On Distance in Graphs and Digraphs

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ON DISTANCE IN GRAPHS AND DIGRAPHS

by

Songlin Tian

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
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ON DISTANCE IN GRAPHS AND DIGRAPHS

Songlin Tian, Ph.D.
Western Michigan University, 1990

One of the most basic concepts associated with a graph is distance. In this dissertation some new definitions of distance in graphs and digraphs are introduced. One principle goal is to extend certain known results involving the standard distance function on graphs to the field of digraphs with an appropriate concept of distance. Several parameters as well as subgraphs and subdigraphs defined in terms of distance are investigated.

Chapter I gives a brief overview of the history of distance and generalized distance in graphs. By presenting a listing of major results in this area, it provides a background for the chapters to follow.

In Chapter II some results concerning distance in graphs are presented. It is proved that, for a graph $G$ and integers $r$ and $d$ with $1 < r < d < 2r$, there exists a connected graph $H$ of radius $r$ and diameter $d$ such that the center of $H$ is isomorphic to $G$. A new distance in graphs, called detour distance, is introduced. A generalized Steiner distance in graphs is discussed as well.

In Chapter III maximum distance in digraphs is introduced. It is proved that maximum distance is a metric. The $m$-radius, $m$-diameter, $m$-center, $m$-periphery and $m$-median, defined in terms of maximum distance, are studied. In particular, it is proved that every oriented graph is isomorphic to the $m$-center of some strong oriented graph.

For an oriented graph $D$, the appendage number of $D$ is defined as the minimum number of vertices required to add to $D$ to produce an oriented graph $H$. 

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such that the m-center of $H$ is isomorphic to $D$. The main result of Chapter IV is a characterization of oriented acyclic graphs having appendage number 2.

In Chapter V sum distance in digraphs is defined. The s-eccentric set, s-radius, s-diameter, s-center, s-appendage number and s-periphery are investigated. In particular, characterizations of s-eccentric sets and s-peripheries of oriented graphs are presented.
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On distance in graphs and digraphs

Tian, Songlin, Ph.D.
Western Michigan University, 1990
To my parents

and to

my lovely wife Qi
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I wish to thank my advisor, Professor Gary Chartrand, for his outstanding guidance and continual encouragement, and for his time spent critiquing and editing this manuscript. I have gained much from his innumerable ideas. I also greatly appreciate his genuine interest in my work and career. I thank Professor Alfred Boals and Professor Naveed Sherwani for their careful and thorough reading of the manuscript. I am very grateful to Professor Yousef Alavi for serving on my committee and always showing interest in my work and career. I would also like to express my gratitude to Professor Erik Schreiner, and Professor Garry Johns of Saginaw Valley State University, for serving on my committee and offering valuable comments. I thank the many faculty members of Western Michigan University for their help and support.

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS .................................................................................................................. iii

CHAPTER

I. PRELIMINARIES ......................................................................................................................... 1
   1.1 Definition and Notation ................................................................................................. 1
   1.2 Historical Review ......................................................................................................... 2

II. SOME RESULTS ON DISTANCE IN GRAPHS ...................................................................... 10
   2.1 On Graphs With Prescribed Center and Other Properties ...................................... 10
   2.2 Detour Distance .......................................................................................................... 15
   2.3 Generalized Steiner Distance ..................................................................................... 23
   2.4 (i, n) Eccentricity and (i, n) Center ........................................................................... 28

III. MAXIMUM DISTANCE IN DIGRAPHS .................................................................................. 32
    3.1 Definition of Maximum Distance and Basic Properties ............................................ 32
    3.2 The m-Centers of Oriented Graphs ............................................................................. 38
    3.3 The m-Peripheries of Oriented Graphs ....................................................................... 49
    3.4 The m-Medians of Oriented Graphs ............................................................................ 53

IV. THE APPENDAGE NUMBER OF AN ORIENTED GRAPH .................................................... 62
    4.1 Introduction .................................................................................................................. 62
    4.2 A Characterization of Oriented Acyclic Graphs with Appendage Number 2 .......... 66

V. SUM DISTANCE IN DIGRAPHS .............................................................................................. 102
    5.1 Definition of Sum Distance and Basic Properties ..................................................... 102
Table of Contents — Continued

5.2 The s-Eccentricity Sets of Oriented Graphs ....................... 106
5.3 The s-Centers and Sum Appendage Numbers of Oriented Graphs ........................................................................... 111
5.4 The s-Peripheries of Oriented Graphs ............................. 115
REFERENCES .................................................................................................................. 121
CHAPTER I

PRELIMINARIES

This chapter begins with some preliminary definitions and notation, followed by an historical review of distance and generalized distance in graphs.

1.1 Definitions and Notation

For a graph $G$, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of $G$, respectively. The number $p(G) = |V(G)|$ denotes the order of a graph $G$, while $q(G) = |E(G)|$ denotes the size of $G$. Similar notation is used for digraphs. If the graph $G$ is clear from the context, we use $p$ and $q$, respectively, for $p(G)$ and $q(G)$. The neighborhood $N(v)$ of a vertex $v$ in $G$ is the set of vertices to which $v$ is adjacent. If $x$ is a vertex of a digraph $D$, then the out-neighborhood $N^+(x)$ of $x$ (the in-neighborhood $N^-(x)$ of $x$) denotes the set of vertices of $D$ adjacent from (to) $x$.

The intersection $G_1 \cap G_2$ of two subgraphs $G_1$ and $G_2$ of a graph $G$ is the subgraph of $G$ whose vertex set is $V(G_1) \cap V(G_2)$ and whose edge set is $E(G_1) \cap E(G_2)$. If $G_1$ and $G_2$ are subgraphs of a connected graph $G$, then we denote by $d_G(G_1, G_2)$ (or, simply, $d(G_1, G_2)$) the minimum distance in $G$ between vertices of $G_1$ and $G_2$.

A digraph $D$ is called an oriented graph if whenever $(u, v)$ is an arc of $D$, then $(v, u)$ is not an arc of $D$. Thus, an oriented graph $D$ can be obtained from a graph $G$ by assigning a direction to each edge of $G$. The underlying graph of a digraph $D$ is that graph $G$ obtained from $D$ by deleting all directions from the arcs of $G$. 

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and deleting an edge from a pair of multiple edges if multiple edges should be produced. If \( U \) is a nonempty subset of the vertex set of a digraph \( D \), then the subdigraph \( <U> \) of \( D \) induced by \( U \) is that digraph having vertex set \( U \) and whose arc set consists of all those arcs of \( D \) joining vertices of \( U \). Similarly, if \( F \) is a nonempty subset of \( E(D) \), then the subdigraph \( <F> \) of \( D \) induced by \( F \) is the digraph whose vertex set consists of those vertices of \( D \) incident to or from at least one arc of \( F \) and whose arc set is \( F \). A digraph \( D \) is called connected or weakly connected if its underlying graph is connected and is strongly connected or strong if for every pair \( u, v \) of vertices of \( D \) there are (directed) \( u\rightarrow v \) paths and \( v\rightarrow u \) paths. A connected component (or, simply, a component) of a digraph \( D \) is that subdigraph induced by a connected component of the underlying graph of \( D \).

An oriented graph \( D \) is an oriented tree if the underlying graph of \( D \) is a tree. If \( v \) is a vertex of an oriented tree \( D \), then a branch of \( D \) at \( v \) is a subdigraph induced by \( v \) and the vertices of a connected component of \( D - v \).

The symbol \((X, Y)\) is the subset of \( E(D) \) defined by
\[
(X, Y) = \{(x, y) \mid x \in X, y \in Y\}.
\]
If \( X = \{x\} \) we often write \((x, Y)\) rather than \((\{x\}, Y)\). Similarly, \((X, y)\) denotes \((X, \{y\})\). We write \( G = G_1 \cup G_2 \cup ... \cup G_n \) to mean \( G \) is the graph induced by the edge sets of \( G_i \) \( (1 \leq i \leq n) \). We will use the symbol "->" to mean all omitted arcs of digraphs shown in figures are from left to right.

### 1.2 Historical Review

Graphs are often used as models for studying structures and relationships in many real-world situations. In general, graphs can be used to model a binary relation system. For a given binary relation system \((V, R)\), the graph \( G \) associated with it is
that graph with $V(G) = V$ and $E(G) = \{ uv \mid u, v \in V \text{ and } uRv \}$. It is natural to ask how close two objects are to each other under a certain binary relation. The closeness then can be measured by the distance between two vertices in the graph associated with the relation.

For a connected graph $G$ and vertices $u$ and $v$ of $G$, we define the distance \( d(u,v) \) between $u$ and $v$ as the length of a shortest $u$-$v$ path in $G$. There are efficient algorithms to find the distance between two vertices in a graph or weighted graph (see [11]). The distance is a metric on the vertex set of $G$, that is:

1. $d(u, u) = 0$ for each $u \in V(G)$;
2. $d(u, v) = d(v, u)$ for all $u, v \in V(G)$; and
3. $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in V(G)$.

The study of distance and of parameters and subgraphs defined in terms of distance has received considerable attention. An entire book [4] has been devoted to the study of distance in graphs.

For a given vertex $v$, the distance between $v$ and a vertex furthest from $v$ is a useful parameter in $G$. This parameter is called the eccentricity $e(v)$ and is formally defined by $e(v) = \max \{ d(v,w) \mid w \in V(G) \}$. The radius of $G$ is $\text{rad } G = \min \{ e(v) \mid v \in V(G) \}$ while the diameter of $G$ is $\text{diam } G = \max \{ e(v) \mid v \in V(G) \}$. The following well known theorem gives a relationship between the radius and the diameter of a connected graph.

**Theorem A** For every connected graph $G$, $\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$.

Ostrand [21] proved the sharpness of this inequality. In Chapter II, we will see a variation of Ostrand's result.

**Theorem B** (Ostrand) For every two positive integers $r$ and $d$ with $r \leq d \leq 2r$, there
exists a connected graph $G$ with $\text{rad } G = r$ and $\text{diam } G = d$. Furthermore, the minimum order of such a graph $G$ is $r + d$.

The center $C(G)$ of a graph $G$ is the subgraph of $G$ induced by those vertices with minimum eccentricity. The first result dealing with centers was obtained by Jordan [16] who determined those graphs that are centers of trees.

**Theorem C** (Jordan) The center of a tree consists of either a single vertex or a pair of adjacent vertices.

Trees form a special family of graphs. It is natural to ask which graphs in general can be the center of some connected graph. This was answered by Kopylov and Timofeev [23] and, independently, by Hedetniemi (see [5]).

**Theorem D** (Kopylov, Timofeev and Hedetniemi) For every graph $G$, there exists a graph $H$ such $C(H) \equiv G$.

By Hedetniemi's construction (see Figure 1.1), there exists such a graph $H$ that exceeds the order of $G$ by at most 4. Buckley, Miller and Slater [5] then defined, for each graph $G$, the parameter $f(G)$ to be the minimum number of vertices required to add to $G$ to produce a graph $H$ having $G$ as its center. We refer to it as the appendage number of $G$. By Theorem D, $f(G) \leq 4$. If $C(H) \equiv G$ and $H \neq G$, then all vertices with maximum eccentricity are in $H - G$. Since there are at least two vertices with maximum eccentricity in every connected graph, we have $|V(H) - V(G)| \geq 2$. Therefore, $f(G) \neq 1$ for each graph $G$. A graph with $f(G) = 0$ is called a self-centered graph. Buckley, Miller and Slater [5] proved that $f(T) \neq 3$ for all trees $T$ and characterized trees $T$ with $f(T) = 0, 2, 4$. There are graphs $G$, however, that are not trees for which $f(G) = 3$ (see [2] and [10]).
Figure 1.1

The concept of the center of a connected graph is useful in solving emergency facility location problem. When finding a good location for an emergency facility, we want to minimize the response time. This is equivalent to minimizing the distance between the facility and a location of a possible emergency. Therefore, a vertex in the center of a graph corresponds to a good location for an emergency facility. If we wish to locate a service facility, then we want to minimize the average distance to the facility from all the households in the community. Another interpretation of the "middle" of the graph is very useful in this situation.

The distance $d(v)$ of a vertex $v$ in a connected graph $G$ is defined as the sum of the distances from $v$ to the vertices of $G$, that is, $d(v) = \sum \{ d(v, w) \mid w \in V(G) \}$. The median $M(G)$ of $G$ is the subgraph of $G$ induced by those vertices having minimum distance. A vertex in the median of a graph then corresponds to a good location for a service facility.

Every graph is isomorphic to the center of some connected graph. There is a similar result involving the median of a connected graph. Hendry [12] obtained the following even stronger result.

**Theorem E (Hendry)** For every two graphs $F$ and $H$, there exists a connected graph $G$ such that $C(G) \cong F$ and $M(G) \cong H$.

The graph $G$ so constructed by Hendry had the property that
d_G(C(G), M(G)) = 1. This result might suggest that C(G) and M(G) must be "close" in G. However, Holbert [15] showed the distance between C(G) and M(G) can be arbitrarily large.

**Theorem F** (Holbert) For every two graphs F and H and positive integer n, there exists a connected graph G such that C(G) \cong F, M(G) \cong H, and d(C(G), M(G)) = n.

Holbert [15] also proved the other extreme case, that is, the center and median of a graph can overlap in a variety of ways.

**Theorem G** (Holbert) For every two graphs F and H and every graph X that is isomorphic to an induced subgraph of both F and H, there exists a connected graph G such that C(G) \cong F, M(G) \cong H, and C(G) \cap M(G) = X.

The center and median are two interpretations of the "middle" of a connected graph. A common definition of the "exterior" of a connected graph G is the periphery of G. Formally, the periphery P(G) of a connected graph G is the subgraph of G induced by those vertices with maximum eccentricity. Every graph is the center of some connected graph but such is not the case for the periphery. Bielak and Syslo [3] characterized those graphs that are the peripheries of connected graphs. The symbol \( \Delta(G) \) represents the maximum degree of the vertices of a graph G.

**Theorem H** (Bielak and Syslo) Let G be a graph of order p. Then G is isomorphic to the periphery of some connected graph if and only if either G is complete or \( \Delta(G) \leq p - 2 \).

Note that, in a connected graph G, \( \deg_G v = p - 1 \) if and only if \( e_G(v) = 1 \). With this observation, Theorem H can be restated as:
Theorem H' A graph $G$ is isomorphic to the periphery of some connected graph if and only if either all vertices of $G$ have eccentricity 1 or no vertices have eccentricity 1.

Chartrand, Johns and Oellermann [6] extended Theorem H in the following manner.

Theorem I Let $n \geq 2$ be an integer. Then a graph $G$ is the periphery of a graph having diameter $n$ if and only if $\text{rad } G \leq n$.

An extension of Theorem I will be given in Chapter II.

Recall that the distance between vertices $u$ and $v$ is the length of a shortest $u$-$v$ path. Equivalently, it is the minimum size of a connected subgraph whose vertex set contains $u$ and $v$. From this observation, Chartrand, Oellermann, Tian and Zou [9] introduced a generalization of the standard distance, called Steiner distance.

Let $G$ be a connected graph and let $S$ be a nonempty subset of $V(G)$. The Steiner distance $d(S)$ of $S$ is the minimum size among all connected subgraphs whose vertex set contains $S$. Since $d(\{u, v\}) = d(u, v)$, Steiner distance is a generalization of the standard distance in graphs.

The problem of finding the Steiner distance of a given set of vertices is called the Steiner problem in graphs. There exist efficient algorithms for finding the distance between a pair of vertices in a graph or weighted graph. Unfortunately, no efficient algorithm is known for finding the Steiner distance $d(S)$ if $|S| \geq 3$. In fact, the Steiner problem for both graphs and weighted graphs is known to be NP-complete.

The $n$-eccentricity $e_n(v)$ of a vertex $v$ of $G$ is defined by $e_n(v) = \max \{d(S) \mid S \subseteq V(G), |S| = n, v \in S\}$. The $n$-radius, $n$-diameter and $n$-center of $G$ are defined
in a natural way. Observe that $e_2(v) = e(v)$, $\text{rad}_2 G = \text{rad} G$ and $\text{diam}_2 G = \text{diam} G$.

Much research has been done to find upper bounds for the $n$-diameter. For a tree $T$, the following result is best possible (see [9]).

**Theorem J** (Chartrand, Oellermann, Tian and Zou) Let $n \geq 2$ be an integer and $T$ a tree of order at least $n$. Then $\text{rad}_n T \leq \text{diam}_n T \leq \frac{n}{n-1} \text{rad}_n T$.

For graphs in general and $2 \leq n \leq 4$, the next result [14] is best possible. The problem is still open for $n \geq 5$.

**Theorem K** (Henning, Oellermann and Swart) If $G$ is a connected graph and $n$ is an integer with $2 \leq n \leq 4$, then $\text{rad}_n G \leq \text{diam}_n G \leq \frac{2n+2}{2n-1} \text{rad}_n G$.

Note that when $n = 2$, both Theorems J and K give us $\text{rad} G \leq \text{diam} G \leq 2 \text{rad} G$. The following result shows that, for any integer $n \geq 3$, every graph is the $n$-center of some connected graph; so it is an extension of Theorem D.

**Theorem L** (Oellermann and Tian) For every integer $n \geq 3$ and every graph $G$, there exists a graph $H$ such that $G$ is isomorphic to the $n$-center of $H$.

A generalized Steiner distance will be discussed in Chapter II, which will employ the concepts of connectivity and hamiltonian graphs.

We now turn to digraphs. The **directed distance** $\bar{d}(u, v)$ from $u$ to $v$ in a digraph $D$ is the length of a shortest $u$-$v$ path in $D$. If there is no $u$-$v$ path in $D$, then $\bar{d}(u, v) = \infty$. The directed distance is not a metric on $V(D)$ since, in general, $\bar{d}(u, v) \neq \bar{d}(u, v)$. The standard distance in graphs satisfies the following properties:

1. $d(u, v) = \min \{d(u, v), d(v, u)\}$ for all $u, v \in V(G)$;
2. $d(u, v) = \max \{d(u, v), d(v, u)\}$ for all $u, v \in V(G)$;
(3) \( d(u, v) = \frac{1}{2} [d(u, v) + d(v, u)] \) for all \( u, v \in V(G) \).

This observation suggests three possible ways to extend the standard distance in graphs to distance in digraphs. For vertices \( u \) and \( v \) in a digraph \( D \), by (1) we can define the minimum distance between \( u \) and \( v \) as \( \min\{d(u, v), d(v, u)\} \). It is clear that the minimum distance is symmetric. However, it does not satisfy the triangle inequality, so it is not a metric on the vertex set of \( D \).

In Chapter III, a distance, called maximum distance, between vertices \( u \) and \( v \) in digraphs will be defined by \( \max\{d(u, v), d(v, u)\} \). It is proved that the maximum distance is a metric on the vertex set of \( D \). The m-radius, m-diameter, m-center, m-periphery and m-median all defined in terms of maximum distance, are investigated. In particular, it is proved that every oriented graph is isomorphic to the m-center of some strong oriented graph. Therefore, we can define, for each oriented graph \( D \), the appendage number of \( D \) to be the minimum number of vertices required to add to \( D \) to produce an oriented graph \( H \) such that the m-center of \( H \) is isomorphic to \( D \).

Chapter IV presents a characterization of oriented acyclic graphs having appendage number 2.

In Chapter V another distance, called sum distance, between vertices \( u \) and \( v \) in digraphs is defined as \( d(u, v) + d(v, u) \). The s-eccentric set, s-radius, s-diameter, s-center, s-appendage number and s-periphery are defined and investigated. In particular, characterizations of s-eccentric sets and s-peripheries of oriented graphs are given.
CHAPTER II

SOME RESULTS ON DISTANCE IN GRAPHS

We begin by presenting an extension of the result that every graph is the center of some connected graph. This result appears in several forms as various definitions of "center" are developed. A new distance, called detour distance, is considered and used to introduce a new class of graphs. We conclude by discussing a generalized Steiner distance in graphs.

2.1 On Graphs With Prescribed Center and Other Properties

Kopylov and Timofeev mentioned in [23] and Hedetniem (see [5]) proved that for every graph $G$, there exists a connected graph $H$ such that $C(H) \equiv G$. The graph $H$ so constructed by Hedetniem has radius 2 and diameter 4. Consequently, in his construction, the subgraph of $H$ induced by those vertices with eccentricity 2 is isomorphic to $G$. We extend this result by showing that for every graph $G$ and integers $r$ and $d$ with $2 < r < d < 2r$, there exists a connected graph $H$ with $\text{rad } H = r$ and $\text{diam } H = d$ such that $C(H) \equiv G$. A connected graph $G$ is unicentered if $G$ has exactly one central vertex. Ostrand [21] proved that for integers $r$ and $d$ with $1 \leq r \leq d \leq 2r$, there exists a connected graph $G$ such that $\text{rad } G = r$ and $\text{diam } G = d$. We strengthen Ostrand's result by showing that if $r \neq d$, then the desired connected graph may be chosen to be unicentered.

**Theorem 2.1** If $r$ and $d$ are integers with $2 \leq r < d \leq 2r$, then there exists a unicentered graph $G$ such that $\text{rad } G = r$ and $\text{diam } G = d$. 

10
Proof. We consider two cases.

Case 1. Assume that $d = 2r$.

Let $G = P_{2r+1} : v_0, v_1, \ldots, v_{2r}$. Then $e(v_i) = \max\{i, 2r - i\}$. Therefore, $\text{rad } G = e(v_r) = r$, $\text{diam } G = e(v_0) = e(v_{2r}) = d$, and $C(G) = \{v_r\}$.

Case 2. Assume that $d < 2r$.

Let $n = 2d - 2r + 2$. Since $d > r$, it follows that $n \geq 4$. Let $H$ be a graph consisting of $n$ copies of $P_r$. Denote $H$ by $Q_1 \cup Q_2 \cup \ldots \cup Q_n$, where $Q_i : v_{i1}, v_{i2}, \ldots, v_{ir}$ $(1 \leq i \leq n)$ is a path of order $r$. We construct the graph $G$ by adding a new vertex $v$ to $H$ and the edges $v_{ir}v_{i+1,r}$, $1 \leq i \leq n - 1$, together with the edges joining $v$ with all vertices $v_{i1}$, $1 \leq i \leq n$ (see Figure 2.1). Then $e(v) = r$. Consider two vertices $v_{ij}$ and $v_{km}$. Without loss of generality, assume that $i < k$. Then the vertices $v_{ij}$ and $v_{km}$ lie on the cycle $C : v, v_{i1}, \ldots, v_{ir}, v_{i+1,r}, \ldots, v_{k1}, v_{kr}, \ldots, v_{km}$. Therefore,

$$d(v_{ij}, v_{km}) \leq \left\lfloor \frac{|V(C)|}{2} \right\rfloor \leq \left\lfloor \frac{2r + k - 1}{2} \right\rfloor \leq r + \left\lfloor \frac{n - 1}{2} \right\rfloor = r + \left\lfloor \frac{2d - 2r + 2 - 1}{2} \right\rfloor = d.$$

On the other hand, since $d(v_{ij}, v_{k,j+1}) = r + 1$, it follows that $r + 1 \leq e(v_{ij}) \leq d$ for all $1 \leq i \leq n$ and $1 \leq j \leq r$. Finally, since $d(v_{11}, v_{k2}) = d$, where $t = \left\lfloor \frac{d}{2} \right\rfloor$ and $s = \left\lceil \frac{d}{2} \right\rceil$, it follows that $\text{rad } G = r$, $\text{diam } G = d$ and $C(G) = \{v\}$. \(\Box\)

![Figure 2.1](image-url)
An extension of Hedetniemi's result follows immediately from Theorem 2.1.

**Corollary 2.2** For every graph $G$ and for integers $r$ and $d$ with $2 \leq r < d \leq 2r$, there exists a connected graph $H$ with $\text{rad} \ H = r$, $\text{diam} \ H = d$ and $C(H) \equiv G$.

**Proof.** By Theorem 2.1, there exists a unicentered graph $H_0$ such that $\text{rad} \ H_0 = r$ and $\text{diam} \ H_0 = d$. We construct the graph $H$ by replacing the central vertex $v$ of $H_0$ by $G$ and joining each vertex of $G$ with all vertices adjacent to $v$ in $H_0$. Clearly, $\text{rad} \ H = r$, $\text{diam} \ H = d$ and $C(H) \equiv G$. □

Ostrand [21] also proved that if $r$ and $d$ are integers with $2 < r < d < 2r - 1$, then the minimum order of a connected graph of radius $r$ and diameter $d$ is $r + d$. It is interesting to know the minimum order of unicentered graphs of radius $r$ and diameter $d$. If $d = 2r$, then since a path of length $d$ is a unicentered graph of radius $r$ and diameter $d$, the minimum order in this special case is $d + 1$. However, a general answer to this problem is unknown.

We now turn to the periphery of a connected graph. As we have seen in Chapter I, a graph $G$ is isomorphic to the periphery of some connected graph if and only if either $G$ is a complete graph or $\Delta(G) \leq p(G) - 2$. If $G$ is complete, then clearly every vertex of $G$ is a central vertex of $G$. If $\text{diam} \ G = 2$ and there exists a connected graph $H (\neq G)$ such that $P(H) \equiv G$, then $C(H)$ is complete. Chartrand, Johns and Oellermann [6] extended this characterization of the periphery of a graph. They showed that a graph $G$ is the periphery of a graph having diameter $n$ if and only if $\text{rad} \ G \geq n$. Our next result shows that for every graph $G$ with $\text{rad} \ G = n \geq 3$, there exists a unicentered graph $H$ with $\text{diam} \ H \leq n$ such that $P(H) \equiv G$.

**Theorem 2.3** Let $F$ be a connected graph with $\text{rad} \ F = n \geq 3$. Then there exists a
unicentered graph $G$ with $\text{diam } G = d$ such that $P(G) \equiv F$ for all $3 \leq d \leq n$.

**Proof.** Let $k = \left\lfloor \frac{d}{2} \right\rfloor$. We first construct a preliminary graph $G_1$ by attaching each vertex of $F$ with a copy of $P_k$ (see Figure 2.2 (a)). For $v \in V(F)$, we denote such an attaching path in $G_1$ by $P_v : v , \ldots , w_v$. Let $G_2$ be the graph obtained from $G_1$ by joining all the vertices $w_v , v \in V(F)$, to a new vertex $w$ (see Figure 2.2 (b)). If $n$ is even, then we let $G = G_1$. It is straightforward to show that $d_G(u, v) \leq d$ for all $u, v \in V(F)$. For $u \in V(F)$, let $v$ be an eccentric vertex of $u$ in $F$. Then, $d_F(u, v) = e(u) \geq d$. Observe that if a shortest $u-v$ path $P$ in $G$ contains the vertex $w$, then the length of $P$ is $2k = d$. Therefore, $d_G(u, v) = d$ so that $e_G(u) \geq d$ for all $u \in V(F)$.

Clearly, the distance between a vertex of $F$ and a vertex not in $F$ is less than $d$. Therefore, $e_G(u) = d$ for all $u \in V(F)$. For $u, v \in V(G) - V(F)$, it follows that $d(u, v) < d(u, w) + d(w, v) < d(u, w) + 2(k - 1) = d - 2 < d$.

Therefore, $\text{diam } G = d$ and $P(G) \equiv F$. Clearly, $e(w) = k = \left\lfloor \frac{d}{2} \right\rfloor$. For $x \in V(P_u) - \{u, w_u\}$, where $u \in V(F)$, let $v$ be an eccentric vertex of $u$ in $F$. Then

$$d_G(x, v) = \min\{d_G(x, u) + d_G(u, v), d_G(x, w) + d_G(w, v)\}$$

$$\geq \min\{e(u) + 1, k + 1\}$$

$$= k + 1 > k = \frac{d}{2}$$

so $x$ is not a vertex of $C(G)$. Therefore, $C(G) = \{w\}$.

Now suppose that $d$ is odd. We construct the connected graph $G$ from $G_2$ by adding the edges $w_uw_v$ for all $u, v \in V(F)$ with $d_F(u, v) = d$ (see Figure 2.2 (c)). Clearly, $e(w) = k + 1 < d$. Let $u, v \in V(F)$. If $d_F(u, v) \geq d$, then

$$d_G(u, v) \leq d_G(u, w_u) + d_G(w_u, w_v) + d_G(w_v, v) = k + 1 + k = d$$

Thus $d_G(u, v) \leq d$ for all $u, v \in V(F)$. Let $u \in V(F)$ and let $v$ be an eccentric vertex of $u$ in $F$. Then $d_F(u, v) = e(u) \geq d$. If a shortest $u-v$ path $P$ contains a
vertex not in F, then the length of P is at least $2k + 1 = d$. Therefore, $d_G(u, v) = d$.

Consider vertices $x \in V(F)$ and $y \in V(G) - V(F) - \{w\}$. Suppose, without loss of generality, that $y \in V(P_v) - \{v\}$. Then $d(w, y) \leq k - 1$, so that

$$d_G(x, y) \leq d(x, w) + d(w, y) \leq k + 1 + k = 2k = d - 1.$$ Combining the above, we see that $e(v) = d$ for all $v \in V(F)$. For $x, y \in V(G) - V(F)$, it follows that

$$d(x, y) \leq d(x, w) + d(w, y) \leq 2k < 2k + 1 = d.$$ Therefore, $diam G = d$ and $P(G) \neq F$. To prove that $C(G) = \{w\}$, it suffices to show that $e(x) > e(w)$ for all $x \in V(G) - V(F) - \{w\}$. Let $u \in V(F)$, and let $v$ be an eccentric vertex of $u$ in $F$. Then $d_F(u, v) = e(u) > d$. Therefore, for $x \in V(P_u) - \{w_u\}$,

$$e(x) \geq d_G(x, v)$$

$$= \min\{d_G(x, u) + d_F(u, v), d_G(x, w_u) + d_G(w_u, w_v) + d_G(w_v, v)\}$$

$$\geq \min\{1 + e(u), 1 + 1 + k\}$$

$$= k + 2 > k + 1 = e(w).$$
For \( u \in V(F) \), let \( v \) be a vertex such that \( d_F(u, v) = d - 1 \). Then \( w_uw_v \in E(G) \). Therefore,
\[
e(w_u) \geq d(w_u, v) = \min\{d(w_u, u) + d_F(u, v), d(w_u, w) + d(w, v)\} \\
\geq \min\{k + d - 1, 1 + k + 1\} = k + 2 > k + 1 = e(w).
\]
This completes the proof. □

The following result is an immediate corollary of Theorem 2.3.

**Corollary 2.4** Let \( F \) be a connected graph with \( \text{rad } F = n \geq 3 \). For every graph \( G \) and integer \( d \) (\( 3 \leq d \leq n \)), there exists a connected graph \( H \) with \( \text{diam } H = d \), \( P(H) \equiv F \) and \( C(H) \equiv G \).

**Proof.** By Theorem 2.3, there exists a unicentered graph \( H_0 \) such that \( \text{diam } H_0 = d \) and \( P(H) \equiv F \). Let \( H \) be the graph obtained by replacing the central vertex \( v \) of \( H_0 \) by \( G \) and joining each vertex of \( G \) with all vertices adjacent to \( v \) in \( H_0 \). Then, \( \text{diam } H = d \), \( P(H) \equiv F \) and \( C(H) \equiv G \). □

It appears very difficult to determine the minimum order of a connected graph of radius \( r \) and diameter \( d \) containing a given graph as its center.

