Robust Rank Analysis for Multivariate Linear Models

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ROBUST RANK ANALYSIS FOR MULTIVARIATE LINEAR MODELS

by

James Buddy Davis

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Submitted to the
Faculty of The Graduate College
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ROBUST RANK ANALYSIS FOR MULTIVARIATE LINEAR MODELS

James Buddy Davis, Ph.D.
Western Michigan University, 1987

Three robust rank procedures are considered to test the multivariate linear hypothesis $\mathbf{HBK} = \mathbf{0}$. It is assumed that an $n \times p$ matrix of observations, $\mathbf{Y}$, can be expressed in terms of a linear model. Of the form $\mathbf{Y} = \mathbf{1}_n \theta' + \mathbf{XB} + \mathbf{E}$. Where $\mathbf{E}$ is an $n \times p$ a random matrix whose rows are iid. In the least squares setting, the Lawley-Hotelling trace criterion is often used. This criterion has an asymptotic $\chi^2$ distribution. This trace criterion can be written in three algebraically equivalent forms: a quadratic test based on full model estimates, a quadratic test based on reduced model residuals and a test based on the drop in the least squares dispersion function. Our work considers robust rank analogues to each of the three forms.

These rank tests are based on the component-wise extension of the R-estimates which Jaeckel (1972) proposed for the univariate linear model. These estimates are based on minimizing a dispersion function which is similar to minimizing the sum of squares in the classical case. The quadratic test procedure we proposed is based on the full model R-estimates. A second procedure is based on the residuals from an R-fit of the reduced model determined by the above linear hypothesis. This procedure is generally called an aligned rank procedure and our test is asymptotically equivalent to the one developed by Puri and Sen. The final analogue, is based on a reduction of dispersion or a drop in dispersion when passing from the reduced to the full model. This is an extension of the univariate procedure proposed by McKean and Hettmansperger (1976).
For asymptotic distribution theory we extend the work of Heiler and Willers (1979) to the multivariate setting. We obtain the asymptotic distribution of each of these procedures under the null hypothesis showing that the three procedures are asymptotically equivalent. Under the hypothesis HBK = 0, the last two procedures are asymptotically equivalent. We also develop the theory under a sequence of contiguous alternatives, from which asymptotic relative efficiency properties are obtained.

Our paper also considers consistent estimates of the covariance and scale parameters which are required by the procedures.
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Robust rank analysis for multivariate linear models

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Western Michigan University, 1987
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James Buddy Davis
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CHAPTER 1

INTRODUCTION

1.1 Statement of Problem

Suppose each repetition of a particular experiment results in a number of measurements. Each repetition can then be expressed as a $1 \times p$ random row vector $Y_i', 1 \leq i \leq n$, where $p$ is the number of measurements per repetition and $n$ is the number of repetitions. Taken together, the sample of $n$ repetitions can be expressed as a $n \times p$ random matrix $Y$. The $i$th row of $Y$ is $Y_i'$. Our interest is in those experiments that can be expressed by the multivariate linear model

$$Y = 1_n \theta' + XB + E. \quad (1.1)$$

Here $X$ is a $n \times q$ matrix of known constants, $B$ is a $q \times p$ matrix of unknown parameters and $E$ is a $n \times p$ random matrix.

The problem is to test the general linear hypothesis of the multivariate setting, $HBK = 0$. Here $H$ and $K$ are known constant matrices.

A least squares solution to this problem is the Lawley-Hotelling trace criterion. It will be discussed in detail in the next section in order to motivate our solution. It is the multivariate extension of the classical univariate $F$ test. Like the
F test, it can be written in three algebraically equivalent forms. Each form corresponds to a well established test procedure. That is, the Lawley-Hotelling procedure can be written in the form of the log likelihood test suggested by Neyman and Pearson (1928), the quadratic test proposed by Wald (1943) or the test based on efficient scores as suggested by Rao (1947).

Our solution to the problem is to develop rank procedures that are analogous to the three forms of the Lawley-Hotelling test. This extends the work of Hettmansperger and McKean (1983). In Section 3 of that paper, they discuss rank procedure analogous to the three forms of the F test.

Our work intersects with the work of Puri and Sen (1985). The aligned rank test, analogous to Rao's test based on efficient scores, is thoroughly discussed in that text. It is a test based on the ranks of the reduced model residuals. Our assumptions are somewhat different than those of Puri and Sen (1985), but our multivariate aligned rank procedure is the same.

When testing the hypothesis $HB = 0$, all three of our procedures are asymptotically equivalent. Two procedures are asymptotically equivalent when testing the hypothesis $HBK = 0$.

Due to the work of Heiler and Willers (1979), our main assumption relating to the design matrix is that Huber's condition holds. That is the same restriction required in least squares asymptotic theory.
1.2 Motivation and Our Solution

First, assume that \( Y \) has the multivariate distribution \( N_{n,p}(X \mu_0, I, \Sigma_1) \). Then, under the assumption \( H_0 \mu = 0 \), the Lawley-Hotelling procedure has the form

\[
\text{tr}(K'Y'(X'X)^{-1} H'(H(X'X)^{-1} H')^{-1} H(X'X)^{-1} X'YK) \\
\times (K' \Sigma_1 K)^{-1}.
\]

(1.2)

Under the assumption that \( \Sigma_1 \) is known, this procedure provides a Uniformly Most Powerful test. If \( \Sigma_1 \) is unknown, then a UMP does not exist. Let \( H \) be a \( r \times q \) matrix, \( r \leq q \), of rank \( r \) and \( K \) be a \( p \times s \) matrix, \( s \leq p \), of rank \( s \). Then, (1.2) has a \( \chi^2(rs) \) distribution.

In least squares, \( B = (X'X)^{-1} X'Y \) and \( B_R = (I - (X'X)^{-1} H'H(X'X)^{-1} H')^{-1} H'B \), where \( B \) and \( B_R \) are the full model and reduced model estimates, respectively. Then, (1.2) can be written in the following three equivalent forms.

\[
\text{tr}(K'((X - XB_R)'(X - XB_R) - (Y - XB)'(Y - XB))K) \\
\times (K' \Sigma_1 K)^{-1}
\]

(1.3)

\[
= \text{tr}(K'B'H'H(X'X)^{-1} H')^{-1}(H \Sigma K)(K' \Sigma_1 K)^{-1}
\]

(1.4)

\[
= \text{tr}(K'(Y - XB_R)'(X'X)^{-1} H'(H(X'X)^{-1} H')^{-1} H(X'X)^{-1} X') \\
\times (Y - XB_R')K(K \Sigma K)^{-1}
\]

(1.5)
These three forms correspond to the test procedures discussed in Section 1.1.

Expression (1.3) will be referred to as the drop in dispersion procedure. The quadratic test based on full model estimates is (1.4). The alligned rank procedure is (1.5). In this thesis, rank procedures will be developed that reflect each least squares procedure.

If the assumption of normality is dropped, then our procedures may be asymptotically \( \chi^2_{(rs)} \) under the hypothesis \( HB_0 K = 0 \). If \( E_i' \) is the \( i \)th row of \( E \), it is required that \( E(E_i') \) and \( \text{COV}(E_i') \) exist. The requirement on the sequence of design matrices is that Huber's condition holds. Huber's condition is that the largest diagonal element of \( X(X'X)^{-1}X' \) goes to 0 as \( n \) goes to infinity. If these requirements are met, our procedures are asymptotically \( \chi^2_{(rs)} \).

The goal of this thesis is to replace the restrictive requirements on the rows of \( E \) with milder ones, replace the least squares estimates and measure of dispersion with robust rank analogues, but maintain our reliance on Huber's practical requirement on the design matrices.

The rank statistics we will consider are related to processes that use the ranks of residuals. Jaeckel (1972) proposed a measure of dispersion that used these ranks. Consider the model \( Y = XB + E \) when \( p = 1 \). Then Jaeckel's dispersion function is

\[
\sum_{i=1}^{n} \text{R}(Y_i - X_i'B) \text{a}(\text{R}(Y_i - X_i'B)),
\]

where \( \text{R}(Y_i - X_i'B) \) is the rank.
of the residual and \( a(\cdot) \) is a function over the integers \( i = 1, \ldots, n \). This provided a geometry to estimation and hypothesis testing similar to least squares. See McKean and Schrader (1980) and Hettmansperger and McKean (1983).

When \( p \geq 1 \), the \( n \times p \) score collection matrix is

\[
A(Y -XB) = [a(R_j(Y_{ij} - X_i' B_j)), i = 1, \ldots, n \text{ and } j = 1, \ldots, p].
\]

Here, \( R_j(Y_{ij} - X_i' B_j) \) represents the rank of the residuals. Jaeckel's dispersion function generalizes to

\[
D(Y -XB) = \text{tr}(Y - XB)'A(Y - XB).
\]

McKean and Hettmansperger (1976) provided a univariate hypothesis testing procedure based on Jaeckel's dispersion function. The procedure compares measures of dispersion based on full and reduced model estimates. It is analogous to the classical \( F \), when written using the \( \ell_2 \) norm. All three rank procedure analogous to the equivalent forms of the \( F \) test are discussed by Hettmansperger and McKean (1983).

A very important random process is

\[
S(Y -XB) = \frac{1}{2} X_c' A(Y - XB),
\]

where \( X_c \) is the centered design matrix. The procedures discussed above required the result of Jureckova (1969) that \( S(Y -XB) \) is asymptotically linear. A drawback to Jureckova's work were the "concordance conditions" placed on the design matrix. However, Heiler and Willers (1979) showed that the result remained true if the "concordance conditions" were replaced by Huber's condition. The work of Heiler and
Willers allowed rank theory to more closely reflect the structure of least squares.

Our thesis generalizes \( A(Y - XB) \), \( D(Y - XB) \) and \( S(Y - XB) \) to the multivariate setting.

Puri and Sen (1969) proposed a multivariate aligned rank procedure to test the hypothesis \( B = 0 \). The work was extended by Sen and Puri (1977) to test the hypothesis \( B_2 = 0 \), when the parameter matrix is partitioned as \( (B_1', B_2')' \). A thorough discussion of their procedures, applied to both the univariate and multivariate setting, is in their text, Puri and Sen (1985). Their estimates are based on Jureckova’s work and her assumptions. A parameter is estimated by selecting \( B \) such that \( S(Y - XB) \) is set to zero. This follows the work of Achichi (1967), but lacks the structure introduced by Jaeckel (1972).

Utts and Hettmansperger (1980) developed a multivariate procedure based on signed ranks. Our work uses signed ranks to estimate the intercept parameter \( \theta \). While that estimate is included in a quadratic test, our primary focus is on procedures using unsigned ranks.
CHAPTER 2

THE ASYMPTOTIC DISTRIBUTIONS OF $S(Y - X B_0)$ AND
OF THE ESTIMATES OF THE REGRESSION PARAMETERS

2.1 Assumptions

As we mentioned before, our multivariate linear model is expressed as $Y = 1 \theta'_n + XB + E$. For convenience, an algebraically equivalent model is actually considered in the development of the theory. Our paper places the model transformation of Heiler and Willers (1979) in the multivariate setting. Our assumptions will be drawn from Jureckova (1971a), Heiler and Willers (1979) and Puri and Sen (1985). Final statements will be made in terms of the original model.

Let $P_n$ be the projection matrix for the space spanned by the $n \times 1$ column of ones $1_n$. Let $I_n$ be the $n \times n$ identity matrix. Then let $X_c = (I_n - P_n)X$ and $\bar{X} = P_n X$. Consider the following notation.

\[(N1)\quad W = (X'_c X_c)^{-1}\]
\[(N2)\quad C = X_c W^{-1}\]
\[(N3)\quad \Delta = WB\]
\[(N4)\quad \alpha' = \theta' + n^{-1} 1'_n X W^{-1} \Delta\]

Each of the above expressions depends on $n$ in a natural fashion. For convenience, the $n$ will be dropped from the
Let \( A_j \) denote the \( j \)th column of \( A \), \( j = 1, \ldots, p \). Let \( C_i' \) denote the \( i \)th row of \( C \), \( i = 1, \ldots, n \). Then we construct a new model in the following manner.

\[
Y = 1_n \theta' + XB + E = 1_n \theta' + (X + X_c)W^{-1}WB + E
\]

\[
= 1_n \theta' + W^{-1} \Delta + CA + E
\]

\[
= 1_n \alpha' + CA + E
\]  

(2.1)

Our first four assumptions are straightforward multivariate extensions of assumptions 1, 5, and 6 of Jureckova (1971a). Jureckova's assumptions also form assumptions I-IV of Heiler and Willers (1979).

(A1) The rows of \( E \) are iid with joint distribution function \( F \).

(A2) The marginal distribution function \( F_j \) has a differentiable density \( f_j \) with finite Fisher's information, \( j = 1, \ldots, p \).

(A3) The scores \( a(i) \) are derived from a score generating function \( \phi \) by either of the following ways:

\[
a(i) = E(\phi(U^{(i)}))
\]

or

\[
a(i) = \phi(\frac{i}{n+1}),
\]
where \( U^{(i)} \) denotes the \( i \)th order statistic in a sample of size \( n \) from a uniform distribution on the interval \((0, 1)\). We also assume \( \sum_{i=1}^{n} a(i) = 0 \).

\[ \phi \] is a non-constant, non-decreasing and square-integrable function on \((0, 1)\). In addition, \( \int \phi = 0 \) and \( \int \phi^2 = 1 \).

Our fifth assumption is assumption V of Heiler and Willers (1979).

\[ \text{(A5)} \] \( X \) and \( X_c \) are of full column rank and Huber's Condition holds for \( CC' \).

Huber's Condition states that the largest diagonal element of the projection matrix \( CC' \) goes to 0 as \( n \) goes to infinity. In practice, care should be taken when using the asymptotic theory for design matrices possessing high-leverage points. See Hoaglin and Welsch (1978) and Belsley, Kuh and Welsch (1980).

Huber (1981) shows that assumption A5 is a necessary and sufficient condition on the design matrix to insure asymptotic normality for the ordinary least square estimate. He also shows that it is sufficient to insure asymptotic normality of M-estimates.

