$, which is the small sample size case
conforming to case $H_3$, the frequency of correct conclusions falls to 85.1% of the trials in the 50% censoring case. The frequency is observed to be about 94% of the trials in the other censoring cases. We conclude that the performance of the three-stage procedure is satisfactory in all simulations except in the 50% censoring case of the small sample size case conforming to case $H_3$ where the frequency of correct conclusions is about 9% lower than the other three censoring cases of case $H_3$.

6.2.7 Type I Error and Power Simulation Results

The estimated simulated Type I error rates of the hypothesis test $H_0$: $\mu_1 = \mu_2 = \mu_3$ versus $H_1: \mu_1 \neq \mu_2 \neq \mu_3$ are presented in Table 11.

In Table 11 we observe that in simulation $A_{20}$ the estimated Type I error rate in the uncensored case, the 10% censoring case and the 25% censoring case is maintained close to $\alpha = .05$ at .052, .052 and .045, respectively. In the 50% censoring case the estimated Type I error rate is slightly higher at .0653. In simulation $A_{100}$ the estimated Type I error rate is maintained close to $\alpha = .05$ at all levels of censoring. The estimated type I error rate is .048 in the uncensored case, .044 in the 10% censoring case, .044 in the 25% censoring case and .0503 in the 50% censoring case. In simulation $C_{20}$ the estimated Type I error rate is much higher than $\alpha = .05$ in the 50% censoring case at .1488, and slightly higher than $\alpha = .05$ in the remaining censoring levels. Specifically, the estimated type I error rate is .055 in the uncensored case, .056 in the 10% censoring case and .059 in the 50% censoring case. In simulation $C_{100}$ the estimated Type I error rate is maintained close to $\alpha = .05$ at all levels of censoring. The 50% censoring case yields the highest estimate of Type I error rate at .0544 and the 10% censoring case yields the lowest at .045. The estimated type I error rate is .048 in the
Table 11
The Estimated Simulated Type I Error Rates

<table>
<thead>
<tr>
<th>Simulation</th>
<th>C.L.</th>
<th>$A_{20}$</th>
<th>$A_{100}$</th>
<th>$C_{20}$</th>
<th>$C_{100}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>.0520</td>
<td>.0480</td>
<td>.0550</td>
<td>.0480</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>.0520</td>
<td>.0440</td>
<td>.0560</td>
<td>.0450</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>.0450</td>
<td>.0430</td>
<td>.0590</td>
<td>.0480</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>.0653</td>
<td>.0503</td>
<td>.1488</td>
<td>.0544</td>
<td></td>
</tr>
</tbody>
</table>

Table 12
The Estimated Simulated Power Rates

<table>
<thead>
<tr>
<th>Simulation</th>
<th>C.L.</th>
<th>$B_{20}$</th>
<th>$B_{100}$</th>
<th>$D_{20}$</th>
<th>$D_{100}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
</tbody>
</table>

uncensored case and the 25% censoring case.

The estimated simulated power rates of the hypothesis test $H_0$: $\mu_1 = \mu_2 = \mu_3$ versus $H_a$: $\mu_1 \neq \mu_2 \neq \mu_3$ for simulations $B_{20}$, $B_{100}$, $D_{20}$ and $D_{100}$ are presented in Table 12. In all simulations the power rates are estimated at the highest possible
value, that is 1.0.

6.2.8 Summary and Conclusions on Type I Error and Power Simulation Results

The estimated Type I error rate is maintained close to the alpha level (.05) at all levels of censoring in the large sample size cases (simulations A_{100} and C_{100}). In the small sample size cases (simulations A_{20} and C_{20}) the estimated Type I error is elevated above the alpha level in the 50% censoring case (.0653 and .1488, respectively) while it is maintained at the alpha level in the other censoring cases. We conclude that the estimated Type I error rate is too high in the 50% censoring case of the small sample size case conforming to case H_3, and slightly high in the 50% censoring case of the small sample size case conforming to case H_1, while it is maintained at the alpha level in all the other censoring cases.

The fact that the estimated power rates are equal to 1.0 when the data simulated conform to either case H_2 or case H_4 leads to the conclusion that there are no errors when the hypothesis of equal medians is rejected.

