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# Jacobi Moments in Applied Mathematics with Computer Applications

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# JACOBI MOMENTS IN APPLIED MATHEMATICS WITH COMPUTER APPLICATIONS

by

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# JACOBI MOMENTS IN APPLIED MATHEMATICS WITH COMPUTER APPLICATIONS

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This work provides solid asymptotic representations, sharp error bounds and stable recurrence methods (both three term and two dimensional) for the Jacobi moments. These moments are currently used in several areas of numerical analysis (numerical integration, integral equations and boundary value problems).

A powerful representation theorem, due to H. Gingold, which uses the Jacobi moments is extended and analyzed. Applications of this theorem to multi-turning point problems and several other areas are given.

For a number of important problems in mathematical physics it is not possible to prove that the currently employed methods of solution converge, or are valid in any sense. In many such cases our methods may be shown to be uniformly convergent and numerically stable.

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John A. Kapenga

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# CHAPTER 1

# INTRODUCTION

## 1.1 Historic Perspective

Starting in the 18th century the development of the techniques of Applied Mathematics was driven by the need to solve, by hand, the problems of Mathematical Physics. Methods that required the use of advanced symbolic operations and moderate arithmetic were preferable to methods requiring large amounts of arithmetical calculation.

Long before the advent of the digital computer many of the problems of Mathematical Physics had already been investigated and numerical solution techniques proposed. Two common techniques were the formal use of power series and orthogonal expansions (e.g. [24] and [80]).

The importance of the use of Chebyshev expansions, known to Chebyshev over 100 years ago, was lost to Applied Mathematics until it was rediscovered by C. Lanczos some 40 years ago. ([78], [79] and [80]). At that time one popular numerical method was to assume a formal power series solution to a problem, then attempt to find conditions which could be used to solve for the coefficients. A partial sum of the power series could then be evaluated to give an approximate numerical value of a solution to the original problem.

Some major problems with formal power series techniques are: almost no verifiable conditions for convergence, non-uniform convergence in cases when convergence occurs and divergence at the endpoints of the interval of interest, even though the solution to the problem under investigation may be continuous there.

The work of C. Lanczos, and others after him ([22], [23], [65] and [74]) provided arguments for using formal Chebyshev series in place of formal power series. The classic text by Fox and Parker ([46]) summarizes many applications of this approach.

The results of using formal Chebyshev series are truely impressive; however the lack of verifiable conditions for the convergence of this method and the fact that divergence may still occur at the endpoints of the interval of interest still remain major problems.

The use of formal Chebyshev series might have replaced the use of formal power series methods in many problems had it not been for the digital computer. The computer developments in the 1950's caused a real drop in the importance of many formal methods. This was due to the fact that simpler methods which required large amounts of arithmetical computation could be used. These methods often have simpler error analysis and convergence criteria than formal methods.

One area where formal methods remains the dominant technique is that of singular perturbation problems (e.g. [98], [123] and [125]) reasons for this are that formal perturbation methods provide the information of interest and these problems are inherently numerically stiff. Thus no simple method can be reliably applied to them.

The use of orthogonal expansions in singular perturbation problems has had very little impact. Their use in several other related areas has led to some important recent work. Some of the areas are:

Singular Integral Equations

([20], [50], [60], [69] and [101])

Cauchy Principal Value Integrals

([67], [68], [114] and [122])

Product Integration

[37], [112], [113] and [119])

Partial Differential Equations, with boundry singularities

([74]p, [89], [99] and [117])

Function Approximation

([62], [83], [84] and [118])

It should be noted that recent work in solving systems of partial differential equations connected with chemical reactions, using the so called decoupled direct method, demonstrated to those doing the research the need for sensitivity coefficients for their systems ([27], [29], [30] and [76]).

These sensitivity coefficients are usually calculated as part of the numerical procedure for solving the system. This can be viewed as an attempt to calculate the first perturbation term of a perturbed system numerically. It appears that formal orthogonal methods, like those to be presented here, will provide sensitivity coefficients far more accurately and easily than the techniques currently employed ([26] and [98]).

The use of orthogonal expansions in the investigation of problems with singularities, or at least great stiffness, is now viewed as an important tool. This use is greater today than it was when the digital computer was initially involved in such problems.

There is a small paradox in the fact that, the digital computer caused a decline in interest in formal methods. The paradox is that because new advanced software allows the computer to compute symbolically as well as numerically, the computer may become an important tool in the application of formal methods.

Already there have been numerous cases where formal methods have been applied to problems where much of the symbolic work was carried out by computer. Some of these are examples of problems that could not have been solved by hand or numerically by the computer ([39], [40], [56], [64], [66], [77], [105] and [130]).

The work to be presented here is a extension of material developed by H. Gingold. The goal is to provide a rigorous basis for the use of formal methods, including basic error estimates and methods of calculating required constants efficiently and accurately.

With the previously presented, admittedly somewhat myopic, view of recent developments it is possible that the work done here may prove to be a foundation for some state of the art numerical techniques, rather than extensions of methods that dropped from importance 25 years ago.

## 1.2 A Summary of Some Results by H. Gingold

The paper [51] by H. Gingold contains a representation theorem which allows the explicit calculation of a sequence of finite Chebyshev series from the power series expansion of a function.

Under mild, often verifiable, conditions the sequence of functions so calculated will converge to the original function. This convergence will tend to be faster and more uniform than the convergence of the original power series, and it can even occur at points on the radius of convergence of the power series where the power series diverges.

The motivation for this work is its application to problems in singular perturbations, as follows.

A solution for a singular perturbation problem is calculated by a formal power series method at a regular point close to the singular point. This power series solution then has a radius of convergence up to the singular point. The representation theorem is then applied to this power series to produce a finite Chebyshev series that is an approximation to the solution; perhaps even at the singular point!

To make these remarks more concrete a few definitions will be given followed by the main representation theorem from H. Gingold ([51]). Then an example from [51] will be discussed with respect to some work in [52]. The work in [52] attempts to compare this method to other summability methods in regular use. The results in Chapter 2 are compared to those in [52] in a set of Remarks (2.4).

### 1.2.1 Notation

Let $U_m(u)$ denote the mth degree Chebyshev polynomial of the second kind. (see Appendix A.1.56)

### 1.2.2 Definition (The Chebyshev Moments)

Let the Chebyshev moments be defined by

$$\phi_{m,n} = \frac{2}{\pi} \int_{-1}^{1} (1-u^2)^{\frac{1}{2}} u^n U_m(u)du, \qquad 1.2.2.1$$

$$\text{for} \quad m,n = 0, 1, \cdots .$$

### 1.2.3 Definition (Cesaro summation of order 2).

For a series

$$\sum_{m=0}^{\infty} A_m , \qquad 1.2.3.1$$

let

$$(C,2) \sum_{m=0}^{M} A_m = \sum_{m=0}^{M} \frac{\binom{M-m+2}{M-m}}{\binom{M+2}{M}} A_m$$

denote its Mth (C,2) partial sum.

In addition let

$$(C,2) \sum_{m=0}^{\infty} A_m = \lim_{M\to\infty} (C,2) \sum_{m=0}^{M} A_m \qquad 1.2.3.2$$

denote the (C,2) sum of 1.2.3.1, if the limit exists.

Now, slightly restated, we present the main expansion theorem of Gingold ([51]).

### 1.2.4 Theorem (Gingold)

Let $y(t,\varepsilon)$ be a holomorphic function of $\varepsilon$ for $\mathrm{Re}\ \varepsilon > 0$ and $t \in [0,1]$.

Let

$$y(t,\varepsilon(u)) = \sum_{n=0}^{\infty} y_n(t)u^n \qquad 1.2.4.1$$

be the power series expansion of $y$ in the disk $|u| < 1$, where

$$u = \frac{\gamma - \varepsilon}{\gamma + \varepsilon} \quad \text{for some fixed } \gamma > 0. \qquad 1.2.4.2$$

Assume that for some $\alpha < 0.5$

$$|y_n(t)| = O(n^{\alpha}). \qquad 1.2.4.3$$

Also assume that $\dfrac{y(t,u)}{\sqrt{1-u^2}}$ is integrable with respect to $u$ on $[-1,1]$.

*Then the Fourier expansion of* $y(t,u)$ *in terms of* $\{U_m(u)\}_{m=0}^{\infty}$ *exists and the* mth *Fourier coefficient,* $a_m(t)$, *is given by the absolutely convergent series*

$$a_m(t) = \sum_{n=m}^{\infty} \phi_{m,n}\, y_n(t), \quad \text{for} \quad m = 0,\ 1,\ \cdots . \qquad 1.2.4.5$$

*Additionally, at all points* $u \in (-1,1)$, *and at* $u = 1$ (or $u = -1$) *if* $y(t,u)$ *is continous there,*

$$y(t,u) = (C,2) \sum_{m=0}^{\infty} a_m(t)\, U_m(u) \quad , \qquad 1.2.4.5$$

$$\textit{for} \quad t \in [0,1].$$

This theorem, roughly speaking, can be applied to problems where any singularities involved have orders less than 0.5.

Gingold ([51]) presents contains some more general results on the existence of the Fourier coefficients and the convergence of 1.2.4.5 in cases where $y(t,u)$ has integrability conditions imposed but may not have a power series expansion or even be continuous for $u \in [-1,1]$.

Gingold also states conditions under which 1.2.4.4 will converge uniformly in $t$.

The fact that closed forms are given in [51] for the Chebyshev moments is important from a computational standpoint since they will be needed numerically for any application of Theorem 1.2.4.

Extensions of these results will be presented in Chapters 2 and 3 replacing the Chebyshev polynomials with Jacobi polynomials.

This allows the restriction $\alpha < 0.5$ to be removed from the theorem corresponding to Theorem 1.2.4. Thus problems with any singularities of finite order may be attacked.

There are a number of practical error bounds presented in Chapter 2 that have no counterparts in [51]. These bounds were obtained as a result of a different, and more direct, method of proof than that employed in [51] or [52]. The remarks in Chapter 2 contain some details on comparing the results there with those in [52].

### 1.3 The Simplest Singular Pertubation Problems.

The simplest singular pertubation problem

$$\varepsilon y' + y = 0; \quad 0 \le t \le 1, \quad \varepsilon > 0 \qquad 1.3.1$$
$$y(0,\varepsilon) = 1$$

was analyzed by Gingold in [51] using Theorem 1.2.4.

We observe that the general solution to 1.3.1 is given by

$$y(t,\varepsilon) = e^{-\frac{t}{\varepsilon}}, \qquad 1.3.2$$

which meets all the conditions of Theorem 1.2.4.

Following Gingold we take

$$\varepsilon = \gamma \frac{1 - u}{1 + u}, \quad \gamma > 0, \qquad 1.3.3$$

which takes 1.3.1 into

$$\gamma(1-u)y' + (1+u)y = 0; \quad 0 \le t \le 1, \ -1 \le u \le 1 \qquad 1.3.4$$
$$y(0,u) = 1$$

Then assuming a power series solution of the form

$$y(t,u) = \sum_{n=0}^{\infty} y_n(t)u^n, \quad |u| < 1 \qquad 1.3.5$$

and substituting it into 1.3.4 we obtain the system

$$\gamma\, y'_0 + y_0 = 0; \quad y_0(0) = 1, \quad 0 \le t \le 1 \qquad 1.3.6$$
$$\gamma\, y'_n + y_n = \gamma\, y'_{n-1} - y_{n-1}; \quad y_n(0) = 0,$$
$$0 \le t \le 1.$$

These equations may be recursively solved for $y_0$, $y_1$, $y_2$ ... in that order.

A recursive solution to the system 1.3.6 is presented in [51]. The $y_n$'s so formed can be used to calculate the $a_m(t)$'s, which can be used in approximating the solution $y(t,u)$, according to Theorem 1.2.4.

It should be noted that, even though

$$\lim_{u \to 1} y(t,u) = \begin{cases} 1: & t = 0 \\ 0: & 0 < t \le 1 \end{cases} \qquad 1.3.7$$

is discontinuous in $t$, the approximation to $y$ gives by 1.2.4.5 will converge for all $t \in [0,1]$ and $u \in [-1,1]$. The power series solution not only converges very slowly near $u = 1$, $t = 0$; but fails to converge at all for $u = 1$. This difference is the real motivation for applying this technique.

In [52] Gingold presents a justification for this approach is based on formal properties of formal expansions. One of the points raised in that paper is that the system 1.3.6 has one unique formal solution, while if a formal Chebyshev expansion is formally substituted into an equation such as 1.3.4, a system can result which has an infinite number of formal solutions, only one of which can correspond to an actual solution to the original problem.

While this objection to the use of formal expansions is valid it is not as important as it might appear. The fact that there will only be one solution among all the formal solutions that converges, or even has $|a_m(t)|$ bounded, follows in most cases, just as it does in the analysis of formal Chebyshev methods, presented in [22, [45], [46] and [74]. This allows formal system to be used in several ways to estimate the Chebyshev coefficients directly. The importance of attempting this follows from the fact that many functions whose Chebyshev expansions converge relatively quickly have formal power series which converge very slowly, forcing an unacceptably large number of the $y_n$'s to be found in order to calculate the first few $a_m$'s.

To see how the formal Chebyshev approach might be applied, let us consider the formal series

$$y(t,u) = \sum_{m=0}^{\infty} a_m(t)U_m(u). \qquad 1.3.8$$

Formally substituting 1.3.8 into 1.3.4, then using the identities

$$u\ U_m(u) = \frac{1}{2}[U_{m-1}(u) + U_{m+1}(u)]\ , \quad m = 1,\ 2,\ \cdots\ , \qquad 1.3.9$$

and

$$u\ U_0(u) = \frac{1}{2}\ U_1(u)$$

we get a formal Chebyshev series. Equating the coefficients of this series to 0 we obtain the system

$$\gamma\ a_0'(t) + a_0(g) - \frac{\gamma}{2}\ a_1'(t) + \frac{1}{2}\ a_1(t) = 0; \qquad 1.3.10$$

$$- \frac{\gamma}{2}\ a_{m-1}'(t) + \frac{1}{2}\ a_{m-1}(t) + \gamma\ a_m'(t) + a_m(t) - \frac{\gamma}{2}\ a_m'(t)$$
$$+ \frac{1}{2}\ a_m(t) = 0,$$

for $m = 1, 2, 3, \cdots$ ;

$a_0(0) = 1;$

and

$a_m(0) = 0$, for $m = 1, 2, 3, \cdots$ .

Letting

$$\bar{a}(t) = [a_0(t),\ a_1(t),\ \cdots\ ]^T \qquad 1.3.11$$

and R and S be the infinite tridiagonal matrices

$$R = \begin{matrix} 1 & -\frac{1}{2} & & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \\ & -\frac{1}{2} & 1 & \ddots \\ 0 & & \ddots & \ddots \end{matrix} \qquad 1.3.12$$

$$S = \begin{matrix} 1 & \frac{1}{2} & & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \\ & \frac{1}{2} & 1 & \ddots \\ 0 & & \ddots & \ddots \end{matrix} \qquad 1.3.13$$

we can write the system 1.3.10 as the matrix system

$$\gamma\, R\, \bar{a}' + S\, \bar{a} = \bar{0}, \quad \text{for} \quad 0 \le t \le 1, \qquad 1.3.14$$

$$\bar{a}(0) = [1,\ 0,\ 0,\ \cdots\ ]^T .$$

where $\bar{0}$ is the zero vector in the form of 1.3.11.

The solution of 1.3.14 can formally be written as,

$$\bar{a}(t) = e^{-\frac{1}{\gamma} R^{-1} S\, t}\, [1,\ 0,\ 0,\ \cdots\ ]^T. \qquad 1.3.15$$

Of course, as noted earlier, this formal solution has little real value.

One method of approximately solving the system 1.3.10 is to use the upper left $M \times M$ subsystem given by

$$\gamma R_M \bar{b}_M' + S_M \bar{b}_M = \bar{0}, \quad \text{for} \quad 0 \le t \le 1, \qquad 1.3.16$$
$$\bar{b}_M(0) = [1, 0, \cdots, 0]^T ,$$

where $R_M$ and $S_M$ are the upper left $M \times M$ submatricies of $R$ and $S$ respectively.

The actual solution of 1.3.16 may be written as

$$\bar{b}_M(t) = e^{-\frac{1}{\gamma} R_M^{-1} S_M t}[1, 0, \cdots, 0]^T \qquad 1.3.17$$

To compare the solution $\bar{b}_M(t)$ to $\bar{a}(t)$, the actual solution of 1.3.10, let

$$\bar{a}_M(t) = [a_0(t), \cdots, a_{M-1}(t)]^T, \qquad 1.3.18$$

then $\bar{a}_M(t)$ satisfies the system

$$\gamma R_M \bar{a}_M' + S_M \bar{a}_M = [0, 0, \cdots, 0, \frac{\gamma}{2} a_M' - \frac{1}{2} a_M]^T, \qquad 1.3.19$$
$$\text{for} \quad 0 \le t \le 1,$$
$$\bar{a}_M(0) = [1, 0, \cdots, 0]^T.$$

Using the standard variation of parameters method we have

$$\bar{a}_M(t) = \bar{b}_M(t) \qquad 1.3.20$$

$$+ \int_0^t e^{\frac{1}{\gamma} R_M^{-1} S_M(s-t)} [0,0,\cdots,0,\tfrac{\gamma}{2} a_M'(S)-\tfrac{1}{2} a_M(S)]^T ds.$$

The fact that the eigenvalues of $R_M^{-1}S_M$ are all positive follows quickly from the definitions of $R_M$ and $S_M$. (It can also be shown that those eigenvalues are distinct). Thus from 1.3.20 we get the estimate

$$|\bar{a}_m(t) - \bar{b}_m(t)| \le \int_0^t |\tfrac{\gamma}{2} a_M'(S) - a_M(S)|ds. \qquad 1.3.21$$

Thus the approximation is good.

In the more general setting of arbitrary linear singular pertabuation problems the same type of analysis may be carried out, the conclusion being that the approximate solution will be very good for solutions whose Chebyshev expansions converge quickly.

## 1.4 Turning Point Problems

One very important area of application for the type of methods considered here will be general turning point problems [126].

As an example consider the problem

$$\varepsilon y'' + q(t,\varepsilon)y' + r(t,\varepsilon)y = s(t,\varepsilon),\ 0 < \varepsilon,\ -1 \le t \le 1, \qquad 1.4.1$$
$$y(-1) = \alpha, \quad y(1) = \beta,$$

which is a generalization of the chemical reaction problem presented in [98].

Assuming $r(t,\varepsilon)$ has no zeros for $-1 \le t \le 1$ then 1.4.1 is said to have a turning point at each of the zeros of $q(t,0)$, for $-1 \le t \le 1$. These turning points play havoc with most standard solution techniques. The result is a group of ad hoc methods that often break down and almost never have any conditions under which they can be proved to converge. For a presentation of several of the such methods to 1.4.1 see Section 7.3 of [88].

The reasons these methods break down stem from the fact that they usually attempt to solve the 'reduced system' corresponding to 1.4.1, namely

$$q(t,0)y' + r(t,0)y = s(t,0) \qquad 1.4.2$$
$$y(-1) = \alpha,\ y(1) = \beta,$$

and then use some formal aspect 1.4.1 to correct the solution of 1.4.2 for small $\varepsilon$. There are several problems in carrying these steps. The system 1.4.2 is singular, while 1.4.1 is regular for any fixed $\varepsilon$. There might be no analytic connection between the reduced system and the full problem. The system 1.4.2 is over determined with respect to having two boundary conditions. The formal correction to the solution of 1.4.2 may have no analytic meaning. None of these problems has a simple solution, or for that matter any solution in general.

The connection of an inner expansion valid near $\varepsilon = 0$ and an outer expansion valid away from $\varepsilon = 0$ has a vast literature. There are no known general numerical methods that can be shown to be valid under reasonable conditions.

Applying the method we have outlined earlier in this section to 1.3.21 has no such difficulties. In addition there are simple conditions for the global convergence of the technique. Turning points do not play nearly as important a role in this setting as they do when standard methods are applied.

There are only two fairly general methods, other than the one suggested here, for attacking multi-turning point problems known to the author.

The work of Olver ([95], [96] and [97]) is considered classical in multi-turning point problems. The complex plane is divided into regions based on the positions and types of the turning points, then an expansion for the solution to the problem is produced for each region. Olver has shown that connection formulae exist for his expansions in a reasonably general setting.

The work of H. Gingold and P. H. Hsieh ([54], [55]) attacks the multi-turning point problem in a very general manner with a fairly complete and satisfactory theory. There are some formal steps in this theory which may prevent it from having any numerical impact. It is a little early to tell.

## 1.5 Chapter Preview

Chapter 2 will contain a fairly complete analytic theory on the use of Jacobi expansions in the same sense that was developed by Gingold in [51]. There are a number of error bounds presented which are quite sharp.

Chapter 3 contains a number of results on using recurrence relations to calculate the modified Jacobi moments and some Jacobi series.

Chapter 4 contains several applications of the material presented in Chapters 2 and 3 to some of the problems mentioned in Section 1.1.

# CHAPTER 2

## ASYMPTOTICS OF THE JACOBI MOMENTS AND REPRESENTATION THEOREMS.

### 2.1 Introduction

This chapter provide a fairly complete analytic and asymptotic analysis of Jacobi moments, which are direct generalizations of the Chebyshev moments presented in Section 1.3. This analysis also provides some sharp bounds on the size of these moments and errors in using asymptotic approximations to them.

Section 2.2 contains some basic definitions and references.

Section 2.3 contains the main theorems of the chapter. The main result of the chapter is Theorem 2.3.4, the Jacobi moment expansion theorem. Section 2.3.7 contains a discussion of these theorems and a number of references to the literature.

Section 2.4 contains a group of basic lemmas that are needed in the proofs of the results of Section 2.3.

Section 2.5 contains the proofs of the theorems of Section 2.3.

### 2.2 Definitions and Notation

For the most part the notation used here is that of Szego ([120]). Appendix A contains a small list of results on orthogonal polynomials and references to some standard sources. Throughout this chapter it will be assumed that $\alpha, \beta > -1$.

### 2.2.1 Definition

Let $P_n^{\alpha,\beta}(u)$ denote the nth degree Jacobi polynomial with parameters $\alpha$ and $\beta$. Let $w^{\alpha,\beta}(u)$ denote the Jacobi weight function

$$w^{\alpha,\beta}(u) = (1-u)^{\alpha}(1+u)^{\beta}. \qquad 2.2.1.1$$

With this notation it is known that $\{P_m^{\alpha,\beta}(u) \mid m = 0, 1, \cdots\}$ is an orthogonal set, namely

$$\int_{-1}^{1} P_m^{\alpha,\beta}(u)\ P_n^{\alpha,\beta}(u)\ w^{\alpha,\beta}(u)du = 0, \quad n \neq m \qquad 2.2.1.2$$

with the normalization constants given by

$$\int_{-1}^{1} [P_m^{\alpha,\beta}(u)]^2\ w^{\alpha,\beta}(u)du = h_m^{\alpha,\beta}, \qquad m = 0, 1, 2, \cdots, \qquad 2.2.1.3$$

where

$$h_m^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}, \quad m = 1, 2, \cdots, \qquad 2.2.1.4$$

and

$$h_0^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

The Jacobi moments $\phi_{m,n}^{\alpha,\beta}$ are defined as

$$\phi_{m,n}^{\alpha,\beta} = \frac{1}{h_m^{\alpha,\beta}} \int_{-1}^{1} u^n\ P_m^{\alpha,\beta}(u) w^{\alpha,\beta}(u)du; \quad m,n = 0, 1, 2, \cdots. \qquad 2.2.1.5$$

Given a function $f(u)$ such that all the integrals

$$a_m = \frac{1}{h_m^{\alpha,\beta}} \int_{-1}^{1} f(u)\, P_m^{\alpha,\beta}(u)\, w^{\alpha,\beta}(u)du \qquad 2.2.1.6$$

$$m = 0, 1, 2, \cdots ,$$

exist the formal Fourier-Jacobi expansion of $f$ is defined as

$$f(u) \sim \sum_{m=0}^{\infty} a_m\, P_m^{\alpha,\beta}(u) . \qquad 2.2.1.7$$

2.2.2 Definition

A series $\sum_{m=0}^{\infty} t_m$ is called $(C,k)$-summable, $k > -1$, with the sum $S$ if

$$\lim_{M\to\infty} \frac{S_M^{(k)}}{C_M^{(k)}} = S \quad , \qquad 2.2.2.1$$

where

$$(1-r)^{-k-1} \sum_{m=0}^{\infty} t_m\, r^m = \sum_{m=0}^{\infty} S_m^{(k)} r^m, \qquad 2.2.2.2$$

and

$$(1-r)^{-k-1} = \sum_{m=0}^{\infty} C_m^{(k)}\, r^m = \sum_{m=0}^{\infty} \binom{m+k}{m}\, r^m. \qquad 2.2.2.3$$

When the limit in 2.2.2.1 exists this will be denoted as

$$(C,k) \sum_{m=0}^{\infty} t_m = S. \qquad 2.2.2.4$$

2.2.3 Remark

Appendix A contains references and remarks on Cesaro summability. It is an easy matter to write the Mth partial Cesaro sum of order $k$ of the series $\sum_{m=0}^{\infty} t_m$ as

$$\frac{S_M^{(k)}}{C_M^{(k)}} = \sum_{n=0}^{M} \frac{C_{M-n}^{(k)}}{C_M^{(k)}} t_M, \qquad 2.2.3.1$$

which would be the normal computational form for an explicitly known series.

