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Generalization of the Form of the T Matrix Operator of Non-Relativistic Scattering Theory

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GENERALIZATION OF THE FORM OF THE
T MATRIX OPERATOR OF NON-RELATIVISTIC
SCATTERING THEORY

by

Francis Leone

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Master of Arts

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Francis Leone
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MASTERS THESIS

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>INTRODUCTION</td>
<td>3</td>
</tr>
<tr>
<td>II</td>
<td>DERIVATION OF THE FORM OF THE T MATRIX OPERATOR</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Potential Scattering</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>General Form of T Matrix Operator for Most General Case which Includes Isospin</td>
<td>16</td>
</tr>
<tr>
<td>III</td>
<td>DETERMINATION OF THE MATRIX ELEMENTS OF THE GENERAL T MATRIX OPERATOR FOR THE CASE IN WHICH THE PROJECTILE AND TARGET ARE INITIALLY IN THEIR GROUND STATES</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>Potential Scattering</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>Most General Case which Includes Isospin</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
<td>37</td>
</tr>
</tbody>
</table>

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CHAPTER I
INTRODUCTION

One of the most powerful methods of solving the two body scattering problem is the Green's function analysis. In such an analysis the geometrical and physical factors of the problem are separated. The physical factor is called the T matrix. The elements of this matrix are directly related to the scattering amplitude and differential cross section and also directly relate the final wave function to the initial incident plane wave function. In the Green's function analysis the elements of the T matrix were calculated for the special case in which both the incident projectile and target were initially in their ground states.

The most general form of the T matrix operator has been formulated and verified. This was done for the simple cases of potential scattering and spinless particle scattering, and for more complicated cases such as a spinless projectile incident on a target having spin, and for the case in which both the target and projectile have spin. Finally, the most general case in which isospin is taken into account was undertaken. Both elastic and inelastic scattering are included in these results. The T matrix operator can be used in all non-relativistic two body scattering problems to obtain the final wave function from the initial one even if the projectile and target are initially in an excited state. The calculations involved in each of the cases mentioned above are done in much the same manner. There-
fore, in what follows, only the case of potential scattering and the most general case will be presented. The matrix elements of the general $T$ operator will be calculated for the case in which the projectile and target are initially in their ground states.
CHAPTER II

DERIVATION OF THE FORM OF THE T MATRIX OPERATOR

Potential Scattering

From the Green's function analysis done by Dr. M. Soga of Western Michigan University the integral equation for the radial parts of the total wave function of the general two body scattering system in terms of the relative coordinate \( r \) is

\[
\Psi_l^\pm(kr) = \pm^l \frac{\partial}{\partial r} \Psi_l^\pm(kr) - iK \int_0^\infty \frac{\partial}{\partial r} \Psi_l^\pm(kr) \hat{h}_l^{(0)}(kr) \Psi_l^\pm(r') \times \hat{h}_l^{(0)}(kr') \, r^2 \, dr'
\]

where,

\[
\Psi_l^\pm(r') = \frac{2\mu}{\hbar^2} V(r')
\]

and,

\( r'' \) = the relative coordinate in the primed notation, in which it designates the scattering range of the scattering center.

\( \mu \) = the reduced mass of the system,

\( V(r) \) = the scattering potential.

\( \hat{j}_l^{(0)}(kr) \) = the spherical Bessel function of order \( l \).

\( \hat{h}_l^{(0)}(kr) \) = the spherical Hankel function of the first kind.

\( r = r_l \) and \( r' = r', \) when \( r \leq r' \)

\( r = r_l \) and \( r = r', \) when \( r > r' \)

\( \Psi_l^\pm(kr) \) = the radial part of the system's total wave function corresponding to \( l \).

\( K \) = the initial wave number of the plane wave of the incident
And \( l \) is simply the orbital angular momentum quantum number, which has the values,

\[ l = 0, 1, 2, \ldots \]

In this analysis only the relative motion of the system under consideration is taken into account, since the system is assumed to have no external forces acting on it, and thus the center of mass moves at a constant velocity and its motion can therefore be ignored here.

By Soga's analysis the radial parts of the total Green's function of the system are

\[ G^r_{2} (r, r') = i k \varphi_l (kr) \, h^l_2 (kr'). \]

So 1) becomes

2) \[ \mathcal{U}(kr) = \varphi_l (kr) - \frac{2M}{\hbar^2} \int_0^\infty G^r_2 (r', r') V(r') \mathcal{U}_2 (kr') \, r'^2 \, dr'. \]

The Hamiltonian operator for the case of potential scattering is

\[ H = T + V(r) = -\frac{\hbar^2}{2M} \left[ \frac{\partial^2}{\partial r^2} \right] + V(r). \]

Now define the radial operator

\[ Q = \frac{1}{r} \frac{2^2}{\partial r^2} r - \frac{\ell (\ell + 1)}{r^2} + k^2 \]

where,

\[ k^2 = \frac{2M \, E}{\hbar^2} \]

and,

\[ E = \text{the energy of the particles in the incident beam}. \]

Operating on 2) with \( Q \) yields

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\[ Q \mathcal{U}_e (kr) = Q \mathcal{U}_e (kr) \]
\[-\frac{2M}{\hbar^2} \int_0^\infty \left[ Q e^+ (r, r') \right] V(r') \mathcal{U}_e (kr') \times r'^2 dr'.\]

\( Q \) can be taken inside the integral sign because it contains only derivatives with respect to \( r \), and the integration is with respect to \( r' \). Now by any intermediate text on quantum mechanics such as Merzbacher, it is shown that the differential equation
\[ Q \mathcal{R}(r) = 0 \]
has the functions \( \mathcal{J}_e (kr) \) as solutions. So,
\[ Q \mathcal{J}_e (kr) = 0 \]
and the first term above in \( 3 \) vanishes. Also, from Soga's analysis or from the treatment of scattering theory presented in Merzbacher or other popular quantum mechanics texts, it can be easily shown that
\[ Q e^+ (r, r') = -\frac{\delta (r-r')}{rr'}. \]
So,
\[-\int_0^\infty \left[ Q e^+ (r, r') \right] V(r') \mathcal{U}_e (kr') \times r'^2 dr' = \int_0^\infty \frac{\delta (r-r') V(r')} {r'} \mathcal{U}_e (kr) r' dr' = V(r) \mathcal{U}_e (kr) \]
And we have the differential equation
\[ 4) \quad Q \mathcal{U}_e (kr) = \frac{2M}{\hbar^2} V(r) \mathcal{U}_e (kr) \]
for \( \mathcal{U}_e (kr) \).