### 2.2 Detour Distance

If \( P \) is a shortest \( u-v \) path in a connected graph \( G \), then \( <V(P)> = P \). This observation suggests a generalization of the standard distance in graphs. Let \( u \) and \( v \) be vertices in a connected graph \( G \). The **detour distance** \( d^*(u, v) \) between \( u \) and \( v \) is the length of a longest \( u-v \) path \( P \) such that \( <V(P)> = P \).

Note that \( d^*(u, v) = d^*(v, u) \) for all vertices \( u, v \) of \( G \). Therefore the detour distance is symmetric. However, in general, the triangle inequality does not hold.

Consider the wheel \( W_n \), where \( n \geq 5 \) (see Figure 2.3). Then
d*(u, v) = n - 2 \geq 2 = d*(u, w) + d*(w, v).

Therefore, in general, the detour distance is not a metric on the vertex set of G.

Let G be a connected graph and let F be a connected induced subgraph of G. Then d_F(u, v) \geq d_G(u, v) for u, v \in V(F). However, for the detour distance, we have the opposite inequality, that is d*_F(u, v) \leq d*_G(u, v), for u, v \in V(F). It is clear that d*(u, v) \geq d(u, v) for all vertices u, v of G. A connected graph G is called a detour graph if d*(u, v) = d(u, v) for all vertices u and v of G. A characterization of detour graphs is given next.

**Theorem 2.5** Let G be a connected graph. Then G is a detour graph if and only if every induced connected subgraph of G is a detour graph.

**Proof.** Suppose G is a detour graph and F is an induced connected subgraph of G. For vertices u and v of F, it follows that

\[ d_F(u, v) \geq d_G(u, v) = d*_G(u, v) \geq d*_F(u, v). \]

On the other hand, we have that d_F(u, v) \leq d*_F(u, v). Therefore, d_F(u, v) = d*_F(u, v) for all vertices u and v of F; so F is a detour graph.

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If every induced connected subgraph of $G$ is a detour graph, then, in particular, $G$ is an induced connected subgraph of itself; so $G$ is a detour graph. □

Since every block is an induced connected subgraph, the following is a stronger characterization of detour graphs.

**Theorem 2.6** Let $G$ be a connected graph. Then $G$ is a detour graph if and only if each block of $G$ is a detour graph.

**Proof.** Suppose $G$ is a detour graph and $u$ and $v$ are vertices in some block $B$ of $G$. Then every induced $u$-$v$ path lies entirely in $B$. Therefore,

$$d^*_B(u, v) = d^*_G(u, v) = d_G(u, v) = d_B(u, v).$$

Since the block $B$ is arbitrary and the vertices $u$ and $v$ are arbitrary in $B$, it follows that every block of $G$ is a detour graph.

We now prove the sufficiency. Consider two arbitrary vertices $u$ and $v$ of $G$. If vertices $u$ and $v$ lie in the same block $B$ of $G$, then $d^*_G(u, v) = d^*_B(u, v) = d_B(u, v) = d_G(u, v)$. Suppose that $u$ and $v$ lie in different blocks of $G$. Let $P$ be a longest $u$-$v$ path with $\langle V(P) \rangle = P$. Without loss of generality, we assume that $P: u = v_1, 1, \ldots, v_1, i_1 = v_2, 1, \ldots, v_2, i_2 = v_3, 1, \ldots, v_{m-1}, i_{m-1} = v_m, 1, \ldots, v_m, i_m = v$, where $v_{k, j} \in V(B_k)$, $1 \leq j \leq i_k$, $1 \leq k \leq m$, and the $B_k$ ($1 \leq k \leq m$) are distinct blocks of $G$. Therefore,

$$d^*_G(u, v) = \sum \{d^*_B(v_{k, 1}, v_{k, i_k}) \mid 1 \leq k \leq m\}$$

$$= \sum \{d_B(v_{k, 1}, v_{k, i_k}) \mid 1 \leq k \leq m\}$$

$$= d_G(u, v).$$

Hence, $G$ is a detour graph. □

The **detour eccentricity** $e^*(v)$ of a vertex $v$ is defined by
\[ e^*(v) = \max \{ d^*(v, w) \mid w \in V(G) \}. \]

The detour eccentricity set \( e^*(G) \) of a connected graph \( G \) is the set consisting of all detour eccentricities of \( G \), that is, \( e^*(G) = \{ e^*(v) \mid v \in V(G) \} \). The difference between the eccentricities of two adjacent vertices is at most 1. However, the difference between the detour eccentricities of two adjacent vertices can be arbitrarily large. In fact, we have the even stronger statement that every nonempty set of integers is the detour eccentricity set of some connected graph.

**Theorem 2.7** Let \( S = \{ s_1, s_2, \ldots, s_k \} \) be a set of integers with \( s_1 < s_2 < \ldots < s_k \). Then there exists a connected graph \( G \) such that \( e^*(G) = S \).

**Proof.** If \( |S| = 1 \) then let \( G = C_{s_1+2} \). It follows that \( e^*(G) = \{ s_1 \} = S \). Suppose that the theorem holds for \( |S| < k \). When \( |S| = k \), we let \( S' = S - \{ s_1 \} \). By the inductive hypothesis, there exists a connected graph \( F \) such that \( e^*(F) = S' \). We construct the connected graph \( G \) by first replacing the vertex \( v_1 \) of \( C_{s_1+2} \) with \( F \). Then we join \( v_2 \) and \( v_{s_1+2} \) with all the vertices in \( F \) (see Figure 2.4). Then, for \( 2 \leq i \leq s_1+2 \) and \( v \in V(F) \), it follows that \( d^*(v_i, v) \leq s_1 \), and \( d^*(v_i, v) = s_1 \) if and only if \( i = 3, s_1+1 \). Clearly, \( d^*(v_i, v_j) \leq s_1 \) for \( 1 \leq i < j \leq s_1+2 \). Furthermore, it follows that \( d^*(v_2, v_{s_1+2}) = s_1 \), and \( d^*(v_i, v_{i-2}) = s_1 \) for \( 4 \leq i \leq s_1+1 \). Therefore, \( e^*(v_i) = s_1 \) for \( 2 \leq i \leq s_1+2 \). Suppose \( v \in V(F) \). Then \( d^*(v, v_i) \leq s_1 < e^*(v) \) for \( 2 \leq i \leq s_1+2 \). It follows that \( d^*_G(v, w) = d^*_F(v, w) \) for all \( v, w \in V(F) \). Therefore, \( e^*_G(v) = e^*_F(v) \) for \( v \in V(F) \). Hence, \( e^*(G) = e^*(F) \cup \{ s_1 \} = S' \cup \{ s_1 \} = S \). \( \square \)

Lesniak [18] defined a nondecreasing sequence \( S: a_1, a_2, \ldots, a_p \) of nonnegative integers to be an eccentric sequence if there exists a connected graph \( G \).
Figure 2.4

whose vertices can be labeled \( v_1, v_2, \ldots, v_p \) so that \( e(v_i) = a_i \) for \( 1 \leq i \leq p \). In this case, \( S \) is said to be the eccentricity sequence of \( G \). We call a nondecreasing sequence \( S: a_1, a_2, \ldots, a_p \) of nonnegative integers a detour eccentric sequence if there exists a connected graph \( G \) whose vertices can be labeled \( v_1, v_2, \ldots, v_p \) so that \( e^*(v_i) = a_i \) for \( 1 \leq i \leq p \). In this case, \( S \) is said to be the detour eccentricity sequence of \( G \).

Lesniak [18] showed that a nondecreasing sequence \( S: a_1, a_2, \ldots, a_p \) with \( m \) distinct values is eccentric if and only if some subsequence with \( m \) distinct values is eccentric. A detour eccentricity sequence may be characterized in an analogous fashion.

**Theorem 2.8** A nondecreasing sequence \( S: a_1, a_2, \ldots, a_p \) of nonnegative integers with \( m \) distinct values is the detour eccentricity sequence of a graph if and only if some subsequence of \( S \) with \( m \) distinct values is the detour eccentricity sequence of some graph.

**Proof.** If \( S \) is a sequence with \( m \) distinct values that is the detour eccentricity
sequence of some graph, then $S$ is a subsequence of itself, that is, $S$ is the detour eccentricity sequence of a graph.

For the converse, suppose that $S'$ is a subsequence of $S$ that has the same $m$ distinct values as $S$ and suppose that $S'$ is the detour eccentricity sequence of some graph $G$. Let $t_1, t_2, \ldots, t_m$ be the distinct values of $S'$. For each $t_i$, $1 \leq i \leq m$, select a vertex $v_i$ of $G$ whose detour eccentricity in $G$ is $t_i$. Let $n_i$ ($1 \leq i \leq m$) be one more than the number of occurrences of $t_i$ in $S$ less the number of occurrences of $t_i$ in $S'$. In $G$ replace $v_1$ with a copy of $K_{n_1}$ and join each vertex of $K_{n_1}$ to all the vertices adjacent to $v_1$ in $G$. Denote this graph by $G_1$. In $G_1$, replace $v_2$ with a copy of $K_{n_2}$ and join each vertex of $K_{n_2}$ to all the vertices adjacent to $v_2$ in $G_1$. We continue in this fashion to obtain the graph $G_m$. Then $S$ is the detour eccentricity sequence of $G_m$. □

The detour radius $\text{rad}^* G$ of $G$ is the minimum detour eccentricity, while the detour diameter $\text{diam}^* G$ of $G$ is the maximum detour eccentricity. Our next result gives upper bounds for the detour radius and detour diameter of a connected graph.

**Theorem 2.9** For every connected graph $G$, $\text{rad}^* G \leq p(G) - \Delta(G)$ and $\text{diam}^* G \leq p(G) - \delta(G)$.

**Proof.** Let $v$ be a vertex in $G$ of maximum degree and let $w$ be a vertex such that $e^*(v) = d^*(v, w)$. Let $P$ be a $v$-$w$ path of length $d^*(v, w)$ such that $\langle V(P) \rangle = P$. Then, $|V(P) \cap N(v)| = 1$ so $|V(P)| \leq p(G) - \deg v + 1 = p(G) - \Delta(G) + 1$. Therefore, $\text{rad}^* G \leq e^*(v) = d^*(v, w) = |V(P)| - 1 \leq p(G) - \Delta(G)$.

Similarly, $\text{diam}^* G \leq p(G) - \delta(G)$. □

The upper bounds given in Theorem 2.9 are sharp. For a wheel $W_p$ ($p \geq 4$),
we see that $\text{rad}^* W_p = 1 = p - \Delta(W_p)$ and $\text{diam}^* W_p = p - 3 = p - \delta(W_p)$. The subgraph induced by those vertices with minimum detour eccentricity is called the detour center of $G$ and is denoted by $C^*(G)$. Harary and Norman [13] proved that the center of every connected graph $G$ lies in a single block of $G$. We now prove an analogue of this result involving detour center.

**Theorem 2.10**  The detour center of every connected graph $G$ lies in a single block of $G$.

**Proof.** Suppose $G$ is a connected graph whose detour center $C^*(G)$ does not lie within a single block of $G$. Then $G$ has a cut-vertex $v$ such that $G-v$ contains components $G_1$ and $G_2$, each of which contains elements of $C^*(G)$. Let $u$ be a vertex such that $d^*(u, v) = e^*(v)$, and let $P_1$ be a $v-u$ path of $G$ having length $e^*(v)$ and $\langle V(P_1) \rangle = P_1$. At least one of $G_1$ and $G_2$, say $G_2$, contains no vertices of $P_1$. Let $w$ be an element of $C^*(G)$ belonging to $G_2$, and let $P_2$ be a $w-v$ path of minimum length. The paths $P_1$ and $P_2$ together form a $u-w$ path $P_3$ with $\langle V(P_3) \rangle = P_3$. Therefore, $e^*(w) \geq |V(P_3)| - 1 > |V(P_1)| - 1 = e^*(v)$, which contradicts the fact that $w \in C^*(G)$. Therefore, $C^*(G)$ lies in a single block of $G$.\[ \square \]

In [5] and [23], it was proved that every graph is the center of some connected graph. This result is also true with respect to the detour center.

**Theorem 2.11** Let $G$ be a graph. Then there exists a connected graph $H$ such that $C^*(H) \equiv G$.

**Proof.** Let $k = p(G)$. We define the graph $H$ by adding $2k$ new vertices $u_i, v_i$ ($1 \leq i \leq k$) to $G$, and the edges $u_i u_{i+1}, v_i v_{i+1}, (1 \leq i \leq k - 1)$ together with the edges joining $u_1$ and $v_1$ with all vertices of $G$ (see Figure 2.5). Then $e^*(u_i) \geq \ldots$
d*(ui, vk) = i + k and e*(vj) ≥ d*(vi, uk) = i + k for 1 ≤ i ≤ k. Clearly, d*(u, v) ≤ k for all u, v ∈ V(G). Furthermore, since d*(w, ui) ≤ d*(w, uk) = k and d*(w, vi) ≤ d*(w, vk) = k for all w ∈ V(G) and 1 ≤ i ≤ k, it follows that C*(H) ≡ G. □

![Figure 2.5](image.png)

The **detour periphery** P*(G) of a connected graph G is the subgraph induced by those vertices with maximum detour eccentricity. The **local detour eccentricity** le*(v) of a vertex v in G is the detour eccentricity of v in the component of G that contains v. A sufficient condition for a graph to be the detour periphery of some connected graph can be given.

**Theorem 2.12** If all vertices of G have the same local detour eccentricity, then there exists a connected graph H such that P*(H) ≡ G.

**Proof.** If G is a complete graph, then P*(G) = G. Suppose G is not complete and le*(v) = k for all vertices v of G. Then, k ≥ 2. We define a connected graph H by adding a new vertex v to G and the edges joining v with all vertices of G. Therefore, e*H(v) = 1 and e*H(w) = le*G(w) = k ≥ 2 for all w ∈ V(G), implying that P*(H) ≡ G. □
2.3 Generalized Steiner Distance

The distance between two vertices $u$ and $v$ of a connected graph $G$ is the minimum size of a connected subgraph $H$ that contains vertices $u$ and $v$. From this point of view, the standard distance between two vertices was extended in [9] to the Steiner distance of a set of vertices. Observe that if the graph $G$ is $k$-connected, then it is possible to require the subgraph $H$ to be $i$-connected for each $i$, $1 \leq i \leq k$. This observation suggests a generalization of the Steiner distance on graphs.

Let $G$ be a $k$-connected graph of order $p \geq 2$ and let $S$ be a nonempty set of vertices of $G$. The $i$-distance $d_i(S)$, $1 \leq i \leq k$, among the vertices of $S$ is the minimum size among all $i$-connected subgraphs whose vertex set contain $S$. Therefore, the $1$-distance of $S$ is simply the Steiner distance $d(S)$ of $S$. A subgraph $H$ of $G$ is called an $i$-subgraph, $1 \leq i \leq k$, of $S$ if $H$ is $i$-connected, $S \subseteq V(H)$ and $d_i(S) = q(H)$. If $G$ is the graph of Figure 2.6 and $S = \{u, v, w\}$, then $d_1(S) = 3$, $d_2(S) = 5$, and $d_3(S) = 10$. Three $i$-subgraphs ($i = 1, 2, 3$) are also shown in Figure 2.6.

![Figure 2.6](Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.)

The Steiner distance satisfies an extended triangle inequality. Let $G$ be a
connected graph, and let \( S, S_1 \) and \( S_2 \) be subsets of \( V(G) \) such that \( \emptyset \neq S \subseteq S_1 \cup S_2 \) and \( |S_1 \cap S_2| \geq 1 \). Then, \( d(S) \leq d(S_1) + d(S_2) \). There is an extension of this property to the i-distance we have defined. Let \( G \) be a k-connected graph and let \( S, S_1 \) and \( S_2 \) be subsets of \( V(G) \) such that \( \emptyset \neq S \subseteq S_1 \cup S_2 \) and \( |S_1 \cap S_2| 
 \geq i \), where \( 1 \leq i \leq k \). Then, \( d_i(S) \leq d_i(S_1) + d_i(S_2) \). To see this, let \( H_j \) (\( j = 1, 2 \)) be an i-subgraph of size \( d_i(S_j) \) such that \( S_j \subseteq V(H_j) \). Let \( H \) be the graph with vertex set \( V(H_1) \cup V(H_2) \) and edge set \( E(H_1) \cup E(H_2) \). Since \( H_1 \) and \( H_2 \) are i-connected and \( |V(H_1) \cap V(H_2)| \geq |S_1 \cap S_2| \geq i \), the graph \( H \) is i-connected. Since \( S \subseteq V(H) \), \( d_i(S) \leq d_i(S) \leq d_i(S_1) + d_i(S_2) \).

It is clear that every i-subgraph \( H \) of a set \( S \) is minimally i-connected. Therefore, \( \delta(H) \geq i \) so that \( d_i(S) = q(H) \geq \left\lceil \frac{i|S|}{2} \right\rceil \). An i-connected graph \( G \) of order \( p \) is called an \((i, n, p)\) graph, where \( 2 \leq i < n \leq p \), if \( d_i(S) = \left\lceil \frac{i|S|}{2} \right\rceil \) for every set of \( n \) vertices of \( G \). A graph \( G \) is \((i, n)\)-connected, \( 1 \leq i < n \), if <\( S \) is i-connected for every set \( S \) of \( n \) vertices of \( G \). It is clear that every \((i, n, p)\) graph is \((i, n)\)-connected.

**Lemma 2.13** Let \( i \) and \( n \) be integers with \( 1 \leq i < n \). A graph \( G \) of order \( p \) is \((i, n)\)-connected if and only if \( G \) is \((p - n + i)\)-connected.

**Proof.** Suppose, to the contrary, that there exists an \((i, n)\)-connected graph that is not \((p - n + i)\)-connected. Then there exists a cut set \( X \) with \( |X| = p - n + i - 1 \) such that \( G - X \) is disconnected. Let \( S = V(G) - X \) and let \( X' = \{x_1, x_2, \ldots, x_{i-1}\} \) be an arbitrary subset of \( i - 1 \) vertices in \( X \). Then \( X' \) is a cut set of <\( S \cup X' \)>, so that <\( S \cup X' \) is not i-connected. Since \( |S \cup X'| = n \), it follows that \( G \) is not \((i, n)\)-connected, a contradiction.

We now prove the sufficiency. Let \( G \) be a \((p - n + i)\)-connected graph and let
S be a subset of V(G) with |S| = n. Suppose that <S> is not i-connected. Then there exists a subset X of S with |X| = i - 1 such that <S> - X is disconnected. Therefore, G - (X ∪ (V(G) - S)) is disconnected. However, since |X ∪ (V(G) - S)| = p - n + i - 1, it follows that G is not (p - n + i)-connected, which is a contradiction. □

**Theorem 2.14** Every (i, n, p) graph, 2 ≤ i < n ≤ p, is (p - n + i)-connected.

**Proof.** Since every (i, n, p) graph is (i, n)-connected, the theorem follows immediately from Lemma 2.3. □

**Corollary 2.15** Let i and p be integers with 2 ≤ i < p. Then every (i, i+1, p) graph is complete.

It is important to note that the converse of Theorem 2.12 is not true. Let i = 2, n = 5 and p = 6; so p - n + i = 3. Then K₃,₃ is (p - n + i)-connected. Since K₃,₃ - v is not hamiltonian for every vertex v, K₃,₃ is not a (2, 5, 6) graph.

For integers i, n and p with 2 ≤ i < n ≤ p, observe that a graph G is an (i, n, p) graph if and only if p(G) = p and <S> contains a spanning i-connected subgraph of size ⌊\frac{1+n}{2}\rfloor, for every set S of n vertices of G. In particular, a graph G is a (2, n, p) graph if and only if p(G) = p and <S> is hamiltonian for every set S of n vertices of G. Therefore, G is a (2, p, p) graph if and only if G is hamiltonian. From this point of view, the (i, n, p) graphs are generalized hamiltonian graphs. Consequently, the problem to determine whether a graph is a (2, n, p) graph is NP-complete.

Chartrand, Kapoor and Lick [7] introduced the concept of n-hamiltonian graphs. A graph G is n-hamiltonian if the removal of any set of n vertices from G
results in a hamiltonian subgraph. Therefore, a graph $G$ is a $(2, n, p)$ graph if and only if $G$ is $(p - n)$-hamiltonian. Wong and Wong [24] studied the minimum size of $n$-hamiltonian graphs (or, $(2, p-n, p)$ graphs). The extreme graph constructed by Wong and Wong is hamiltonian. We ask the following question: Does there exist a nonhamiltonian $(2, n, p)$ graph? The circumference $c(G)$ of a graph $G$ is the length of a longest cycle in $G$. With the aid of following lemma, we can give upper and lower bounds for $c(G)$ for a $(2, n, p)$ graph $G$.

**Lemma 2.16** Let $G$ be a $(2, n, p)$ graph with $3 < n < p$. If $S$ is a subset of $V(G)$ with $|S| > n$, then $<S>$ is a $(2, n, |S|)$ graph.

We now establish an upper bound for the circumference of nonhamiltonian $(2, n, p)$ graphs.

**Theorem 2.17** Let $G$ be a $(2, n, p)$ graph with $3 < n < p$. If $G$ is not hamiltonian, then $c(G) < 2n - 6$.

**Proof.** Let $C$ be a longest cycle in $G$ and let $H = <V(C) \cup \{v\}>$, where $v$ is a vertex not on $C$. Since $G$ is a $(2, n, p)$ graph, it follows that $c(G) \geq n$. By Lemma 2.9, $H$ is a $(2, n, c(G)+1)$ graph. It follows from Theorem 2.4 that $H$ is $(c(G)+1-n+2)$-connected. Therefore, $\deg_H(v) \geq \delta(H) \geq c(G) + 1 - n + 2 = c(G) - n + 3$. Since $C$ is a longest cycle in $G$, it follows that $c(G) - n + 3 \leq \frac{c(G)}{2}$, implying that $c(G) \leq 2n - 6$. □

A lower bound for the circumference of a nonhamiltonian $(2, n, p)$ graph can be given as follows.

**Theorem 2.18** If $G$ is an $(2, n, p)$ graph with $3 \leq n \leq p$, then $c(G) \geq p - \left\lfloor \frac{p}{2} \right\rfloor + 1$. 

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Proof. If $G$ is hamiltonian, then $c(G) = p > p - \left\lfloor \frac{n}{2} \right\rfloor + 1$. Suppose $G$ is not hamiltonian. Let $C$ be a longest cycle in $G$ and let $X = V(G) - V(C)$. Then $X \neq \emptyset$. We prove that $|X| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Suppose, to the contrary, that $|X| > \left\lfloor \frac{n}{2} \right\rfloor - 1$. Define $m = n - 2$ if $|X| \geq n - 2$, and $m = |X|$ otherwise. Let $V_1 \subseteq X$ be a subset of cardinality $m$. Let $P: u = v_1, v_2, ..., v_{n-m} = w$ be a subpath of $C$. Since $G$ is a $(2, n, p)$ graph and $|V_1 \cup V(P)| = n$, the graph $\langle V_1 \cup V(P) \rangle$ is hamiltonian. Suppose $C_1$ is a hamiltonian cycle in $\langle V_1 \cup V(P) \rangle$. Since $C_1$ contains vertices $u$ and $w$, the cycle $C_1$ produces two edge-disjoint $u$-$w$ paths $P_1$ and $P_2$. Clearly, at least one of them, say $P_1$, has length at least $\left\lceil \frac{n}{2} \right\rceil$. Therefore, by the choice of $m$,

$$|V(P_1) \cup (V(C) - V(P))| = \left\lceil \frac{n}{2} \right\rceil + 1 + c(G) - n + m$$

$$= c(G) + m - \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$\geq c(G) + 1.$$ 

Since $V(P_1) \cap (V(C) - V(P)) = \emptyset$, the induced graph $\langle V(P_1) \cup (V(C) - V(P)) \rangle$ is hamiltonian. Therefore, the graph $G$ contains a cycle of length exceeding $c(G)$, a contradiction. \(\square\)

For integers $p \geq 3$, we define the parameter $f(p)$ to be the minimum $n$, $3 \leq n \leq p$, such that there exists an nonhamiltonian $(2, n, p)$ graph. The parameter $f(p) = \infty$ if no $(2, n, p)$ graph exists for all $n$, $3 \leq n \leq p$. A lower bound for the parameter $f(p)$ is given in the following corollary.

**Corollary 2.19** For all integers $p \geq 3$, $f(p) \geq \left\lceil \frac{2p-1}{5} \right\rceil + 3$.

**Proof.** Suppose $f(p)$ is finite. Let $G$ be a nonhamiltonian $(2, n, p)$ graph with $3 \leq n \leq p$. Combining Theorem 2.17 and 2.18, we have $p - \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq c(G) \leq 2n - 6$, that is, $n \geq \left\lceil \frac{2p-1}{5} \right\rceil + 3$. Therefore, $f(p) \geq \left\lceil \frac{2p-1}{5} \right\rceil + 3$ for all $p \geq 3$. \(\square\)
A graph $G$ is hypohamiltonian if $G$ is not hamiltonian and $G - v$ is hamiltonian for all vertices $v$ of $G$. Therefore, a nonhamiltonian $(2, p-1, p)$ graph is then a hypohamiltonian graph. Much study has been done on the existence of hypohamiltonian graphs. Thomassen [22] showed that there exists a hypohamiltonian graph of order $p$ for all $p > 13$ except for $p = 14, 17, 19$. Thus, $f(p) \leq p - 1$ for $p \geq 13$ and $p \neq 14, 17, 19$.

2.4 $(i, n)$ Eccentricity and $(i, n)$ Center

Let $G$ be a $k$-connected graph of order $p$ and let $i$ and $n$ be integers with $1 \leq i \leq k$ and $1 \leq n \leq p$. The $(i, n)$-eccentricity $e_{i,n}(v)$ of a vertex $v$ of $G$ is defined by

$$e_{i,n}(v) = \max\{d_{i}(S) \mid v \in S \subseteq V(G) \text{ and } |S| = n\}.$$ 

Observe that, for $v \in V(G)$,

1. $e_{1,1}(v) = 0$;
2. $e_{1,2}(v) = e(v)$, the standard eccentricity;
3. $e_{i,n}(v) = e_{n}(v)$, the Steiner $n$-eccentricity; and
4. $e_{2,n}(v)$ is the minimum size among all cycles of $G$ containing $v$.

We call a nondecreasing sequence $S: a_1, a_2, \ldots, a_p$ of nonnegative integers an $(i, n)$-eccentricity sequence if there exists an $i$-connected graph $G$ whose vertices can be labeled as $v_1, v_2, \ldots, v_p$ so that $e_{i,n}(v_j) = a_j$ for $1 \leq j \leq p$. A characterization of $(2, 2)$-eccentricity sequences similar to Theorem 2.8 can be given.

Theorem 2.20 A nondecreasing sequence $S: a_1, a_2, \ldots, a_p$ with $m$ distinct values is the $(2, 2)$-eccentricity sequence of a graph if and only if some subsequence of $S$ with $m$ distinct values is the $(2, 2)$-eccentricity sequence of some graph.
We omit the proof since it is similar to the proof of Theorem 2.8. The (i, n)-center $C_{i,n}(G)$ of an i-connected graph $G$ is the subgraph induced by those vertices with minimum (i, n)-eccentricity. We now prove that for integers $i$ and $n$ with $i, n > 1$ and for a given graph $G$, there exists an i-connected graph $H$ such that the (i, n)-center of $H$ is isomorphic to $G$.

**Theorem 2.21** Let $i$ and $n$ be integers greater than 1 and let $G$ be a graph. Then there exists an i-connected graph $H$ such that $C_{i,n}(H) \cong G$.

**Proof.** Let $k = \max \{\lfloor \frac{i}{2} \rfloor, i - p(H)\}$, and let $m \geq 3$ be an integer such that $\left\lceil \frac{imn}{2} \right\rceil > q(H)$. We first define a preliminary graph $G_1$ by

$$V(G_1) = \{v\} \cup \{v_i j | 1 \leq i \leq n + 1, 1 \leq j \leq m\}$$

and

$$E(G_1) = \{v_i j v_{i+1} j | 1 \leq i \leq n + 1, 1 \leq j \leq m - 1\} \cup \{v_{n+1} m v_{1 m}\}$$

$$\cup \{v_i m v_{i+1} m | 1 \leq i \leq n\} \cup \{v v_{1} 1 | 1 \leq i \leq n+1\}.$$ 

Let $G$ be the graph obtained from $G_1$ by first replacing the vertex $v$ with $V(H)$ and replacing the vertex $v_i j$ with vertices $V_i j = \{v^1_i j, v^2_i j, \ldots, v^k_i j\}$, where $1 \leq i \leq n + 1$ and $1 \leq j \leq m$. To define the edge set of $G$, we let $\varphi : V(G) \rightarrow V(G_1)$ be a mapping defined by $\varphi(w) = v$ for $w \in V(H)$ and $\varphi(v^s_i j) = v_i j$, where $1 \leq i \leq n + 1, 1 \leq j \leq m$ and $1 \leq s \leq k$. We define then the edge set of $G$ by

$$E(G) = \{xy | x, y \in V(G) \text{ and } \varphi(x)\varphi(y) \in E(G_1)\}$$

(see Figure 2.7).

Let $V_i = \bigcup_{j=1}^{m} V^j i j$. By the structure of $G$, it follows that every i-connected subgraph $F$ of $G$ has the property

(*) if $v^s_i j \in V(F)$ for some $1 \leq s \leq k$ and $1 \leq j \leq m - 1$, then $V_i \subseteq V(F)$.
Further, every $i$-connected subgraph containing $V_i$ has size greater than $\left\lceil \frac{ikm}{2} \right\rceil$ ( $>$ $q(H)$).

To prove that $e_{in}(v) = e_{in}(w)$ for all $v, w \in V(H)$, let $v \in V(H)$. Suppose that $S \subseteq V(G) - \{v\}$ is a set of $n-1$ vertices and $F$ is an $i$-connected subgraph of $G$ with $\{v\} \cup S \subseteq V(F)$ such that $e_{in}(v) = d_{ij}(\{v\} \cup S) = q(F)$. We claim that $S \cap V(H) = \emptyset$. Suppose, to the contrary, that $w \in S \cap V(H)$. Since $|S| = n - 1$, there exists $V_i$ such that $S \cap V_i = \emptyset$. Let $S' = (S - \{w\}) \cup \{v^1_{ij}\}$. Then,

$$d_{ij}(\{v\} \cup S') \geq d_{ij}(\{v\} \cup S) + \left\lceil \frac{ikm}{2} \right\rceil - q(H)$$

a contradiction. Therefore, $S \subseteq \bigcup_{i=1}^{n+1} V_i$. Assume, without loss of generality, that $v^5_{i1} \in V(F)$. Then $v^5_{i1}$ is adjacent to at least $i - m$ vertices of $V(H)$ in $F$. Therefore, $|V(F) \cap V(H)| \geq i - k$. Since $F$ is a minimal $i$-connected graph containing $\{v\} \cup S$, we have $|V(F) \cap V(H)| = i - k$. Further, the induced subgraph $<V(F) \cap V(H)>$ of $F$ is empty. Therefore, $e_{in}(v)$ in $G$ is independent of the choice of $v$ from $H$. Hence, $e_{in}(v) = e_{in}(w)$ for all $v, w \in V(H)$.