2.2.4 Definition

For a given set $U$ and $u \varepsilon U$ a series $\sum_{m=0}^{\infty} t_m(u)$ will be called uniformly $(C,k)$-summable over $U$ (uniformly $(C,k)$-summable if $U$ is understood), if $\frac{S_M^{(k)}(u)}{C_M^{(k)}}$ converges uniformly to a limit $S(u)$.

2.2.5 Definition

For sequences $\{t_n\}$, $\{B_n\}$ such that

$$t_n \le B_n, \quad n = 0, 1, 2, \cdots \tag{2.2.5.1}$$

$B_n$ is said to be an upper bound for $t_n$. If in addition

$$\lim_{n\to\infty} \frac{t_n}{B_n} = 1 \tag{2.2.5.2}$$

then $B_n$ is called an asymptotically sharp upper bound for $t_n$.

Similar notation is used for lower bounds. If $t_n$ and $B_n$ both depend on a parameter $u$ and the limit in 2.2.5.2 is uniform over a set $U$ then the bound is said to be uniformly asymptotically sharp.

2.2.6 Remark

If $B_n$ is an upper bound for $t_n$; $r_n \sim t_n$, as $n \to \infty$; and

$$\lim_{n\to\infty} \frac{r_n}{B_n} = A \ne 0.$$

then it follows that

$$\lim_{n\to\infty} \frac{t_n}{B_n} = A \ne 0.$$

Hence if $A = 1$ then $B_n$ is an asymptotically sharp upper bound for $t_n$.

## 2.3 The Main Expansion Theorems

Lemma 2.3.1 and Propositions 2.3.2 and 2.3.3 give a fairly complete analysis of the Jacobi moments. This material is used in the proof of Theorems 2.3.4 - 2.3.6. The Jacobi moment expansion Theorem 2.3.4 is the basic theorem of this chapter. It shows when and how a power series may be used to produce a Jacobi expansion. Theorems 2.3.5 and 2.3.6 are collections of results on (C,k)-summability of Jacobi series that show how to use Jacobi expansions to produce results valid over the entire interval (-1, 1) and sometimes at one or both endpoints. This is the main reason for such usage of the power series, since very little can be said about the convergence of the power series itself at the endpoints. There is a set of remarks at 2.3.7 describing these results and giving some references. The proofs for these results will be given in Section 2.5.

<u>2.3.1 Lemma (The asymptotics of the Jacobi moments)</u>.

<u>If</u> $m \neq 0$ <u>an asymptotic expression of</u> $\phi^{\alpha,\beta}_{m,n}$ <u>as</u> $n\to\infty$ <u>is given by</u>:

$$\phi^{\alpha,\beta}_{m,n} = 0, \quad \underline{\text{for}} \quad n < m \qquad 2.3.1.1$$

$$\underline{\text{or}} \quad n - m \quad \underline{\text{even and}} \quad \alpha = \beta,$$

$$\phi_{m,n}^{\alpha,\beta} \sim s \frac{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)}{2^{\delta+1}\Gamma(m+\gamma+1)} \, 2 \, (\frac{1}{n})^{\delta+1} , \qquad 2.3.1.2$$

<u>for</u> $\alpha = \beta$ <u>and</u> $n - m$ <u>odd</u>,

$$\phi_{m,n}^{\alpha,\beta} \sim s \frac{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)}{2^{\delta+1}\Gamma(m+\gamma+1)} (\frac{1}{n})^{\delta+1} , \qquad 2.3.1.3$$

<u>for</u> $\alpha \neq \beta$,

<u>where</u>

$$\delta = \min(\alpha,\beta), \qquad 2.3.1.4$$

$$\gamma = \max(\alpha,\beta), \qquad 2.3.1.5$$

and

$$s = \begin{cases} 1, & \text{if } \alpha > \beta \\ (-1)^{n-m}, & \text{if } \alpha < \beta . \end{cases} \qquad 2.3.1.6$$

<u>In the case</u> $m = 0$ <u>the above relations hold for</u> $\phi_{o,n}^{\alpha,\beta}$ <u>when</u> $(2m+\alpha+\beta+1)\Gamma(\alpha+m+1)$ <u>is replaced by</u> $\Gamma(\alpha+\beta+2)$ <u>in</u> 2.3.1.2 <u>and</u> 2.3.1.3.

The proof for Lemma 2.3.1 is given in 2.5.1. The following corollary follows immediately. The Corollary to Lemma 2.3.1 is:

$$\left|\phi_{m,n}^{\alpha,\beta}\right| = O((\frac{1}{n})^{\delta+1}), \quad \text{as} \quad n \to \infty. \qquad 2.3.1.7$$

<u>2.3.2 Proposition (Bounds for the absolute values of the Jacobi moments)</u>.

<u>The following bound for</u> $|\phi^{\alpha,\beta}_{m,n}|$ <u>holds:</u>

$$|\phi^{\alpha,\beta}_{m,n}| \le \begin{cases} \dfrac{\binom{n}{m}}{h^{\alpha,\beta}_m}\,[B^{\alpha,\beta}_{m,n} + B^{\beta,\alpha}_{m,n}], & \underline{\text{for}}\ \ n - m\ \ \underline{\text{even}}\ , \\[2ex] \dfrac{\binom{n}{m}}{h^{\alpha,\beta}_m}\,\mathrm{Max}\{B^{\alpha,\beta}_{m,n}\ ,\ B^{\beta,\alpha}_{m,n}\}, & \underline{\text{for}}\ \ n - m\ \ \underline{\text{odd}}\ , \end{cases} \qquad 2.3.2.1$$

<u>where</u> $B^{\alpha,\beta}_{m,n}$ <u>is given by</u>

$$B^{\alpha,\beta}_{m,n} = \begin{cases} 2^{\beta}\ \dfrac{\Gamma(\alpha+m+1)}{(n-m+1)^{\alpha+m+1}}\ , & \underline{\text{for}}\ \ m \ge 0\ \ \underline{\text{or}}\ \ m = 0\ \ \underline{\text{and}}\ \alpha\ge 0,\ \beta\ge 0, \\[2ex] 2^{\beta}\,\dfrac{\Gamma(\alpha+1)}{n^{\alpha+1}}\ , & \underline{\text{for}}\ \ m = 0\ \ \underline{\text{and}}\ \ \alpha < 0,\ \ \beta \ge 0, \\[2ex] \dfrac{\Gamma(\alpha+1)}{(n+1)^{\alpha+1}}\ , & \underline{\text{for}}\ \ m = 0\ \ \underline{\text{and}}\ \ \alpha \ge 0,\ \ \beta < 0, \\[2ex] \dfrac{\Gamma(\alpha+1)}{n^{\alpha+1}}\ , & \underline{\text{for}}\ \ m = 0\ \ \underline{\text{and}}\ \ \alpha < 0,\ \ \beta < 0. \end{cases} \qquad 2.3.2.2$$

<u>Furthermore</u>, <u>when</u> $\alpha \ne \beta$ <u>or</u> $m$ <u>is even</u>, <u>the bound in</u> 2.3.2.1 <u>is asymptotically sharp if</u> $\beta \ge 0$ <u>and</u> $\alpha \ge 0$.

The proof of Proposition 2.3.2 may be found in 2.5.2.

<u>2.3.3 Proposition (An error bound on the first asymptotic term for the Jacobi moments)</u>.

<u>Let</u>

$$\hat{\phi}^{\alpha,\beta}_{m,n} = \frac{\binom{n}{m}}{h^{\alpha,\beta}_{m}} \left[2^{\beta} \frac{\Gamma(\alpha+m+1)}{n^{\alpha+m+1}} + (-1)^{n-m}\, 2^{\alpha} \frac{\Gamma(\beta+m+1)}{n^{\beta+m+1}}\right] . \qquad 2.3.3.1$$

<u>The error in using</u> $\hat{\phi}^{\alpha,\beta}_{m,n}$ <u>to approximate</u> $\phi^{\alpha,\beta}_{m,n}$ <u>is bounded by</u>

$$|\hat{\phi}^{\alpha,\beta}_{m,n} - \phi^{\alpha,\beta}_{m,n}| \le \frac{\binom{n}{m}}{h^{\alpha,\beta}_{m}}\, c_0\left[2^{\beta} \frac{\Gamma(\alpha+m+2)}{(n-d_0)^{\alpha+m+2}} + 2^{\alpha} \frac{\Gamma(\beta+m+2)}{(n-d_0)^{\beta+m+2}}\right], \qquad 2.3.3.2$$

<u>where</u>

$$d_0 = \begin{cases} 0, & \underline{\text{for}} \quad m = 0, 1, \\ m-1, & \underline{\text{for}} \quad m = 2, 3, \cdots , \end{cases} \qquad 2.3.3.3$$

<u>and</u>

$$c_0 = \begin{cases} \frac{\alpha+\beta}{2} + 1, & \underline{\text{for}} \quad m = 0, 1 , \\ \max(\frac{1}{2}, \frac{\alpha+\beta}{2} + 1), & \underline{\text{for}} \quad m = 2, 3, 4, \cdots . \end{cases} \qquad 2.3.3.4$$

<u>Furthermore if</u> $c_0 = \frac{\alpha+\beta}{2} + 1$, <u>then the bound in</u> 2.3.3.2 <u>is asymptotically sharp</u>. The Corollary (to Proposition 2.3.3) is: <u>With</u> $\delta$ <u>and</u> $\gamma$ <u>as defined in Lemma</u> 2.3.1,

$$\left|\phi_{n,m}^{\alpha,\beta}\right| \le \frac{\binom{n}{m}2^{\gamma+1}\Gamma(\gamma+m+1)}{h_m^{\alpha,\beta}} \left[ \frac{1}{n^{\delta+m+1}} + \frac{c_0(\gamma+m+1)}{(n-d_0)^{\delta+m+2}} \right]. \qquad 2.3.3.5$$

The proof for Proposition 2.3.3 and it's corollary may be found in 2.5.3.

<u>2.3.4 Theorem (Jacobi moment expansion theorem)</u>.

<u>Let</u> $J$ <u>be an arbitrary set and</u> $B_0 = \{u: |u| < 1\}$. <u>Suppose that</u> $y(x,u)$ <u>is holomorphic in</u> $u$ <u>on</u> $J \times B_0$ <u>and admits an expansion</u>

$$y(x,u) = \sum_{n=0}^{\infty} y_n(x)u^n, \quad |u| < 1, \quad x \in J; \qquad 2.3.4.1$$

<u>furthermore assume that</u>:

$$\alpha,\ \beta > -1, \qquad 2.3.4.2$$

$$\delta = \min(\alpha,\beta), \qquad 2.3.4.3$$

$$\gamma = \max(\alpha,\beta), \qquad 2.3.4.4$$

$$q < \delta \qquad 2.3.4.5$$

<u>and for some function</u> $M(x)$, <u>independent of</u> $n$,

$$|y_n(x)| \le M(x)n^q, \quad \text{for} \quad x \in J, \quad n = 0, 1, \cdots . \qquad 2.3.4.6$$

<u>Then</u> $y(x,u)$ <u>admits a Jacobi expansion</u>

$$y(x,u) \sim \sum_{m=0}^{\infty} a_m(x) P_m^{\alpha,\beta}(u), \quad \text{for} \quad x \in J , \qquad 2.3.4.7$$

<u>where</u> $a_m(x)$ <u>may be calculated by the absolutely convergent series</u>

$$a_m(x) = \sum_{n=m}^{\infty} \phi_{m,n}^{\alpha,\beta} y_n(x), \quad \underline{\text{for}} \quad x \in J \quad \underline{\text{and}} \qquad 2.3.4.8$$

$$m = 0, 1, 2, \cdots .$$

<u>Moreover, the truncation error</u>

$$T_{m,N}^{\alpha,\beta} = |a_m(x) - \sum_{n=m}^{N} \phi_{m,n}^{\alpha,\beta} y_n(x)| \qquad 2.3.4.9$$

<u>can be approximated by</u>

$$T_{m,N}^{\alpha,\beta} \approx M(x) \frac{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)}{2^{\delta}\Gamma(m+\gamma+1)(\delta-q)} \left(\frac{1}{N-1}\right)^{\delta-q} , \qquad 2.3.4.10$$

<u>where for</u> $m = 0$ <u>the factor</u> $(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)$ <u>must be replaced by</u> $\Gamma(\alpha+\beta+2)$.

<u>Furthermore,</u> $T_{m,n}^{\alpha,\beta}$ <u>has the following upper bound:</u>

$$T_{m,N}^{\alpha,\beta} \le \frac{2^{\gamma+1}\Gamma(\gamma+m+1)}{h_m^{\alpha,\beta}} \left[ \frac{1}{(\delta-q)m!} \cdot \frac{1}{N^{\delta-q}} + \frac{\binom{N}{m} c_0(\gamma+m+1)}{(N-d_0)^m (\delta-q+1)} \cdot \frac{1}{(N-d_0)^{\delta-q+1}} \right], \qquad 2.3.4.11$$

where $d_0$ and $c_0$ are defined by 2.3.3.3 and 2.3.3.4.

The proof of Theorem 2.3.4 may be found in 2.5.4. Trivially, we have Corollary (to Theorem 2.3.4) is:

$$T_{m,N}^{\alpha,\beta} = O(M(t) \cdot \frac{1}{N^{\delta-q}}) \text{ .} \qquad 2.3.4.12$$

2.3.5 Theorem. (Summability theorem for Jacobi expansions of of functions continuous on [-1,1]).

Fix $x \in J$ and $\tau$ such that $0 < \tau < 1$. Assume the conditions of Theorem 2.3.4 hold and in addition suppose that $y(x,u)$ is continuous in $u$ for $u \in [-1, 1]$.

Then:

1) the expansion 2.3.4.7 is uniformly (C,k)-summable on $[-1+\tau, 1-\tau]$ for any $k \geq 0$,

2) if $k > \alpha + \frac{1}{2}$ then 2.3.4.7 is uniformly (C,k)-summable on $[-1+\tau, 1]$,

3) if $k > \beta + \frac{1}{2}$ then 2.3.4.7 is uniformly (C,k)-summable on $[-1, 1-\tau]$,

4) if $k > \max(\alpha + \frac{1}{2}, \beta + \frac{1}{2})$ then 2.3.4.7 is uniformly (C,k)-summable on $[-1, 1]$.

Furthermore, 2), 3) and 4) fail if the inequalities in $k$ are replaced by equalities.

The proof of Theorem 2.3.5 may be found in 2.5.5.

2.3.6 Theorem. (Summability Theorem for Jacobi expansions of functions continuous at one end point).

Fix $x \in J$ and $\tau$ such that $0 < \tau < 1$. Assume the conditions of Theorem 2.3.4 hold and in addition suppose that $y(x,u)$ is continuous from the left at $u = 1$.

Then:

1) If $k \geq 0$ and

$$\int_{-1}^{1} |y(x,u)| w^{\alpha,\beta}(u)du \quad \text{exists}, \qquad 2.3.6.1$$

then the expansion 2.3.4.7 is uniformly (C,k)-summable on $[-1 + \tau, 1 - \tau]$.

2) If: condition 2.3.6.1 holds,

$$\alpha + \frac{1}{2} < k < \alpha + \beta + 1, \qquad 2.3.6.2$$

$$-\frac{1}{2} < \beta \qquad 2.3.6.3$$

and the integral

$$\int_{-1}^{0} |y(x,u)|(1+u)^{\frac{\beta}{2} - \frac{1}{4}}du \quad \text{exists} \qquad 2.3.6.4$$

(*the antipole condition*), *then the expansion* 2.3.4.6 *is uniformly* (C,k)-summable *for* $u \in [-1 + \tau, 1]$.

3) *If condition* 2.3.6.2 *in* 2) *is replaced by*

$$\alpha + \beta + 1 \leq k \qquad 2.3.6.5$$

*then condition* 2.3.4.2 *may be dropped and the same conclusion holds*.

*If* $k \leq \alpha + \frac{1}{2}$ *or if* 2.3.6.1 *holds without* 2.3.6.4, *then the expansion* 2.3.4.7 *is in general not* (C,k)-summable *at* $u = 1$.

It should be noted that similar results hold at $u = -1$ with the conditions on $\alpha$ and $\beta$ interchanged. The proof of Theorem 2.3.6 may be found in 2.5.6

## 2.3.7 Remarks

Results similar to 2.3.1 - 2.3.6 were presented for $\alpha = \beta = \frac{1}{2}$ (Chebyshev polynomials of the second kind, see A.1.5.6) by Gingold in ([51]). After that work it was suggested to the author by H. Gingold that those results might be generalized to include the cases of Jacobi, Laguerre and Hermite polynomials (see A.1.5.1 - A.1.5.7).

An attempt was made in [52] to carry out such an extension to the case of Jacobi polynomials. That work was only partly successful because the main summability theorem in [51] could not be carried out. Instead summability theorems of Szego were quoted (see A.2.5 and A.2.6). The reason the original summability result could not be extended was the lack of a suitable form for the Cesaro kernel, which is a well known problem in summability theory (e.g. [3], [4], [5], [14] and [15]).

The fact that a reasonably simple form for the Cesaro kernel is known for the case of expansions in ultraspherical polynomials ([14]) leaves the possibility of extending the results of Gingold ([51]) to include that case. In the proof of Theorem 9.1.3 of Szego [120] a form for the Cesaro kernel for jacobi expansions is used. Unfortunately,this form appears to be unsuitable for attempting to extend the theorems in [51] to include Jacobi expansions.

The main tool used by Gingold in [51] and [52] is Laplace's method (e.g. see [108]. Instead of Laplace's method this chapter uses Watson's Lemma (A.2.8) and an interesting theorem on the change in order of integration and summation (A.2.1). This approach is more direct and it's benefits include: a well defined method for generating higher order terms, asymptotic and actual error bounds and a direct extension to Laguerre and Hermite expansions.

The theory of error bounds and estimates from Watson's Lemma is due to Olver ([95] and [96]). The results included here on error bounds and estimates are not contained in Gingold [52].

There are many relationships between the Jacobi moments and the hypergeometric functions. One such relationship is shown in 3.2.1.1. Some authors use similar relations in connection with so called modified moments to show the rates at which the moments tend to zero.

We chose not to follow the approach using the relationship between the Jacobi moments and the hypergeometric functions (3.2.1.1) for two reasons. First the result would only yield the first term of the asymptotic expansion, with no error bounds; and second the results would rely on the fact that in this special case several identities in hypergeometric functions hold outside their normal range of validity. For these reasons the analysis of the Jacobi moments by using the fact that they can be expressed as hypergeometric function was not presented here.

Asymptotic results for quantities such as $\phi_{m,n}^{-\frac{1}{2},-\frac{1}{2}}$ have to be studied in connection with the classical method of moments (e.g. [111] and [124]).

More recently such quantities have been used in connection with numerical integration formulae, where they are called modified moments. In this setting asymptotic results are used to provide error estimates and justify extrapolation methods ([11], [100] and [116]).

Currently many types of modified moments are under study in connection with product integration rules. These rules allow efficient integration of integrands with many types of singularities. This work is interesting in its own right, but its real importance will come if it can be applied to integral equations with singular kernels. The results and methods of this chapter and the next fit in very closely with the current work in this area.

Numerical aspects of formal methods using power series and Chebyshev series are presented in [46] and [73]. There are two main weaknesses of these methods. No conditions exist for the convergence of a solution (especially at the end points of the interval of interest); and if a formal power series solution does converge, it may converge slowly (even in cases where the orthogonal expansion converges quickly). Theorems 2.3.1 -2.3.3 answer the first of these objections for a wide class of problems. The second objection is inherent in the method, although the application of the $(C,k)$ summation process may indeed help.

The interrelation between the selection of $\delta$ in Theorem 2.3.4 and $k$ in Theorem 2.3.5 (or 2.3.6) cannot be treated analytically, but some general observations can be made. The larger $\delta$ is chosen, the smaller the bound on the truncation error in the series for $a_m(t)$ (2.3.1.8) becomes. Thus the actual rate of convergence may be improved by choosing a larger $\delta$. Unfortunately a larger $\delta$ forces a larger $k$ in Theorem 2.3.2 or (2.3.3).

The selection of $k$ has the following consequences:

1) Selecting $k$ close to its minimum (say close to $\alpha + \frac{1}{2}$ in case 2 of Theorem 2.3.2) may make the calculation of the Cesaro partial sums a numerically unstable process. This follows since for some functions there can be a divergence of the sums when $k \leq \alpha + \frac{1}{2}$.

2) For all but terminating series there is some point at which a larger $k$ will slow down the rate of convergence of the Cesaro partial sums.

It follows from these two facts that $\delta$ should be selected larger than its minimum value, but not too much larger, and then $k$ should be selected somewhat larger than its minimum value as well. An excess value of $\frac{1}{2}$ for both $k$ and $\delta$, over their minimum values, has been used successfully.

Since the summability is needed only at the endpoints of the interval $[-1,1]$ in Theorems 2.3.2 and 2.3.3, it is natural to ask about using a $k$ that depends on $u \in [-1,1]$ in a continuous way. This still produces a continuous representation over $[-1,1]$ and may provide better convergence, say in the $L^1$ norm, than a constant $k$ could. This is true in examples but no theory will be given. The function

$$K(u) = k \cdot (u^2 + \tfrac{1}{2}); \quad \text{where} \quad k = \alpha + \tfrac{1}{2}$$

was used in connection with case 2 of Theorem 2.3.1.

The reason that little analytic information is available about the rate of convergence of Cesaro partial sums is that in general the Cesaro summation process itself admits no such analysis. There has been active work on the question of the rate of convergence for Cesaro and other summability methods in, the last decade ([12], [53], [61] and [110]). The best hope for improvements in the theory presented here, in terms of getting analytic information on error terms of the final representa tion, would seem to be to use some other summability method; one that allows asymptotic information about the terms of the series to provide information about the asymptotic truncation error. The ε-algorithm and several other summation (often called convergence accellerating) methods have such features. (See [53] and [61]).

The problem with using these methods to form another representation theory is that they do not seem to support as solid a convergence theory, with respect to orthogonal expansions, as the Cesaro summation method.

If the detailed information on the error terms presented in this section is not needed, a greatly simplified proof of the convergence of the expansion for the Fourier-Jacobi coefficient $a_m(t)$ may be presented.

This presentation would prove the corollary to Lemma 2.3.1 directly by using the formulae for the asymptotic ratio of two Gamma functions, the connection between the Gamma and Beta functions and the integral representations of the Beta function. The corollary to Lemma 2.3.1 could then be used to prove that results 2.3.4.7 and 2.3.4.8 in Theorem 2.3.4 and the corollary to Theorem 2.3.4 hold.

There is current work using factorial series to sum divergent asymptotic expansions in connection with perturbation theory. ([6], [7] and [125). The theory usually presented is in terms of Laplace transforms. The results there and here are related but neither theory dominates the other.

To apply Thoorem 2.3.4 equation 2.3.4.6 must hold. Sometimes such a relationship may be known from the problem under consideration ([6], [51], and [52] and [68]).

There are some general results that are useful in showing 2.4.3.6 holds. One such general type of result relates to functions with algebraic singularities. We will now examine a few results in this direction.