Now the general solution of \( 4 \) can be found in the following way. Let the inverse of the operator \( Q \) be denoted by
\[ Q^{-1} = \frac{1}{Q} \]

So an obvious particular solution of 4) is
\[ \mathcal{U}_e^0(kr) = \frac{1}{Q} \frac{2M}{\hbar^2} V(r) \mathcal{U}_e(kr). \]

And the general solution of the homogeneous case of 4) is \( \mathcal{U}_e^0(kr) \) and the general solution of 4) is
\[ \mathcal{U}_e(kr) = \mathcal{U}_e^0(kr) + \frac{1}{Q} \frac{2M}{\hbar^2} V(r) \mathcal{U}_e(kr). \]

Now let
\[ h_\varepsilon(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r - \frac{2\varepsilon(r+1)}{r^2}. \]

So,
\[ Q = h_\varepsilon(r) + k^2. \]

And,
\[ 5) \mathcal{U}_e(kr) = \mathcal{U}_e^0(kr) + \frac{1}{h_\varepsilon(r) + k^2} \frac{2M}{\hbar^2} V(r) \mathcal{U}_e(kr). \]

Now, as Soga emphasizes in his analysis, the elements of the T matrix, \( T_e \), served to give the final wave function \( \psi_k^+(F) \) in terms of the initial plane wave function \( c^{k,F} = \phi_{in} \). So the first step in deriving the form of the generalized T operator is to define the operator \( \mathcal{A}(k,F) \) which gives the final wave function when it operates on the initial incident plane wave function \( c^{k,F} \). Or,
\[ 5-1) \; \psi_k^+(F) = \mathcal{A}(k,F) \phi_{in} \]
Now since $J$ and $\ell$ are good quantum numbers, the operator that produces $U_{\ell}(k\ell)$ from the initial $E_{\ell}(k\ell)$ is

$$
\int_{0}^{2\pi} \int_{0}^{\pi} Y_{\ell m}^*(\theta, \phi) \lambda_0(r, \theta, \phi) Y_{\ell m}(\theta, \phi) \sin \theta \, d\theta \, d\phi =
$$

$$
\delta_{\ell m} \Lambda^2(r) = \Lambda^2(r).
$$

And,

$$
\int_{0}^{2\pi} \int_{0}^{\pi} Y_{\ell m}^*(\theta, \phi) \lambda_0(r, \theta, \phi) Y_{\ell m}(\theta, \phi) \sin \theta \, d\theta \, d\phi =
$$

$$
\Lambda^2(r) E_{\ell}(k\ell) = E_{\ell}(k\ell)
$$

Now, since the $Y_{\ell m}(\theta)$ are an orthonormal set

$$
E_{\ell}(k\ell) = \int_{0}^{2\pi} \int_{0}^{\pi} Y_{\ell m}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) \sin \theta \, d\theta \, d\phi
$$

$$
= \int_{0}^{2\pi} \int_{0}^{\pi} Y_{\ell m}^*(\theta, \phi) \lambda^2(r) Y_{\ell m}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \Lambda^2(r) E_{\ell}(k\ell)
$$

And substituting these in 5) we have

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\[ \int_{r(r)} Y_2^{*}(r, \alpha) Y_2(r, \alpha) \, dr \approx \left( I - \frac{1}{2mE} \right) Y_2(r, \alpha) \, dr = \frac{1}{2mE} V(r) Y_2(r, \alpha) \, dr \]

Or,

\[ \Lambda^2(r) \dot{\varphi}_e(kr) = \delta_{ee} \delta_{mm} \Gamma \dot{\varphi}_e(kr) + \left( \frac{1}{h^2} + \frac{2mE}{\hbar^2} V(r) \right) \Lambda^2(r) \dot{\varphi}_e(kr) = \Gamma \dot{\varphi}_e(kr) \]

And,

\[ \Lambda^2(r) \dot{\varphi}_e(kr) = \left[ I + \frac{1}{h^2} + \frac{2mE}{\hbar^2} V(r) \right] \dot{\varphi}_e(kr) \]

which suggests the identity

\[ \Lambda^2(r) = I + \frac{1}{h^2} + \frac{2mE}{\hbar^2} V(r) \Lambda^2(r) \]

for the radial parts of the operator \( \Lambda(r, \alpha) \).

Now let

\[ H_0'(r, \alpha) = \frac{1}{r^2} \frac{d^2}{dr^2} r - \frac{1}{h^2} \]

which in turn suggests the more general identity

\[ \Lambda(r, \alpha) = I + \frac{1}{H_0'(r, \alpha) + \frac{2mE}{\hbar^2} V(r)} \Lambda(r, \alpha) \]

The validity of this identity can be illustrated by operating with it on \( \Phi_{in} \), and then integrating with respect to the angle variables. If the expression 5) can be obtained by this process, the postulated identity is valid. Now
\[ \sum_{n} (r, \ell, m) \Phi_{in} = \Phi_{in} + \frac{1}{H_0(r, \ell, m) + \frac{2mE}{\hbar^2}} \] 
and thus 
\[ \int Y_{\ell m}^* (\mathbf{r}) \nabla (r, \ell, m) \Phi_{in} \, dr = \int Y_{\ell m}^* (\mathbf{r}) \Phi_{in} \, dr + \int Y_{\ell m}^* (\mathbf{r}) \frac{1}{H_0(r, \ell, m) + \frac{2mE}{\hbar^2}} V(r) \nabla (r, \ell, m) \Phi_{in} \, dr \]
At this point \( \Phi_{in} \) and \( \nabla (r, \ell, m) \Phi_{in} \) must be expanded in terms of their radial parts and the spherical harmonics \( Y_{\ell \ell} (\mathbf{r}) \).