6.3 Mean Square Errors and Bias Simulation Results

Tables 13 and 14 contain the estimated simulated mean square errors (MSE) of the ML estimates of the lognormal parameters under H_1 for the small sample size case and the large sample size case, respectively. Tables 15 and 16 contain the estimated simulated biases (ESB) of the ML estimates of the lognormal parameters under H_1 for the small sample size case and the large sample size case, respectively. Tables 17 and 18 contain MSE of the ML estimates under H_2 for the small sample size case and the large sample size case, respectively. Tables 19 and 20 contain the ESB of the ML estimates under H_2 for the small sample size case and the large sample size case, respectively. Tables 21 and 22 contain
the MSE of the ML estimates under $H_3$ for the small sample size case and the large sample size case, respectively. Tables 23 and 24 contain the ESB of the ML estimates under $H_3$ for the small sample size case and the large sample size case, respectively. Tables 25 and 26 contain the MSE of the ML estimates under $H_4$ for the small sample size case and the large sample size case, respectively. Tables 27 and 28 contain the ESB of the ML estimates under $H_4$ for the small sample size case and the large sample size case, respectively.

6.3.1 Mean Square Error and Bias Simulation Results for Case $H_1$

In simulation cases $A_{20}$ (small sample size case) and $A_{100}$ (large sample size case) data are created from three independent lognormal distributions with the same median and skewness parameters, $\mu = 0$ and $\sigma = .1$, respectively. In both of these simulations the EM algorithm yields the ML estimates of $\mu$ and $\sigma$ as $\hat{\mu}$ and $\hat{\sigma}$, respectively. The MSE of $\hat{\mu}$ and $\hat{\sigma}$ were computed using (6.3). The average MSE values are reported in Table 13 for simulation $A_{20}$ and in Table 14 for simulation $A_{100}$ for each of the four censoring cases. From Table 13 we conclude that the MSE of both $\hat{\mu}$ and $\hat{\sigma}$ increase as the level of censoring increases. Specifically, the MSE of $\hat{\mu}$ increases from $1.677 \times 10^{-4}$ in the uncensored case to $2.939 \times 10^{-4}$ in the 50% censoring case. The MSE of $\hat{\sigma}$ increases from $8.825 \times 10^{-5}$ in the uncensored case to $2.172 \times 10^{-4}$ in the 50% censoring case. The same trend is observed in the MSE of $\hat{\mu}$ and $\hat{\sigma}$ in simulation $A_{100}$ which are presented in Table 14. Specifically, the MSE of $\hat{\mu}$ increases from $3.213 \times 10^{-8}$ in the uncensored case to $5.5464 \times 10^{-8}$ in the 50% censoring case. The MSE of $\hat{\sigma}$ increases from $1.759 \times 10^{-5}$ in the uncensored case to $4.453 \times 10^{-5}$ in the 50% censoring case.

The estimated simulated biases (ESB) of the ML estimates of the lognormal parameters in simulations $A_{20}$ and $A_{100}$ were computed using (6.4). Unlike the
Table 13
The MSE $\times 10^{-4}$ of the ML Estimates of the Lognormal Parameters in Simulation $A_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE($\hat{\mu}$)</th>
<th>MSE($\hat{\sigma}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>1.677</td>
<td>.8825</td>
</tr>
<tr>
<td>10%</td>
<td>1.727</td>
<td>1.060</td>
</tr>
<tr>
<td>25%</td>
<td>1.926</td>
<td>1.333</td>
</tr>
<tr>
<td>50%</td>
<td>2.939</td>
<td>2.172</td>
</tr>
</tbody>
</table>

Table 14
The MSE $\times 10^{-4}$ of the ML Estimates of the Lognormal Parameters in Simulation $A_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE($\hat{\mu}$)</th>
<th>MSE($\hat{\sigma}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>.3213</td>
<td>.1759</td>
</tr>
<tr>
<td>10%</td>
<td>.3304</td>
<td>.2058</td>
</tr>
<tr>
<td>25%</td>
<td>.3643</td>
<td>.2840</td>
</tr>
<tr>
<td>50%</td>
<td>.5464</td>
<td>4553</td>
</tr>
</tbody>
</table>

MSE of the ML estimates, the ESB of the ML estimates which are presented in Tables 15 and 16 for simulation $A_{20}$ and $A_{100}$, respectively, do not follow a particular order of magnitude as the level of censoring increases. It is noted that the ESB of the ML estimates are maintained at low levels in both simulations.