For notation assume $y(u)$ is analytic for $|u| < 1$ and has the power series expansion

$$y(u) = \sum_{n=0}^{\infty} y_n u^n, \ |u| < 1. \qquad 2.3.7.1$$

If $y$ has a singularity at $u = u_0$ that singularity is said to be algebraic if we can write

$$y(u) = f_0(u) + \sum_{i=1}^{N} \left(1 - \frac{u}{u_0}\right)^{-w_i} f_i(u); \qquad 2.3.7.2$$

where $f_i$, for $i = 0, 1, \cdots$, are analytic in a neighborhood of $u_0$,

$$f_i(u_0) \neq 0, \ \text{for} \ i = 1, 2, \cdots, \ \text{and}$$

$w_i$, for $i = 1, 2, \cdots, k$, are complex numbers not equal to $0, -1, -2, \cdots$.

Asymptotic formulaes for power series that have only algebraic singularities can be found as early as 1878 in the work of Darboux.

There is useful corollary to a theorem of Hardy and Littlewood presented in [103] (page 33). A slight restatement of that result is the following

2.3.7.4 Theorem

Suppose $y_0 \geq y_1 \geq y_2 \geq \cdots \geq 0$ and

$$y(u) \sim \frac{1}{(1-u)^r}, \quad \text{as} \quad u \to 1- ;$$

where $0 < r < 1$.

Then

$$\lim_{n\to\infty} \Gamma(r) n^{1-r}\, y_n = 1$$

(thus $y_n = O(n^{r-1})$).

Results of Heilbronn and Landau (see [103]) show the following

2.3.7.5 Theorem

Suppose $y(u)$ is regular at $u = 1$ and

$$y_n \geq -r(n+1)^{\alpha}, \quad \text{where}$$

$r > 0$ _is a constant and_ $\alpha \geq -1$. _Then_

$$|y_n| = O(n^{\alpha}).$$

Another attractive result is contained within the proof of Tsuji's Theorem presented in [103] (page 128). That result gives us the

2.3.7.6 _Theorem_

_Suppose_ $y$ _has_ $M$ _algebraic singularities_ $\{u_j\}_{j=1}^{M}$ _on_

$|u| = 1$ _and all the_ $w_{i,j}$'s, $i = 1, 2, \cdots N_j$, _in expression_ 2.3.7.2 _are rational, at each singular point,_ $u_j$,

_Then letting_ $\mu = \max\limits_{\substack{1 \leq i \leq N_j \\ 1 \leq j \leq M}} \{w_i\}$ _it follows that_

$$|y_n| = O(n^{\mu-1}) .$$

If the function under consideration is known to be univalent then the recently proved Bieberbach conjecture ([133]) provides a bound on the growth role of the coefficients.

_Theorem (de Branges)_

_Suppose_ $y$ _is a one-to-one analytic function on the unit disk for which_ $y_0 = 0$ _and_ $y_1 = 1$, then $|y_n| \leq n$, for $n = 2, 3, \cdots$ .

As a final result in this direction consider the case where $y$ has a pole of order $w$ at $u = 1$ as its only singularity in the disk $|u| \leq 1 + \varepsilon$, for some $\varepsilon > 0$.
Then we may write

$$y(u) = (1-u)^{-w}\, g(u) \qquad 2.3.7.7$$

with $g$ analytic on $|u| < 1 + \varepsilon$ and $w$ real. The expansion

$$(1-u)^{-w} = \sum_{n=0}^{\infty} \binom{n+w-1}{n} u^n$$

and the condition

$$|g_n| = O(\tfrac{1}{n}), \qquad \text{where}$$

$$g(u) = \sum_{n=0}^{\infty} g_n\, u^n$$

can be used to show that

$$|y_n| = O(n^{w-1}).$$

This last result is of interest considering that Equation 2.3.7.7 shows that $\alpha \geq w - 1$ is required for the existence of the Jacobi-Fourier coefficient (2.2.1.6). This is the same condition required for the selection of $\delta$ in Theorem 2.3.4. Thus the condition on $\delta$ in Theorem 2.3.4 is the best possible.

As final references, the recent work in Hunter and Guerrieu ([68]) may prove useful in relating the properties of the singularities of analytic functions to growth rates of the coefficients.

## 2.4 Some Basic Lemmas

This section contains a series of Lemmas followed by their proofs. These results are used in Section 2.5 to prove the results of Section 2.3.

Analysis of the Jacobi moments is the key to the Jacobi moment expansion Theorem 2.3.4. Lemma 2.4.2 shows that these moments may be expressed as a sum of two terms involving integrals of the form $I^{\alpha,\beta}_{m,n}$, as given in the following definition.

2.4.1 Definition Denote

$$I^{\alpha,\beta}_{m,n} = \int_0^\infty e^{(-n+m-1)t}\, t^{\alpha+m}\, 2^{\beta+m} \left(\frac{1-e^{-t}}{t}\right)^{\alpha+m} \left(\frac{1+e^{-t}}{2}\right)^{\beta+m} dt. \qquad 2.4.1.1$$

2.4.2 Lemma

$$\phi^{\alpha,\beta}_{m,n} = \frac{\binom{n}{m}}{h^{\alpha,\beta}_m 2^m} \left[I^{\alpha,\beta}_{m,n} + (-1)^{n-m}\, I^{\beta,\alpha}_{m,n}\right]. \qquad 2.4.2.1$$

The proof of Lemma 2.4.2 is given in 2.4.8.

The next lemma collects some results about two functions that occur within the integrand for $I^{\alpha,\beta}_{m,n}$ in 2.4.1.1. These results will be used in the proofs of Lemmas 2.4.4 and 2.4.6.

<u>2.4.3 Lemma</u>

<u>Let</u>

$$f_1(t) = \frac{1 - e^{-t}}{t}, \quad 0 \le t, \qquad 2.4.3.1$$

<u>and</u>

$$f_2(t) = \frac{1 + e^{-t}}{2}, \quad 0 \le t. \qquad 2.4.3.2$$

<u>The following facts about</u> $f_1$ <u>and</u> $f_2$ <u>hold for</u> $t \ge 0$:

$f_1$ <u>and</u> $f_2$ <u>are analytic</u> (<u>without restrictions on</u> <u>t</u>); 2.4.3.3

$$f_1(0) = 1, \qquad f_2(0) = 1; \qquad 2.4.3.4$$

$$0 < f_1(t) \le 1, \quad \frac{1}{2} < f_2(t) \le 1; \qquad 2.4.3.5$$

$$f_1'(0) = -\frac{1}{2}, \qquad f_2'(0) = -\frac{1}{2}; \qquad 2.4.3.6$$

$$-\frac{1}{2} \le f_1'(t) < 0, \quad -\frac{1}{2} \le f_2'(t) < 0; \qquad 2.4.3.7$$

$$f_1'(t) + \frac{1}{2} f_1(t) \ge 0, \quad f_2'(t) + \frac{1}{2} f_2(t) \ge 0; \qquad 2.4.3.8$$

$$\lim_{t \to 0} \frac{f_1'(t) + \frac{1}{2} f_1(t)}{t} = \frac{1}{12}, \quad \lim_{t \to 0} \frac{f_2'(t) + \frac{1}{2} f_2(t)}{t} = \frac{1}{4}; \qquad 2.4.3.9$$

$$\frac{d}{dt}\,\frac{f_1'(t) + \frac{1}{2} f_1(t)}{t} \le 0, \quad \frac{d}{dt}\,\frac{f_2'(t) + \frac{1}{2} f_2(t)}{t} \le 0 \,. \qquad 2.4.3.10$$

Here the expressions involving $f_1$ and $f_1'$ at $t = 0$ in 2.4.3.4-2.4.3.8 and 2.4.3.10 should be taken as the limit as $t \to 0^+$.

This Lemma can be shown easily using elementary calculus.

Watson's Lemma may be applied to the integral for $I^{\alpha,\beta}_{m,n}$ in 2.4.1.1 to show that the asymptotic behavior of $I^{\alpha,\beta}_{m,n}$ as $n \to \infty$ is given by,

2.4.4 Lemma

$$I^{\alpha,\beta}_{m,n} \sim 2^{\beta+m}\,\Gamma(\alpha+m+1)\left(\frac{1}{n}\right)^{\alpha+m+1} - 2^{\beta+m}\,\Gamma(\alpha+m+2)\left(\frac{\alpha+\beta}{2} + 1\right)\left(\frac{1}{n}\right)^{\alpha+m+2} + \cdots \,, \qquad 2.4.4.1$$

as $n \to \infty$.

Details of the proof are given in 2.4.9.

Lemma 2.4.5 provides a bound on the integral $I^{\alpha,\beta}_{m,n}$ that is an easy consequence of using Watson's Lemma to prove 2.4.4.

2.4.5 Lemma

Let

$$B^{\alpha,\beta}_{m,n} = \begin{cases} 2^{\beta+m} \dfrac{\Gamma(\alpha+m+1)}{(n-m+1)^{\alpha+m+1}} & ; \ \underline{\text{for}} \ \ m > 0 \\[2ex] 2^{\beta} \ \dfrac{\Gamma(\alpha+1)}{(n+1)^{\alpha+1}} & ; \ m = 0, \ \ \alpha \geq 0, \ \beta \geq 0, \\[2ex] 2^{\beta} \dfrac{\Gamma(\alpha+1)}{n^{\alpha+1}} & ; \ m = 0, \ \alpha < 0, \ \ \beta \geq 0 \\[2ex] \dfrac{\Gamma(\alpha+1)}{(n+1)^{\alpha+1}} & ; \ m = 0, \ \ \alpha \geq 0, \ \ \beta < 0, \\[2ex] \dfrac{\Gamma(\alpha+1)}{n^{\alpha+1}} & ; \ m = 0, \ \ \alpha < 0, \ \ \beta < 0. \end{cases} \qquad 2.4.5.1$$

<u>Then</u>

$$0 < I^{\alpha,\beta}_{m,n} \leq B^{\alpha,\beta}_{m,n} , \qquad \underline{\text{for}} \ \ m \leq n. \qquad 2.4.5.2$$

<u>Furthermore</u>, <u>in</u> <u>the</u> <u>cases</u> <u>where</u> $\beta \geq 0$ <u>the</u> <u>bound</u> <u>is</u> <u>asymptotically</u> <u>sharp</u>. (<u>When</u> $\beta < 0$ ,

$$\lim_{n\to\infty} \frac{I^{\alpha,\beta}_{m,n}}{B^{\alpha,\beta}_{m,n}} = 2^{\beta} \ )$$

The proof of Lemma 2.4.3 is given in 2.4.10.

As suggested by Olver ([95] and [96]) it is often possible to produce error bounds for asymptotic expansions derived by Watson's Lemma. This is accomplished by first showing a specific type of inequality related to the asymptotic expansion. The next lemma is a delicate inequality that is used in Lemma 2.4.7 in this manner.

2.4.6 Lemma

The inequality

$$\left|\left(\frac{1 - e^{-t}}{t}\right)^{\alpha+m} \left(\frac{1 + e^{-t}}{2}\right)^{\beta+m} e^{(m-1)t} - 1\right| \le c_0\ t\ e^{d_0 t} \qquad 2.4.6.1$$

holds for $\alpha > -1,\ \beta > -1,\ 0 \le t,\ m = 0, 1, 2, \cdots,$

where $d_0$ and $c_0$ are given by 2.3.3.3 and 2.3.3.4.

Furthermore,

$$d_0 = \min_{d,c} \left\{ d:\ \left|\left(\frac{1-e^{-t}}{t}\right)^{\alpha+m} \left(\frac{1+e^{-t}}{2}\right)^{\beta+m} e^{(m-1)t} - 1\right| \le c\ t\ e^{dt},\ \text{for all}\ t \ge 0 \right\}, \qquad 2.4.6.2$$

and if $c_0 = \frac{\alpha+\beta}{2} + 1$ then

$$c_0 = \min_{d,c} \left\{ c:\ \left|\left(\frac{1-e^{-t}}{t}\right)^{\alpha+m} \left(\frac{1+e^{-t}}{2}\right)^{\beta+m} e^{(m-t)} - 1\right| \le c\ t\ e^{dt},\ \text{for all}\ t \ge 0 \right\}. \qquad 2.4.6.3$$

Finally note that $c_0 = \frac{\alpha+\beta}{2} + 1$ does not suffice for all $-1 < \alpha$ and $-1 < \beta$ in 2.4.4.1. (If $\alpha + \beta \ge -1$ then $c_0 = \frac{\alpha+\beta}{2} + 1$).

The proof of Lemma 2.4.6 is given in 2.4.11.

2.4.7 Lemma

The inequality

$$\left| I^{\alpha,\beta}_{n,m} - 2^{\beta+m} \frac{\Gamma(\alpha+m+1)}{n^{\alpha+m+1}} \right| \le 2^{\beta+m} c_0 \frac{\Gamma(\alpha+m+2)}{(n-d_0)^{\alpha+m+2}} , \qquad 2.4.7.1$$

holds, where $d_0$ and $c_0$ are given by 2.3.3.3 and 2.3.3.4.

The proof of Lemma 2.4.7 is given in 2.4.12.

2.4.8 Proof of Lemma 2.4.2

By applying the second corollary to the Jacobi reduction formula (A.1.22) to the integral in 2.2.1.5 it follows that

$$\phi^{\alpha,\beta}_{m,n} = \frac{\binom{n}{m}}{h^{\alpha,\beta}_{m} 2^m} \int_{-1}^{1} w^{\alpha+m,\beta+m}(u)\, u^{n-m} du . \qquad 2.4.8.1$$

Breaking the integral into two ranges [-1,0] and [0,1] then applying the transformation $v = -u$ on the integral over [-1,0], we have the equation

$$\phi^{\alpha,\beta}_{m,n} = \frac{\binom{n}{m}}{h^{\alpha,\beta}_{m} 2^m} \Big[ \int_{0}^{1} w^{\alpha+m,\beta+m}(u) u^{n-m} du \qquad 2.4.8.2$$

$$+ (-1)^{n-m} \int_{0}^{1} w^{\beta+m,\alpha+m}(v) v^{n-m} dv \Big].$$

Applying the transformation $u = e^{-t}$ to the first integral in 2.4.8.2 and $v = e^{-t}$ to the second, by 2.2.1.1 and 2.4.1.1, the lemma follows.

2.4.9 Proof of Lemma 2.4.4

Regrouping under the integral 2.4.1.1, $I_{m,n}^{\alpha,\beta}$ can be rewritten as

$$I_{m,n}^{\alpha,\beta} = \int_0^\infty e^{-nt} q(t)dt, \qquad 2.4.9.1$$

where

$$q(t) = (1 - e^{-t})^{\alpha+m}(1 + e^{-t})^{\beta+m}\, e^{(m-1)t} \quad . \qquad 2.4.9.2$$

Factoring 2.4.9.2, $q(t)$ can be written as

$$q(t) = t^{\alpha+m} 2^{\beta+m} h(t), \qquad 2.4.9.3$$

where

$$h(t) = \left(\frac{1-e^{-t}}{t}\right)^{\alpha+m}\left(\frac{1-e^{-t}}{2}\right)^{\beta+m} e^{(m-1)t} \qquad 2.4.9.4$$

$$= (f_1(t))^{\alpha+m}(f_2(t))^{\beta+m}\, e^{(m-1)t},$$

with $f_1$ and $f_2$ given in Lemma 2.4.3. Thus, by 2.4.3.3 and 2.4.3.5, $h(t)$ is analytic at $t = 0$. Differentiating 2.4.9.4, then using 2.4.3.4 and 2.4.3.6, it follows that

$$h(0) = 1 \qquad 2.4.9.5$$

and

$$h'(0) = -\left(\frac{\alpha+\beta}{2} + 1\right) . \qquad 2.4.9.6$$

Now, since $h$ is analytic at $t = 0$, using 2.4.9.5 and 2.4.9.6 with 2.4.9.3, we have the asymptotic expansion of $q(t)$,

$$q(t) \sim 2^{\beta+m} t^{\alpha+m} - (\frac{\alpha+\beta}{2} + 1)2^{\beta+m} t^{\alpha+m+1} + \dots , \qquad 2.4.9.7$$

$$\text{as } t \to 0.$$

Watson's Lemma ([96]) can now be applied to 2.4.9.1 using 2.4.9.7. This result is 2.4.4.1, the conclusion of the Lemma.

2.4.10 Proof of Lemma 2.4.5

The inequality $0 < I^{\alpha,\beta}_{m,n}$ follows immediately since the integrand in 2.4.1.1 is positive. To show $I^{\alpha,\beta}_{m,n} \leq B^{\alpha,\beta}_{m,n}$ for $t \geq 0$ let

$$H(t) = \begin{cases} e^{(m-1)t}; & \text{for } m > 0 , \\ e^{-t}; & \text{for } m = 0,\ \alpha \geq 0,\ \beta \geq 0 , \\ 1; & \text{for } m = 0,\ \alpha < 0,\ \beta \geq 0 , \\ (\frac{1}{2})^{\beta} e^{-t}; & \text{for } m = 0,\ \alpha \geq 0,\ \beta < 0 , \\ (\frac{1}{2})^{\beta}; & \text{for } m = 0,\ \alpha < 0,\ \beta < 0. \end{cases} \qquad 2.4.10.1$$

Using 2.4.3.5 it follows that

$$h(t) \leq H(t), \quad \text{for } t \geq 0, \qquad 2.4.10.2$$

where $h(t)$ is given by 2.4.9.4.

Applying 2.4.10.2 on 2.4.9.3 we have

$$q(t) \le 2^{\beta+m}\, t^{\alpha+m}\, H(t), \quad \text{for} \quad t \ge 0 \qquad 2.4.10.3$$

Substituting the bound 2.4.10.3 into equation 2.4.9.1 and integrating, using the integral representation of the gamma function, 2.4.5.2 follows. The fact that $I^{\alpha,\beta}_{m,n} < B^{\alpha,\beta}_{m,n}$ unless $m = \alpha = \beta = 0$ may also be observed.

To justify the statement that when $\beta \ge 0$ $B^{\alpha,\beta}_{m,n}$ is an asymptotically sharp upper bound for $I^{\alpha,\beta}_{m,n}$, Remark 2.2.6 together with Lemma 2.4.4 may be used. Only the first term in 2.4.4.1 needs to be considered.

## 2.4.11 Proof of Lemma 2.4.6

In order to show 2.4.6.1 define $\psi_1(t)$ and $\psi_2(t)$, for $t \ge 0$, by

$$\psi_1(t) = \left(\frac{1-e^{-t}}{t}\right)^{\alpha+m}\left(\frac{1+e^{-t}}{2}\right)^{\beta+m} e^{(m-1)t} - 1 - c_0 + e^{d_0 t} \qquad 2.4.11.1$$

$$= f_1(t)^{\alpha+m}\, f_2(t)^{\beta+m}\, e^{(m-1)t} - 1 - c_0 + e^{d_0 t}$$

and

$$\psi_2(t) = 1 - \left(\frac{1-e^{-t}}{t}\right)^{\alpha+m}\left(\frac{1+e^{-t}}{2}\right)^{\beta+m} - c_0 + e^{d_0 t} \qquad 2.4.11.2$$

$$= 1 - f_1(t)^{\alpha+m}\, f_2(t)^{\beta+m}\, e^{(m-1)t} - c_0 + e^{d_0 t},$$

where $f_1$ and $f_2$ were defined in Lemma 2.4.3 by 2.4.3.1 and 2.4.3.2.

By 2.4.3.4 it follows that

$$\psi_1(0) = 0, \qquad 2.4.11.3$$

$$\psi_2(0) = 0. \qquad 2.4.11.4$$

It will be shown later that

$$\psi_1'(t) \le 0, \quad \text{for} \quad t \ge 0 , \qquad 2.4.11.5$$

and

$$\psi_2'(t) \le 0, \quad \text{for} \quad t \ge 0. \qquad 2.4.11.6$$

Because of 2.4.11.3 and 2.4.11.4 the inequalities 2.4.11.5 and 2.4.11.6 imply that

$$\psi_1(t) \le 0, \quad \text{for} \quad t \ge 0 , \qquad 2.4.11.7$$

and

$$\psi_2(t) \le 0, \quad \text{for} \quad t \ge 0. \qquad 2.4.11.8$$

Then the proof is complete for it is easy to see that 2.4.11.7 and 2.4.11.8 together imply 2.4.6.1.

The inequalities 2.4.11.5 and 2.4.11.6 will be verified in three cases $m = 0$, $m = 1$, $m = 2, 3, 4, \cdots$ .

Note that, since $\alpha,\beta > -1$, we have

$$\left(\frac{\alpha+\beta}{2} + 1\right) > 0 , \qquad 2.4.11.9$$

$$(\frac{\alpha}{2} + 1) > 0 , \qquad 2.4.11.10$$

$$(\frac{\beta}{2} + 1) > 0. \qquad 2.4.11.11$$

<u>Case 1</u> $(m = 0)$

First 2.4.11.5 will be shown to hold. Differentiating 2.4.11.1 yields

$$\psi_1'(t) = [\alpha\ f_1(t)^{\alpha-1}\ f_2(t)^{\beta}\ f_1'(t) \qquad 2.4.11.12$$

$$+ \beta\ f_1(t)^{\alpha}\ f_2(t)^{\beta-1}\ f_2'(t) - f_1(t)^{\alpha}\ f_2(t)^{\beta}]e^{-t} - (\frac{\alpha+\beta}{2} + 1),$$

Using 2.4.11.12, 2.4.11.5 will be proved in four cases depending on $\alpha$ and $\beta$. If $\alpha \geq 0$ and $\beta \geq 0$, then by 2.4.3.5, 2.4.3.7 and 2.4.11.9 it follows that each of the four terms in 2.4.11.12 is nonpositive, hence 2.4.11.5 holds. If $\alpha < 0$ and $\beta < 0$ rewrite 2.4.11.12 as

$$\psi_1'(t) = [\alpha\ f_1(t)^{\alpha-1}\ f_2(t)^{\beta}[f_1'(t) + \frac{1}{2}\ f_1(t)] \qquad 2.4.11.13$$

$$+ \beta\ f_1(t)^{\alpha}\ f_2(t)^{\beta-1}[f_2'(t) + \frac{1}{2}\ f_2(t)]$$

$$- (\frac{\alpha+\beta}{2} + 1)\ f_1^{\alpha}(t)\ f_2^{\beta}(t)]e^{-t} - (\frac{\alpha+\beta}{2} - 1).$$

Then by 2.4.3.5, 2.4.3.8 and 2.4.11.9 it follows that $\psi_1'$ is the sum of four negative terms, hence 2.4.11.5 holds. If $\alpha \geq 0$ and $\beta < 0$, rewrite 2.4.11.12 as

$$\psi_1'(t) = [\alpha\, f_1(t)^{\alpha-1}\, f_2(t)^{\beta}\, f_1'(t) \tag{2.4.11.14}$$

$$+ \beta\, f_1(t)^{\alpha}\, f_2(t)^{\beta-1}[f_2'(t) + \tfrac{1}{2}\, f_2(t)]$$

$$- (\tfrac{\beta}{2} + 1)\, f_1(t)^{\alpha}\, f_2(t)^{\beta}]e^{-t} - (\tfrac{\alpha+\beta}{2} + 1)\ .$$

Then by 2.4.3.5, 2.4.3.7, 2.4.3.8, 2.4.11.9 and 2.4.11.11 it follows that 2.4.11.5 holds. If $\alpha < 0$ and $\beta \geq 0$ then rewrite 2.4.11.12 in the form

$$\psi_1'(t) = [\alpha\, f_1(t)^{\alpha-1}\, f_2(t)^{\beta}[f_1'(t) + \tfrac{1}{2}\, f_1(t)] \tag{2.11.15}$$

$$+ \beta\, f_1(t)^{\alpha}\, f_2(t)^{\beta-1}[f_2'(t) - \tfrac{1}{2}\, f_2(t)]$$

$$- (\tfrac{\alpha}{2} + 1)\, f_1(t)^{\alpha}\, f_2(t)^{\beta}\ ]e^{-t} - (\tfrac{\alpha+\beta}{2} + 1)$$

and again it follows that 2.4.11.5 holds. Thus for all possible $\alpha,\beta$ 2.4.11.5 holds. This completes the first half of the case $m = 0$.