Or,
\[ \Phi_{in} = e^{iK \cdot \mathbf{r}} F \]
\[ \sum_{\ell, m} \int Y_{\ell m}^* (\mathbf{r}) \nabla (r, \ell, m) \Phi_{in} \, dr = \sum_{\ell, m} \int Y_{\ell m}^* (\mathbf{r}) \Phi_{in} \, dr + \sum_{\ell, m} \int Y_{\ell m}^* (\mathbf{r}) \frac{1}{H_0(r, \ell, m) + \frac{2mE}{\hbar^2}} V(r) \nabla (r, \ell, m) \Phi_{in} \, dr \]
And,
\[ \nabla (r, \ell, m) \Phi_{in} = \Psi_K^+(F) = \sum_{\ell, m} \int Y_{\ell m}^* (\mathbf{r}) \nabla (r, \ell, m) \Phi_{in} \, dr \]
Then,
\[ \int Y_{\ell m}^* (\mathbf{r}) \left[ \sum_{\ell, m} \int Y_{\ell m}^* (\mathbf{r}) \nabla (r, \ell, m) \Phi_{in} \, dr \right] \, dr + \int Y_{\ell m}^* (\mathbf{r}) \left[ \sum_{\ell, m} \int Y_{\ell m}^* (\mathbf{r}) \frac{1}{H_0(r, \ell, m) + \frac{2mE}{\hbar^2}} V(r) \nabla (r, \ell, m) \Phi_{in} \, dr \right] \, dr 
\]
This is in the form \( \sum_{m} \mathbf{I}_0 = I_1 + I_2 \). And,

\[
I_0 = \int_{-\pi}^{\pi} Y_{l,m}^*(\hat{\mathbf{k}}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

\[
= \sum_{l',m'} 4\pi i e^{il'} \int_{0}^{\pi} \sin \theta \, Y_{l,m}^*(\hat{\mathbf{k}}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

\[
= \sum_{l',m'} 4\pi i e^{il'} \int_{0}^{\pi} \sin \theta \, Y_{l,m}^*(\hat{\mathbf{k}}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

since the spherical harmonics are orthonormal. By carrying out the summations, and recalling the definition of the Kronecker delta

\[
\delta_{ij}, \text{ it follows that}
\]

\[
I_0 = 4\pi i e^{il} Y_{l,m}(\hat{\mathbf{k}}) \mathbf{J}_l(\mathbf{k} r)
\]

Also,

\[
I_1 = \int_{0}^{\pi} \sin \theta \, Y_{l,m}^*(\hat{\mathbf{k}}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

\[
= \sum_{l',m'} 4\pi i e^{il'} \int_{0}^{\pi} \sin \theta \, Y_{l,m}^*(\hat{\mathbf{k}}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

\[
= \sum_{l',m'} 4\pi i e^{il'} \int_{0}^{\pi} \sin \theta \, Y_{l,m}^*(\hat{\mathbf{k}}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

So,

\[
I_1 = 4\pi i e^{il} Y_{l,m}(\hat{\mathbf{k}}) \mathbf{J}_l(\mathbf{k} r)
\]

Now,

\[
I_2 = \int_{0}^{\pi} \sin \theta \, Y_{l,m}^*(\hat{\mathbf{k}}) \left[ \frac{1}{H_0'/(\mathbf{k} r)} + \frac{2\pi e}{H^2} \right] \int_{0}^{\pi} \sin \theta \, Y_{l,m}(\theta, \phi) \, d\theta \, d\phi
\]

\[
\times \mathbf{V}(\mathbf{r}) \int_{0}^{\pi} \sin \theta \, Y_{l',m'}(\theta, \phi) \, d\theta \, d\phi
\]

\[
\times \frac{1}{H_0'/(\mathbf{k} r)} + \frac{2\pi e}{H^2} \int_{0}^{\pi} \sin \theta \, Y_{l,m}(\theta, \phi) \, d\theta \, d\phi
\]

At this point, it is necessary to prove a helpful theorem. Let
the eigenvalue equation

$$H \psi = \varepsilon \psi$$

be given, $\varepsilon$ being the eigenvalue of $H$ corresponding to $\psi$. If $F(H)$ is a series in $H$, that is, if $F(H) = \sum_{k=0}^{\infty} a_k H^k$ (the $a_k$ being constants) then

$$F(H) \psi = \sum_{k=0}^{\infty} a_k (\varepsilon^k \psi) = \sum_{k=0}^{\infty} a_k (H^k \psi).$$

By the eigenvalue equation it follows that

$$H^k \psi = \varepsilon^k \psi.$$ 

Therefore,

$$F(H) \psi = \sum_{k=0}^{\infty} a_k (\varepsilon^k \psi) = \sum_{k=0}^{\infty} (a_k \varepsilon^k) \psi$$

Because of the nature of the function $F$, it follows that

$$\sum_{k=0}^{\infty} a_k \varepsilon^k = F(\varepsilon).$$

Therefore the theorem

6) $F(H) \psi = F(\varepsilon) \psi$

is proven.

Now,

$$I_2 = \sum_{\ell, m} \int \psi^* Y_{\ell}^{m'} A_{\ell, m'}^{m'}(r) A_{\ell, m'}^{m'}(r) \psi \, dr$$

where,

$$A_{\ell, m'}^{m'}(r) = \frac{1}{\sqrt{\pi}} \int_0^{4\pi} Y_{\ell}^{m'}(\theta) \frac{1}{H_0'(\ell, \gamma, 2 \ell \omega, \frac{2 \pi E}{E}} \frac{1}{H_0'(\ell, \gamma, 2 \ell \omega, \frac{2 \pi E}{E}} \psi \, d\theta$$

The operator $\frac{1}{r} \frac{\partial}{\partial r^2} r$ is Hermitian, since

$$H_0'(\ell, \gamma, 2 \ell \omega, \frac{2 \pi E}{E}) = \frac{1}{r} \frac{\partial}{\partial r^2} r - \frac{\ell^2 \omega}{\hbar^2 r^2}.$$
is such, and so is \( \frac{2mE}{\hbar^2} \), and therefore their sum in the denominator is Hermitian. Here \( T \) is the kinetic energy operator for the system. Then,

\[
A_{\ell \ell'}^{mm'}(r) = \left[ \int_0^{2\pi} \left( \frac{1}{H_0' \left( \frac{1}{r} \right) + \frac{2mE}{\hbar^2}} \right) Y_{\ell m}(r) Y_{\ell' m'}(r) \, dr \right] \frac{2\mu}{\hbar^2} V(r) \mathcal{M}_{\ell \ell'}(kr)
\]