Assumption A5 replaces the "concordance conditions" (Assumptions 3a-3c) of Jureckova (1971a). Besides being hard to check, Heiler and Willers note that these conditions "lead to a proper restriction of the set of sequences of design matrices as compared..."
to those admissible in the ordinary least squares case." Such a proper restriction does not exist when Huber's Condition is used.

The following two assumptions are very similar to those used extensively in the text by Puri and Sen (1985).

(A6) \( \text{Cov}(\hat{\phi}(F_j(e_{ij})), \hat{\phi}(F_j, (e_{ij}'))) = \sigma_{jj} < \infty \) for \( j, j' = 1, \ldots, p \) and \( i = 1, \ldots, n \). Also, \( \sum = [\sigma_{jj}] \) is positive definite.

(A7) \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} = \frac{1}{2} \), where \( V^2 \) is finite and positive definite.

Note that assumption 4 implies \( \sigma_{jj} = 1 \).

These assumptions form the foundation of our work. There are a few more assumptions required for specific aspects and these will be introduced at the appropriate time.

2.2 The Asymptotic Distribution of \( S(Y - XB_0) \)

In this section, we will discuss the asymptotic distribution of \( S(Y - XB_0) \), where \( B_0 \) is the true parameter matrix. We are assuming there exists a \( 1 \times p \) vector \( \theta_0' \) and a \( q \times p \) matrix \( B_0 \) such that \( Y - 1_n \theta_0' - XB_0 = E \), where \( E \) satisfies Al. Note that the linear rank statistic is location invariant so that \( S(Y - 1_n \theta_0' - XB_0) = S(Y - XB_0) \). We can assume without loss of generality that the elements of \( \theta_0' \) are zero. In the proof of the theorem, the random process is solely described by the random matrix.
E. Therefore, we can also assume $B_0 = 0$ without loss of generality.

When applying the score-collection matrix $A(Y - XB)$ and the linear rank statistic $S(Y - XB)$ to model (2.1), the following notation will be used.

\begin{align*}
(N5) \quad A(\Delta) &= A(Y - C\Delta) \\
(N6) \quad S(\Delta) &= S(Y - C\Delta) = C'A(\Delta)
\end{align*}

In terms of notation N6, our interest is in the asymptotic distribution of $S(0)$.

First, consider the case where $S(0)$ is a random variable. With $S(0)$ written as $C'A(0)$, $C$ is a $n \times 1$ vector of known constants and $A(0)$ is a $n \times 1$ random vector generated by the ranks of $n$ iid random variables. Using assumptions similar to our A1-A5, Theorem V.1.5.a of Hajek and Sidak (1967) demonstrates that $S(0)$ is asymptotically normal. Elements of their proof of that theorem and of their Theorem V.1.6.a will be used in our work.

Jureckova (1971a) extended the above Theorem V.1.5.a to develop the distribution of the $q \times 1$ random vector $S(0) = C'A(0)$, where $C$ is a $n \times q$ matrix of known constants. She uses an additional assumption similar to our A7. Our proof of the asymptotic matrix normality of $S(0)$ will use assumptions A1-A6.

In a somewhat different setting, Puri and Sen (1985) come up with the same result in their Theorem 5.4.2. However, they wish $S(0)$ to be conditionally distribution free for all finite $n$. That
is not required in the following theorem. Our work is based on a proper subset of their assumptions.

Let $E = \{e_{ij}\}$ and $C = \{c_{ik}\}$, where $i = 1, \ldots, n$, $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

**Theorem 2.1** Under assumptions A1-A6, $S(0) \xrightarrow{L} \mathbb{N}_{q,p}(0, I, \Sigma)$.

**Proof.** Note that $S(0) = \left[ \sum_{i=1}^{n} c_{ik} a(R_j(e_{ij})) \right]$. We will consider an equivalent random process where a uniform $(0, 1)$ random variable $U_{ij}$ replaces $e_{ij}$. We can do this because $a(R_j(e_{ij})) = a(R_j(F_j(e_{ij}))) = a(R_j(U_{ij}))$.

Consider the matrix $T = C'\Phi$, where $\Phi = \{\phi(F_j(e_{ij}))\} = \{\phi(U_{ij})\}$. Note that $T = \left[ \sum_{i=1}^{n} c_{ik} \phi(U_{ij}) \right]$. The rows of $\Phi$ are iid and the covariance matrix of each row is $\Sigma$, defined in A6. Note that A3 implies that $E(S(0)) = 0$ and A4 implies $E(T) = 0$. If $S(0) = [S_{kj}]$ and $T = [T_{kj}]$, we wish to show that

$$\lim_{n \to \infty} \frac{\text{Var}(S_{kj} - T_{kj})}{\text{Var}(T_{kj})} = 0$$

over all $k$ and $j$. If that is true and if it can be shown that $T \xrightarrow{L} \mathbb{N}_{q,p}(0, I, \Sigma)$, it will follow from Lemma 4.2.3 of Puri and Sen (1985) that $S(0) \xrightarrow{L} \mathbb{N}_{q,p}(0, I, \Sigma)$.

Note that $C'C = I_q$. If $C_k$ is the $k$th column of $C$ and $\Phi_j$ is the $j$th column of $\Phi$, then $T_{kj} = C_k'\Phi_j$. Then

$$\text{Var}(T_{kj}) = C_k'\text{Var}(\Phi_j)C_k = C_k'I_nC_k = 1.$$  

Hence we need only show that

$$\lim_{n \to \infty} \text{Var}(S_{kj} - T_{kj}) = 0.$$
Following the notation of Hajek and Sidak (1967), define
\[ a^\phi(i) = E(\phi(U(i))). \] Then let
\[ S^\phi(0) = \left[ \sum_{i=1}^{n} c_{ik} a^\phi(R_j(U_{ij})) \right]^2 [S^\phi]. \]
Hence, \( \text{Var}(S_{kj} - T_{kj}) \leq 2(\text{Var}(S_{kj} - S^\phi_{kj}) + \text{Var}(S^\phi_{kj} - T_{kj})). \)

Using the same reasoning as in proving V.1.5.10 of Hajek and Sidak, we have
\[
\text{Var}(S^\phi_{kj} - T_{kj}) \leq \frac{n}{n-1} \sum_{i=1}^{n} c_{ik}^2 (E(a^\phi(R_j(U_{ij})))^2 - \phi(U_{ij}))^2)
\]
\[ = \frac{n}{n-1} E(a^\phi(R_j(U_{ij})) - \phi(U_{ij}))^2. \]
The last expression goes to 0 by Theorem V.1.4.a of Hajek and Sidak (1967).

Similarly, \( \text{Var}(S_{kj} - S^\phi_{kj}) \leq \frac{n}{n-1} \sum_{i=1}^{n} c_{ik}^2 \int (a^\phi(1 + [un]) - a(1 + [un]))du = \frac{n}{n-1} \int (a^\phi(1 + [un]) - a(1 + [un]))du, \) if we use the same reasoning as in proving statement V.1.6.6 of Hajek and Sidak. The last expression goes to 0 by the argument following statement V.1.6.6. Hence, \( \lim_{n \to \infty} \text{Var}(S_{kj} - T_{kj}) = 0. \)

The asymptotic normality of \( T \) will follow by applying Theorem 19.1.5 of Arnold (1981). That is, if
\( \text{tr } M'T \overset{L}{\rightarrow} N_q(0, \text{tr } M \Sigma M') \) for all \( q \times p \) constant matrices \( M, \)
then \( T \overset{L}{\rightarrow} N_{q,p}(0, I, E). \)

For an arbitrary matrix \( Z = [Z_{gh}], \) let \( m(Z) = \max_{g,h} |Z_{gh}|. \) We can write \( \text{tr } M'T = \text{tr}(CM)'\phi. \) Then Theorem 19.16 of Arnold states if \( \text{tr } C M \Sigma M'C' = c, \) a constant, for all \( n \) and \( m(CM) \to 0, \)
then \( \text{tr} H' T \xrightarrow{L} N_1(0, c) \). The first condition holds since
\[
\text{tr} C M E M' C' = \text{tr} M E M' C' C = \text{tr} M E M' C',
\]
a constant. The second condition holds since
\[
(qm(C))^2 \leq (qm(C)m(M))^2 \leq (qm(M))^2 m(CC') \to 0,
\]
by A5.

Therefore, \( S(0) \xrightarrow{L} N_{q,p}(0, I, \Sigma) \).
q.e.d.

In the univariate setting, \( S(0) \) is asymptotically distribution free and also for finite \( n \). In the multivariate setting, \( S(0) \) is never unconditionally distribution free. Asymptotically, \( S(0) \) depends on \( F \) through \( \Sigma \). Rank estimates for \( \Sigma \) will appear in Chapter 4.

2.3 The Asymptotic Distribution of the Estimates of the Regression Parameters

In this section, we assume there exists a \( 1 \times p \) vector \( \alpha'_0 \) and a \( q \times p \) matrix \( \Delta_0 \) such that \( Y - 1_n \alpha'_0 - CA_0 = E \). We will consider the joint asymptotic distribution of the estimates of \( \alpha'_0 \) and \( \Delta_0 \).

The vector \( \alpha'_0 \) will be estimated in the manner of Hodges and Lehmann (1963) and Adichie (1967). The matrix \( \Delta_0 \) will be estimated following the work of Jaeckel (1972). In either case, the estimates are translation invariant. Therefore, it can be assumed without loss of generality that the elements of \( \alpha'_0 \) and \( \Delta_0 \) are all zero.

Our primary task in this section is to extend the work done in the univariate setting to our multivariate model. The following
will be used to denote the columns of matrices defined earlier.

Assuming \( j = 1, \ldots, p \),

(N8) Let \( a_j \) be the \( j \)th element of \( a' \).

(N9) Let \( A_j(\Delta) \) be the \( j \)th column of \( A(\Delta) \).

(N10) Let \( S_j(\Delta) \) be the \( j \)th column of \( S(\Delta) \).

Jaeckel's dispersion function restricted to the \( j \)th column would then be as follows.

(N11) \[ D_j(\Delta_j) = (Y_j - CA_j)'A_j(\Delta_j). \]

The multivariate version of the dispersion function is written in the following manner.

(N12) \[ D(\Delta) = D(Y - CA) = \text{tr}(Y - CA)'A(\Delta) = \sum_{i=1}^{p} D_j(\Delta_j). \]

In the univariate setting, Jaeckel (1972) showed that \( D(\Delta) \) was nonnegative, continuous and convex. Since \( D \) is a finite sum of such functions, the same properties are true in the multivariate setting.

Jaeckel's dispersion function is translation invariant in the sense that \( D(Y - 1_n \ a' - CA) = D(Y - CA) \). Estimates of \( \Delta_0 \) will be obtained by minimizing \( D(\Delta) \), but estimates of \( a_0' \) will require work with signed-rank procedures. Estimation of \( \Delta_0 \) will be considered first.
Let $B_j = \{\Delta_j^{Qx1} | D_j(\Delta_j) = \min\}$. Jaeckel (1972) showed that $B_j$ is nonempty and bounded. Therefore, the following notation is meaningful.

(N13) $B = \{\Delta_j^{QxP} | D(\Delta) = \min\}$.

(N14) $\Delta \in B$.

Jureckova (1971a) follows the manner of Hodges and Lehmann (1963) and Adichie (1967) in estimating $\Delta_0$ in the univariate setting. However, Jaeckel shows that his estimates are asymptotically equivalent. He uses Jureckova's "concordance conditions" in showing the asymptotic distribution of his estimates and in showing the asymptotic equivalence to Jureckova's estimates. Heiler's and Willers (1979) replaced the "concordance conditions" with Huber's condition. They show that Jaeckel's estimates continue to be asymptotically normal and continue to be asymptotically equivalent to the Hodges-Lehmann type estimate.

Puri and Sen (1985) extend Jureckova's work to the multivariate setting. We will extended the work of Heiler and Willers. We introduce the gamma parameter at this time. Assuming $j = 1, \ldots, p$,

(N15) Let $\gamma_j = \gamma_j(F_j) = \int \phi(\mu) \left( - \frac{f'_j(F_j^{-1}(\mu))}{f_j(F_j^{-1}(\mu))} \right) d\mu$.

(N16) Let $\Gamma$ be a diagonal matrix whose $j$th diagonal element is $\gamma_j$. 

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The parameter $\gamma_j$ varies inversely with scale. That is, if $Z_{ij} = b Y_{ij}$ and $b > 0$, then $\gamma_j(F_{Z_{ij}}) = b^{-1} \gamma_j(F_{Y_{ij}})$. We shall assume $\gamma_j(F_j) > 0$.

The distribution of $\Delta$ is immediate after the following results.

**Theorem 2.2 (Heiler and Willers)** Under assumptions $A1 - A5$,

$$\lim_{n \to \infty} P\left\{ \sup_{\Delta \in B} \| \Delta - S(0) \Gamma^{-1} \| \geq \epsilon \right\} = 0$$

for all $j = 1, \ldots, p$ and $\epsilon > 0$.

**Corollary 2.2a** Under assumptions $A1 - A5$,

$$\lim_{n \to \infty} P\left\{ \sup_{\Delta \in B} \| \Delta - S(0) \Gamma^{-1} \| \geq \epsilon \right\} = 0$$

for all $\epsilon > 0$.

**Proof.** Let $\epsilon > 0$. Note that for every $\Delta$, $\| \Delta - S(0) \Gamma^{-1} \|

\leq \sum_{i=1}^{p} \| \Delta_j - \gamma_j^{-1} S_j(0) \|$. Therefore,

$$P\left\{ \sup_{\Delta \in B} \| \Delta - S(0) \Gamma^{-1} \| \geq \epsilon \right\}$$

$$\leq P\left\{ \sum_{j=1}^{p} \| \Delta_j - \gamma_j^{-1} S_j(0) \| \geq \epsilon \right\}$$

$$\leq P\left\{ \sum_{j=1}^{p} \sup_{\Delta_j \in B_j} \| \Delta_j - \gamma_j^{-1} S_j(0) \| \geq \epsilon \right\}$$

$$\leq \sum_{j=1}^{p} P\left\{ \sup_{\Delta_j \in B_j} \| \Delta_j - \gamma_j^{-1} S_j(0) \| \geq \epsilon \right\}$$

$$\leq \left( \sup_{j=1}^{p} \right) \left\{ \sup_{\Delta_j \in B_j} \| \Delta_j - \gamma_j^{-1} S_j(0) \| \geq \frac{\epsilon}{p} \right\}$$

$$\leq \sum_{j=1}^{p} P\left\{ \sup_{\Delta_j \in B_j} \| \Delta_j - \gamma_j^{-1} S_j(0) \| \geq \frac{\epsilon}{p} \right\}.$$
The result follows since every term in the last expression goes to 0 by Theorem 2.2.