Now to show 2.4.11.6 holds. Differentiating 2.4.11.2 yields

$$\psi_2'(t) = [-\alpha\, f_1(t)^{\alpha-1}\, f_2(t)^{\beta}\, f_1'(t) \tag{2.4.11.16}$$

$$-\beta\, f_1(t)^{\alpha}\, f_2(t)^{\beta-1}\, f_2'(t) + f_1(t)^{\alpha}\ \ f_2(t)^{\beta}]e^{-t} - (\tfrac{\alpha+\beta}{2} - 1).$$

Breaking up the analysis of $\psi_2'(t)$ into four cases similar to 2.4.11.12, 2.4.11.13, 2.4.11.14, and 2.4.11.15 it follows that in all cases $\psi_2'(t)$ may be written as the sum of four terms. The first three of these terms are positive and decreasing, the fourth term is $-(\frac{\alpha+\beta}{2} - 1)$. It follows that the maximum value of $\psi_2'(t)$ is $\psi_2'(0)$. This proves that 2.4.11.6 holds. This completes the proof for the case $m = 0$.

Case 2 $(m = 1)$

First 2.4.11.5 will be shown to hold. Differentiating 2.4.11.1 yields

$$\psi_1'(t) = (\alpha+1)\, f_1(t)^{\alpha}\, f_2(t)^{\beta+1}\, f_1'(t) \qquad 2.4.11.17$$

$$+ (\beta+1)\, f_1(t)^{\alpha+1}\, f_2(t)^{\beta}\, f_2'(t) - (\frac{\alpha+\beta}{2} + 1).$$

Since, $\alpha,\beta > -1$ by 2.4.3.5, 2.4.3.7 and 2.4.11.9 it follows that each of the three terms in 2.4.11.17 is negative, hence 2.4.11.5 holds.

Now to show 2.4.11.6 holds. Differentiating 2.4.11.2 yields

$$\psi_2'(t) = -(\alpha+1)\, f_1(t)^{\alpha}\, f_2(t)^{\beta-1}\, f_1'(t) \qquad 2.4.11.18$$

$$-(\beta+1)\, f_1(t)^{\alpha+1}\, f_2(t)^{\beta}\, f_2'(t) - (\frac{\alpha+\beta}{2} + 1).$$

2.4.11.18 can be rewritten as

$$\psi_2'(t) = -(\alpha+1)\ f_1(t)^{\alpha}\ f_2(t)^{\beta+1}[f_1'(t) + \tfrac{1}{2}\ f_1(t)] \qquad 2.4.11.19$$

$$-(\beta+1)\ f_1(t)^{\alpha+1}\ f_2(t)^{\beta}\ [f_2'(t) + \tfrac{1}{2}\ f_2(t)]$$

$$+ (\tfrac{\alpha+\beta}{2} + 1)\ f_1^{\alpha+1}(t)\ f_2^{\beta+1}(t) - (\tfrac{\alpha+\beta}{2} - 1).$$

By 2.4.3.5 and 2.4.3.8 it follows that the first two terms in 2.4.11.19 are negative. By 2.4.3.5,

$$0 < f_1(t)^{\alpha+1}\ f_2(t)^{\beta+1} \leq 1. \qquad 2.4.11.20$$

So that the sum of the last two terms in 2.4.11.19 must be non-positive. This shows that 2.4.11.6 holds and completes the proof for the case $m = 1$.

<u>Case 3</u> ($m = 2, 3, 4, \cdots$)

First 2.4.11.5 will be shown to hold. Differentiating 2.4.11.1 yields

$$\psi_1'(t) = [(\alpha+m)\ f_1(t)^{\alpha+m-1}\ f_2(t)^{\beta+m}\ f_1'(t) \qquad 2.4.11.21$$

$$+ (\beta+m)\ f_1(t)^{\alpha+m}\ f_2(t)^{\beta+m-1} f_2'(t) - c_0$$

$$+ (m-1)[f_1(t)^{\alpha+m}\ f_2(t)^{\beta+m} - c_0 t]]e^{(m-1)t}.$$

Now, let

$$\eta(t) = f_1(t)^{\alpha+m}\ f_2(t)^{\beta+m} - c_0 t. \qquad 2.4.11.22$$

By 2.4.3.7 and the fact that $\alpha + m, \beta + m > 0$ it follows that

$$f_1(t)^{\alpha+m} f_2(t)^{\beta+m} \text{ is strictly decreasing.} \qquad 2.4.11.23$$

Thus,

$$\eta(t) \text{ is strictly decreasing.} \qquad 2.4.11.24$$

By 2.4.3.4

$$\eta(0) = 1 \qquad 2.4.11.25$$

and by 2.4.3.5 and 2.3.3.4 it follows that:

$$\lim_{t\to\infty} \eta(t) = -\infty. \qquad 2.4.11.26$$

From 2.4.11.24, 2.4.11.25 and 2.4.11.26 it follows that there is a unique $\xi$ such that $0 < \xi < \infty$ and

$$\eta(\xi) = 0. \qquad 2.4.11.27$$

Furthermore,

$$\eta(t) > 0 \text{ on } [0,\xi), \qquad 2.4.11.28$$

while

$$\eta(t) < 0 \text{ on } (\xi,\infty). \qquad 2.4.11.29$$

Since $\alpha + m, \beta + m > 0$, using 2.4.3.5, 2.4.3.7 and 2.3.3.4, it follows that the first three terms in 2.4.11.21 are negative.

From 2.4.11.27 and 2.4.11.29 it follows that on $[\xi,\infty)$ the fourth term in 2.4.11.21 is

$$(m-1)\ \eta(t)\ e^{(m-1)} \le 0, \quad \text{for} \quad t \in [\xi,\infty). \qquad 2.4.11.30$$

Thus each term in 2.4.11.21 is nonpositive on $[\xi,\infty)$, therefore

$$\psi_1(t) \le 0, \quad \text{for} \quad t \in [\xi,\infty). \qquad 2.4.11.31$$

Now to show 2.4.11.5 for $t \in [0,\xi)$, write 2.4.11.21 as

$$\psi_1'(t) = [(\alpha+m)f_1(t)^{\alpha+m-1}f_2(t)^{\beta+m}\ [\frac{f_1'(t) + \frac{1}{2}\ f_1(t)}{t} - \frac{c_0}{2}]t \qquad 2.4.11.32$$

$$+ (\beta+m)f_1(t)^{\alpha+m}f_2(t)^{\beta+m-1}[\frac{f_2'(t) + \frac{1}{2}\ f_2(t)}{t} - \frac{c_0}{2}]t - c_0$$

$$- (\frac{\alpha+\beta}{2} + 1)[f_1(t)^{\alpha+m}\ f_2(t)^{\beta+m} - c_0t]]e^{(m-1)t}.$$

It will be shown that each of the four terms in 2.4.11.32 is nonpositive. Let

$$\eta_1(t) = \frac{f_1'(t) + \frac{1}{2}\ f_1(t)}{t} - \frac{c_0}{2}\ . \qquad 2.4.11.33$$

By 2.4.3.10 it follows that

$$\eta_1'(t) \le 0, \quad \text{for} \quad t \ge 0, \qquad 2.4.11.34$$

and by 2.4.3.9

$$\eta_1(0) = \frac{1}{12} - \frac{c_0}{2} . \qquad 2.4.11.35$$

Since $c_0 \geq \frac{1}{2}$ by 2.3.3.4 it follows that $\frac{1}{12} - \frac{c_0}{2} < 0.$ This with 2.4.11.35 and 2.4.11.34 shows that

$$\eta_1(t) \leq 0, \quad \text{for} \quad t \geq 0 \qquad 2.4.11.36$$

Now conisder,

$$\eta_2(t) = \frac{f_2'(t) + \frac{1}{2} f_2(t)}{t} - \frac{c_0}{2} , \qquad 2.4.11.37$$

in a similar manner to 2.4.11.34 - 2.4.11.36 it follows that:

$$\eta_2'(t) \leq 0, \quad \text{for} \quad t \geq 0 \qquad 2.4.11.38$$

$$\eta_2(0) = \frac{1}{4} - \frac{c_0}{2} ; \quad \text{and} \qquad 2.4.11.39$$

$$\eta_2(t) \leq 0, \quad \text{for} \quad t \geq 0 . \qquad 2.4.11.40$$

Now by 2.4.11.36 and 2.4.11.40 the first two terms in 2.4.11.32 are nonpositive. By 2.3.3.4 the third term is negative and by 2.4.11.28 the final term is negative on $[0,\xi)$. Thus

$$\psi_1(t) \leq 0, \quad \text{for} \quad t \in [0,\xi). \qquad 2.4.11.41$$

This together with 2.4.11.31 gives 2.4.11.5. This completes the first half of the case $m = 2, 3, 4, \cdots$ .

Now it will be shown that 2.4.11.6 holds. Differentiating 2.4.11.2 and regrouping yields

$$\psi_2'(t) = [-(\alpha+m)f_1(t)^{\alpha+m-1}f_2(t)^{\beta+m}[f_1'(t) + \frac{1}{2} f_1(t)] \qquad 2.4.11.42$$

$$- (\beta+m)f_1(t)^{\alpha+m}f_2(t)^{\beta+m-1}[f_2'(t) + \frac{1}{2} f_2(t)] - c_0$$

$$- (\frac{\alpha+\beta}{2} + 1) f_1(t)^{\alpha+m} f_2(t)^{\beta+m} - c_0(m-1)t]e^{(m-1)t}.$$

By 2.4.3.5 and 2.4.3.8 the first two terms in 2.4.11.42 are nonpositive. By 2.3.3.4 the last term is nonpositive. Consider,

$$\eta_3(t) = - c_0 - (\frac{\alpha+\beta}{2} + 1)f_1(t)^{\alpha+m} f_2(t)^{\beta+m}. \qquad 2.4.11.43$$

It follows from 2.3.3.4 and 2.4.3.5 that $\eta_3(t)$ is nonpositive. Thus 2.4.11.6 holds. This completes the proof for the case $m = 2, 3, \cdots$. Thus 2.4.6.1 is verified in all cases.

To prove 2.4.6.2 first consider the cases for $m = 0,1$. Let $d < d_0 = 0$. Then,

$$\lim_{t\to\infty}[1 - f_1(t)^{\alpha+m}f_2(t)^{\beta+m} e^{(m-1)t} - c\ t\ e^{dt}] = 1, \qquad 2.4.11.44$$

for $m = 0, 1$.

For $m = 2, 3, 4, \cdots$, let $d < d_0 = m - 1$. Then

$$\lim_{t\to\infty}[f_1(t)^{\alpha+m} f_2(t)^{\beta+m} e^{(m-1)t} - 1 - c\ t\ e^{dt}] = \infty, \qquad 2.4.11.45$$

where $c$ is any constant. Thus in either case 2.4.6.2 fails to hold at $\infty$.

Taking a limit as $t \to 0$ one can prove 2.4.6.3 in a similar manner, by considering

$$\lim_{t \to 0} \frac{1 - f_1(t)^{\alpha+m} f_2(t)^{\beta+m} e^{(m-1)t}}{t} = \left(\frac{\alpha+\beta}{2} - 1\right).$$

The final claim may be verified directly with the example

$$\alpha = \beta = -.99 \quad \text{and} \quad m = 2$$

for then

$$\psi_2(t) > 6.$$

<u>2.4.12 Proof of Lemma 2.4.7</u>.

For $\psi_1$ and $\psi_2$ given by 2.4.11.1 and 2.4.11.2 it has been shown that 2.4.11.7 and 2.4.11.8 hold.

Multiplying 2.4.11.7 by

$$2^{\beta+m} t^{\alpha+m} e^{-nt} \tag{2.4.12.1}$$

and integrating from 0 to $\infty$, then using 2.4.11.1 we have

$$\int_0^\infty (1 - e^{-t})^{\alpha+m} (1 + e^{-t})^{\beta+m} e^{-(n-m+1)t} dt \tag{2.4.12.2}$$

$$- \int_0^\infty 2^{\beta+m} t^{\alpha+m} e^{-nt} dt - \int_0^\infty c_0 \, 2^{\beta+m} t^{\alpha+m+1} e^{-(n-d_0)t} dt$$

$$\le 0.$$

Then using the formula

$$\int_0^\infty t^w e^{-\theta t}dt = (\tfrac{1}{\theta})^{w+1} \Gamma(w+1), \qquad 2.4.12.3$$

which follows from the integral representation of the Gamma function, on the second and third terms in 2.4.12.2 and using 2.4.1.1 to identify the first term, it follows that

$$I^{\alpha,\beta}_{m,n} - 2^{\beta+m} \frac{\Gamma(\alpha+m+1)}{\alpha+m+1_n} - c_0\, 2^{\beta+m} \frac{\Gamma(\alpha+m+2)}{\alpha+m+2_{(n-d_0)}} \le 0. \qquad 2.4.12.4$$

Taking similar the same steps for 2.4.11.8 and using 2.4.12.4, yields

$$2^{\beta+m} \frac{\Gamma(\alpha+m+1)}{\alpha+m+1_n} - I^{\alpha,\beta}_{m,n} - c_0\, 2^{\beta+m} \frac{\Gamma(\alpha+m+2)}{\alpha+m+2_{(n-d_0)}} \le 0. \qquad 2.4.12.5$$

Inequalities 2.4.12.4 and 2.4.12.5 prove the Lemma.

## 2.5 Proofs of the Main Theorems

### 2.5.1 Proof of Lemma 2.3.1.

2.3.1.1 follows directly from 2.4.2.1. Applying 2.4.4.1, using only the first term in the asymptotic expansion, to equation 2.3.1.1, both 2.3.1.2 and 2.3.1.3 follow.

It should be noted that using both terms in 2.4.4.1, rather than just the first, on 2.3.1.1 produces at least the first two terms of the expansion of $\phi^{\alpha,\beta}_{m,n}$. Higher order terms could be produced in this manner with relative ease.

## 2.5.2 Proof of Proposition 2.3.2

This proposition is a direct application of Lemma 2.4.5 to $\phi^{\alpha,\beta}_{m,n}$, given by equation 2.4.2.1. The comments on asymptotic sharpness follow from Remark 2.26 by 2.3.2.1 and Lemma 2.3.1.

## 2.5.3 Proof of Proposition 2.3.3

Using 2.4.5.4 and 2.11.2.1, then regrouping, noticing that $h^{\alpha,\beta}_m = h^{\beta,\alpha}_m$, we obtain

$$|\phi^{\alpha,\beta}_{m,n} - \hat{\phi}^{\alpha,\beta}_{m,n}| = \tag{2.5.3.1}$$

$$\left|\frac{\binom{n}{m}}{h^{\alpha,\beta}_m 2^m}\left[I^{\alpha,\beta}_{m,n} - 2^{\beta+m}\frac{\Gamma(\alpha+m+1)}{n^{\alpha+m+1}}\right] + (-1)^{n-m}\frac{\binom{n}{m}}{h^{\alpha,\beta}_m 2^m}\left[I^{\beta,\alpha}_{m,n} - 2^{\alpha+m}\frac{\Gamma(\beta+m+1)}{n^{\beta+m+1}}\right]\right|.$$

Using the triangle inequality and Lemma 2.4.7, the inequality 2.3.3.2 follows.

The corollary to Proposition 2.3.3 is proved by using the reverse triangle inequality on 2.3.3.2, then using 2.3.3.1 and maximizing each term with respect to $\alpha$ and $\beta$.

## 2.5.4 Proof of Theorem 2.3.4

Using 2.2.1.7 and 2.3.4.1, the Fourier coefficient of the Jacobi expansion may be expressed in the following steps, if the final integral exists. From 2.3.4.7 and 2.2.1.1,

$$a_m(x) = \frac{1}{h_m^{\alpha,\beta}} \int_{-1}^{1} w^{\alpha,\beta}(u)\; y(x,u)\; P_m^{\alpha,\beta}(u)\; du \qquad 2.5.4.1$$

$$= \frac{1}{h_m^{\alpha,\beta}} \int_{-1}^{1} w^{\alpha,\beta}(u)\; [\sum_{n=0}^{\infty} y_n(x)u^n]\; P_m^{\alpha,\beta}(u)\; du$$

$$= \int_{-1}^{1} [\sum_{n=0}^{\infty} F_n(u)]du,$$

where

$$F_n(u) = \frac{1}{h_m^{\alpha,\beta}} w^{\alpha,\beta}(u)\; u^n\; P_m^{\alpha,\beta}(u). \qquad 2.5.4.2$$

Theorem A.2.1 in Appendix A shows that if

$$\sum_{n=0}^{\infty} [\int_{-1}^{1} F_n(u)du] \quad \text{exists} \qquad 2.5.4.3$$

and

$$\int_{-1+\varepsilon}^{1-\varepsilon} [\sum_{n=0}^{\infty} F_n(u)]du = \sum_{n=0}^{\infty} [\int_{-1+\varepsilon}^{1-\varepsilon} F_n(u)du] \qquad 2.5.4.4$$

for all $\varepsilon$ $(0 < \varepsilon < 1)$,

then 2.5.4.1 exists and the order of summation and integration in 2.5.4.1 can be interchanged. Thus

$$a_m(x) = \sum_{n=0}^{\infty} [\int_{-1}^{1} F_n(u)du]. \qquad 2.5.4.5$$

We first show that 2.5.4.3 holds. By 2.5.4.3, 2.5.4.2 and 2.2.1.5

$$\sum_{n=0}^{\infty} [\int_{-1}^{1} F_n(u)du] = \sum_{n=0}^{\infty} [\frac{1}{h_m^{\alpha,\rho}} \int_{-1}^{1} u^n w^{\alpha,\beta}(u)P^{\alpha,\beta}(u)du]\ y(x) \qquad 2.5.4.6$$

$$= \sum_{n=0}^{\infty} \phi_{m,n}^{\alpha,\beta}\ y_n(x).$$

We now show the series 2.5.4.6 converges. By the corollary to Lemma 2.3.1, 2.3.1.7 and 2.3.4.6 it follows that

$$|\phi_{m,n}^{\alpha,\beta}\ y_n(x)| = O(M(x)n^{-\delta-1+q}). \qquad 2.5.4.7$$

Since 2.3.4.5 implies that

$$-\delta - 1 + q < -1 \qquad 2.5.4.8$$

2.5.4.7 shows that 2.5.4.6 converges absolutely.

Next to show that 2.5.4.4 holds, let $\varepsilon$ $(0 < \varepsilon < 1)$ be fixed. Since

$$\frac{1}{h_m^{\alpha,\beta}} w^{\alpha,\beta}(u)\ P_m^{\alpha,\beta}(u) \qquad 2.5.4.9$$

is continuous on $[-1 + \varepsilon, 1 - \varepsilon]$ there exists a constant $L$ such that

$$\frac{1}{h_m^{\alpha,\beta}} w^{\alpha,\beta}(u)\ P_m^{\alpha,\beta}(u) \le L, \quad \text{for} \quad u \in [-1 + \varepsilon, 1 - \varepsilon]. \qquad 2.5.4.10$$

Now by 2.5.4.2, 2.3.4.6 and 2.5.4.10

$$|F_n(u)| \leq L\ M(x) n^q (1 - \varepsilon)^n, \quad \text{for} \quad u \varepsilon [-1 + \varepsilon, 1 - \varepsilon]. \qquad 2.5.4.11$$

The series

$$\sum_{n=0}^{\infty} n^q (1 - \varepsilon)^n \qquad 2.5.4.12$$

converges for all $q$ and $\varepsilon$ $(0 < \varepsilon < 1)$. Using the Weierstrass M-test, 2.5.4.11 and 2.5.4.12 we obtain that

$$\sum_{n=0}^{\infty} F_n(u) \qquad 2.5.4.13$$

converges uniformly on $[-1 + \varepsilon, 1 - \varepsilon]$. Consequently 2.5.4.4 holds.

Now that 2.5.4.3 and 2.5.4.4 hold we have that 2.5.4.5 holds. 2.5.4.5 and 2.5.4.6 together prove 2.3.4.8, which is the first conclusion of the theorem.

The estimate of $T^{\alpha,\beta}_{m,N}$ Given by 2.3.4.10 follows from 2.3.4.9 since by 2.3.4.8

$$T^{\alpha,\beta}_{m,N} = \left| \sum_{n=N+1}^{\infty} \phi^{\alpha,\beta}_{m,n}\, y_n(x) \right| . \qquad 2.5.4.14$$

Replacing $\phi^{\alpha,\beta}_{m,n}$ by the asymptotic expression 2.3.1.3 and $y_n(x)$ by it's bound, 2.3.4.6 produces

$$T^{\alpha,\beta}_{m,N} \dot{\approx} \left| \sum_{n=N+1}^{\infty} S^{\alpha,\beta}_{n,m} \frac{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)}{2^{\delta+1}\ \Gamma(m+r+1)} M(x) \left(\frac{1}{n}\right)^{\delta+1-q} \right|, \quad 2.5.4.15$$

where

$$S^{\alpha,\beta}_{n,m} = \begin{cases} 0, & \text{for } \alpha = \beta \text{ and } n-m \text{ odd,} \\ 2, & \text{for } \alpha = \beta \text{ and } n-m \text{ even,} \\ 1, & \text{for } \alpha > \beta, \\ (-1)^{n-m}, & \text{for } \beta > \alpha\ , \end{cases} \quad 2.5.4.16$$

and in the case $m = 0$, $(2m+\alpha+\beta+1)\Gamma(\alpha+\beta+1)$ must be replaced by $\Gamma(\alpha+\beta+2)$ in 2.5.4.15. The notation

$$t_N \dot{\approx} B_N \text{ means that} \quad 2.5.4.17$$

for any $\varepsilon > 0$ there exists $N_0 > 0$ such that for all $N > N_0$

$$t_N \le B_N + \varepsilon\ .$$

The bound

$$|S^{\alpha,\beta}_{n,m}| \le 2 \quad , \quad 2.5.4.18$$

which follows from 2.5.4.14, and the bound

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{N}\right)^{\alpha} \le \frac{1}{\alpha-1} \left(\frac{1}{N}\right)^{\alpha-1} \quad 2.5.4.19$$

together applied to 2.5.4.15 produce 2.3.4.10.

It should be noted that in some cases 2.3.4.6 might be much larger than is needed for large $n$. If relationships of the form

$$|y_n(x)| \le M_N(x)n^{q_N}, \quad \text{for} \quad n \ge N > 0,$$

or

$$y_n(x) \sim \mathcal{M}(x)n^{q_m}, \quad \text{as} \quad n \to \infty$$

are known they could produce much better approximations than 2.3.4.10 for large $N$ by replacing $q$ and $M(x)$ by $q_N$ $(q_m)$ and $M_N(x)$ $(\mathcal{M}(x))$.

The proof of the bound 2.3.4.11 follows from 2.5.4.14 by using 2.3.3.5 and 2.3.4.6 to produce

$$T^{\alpha,\beta}_{m,N} \tag{2.5.4.20}$$

$$\le \sum_{n=N+1}^{\infty} \frac{\binom{n}{m} 2^{\gamma+1}\Gamma(\gamma+m+1)}{h^{\alpha,\beta}_m}\left[\frac{1}{n^{\delta+m+1}}+\frac{c_0(\gamma+m+1)}{(n-d_0)^{\delta+m+2}}\right]M(x)n^q,$$

where $c_0$ and $d_0$ are defined by 2.3.3.3 and 2.3.3.4.

Using

$$\binom{n}{m} \le \left(n - \frac{m-1}{2}\right)^m \tag{2.5.4.21}$$

and 2.5.4.20 together with 2.5.4.19 the result 2.3.4.11 follows.

The same remarks following the proof of 2.3.4.11 apply here. This completes the proof of Theorem 2.3.4

2.5.5 Proof of Theorem 2.3.5

Let

$S_n(u,x)$ denote the nth partial sum of the Jacobi expansion of $y(u,x)$ and 2.5.5.1

$s_n(u,x)$ denote the nth partial sum of the Fourier cosine expansion of 2.5.5.2

$$(1 - \cos(\theta))^{\frac{\alpha}{2} + \frac{1}{4}} (1 + \cos(\theta))^{\frac{\beta}{2} + \frac{1}{4}} y(\cos(\theta), x). \qquad 2.5.5.3$$

Theorem A. 1.23 of Appendix A will be used, With $f(u)$ in Theorem A.1.23 replaced by $y(u,x)$. The continuity of $y(u,x)$ implies that conditions A.1.23.1, A.1.23.2 and A.1.23.3 all hold. Then by A.1.23.6 it follows that

$$\lim_{n\to\infty} |S_n(u,x) - (1-u)^{-\frac{\alpha}{2} - \frac{1}{4}} (1+u)^{-\frac{\beta}{2} - \frac{1}{4}} s_n(u)| = 0 \qquad 2.5.5.4$$

uniformly on $u \varepsilon [-1 + \varepsilon, 1 - \varepsilon]$.