By comparing the MSE of $\hat{\mu}$ and $\hat{\sigma}$ in Table 13 with the corresponding MSE in Table 14 for each of the censoring cases separately, we conclude that the
MSE of the ML estimates decrease as the sample sizes increase. For example, let us consider the 50% censoring case. From Table 13, in the small sample size case the MSE of $\hat{\mu}$ is $2.939 \times 10^{-4}$ and the MSE of $\hat{\sigma}$ is $2.172 \times 10^{-4}$. From Table 14, in the large sample size case the MSE of $\hat{\mu}$ is $5.464 \times 10^{-5}$ and the MSE of $\hat{\sigma}$ is $4.553 \times 10^{-5}$. The same tendency occurs in the other censoring cases.

The same trend is observed in the ESB of the ML estimates. That is, the
ML estimates of the lognormal parameters are less biased in the large sample size case. For example, let us consider the 50% censoring case. From Table 15, in the small sample size case the ESB of $\hat{\mu}$ is $-6.009 \times 10^{-4}$ and the ESB of $\hat{\sigma}$ is $-2.794 \times 10^{-3}$. From Table 16, in the large sample size case the ESB of $\hat{\mu}$ is $-1.190 \times 10^{-4}$ and the ESB of $\hat{\sigma}$ is $-5.830 \times 10^{-4}$.

6.3.2 Mean Square Error and Bias Simulation Results for Case $H_2$

In simulation cases $B_{20}$ (small sample size case) and $B_{100}$ (large sample size case) data are created from three independent lognormal distributions with different median parameters ($\mu_1 = -1$, $\mu_2 = 0$, $\mu_3 = 1$) and with the same skewness parameter ($\sigma = .1$). In both of these simulations the EM algorithm yields the ML estimates of $\mu_1$, $\mu_2$, $\mu_3$ and $\sigma$ as $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\sigma}$, respectively. The MSE of these four ML estimates were computed using (6.3). The average MSE values are reported in Table 17 for simulation $B_{20}$ and in Table 18 for simulation $B_{100}$ for each of the four censoring cases. From Table 17 we conclude that the MSE of the four ML estimates increase as the level of censoring increases. Specifically, the MSE of $\hat{\mu}_1$ increases from $4.929 \times 10^{-4}$ in the uncensored case to $8.167 \times 10^{-4}$ in the 50% censoring case; the MSE of $\hat{\mu}_2$ increases from $4.902 \times 10^{-4}$ in the uncensored case to $8.298 \times 10^{-4}$ in the 50% censoring case; the MSE of $\hat{\mu}_3$ increases from $4.935 \times 10^{-4}$ in the uncensored case to $9.385 \times 10^{-4}$ in the 50% censoring case. The MSE of $\hat{\sigma}$ increases from $8.825 \times 10^{-5}$ in the uncensored case to $2.172 \times 10^{-4}$ in the 50% censoring case. The same trend is observed in the MSE of $\hat{\mu}$ and $\hat{\sigma}$ in the large sample size case which are presented in Table 18. Specifically, the MSE of $\hat{\mu}_1$ increases from $9.871 \times 10^{-5}$ in the uncensored case to $1.551 \times 10^{-4}$ at the 50% censoring case; the MSE of $\hat{\mu}_2$ increases from $9.999 \times 10^{-5}$ at the uncensored case to $1.654 \times 10^{-4}$ in the 50% censoring case; the MSE of $\hat{\mu}_3$...

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Table 17
The MSE×10⁻⁴ of the ML Estimates of the Lognormal Parameters in Simulation B₂₀

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE(μ₁)</th>
<th>MSE(μ₂)</th>
<th>MSE(μ₃)</th>
<th>MSE(φ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>4.929</td>
<td>4.902</td>
<td>4.935</td>
<td>.8825</td>
</tr>
<tr>
<td>10%</td>
<td>5.150</td>
<td>5.054</td>
<td>5.100</td>
<td>1.060</td>
</tr>
<tr>
<td>25%</td>
<td>5.522</td>
<td>5.544</td>
<td>5.765</td>
<td>1.333</td>
</tr>
<tr>
<td>50%</td>
<td>8.617</td>
<td>8.298</td>
<td>9.385</td>
<td>2.172</td>
</tr>
</tbody>
</table>

Table 18
The MSE×10⁻⁴ of the ML Estimates of the Lognormal Parameters in Simulation B₁₀₀

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE(μ₁)</th>
<th>MSE(μ₂)</th>
<th>MSE(μ₃)</th>
<th>MSE(φ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>.9781</td>
<td>.9999</td>
<td>.8848</td>
<td>.1759</td>
</tr>
<tr>
<td>10%</td>
<td>.9948</td>
<td>1.032</td>
<td>.9017</td>
<td>.2058</td>
</tr>
<tr>
<td>25%</td>
<td>1.057</td>
<td>1.106</td>
<td>.9899</td>
<td>.2840</td>
</tr>
<tr>
<td>50%</td>
<td>1.551</td>
<td>1.654</td>
<td>1.375</td>
<td>.4553</td>
</tr>
</tbody>
</table>