It will be useful to quote a theorem from Rao (1973).

**Theorem 2.4(ix) (Rao)** Let \( \{J_n, L_n\}, n = 1, 2, \ldots \), be a sequence of pairs of variables, vectors, or matrices. Then if
\[
\lim_{n \to \infty} P\{\|J_n - L_n\| \geq \epsilon\} = 0, \quad \text{and} \quad L_n \xrightarrow{L} L, \quad \text{then} \quad J_n \xrightarrow{L} L.
\]

The asymptotic distribution of \( \Delta \) follows from Theorems 2.1, 2.2, and the above theorem from Rao.

**Theorem 2.3** Under assumptions A1 - A6,

\[
\Delta \xrightarrow{L} N_{q,p}(0, I, \Gamma^{-1} \Sigma \Gamma^{-1}).
\]

The estimate of the \( j \)th element of \( \alpha_0 \) will be obtained by applying a Hodges-Lehmann type procedure to the following signed-rank process. Assuming \( j = 1, \ldots, p \), let

\[
S_j^+(Y_j - 1_n \alpha_j - C\Delta_j) = \frac{1}{2n} \sum_{i=1}^{n} a^+(R_j(\{Y_{ij} - \alpha_j - C_i \Delta_j\})) \text{sgn}(Y_{ij} - \alpha_j - C_i \Delta_j),
\]

where \( a^+(i) = \phi^+(\frac{i}{n+1}) \) and \( \phi^+(u) = \phi(\frac{u+1}{2}) \).

A Hodges-Lehmann type of estimate of \( \alpha_j \) would be a value \( \tilde{\alpha}_j \) such that \( S_j^+(Y_j - 1_n \tilde{\alpha}_j - C\tilde{\alpha}_j) \) is approximately zero. Formally,
we have the following,

\[(N18) \quad \tilde{c}_j = \frac{1}{2}(a_{1j}(0) + a_{2j}(0)), \quad \text{where} \]
\[\alpha_{1j}(\eta) = \inf\{a_j|S_j^+(Y_j - \ln n - \alpha_j - C\tilde{\alpha}_j) < \eta\} \]
and \[\alpha_{2j}(\eta) = \sup\{a_j|S_j^+(Y_j - \ln n - \alpha_j - C\tilde{\alpha}_j) > \eta\}.\]

\[(N19) \quad \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p).\]

Recall that \(\Phi\) is defined in the proof of Theorem 2.1.

In order to study the joint distribution of \(\tilde{\alpha}'\) and \(\tilde{\Delta}\), it is necessary to show that \(\frac{1}{n} \tilde{\alpha}' \Gamma\) is asymptotically equivalent to \(\frac{1}{n} \tilde{\alpha}' \Gamma\).

While working with the intercept parameters, we will assume the marginal density functions are symmetric about 0. Also, we need to introduce another assumption.

\[(A8) \quad \text{The score generating function } \phi(\mu) = \phi_1(\mu) - \phi_2(\mu), 0 < \mu < 1, \text{ where } \phi_j(\mu), \ j = 1, 2, \text{ are monotonic, absolutely continuous and square-integrable inside (0, 1).}\]

**Theorem 2.4** If each density \(f_j\) is symmetric about 0, \(j = 1, \ldots, p\), and assumptions A1 - A6, A8 hold, then

\[\lim_{n \to \infty} P\{|n^{\frac{1}{2}} \tilde{\alpha}' \Gamma - n^{\frac{1}{2}} 1' \tilde{\Phi} \geq \epsilon\} = 0, \text{ for all } \epsilon > 0.\]
Proof. Let \( \varepsilon > 0 \) be given. As in the proof of Corollary 2.2a, it suffices to show \( \lim_{n \to \infty} \mathbb{P}(|\sum_{j=1}^{n} \phi_j \gamma_j - \sum_{j=1}^{n} \phi_j \gamma_j - \frac{1}{2} \mathbf{1}_n \phi_j | \geq \varepsilon) = 0 \). Due to the symmetry of each \( f_j \) and the relationship between \( \phi \) and \( \phi^+ \), McKean and Hettmansperger (1978) showed that

\[
\sum_{j=1}^{n} \phi_j = n - \frac{1}{2} \sum_{j=1}^{n} \phi^+(U_{ij}^+ \text{sgn}(Y_{ij})) = n - \frac{1}{2} \sum_{j=1}^{n} \phi_j \text{sgn}(Y_{ij}),
\]

where \( U_{ij}^+ = F_j^+(|Y_{ij}|) \), \( F_j^+ \) being the distribution function of \( |Y_{ij}| \). Let the sum on the right be denoted \( T_j^+ \).

Therefore, note that

\[
\mathbb{P}(|n^{\frac{1}{2}} \sum_{j=1}^{n} \phi_j \gamma_j - T_j^+ | \geq \varepsilon) \leq \mathbb{P}(|n^{\frac{1}{2}} \sum_{j=1}^{n} \phi_j \gamma_j - \sum_{j=1}^{n} \phi_j (Y_{ij}) | \geq \frac{\varepsilon}{2}) + \mathbb{P}(|\sum_{j=1}^{n} \phi_j (Y_{ij}) - T_j^+ | \geq \frac{\varepsilon}{2}).
\]

The second term is shown to go to zero on page 167 of Hajek and Sidak (1967). Assumption A1 - A3 are sufficient to quote that result.

The first term is dominated by
The first of the immediately preceding terms goes to 0 by an application of Lemma 3.2 of McKeans and Hettmansperger (1976).
Assumptions A1 - A6 and A8 are sufficient. The last term goes to 0 by an application of result (5.10) of Jureckova (1971b).
Assumptions A1 - A6 are sufficient.

The middle term requires a bit of work. Let \( Z_{\frac{m}{2}} \) have the conventional Z-score meaning. That is, the area under the standard normal curve to the right of \( Z_{\frac{m}{2}} \) is \( \frac{m}{2} \). Pick \( \pi, 0 < \pi < 1, \) such that \( \frac{\pi}{6} > Z_{\frac{m}{2}} > 0 \). Let \( a^{*}_j = \)

\[
\{a_j \mid \max(|S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)|, |S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)|, |S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)|) \geq \frac{\pi}{6} \}.
\]

Then, we have

\[
P\{|S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)| \geq \frac{\pi}{6}\}
\leq P\{\max(|S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)|, |S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)|) \geq \frac{\pi}{6}\}
\leq P\{|S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)| \geq \frac{\pi}{6}\}
\leq P\{Z_{\frac{m}{2}} + \frac{|S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)| - Z_{\frac{m}{2}} | \geq \frac{\pi}{6}\}
= P\{|S^+_j(Y_j - 1_n \tilde{a}_j - C\tilde{a}_j)| - Z_{\frac{m}{2}} | \geq \frac{\pi}{6} - Z_{\frac{m}{2}}\}.
\]
The immediately preceding expression goes to 0 by an application of Lemma 3.5 of McKean and Hettmansperger (1976).

Assumptions A1 - A6 and A8 are sufficient. q.e.d.

From Theorem 2.1, Corollary 2.2a and Theorem 2.4, one can conclude that $\left(\frac{1}{n^2} \tilde{a}, \tilde{A}'\right)'$ and $\left(\frac{-1}{n} 1_n, C\right)' \Phi$ have the same asymptotic distribution. The asymptotic matrix normality follows in the same manner as in the last part of the proof of Theorem 2.1.

**Theorem 2.5** Under the assumptions of Theorem 2.4,

$$\frac{1}{n^2} \left(\frac{1}{n} \tilde{a}, \tilde{A}'\right)' \xrightarrow{L} N_{(q+1),p}(0, I, \Gamma^{-1} \Sigma \Gamma^{-1}).$$

**Proof.** The proof of Theorem 2.1 shows that $\lim_{n \to \infty} E(\Delta(0) - C' \Phi)^2 = 0$. An application of Corollary 2.2a produces the fact that $\lim_{n \to \infty} P\{\|\Delta - C' \Phi\| > \varepsilon\} = 0$. Combining the results of Theorem 2.4,

$$\lim_{n \to \infty} P\{\|\left(\frac{1}{n^2} \tilde{a}, \tilde{A}'\right)'\Gamma - \left(\frac{-1}{n} 1_n, C\right)' \Phi\| > \varepsilon\} = 0.$$

In the same fashion as in the proof of Theorem 2.1, it is a straightforward application of Arnold’s (1981) Theorem 19.15 and Theorem 19.16 to show that

$$\left(\frac{-1}{n^2} 1_n, C\right)' \Phi \xrightarrow{L} N_{(q+1),p}(0, I, \Sigma).$$

It sufficient to note that
\[(n - \frac{1}{2} l_n, C)'(n - \frac{1}{2} l_n, C) = I_{q+1} \text{ and } (n - \frac{1}{2} l_n, C)(n - \frac{1}{2} l_n, C)' = C C' + P_n = O, \text{ where } P_n \text{ is the projection matrix on the space spanned by } l_n.\] q.e.d.

Finally, we would like to express the joint asymptotic distribution of our estimates in terms of the original model, 
\(Y = l_n \theta' + XB + E.\) Note that \(\tilde{B}\) would minimize \(D(Y - XB)\) and \(\tilde{\theta}_j\) would be the Hodges-Lehmann type estimate using 
\(S_j(Y_j - l_n \theta'_j - XB).\) It would follow that \(\tilde{B} = \tilde{W}' \tilde{A}\) and 
\(\tilde{\theta}' = \tilde{a}' - n^{-1} l_n' X \tilde{B}.\)

In the manner of Adiche (1967), we must make an additional assumption.

\[(A9) \lim_{n \to \infty} n^{-1} l_n' X = v'.\]

**Theorem 2.6** Under the assumptions of Theorem 2.4, A7 and A9,

\[
\begin{pmatrix}
\frac{1}{l_n} \\
\frac{1}{l_n^2} & \tilde{\theta}' \\
\frac{1}{n^2} & \tilde{B}
\end{pmatrix}
\xrightarrow{L}
N_{(q+1),p}(0, (1 + v' V^{-1} v - v' v^{-1}) \Gamma^{-1} \Sigma \Gamma^{-1}),
- V^{-1} v

\text{where } V = V^2 V^2.

**Proof.** Let \(0_q\) denote a \(q \times 1\) vector of zeros. Note that
\[
\begin{pmatrix}
1 - n^{-\frac{1}{2}} l_n' X W^{-1} \\
\tilde{o}_q
\end{pmatrix}
\begin{pmatrix}
\frac{1}{n^2} \tilde{a}' \\
\tilde{\Lambda}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{n^2} \tilde{\theta}' \\
\tilde{B}
\end{pmatrix}.
\]

The result follows from the fact that \(-n^{-\frac{1}{2}} l_n X W^{-1} =\)
\[-n^{-1} l_n' X(n^{-\frac{1}{2}} W^{-1}) \rightarrow v' v^{-\frac{1}{2}} \quad \text{and} \quad n^{-\frac{1}{2}} W^{-1} \rightarrow V^{-\frac{1}{2}}.\]
q.e.d.

Because the estimates are translation invariant, we have the following corollary.

**Corollary 2.6a** Under the model \(Y = l_n' \theta_o + X B_o + E\) and under the assumptions of Theorem 2.4, A7 and A9,

\[
\begin{pmatrix}
\frac{1}{n^2} (\tilde{\theta} - \tilde{\theta}_o') \\
\frac{1}{n^2} \tilde{B} - \tilde{B}_o
\end{pmatrix}
\rightarrow^L N(q+1),p(0, (1 + v' v^{-1} v' v^{-1}), \Gamma^{-1} \Sigma \Gamma^{-1}), -v^{-1} v v^{-1}
\]

where \(v = v^2 v^2\).
CHAPTER 3

THE NULL ASYMPTOTIC DISTRIBUTION OF
THE TEST PROCEDURES

3.1 Introduction

In general, the multivariate linear hypothesis is of the form $H_{B_0} K = 0$. In this chapter, our interest is in the hypothesis $H_{B_0} = 0$, that is, when $K$ is an identity matrix. A quadratic test, an aligned rank test and a drop in dispersion test are developed. All three tests are shown to be asymptotically equivalent. These procedures are extended to test the general hypothesis $H_{B_0} K = 0$ in Chapter 6. The quadratic procedure will also be used to test the hypothesis $H_1((\theta_0, B_0')' = 0$ where $\theta_0$ is the vector of intercept parameters.

Here, $H$ is a $r \times q$ matrix, $r \leq q$, of rank $r$. Also, $H_1$ is a $r \times (q + 1)$ matrix, $r \leq (q + 1)$, of rank $r$. Note that the symbol $r$ denotes the number of rows in $H$ and $H_1$. The two cases are handled separately, so additional notation is not necessary.

The theory will be developed in terms of model (2.1). The following new notation is needed:

(N20) $G = HW^{-1}$
Expressions of the form $G \Delta$ and $G_1((n^2 \alpha, \Lambda')')$ will be used in the development of our theory. However, the actual hypothesis being tested is either $H_B = 0$ or $H_1((\theta_0, B_0')') = 0$.

The reliance on those hypotheses will be shown when the tests are expressed in terms of the model $Y = X\theta_0 + XB_0 + E$.

Without loss of generality, the theory will be developed under the assumptions that the elements of $\theta_0'$ and $B_0$ are all zero. As in the last chapter, this assumption will be dropped eventually.

We need the following notation and technical lemmas.