By 6),

\[
\frac{1}{H_0' \left( \frac{1}{r} \right) + \frac{2mE}{\hbar^2}} Y_{\ell m}(r) = \frac{1}{L \frac{d^2}{dr^2} - \frac{L^2}{\hbar^2} r^2 + \frac{2mE}{\hbar^2}} Y_{\ell m}(r)
\]

Therefore,

\[
A_{\ell \ell'}^{mm'}(r) = \left[ \int_0^{2\pi} \left( \frac{1}{h\varepsilon(r) + \frac{2mE}{\hbar^2}} \right) Y_{\ell m}(r) Y_{\ell' m'}(r) \, dr \right] \frac{2\mu}{\hbar^2} V(r) \mathcal{M}_{\ell \ell'}(kr) = \int_0^{2\pi} Y^\ast_{\ell m'}(r) \frac{1}{h\varepsilon(r) + \frac{2mE}{\hbar^2}} \frac{2\mu}{\hbar^2} V(r) \mathcal{M}_{\ell \ell'}(kr) Y_{\ell m}(r) \, dr = \frac{1}{h\varepsilon(r) + \frac{2mE}{\hbar^2}} \frac{2\mu}{\hbar^2} V(r) \mathcal{M}_{\ell \ell'}(kr) \int_0^{2\pi} Y^\ast_{\ell m'}(r) Y_{\ell m}(r) \, dr = \frac{1}{h\varepsilon(r) + \frac{2mE}{\hbar^2}} \frac{2\mu}{\hbar^2} V(r) \mathcal{M}_{\ell \ell'}(kr) \frac{\delta_{\ell \ell'} \delta_{mm'}}{2\pi}
\]
Then,

\[ I_z = \]

\[ 4\pi \int \frac{\mathcal{E}_{l,m}(r,k) 1}{\hbar \varepsilon(r) + \frac{2mE}{\hbar^2}} V(r) \mathcal{N}_l(kr) \mathcal{N}_m(kr) \]

\[ = 4\pi \int \mathcal{E}_{l,m}(r,k) \mathcal{N}_l(kr) \mathcal{N}_m(kr) \mathcal{N}_l(kr) \mathcal{N}_m(kr) \]

And substituting in \( \star \), it follows that

\[ 4\pi i \mathcal{E}_{l,m}(r,k) \mathcal{N}_l(kr) = \]

\[ 4\pi i \mathcal{E}_{l,m}(r,k) \mathcal{N}_l(kr) \mathcal{N}_l(kr) \]

\[ = 4\pi i \mathcal{E}_{l,m}(r,k) \mathcal{N}_l(kr) \mathcal{N}_l(kr) \left[ \frac{1}{\hbar \varepsilon(r) + \frac{2mE}{\hbar^2}} V(r) \mathcal{N}_l(kr) \mathcal{N}_l(kr) \right] \]

This simplifies to

\[ \mathcal{N}_l(kr) = \mathcal{E}_l(kr) + \frac{1}{\hbar \varepsilon(r) + \frac{2mE}{\hbar^2}} V(r) \mathcal{N}_l(kr) \mathcal{N}_l(kr) \]

which is identical to \( 5 \), so the identity

\[ 7) \mathcal{N}(15\mathcal{N}) = I + \frac{1}{\hbar \varepsilon(15\mathcal{N}) + \frac{2mE}{\hbar^2}} V(15\mathcal{N}) \mathcal{N}(15\mathcal{N}) \]

is a valid one.

Thus, by \( 5, 1 \) and \( 7 \)

\[ \Psi_k^+(F) = \mathcal{N}(15\mathcal{N}) \Phi_{in} = \]

\[ \left[ I + \frac{1}{\hbar \varepsilon(15\mathcal{N}) + \frac{2mE}{\hbar^2}} V(15\mathcal{N}) \mathcal{N}(15\mathcal{N}) \right] \Phi_{in} \]

Define the general \( T \) operator by
and by the above, the final wave function is given by
\[ \Phi_k^+(\mathbf{r}) = \left[ I + \frac{1}{\mathcal{H}_0(\mathbf{r}, \mathbf{\rho}) + \frac{2mE}{\hbar^2}} \mathcal{V}(\mathbf{r}, \mathbf{\rho}) \right] \phi_{in} \]
in terms of the initial plane wave function.

Now,
\[ \mathcal{H}_0'(\mathbf{r}, \mathbf{\rho}) = \frac{1}{r} \frac{\partial}{\partial r} \left( - \frac{1}{r^2} \right) \]
and let
\[ \mathcal{H}_0(\mathbf{r}, \mathbf{\rho}) = -\frac{\hbar^2}{2m} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r - \frac{1}{r^2} \right) \right] = T \]
so that,
\[ \mathcal{H}_0'(\mathbf{r}, \mathbf{\rho}) = -\frac{2mE}{\hbar^2} \mathcal{H}(\mathbf{r}, \mathbf{\rho}) \]
and thus,
\[ \mathcal{H}_0'(\mathbf{r}, \mathbf{\rho}) + \frac{2mE}{\hbar^2} = \frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2} \mathcal{H}_0(\mathbf{r}, \mathbf{\rho}) = \frac{2m}{\hbar^2} \left[ E - \mathcal{H}_0(\mathbf{r}, \mathbf{\rho}) \right] \]
So by 7-1) and 8) it follows that
\[ \mathcal{V}(\mathbf{r}, \mathbf{\rho}) = I + \frac{1}{E - \mathcal{H}_0(\mathbf{r}, \mathbf{\rho})} \mathcal{V}(\mathbf{r}, \mathbf{\rho}). \]
By multiplying both sides of this expression on the left by \( \mathcal{V}(\mathbf{r}) \) and using 7-1) it follows that
\[ 9) \mathcal{V}(\mathbf{r}, \mathbf{\rho}) = \mathcal{V}(\mathbf{r}) + \frac{1}{E - \mathcal{H}_0(\mathbf{r}, \mathbf{\rho})} \mathcal{V}(\mathbf{r}, \mathbf{\rho}). \]
Multiplying both sides of expression by the operator \( \frac{1}{E - \mathcal{H}_0(\mathbf{r}, \mathbf{\rho})} \) on the left it follows that