The estimated simulated biases (ESB) of the ML estimates of the lognormal parameters in simulations B₂₀ and B₁₀₀ were computed using (6.4). Unlike the MSE of the ML estimates, the ESB of the ML estimates which are presented in
### Table 19

The ESB × 10^{-4} of the ML Estimates of the Lognormal Parameters in Simulation $B_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>ESB($\hat{\mu}_1$)</th>
<th>ESB($\hat{\mu}_2$)</th>
<th>ESB($\hat{\mu}_3$)</th>
<th>ESB($\hat{\phi}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>3.624</td>
<td>2.605</td>
<td>.7985</td>
<td>-31.04</td>
</tr>
<tr>
<td>10%</td>
<td>-2.571</td>
<td>-2.985</td>
<td>-4.549</td>
<td>-27.22</td>
</tr>
<tr>
<td>50%</td>
<td>-16.90</td>
<td>-13.76</td>
<td>-20.48</td>
<td>-27.94</td>
</tr>
</tbody>
</table>

### Table 20

The ESB × 10^{-4} of the ML Estimates of the Lognormal Parameters in Simulation $B_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>ESB($\hat{\mu}_1$)</th>
<th>ESB($\hat{\mu}_2$)</th>
<th>ESB($\hat{\mu}_3$)</th>
<th>ESB($\hat{\phi}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>1.430</td>
<td>- .3155</td>
<td>-.5002</td>
<td>-5.797</td>
</tr>
<tr>
<td>10%</td>
<td>.1238</td>
<td>-1.115</td>
<td>-.8822</td>
<td>-5.675</td>
</tr>
<tr>
<td>25%</td>
<td>-.6329</td>
<td>-2.052</td>
<td>-1.721</td>
<td>-5.606</td>
</tr>
<tr>
<td>50%</td>
<td>-2.828</td>
<td>-4.469</td>
<td>-2.227</td>
<td>-5.828</td>
</tr>
</tbody>
</table>

Tables 19 and 20 for simulation $B_{20}$ and $B_{100}$, respectively, do not follow a particular order of magnitude as the level of censoring increases. It is noted that the ESB of the ML estimates are maintained at low levels in both simulations.

By comparing the MSE of the ML estimates in Table 17 with the corresponding MSE in Table 18 for each of the censoring cases separately, we conclude that the MSE of the ML estimates decrease as the sample sizes increase. For
example, let us consider the 25% censoring case. From Table 17, in the small sample size case the MSE of $\hat{\mu}_1$ is $5.522 \times 10^{-4}$, the MSE of $\hat{\mu}_2$ is $5.544 \times 10^{-4}$, the MSE of $\hat{\mu}_3$ is $5.675 \times 10^{-4}$ and the MSE of $\hat{\sigma}$ is $1.333 \times 10^{-4}$. From Table 18, in the large sample size case the MSE of $\hat{\mu}_1$ is $1.057 \times 10^{-4}$, the MSE of $\hat{\mu}_2$ is $1.106 \times 10^{-4}$, the MSE of $\hat{\mu}_3$ is $9.899 \times 10^{-5}$ and the MSE of $\hat{\sigma}$ is $2.840 \times 10^{-5}$.

The same trend occurs in the other censoring cases.

The same trend is observed in the ESB of the ML estimates. That is, the ML estimates of the lognormal parameters are less biased in the large sample size case than in the small sample size case. For example, let us consider the 25% censoring case. From Table 19, in the small sample size case the ESB of $\hat{\mu}_1$ is $-6.847 \times 10^{-4}$, the ESB of $\hat{\mu}_2$ is $-7.634 \times 10^{-4}$, the ESB of $\hat{\mu}_3$ is $-9.359 \times 10^{-4}$ and the ESB of $\hat{\sigma}$ is $-2.655 \times 10^{-3}$. From Table 20, in the large sample size case the ESB of $\hat{\mu}_1$ is $-6.329 \times 10^{-5}$, the ESB of $\hat{\mu}_2$ is $-2.052 \times 10^{-4}$, the ESB of $\hat{\mu}_3$ is $-1.721 \times 10^{-4}$ and the ESB of $\hat{\sigma}$ is $-5.606 \times 10^{-4}$.