(N21) $G_1 = H_1 \left( \begin{array}{ccc} -\frac{1}{n} & -n^{-1} \frac{1}{n} \times W^{-1} \\ 0_q & W^{-1} \end{array} \right)$.

(N22) $Q = G'(GG')^{-1}G$.

(N23) $Q_1 = G_1'(G_1G_1')^{-1}G_1$.

(N24) $(S^+(0))' = (S^+(Y_1), \ldots, S^+(Y_p))$, where

$S^+_j(Y_j), j = 1, \ldots, p$, is defined by N17.

It will be useful to quote the following theorem from Rao (1973).

**Theorem 2c.4(xiv) (Rao)** Let $\{J_n, L_n\}, n = 1, 2, \ldots$, be a sequence of pairs of variables, vectors, or matrices. Let $g$ be a continuous function. If $\lim_{n \to \infty} P(\|J_n - L_n\| \geq \epsilon) = 0$ and $L_n \xrightarrow{L} L$,
then \( \lim_{n \to \infty} P\{|g(J_n) - g(L_n)| \geq \varepsilon \} = 0. \)

**Lemma 3.1** Under assumptions A1 - A7,

\[
\text{tr}(S(0))^T Q S(0) \sum^{-1} \overset{L}{\to} \chi^2(rp).
\]

**Proof.** From A7, we know that

\[
Q = G'(GG')^{-1}G = W^{-1}H'(H W^{-2} H')^{-1}H W^{-1}
\]

\[
= (n^2 W^{-1})H'(H(n W^{-2})H')^{-1}(n^2 W^{-1}) \rightarrow -\frac{1}{2} H'(H V^{-1} H')^{-1} H V^{-1} \frac{1}{2},
\]

as \( n \) goes to infinity. Therefore,

\[
(Q - V^2 H'(H V^{-1} H')^{-1} H V^{-1}) S(0)
\]

\[
= Q S(0) - (V^2 H'(H V^{-1} H')^{-1} H V^{-1}) S(0) \overset{P}{\to} 0.
\]

Note that the term involving the analytic limit of \( Q \) has an asymptotic, singular matrix normal distribution.

Note that \( g(Z) = \text{tr} Z'Z^{-1} \) is a continuous function over all \( q \times p \) matrices. Hence,

\[
\text{tr}(S(0))^T Q S(0) \sum^{-1}
\]

\[
- \text{tr} (S(0))^T (V^2 H'(H V^{-1} H')^{-1} H V^{-1}) S(0) \sum^{-1} \overset{P}{\to} 0.
\]

Note that \( H V \frac{1}{2} S(0) \overset{L}{\to} N_{r,p}(0, H V^{-1} H', \Sigma). \) Therefore, the
result follows from the fact that
\[ \text{tr}(S(0))'((V^\frac{1}{2}H(H^{-1}H')^{-1}HV^\frac{1}{2}))S(0)\Sigma^{-1} \xrightarrow{L} \chi^2(rp). \]

q.e.d.

Lemma 3.2 If each marginal density \( f_j \) is symmetric about 0, \( j = 1, \ldots, p \), and assumptions A1 - A9 hold, then
\[ (S^+(0), (S(0))')' \xrightarrow{L} N_q(0, I, \Sigma). \]

Proof. Let \( \varepsilon > 0 \). Note that
\[ P\{\|S^+(0)\| - n^{-\frac{1}{2}}l_n^t \phi \geq \varepsilon\} \leq \sum_{j=1}^{p} P\{\|S_j^+(Y_j) - n^{-\frac{1}{2}}l_n^t \phi_j \geq \varepsilon\}. \]

as in the manner of the proof of Corollary 2.2a. Each of the terms in the sum was shown to go to zero in the proof of Theorem 2.4. The proof of Theorem 2.1 shows that \( \lim_{n \to \infty} E(S(0) - C^t \phi)^2 = 0 \). It follows that \( \lim_{n \to \infty} P\{\|S^+(0), (S(0))'\| - (n^{-\frac{1}{2}}l_n, C)' \phi \geq \varepsilon\} = 0 \). In Theorem 2.5, it was shown that \( (n^{-\frac{1}{2}}l_n, C)' \phi \xrightarrow{L} N_q(0, I, \Sigma). \)
The result is obtained by combining the last two statements.

q.e.d.

Lemma 3.3 If each marginal density \( f_j \) is symmetric about 0, \( j = 1, \ldots, p \), and assumptions A1 - A9 hold, then
\[ \text{tr}(S^+(0), (S(0))')Q_1((S^+(0))', S(0))\Sigma^{-1} \xrightarrow{L} \chi^2(rp). \]
Proof. From assumptions A7 and A9, we know that

\[
Q_1 = C_1'(G_1 C_1')^{-1} G_1 =
\]

\[
= \begin{pmatrix}
\frac{-1}{2} & 0 \\
0 & q \\
-w^{-1} x' & w^{-1}
\end{pmatrix}
\]

\[
= H_1'
\]

\[
X \left( \begin{pmatrix}
-1 & 0 \\
0 & q \\
-w^{-2} x' & w^{-2}
\end{pmatrix} H_1
\right)^{-1}
\]

\[
X H_1 \left( \begin{pmatrix}
\frac{-1}{2} & -1 \\
-1' & w^{-1}
\end{pmatrix}
\right) = \left( \begin{pmatrix}
1 & 0 \\
-\frac{1}{2} w^{-1} x' & 1' w^{-1}
\end{pmatrix} H_1
\right)
\]

\[
X H_1 \left( \begin{pmatrix}
1 & -1 \\
-\frac{1}{2} x' & 1
\end{pmatrix}
\right) = \left( \begin{pmatrix}
1 & 0 \\
-\frac{1}{2} w^{-1} x' & 1' w^{-1}
\end{pmatrix} H_1
\right)
\]

\[
X H_1 \left( \begin{pmatrix}
1 & -\frac{1}{2} \\
-1' & w^{-1}
\end{pmatrix}
\right)
\]

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Using Lemma 3.2, we obtain the desired result by following the same procedure as in the proof of Lemma 3.1.

q.e.d.

3.2 The Quadratic Test

The quadratic test is a straightforward application of the result on the asymptotic distribution of the estimates. Nevertheless, in order to establish the asymptotic equivalence of our three tests, the lemmas of the preceding section will be used to obtain the results.

We will first consider the test of the hypothesis \( H_B^0 = 0 \). However, our initial result will be in terms of (2.1) and under the assumption that the true parameter matrix is zero.
Theorem 3.4  Under assumptions A1 - A7,

\[ \text{tr}(G\Delta)'(GG')^{-1}(G\Delta)(\Gamma \Gamma^{-1}) \xrightarrow{L} \chi^2(rp). \quad (3.1) \]

Proof. Let \( \epsilon > 0 \). Note that

\[ P\{\sup_{\Delta \in B} \|Q\Delta - QS(0)\| \geq \epsilon\} \]
\[ = P\{\sup_{\Delta \in B} \|Q(\Delta - S(0)\Gamma^{-1})\| \geq \epsilon\} \]
\[ \leq P\{\sup_{\Delta \in B} \|\Delta - S(0)\Gamma^{-1}\| \geq \epsilon\} \]
\[ = P\{\sup_{\Delta \in B} \|\Delta - S(0)\Gamma^{-1}\| \geq \epsilon \frac{1}{2}(\|\Gamma\|^{-1})\}. \]

The last expression goes to zero by an application of Corollary 2.2a.

As in Lemma 3.1, we use the continuous function \( g(Z) = \text{tr} Z'Z \Gamma^{-1} \) to obtain for \( \tilde{\Delta} \in B, \)

\[ \text{tr} (G\tilde{\Delta})'(G\tilde{\Delta})^{-1}(G\tilde{\Delta})(\Gamma \Gamma^{-1}) = \text{tr}(S(0))'Q S(0)\Gamma^{-1} \xrightarrow{P} 0. \]

The result is immediate.

q.e.d.

Corollary 3.4a  Under the model \( Y = \mathbf{1}_n \theta_0' + \mathbf{X}_B \theta + \mathbf{E}, \) assumptions A1 - A7, and the assumption \( HB_0 = 0, \)

\[ \text{tr} (\tilde{H}\tilde{B})'(\mathbf{X}_c'\mathbf{X}_c)^{-1} \tilde{H}'(\tilde{H}\tilde{B})(\Gamma \Gamma^{-1}) \xrightarrow{L} \chi^2(rp). \quad (3.2) \]
Proof. Under the assumption that $B_0 = 0$, (3.1) and (3.2) are algebraically equivalent. For an arbitrary $B_0$, $\tilde{A} = W(\tilde{B} - B_0)$.

Hence, if $HB_0 = 0$, $G\tilde{A} = H(\tilde{B} - B_0) = H\tilde{B}$.

q.e.d

We will now consider the test of the hypothesis

$H_1((\theta_0, B_0')) = 0$.

**Theorem 3.5** If each marginal density $f_j$ is symmetric about 0, $j = 1, \cdots, p$, and assumptions A1 - A9 hold, then

$$
\frac{1}{\text{tr}(n^{-2}a, \tilde{\alpha}'G_1(G_1G_1')^{-1}G_1(n^{-2}a, \tilde{\alpha}')') \Gamma^{-1} \Gamma \xrightarrow{L} \chi^2(rp)} (3.3)
$$

Proof. Let $\epsilon > 0$. Note that

$$
P\left\{\|Q_1(n^{-2}a, \tilde{\alpha}')\Gamma - Q_1(S^+(0), (S(0))')\| \geq \epsilon\right\}
\leq P\{\|Q_1\| \|(n^{-2}a, \tilde{\alpha}')\Gamma - (S^+(0), (S(0))')\| \geq \epsilon\}
\leq P\{\|(n^{-2}a, \tilde{\alpha}')\| \Gamma - (n^{-2}1_n, C)'\phi \geq \frac{\epsilon}{2} (r - \frac{1}{2})\}
+ P\{\|(n^{-2}1_n, C)'\phi - (S^+(0), (S(0))')\| \geq \frac{\epsilon}{2} (r - \frac{1}{2})\}.
$$

With respect to the preceding expression, the first term is shown to go to zero in the proof of Theorem 2.5. The second term is shown to go to zero in the proof of Lemma 3.2. The result follows in the same manner as in the proof of Theorem 3.4.

q.e.d.
Corollary 3.5a If each marginal density \( f_j, j = 1, \ldots, p, \) is symmetric about 0, assumptions A1 - A9 hold, and the assumption \( H_1((\theta_0, B_0')') = 0 \) holds,

\[
\text{tr}(\hat{\theta}, B')H_1^{-1} \left( \begin{pmatrix} \left( n^{-1} + n^{-2} & 1_n'XW^{-1}X'1_n \right)^{-1} & -1_n'XW^{-2} \\ -n^{-1}W^{-2}X'1_n & W^{-2} \end{pmatrix} H_1' \right)^{-1} \\
XH_1^2(\hat{\theta}, \tilde{B}')' \left( \Gamma \sum_{-1}\Gamma \right) \overset{L}{\rightarrow} \chi^2(rp).
\]

(3.4)

Proof. Under the assumption that \( \alpha_0^0 \) and \( B_0 \) are composed of zeros, (3.3) and (3.4) are algebraically equivalent. For an arbitrary \( B_0, \tilde{\Delta} = W(\tilde{B} - B_0). \) For an arbitrary \( \theta_0', \)

\[
\tilde{\alpha} = (\tilde{\theta} - \theta_0)' + n^{-1}1_n'X(\tilde{B} - B_0).
\]

Hence, if \( H_1((\theta_0, B_0')') = 0, \)

\[
\frac{1}{n}G_1(\tilde{\alpha}, \tilde{\alpha}') = H_1((\tilde{\theta} - \theta_0), (\tilde{B} - B_0')') = H_1((\tilde{\theta}, \tilde{B}')').
\]

q.e.d.

3.3 The Aligned Rank Test

In this section, the aligned rank procedure will be considered to test the hypothesis \( H B_0 = 0. \) While the quadratic test employs only full model estimates, the aligned rank test uses only reduced model estimates. Our initial focus will then be on the theory involving reduced model estimates.

Our theory continues to reflect the basic structure of normal, least squares theory. The reduced model estimate is an asymptotic,
linear function of the full model estimate. It has an asymptotic, singular matrix normal distribution.

Consider the following notation.

\[(N25) \quad B_R = \{\Delta^{q \times p} \mid D(\Delta) = \min, G\Delta = 0\}\]

\[(N26) \quad \Delta_R \in B_R.\]

In terms of model (2.1), our aligned rank procedure becomes
\[\text{tr}(S(\Delta_R)^T Q S(\Delta_R)^{-1}).\]

We will rely on the assumptions and the result of Lemma 3.1 for our distribution theory. We will initially assume that the true parameter matrix is zero.

The asymptotic linearity result of Jureckova (1969) is essential to the development of this procedure. However, her work uses the "concordance conditions" discussed earlier. We will use the result as developed by Heiler and Willers (1979).

**Theorem 3.6 (Heiler and Willers)** Under assumptions Al - A5,

\[\lim_{n \to \infty} P\{ \sup_{\|\Delta_j\| \leq c} \|S_j(\Delta_j) - S_j(0) + \gamma_j \Delta_j\| \geq \varepsilon \} = 0\]

for all \(c > 0, \varepsilon > 0\) and \(j = 1, \ldots, p\).

**Corollary 3.6a** Under assumptions Al - A5,

\[\lim_{n \to \infty} P\{ \sup_{\|\Delta\| \leq c} \|S(\Delta) - S(0) + \Delta \| \geq \varepsilon \} = 0\]

for all \(c > 0\) and \(\varepsilon > 0\).
The proof of the preceding corollary is essentially the same as for Corollary 2.2a.

Under the restriction $GA = 0$, $CA = C(Q + (I_q - Q))A = C(I_q - Q)A$. Let $C_1$ be an orthonormal, centered, full rank, $n \times (q - r)$ matrix such that $\text{rng } C_1 = \text{rng } C(I_q - Q)$. From page 39 of Arnold (1981), we know that $C_1 C_1' = C(I_q - Q)C'$.