We now prove that $e_{in}(v) < e_{in}(w)$ for all $v \in V(H)$ and $w \in V(G) - V(H)$,
from which it will follow that $C_{1n}(H) \equiv G$. Consider again a vertex $v \in V(H)$. Let $X$ be a set of $i-m$ vertices of $V(H)$ that contains $v$. Then, by the above, $e_{i,n}(v) = d_i(v, v^{1}_{11}, v^{1}_{31}, \ldots, v^{1}_{n1})$. The value of $e_{i,n}(v)$ is the size of a minimal $i$-connected subgraph $F$ of $G$ with vertex set $V(F) = X \cup V_{2m} \cup (\cup \{V_i \mid 1 \leq i \leq n, i \neq 2\})$ if $n \geq 3$. If $n = 2$, then $V(F) = X \cup V_1 \cup V_3$. We now consider the $(i, n)$-eccentricity of $v^{1}_{11}$. Let $S = \{v^{1}_{21}, v^{1}_{31}, \ldots, v^{1}_{n1}\}$. Then, any minimal $i$-connected subgraph $F$ containing $\{v^{1}_{11}\} \cup S$ contains $\bigcup_{i=1}^{n} V_i$ and a set $Y$ of $i-m$ vertices of $V(H)$. Without loss of generality, assume that $Y = X$. Therefore, $e_{i,n}(v^{1}_{11}) = d_i((v^{1}_{11}) \cup S) = q(F) > e_{i,n}(v)$. By the symmetry of $G$ and $(*), e_{i,n}(v^{s}_{ij}) > e_{i,n}(v)$ for all $i, j$ and $s$ with $1 \leq i \leq n+1, 1 \leq j \leq m-1$ and $1 \leq s \leq k$. Similarly, the $(i, n)$-eccentricity of $v^{1}_{i,m}$ equals the size of a minimal $i$-connected subgraph $F_1$ with vertex set $V(F_1) = X \cup V_{1m} \cup V_{n+1,m} \cup (\cup_{i=2}^{n} V_i)$ if $n \geq 3$. If $n = 2$, then $V(F_1) = X \cup V_{1m} \cup V_2 \cup V_3$. Therefore, $e_{i,n}(v^{1}_{i,m}) > e_{i,n}(v)$. By symmetry, we have $e_{i,n}(v^{s}_{im}) > e_{i,n}(v)$ for all $1 \leq i \leq n+1$ and $1 \leq s \leq k$. This completes the proof. \qed
CHAPTER III

MAXIMUM DISTANCE IN DIGRAPHS

In this chapter, we define a new distance on digraphs, called maximum distance. Then we explore some results concerning oriented graphs and maximum distance.

3.1 Definition of Maximum Distance and Basic Properties

We have seen in Chapter I that neither directed distance nor minimum distance, defined on the vertex set of a digraph, is a metric, although both inherit some characteristics from the definition of distance between two vertices of a graph. In this section we give another definition of the distance between two vertices in a digraph. It will be shown that this new distance is a metric on the vertex set of a digraph.

Recall, for vertices u and v in a connected graph G, that the distance between u and v is the length of a shortest u-v path in G. Equivalently, the distance between u and v is the maximum of the lengths of a shortest u-v path and a shortest v-u path. This observation suggests an extension of the distance from graphs to digraphs. For vertices u and v in a digraph D, we define the maximum distance (or m-distance) md(u, v) between u and v as

\[ \text{md}(u, v) = \max\{ \overrightarrow{d}(u, v), \overleftarrow{d}(v, u) \}. \]

The m-distance \( \text{md}(u, v) = \infty \) if at least one of \( \overrightarrow{d}(u, v) \) and \( \overleftarrow{d}(v, u) \) is \( \infty \). For the digraph shown in Figure 3.1, it follows that \( \text{md}(u, v) = 4 \) and \( \text{md}(v, w) = \infty \).

We now show that the m-distance is a metric on \( V(D) \) if D is a strong digraph. It is clear that \( \text{md}(u,u) = 0 \) and \( \text{md}(u,v) = \text{md}(v,u) \). To prove that the
m-distance satisfies the triangle inequality, we assume, without loss of generality, that 
\[ md(u,v) = \overrightarrow{d}(u,v). \]
Considering an arbitrary vertex \( w \) in \( D \), we have
\[
md(u,v) = \overrightarrow{d}(u,v) \\
\leq \overrightarrow{d}(u,w) + \overrightarrow{d}(w,v) \\
\leq \max\{\overrightarrow{d}(u,w), \overrightarrow{d}(w,u)\} + \max\{\overrightarrow{d}(w,v), \overrightarrow{d}(v,w)\} \\
= md(u,w) + md(w,v).
\]
Therefore, the m-distance is, in fact, a metric on \( V(D) \).

![Figure 3.1](image)

The converse \( D^\circ \) of a digraph \( D \) is the digraph obtained by reversing all the
arcs of \( D \). Our first result shows that the m-distances between two vertices in a strong
digraph and its converse are equal.

**Theorem 3.1** If \( D \) is a strong digraph then \( md(u,v) = md_D(u,v) \) for all vertices
\( u, v \in V(D) \).

**Proof.** Let \( u, v \in V(D) \). Since \( \overrightarrow{d}(u,v) = \overrightarrow{d}_D(v,u) \), it follows that \( md(u,v) = \max\{\overrightarrow{d}(u,v), \overrightarrow{d}_D(v,u)\} = \max\{\overrightarrow{d}_D(v,u), \overrightarrow{d}_D(u,v)\} = md_D(u,v). \)

For each vertex \( v \) of a digraph \( D \), we define the m-eccentricity \( me(v) \) of \( v \) as
\[ me(v) = \max\{md(u,v) | u \in V(D)\}. \] A vertex \( v \) is said to be the m-eccentric vertex of
u if \( md(u, v) = me(u) \). It is well known for a connected graph \( G \) that if \( uv \in E(G) \), then \( |e(u) - e(v)| \leq 1 \). However, in a digraph \( D \), the difference between the m-eccentricities of two adjacent vertices can be arbitrarily large. For an integer \( n \geq 2 \), we define the digraph \( D \) as follows:

\[
V(D) = \{ w, u_1, v_1, \ldots, u_n, v_n \} \quad \text{and} \quad E(D) = \{(w, u_1), (w, v_1), (u_n, w), (v_n, w)\}
\]

\[\cup \{(u_i, u_{i+1}), (v_i, v_{i+1}) \mid 1 \leq i \leq n - 1\}.\]

Then it follows that \( me(w) = n \) and \( me(u_1) = me(v_1) = me(u_n) = me(v_n) = 2n \). Therefore, \( |me(v) - me(w)| = n \) for all vertices \( v \) adjacent to or from \( w \).

The m-radius of \( D \) is \( m\)-rad \( D = \min\{me(v) \mid v \in V(D)\} \) and the m-diameter of \( D \) is \( m\)-diam \( D = \max\{me(v) \mid v \in V(D)\} \). Note that \( md(u, v) = 1 \) if and only if there exist symmetric arcs \( (u, v) \) and \( (v, u) \) in \( D \). Therefore, for all oriented graphs \( D \), \( m\)-rad \( D \geq 2 \).

Clearly, a graph \( G \) has diameter 1 if and only if \( G \) is a complete graph. We now consider the oriented graphs of m-diameter 2. For convenience, we call such an oriented graph an m-dense oriented graph. We now give a lower bound for the minimum degree of an m-dense oriented graph.

**Theorem 3.2** The minimum degree of any m-dense oriented graph of order \( p \geq 5 \) is at least 4.

**Proof.** Let \( D \) be an m-dense oriented graph. Since \( D \) is strong, it follows that \( id(v), od(v) \geq 1 \) for all vertices \( v \) of \( D \). If there exists a vertex \( v \) with \( id(v) = od(v) = 1 \), then let \( u, w \) be the vertices such that \( (u, v), (v, w) \in E(D) \). Since \( d(w, v) = 2 \), it follows that \( (w, u) \in E(D) \). For \( x \in V(D) - \{u, v, w\} \), since \( d(x, v) = d(v, x) = 2 \), it follows that \( (x, u), (w, x) \in E(D) \). Thus \( d(x, w) = 3 \), a
contradiction. Therefore $\delta(D) \geq 3$. Suppose that $\delta(D) = 3$. Let $v$ be a vertex with $\deg(v) = 3$. Suppose without loss of generality that $\text{id}(v) = 1$ and $\text{od}(v) = 2$. Let $u$ be the vertex adjacent to $v$, and let $w, x$ be the vertices to which $v$ is adjacent. Then $(y, u) \in E(D)$ for all $y \in V(D) - \{u, v\}$. Since $p \geq 5$, there exists $y \in V(D) - \{u, v, w, x\}$. Either $(w, y)$ or $(x, y) \in E(D)$, say $(w, y) \in E(D)$. Then $\overrightarrow{d}(y, w) = 3$, again a contradiction. Therefore $\delta(D) \geq 4$.

The only $m$-dense oriented graph of order 3 is the cyclic tournament with three vertices. It can be easily shown that there is no $m$-dense oriented graph of order 4. By Theorem 3.2, every $m$-dense oriented graph of order 5 is again a tournament. In fact, there exist $m$-dense tournaments of order $n$ for all $n \geq 5$. An $m$-dense tournament of order 6 is given in Figure 3.2 (a). Figure 3.2 (b) shows how to construct an $m$-dense tournament $D_n$ of order $n$ from an $m$-dense tournament $D_{n-2}$ of order $n-2$ for $n \geq 5$ and $n \neq 6$. However, our next result shows that an $m$-dense oriented graph need not necessarily be a tournament.

Theorem 3.3 There exist $m$-dense oriented graphs of order $p$ that are not tournaments for all $p \geq 6$. 

Figure 3.2
Proof. If $D_1$ and $D_2$ are two m-dense oriented graphs, then an m-dense oriented graph $D$ that is not a tournament can be constructed as shown in Figure 3.3 (a). Since there exist m-dense oriented graphs of order 1, 3 and 5, it is straightforward to see that an m-dense oriented graph of order $p$ that is not a tournament can be constructed in this manner for all $p \geq 6$ and $p \neq 7, 9$. Since the m-dense oriented graph of order $p = 6, 8$ contains $\frac{p}{2}$ copies of oriented $K_2$, it follows that an m-dense oriented graph $D$ of order $p+1$, that is not a tournament, can be constructed as shown in Figure 3.3 (b) □

![Figure 3.3](image)

Define $q(n)$ to be the minimum size of an m-dense oriented graph of order $n$, and $q(n) = \infty$ if there is no m-dense oriented graph of order $n$. Then

\[ q(1) = 0, \quad q(2) = \infty, \quad q(3) = 3, \quad q(4) = \infty \text{ and } q(5) = 10. \]

Combining Theorem 3.2 and the construction in the proof of Theorem 3.3, we see that $q(6) = 12$. As a corollary of Theorem 3.3, our next result gives an upper bound of the parameter $q(n)$, where $n \geq 1$, $n \neq 2, 4$.

Corollary 3.4 Let $n \geq 1$ and $n \neq 2, 4$. If $n = 5d + r$ where $d \geq 0$ and $r \in \{1, 3, 5, 7, 9\}$, then $q(n) \leq q(r) + 2d(n + r - 1)$. 

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Proof. The inequality holds for \( n \in \{1, 3, 5, 6, 7, 9\} \). Suppose that \( n = 8 \) or \( n \geq 10 \). Let \( H \) be an \( m \)-dense oriented graph of order \( n - 5 \) and size \( q(n - 5) \). Let \( D \) be the \( m \)-dense oriented graph obtained by replacing \( D_1 \) and \( D_2 \) in Figure 3.3 by \( H \) and \( K_1 \), respectively. Since \( p(D) = n \), it follows that

\[
q(n) \leq q(D) = q(n - 5) + 4(n - 5) + 8.
\]

\[
= q(r) + 4(n - 5d) + \ldots + 4(n - 5) + 8d
\]

\[
= q(r) + 4nd - 10d^2 - 2d
\]

\[
= q(r) + 2d(n + r - 1).
\]

□

Observe that the upper bound in the corollary is sharp when \( n = 6 \). From the construction in the previous proof, it follows that for every integer \( n \geq 6 \) and \( n \neq 7, 9 \), there exists an \( m \)-dense oriented graph of minimum degree \( 4 \). Therefore, Theorem 3.2 is best possible. It is interesting to know the exact value of \( q(n) \) for \( n \geq 7 \).

Our next result gives familiar bounds for \( m \)-diam \( D \).

**Theorem 3.5** Let \( D \) be a digraph. Then \( m \)-rad \( D \leq m \)-diam \( D \leq 2 m \)-rad \( D \).

**Proof.** The first inequality follows from the definitions. For the second inequality, let \( u \) and \( v \) be vertices of \( D \) such that \( md(u, v) = m \)-diam \( D \) and let \( w \in V(D) \) such that \( me(w) = m \)-rad \( D \). Then \( m \)-diam \( D = md(u, v) \leq md(u, w) + md(w, v) \leq 2 me(w) = 2 m \)-rad \( D \). □

We now consider the sharpness of these bounds.

**Theorem 3.6** Let \( r \) and \( d \) be integers with \( 2 \leq r \leq d \leq 2r \). Then there exists an oriented graph \( D \) such that \( m \)-rad \( D = r \) and \( m \)-diam \( D = d \).
Proof. Let $D$ be an oriented graph with $V(D) = \{v_0, v_1, \ldots, v_r, v_{r+1}, \ldots, v_d\}$ and $E(D) = \{(v_r, v_0)\} \cup \{(v_i, v_{i+1}) | 0 \leq i \leq d-1\} \cup \{(v_i, v_0) | r+1 \leq i \leq d\}$ (see Figure 3.4). It is straightforward to verify that $me(v_r) = \max\{r, i, d - i\}, 0 \leq i \leq d$. Therefore, $m$-rad $D = r$ and $m$-diam $D = d$. □

\[\begin{align*}
\text{D:} \\
\begin{array}{ccccccc}
v_0 & \rightarrow & \cdots & \rightarrow & v_r & \rightarrow & v_{r+1} & \rightarrow & \cdots & \rightarrow & v_d \\
\end{array}
\end{align*}\]

Figure 3.4

3.2 The $m$-Centers of Oriented Graphs

A vertex of minimum $m$-eccentricity is called an $m$-central vertex and the $m$-center $mC(D)$ of a digraph $D$ is the subdigraph induced by its $m$-central vertices. Harary and Norman [13] proved that the center of every connected graph $G$ lies in a single block of $G$. For the $m$-center of oriented graphs, we have the following result.

Theorem 3.7 The $m$-center of every strong oriented graph $D$ lies in a single block of the underlying graph of $D$.

Proof. Suppose $D$ is a strong oriented graph whose $m$-center $mC(D)$ does not lie within a single block of the underlying graph $G$ of $D$. Then $G$ has a cut-vertex $v$ such that $G - v$ contains components $G_1$ and $G_2$, each of which contains elements of $mC(D)$. Let $u$ be a vertex such that $md(u, v) = me(v)$, and without loss of generality, suppose that $md(u, v) = d(u, v)$. Let $P_1$ be a $u$-$v$ path of $D$ having length $me(v)$. At least one of $G_1$ and $G_2$, say $G_2$, contains no vertices of $P_1$. Let
w be an element of \( mC(D) \) belonging to \( G_2 \), and let \( P_2 \) be a \( v-w \) path of minimum length. Then paths \( P_1 \) and \( P_2 \) together form a \( u-w \) path \( P_3 \), which is necessarily a \( u-w \) path of length \( \tilde{d}(u, w) \). Thus, \( me(w) \geq \max(\tilde{d}(u, w), \tilde{d}(w, u)) \geq \tilde{d}(u, w) > \tilde{d}(u, v) = me(v) \), which contradicts the fact that \( w \in mC(D) \). Therefore, \( mC(D) \) lies in a single block of \( G \).

Recall that every graph is the center of some graph. Our next result considers a parallel situation for oriented graphs under \( m \)-distance.

**Theorem 3.8** Let \( D \) be an oriented graph. Then there exists a strong oriented graph \( H \) such that \( mC(H) \equiv D \) and \( p(H) \leq p(D) + 4 \).

**Proof.** Let \( V_1 = \{ v \mid v \in V(D), me_D(v) \geq 3 \} \) and \( V_2 = \{ v \mid v \in V(D), me_D(v) = 2 \} \). Then \( V(D) = V_1 \cup V_2 \). We consider the following cases:

**Case 1. Assume that** \( V_1 = \emptyset \).

In this case, \( me_D(v) = 2 \) for every vertex \( v \) of \( D \); so \( mC(D) = D \). Then if we let \( H = D \), the digraph \( H \) has the desired property.

**Case 2. Assume that** \( V_2 = \emptyset \).

In this case, we construct an oriented graph by adding four new vertices \( v_1, v_1', v_2, v_2' \) to \( D \), the arcs \((v_1, v_2)\) and \((v_1', v_2')\), together with the arcs joining all vertices of \( D \) to \( v_1 \) and \( v_1' \) and joining all vertices of \( D \) from \( v_2 \) and \( v_2' \) (see Figure 3.5).

It suffices to prove that \( me_H(v_i') = me_H(v_i) = 4 \) \( (i = 1, 2) \) and \( me_H(x) = 3 \) for all \( x \in V(D) \). We verify the second of these two claims first. We do this by showing that \( md_H(x, z) \leq 3 \) for all \( z \) in \( H \) and, moreover, that \( md_H(x, z) = 3 \) for at least one \( z \) in \( D \), from which it will follow that \( me_H(x) = 3 \). First observe that for an arbitrary vertex \( x \in V(D) \), \( md_H(x, y) \leq 3 \) for all \( y \in V(D) \). If \( x' \) is an
m-eccentric vertex of $x$ in $D$, then $m_d(x, x') = m_e(x) \geq 3$. Since $x, v_1, v_2, x'$ is a shortest $x-x'$ path in $H$ and $x', v_1, v_2, x$ is a shortest $x'-x$ path in $H$, it follows that $md_H(x, x') = 3$. Furthermore, $md_H(x, v_i) = md_H(x, v_i') = 2 \ (i = 1, 2)$. This completes the argument that $me_H(x) = 3$ for $x \in V(D)$.

Next we show that $me_H(v_i) = me_H(v_i') = 4 \ (i = 1, 2)$. This follows from the observations that

1. $md_H(v_1, v_2) = md_H(v_1', v_2') = 2$,
2. $md_H(v_1, v_1') = md_H(v_2, v_2') = 3$, and
3. $md_H(v_1, v_2') = me_H(v_1', v_2) = 4$.

Therefore $me_H(v_i) = me_H(v_i') = 4 \ (i = 1, 2)$.

Case 3. Assume that $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$.

In this case, we define

$S_1 = \{v \mid v \in V_2, \text{ there exists } u \in V_1 \text{ such that } (u, v) \in E(D)\}$,
and

$S_2 = \{v \mid v \in V_2, \text{ there exists } w \in V_1 \text{ such that } (v, w) \in E(D)\}$.

Let

$V_1 0' = S_1 \cap S_2, \ V_1 1' = S_1 \subset (S_1 \cap S_2), \ V_1 2' = S_2 - (S_1 \cap S_2), \text{ and } V_1 3 = V_2 - (S_1 \cup S_2)$.
Note that $V(D) = V_1 \cup V_1^0 \cup V_1^1 \cup V_1^2 \cup V_1^3$, where this union is a disjoint union. We distinguish two subcases.

**Subcase 3.1** Assume that $V_1^3 = \emptyset$. Let $H$ be the oriented graph defined by adding four new vertices $v_1, v_1', v_2, v_2'$ to $D$, the arcs $(v_1, v_2)$ and $(v_1', v_2')$, together with the arcs joining all vertices in $V_1 \cup V_1^1$ to $v_1$ and $v_1'$ and joining all vertices in $V_1 \cup V_1^2$ from $v_2$ and $v_2'$ (see Figure 3.6).

Since $v_i$ and $v_i'$ ($i = 1, 2$) are symmetric in $H$, it suffices to prove that $\text{me}_H(v_1) = \text{me}_H(v_2) = 4$ and $\text{me}_H(x) = 3$ for all $x \in V(D)$. First we show that $\text{me}_H(v_1) = \text{me}_H(v_2) = 4$. If $x \in V_1$, then $\text{md}_H(v_1, x) = \text{md}_H(v_2, x) = \text{md}_H(v_1, v_2) = 2$. If $x \in V_1^0$, then by the definition of $V_1^0$, there exist $u, w \in V_1$ such that $(u, x) \in E(D)$ and $(x, w) \in E(D)$. Therefore, $\overrightarrow{d}(x, v_1) = 2$, $\overrightarrow{d}(v_1, x) = 3$, $\overrightarrow{d}(x, v_2) = 3$, and $\overrightarrow{d}(v_2, x) = 2$, so that $\text{md}_H(v_1, x) = \text{md}_H(v_2, x) = 3$. If $x \in V_1^1$, then by the definition of $V_1^1$ there exists $u \in V_1$ such that $(u, x) \in E(D)$. Therefore $\overrightarrow{d}(x, v_1) = 1$, $\overrightarrow{d}(v_1, x) = 3$ and $\overrightarrow{d}(x, v_2) = \overrightarrow{d}(v_2, x) = 2$, so that
md_H(v_1, x) = 3 and md_H(v_2, x) = 2. If x \in V_1', then there exists w \in V_1 such that (x, w) \in E(D). By a similar argument, md_H(v_1, x) = 2 and md_H(v_2, x) = 3. Furthermore, since md_H(v_1, v_2') = md_H(v_1', v_2) = 4 and md_H(v_1, v_1') = md_H(v_2, v_2') = 3, it follows that me_H(v_1) = me_H(v_2) = 4.

Next we show that me_H(x) = 3 for all x \in V(D). By a similar argument to Case 2, me_H(x) = 3 for all x \in V_1. If x \in V_1 0', then since me_D(x) = 2, md_H(x, w) = 2 for all w \in V(D) - \{x\}. Recall that md_H(x, v_1) = md_H(x, v_2) = 3. Therefore, me_H(x) = 3 for all x \in V_1 0'. Similarly, me_H(x) = 3 for all x \in V_1 1' \cup V_1 2'. Therefore me_H(x) = 3 for all x \in V(D).

Subcase 3.2 Assume that V_1 3 \neq \emptyset. Since D is strong and there is no arc between V_1 and V_1 3, it follows that V_1 0' \cup V_1 1' \cup V_1 2' \neq \emptyset. We first claim that there exist disjoint nonempty sets V_{1 1} and V_{1 2} such that V_{1 1}' \subset V_{1 1}, V_{1 2}' \subset V_{1 2}, and V_{1 1} \cup V_{1 2} = V_1 0' \cup V_1 1' \cup V_1 2'. If |V_1 0'| \geq 2, say u \in V_1 0', then V_{1 1} = V_{1 1}' \cup \{u\} and V_{1 2} = V_{1 2}' \cup (V_1 0' - \{u\}) have the desired properties. If |V_1 0'| = 1, say V_1 0' = \{u\}, then either V_{1 1}' \neq \emptyset or V_{1 2}' \neq \emptyset. Otherwise, let x \in V_1 and w \in V_1 3 with (w, u) \in E(D). Then \overline{d}(x, w) \geq 3, which contradicts me_D(w) = 2. Suppose V_{1 1}' \neq \emptyset. Then V_{1 1} = V_{1 1}' and V_{1 2} = V_1 0' \cup V_{1 2}' have the desired properties. If |V_1 0'| = 0, then V_{1 1}' \neq \emptyset and V_{1 2}' \neq \emptyset. Therefore V_{1 1} = V_{1 1}' and V_{1 2} = V_{1 2}' satisfy the properties. Define

S_{2 1} = \{v \mid v \in V_{1 3}, \text{there exists } u \in V_{1 2} \text{ such that } (u, v) \in E(D)\},

and

S_{2 2} = \{v \mid v \in V_{1 3}, \text{there exists } w \in V_{1 1} \text{ such that } (v, w) \in E(D)\}.

Let V_{2 1} = S_{2 1}, V_{2 2} = S_{2 2} - S_{2 1}, and V_{2 3} = V_{1 3} - (S_{2 1} \cup S_{2 2}). Therefore V(D) = V_1 \cup V_{1 1} \cup V_{1 2} \cup V_{2 1} \cup V_{2 2} \cup V_{2 3}, where this union is a disjoint union.
In general, suppose $V_k^1$, $V_k^2$, and $V_k^3$ have been defined. If $V_k^1 \cup V_k^2 \neq \emptyset$ and $V_k^3 \neq \emptyset$, then we define

$$S_{k+1,1} = \{v \mid v \in V_k^3, \text{there exists } u \in V_k^2 \text{ such that } (u,v) \in E(D)\},$$

and

$$S_{k+1,2} = \{v \mid v \in V_k^3, \text{there exists } w \in V_k^1 \text{ such that } (v,w) \in E(D)\}.$$ 

Note that if $V_k^2 = \emptyset$, then $S_{k+1,1} = \emptyset$, while if $V_k^1 = \emptyset$, then $S_{k+1,2} = \emptyset$. Let $V_{k+1,1} = S_{k+1,1}$, $V_{k+1,2} = S_{k+1,2} - S_{k+1,1}$, and $V_{k+1,3} = V_k^3 - (S_{k+1,1} \cup S_{k+1,2})$. If $V_{k+1,1} \cup V_{k+1,2} = \emptyset$ or $V_{k+1,3} = \emptyset$, then we stop the partition process; otherwise a similar partition of $V_{k+1,3}$ is performed. It is clear that $|V_k^3| > |V_{k+1,3}|$ if $V_{k+1,1} \cup V_{k+1,2} \neq \emptyset$. Since $D$ is a finite digraph, the process will end in a finite number of steps, that is, there exists an integer $m (\geq 2)$ such that either $V_m^1 \cup V_m^2 = \emptyset$ or $V_m^3 = \emptyset$. Note that $V_m^1 \cup V_m^2 \cup V_m^3 = V_m^1 \cup V_m^2 \cup V_m^3 = V_m^1 \cup V_m^2 \cup V_m^3 = \emptyset$. We now consider two subcases.

**Subcase 3.2.1** Assume that $V_m^1 \cup V_m^2 = \emptyset$. In this case $V(D) = V_1 \cup V_1 \cup V_1 \cup V_1 \cup V_2 \cup V_2 \cup \ldots \cup V_m \cup V_m \cup V_m$, where this union is a disjoint union. Note that $V_1 \neq \emptyset$, $V_1 \neq \emptyset$, $V_1 \neq \emptyset$, $V_m^1 = \emptyset$, $V_m^2 = \emptyset$, and $V_m^3 \neq \emptyset$. By the definition of $V_k^i$ ($i = 1, 2$), if $V_k^i = \emptyset$, then $V_{k+1,j} = \emptyset$, where $j = 3 - i$.

We now construct an oriented graph by first adding four new vertices $v_1, v_2, v_3, v_4$ to $D$, and the arcs $(v_2, v_1), (v_2, v_3), (v_4, v_2)$. We then join all vertices of $V_1$ to $v_1$ and join all vertices of $V_{12}$ from $v_1$. Next we join all vertices of $V_{11}$ to $v_4$ and join all vertices of $V_1$ from $v_4$. We complete the oriented graph $H$ by adding the arcs joining all vertices in $V_1 \cup \bigcup_{i=1}^{m-1} V_i$ to $v_2$ and joining all vertices of $V_1 \cup \bigcup_{i=1}^{m-1} V_i$ to $v_3$, together with the arcs joining all vertices of $V_m^3$ from $v_2$ and to $v_3$ (see Figure 3.7).
We claim that $m_{dH}(v_1, x) \leq 3$ for all $x \in V(D)$, $m_{dH}(v_1, v_3) = 4$, and $m_{eH}(v_1) = 4$. Choose $u \in V_1$, $v \in V_1$ such that $(u, v) \in E(D)$. Let $x$ be an arbitrary vertex of $V(D)$. We first verify that $m_{dH}(v_1, x) \leq 3$. If $x \in V_1$ then $d_H(x, v_1) = 1$. Since $m_{eD}(u) = 2$, $m_{dD}(u, x) = 2$. Noting that $(v_1, u) \in E(H)$ and $d_H(u, x) \leq m_{dH}(u, x) \leq m_{dD}(u, x) = 2$, we have $d_H(v_1, x) \leq d_H(v_1, u) + d_H(u, x) \leq 3$. Therefore $m_{dH}(v_1, x) \leq 3$ for all $x \in V_1$. If $x \in V_{12}$ then $(v_1, x) \in E(H)$. By the definition of $V_{12}$ there exists $y \in V_1$ such that $(x, y) \in E(D)$. Since $(y, v_1) \in E(H)$, $d_H(x, v_1) = 2$. Therefore $m_{dH}(v_1, x) = 2$ for all $x \in V_{12}$. If $x \in V(D) - (V_1 \cup V_{12})$, then $m_{eD}(x) = 2$. Therefore $d_H(x, v) \leq 2$, so that $d_H(x, v_1) \leq 3$. Furthermore, since $d_H(u, x) \leq 2$ and $(v_1, u) \in E(H)$, it follows that $d_H(v_1, x) \leq 3$. Therefore $m_{dH}(v_1, x) \leq 3$ for all $x \in V(D) - (V_1 \cup V_{12})$. This completes the argument that $m_{dH}(v_1, x) \leq 3$ for all $x \in V(D)$.

We now verify that $m_{dH}(v_1, v_2) \leq 3$, $m_{dH}(v_1, v_3) = 4$ and $m_{dH}(v_1, v_4) \leq 4$, from which it will follow that $m_{eH}(v_1) = 4$. Since there is a cycle $v_1, u, v, v_2, v_1$ in $H$. 

Figure 3.7.
\( m_{dl}(v_1, v_2) \leq 3 \). It is clear that \( d_H(v_3, v_1) = 2 \). Since the in-neighborhood \( N^-(v_3) = V_{m3} \cup \{v_2, v_4\} \) and the out-neighborhood \( N^+(v_1) = V_{12} \), it follows that there is no arc from \( N^+(v_1) \) to \( N^-(v_3) \). Therefore \( d_H(v_1, v_3) \geq 4 \). On the other hand, since \( v_1, u, v, v_2, v_3 \) is a \( v_1-v_3 \) path in \( H \), it follows that \( d_H(v_1, v_3) = 4 \). Therefore \( m_{dl}(v_1, v_3) = 4 \). Observe that \( d_H(v_4, v_1) = 2 \). To show \( d_H(v_1, v_4) \leq 4 \), we choose a vertex \( w \in V_{11} \). Since \( m_{eh}(w) = 2 \), \( d_H(u, w) \leq d_D(u, w) \leq 2 \). Furthermore, since \( (v_1, u), (w, v_4) \in E(H) \), it follows that \( d_H(v_1, v_4) \leq d_H(v_1, u) + d_H(u, w) + d_H(w, v_4) \leq 4 \). Therefore \( m_{dl}(v_1, v_4) \leq 4 \). This completes the argument that \( m_{eh}(v_1) = 4 \).

By a similar argument, it follows that \( m_{dl}(v_4, x) \leq 3 \) for all \( x \in V(D) \), \( m_{dl}(v_4, v_2) = 4 \) and \( m_{eh}(v_4) = 4 \).

Next we claim that:

(i) \( m_{dl}(v_2, x) = 2 \) for all \( x \in V_1 \cup \bigcup_{i=1}^{m-1} V_{i2} \);

(ii) \( m_{dl}(v_2, x) = 3 \) for all \( x \in V_{m3} \cup \bigcup_{i=1}^{m-1} V_{i1} \); and

(iii) \( m_{eh}(v_2) = 4 \).