### 6.3.3 Mean Square Error and Bias Simulation Results for Case $H_3$

In simulation cases $C_{20}$ (small sample size case) and $C_{100}$ (large sample size case) data are created from three independent lognormal distributions with the same median parameter ($\mu = 0$) and with different skewness parameters ($\sigma_1 = .1$, $\sigma_2 = .33$, $\sigma_3 = 1$). In both of these simulations the EM algorithm yields the ML estimates of $\mu$, $\sigma_1$, $\sigma_2$ and $\sigma_3$ and $\sigma$ as $\hat{\mu}$, $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$, respectively. The MSE of these four ML estimates were computed using (6.3). The average MSE values are reported in Table 21 for simulation $C_{20}$ and in Table 18 for simulation $C_{100}$ for each of the four censoring cases. From Table 21 we observe that the MSE of $\hat{\mu}$ decreases slightly from $6.064 \times 10^{-3}$ in the uncensored case to $5.991 \times 10^{-3}$ in the 10% censoring case to $5.880 \times 10^{-3}$ in the 25% censoring case. It then increases.
Table 21
The MSE×10⁻⁴ of the ML Estimates of the Lognormal Parameters in Simulation C₂₀

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE(( \mu ))</th>
<th>MSE(( \hat{\sigma}_1 ))</th>
<th>MSE(( \hat{\sigma}_2 ))</th>
<th>MSE(( \hat{\sigma}_3 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>60.64</td>
<td>2.412</td>
<td>27.05</td>
<td>258.9</td>
</tr>
<tr>
<td>10%</td>
<td>59.91</td>
<td>2.970</td>
<td>34.03</td>
<td>303.9</td>
</tr>
<tr>
<td>25%</td>
<td>58.80</td>
<td>3.695</td>
<td>42.35</td>
<td>389.8</td>
</tr>
<tr>
<td>50%</td>
<td>76.33</td>
<td>5.767</td>
<td>67.24</td>
<td>662.8</td>
</tr>
</tbody>
</table>

Table 22
The MSE×10⁻⁴ of the ML Estimates of the Lognormal Parameters in Simulation C₁₀₀

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE(( \mu ))</th>
<th>MSE(( \hat{\sigma}_1 ))</th>
<th>MSE(( \hat{\sigma}_2 ))</th>
<th>MSE(( \hat{\sigma}_3 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>11.17</td>
<td>.4939</td>
<td>5.685</td>
<td>49.97</td>
</tr>
<tr>
<td>10%</td>
<td>11.09</td>
<td>.5808</td>
<td>6.726</td>
<td>58.64</td>
</tr>
<tr>
<td>25%</td>
<td>11.08</td>
<td>.7453</td>
<td>8.951</td>
<td>73.40</td>
</tr>
<tr>
<td>50%</td>
<td>18.12</td>
<td>1.191</td>
<td>15.47</td>
<td>119.3</td>
</tr>
</tbody>
</table>

to 7.633 × 10⁻³ in the 50% censoring case which is the highest MSE among all the censoring cases. The MSE of the ML estimates of the skewness parameters increase as the level of censoring increases. Specifically, the MSE of \( \hat{\sigma}_1 \) increases from 2.412×10⁻⁴ in the uncensored case to 5.767×10⁻⁴ in the 50% censoring case; the MSE of \( \hat{\sigma}_2 \) increases from 2.705×10⁻³ in the uncensored case to 6.724×10⁻³ in the 50% censoring case; the MSE of \( \hat{\sigma}_3 \) increases from 2.589×10⁻² at the...
uncensored case to $6.628 \times 10^{-2}$ in the 50% censoring case. The same situation prevails in the large sample size case. From Table 22 the MSE of $\hat{\mu}$ decreases slightly from $1.117 \times 10^{-3}$ in the uncensored case to $1.109 \times 10^{-3}$ in the 10% censoring case to $1.108 \times 10^{-3}$ in the 25% censoring case. It then increases to $1.812 \times 10^{-3}$ in the 50% censoring case which is the highest MSE among all the censoring cases. The MSE of $\hat{\sigma}_1$ increases from $4.939 \times 10^{-5}$ in the uncensored