Let $B_1 = \{\Delta^{(q-r)\times p} | D(Y - C_1 \Delta_1) = \min \}$ and let $\Delta_1$ be any element of $B_1$. Let $B_2 = \{\Delta^{q\times p} | \Delta = C'C_1 \Delta_1 \}$. We wish to show $B_R = B_2$. Such a characterization will help us in the development of our theory.

Let $C'C_1 \Delta_1 \in B_2$. We know there exists a $q \times p$ matrix $\tilde{\Delta}$ such that

$$C(I_q - Q)\tilde{\Delta} = C_1 \Delta_1.$$ 

So,

$$C'C_1 \Delta_1 = C'C(I_q - Q)\tilde{\Delta} = (I_q - Q)\tilde{\Delta}.$$ 

So

$$G C'C_1 \Delta_1 = 0.$$ 

Note that

$$D(Y - C_1 \tilde{\Delta}_R) = D(Y - C(I_q - Q)\tilde{\Delta}_R)$$

$$= D(Y - C(I_q - Q)C'C_1 \tilde{\Delta}_R) = D(Y - C_1 C_1' \tilde{\Delta}_R)$$

$$\geq D(Y - C_1 \tilde{\Delta}_1) = D(Y - CC'C_1 \tilde{\Delta}_1) \geq D(Y - C\tilde{\Delta}_R).$$

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Therefore,
\[ C'C_1\tilde{\Delta}_1 \in B_R. \]

Let \( \tilde{\Delta}_R \in B_R \). There exists a \((q - r) \times p\) matrix such that
\[ C_1\tilde{\Delta}_1 = C(I_q - Q)\tilde{\Delta}_R. \]

Note that
\[
D(Y - C_1\tilde{\Delta}_1) = D(Y - C_1C_1'\tilde{\Delta}_1)
\]
\[
= D(Y - C(I_q - Q)C_1'\tilde{\Delta}_1) \geq D(Y - C\tilde{\Delta}_R)
\]
\[
= D(Y - C(I_q - Q)\tilde{\Delta}_R) = D(Y - C_1\tilde{\Delta}_1) \geq D(Y - C_1\tilde{\Delta}_1).
\]

So, \( \tilde{\Delta}_1 \in B_1 \). Therefore,
\[ C'C_1\tilde{\Delta} = C'C(I_q - Q)\tilde{\Delta}_R = (I_q - Q)\tilde{\Delta}_R = \tilde{\Delta}_R \in B_2. \]

Therefore, \( B_R = B_2 \).

It should be noted that \( B_R \) is bounded. We know that \( B_1 \) is bounded from Jaeckel (1972). Since \( \|\tilde{\Delta}_R\| = \|C'C_1\tilde{\Delta}_1\| \)
\[
\leq (\text{tr} C'C_1C_1'C)\frac{1}{2}\|\tilde{\Delta}_1\| = (\text{tr}(I_q - Q))\frac{1}{2}\|\tilde{\Delta}_1\| = (q - r)^2\frac{1}{2}\|\tilde{\Delta}_1\|,
\]
the initial statement is shown to be true.

We now show that \( \tilde{\Delta}_R \) is asymptotically equivalent to
\( (I_q - Q)\tilde{\Delta} \).
Theorem 3.7 Under assumptions A1 - A5 and A7,

$$\lim_{n \to \infty} P\{ \sup_{\Delta \in B_R} \| (I_q - Q)\Delta \| \geq \varepsilon \} = 0 \text{ for all } \varepsilon > 0.$$ 

Proof. The result will follow if it can be shown that

$$\lim_{n \to \infty} P\{ \sup_{\Delta \in B_R} \| (I_q - Q)S(0)^{-1} \| \geq \varepsilon \}$$

for all $\varepsilon > 0$.

Let $\varepsilon > 0$. Recall that $S(0) = C'A(0)$. Consider the following sequence of statements.

$$P\{ \sup_{\Delta \in B_R} \| (I_q - Q)C'A(0)^{-1} \| \geq \varepsilon \}$$

$$= P\{ \sup_{\Delta \in B_1} \| (I_q - Q)C'A(0)^{-1} \| \geq \varepsilon \}$$

$$= P\{ \sup_{\Delta \in B_1} \| C'C_1 - C'(I_q - Q)C'A(0)^{-1} \| \geq \varepsilon \}$$

$$= P\{ \sup_{\Delta \in B_1} \| C'C_1 - C'C_1 A(0) \| \geq \varepsilon \}$$

$$= P\{ \sup_{\Delta \in B_1} \| C'C_1 (\Delta_1 - C'A(0)^{-1}) \| \geq \varepsilon \}$$

$$\leq P\{ \sup_{\Delta \in B_1} \| C'C_1 \| \| \Delta_1 - C'A(0)^{-1} \| \geq \varepsilon \}$$

$$= P\{ \sup_{\Delta \in B_1} \| \Delta_1 - C'A(0)^{-1} \| > (q - r)^{1/2} \varepsilon \}.$$
If $C_1C_1^\prime$ satisfies Huber's condition, the last expressions go to 0 by an application of Corollary 2.2a. We have that result since

$$m(C_1C_1^\prime) = m(C(I_q - Q)C') \leq q(m(C(I_q - Q)))^2$$
$$\leq q^2 (m(C)m(I - Q))^2$$
$$\leq q^2 (m(I_q - Q))^2 m(CC')$$
$$\rightarrow q^2 (m(I_q - V - \frac{1}{2}H'(HV^{-1}H')^{-1}HV - \frac{1}{2}))^2(0) = 0.$$

q.e.d

The asymptotic distributions of $\tilde{\Delta}_R$ is immediate.

**Theorem 3.8** Under assumptions A1 - A7,

$$\tilde{\Delta}_R \xrightarrow{L} N_q,\rho (0, I - V - \frac{1}{2}H'(HV^{-1}H')^{-1}HV - \frac{1}{2}, \Gamma^{-1}\Sigma \Gamma^{-1}).$$

The asymptotic distribution of our aligned rank procedure is straightforward.

**Theorem 3.9** Under assumptions A1 - A7,

$$\text{tr}(S(\tilde{\Delta}_R))'Q S(\tilde{\Delta}_R)^{-1} \xrightarrow{L} \chi^2(rp). \quad (3.5)$$

**Proof.** Let $\varepsilon > 0$. Note that
\[ P \{ \sup_{\Delta R \in B_R} \| S(\Delta_R) - Q S(0) \| \geq \varepsilon \} \leq P \{ \sup_{\Delta R \in B_R} \| S(\Delta_R) - S(0) + \Delta R \| \geq \frac{\varepsilon}{2} \} + P \{ \sup_{\Delta R \in B_R} \| (I_q - Q) S(0) - \Delta R \| \leq \frac{\varepsilon}{2} \}. \]

The first term goes to zero by an application of Corollary 3.6a. The second term was shown to go to zero in Theorem 3.7.

Therefore, we have

\[ P \{ \sup_{\Delta R \in B_R} \| Q S(\Delta_R) - Q S(0) \| \geq \varepsilon \} = P \{ \sup_{\Delta R \in B_R} \| Q S(\Delta_R) - Q^2 S(0) \| \geq \varepsilon \} \leq P \{ \sup_{\Delta R \in B_R} \| S(\Delta_R) - Q S(0) \| \geq \varepsilon \left( r^{-\frac{1}{2}} \right) \}. \]

The result follows in the same manner as in the proof of Theorem 3.4.

q.e.d.

In order to compare our aligned rank procedure with the work of others, it will be necessary to express (3.5) in terms of projection matrices. Let \( Q = \text{rng } C \) and \( \omega = \text{rng } C_1 \). Then let \( P_Q \) and \( P_\omega \) denote the projection matrices for \( Q \) and \( \omega \), respectively. Thus, \( P_Q = C C' \) and \( P_\omega = C_1 C_1' = C (I_1 - Q) C' \).

The subspace of \( Q \) whose elements are orthogonal to the elements
of \( \omega \) is denoted \( \Omega|\omega \). It is well known that \( P_{\Omega|\omega} = P_{\Omega} - P_{\omega} \). Thus, \( P_{\Omega|\omega} = CQC' \). Therefore, \( (3.5) \) can be expressed as
\[
\text{tr} A((\tilde{\Delta}_R))'P_{\Omega|\omega} A(\tilde{\Delta}_R)\Sigma^{-1}
\]

If \( p = 1 \), this would be the univariate aligned rank procedure considered by Hettmansperger and McKean (1983). Thus, \( (3.5) \) is a multivariate extension of their result.

The translation property of our estimates hold under linear constraints. For an arbitrary model \( Y = X \theta_0 + XB_0 + E \), \( \tilde{\Delta}_R = W(\tilde{E}_R - B_0) \), where \( \tilde{E}_R \) minimizes \( D(Y - XB) \) subject to the constraint \( HB = 0 \). Under the assumptions the true parameter matrix is zero, \( S(\tilde{\Delta}_R) = S(E - \tilde{C}\tilde{\Delta}_R) = S(E - X_0 \tilde{E}_R) = S(E - X \tilde{E}_R) \). Thus under an arbitrary linear model, we have \( S(Y - XB_0 - X(B_R - B_0)) = S(Y - X\tilde{E}_R) \).

**Corollary 3.9a** Under the model \( Y = X \theta_0 + XB_0 + E \), assumptions \( A1 - A7 \), and the assumptions \( HB_0 = 0 \),
\[
\text{tr}(S(Y - X\tilde{E}_R))'Q S(Y - X\tilde{E}_R)\Sigma^{-1} \overset{L}{\rightarrow} \chi^2(rp). \quad (3.6)
\]

**Proof.** Consider the inequality of Theorem 3.9. The first term on the right hand side will always go to zero. The second term requires the assumption \( \tilde{G}\tilde{\Delta}_R = 0 \). Note that \( \tilde{H}\tilde{E}_R = 0 \). Since \( \tilde{G}\tilde{\Delta}_R = H(\tilde{E}_R - B_0) = -HB_0 \), the assumption \( HB_0 = 0 \) is required.

q.e.d.
Suppose our aligned rank procedure is used to test the hypothesis \( B_2 = 0 \), where \((B_1', B_2')'\) is a partitioned parameter matrix with a centered design matrix \((X_1, X_2)\). Then \( \omega = \text{rang } X_1 \) and \( P_\omega = X_1(X'_1X_1)^{-1}X'_1 \). Let \( P_\omega' = I_n - P_\omega \). Then, \( P_\Omega|_\omega = (P_\omega X_2(X'_2P'_2X_2)^{-1}X'_2(P'_2) - P_\omega X_2(X'_2P'_2X_2)^{-1}X'_2(P'_2)) \).

From Hettmansperger and McKean (1983), a property of the matrix \( A(Y - X \tilde{B}_R) \) is that \( (P_\omega A)(Y - X \tilde{B}_R) = 0 \). Here, \( \tilde{B}_R = [\tilde{B}_1 : 0] \), where \( \tilde{B}_1 \) minimizes \( D(Y - X_1B_1) \). Thus, (3.6) becomes essentially
\[
\text{tr } A'(Y - X_1\tilde{B}_1)X_2(X'_2P'_2X_2)^{-1}X'_2A(Y - X_1\tilde{B}_1)\Sigma^{-1}.
\]
This is the same result as obtained by Sen and Puri (1977).

### 3.4 A Test Based on the Drop in Dispersion

The present section extends the work of McKean (1975). Recall that Jaeckel's dispersion function is a measure of how well a given model describes an experimental process. The model that generates the least dispersion is used to estimate the parameters of the process. The current procedure compares the dispersion generated by estimates that satisfy the constraints of the linear hypothesis and the dispersion generated by unrestricted estimates. McKean's drop in dispersion procedure for the \( j \)th column and in terms of model (2.1) is as follows.

\[
(N27) \quad 2\gamma_j(D_j(\tilde{\Lambda}_R) - D_j(\tilde{\Lambda}_j))
\]
The multivariate extension of the drop in dispersion will rely on the matrix $S = [S_{jj'}]$, $jj' = 1, \ldots, P$.

\[(N28) \quad S_{jj'} = \begin{cases} 
2\gamma_j (D_j(\bar{\alpha}_{Rj}) - D_j(\bar{\alpha}_j)), & \text{if } j = j'. \\
(S_j(\bar{\alpha}_{Rj}))'Q S_j(\bar{\alpha}_{Rj}), & \text{if } j \neq j'. 
\end{cases}\]

The multivariate drop in dispersion is then written as $\text{tr} \ S \ \Sigma^{-1}$. Note that it is created by first replacing the diagonal elements of $(S(\bar{\alpha}_R))'Q S(\bar{\alpha}_R)$ with McKean's univariate drop in dispersion procedure. Our process is the trace of the product of the new matrix and $\Sigma^{-1}$. The asymptotic distributions will follow from the fact that $\text{tr} \ S \ \Sigma^{-1}$ is asymptotically equivalent to $\text{tr}(S(\bar{\alpha}_R))'Q S(\bar{\alpha}_R)\Sigma^{-1}$.

We will again begin by assuming the true parameter matrix is zero.

In order to show the asymptotic equivalence of the two procedures, we need Lemma 1 of Jaeckel (1972). He shows that his dispersion function is asymptotically equivalent to a quadratic function. The proof follows from Jureckova's asymptotic linearity result. Due to the work of Heller and Willers (1979), we know Jaeckel's lemma holds under our assumptions.
Theorem 3.10 (Jaeckel) Under assumptions A1-A5,

\[
\lim_{n \to \infty} P\{ \sup_{\| \Delta \| \leq C} |D_j(\Delta_j) - (\frac{1}{2} \gamma_j \Delta_j' \Delta_j - \Delta_j' S_j(0) + D_j(0))| \geq \epsilon \} = 0
\]

for $C > 0$, $\epsilon > 0$ and $j = 1, \ldots, p$.

Following a procedure similar to the proof of Corollary 2.2a, we obtain this corollary.

Corollary 3.10a Under assumptions A1 - A5,

\[
\lim_{n \to \infty} P\{ \sup_{\| \Delta \| \leq C} |D(\Delta) - (\frac{1}{2} \text{tr} \Delta' \Delta - \text{tr} \Delta' S(0) + D(0))| \geq \epsilon \} = 0
\]

for all $C > 0$ and $\epsilon > 0$.