First we verify (i). If \( x \in V_1 \), then \( v_2, v_3, x, v_2 \) is a triangle in \( H \). Thus \( m_{dl}(v_2, x) = 2 \) for all \( x \in V_1 \). Let \( x \) be a vertex in \( \bigcup_{i=1}^{m-1} V_{i2} \). Observe that \( d_H(v_2, x) = 2 \). If \( x \in V_{12} \), then there exists \( w \in V_1 \) such that \( (x, w) \in E(D) \). Thus \( d_H(x, v_2) = 2 \). If \( x \in \bigcup_{i=1}^{m-1} V_{i2} \), say \( x \in V_{k2} \), then by the definition of \( V_{k2} \), there exists \( w \in V_{k-1} \) such that \( (x, w) \in E(D) \). Since \( (w, v_2) \in E(D) \), \( d_H(x, v_2) = 2 \). Therefore, \( m_{dl}(v_2, x) = 2 \) for all \( x \in \bigcup_{i=1}^{m-1} V_{i2} \). This completes the proof of (i).

If \( x \in V_{m3} \), then \( d_H(v_2, x) = 1 \). Observe that there is no arc from \( V_{m3} \) to \( N^-(v_2) = V_1 \cup \bigcup_{i=1}^{m-1} V_{i1} \). Thus \( d_H(x, v_2) \geq 3 \). Since \( x, v_3, w, v_2 \) is an \( x-v_2 \) path
in \( H \), where \( w \in V_1 \), it follows that \( d_H(x, v_2) \leq 3 \). Therefore, \( md_H(v_2, x) = 3 \) for all \( x \in V_{m3} \). If \( x \in \bigcup_{i=1}^{m-1} V_i \), then \( d_H(x, v_2) = 1 \). Since \( N^+(v_2) = V_{m3} \cup \{v_1, v_3\} \), there is no arc from \( N^+(v_2) \) to \( x \). Thus \( d_H(v_2, x) \geq 3 \). On the other hand, choose \( w \in V_{m3} \). Since \( me_D(w) = 2 \), \( d_D(w, x) \leq 2 \). It follows that \( d_H(v_2, x) \leq d_H(v_2, w) + d_H(w, x) \leq 3 \). Therefore \( d_H(v_2, x) = 3 \), from which it follows that \( md_H(v_2, x) = 3 \) for all \( x \in V_{m3} \cup (\bigcup_{i=1}^{m-1} V_i) \). This completes the proof of (ii).

To prove \( me(v_2) = 4 \), it suffices to show that \( md_H(v_2, v_3) \leq 4 \). Let \( v \in V_1 \). Since \( v_2, v_3, v, v_2 \) is a triangle in \( H \), it follows that \( md_H(v_2, v_3) = 2 \). This completes the proof of the previous claim.

By a similar argument it follows that

(i) \( md_H(v_3, x) = 2 \) for all \( x \in V_1 \cup \bigcup_{i=1}^{m-1} V_i \),

(ii) \( md_H(v_3, x) = 3 \) for all \( x \in V_{m3} \cup (\bigcup_{i=1}^{m-1} V_i) \), and

(iii) \( me(v_3) = 4 \).

Now we show that \( me_H(x) = 3 \) for all \( x \in V(D) \). By an argument similar to that used in Case 2, \( me_H(x) = 3 \) for all \( x \in V_1 \). Observe that
\[
V(D) - V_1 = V_2 = V_{m3} \cup (\bigcup_{i=1}^{m-1} (V_1 \cup V_i))
\]
Since all vertices in \( V_2 \) have \( m \)-eccentricities 2 in \( D \), it suffices to prove that \( md_H(x, v_i) \leq 3 \) \((i = 1, 2, 3, 4)\) for \( x \in V_2 \) and, moreover, that either \( md_H(x, v_2) = 3 \) or \( md_H(x, v_3) = 3 \). However, this has been done in the previous arguments. Therefore \( me_H(x) = 3 \) for all \( x \in V(D) \). This completes the proof of the subcase.

**Subcase 3.2.2** Assume that \( V_{m3} = \emptyset \). Then \( V_{m1} \cup V_{m2} \neq \emptyset \). We now construct an oriented graph by first adding four new vertices \( v_1, v_2, v_3, v_4 \) to \( D \) together with the arcs \( (v_2, v_1), (v_2, v_3), \) and \( (v_4, v_3) \). We then join all vertices of \( V_1 \).
to \( v_1 \) and join all vertices of \( V_{12} \) from \( v_1 \). Next we join all vertices of \( V_{11} \) to \( v_4 \) and join all vertices of \( V_1 \) from \( v_4 \). We complete the oriented graph \( H \) by adding the arcs joining all vertices in \( V_1 \cup (\bigcup_{i=1}^{m} V_{i1}) \) to \( v_2 \) and joining all vertices of \( V_1 \cup (\bigcup_{i=1}^{m} V_{i2}) \) from \( v_3 \) (see Figure 3.8).

\[
H: \\
[\text{Diagram of } H] \\
\text{Figure 3.8.}
\]

The remainder of the proof in this subcase is similar to the previous proof and so will be omitted. \( \square \)

The type of construction used in Figure 3.5 does not work in the proof of Subcase 3.2.1. However, an alternative construction can be given for the proof of Subcase 3.2.2 (see Figure 3.9).

If the digraphs in Theorem 3.8 are not required to be oriented, then a desired digraph \( H \) can have smaller order.
Theorem 3.9 For every digraph $D$, there exists a strong digraph $H$ with $p(H) = p(D) + 3$ such that $mC(H) = D$.

Proof. We construct a digraph $H$ by adding three new vertices $u$, $v$, $w$ to $D$ along with the arcs $(u, v)$, $(u, w)$ and $(w, v)$, together with the symmetric arcs (indicated by bold edges in Figure 3.10) joining all vertices of $D$ to and from $u$ and $v$. It is routine to verify that $me(u) = me(v) = me(w) = 3$ and $me(x) = 2$ for all $x \in V(D)$. Therefore, $mC(H) = D$. □
3.3 The m-Peripheries of Oriented Graphs

We now turn to m-peripheries of digraphs. A vertex of maximum m-eccentricity is called an m-peripheral vertex. The m-periphery \( mP(D) \) of a digraph \( D \) is the subdigraph induced by its m-peripheral vertices. Our next theorem, which is an analogue of Theorem H', gives a characterization of m-peripheries of oriented graphs.

**Theorem 3.10** Let \( D \) be an oriented graph. Then \( D \) is isomorphic to the m-periphery of some oriented graph if and only if either all vertices of \( D \) have m-eccentricity 2 or no vertices have m-eccentricity 2.

**Proof.** We begin by proving the necessity.

Suppose, to the contrary, that \( D \cong mP(H) \) and there exist vertices \( u, v \in V(D) \) with \( me_D(u) = 2 \) and \( me_D(v) > 2 \). Without loss of generality, assume that \( D \) is an induced subdigraph of \( H \) (\( mP(H) = D \)). Let \( w \) be a vertex of \( H \) such that \( md(u, w) = me_H(u) \). Since \( me_H(w) \geq md_H(u, w) = me_H(u) \) and \( u \in mP(H) \), we have \( w \in mP(H) \).

Next we prove that \( w \not\in V(D) \). We claim \( me_H(u) > 2 \); for otherwise, \( me_H(u) = 2 \) and \( u \) is an m-peripheral vertex of \( H \) imply that \( me_H(x) = 2 \) for every vertex \( x \in V(H) \), namely \( D = mP(H) = H \). Therefore, \( 2 = me_H(u) = me_H(v) = me_D(v) > 2 \), a contradiction. Suppose \( w \in V(D) \). Then

\[
2 = me_D(u) \geq md_D(u, w) \geq md_H(u, w) = me_H(u) > 2,
\]

again a contradiction. Since \( w \in mP(H) \) and \( w \not\in V(D) \), \( mP(H) \neq D \), a contradiction to the assumption.

We now prove the sufficiency. If all vertices of \( D \) have m-eccentricity 2, then \( H = D \) possesses the desired property. If all vertices of \( D \) have m-eccentricity
greater than 2, then the oriented graph $H$ can be constructed as shown in Figure 3.11. It is clear that $mP(H) \equiv D$. □

We now consider the relationship between the $m$-center and the $m$-periphery of an oriented graph. In particular, what can be said about the $m$-center of an oriented graph with prescribed $m$-periphery? Our next result shows that we can say very little if the $m$-periphery itself is to have $m$-radius greater than 3, that is, there are no restrictions as to what the $m$-center can be.

![Diagram of oriented graph H](image)

**Figure 3.11**

**Theorem 3.11** Let $D$ be an oriented graph with $m\text{e}_D(v) \geq 4$ for all $v \in V(D)$. Then, for any oriented graph $F$, there exists an oriented graph $H$ such that $mP(H) \equiv D$ and $mC(H) \equiv F$.

**Proof.** We distinguish two cases according to the value of $p(F)$.

**Case 1. Assume that** $p(F) = 1$.

Suppose that $V(F) = \{v\}$. We define $H$ as follows:

$V(H) = V(D) \cup V(F) \cup \{u, w\}$,

and

$E(H) = E(D) \cup \{(u, v), (v, w), (w, u)\} \cup \{(x, u), (w, x) \mid x \in V(D)\}$.
It is obvious that $mP(H) = D$ and $mC(H) = F$.

**Case 2. Assume that** $p(F) \geq 2$.

Let $V(F) = \{v_1, v_2, ..., v_n\}$. We first construct an oriented graph $H_0$ with $V(H_0) = V(F) \cup \{x_i, y_i \mid 1 \leq i \leq n\}$. The arc set of $H_0$ consists of the arcs of $F$ together with several additional arcs. First we add arcs $(x_i, y_j)$, $(x_i, v_j)$, and $(v_i, y_i)$ for all $1 \leq i \leq n$. For $1 \leq i \leq n$, we join $x_i$ from all the vertices $v_j$ where $j \neq i$, and we complete the construction of $H_0$ by joining $y_i$ to all vertices $v_j$ where $j \neq i$.

We now define the oriented graph $H$ as follows:

$$V(H) = V(D) \cup V(H_0) \cup \{u_0, v_0\},$$

and

$$E(H) = E(D) \cup E(H_0) \cup \{(w, u_0), (v_0, w) \mid w \in V(D)\}$$

$$\cup \{(u_0, w), (w, v_0) \mid w \in V(F)\}$$

$$\cup \{(v_0, x_i), (y_i, u_0) \mid 1 \leq i \leq n\}$$

(see Figure 3.12).

Figure 3.12

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First we show that $\text{me}_H(v_i) = 2$ for $1 \leq i \leq n$. It is clear that $\text{md}_H(v_i, w) = 2$ for all $w \in V(D)$. Observe that $H$ contains the following triangles:

$v_i, y_i, u_0, v_i; v_i, v_0, x_i, v_i;$ and $v_i, x_j, y_j, v_i$, where $j \neq i$.

Therefore, $\text{md}_H(v_i, w) = 2$ for all $w \in \{u_0, v_0\} \cup \{x_i, y_i \mid 1 \leq i \leq n\}$. Choosing a vertex $v_j \in V(F)$ where $j \neq i$, we can form the paths $v_i, y_i, v_j$ and $v_j, y_j, v_i$. Thus $\text{md}_H(v_i, v_j) = 2$ for $1 \leq i < j \leq n$. Combining the above, we have $\text{me}_H(v_i) = 2$ for $1 \leq i \leq n$.

Next we show that $\text{me}_H(x_i) = \text{me}_H(y_i) = 3$ for $1 \leq i \leq n$. Observe that the following paths exist in $H$:

$x_i, y_i; y_i, v_j, x_i; x_i, v_i, x_j;$ and $y_i, v_j, y_j$, where $j \neq i$.

Therefore, $\text{md}_H(x_i, y_i) = 2$ for $1 \leq i \leq n$, and $\text{md}_H(x_i, x_j) = \text{md}_H(y_i, y_j) = 2$ for $1 \leq i < j \leq n$. Since there exist the paths $x_i, v_i, x_j, y_j$ and $y_j, u_0, v_j, x_j$ in $H$, $\text{md}_H(x_i, y_j) \leq 3$ for $1 \leq i < j \leq n$. Further, note that

$x_i, y_i, u_0; u_0, v_j, x_i; u_0, v_i, y_i;$ and $y_i, u_0$, where $j \neq i$ are paths in $H$. It follows that $\text{md}_H(x_i, u_0) = \text{md}_H(y_i, u_0) = 2$ for $1 \leq i \leq n$. Similarly, $\text{md}_H(x_i, v_0) = \text{md}_H(y_i, v_0) = 2$ for $1 \leq i \leq n$. Let $w \in V(D)$. Then $x_i, v_i, v_0, w$ and $w, u_0, v_j, x_i$ are two shortest paths. Therefore $\text{md}_H(x_i, w) = 3$. By a similar argument, $\text{md}_H(y_i, w) = 3$. Therefore $\text{me}_H(x_i) = \text{me}_H(y_i) = 3$ for $1 \leq i \leq n$.

Note that $\text{md}_H(u_0, v_0) = 2$, and $\text{md}_H(u_0, w) = \text{md}_H(w, v_0) = 3$ for all $w \in V(D)$. Further, by the conclusions above, $\text{me}_H(u_0) = \text{me}_H(v_0) = 3$.

Finally, since $\text{me}_D(w) \geq 4$ for $w \in V(D)$ and since, for all $w_1, w_2 \in V(D)$, there is a shortest $w_1-w_2$ path, namely $w_1, u_0, v_1, v_0, w_2$, in $H$, it follows that $\text{me}_H(w) = 4$ for all $w \in V(D)$.

Therefore, $\text{mP}(H) \equiv D$ and $\text{mC}(H) \equiv F$. ✷
3.4 The m-Medians of Oriented Graphs

The m-distance $md(v)$ of a vertex $v$ in a strong digraph $D$ is defined by

$$md(v) = \sum_{v \in V(D)} md(v, w).$$

A vertex with minimum m-distance is called a m-median vertex (or, simply, median vertex) of $D$. The m-median $mM(D)$ of $D$ is the subdigraph induced by its median vertices. For subdigraphs $F_1$ and $F_2$ of $D$, the m-distance between $F_1$ and $F_2$ is defined by

$$md_{D}(F_1, F_2) = \min \{md_D(u, v) | u \in V(F_1), v \in V(F_2) \}.$$ 

For example, we have $md_D(F_1, F_2) = md(u, v) = 3$ in Figure 3.13.

Hendry [12] and Holbert [15] studied the relative location of the center and the median of a connected graph. Holbert proved that the center and the median of a connected graph can be arbitrarily far apart from each other, they can also be close in any possible way. In this section, we will present two similar results about the relative location of the m-center and m-median of a strong oriented graph. For a given oriented graph $D$, our next lemma shows that any subdigraph $F$ of $D$, whose vertices have the same m-distance in $D$, can be the m-median of some oriented graph that contains $D$ as an induced subdigraph.

---

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Lemma 3.12 Let $D$ be an strong oriented graph and let $F$ be a subdigraph of $D$ with $md_D(u) = md_D(v)$ for all $u, v \in V(F)$. Then there exists an oriented graph $H$ having $D$ as an induced subdigraph such that $mM(H) = F$.

Proof. Suppose $md_D(v) = k$ for all $v \in V(F)$. Let $n = \left\lceil \frac{p(F) + k}{2} \right\rceil - p(D) + 1$.

We construct an oriented graph $H$ by adding $2n$ new vertices $u_i, v_i$ ($1 \leq i \leq n$) to $D$ and the arcs joining all vertices of $F$ to $u_i$ and from $v_i$ for $1 \leq i \leq n$ (see Figure 3.14).

![Diagram](image)

Figure 3.14

Then,

$$md_H(v) = \sum_{1 \leq i \leq n} (md(v, u_i) + md(v, v_i)) + \sum_{x \in V(D)} md_H(v, x)$$

$$\leq 4n + \sum_{x \in V(D)} md_D(v, x)$$

$$= 4n + md(v) = 4n + k, \text{ for } v \in V(F).$$

For $1 \leq i \leq n$, it follows that

$$md(u_i) = \sum_{1 \leq j \leq n} (md(u_i, u_j) + md(u_i, v_j)) + md(u_i, v_j) + \sum_{x \in V(F)} md_H(u_i, x) + \sum_{x \in V(D) - V(F)} md_H(u_i, x)$$
\[ \geq 7(n - 1) + 2 + 2p(F) + 3(p(D) - p(F)) \]
\[ = 7n + 3p(D) - p(F) - 5. \]

Similarly, \( m_d(v_i) \geq 7n + 3p(D) - p(F) - 5 \) for \( 1 \leq i \leq n \). If \( v \in V(D) - V(F) \), then
\[
m_d_H(v) = \sum_{1 \leq i \leq n} (m_d(v, u_i) + m_d(v, v_i)) + \sum_{x \in V(D)} m_d_H(v, x) \geq 6n + 2(p(D) - 1). \]

Since \( n = \left\lceil \frac{p(F) + k}{2} \right\rceil - p(D) + 1 \), it follows that
\[ 7n + 3p(D) - p(F) - 5 > 4n + k \quad \text{and} \quad 6n + 2(p(D) - 1) > 4n + k. \]
Therefore, \( m_M(H) \equiv F. \]

In order to apply Lemma 3.12 to any subdigraph \( F \) of \( D \), we prove that under certain conditions the oriented graph \( D \) can be imbeded into an oriented graph \( H \) such that all vertices of \( F \) have the same \( m \)-distance in \( H \).

**Lemma 3.13** Let \( D \) be a strong oriented graph and let \( F \) be a subdigraph of \( D \) with \( \max\{m_d(D, u, v) \mid u, v \in V(F)\} \leq 3 \). Then there exists an oriented graph \( H \) containing \( D \) as an induced subdigraph such that

1. if \( V(H) \neq V(D) \) then \( \max\{m_d_H(u, v) \mid u \in V(F), v \in V(H) - V(D)\} = 3 \), and
2. \( m_d_H(u) = m_d_H(v) \) for all \( u, v \in V(F) \).

**Proof.** Let \( m_\Delta(D) = \max\{m_d(x) \mid x \in V(F)\} \), \( m_\delta(D) = \min\{m_d(x) \mid x \in V(F)\} \) and \( n = m_\Delta(D) - m_\delta(D) \). If \( n = 0 \), then \( m_d(D, u) = m_d(D, v) \) for all \( u, v \in V(F) \). Let \( H = D \). Then the oriented graph \( H \) has the desired property. If \( n \geq 1 \), then we denote \( S_\Delta(D) = \{x \in V(F) \mid m_d(x) = m_\Delta(D)\} \). Define an oriented graph \( H_1 \) by
\[
V(H_1) = V(D) \cup \{w_1, x_1, y_1\}
\]

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and

\[ E(H_1) = E(D) \cup \{(w_1, x_1), (x_1, y_1), (y_1, w_1)\} \]
\[ \cup \{(w_1, z), (z, y_1) \mid z \in S_\Delta(D)\} \]
\[ \cup \{(x_1, z), (z, y_1) \mid z \in V(F) - S_\Delta(D)\} \]

(see Figure 3.15).

\[ \text{Figure 3.15} \]

Clearly, \( D \) is an induced subdigraph of \( H_1 \) and \( \max \{ m_{dH_1}(u, v) \mid u \in V(F), v \in V(H_1) - V(D)\} = 3 \). Since \( m_{dD}(z_1, z_2) \leq 3 \) for all \( z_1, z_2 \in V(F) \), it follows that \( m_{dH_1}(z, t) = m_{dD}(z, t) \) for all \( z \in V(F) \) and \( t \in V(D) \). In particular, \( m_{dH_1}(z_1, z_2) = m_{dD}(z_1, z_2) \leq 3 \) for all \( z_1, z_2 \in V(F) \). Therefore, for \( z \in S_\Delta(D) \),
\[ m_{dH_1}(z) = m_{dD}(z) + m_{dH_1}(z, w_1) + m_{dH_1}(z, x_1) + m_{dH_1}(z, y_1) \]
\[ = m_{dD}(z) + 2 + 3 + 2 = m_{dH_1}(z) + 7. \]

Similarly, \( m_{dH_1}(z) = m_{dD}(z) + 8 \) for \( z \in V(F) - S_\Delta(D) \). Define \( m_\Delta(H_1) = \max \{ m_{dH_1}(x) \mid x \in V(F) \} \) and \( m_\delta(H_1) = \min \{ m_{dH_1}(x) \mid x \in V(F) \} \). Then \( m_\Delta(H_1) = m_\Delta(D) + 7 \) and \( m_\delta(H_1) = m_\delta(D) + 8 \). Therefore, \( m_\Delta(H_1) - m_\delta(H_1) = (m_\Delta(D) + 7) - (m_\delta(D) + 8) = m_\Delta(D) - m_\delta(D) - 1 \). Let \( S_\Delta(H_1) = \{ x \in V(F) \mid m_{dH_1}(x) = m_\Delta(H_1) \} \).

We define an oriented graph \( H_2 \) by
\[ V(H_2) = V(H_1) \cup \{w_2, x_2, y_2\} \]

and

\[ E(H_2) = E(H_1) \cup \{(w_2, x_2), (x_2, y_2), (y_2, w_2)\} \]

\[ \cup \{(w_2, z), (z, y_2) \mid z \in S_\Delta(H_1)\} \]

\[ \cup \{(x_2, z), (z, y_2) \mid z \in V(F) - S_\Delta(H_1)\}. \]

By a similar argument, it follows that \( m_\Delta(H_2) - m_\mathcal{S}(H_2) = m_\Delta(D) - m_\mathcal{S}(D) - 2 \). We repeat this process \( n - 1 \) times. Let \( H = H_n \). Then \( m_\Delta(H) = m_\mathcal{S}(H) \), namely, \( m_\Delta(H)(u) = m_\mathcal{S}(H)(v) \) for all \( u, v \in V(F) \). In addition, by the construction of \( H_n \), it follows that \( D \) is an induced subdigraph of \( H \) and \( \max\{m_\Delta(H)(u, v) \mid u \in V(F), v \in V(H) - V(D)\} = 3 \). □

With the aid of Lemma 3.12 and 3.13, we now are ready to prove that for every pair of oriented graphs \( D_1 \) and \( D_2 \) there exists an oriented graph \( H \) such the m-center and m-median are isomorphic to \( D_1 \) and \( D_2 \), respectively. Furthermore, the m-distance between \( D_1 \) and \( D_2 \) in \( H \) can be arbitrarily prescribed.

**Theorem 3.14** Let \( D_1 \) and \( D_2 \) be oriented graphs. For all integers \( k \geq 2 \), there exists a strong oriented graph \( H \) such that \( \text{mC}(H) \equiv D_1 \), \( \text{mM}(H) \equiv D_2 \) and \( \text{md}_H(\text{mC}(H), \text{mM}(H)) = k \).

**Proof.** We first define an oriented graph \( H_0 \) by adding two new vertices \( u \) and \( v \) to \( D_2 \) and the arc \((u, v)\) together with the arcs joining all the vertices of \( D_2 \) to \( u \) and from \( v \). Clearly, \( H_0 \) is strong and \( \text{md}_H(0, x, y) \leq 3 \) for all \( x, y \in V(D_2) \). By Lemma 3.13, there exists an oriented graph \( H_1 \) containing \( H_0 \) as an induced subdigraph such that (i) if \( V(H_1) \neq V(H_0) \), then \( \max\{\text{md}_{H_1}(x, y) \mid x \in V(D_2), y \in V(H_1) - V(H_0)\} = 3 \), and (ii) \( \text{md}_{H_1}(x) = \text{md}_{H_1}(x) \) for all \( x, y \in V(D_2) \). Let \( n_1 = \max\{d_{H_1}(x, y) \mid x \in V(D_2), y \in V(H_1) - V(D_2)\} \) and \( n_2 = \max\{d_{H_1}(y, x) \mid x \in V(D_2), y \in V(H_1) - V(D_2)\} \).
V(D2)). Since $H_1$ is strong, it follows that $n_1, n_2 \geq 2$. By the construction of $H_1$, if $H_1 \neq H_0$, then $n_1 = n_2 = 3$. Further, if $H_1 = H_0$, then $n_1 = n_2 = 2$. Therefore, $n_1 = n_2$. Let $t = \max\{3, n_1\}$. We define the oriented graph $H_2$ by

$$V(H_2) = V(H_1) \cup V(D_1) \cup \{u_i \mid 0 \leq i \leq k - 1\} \cup \{v_i \mid 0 \leq i \leq k + t\}$$

and

$$E(H_2) = E(H_1) \cup E(D_1) \cup \{(u_0, v_0)\} \cup \{(x, u_0), (v_0, x) \mid x \in V(D_1)\} \cup \{(u_i, u_{i+1}) \mid 1 \leq i \leq k - 2\} \cup \{(v_i, v_{i+1}) \mid 1 \leq i \leq k + t - 1\} \cup \{(x, u_1), (x, v_1), (v_{k+t}, x) \mid x \in V(D_1)\} \cup \{(u_{k-1}, x) \mid x \in V(D_2)\} \cup \{(x, y) \mid x \in V(D_1), y \in V(D_2)\}$$

(see Figure 3.16).

![Diagram of $H_2$](image)

Figure 3.16

We now show that $mC(H_2) \equiv D_1$. Let $x \in V(D_1)$. First observe that

(i) $md_{H_2}(x, y) \leq 3$ for all $y \in V(D_1)$;
(ii) $md_{H_2}(x, u_0) = md_{H_2}(x, v_0) = 2$;
(iii) $md_{H_2}(x, u_i) \leq k$ for $1 \leq i \leq k - 1$;
(iv) $md_{H_2}(x, v_i) \leq k + t$ for $1 \leq i \leq k + t$; and
(v) $md_{H_2}(x, v_1) = k + t$.

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For \( y \in V(H_1) \), it follows that
\[
md_{H_2}(x, y) = \max \{ d_{H_2}(x, y), d_{H_2}(y, x) \}
\leq \max \{ d_{H_2}(x, z) + d_{H_2}(z, y), d_{H_2}(y, z) + d_{H_2}(z, x) \}
\leq \max \{ k + d_{H_2}(z, y), d_{H_2}(y, z) + 1 \}
\leq \max \{ k + d_{H_1}(z, y), 1 + d_{H_1}(y, z) \}, \text{where } z \in V(D_2).
\]
Observe that 
\[
d_{H_1}(z, y) \leq \max \{ d_{H_1}(z, y') \mid y' \in V(H_1) \} = \max \{ \max \{ d_{H_1}(z, y') \mid y' \in V(D_2) \}, \max \{ d_{H_1}(z, y') \mid y' \in V(H_1) - V(D_2) \} \} \leq \max \{ 3, n_1 \} = t.
\]
Similarly, 
\[
d_{H_1}(y, z) \leq \max \{ 3, n_2 \} = t.
\]
Therefore,
\[
md_{H_2}(x, y) \leq \max \{ k + t, 1 + t \} = k + t \text{ for all } y \in V(H_1).
\]
Hence, \( me_{H_2}(x) = k + t \), for all \( x \in V(D_1) \). It is obvious that \( me_{H_2}(x) > k + t \), for all \( x \in V(H_2) - V(D_1) \). Thus \( mC(H_2) \equiv D_1 \).

Since \( k \geq 2 \), it follows that 
\[
md_{H_2}(x, y) = md_{H_1}(x, y), \text{ for all } x \in V(D_2), y \in V(H_1).
\]
It follows also that 
\[
md_{H_2}(x, z) = md_{H_2}(y, z), \text{ for all } x, y \in V(D_2), z \in V(H_2) - V(H_1).
\]
Therefore,
\[
md_{H_2}(x) = md_{H_1}(x) + \sum_{z \in V(H_2) - V(H_1)} md_{H_2}(x, z)
= md_{H_1}(y) + \sum_{z \in V(H_2) - V(H_1)} md_{H_2}(y, z)
= md_{H_2}(y),
\]
for all \( x, y \in V(D_2) \). Hence, by Lemma 3.12, there exists an oriented graph \( H \) containing \( H_2 \) as an induced subdigraph such that \( mM(H) \equiv D_2 \). Further, by the construction of \( H \) in the proof of Lemma 3.12, it follows that \( md_H(x, y) = 2 \) for all \( x \in V(D_2), y \in V(H) - V(H_2) \). Therefore \( mC(H) = mC(H_2) \equiv D_1 \). It is obvious that \( md_H(mC(H), mM(H)) = k \). □

We now prove the other extreme case where the m-center and m-median of an
oriented graph can be overlaped on any common part of them.

**Theorem 3.15** Let \( D_1, D_2 \) be oriented graphs. Let \( K \) be a nonempty oriented graph isomorphic to an induced subgraph of both \( D_1 \) and \( D_2 \). Then there exists an oriented graph \( H \) such that \( mC(H) \equiv D_1, mM(H) \equiv D_2 \) and \( mC(H) \cap mM(H) \equiv K \).

**Proof.** Suppose \( V(D_1) = \{u_1, u_2, \ldots, u_{p_1}\} \) and \( V(D_2) = \{v_1, v_2, \ldots, v_{p_2}\} \). Without loss of generality, we assume that \( p(K) = k, \langle u_1, u_2, \ldots, u_k \rangle \equiv \langle v_1, v_2, \ldots, v_k \rangle \), and that \( u_j \rightarrow v_j \) (\( j = 1, 2, \ldots, k \)) is an isomorphism between \( \langle u_1, u_2, \ldots, u_k \rangle \) and \( \langle v_1, v_2, \ldots, v_k \rangle \). We first construct an oriented graph \( H_0 \) by identifying \( u_j \) and \( v_j \), and labeling the resulting vertex again by \( u_j \) for \( 1 \leq j \leq k \).

We now define an oriented graph \( H_1 \) by
\[
V(H_1) = V(H_0) \cup \{u, v\} \cup \{w_i, w_i' \mid 1 \leq i \leq 6\}
\]
and
\[
E(H_1) = E(H_0) \cup \{(u, v)\} \cup \{(w_i, w_{i+1}), (w_i', w_{i+1}') \mid 1 \leq i \leq 5\}
\]
\[
\cup \{(x, u), (v, x) \mid x \in V(H_0)\}
\]
\[
\cup \{(u_i, w_1), (w_6, u_i), (u_i, w_1'), (w_6, u_i) \mid 1 \leq i \leq p_1\}
\]
(see Figure 3.17).

It is clear that \( m\text{-rad } H_1 = 6 \) and \( mC(H_1) \equiv D_1 \). By Lemma 3.13, there exists an oriented graph \( H_2 \) containing \( H_1 \) as a subdigraph such that (i) If \( V(H_2) \neq V(H_1) \), then \( \max\{m\text{d}_{H_2}(x, y) \mid x \in V(D_2), y \in V(H_2) - V(H_1)\} = 3 \) and (ii) \( m\text{d}_{H_2}(x) = m\text{d}_{H_2}(y) \) for all \( x, y \in V(D_2) \). Thus \( m\text{d}_{H_2}(x, y) \leq 6 \) for \( x \in V(D_1), y \in V(H_2) - V(H_1) \), from which it is easy to see that \( m\text{-rad } H_2 = m\text{-rad } H_1 = 6 \) and \( mC(H_2) = mC(H_1) \equiv D_1 \). By Lemma 3.12, there exists an oriented graph \( H \) containing \( H_2 \) as an induced subdigraph such that \( mM(H) \equiv D_2 \). The construction of

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$H$ in the proof of Lemma 3.12 implies that $md_H(x, y) = 2$ for $x \in V(D_2), y \in V(H) - V(H_2)$. Therefore $mC(H) = mC(H_2) \equiv D_1$. $\square$
CHAPTER IV

THE APPENDAGE NUMBER OF AN ORIENTED GRAPH

The appendage number of an oriented graph is defined in this chapter. The principle result is to characterize those oriented acyclic graphs with appendage number 2.