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\[
\frac{1}{E - H_0(\tau, \nu)} \quad \mathcal{P}(\tau, \nu) = \frac{1}{E - H_0(\tau, \nu)} V(\tau)
\]

\[+ \frac{1}{E - H_0(\tau, \nu)} V(\tau) \frac{1}{E - H_0(\tau, \nu)} \mathcal{P}(\tau, \nu).\]

Therefore,
\[
\left[1 - \frac{1}{E - H_0(\tau, \nu)} V(\tau) \right] \frac{1}{E - H_0(\tau, \nu)} \mathcal{P}(\tau, \nu)
\]
\[= \frac{1}{E - H_0(\tau, \nu)} V(\tau)\]

Or,
\[
\frac{1}{E - H_0(\tau, \nu)} \left[ E - H_0(\tau, \nu) - V(\tau) \right] \frac{1}{E - H_0(\tau, \nu)} \mathcal{P}(\tau, \nu)
\]
\[= \frac{1}{E - H_0(\tau, \nu)} V(\tau)\]

And this becomes
\[
\frac{1}{E - H_0(\tau, \nu)} \mathcal{P}(\tau, \nu) = \frac{1}{E - H_0(\tau, \nu) - V(\tau)} V(\tau).
\]

And substituting in \( \mathcal{P} \), it follows that
\[
\mathcal{P}(\tau, \nu) = V(\tau) + V(\tau) \frac{1}{E - H_0(\tau, \nu) - V(\tau)} V(\tau).
\]
which is the general form of the T operator for the case of potential scattering.

General Form of T Matrix Operator for Most General Case which includes Isospin

In this case it is assumed that both the target and projectile have internal structure, and that the spins of both are nonzero before and after the interaction. The process considered here is inelastic scattering in which the incident particle and the target are initially in an excited state. Therefore, the orbital angular momentum of the projectile is no longer a constant of the motion.

For the target there is the eigenvalue equation,

$$H_T (T) X_{I\alpha M_{\alpha}} (\bar{T}) = \lambda X_{I\alpha M_{\alpha}} (\bar{T})$$

and for the projectile,

$$H_P (\bar{T}) X_{I'\beta M_{\beta}} (\bar{T}) = \beta X_{I'\beta M_{\beta}} (\bar{T})$$

where,

$$\lambda$$ = the energy eigenvalue of the target corresponding to the eigenstate $$X_{I\alpha M_{\alpha}} (T)$$ in spin state $$I_{\alpha}$$, and

$$\beta$$ = the energy eigenvalue of the projectile corresponding to the eigenstate $$X_{I'\beta M_{\beta}} (\bar{T})$$ in spin state $$I_{\beta}$$.

By the author's extension of Soga's Green's function analysis it has been shown that the integral equation for the radial parts,

$$\mathcal{U}_{I'\beta T'} (k_{\alpha B})$$

of the total wave function of the system is
in the special case in which both the target and projectile are
initially in their ground states.

Here,

\( \mathcal{E} \) = the initial value of the quantum number of the orbital
angular momentum of the projectile.

\( \mathcal{E}^\prime \) = the final value of the quantum number of the orbital angular
momentum of the projectile.

\( \mathcal{J} \) = the initial value of the quantum number of the total angular
momentum of the projectile.

\( \mathcal{J}^\prime \) = the final value of the quantum number of the total angular
momentum of the projectile.

\( I_0 \) = the initial value of the quantum number of the spin of the projectile.

\( I_\beta \) = the final value of the quantum number of the spin of the projectile.

\( I_0 \) = the initial value of the spin quantum number of the target.

\( I_\alpha \) = the final value of the spin quantum number of the target.

\( J \) = the value of the quantum number of the total angular momentum of the projectile and target system.

The total angular momentum of the system is a conserved quantity throughout the scattering process.

Also,

\( T \) = the value of the quantum number of the total isospin of the projectile and target system, which is also conserved in the scattering process.

\( T_0 \) = the initial value of the quantum number of the isospin of the projectile.

\( T_\rho \) = the final value of the quantum number of the isospin of the projectile.

\( T_0 \) = the initial value of the quantum number of the isospin of the target.

\( T_r \) = the final value of the quantum number of the isospin of the target.

Again,
\[ Q = \frac{1}{\gamma^{2}} \frac{\partial^{2} \gamma}{\partial r^{2}} - \frac{e'(\gamma')}{\gamma' - 2} + \frac{k_{\alpha}^{2}}{\gamma^{2}} = \]

\[ \hbar e'(1) + k_{\alpha}^{2} \]

where,

\[ k_{\alpha}^{2} = \frac{2\gamma}{\hbar^{2}} (E - \alpha - \beta) \]

Here \( \alpha' \) is used, since it is the final orbital angular momentum of the projectile that is of concern in this inelastic process. As before we get a differential equation

\[ 2) \quad Q \left[ \frac{J^T}{2} (k_{\alpha} \alpha') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

\[ = \quad \frac{2\gamma}{\hbar^{2}} \]

\[ \left[ \frac{J^T}{2} (\beta') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

\[ \left[ \frac{J^T}{2} (\beta') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

with general solution,

\[ 2-I) \quad Q \left[ \frac{J^T}{2} (k_{\alpha} \alpha') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

\[ \left[ \frac{J^T}{2} (\beta') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

\[ \left[ \frac{J^T}{2} (\beta') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

\[ \left[ \frac{J^T}{2} (\beta') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]

\[ \left[ \frac{J^T}{2} (\beta') \right] \quad e' \left[ \frac{J^T}{2} (\beta') \right] \quad I_{\alpha}^{(2)} \]

\[ \frac{I_{\alpha} + I_{\alpha}^{(2)}}{I_{\alpha} + I_{\alpha}^{(2)}} \]
At this point, a couple things must be noted. The expression obtained for \( \mathcal{G}(\mathbf{K}, \mathbf{R}, \mathbf{r}, \mathbf{t}) \) in the Green's function analysis pertained to the restricted case in which the target and projectile were assumed to be in their ground states. The general solution 2-1) will be more general than that expression, since it will be the complete solution of the general differential equation above, that all possible \( \mathcal{H}(\mathbf{K}, \mathbf{R}) \) must satisfy. This is why the method of finding the general \( T \) matrix operator is suited to the general case in which both particles are initially in an excited state. The method will give the general solution of the above equation, which includes all possible cases.