We will need Corollary 3.10a to prove the next theorem. The statement and method of proof of that theorem are multivariate extensions of Theorem 2.2 of McKean (1975). The following two theorems are critical in showing the asymptotic equivalence of the alligned rank procedure and the drop in dispersion procedure.

Theorem 3.11 Under assumptions A1 - A7,

\[
\lim_{n \to \infty} \mathbb{P}\{|(D(\bar{A}_R) - D(\bar{A})) - \frac{1}{2} \text{tr} (S(0)'Q S(0)\Gamma^{-1})| \geq \epsilon \} = 0
\]

for $\epsilon > 0$. 

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Proof Let \( \varepsilon > 0 \). Let \( \Delta = S(0)\Gamma^{-1} \) and \( \Delta_R = (I_q - Q)S(0)\Gamma^{-1} \).

Let \( h(\Delta) = \frac{1}{2} \text{tr} \Delta' \Delta - \text{tr} \Delta' S(0) + D(0) \).

Note that

\[
|D(\Delta_R) - D(\Delta)| = \left| \frac{1}{2} \text{tr} (S(0))'Q S(0)\Gamma^{-1} \right|
\]

\[
\leq |D(\Delta_R) - h(\Delta_R)| + |h(\Delta_R) - h(\Delta)|
\]

\[
+ |h(\Delta_R) - h(\Delta) - \frac{1}{2} \text{tr} (S(0))'Q S(0)\Gamma^{-1}|
\]

\[
+ |h(\Delta) - h(\Delta) - D(\Delta)|.
\]

The result follows if each term goes to 0 in probability.

Since \( B \) and \( B_R \) are bounded, the first and the last term go to 0 by Corollary 3.10a. The middle term is identically 0 by the fact that

\[
h(\Delta) = -\frac{1}{2} \text{tr} (S(0))' S(0)\Gamma^{-1} + D(0)
\]

and

\[
h(\Delta_R) = -\frac{1}{2} \text{tr}(S(0))' (I_q - Q) S(0)\Gamma^{-1} + D(0)
\]

\[
= -\frac{1}{2} \text{tr}(S(0))' S(0)\Gamma^{-1} + \frac{1}{2} \text{tr} (S(0))'Q S(0)\Gamma^{-1} + D(0).
\]

Note that \( h(\Delta) \) is continuous over all \( q \times p \) matrices.

Since both \( \Delta \) and \( \Delta_R \) have an asymptotic, matrix normal distribution, the second and fourth terms go to 0 by applying Rao's Theorem 3.c.4(xiv) to Corollary 2.2a and to Theorem 3.7.

q.e.d.
Theorem 3.12 Under assumptions A1 - A7,

\[ \lim_{n \to \infty} P\{ \frac{1}{2} \text{tr} (S(\hat{\Delta}_R))'Q S(\hat{\Delta}_R) \Gamma^{-1} - \frac{1}{2} \text{tr} (S(0))'Q S(0) \Gamma^{-1} \geq \varepsilon \} = 0 \]

for all \( \varepsilon > 0 \).

Proof Let \( \varepsilon > 0 \). In the proof of Theorem 3.9, it was shown that 

\[ P\{ \sup_{\Delta \in \Delta_R} \| Q S(\Delta) - Q S(0) \| \geq \varepsilon \} = 0. \]

The needed result can then be obtained by an application of Rao's Theorem 2.3 (xiv). This can be done by noting that \( \frac{1}{2} \text{tr} Z'Z \Gamma^{-1} \) is a continuous function over all \( q \times p \) matrices and that \( Q S(0) \) has an asymptotic, singular, matrix normal distribution

q.e.d.

We obtain the asymptotic distribution of the drop in dispersion procedure in the next theorem.

Theorem 3.13 Under assumptions A1 - A7,

\[ \text{tr } \Sigma^{-1} \xrightarrow{L} \chi^2(rp). \quad (3.7) \]

Proof The asymptotic distribution follows from the fact that 

\[ \text{tr } \Sigma^{-1} \text{ and } \text{tr} (S(\hat{\Delta}_R))'Q S(\hat{\Delta}_R) \Sigma^{-1} \text{ are asymptotically equivalent.} \]

Denote \( \Sigma^{-1} = [\Sigma_{jj}^{-1}] \).

Note that
\[
\text{tr} \sum^{-1} - \text{tr} (S(\tilde{\Delta}_R))'Q S(\tilde{\Delta}_R) \sum^{-1} \\
= |\text{tr} (\sum - (S(\tilde{\Delta}_R))'Q S(\tilde{\Delta}_R)) \sum^{-1} | \\
= \left| \sum_{j=1}^{p} (\sum^{-1})_j (2\gamma_j (D_j(\tilde{\Delta}_{R,j}) - D_j(\tilde{\Delta}_j)) - (S_j(\tilde{\Delta}_{R,j}))'Q S_j(\tilde{\Delta}_{R,j}) | \\
\leq \sum_{j=1}^{p} 2(\sum^{-1})_j |D_j(\tilde{\Delta}_{R,j}) - D_j(\tilde{\Delta}_j)) - \frac{1}{2} \gamma_j^{-1} (S_j(\tilde{\Delta}_{R,j}))'Q S_j(\tilde{\Delta}_{R,j}) | \\
\leq \max_j \{ 2(\sum^{-1})_j \} |D(\tilde{\Delta}_R) - D(\tilde{\Delta}) - \frac{1}{2} \text{tr} (S(\tilde{\Delta}_R))'Q S(\tilde{\Delta}_R) \Gamma^{-1} | \\
\leq \max_j \{ 2(\sum^{-1})_j \} |(D(\tilde{\Delta}_R) - D(\tilde{\Delta})) - \frac{1}{2} \text{tr} (S(0))'Q S(0) \Gamma^{-1} | \\
+ \frac{1}{2} \text{tr} (S(0))'Q S(0) \Gamma^{-1} - \frac{1}{2} \text{tr} (S(\tilde{\Delta}_R))'Q S(\tilde{\Delta}_R) \Gamma^{-1} | 
\]

The last expression goes to 0 in probability by applying Theorem 3.11 and Theorem 3.12. By then applying Theorem 3.9, we have the desired result.

\text{q.e.d.}

We now consider the drop in dispersion procedure for an arbitrary linear model \( Y = \theta_0 + X B_0 + E \). When \( B_0 \) is zero, we have \( D(\tilde{\Delta}) = D(E - C\tilde{\Delta}) = D(E - X_C \tilde{B}) = D(E - X \tilde{B}) \). For an arbitrary \( B_0 \), we then have \( D(\tilde{\Delta}) = D(Y - X B_0 - X(B - B_0)) = D(Y - X \tilde{B}) \). In a similar fashion, \( D(\tilde{\Delta}_R) = D(Y - X \tilde{B}_R) \) and \( S(\tilde{\Delta}_R) = S(Y - X \tilde{B}_R) \) for an arbitrary \( B_0 \). Therefore, \( \sum_{j,j'} \) becomes \( (S_j(Y_j - X \tilde{B}_{R,j}))'Q S_j(Y_j, - X \tilde{B}_{R,j}) \), when \( j \neq j' \). We then have the corollary to Theorem 3.13.
Corollary 3.13a  Under the model $Y = \ln \theta_0 + X B_0 + \varepsilon$, assumptions $A_1 - A_7$, and the assumption $H B_0 = 0$, $\text{tr} \sum^{-1} \xrightarrow{L} \chi^2(rp)$, where $\sum$ is defined in the previous paragraph.
CHAPTER 4

CONSISTENT ESTIMATES OF $\Sigma$ AND $\Gamma$

4.1 A Consistent Estimate of $\Sigma$

Our estimate of $\Sigma$ is $\text{tr}(n - 1)^{-1}(A(Y - CA))' A(Y - CA)$. It is a natural analogue to the least squares estimate of the covariance matrix. Also, it makes practical sense in all three of our procedures. Following the suggestion of Pillai (1955), Puri and Sen (1985) use reduced model estimates.

Again, assume that the true parameter matrix is $0$.

Puri and Sen give us the following asymptotic relationship.

Theorem 4.1 (Puri and Sen) Under assumptions A1, A3 and A8,

$$(n - 1)^{-1} (A(0))' A(0) \overset{P}{\rightarrow} \Sigma.$$

Using the "concordance conditions" of Jureckova, Lemma 7.4.1 of Puri and Sen show that their estimate is a consistent for $\Sigma$. We wish to prove the lemma under the assumptions of this paper and for full model estimates. Define,

$$(N29) \quad \tilde{\Sigma} = (n - 1)^{-1}(A(Y - \tilde{C}A))' A(Y - \tilde{C}A).$$

Theorem 4.2 Under assumptions A - A6 and A8, $\tilde{\Sigma} \overset{P}{\rightarrow} \Sigma$.

Proof It is sufficient to show that statement 7.4.34 of Puri and Sen (1985) holds under the current assumptions. That is, we need to
show \( \max_{i=1,\ldots,n} \|Y_i' - (Y_i' - C_i\hat{A})\| \xrightarrow{P} 0 \), where \( Y_i' \) is the \( i \)th row of \( Y \) and \( C_i' \) is the \( i \)th row of \( C \).

Note that

\[
P\{ \max_{i=1,\ldots,n} \|Y_i' - (Y_i' - C_i\hat{A})\| \geq \varepsilon \} = P\{ \max_{i=1,\ldots,n} \|\hat{A}\| \geq \varepsilon \}
\]

\[
\leq P\{ \max_{i=1,\ldots,n} (C_i' C_i)^{\frac{1}{2}} \|\Delta\| \geq \varepsilon \}
\]

\[
= P\{ \|\Delta\| \geq ( \max_{i=1,\ldots,n} (C_i' C_i)^{\frac{1}{2}} )^{-1} \varepsilon \}.
\]

Assumption A5 implies \(( \max_{i=1,\ldots,n} (C_i' C_i)^{\frac{1}{2}} )^{-1} \xrightarrow{} \infty \). The result follows by noting that \( \hat{A} \) has an asymptotic normal distribution under assumptions A1 - A6.

q.e.d.

**Corollary 4.2.a** Under assumptions A1 - A6, A8 and model

\[ Y = \begin{pmatrix} \theta_0' \\ X B_0 + E \end{pmatrix}, \quad \Sigma = (n - 1)^{-1}(A(Y - XB))' A(Y - XB) \xrightarrow{P} \Sigma. \]

### 4.2 Consistent Estimates of \( \Gamma \)

Because \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_p) \), our estimate \( \hat{\Gamma} \) is just a simple extension of the estimates of \( \gamma_j, j = 1, \ldots, p \). We will consider two consistent estimates. The first is based on McKean and
Hettmansperger (1976) and assumes each $f_j$ is symmetric. The second is based on Koul, Sievers and McKean (1987) and does not assume the symmetry of each $f_j$. Assume the true parameter matrices are zero.

McKean and Hettmansperger's estimate relies on the signed rank procedure.

$$\tilde{\gamma}_{1j} = 2z_{n} \sqrt{n} \frac{1}{2} \left( a_{2j} - a_{1j} \right) \left( Z_{n} \right) - a_{1j} \left( Z_{n} \right),$$

where $a_{2j}(n)$ and $a_{1j}(n)$ are defined in N18.

**Theorem 4.3** (McKean and Hettmansperger) Assume A1 - A8 hold. For $j = 1, \ldots, p$, assume that $f_j$ is absolutely continuous and symmetric. Then for each $j$, \[
\lim_{n\to\infty} P\{|\tilde{\gamma}_{1j} - \gamma_j| \geq \epsilon\} = 0,
\]
for $\epsilon \geq 0$.

**Corollary 4.3a** Let $\tilde{\Gamma} = \text{diag}\{\tilde{\gamma}_{11}, \ldots, \tilde{\gamma}_{1p}\}$. If the conditions of Theorem 4.3 hold, $\tilde{\Gamma} \xrightarrow{P} \Gamma$.

The estimate of Koul, Sievers and McKean utilizes the order residuals within each column. Let $Z = Y - CA$. Let $Z_j(1) \leq \cdots \leq Z_j(n)$ be the ordered residuals within the $j$th column of $Z$. For $j = 1, \ldots, p$, define the function $\Pi_j(y)$ to be

$$n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \phi \left( \frac{Z_{i}^{j}}{n} \right) - \phi \left( \frac{Z_{j}^{j-1}}{n} \right) \right) I(|Z_{j}(1) - Z_{j}(k)| \leq y),$$

where $y \geq 0$ and $I(A)$ is an indicator function. For $\tau$, 

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0 < \tau < 1, let t^j(\tau) be the \tau th quantile of H^j. We then have our second estimate.

(N31) \tilde{\gamma}_{2j} = H_j(n^{-\frac{1}{2}}t^j(\tau))(2n^{-\frac{1}{2}}t_h(\tau))^{-1}.

The consistency of the second estimate requires a few refinements of our original assumptions.

(A2') In addition to A2, each f^j is uniformly continuous and bounded. Also, each f^j > 0 a.e. in Lebesgue measure and f^j(y) \to 0 as y \to \pm \infty.

(A4') In addition to A4, \phi is bounded.

**Theorem 4.4** (Koul, Sievers and McKean) Assume that A1, A2', A3, A4', A5-A7 hold. Then for each j, \lim_{n \to \infty} \Pr\{|\tilde{\gamma}_{2j} - \gamma_j| \geq \varepsilon\} = 0, for \varepsilon > 0.

**Corollary 4.4a** Let \tilde{\Gamma}_2 = \text{diag}\{\tilde{\gamma}_{21}, \ldots, \tilde{\gamma}_{2p}\}. If the conditions of Theorem 4.4 hold, \tilde{\Gamma}_2 \xrightarrow{p} \Gamma.