4.1 Introduction

Buckley, Miller and Slater [5] defined, for each graph G, the parameter

\[ A(G) = \min\{ p(H) - p(G) \mid C(H) = G \}. \]

We refer to the number \( A(G) \) as the appendage number of G. Recall that every graph G is the center of some graph H whose order exceeds the order of G by at most 4. Thus \( A(G) \leq 4 \) for all graphs G. A graph G with \( A(G) = 0 \) is therefore a self-centered graph. Buckley, Miller and Slater [5] characterized those trees with appendage number 2 and proved that no tree has appendage number 3.

We have seen in Chapter III that for every oriented graph D there exists an oriented graph H such that the m-center of H is isomorphic to D. Therefore the parameter \( A(G) \) of a graph has a counterpart with respect to m-distance in digraphs. For each oriented graph D, we define the appendage number \( A(D) \) of D to be the minimum number of vertices required to add to D to produce an oriented graph H having D as its m-center, that is, \( A(D) = \min\{ p(H) - p(D) \mid mC(H) = D \} \).

If \( A(D) \neq 0 \) (that is \( mC(D) \neq D \)), then \( V(H) - V(D) \) contains all vertices having maximum m-eccentricity. Since in any oriented graph there are at least two vertices having maximum m-eccentricity, it follows that \( A(D) \neq 1 \) for all oriented graphs.
graphs. As an immediate consequence of Theorem 3.8, we have the following upper
bound for the appendage number of an oriented graph.

**Theorem 4.1** If $D$ is an oriented graph, then $A(D) \leq 4$.

It is interesting to note that if an oriented graph $D$ has appendage number then
equal to 2 or 3 then the $m$-diameter and $m$-radius of every oriented graph $H$ of order
$p(D) + A(D)$ that contains $D$ as its $m$-center are very close.

**Lemma 4.2** Let $D$ be an oriented graph with $2 \leq A(D) \leq 3$. If $H$ is an oriented
graph with $mC(H) \equiv D$ and $p(H) = p(D) + A(D)$, then $m$-diam $H = m$-rad $H + 1$.

**Proof.** Since $p(H) = p(D) + A(D) \geq p(D) + 2$ and $mC(H) \equiv D$, it follows that
$m$-diam$(H) \geq m$-rad $H + 1$ and $V(mP(H)) \subset V(H) - V(D)$. Let $v \in mP(H)$. Suppose that $me(v) = \overrightarrow{d}(v, w)$ and $v = v_0, v_1, ..., v_{me(v)} = w$ is a shortest $v$-$w$
path. Then $me(w) \geq md(v, w) \geq \overrightarrow{d}(v, w) = me(v) = m$-diam $H$, from which it
follows that $w \in mP(H)$. Since $v, w \in V(H) - V(D)$ and $|V(H) - V(D)| = A(D) \leq 3$, it follows that at most one $v_i$ ($1 \leq i \leq me(v) - 1$) is in $V(H) - V(D)$. Since
$me(v) = m$-diam $H \geq m$-rad $H + 1 \geq 3$, either $v_1 \in V(D)$ or $v_{me(v)} - 1 \in V(D)$. We
assume, without loss of generality, that $v_1 \in V(D)$. Then $m$-rad $H = me_H(v_1) \geq \overrightarrow{d}(v_1, w) = \overrightarrow{d}(v, w) - 1 = m$-diam $H - 1$, that is, $m$-diam $H \leq m$-rad $H + 1$.

Therefore, $m$-diam $H = m$-rad $H + 1$. \(\square\)

The next lemma plays a key role in this Chapter. With the aid of this result, we
will be able to show that the upper bound for appendage number given in Theorem 4.1
is best possible. We will employ it in the next section as well to characterize those
oriented acyclic graphs $D$ with appendage number 2.
Lemma 4.3 Let $D$ be an oriented graph with $A(D) = 2$ and let $H$ be an oriented graph with $mC(H) \cong D$ and $p(H) = p(D) + A(D)$. If $\bar{d}_H(v, w) = m$-diam $H$ then for all $x \in N^+(v)$ and $y \in N^-(w)$, any shortest $x$-$y$ path lies entirely in $D$.

Proof. Observe that $m$-diam $H = \max\{m(x) \mid x \in V(H)\} = \text{md}(y, z)$, where $y, z \in V(H) - V(D)$. Since $A(D) = 2$ and $\text{md}(v, w) = \text{m-diam} H$, it follows that $V(H) = V(D) \cup \{v, w\}$ and $\text{me}_H(u) = \text{m-rad} H < \text{m-diam} H$ for all $u \in V(D)$. Suppose, to the contrary, that there exist vertices $x \in N^+(v)$ and $y \in N^-(w)$ such that a shortest $x$-$y$ path $P$ contains $v$ or $w$. If $v$ lies on $P$, then $\bar{d}(x, y) = \bar{d}(x, v) + \bar{d}(v, y)$. Since $(y, w) \in E(D)$ and $(v, x) \in E(D)$, it follows that $\bar{d}(y, w) = 1$ and $\bar{d}(x, v) > 1$. Therefore,

$$m \text{-diam } H = \bar{d}(v, w) \leq \bar{d}(v, y) + \bar{d}(y, w) \leq \text{md}(x, y) \leq m \text{e}_H(x),$$

which contradicts the fact that $\text{me}(x) < \text{m-diam } H$. The proof is similar if $w$ lies on a shortest $x$-$y$ path in $H$. □

The next result shows that the upper bound for $A(D)$ in Theorem 4.1 is sharp.

Theorem 4.4 Let $n \geq 2$ be an integer. Then $A(K_n^*) = 4$.

Proof. It is clear that $A(K_n^*) \neq 0$. Since $K_n^*$ is acyclic, by Lemma 4.3, $A(K_n^*) \geq 3$. Since $A(K_n^*) \leq 4$, it suffices to prove that $A(K_n^*) \neq 3$. Suppose, to the contrary, that $A(D) = 3$ where $D \equiv K_n^*$. Let $H$ be an oriented graph with $V(H) = V(D) \cup \{u, v, w\}$ and $mC(H) \equiv D$. We assume, without loss of generality, that $\bar{d}_H(u, v) = m$-diam $H$ and $P$ is a shortest $u$-$v$ path in $H$. Since $E(D) = \emptyset$ and
$|V(H) - V(D)| = 3$, $P$ contains at most two vertices of $D$. On the other hand, since $m$-$\text{diam} \ H \geq 3$, $P$ contains at least one vertex of $D$. Therefore, $\tilde{d}_H(u, v) = 3$ or 4.

We consider the following cases:

**Case 1. Assume that** $\tilde{d}_H(u, v) = 3$ **and** $P: u, x, w, v$ **for some** $x \in V(D)$.

Let $y \in V(D) - \{x\}$. Since $m_e(z) = 2$ for all $z \in V(D)$, it follows that $\tilde{d}(x, y) = 2$. Noting that $\tilde{d}(u, v) = 3$, it follows that $(w, y) \in E(H)$. However, this implies that $\tilde{d}(y, w) \geq 3$, a contradiction.

**Case 2. Assume that** $\tilde{d}_H(u, v) = 3$ **and** $P: u, w, x, v$ **for some** $x \in V(D)$.

Let $y \in V(D) - \{x\}$. Since $H$ contains a $w$-$y$ path of length at most 2 and $(w, u), (w, v) \not\in E(H)$, it follows that $(w, y) \in E(H)$. Similarly, since $H$ contains a $y$-$v$ path of length at most 2, it follows that $(y, v) \in E(H)$. Furthermore, since $\tilde{d}_H(u, v) = 3$, it follows that $(u, y) \not\in E(D)$. Therefore, $\tilde{d}_H(x, y) \geq 3$, a contradiction.

**Case 3. Assume that** $\tilde{d}_H(u, v) = 4$ **and** $P: u, x, w, y, v$ **for some vertices** $x, y \in V(D)$.

It follows that $(v, u) \not\in E(H)$; for otherwise $m$-$\text{diam} \ H = m_e(w) = \max\{md(w, u), md(w, v)\} = 3$, a contradiction. We claim that $\tilde{d}_H(v, u) = 2$. Suppose to the contrary that $\tilde{d}_H(v, u) \geq 3$. Then $w$ lies on every shortest $v$-$u$ path. Furthermore, since $\tilde{d}_H(v, u) \leq 4$, it follows that $\tilde{d}_H(w, u) \leq 3$ and $\tilde{d}_H(v, w) \leq 3$. By noting that $\tilde{d}_H(u, w) = 2$ and $\tilde{d}_H(w, v) = 2$, we conclude that $m$-$\text{diam} \ H = m_e(w) = \max\{md(w, u), md(w, v)\} \leq 3$, a contradiction. Therefore, there exists a vertex $z \in V(D) - \{x, y\}$ such that $(v, z), (z, u) \in E(H)$. Since $(x, v) \not\in E(H)$ and $\tilde{d}_H(x, z) \leq 3$, we have $(w, z) \in E(H)$. Therefore, $\tilde{d}_H(z, y) = 4$, a contradiction. □

Clearly, there exist oriented graphs with appendage number 0. We will see later that there exist oriented trees with appendage number 2 or 3.
4.2 A Characterization of Oriented Acyclic Graphs with Appendage Number 2

We now turn to oriented acyclic graphs with appendage number 2. Before characterizing these oriented graphs, we need several lemmas. The next result gives information about an oriented graph whose m-center is an oriented acyclic graph with appendage number 2.

**Lemma 4.5** Let D be an acyclic oriented graph with A(D) = 2. Further, let H be an oriented supergraph of D with V(H) = V(D) ∪ {u, v} and mC(H) = D. If \( \tilde{d}(u, v) = \text{m-diam } H \), then \( N^+(u) = \{ w \mid w \in V(D) \text{ and } id_D(w) = 0 \} \) and \( N^-(v) = \{ w \mid w \in V(D) \text{ and } od_D(w) = 0 \} \).

**Proof.** Since \( mC(H) = D \), it follows that \( me_H(u) = me_H(v) = \text{m-diam } H \) and \( me_H(w) = \text{m-rad } H \) for all \( w \in V(D) \). Furthermore, by Lemma 4.3, \( \text{m-rad } H = \text{m-diam } H - 1 \).

First we prove that \( \{ w \mid w \in V(D) \text{ and } id_D(w) = 0 \} \subseteq N^+(u) \). Suppose \( w \in V(D) \), \( id_D(w) = 0 \) and \( (u, w) \notin E(H) \). Since \( id_H(w) \neq 0 \), it follows that \( (v, w) \in E(D) \) and that \( v \) lies on every shortest \( u-w \) path. Therefore, \( \tilde{d}(u, w) = \tilde{d}(u, v) + \tilde{d}(v, w) = \tilde{d}(u, v) + 1 \), so that \( me_H(w) \geq \tilde{d}(u, w) = \tilde{d}(u, v) + 1 = \text{m-diam } H + 1 \), a contradiction. Hence \( \{ w \mid w \in V(D) \text{ and } id_D(w) = 0 \} \subseteq N^+(u) \).

Next we show that \( N^+(u) \subseteq \{ w \mid w \in V(D) \text{ and } id_D(w) = 0 \} \). Suppose, to the contrary, that \( (u, w) \in E(H) \) and \( id_D(w) \neq 0 \), say \( (w', w) \in E(D) \). Since D is an acyclic digraph, there is no \( w-w' \) path in D. Thus, by Lemma 4.3, \( (w', v) \notin E(H) \). Let \( P \) be a shortest \( w'-v \) path in H. Then, \( u \notin V(P) \); for otherwise,

\[
\tilde{d}_H(w', v) = \tilde{d}_H(w', u) + \tilde{d}_H(u, v) \\
\geq 1 + \tilde{d}_H(u, v) = 1 + \text{m-diam } H,
\]
a contradiction. We show that $w$ lies on $P$. Suppose $w \notin V(P)$ and $x$ is a vertex on the path $P$ such that $(x, v) \in E(D)$. By Lemma 4.3, there is a shortest $w-x$ path $Q$ in $D$. Let $H_1 = P_1 \cup Q + (w', w)$, where $P_1$ is the $w'-x$ subpath in $P$. Since $H_1 \subseteq D$, it follows that the underlying graph of $D$ contains a cycle, a contradiction to the fact that $D$ is acyclic. Therefore, $w$ lies on every shortest $w'-v$ path. It follows that

$$d_H(w', v) = d_H(w', w) + d_H(w, v)$$

$$\geq 1 + d_H(u, v) - 1$$

$$= d_H(u, v) = m \text{-diam } H.$$  

Thus, $mH(w') \geq d_H(w', v) \geq m \text{-diam } H$, a contradiction. Hence

$$N^+(u) = \{w \mid w \in V(D) \text{ and } d_D(w) = 0\}.$$  

Now we prove that $N^-(v) = \{w \mid w \in V(D) \text{ and } o_D(w) = 0\}$. By Theorem 3.1, $mC(H) = D$, $d_H(v, u) = d_H(u, v) = m \text{-diam } H = m \text{-diam } H$, and $o_D(w) = 0$ if and only if $d_D(w) = 0$. Since $N^-(v; H) = N^+(v; H)$,

$$N^-(v) = N^-(v; H) = N^+(v; H)$$

$$= \{w \mid w \in V(D) \text{ and } d_D(w) = 0\}$$

$$= \{w \mid w \in V(D) \text{ and } o_D(w) = 0\}. \square$$

Combining Lemmas 4.3 and 4.5, we obtain the following lemma.

**Lemma 4.6** Let $D$ be an oriented acyclic graph with $A(D) = 2$. If $d(u) = 0$ and $o(v)$, then there exists a unique directed $u-v$ path in $D$.

The following result is an immediate consequence of Lemma 4.6.

**Corollary 4.7** If $D$ is an oriented acyclic graph with $A(D) = 2$, then $D$ is an oriented tree.
Our next result gives more structure about oriented trees with appendage number 2.

**Lemma 4.8** Let $T$ be an oriented tree with $A(T) = 2$. Then there exists an induced directed path $P: v_0, v_1, \ldots, v_{n-1}, v_n$ such that $id_T(v_i) = od_T(v_i) = 1$ for all $i$ ($1 \leq i \leq n-1$). Further, every path from a vertex of indegree 0 to a vertex of outdegree 0 contains $P$ as a subpath.

**Proof.** Define

$$V_1 = \{ v | v \in V(T) \text{ and } id(v) = 0 \}$$

and

$$V_0 = \{ v | v \in V(T) \text{ and } od(v) = 0 \}.$$

Define $w_1$ as any element $v \in V(T)$ for which $\sum_{x \in V_1} d(x, v)$ is a minimum.

By Lemma 4.6, $w_1$ is well defined. Similarly, $w_2$ is defined as any element $v \in V(T)$ for which $\sum_{x \in V_0} d(v, x)$ is minimum.

We claim that $id(w_1) \neq 1$; for otherwise, suppose $w$ is the only vertex such that $(w, w_1) \in E(T)$. Thus $w$ lies on every path from a vertex in $V_1$ to $w_1$. Therefore

$$\sum_{u \in V_1} d(u, w) < \sum_{u \in V_1} d(u, w_1),$$

which contradicts the choice of $w_1$. Similarly, it follows that $od(w_2) \neq 1$. We next claim that $w_1$ and $w_2$ lie on every path connecting a vertex of $V_1$ to a vertex of $V_0$. To prove the claim, we consider the following cases.

**Case 1.** Assume that $id(w_1) = od(w_2) = 0$. 

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In this case, $V_1 = \{w_1\}$ and $V_0 = \{w_2\}$. Then the claim follows immediately from Lemma 4.6.

**Case 2. Assume that** $\text{id}(w_1) = 0$ and $\text{od}(w_2) \geq 2$.

It is clear that $V_i = \{w_1\}$. Since $\text{od}(w_2) \geq 2$, it follows that $|V_0| \geq 2$. Let $x_1$ and $x_2$ be two distinct vertices in $V_0$. By Lemma 4.6, there exist $w_1-x_1$ and $w_1-x_2$ paths, say $P_1$ and $P_2$, in $T$. Since $\tilde{d}(w_2,x_1) < \infty$ and $\tilde{d}(w_2,x_2) < \infty$, there exists a $w_2-x_1$ path $P_3$ and a $w_2-x_2$ path $P_4$ in $T$. Let

$$H = P_1 \cup P_2 \cup P_3 \cup P_4.$$  

Since $H$ is a subdigraph of an oriented tree, it follows that the underlying graph of $H$ is acyclic. Therefore, $P_3 \subseteq P_1$ and $P_4 \subseteq P_2$. Since $x_1$ and $x_2$ are arbitrary vertices in $V_0$ and $V_1 = \{w_1\}$, it follows that $w_1$ and $w_2$ lie on every path from $V_i$ to $V_0$.

**Case 3. Assume that** $\text{id}(w_1) \geq 2$ and $\text{od}(w_2) = 0$.

The proof is similar to Case 2. It will be omitted.

**Case 4. Assume that** $\text{id}(w_1) \geq 2$ and $\text{od}(w_2) \geq 2$.

Let $x_1, x_2$, and $y_1, y_2$ be distinct vertices in $V_1$ and $V_0$, respectively. By Lemma 4.6, there exists an $x_1-y_1$ path $P_1$ and an $x_2-y_2$ path $P_2$. By the definition of $w_1$ and $w_2$, there exists a $w_2-y_1$ path $P_3$, a $w_2-y_2$ path $P_4$, an $x_1-w_1$ path $P_5$ and an $x_2-w_1$ path $P_6$, all of which are in $T$.

Let

$$H = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6.$$  

Since the underlying graph of $T$ is acyclic, the underlying graph of $H$ is also acyclic. Therefore, $w_1$ and $w_2$ lie on both $P_1$ and $P_2$. Observing that $x_1$ and $y_1$ are arbitrary vertices in $V_1$ and $V_0$ respectively, we see that $w_1$ and $w_2$ lie on every path from $V_i$ to $V_0$, which completes the proof of the claim.
Let $P: w_1 = v_0, v_1, ..., v_m = w_2$ be the $w_1$-$w_2$ path in $T$. We now show that either $m = 0$ (that is $w_1 = w_2$), or that $m \geq 1$, $\text{od}(v_i) = 1\ (0 \leq i \leq m-1)$, and $\text{id}(v_i) = 1\ (1 \leq i \leq m)$. Suppose $m \geq 1$ and there exists a vertex $v_k\ (0 \leq k \leq m-1)$ such that $\text{od}(v_k) \geq 2$. Choose a vertex $x$ with $\text{od}(x) = 0$ such that $x$ and $w_2$ lie on different branches at $v_k$. Then, there is no $w_2$-$x$ path in $D$, that is $\overrightarrow{d}(w_2, x) = \infty$, which is a contradiction to the choice of $w_2$. Therefore, $\text{od}(v_i) = 1$ for all $i\ (0 \leq i \leq m-1)$. A similar argument gives $\text{id}(v_i) = 1\ 1 \leq i \leq m$, which completes the proof. □

As our next lemma shows, the directed distance from a vertex of in-degree 0 to a vertex of out-degree 0 is a constant in any oriented tree of appendage number 2.

**Lemma 4.9** Let $T$ be an oriented tree with appendage number 2. If $V_i = \{ v \mid v \in V(T), \text{id}(v) = 0 \}$ and $V_o = \{ v \mid v \in V(T), \text{od}(v) = 0 \}$, then $\overrightarrow{d}(u, v) = \overrightarrow{d}(u', v')$ for all $u, u' \in V_i$ and $v, v' \in V_o$.

**Proof.** Since $A(T) = 2$, there exists an oriented graph $D$ with $mC(D) = T$ and $V(D) = V(T) \cup \{ x, y \}$. Assume, without loss of generality, that $\overrightarrow{d}(x, y) = m\text{-diam } D$. By Lemma 4.5, it follows that $N^+(x) = V_i$ and $N^-(y) = V_o$. Therefore, for all $u \in V_i$ and $v \in V_o$, $\overrightarrow{d}(u, v) \geq \overrightarrow{d}(x, y) - 2 = m\text{-diam } D - 2$. Since $\overrightarrow{d}(u, v) \leq \text{med}(u) \leq m\text{-diam } D - 1$ for all $u \in V_i$, there exists $u_0 \in V_i$ and $v_0 \in V_o$ such that $\overrightarrow{d}(u_0, v_0) = m\text{-diam } D - 2$. We now prove that $\overrightarrow{d}(u, v) = m\text{-diam } D - 2$ for all $u \in V_i$ and $v \in V_o$. Suppose that $\overrightarrow{d}(u', v') > m\text{-diam } D - 2$ for some $u' \in V_i$ and $v' \in V_o$. Let $w_1$ and $w_2$ be the vertices in Lemma 4.8. Then $w_1$ and $w_2$ lie on every path from $V_i$ to $V_o$. Therefore

$$\overrightarrow{d}(u', v') = \overrightarrow{d}(u', w_1) + \overrightarrow{d}(w_1, w_2) + \overrightarrow{d}(w_2, v')$$

and
\[ d(u_0, v_0) = d(u_0, w_1) + d(w_1, w_2) + d(w_2, v_0). \]

By the choice of \( u_0 \) and \( v_0 \), it follows that \( d(u', w_1) \geq d(u_0, w_1) \) and \( d(w_2, v') \geq d(w_2, v_0) \). Since \( d(u', v) > m-diam D - 2 = d(u_0, v_0) \), it follows that either \( d(u', w_1) > d(u_0, w_1) \) or \( d(w_2, v') > d(w_2, v_0) \). If \( d(u', w_1) > d(u_0, w_1) \), then

\[
me(u') \geq d(u', y) = d(u', w_1) + d(w_1, y) \\
\geq d(u_0, w_1) + 1 + d(w_1, y) \\
= d(u_0, y) + 1 = m-diam D,
\]
a contradiction. Similarly, if \( d(w_2, v') > d(w_2, v_0) \), then

\[
me(v') \geq d(x, v') = d(x, w_2) + d(w_2, v') \\
\geq d(x, w_2) + d(w_2, v_0) + 1 \\
= d(x, v_0) + 1 = m-diam D,
\]
again a contradiction. The proof is complete. \( \square \)

Combining previous results, we have the following lemma which gives an indication of the structure of an oriented tree with appendage number 2.

**Theorem 4.10** If \( T \) is an oriented tree with \( A(T) = 2 \), then \( V(T) \) can be partitioned as \( V(T) = S_1 \cup S_2 \cup \ldots \cup S_n \) such that

1. \( n(T) = n = \min \{ d(u, v) \mid \text{id}(u) = 0, \text{od}(v) = 0 \} + 1; \)
2. \( S_1 = \{ v \mid v \in V(T), \text{id}(v) = 0 \} \) and \( S_n = \{ v \mid v \in V(T), \text{od}(v) = 0 \} \);
3. \( d(u, v) = n - 1 \) for all \( u \in S_1 \) and \( v \in S_n \);
4. \( E(T) \subseteq \bigcup_{i=1}^{n-1} (S_i, S_{i+1}), \) and \( (S_i, S_{i+1}) \neq \emptyset \) for all \( 0 \leq i \leq n - 1; \)
5. there exist integers \( k(T) = k \) and \( m(T) = m \) with \( 1 \leq k \leq m \leq n \) such that \( |S_i| = 1 \) if and only if \( k \leq i \leq m; \) and
6. \( \text{od}(v) = 1 \) for all \( v \in \bigcup_{i=1}^{k-1} S_i, \) and \( \text{id}(v) = 1 \) for all \( v \in \bigcup_{i=m+1}^{n} S_i. \)
Proof. Let \( n = \min\{ \overline{d}(u, v) \mid \text{id}(u) = 0, \text{od}(v) = 0 \} + 1 \). Clearly \( n \) is finite. We define \( S_1 = \{ v \mid v \in V(T), \text{id}(v) = 0 \} \) and \( S_n = \{ v \mid v \in V(T), \text{od}(v) = 0 \} \). Then (1) and (2) are satisfied. By Lemma 4.9, (3) holds. For \( 2 \leq i \leq n - 1 \), we define

\[
S_i = \{ v \mid v \in V(T), \text{there exists } u \in S_1 \text{ such that } \overline{d}(u, v) = i - 1 \}.
\]

It is clear that \( S_i \neq \emptyset \) (1 \( \leq i \leq n \)) and \( \bigcup_{i=1}^{n} S_i = V(T) \). To prove \( \{ S_1, S_2, \ldots, S_n \} \) is a partition of \( V(T) \), we next show that \( S_i \cap S_j = \emptyset \) for \( 1 \leq i < j \leq n \). Since \( \rho(T) \geq 2 \), \( S_1 \cap S_n = \emptyset \). By the definition of \( S_i \), it follows that \( S_1 \cap S_i = \emptyset \) for \( 2 \leq i \leq n - 1 \). By (3), it follows that \( S_i \cap S_n = \emptyset \) for \( 2 \leq i \leq n - 1 \). We now prove by contradiction that \( S_i \cap S_j = \emptyset \) for \( 2 \leq i < j \leq n - 1 \). Suppose \( S_i \cap S_j \neq \emptyset \) for some \( 2 \leq i < j \leq n - 1 \). Let \( w \in S_i \cap S_j \). Then there exist distinct vertices \( v, v' \in S_1 \) such that \( \overline{d}(v, w) = i - 1 \) and \( \overline{d}(v', w) = j - 1 \). Let \( x \in S_n \).

By Lemma 4.6 again, there are \( v-x \) and \( v'-x \) paths, say \( P \) and \( Q \) respectively, in \( T \). Assume \( R \) and \( S \) are shortest \( v-w \) and \( v'-w \) paths respectively. Then \( P \cup Q \cup R \cup S \) is a subdigraph of \( T \). Since the underlying graph of \( T \) is acyclic, it must be true that \( R \subset P \) and \( S \subset Q \). Note that between any two vertices of \( T \), there exists at most one directed path. Therefore,

\[
\overline{d}(v, x) = \overline{d}(v, w) + \overline{d}(w, x) = i - 1 + \overline{d}(w, x) = j - 1 + \overline{d}(w, x) = \overline{d}(v', w) + \overline{d}(w, x) = \overline{d}(v', x),
\]

which contradicts (3). Therefore, \( \{ S_1, S_2, \ldots, S_n \} \) is a partition of \( V(T) \).

We now verify (4). Since \( S_i \cap S_j = \emptyset \) (i \( \neq j \)), it follows that \( (S_i, S_j) = \emptyset \) (i \( > j \)) where \( (S_i, S_j) \) denotes the set of all arcs from the vertices of \( S_i \) to the vertices of \( S_j \). For \( 2 \leq i + 1 < j \leq n \), since \( S_{i+1} \cap S_j = \emptyset \), it follows from the definition that \( (S_i, S_j) = \emptyset \). Since the underlying graph of \( T \) is connected, \( (S_i, S_{i+1}) \neq \emptyset \) for all \( 0 \leq i \leq n - 1 \). Thus, to prove \( E(T) \subseteq \bigcup_{i=1}^{n-1} (S_i, S_{i+1}) \), it suffices to show that \( S_i \)
is an independent set for all $1 \leq i \leq n$. First, it is clear that $S_1$ and $S_n$ are independent sets. Suppose $u, w \in S_i$ ($2 \leq i \leq n-1$) such that $(u, w) \in E(T)$. Let $u', w' \in S_1$ such that $d(u', u) = d(w', w) = i - 1$. Let $x \in S_n$. Then by Lemma 4.6, there exist unique $u'\rightarrow x$ and $w'\rightarrow x$ paths in $T$. Since the underlying graph of $T$ is a tree, it follows that $w$ lies on the $w'\rightarrow x$ path, and $u, w$ both lie on the $u'\rightarrow x$ path. Therefore,

$$
\begin{align*}
\tilde{d}(u', x) &= \tilde{d}(u', u) + \tilde{d}(u, w) + \tilde{d}(w, x) \\
&= i - 1 + 1 + \tilde{d}(w, x) \\
&= \tilde{d}(w', w) + \tilde{d}(w, x) + 1 \\
&= \tilde{d}(w', x) + 1,
\end{align*}
$$

which contradicts (3) again. Thus (4) holds. Observe that (5) is an immediate consequence of Lemma 4.8. To show (6), we observe that if $\text{od}(v) > 1$ for some $v \in \bigcup_{i=1}^{k-1} S_i$, then the underlying graph of $T$ contains a cycle, which is a contradiction. Therefore, $\text{od}(v) = 1$ for all $v \in \bigcup_{i=1}^{k-1} S_i$. Similarly, $\text{id}(v) = 1$ for all $v \in \bigcup_{i=m+1}^{n} S_i$. □

For an oriented tree $T$ of appendage number 2, we call a partition of $V(T)$ that satisfies Theorem 4.10 a standard partition of $T$. The numbers $k(T)$, $m(T)$, and $n(T)$ described in the statement of Theorem 4.10 are referred as the parameters associated with the standard partition of $T$, and the sets $S_i(T)$, $i = 1, 2, \ldots, n(T)$, are called the standard partition sets of $T$. We will use the more convenient notation $k$, $m$, $n$, and $S_i$, if there is no confusion. We denote by $T(k, m, n)$ the family of oriented trees whose standard partition parameters are $k$, $m$, and $n$ respectively.

In the remainder of this Chapter, we denote by $T$ an oriented tree in $T(k, m, n)$ with appendage number 2. Denote by $D$ an oriented supergraph of $T$ with $V(D) = V(T) \cup \{u, w\}$ and m-diam $D = \tilde{d}_D(u, w)$. Let $\{S_1, S_2, \ldots, S_n\}$.
be a standard partition of $T$. Since $|S_i| = 1$, where $i = k, \ldots, m$, we assume that $S_i = \{v_i\}$ where $k \leq i \leq m$. By Lemma 4.5, $N^+(u) = S_1$, $N^-(w) = S_n$ and $m\text{-diam } D = m\text{-rad } D + 1 = n + 1$ (see Figure 4.1).

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure4.1.png}
\caption{Figure 4.1}
\end{figure}

By Theorem 4.10, if $A(T) = 2$ then $T \in \mathcal{T}(k, m, n)$ for some integers $k$, $m$ and $n$ with $1 \leq k \leq m \leq n$. It is important to note that the converse of this statement is not true. Our next result characterizes all oriented trees in $\mathcal{T}(k, m, n)$ with appendage number 2 and $\max\{k - 1, n - m\} \leq 2$. We denote by $\mathcal{T}_1'$ the union of the families of oriented trees given in Figure 4.2. Let $\mathcal{T}_1$ the set of all oriented trees in $\mathcal{T}(k, m, n)$ with $\max\{k - 1, n - m\} \leq 2$ that are not in $\mathcal{T}_1'$.

**Theorem 4.11** Let $T \in \mathcal{T}(k, m, n)$ where $\max\{k - 1, n - m\} \leq 2$. Then $A(T) = 2$ if and only if $T \in \mathcal{T}_1$.