A well known corollary to Fejer's Theorem is that the Fourier series of $f(u)$ converges uniformly in any interval interior to an interval where $f$ is continuous and of bounded variation. Since $y(u,x)$ is analytic it follows from this corollary that

$$\lim_{n\to\infty} |(1-u)^{\frac{\alpha}{2}+\frac{1}{4}}(1+u)^{\frac{\beta}{2}+\frac{1}{4}} y(u,x) - s_n(u)| = 0 \qquad 2.5.5.5$$

uniformly for $u \in [-1+\varepsilon, 1-\varepsilon]$.

Multiplying 2.5.5.5 by $(1-u)^{-\frac{\alpha}{2}-\frac{1}{4}}(1-u)^{-\frac{\alpha}{2}-\frac{1}{4}}$, which is continuous on $[-1+\varepsilon, 1-\varepsilon]$, shows

$$\lim_{n\to\infty} |u(u,x) - (1-u)^{-\frac{\alpha}{2}-\frac{1}{4}}(1+u)^{-\frac{\beta}{2}-\frac{1}{4}} s_n(u)| = 0 \qquad 2.5.5.6$$

uniformly on $[-1+\varepsilon, 1-\varepsilon]$. Now 2.5.5.4 and 2.5.5.6 together show that 1) holds for $k = 0$. A simple proposition is that Cesaro summability is uniformly regular. This then completes the proof of 1). It might be noted that this result may hold for $0 < k < 1$ as well.

To show 2) and 3), Theorem A.1.24 of Appendix A may be taken. Result 4) is proved by Theorem A.1.26 of the Appendix. It is interesting to note that the condition in Theorem A.1.26 $(\alpha,\beta \geq -\frac{1}{2})$ is the same as that found for inequality 2.4.6.1 to hold. This completes the proof of Theorem 2.3.5.

Proof of Theorem 2.3.6

The proof of 1) follows as the proof of 1) in Theorem 2.3.5. The proofs of 2) and 3) follow directly from Theorem A.1.2.6 of Appendix A.

CHAPTER 3

# RECURRENCE RELATIONS FOR JACOBI AND MODIFIED JACOBI MOMENTS

## 3.1 Introduction

This chapter provides stable (see Appendix R) computational tools that may be used in applying the theoretical results of Chapter 2. Appendix J contains descriptions of several Fortran subroutines from the JLIB library ([72]) that is based on the results of this chapter.

Section 3.2 presents closed forms for the Jacobi moments. These closed forms are interesting, but from a computational point of view they are not satisfactory. Their main use is in deriving the recurrence relations in Section 3.3

Section 3.3 presents a stable three term recurrence relation, in $n$, for the Jacobi moments $\phi_{m,n}^{\alpha,\beta}$. This relation is then used with a standard result to derive a three term recurrence relation, in $n$, that allows the stable calculation of the sum $\sum_{n=0}^{N} \phi_{m,n}^{\alpha,\beta} y_n$ without calculating the moments, $\phi_{m,n}^{\alpha,\beta}$.

In section 3.4 we investigate an interesting 'two dimensional' recurrence relation for the Jacobi moments.

## 3.2 Some Closed Forms for Jacobi Moments

Closed forms for the Jacobi moments are given by the following theorem. Remarks on Theorems 3.2.1 and 3.2.2 are contained in 3.2.3.

Let ${}_2F_1(\quad)$ be the hypergeometric functon and $(n+k)_n =$ $n(n-1)\cdots(n-k+1)$.

3.2.1 Theorem

The following closed forms for the Jacobi moments hold.

$$\phi_{m,n}^{\alpha,\beta} = (-1)^{n-m}2^m \frac{(n-m+1)_m}{(m+\alpha+\beta+1)_m} \; {}_2F_1(-(n-m),\beta+m+1;\alpha+\beta+2m+2;2) \qquad 3.2.1.1$$

and

$$\phi_{m,n}^{\alpha,\beta} = (-1)^{n-m}2^m \frac{(n-m+1)_m}{(m+\alpha+\beta+1)_m} \sum_{k=0}^{n-m} \frac{(-(n-m))_k(\beta+m+1)_k}{(\alpha+\beta+2m+2)_k \, k!} \, 2^k , \qquad 3.2.1.2$$

for $n = m, m+1, \cdots$ .

In addition

$$\phi_{m,n}^{\alpha,\beta} = (-1)^{n-m} \, \phi_{m,n}^{\beta,\alpha} \quad . \qquad 3.2.1.3$$

The proof of Theorem 3.2.1 is in Section 3.2.4.

The next theorem provides a representation of the Jacobi moments in terms of the Beta function when $\alpha = \beta$, which corresponds to the ultraspherical polynomials.

3.2.2 Theorem

Let $\gamma > -1$. It follows that

$$\phi_{m,n}^{\gamma,\gamma} = \frac{\binom{n}{m}}{h_m^{\gamma,\gamma} \, 2^m} \, B\left(\frac{n-m+1}{2}, \gamma + m + 1\right). \qquad 3.2.2.1$$

The proof of Theorem 3.2.2 is contained in 3.2.5.

## 3.3 Remarks

Although Theorem 3.2.3 expresses the Jacobi moments in terms of the hypergeometric function ${}_2F_1$ this is not a real aid in their calculation. The reason for this is twofold. First the calculation of ${}_2F_1$ is a well studied problem; but very difficult. The solution is usually obtained only for special cases. Second, when one attempts to evaluate ${}_2F_1$ at $z = 2$, which is outside the normal radius of convergence, $|z| < 1$, most commonly used methods are not be applicable at all.

The expression 3.2.1.2 can be used to calculate the Jacobi moments; however it is both slow and prone to considerable roundoff for $n >> m$.

The main importance of Theorem 3.2.1 is that it allows the calculation of recurrence relations and other formal relationships.

Theorem 3.2.2 is a somewhat different matter since the quanities involved $\binom{n}{m}$, $h_m^{\alpha,\beta}$, $2^m$ and $B(\frac{n-m+1}{2}$ , $r + m + 1)$ all can be expressed as well defined products. (For the Beta function and Gamma function see [83]. The only problems in performing the calculation numerically is to avoid overflow and underflow. The time required to calculate the moments in this manner is still large, so that the recurrence relations to be presented in the next sections provide a better means of calculating these moments even in the ultraspherical case.

As a last observation we note that Equation 2.4.8.1 makes a good choice for a means of evaluating the Jacobi moments if numerical quadrature routines are available and an assured error on the calculations is desired (e.g. [102]).

### 3.2.4 Proof of Theorem 3.2.1.

Applying the transformation $v = \frac{1+u}{2}$ to 2.4.8.1 we have

$$\phi_{m,n}^{\alpha,\beta} = \frac{\binom{n}{m}}{h_m^{\alpha,\beta}\, 2^m} (-1)^{n-m}\, 2^{\alpha+\beta+2m+1} \int_0^1 (1-v)^{\alpha+m} v^{\beta+m} (1-2v)^{n-m} dv. \quad 3.2.4.1$$

A standard integral representation of the Hypergeometric function ${}_2F_1$ is

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \quad 3.2.4.2$$

for $Re(c) > Re(b) > 0$ and $|z| < 1$. The proof of 3.2.4.2 [104] shows that if a is non-positive the restriction $|z| < 1$ may be dropped.

Using 3.2.4.1 and 3.2.4.2 with

$$\begin{aligned} z &= 2, \quad 3.2.4.3 \\ a &= -(n-m) \\ b &= \beta + m + 1, \\ c &= \alpha + \beta + 2m + 2 \end{aligned}$$

we have

$$\phi_{m,n}^{\alpha,\beta} = \frac{\binom{n}{m}}{h_m^{\alpha,\beta}\, 2^m} (-1)^{n-m} 2^{\alpha+\beta+2m+1} \frac{\Gamma(\beta+m+1)(\Gamma(\alpha+m+1)}{\Gamma(\alpha+\beta+2m+2)} \qquad 3.2.4.4$$

$$_2F_1\,(-(n-m),\beta+m+1;\ \alpha+\beta+2m+2;2)\ .$$

Now using the identity

$$\frac{\Gamma(\delta+N)}{\Gamma(\delta)} = (\delta)_N, \quad \text{for} \quad \delta \neq 0, -1, -2, \cdots \qquad 3.2.4.5$$

$$\text{and} \quad N = 0,\ 1,\ 2,\ \cdots$$

3.2.4.4 may be simplified into the result 3.2.2.1.

To show that 3.2.1.2. holds, recall

$$_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\ k!} z^k\ , \qquad 3.2.4.6$$

$$c \neq 0,\ -1,\ -2,\ \cdots \quad \text{and} \quad |z| < 1.$$

As with 3.2.4.2, under some cases the restriction $|z| < 1$ may be dropped. One such case is when $a$ is a negative integer, for then the sum terminates at the $-a$ th term.

Using the terminating expansion 3.2.4.6 on 3.2.1.1, the result 3.2.1.2 follows.

Note that the sum in 3.2.1.2 may be written as

$$\sum_{k=0}^{n-m} R_k,$$

where

$$R_0 = 1$$

and

$$R_k = \frac{(-(n-m)+k-1)(\beta+m+k)}{(\alpha+\beta+2m+k+1)k} \, 2 \, R_{k-1} \, .$$

To prove 3.2.1.3 refer to Lemma 2.4.2. This completes the proof of Theorem 3.2.1

3.2.5 Proof of Theorem 3.2.2

Letting $\gamma = \alpha = \beta$ in 2.4.8.1 and using 2.2.1.1

$$\phi^{\gamma,\gamma}_{n,m} = \frac{\binom{n}{m}}{h^{\gamma,\gamma}_m \, 2^m} \int_{-1}^{1} (1 - u^2)^{\gamma+m} \quad u^{n-m} du. \qquad 3.2.5.1$$

When $n - m$ is even, the integrand is positive and an even function.

Using this and the transformation $v = u^2$, we have

$$\phi^{\gamma,\gamma}_{n,m} = \frac{\binom{n}{m}}{h^{\gamma,\gamma}_m \, 2^m} \, 2\int_0^1 (1-v)^{\gamma+m} \, v^{\frac{n-m-1}{2}} \quad \frac{1}{2} \, dv \, . \qquad 3.2.5.2$$

Using the integral representation of the Beta function on 3.2.5.1 we obtain 3.2.2.1.

## 3.3 A Three Term Recurrence Relation for the Jacobi Moments

The main result is the following

### 3.3.1 Theorem

*The three term recurrence relation*

$$\phi^{\alpha,\beta}_{m,n} = r^{\alpha,\beta}_{n,m}\,\phi^{\alpha,\beta}_{m,n-1} + s^{\alpha,\beta}_{n,m}\,\phi^{\alpha,\beta}_{m,n-2}, \quad \underline{\text{for}} \quad n \geq m + 2, \qquad 3.3.1.1$$

*holds, where*

$$r^{\alpha,\beta}_{n,m} = \frac{n(\alpha-\beta)}{(n+\alpha+\beta+m+1)(n-m)} \qquad 3.3.1.2$$

*and*

$$s^{\alpha,\beta}_{n,m} = \frac{n(n-1)}{(n+\alpha+\beta+m+1)(n-m)} \quad , \qquad 3.3.1.3$$

The proof of Theorem 3.3.1 may be found in 3.3.5.

The fact that, unlike the closed forms in Section 3.2, the recurrence relation in Theorem 3.3.1 may be used computationally is indicated by the next Theorem.

### 3.3.2 Theorem

*The recurrence relation* 3.3.1.1 *is weakly stable, in both the absolute* (See Appendix R) *and the relative sense, for calculating the Jacobi moments.*

The proof of Theorem 3.3.2 can be found in 3.3.6.

The next theorem is the result of a well known recurrence method applied to the recurrence relation 3.3.1.1. It should be noted that this algorithm is appealing, as it does not require the calculation of the moments $\phi^{\alpha,\beta}_{m,n}$ or the storage of any vectors.

### 3.3.3 Theorem

*The partial sum of* 2.3.4.7,

$$a_{m,n} = \sum_{n=0}^{N} \phi_{m,n} \, y_n, \qquad 3.3.3.1$$

*can be computed by the following recursive scheme.*
*Compute* $B(n)$, *for* $0 \le h \le N + 2$ *from*

$$B(n) = - r^{\alpha,\beta}_{m,n+1} \, B(n+1) - s^{\alpha,\beta}_{m,n+2} \, B(n+2) + y_n \qquad 3.3.3.2$$

*in the order of* $n = N, \; N - 1, \cdots, 0,$ *starting with*

$$B(N+1) = B(N+2) = 0. \qquad 3.3.3.3$$

*Then*

$$a_{m,n} = B(0)\phi^{\alpha,\beta}_{m,0} + B(1)[\phi^{\alpha,\beta}_{m,1} + r^{\alpha,\beta}_{m,1} \, \phi^{\alpha,\beta}_{m,0}] \; . \qquad 3.3.3.4$$

*Proof*. This is a direct application of a result known as the Clenshaw method (eg. [21] and [94]).

### 3.3.4 Remarks

The recurrence relation in Theorem 3.3.1 is surprisingly simple, considering the complicated nature of the definitions of the

Jacobi moments and the algebra needed to derive the relation. The fact that it is computationally stable is even more pleasing.

A fair amount of numerical exploration with the recurrence relation in Theorem 3.3.1 was done and confirms the stability indicated by Theorem 3.3.2.

Starting values for the recurrence may be determined from 3.2.1.2, which only becomes unstable as $n$ gets much larger than $m$.

Starting values may also be determined from the recurrence relations for the moments $(\phi_{m,m}^{\alpha,\beta})_{m=0}^{\infty}$ and $(\phi_{m,m+1}^{\alpha,\beta})_{m=0}^{\infty}$ to be presented in Section 3.4.

As a third method for determining starting values, direct numerical integration of equation 2.4.8.1 can be used, though great care must be taken to avoid underflows overflows and roundoff error, as $m$ and $n$ become large.

It was pointed out in Chapter 2 that the behavior of $\phi_{m,n}^{\alpha,\beta}$ is a little strange even is the cases $\alpha, \beta > 0$, where $\lim_{n\to\infty} \phi_{m,n}^{\alpha,\beta} = 0$. Formally if we let

$$U(m) = \max_{n\in z^+} \{|\phi_{m,n}^{\alpha,\beta}|\} \quad \text{and} \qquad 3.3.4.1$$

$$N(m) = \min_{n\in z^+} \{n: |\phi_{m,n}^{\alpha,\beta}| = U(m)\} \qquad 3.3.4.2$$

then we may observe that

$$\lim_{m\to\infty} \left|\frac{\phi_{m,0}^{\alpha,\beta}}{U(m)}\right| = 0 \qquad \text{and} \qquad 3.3.4.3$$

$$\lim_{m\to\infty} N(m) = \infty \quad . \qquad 3.3.4.4$$

We may estimate $N(m)$ from equation 3.3.1.1 as

$$N(m) = \frac{m(m+\alpha+\beta+1)}{2\sigma} \quad , \qquad 3.3.4.5$$

where

$$\sigma = \min(\alpha,\beta) \quad . \qquad 3.3.4.6$$

These remarks show that for a large $m$ backward recurrence or the Miller algorithm must be used to calculate the moments in a stable manner.

No error analysis for Theorem 3.3.3 is presented because it is a little more complicated than Theorem 3.3.1. However, Elliott ([35]) in the general case and Oliver ([91] and [92]) in the case of applying the Clenshaw method to Chebyshev polynomials (note that we are not applying the Clenshaw method to Jacobi polynomials, but rather to the Jacobi moments) have shown that this method will tend to be stable providing the solution being calculated is not dominated by any other solution. From Theorem 3.3.2 we know that is the case here.

Finally we should remark that if we have two expansions for the same Nth degree polynomial $f_N$

$$f_N(x) = \sum_{n=0}^{N} A_n P_n(x) \qquad 3.3.4.7$$

and

$$f_N(x) = \sum_{n=0}^{N} A_n^* P_n^*(x) \quad , \qquad 3.3.4.8$$

where each set $\{P_n\}$ and $\{P_n^*\}$ obeys a three term recurrence relation (but is not necessarily orthogonal) and has $P_n$ and $P_n^*$ each a polynomial of degree $n$ then there exists a fifth degree (or lower) recurrence scheme for calculating $A_n^*$ from $A_n$. ([107])

In particular this allows the expansion in terms of Jacobi polynomials to be calculated from the power series expansion. The stability in this method for Chebyshev expansions is investigated in [127] and [128].

By producing a three term recurrence relation for

$$\frac{C_{m-n}^{(k)}}{C_m^{(k)}} P_m^{\alpha,\beta}(u) \qquad 3.3.4.9$$

we were able to generate a recurrence method to produce the Nth Cesaro sum of order $k$ of the Jacobi expansion of a function given its power series expansion. That of course is the sum of interest in Theorems 2.3.5 and 2.3.6.

Numerical results on this look promising, but the error analysis is not complete at this point.

Should further work on this material produce viable results it will in no way diminish the importance of the theorems of Chapter 2 or 3. It is expected that the theorems in Chapter 2 would prove to be crucial in extending results in this direction.

### 3.3.5 Proof of Theorem 3.3.1

The proof will be based on Fasemyer's technique ([41] and [104]).

Formally let

$$f_n = \sum_{k=0}^{\infty} \varepsilon(n,k), \tag{3.3.5.1}$$

where

$$\varepsilon(n,k) = \frac{(-1)^{n-m} 2^m (n-m+1)_m (-(n-m))_k (\beta+m+1)_k}{(m+\alpha+\beta+1)_m (\alpha+\beta+2m+2)_k \, k!} z^k \tag{3.3.5.2}$$

(compare this to 3.2.1.2).

By 3.3.5.1 and 3.3.5.2 each of the following hold:

$$f_{n-1} = \sum_{k=0}^{\infty} \frac{-n+m+k}{n} \varepsilon(n,k), \tag{3.3.5.3}$$

$$z\, f_{n-1} = \sum_{k=0}^{\infty} \frac{(\alpha+\beta+2m+k+1)k}{n(\beta+m+k)} \varepsilon(n,k), \tag{3.3.5.4}$$

$$f_{n-2} = \sum_{k=0}^{\infty} \frac{(n-m-k)(n-m-k-1)}{n(n-1)} \varepsilon(n,k) \tag{3.3.5.5}$$

and

$$z\, f_{n-2} = \sum_{k=0}^{\infty} \frac{(-n+m+k)(\alpha+\beta+2m+k+1)k}{n(n-1)(\beta+m+k)} \varepsilon(n,k). \tag{3.3.5.6}$$

The next step is to let $c_1$, $c_2$, $c_3$ and $c_4$ be arbitrary functions of $\alpha$, $\beta$, $m$ and $n$, independent of $k$, and then find conditions such that

$$f_n + c_1 f_{n-1} + c_2 Z f_{n-1} + c_3 f_{n-2} + c_4 Z f_{n-2} = 0. \qquad 3.3.5.6$$

Equation 3.3.5.6 may be satisfied by using 3.3.5.1 and 3.3.5.3 - 3.3.5.6, forcing each coefficient of $\varepsilon(n,k)$ to sum to 0. The least common denominator of the coefficients of $\varepsilon(n,k)$ in 3.3.5.1 and 3.3.5.3 - 3.3.5.6 is

$$n(n-1)(\beta+m+k). \qquad 3.3.5.7$$

Using this and 3.3.5.6 we have the following equation for $c_1 - c_4$.

$$\begin{aligned} & n(n-1)(\beta+m+k) + c_1(-n+m+k)(n-1)(\beta+m+k) \qquad 3.3.5.8 \\ & + c_2(\alpha+\beta+2m+k+1)k(n-1) \\ & + c_3(n-m-k)(n-m-k-1)(\beta+m+k) \\ & + c_4(-n+m+k)(\alpha+\beta+2m+k+1)k \\ & = 0. \end{aligned}$$

The degrees in $k$ of the terms in 3.3.5.8 are 1, 2, 2, 3 and 3 respectively. This insures that we can solve 3.3.5.8 for $c_1 - c_4$.

A solution for 3.3.5.8 may be found by setting $k = 0$, $n - m$, $n - m - 1$ and finding the coefficient of $k^3$. This gives a system of four equations for the four unknowns $c_1 - c_4$.

The values for $c_1 - c_4$ are then used in 3.3.5.8. The identification of the fact that

$$f_n = \phi_{m,n}^{\alpha,\beta}, \quad \text{when} \quad Z = 2, \qquad 3.3.5.9$$

which follows from 3.2.1.2, and using the relations

$$r_{m,n}^{\alpha,\beta} = c_1 + 2c_2 \qquad 3.3.5.10$$

and

$$s_{m,n}^{\alpha,\beta} = c_3 + 2c_4 \qquad 3.3.5.11$$

on 3.3.5.6 completes the proof

It is noteworthy that Fusemyer's technique usually requires that $f_{n-3}$ be included in 3.3.5.6. However this was not needed here.

### 3.3.6 Proof of Theorem 3.3.2

From equations 3.3.1.2 - 3.3.1.3 it follows that

$$\lim_{n\to\infty} r_{m,n}^{\alpha,\beta} = 0 \quad \text{and} \qquad 3.3.6.1$$

$$\lim_{n\to\infty} s_{m,n}^{\alpha,\beta} = 1 \quad . \qquad 3.3.6.2$$

Thus the recurrence relation 3.3.1.1 is of Poincare type and has characteristic equation (see Appendix R.3)

$$\lambda^2 - 1 = 0, \qquad 3.3.6.3$$

which has roots

$$\lambda = \pm 1 \, . \qquad 3.3.6.4$$

It follows from Appendix R that the recurrence relation is weakly stable in the absolute sense.

To prove weak stability in the relative sense some detailed information on the general solutions to the recurrence relation is needed. Neither Poincare's Theorem nor the Perron-Kreusev Theorem (e.g. [87]) provides specific enough information for our purpose.

By the existence theorem for Birkhoff series (see Appendix R.10) we know a general form for an asymptotic series representing any solution to the recurrence relation. In particular this representation will hold for any fundamental set of solutions $\{\phi_n^{(1)}, \phi_n^{(2)}\}$. This follows since any solution to the recurrence relation $\phi_n$ may be written as

$$\phi_n = a\,\phi_n^{(1)} + b\,\phi_n^{(2)} \qquad 3.3.6.5$$

for some constants $a$ and $b$.

We now recall the proof of Lemma 2.3.1, where it was noted that

$$\phi_{m,n}^{\alpha,\beta} \approx a_0\left(\frac{1}{n}\right)^{\alpha+1} + a_1\left(\frac{1}{n}\right)^{\alpha+2} + \cdots \qquad 3.3.6.6$$

$$+ b_0\left(\frac{1}{n}\right)^{\beta+1} + b_1\left(\frac{1}{n}\right)^{\beta+2} + \cdots \quad .$$

Since $\phi_{m,n}^{\alpha,\beta}$ is indeed a solution to the recurrence relation it may be written in the form 3.3.6.5. The existence theorem for Birkhoff series (see Appendix R.10) then implies we may choose the fundamental set $\{\phi_n^{(1)}, \phi_n^{(2)}\}$ such that

$$\phi_n^{(1)} \approx (\frac{1}{n})^{\alpha+1} \quad \text{and} \qquad 3.3.6.7$$

$$\phi_n^{(2)} \approx (\frac{1}{n})^{\beta+1} \quad . \qquad 3.3.6.8$$

From 3.3.6.6 we conclude that there is a constant $C_m$ such that for all $n$

$$\left|\frac{\phi_{m,n}^{\alpha,\beta}}{\phi_n^{(i)}}\right| < C_m \; ; \quad \text{for } i = 1, 2.$$

This, according to Appendix R.6, proves that the recurrence relation is weakly stable in the relative sense for forward recurrence.

It is noteworthy that the generally difficult process of substituting the Birkhoff series into the recurrence relation to try and determine its form was totally avoided in the proof.

## Section 3.4 Two Dimensional Recurrence Relations for the Jacobi Moments

The definition of a two dimensional recurrence relation is given in Theorem 3.4.1 followed by two important examples connecting the Jacobi moments, Theorem 3.4.2 and its Corollary 3.4.3. The remainder of the section provides a stability analysis for these recurrence relations.