No new postulate the identity

\[
\mathcal{N}(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t}) = \int \frac{1}{E - H_0(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t})} \mathcal{N}(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t}) \mathcal{X} \mathcal{N}(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t})
\]

No new coordinates are involved here, and,

\[
\psi_{E}^{+}(\mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}) = \mathcal{N}(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t}) \phi_{in} = \phi_{in} + \frac{1}{E - H_0(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t})} \mathcal{V}(\mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t}) \mathcal{N}(1, \mathbf{F}, \mathbf{r}, \mathbf{t}, \mathbf{R}, \mathbf{r}, \mathbf{t}) \phi_{in}.
\]

Here,

- \( m_o \) = the quantum number of the projectile's initial spin orientation.
- \( M_o \) = the quantum number of the target's initial spin orientation.
- \( m_o \) = the quantum number of the projectile's initial isospin orientation.
- \( N_o \) = the quantum number of the orientation of the target's
initial isospin.

$N_p$ is the quantum number of the orientation of the projectile's final isospin.

$N^o_T$ is the quantum number of the target's final isospin orientation.

Then,

$$\Psi = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \phi_{in} = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

$$\Psi = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

$$\Psi = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

$$\Psi = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

Here,

$N^f_T = \text{the quantum number of the total isospin's orientation, and}$

the $\Psi = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$ are harmonics that are eigenfunctions of $L^2, H_F \mid T \rangle, H_T$, that take isospin into account. They form an orthonormal set. Now,

$$\phi_{in} = \text{E}^{i2\pi T} X_{i0\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

$$\phi_{in} = \text{E}^{i2\pi T} X_{i0\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

$$\phi_{in} = \text{E}^{i2\pi T} X_{i0\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} = \sum_{E10\text{mTo} \rightarrow \text{NTo} \text{NTo} \text{NTo}} \text{E}^{i2\pi T}$$

So again we have
\[
\begin{align*}
&\frac{1}{4\pi} \int \int \int_{\Omega} \sum_{\alpha} \sum_{\gamma} \left[ \nabla \left( \phi_{\text{in}} \right) \right] \cdot \mathbf{N} \\
&\quad \times d\tau \ d\tau \ d\tau = \frac{1}{4\pi} \int \int \int_{\Omega} \left\{ \nabla \left( \phi_{\text{in}} \right) \right\} \cdot \mathbf{N} \\
&\quad \times \left[ \sum_{\alpha} \sum_{\gamma} \phi_{\text{in}} \right] \ d\tau \ d\tau \ d\tau
\end{align*}
\]

or, 3) \( I_0 = I_1 + I_2 \)

Then,

\[
I_0 = \int \int \int_{\Omega} \sum_{\alpha} \sum_{\gamma} \mathbf{N} \cdot \mathbf{J} \left( \frac{\tau}{\mathbf{N}} \right) \mathbf{E} \left( \frac{\tau}{\mathbf{N}} \right) \ d\tau \ d\tau \ d\tau
\]
And,

\[ I_x = \sum_{n} \sum_{m} f_{\kappa-n} \left[ \frac{1}{e^{-i E T_x (\phi) I_{xT}} E - H_0 (E_T, \Phi_x)} \right] Y_{\kappa m} \frac{4 \pi i^2}{X} \left( \frac{2}{T' L_{T'x}} \right) \]

\[ x \left( \sum_{n} \sum_{m} f_{\kappa-n} \left[ \frac{1}{e^{-i E T_x (\phi) I_{xT}} E - H_0 (E_T, \Phi_x)} \right] Y_{\kappa m} \frac{4 \pi i^2}{X} \left( \frac{2}{T' L_{T'x}} \right) \right) \]

\[ X \left( \sum_{n} \sum_{m} f_{\kappa-n} \left[ \frac{1}{e^{-i E T_x (\phi) I_{xT}} E - H_0 (E_T, \Phi_x)} \right] Y_{\kappa m} \frac{4 \pi i^2}{X} \left( \frac{2}{T' L_{T'x}} \right) \right) \]

And let,

\[ \frac{1}{E - H_0 (E_T, \Phi_x)} \left[ \frac{1}{e^{-i E T_x (\phi) I_{xT}} E - H_0 (E_T, \Phi_x)} \right] Y_{\kappa m} \frac{4 \pi i^2}{X} \left( \frac{2}{T' L_{T'x}} \right) \]

\[ x \left( \sum_{n} \sum_{m} f_{\kappa-n} \left[ \frac{1}{e^{-i E T_x (\phi) I_{xT}} E - H_0 (E_T, \Phi_x)} \right] Y_{\kappa m} \frac{4 \pi i^2}{X} \left( \frac{2}{T' L_{T'x}} \right) \right) \]

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\[ E = \sum_{\mathcal{A}} \sum_{\mathcal{B}} \sum_{\mathcal{C}} \int_{\mathcal{A}} \int_{\mathcal{B}} \int_{\mathcal{C}} \frac{1}{E - H_0(\mathcal{A}, \mathcal{B}, \mathcal{C})} \left( \frac{J_{\mathcal{A}}(\mathcal{A}, \mathcal{B}, \mathcal{C})}{I_{\mathcal{A}} I_{\mathcal{B}} I_{\mathcal{C}}} \right) \]

\[ \times V(\mathcal{A}, \mathcal{B}, \mathcal{C}) \frac{1}{E - H_0(\mathcal{A}, \mathcal{B}, \mathcal{C})} \left( \frac{J_{\mathcal{A}}(\mathcal{A}, \mathcal{B}, \mathcal{C})}{I_{\mathcal{A}} I_{\mathcal{B}} I_{\mathcal{C}}} \right) \]