<table>
<thead>
<tr>
<th>C.L.</th>
<th>ESB($\hat{\mu}$)</th>
<th>ESB($\hat{\sigma}_1$)</th>
<th>ESB($\hat{\sigma}_2$)</th>
<th>ESB($\hat{\sigma}_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>6.763</td>
<td>-38.89</td>
<td>-128.7</td>
<td>-404.3</td>
</tr>
<tr>
<td>10%</td>
<td>-112.9</td>
<td>-27.93</td>
<td>-95.78</td>
<td>-303.9</td>
</tr>
<tr>
<td>25%</td>
<td>-278.9</td>
<td>-24.54</td>
<td>-83.79</td>
<td>-273.3</td>
</tr>
<tr>
<td>50%</td>
<td>-583.9</td>
<td>-42.15</td>
<td>-138.9</td>
<td>-289.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C.L.</th>
<th>ESB($\hat{\mu}$)</th>
<th>ESB($\hat{\sigma}_1$)</th>
<th>ESB($\hat{\sigma}_2$)</th>
<th>ESB($\hat{\sigma}_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>-1.541</td>
<td>-7.430</td>
<td>-25.05</td>
<td>-73.68</td>
</tr>
<tr>
<td>10%</td>
<td>-115.5</td>
<td>-5.121</td>
<td>-20.94</td>
<td>-67.65</td>
</tr>
<tr>
<td>25%</td>
<td>-279.4</td>
<td>-4.637</td>
<td>-18.74</td>
<td>-61.41</td>
</tr>
<tr>
<td>50%</td>
<td>-761.8</td>
<td>-4.672</td>
<td>-16.46</td>
<td>-76.68</td>
</tr>
</tbody>
</table>
case to $1.191 \times 10^{-4}$ in the 50% censoring case; the MSE of $\hat{\sigma}_2$ increases from $5.685 \times 10^{-4}$ in the uncensored case to $1.547 \times 10^{-3}$ in the 50% censoring case; the MSE of $\hat{\sigma}_3$ increases from $4.997 \times 10^{-3}$ in the uncensored case to $1.193 \times 10^{-2}$ in the 50% censoring case.

The estimated simulated biases (ESB) of the ML estimates of the lognormal parameters in simulations $C_{20}$ and $C_{100}$ were computed using (6.4). The ESB of the ML estimates of the skewness parameters do not follow a particular order as the level of censoring increases. However, $\hat{\mu}$ becomes unusually more biased as soon as censoring is introduced in both simulations $C_{20}$ and $C_{100}$. From Table 23, in simulation $C_{20}$ the ESB of $\hat{\mu}$ is $6.763 \times 10^{-4}$ in the uncensored case. At 10% censoring the ESB of $\hat{\mu}$ is $-1.129 \times 10^{-2}$, at 25% censoring the ESB of $\hat{\mu}$ is $-2.789 \times 10^{-2}$ and at 50% censoring the ESB of $\hat{\mu}$ is $-5.839 \times 10^{-2}$. From Table 24, in simulation $C_{100}$ the ESB of $\hat{\mu}$ is $-1.541 \times 10^{-4}$. At 10% censoring the ESB of $\hat{\mu}$ is $-1.155 \times 10^{-2}$, at 25% censoring the ESB of $\hat{\mu}$ is $-2.794 \times 10^{-2}$ and at 50% censoring the ESB of $\hat{\mu}$ is $-7.618 \times 10^{-2}$.

By comparing the MSE of the ML estimates in Table 23 with the corresponding MSE in Table 24 for each of the censoring cases separately, we conclude that the MSE of the ML estimates decrease as the sample sizes increase. For example, let us consider the 10% censoring case. From Table 23, in the small sample size case the MSE of $\hat{\mu}$ is $5.991 \times 10^{-3}$, the MSE of $\hat{\sigma}_1$ is $2.970 \times 10^{-4}$, the MSE of $\hat{\sigma}_2$ is $3.403 \times 10^{-3}$ and the MSE of $\hat{\sigma}_3$ is $3.039 \times 10^{-3}$. From Table 24, in the large sample size case the MSE of $\hat{\mu}$ is $1.109 \times 10^{-3}$, the MSE of $\hat{\sigma}_1$ is $5.808 \times 10^{-5}$, the MSE of $\hat{\sigma}_2$ is $6.726 \times 10^{-4}$ and the MSE of $\hat{\sigma}_3$ is $5.864 \times 10^{-3}$. The same trend occurs in the other censoring cases.