**Proof.** To prove the necessity, we first show that $A(T) \geq 3$ for all $T \in \mathcal{T}_1'$. Suppose $T \in \mathcal{T}(1, 1, 2)$. Clearly $A(T) \geq 2$. If $A(T) = 2$, then the structure of $D$ (shown in Figure 4.1) becomes that indicated in Figure 4.3. Let $v$ and $v'$ be two distinct vertices in $S_2$. Then $\overline{d}(v, v') \geq 3 = m\text{-diam } D$, a contradiction. Therefore
$T(1, 1, 2)$: \hspace{1cm} $T(2, 2, 3)$: \hspace{1cm} $T(2, 2, 2)$: \\
$T(1, 1, 3)$: \hspace{1cm} $T(3, 3, 3)$:

Figure 4.2

\[ A(T) \geq 3 \] for all $T \in \mathcal{T}(1, 1, 2)$. Similarly, it follows that $A(T) \geq 3$ for all $T \in \mathcal{T}(2, 2, 2)$.

D:

\[ \begin{array}{c}
\text{u} \\
\text{v}_1 \\
\text{S}_2 \\
\text{v} \\
\text{w}
\end{array} \]

Figure 4.3

Suppose $T \in \mathcal{T}(1, 1, 3)$ with $A(T) = 2$. It follows that any oriented supper graph $D$ associated with $T$ has the structure in Figure 4.4. Choose $v \in S_2$ and $v' \in S_3$ such that $(v, v') \notin E(T)$. Then $\delta(v, v') \geq 4 = m\text{-diam } D$, a contradiction. A similar argument gives us that $A(T) \geq 3$ for $T \in \mathcal{T}(3, 3, 3)$. 

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We now prove that $A(T) \geq 3$ for every $T \in \mathcal{I}(2, 2, 3)$. Suppose $A(T) = 2$ for some $T \in \mathcal{I}(2, 2, 3)$. Then the superdigraph $D$ of $T$ has the structure in Figure 4.5. Observe that $m\text{-diam } D = 4$ and $m\text{-rad } D = 3$. Let $v_1$ and $v_1'$ be distinct vertices in $S_1$. Since $\overline{d}(v_1, v_1') \leq m\text{-rad } D = 3$, it follows that $(v_2, u) \in E(D)$. By considering the $m$-distance between two vertices in $S_3$, we see that $(w, v_2) \in E(D)$. However, this implies that $m\text{e}(v_2) = 2 < 3 = m\text{-rad } D$, a contradiction. This completes the proof of the necessity.
We now prove the sufficiency, namely $A(T) = 2$ for all $T \in \mathcal{T}_1$. Consider the following three cases.

**Case 1. Assume that** $n = 2$.

Since $\max\{k-1, n-m\} \leq 2$ and $1 \leq k \leq m \leq n$, it follows that $k = 1$ and $m = n = 2$. Therefore, $T$ is the oriented $K_2$, so $A(T) = 2$ follows immediately from the construction in Figure 4.6.

$$D:$$

![Figure 4.6](image)

**Case 2. Assume that** $n = 3$.

In this case, it follows that $T \in \mathcal{T}(1, 2, 3) \cup \mathcal{T}(1, 3, 3) \cup \mathcal{T}(2, 3, 3)$. Suppose that $T \in \mathcal{T}(1, 2, 3)$. Let $D$ be the oriented graph in Figure 4.7. It follows that $m \subset (D) \equiv T$. Thus $A(T) = 2$. Observe that if $T \in \mathcal{T}(2, 2, 3)$ then $\bar{T} \in \mathcal{T}(1, 2, 3)$. Therefore, $A(T) = A(\bar{T}) = 2$. The construction of $D$ for $T \in \mathcal{T}(1, 3, 3)$ is a special case in Figure 4.7, where $|S_3| = 1$. Therefore the construction of Figure 4.7 is, in fact, the situation when $T \in \mathcal{T}(1, 3, 3)$. Hence $A(T) = 2$ for $T \in \mathcal{T}(1, 3, 3)$.

**Case 3. Assume that** $n \geq 4$.

Let $D$ be the oriented graph with

$$V(D) = V(T) \cup \{u, w\}$$

and
Figure 4.7

$$E(D) = E(T) \cup \{(u, v), (w, v) \mid v \in S_1\} \cup \{(v, u), (v, w) \mid v \in S_n\}$$
$$\cup \{(v, u) \mid v \in S_2\} \cup \{(w, v) \mid v \in S_{n-1}\}.$$ 

Since $\max\{k-1, n-m\} \leq 2$, it follows that $k \leq 3$ and $m \geq n - 2$. Therefore $|S_i| = 1$ for all $i$ $(3 \leq i \leq n-2)$. Note that it may be the case that $|S_j| = 1$ for some $j = 1, 2, n-1, n$ (see Figure 4.8). It is routine to check that $mC(D) \equiv T$. □

Before characterizing $A(T) = 2$ for $T \in \mathcal{T}(k, m, n)$ where $\max\{k-1, n-m\} \geq 3$, we present three useful lemmas.

**Lemma 4.12** Let $T$ be an oriented tree with $A(T) = 2$. Further, let $D$ be an oriented supergraph of $T$ with $mC(D) = T$ and $V(D) = V(T) \cup \{u, w\}$. If $d(u, w) = m \cdot \text{diam } D$, then either $(w, u) \not\in E(D)$ or $D - (w, u)$ is strong.

**Proof.** Suppose $(w, u) \in E(D)$. Let $H = D - (w, u)$. Since $D$ has the structure in Figure 4.1, it suffices to show that there is a $w-u$ path in $H$. We first claim that
either $\text{id}_H(u) \neq 0$ or $\text{od}_H(w) \neq 0$. Suppose that $\text{id}_H(u) = \text{od}_H(w) = 0$. Let $v$ be a vertex with $\text{id}_T(v) = 0$. Then, by Lemma 4.5, $(u, v) \in E(D)$. Therefore,

$$m\text{-diam } D > m\varepsilon(v) \geq \bar{d}(v, u) = \bar{d}(v, w) + \bar{d}(w, u) \geq \bar{d}(v, w) + 1 = \bar{d}(u, v) + \bar{d}(v, w) \geq \bar{d}(u, w) = m\text{-diam } D,$$

a contradiction. We now consider two cases.

**Case 1. Assume that $\text{id}_H(u) \neq 0$.**

Let $v$ be a vertex with $(v, u) \in E(D)$ such that $\bar{d}(u, v)$ is as large as possible. If $\text{od}_T(v) = 0$ then, since $\bar{d}(u, w) = m\text{-diam } D$ and $\bar{d}(w, v) \leq m\varepsilon(v) < m\text{-diam } D$, there exists an $w$--$v$ path in $H$. Therefore there is an $w$--$u$ path in $H$. If $\text{od}_T(v) \neq 0$, then let $v'$ be a vertex with $(v, v') \in E(T)$. By the choice of $v$, it follows that $w$ lies on every $v'$--$v$ path in $D$. Since $\bar{d}(v', w) + \bar{d}(w, v) = \bar{d}(v', v) \leq m\varepsilon(v') < m\text{-diam } D$, there exists an $w$--$v$ path in $H$. Therefore there is an $w$--$u$ path in $H$.

**Case 2. Assume that $\text{od}_H(w) \neq 0$.**
Let \( v \) be a vertex with \((w, v) \in E(D)\) such that \( \delta(w, v) \) is as small as possible. If \( \text{id}_T(v) = 0 \), then, since \( \delta(w, v) \leq \text{me}(v) < m\cdot\text{diam } D \), there exists a \( w-u \) path in \( H \). Therefore, there is an \( w-u \) path in \( H \). If \( \text{id}_T(v) \neq 0 \), then let \( v' \) be a vertex with \((v', v) \in E(T)\). By the choice of \( v \), it follows that \( u \) lies on every \( v-v' \) path in \( D \). Since \( \delta(v, u) + \delta(u, v') = \delta(v, v') \leq \text{me}(v) < m\cdot\text{diam } D \), there exists a \( v-u \) path in \( H \). Therefore, there is a \( w-u \) path in \( H \). □

The next lemma will be helpful for the case \( k \geq 4 \).

**Lemma 4.13** Let \( v_2 \in S_2 \). If \( k \geq 4 \), then \( \delta_D(v_2, x) < n \) for all \( x \in V(D) \), and \( \delta_D(x, v_2) < n \) for all \( x \in \{u\} \cup S_1 \cup S_2 \).

**Proof.** Let \( v_1 \in S_1 \) such that \((v_1, v_2) \in E(T)\). We first prove that \( \delta(v_2, x) < n \) for all \( x \in V(D) \). Since \( v_2 \) is the only vertex to which \( v_1 \) is adjacent, it follows, for \( x \in V(D) - \{v_1\} \), that every shortest \( v_1-x \) path contains \( v_2 \). Therefore \( \delta(v_2, x) = \delta(v_1, x) - 1 \leq n - 1 < n \) for all \( x \in V(D) - \{v_1\} \). To prove \( \delta(v_2, v_1) < n \), we consider a vertex \( v \in S_1 - \{v_1\} \). Observe that \( u \) lies on every shortest \( v_1-v \) path and \( v_2 \) lies on every shortest \( v_1-u \) and \( v_1-v \) paths. Therefore,

\[
\delta(v_2, v_1) \leq 1 + \delta(v_2, u) = \delta(v_1, v_2) + \delta(v_2, u) = \delta(v_1, u) + \delta(v_1, v) - 1 \leq n - 1 < n.
\]

We now prove that \( \delta(x, v_2) < n \) for all \( x \in \{u\} \cup S_1 \cup S_2 \). It is clear that \( \delta(u, v_2) = 2 < n \). Suppose \( x \in S_1 \). If \( (x, v_2) \in E(T) \), then \( \delta(x, v_2) = 1 < n \). If \( (x, v_2) \notin E(T) \), then, since \( k \geq 4 \), there exists \( x' \in S_3 \) such that \( x \) and \( x' \) lie on different branches of \( T \). Therefore, every shortest \( x-x' \) path passes through \( u \). Observe that \( u \) lies on every shortest \( x-v_2 \) path. Then

\[
\delta(x, v_2) = \delta(x, u) + \delta(u, v_2) = \delta(x, u) + \delta(u, v_2) + 1 - 1 = \delta(x, u) + \delta(u, x') - 1 = \delta(x, x') - 1 \leq n - 1.
\]
Therefore, $d(x, v_2) \leq n - 1$ for all $x \in S_1$. If $x \in S_2 - \{v_2\}$, then there exists $x' \in S_1$ with $(x', x) \in E(T)$. By (6) of Theorem 4.10, it follows that $od(x') = 1$. Thus $x$ lies on every $x' - v_2$ path. Therefore

$$d(x, v_2) \leq d(x, u) - 1 \leq n - 1$$

for all $x \in S_2 - \{v_2\}$, which completes the proof. □

**Lemma 4.14** If $\left(\{w\}, \bigcup_{i=2}^{k-1} S_i\right) \neq \emptyset$, where $k \geq 4$, then either $n = m = k + 1$ or $k = m < n$ and $n - m \leq 2$.

**Proof.** Since $k \geq 4$ and $A(T) \geq 3$ for all $T \in T(k, k+1, k+1)$, where $k \geq 2$, it follows that $k, m,$ and $n$ are not all equal. We first prove that if $k < m$, then $n = k + 1$. Suppose, to the contrary, that $n \geq k + 2$. We now prove that $me(v_{k+1}) < n$, which will produce a contradiction. Since $(w, \bigcup_{i=2}^{k-1} S_i) \neq \emptyset$, it follows that $\overline{d}(v_{k+1}, v_k) < n$ and $\overline{d}(v, v_{k+1}) < n$ for all $v \in V(D)$. Since $v_{k+1}$ lies on every $v - v$ path for all $v \in V(D) - \{v_k\}$, we have that $\overline{d}(v_{k+1}, v) < \overline{d}(v_k, v) \leq n$ for all $v \in V(D) - \{v_k\}$. Therefore $me(v_{k+1}) < n$, a contradiction.

We now prove that if $k = m < n$, then $n - m \leq 2$. Suppose, to the contrary, that $k = m < n$ and $n \geq m + 3$. Let $w_1, w_2,$ and $w_3$ be vertices in $S_{n-2}, S_{n-1},$ and $S_n$, respectively, such that $(w_1, w_2), (w_2, w_3) \in E(T),$ and $(w_3, u) \in E(D)$ if $(S_n, u) \neq \emptyset$ (see Figure 4.9). We now produce a contradiction by showing that $me(w_2) < n$. Since $w$ is adjacent to some vertex in $\bigcup_{i=2}^{k-1} S_i$, it follows that $\overline{d}(w_3, w_2) < n$. Since $w_2$ is the only vertex adjacent to $w_3$, the vertex $w_2$ lies on every $x-w_3$ path for all $x \in V(D) - \{w_3\}$. Therefore

$$\overline{d}(x, w_2) = \overline{d}(x, w_3) - 1 \leq n - 1$$

for all $x \in V(D) - \{w_3\}$. 

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Next we prove that $\bar{d}(w_2, x) \leq n - 1$ for all $x \in V(D)$. Since $(w, \bigcup_{i=2}^{k-1} S_i) \neq \emptyset$, it follows that $\bar{d}(w_2, w_1) < n$. Let $V_1$ be the set of all vertices on the branches of $T$ at $w_1$ that contains a vertex in $S_1$. Let $V_2$ be the set of vertices on all branches of $T$ at $w_1$ that contains all vertices other than $w_2$, that are adjacent from $w_1$. Let $V_3$ be the set of vertices on that branch of $T$ at $w_1$ that contains the vertex $w_2$. Observe that $V(D) = V_1 \cup V_2 \cup V_3 \cup \{u, w, w_1\}$. It follows, for all $x \in V_3 \cup \{w\}$, that $\bar{d}(w_2, x) = \bar{d}(w_1, x) - 1 \leq n - 1$. Since $(w, \bigcup_{i=2}^{k-1} S_i) \neq \emptyset$, it follows that $(\bigcup_{i=k+1}^{n-1} S_i, u) = \emptyset$. If $(S_n, u) = \emptyset$, then $w$ lies on every $w_1$-$x$ path and every $w_2$-$x$ path for all $x \in V_1 \cup \{u\}$. Therefore,

\[
\bar{d}(w_2, x) = \bar{d}(w_2, w) + \bar{d}(w, x) = \bar{d}(w_1, w) - 1 + \bar{d}(w, x) = \bar{d}(w_1, x) - 1 \leq n - 1
\]

for all $x \in V_1 \cup \{u\}$. If $(S_n, \{u\}) \neq \emptyset$, then by the choice of $w_3$, it follows that $(w_3, u) \in E(D)$. Thus, for all $x \in \{u\} \cup V_1$, there exist shortest $w_1$-$x$ and $w_2$-$x$ paths such that they both contain $w_3$. Therefore,

\[
\bar{d}(w_2, x) = \bar{d}(w_2, w_3) + \bar{d}(w_3, x) = \bar{d}(w_1, w_3) - 1 + \bar{d}(w_3, x) = \bar{d}(w_1, x) - 1 \leq n - 1
\]
for all \( x \in V_1 \cup \{u\} \).

It remains to prove that \( \overline{d}(w_2, x) \leq n - 1 \) for all \( x \in V_2 \). Let \( w_{1}' \in S_{n-2} - \{w_1\} \). Observe that \( w \) lies on every \( w_{1}'-x \) path. Therefore,

\[
\overline{d}(w_2, x) \leq \overline{d}(w_2, w) + \overline{d}(w, x) = \overline{d}(w_{1}', w) - 1 + \overline{d}(w, x) = \overline{d}(w_{1}', x) - 1 \leq n - 1
\]

for all \( x \in V_2 \). Hence, \( \overline{d}(w_2, x) \leq n - 1 \) and \( \overline{d}(x, w_2) \leq n - 1 \) for all \( x \in V(D) \), that is, \( m_e(w_2) < n = m_{\text{rad}} D \), a contradiction. \(
\)

Let us now define two families of oriented trees as follows:

\[ T_2 = \mathcal{T}(4, 4, 6) \quad \text{and} \quad T_3 = \{ T \mid T \in \mathcal{T}(5, 5, 6) \text{ and } |S_4| \geq 3 \}. \]

Before defining another family, we introduce an oriented tree operation. Let \( T_1 \) be an oriented tree in \( \mathcal{T}(k, k, k+1) \) where \( k \geq 4 \). Let \( T_2 \in \mathcal{T}(3, k-1, k-1) \). We define \( T_2 \rightarrow T_1 \) to be the oriented tree obtained by joining the vertex \( v_{k-1} \) in \( T_2 \) with the vertex \( v_k \) in \( T_1 \) (see Figure 4.10). We denote by \( T_4 \) the family of all oriented trees \( T_2 \rightarrow T_1 \), where \( T_1 \in \mathcal{T}(k, k, k+1) \), \( T_2 \in \mathcal{T}(3, k-1, k-1) \) and \( k \geq 4 \). Let \( T_i \in \mathcal{T}_i \), \( i = 2, 3, 4 \).

For an oriented tree \( T \in \mathcal{T}(k, m, n) \) and an oriented supergraph \( D \) associated with \( T \), we define \( H = D - (w, u) \) if \( (w, u) \in E(D) \), and \( H = D \) otherwise. Let

\[
s = \min \{ i \mid 1 \leq i \leq n, \, x \in S_i, \, (w, x) \in E(H) \}
\]

and

\[
t = \max \{ i \mid 1 \leq i \leq n, \, x \in S_i, \, (x, u) \in E(H) \}.
\]

By Lemma 4.12, \( H \) is strong. Thus, \( 1 \leq s \leq t \leq n \). We are now ready to characterize those oriented trees with appendage number 2. The following five propositions lead to a necessary condition for an oriented tree with appendage number 2.
Proposition 4.15 Let \( T \in \mathcal{T}(k, m, n) \), where \( \max\{k - 1, n - m\} \geq 3 \), be an oriented tree with appendage number 2. If \( s = 1 \) and \( k \geq 4 \), then \( T \in \mathcal{T}_2 \cup \mathcal{T}_4 \).
Proof. Suppose $v_1 \in S_1$ such that $(w, v_1) \in E(D)$. Let $v_2 \in S_2$ with $(v_1, v_2) \in E(T)$. By Lemma 4.13, it follows that $d'(v_2, v) < n$ for all $v \in V(D)$ and $d'(v, v_2) < n$ for all $v \in \{u\} \cup S_1 \cup S_2$. Since $(w, v_1), (v_1, v_2) \in E(D)$, it follows that $d'(v, v_2) < n$ for all $v \in \{w\} \cup \bigcup_{i=4}^{n} S_i$. Since $me(v_2) = n$, there exists a vertex $v_3 \in S_3$ such that $d'(v_3, v_2) = n$. Suppose $P$ is a shortest $v_3-V_2$ path. Then $P$ contains a subpath $v_3, v_4, ..., v_{k-1}$ such that $v_i \in S_i$, $3 \leq i \leq k-1$ (see Figure 4.11). Since $d'(v_3, v_2) = n$, we see that $(v_i, u) \notin E(D)$ for all $i$ ($3 \leq i \leq k-1$). We now prove that $(w, S_2) = \emptyset$. Suppose $(w, x) \in E(D)$ for some $x \in S_2$. Then $d'(y, x) \leq n - 1$ for all $y \in \{w\} \cup \bigcup_{i=3}^{n} S_i$. Then, by Lemma 4.13, $me(x) \leq n - 1$, which is a contradiction. Therefore, $(w, v) \notin E(D)$ for all $v \in S_2$.

Next we show, for all $v \in S_3 - \{v_3\}$, that $(w, v) \in E(D)$. We first claim that $w$ lies on every shortest $v_3-V$ path; for otherwise, there exists a shortest $v_3-V$ path that passes through $u$. Therefore,

$$d'(v_3, v_2) \leq d'(v_3, u) + d'(u, v_2) = d'(v_3, u) + d'(u, v) - 1$$

$$= d'(v_3, v) - 1 \leq n - 1,$$

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which is a contradiction. Since every $v_3$–$v$ path contains $w$ and $(w, S_2) = \emptyset$, it follows that $(w, v) \in E(D)$ for all $v \in S_3 - \{v_3\}$. Therefore, by Lemma 4.14, either $n = m = k + 1$ or $k = m < n$ and $n - k \leq 2$. (see Figure 4.12). Since $me(v_k) = n$, the arc $(v, u) \notin E(D)$ for all $v \in S_i$ $(k \leq i \leq n-1)$.

![Figure 4.12](image-url)

We now prove that $(w, v_i) \notin E(D)$ for all $3 \leq i \leq k - 1$. Suppose that $(w, v_i) \in E(D)$ for some $i$ $(3 \leq i \leq k - 1)$. Then $d(v, v_k) \leq n - 1$, for all $v \in V(D)$. Clearly, $d(v_k, v_{k-1}) \leq n - 2$. For $v \in V(D) - \{v_{k-1}\}$, since $v_k$ lies on every $v_{k-1}$–$v$ path, $d(v_k, v) \leq d(v_{k-1}, v) - 1 \leq n - 1$. Combining these inequalities, it follows that $me(v_k) < n$, which is a contradiction. Therefore, $(w, v_i) \notin E(D)$ for all $3 \leq i \leq k - 1$.

We show that $id(v_i) = 1$ for all $i$ $(4 \leq i \leq k - 1)$. If $id(v_i) \geq 2$ for some $i$ $(4 \leq i \leq k - 1)$, then there exists a path from a vertex in $S_3 - \{v_3\}$ to $v_{k-1}$ in $T$. Since $(w, v) \in E(D)$ for all $v \in S_3 - \{v_3\}$, we have that $d(v_k, v_{k-1}) \leq n - 2$. A similar argument will produce a contradiction that $me(v_k) < n$. Therefore, $id(v_i) = 1$.
for all $i$ $(4 \leq i \leq k-1)$. By (6) of Theorem 4.10, it follows that $\od(v_i) = 1$ for all $i$ $(4 \leq i \leq k-1)$. Therefore, if $n = k + 1$, then $T \in T_4$.

We now assume that $n = k + 2$. Observe that if $k = 4$ then $T \in T_2$. It suffices to prove that $n = k + 2$ implies that $k = 4$. We first show that $(v, u) \not\in E(D)$ for all $v \in S_3$. It is clear that $(v_3, u) \not\in E(D)$. Suppose that there exists $v_3' \in S_3 - \{v_3\}$ such that $(v_3', u) \in E(D)$. Then $d(v_3', x) < n$ for all $x \in V(D)$. Observe that $d(x, v_3') < n - 3 + 2 < n$ for all $x \in \{w\} \cup (\bigcup_{i=3}^{n} S_i)$. Before showing $d(x, v_3') < n$ for all $x \in S_1$, we first prove that $d(x, u) \leq 3$ for all $x \in S_1$. For $x \in S_1$, we choose $y \in S_{k-1}$ such that $x$ and $y$ lie on different branches of $T$ at $v_k$. Then every shortest $x-y$ path in $D$ contains $u$. Then $d(x, y) = d(x, u) + d(u, y) = d(x, u) + k - 1$. Since $n - k = 2$, it follows that $d(x, y) \leq \me(x) \leq n = k + 2$. Therefore, $d(x, u) + k - 1 \leq k + 2$, that is, $d(x, u) \leq 3$. Therefore, $d(x, u) \leq 3$ for all $x \in S_1$. Hence,

$$d(x, v_3') \leq d(x, u) + d(u, v_3') \leq 3 + 3 = 6 < n$$

for all $x \in S_1$. For $x \in S_2$, let $y \in S_1$ such that $(y, x) \in E(T)$. Then $d(x, v_3') = d(y, v_3') - 1 \leq 6 - 1 < n$. Clearly, $d(u, v_3') = 3 < n$. Therefore, $\me(v_3') < n$, which is a contradiction. Hence $(v, u) \not\in E(D)$ for all $v \in S_3$.

Next we show that $(v, u) \in E(D)$ for all $v \in S_2$. For $v \in S_2$, choose $v' \in S_1$ such that $(v', v) \in E(D)$. Note that $(v', u) \not\in E(D)$ and $(S_3, u) = \emptyset$. Since $d(v', u) \leq 3$, it follows that $(v, u) \in E(D)$. Therefore $(v, u) \in E(D)$ for all $v \in S_2$ (see Figure 4.13).

We now prove that $k \leq 5$. Let $v_{k+1} \in S_{k+1}$ and $v_{k+2} \in S_{k+2}$ such that $(v_{k+1}, v_{k+2}) \in E(T)$ and $(v_{k+2}, u) \in E(T)$ if $(S_{k+2}, u) \neq \emptyset$. Then, for $v \in V(D) - \{v_{k+2}\}$, $d(v, v_{k+1}) < d(v, v_{k+2}) \leq k + 2$. Clearly $d(v_{k+2}, v_{k+1}) \leq k$. Since $d(v_3, v) \leq k + 2$ for all $v \in S_2$, it follows that either $(S_{k+2}, u) \neq \emptyset$ or

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(w, x) ∈ E(D) for all x ∈ S_1. In both cases, it follows that \( \overline{d}(v_{k+1}, v) < n \) for all v ∈ \( \bigcup_{i=3}^{k-1} S_i \). It is clear that \( \overline{d}(v_{k+1}, v) < n \) for all v ∈ \( \{v_k\} \cup S_{k+1} \). Note that if k ≥ 6 then \( \overline{d}(v_{k+1}, v) < n \) for all v ∈ \( \{u\} \cup S_1 \cup S_2 \). Since me(\( v_{k+1} \)) = k + 2, it follows that \( \overline{d}(v_{k+1}, v) = k+2 \) for some v ∈ \( S_{k+2} \). This implies that (w, \( \bigcup_{i=4}^{k} S_i \)) = \( \emptyset \). Let \( v_{k-1}' \in S_{k-1} - \{v_{k-1}\} \). Consider a shortest \( v_3-v_{k-1}' \) path P. Since \( \overline{d}(v_3, v_2) = k + 2 \), the path P contains w. Then \( \overline{d}(v_3, v_{k-1}') = \overline{d}(v_3, w) + \overline{d}(w, v_{k-1}') = k + 1 + (k - 1) - 3 = 2k - 3 \). Since \( \overline{d}(v_3, v_{k-1}') \leq me(v_3) = k + 2 \), we have that 2k - 3 ≤ k + 2, that is, k ≤ 5.

We finally prove that k ≠ 5, for suppose, to the contrary, that k = 5. Then n = k + 2 = 7. Let v ∈ S_4 - \{v_4\}. Since \( \overline{d}(v, v_4) \leq 7 \) and (x, u) ∈ E(D) for all x ∈ \( \{v_5\} \cup S_6 \), and since, (w, y) ∈ E(D) for all y ∈ \( \{v_5, v_4\} \cup S_2 \), the arc (v, u) ∈ E(D). Therefore, (v, u) ∈ E(D) for all v ∈ S_4 - \{v_4\}. Let \( v' \in S_3 - \{v_3\} \) (see Figure 4.14). It is easy to verify that me(\( v' \)) < 7, which is a contradiction. Therefore k ≠ 5. □
Proposition 4.16 Let $T \in \mathcal{T}(k, m, n)$, where $\max\{k - 1, n - m\} \geq 3$, be an oriented tree having appendage number 2. If $s > 1$, $k \geq 4$, $n = m + 2$, and $(v_n, u) \in E(D)$ for some $v_n \in S_n$, then $T \in \mathcal{T}_2$.

Proof. Let $v_{n-1} \in S_{n-1}$ such that $(v_{n-1}, v_n) \in E(T)$. For $v \in V(D) - \{v_n\}$, since $v_{n-1}$ lies on every $v-v_n$ path, it follows that $d'(v, v_{n-1}) = d(v, v_n) - 1 \leq n - 1$. Consider a vertex $v \in S_n - \{v_n\}$. Then every shortest $v-v_n$ path contains both $w$ and $v_{n-1}$. Therefore,

$$d'(v_n, v_{n-1}) \leq d'(v_n, w) + d'(w, v_{n-1}) = d(v_n, w) + d(w, v_{n-1}) - 1 = d(v_n, v_{n-1}) - 1 \leq n - 1.$$ 

Observe that $d'(v_{n-1}, v) \leq n - 1$ for all $v \in \{u, w\} \cup \bigcup_{i=1}^{m-1} S_i$. For a vertex $v \in S_{n-1} - \{v_{n-1}\}$, there exists $v' \in S_n$ such that $(v, v') \in E(T)$. Therefore, $d'(v_{n-1}, v) = d'(v_{n-1}, v') - 1 \leq n - 1$ for all vertices $v \in S_{n-1} - \{v_{n-1}\}$. Combining these inequalities, since $m(v_{n-1}) = n$, we have that either $d'(v_{n-1}, v_m) = n$, where $\{v_m\} = S_m$, or $d'(v_{n-1}, x) = n$ for some $x \in S_n$. We distinguish two cases.

Case 1. Assume that $d'(v_{n-1}, v_m) = n$. 

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Since \( s > 1 \), \((w, v) \not\in E(D)\) for all \( v \in S_1 \). Furthermore, since \( \bar{d}(v_{n-1}, v_m) = n \), \((w, v) \not\in E(D)\) for all \( v \in \bigcup_{i=2}^{m} S_i \). Since \( k \geq 4 \), applying Lemma 4.14 on \( T \), \((v, u) \not\in E(D)\) for all \( v \in S_{n-1} \). We claim that \((w, u) \in E(D)\). Since \( od(w) \geq 1 \), it follows that if \((w, u) \not\in E(D)\), then all vertices that are adjacent from \( w \) belong to \( S_{n-1} \). Furthermore, for \( v \in S_{n-1} \), since \((v, u) \not\in E(D)\), it follows that \( d(v, v_m) = n \). Therefore, \( \bar{d}(w, v_m) = n + 1 \), which is a contradiction. So \((w, u) \in E(D)\). Let \( v_2 \in S_2 \). By Lemma 4.13, \( \bar{d}(v_2, v) < n \) for all \( v \in V(D) \) and \( \bar{d}(v, v_2) < n \) for all \( v \in \{u\} \cup S_1 \cup S_2 \). Since \((w, u) \in E(D)\), it follows that \( \bar{d}(v, v_2) < n \) for all \( v \in \{w\} \cup \bigcup_{i=4}^{n} S_i \). Since \( me(v_2) = n \), there exists a vertex \( v_3 \in S_3 \) such that \( \bar{d}(v_3, v_2) = n \).

Suppose \( P \) is a shortest \( v_3-v_2 \) path. Then \( P \) contains a subpath \( v_3, v_4, ..., v_{k-1} \) such that \( v_i \in S_i \), \( 3 \leq i \leq k-1 \). Then \((v_i, u) \not\in E(D)\) for all \( i \) \((3 \leq i \leq k-1)\). By a similar argument to the proof of Proposition 4.15, it follows that \((w, v) \in E(D)\) for all \( v \in S_3 - \{v_3\} \), and \( id(v_i) = od(v_i) = 1 \) for all \( i \) \((4 \leq i \leq k-1)\) (see Figure 4.14). By Lemma 4.14, \( k = m \). A similar proof to Proposition 4.15 gives us that \( k = 4 \). Therefore \( T \in T_2 \).