By 6) of the last section, and

\[ H_0(\mathcal{A}, \mathcal{B}, \mathcal{C}) = T + H_p(\mathcal{C}) + H_r(\mathcal{T}) \]

it follows that

\[ \frac{1}{E - T - H_p(\mathcal{C}) - H_r(\mathcal{T})} \frac{J_{\mathcal{A}}(\mathcal{A}, \mathcal{B}, \mathcal{C})}{I_{\mathcal{A}} I_{\mathcal{B}} I_{\mathcal{C}}} = \]

\[ \frac{1}{h^2} \frac{h\xi^r(\mathcal{r}) + E - \alpha - B}{h^2} \frac{J_{\mathcal{A}}(\mathcal{A}, \mathcal{B}, \mathcal{C})}{I_{\mathcal{A}} I_{\mathcal{B}} I_{\mathcal{C}}} \]

So substituting in 4) yields
\[ B = \frac{1}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \left( \frac{2\pi}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \right) \sum_{n=0}^{\infty} N(\lambda, \mu, r, T) \times \frac{J_0(r) + J_1(r) \frac{2\pi}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \sqrt{J_1(r)}}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \text{d}r \text{d}T \right. \\
\left. \times \frac{1}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \sqrt{J_1(r)} \right) dT. \\
\]

Therefore,

\[ I_2 = \frac{1}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \left( \frac{2\pi}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \right) \sum_{n=0}^{\infty} N(\lambda, \mu, r, T) \times \frac{J_0(r) + J_1(r) \frac{2\pi}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \sqrt{J_1(r)}}{h_{\text{eff}}(r) + k_{\text{eff}}^2} \text{d}r \text{d}T \right. \\
\left. \times J_0(T) \text{d}T. \\
\]

And substituting \( I_0, I_1, \) and \( I_2 \) in 3) gives

\[ \]

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\[ \frac{1}{4\pi i \varepsilon \left( \lambda m_{\text{io}} \sigma_{\text{mt}} \right)} \cdot \frac{1}{\hbar^2 \left( \varepsilon + \frac{i \hbar}{2\pi} \right)} \cdot \frac{i}{\varphi_{\text{io}} \varphi_{\text{im}}} \cdot \frac{1}{\varphi_{\text{io}} \varphi_{\text{im}}} \]

Now the quantities

\[ \frac{1}{4\pi i \varepsilon \left( \lambda m_{\text{io}} \sigma_{\text{mt}} \right)} \cdot \frac{1}{\hbar^2 \left( \varepsilon + \frac{i \hbar}{2\pi} \right)} \cdot \frac{i}{\varphi_{\text{io}} \varphi_{\text{im}}} \cdot \frac{1}{\varphi_{\text{io}} \varphi_{\text{im}}} \]

are linearly independent, so it follows that

\[ i^2 \varepsilon \cdot \varepsilon = i^2 \delta \varepsilon \cdot \varepsilon \]

has been used.
which is identical to 2-1. So the identity

\[ \psi (\Gamma, \vec{n}, \vec{k}, \vec{r} \mid \vec{r}_0) = I + \frac{1}{E - H_0 (\Gamma, \vec{n}, \vec{k}, \vec{r})} V (\vec{r}, \vec{n}, \vec{k}, \vec{r}_0) \psi (\Gamma, \vec{n}, \vec{k}, \vec{r} \mid \vec{r}_0) \]

is valid, and in the same way as was done for potential scattering, it can be shown that

\[ \psi (\Gamma, \vec{n}, \vec{k}, \vec{r}) = \]

\[ V (\vec{r}, \vec{n}, \vec{k}, \vec{r}) + V (\vec{r}, \vec{n}, \vec{k}, \vec{r}) \frac{1}{E - H_0 (\Gamma, \vec{n}, \vec{k}, \vec{r}) - V (\vec{r}, \vec{n}, \vec{k})} V (\vec{r}, \vec{n}, \vec{k}, \vec{r}) \]

which is the most general form of the T matrix operator.
CHAPTER III
DETERMINATION OF THE MATRIX ELEMENTS
OF THE GENERAL TMATRIX OPERATOR FOR
THE CASE IN WHICH THE PROJECTILE AND
TARGET ARE INITIALLY IN THEIR GROUND
STATES

Potential Scattering

For this case, \( \Psi \) is the matrix element of \( \mathcal{H}(r, \nu) \)
corresponding to \( E = \frac{\hbar^2}{2m} \int_0^\infty \int_0^{2\pi} \int_0^\pi j_x(kr) J_{\ell}(\rho) j_x(kr) \rho \cos \theta \, d\theta \, d\rho \, dr \).

Here \( j_x(kr) \) is used because the \( j_x(kr) \) served as a basis for the total wave function. Now the completeness relation for the \( j_x(kr) \) is

\[
\sum_{m} j_{x}^{*}(m)(\nu) j_{x}(m', \nu) = \delta(\nu - \nu').
\]

So with,

\[
\Psi(r, \nu) = \mathcal{V}(r) j_{x}(r, \nu)
\]

it follows that

\[
\Psi = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi j_x(kr) J_{\ell}(\rho) \mathcal{V}(r, \nu) j_x(kr) \rho \cos \theta \, d\theta \, d\rho \, dr \\
= \frac{1}{2\pi} \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi j_x^{*}(m)(\nu) j_x(m', \nu) V(r) Y_{\ell m}(\nu) Y_{\ell m'}(\nu') \rho \cos \theta \, d\theta \, d\rho \, dr \\
X \rho \cos \theta \, d\theta \, d\rho \, dr.
\]

Now it is true that
\[
\int_0^{\pi} \phi_{\ell m}^*(r) Y_{\ell m}(\hat{\mathbf{r}}') V(r') Y_{\ell m}(\hat{\mathbf{r}}) \, d\hat{\mathbf{r}}' =
\int_0^{\pi} \phi_{\ell m}^*(r) \int_0^{\pi} Y_{\ell m}(\hat{\mathbf{r}}') Y_{\ell m}(\hat{\mathbf{r}}) \, d\hat{\mathbf{r}}' \, d\Omega' =
\int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega'
\]

where,

\[
V(r) = \frac{\hbar^2}{2m} \, U_e(r)
\]