### 3.4.1 Definition

Let $N = \{0, 1, 2, \cdots\}$ and $C$ be the complex plane. Suppose $\phi$: $N \times N \to C$. If there exist constants $i_0$, $j_0$,

$$g_{n,m} \text{ and } a_{i,j,n,m} \text{ ; for } \begin{array}{l} i = 0, 1, \cdots i_0 ; \\ j = 0, 1, \cdots , j_0 , \\ n,m = 0, 1, 2, \cdots ; \end{array} \tag{3.4.1.1}$$

such that

$$\sum_{i=0}^{i_0} \sum_{j=0}^{j_0} a_{i,j,n,m} \ \phi(n+i,\ m+j) = g_{n,m} ; \tag{3.4.1.2}$$

$$\text{for } n,m = 0, 1, 2, \cdots ,$$

Then 3.4.1.1 is said to be a finite two dimensional linear recurrence relation for $\phi$.

If $g_{n,m} = 0$; for $n,m = 0, 1, \cdots$ ; then the recurrence relation is said to be homogeneous.

### 3.4.2 Theorem

The Jacobi moments have the following two dimensional recurrence relation

$$\phi^{\alpha,\beta}_{m,n+1} = b^{\alpha,\beta}_{1,m}\,\phi^{\alpha,\beta}_{m-1,n} + b^{\alpha,\beta}_{2,m}\,\phi^{\alpha,\beta}_{m+1,n} + b^{\alpha,\beta}_{3,m}\,\phi^{\alpha,\beta}_{m,n}\ ; \qquad 3.4.2.1$$

<u>where</u>

$$b^{\alpha,\beta}_{1,m} = \frac{2m(m+\alpha+\beta)}{(2m+\alpha+\beta+1)(2m+\alpha+\beta)}\ , \qquad 3.4.2.2$$

$$b^{\alpha,\beta}_{2,m} = \frac{2(m+\alpha+2)(m+\beta+1)}{(2m+\alpha+\beta+3)(2m+\alpha+\beta+2)} \quad \text{and} \qquad 3.4.2.3$$

$$b^{\alpha,\beta}_{3,m} = \frac{-(\alpha^2 - \beta^2)}{(2m+\alpha+\beta)(2m+\alpha+\beta+2)}\ ; \qquad 3.4.2.4$$

for $m = 1, 2, 3, \cdots$ and $n = 0, 1, 2, \cdots$ .

The proof of Theorem 3.4.1 may be found in 3.4.11.

The following corollary may be used to calculate the diagonal and first super diagonal elements of the matrix $\{\phi^{\alpha,\beta}_{n,m}\}^{\infty}_{n,m=0}$, if starting values are known.

### <u>3.4.3 Corollary</u>

<u>The following recurrence relations hold</u>

$$\phi^{\alpha,\beta}_{m,m} = b^{\alpha,\beta}_{1,m}\,\phi^{\alpha,\beta}_{m-1,m-1}\ , \quad \text{for } m = 1, 2, \cdots \qquad 3.4.3.1$$

and

$$\phi^{\alpha,\beta}_{m,m+1} = b^{\alpha,\beta}_{1,m}\,\phi^{\alpha,\beta}_{m-1,m} + b^{\alpha,\beta}_{3,m}\,\phi^{\alpha,\beta}_{m,m}\ , \quad \text{for } m = 1, 2, \cdots\ ; \qquad 3.4.3.2$$

<u>where</u> $b^{\alpha,\beta}_{1,m}$ <u>and</u> $b^{\alpha,\beta}_{3,m}$ <u>are given by</u> 3.4.2.2 <u>and</u> 3.4.2.4.

Proof. The proof follows at once from 2.3.1.1.

There is one more two dimensional recurrence relation which may easily be obtained for the Jacobi moments. This is given by the next

3.4.4 Theorem

The following recurrence relation holds

$$\phi^{\alpha,\beta}_{0,n+1} = \frac{2(\alpha+1)(\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)} \phi^{\alpha,\beta}_{1,n} - \frac{(\alpha-\beta)}{(\alpha+\beta+2)} \phi^{\alpha,\beta}_{0,n} , \qquad 3.4.4.1$$

for $n = 0, 1, 2, \cdots$ .

Proof. The proof follows from A.1.8.6 and A.1.8.7 in a manner similar to that used in the proof of Theorem 3.4.2.

We are now in a position to describe the means intended for calculating the Jacobi moments using these two dimensional recurrence relations.

3.4.5 Method for Calculating the Jacobi Moments.

$$\phi^{\alpha,\beta}_{m,n}: \ 0 \le m \le M \ \text{and} \ m \le n \le m + N ,$$

where $M$ and $N$ are positive integers.

Step 1. Calculate $\phi^{\alpha,\beta}_{0\,0}$ and $\phi^{\alpha,\beta}_{0\,1}$ using 3.2.1.2.

Step 2. Calculate $\phi^{\alpha,\beta}_{0,n}$, for $n = 2, 3, \cdots N$ using 3.3.1.1.

<u>Step 3</u>. <u>Calculate</u> $\phi^{\alpha,\beta}_{m,m}$, <u>for</u> $m = 2, 3, \cdots M$ <u>using</u> 3.4.3.1.

<u>Step 4</u>. <u>Calculate</u> $\phi^{\alpha,\beta}_{m,m+1}$, <u>for</u> $m = 2, 3, \cdots M$ <u>using</u> 3.4.3.2.

<u>Step 5</u>. <u>Calculate</u> $\phi^{\alpha,\beta}_{m,m+k}$, <u>for</u> $m = 2, 3, \cdots M$ <u>using</u> 3.4.2.1.

The recurrence in step two is intended to be forward recurrence in $n$ while that in steps three, four and five is intended to proceed down the diagonals.

The following two definitions attempt to provide insight into the stability of the proposed method for calculating the Jacobi moments given in Method 3.4.5.

<u>3.4.6 Definition</u>

<u>If a two dimensional linear recurrence relation</u> (Definition 3.4.1) <u>is used in some method to calculate</u> $\phi(n+i_0', m+j_0')$, <u>where</u> $0 \le i_0' \le i_0$ <u>and</u> $0 \le j_0' \le j_0$, <u>at each step as</u>

$$\phi(n+i_0', m+j_0') = \sum_{\substack{i=0 \\ i\ne i_0'}}^{i_0} \sum_{\substack{j=0 \\ j\ne j_0'}}^{j_0} \frac{a_{i,j,n,m}}{a_{i_0',j_0',n,m}} \phi(n+i, m+j) + g_{n,m} \qquad 3.4.6.1$$

<u>and if</u>

$$\lim_{|n|+|m|\to\infty} \sum_{\substack{i=0 \\ i\neq i_0'}}^{i_0} \sum_{\substack{j=0 \\ j\neq j_0'}}^{j_0} \left|\frac{a_{i,j,n,m}}{a_{i_0',j_0',n,m}}\right| \le 1, \qquad 3.4.6.2$$

<u>then the recurrence method</u> 3.4.6.1 <u>is said to have type 1 weak stability in the absolute sense in the forward direction</u>.

<u>If</u>

$$\sum_{\substack{i=0 \\ i\neq i_0}}^{i_0} \sum_{\substack{j=0 \\ j\neq j_0}}^{j_0} \left|\frac{a_{i,j,n,m}}{a_{i_0',j_0',n,m}}\right| \le 1 ,$$

$$, \quad \text{for} \quad n,m = 0,\ 1,\ 2,\ \cdots ;$$

<u>then the recurrence method</u> 3.4.6.1 <u>is said to have type 1 stability in the absolute sense in the forward directions</u>.

### <u>3.4.7 Definition</u>

<u>If we can write</u> 3.4.6.1 <u>as</u>

$$\phi(n+i_0',\ m+j_0') = -\sum_{k=1}^{k_0} b_{k,n,m}\,\phi(n+i_0'-k,\ m+j_0'-k) \qquad 3.4.7.1$$

$$-\sum_{\substack{i=0 \\ i-j<i_0-j_0}}^{i_0} \sum_{j=0}^{j_0} b_{i,j,n,m}\,\phi(n+i,\ n+j) + g_{n,m},$$

$$\text{for} \quad n,m = 0,\ 1,\ 2,\ \cdots ;$$

then we say that the recurrence method is diagonal.

If the one dimensional recurrence relation

$$\phi(n+i_0', m + j_0') = - \sum_{k=1}^{k_0} b_{k,n,m} \phi(n+i_0' - k, m+j_0'-k) \qquad 3.4.7.2$$

has any kind of stability (weak, absolute, relative, etc.) then 3.4.7.1 is said to have diagonal stability of the same type.

With these definitions in hand the following theorem is immediate.

3.4.8 Theorem

The diagonal recurrence relation 3.4.2.1, is weakly stable in the absolute sense in the forward direction.

It is also diagonally stable in the relative sense.
There is one more result on the error propagation of 3.4.2.1 that can be observed. If a more detailed general theory of error propagation for two dimensional recurrence were presented here this result would provide the means of integrating the stability analysis of 3.4.2.1 into that theory.

3.4.9. Theorem

The two dimensional recurrence relation 3.4.2.1 is of Poincare type with limiting recurrence relation

$$\tilde{\phi}^{\alpha,\beta}_{m,n+1} = \frac{1}{2} \tilde{\phi}^{\alpha,\beta}_{m-1,n} + \frac{1}{2} \tilde{\phi}^{\alpha,\beta}_{m+1,n} \quad , \qquad 3.4.9.1$$

for $m = 1, 2, \cdots$ and $n = 0, 1, \cdots$ .

If an error $\varepsilon_{m,n}$ is added to $\tilde{\phi}^{\alpha,\beta}_{m,n}$, and exact arithmetic is used from then on in 3.4.9.1 thenthat error propagates as $\varepsilon_{m+k_1,n+k_2}$ in $\tilde{\phi}_{m+k_1,n+k_2}$,

where $k_2 > 0,$

$$|k_1| \leq k_2,$$

$$\varepsilon_{m+k_1,n+k_2} = \begin{cases} 0 & ; \text{ for } k_2 - |k_1| \text{ odd} \\ \dfrac{\binom{k_2}{k_2-|k_1|}}{2^{k_1}} \varepsilon_{m,n}; & \text{for } k_2 - |k_1| \text{ even} \end{cases} \qquad 3.4.9.2.$$

Proof. 3.4.9.2 is obtained by a simple induction.

3.4.10 Remarks

It is tempting to try to extend the general theory of one dimensional recurrence relations to a general theory of two dimensional recurrence relations. Many of the definitions in one dimension may be extended in reasonable ways to two dimensions.

This was not done for two reasons. First, the general nature of such a theory would be better based after more examples of practical importance are analyzed; and second, any general theory would make this special case less clear.

The two dimensional recurrence relation 3.4.2.1 is a very good one. It in fact is better than the theorems here would indicate.

This can be seen in 3.4.2.1 since not only do $b_{1,m}^{\alpha,\beta}$, $b_{2,m}^{\alpha,\beta}$ and $b_{3,m}^{\alpha,\beta}$ quickly reach their limiting values, independent of $n$, the sum of their absolute values is never much larger than 1 and is quickly less than 1, again independent of $n$.

In extending the error analysis of theorem 3.4.8 further one might compute the asymptotic of $\phi_{m,m+k}^{\alpha,\beta}$ as $m \to \infty$ and compare that to a solution of the diagonal part of 3.4.2.1, whose limiting form is

$$\lim_{m\to\infty} \left|\frac{\phi_{m,m+k}^{\alpha,\beta}}{\phi_{m,m+k}}\right| \neq 0 .$$

This shows so that any error does not grow in the relative sense along the kth upper diagonal, where that error is made. Further analysis based on this and Theorem 3.4.9 shows that in some sense the relative error does not increase in later diagonals (i.e. the $k+1$, $k+2 \cdots$ diagonals).

This diagonal recurrence scheme improves over the three term recurrence in the last section which becomes numerically less useful as $m$ increases. The reason this method works is that, while $\left|\phi_{m,n+k}^{\alpha,\beta}\right|$ decreases eventually as $k \to \infty$ it does not decrease as quickly as $\left|\phi_{m+k,n+k}^{\alpha,\beta}\right|$. In fact, for $\alpha, \beta \geq 0$, the decrease appears monotonic, although no proof of this monotonicity is known.

This type of behavior should be expected in most modified moments of practical importance. For such moments the general idea of diagonal 2 dimensional recursion should have some merit.

Two dimensional recurrence relations are being investigated in connection with calculating modified moments for product integration rules ([13]. Unfortunatley, there has not been any large degree of success to date, other then the results presented here.

In attempting to formulate a general theory for two dimensional recurrence the results of finite differences used in estimating PDE's have a superficial resemblence to the recurrence relations here. Superificial in the sense that there the asymptotics are computed as the mesh size goes to zero over a constant area, while here the mesh size is constant while the area is increasing. It is felt by the author that little of that work will be useful here.

### 3.4.11 Proof of Theorem 3.4.2

The proof is very simple. Starting with the three term recurrence relation, in $n$, for the Jacobi polynomials $P_m^{\alpha,\beta}(u)$ (see A.1.8) multiply by the factor

$$w^{\alpha,\beta}(u)\ u^n \qquad 3.4.11.1$$

and integrate each term from $-1$ to $1$. The use of 2.2.1.5 then allows the identification of the following equation.

$$h_{m+1}^{\alpha,\beta}\ a_{1,m}\ \phi_{m+1,n}^{\alpha,\beta} = h_m^{\alpha,\beta}\ a_{2,m}\ \phi_{m,n}^{\alpha,\beta} \qquad 3.4.11.2$$

$$+\ h_m^{\alpha,\beta}\ a_{3,m}\ \phi_{m,n+1}^{\alpha,\beta} - h_{m-1}^{\alpha,\beta}\ a_{4,m}\ \phi_{m-1,n}^{\alpha,\beta},$$

$$\text{for } m = 1, 2, \cdots ;$$

where $a_{1,m}$, $a_{2,m}$, $a_{3,m}$ and $a_{4,m}$ are given by A.1.8.2 - A.1.8.5.

The use of Equation 2.2.1.4 then completes the proof.

## 3.5 Jacobi Moments with Logarithmic Singularities

In general the closer the nature of the singular point of a modified moment is to that of a function to be expanded with those moments, the better the expansion will be. With this in mind, the definition of the Jacobi moments will now be extended to include log singularities at the end points, after which some recurrence relations for these moments will be presented.

### 3.5.1 Definition

The moment $\phi^{\alpha,\beta}_{m,n,i,j}$ is defined by

$$\phi^{\alpha,\beta}_{m,n,i,j} = \frac{1}{h^{\alpha,\beta}_m} \int_{-1}^{1} [(1-u)^{\alpha}(\mathrm{Ln}(1-u))^{i}(1+u)^{\beta}(\mathrm{Ln}(1+u))^{j} \qquad 3.5.1.1$$

$$u^n\, P^{\alpha,\beta}_m(u)]\, du,$$

x for $\alpha,\beta > -1$ and $n,m,i,j = 0, 1, 2, \cdots$ .

The key to the recurrence relations that will be presented is given in the following lemma, whose proof is immediate.

### 3.5.2 Lemma

The following equations hold

$$\frac{\partial}{\partial\alpha}\, \phi^{\alpha,\beta}_{0,n,i,j} = \phi^{\alpha,\beta}_{0,n,i+1,j} \qquad 3.5.2.1$$

<u>and</u>

$$\frac{\partial}{\partial\beta}\,\phi^{\alpha,\beta}_{0,n,i,j} = \phi^{\alpha,\beta}_{0,n,i,j+1} \qquad 3.5.2.2$$

Lemma 2.5.2 may now be used to provide a recurrence relation for the moments $\{\phi_{0,n,i,j}\}_{n,i,j} = 0,1,\cdots$ .

### 3.5.3 Theorem

<u>The following recurrence relations hold.</u>

$$\phi^{\alpha,\beta}_{0,n,i,j} = r^{\alpha,\beta}_{0,n}\,\phi^{\alpha,\beta}_{0,n-1,i,j} + s^{\alpha,\beta}_{0,n}\,\phi^{\alpha,\beta}_{0,n-2,i,j} + \frac{i}{n+\alpha+\beta+1}(\phi^{\alpha,\beta}_{0,n-1,i,j} + \phi^{\alpha,\beta}_{0,n-1,i-1,j}) \qquad 3.5.3.1$$

<u>and</u>

$$\phi^{\alpha,\beta}_{0,n,i,j} = r^{\alpha,\beta}_{0,n}\,\phi^{\alpha,\beta}_{0,n-1,i,j} + s^{\alpha,\beta}_{0,n}\,\phi^{\alpha,\beta}_{0,n-2,i,j} - \frac{j}{n+\alpha+\beta+1}(\phi^{\alpha,\beta}_{0,n,i,j-1} + \phi^{\alpha,\beta}_{0,n-1,i,j-1}), \qquad 3.5.3.2$$

<u>where</u> $r^{\alpha,\beta}_{0,n}$ <u>and</u> $s^{\alpha,\beta}_{0,n}$ <u>are given by</u> 3.3.1.2 <u>and</u> 3.3.1.3.

<u>Proof</u>

First we prove 3.5.3.1. The proof is by induction on $i$. The induction is grounded at $i = 0$ by Equation 3.3.1.1. For the inductive Step, 3.5.3.1 is multiplied through by $n + \alpha + \beta + 1$ and then the partial derivative with respect to $\alpha$ is taken. Using 3.3.1.2, 3.3.1.3 and Lemma 3.5.2, the inductive step is completed.

The proof of 3.5.3.2 is similar, taking partials with respect to $\beta$ in the inductive step.

### 3.5.4 Remarks

Theorem 3.5.3 provides a very stable way of calculating the moments $\phi^{\alpha,\beta}_{0,n,i,j}$, providing only small values of $i$ and $j$ are needed; say $i, j = 0, 1, 2, 3$ .

The recurrence relations 3.5.3.1 and 3.5.3.2 can be used as non-homogeneous three term recurrence relations in $n$,

That is if we want $\{\phi^{\alpha,\beta}_{0,n,i,j}\}^N_{n=0}$ we could calculate the following sets can be calculated, in order, using Theorem 3.5.3:

$$\{\phi^{\alpha,\beta}_{0,n,0,0}\}^N_{n=0}, \{\phi^{\alpha,\beta}_{0,n,1,0}\}^N_{n=0} \cdots \{\phi^{\alpha,\beta}_{0,n,i,0}\}^N_{n=0},$$

$$\{\phi^{\alpha,\beta}_{0,n,i,1}\}^N_{n=0}, \{\phi^{\alpha,\beta}_{0,n,i,2}\}^N_{n=0} \cdots \{\phi^{\alpha,\beta}_{0,n,i,j}\}^N_{n=0} .$$

This approach is reasonable for small values of $i$ and $j$.

We can provide an error analysis of this technique based on the analysis of the homogeneous recurrence relation in Theorem 3.3.1.

Such an error analysis would not take into account error propagation from one set to another but, only within a given set. This proves no problem for the small $i$ and $j$'s mentioned but would cause disaster if used very far in the $i$ or $j$ directions.

These moments, and many others, obey the same simple two dimensional recurrence relation as the regular Jacobi moments, as the following theorem indicates.

<u>3.5.5 Theorem</u>

<u>For</u> $g(u)$: $(-1,1) \to C$

<u>Assume the existence of</u>

$$\psi_{m,n} = \frac{1}{h_m^{\alpha,\beta}} \int_{-1}^{1} g(u) u^m P_m^{\alpha,\beta}(u) du \ ;$$

$$\underline{\text{for}} \quad n,m = 1,\ 2,\ \cdots \quad .$$

<u>Then</u>

$$\psi_{m,n+1} = b_{1,m}^{\alpha,\beta}\ \psi_{m-1,n} + b_{2,m}^{\alpha,\beta}\ \psi_{m+1,n} + b_{3,m}^{\alpha,\beta}\ \psi_{m+1,n}\ ,$$

<u>where</u> $b_{1,m}^{\alpha,\beta}$, $b_{2,m}^{\alpha,\beta}$ <u>and</u> $b_{3,m}^{\alpha,\beta}$ <u>are given by</u> 3.4.2.2 - 3.4.2.4.

<u>Proof</u>

The proof follows the same line as the proof of Theorem 3.4.2.

<u>Remarks 3.5.5.1</u>

As a corollary to Theorem 3.5.5 we may calculate the moments

$$\{\phi_{m,n,i,j}^{\alpha,\beta}:\ m + n \le N\}$$

from the moments $\{\phi_{0,n,i,j}^{\alpha,\beta}:\ n = 0,\ 1,\ \cdots\ N\}$ and $\{\phi_{m,0,i,j}^{\alpha,\beta}:\ m = 0,\ 1,\ 2,\ \cdots\ N\}$. Furthermore this calculation will be reasonably stable, as the analysis for the recurrence relation in

Theorem 3.4.2 shows (the extra log terms do not greatly effect the analysis).

# CHAPTER 4

# APPLICATIONS

## 4.1 Introduction

This chapter will present a number of applications of the results in Chapters 2 and 3. These applications will be in areas others then the perturbation problems mentioned in Chapter 1.

These sections are intended to be a very light set of remarks. Some numerical work has been done for the new methods mentioned, and as can be observed from the comments made, many convergence theorems, whose proofs are very short, can be produced.

It should be noted that, in general, corresponding convergence criteria for more traditional formal methods do not exist.

The sections will be:

4.2 Numerical Integration

4.3 Cauchy Principal Values

4.4 Ordinary Differential Equations

4.5 Integral Equations

4.6 Partial Differential Equations.

## 4.2 Numerical Integration

The most immediate application of the work in Chapters 2 and 3 is in the area of numerical integration. Two cases will be explained. The first is the integration of a function that can be expanded in a rapidly converging modified Jacobi expansion and the second is the more interesting case of product integration.

<u>4.2.1 Theorem</u>

<u>Suppose</u>

$$f(u) = \sum_{n=0}^{N} \sum_{m=0}^{M} (1-u)^{\alpha}(1+u)^{\beta}\, u^{n}\, P_{m}^{\alpha,\beta}(u)\; d_{n,m} \qquad 4.2.2.1$$

$$+\; r_{N,M}(u), \quad -1 < u < 1.$$

<u>Then</u>

$$\int_{-1}^{1} f(u)du = \sum_{n=0}^{N} \sum_{m=0}^{M} \phi_{n,m}\, d_{n,m} + \int_{-1}^{1} r_{N,M}(u)du. \qquad 4.2.1.2$$

<u>Additionally if</u>

$$|r_{N,M}(u)| \le e_{N,M}(1-u)^{\alpha}(1+u)^{\beta}\, u^{n} \qquad 4.2.1.3$$

<u>then a bound for the error term in</u> 4.2.1.2 <u>is given by</u>

$$\left|\int_{-1}^{1} r_{N,M}(u)du\right| \le e_{N,M}\left|\phi_{0,n}^{\alpha,\beta}\right| . \qquad 4.2.1.4$$

Since the results of Chapter 3 provide an accurate and efficient means for calculating the modified Jacobi moments, this theorem gives an easy means of calculating the integral of a function with endpoint singularities.

The integration of a function such as

$$f(u) = (1-u)^{-\frac{3}{4}}(1+u)^{-\frac{2}{3}}(1+u-2u^2)\cos(u) \qquad 4.2.2$$

can be carried out by simply expanding the $\cos(u)$ in terms of $\{J_m^{-\frac{3}{4}, -\frac{2}{3}}(u)\}$, which is a rapidly converging series, and then using Theorem 4.1.1.

It should be noted that this method of integration will be numerically stable for functions with rapidly converging modified Jacobi series.

The special cases $M = 0$ or $N = 0$ in Theorem 4.2.1 should also be noted.

An extension of Theorem 4.2.1 allows the inclusion of the factor $[\mathrm{Ln}(1-u)]^i[\mathrm{Ln}(1+u)]^j$ in the integrand in 4.2.1.2. This is covered by the use of the moments $\phi_{n,m,i,j}^{\alpha,\beta}$ in 3.5.1.1, which gives a result similar to 4.2.1.1. For results in this direction using the moments $\phi_{0,m,1,1}^{-\frac{1}{2}, -\frac{1}{2}}$ see [100]).

The integration of functions by Theorem 4.2.1 requires some analytic work. The work done recently on product integration by a number of authors does not require any analytic effort and also provides the values of integrals whose integrands have singularities within the range of integration.

The key to most of the work on product integration may be presented in terms of a special form of an interpolation formula. To give this formula we need

### 4.2.3 Notation

Let $u_{m,i}$, $1 \le i \le m$, be the zeros of $P_m^{\alpha,\beta}(u)$. Also let $\{\mu_{m,i}\}_{i=1}^{m}$ be the Christoffel numbers of the Gauss Jacobi integration rule

$$\int_{-1}^{1} f(u)\,du \approx \sum_{i=1}^{m} \mu_{m,i}\, f(u_{m,i}). \qquad 4.2.3.1$$

Recall that the $\mu_{m,i}$'s are chosen to make the rule 4.2.4.1 exact for all polynomials of degree less than $2m$.