**Case 2.** Assume that \( \bar{d}(v_{n-1}, v_m) < n \) and \( \bar{d}(v_{n-1}, v) = n \) for some \( v \in S_n \).

Let \( P \) be a shortest \( v_{n-1}-v \) path. Then \( P \) contains \( w \). Since \( \bar{d}(v_{n-1}, v) = n \), we have \((w, x) \not\in E(D)\) for all \( x \in \bigcup_{i=2}^{m} S_i \). Furthermore, there exists \( x \in S_3 \) such that \((w, x) \in E(D)\). Since \( n - m = 2 \), by Lemma 4.14, it follows that \( m = k \). Since \( me(v_k) = n \), there exists \( v_{k-1} \in S_{k-1} \) such that \( \bar{d}(v_k, v_{k-1}) = n \); for otherwise, we have \( me(v_k) < n \), a contradiction. We now prove that either \((w, u) \in E(D)\) or \((v, u) \in E(D)\) for all \( v \in S_n \). Suppose, to the contrary, that \((w, u) \not\in E(D)\) and there exists \( v_n' \in S_n - \{v_n\} \) such that \((v_n', u) \not\in E(D)\). Let \( v_{n-1}' \in S_{n-1} \) such that \((v_{n-1}', v_n') \in E(D)\). Let \( P \) be a shortest \( v_{n-1}'-v_{k-1} \) path. Observe that there is no
arc from \( \{v_{n-1}', v_n', w\} \) to \( u \), and from \( w \) to any vertex in \( S_1 \). Furthermore, since \( \overline{d}(v_k, v_{k-1}) = n \), every \( w-v_{k-1} \) path contains \( u \). Therefore, \( P \) contains both \( w \) and \( u \). Hence,
\[
\overline{d}(v_{n-1}', v_k) = \overline{d}(v_{n-1}', w) + \overline{d}(w, u) + \overline{d}(u, v_k) \geq n + 1,
\]
a contradiction. Therefore, either \( (w, u) \in E(D) \) or \( (v, u) \in E(D) \) for all \( v \in S_n \). Furthermore, if \( (w, u) \notin E(D) \), then since \( (w, \bigcup_{i=1}^{k} S_i) = \emptyset \), we have \( N^+(w) \subseteq S_{n-1} \).

Since \( \od(w) \neq 0 \), there exists \( v^* \in S_{n-1} \) such that \( (w, v^*) \in E(D) \). It is clear, in each case, that \( \overline{d}(v, v_2) < n \) for all \( v \in \{w\} \cup \bigcup_{i=4}^{n} S_i \). By Lemma 4.13, there exists \( v_3 \in S_3 \) such that \( \overline{d}(v, v_2) = n \). Following an argument similar to the proof of Proposition 4.15, we conclude that \( k = 4 \), so \( T \in \mathcal{I}_2 \). □

**Proposition 4.17** Let \( T \in \mathcal{I}(k, m, n) \), where \( \max\{k - 1, n - m\} \geq 3 \), be an oriented tree with appendage number 2. If \( s > 1 \), \( k > 4 \), \( n - m = 1 \), and \( (v_n, u) \in E(D) \) for some \( v_n \in S_n \), then \( T \in \mathcal{T}_4 \).

**Proof.** We prove that either \( (w, u) \in E(D) \) or \( (v, u) \in E(D) \) for all \( v \in S_n \). Suppose that \( (w, u) \in E(D) \). Since \( s > 1 \) and \( \od(w) \geq 1 \), it follows that \( w \) is adjacent to some vertex in \( \bigcup_{i=2}^{m} S_i \). Since \( \me(v_m) = n \), \( (w, \bigcup_{i=1}^{m-1} S_i) = \emptyset \). Therefore,
\[
(w, \{v_m\} \cup \bigcup_{i=2}^{k-1} S_i) \neq \emptyset.
\]
We claim that \( (w, \bigcup_{i=2}^{k-1} S_i) \neq \emptyset \). If \( (w, \bigcup_{i=2}^{k-1} S_i) = \emptyset \), then \( N^+(w) = \{v_m\} \). Since \( \me(v_m) = n \), it follows that \( (v_m, u) \in E(D) \). Let \( v_{m-1} \in S_{m-1} \). Then \( \overline{d}(w, v_{m-1}) = \overline{d}(w, v_m) + \overline{d}(v_m, v_{m-1}) = n + 1 \), a contradiction. Therefore, \( (w, \bigcup_{i=2}^{k-1} S_i) \neq \emptyset \). By Lemma 4.14, \( m = k \). Note that \( (w, u) \notin E(D) \), and \( (w, v) \notin E(D) \) for all \( v \in S_1 \). For \( v \in S_{k-1} \), if \( u \) lies on a shortest \( w-v \) path, then
\[
\overline{d}(w, v) = \overline{d}(w, u) + \overline{d}(u, v) = 2 + k - 1 = k - 1.
\]
Otherwise, \( \overline{d}(w, v) \leq k - 2 \). However, if \( \overline{d}(w, v) \leq k - 2 \) for all \( v \in S_{k-1} \), then \( \overline{d}(v_k, v) \leq (k - 2) + 2 = k <
\[ k + 1 = n \] for all \( v \in S_{k-1} \). This implies that \( me(v_k) < n \), which is a contradiction. Therefore, there exists a vertex \( v_{k-1} \in S_{k-1} \) such that \( \tilde{d}(w, v_{k-1}) = n \). This implies that \( (v, u) \in E(D) \) for all \( v \in S_n \). Also, every shortest \( w-v_{k-1} \) path contains \( u \). Therefore, there exists \( x \in \bigcup_{i=4}^{m-1} S_i \) such that \( (w, x) \in E(D) \) and \( (x, u) \in E(D) \). Therefore, \( \tilde{d}(v, v_2) < n \) for all \( v \in \{w\} \cup \bigcup_{i=4}^{n} S_i \) and \( v_2 \in S_2 \). By Lemma 4.13, \( \tilde{d}(v_2, v) < n \) for all \( v \in V(D) \) and \( \tilde{d}(v, v_2) < n \) for all \( v \in \{u\} \cup S_1 \cup S_2 \). Since \( me(v_2) = n \), there exists \( v_3 \in S_3 \) such that \( \tilde{d}(v_3, v_2) = n \). By an argument similar to the proof of Proposition 4.15, it follows that \( T \in T_4 \). \( \square \)

**Proposition 4.18** Let \( T \in T(k, m, n) \), where \( \max\{k - 1, n - m\} \geq 3 \), be an oriented tree having appendage number 2. If \( s > 1, k \geq 4, n = m \), and \( (v_n, u) \in E(D) \) for some \( v_n \in S_n \), then \( T \in T_4 \).

**Proof.** We prove that either \( (w, u) \in E(D) \) or there exists \( v^* \in \bigcup_{i=2}^{m} S_i \) such that \( (w, v^*), (v^*, u) \in E(D) \). Suppose that \( (w, u) \in E(D) \). Since \( od(w) \geq 1 \), \( w \) is adjacent to some vertex of \( D \). Since \( s > 1 \), we have \( (w, S_1) = \emptyset \). Furthermore, since \( me(v_{n-1}) = n \) and \( (v_n, u) \in E(D) \), it follows that \( (w, \{v_{k}, v_{k+1}, \ldots, v_{n-2}\}) = \emptyset \). Therefore, \( N^+(w) \subseteq \{v_{n-1}\} \bigcup_{i=1}^{m-1} S_i \). We claim that \( (w, \bigcup_{i=2}^{k-1} S_i) = \emptyset \). If \( (w, \bigcup_{i=2}^{k-1} S_i) = \emptyset \), then \( N^+(w) = \{v_{n-1}\} \). Since \( me(v_{n-1}) = n \), it follows that \( (v_{n-1}, u) \notin E(D) \). Let \( v_{n-2} \in S_{n-2} \). Then \( \tilde{d}(w, v_{n-2}) = \tilde{d}(w, v_{n-1}) + \tilde{d}(v_{n-1}, v_{n-2}) = n + 1 \), a contradiction. Therefore, \( (w, \bigcup_{i=2}^{k-1} S_i) \neq \emptyset \). By Lemma 4.14, it follows that \( n = m = k + 1 \). Note that \( (w, u) \notin E(D) \) and \( (w, v) \notin E(D) \) for all \( v \in S_1 \). For \( v \in S_{k-1} \), if \( u \) lies on a shortest \( w-v \) path then \( \tilde{d}(w, v) = \tilde{d}(w, u) + \tilde{d}(u, v) = 3 + k - 1 = k + 2 = n + 1 \), a contradiction. Therefore, \( \tilde{d}(w, v) \leq k - 2 \). However, if \( \tilde{d}(w, v) \leq k - 2 \) for all \( v \in S_{k-1} \), then \( \tilde{d}(v_k, v) \leq (k - 2) + 2 = k < k + 1 = n \) for all \( v \in S_{k-1} \). This
implies that \( m(e(v_k)) < n \), which is a contradiction. Therefore, there exists a vertex \( v_{k-1} \in S_{k-1} \) such that \( d(w, v_{k-1}) = n \). However, this is possible only if there exists a vertex \( v* \in \bigcup_{i=2}^{m} S_i \) such that \( (w, v*), (v*, u) \in E(D) \). The remainder of the proof is similar to the proof of Proposition 4.17 and is therefore omitted. □

**Proposition 4.19** Let \( T \in T(k, m, n) \), where \( \max\{k - 1, n - m\} \geq 3 \), be an oriented tree of appendage number 2. If \( 2 \leq s \leq t < k \), then \( T \in T_2 \cup T_3 \cup T_4 \).

**Proof.** By Lemma 4.14, either \( n = m = k + 1 \) or \( n > m = k \) and \( n - k \leq 2 \). Since \( k - 1 \geq 3 \), we have \( k \geq 4 \). Clearly if \( k = 4 \), then \( T \in T_2 \cup T_4 \). Suppose that \( k \geq 5 \).

Let \( v_2 \in S_2 \). Then by Lemma 4.13, \( d(v_2, v) < n \) for all \( v \in V(D) \) and \( d(v, v_2) < n \) for all \( v \in \{u\} \cup S_1 \cup S_2 \). If \( (w, v_2) \in V(D) \), where \( v_2 \in S_2 \), then \( d(v, v_2) < n \) for all \( v \in \{w\} \cup \bigcup_{i=4}^{n} S_i \), implying that \( m(e(v_2)) < n \), a contradiction. Therefore, \( s > 2 \), that is, \( (w, v) \notin E(D) \) for all \( v \in S_1 \cup S_2 \). Next we prove that \( d(v, v_2) < n \) for all \( v \in S_3 \). If a \( v-v_2 \) path contains \( w \), then since \( (w, S_1 \cup S_2) = \emptyset \), it must pass through both \( w \) and \( u \). Then the length of this path is at least \( n + 1 \), which is not possible. Therefore, every \( v-v_2 \) path contains \( u \) but not \( w \). Since \( t < k \), it follows that \( d(v, v_2) \leq d(v, u) + d(u, v_2) \leq k - 3 + 2 = k - 1 < n \). For \( v \in \{w\} \cup \bigcup_{i=k}^{n} (S_i) \), every \( v-v_2 \) path contains \( w \). Therefore, \( d(v, v_2) < d(v_k, v_2) \leq n \) for all \( v \in \{w\} \cup \bigcup_{i=k}^{n} S_i \). Since \( m(e(v_2)) = n \), there exists a vertex \( v* \in \bigcup_{i=4}^{k} S_i \) such that \( d(v*, v_2) = n \). Since \( d(v*, v_3) \leq m(e(v*)) = n \), it follows that \( (w,v_3) \in E(D) \) for all \( v_3 \in S_3 \). Similarly, since \( d(v_{k-1}, v_2) \leq m(e(v_{k-1})) = n \) for all \( v_{k-1} \in S_{k-1} \), it follows that \( (v_{k-1}, u) \in E(D) \) for all \( v_{k-1} \in S_{k-1} \) (See Figure 4.15).

We now prove that \( n = k + 1 \). Suppose that \( n = k + 2 \). If there exists a vertex \( x \in \bigcup_{i=4}^{k-1} S_i \) such that \( (w,x) \in E(D) \) then choose \( v_{n-1} \in S_{n-1} \). Then, since \( n - 1 > k \),
$d(v_{n-1}, v) < d(v_k, v) \leq n$ for all $v \in \{u\} \cup \bigcup_{i=1}^{k-1} S_i$. Since $k \geq s$, it follows that $d(v_{n-1}, v) < n$ for all $v \in \{w\} \cup \bigcup_{i=k}^{n} S_i$. Similarly, by noting that $(w, \bigcup_{i=k}^{n} S_i) \neq \emptyset$, it follows that $d(v, v_{n-1}) < n$ for all $v \in V(D)$. Therefore, $me(v_{n-1}) < n$, a contradiction. Therefore, $(w, \bigcup_{i=1}^{k-1} S_i) = \emptyset$. Observe that $d(v_k, v_2) = n$ for some $v_2 \in S_2$. Hence, $(\bigcup_{i=4}^{k-1} S_i, u) = \emptyset$ and there exists a vertex $v_{k-2} \in S_{k-2}$ such that $(v_{k-2}, u) \in E(D)$. Since $k \geq 5$, $d(v_{k-2}, v) < n$ for all $v \in V(D)$ and $d(v, v_{k-2}) < n$ for all $v \in \{u, w\} \cup \bigcup_{i=4}^{n} S_i$. Let $x \in \bigcup_{i=4}^{k-3} S_i$. Let $y$ be a vertex in $S_{k-1}$ such that $x$ and $y$ lie on different branches of $T$ at $v_k$. We claim that every shortest $x$-$y$ path contains $u$; for otherwise, it must contain $w$. Then

$d(x, y) = d(x, w) + d(w, y) = n + 1 - (k - 3) + 1 + (k - 1) - 3 = n - 1$,

a contradiction. Therefore $d(x, y) = d(x, u) + d(u, y) > d(x, u) + d(u, v_{k-2}) \geq d(x, v_{k-2})$. Hence $d(v, v_{k-2}) < n$ for all $v \in \bigcup_{i=1}^{k-3} S_i$. Combining these inequalities, we arrive at $me(v_{k-2}) < n$, which is a contradiction. Therefore $n = k + 1$.

We now prove that $k < 6$. Suppose, to the contrary, that $k \geq 6$. We claim that $(S_3, u) = \emptyset$; for otherwise, suppose that $(v_3, u) \in E(D)$. Then
\[ n = \overline{d}(v_k, v_2) \leq \overline{d}(v_k, v_3) + \overline{d}(v_3, v_2) = 3 + 3 = 6, \]

implying that \( k = 5 \). Let \( v_3 \in S_3 \). Choose \( v_{k-1} \in S_{k-1} \) such that \( v_3 \) and \( v_{k-1} \) lie on different branches at \( v_k \). Let \( P \) be a shortest \( v_3 - v_{k-1} \) path. Then \( P \) contains either \( u \) or \( w \). Since \( \overline{d}(v_3, v_{k-1}) \leq n \), if \( u \in V(P) \) then \( (v_3, v_4) \in E(D) \) and \( (v_4, u) \in E(D) \), where \( v_4 \) is the vertex in \( S_4 \) to which \( v_3 \) is adjacent. However, since \( \overline{d}(v_k, v_2) = n \), it follows that \( n = \overline{d}(v_k, v_2) \leq \overline{d}(v_k, v_3) + \overline{d}(v_3, v_2) = 3 + 4 = 7 \), implying that \( k = 6 \). Similarly, if \( w \in V(P) \), then there exists \( v_{k-2} \in S_{k-2} \) such that \((w, v_{k-2}) \in E(D) \) and \((v_{k-2}, v_{k-1}) \in E(D) \). Then \( n = \overline{d}(v_k, v_2) \leq 7 \), implying that \( k = 6 \). Hence \( k = 6 \), so \( n = 7 \). To produce a contradiction, we prove next that \( k = 6 \) is not possible. Observe that there exists no vertex \( v_4 \in S_4 \) such that \((w, v_4) \in E(D) \) and \((v_4, u) \in E(D) \); for otherwise, \( \text{me}(v_4) < n \), a contradiction. However, by the above argument, there exists \( v_4 \in S_4 \) such that either \((w, v_4) \in E(D) \) or \((v_4, u) \in E(D) \), but not both. If \((w, v_4) \in E(D) \) then we show that \((w, v) \in E(D) \) for all \( v \in S_4 \). Select a vertex \( v_3 \in S_3 \) such that \((v_3, v_4) \in E(D) \). Suppose \( v \in S_4 - \{v_4\} \) and \( v_5 \) is the vertex in \( S_5 \) to which \( v \) is adjacent. Since \((v, u), (v_4, u) \notin E(D) \) and \( \overline{d}(v_3, v_5) \leq n = 7 \), it follows that \((w, v) \in E(D) \). Therefore, \((w, v) \in E(D) \) for all \( v \in S_4 \). Then it follows that \( \overline{d}(v_4, v) < n \) for all \( v \in \{w\} \cup (\bigcup_{i=1}^{3} S_i) \). Furthermore, since \( \overline{d}(v_4, v) < n \) for all \( v \in \{u\} \cup (\bigcup_{i=1}^{3} S_i) \), and \( \overline{d}(v, v_4) < n \) for all \( v \in V(D) \), it follows that \( \text{me}(v_4) < n \), a contradiction. Similarly, if \((v_4, u) \in E(D) \) then \( \text{me}(v_4) < n \), again a contradiction. Therefore \( k < 6 \). Since \( k \geq 5 \), we conclude that \( k = 5 \).

We prove that \((S_3, u) = \emptyset \). Suppose that \((v_3, u) \in E(D) \) for some \( v_3 \in S_3 \). Then clearly \( \overline{d}(v, v_3) < n \) for all \( v \in V(D) \). Observe that \( \overline{d}(v_4, v) < n \) for all \( v \in V(D) \). Therefore, \( \text{me}(v_3) < n \), a contradiction. Since \( \overline{d}(v_5, v_2) = n \) for some \( v_2 \in S_2 \), it follows that there exists \( v_4 \in S_4 \) such that \((w, v_4) \in E(D) \). Since \( k = 5 \), we
have that $|S_4| \geq 2$. However, if $|S_4| = 2$, say $S_4 = \{v_4, v_4'\}$, then choose $v_3 \in S_3$ such that $(v_3, v_4') \in E(D)$. Then $\text{me}(v_3) < n$, a contradiction. Therefore, $|S_4| \geq 3$, and by the definition of $T_3$, we conclude that $T \in T_3$. 

Let $\mathcal{F} = T_2 \cup T_3 \cup T_4 \cup \overline{T_2} \cup \overline{T_3} \cup \overline{T_4}$. Observe that $T \in \mathcal{F}$ if and only if $\overline{T} \in \mathcal{F}$. Combining the above five properties, we have the following necessary condition.

**Theorem 4.20** If $T \in T(k, m, n)$, where $\max \{k - 1, n - m\} \geq 3$, is an oriented tree having appendage number 2, then $T \in \mathcal{F}$.

**Proof.** Let $s$ and $t$ be the parameters defined above. Since $1 \leq s \leq t \leq n$ and $\text{me}(v_i) = n$, where $\{v_i\} = S_i$, for all $i$ ($k \leq i \leq m$), it follows that if $k \leq s \leq m$, then $t = n$. Similarly, if $2 \leq s < k$, then either $s \leq t < k$ or $t = n$. Therefore, there are four possibilities: (i) $s = 1$, (ii) $2 \leq s \leq t < k$, (iii) $m < s \leq t \leq n - 1$, and (iv) $t = n$. By Theorem 3.1, $A(T) = 2$ if and only if $A(\overline{T}) = 2$. In addition, $\text{mC}(D) = T$ if and only if $\text{mC}(\overline{D}) = \overline{T}$. Note that $s(D) = 1$ if and only if $t(D) = n(D)$ and also that $2 \leq s(D) \leq t(D) < k(D)$ if and only if $m(D) < s(D) \leq t(D) \leq n(D) - 1$. Therefore, it suffices to prove $T \in \mathcal{F}$ for the cases $s = 1$ and $2 \leq s \leq t < k$.

**Case 1.** Assume that $s = 1$.

We consider four possibilities according to the value of $k(T)$. If $k(T) \geq 4$, then, by Proposition 4.15, $T \in T_2 \cup T_4 \subset \mathcal{F}$.

Suppose that $k(T) = 3$. It suffices to prove that $\overline{T} \in \mathcal{F}$. Consider a standard partition of $\overline{T}$ and an oriented supergraph $D'$ associated with $\overline{T}$ as shown in Figure 4.1. Since $k(T) = 3$ and $\max \{k(T) - 1, n(T) - m(T)\} \geq 3$, it follows that $n(T) - m(T) \geq 3$. Furthermore, since $k(\overline{T}) = n(T) - m(T) + 1$, $k(\overline{T}) \geq 4$. Since $n(\overline{T}) - m(\overline{T}) = k(T) - 1$ and $k(T) = 3$, it follows that $n(\overline{T}) = m(\overline{T}) + 2$. If $s(\overline{T}) = 1$, then,
applying Proposition 4.15 to $\bar{T}$, we have $\bar{T} \in \mathcal{T}_{2} \cup \mathcal{T}_{4} \subset \mathcal{F}$. Suppose that $s(\bar{T}) > 1$. Observe that $s(T) = 1$ if and only if there exists a vertex $v_{n} \in S_{n}(\bar{T})$ such that $(v_{n}, u') \in E(D')$, where $u'$ is the vertex in $V(D') - V(\bar{T})$ such that $N^{+}(u'; D') = S_{1}(\bar{T})$ (see Figure 4.1). Therefore, $\bar{T}$ satisfies the condition of Proposition 4.16, so that $\bar{T} \in \mathcal{T}_{2} \subset \mathcal{F}$.

When $k(T) = 1$ or 2, we can similarly apply Proposition 4.17 or 4.18 to $\bar{T}$ and conclude that $\bar{T} \in \mathcal{T}_{2} \cup \mathcal{T}_{3} \cup \mathcal{T}_{4} \subset \mathcal{F}$.

**Case 2.** Assume that $2 \leq s \leq t < k$.

By Proposition 4.19, $T \in \mathcal{T}_{2} \cup \mathcal{T}_{3} \cup \mathcal{T}_{4} \subset \mathcal{F}$. □

As our next result shows, the condition in Theorem 4.20 is also sufficient.

**Theorem 4.21** If $T \in \mathcal{F}$, then $A(T) = 2$.

**Proof.** Since $A(T) = 2$ if and only if $A(\bar{T}) = 2$, it suffices to prove that $A(T) = 2$ for all trees $T \in \mathcal{T}_{2} \cup \mathcal{T}_{3} \cup \mathcal{T}_{4}$. We proceed by cases.

**Case 1.** Assume that $T \in \mathcal{T}_{2}$.

Let $D$ be the oriented graph of Figure 4.16. Then $\bar{d}(u, w) = 7$. Since $\bar{d}(v, u) \leq 5$ for all $v \in V(D)$, we have $\bar{d}(u, v) \leq 6$ for all $v \in V(D) - \{w\}$; so $me(u) = 7$. Similarly, $me(w) = 7$. We next prove that $me(v) \geq 6$ for all $v \in V(T)$. Suppose $v \in S_{1}$. Since $\bar{d}(v, w) \leq 6$, it follows that $me(v) \geq 6$. Let $v \in S_{6}$. Since $\bar{d}(u, v) \leq 6$, we have $me(v) \geq 6$. Note that $v \in S_{2}$. Then $\bar{d}(v_{4}, v) \leq 6$. Therefore, $me(v_{4}) \geq 6$ and $me(v) \geq 6$ for all $v \in S_{3}$. If $v \in S_{3}$, then choose $v_{1} \in S_{1}$ such that $v$ and $v_{1}$ lie on different branches of $T$ at $v_{4}$. Therefore, $\bar{d}(v, v_{4}) = 6$, implying that $me(v) \geq 6$ for all $v \in S_{3}$. Finally, suppose that $v \in S_{5}$. Let $v_{6} \in S_{6}$ such that $(v, v_{6}) \notin E(T)$. Then $\bar{d}(v, v_{6}) = 6$, from which it follows that $me(v) \geq 6$ for all
Figure 4.16

$v \in S_5$. Since it is routine to check that $me(v) \leq 6$ for all $v \in V(T)$, the proof of this is omitted.

**Case 2. Assume that** $T \in \mathcal{T}_3$.

Let $D$ be the oriented graph of Figure 4.17.

Figure 4.17

By an argument similar to that used in Case 1, it follows that $me(u) = me(w) = 7$, and $me(v) \geq 6$ for all $v \in S_1 \cup S_6$. For $v \in S_2$, we have $\tilde{d}(v_5, v) = 6$. Therefore,
me(v) ≥ 6 and me(v) ≥ 6 for all v ∈ S. If v ∈ S4, then choose v1 ∈ S1 such that v and v1 lie on different branches of T at v5. Therefore, d(v1, v) = 6, implying that me(v) ≥ 6 for all v ∈ S4. Suppose v ∈ S3. Since |S4| ≥ 3, it follows that |S4 - {v4}| ≥ 2. There exists v4' ∈ S4 - {v4} such that (v, v4') ∉ E(T). Then d(v, v4') = 6. Therefore, me(v) ≥ 6 for all v ∈ S3. We omit the proof that me(v) ≤ 6 for all v ∈ V(T).

**Case 3. Assume that T ∈ T4.**

Suppose that T = T1 → T2. Let D be the oriented graph (see Figure 4.18) with

\[ V(D) = V(T) \cup \{u, w\} \]

and

\[ E(D) = E(T) \cup \{(u, v), (w, v) \mid v \in S_1(T_1) \cup S_1(T_2)\} \]
\[ \cup \{(v, u) \mid v \in S_2(T_1) \cup S_2(T_2)\} \]
\[ \cup \{(w, v) \mid v \in S_{2i-1}(T_1), 2 \leq i \leq \left\lceil \frac{k-1}{2} \right\rceil \} \]
\[ \cup \{(v, u) \mid v \in S_{2i}(T_1), 2 \leq i \leq \left\lceil \frac{k-1}{2} \right\rceil \} \]
\[ \cup E_r \]

where \( E_r = \{(w, v) \mid v \in S_{k-1}(T_1)\} \) if k is odd and \( E_r = \{(v, u) \mid v \in S_{k-1}(T_1)\} \) otherwise.

It is clear that me(u) = me(w) = k + 2. We now show that me(v) ≥ k + 1 for all v ∈ V(T). For every vertex v ∈ S_{k+1}, it follows that me(v) ≥ k + 1 since d(u, v) = k + 1. Similarly, d(v, w) = k + 1, where v ∈ S1, implies that me(v) ≥ k + 1 for all v ∈ S1. Since d(v3, x) = k + 1, it follows that me(v3) ≥ k + 1 and me(x) ≥ k + 1, for x ∈ S_{2}(T1) ∪ S_{2}(T2). Observe that d(vi, vi-1) = k + 1, i = 4, 5, ..., k, so me(vi) ≥ k + 1, i = 4, 5, ..., k. Suppose v ∈ S_{k-1}(T1) and v' ∈ S_{1}(T2). Then d(v', v) = k + 1, so me(v) ≥ k + 1 for all v ∈ S_{k-1}(T1). Consider a vertex v ∈ S_{2i-1}(T1), where 2 ≤ i < \left\lceil \frac{k-1}{2} \right\rceil. Then d(v, v_{k-1}) = k + 1. Therefore, me(v) ≥ k + 1 for all v ∈ S_{2i-1}(T1), 2 ≤ i < \left\lceil \frac{k-1}{2} \right\rceil. Similarly, d(v3, v) = k + 1.
implies that $m_e(v) \geq k + 1$ for all $v \in S_{2i}(T_1)$, $2 \leq i < \left[\frac{k-1}{2}\right]$. Therefore, $m_e(v) \geq k + 1$ for all $v \in V(T)$. The proof of $m_e(v) \leq k + 1$ for all $v \in V(T)$ is again omitted. □

We now summarize this Chapter with the following theorem.

**Theorem 4.22** Let $D$ be an oriented acyclic graph. Then $A(D) = 2$ if and only if $D \in T_1 \cup F$.

We conclude this chapter with the following remark. Recall that no tree has appendage number 3. However, there exist oriented trees with appendage number 3. If we let $D$ and $T$ be the oriented graphs in Figure 4.19, then $m_C(D) \equiv T$. Therefore, $A(T) \leq 3$. By Theorem 4.11, an "in-star" $T \in T(1, 1, 2)$ has appendage number at least 3. Therefore, $A(T) = 3$. Similarly, all "out-stars" in $T(2, 2, 2)$ also have appendage number 3. It is natural to inquire about the structure of oriented acyclic graphs having appendage number 3. This is, however, an open problem.
Figure 4.19
CHAPTER V

SUM DISTANCE IN DIGRAPHS

In this chapter, we define another distance on digraphs, called sum distance. We then explore some results concerning sum distance in oriented graphs.

5.1 Definition of Sum Distance and Basic Properties

We have seen in Chapter III that the standard distance on graphs can be extended to m-distance on digraphs such that the metric property of distance is preserved. Since the m-distance \( md(u, v) = \max\{d(u, v), d(v, u)\} \), it can be viewed as the "worst case" distance between two vertices \( u \) and \( v \) of a digraph. It is natural to also define an "average" distance between two vertices of a strong digraph by taking the average value of \( d(u, v) \) and \( d(v, u) \). More formally, for vertices \( u \) and \( v \) in a strong digraph \( D \), we define the average distance \( ad(u, v) \) between \( u \) and \( v \) as

\[
ad(u, v) = \frac{d(u, v) + d(v, u)}{2}.
\]

Let \( G \) be a connected graph and \( G^* \) the digraph obtained by replacing each edge of \( G \) by a symmetric pair of arcs. Then \( ad_{G^*}(u, v) = d_G(u, v) \). In this sense, we can say that the average distance of two vertices in strong digraphs is an extension of the standard distance in graphs. We define the sum distance (or s-distance) \( sd(u, v) \) between vertices \( u \) and \( v \) in a strong digraph \( D \) as

\[
sd(u, v) = d(u, v) + d(v, u).
\]

Clearly, \( sd(u, v) = 2 \ ad(u, v) \) for all vertices \( u \) and \( v \) in \( D \). Therefore, \( sd(u, v) \) and \( ad(u, v) \) are isomorphic metrics on the vertex set of a strong digraph. For the sake of
convenience we consider only the $s$-distance in this Chapter.

The following observations will be used throughout this chapter. For $u, v,$ and $w \in V(D)$:

(a) $sd(u, u) = 0$ for each $u \in V(D)$;

(b) $sd(u, v) \neq 1$ for all $u, v \in V(D)$;

(c) $sd(u, v) = 2$ if and only if $(u, v), (v, u) \in E(D)$;

(d) $sd(u, v) = sd(v, u)$; and 

(e) $sd(u, w) = d(u, w) + d(w, u)$

\[ \leq [d(u, v) + d(v, w)] + [d(w, v) + d(v, u)] \]

\[ = [d(u, v) + d(v, u)] + [d(v, w) + d(w, v)] \]

\[ = sd(u, v) + sd(v, w). \quad \text{[Triangle Inequality]} \]

Therefore, the $s$-distance is a metric on $V(D)$.

The $s$-distances in a strong digraph $D$ and in its converse $\overline{D}$ are identical. We state this without proof as our first result of this Chapter.

**Theorem 5.1** If $D$ is a strong digraph, then $sd_D(u, v) = sd_{\overline{D}}(u, v)$ for all vertices $u, v \in V(D)$.

For each vertex $v$ in a strong digraph $D$, we define the $s$-eccentricity $se(v)$ of $v$ as

\[ se(v) = \max \{ sd(u, v) \mid u \in V(D) \}. \]

It is well known for a connected graph $G$ that if $uv \in E(G)$, then $|e(u) - e(v)| \leq 1$. However, the difference between $s$-eccentricities of two adjacent vertices in a strong digraph can be arbitrarily large. For an integer $n \geq 2$, let $D$ be the digraph in Figure 5.1. Then $se(w) = n$ and $se(u_1) = se(v_1) = se(u_{n-1}) = se(v_{n-1}) = 2n$. Therefore, $|se(v) - se(w)| = n$ for all vertices $v$ adjacent to or from $w$. 