Our estimates of \Gamma would not change if they were constructed under the assumptions that the true model was of the form

Y = 1_n \theta_0' + X B_0 + \varepsilon.
CHAPTER 5

ASYMPTOTIC RELATIVE EFFICIENCY

Asymptotic relative efficiency is a method of comparing our procedures with least squares. This paper follows the work of Pitman (1948) and Hannan (1956). Over a sequence of alternative hypotheses, our procedures have noncentral $\chi^2$ distributions. The least squares procedure also has a noncentral $\chi^2$ distribution over the same sequence. Hannan showed that the Pitman efficiency can be obtained by taking the ratio of the two noncentrality parameters, when the $\chi^2$ distributions have the same degrees of freedom. According to Hannan, this "gives the asymptotic ratio of the sample sizes required for the two tests to give the same power against the same sequence of alternatives."

Consider the sequence of alternative hypotheses $H_n : B_0 = W^{-1}A$, where $A$ is a constant $q \times p$ nonzero matrix with no zero columns. These converge to the null hypothesis $H_0 : B_0 = 0$.

Under $H_n$, we have

$$Y = 1_n' \theta_0 + X W^{-1}A + E$$

$$Y = 1_n' \theta_0 + \overline{x} W^{-1}A + CA + E.$$

Assume the intercept matrix is zero. Therefore, under $H_n$, we have

$$Y = CA + E.$$
Our previous results have been based on a null hypothesis. The concept of contiguity is necessary to show that they also hold under the current alternative hypotheses. Contiguity expresses a relationship between the densities under \( H_0 \) and under the sequence \( H_n \). Our work draws from Hajek and Sidak (1967).

**Definition (Hajek and Sidak)** A sequence of densities \( q_n \) is contiguous to another sequence of densities \( p_n \) if for any sequence of events \( A_n \), \( \lim_{n \to \infty} P(A_n) = 0 \) implies \( \lim_{n \to \infty} Q(A_n) = 0 \), where 

\[
\begin{align*}
dP &= p_n \, d\mu_n, \\
dQ &= q_n \, d\mu_n \quad \text{and} \quad \mu_n \text{ is the Lebesgue measure in } \mathbb{R}^n.
\end{align*}
\]

As Hajek and Sidak point out, the following is an immediate consequence of the definition.

**Lemma 5.1 (Hajek and Sidak)** Suppose the densities \( q_n \) are contiguous to \( p_n \) and \( J_n \) is a sequence of random variables. If 

\[
J_n \xrightarrow{P} 0 \quad \text{under } p_n, \\
\text{then } J_n \xrightarrow{P} 0 \quad \text{under } q_n.
\]

We wish to show our marginal densities are contiguous. If that is true, then Theorem 3.6 holds under \( H_n \). Hence, Corollary 3.6a holds under \( H_n \). In the same fashion, all our expressions that go to zero in probability under \( H_0 \), go to zero under \( H_n \).

Let \( f_j \), \( j = 1, \ldots, p \), be a density satisfying assumption A2.

Under \( H_0 \), the density of \( Y_j \) is \( \prod_{i=1}^{n} f_i(y_{ij}) \). Under \( H_n \), the
density becomes \( \prod_{i=1}^{n} f_j(y_{ij} - C_i^t A_j) \), where \( C_i^t \) is the \( i \)th row of \( C \) and \( A_j \) is the \( j \)th column of \( A \).

**Lemma 5.2** For \( j = 1, \ldots, p \), assume A2 and A5 hold. Then
\[
\prod_{i=1}^{n} f_j(y_{ij} - C_i^t A_j) \text{ is contiguous to } \prod_{i=1}^{n} f_j(y_{ij}).
\]

**Proof** Let \( d_{ij} = C_i^t A_j \). Note that \( n^{-1} \sum_{i=1}^{n} d_{ij} = 0 \). By Theorem 5.2.1 of Hajek and Sidak (1967), it is sufficient to show that
\[
\lim_{n \to \infty} (\max_{i \leq n} d_{ij}^2) = 0 \quad \text{and} \quad \lim_{n \to \infty} \left( \sum_{i=1}^{n} d_{ij}^2 \right) \text{ converges to a positive number.}
\]

The first requirement is satisfied since
\[
\max_{i \leq n} d_{ij}^2 = \max_{i \leq n} (C_i^t A_j)^2 \leq \max_{i \leq n} (C_i^t C_i)(A_j^t A_j) \to 0, \quad \text{by A5.}
\]

The second limit converges because
\[
\sum_{i=1}^{n} d_{ij}^2 = \sum_{i=1}^{n} (C_i^t A_j)^2 \\
\leq \sum_{i=1}^{n} (C_i^t C_i)(A_j^t A_j) = (A_j^t A_j) \text{ tr } C C'
\]

since \( A_j \) contains at least one nonzero element. It converges to a positive number because it is assumed that the sequence \( \{d_{ij}\} \) is not a sequence of zeros.

\[\text{q.e.d.}\]
We will consider the quadratic procedure (3.1) under $H_n$. Note that due to contiguity, (3.1), (3.5) and (3.7) remain asymptotically equivalent under $H_n$. Our quadratic procedure will be compared to the least squares quadratic procedure. In least squares, the quadratic procedure is always algebraically equivalent to the procedures based on reduced model residuals and on the drop in dispersion.

**Theorem 5.3** Under assumptions $A_1 - A_6$ and $H_n$,

$$\tilde{\Delta} \xrightarrow{L} N_{q,p}(A, I, \Gamma^{-1} \Sigma \Gamma^{-1}).$$

**Proof** Lemma 5.1 and Lemma 5.2 imply that the result of Corollary 3.6a holds under $H_n$. That is, we have that $S(A) - S(0) + \Delta \Gamma \xrightarrow{P} H_n$. Since $S(A) \xrightarrow{L} N_{q,p}(0, I, \Sigma)$, we have

$$S(0)\Gamma^{-1} \xrightarrow{L} N_{q,p}(A, I, \Gamma^{-1} \Sigma \Gamma^{-1}).$$

As with Corollary 3.6a, Corollary 2.2a implies $\tilde{\Delta} - S(0)\Gamma^{-1} \xrightarrow{P} H_n$. The result is immediate. q.e.d.

**Corollary 5.3a** Under assumptions $A_1 - A_6$ and $H_n$,

$$\text{tr}(\tilde{\Delta}' \tilde{\Delta})(\Gamma \Sigma^{-1} \Gamma) \xrightarrow{L} \chi^2(qp, \text{tr}(\Lambda' \Lambda)(\Gamma \Sigma^{-1} \Gamma)). \tag{5.1}$$

In order to consider least squares, it will be necessary to assume certain expectations exist. For the random matrix $E$, assume $E(E) = 0$ and $\text{COV}(E) = \Sigma_1$, where $\Sigma_1$ is positive definite.
Theorem 5.4 Assume A1 and A5 hold. If \( \text{E}(\text{E}) = 0 \) and \( \text{COV}(\text{E}) = \Sigma_1 \), then
\[
\text{C}' \text{E} \xrightarrow{L} N_{q,p}(0, \text{I}, \Sigma_1).
\]

Proof The proof will be based on Theorem 19.15 and Theorem 19.16 of Arnold (1981). Let \( M \) be any constant \( q \times p \) matrix. We will show that \( \text{tr} M' C' \text{E} \) has an asymptotic distribution by applying Theorem 19.16. It suffices to show that \( \text{tr} CM \Sigma_1 M' C' \) equals a constant and \( m(CM) \rightarrow 0 \).

The first is easy since \( \text{tr} CM \Sigma_1 M' C' = \text{tr} M \Sigma_1 M' C' = \text{tr} M \Sigma_1 M' \) is a constant.

The second follows from the fact that \( m(CM)^2 \leq q^2 (m(C))^2 (m(M))^2 \leq q^2 (m(M))^2 m(C C') \rightarrow 0 \), by A5.

Hence, \( \text{tr} M' C' \text{E} \xrightarrow{L} N_{q,p}(0, \text{I}) \). Since \( M \) is arbitrary, Theorem 19.15 implies that \( \text{C}' \text{E} \xrightarrow{L} N_{q,p}(0, \text{I}, \Sigma_1) \).

q.e.d.

In terms of our current model, the least squares dispersion function is \( \text{tr} (Y - \text{CA})' (Y - \text{CA}) \). Hence, \( \Delta_L = \text{C}' Y \) is our full model estimate.

Corollary 5.4a Assume A1 and A5 hold. If \( \text{E}(\text{E}) = 0 \) and \( \text{COV}(\text{E}) = \Sigma_1 \), then \( \Delta_L \xrightarrow{H_n} N_{q,p}(A, \text{I}, \Sigma_1) \).

Proof Under \( H_n \), \( Y = \text{CA} + \text{E} \). Therefore \( \text{C}'Y - A = \text{C}'\text{E} \). The result is immediate.

q.e.d.
Corollary 5.4b  Under the assumptions of Corollary 5.4a,

\[ \text{tr} \left( \begin{array}{c} \tilde{\Delta}^L \\ \tilde{\Delta}^L \end{array} \right) \Sigma^{-1} L n H_n \rightarrow \chi^2(q_p, \text{tr} A' A \Sigma^{-1}). \]  (5.2)

The asymptotic relative efficiency of the two procedures is then immediate. The result is the same as found in Section 5.8 of Puri and Sen (1985).

Theorem 5.5 Assume A1 - A6 hold. If \( E(E) = 0 \) and \( \text{COV}(E) = \Sigma_1 \), then the asymptotic relative efficiency of (5.1) with respect to (5.2) is

\[ \frac{\text{tr}(A' A)(\Gamma \Sigma^{-1} \Gamma)}{\text{tr}(A' A)\Sigma^{-1}}. \]
In this chapter, the results of Chapter 3 are extended to test the hypothesis $H_{0,K} = 0$. Among other things, these procedures allow us to test for parallelism. The new procedures are not asymptotically equivalent because the quadratic procedure requires us to use the factor $G_A K$, instead of $G_A I K$. This is to insure $G_A K$ becomes $H_B K$ under the model $Y = l_n \cdot \theta_0 + X B_0 + E$ and the hypothesis $H_{0,K} = 0$. The expression $G_A I K$ does not necessarily become $H_B I K$.

We will consider the quadratic procedure for testing the hypothesis $H_1((\theta_0, B_0')')K = 0$ at the end of the chapter. Here, $K$ is a $p \times s$ matrix, $s \leq p$, of rank $s$. The matrix $H$ is the same as before.

The following lemma and theorems assume the true parameter matrix is zero. As before, the results will first be obtained in terms of model (2.1). The theorems will be followed by corollaries that express the results in terms of an arbitrary linear model.

The theory will be obtained by making minor changes in the proofs of Chapter 3. The following lemma follows from Lemma 3.1.

**Lemma 6.1** Under assumptions A1 - A7,

$$\text{tr}(S(0)\Gamma^{-1}K)^{-1} Q(S(0)\Gamma^{-1}K)(K'\Gamma^{-1}S^{-1}K)^{-1} \overset{L}{\rightarrow} \chi^2(rs). \quad (6.1)$$
Proof Note that \( \frac{1}{2} S(0) \Gamma^{-1} K \xrightarrow{L} N_{r,s}(0, HV^{-1} H', K' \Gamma^{-1} \sum K^{-1} K) \).

The proof is identical to the proof of Lemma 3.1 if \( S(0) K \) replaces \( S(0) \) and \( \text{tr} Z' Z (K' \Gamma^{-1} \sum K^{-1} K)^{-1} \) replaces \( \text{tr} Z' Z \Sigma^{-1} \).

q.e.d.

The quadratic test procedure is obtained from Lemma 6.1.

**Theorem 6.2** Under assumptions A1 - A6,

\[
\text{tr}(G\tilde{A}K)'(G'G)^{-1}(G\tilde{A}K)(K' \Gamma^{-1} \sum K^{-1} K)^{-1} \xrightarrow{L} \chi^2(rs).
\]  

(6.2)

Proof We can show

\[
\lim_{n \to \infty} P\{\sup_{H \in B} |QAK - Q S(0) \Gamma^{-1} K| \geq \epsilon\} = 0
\]

in the same manner as in the proof of Theorem 3.4. Using the continuous function \( g(Z) = \text{tr} Z' Z (K' \Gamma^{-1} \sum K^{-1} K)^{-1} \), we can show that (6.2) is asymptotically equivalent to (6.1). The results is immediate.

q.e.d.

From the proof of Corollary 3.4a, the following is immediate.

**Corollary 6.2a** Under the model \( Y = \theta' + X \beta_0 + E \), assumptions A1-A7, and the assumption \( H\beta_0 K = 0 \),

\[
\text{tr}(H\tilde{B}K)'(H(X'X)^{-1}H')^{-1}(H\tilde{B}K)(H' \Gamma^{-1} \sum \Gamma^{-1} K)^{-1} \xrightarrow{L} \chi^2(rs).
\]  

(6.3)
The distribution of the aligned rank procedure follows in a similar manner.

**Theorem 6.3** Under assumptions A1 - A7,

\[
\text{tr}(S(\tilde{\Delta}_R)^{\text{K}})'Q(S(\tilde{\Delta}_R)^{\text{K}})(K' \sum K)^{-1} \xrightarrow{L} \chi^2(rs). \quad (6.4)
\]

**Proof** In a manner similar to the proof of Lemma 6.1, it can be shown that \(\text{tr}(S(0)^{\text{K}})' Q(S(0)^{\text{K}})(K' \sum K)^{-1} \xrightarrow{L} \chi^2(rs)\). We can show

\[
\lim_{n \to \infty} P\{ \sup_{\Delta R \in B_R} \|Q(S(\tilde{\Delta}_R)^{\text{K}}) - Q(S(0)^{\text{K}})\| \geq \varepsilon \} = 0
\]

in the same manner as in the proof of Theorem 3.9. Using the continuous function \(g(Z) = \text{tr} Z'Z(K' \sum K)^{-1}\), the result follows in the same manner as in the proof of Theorem 6.2. q.e.d.