It should be noted that the very important problem of generating the sets $\{u_{m,i}\}_{i=1}^{m}$ and $\{\mu_{m,i}\}_{i=1}^{m}$ has received a great amount of attention and many effective computational methods are known (see [25], [57], and [81]).

### 4.2.4 Theorem

The polynomial of degree $m - 1$ which interpolates $f\{(u_{m,i}^{m})\}_{i=1}$ at the nodes $\{u_{m,i}\}_{i=1}^{m}$ is given by

$$4.2.5.1 \qquad L_m(f,u) = \sum_{\ell=0}^{m-1} \sum_{i=1}^{m} \frac{\mu_{m,i}}{h_\ell^{\alpha,\beta}} f(u_{m,i}) P_\ell^{\alpha,\beta}(u_{m,i}) P_\ell^{\alpha,\beta}(u) \quad .$$

This result is fairly direct, although it only seems to be recently known (e.g. [116]). The possible use of 4.2.4.1 in many settings should be apparent. It will be mentioned again in Section 4.4 in connection with integral equations.

One type of product integration rule is of the form

$$\int_{-1}^{1} k(u)f(u)du \approx \sum_{i=1}^{m} w_{m,i}\ f(u_{m,i}) \qquad 4.2.5$$

where the $w_{m,i}$ are to be determined.

Smith and Sloan ([116]) suggest one choice of the weights $\{w_{m,i}\}_{i=1}^{m}$ as

$$w_{m,i} = \mu_{m,i} \sum_{\ell=0}^{m-1} b_\ell\ P_\ell^{\alpha,\beta}(u_{m,i}),\quad 1 \le i \le m, \qquad 4.2.6$$

where

$$b_\ell = \frac{1}{h_\ell^{\alpha,\beta}} \int_{-1}^{1} k(u)\ P_\ell^{\alpha,\beta}(u)du,\quad \ell \ge 0. \qquad 4.2.8$$

With this choice of weights Smith and Sloan produce several powerful convergence theorems of the form

$$\lim_{m\to\infty} \sum_{i=1}^{m} w_{m,i}\ f(u_{m,i}) = \int_{-1}^{1} k(u)\ f(u)du. \qquad 4.2.9$$

Their results appear in a series of papers ([112], [113], [114] and [116]).

A similar theorem, with fairly simple conditions, is the following

4.2.9 Theorem

Let $\alpha, \beta > -1$,

$$\alpha_0 = -\max(\frac{2\alpha+1}{4}, 0) \qquad 4.2.9.1$$

and

$$\beta_0 = -\max(\frac{1\beta+1}{4}, 0). \qquad 4.2.9.2$$

Suppose

$$\int_{-1}^{1} |k(u)(1-u)^{\alpha_0}(1+u)^{\beta_0}|^p \, du < \infty \qquad 4.2.9.3$$

for some $p > 1$, then 4.2.8 holds for all continous functions $f$.

This is a very powerful theorem which allows many types of singularities in $k$ both inside the interval $[-1,1]$ and at the endpoints $u = -1, 1$.

One major weakness to the whole area of product integration has been the work required to evaluate 4.2.7. This point is usually not brought up by the papers on product integration. The relation between $b_\ell$ in 4.2.7 and the moments $\phi^{\alpha,\beta}_{m,n,i,j}$ should be immediately apparent. Thus the recurrence relations of Chapter 3 provide an important tool for the area of product integration. It is also true that Theorem 4.2.1 and its extension should be the first application of the modified Jacobi moments to calculating integrals such as 4.2.7.

There are a number of other convergence theorems for product integration in the literature, some with error bounds. The main application may well be integral equations, and the only thing so far preventing its practical application there has been the difficulty in obtaining the $b_\ell$'s. Since, as noted, the Jacobi moments form a broad class of $b_\ell$'s it is clear that their evaluation by the methods of Chapter 2 is an important contribution.

## 4.3 Cauchy Principal Values

The problem of calculating a Cauchy Principal Value of a function is an important problem of Mathematical Physics. It can be transformed into the calculation of

$$P_0 = \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon} f(u)du + \int_{\varepsilon}^{1} f(u)du\right], \qquad 4.3.1$$

where the proper integral

$$\int_{-1}^{1} f(u)du \qquad 4.3.2$$

may not exist.

A number of authors have shown that the use of some product integration rules with nodes placed at the Jacobi zeros actually give rules which converge for Cauchy principal value problems ([116]). Thus the whole of Chapters 2 and 3, as suggested in 4.1, may be applied to these problems.

There is also another approach to calculating Cauchy principal values which may be taken. That is to consider

$$P(u) = \int_{-1}^{u} [f(\tfrac{1}{2}v - \tfrac{1}{2}) - f(-\tfrac{1}{2}v + \tfrac{1}{2})]\tfrac{1}{2}\, dv. \qquad 4.3.3$$

Noting that

$$P_0 = \lim_{u \to 1} P(u) \ , \qquad 4.3.4$$

we see that the fundamental expansion theorem from Chapter II may be applied to yield the value $P(0) = P_0$. All that is needed is the calculation of $\{P^{(n)}(0)\}_{n=0}^{N}$ and some bound of their growth rate.

We can see that

$$P(0) = \int_{-1}^{0} [f(\tfrac{1}{2}v - \tfrac{1}{2}) - f(-\tfrac{1}{2}v + \tfrac{1}{2})]\tfrac{1}{2}\, dv, \qquad 4.3.5$$

$$P^{(n)}(0) = [f^{(n-1)}(-\tfrac{1}{2}) + (-1)^n f^{(n-1)}(\tfrac{1}{2})](\tfrac{1}{2})^n,$$

for $n \geq 1$.

With an appropriate transformation the points where the derivative in $f$ is needed could have been moved to points other than $\frac{1}{2}$ and $-\frac{1}{2}$.

Perhaps it is time to make an observation about the use of the Jacobi polynomials. Just as with the Chebyshev series we may define shifted Jacobi series, that is, polynomials $\{P_m^{*\alpha,\beta}(u)\}_{m=0}^{\infty}$ orthogonal over $[0,1]$ with a weight function $u^{\alpha}(1-u)^{\beta}$. All the work in Chapters 2 and 3 can be repeated, including the generation of modified shifted Jacobi moments. The advantage of this approach from a computational standpoint is the simplification of the Transformation 1.2.4.3 into a simpler form, in particular, when the range of interest is finite.

### 4.4 Ordinary Differential Equations

In the area of ordinary differential equations, without considering perturbations, the representation theorem from Chapter 2 provides a means of finding numerical solutions with a guarantee of success. The classical formal powers series methods can not offer such a guarantee.

To show how this material might be applied, consider the boundary value problem

$$y' + y = f(u) \tag{4.4.1}$$

$$y(-1) + 2y(0) - y(1) = 0 \tag{4.4.2}$$

Assume that we may write

$$f(u) = \sum_{n=0}^{\infty} f_n u^n, \quad |u| < 1. \tag{4.4.3}$$

As a first method of solving 4.4.1 we substitute in a solution of the form

$$y(u) = \sum_{n=0}^{\infty} y_n u^n \tag{4.4.4}$$

into 4.4.1. This gives conditions

$$y_{n+1}(n+1) + y_n = f_n, \quad n \geq 0. \qquad 4.4.5$$

The problem with getting conditions from 4.4.2 to go with those in 4.4.5 is that $\sum_{n=0}^{\infty} y_n$ or $\sum_{n=0}^{\infty} y_n(-1)^n$ might not converge. To solve this problem we refer to the representation theorem, where we see we can use the polynomial

$$Y_M(u) = \sum_{m=0}^{N} \frac{\binom{N-m+k}{N-m}}{\binom{M+k}{M}} \left[ \sum_{n=m}^{N} y_n \, \phi_{n,m}^{\alpha,\beta} \right] P_m^{\alpha,\beta}(u). \qquad 4.4.6$$

over all of $[-1,1]$. This follows since we know that for

$$\varepsilon_M(u) = y(u) - Y_M(u) \qquad 4.4.7$$

we have $|\varepsilon_M(u)| \to 0$ on $[-1,1]$, under the conditions of Theorem 2.3.5.

Using 4.4.6 in 4.4.2 gives one linear condition on $\{y_n\}_{n=0}^{N}$. Then adding the first $N$ conditions from 4.4.4 gives $N + 1$ linear conditions on $\{y_n\}_{n=0}^{N}$, which may be solved for the $y_n$'s.

Because of 4.4.7 we may in general expect this solution to converge to the true solution provided the original equation (4.4.1) is regular with respect to a perturbation in $f$.

Modifications of this approach that are still supported by the material presented here include using a formal Jacobi series in place of 4.4.4 or using $L_m(y,u)$ (see 4.2.4.1) in place of 4.4.4 and 4.4.6.

In the latter case the linear system would be solved for the discrete values $\{y(u_{m,i})\}_{i=1}^{M}$ of $y$.

The use of this method for ordinary differential equations has some merit, despite the large amount of numerical software already in existence for their solution. This is because this method can provide solutions to problems with very general multipoint boundary value conditions, which are not covered by most of the current numerical methods ([73] and [46]).

The use of the (C,k)-summability process on the formal Jacobi expansion provides an immediate extension to the results on using Chebyshev expansions to solve ordinary differential equations developed in [74].

### 4.5 Integral Equations

One major motivation for the work being done in product integration has been to apply these results to the numerical solution of integral equations.

Since most integral equations of practical importance involve at least a mild singularity (and may even require interpretions as Cauchy principal values) it follows that methods which solve these problems may be able to induce methods which can solve integral equations. Thus the work here provides a new set of tools for use on such integral equations.

A slightly more direct approach is also possible as well. That is to formally substitute 4.2.5.1 or 4.4.6 into the problem then generate a set of linear equations for the unknown by selecting a set of points in the interval. This allows the solution of a set of linear equations to provide approximate values of the solution at these selected points.

Because of the strong convergence theory of the expansions 4.2.4.1 and 4.4.6, fairly mild conditions on the distribution of the node points and a regularity condition on the integral equation itself, with respect to an additive perturbution, produce a very reasonable convergence theorem.

Expansion of singular kernels with modified Jacobi series is also being currently investigated as a solution technique ([20]). Within this work one also finds the need to calculate these moments, and, as mentioned earlier, little is said about their calculation.

### 4.6 Partial Differential Equations

Formal Expansions of solutions of partial differential equations (pde's) is an old and valued method. As with ordinary differential equations with boundary conditions, we may apply the methods of this work to pde's.

One interesting example of why the adjustments to standard formal methods are needed can be found in the paper ([99]) where a parabolic pde is solved by a formal Legendre expansion method. In that paper the boundary conditions were continous but not smooth at one point. This caused their solution to be 'slightly irregular' near the point of discontinuity in the derivative of the boundary condition.

This singularity showed up as a small ripple in an otherwise very smooth and accurate solution. As more terms were calculated the ripple still remained.

The reason for the ripple can be obtained by considering the Jacobi expansion theorem (2.3.4) and the remarks in 2.3.7. The method being applied corresponded to using $\alpha = \beta = 0$, $k = 0$. According to the remarks in 2.3.7 one should not expect convergence in this case.

For the pde and boundary conditions given it can be shown that taking $\alpha = \beta = 1$ and $k = \frac{3}{2}$ will result in an approximation which converges to the true solution (selecting $\alpha = \beta = \frac{1}{2}$, $k = \frac{3}{4}$ also worked, with faster convergence, but I could not verify the growth rate of the power series coefficients was less then $O(n)$).

## APPENDIX A

### A.1 Basic Information on the Classical Orthogonal Polynomials

Expansions of functions in orthogonal polynomials is a very developed subject area. Because of the differences in notation used by various authors it is important to establish the definitions being used. This will be done in this section. Whenever possible the notation of Szego ([120]) is followed.

In what follows $d\alpha(u)$ will be assumed to be a measure over $[a,b]$ with $\alpha$ nonconstant and nondecreasing. If $a = -\infty$ $(b = \infty)$ it will be assumed that $\alpha(-\infty) = \lim_{u\to-\infty} \alpha(u)$ $(\alpha(\infty)\lim_{u\to\infty} \alpha(u))$ exists.

For the classical orthogonal polynomials the measure $d\alpha(x)$ may be expressed as $d\alpha(u) = w(u)du$. In this case it will be assumed that $w$ is nonnegative, $L^1$-measurable and $\int_a^b w(u)du > 0$.

#### A.1.1 Definition

The inner product is defined for any two functions $\psi_1, \psi_2 \in L_\alpha^2[a,b]$ as

$$\langle\psi_1, \psi_2\rangle = \int_a^b \psi_1(u)\, \overline{\psi_2(u)}\, d\alpha(w). \qquad \text{A.1.1.1}$$

#### A.1.2 Definition

A set of functions $\{\psi_i\}_{i=0}^{L}$, where $L$ may be finite or infinite, with

$$\psi_i \in L_\alpha^2[a,b] \text{ for } i = 0, 1, 2, \cdots, L,$$

is said to be orthogonal if

$$\langle \psi_i, \psi_j \rangle = 0, \quad \text{for } i, j = 0, 1, 2, \cdots, L,$$
$$\text{and } i \neq j.$$

If in addition

$$\langle \psi_i, \psi_i \rangle = 1, \quad \text{for } i = 0, 1, 2, \cdots, L,$$

then the functions are said to be orthonormal.

A.1.3 Definition

For any function $f$ in $L^2_\alpha[a,b]$ the formal fourier expansion of $f$ in terms of the orthogonal system $\{\psi_i\}_{i=0}^{L}$ is given by

$$f(u) \sim \sum_{m=0}^{L} a_m \psi_m(u), \qquad \text{A.1.3.1}$$

where the fourier coefficient $a_m$, $m = 0, 1, 2, \cdots, L$, are defined by

$$a_m = \frac{1}{h_m} \langle f, \psi_m \rangle \qquad \text{A.1.3.2}$$

with

$$h_m = \langle \psi_m, \psi_m \rangle. \qquad \text{A.1.3.3}$$

### A.1.4 Definition

The moments of an orthogonal system $\{\psi_i\}_{i=1}^{L}$ are defined as

$$\phi_{m,n} = \frac{1}{h_m} \langle u^n, \psi_m(u) \rangle, \quad n = 0, 1, 2, \cdots, L, \qquad \text{A.1.4.1}$$

whenever these integrals exist.

For a given weight function $w(u)$ the Gram-Schmidt process shows how to construct a system of polynomials $\{p_m\}_{m=0}^{\infty}$ such that the system is orthogonal, with respect to $w(u)$, and each $p_m$ is of exact degree $m$.

With the notation developed the classical orthogonal polynomials may now be defined.

### A.1.5 Definition

The Jacobi polynomials $p_m(u) = P_m^{\alpha,\beta}(u)$ have $a = -1$, A.1.5.1
$b - 1$,

$$w(u) = (1 - u)^{\alpha}(1 + u)^{\beta},$$

$$\alpha > -1, \quad \beta > -1$$

and

$$h_m = h_m^{\alpha,\beta} = \begin{cases} \dfrac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}, & \text{for } m = 1, 2, 3, \cdots \\[2ex] \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{for } m = 0. \end{cases}$$

The Laguerre polynomials $p_m(u) = L_m^{(\alpha)}(u)$ have A.1.5.2

$$a = 0, \; b = +\infty,$$

$$w(u) = e^{-u} u^{\alpha},$$

$$\alpha > -1,$$

<u>and</u>

$$h_m = h_m^{(\alpha)} = \Gamma(\alpha+1)\binom{m+\alpha}{m} .$$

<u>The Hermite polynomials</u> $p_m(u) = H_m(u)$ <u>have</u> A.1.5.3

$$a = -\infty, \; b = \infty,$$

$$w(u) = e^{-u^2}$$

<u>and</u>

$$h_m = \pi^{\frac{1}{2}} 2^m m! .$$

<u>The ultraspherical polynomials</u> $p_m(u) = C_m^{(\alpha)}(u)$ <u>have</u> A.1.5.4

$$a = -1, \; b = -1$$

$$w(u) = (1 - u^2)^{\alpha - \frac{1}{2}}$$

$$\alpha > -\frac{1}{2}$$

$$h_m = \begin{cases} \dfrac{\pi\, 2^{1-2\alpha}\Gamma(m+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}, & \alpha \neq 0 \\ \dfrac{2\pi}{n^2}, & \alpha = 0 \end{cases} .$$

<u>The Chebyshev polynomials of the first kind</u> $p_m(u) = T_m(u)$ A.1.5.5 <u>have</u>

$$a = -1, \quad b = 1,$$

$$w(u) = (1 - u^2)^{-\frac{1}{2}},$$

and

$$h_m = \begin{cases} \frac{\pi}{2}, & m \neq 0 \\ \pi, & = 0. \end{cases}$$

<u>The Chebyshev polynomials of the second kind</u> $p_m = U_m(u)$ A.1.5.6 <u>have</u>

$$a = -1, \quad b = 1$$

$$w(u) = (1 - u^2)^{\frac{1}{2}},$$

<u>and</u>

$$h_m = \frac{\pi}{2}.$$

<u>The Legendre polynomials</u> $p_m(u) = P_m(u)$ <u>have</u> A.1.5.7

$$a = -1, \quad b = 1,$$

$$w(u) = 1$$

<u>and</u>

$$h_m = \frac{2}{2m+1} .$$

<u>A.1.6 Theorem</u>

<u>The following relationships hold</u>

$$C_m^{(\alpha)}(u) = \begin{cases} \dfrac{\Gamma(\alpha+m+1)\Gamma(2\alpha\ m)}{\Gamma(2\alpha)\Gamma(\alpha+m+\frac{1}{2})} P_m^{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(u), & \alpha \neq 0 \\[2ex] \dfrac{(m-1)!\sqrt{\pi}}{\Gamma(m+\frac{1}{2})} P_m^{-\frac{1}{2},-\frac{1}{2}}(u), & \alpha = 0 \end{cases} \qquad \text{A.1.6.1}$$

$$T_m(u) = \frac{n}{2} C_m^{(0)}(u) = \frac{m!\sqrt{\pi}}{\Gamma(m+\frac{1}{2})} P_m^{-\frac{1}{2},\ -\frac{1}{2}}(u), \qquad \text{A.6.2.}$$

$$U_m(u) = C_m^{(1)}(u) = \frac{(m+1)!\sqrt{\pi}}{2\Gamma(m+\frac{3}{2})} P_m^{\frac{1}{2},\ \frac{1}{2}}(u), \qquad \text{A.1.6.3}$$

<u>and</u>

$$P_m(u) = C_m^{(\frac{1}{2})}(u) = P_m^{0,0}(u). \qquad \text{A.1.6.4}$$

It is clear that the relationships presented in A.1.6 allow any results about Jacobi expansions, in particular results about the Jacobi moments, to give corresponding results about Ultraspherical, Chebyshev and Legendre polynomials.

The next theorem, known as the Rodrigues formula, is sometimes used as the definition of the classical orthogonal polynomials.

A.1.7 Theorem

The Jacobi, Laguerre and Hermite polynomials each satisfy a Rodrigues' formula

$$p_m(u) = \frac{1}{a_m w(u)} \frac{d^m}{du^m} [w(u)(g(u))^m], \qquad \text{A.1.7.1}$$

where:

for the Jacobi polynomials

$$w(u) = (1-u)^{\alpha}(1+u)^{\beta},$$

$$a_m = (-1)^m \, 2^m \, m! \quad \text{and}$$

$$g(u) = 1 - u^2 \qquad ,$$

for the Laguerre polynomials

$$w(u) = e^{-u}u^{\alpha},$$

$$a_m = m! \quad \underline{\text{and}}$$

$$g(u) = u$$

<u>and for the Hermite polynomials</u>

$$w(u) = e^{-u^2}$$

$$a_m = (-1)^m \quad \underline{\text{and}}$$

$$g(u) = 1 \quad .$$

<u>A.1.8 Theorem</u>

Any three consecutive orthogonal polynomials $P_{m-1}$, $P_m$ and $P_{m+1}$ obey a recurrence relation

$$a_{1,m}\, P_{m+1}(u) = (a_{2,m} + a_{3,m}u)P_m(u) - a_{4,m}\, P_{m-1}(u), \qquad \text{A.1.8.2}$$

where $a_{i,m}$; for $i = 0, 1, 3, 4,$ and $m = 1, 2, 3, \cdots$ ; are independent of $u$.

For the Jacobi polynomials these constants are:

$$a_{1,m} = 2(m+1)(m+\alpha+\beta+1)(2m+\alpha+\beta), \qquad \text{A.1.8.2}$$

$$a_{2,m} = (2m+\alpha+\beta+1)(\alpha^2-\beta^2), \qquad \text{A.1.8.3}$$

$$a_{3,m} = (2m+\alpha+\beta)(2m+\alpha+\beta+1)(2m+\alpha+\beta+2), \qquad \text{A.1.8.4}$$

$$a_{4,m} = 2(m+\alpha)(m+\beta)(2m+\alpha+\beta+2); \qquad \text{A.1.8.5}$$

with

$$P_0^{\alpha,\beta}(u) = 1 \quad \text{and} \qquad \text{A.1.8.6}$$

$$P_1^{\alpha,\beta}(u) = \frac{1}{2}(\alpha+\beta+2)u + \frac{1}{2}(\alpha-\beta) \qquad \text{A.1.8.7}$$

For the Laguerre polynomials these constants are:

$$a_{1,m} = m + 1, \qquad \text{A.1.8.8}$$

$$a_{2,m} = 2m + \alpha + 1, \qquad \text{A.1.8.9}$$

$$a_{3,m} = -1, \qquad \text{A.1.8.10}$$

$$a_{4,m} = m + \alpha, \qquad \text{A.1.8.11}$$

with

$$L_0^{(\alpha)}(u) = 1 \qquad \text{and} \qquad \text{A.1.8.12}$$

$$L_1^{(\alpha)}(u) = -u + \alpha + 1. \qquad \text{A.1.8.13}$$

For the Hermite polynomials these constants are

$$a_{1,m} = 1, \qquad \text{A.1.8.14}$$

$$a_{2,m} = 0, \qquad \text{A.1.8.15}$$

$$a_{3,m} = 2, \qquad \text{A.1.8.16}$$

$$a_{4,m} = 2m, \qquad \text{A.1.8.17}$$

with

$$H_0(u) = 0 \quad \text{and} \qquad \text{A.1.8.18}$$

$$H_1(u) = 2u. \qquad \text{A.1.8.19}$$

## A.2 Some Useful Results

This section contains a group of useful theorems. Most are rather well known with the exception of A.2.7, the Jacobi reduction theorem and its two corollaries. That theorem seems to be reproved by many authors for special cases.

Theorems A.2.1-A.2.3 are useful for calculating Fourier coefficients of functions whose power series representation is uniformly convergent over all subintervals of the interval of integration.

### A.2.1 Theorem

<u>If</u>

$$\int_{-1+\varepsilon}^{1-\varepsilon} \Big[\sum_{n=0}^{\infty} l_n(x)\Big]dx = \sum_{n=0}^{\infty} \Big[\int_{-1+\varepsilon}^{1-\varepsilon} l_n(x)dx\Big]$$

<u>for all</u> $1 > \varepsilon \geq 0$

<u>Then, if either side of the following equation converges absolutely so does the other, and the limits are equal.</u>

$$\int_{-1}^{1}[\sum_{n=0}^{\infty} l_n(x)]dx = \sum_{n=0}^{\infty}[\int_{-1}^{1} l_n(x)dx] .$$

Note that this does not require uniform converges of the series over [-1,1].

<u>A.2.2 Theorem</u>

<u>If</u>

$$\int_{\varepsilon}^{m} [\sum_{n=0}^{\infty} l_n(x)]ds = \sum_{n=0}^{\infty} [\int_{\varepsilon}^{m} l_n(x)dx]$$

<u>for all</u> $m > \varepsilon > 0$

<u>then if either side of the following equation converges absolutely then so does the other, and the limits are equal.</u>

$$\int_{0}^{\infty} [\sum_{n=0}^{\infty} l_n(x)]dx = \sum_{n=0}^{\infty} [\int_{0}^{\infty} l_n(x)dx] .$$

<u>A.2.3 Theorem</u>

<u>Theorem A.2.2 also holds for doubly infinite integrals.</u>

The proof of Theorems A.2.1–A.2.3 may be found in [12]).