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The \textit{s-radius} of a strong digraph $D$ is $s\text{-}rad(D) = \min\{s\text{-}r(v) \mid v \in V(D)\}$ and the \textit{s-diameter} of $D$ is $s\text{-}diam(D) = \max\{s\text{-}r(v) \mid v \in V(D)\}$. Since $sd(u, v) \geq 2$ for all $u \neq v$ and since $sd(u, v) = 2$ if and only if there exist symmetric arcs $(u, v)$ and $(v, u)$ in $D$, it follows for all oriented graphs $D$ that $s\text{-}rad(D) \geq 3$.

Observe that a graph has diameter 1 if and only if it is a complete graph. Our next result gives some information about those strong oriented graphs with minimum s-diameter, namely s-diameter 3.

\textbf{Theorem 5.2} A strong oriented graph $D$ has s-diameter 3 if and only if any two vertices of $D$ lie on a common triangle.

\textbf{Proof.} Let $u, v \in V(D)$. Since $D$ is an oriented graph, it follows that $d(u, v) + d(v, u) \geq 3$. On the other hand, $d(u, v) + d(v, u) = sd(u, v) \leq s\text{-}diam(D) = 3$. Therefore, $d(u, v) + d(v, u) = 3$. Let $P$ be a shortest $u$-$v$ path and $Q$ a shortest $v$-$u$ path. Then $P \cup Q$ is a triangle that contains both $u$ and $v$.

The proof of the sufficiency is obvious, and is therefore omitted. $\Box$
A vertex $u$ in a tournament $T$ is called a **king** if $\hat{d}(u, v) \leq 2$ for all $v \in V(T)$. In [17] it was shown that every vertex of maximum outdegree in a tournament is a king. A tournament in which every vertex is a king is called a **king tournament**. By Theorem 5.2, $s$-diam$(D) = 3$ if and only if $D$ is a king tournament.

We have seen in Chapter III that there exist oriented graphs $D$ that are not tournaments and that have $m$-diameter 2. Therefore, $m$-diam$(D) = 2$ does not imply that $s$-diam$(D) = 3$. However, the implication does hold if $D$ is a tournament.

**Theorem 5.3** Let $D$ be a tournament. Then $m$-diam$(D) = 2$ if and only if $s$-diam$(D) = 3$.

**Proof.** Suppose $D$ is a tournament with $m$-diam$(D) = 2$. Let $u$ and $v$ be vertices of $D$. Without loss of generality, we assume that $(u, v) \in E(D)$. Then $(v, u) \notin E(D)$ so $\hat{d}(v, u) \geq 2$, implying that $sd(u, v) = \hat{d}(u, v) + \hat{d}(v, u) = 1 + 2 = 3$. Since $u$ and $v$ are arbitrary, it follows that $s$-diam$(D) = 3$. The proof of the sufficiency is omitted. □

Our next result gives familiar bounds for $s$-diam$(D)$.

**Theorem 5.4** If $D$ is a strong digraph, then $s$-rad$(D) \leq s$-diam$(D) \leq 2 s$-rad$(D)$.

**Proof.** The first inequality follows from the definitions. For the second inequality, let $u$ and $v$ be vertices of $D$ such that $sd(u, v) = s$-diam$(D)$ and let $w \in V(D)$ such that $se(w) = s$-rad$(D)$. Then $s$-diam$(D) = sd(u, v) \leq sd(u, w) + sd(w, v) \leq 2 se(w) = 2 s$-rad$(D)$. □

It will be shown in the next section that these bounds are sharp.
5.2 The $s$-Eccentric Sets of Oriented Graphs

The eccentricity set of a connected graph $G$ is simply the set of eccentricities of the vertices of $G$. The \textit{s-eccentricity set} $se(D)$ of a strong digraph $D$ is similarly defined as $\{se(v) | v \in V(D)\}$. Recall, for every connected graph $G$, that $rad(G) \leq diam(G) \leq 2 \ rad(G)$. Therefore, not every set of consecutive integers can be the eccentricity set of some connected graph. Behzad and Simpson [1] characterized the eccentricity set of connected graphs. It turns out that $rad(G) \leq diam(G) \leq 2 \ rad(G)$ is the only additional restriction. Since the difference between $s$-eccentricities of two adjacent vertices can be arbitrarily large, one might expect that the $s$-eccentricity set need not be a set of consecutive integers. This is indeed the case. Before presenting a characterization of the $s$-eccentricity set of an oriented graph, we need two additional definitions. A vertex $w$ is an \textit{$s$-eccentric vertex} of $v$ if $sd(v, w) = se(v)$. An oriented graph $D$ is \textit{upper $s$-eccentric} (or, more simply, \textit{upper eccentric}) if for every vertex $v$ of $D$ there exists an $s$-eccentric vertex $w$ of $v$ such that $\min\{d(v, w), d(w, v)\} = \left\lceil \frac{se(v)}{2} \right\rceil$. Since $se(v) = d(v, w) + d(w, v), \left\lceil \frac{se(v)}{2} \right\rceil$ is an upper bound of $\min\{d(v, w), d(w, v)\}$. It is clear that if $se(v) = 3$ and $w$ is a $s$-eccentric vertex of $v$ then $\min\{d(v, w), d(w, v)\} = \left\lceil \frac{se(v)}{2} \right\rceil$. Therefore, any oriented graph of $s$-diameter 3 is upper eccentric.

\textbf{Theorem 5.5} Let $S = \{e_1, e_2, ..., e_k\}$ be a set of integers with $3 \leq e_1 < e_2 < ... < e_k \leq 2e_1$. Then there exists an oriented graph $D$ such that $se(D) = S$.

\textbf{Proof.} We actually show that there exists an upper eccentric oriented graph $D$ with $se(D) = S$. We proceed by induction on $k$. If $k = 1$, then for $D$ a directed cycle of length $e_1$, we have $se(D) = \{e_1\}$. Clearly, $D$ is upper eccentric. Assume the result is
true for $k = n - 1$ and assume $k = n$. By the inductive hypothesis, there exists an upper eccentric oriented graph $H$ with $se(H) = \{e_2, e_3, ..., e_n\}$. We construct an oriented graph $D$ by replacing the vertex $v_0$ of the directed cycle: $v_0, v_1, ..., v_m$ by $H$, where $m = e_1 - 1$ (see Figure 5.2).

First observe that $se(v_i) = e_i$ for $1 \leq i \leq m$. We claim that $se_D(v) = se_H(v)$ for all $v \in V(H)$, from which it will follow that $se(D) = S$. Let $v \in V(H)$. It is clear that $sd_D(v_i) = e_i$ for $1 \leq i \leq m$. Since $H$ is upper eccentric, there exists an s-eccentric vertex $w$ of $v$ such that $\min\{d_H(v, w), d_H(w, v)\} = \lceil\frac{se_H(v)}{2}\rceil$, which implies that $\max\{d_H(v, w), d_H(w, v)\} = \lceil\frac{se_H(v)}{2}\rceil$. Since $se_H(v) \leq e_n \leq 2 e_1$, it follows that $\max\{d_H(v, w), d_H(w, v)\} = \lceil\frac{se_H(v)}{2}\rceil \leq e_1$. Therefore,

$$d_D(v, w) = \min\{d_H(v, w), e_1\} = d_H(v, w)$$

and

$$d_D(w, v) = \min\{d_H(w, v), e_1\} = d_H(w, v).$$

So $sd_D(v, w) = sd_H(v, w) = se_H(v) \geq e_2 > e_1$.

We denote by $S(v)$ the set of all s-eccentric vertices $w$ of $v$ in $H$. 

![Figure 5.2](image-url)
Therefore,
\[ \text{se}_D(v) = \max \{ s_dD(v, x) \mid x \in V(D) \} \]
\[ = \max \{ \max \{ s_dD(v, x) \mid x \in V(H) \}, \max \{ s_dD(v, v_i) \mid i = 1, 2, \ldots, m \} \} \]
\[ = \max \{ \max \{ s_dD(v, x) \mid x \in S(v) \}, \max \{ s_dD(v, x) \mid x \in V(H) - S(v) \}, e_1 \} \]
\[ = \max \{ \max \{ s_dH(v, x) \mid x \in S(v) \}, e_1 \} = \text{se}_H(v). \]

Hence, \( \text{se}(D) = \text{se}(H) \cup \{ e_1 \} = S. \)

We now show that \( D \) is upper eccentric. Observe, for \( v \in V(H) \), that a vertex \( w \) is an \( s \)-eccentric vertex of \( v \) in \( D \) if and only if \( w \) is an \( s \)-eccentric vertex of \( v \) in \( H \). By the inductive hypothesis, for \( v \in V(H) \) there exists an \( s \)-eccentric vertex \( w \) of \( v \) such that \( \min \{ d_D(v, w), d_H(w, v) \} = \frac{\text{se}(v)}{2} \). Choose \( v_0 \in V(H) \). Then there exists a directed cycle \( C: v_0, v_1, \ldots, v_m \) in \( D \). For an integer \( i \), \( 1 \leq i \leq m \), let \( v_j \) be an opposite diagonal vertex \( v_i \) on cycle \( C \). Therefore, \( v_j \) is an \( s \)-eccentric vertex of \( v_i \) satisfying
\[ \min \{ d_D(v_i, v_j), d_H(v_j, v_i) \} = \frac{\text{se}(v_i)}{2}. \]

Hence, \( D \) is upper eccentric and the proof is complete. \( \square \)

The digraph \( D \) constructed in the proof of Theorem 5.5 for which \( \text{se}(D) = S \) has especially large order. We are able to reduce this order significantly in the case where \( |S| \leq 2 \).

**Theorem 5.6** Let \( S = \{ r, d \} \) be a set of integers with \( 3 \leq r \leq d \leq 2r \). Then there exists an oriented graph \( D \) of order at most \( r+2 \) such that \( \text{se}(D) = S \).

**Proof.** We proceed by cases.

**Case 1.** Assume that \( d = r \).

Let \( D \) be a directed cycle of length \( r \). Then \( s\text{-rad}(D) = s\text{-diam}(D) = r \),
so \( se(D) = S \).

**Case 2. Assume that** \( d = r + 1 \).

We construct the oriented graph \( D \) by first identifying the vertex \( v \) of the triangle \( v, w, x \) with the vertex \( v_1 \) of the directed cycle: \( v_1, v_2, ..., v_r \) and label the new vertex as \( v_1 \). Then we join \( v_3 \) from \( x \), and \( v_{r-1} \) to \( w \) (see Figure 5.3).

**Figure 5.3**

Observe that \( se(v_1) = r \) and \( se(v_i) = r \) for \( 3 \leq i \leq r - 1 \). Since \( d(v_2, w) = r - 2 \) and \( d(w, v_2) = 3 \), it follows that \( sd(v_2, w) = d(v_2, w) + d(w, v_2) = r - 2 + 3 = r + 1 \). Similarly, \( sd(v_2, x) = r + 1 \). Therefore, \( se(v_2) = r + 1 \). Similarly, \( se(v_r) = se(w) = se(x) = r + 1 \). Therefore, \( se(D) = S \). Clearly, \( p(D) = d + 1 \).

**Case 3. Assume that** \( d = r + 2 \).

Let \( D \) be the oriented graph shown in Figure 5.4. It is clear that \( se(v_1) = se(v_2) = r \). Observe that \( d(v_i, w) = r - i + 1 \) and \( d(w, v_i) = i + 1 \) for \( 2 \leq i \leq r - 1 \). Therefore, \( sd(v_i, w) = r + 2 \) and, similarly, \( sd(v_i, x) = r + 2 \) for \( 2 \leq i \leq r - 1 \). Furthermore, since \( sd(v_i, v_j) = r \) for \( 1 \leq i < j \leq r \), it follows that \( se(v_i) = se(w) = se(x) = r + 2 \), where \( 2 \leq i \leq r - 1 \). Therefore, \( se(D) = S \). In this case, \( p(D) = d \).
Case 4. Assume that \( d \geq r + 3 \).

Let \( D \) be the oriented graph given in Figure 5.5, where \( k = d - r - 2 \). Since \( d \geq r + 3 \) and \( d \leq 2r \), it follows that \( k \geq 1 \) and \( k \leq r - 2 \). It is clear that \( se(v_i) = r \) for \( 0 \leq i \leq k \). Observe that \( sd(w, v_i) = k + 2 + r = d \) for \( k + 1 \leq i \leq r - 1 \). Therefore, \( se(w) = se(v_i) = d \) for \( k + 1 \leq i \leq r - 1 \), so that \( se(D) = S \). Further, \( p(D) = r + 1 < d \). \( \square \)
This result can be restated as Theorem 5.6' from which it will follow that the inequalities in Theorem 5.3 are sharp.

**Theorem 5.6'** For integers \( r \) and \( d \) with \( 3 \leq r \leq d \leq 2r \), there exists an oriented graph \( D \) such that \( s\text{-rad}(D) = r \) and \( s\text{-diam}(D) = d \).

Let \( f(r, d) \) denote the minimum order of an oriented graph of \( s\)-radius \( r \) and \( s\)-diameter \( d \), where \( 3 \leq r \leq d \leq 2r \). By Theorem 5.6, \( f(r, d) \leq r + 2 \) for \( 3 \leq r \leq d \leq 2r \). From the construction in the proof of Theorem 5.6, it follows that \( f(r, r) \leq r \) for all \( r \geq 3 \). However, we do not know the exact value of \( f(r, r) \), where \( r \geq 3 \).

### 5.3 The \( s \)-Centers and Sum Appendage Numbers of Oriented Graphs

A vertex of minimum \( s \)-eccentricity is called an \textit{s-central vertex} and the \textit{s-center} \( sC(D) \) of a digraph \( D \) is the subdigraph induced by its \( s \)-central vertices. The following result is an analogy of Theorem 3.7. We state it without the proof.

**Theorem 5.7** The \( s \)-center of every strong oriented graph \( D \) lies in a single block of the underlying graph of \( D \).

As was the case involving \( m \)-distance every oriented graph is the \( s \)-center of some strong oriented graph.

**Theorem 5.8** Let \( D \) be an oriented graph. Then there exists an oriented graph \( H \) such that \( sC(H) \equiv D \) and \( p(H) \leq p(D) + 6 \).

**Proof.** If \( sC(D) = D \), then the oriented graph \( H = D \) has the desired property. If \( sC(D) \neq D \), then we construct the oriented graph \( H \) by adding to \( D \) the six vertices \( v_i \) \((i = 1, 2, ..., 6) \) to \( D \), the arcs \( (v_1, v_2), (v_2, v_3), (v_3, v_1), (v_4, v_5), (v_5, v_6) \) and
(v_6, v_4), together with the arcs joining all vertices of D to v_1 and v_4 and all arcs joining the vertices of D from v_2 and v_5 (see Figure 5.6).

![Diagram](image)

Figure 5.6

It suffices to prove that \( se(v_i) \geq 9 \) for \( i = 1, 2, \ldots, 6 \) and \( se_H(x) = 6 \) for \( x \in V(D) \). We first verify that \( se(v_i) \geq 9 \) (\( i = 1, 2, \ldots, 6 \)). This follows from the observation that

\[
sd(v_1, v_6) = sd(v_2, v_6) = sd(v_3, v_4) = sd(v_3, v_5) = 9.
\]

We now verify the second claim. Suppose \( u, v \in V(D) \). Since \( \bar{d}_H(u, v) \leq 3 \), it follows that \( sd_H(u, v) \leq 6 \). Clearly, there exist triangles \( v_1, v_2, v \) and \( v_4, v_5, v \) in H. Therefore, \( sd(v, v_i) = 3 \) for \( i = 1, 2, 4, 5 \). Finally, since \( sd(v, v_3) = sd(v, v_6) = 6 \), we have \( se(v) = 6 \) for \( v \in V(D) \). The proof is complete. \( \square \)

Theorem 5.8 then suggests the following definition. The **sum appendage number** \( A_5(D) \) of an oriented graph \( D \) is the minimum number of vertices that must be added to \( D \) to produce an oriented graph \( H \) such that \( sC(H) = D \). According to Theorem 5.8, then, \( A_5(D) \) is defined and bounded above by 6 for every oriented graph \( D \). However, we do not know whether the bound 6 is best possible. An
improved bound can be given, however, if $D$ is an oriented graph with $s\text{-rad}(D) \geq 6$.

**Theorem 5.9** If $D$ is an oriented graph with $s\text{-rad}(D) > 6$ then $A_5(D) \leq 5$.

**Proof.** If $sC(D) = D$, then we let $H = D$. Suppose $sC(D) \neq D$. Define $H$ as in Figure 5.7. Then, a similar argument to the proof of Theorem 5.4 implies that $se(v_i) = 7$ (1 ≤ $i$ ≤ 5) and $se_H(v) = 6$ for $v \in V(D)$. Therefore, $sC(H) \equiv D$. Further, $p(H) = p(D) + 5$. □

If $D$ is a tournament, then an even better bound for $A_5(D)$ can be given.

![Figure 5.7](image)

**Theorem 5.10** If $D$ is a tournament, then $A_5(D) \leq 4$.

**Proof.** Define $S = \{v \in V(D) \mid se(v) = 3\}$. We consider two cases.

**Case 1. Assume that** $S = \emptyset$.

Let $H$ be the oriented graph as in Figure 5.8. By a similar argument to the proof of Theorem 5.4, it follows that $se(v_i) = 6$ and $se_H(v) = 4$ for all $v \in V(D)$. Therefore, $sC(H) \equiv D$ and $p(H) = p(D) + 4$. 

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Case 2. Assume that \( S \neq \emptyset \).

Let

\[
S_1 = \{ v \in S \mid (v, w) \in E(D) \text{ for some } w \in V(D) - S \}
\]

and let

\[
S_2 = S - S_1.
\]

We construct the oriented graph \( H \) by adding to \( D \) the four vertices \( v_i \) (1 ≤ \( i \) ≤ 4), the arcs \((v_1, v_2)\) and \((v_3, v_4)\), together with the arcs joining \( v_2 \) and \( v_4 \) to all vertices in \( S \cup S_1 \) and the arcs joining \( v_1 \) and \( v_3 \) from all vertices in \( S \cup S_2 \) (see Figure 5.9).
We show that \( se_H(v) = 4 \) for \( v \in V(D) \). Suppose first that \( v \in V(D) - S \). Then \( sd(v, v_i) = 3 \) \( (1 \leq i \leq 4) \). Let \( w \in S_1 \cup S_2 \). Since \( se_D(w) = 3 \), it follows that \( sd_H(v, w) \leq sd_D(v, w) \leq se_D(w) = 3 \). For \( w \in V(D) - S - \{ v \} \), we assume, without loss of generality, that \( (v, w) \in E(D) \). Then \( H \) contains a 4-cycle \( v, w, v_1, v_2, v \). Therefore, \( sd_H(v, w) \leq 4 \). Further, let \( v' \in V(D) - S \) be an \( s \)-eccentric vertex of \( v \) in \( D \). Then \( sd_H(v, v') = 4 \). Therefore, \( se_H(v) = 4 \) for \( v \in V(D) - S \).

Consider a vertex \( v \in S_1 \). It follows that \( sd_H(v, x) \leq sd_D(v, x) \leq se_D(v) = 3 \) for all \( x \in V(D) \). It is clear that \( sd_H(v, v_i) = 4 \) \( (1 \leq i \leq 4) \). Therefore, \( se_H(v) = 4 \) for \( v \in S_1 \). Similarly, \( se_H(v) = 4 \) for \( v \in S_2 \). Observe that \( sd(v_1, v_3) = sd(v_2, v_4) = 6 \). Therefore, \( sC(H) \equiv D \). Clearly, \( p(H) = p(D) + 5 \). □

5.4 The \( s \)-Peripheries of Oriented Graphs

We now turn to \( s \)-peripheries of digraphs. A vertex of maximum \( s \)-eccentricity is called an \( s \)-peripheral vertex. The \( s \)-periphery \( sP(D) \) of a digraph \( D \) is the subdigraph induced by its \( s \)-peripheral vertices. Our next theorem gives a characterization of \( s \)-peripheries of oriented graphs.

**Theorem 5.11** Let \( D \) be an oriented graph. Then \( D \) is isomorphic to the \( s \)-periphery of some oriented graph if and only if either all vertices of \( D \) have \( s \)-eccentricity 3 or no vertices have \( s \)-eccentricity 3.

**Proof.** Suppose, to the contrary, that \( D \equiv sP(H) \) and there exist vertices \( u, v \in V(D) \) with \( se_D(u) = 3 \) and \( se_D(v) \geq 4 \). Without loss of generality, we assume that \( D \) is an induced subdigraph of \( H \) (that is, \( D = sP(H) \)). Let \( w \) be an eccentric vertex of \( u \) in \( H \). Then \( se_H(w) \geq sd_H(u, w) = se_H(u) = s-diam(H) \), so \( w \in V(D) \). Since \( D \) is an induced subdigraph of \( H \), it follows that \( sd_H(u, w) \leq sd_D(u, w) \). Therefore \( s-diam(H) = \)
\[ s_{eH}(u) = s_{dH}(u,w) \leq s_{dD}(u,w) \leq s_{eD}(u) = 3 \leq s\text{-rad}(H), \] from which it follows that
\[ H = sP(H) = D \] and \[ s\text{-rad}(D) = s\text{-diam}(D) = 3. \] Therefore, \[ s_{eD}(w) = 3 \] for all \( w \in V(D) \). This contradicts the assumption that \( s_{eD}(v) \geq 4 \).

To prove the sufficiency, we consider two cases.

**Case 1.** Assume that \( s_{eD}(v) = 3 \) for all \( v \in V(D) \).

In this case \( sP(D) = D \).

**Case 2.** Assume that \( s_{eD}(v) \geq 4 \) for all \( v \in V(D) \).

Define \( S = \{ v \in V(D) \mid \deg_D v < p(D) - 1 \} \). We consider two subcases.

**Subcase 2.1** Assume that \( S = \emptyset \).

Let \( H \) be the oriented graph in Figure 5.10. For \( w \in V(D) \), there exists a triangle \( u, v, w, u \) in \( H \). Therefore, \( s_e(u) = s_e(v) = 3 \). Since \( S = \emptyset \), it follows that \( D \) is a tournament. Suppose \( w' \in V(D) \) and, without loss of generality, that \( (w, w') \in E(D) \). Then there exists a 4-cycle \( u, v, w, w', u \) in \( H \), so \( s_{dH}(w, w') \leq 4 \). Therefore, \( s_{eH}(w) \leq 4 \) for all \( w \in V(D) \). To show that \( s_{eH}(w) \geq 4 \), we let \( w' \) be an s-eccentric vertex of \( w \) in \( D \). Then \( s_{dD}(w, w') = s_{eD}(w) \geq 4 \). Observe that every \( w-w' \) circuit that contains \( u \) has length 3. It follows that \( s_{dH}(w, w') \geq 4 \). Therefore, \( s_{eH}(w) = 4 \) for all \( w \in V(D) \).

\[ H: \]

![Figure 5.10](image_url)
Subcase 2.2 Assume that $S \neq \emptyset$.

Suppose $u \in S$. Since $\deg_D u < p(D) - 1$, there exists $v \in V(D)$ such that $(u, v), (v, u) \notin E(D)$. Thus $\deg_D v < p(D) - 1$, so $v \in S$. Therefore, $|S| \geq 2$. Let $n = (|S|^2)$. So $n \geq 1$. Clearly, the set $S$ has $n$ different subsets of cardinality 2. Suppose that there are $m$ of them each induces an empty digraph in $D$. Then $m \geq 1$. We denote these subsets by

$$S_i = \{v_{i1}, v_{i2}\}, \; i = 1, 2, \ldots, m.$$ 

Let $H_0 = D$. For $1 \leq k \leq m$, we recursively define $H_k$ to be the oriented graph with

$$V(H_k) = V(H_{k-1}) \cup \{w_{k1}, w_{k2}, w_{k3}, w_{k4}\}$$

and

$$E(H_k) = E(H_{k-1}) \cup \{(v, w_{k2}), (v, w_{k3}), (w_{k2}, v), (w_{k3}, v) \mid v \in V(H_{k-1}) - S_k\}$$

$$\cup \{(w_{k1}, w_{k2}), (w_{k3}, w_{k4}), (w_{k1}, v_{k1}), (w_{k1}, v_{k2}), (v_{k1}, w_{k1}), (w_{k2}, v_{k1}), (w_{k3}, v_{k2}), (w_{k4}, v_{k2})\}$$

(see Figure 5.11).

We claim that

(a) if $\sigma_E(H_{k-1})(v) = 3$, then $\sigma_E(H_k)(v) = 3$;

(b) $\sigma_d(H_k)(v_1, v_2) = 4$ and $\sigma_E(H_k)(w_{ki}) = 3$ for $i = 1, 2, 3, 4$; and

(c) if $\sigma_E(H_{k-1})(v) \geq 4$, then $\sigma_E(H_k)(v) \geq 4$.

Since $H_{k-1}$ is an induced subdigraph of $H_k$, it follows that $\sigma_E(H_{k-1})(v) \geq \sigma_E(H_k)(v)$. Therefore, if $\sigma_E(H_k)(v) = 3$, then $3 \leq \sigma_E(H_k)(v) \leq \sigma_E(H_{k-1})(v) = 3$, that is $\sigma_E(H_k)(v) = 3$. This proves (a). For $v \in V(H_k) - S_k$, there exist the following triangles in $H_k$:

$$v, w_{k1}, w_{k2}, v; \; v, w_{k3}, w_{k4}, v; \; v_{k1}, w_{k1}, v_{k2}, v_{k3}; \; v_{k1}, w_{k2}, v_{k3}, v_{k4}.$$
Since $v$ is an arbitrary vertex in $V(H_k) - S_k$, it follows that $w_{k_1}$ and $z$ lie on a common triangle in $H_k$ for all $z \in V(H_k) - \{w_{k_1}\}$, where $i = 1, 2, 3, 4$. Therefore, $se_{H_k}(w_{k_i}) = 3$ for $i = 1, 2, 3, 4$. Since there is no arc between $v_{k_1}$ and $v_{k_2}$ in $H_k$, we have $sd_{H_k}(v_{k_1}, v_{k_2}) \leq 4$. Furthermore, since $H_k$ contains a 4-cycle: $v_{k_1}, w_{k_3}, v_{k_2}, w_{k_4}, v_{k_1}$ it follows that $sd_{H_k}(v_{k_1}, v_{k_2}) = 4$. To prove (c), we assume that $se_{H_k}(v) \geq 4$. If $v = v_{k_1}$ or $v = v_{k_2}$, then $se_{H_k}(v) \geq sd_{H_k}(v_{k_1}, v_{k_2}) = 4$. If $v \in V(H_k) - S_k$, we let $v'$ be an s-eccentric vertex of $v$ in $H_{k-1}$. If $v' = v_{k_1}$ or $v' = v_{k_2}$, then $v$ and $v'$ lie on a 4-cycle in $H_k$: $v, w_{k_3}, v', w_{k_1}$. Therefore, $se_{H_k}(v) \geq sd_{H_k}(v, v') = 4$. If $v' \in V(H_k) - S_k$, then $se_{H_k}(v) \geq \min\{sd_{H_k}(v, v'), 3 + d_{H_{k-1}}(v, v')\} \geq 4$. Therefore, the claim is true.

Let $H = H_m$. We now verify $sP(H) \equiv D$ by proving that $se_H(x) = 4$ for all $x \in V(D)$, and that $se_H(y) = 3$ for all $y \in V(H) - V(D)$. We show that $se_{H_1}(v) = 4$ for all $v \in V(D) - S$. Suppose $v \in V(D) - S$. It is clear that $sd_{H_1}(v, x) \leq 4$ for $x \in$
Furthermore, since \( v \notin S \), it follows that \( \deg_D v = p(D) - 1 \), that is \( v \) is adjacent to or from every other vertex in \( D \). Suppose \( w \in V(D) - S - \{v\} \) with \((v, w) \in E(D)\). Then the 4-cycle \( v, w, w_1, w_2 \), contains both \( v \) and \( w \), so \( \text{sd}_{H_1}(v, w) \leq 4 \). Therefore, \( \text{sd}_{H_1}(v, x) \leq 4 \) for all \( x \in V(H_1) \). Let \( v' \) be an \( s \)-eccentric vertex of \( v \) in \( D \). Then \( \text{sd}_D(v, v') = \text{se}_D(v) \geq 4 \), implying that \( \text{sd}_{H_1}(v, v') = 4 \). Hence \( \text{se}_{H_1}(v) = 4 \) for \( v \in V(D) - S \).

Note that \( V(H) = V(D) \cup \{w_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq 4\} \). By the claim, we have \( \text{se}_H(w_{ij}) = 3 \), where \( 1 \leq i \leq m \) and \( 1 \leq j \leq 4 \). Since \( \text{se}_{H_1}(v) = 4 \) for \( v \in V(D) - S \), it follows by the claim that \( 4 \leq \text{se}_H(v) \leq \text{se}_{H_1}(v) = 4 \), that is, \( \text{se}_H(v) = 4 \) for \( v \in V(D) - S \). Suppose now that \( v \in S \). It is clear that \( \text{sd}_H(v, x) \leq 4 \) for \( x \in (V(D) - S) \cup \{w_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq 4\} \). Since \( v \in S \), there exists an integer \( i \) \((1 \leq i \leq m)\) such that \( S_i = \{v, v'\} \) for some \( v' \in S \). For \( w \in S - \{v, v'\} \), the oriented graph \( H_i \) contains a 4-cycle: \( v, w, w_1, v \). Therefore, \( \text{sd}_{H_i}(v, w) \leq 4 \), so \( \text{sd}_H(v, w) = 4 \). Observe that \( \text{sd}_{H_i}(v, v') = 4 \), implying that \( \text{sd}_H(v, v') = 4 \). Therefore, \( \text{se}_H(v) = 4 \) for \( v \in S \). The proof is complete. \( \square \)

It is important to note that there exists an oriented graph that is isomorphic to the \( s \)-periphery of some oriented graph but not to the \( m \)-periphery of an oriented graph. For example, let \( D \) be the oriented graph shown in Figure 5.12 (a). Then \( s \text{-rad}(D) = s \text{-diam}(D) = 4 \), so \( sP(D) = D \). But, since \( \text{me}(u) = 2 \) and \( \text{me}(v) = 3 \), by Theorem 3.10 \( D \) cannot be the \( m \)-periphery of an oriented graph. Similarly, consider the oriented graph \( D \) in Figure 5.12 (b). Then \( m \text{-diam}(D) = 2 \), so \( mP(D) = D \). However, since \( \text{se}(u) = 3 \) and \( \text{se}(v) = 4 \), by Theorem 5.4 \( D \) cannot be the \( s \)-periphery of an oriented graph. There also exist oriented graphs \( D \) such that \( D \) is neither the \( m \)-periphery nor the \( s \)-periphery of any oriented graph (see Figure 5.12 (c)).
Note also that if \( m\text{-rad}(D) \geq 3 \), then \( s\text{-rad}(D) \geq 4 \) and that if \( s\text{-diam}(D) = 3 \), then \( m\text{-diam}(D) = 2 \). Therefore, if \( m\text{-rad}(D) \geq 3 \), then there exist oriented graphs \( H_1 \) and \( H_2 \) such that \( mP(H_1) = sP(H_2) = D \). Clearly, if \( s\text{-diam}(D) = 3 \), then \( mP(D) = sP(D) = D \).

Figure 5.12

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REFERENCES