The same trend is observed in the ESB of the ML estimates of the skewness parameters but not for the ESB of $\hat{\mu}$. That is, $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$ are less
biased in the large sample size case than in the small sample size case while \( \hat{\mu} \) maintains about the same level of bias in the two sample size cases. For example, let us consider the 25% censoring case. From Table 23, in the small sample size case the ESB of \( \mu \) is \(-2.789 \times 10^{-2}\) while from Table 24, in the large sample size case the ESB of \( \mu \) is \(-2.794 \times 10^{-2}\). From Table 23, in the small sample size case the ESB of \( \hat{\sigma}_1 \) is \(-2.454 \times 10^{-3}\), the ESB of \( \hat{\sigma}_2 \) is \(-8.379 \times 10^{-3}\) and the ESB of \( \hat{\sigma}_3 \) is \(-2.733 \times 10^{-2}\). From Table 24, in the large sample size case the ESB of \( \hat{\sigma}_1 \) is \(-4.637 \times 10^{-4}\), the ESB of \( \hat{\sigma}_2 \) is \(-1.874 \times 10^{-3}\) and the ESB of \( \hat{\sigma}_3 \) is \(-6.141 \times 10^{-4}\).

6.3.4 Mean Square Error and Bias Simulation Results for Case \( H_4 \)

In simulation cases \( D_{20} \) (small sample size case) and \( D_{100} \) (large sample size case) data are created from three independent lognormal distributions with different median parameters \((\mu_1 = -1, \mu_2 = 0, \mu_3 = 1)\) and with different skewness parameters \((\sigma_1 = .1, \sigma_2 = .33, \sigma_3 = 1)\). In both of these simulations the EM algorithm yields the ML estimates of \( \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2 \) and \( \sigma_3 \) and as \( \hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\sigma}_1, \hat{\sigma}_2 \) and \( \hat{\sigma}_3 \), respectively. The MSE of these six ML estimates were computed using (6.1). The average MSE values are reported in Table 25 for simulation \( D_{20} \) and in Table 26 for simulation \( D_{100} \) for each of the four censoring cases. For both of these simulations the MSE of the ML estimates increase as the level of censoring increases. The estimated simulated biases (ESB) of the ML estimates of the lognormal parameters in simulations \( D_{20} \) and \( D_{100} \) were computed using (6.4). Unlike the MSE of the ML estimates, the ESB of the ML estimates which are presented in Tables 27 and 28 for simulation \( D_{20} \) and \( D_{100} \), respectively, do not follow a particular order of magnitude as the level of censoring increases. It is noted that the ESB of the ML estimates are maintained at low levels in both simulations.
Table 25
The MSE\times10^{-4} of the ML Estimates of the Lognormal Parameters in Simulation $D_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE($\hat{\mu}_1$)</th>
<th>MSE($\hat{\mu}_2$)</th>
<th>MSE($\hat{\mu}_3$)</th>
<th>MSE($\hat{\sigma}_1$)</th>
<th>MSE($\hat{\sigma}_2$)</th>
<th>MSE($\hat{\sigma}_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>4.929</td>
<td>54.47</td>
<td>493.5</td>
<td>2.412</td>
<td>27.05</td>
<td>258.9</td>
</tr>
<tr>
<td>10%</td>
<td>5.150</td>
<td>56.16</td>
<td>495.4</td>
<td>2.970</td>
<td>34.03</td>
<td>303.9</td>
</tr>
<tr>
<td>25%</td>
<td>5.522</td>
<td>61.60</td>
<td>576.5</td>
<td>3.695</td>
<td>42.35</td>
<td>389.8</td>
</tr>
<tr>
<td>50%</td>
<td>8.618</td>
<td>92.21</td>
<td>938.6</td>
<td>5.767</td>
<td>71.18</td>
<td>681.3</td>
</tr>
</tbody>
</table>

Table 26
The MSE\times10^{-4} of the ML Estimates of the Lognormal Parameters in Simulation $D_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>MSE($\hat{\mu}_1$)</th>
<th>MSE($\hat{\mu}_2$)</th>
<th>MSE($\hat{\mu}_3$)</th>
<th>MSE($\hat{\sigma}_1$)</th>
<th>MSE($\hat{\sigma}_2$)</th>
<th>MSE($\hat{\sigma}_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>.9781</td>
<td>11.11</td>
<td>88.48</td>
<td>.4939</td>
<td>5.685</td>
<td>49.97</td>
</tr>
<tr>
<td>10%</td>
<td>.9948</td>
<td>11.47</td>
<td>90.17</td>
<td>.5808</td>
<td>6.726</td>
<td>58.64</td>
</tr>
<tr>
<td>25%</td>
<td>1.057</td>
<td>12.29</td>
<td>98.99</td>
<td>.7453</td>
<td>8.951</td>
<td>73.40</td>
</tr>
<tr>
<td>50%</td>
<td>1.551</td>
<td>18.38</td>
<td>124.8</td>
<td>1.192</td>
<td>15.40</td>
<td>119.5</td>
</tr>
</tbody>
</table>