**Corollary 6.3a** Under the model \(Y = l_n \Theta_0' + X\eta_0 + \epsilon\), assumptions A12 - A7, and the assumption \(HB_0 K = 0\),

\[
\text{tr}(S(Y - XB_{R})^{\text{K}})Q(S(Y - XB_{R})^{\text{K}})(K' \sum K)^{-1} \xrightarrow{L} \chi^2(rs). \quad (6.5)
\]

**Proof** Following a procedure similar to Corollary 3.9a, it is necessary to show \(C(\tilde{\Delta}_R)^{\text{K}} = C(I - Q)\tilde{\Delta}_R^{\text{K}}\). Recall that in the present situation \(\tilde{\Delta}_R = W(\tilde{B}_R - B_0)\). The condition is satisfied if \(G\tilde{\Delta}_R K = 0\). Since \(HB_{R} = 0\), \(G\tilde{\Delta}_R K = H(\tilde{B}_R - B_0)K = -HB_0 K\). The corollary assumes \(HB_0 K = 0\). q.e.d.
We obtain the distribution of the drop in dispersion procedure by noting its relation to the aligned rank procedure.

**Theorem 6.4** Under assumptions A1 - A7,

\[ \text{tr}(K' \Sigma K)^{-1} \xrightarrow{L} \chi^2(rs). \]  

**Proof** Note that

\[ |\text{tr}(K' \Sigma K)^{-1} - \text{tr}(S(\Delta R)'Q(S(\Delta R))(K' \Sigma K)^{-1})| = (\text{tr}(E - (S(\Delta R)'Q(S(\Delta R))(K' \Sigma K)^{-1}K'))|. \]

The expression goes to zero in probability if one applies the procedure used in Theorem 3.13. q.e.d.

**Corollary 6.4a** Under the model \( Y = \eta' + \xi + \epsilon \), assumptions A1 - A7, and the assumption \( H_{q_b}K = 0 \),

\[ \text{tr}(K' \Sigma K)^{-1} \xrightarrow{L} \chi^2(rs), \]  

where \( E \) is defined in the paragraph preceding Corollary 3.13a. (6.7)

The quadratic procedure for testing the hypothesis \( H_1((\theta_0', B_0')'K = 0 \) follows from Lemma 3.2 in a manner that parallels the development of (6.3) from Lemma 3.1.

**Theorem 6.5** Assume that \( Y = \eta' + \xi + \epsilon \). If each marginal density \( f_j, j = 1, \ldots, P \), is symmetric about 0, Assumptions A1 - A9 hold, and the Assumptions \( H_1((\theta_0', B_0')'K = 0 \) holds
\[
\text{tr}(K'(\tilde{\theta}, \tilde{B}')H_1')
\]

\[
\left(\begin{array}{cc}
\frac{n^{-1} + n^{-2} 1'}n & -n^{-1}1'Xw^{-2}n \\
-n^{-1}w^{-2}X1'n & w^{-2}
\end{array}\right)^{-1}
\]

\[
\chi\left(H_1\left((\tilde{\theta}, \tilde{B}')'K\right) \left(K'\Gamma^{-1}\Sigma^{-1}K\right)^{-1}\right) \rightarrow \chi^2(\tau_0). \tag{6.8}
\]
CHAPTER 7

SOME EXAMPLES

In this chapter, our interest is in the application of our procedures. Two problems are taken from Timm (1975). Each comes from the area of profile analysis. The second problem is also considered when a data point is replaced with a multivariate outlier.

In profile analysis, \( n \) subjects are first assigned to \( q \) experimental groups. Then, the responses to \( p \) conditions are observed for each subject. In our first example, \( q = 2 \) and \( p = 5 \). The \( n = 20 \) subjects are evenly divided between the two groups. The experimental process is expressed in terms of the following linear model. It is

\[
Y = \begin{pmatrix}
1_{10} & 0_{10}
\end{pmatrix}
\begin{pmatrix}
\mu_1^1 \\
\mu_2^1 
\end{pmatrix}
+ E, 
(7.1)
\]

where \( \mu_k = (\mu_{k1}, \ldots, \mu_{kp}), \ k = 1, \ldots, q. \)

In order that Assumption A5 is satisfied, we will work with the following equivalent model.

\[
Y = \frac{1}{120} \left( \frac{1}{2} (\mu_1^1 + \mu_2^1) \right) + \begin{pmatrix}
1_{10} \\
-1_{10}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} (\mu_1^1 - \mu_2^1) \\
\frac{1}{2} (\mu_1^1 - \mu_2^1)
\end{pmatrix}
+ E
(7.2)
\]
In terms of the model $Y = \frac{1}{n} \sum Y_i + \beta X + E$, $\theta' = \left( \frac{1}{2} (\mu_1' + \mu_2') \right)$ and $B = \left( \frac{1}{2} (\mu_1' - \mu_2') \right)$. Note that

$$
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
\theta' \\
B
\end{pmatrix}
= 
\begin{pmatrix}
\mu_1' \\
\mu_2'
\end{pmatrix}.
$$

All estimates and assignments of scores come from a program written by Dr. Joseph W. McKean, Department of Mathematics and Statistics, Western Michigan University. Discussion of the techniques can be found in Hettmansperger and McKean (1983) and McKean and Hettmansperger (1978).

Our first example is Example 3.16 of Timm (1975). See Table 7.1. The least squares and rank procedure estimates of central tendency follow. They are expressed graphically in Figures 7.1 and 7.2.

### Estimates - Example 3.16

#### Least Squares

<table>
<thead>
<tr>
<th>Conditions</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1</td>
<td>42.0</td>
<td>28.4</td>
<td>41.2</td>
<td>31.2</td>
<td>33.4</td>
</tr>
<tr>
<td>Group 2</td>
<td>50.9</td>
<td>35.0</td>
<td>49.6</td>
<td>37.9</td>
<td>44.9</td>
</tr>
</tbody>
</table>

#### Rank Procedures

<table>
<thead>
<tr>
<th>Conditions</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1</td>
<td>42.0</td>
<td>27.7</td>
<td>41.0</td>
<td>31.0</td>
<td>33.0</td>
</tr>
<tr>
<td>Group 2</td>
<td>50.0</td>
<td>34.3</td>
<td>49.0</td>
<td>37.0</td>
<td>47.0</td>
</tr>
</tbody>
</table>
Profile analysis involves testing for parallelism (group-by-condition interaction), equality among conditions and equality among groups. If the hypothesis of parallelism is not rejected, the second test is then for equality among the $p$ average conditions. Wilks' $A$ criterion will be used to test the hypotheses in the least squares setting. The result of the aligned rank procedure will be denoted by $A$; the drop in dispersion procedure, by $D$; and the quadratic procedure, by $Q$. A value that leads to the rejection of the hypothesis with $\alpha = .05$ will be designated by an asterisk (*).

The results of the testing procedures for Example 3.16 are as follows.

$H_{01}$: The profiles of the two groups are parallel.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 \\
\end{bmatrix} = 0
\]

Least Squares: $A = .8391$

Rank Procedures:

$A = .9176$

$D = 4.6909$

$Q = 4.7561$
\[ H_{02}: \text{The average conditions are equal.} \]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

(1) \( \Theta' \)

Least Squares:

\[ A = .2197^* \]

Rank Procedures:

\[ Q = 42.2808^* \]

\[ H_{03}: \text{The group are equal.} \]

(1) \( B I_5 = 0 \)

Least Squares:

\[ A = .5561 \]

Rank Procedures:

\[ A = 9.2022 \]

\[ D = 13.3079^* \]

\[ Q = 16.7263^* \]

We now consider Exercise 5.13 of Timm. See Table 7.2. In this case, \( q = 3 \) and \( p = 4 \). The \( n = 15 \) subjects are evenly divided into the three groups. Generally, this experimental procedure is expressed as

\[
Y = \begin{pmatrix}
1_5 & 0_5 & 0_5 \\
0_5 & 1_5 & 0_5 \\
0_5 & 0_5 & 1_5 \\
\end{pmatrix}
\begin{pmatrix}
\mu_1' \\
\mu_2' \\
\mu_3' \\
\end{pmatrix} + E.
\]

\[ (7.3) \]
We will work with the following equivalent model.

\[ Y = 1_{15} \left( \frac{1}{3} (\mu_1' + \mu_2' + \mu_3') \right) \]

\[ + \left( \begin{array}{ccc} 1_5 & 1_5 & 1_5 \\ 1_5 & -2_5 & 1_5 \\ 1_5 & 1_5 & -2_5 \end{array} \right) \left( \begin{array}{c} \frac{1}{3} (\mu_1' - \mu_2') \\ \frac{1}{3} (\mu_1' - \mu_3') \end{array} \right) + E \]  

(7.4)

In terms of the model

\[ Y = 1_n \Theta' + XB + E, \quad \Theta' = \left( \frac{1}{3} (\mu_1' + \mu_2' + \mu_3') \right) \]

and \( B = \left( \begin{array}{c} \frac{1}{3} (\mu_1' - \mu_2') \\ \frac{1}{3} (\mu_1' - \mu_3') \end{array} \right) \). Note that

\[ \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{array} \right) \left( \begin{array}{c} \Theta' \\ B \end{array} \right) = \left( \begin{array}{c} \mu_1' \\ \mu_2' \\ \mu_3' \end{array} \right) . \]

The least squares and rank procedure estimates of central tendency follow. They are expressed graphically in Figures 7.3 and 7.4.
### Estimates - Exercise 5.13

#### Least Squares

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Group</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.0</td>
<td>17.4</td>
<td>15.2</td>
<td>30.4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>28.2</td>
<td>14.2</td>
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#### Rank Procedures

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The results of the testing procedures for Exercise 5.13 are as follows.

**H\(_0\)_1**: The profiles of the three groups are parallel.

\[
I_2 B \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = 0
\]

**Least Squares:**

\[A = 0.2179^*\]

**Rank Procedures:**

\[A = 10.0167\]

\[B = 44.3284^*\]

\[C = 44.2903^*\]
$H_{02}$: The conditions are equal

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
\beta'
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -1 & -1
\end{pmatrix} = 0
$$

Least Squares: $A = .1664^*$

Rank Procedures: $Q = 58.0501^*$

$H_{03}$: The groups are equal.

$$I_2 B I_3 = 0$$

Least Squares: $A = .1414^*$

Rank Procedures:

- $A = 13.7525$
- $B = 60.2926^*$
- $Q = 80.3902^*$

Finally, we reconsider Exercise 5.13 after the last line has been replaced with a multivariate outlier. That is (11, 20, 17, 6) in Group 3 is replaced by (99, 5, 5, 99). The estimates are given below. They are expressed graphically in terms of Figures 7.5 and 7.6.
Estimates - Exercise 5.13 with outlier

### Least Squares

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### Rank Procedures

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<td>15.4</td>
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The results of the testing procedures for the modified Exercise 5.13 are as follows.

- **$H_{01}$**: Least Squares
  - $A = .8985$
  - $A^* = 5.3266$
  - $D = 45.2972^*$
  - $Q = 28.0079^*$

- **$H_{02}$**: Least Squares
  - $A = .6819$
  - $Q = 37.8872^*$

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\[ H_{03}^i \text{ Least Squares} \]
\[ A = 0.7452 \]

\[ \text{Rank Procedures} \]
\[ A = 12.2631 \]
\[ D = 68.8237^* \]
\[ Q = 118.4548^* \]
Table 7.1 Example 3.16, Timm (1975)

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Table 7.2 Example 5.13, Timm (1975)

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Figure 7.1 Least Squares Estimates, Example 3.16, Timm (1975)

Figure 7.2 Rank Procedures Estimates, Example 3.16, Timm (1975)
Figure 7.3 Least Squares Estimates, Exercise 5.13, Timm (1975)

Figure 7.4 Rank Procedures Estimates, Exercise 5.13, Timm (1975)

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Figure 7.5 Least Squares Estimates, Exercise 5.13 with outlier, Timm (1975)

Figure 7.6 Rank Procedure Estimates, Exercise 5.13 with outlier Timm (1975)

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CHAPTER 8
CONCLUSION

In this thesis, a comprehensive theory has been developed for testing the multivariate linear hypothesis by robust rank procedures. The theory reflects the structure of least squares. Our procedures are analogous to the Lawley-Hotelling trace criterion. Procedures that are analogous to Wilks' A criterion and to Roy's largest root test should be considered in further research.

Investigation into the practical nature of our procedures is needed. A Monte Carlo study can be done. It would be interesting to compare the behavior of our tests over a large number of trials and over different designs. Practical critical values for each of our procedures could result from such a study.

The following is a summary of our thesis.

In Chapter 1, robust rank procedures are proposed for testing the multivariate linear hypothesis. They are rank analogues to the three forms of the Lawley-Hotelling trace criterion. The procedures are developed under a multivariate generalization of the assumptions of Heiler and Willers (1979). Primarily, that means assuming Huber's condition holds on our design matrices.

The asymptotic distribution of \( S(Y - XB_0) \) is developed in Theorem 2.1 of Chapter 2. The distribution of \( \hat{\beta} \) follows from that theorem. In addition, the distribution of \( \hat{\theta} \) is shown to be closely related to the processes developed in Theorem 2.1. The
joint distribution of \((\bar{b} - 0_{0}), (\bar{b} - B_{0})'\)' concludes Chapter 2.

Our three procedures for testing the hypothesis \(HB_{0} = 0\) are put forth in Chapter 3. A quadratic involving \(S(Y - XB_{0})\) is first developed. Then, the three procedures are each constructed and are shown to be asymptotically equivalent to this quadratic.

In Chapter 4, estimates of \(\Sigma\) and \(\Gamma\) are considered. We obtain an estimate of \(\Sigma\) by making some adjustments in the estimate of Puri and Sen (1985). We have two estimates of \(\Gamma\). The first is a generalization of the work of McKean and Hettmansperger (1976); the second, a generalization of the work of Koul, Sievers and McKean (1987).

In Chapter 5, the asymptotic relative efficiency of our procedure as compared with least squares is considered. Our results are the same as for the aligned rank procedure of Puri and Sen (1985).

The results of Chapter 3 are extended in Chapter 6 to test the hypothesis \(HB_{0}K = 0\). However, the quadratic procedure is no longer asymptotically equivalent to the aligned rank and drop in dispersion procedures.

Some examples are considered in Chapter 7. Two problems in profile analysis and taken from Timm (1975). One of the problems is reconsidered after a multivariate outlier has been placed in the data.

Chapter 8 concludes the thesis.
BIBLIOGRAPHY


[22] Pitman, E. (1948), Notes for lectures on Non-Parametric Inference given at University of North Carolina.