Theorems A.2.5 - A.2.7 are fairly recent results on summability of Jacobi series. It appears that very little in the way of practical numerical applications of these theorems has been made to date.

A.2.4 Theorem (Equiconvergence Theorem for Jacobi Series in the Interior of the Interval [-1, 1]).

Let:

$$f(u) \text{ be Lebesgue-measurable in } [-1, 1], \tag{A.2.4.1}$$

$$\int_{-1}^{1} (1-u)^{\alpha}(1+u)^{\beta}|f(u)|du \text{ exist} \tag{A.2.4.2}$$

and

$$\int_{-1}^{1} (1-u)^{\frac{\alpha}{2}-\frac{1}{4}} (1+u)^{\frac{\beta}{2}-\frac{1}{4}} |f(u)|du \text{ exist .} \tag{A.2.4.2}$$

If $s_n(u)$ denotes the nth partial sum of the expansion of $f(u)$ in a Jacobi series, and $\mathcal{s}_n(\cos(\theta))$ the nth partial sum of the sum of the Fourier (cosine) series of

$$(1-\cos(\theta))^{\frac{\alpha}{2}-\frac{1}{4}} (1+\cos(\theta))^{\frac{\beta}{2}-\frac{1}{4}} f(\cos\theta) \tag{A.2.4.4}$$

then for $-1 < u < 1$,

$$\lim_{n\to\infty} |s_n(u) - (1-u)^{-\frac{\alpha}{2}-\frac{1}{4}} (1+u)^{-\frac{\beta}{2}-\frac{1}{4}} \mathcal{s}_n(u)| = 0, \tag{A.2.4.5}$$

uniformly in $-1 + \varepsilon \le u \le 1 - \varepsilon$, where $\varepsilon$ is fixed $(0 < \varepsilon < 1)$. This is Theorem 9.1.2 of [120].

A.2.5 Theorem (Summability Theorem for Jacobi Series at the End Points $\pm$ 1).

Let $f(x)$ be continuous on the closed segment $[-1, 1]$. The expansion of $f(x)$ in a Jacobi series is $(C,k)$-summable at $x = +1$ provided $k > \alpha + \frac{1}{2}$. This is in general not true if $k = \alpha + \frac{1}{2}$. An analogous statement holds for $x = -1$, $\alpha$ being replaced by $\beta$.

This is Theorem 9.1.3 of [120].

A.2.6 Theorem (Generalized Summability Theorem for Jacobi Series).

Let: $f(u)$ be Lebesque-Measurable in $[-1, 1]$, $f(u)$ be continuous at $u = 1$ and A.2.6.1

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}|f(u)|du \quad \text{exist.}$$

Then the Jacobi series is $(C,k)$-summable $k > \alpha + \frac{1}{2}$, at $u = 1$, provided that is the case

$$\beta > -\frac{1}{2}, \ \alpha + \frac{1}{2} < k < \alpha + \beta + 1, \qquad \text{A.2.6.4}$$

the following additional "antipole condition" is satisfied: the integral

$$\int_{-1}^{0} (1 + u)^{\frac{\beta}{2} - \frac{1}{4}} |f(u)|du \quad \text{exists.} \qquad \text{A.2.6.5}$$

(*for* $k \geq \alpha s + \beta + 1$ *no antipole condition is needed.*) *For* $k \leq \alpha + \frac{1}{2}$ *or* $\alpha + \frac{1}{2} < k < \alpha + \beta + 1$, *but without the antipole condition, the statement is not true.*

This is Theorem 9.1.4 of Szego [120].

A.2.7 Theorem

*The Jacobi series of a continuous function is uniformly* (C,k)-summable *for* $k > \max(\alpha + \frac{1}{2}, \beta + \frac{1}{2})$ *where*

$$\alpha, \beta \geq -\frac{1}{2}$$

or

$$\alpha + \beta \geq -1 .$$

This theorem is suggested in [5] as a corollary to a general result.

The next result, Theorem A.2.8, is known as Watson's Lemma. It has been studied and presented with error bounds by Olver ([96]). This theorem provides a fairly direct method for getting asymptotic expansions and error bounds whenever it may be applied.

A.2.8 Lemma (Watson)

*Fix* $\delta$ $(\delta > 0)$. *Assume that:*

q(t) <u>is a real or complex function of the positive variable</u> t <u>with a finite number of discontinuities and infinities;</u> A.2.8.1

$$q(t) \sim \sum_{s=0}^{\infty} a_s \, t^{(s+1)-u)/u}, \quad \text{as} \quad t \to +0, \tag{A.2.8.2}$$

<u>where</u> u <u>is a positive constant and</u> λ <u>is a real or complex constant such that</u> $Re(\lambda) > 0$

<u>and</u>

<u>The abscissa of convergence of the integral</u> A.2.8.3

$$I(z) = \int_0^{\infty} e^{-zt} q(t)dt$$

<u>is not</u> $-\infty$.

<u>Then</u>

$$I(z) \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{u}\right) \frac{a_s}{z^{(s+\lambda/u)}}, \quad \text{as} \quad z \to \infty \tag{A.2.8.4}$$

<u>in the sector</u>

$$|ph(z)| \leq \frac{1}{2}\pi - \delta,$$

<u>where</u> $z^{(s+\lambda)/u}$ <u>has its principal value</u>.

The proof of this theorem may be found in Olver [96].

Next the Jacobi reduction theorem and two corollaries will be given followed by a proof.

## A.2.9 Theorem (Jacobi Reduction)

*If* $h \in C^k$ *then for* $m = k, k+1, \cdots,$

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}h(u)P_m^{\alpha,\beta}(u)du \tag{A.2.9.1}$$

$$= \frac{(m-k)!}{2^k\, m!}\int_{-1}^{1}(1-u)^{\alpha+k}(1+u)^{\beta+k}h^{(k)}(u)P_{m-k}^{\alpha+k,\beta+k}(u)du.$$

## A.2.10 Corollary to Theorem A.2.9

*If* $h \in C^m$ *then*

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}h(u)P_m^{\alpha,\beta}(u)du \tag{A.2.10.1}$$

$$= \frac{1}{2^m\, m!}\int_{-1}^{1}(1-u)^{\alpha+m}(1+u)^{\beta+m}h^{(m)}(u)du.$$

## A.2.11 Corollary to the Corollary A.2.10

*For* $n \geq m$ ($n$ *need not be integer*)

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}u^{n}\ P_m^{\alpha,\beta}(u)du \tag{A.2.11.1}$$

$$= \frac{\binom{n}{m}}{2^m}\int_{-1}^{1}(1-u)^{\alpha+m}(1+u)^{\beta+m}\ u^{n-m}du\ .$$

## A.2.12 Proof of Theorem A.2.9 and Corollaries A.2.10 - A.2.11.

Using the Rodrigues' formula A.1.7.1 for the Jacobi polynomals

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}h(u)\ P_m^{\alpha,\beta}(u)du \qquad \text{A.2.12.1}$$

$$= \int_{-1}^{1}\ h(u)\ \frac{1}{(-1)^m\ 2^m\ m!}\ \frac{\partial^m}{\partial u^m}\ [(1-u)^{\alpha+m}(1+u)^{\beta+m}]du.$$

Applying integration by parts to A.2.12.1

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}h(u)P_m^{\alpha,\beta}(u)du \qquad \text{A.2.12.2}$$

$$= \frac{h(u)}{(-1)^m\ 2^m\ m!}\ \frac{d^{m-1}}{du^{m-1}}\ [(1-u)^{\alpha+m}(1+u)^{\beta+m}]\Big|_{-1}^{1}$$

$$- \int_{-1}^{1}\ h^{(1)}(u)\ \frac{1}{(-1)^m\ 2^m\ m!}\ \frac{d^{m-1}}{du^{m-1}}\ [(1-u)^{\alpha+m}(1+u)^{\beta+m}]du.$$

The differentiation formula

$$[RS]^{(m)} = \sum_{\ell=0}^{m} \binom{m}{\ell}\ R^{(\ell)}S^{(m-\ell)} \qquad \text{A.2.12.3}$$

applied to the function

$$f(u) = \frac{d^{m-1}}{du^{m-1}}\ [(1-u)^{\alpha+m}(1+u)^{\beta+m}] \qquad \text{A.2.12.4}$$

shows

$$f(u) = \sum_{\ell=0}^{m-1} \binom{m}{\ell}\ [\frac{d^{\ell}}{du^{\ell}}\ (1-u)^{\alpha+m}][\frac{d^{m-\ell-1}}{du^{m-\ell-1}}\ (1+u)^{\beta+m}]. \qquad \text{A.2.12.5}$$

It is now easy to see that each term in A.2.12.5 must be zero at 1 and at -1. Thus $f(1) = f(-1) = 0$. This in turn shows that from A.2.12.2

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}h(u)P_m^{\alpha,\beta}(u)du \qquad \text{A.2.12.6}$$

$$= \frac{1}{2m}\int_{-1}^{1}h^{(1)}(u)\frac{1}{(-1)^{m-1}2^{m-1}(m-1)!}\frac{d^{m-1}}{du^{m-1}}[(1-u)^{(\alpha+1)+(m-1)}(1+u)^{(\beta+1)+(m-1)}]du$$

Now the Rodrigues formula A.1.7.1 may be applied to A.2.12.6 to produce

$$\int_{-1}^{1}(1-u)^{\alpha}(1+u)^{\beta}h(u)P_m^{\alpha,\beta}(u)du \qquad \text{A.2.12.7}$$

$$= \frac{1}{2m}\int_{-1}^{1}(1-u)^{\alpha+1}(1+u)^{\beta+1}h^{(1)}(u)P_{m-1}^{\alpha+1,\beta+1}(u)du.$$

An easy induction now shows that A.2.9.1 holds, which completes the proof of Theorem A.2.9.

Corollary A.2.10 follows at once since $P_0^{\alpha,\beta}(u) \equiv 1$, for all $\alpha,\beta > -1$. Corollary A.2.11 is immediate.

## APPENDIX B

### The JLIB Subroutine Library

The JLIB subroutine library (KAPENGA 85) is a set of FORTRAN subroutines that were produced to aid in the application of the methods advocated in this work. They served as a check of the validity of many of the theorems and statements made in this work.

Some of the subroutines in the JLIB library call functions from the NAG, QUADPACK and LIBRARY libraries. The calls to NAG are for special function values, such as the gamma and beta functions. The calls to QUADPACK are used to generate starting values for the recurrence relations and high accuracy single values for the Jacobi moments. The calls to LINPACK are only in some routines which attempt to solve three term recurrence relations using mixed and conditions (those routines are not central to the problems involving Jacobi moments.

The JLIB library currently includes a number of routines that are direct results of this work. Some of these routines are:

JMIJU1 A HIGH ACCURACY ROUTINE FOR A SINGLE JACOBI MOMENT.

JMIJNM A ROUTINE THAT A ROW OF JACOBI MOMENTS USING THE THREE TERM RECURRENCE RELATION.

CMIJOM A ROUTINE THAT CALCULATES A JACOBI-FOURIER COEFFICIENTS FROM THE POWER SERIES COEFFICIENTS USING SALZER'S METHOD.

JMIJMN A ROUTINE THAT CALCULATES AN ARRAY OF JACOBI MOMENTS USING THE TWO DIMENSIONAL RECURRENCE RELATION.

CKSUM A ROUTINE THAT CALCULATES A PARTIAL CESARO SUM OF A SERIES.

HG01 A ROUTINE THAT RETURNS VALUES TO BE USED BY HG02 TO EVALUATE A FUNCTION GIVEN BY IT'S POWER SERIES. HG02 A ROUTINE THAT EVALUATES A FUNCTION USING THE VALUES GENERATED BY HG01.

In general these routines attempt to provide error bounds and stable results. There are also a few programs in JLIB that allow display of results using the NAG GRAPHICS LIBRARY.

## APPENDIX C

### Recurrence Relations

This is a small collection of Basic Results on linear recurrence relations. The text Milne-Thomson [87] together with the fundamental papers of Oliver [91] and [92] must be considered basic references. The recent text by J. Wimp [127] gives very good coverage of some of the work done since the 1960's.

### C.1 Definition

Let $\sigma$ be a fixed positive integer. Suppose $y: Z^0 \to C$, where $Z^0 = \{0, 1, 2, \cdots\}$ and $C$ is the complex plane. The operator

$$\mathcal{L}(y(n)) = \sum_{\gamma=0}^{\sigma} A_\gamma(n)\, y(n+\gamma), \qquad \text{C.1.1}$$

where $\{A_\gamma(n): \gamma = 0, 1, 2, \cdots, \sigma$ and $n = 0, 1, \cdots\}$ is a set in $C$ such that

$$A_0(n) \cdot A_\sigma(n) \neq 0, \quad \text{for } n = 0, 1, 2, \cdots .$$

is called a linear recurrence operator of order $\sigma$.

The equation

$$\mathcal{L}(y(n)) = f(n), \quad \text{for } n \geq 0, \qquad \text{C.1.2}$$

is called a linear recurrence relation of order $\sigma$.

The recurrence relation

$$\mathcal{L}(y(n)) = 0, \quad \text{for} \quad n \geq 0, \qquad \text{C.1.3}$$

is called a homogeneous recurrence relation.

C.2. Definition

A set of functions;

$$y_k\text{:} \quad Z^0 \to C, \quad \text{for} \quad k = 1, 2, \cdots \delta;$$

is called linearly dependent if and only if there exist complex constants $c_1, c_2, \cdots, c_\delta$; not all zero; such that

$$c_1 \, y_1(n) + c_2 \, y_2(n) + \cdots + c_\delta \, y_\delta(n) = 0, \qquad \text{C.2.1}$$

$$\text{for} \quad n = 0, 1, \cdots \quad .$$

Most stability analysis techniques for linear recurrence relations are based on the assumption that the limiting values of $A_y(n)$, as $n \to \infty$, in R.1.1 are known. Then the well developed theory of recurrence relations with constant coefficients may be used. Of course any such analysis is only of practical value if the limiting behaviour is fairly quick and no degenerate cases occur on the way.

The next two definition makes clear which type of recurrence relation this type of analysis may be applied to.

### C.3 Definition

Suppose that for each function $A_\gamma$ in R.1.32 the limit

$$\lim_{n\to\infty} A_\gamma(n) = \mu_\gamma \qquad \text{C.3.1}$$

exists and $\mu_0 \neq 0$. Then the recurrence relation R.1.3 is said to be of Poincare type with characteristic equation

$$\sum_{\gamma=0}^{\sigma} \mu_\gamma \lambda^\gamma = 0 \qquad \text{C.3.2}$$

### C.4 Definition

Suppose that in R.1.3 we have

$$A_0(n) = 1, \quad \text{for} \quad n = 0, 1, \cdots \;;$$

$$A_\delta(n) \neq 0, \quad \text{for} \quad n = 0, 1, \cdots \;; \qquad \text{C.4.1}$$

$$\delta \geq 2$$

and each $A_\gamma(n)$ has the following type of asymptotic representation

$$A_\gamma(n) \sim n^{k_\gamma/w}[a_{0,\gamma} + a_{1,\gamma}n^{-1/w} + a_{2,\gamma}n^{-2/w} + \cdots ] \qquad \text{C.4.2}$$

$$, \text{ as } n \to \infty \;;$$

where $k_\gamma$ is an integer, $w$ is an integer $w \geq 1$ and $a_{0,\gamma} \neq 0$ unless $A_\gamma(n) \equiv 0$.

We then say the recurrence relation R.1.3 is of Birkhoff type.

Note that all recurrence relations with coefficients which are rational in $n$ are of Birkhoff type.

We now list a few basic existence and uniqueness theorems.

C.5 Theorem

For any fixed constants $\{b_\gamma\}_{\gamma=0}^{\delta-1}$ and $k \geq 0$ the recurrence relation R.1.2 has a unique solution $y$ such that

$$y(\gamma + k) = b_\gamma, \quad \text{for} \quad \gamma = 0, 1, \cdots, \delta - 1. \qquad \text{C.5.1}$$

C.6 Theorem

The homogeneous recurrence relation R.1.3 has a set of $\delta$ linearly independent solutions $\{y_k\}_{k=1}^{\delta}$. Furthermore any solution $y$ to the recurrence relation may be written as

$$y(n) = \sum_{\gamma=1}^{\delta} c_\gamma \, y_\gamma \ (n).$$

for some constants $\{c_\gamma\}_{\gamma=1}^{\delta}$. Furthermore these constants are unique.

The set of solutions in the last theorem is called a fundamental set of the recurrence relation.

C.7 Theorem

If $Y_p$ is a solution of R.1.2 and $\{y_\gamma\}_{\gamma=1}^{\delta}$ is a fundamental set of the corresponding homogenous recurrence relation, then any other solution $y$ of R.1.2 may be written as:

$$y(n) = \sum_{\gamma=1}^{\delta} c_\gamma y_\gamma(n) + Y_p(n), \qquad \text{C.7.1}$$

for some constants $\{c_\gamma\}_{\gamma=1}^{\delta}$. Furthermore these constants are unique.

C.8 Definition

Let $t$ be a fixed integer, $t \geq 0$,

$$Q(\rho,n) = \mu_0\, n\, Ln(n) + \sum_{i=0}^{\rho} \mu_i\, n^{(\rho+1-i)/\rho}, \qquad \text{C.8.1}$$

$$S(\rho,n) = n^{\theta} \sum_{i=0}^{t} (Ln(n))^{i}\, n^{r_{t-i}/\rho}\, q_i(\rho,n) \qquad \text{C.8.2.}$$

and

$$q_i(\rho,n) = \sum_{s=0}^{\infty} b_{s,i}\, n^{-s/\rho}; \qquad \text{C.8.3}$$

<u>where</u> $\rho$, $r_i$ and $\mu_0\rho$ <u>are</u> <u>integers</u>; C.8.4

$\rho \geq 1$;

$\mu_i$, $\theta$ <u>and</u> $b_{s,i}$ <u>are</u> <u>complex</u>;

$b_{0,i} \neq 0$ <u>unless</u> $b_{s,i} = 0$ <u>for</u> <u>all</u> $s$;

$r_0 = 0$ <u>and</u>

$- \pi \leq \mathrm{Im}(\mu_1) < \pi$.

<u>Then</u> <u>the</u> <u>series</u>

$$e^{Q(\rho,n)}S(\rho,n) \qquad \text{C.8.5}$$

<u>is</u> <u>called</u> <u>a</u> <u>formal</u> <u>series</u> (fs).

<u>The</u> <u>formal</u> <u>series</u> R.8.5 <u>is</u> <u>said</u> <u>to</u> <u>be</u> <u>a</u> <u>formal</u> <u>solution</u> <u>of</u> <u>a</u> <u>Birkhoff</u> <u>type</u> <u>recurrence</u> <u>relation</u> <u>if</u> <u>it</u> <u>may</u> <u>be</u> <u>formally</u> <u>substituted</u> <u>into</u> <u>the</u> <u>equation</u> <u>and</u> <u>the</u> <u>coefficients</u> <u>of</u> <u>the</u> <u>quantities</u> $n^{\theta+(r/e)+(s/w)}(\mathrm{Ln}(n))^j$, <u>for</u> $j = 0, 1, \cdots, t$ <u>and</u> $r,s = 0, \pm 1, \pm 2, \cdots$, <u>are</u> <u>all</u> <u>equal</u> <u>to</u> <u>zero</u>.

The formal equality of two formal series may be defined in a similar manner.

We now define what it means for a formal series to be an asymptotic representation for a function.

## C.9 Definition

Let $\{f(n)\}_{n=0}^{\infty}$ be a complex sequence;

then

$$f(n) \sim \rho^{Q(\rho,n)} s(\rho,n), \quad \text{as} \quad n \to \infty, \qquad \text{C.9.1}$$

means that for every $k \geq 1$ we can determine functions $A_{k,i}(n)$, for $i = 0, 1, \cdots, t$; such that

$$e^{-Q(\rho,n)} n^{-\theta} f(n) = \sum_{i=0}^{t} (\mathrm{Ln}(n))^{i} n^{r_{t-i}/\rho} \sum_{s=0}^{k-1} b_{s,i} n^{-s/\rho} \qquad \text{C.9.2}$$

$$+ n^{-k/\rho} \sum_{i=0}^{t} (\mathrm{Ln}(n))^{j} n^{r_{t-i}/\rho} A_{k,i}(n),$$

where $|A_{k,i}^{(n)}|$ is bounded as $n \to \infty$.

The representation of R.9.1 can be seen to be unique.

## C.10 Definition

Let $W_k$ denote the determinant with entries

$$e^{Q_i(\rho,n+j)} S_i(\pi,n+j), \quad \text{for} \quad 1 \leq i,\ j \leq k \qquad \text{C.10.1}$$

we can write

$$W_k = e^{\sum_{i=1}^{n} Q_i(\rho,n)} \overline{S}(\rho,n). \qquad \text{C.10.2}$$

<u>We say the</u> k <u>formal series</u>

$$\{e^{Q_i(\rho,n)} s_i(\rho,n)\} \qquad \text{C.10.3}$$

<u>are formally linearly independent if</u> $\bar{s}(n) \neq 0$, <u>otherwise they are formally dependent</u>.

<u>C.11 Theorem (Birkhoff - Trjitzinsky)</u>

<u>Every homogeneous linear recurrence relation of Birkhoff type has</u> $\delta$ <u>formally linearly independent formal solutions. Each of the</u> $\delta$ <u>formally independent solutions represents an actual solution asymptotically. The</u> $\delta$ <u>actual solutions so represented make up a fundamental set for the recurrence relation</u>.

The fundamental sets in the last theorem are called Birkhoff sets. The solutions and series are called Birkhoff solutions and Birkhoff series respectively.

The proof of the last theorem and examples of the use of Birkhoff series may be found in [8] and [9].

We will now give two definitions of stability for the recurrence relation C.1.2. The first definition, for absolute stability, is standard fare. The more interesting, and practical, definition of relative stability follows Gavtschi see [127]).

C.12 DEFINITION

*The linear recurrence relation* C.1.2 *is said to be stable in the absolute sense in the forward direction if for all the roots of the characteristic equation* C.3.2, *denoted by* $\{\partial_i : i = 1, \cdots, \sigma\}$ *we have*

$$|\partial_i| < 1, \quad \text{for} \quad i = 1, 2, \cdots, \sigma. \qquad \text{C.12.1}$$

*If* C.12.1 *is replaced by*

$$|\partial_i| \le 1, \quad \text{for} \quad i = 1, 2, \cdots, \sigma \qquad \text{C.12.2}$$

*the* C.1.2 *is said to be weakly stable in the absolute sense in the forward direction.*

We note that the $\sigma$ functions $\{y_k\}_{k=1}^{\sigma}$, which are solutions to C.1.3, are called complementary solutions to C.1.2.

C.1.13 DEFINITION

*Let* w *be a solution to* C.1.2 *and* $\{y_k\}_{k=1}^{\sigma}$ *be a set of complementary solutions* C.1.2. *Denote*:

$$u_k(n) = [y_k(n), y_k(n+1), \cdots, y_k(n+\sigma-1)]^T$$
$$, \text{ for } k = 1, 2, \cdots, \sigma \text{ and } n = 0, 1, 2, \cdots, .$$

$$U(n) = [u_1(n), u_2(n), \cdots, u_\sigma(n)]$$
$$, \text{ for } n = 0, 1, \cdots ;$$

$$V(n) = [w(n),\ w(n+1),\ \cdots,\ w(n+\sigma-1)]^T$$

$$, \quad \text{for} \quad n = 0,\ 1,\ 2,\ \cdots \quad .$$

We now define

$$\alpha(n,k) = \frac{\|V(k)\|}{\|V(n)\|}\ \|U(n)U^{-1}(k)\|$$

$$, \quad \text{for} \quad 0 \le k < n.$$

We remark that $\alpha(n,k)$ is independent of the choice of the complementary solutions chosen and is used in a manner independent of the norms chosen.

C.14 DEFINITION

The recurrence relation C.1.2 is said to be stable in the relative sense for the computation of $w(n)$ in the forward direction if

$$\sup_{0<k<n} \alpha(n,k) < \infty. \qquad \text{C.14.1}$$

The relation is said to be weakly stable in the relative sense for the computation of $w(n)$ in the forward direction if for each non-negative integer $K$.

$$\sup_{\substack{k<n \\ 0<k\le K}} \alpha(n,k) < \infty. \qquad \text{C.14.2}$$

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