Therefore,

\[
\phi_{\ell m} = \frac{\hbar^2}{2m} \, U_e(r) \int_0^{\pi} \phi_{\ell m}^*(r) \, d\Omega \, d\Omega' \int_0^{\pi} Y_{\ell m}(\hat{\mathbf{r}}') Y_{\ell m}(\hat{\mathbf{r}}) \, d\hat{\mathbf{r}}' \, d\Omega' \int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega'
\]

\[
X \left[ \int_0^{\pi} Y_{\ell m}^*(\hat{\mathbf{r}}) \mathcal{L} (r, \hat{\mathbf{r}}) Y_{\ell m}(\hat{\mathbf{r}}) \, d\Omega \right] \phi_{\ell m}(r)
\]

\[
\int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega' \int_0^{\pi} Y_{\ell m}(\hat{\mathbf{r}}') Y_{\ell m}(\hat{\mathbf{r}}) \, d\hat{\mathbf{r}}' \, d\Omega' =
\int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega'
\]

but,

\[
\int_0^{\pi} Y_{\ell m}^*(\hat{\mathbf{r}}) \mathcal{L} (r, \hat{\mathbf{r}}) Y_{\ell m}(\hat{\mathbf{r}}) \, d\Omega \phi_{\ell m}(r) = \mathcal{M}_e(kr)
\]

so,

\[
\phi_{\ell m} =
\frac{\hbar^2}{2m} \int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega'
\]

\[
\int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega' \int_0^{\pi} Y_{\ell m}(\hat{\mathbf{r}}') Y_{\ell m}(\hat{\mathbf{r}}) \, d\hat{\mathbf{r}}' \, d\Omega' =
\int_0^{\pi} \phi_{\ell m}(r) \, d\Omega \, d\Omega'
\]

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However, from Soga's analysis,
\[ \int_0^\infty J_\ell(kr) U_{\ell \chi}(r) U_{\ell \chi}(kr) r^2 \, dr = -\frac{1}{\hbar} T_\ell \]
where,
\[ T_\ell = \text{the } \ell^{th} \text{ element of the } T \text{ matrix.} \]

Therefore,
\[ J_\ell = -\frac{1}{\hbar} \frac{\hbar^2}{2\mu} T_\ell \]

**Most General Case which includes Isospin**

In this case the desired matrix element is in the same way as before

\[ J_{\ell_1} J_{\ell_2} \frac{1}{\hbar} \int_{T_{\ell_1}}^{T_{\ell_2}} \int_{T_{\ell_2}}^{T_{\ell_1}} \int_{0}^{\infty} J_{\ell_1}(kr) J_{\ell_2}(kr) \frac{1}{\hbar} \int_{T_{\ell_1}}^{T_{\ell_2}} \int_{T_{\ell_2}}^{T_{\ell_1}} \int_{0}^{\infty} \]

The completeness relation for the harmonics
\[ \frac{1}{\hbar} \int_{T_{\ell_1}}^{T_{\ell_2}} \int_{T_{\ell_2}}^{T_{\ell_1}} \int_{0}^{\infty} \]

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Therefore, using this relation

\[ J I = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (p - p') I_{p'} T_1 \]

\[ \times \left( \frac{\gamma - 1}{\sqrt{\gamma}} \right) \Delta^2 (\tau - \tau') X \]

Here \( |F'| = r \), since only the angular part of \( F' \) is primed.
Thus,

\[ J \int_{\mathcal{C}} \int_{\mathcal{T}_{0}} \left[ f_{E}(k, \omega) \right] \cdot \mathbf{E}(k, \omega) \, dt \, \, \text{d}k \]

Now recall that

\[ \int_{\mathcal{C}} \mathbf{E}(k, \omega) \cdot \mathbf{E}(k, \omega) \, dt \, \, \text{d}k = \]

\[ \sum_{m=1}^{\infty} \frac{1}{m^{2}} \left( \mathbf{m} \cdot \mathbf{m} \right) \mathbf{Y}_{m, \omega} \times \mathbf{Y}_{m, \omega} \]

\[ \mathbf{E}(k, \omega) \times \mathbf{H}(k, \omega) = \]

\[ \int_{\mathcal{C}} \mathbf{E}(k, \omega) \cdot \mathbf{E}(k, \omega) \, dt \, \, \text{d}k \]
Multiply both sides of the above by 

\[ \mathbf{J} \] 

and integrate with respect to \( \mathbf{r} \), \( \mathbf{r} \), and \( \mathbf{T} \) to get

\[ X \mathbf{Y} (\mathbf{r}, \mathbf{r}) \mathbf{J} = \mathbf{J} \mathbf{Y} (\mathbf{r}, \mathbf{r}) X \]

or,

\[ \int \int \int X \mathbf{Y} (\mathbf{r}, \mathbf{r}) \mathbf{J} (\mathbf{r}, \mathbf{r}, \mathbf{T}) \mathbf{J} (\mathbf{r}, \mathbf{r}, \mathbf{T}) d\mathbf{r} d\mathbf{r} d\mathbf{T} = X \mathbf{Y} (\mathbf{r}, \mathbf{r}) \mathbf{J} (\mathbf{r}, \mathbf{r}, \mathbf{T}) \mathbf{J} (\mathbf{r}, \mathbf{r}, \mathbf{T}) d\mathbf{r} d\mathbf{r} d\mathbf{T} \]
By the linear independence of the
it follows that

So it follows that again for $J = J', M = M', T = T'$, and $N = N'$ it is true that
Thus, for the most general case the desired matrix element is

\[
\langle \mathcal{F}, T' \mathcal{F} \rangle = \frac{i e^2}{2\pi \alpha} \int d^2 x \frac{\mathcal{T}}{\mathcal{F}} \langle \mathcal{F}, T' \mathcal{F} \rangle \frac{\mathcal{T}}{\mathcal{F}} = \frac{i e^2}{2\pi \alpha} \int d^2 x \frac{\mathcal{T}}{\mathcal{F}} \langle \mathcal{F}, T' \mathcal{F} \rangle \frac{\mathcal{T}}{\mathcal{F}}.
\]
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