By comparing the MSE from Table 25 with the MSE from Table 26 at each of the four censoring levels we observe that the MSE are larger in the small sample size case. Therefore we conclude that the MSE of the ML estimates increase when the sample sizes increase. The same conclusion is drawn for the ESB of the ML estimates by comparing the ESB from Table 27 with the ESB from Table 28. That is, the ML estimates of the lognormal parameters are less biased in the large
Table 27

The $ESB \times 10^{-4}$ of the ML Estimates of the Lognormal Parameters in Simulation $D_{20}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$ESB(\hat{\mu}_1)$</th>
<th>$ESB(\hat{\mu}_2)$</th>
<th>$ESB(\hat{\mu}_3)$</th>
<th>$ESB(\hat{\sigma}_1)$</th>
<th>$ESB(\hat{\sigma}_2)$</th>
<th>$ESB(\hat{\sigma}_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>3.624</td>
<td>8.682</td>
<td>7.985</td>
<td>-38.89</td>
<td>-128.7</td>
<td>-404.3</td>
</tr>
<tr>
<td>10%</td>
<td>-2.571</td>
<td>-9.953</td>
<td>-45.50</td>
<td>-27.93</td>
<td>-95.78</td>
<td>-303.9</td>
</tr>
<tr>
<td>25%</td>
<td>-6.848</td>
<td>-25.45</td>
<td>-93.62</td>
<td>-24.54</td>
<td>-83.79</td>
<td>-273.3</td>
</tr>
<tr>
<td>50%</td>
<td>-16.91</td>
<td>-45.90</td>
<td>-204.9</td>
<td>-23.41</td>
<td>-89.81</td>
<td>-258.9</td>
</tr>
</tbody>
</table>

Table 28

The $ESB \times 10^{-4}$ of the ML Estimates of the Lognormal Parameters in Simulation $D_{100}$

<table>
<thead>
<tr>
<th>C.L.</th>
<th>$ESB(\hat{\mu}_1)$</th>
<th>$ESB(\hat{\mu}_2)$</th>
<th>$ESB(\hat{\mu}_3)$</th>
<th>$ESB(\hat{\sigma}_1)$</th>
<th>$ESB(\hat{\sigma}_2)$</th>
<th>$ESB(\hat{\sigma}_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNC.</td>
<td>1.430</td>
<td>-1.052</td>
<td>-5.002</td>
<td>-7.430</td>
<td>-25.05</td>
<td>-73.68</td>
</tr>
<tr>
<td>10%</td>
<td>.1217</td>
<td>-3.724</td>
<td>-8.838</td>
<td>-5.118</td>
<td>-20.93</td>
<td>-67.64</td>
</tr>
<tr>
<td>25%</td>
<td>-.6349</td>
<td>-6.848</td>
<td>-17.24</td>
<td>-4.637</td>
<td>-18.74</td>
<td>-61.41</td>
</tr>
<tr>
<td>50%</td>
<td>-2.833</td>
<td>-14.92</td>
<td>-22.36</td>
<td>-4.103</td>
<td>-16.69</td>
<td>-72.23</td>
</tr>
</tbody>
</table>

sample size case than in the small sample size case.

6.3.5 Summary and Conclusions on MSE and ESB Simulation Results

The estimated simulated mean square errors of the ML estimates of the lognormal parameters increase as the level of censoring increases. The only exception to this rule is observed in the MSE of the of the ML estimate of the
common median parameter ($\mu$) in case $H_3$ in both sample size cases. In this case the MSE($\hat{\mu}$) decreases slightly as the level of censoring increases from 0% to 25% and then increases at 50% censoring where it obtains the highest value. Also the MSE of the ML estimates decrease as the sample sizes increase.

The estimated simulated biases (ESB) of the ML estimates are maintained at low levels and do not follow a particular order of magnitude as the level of censoring increases. The only exception is observed in the ESB of the the ML estimate of the common median parameter ($\mu$) in case $H_3$ in both sample size cases. In this case $\hat{\mu}$ becomes unusually more biased as censoring is introduced at 10% and continues to become more biased as the level of censoring increases. Also the ESB of the ML estimates of the lognormal parameters are less biased in the large sample size case. The only exception is observed in in the ESB($\hat{\mu}$) in case $H_3$ which remains the same in both sample size cases.
BIBLIOGRAPHY


